Minimum-cost Flow

Advanced Algorithms 2013 Project 1

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This is the first of two mandatory assignments for the course Advanced Algorithms 2013. The Assignment should be solved in groups of 2 or 3 students. Permission for individual assignments is only granted in special cases. The assignment is due May 12th at 22:00. To pass the assignment the group must have completed most of the questions satisfactory. To be allowed to resubmit the group must have made a reasonable attempt at answering most of the questions. The resubmission is due Mon. 20th of May at 22:00. Completed assignments must be uploaded to Absalon (this page). All descriptions and arguments should be kept concise while still containing relevant points.

Exercise 1: b-flow

Let $G = (V, E)$ be a directed graph. For each vertex $v \in V$ $\delta^+(v)$ be the set of outgoing edges from v and $\delta^-(v)$ be the set of incoming edges to v.

Each edge $e \in E$ has a *capacity* $u_e \in \mathbb{R} \cup \{\infty\}$, and each vertex $v \in V$ has a so called *demand* $b_v \in \mathbb{R}$.

Our goal (if possible) is to find a *b-flow*, which is an assignment of real numbers x_e to each edge $e \in E$, such that

$$\sum_{e \in \delta^-(v)} x_e - \sum_{e \in \delta^+(v)} x_e = b_v, \forall v \in V,$$
$$0 \leq x_e \leq u_e, \forall e \in E.$$

Find, for each of the two graphs in Figure 1, a b-flow for the graph or argue that it does not have any b-flow.
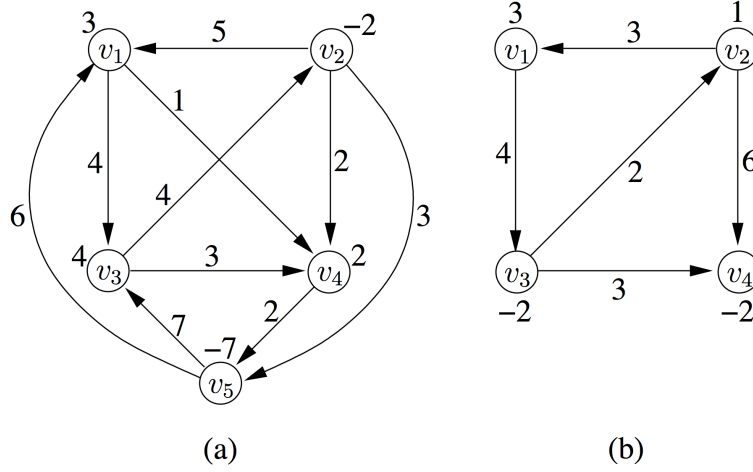
Figure 1: Graphs with capacities and demands (for instance vertex v_2 has demand -2 and the edge (v_1, v_3) has capacity 4 in (a)).

Exercise 2: Minimum-cost flow problem

Now we consider a graph G with capacities and demands as in Exercise 1, as well as a cost $c_e \in \mathbb{R}$ for each edge $e \in E$. Given a b-flow $\{x_e | e \in E\}$ in G, we define its cost as $\sum_{e \in E} c_e x_e$.

In the *minimum-cost flow-problem (MCFP)* we are looking for a b-flow in G of minimum cost.

Write the LP-formulation in standard form of the MCFP instance in Figure 1(a) with the following cost function: $c_{v_1 v_3} = 1$, $c_{v_1 v_4} = 2$, $c_{v_2 v_1} = 3$, $c_{v_2 v_4} = 4$, $c_{v_2 v_5} = 5$, $c_{v_3 v_2} = 6$, $c_{v_3 v_4} = 7$, $c_{v_4 v_5} = 8$, $c_{v_5 v_1} = 9$ and $c_{v_5 v_3} = 10$.

Formulate the dual problem of your LP-formulation.

Exercise 3: An application of MCFP: rectilinear planar embedding

We will now look at an application of MCFP to a geometric problem in the plane. In the following we make the following assumptions:

- $G = (V, E)$ is an undirected graph.

- For technical reasons, we assume that G is connected, consists of at least three vertices, and has no cut vertices, i.e., vertices whose removal yields a disconnected graph (you may simply ignore these details).

- No vertices in G have degree greater than 4.

We define a *planar embedding* of G as a drawing of G in the plane such that no two edges intersect (except possibly at their endpoints). An example is given in Figure 2. If in addition the graph is drawn such that each edge consists exclusively of vertical and horizontal line segments, we say that the embedding is *rectilinear* (see Figure 3). For such an embedding we define *break points* to be the points where the horizontal and vertical line segments from the same edge meet.

