Name:

NetId:

Honors Ordinary Differential Equations

Midterm Exam, Fall 2021

DO NOT OPEN YET

...and wait until the proctor announces that it is time to start.

In the mean time, please write your name and NetID legibly,
and **read the instructions below carefully**.

* Please do not fold or damage the exam papers. After you finish your exam, please put the pages in correct order back into the sleeve.

* There are 5 questions in this exam, the sleeve should contain 8 pieces of paper.

* The scratch paper is included: three last papers are blank. If your solution continues on scratch paper, please clearly indicate it.

* For questions asking to prove a result, the clarity of the mathematical argument will be taken into account in the score.

* Questions formulated in terms of real functions should be answered with real functions.

* Question marked with (†) is challenge.

Good luck!

1. Answer whether the statement is **TRUE** or **FALSE**.

   (a) **TRUE / FALSE:** $ty'' + t^2y' = t^3$ is a second order differential equation.

   (b) **TRUE / FALSE:** $y' + \sin(t)y = e^t$ is a non-linear differential equation.

   (c) **TRUE / FALSE:** If y_1 and y_2 are solutions to the differential equation

   $$y'' - t^2y' + 5y = e^t \sin(t),$$

   then $y_1 + y_2$ is also a solution of this equation.

   (d) **TRUE / FALSE:** The initial value problem $y'' + y = \sin(10t)$ has a unique solution on $\mathbb{R}$ for $y(0) = 0, y'(0) = 1$.

   (e) **TRUE / FALSE:** The initial value problem $y' - \frac{1}{1+|t|}y = t^{\frac{1}{3}},\ y(0) = 0,\ t \geq 0$ has more than one solution.

   (f) **TRUE / FALSE:** Assume that p and q are continuous and that the functions y_1 and y_2 are solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ on an open interval I. If y_1 and y_2 are zero at the same point in I, then $y_1 = \lambda y_2$ for some $\lambda \in \mathbb{R}$.

   (g) **TRUE / FALSE:** If the power series $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ has radius of convergence ρ, then $\sum_{k=0}^{\infty} ka_k(x - x_0)^k$ also has a radius of convergence ρ.

   (h) **TRUE / FALSE:** If the power series $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ has radius of convergence ρ, then $\{\rho^n a_n\}_{n\in\mathbb{N}}$ is bounded.

2. Consider the differential equation

$$y' = p(t)y + \cos t, \tag{1}$$

where p is a continuous function on $\mathbb{R}$.

(a) Show carefully that there exists a unique solution for that equation with initial value $y(0) = y_0$, and determine it explicitly.

(b) (($\dagger\dagger$), could be considered later in the exam) Prove that, if $y_0 > 1$, $p(t) \geq 1$ on $\mathbb{R}$, then $\lim_{t\to+\infty} y(t) = +\infty$.

(c) Now assume that p is the constant function $p = 1$. Show that there exists $y_0 \in \mathbb{R}$ such that $y(0) = y_0$ and y is a periodic solution, and prove carefully such y_0 is unique.

(a) $(1) \Longleftrightarrow$

$\Longleftrightarrow \left(y(t)\exp\left(-\int_0^t p(s)\,ds\right)\right)' = \left(y'(t) - p(t)y(t)\right)\exp\left(-\int_0^t p(s)\,ds\right)$

$= \exp\left(-\int_0^t p(s)\,ds\right)\cos t$

$\Longleftrightarrow y(t)\exp\left(-\int_0^t p(s)\,ds\right) = y(0) + \int_0^t \exp\left(-\int_0^s p(u)\,du\right)\cos s\,ds$

$\Longleftrightarrow y(t) = \exp\left(\int_0^t p(s)\,ds\right)\left(y_0 + \int_0^t \exp\left(-\int_0^s p(u)\,du\right)\cos s\,ds\right)$

(b) Let $s = \inf\left\{t \geq 0 : y(t) = 1 + \frac{y_0 - 1}{2}\right\}$.

Then, for all $t \leq s$,

$y'(t) \geq y(t)\left(1 + \frac{y_0 - 1}{2}\right) + \cos t \geq \frac{y_0 - 1}{2}.$

Since $p(t) \geq 1$, $\cos t \geq -1$.

Hence $y(t) \geq y_0 + \left(\frac{y_0 - 1}{2}\right)t$ for all $t \leq s$

$\Rightarrow s = \infty$ and $\lim_{t\to\infty} y(t) = \infty$.

