Name: 　　　　　　　　　　 **NetId:**

Probability Limit Theorems

Final Exam, Fall 2021

DO NOT OPEN YET

...and wait until the proctor announces that it is time to start.

In the mean time, please write your name and NetID legibly,
and **read the instructions below carefully**.

* Please do not fold or damage the exam papers. After you finish your exam, please put the pages in correct order back into the sleeve.

* There are 6 questions in this exam, the sleeve should contain 8 pieces of paper.

* The scratch paper is included: three last papers are blank. If your solution continues on scratch paper, please clearly indicate it.

* For questions asking to prove a result, the clarity of the mathematical argument will be taken into account in the score.

* Questions formulated in terms of real functions should be answered with real functions.

Good luck!

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let X be a real-valued random variable, $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables in this probability space.

   (Throughout this exercise, you may use elementary properties of measures and measurable functions without proof provided they are clearly stated.)

   (a) i. What does it mean to say $(X_n)_{n\in\mathbb{N}}$ converges to X almost surely ? What does it mean to say X_n converges to X in probability ?

   ii. Show, using Reverse Fatou's Lemma, that if $(X_n)_{n\in\mathbb{N}}$ converges to X almost surely, then $(X_n)_{n\in\mathbb{N}}$ converges to X in probability.

   (b) i. What does it mean to say that random variables $X_n, n \in \mathbb{N}$ are independent ? State the second Borel-Cantelli Lemma on $(\Omega, \mathcal{F}, \mathbb{P})$. [No proof is required.]

   ii. Show, using the first Borel-Cantelli Lemma that if X_n converges to X in probability, then there exists a strictly increasing sequence $n_k \xrightarrow{k\to\infty} \infty$ such that $(X_{n_k})_{k\in\mathbb{N}}$ converges to X almost surely.

   iii. A sequence of real-valued random variables $(X_n)_{n\in\mathbb{N}}$ is said to be completely convergent if

$$\sum_{n=1}^{\infty} \mathbb{P}[|X_n - X| > \epsilon] < \infty,$$

   for all $\epsilon > 0$. Show that for sequence of independent random variables, complete convergence is equivalent to almost surely convergence.

   iv. Find a sequence of (dependent) random variables which converge almost surely but not completely.
   [Hint: you may choose $X_n = a_n X$ for an adequate random variable X and real sequence $(a_n)_{n\in\mathbb{N}}$.]

Solution:

(a) i. $X_n \xrightarrow{n\to\infty} X$ almost surely $\iff \mathbb{P}[\limsup_{n\to\infty} |X_n - X| \neq 0] = 0$.
$X_n \xrightarrow{\mathbb{P}} X \iff$ for any $\epsilon > 0$, $\limsup_{n\to\infty} \mathbb{P}[|X_n - X| \geq \epsilon] = 0$.

ii. Using Reverse Fatou's Lemma, for any $\epsilon > 0$, we have

$$\mathbb{P}[\limsup_{n\to\infty}\{|X_n - X| \geq \epsilon\}] \geq \limsup_{n\to\infty} \mathbb{P}[\{|X_n - X| \geq \epsilon\}].$$

Notice the fact that $X_n \xrightarrow{n\to\infty} X$ almost surely, we have

$$\mathbb{P}[\limsup_{n\to\infty}\{|X_n - X| \geq \epsilon\}] = 0,$$

which implies that

$$\limsup_{n\to\infty} \mathbb{P}[\{|X_n - X| \geq \epsilon\}] \leq 0.$$

This is the convergence in probability.

(b) i. The random variables $(X_n)_{n\in\mathbb{N}}$ are independent iff $\sigma(X_n)_{n\in\mathbb{N}}$ are independent σ-algebra.
Second Borel-Cantelli Lemma: for independent events $(A_n)_{n\in\mathbb{N}}$, if $\sum_{n=1}^{\infty} \mathbb{P}[A_n] = \infty$, then $\mathbb{P}[i.o. A_n] = 1$.

ii. $X_n \xrightarrow{\mathbb{P}} X$ implies that for all $k \in \mathbb{N}$, there exists an increasing n_k such that

$$\mathbb{P}[|X_{n_k} - X| \geq 2^{-k}] \leq 2^{-k}.$$

Then $\sum_{k=1}^{\infty} \mathbb{P}[|X_{n_k} - X| \geq 2^{-k}] < \infty$ implies that

$$\mathbb{P}[i.o.\{|X_{n_k} - X| \geq 2^{-k}\}] = 0.$$

Then $X_{n_k} \to X$ almost surely.

