Name: ____________ **NetId:** ____________

Honors Ordinary Differential Equations

Final Exam, Fall 2021

DO NOT OPEN YET

...and wait until the proctor announces that it is time to start.

In the mean time, please write your name and NetID legibly,
and **read the instructions below carefully**.

* Please do not fold or damage the exam papers. After you finish your exam, please put the pages in correct order back into the sleeve.

* There are 5 questions in this exam, the sleeve should contain 8 pieces of paper.

* The scratch paper is included: three last papers are blank. If your solution continues on scratch paper, please clearly indicate it.

* For questions asking to prove a result, the clarity of the mathematical argument will be taken into account in the score.

* Questions formulated in terms of real functions should be answered with real functions.

* Question marked with (†) is challenge.

Good luck!

1. Determine the order of the following differential equations, and whether they are: partial or ordinary, and linear or non-linear. If they are ordinary and first order, determine whether they are autonomous or not, and separable or not. If they are linear, determine whether they are homogeneous.

   (a) $y'' + (y')^2 = 0$.

   (b) $\frac{\partial}{\partial t} y = \frac{\partial^2}{\partial x^2} y + \frac{\partial^2}{\partial t^2} y$.

   (c) $y' = y^2 + y^3$.

   (d) $(t^4 + 1)y' + 100y = \sin(t)$.

Solution.

(a) Second order, ODE, non-linear.

(b) Second order, PDE. (First order on time, second order on space.)

(c) First order, ODE, non-linear, autonomous, separable.

(d) First order, ODE, linear, not autonomous, not separable, inhomogeneous.

2. In this question, we study an example of numerical approximation of solution to ODE. Consider the initial value problem

$$y' = 1 - t + y, \quad y(t_0) = y_0.$$

   (a) Give the solution $y(t)$ with exact expression.

   (b) Using the discrete approximation: setting step size $h > 0$ and $t_k := t_0 + kh$

$$y_k = (1+h)y_{k-1} + h - ht_{k-1}, \quad k = 1, 2, \cdots$$

   Show by induction that

$$y_n = (1+h)^n(y_0 - t_0) + t_n, \tag{1}$$

   for each positive integer n.

   (c) Consider a fixed $t > t_0$ and, for a given n choose $h = (t - t_0)/n$. Show that for y_n in (1) and $y(t)$ in Question (a), we have $y_n \to y(t)$ as $n \to \infty$.

Solution.

(a) From the equation, we have

$$\begin{aligned} &\frac{d}{dt}(y - t) = (y - t) \\ &\Rightarrow y(t) - t = (y_0 - t_0)e^{t-t_0} \\ &\Rightarrow y(t) = (y_0 - t_0)e^{t-t_0} + t. \end{aligned}$$

(b) From $y_k = (1+h)y_{k-1} + h - ht_{k-1}$, we obtain that

$$y_k - t_k = (1+h)(y_{k-1} - t_{k-1}),$$

which implies by induction that

$$y_n = (1+h)^n(y_0 - t_0) + t_n.$$

(c) Put $h = \frac{t-t_0}{n}$ into the expression of y_n that

$$y_n = \left(1 + \frac{t - t_0}{n}\right)^n (y_0 - t_0) + t \xrightarrow{n\to\infty} e^{t-t_0}(y_0 - t_0) + t = y(t),$$

where we use the limit $\lim_{n\to\infty}\left(1 + \frac{a}{n}\right)^n = e^a$.

3. Consider the differential equation

$$x^3 y'' + \alpha x y' + \beta y = 0, \tag{2}$$

where α and β are real constants and $\alpha \neq 0$. In this question, we attempt to find its solution of form $\sum_{n=0}^{\infty} a_n x^{r+n}$.

(a) Show that $x = 0$ is an irregular singular point.

(b) Using the formal series solution to write down (2) as

$$F(r) a_0 x^r + \sum_{n=1}^{\infty} c_n x^{r+n} = 0.$$

Express $F(r)$ in function of r, and c_n in function of $(a_n)_{n \in \mathbb{N}}, r, \alpha, \beta$.

(c) Show that $F(r) = 0$ only has one root, and consequently there is only one possible formal solution of the assumed form. Write down the recurrence of a_n under this condition.

(d) Show that if $\beta/\alpha \in \{-1, 0, 1, 2, ...\}$, then only finite terms in $(a_n)_{n \in \mathbb{N}}$ are non-zero, and therefore it is an actual solution.

