Name: | | **NetId:** | |

Probability Limit Theorems

Midterm Exam, Fall 2021

DO NOT OPEN YET

...and wait until the proctor announces that it is time to start.

In the mean time, please write your name and NetID legibly,
and **read the instructions below carefully**.

* Please do not fold or damage the exam papers. After you finish your exam, please put the pages in correct order back into the sleeve.

* There are 6 questions in this exam, the sleeve should contain 8 pieces of paper.

* The scratch paper is included: three last papers are blank. If your solution continues on scratch paper, please clearly indicate it.

* For questions asking to prove a result, the clarity of the mathematical argument will be taken into account in the score.

* Questions formulated in terms of real functions should be answered with real functions.

Good luck!

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0 \text{ or } 1\}$. Show that $\mathcal{G}$ is a σ-algebra.

   Solution: We check the terms one by one in the definition of σ-algebra.

   (a) $\Omega \in \mathcal{G}$: because $\mathbb{P}(\Omega) = 1$, $\Omega \in \mathcal{G}$ by definition of $\mathcal{G}$.

   (b) Close by complement: for $A \in \mathcal{G}$, we have

   $$\mathbb{P}(A^c) = \begin{cases} 1 & \text{if } \mathbb{P}(A) = 0; \\ 0 & \text{if } \mathbb{P}(A) = 1. \end{cases}$$

   Therefore, $A \in \mathcal{G}$ by definition.

   (c) Close by countable union: if for any $n \in \mathbb{N}$, $A_n \in \mathcal{G}$, then we study two cases:

   i. For all $n \in \mathbb{N}$, $\mathbb{P}(A_n) = 0$. Then we have

   $$0 \leq \mathbb{P}(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n) = 0,$$

   which implies that $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = 0$ and thus $\cup_{n=1}^{\infty} A_n \in \mathcal{G}$.

   ii. There exists some $k \in \mathbb{N}$, such that $\mathbb{P}(A_n) = 1$. Then we have

   $$1 \leq \mathbb{P}(A_k) \leq \mathbb{P}(\cup_{n=1}^{\infty} A_n) \leq \mathbb{P}(\Omega) = 1,$$

   which implies that $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = 1$ and thus $\cup_{n=1}^{\infty} A_n \in \mathcal{G}$.

2. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider the space of real-valued random variables modulo a.s. equivalence, i.e., $X_1 \sim X_2$ if $\mathbb{P}(X_1 = X_2) = 1$. Define a metric on this space by

$$d(X, Y) = \mathbb{E}(\min\{1, |X - Y|\}).$$

Show that $X_n \xrightarrow{\mathbb{P}} X$ if and only if $d(X_n, X) \xrightarrow{n\to\infty} 0$.

Solution: For any $0 < \varepsilon < 1$,

$$\begin{aligned} d(X, Y) &= \mathbb{E}(\min\{1, |X - Y|\}\mathbf{1}_{|X-Y|>\epsilon} + \mathbf{1}_{|X-Y|\leq\epsilon}) \\ &\leq \mathbb{P}(|X - Y| > \epsilon) + \epsilon, \end{aligned}$$

and the equation above also implies that

$$d(X, Y) \geq \epsilon\mathbb{P}(|X - Y| > \epsilon). \tag{1}$$

Direction $X_n \xrightarrow{\mathbb{P}} X \Rightarrow d(X_n, X) \xrightarrow{n\to\infty} 0$: for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ and for every $n \geq n_0$, we have $\mathbb{P}(|X - X_n| > \epsilon) \leq \epsilon$, whcih implies that

$$d(X, X_n) \leq 2\epsilon.$$

Direction $d(X_n, X) \xrightarrow{n\to\infty} 0 \Rightarrow X_n \xrightarrow{\mathbb{P}} X$: from the inequality (1),

$$\mathbb{P}(|X - X_n| > \epsilon) \leq \frac{d(X, X_n)}{\epsilon} \xrightarrow{n\to\infty} 0.$$

3. If $f \in L^1(\mathbb{R})$, show that

$$\lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} f(x)\, dx = 0.$$

Solution: Define $f_n = \frac{1}{2n}\mathbf{1}_{[-n,n]}f$. Note that $f_n \to 0$ when $f \neq \pm\infty$. Then $f_n \to 0$ a.s. in $\mathbb{R}$. Since

$$|f_n| \leq \frac{1}{2n}|f| \in L^1(\mathbb{R}), \tag{2}$$

by Dominated Convergence Theorem,

$$\lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} f \,\mathrm{d}x = \int_{-\infty}^{\infty} f \,\mathrm{d}x = 0. \tag{3}$$

4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\{A_n\}_{n\geq 1}$ be a sequence of measurable subsets.

(a) Suppose that

$$\lim_{n\to\infty} \mathbb{P}(A_n) = 0 \text{ and } \sum_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty.$$

Show, without using Borel-Cantelli lemma, that $\mathbb{P}(A_n \text{ occurs infinitely often}) = 0$.

(b) Find an example of a sequence $\{A_n\}_{n\geq 1}$ to which the result in (a) can be applied but the Borel-Cantelli lemma cannot.