(c) $p = 1$, $y' = y + \cos t$.

$$\iff (ye^{-t})' = e^{-t}\cos t$$

$$\iff ye^{-t} = y_0 + \frac{e^{-t}}{2}(\sin t - \cos t) + \frac{1}{2}$$

$$\iff y(t) = \left(y_0 + \frac{1}{2}\right)e^{t} + \frac{\sin t - \cos t}{2}$$

If $y_0 = -\frac{1}{2}$, then $y(t) = \frac{\sin t - \cos t}{2} = \frac{\sqrt{2}}{2}\sin\left(x - \frac{\pi}{4}\right)$

has period 2π.

Otherwise, $y(t) \to \pm\infty$ (depending on the sign of $y_0 + \frac{1}{2}$), so it cannot be periodic.

3. Given continuous functions p and q on an open interval I containing t_0, let y_1 and y_2 be two solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

on that open interval I.

(a) Define the Wronskian of the two solutions y_1 and y_2.

(b) State the theorem relating Wronskian to Fundamental Set of Solutions (FSS).

(c) Write down (without proof) the expression of $W_{[y_1,y_2]}(t)$ as a function of $W_{[y_1,y_2]}(t_0)$, for $t_0, t \in I$.

(d) If $y_1(t_0) = 1, y_1'(t_0) = 1, y_2(t_0) = 1, y_2'(t_0) = -1$, write down the expression of the solution y of the initial value problem (2) with $y(t_0) = a, y'(t_0) = b$. Justify your answer.

(e) Assume that, given $\alpha \in \mathbb{R}$, the functions p and q are constant equal to

$$p(t) = -4,$$
$$q(t) = -\alpha^2 + 2\alpha + 3.$$

Compute the general solution. Determine the set of $\alpha \in \mathbb{R}$ such that there exists a bounded non-zero solution to the ordinary differential equation (2).

(a) $W_{[y_1,y_2]} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

(b) y_1, y_2 FSS $\Longleftrightarrow W_{[y_1,y_2]}(t_0) \neq 0$.

(c) $W_{[y_1,y_2]}(t) = W_{[y_1,y_2]}(t_0) \exp\left(-\int_{t_0}^{t} p(s)ds\right)$

(d) Let $u_1 = \frac{y_1+y_2}{2}$, $u_2 = \frac{y_1-y_2}{2}$.

Then $u_1(t_0) = 1$, $u_1'(t_0) = 0$

$u_2(t_0) = 0$, $u_2'(t_0) = 1$.

Now u_1, u_2 are FSS, since $W_{[u_1,u_2]}(t_0) = 1$.

Hence $y = c_1 u_1 + c_2 u_2$. Now $y(t_0) = c_1$ and $y'(t_0) = c_2$, so that

$$y = y(t_0)\left(\frac{y_1 + y_2}{2}\right) + y'(t_0)\left(\frac{y_1 - y_2}{2}\right).$$

(e) The characteristic equation is

$$r^2 - 4r - \alpha^2 + 2\alpha + 3 = 0$$

$$\iff (r-2)^2 = (\alpha - 1)^2 \iff r = \alpha + 1 \text{ or } 3 - \alpha.$$

hence the general solution is

$$y(t) = c_1 e^{(\alpha+1)t} + c_2 e^{(3-\alpha)t}$$

$\exists$ bounded nonzero solution

$\iff \alpha + 1 \leq 0$ or $3 - \alpha \leq 0$.

$\iff \alpha \leq -1$ or $\alpha \geq 3$.

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4. The aim of this question is to prove Grönwall's inequality, which can be stated as follows.

Let $a, y : \mathbb{R} \to [0, \infty)$ be two continuous nonnegative functions and suppose that, for any $t \geq 0$,

$$y(t) \leq 1 + \int_0^t a(s)y(s)\, ds. \tag{3}$$

Then

$$y(t) \leq \exp\left(\int_0^t a(s)\, ds\right). \tag{4}$$

In the rest of the exercise, we assume (3), with the aim to prove (4).

(a) Define $w(t) := 1 + \int_0^t a(s)y(s)\, ds$. Prove that w is a differentiable function, and compute $w'(t)$.

(b) Plugging (3) in the expression for $w'(t)$, write down a differential inequality using only $w'(t), w(t), a(t)$.

(c) Deduce from (b) an upper bound estimate for $w(t)$.

(d) Conclude (4).