(e) Show that if $\beta/\alpha \notin \{-1, 0, 1, 2, ...\}$, show that the formal series solution has a zero radius of convergence and so does not represent an actual solution in any interval.

Solution.

(a) Because $\lim_{x \to 0} \frac{\alpha x}{x^3} \times x = \infty$, 0 is not a regular point.

(b) Suppose that $y = \sum_{n=0}^{\infty} a_n x^{r+n}$, then we have

$$\begin{aligned}
x^3 y'' &= \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n+1}, \\
\alpha x y &= \sum_{n=0}^{\infty} \alpha (r+n) a_n x^{r+n}, \\
\beta y &= \sum_{n=0}^{\infty} \beta a_n x^{r+n}.
\end{aligned}$$

We put this equation into (2), and obtain

$$\begin{aligned}
F(r) &= \alpha r + \beta, \\
c_n &= (\alpha(r+n) + \beta) a_n + (r+n-1)(r+n-2) a_{n-1}.
\end{aligned}$$

(c) From the expression above, $F(r) = 0$ contains only one solution $r = -\frac{\beta}{\alpha}$. Then we use the expression of c_n to get the recurrence of a_n

$$a_n = -\frac{(-\beta/\alpha + n - 1)(-\beta/\alpha + n - 2)}{\alpha n} a_{n-1}, \qquad n \geq 1. \tag{3}$$

(d) From (3), if $\beta/\alpha \in \{-1, 0, 1, 2, ...\}$, there is necessarily one term n such that

$$-\beta/\alpha + n - 2 = 0.$$

Then from this term a_n becomes identically zero.

(e) Otherwise, if $\beta/\alpha \notin \{-1, 0, 1, 2, ...\}$, the fraction in (3) never vanishes, then we use ratio test

$$\limsup_{n\to\infty} \left|\frac{a_n}{a_{n-1}}\right| = \limsup_{n\to\infty} \left|\frac{(-\beta/\alpha + n - 1)(-\beta/\alpha + n - 2)}{\alpha n}\right| = \infty,$$

which implies that the radius of convergence is 0.

4. The convolution operator $\star$ on $\mathbb{R}_+$ is defined for two functions f, g as

$$\forall t \geq 0, \qquad (f \star g)(t) = \int_0^t f(t-u)g(u)\, du,$$

when this integral is well-defined.

(a) Prove that $f \star g = g \star f$.

(b) Let $\mathcal{L}$ be Laplace transform. Prove that $\mathcal{L}[f \star g] = \mathcal{L}[f]\mathcal{L}[g]$.

(c) We use Laplace transform and the convolution to study the following differential equation

$$y'' + ay' + by = f, \qquad y(0) = 0, y'(0) = 0,$$

where a and b are constants, while f is a bounded continuous function.

i. Give the expression of $\mathcal{L}[y](s)$ in function of $\mathcal{L}[f](s), a, b, s$.

ii. Show that y has the solution $y = w \star f$, where $w(t)$ is the solution of

$$w'' + aw' + bw = 0, \qquad w(0) = 0, w'(0) = 1.$$

[Hint: use (b) and the uniqueness theorem of Laplace transform.]

Solution.

(a) This is just a change of variable $v := t - u$

$$(f \star g)(t) = \int_0^t f(t-u)g(u)\, du = \int_0^t f(v)g(t-v)\, dv = (g \star f)(t).$$

(b) By a direct calculation

$$\begin{aligned}
\mathcal{L}[f \star g](s) &= \int_0^\infty \left(\int_0^t f(t-u)g(u)\, du \right) e^{-st}\, dt \\
&= \int_0^\infty \left(\int_0^t f(t-u)e^{-s(t-u)}g(u)e^{-su}\, du \right) dt \\
&= \int_0^\infty \left(\int_0^t f(t-u)e^{-s(t-u)}\, dt \right) g(u)e^{-su}\, du.
\end{aligned}$$

By a change of variable $v = t - u$, and we change the order of integration, we have

$$\begin{aligned}
\mathcal{L}[f \star g](s) &= \left(\int_0^\infty g(u)e^{-su}\, du \right) \left(\int_0^\infty f(v)e^{-sv}\, dv \right) \\
&= \mathcal{L}[f](s)\mathcal{L}[g](s).
\end{aligned}$$

(Remark: a very rigorous proof also requires the integrability of functions f, g, and here we skip the details but just suppose all the integrals are well-defined.)