Solution:

(a) Note that

$$\bigcup_{m=n}^{\infty} A_m = A_n \cup \bigcup_{m=n}^{\infty} A_{m+1} \setminus A_m. \tag{4}$$

Then

$$\mathbb{P}\left(\bigcup_{m=n}^{\infty} A_n\right) \leq \mathbb{P}(A_n) + \sum_{m=n}^{\infty} \mathbb{P}\left(A_{m+1} \setminus A_m\right) \xrightarrow{n\to\infty} 0. \tag{5}$$

Therefore,

$$0 = \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{m=n}^{\infty} A_n\right) = \mathbb{P}(A_n \,\text{i.o.}). \tag{6}$$

(b) Let U be a uniform random variable defined in $[0, 1]$, and let $A_n = \{U \in [0, \frac{1}{n}]\}$. Then it is clear that

$$\mathbb{P}[A_n] = \frac{1}{n}, \quad \mathbb{P}[A_n^c \cap A_{n+1}] = 0,$$

which implies that

$$\sum_{n=1}^{\infty} \mathbb{P}[A_n] = \infty,$$

$$\lim_{n\to\infty} \mathbb{P}[A_n] = 0, \quad \sum_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) = 0 < \infty.$$

Thus, for this example we have $\mathbb{P}[i.o. A_n] = 0$ using (a), but Borel–Cantelli does not apply.

5. Let U be a random variable with uniform distribution on $[0,1]$, i.e. $\mathbb{P}(U \in [a,b)) = b-a$ for all $0 \leq a \leq b \leq 1$. Consider $(X_n)_{n\geq 1}$ the random variables defined by

$$X_n = \sin(2\pi n U).$$

Show that, for every $n, m \geq 1$, $n \neq m$,

$$\mathbb{E}(X_n X_m) = \mathbb{E}(X_n)\mathbb{E}(X_m),$$

and that the random variables $(X_n)_{n\geq 1}$ are not independent.

Solution: Let us compute the expectation of X_n,

$$\mathbb{E}(X_n) = \int_0^1 \sin(2\pi n x)\,\mathrm{d}x = \int_0^{2\pi n} \frac{\sin(x)}{2\pi n}\,\mathrm{d}x = -\frac{\cos(2\pi n)}{2\pi n} + \frac{1}{2\pi n} = 0. \tag{7}$$

Now, let us compute $\mathbb{E}(X_n X_m)$. We use the product-to-sum identity

$$\sin(\theta)\sin(\varphi) = \frac{1}{2}\left(\cos(\theta - \varphi) - \cos(\theta + \varphi)\right),$$

then for $n \neq m$ we have

$$\begin{aligned}
\mathbb{E}(X_n X_m) &= \int_0^1 \sin(2\pi n x)\sin(2\pi m x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 (\cos(2\pi(n-m)x) - \cos(2\pi(n+m)x))\,\mathrm{d}x \\
&= \frac{1}{2\pi(n-m)}(\sin(2\pi(n-m)) - 0) - \frac{1}{2\pi(n+m)}(\sin(2\pi(n+m)) - 0) \\
&= 0.
\end{aligned}$$

Thus

$$\mathbb{E}(X_n X_m) = \int_0^1 \sin(2\pi n x)\sin(2\pi m x)\,\mathrm{d}x = 0. \tag{8}$$

Now, let us show that $(X_n)_{n\geq 1}$ are not independent. Let $n = 1$, $m = 2$, and the intervals

$$A = \left(0, \frac{\sqrt{2}}{2}\right) \quad \text{and} \quad B = \left(0, \frac{1}{2}\right). \tag{9}$$

We have

$$\mathbb{P}(X_1 \in A) = \mathbb{P}\left(\sin(2\pi U) \in \left(0, \frac{\sqrt{2}}{2}\right)\right) = \frac{\pi}{4}, \tag{10}$$

and

$$\mathbb{P}(X_2 \in B) = \mathbb{P}\left(\sin(4\pi U) \in \left(0, \frac{1}{2}\right)\right) = \frac{\pi}{6}. \tag{11}$$

On the other hand,

$$\mathbb{P}([X_1 \in A] \cap [X_2 \in B]) = \mathbb{P}\left(\left(0, \frac{\pi}{4}\right) \cap \left(0, \frac{\pi}{6}\right)\right) = \frac{\pi}{6}. \tag{12}$$

6. (a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability measure, and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Given $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, define $Y = \mathbb{E}(X|\mathcal{G})$. Show that, if $\mathbb{E}(X^2) = \mathbb{E}(Y^2)$, then $X = Y$ a.s. .

   (b) Suppose X and Y are square-integrable random variables such that $\mathbb{E}(X|Y) = Y$ and $\mathbb{E}(Y|X) = X$. Show that $X = Y$ almost surely. (Hint: use the result in (a).)

Solution:

(a) Let us compute $\mathbb{E}((X-Y)^2)$.

$$\begin{aligned}\mathbb{E}((X-Y)^2) &= \mathbb{E}(X^2 - 2XY + Y^2)\\ &= \mathbb{E}(X^2) - 2X\mathbb{E}(\mathbb{E}(XY|\mathcal{G})) + \mathbb{E}(X^2)\\ &= 2\mathbb{E}(X^2) - 2\mathbb{E}(Y\mathbb{E}(X|\mathcal{G}))\\ &= 2\mathbb{E}(X^2) - 2\mathbb{E}(Y^2).\end{aligned}$$

Therefore $\mathbb{E}((X-Y)^2) = 0$, concluding $X = Y$ a.s..

(b) We calculate the expectation $\mathbb{E}(X^2)$ that

$$\begin{aligned}\mathbb{E}(X^2) &= \mathbb{E}(X\mathbb{E}(Y|X))\\ &= \mathbb{E}(\mathbb{E}(XY|X))\\ &= \mathbb{E}(XY).\end{aligned}$$

From the first line to the second line, we use the definition and from the second line to third line, we use the fact that X is X-measurable. Similar identity works for $\mathbb{E}(Y^2)$, so we have

$$\mathbb{E}(X^2) = \mathbb{E}(XY) = \mathbb{E}(Y^2),$$

and we can apply the result from (a).