Exercises for the lecture: Representations of uncertainty

October 25, 2017

1 Simple uncertainty representation

Let the possible outcomes (events) of rolling a six-sided die be $W = \{w_1, w_2, \ldots, w_6\}$. Define the events

- $O =$"die shows an odd number".
- $E =$"die shows an even number".
- $I =$"die shows a number between 3 and 5 (both inclusive)".

E1. (**1p**) What are the set representations of O,E and I ?

E2. (**1p**) Are O,E, I possible, impossible or certain ?

E3. (**1p**) Assume you learn that $W' = \{w_1, w_2\}$. Are O,E, I possible, impossible or certain given W'?

E4. (**1p**) What is the set representation of the proposition: Z="it is not true that the die shows an even number or a number ≥ 3 and ≤ 5" ?

E5. (**1p**) Is Z possible, impossible or certain given W'?

E6. (**1p**) What is the set representation of the proposition: Y="it is true that the die shows an even number and an odd number'" ?

2 Probability

E7. (**2p**) In the lecture, it was claimed that σ-algebras are closed under intersection. Prove this claim.

Let $(W, \mathcal{F}, P)$ be a probability space, where $\mathcal{F}$ is a σ-algebra over W and $P : \mathcal{F} \to [0, 1]$ is the probability measure, with the properties

1. $P(W) = 1$
2. If $U, V \in \mathcal{F}$ and $U \cap V = \emptyset$, then $P(U \cup V) = P(U) + P(V)$.

E8. (**3p**) Prove that this definition implies $P(U \cup V) = P(U) + P(V) - P(U \cap V)$ in general, i.e. when $U \cap V \neq \emptyset$

In the lecture, we discussed Ramsey's rationality requirements RAT1-RAT4 for betting under uncertainty. A bet (U, α) is: 'I win $1 - \alpha$ if U happens, and lose

α otherwise'. RAT1 required that $(U,\alpha) \succeq (V,\beta)$, i.e. (U,α) is preferred to (V,β) if the payoff from (U,α) was guaranteed to be at least as large as the payoff from (V,β).

E9. (**2p**) Under what conditions is the payoff from (U,α) guaranteed to be at least as large as the payoff from (V,β) ? *Hint*: consider the events $U \cap V$, $\bar{U} \cap V$, $U \cap \bar{V}$ and $\bar{U} \cap \bar{V}$, and derive conditions on α and β.

We discussed in the lecture that $(U,0)$ is a "can't lose" bet, whereas $(U,1)$ is a "can't win" bet when betting on U. Likewise, $(\bar{U},1)$ is a "can't win" bet when betting on $\bar{U}$. Intuitively, that means that there should be an α_U such that for all $\alpha < \alpha_U$, you prefer (U,α) to $(\overline{U},1-\alpha)$, whereas for $\alpha > \alpha_U$, you prefer $(\overline{U},1-\alpha)$ to (U,α). Denote with $\alpha_U = \sup\{\alpha : (U,\alpha) \succeq (\overline{U},1-\alpha)\}$ the supremum, i.e. the smallest upper bound on α such that (U,α) is preferred to $(\overline{U},1-\alpha)$.

E10. (**3p**) Use RAT1-RAT3 to show that this supremum exists, and that preferences can't change more than once as you increase α from 0 to 1.

Since α_U is the largest α for which you would prefer (U,α) to $(\overline{U},1-\alpha)$, it can be seen as a measure of your uncertainty about U. For it to be a probability, it remains to show that P2 is fulfilled, i.e. for events U, V such that $U \cap V = \emptyset$, it holds that $\alpha_U + \alpha_V = \alpha_{U\cup V}$

E11. (**5p**) Show that this is true. *Hint*: proof by contradiction, using RAT4. Show that if $\alpha_U + \alpha_V > \alpha_{U\cup V}$, then there are sets of bets where you are guaranteed to lose, even though you obey RAT1-RAT4 otherwise. Likewise for the other direction of the inequality. You may assume that the payoffs of a set of bets are additive.