(c) i. We apply the Laplace transform of derivative that

$$\begin{aligned}
\mathcal{L}[y'](s) &= s\mathcal{L}[y](s) - y(0) = s\mathcal{L}[y](s), \\
\mathcal{L}[y''](s) &= s^2\mathcal{L}[y](s) - sy(0) - y'(0) = s^2\mathcal{L}[y](s), \\
\Rightarrow \mathcal{L}[y'' + ay' + by](s) &= (s^2 + as + b)\mathcal{L}[y], \\
\Rightarrow \mathcal{L}[y] &= \frac{\mathcal{L}[f]}{(s^2 + as + b)}.
\end{aligned}$$

ii. By a similar calculation for w

$$\begin{aligned}
\mathcal{L}[w'](s) &= s\mathcal{L}[w](s) - w(0) = s\mathcal{L}[w](s), \\
\mathcal{L}[w''](s) &= s^2\mathcal{L}[w](s) - sw(0) - w'(0) = s^2\mathcal{L}[w](s) - 1, \\
\Rightarrow \mathcal{L}[w'' + aw' + wy](s) &= (s^2 + as + b)\mathcal{L}[w] - 1 = 0, \\
\Rightarrow \mathcal{L}[w] &= \frac{1}{(s^2 + as + b)}.
\end{aligned}$$

Combine this and the Laplace transform of y, we obtain that

$$\mathcal{L}[y] = \mathcal{L}[w]\mathcal{L}[f].$$

Moreover, from (b) we known that $\mathcal{L}[w \star f] = \mathcal{L}[w]\mathcal{L}[f]$. Using the uniqueness theorem of Laplace transform, it implies that $y = w \star f$.

5. Consider a differential equation

$$y^{(n)} = a_0 y + a_1 y^{(1)} + a_2 y^{(2)} + \cdots + a_{n-1} y^{(n-1)}, \tag{4}$$

where $a_1, a_2, \cdots a_{n-1}$ are continuous functions on interval $I \subset \mathbb{R}$.

(a) Show that the vector $\begin{pmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \cdots \\ y^{(n-1)} \end{pmatrix}$ is solution of an ODE of order 1.

(b) Deduce carefully there exists a unique solution to (4) with initial condition $y(t_0) = y_0$, $y^{(1)}(t_0) = y_0^{(1)}, \cdots, y^{(n-1)}(t_0) = y_0^{(n-1)}$.

(c) Under which assumption can we ensure that the solutions $y_1, \cdots, y_n$ of (4) are a FSS? Carefully justify and prove your answer. (You are allowed to use the theorems from the courses.)

(d) Given $y_1, \cdots, y_n$ FSS of (4) and y solution to (4), write y in terms of $y_1, \cdots, y_n$ and $y^{(j)}(t_0)$, $y_i^{(j)}(t_0)$, for $1 \le i \le n$, $0 \le j \le n-1$. (The operations of vector/matrix like product, inverse and determinant are allowed in the expression.)

Solution.

(a) We can show that

$$\frac{d}{dt}\begin{pmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \cdots \\ y^{(n-1)} \end{pmatrix} = \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \\ \cdots \\ y^{(n)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \end{pmatrix} \begin{pmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \cdots \\ y^{(n-1)} \end{pmatrix}. \tag{5}$$

In order to simplify the notation, we denote by $\mathbf{y}(t) = (y(t), y^{(1)}(t), y^{(2)}(t), \cdots, y^{(n-1)}(t))^T$ and $\mathbf{A}(t)$ for the matrix above. Then we have

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}(t)\mathbf{y}(t). \tag{6}$$

(b) From question (a), the solution y of (4) with initial value $y(t_0) = y_0, \cdots, y^{(n-1)}(t_0) = y_0^{(n-1)}$ implies that $\mathbf{y}(t) = (y(t), \cdots, y^{(n-1)}(t))$ is a solution for (6) with initial condition

$$\mathbf{y}(t_0) = \begin{pmatrix} y_0 \\ y_0^{(1)} \\ y_0^{(2)} \\ \cdots \\ y_0^{(n-1)} \end{pmatrix}.$$

For the latter, thanks to the fact $\mathbf{A}$ is continuous, by the existence and uniqueness theorem of first order system, it admits a unique solution. Then, the first component of $\mathbf{y}$ is the solution of (4) for the associated initial condition.