iii. Completely convergence is a strong notation as for a completely convergent sequence $(X_n)_{n\in\mathbb{N}}$

$$\forall \epsilon > 0, \qquad \sum_{n=1}^{\infty} \mathbb{P}[|X_n - X| \geq \epsilon] < \infty,$$

then by the first Borel-Cantelli Lemma

$$\mathbb{P}[i.o.\{|X_n - X| \geq \epsilon\}] = 0,$$

which implies the almost surely convergence.
On the other hand, if $(X_n)_{n\in\mathbb{N}}$ are independent and they are not completely convergent, then there exists $\epsilon > 0$, such that

$$\sum_{n=1}^{\infty} \mathbb{P}[|X_n - X| \geq \epsilon] = \infty.$$

Using the second Borel-Cantelli Lemma

$$\mathbb{P}[i.o.\{|X_n - X| \geq \epsilon\}] = 1,$$

and then it does not have almost surely convergence.

iv. Take a positive random variable $\mathbb{E}[X] = \infty$ and let $X_n = \left(1 + \frac{1}{n}\right) X$. Then pointwisesly, we have $X_n \xrightarrow{n \to \infty} X$. However, for any $\epsilon > 0$

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}[|X_n - X| \geq \epsilon] &= \sum_{n=1}^{\infty} \mathbb{P}\left[\frac{1}{n} X \geq \epsilon\right] \\
&= \sum_{n=1}^{\infty} \mathbb{P}\left[X \geq n\epsilon\right] \\
&\simeq \frac{1}{\epsilon}\mathbb{E}[X] = \infty.
\end{aligned}$$

This is a desired example of almost surely convergence but no completely convergence.

2. (a) Let Z be random variable of law Poisson(λ), i.e.

$$\forall k \in \mathbb{N}, \qquad \mathbb{P}[Z = k] = e^{-\lambda}\frac{\lambda^k}{k!}. \tag{1}$$

Calculate $\mathbb{P}[Z = 0]$, $\mathbb{P}[Z = 1]$ and $\mathbb{E}[Z]$.

(b) Compute the characteristic function of Z.

(c) State Lévy's convergence theorem.

(d) Let $(X_{n,i})_{n,i\in\mathbb{N}}$ be independent Bernoulli random variables with

$$\mathbb{P}[X_{n,i} = 1] = p_{n,i}, \qquad \mathbb{P}[X_{n,i} = 0] = 1 - p_{n,i}.$$

Suppose that

- $\sum_{i=1}^n p_{n,i} \xrightarrow{n\to\infty} \lambda \in (0, \infty)$;
- $\max_{1\le i\le n} p_{n,i} \xrightarrow{n\to\infty} 0$.

Let $S_n := \sum_{i=1}^n X_{n,i}$, prove that $S_n \xrightarrow[n\to\infty]{w} Z$, where Z has a law Poisson(λ) defined in (1).

Solution:

(a) $\mathbb{P}[Z = 0] = e^{-\lambda}$, $\mathbb{P}[Z = 1] = \lambda e^{-\lambda}$. The expectation is

$$\begin{aligned}
\mathbb{E}[Z] &= \sum_{k=0}^{\infty} \mathbb{P}[Z = k]k \\
&= \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} \times k \\
&= \lambda \sum_{k=1}^{\infty} e^{-\lambda}\frac{\lambda^{(k-1)}}{(k-1)!} \\
&= \lambda.
\end{aligned}$$

(b)

$$\begin{aligned}
\phi_Z(\theta) &= \mathbb{E}[\exp(i\theta Z)] \\
&= \sum_{k=0}^{\infty} \mathbb{P}[Z = k]\exp(ik\theta) \\
&= \sum_{k=0}^{\infty} e^{-\lambda}\frac{(e^{i\theta}\lambda)^k}{k!} \\
&= \exp(\lambda(e^{i\theta} - 1)).
\end{aligned}$$

(c) Lévy's convergence theorem: for a sequence of random variables X_n, if their characteristic functions ϕ_{X_n} converges pointwise to a function ϕ, which is continuous at 0, then there exists a random variable X such that $X_n \xrightarrow{w} X$, and ϕ is the characteristic function of X.