A planar embedding divides the plane into a number of regions called *faces*; see Figure 2(b). Exactly one of these faces is unbounded, and is called
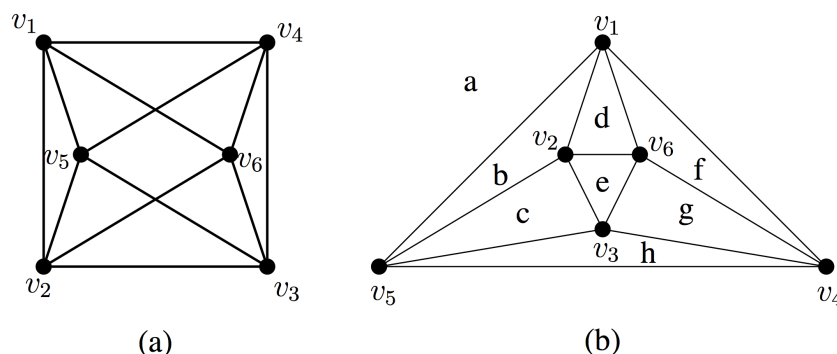

Figure 2: (a): A graph and (b): A planar embedding of this graph. The embedding divides the plane into faces a, b, c, d, e, f, g and h, where a is external.

the *external face*. The vertices and edges of a face form a simple cycle (this will not be shown but follows from one of our assumptions above). We call such a cycle a *boundary cycle*.

Suppose now that we have a planar embedding of G. Then define a *rectilinear layout* of G to be a rectilinear embedding of G, containing the same boundary cycles as the given embedding.

The problem considered in this exercise is to find a rectilinear layout of G with a minimum number of break points. The idea is to formulate the problem as an instance of MCFP by introducing variables that keep track of the number of breakpoints while maintaining the feasibility. In the following we will consider a rectilinear layout which we do not know explicitly; we only know the associated planar embedding. Given a boundary cycle f of this rectilinear layout and a vertex v in f, we say that f makes an *inner turn* in v if the two edges in f incident to v form an angle of strictly less than $180°$ viewed from f. If the angle is strictly greater than $180°$, f is said to make an outer turn in v. An example is given in Figure 3. We define inner and outer turns of break points similarly. For each boundary cycle f and each vertex v in f we introduce a variable x_{vf} whose value is defined as follows:

$$x_{vf} = \begin{cases} 1 & \text{if } f \text{ makes an inner turn in } v \\ -1 & \text{if } f \text{ makes an outer turn in } v \\ 0 & \text{otherwise.} \end{cases}$$

Let f and g be two boundary cycles and let E_{fg} be the (possibly empty) set of edges shared by f and g. We define the variable z_{fg} to be the number of inner turns that f makes in the break points of edges of E_{fg}. Note that z_{fg} and z_{gf} are two different variables.
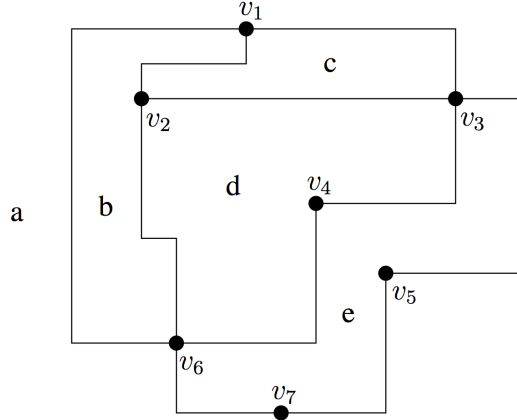

Figure 3: A rectilinear layout of a graph. Example: boundary cycle d makes an inner turn in v_2, v_3 and v_6 and an outer turn in v_4.

Exercise 3.1

Give the values of all variables z_{fg} and x_{vf} for the rectilinear layout in Figure 3. How many break points are there?

Draw a rectilinear layout of the graph from Figure 2(a) given the planar embedding in Figure 2(b).

Exercise 3.2

Let f be a boundary cycle in a rectilinear layout. It can be shown that the number of inner turns minus the number of outer turns that f makes in its vertices and break points is -4 if f corresponds to the external face and 4 otherwise.

State these constraints as linear constraints in the previously introduced variables.

Verify that the constraints hold for boundary cycles a and e in Figure 3.

Exercise 3.3

In the beginning of the exercise, we assumed that no vertex in G has degree greater than 4. Why is this assumption necessary?

It also follows from our assumptions about G that no vertex has degree less than 2 (You are not required to show this). Show that the following holds for each vertex v:

$$\sum_f x_{vf} = \begin{cases} 0 & \text{if } v \text{ has degree 2} \\ 2 & \text{if } v \text{ has degree 3} \\ 4 & \text{if } v \text{ has degree 4,} \end{cases} \tag{1}$$

where the sum is over all boundary cycles f containing v.

Exercise 3.4

We want to minimize the number of break points. Formulate an objective function whose minimization ensures this. The objective function must be expressed in the previously introduced variables.

Write a linear program with this objective function and the following constraints:

1. The constraints from Exercise 3.2.

2. The constraints from Exercise 3.3.

3. All variables z_{fg} are non-negative.

The following theorem can be used without proof: *An optimal solution to the given linear program has integer values for the variables x_{vf} and z_{fg}.*

Exercise 3.5

Rewrite the linear program from Exercise 3.4 to an instance of MCFP. It is important to define the constraints such that they are on the form specified in Exercise 1.

Describe which vertices and edges belong to the associated graph, and give the demands for the vertices and for the capacities and costs of the edges.

Next, write the instance of MCFP corresponding to the embedding in Figure 2(b).

Hint: make an appropriate substitution of certain variables in order to ensure that they are non-negative.

Comments on Exercise 3

It can be shown that an optimal solution to the above instance of MCFP corresponds to a rectilinear layout, with a minimum number of break points. In addition, it can be shown that it is relatively easy to find this layout, when the optimal solution for MCFP-instance is found. To limit the scope of the assignment we will not focus on this.