(a) By Fundamental theorem of Calculus (FTC), w is differentiable with derivative

$$w'(t) = a(t)y(t).$$

(b) $w'(t) = a(t)y(t) \leq a(t)w(t)$.

(c) Note that (b) is equivalent to

$$\left(w(t)\exp\left(-\int_0^t a(s)\,ds\right)\right)'$$

$$= \exp\left(-\int_0^t a(s)\,ds\right)\left(w'(t) - a(t)w(t)\right)$$

$$\leq 0.$$

$\iff w(t) \leq \exp\left(\int_0^t a(s)\,ds\right)$ since $w(0)=1$.

(d) Apply (3) and (c):

$$y(t) \leq w(t) \leq \exp\left(\int_0^t a(s)\,ds\right).$$

5. (a) Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two series with respectively radii of convergence ρ_1 and ρ_2. Prove that the radius of convergence R of $\sum_{n=0}^{\infty} a_n b_n x^n$ satisfies $R \geq \rho_1 \rho_2$. Do we always have equality ?

   (b) Consider the series

$$f(x) \to \sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right) x^n. \tag{5}$$

   i. Determine the radius of convergence ρ of this series.

   ii. Study if this series converges at ρ and $-\rho$.

   iii. (($\dagger$), could be considered later in the exam) Prove that, for any $M > 0$, there exists $\delta > 0$ and $N \in \mathbb{N}$ such that for any $x \in (1-\delta, 1)$, we have

$$\sum_{n=1}^{N} \sin\left(\frac{1}{\sqrt{n}}\right) x^n \geq M. \tag{6}$$

   iv. Compute the limit $\lim_{x \to 1-} f(x)$.

(a) $R = \sup\{r > 0 : a_n b_n r^n \text{ bounded}\}$.

If $r < \rho_1 \rho_2$, it can be written as $r = r_1 r_2$ with $r_1 < \rho_1$ and $r_2 < \rho_2$. Now, by definition of ρ_1 and ρ_2, $a_n r_1^n$ and $b_n r_2^n$ are bounded

$\Rightarrow a_n b_n r^n = a_n b_n (r_1 r_2)^n$ bounded.

Example of $R > \rho_1 \rho_2$: $\forall n \in \mathbb{N}$, $a_{2n} = b_{2n+1} = 1$, $b_{2n} = a_{2n+1} = 0$.

Then $R = \infty$, $\rho_1 = \rho_2 = 1$.

(b) (i) $\rho = \sup\{r > 0 : \sin\left(\frac{1}{\sqrt{n}}\right) r^n \text{ bounded}\}$

By L'Hôpital's rule, $\sin\left(\frac{1}{\sqrt{n}}\right)\sqrt{n} \underset{n\to\infty}{\longrightarrow} 1$,

(or simply because $(\sin')(0) = 1$),

and $\frac{r^n}{\sqrt{n}} = \frac{\exp(n \log r)}{\sqrt{n}}$. Now exponentials always prevail over polynomials, so that $\frac{r^n}{\sqrt{n}}$ is not bounded if $r > 1$, and is bounded if $r < 1$. Hence $\rho = 1$.

(ii) $\sin(x) \geq \frac{x}{2}$ for small x, since $\sin'(0) = 1$

But $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right)$ diverges

Now $\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right)(-1)^n$ is an alternating series, hence it converges.

(iii) For $N \geq n_0$ large, $x > 1-\delta$,

$$\sum_{n=n_0}^{N} \sin\left(\frac{1}{\sqrt{n}}\right) x^n \geq \sum_{n=n_0}^{N} \frac{x^n}{2\sqrt{n}} \geq (1-\delta)^N \sum_{n=n_0}^{N} \frac{1}{2\sqrt{n}}$$

$$\overset{*}{\geq} (1-\delta)^N (\sqrt{N+1} - \sqrt{n_0}) \geq \frac{\sqrt{N+1} - \sqrt{n_0}}{2} \geq M, \text{ if}$$

we choose $N \geq (\sqrt{n_0} + 4M)^2$ and $\delta > 0$ s.t. $(1-\delta)^N > \frac{1}{2}$.

In $*$ we used $\sqrt{n+1} - \sqrt{n} = (\sqrt{n} + \sqrt{n+1})^{-1} \leq \frac{1}{2\sqrt{n}}$.

(iv) It follows from (6) that, $\forall M > 0$, $\exists \delta > 0$ $f(x) \geq M$ for $x \geq 1-\delta$.

Hence $\lim_{x \to 1^-} f(x) = \infty$.