(d) We calculate the characteristic function of S_n

$$\begin{aligned}\phi_{S_n}(\theta) &= \mathbb{E}[\exp(i\theta S_n)] \\ &= \mathbb{E}[\exp(i\theta(\sum_{j=1}^{n} X_{n,j}))] \\ &= \prod_{j=1}^{n} \mathbb{E}[\exp(i\theta X_{n,j})] \\ &= \prod_{j=1}^{n}(1 + p_{n,j}(e^{i\theta} - 1)).\end{aligned}$$

Observing that

$$1 + p_{n,j}(e^{i\theta} - 1) = \exp(\ln(1 + p_{n,j}(e^{i\theta} - 1))) = \exp(p_{n,j}(e^{i\theta} - 1) + O((p_{n,j})^2)),$$

which implies that

$$\phi_{S_n} = \exp\left(\sum_{j=1}^{n} \left(p_{n,j}(e^{i\theta} - 1) + O((p_{n,j})^2)\right)\right).$$

We take a limit and with the conditions $\sum_{i=1}^{n} p_{n,i} \xrightarrow{n\to\infty} \lambda$ and $\max_{1\le i\le n} p_{n,i} \xrightarrow{n\to\infty} 0$,

$$\begin{aligned}&\sum_{j=1}^{n} p_{n,j}(e^{i\theta} - 1) \xrightarrow{n\to\infty} \lambda(e^{i\theta} - 1), \\ &\sum_{j=1}^{n} O((p_{n,j})^2)) \le O\left(\max_{1\le j\le n} p_{n,j}(\sum_{j=1}^{n} p_{n,j})\right) \xrightarrow{n\to\infty} 0.\end{aligned}$$

Therefore, we have proved that

$$\lim_{n\to\infty} \phi_{S_n}(\theta) = \exp(\lambda(e^{i\theta} - 1)),$$

which is the characteristic function of Z. We invoke Lévy's convergence theorem and proves $S_n \xrightarrow{w} Z$.

3. Let X_n be a sequence of independent real random variables which converges in probability to the limit X. Show that X is almost surely constant.

   Solution: We can extract a subsequence X_{n_k} such that $X_{n_k} \xrightarrow{k\to\infty} X$ almost surely. We observe that the event

$$\{X \le s\} = \left\{\lim_{k\to\infty} X_{n_k} \le s\right\},$$

   is a tail event, then by Kolmogorov's zero-one theorem $\mathbb{P}[X \le s] \in \{0, 1\}$. Then we look at the distribution function of X and define

$$a = \sup_{s\in\mathbb{R}}\{s : \mathbb{P}[X \le s] = 0\},$$

   which is well-defined as $s \mapsto \mathbb{P}[X \le s]$ is increasing. By the definition, we notice for $s < a, \mathbb{P}[X \le s] = 0$ and for $s \ge a, \mathbb{P}[X \le s] = 1$, which implies $X = a$ almost surely.

4. Let X be an L^1 real random variable, and for $\delta > 0$, set

$$I_X(\delta) = \sup\{\mathbb{E}[|X|\mathbf{1}_A] : A \in \mathcal{F}, \mathbb{P}[A] \leq \delta\}.$$

Using Dominated Convergence Theorem, show that

$$\lim_{\delta \to 0} I_X(\delta) = 0.$$

Solution: Suppose, by contradiction, that there exists $\epsilon > 0$ such that, for every $n \geq 1$, there exists $A_n \in \mathcal{F}$ such that $\mathbb{P}(A_n) \leq 2^{-n}$ and

$$\mathbb{E}(|X|\mathbf{1}_{A_n}) > \epsilon.$$

Since

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty,$$

by the First Borel-Cantelli Lemma, $\mathbb{P}(A_n i.o.) = 0$. However, since

$$\lim_{m \to \infty} \left[\bigcup_{n=m}^{\infty} A_n \right] = [A_n i.o.]$$

and $X \in L^1$, by Dominated Convergence Theorem,

$$\epsilon \leq \limsup_{m \to \infty} \mathbb{E}\left[|X|\mathbf{1}_{\bigcup_{n=m}^{\infty} A_n}\right] = \mathbb{E}\left[|X|\mathbf{1}_{[A_n i.o.]}\right] = 0,$$

a contradiction.