(c) Let $\mathbf{y}_i = (y_i, y_i^{(1)}, y_i^{(2)}, \cdots, y_i^{(n-1)})^T$. We claim that $(y_i)_{1\le i\le n}$ are FSS iff $(\mathbf{y}_i(t_0))_{1\le i\le n}$ are linear independent vectors for any $t_0 \in I$, and we provt it in the following paragraphs.

- $(y_i)_{1\le i\le n}$ are FSS $\Rightarrow$ $(\mathbf{y}_i(t_0))_{1\le i\le n}$ are linear independent vectors for any $t_0 \in I$: otherwise, there is a point $t_0 \in I$ and non zero coefficient $(\alpha_i)_{1\le i\le n-1}$ such that
$$\mathbf{y}_n(t_0) = \sum_{i=1}^{n-1} \alpha_i \mathbf{y}_i(t_0).$$
Then, by question (a), $\mathbf{y}_n$ and $\sum_{i=1}^{n-1} \alpha_i \mathbf{y}_i$ are all solution of (6) with the same initial condition at t_0. Using the existence and uniqueness theorem of first order system, this implies that
$$\forall t \in I, \qquad \mathbf{y}_n(t) = \sum_{i=1}^{n-1} \alpha_i \mathbf{y}_i(t).$$
We look at its first component that
$$\forall t \in I, \qquad y_n(t) = \sum_{i=1}^{n-1} \alpha_i y_i(t),$$
which contradicts the condition $(y_i)_{1\le i\le n}$ are FSS.
- $(\mathbf{y}_i(t_0))_{1\le i\le n}$ are linear independent vectors for any $t_0 \in I \Rightarrow (y_i)_{1\le i\le n}$ are FSS : assume that $(y_i)_{1\le i\le n}$ are not FSS, then there exists $(\alpha_i)_{1\le i\le n-1}$ such that
$$\forall t \in I, \qquad y_n(t) = \sum_{i=1}^{n-1} \alpha_i y_i(t).$$
This implies that
$$\begin{aligned}\forall t_0 \in I, \qquad y_n'(t_0) &= \lim_{t\to t_0} \frac{y_n(t) - y_n(t_0)}{t - t_0} \\ &= \lim_{t\to t_0} \frac{\sum_{i=1}^{n-1} \alpha_i y_i(t) - \sum_{i=1}^{n-1} \alpha_i y_i(t_0)}{t - t_0} \\ &= \sum_{i=1}^{n-1} \alpha_i y_i'(t).\end{aligned}$$
By induction, this also implies that $(\mathbf{y}_i(t_0))_{1\le i\le n}$ are not linear independent, which contradicts the condition.

Furthermore, $(\mathbf{y}_i(t))_{1\le i\le n}$ are linear independent vector for any $t \in I$ is equivalent to its Wronskian $W_{[\mathbf{y}_1,\cdots,\mathbf{y}_n]}(t) \neq 0$ for all $t \in I$. Especially, since it is a linear system, it suffices $W_{[\mathbf{y}_1,\cdots,\mathbf{y}_n]}(t_0) \neq 0$ for one $t_0 \in I$.

(d) Suppose $y = \sum_{i=1}^n c_i y_i$, then we have

$$\begin{pmatrix} y_0 \\ y_0^{(1)} \\ y_0^{(2)} \\ \cdots \\ y_0^{(n-1)} \end{pmatrix} = \begin{pmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1^{(1)}(t_0) & y_2^{(1)}(t_0) & \cdots & y_n^{(1)}(t_0) \\ y_1^{(2)}(t_0) & y_2^{(2)}(t_0) & \cdots & y_n^{(2)}(t_0) \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \cdots \\ c_n \end{pmatrix}.$$

We deduce from this that

$$\begin{aligned} y &= (y_1, y_2, y_3, \cdots, y_n) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \cdots \\ c_n \end{pmatrix} \\ &= (y_1, y_2, y_3, \cdots, y_n) \begin{pmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1^{(1)}(t_0) & y_2^{(1)}(t_0) & \cdots & y_n^{(1)}(t_0) \\ y_1^{(2)}(t_0) & y_2^{(2)}(t_0) & \cdots & y_n^{(2)}(t_0) \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix}^{-1} \begin{pmatrix} y_0 \\ y_0^{(1)} \\ y_0^{(2)} \\ \cdots \\ y_0^{(n-1)} \end{pmatrix}. \end{aligned}$$