5. Let X_n be i.i.d. random variables such that

$$\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = \frac{1}{2},$$

and $S_n := \sum_{i=1}^n X_i$. We also define the natural filtration $\mathcal{F}_n = \sigma((X_i)_{1\leq i \leq n})$ and a quantity that

$$\forall a \in \mathbb{Z}_+, \tau_a = \min\{n \geq 0 : S_n = -a\}.$$

(a) Check that τ_a is a stopping time with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

(b) Check that for any $\theta \in \mathbb{R}$, $Y_n := \exp(\theta S_n)\left(\frac{e^\theta + e^{-\theta}}{2}\right)^{-n}$ is a martingale.

(c) Prove that, when $\theta < 0$, the martingale $(Y_{n\wedge\tau_a})_{n\in\mathbb{N}}$ is bounded.

(d) Define

$$A_+ = \left\{\limsup_{n\to\infty} S_n = +\infty\right\}, \qquad A_- = \left\{\liminf_{n\to\infty} S_n = -\infty\right\}.$$

i. Prove that $\mathbb{P}[A_+], \mathbb{P}[A_-] \in \{0, 1\}$.

ii. Show, using the Central Limit Theorem that, $\mathbb{P}[A_+ \cup A_-] = 1$.

iii. Conclude that, for all $a \in \mathbb{Z}_+$, $\mathbb{P}[\tau_a < \infty] = 1$.

(e) Prove that, for every $s \in (0, 1)$, one has

$$\mathbb{E}[s^{\tau_a}] = \left(\frac{1 - \sqrt{1 - s^2}}{s}\right)^a.$$

(f) For $a = 1$, use the formula above to compute explicitly the probabilities $\mathbb{P}[\tau_a = 2k - 1]$ for $k \geq 1$.

Solution:

(a) From the definition, it is clear S_n is $\mathcal{F}_n$-measurable. Then as every step is of size 1

$$\{\tau_a \le n\} = \cup_{i=1}^n \{S_i = -a\},$$

which is also $\mathcal{F}_n$-measurable. Therefore, it is a stopping time with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

(b) We calculate the conditional expectation

$$\begin{aligned}
\mathbb{E}[Y_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[\exp(\theta S_{n+1})\left(\frac{e^\theta + e^{-\theta}}{2}\right)^{-(n+1)}\Big|\mathcal{F}_n\right] \\
&= \exp(\theta S_n)\left(\frac{e^\theta + e^{-\theta}}{2}\right)^{-(n+1)}\mathbb{E}\left[\exp(\theta X_{n+1})\Big|\mathcal{F}_n\right] \\
&= \exp(\theta S_n)\left(\frac{e^\theta + e^{-\theta}}{2}\right)^{-(n+1)} \times \left(\frac{e^\theta + e^{-\theta}}{2}\right) \\
&= Y_n.
\end{aligned}$$

Here from the first line to the second line, we use the fact that S_n is $\mathcal{F}_n$-measurable, and from the second line to the third line, we apply the independence. This proves that $(Y_n)_{n\in\mathbb{N}}$ is a martingale.

(c) Notice that $S_{n\wedge\tau_a} \ge -a$, thus $\theta S_{n\wedge\tau_a} \le -\theta a$. Moreover, we have

$$\frac{e^\theta + e^{-\theta}}{2} \ge 1,$$

which implies that $\left(\frac{e^\theta+e^{-\theta}}{2}\right)^{-n} \le 1$ and $Y_{n\wedge\tau_a} \in (0, \exp(-\theta a))$ is bounded.

(d) i. A_+ and A_- are all tail events. By Kolmogorov 0-1 law, $\mathbb{P}[A_+], \mathbb{P}[A_-] \in \{0,1\}$.

ii. We have

$$\mathbb{P}[A_- \cup A_+] = \lim_{K\to\infty} \mathbb{P}\left[\limsup_{n\to\infty} |S_n| \ge K\right].$$

Using Reverse Fatou's Lemma and the Central Limit Theorem, we have

$$\begin{aligned}
\mathbb{P}\left[\limsup_{n\to\infty} |S_n| \ge K\right] &\ge \limsup_{n\to\infty} \mathbb{P}\left[|S_n| \ge K\right] \\
&= \limsup_{n\to\infty} \mathbb{P}\left[\frac{|S_n|}{\sqrt{n}} \ge \frac{K}{\sqrt{n}}\right] \\
&\ge \limsup_{n\to\infty} \mathbb{P}\left[\frac{|S_n|}{\sqrt{n}} \ge \epsilon\right] \\
&= \mathbb{P}[|N| \ge \epsilon].
\end{aligned}$$

Here ϵ is a arbitrary positive number. This proves that

$$\mathbb{P}\left[\limsup_{n\to\infty} |S_n| \ge K\right] = 1,$$

so we have $\mathbb{P}[A_- \cup A_+] = 1$.

iii. By the symmetry of X_n, we have $\mathbb{P}[A_+] = \mathbb{P}[A_-]$. From Question d(i), if they are all 0, it contradicts d(ii). Therefore, we have $\mathbb{P}[A_+] = \mathbb{P}[A_-] = 1$, and this implies that for all $a \in \mathbb{Z}_+$, $\mathbb{P}[\tau_a < \infty] = 1$

(e) Now, since $(Y_{n\wedge\tau_a})_{n\in\mathbb{N}}$ is bounded martingale, $n \wedge \tau_a \leq n$, we can apply optional stopping time theorem

$$\mathbb{E}[Y_{n\wedge\tau_a}] = \mathbb{E}[Y_0] = 1.$$

$Y_{n\wedge\tau_a}$ is bounded and τ_a is almost surely finite, thus

$$\lim_{n\to\infty} Y_{n\wedge\tau_a} = Y_{\tau_a}, \qquad \text{almost surely .}$$

Using the Dominated Convergence Theorem, we have

$$1 = \lim_{n\to\infty} \mathbb{E}[Y_{n\wedge\tau_a}] = \mathbb{E}[Y_{\tau_a}] = \mathbb{E}\left[\left(\frac{e^{\theta}+e^{-\theta}}{2}\right)^{-\tau_a} e^{-\theta a}\right].$$

By a change of variable $s = \left(\frac{e^{\theta}+e^{-\theta}}{2}\right)^{-1}$, we obtain the desried result.

(f) We develop the equation $\mathbb{E}[s^{\tau_1}] = \left(\frac{1-\sqrt{1-s^2}}{s}\right)$ in series of s on the two sides. On the left-hand side, we calculate that

$$\mathbb{E}[s^{\tau_1}] = \sum_{n=0}^{\infty} \mathbb{P}[\tau_1 = n] s^n.$$

On the right-hand side, we develop

$$\sqrt{1-s^2} = 1 - \sum_{k=1}^{\infty} \frac{1}{k}\binom{2k-2}{k-1} 2^{-(2k-1)} s^{2k},$$

thus

$$\frac{1-\sqrt{1-s^2}}{s} = \sum_{k=1}^{\infty} \frac{1}{k}\binom{2k-2}{k-1} 2^{-(2k-1)} s^{2k-1}.$$

We compare the two equations, and conclude that

$$\mathbb{P}[\tau_1 = 2k-1] = \frac{1}{k}\binom{2k-2}{k-1} 2^{-(2k-1)}.$$

(Remark: $\frac{1}{k+1}\binom{2k}{k}$ is known as Catalan number.)

6. Assume that (f_n), (g_n), f, $g \in L^1(\mathbb{R})$, $f_n \to f$ a.s., $g_n \to g$ a.s., $|f_n| \leq g_n$ a.s. and

$$\int_{\mathbb{R}} g_n d\mu \to \int_{\mathbb{R}} g d\mu.$$

Show that

$$\int_{\mathbb{R}} f_n d\mu \to \int_{\mathbb{R}} f d\mu.$$

Solution: Since $|f_n - f| \leq g_n + |f|$, then $g_n + |f| - |f_n - f| \geq 0$. By Fatou's Lemma,

$$\begin{aligned}
\int_{\mathbb{R}^n} g d\mu + \int_{\mathbb{R}^n} |f| d\mu &= \int_{\mathbb{R}^n} (g + |f|) d\mu \\
&= \int_{\mathbb{R}^n} \liminf_{n\to\infty} (g_n + |f| - |f_n - f|) d\mu \\
&\leq \liminf_{n\to\infty} \int_{\mathbb{R}^n} (g_n + |f| - |f_n - f|) d\mu \\
&= \int_{\mathbb{R}^n} g d\mu + \int_{\mathbb{R}^n} |f| d\mu - \limsup_{n\to\infty} \int_{\mathbb{R}^n} |f_n - f| d\mu.
\end{aligned}$$

This implies

$$\limsup_{n\to\infty} \int_{\mathbb{R}^n} |f_n - f| d\mu \leq 0,$$

which guarantees

$$\int_{\mathbb{R}^n} |f_n - f| d\mu \to 0.$$