第四次作业

洪艺中 12335025

2024 年 3 月 31 日

0.1 133 页习题 11

题目 1. 设 (M_1, g_1), (M_2, g_2) 均为 Riemann 流形. $\nabla^{(1)}$, $\nabla^{(2)}$ 分别为它们的 Riemann 联络. $F\colon M_1 \to M_2$ 为等距微分同胚, 即 $g_1 = F^* g_2$. 证明 $F_*(\nabla^{(1)}_X Y) = \nabla^{(2)}_{F_* X} F_* Y$, $\forall X, Y \in \mathscr{X}(M_1)$.

解答. 因为 F 是微分同胚, 所以 M_1 和 M_2 是同维数流形. 因此 F_p^* 是切空间 T_pM 到 $T_{F(p)}M$ 的同构. 故要证明 $F_*(\nabla^{(1)}_X Y) = \nabla^{(2)}_{F_* X} F_* Y$, $\forall X, Y \in \mathscr{X}(M_1)$, 只需要证明任取 $Z \in \mathscr{X}(M_1)$,

$$g_1((\nabla^{(1)}_X Y), Z) = g_2(F_*(\nabla^{(1)}_X Y), F_* Z) = g_2(\nabla^{(2)}_{F_* X} F_* Y, \forall X, Y \in \mathscr{X}(M_1), F_* Z). \tag{$*$}$$

而利用 Riemann 联络的唯一性构造, 联络 ∇ 和度量 $\langle \cdot, \cdot \rangle$ 满足

$$2\langle \nabla_X Y, Z \rangle = X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle + \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle,$$

要证明 $(*)$ 式, 只需要证明: 任取 $X, Y, Z \in \mathscr{X}(M_1)$,

$$X(g_1(Y, Z)) = (F_* X)(g_2(F_* Y, F_* Z))$$

和

$$g_1([X, Y], Z) = g_2([F_* X, F_* Y], F_* Z).$$

利用 $g_1 = F^* g_2$ 和 Lie 括号与切映射交换, 计算可得:

$$(F_* X)(g_2(F_* Y, F_* Z)) = X\left(g_2(F_* Y, F_* Z) \circ F\right) = X((F^* g_2)(Y, Z)) = X(g_1(Y, Z)),$$

以及

$$g_2([F_* X, F_* Y], F_* Z) = g_2(F_*[X, Y], F_* Z) = (F^* g_2)([X, Y], Z) = g_1([X, Y], Z).$$

所以题目得证.

0.2 113 页习题 12

题目 2. 设 (M^m, g) 为连通 Riemann 流形, ∇ 为 Riemann 联络, A 为二阶对称张量且 $\nabla A = 0$. 定义线性映射 $A^*\colon T_pM \to T_pM, \forall p \in M$ 如下: 对任意的 $X, Y \in T_p(M)$

$$\langle A^*(X), Y \rangle_p := A(X, Y)(p),$$

设 ρ_i 为 A^* 的特征值, $\tilde{e}_i$ 为其相应的单位特征向量, 证明:

1. 所有特征值在 M 上均为常数;
2. 若 $\rho_h \neq \rho_k$, 则 $\langle e_h, e_k\rangle = 0$. 设 $\{\tilde{e}_i\}$ 为 $A^\star$ 的特征向量标架, 使得 $\langle \tilde{e}_i, \tilde{e}_j\rangle = \delta_{ij}$, 则 $\rho_h \neq \rho_k$ 时, 有
$$\langle \nabla_{\tilde{e}_i}\tilde{e}_h, \tilde{e}_k\rangle = 0, \quad h, i, k = 1, \cdots, m;$$
3. 设 ρ_i 为 r 重根, 对应特征向量为 $\tilde{e}_1, \cdots, \tilde{e}_r$, 则 $\tilde{e}_{r+1}, \cdots, \tilde{e}_m$ 生成的分布 $\mathscr{D}$ 是完全可积的.

解答.

1. 取 $X, Y \in \mathscr{X}(M)$, 则根据 $\nabla A = 0$,
$$X(A(\tilde{e}_i, Y)) = A(\nabla_X \tilde{e}_i, Y) + A(\tilde{e}_i, \nabla_X Y)$$
利用 g 也关于联络平行,
$$X(A(\tilde{e}_i, Y)) = X(\rho_i\langle \tilde{e}_i, Y\rangle) = X(\rho_i)\langle \tilde{e}_i, Y\rangle + \rho_i\langle \nabla_X \tilde{e}_i, Y\rangle + \rho_i\langle \tilde{e}_i, \nabla_X Y\rangle.$$
所以
$$A(\nabla_X \tilde{e}_i, Y) = X(\rho_i)\langle \tilde{e}_i, Y\rangle + \rho_i\langle \nabla_X \tilde{e}_i, Y\rangle,$$
因为 $\tilde{e}_i$ 是单位向量, 所以 $\langle \nabla_X \tilde{e}_i, \tilde{e}_i\rangle = 0$, 因此在上式代入 $Y = \tilde{e}_i$, 得到
$$\begin{aligned} A(\nabla_X \tilde{e}_i, \tilde{e}_i) =& X(\rho_i)\langle \tilde{e}_i, \tilde{e}_i\rangle + \rho_i\langle \nabla_X \tilde{e}_i, \tilde{e}_i\rangle \\ =& X(\rho_i), \end{aligned}$$
而左边又有 $X(\rho_i) = X(A(\tilde{e}_i, \tilde{e}_i)) = 2A(\nabla_X \tilde{e}_i, \tilde{e}_i)$, 于是
$$\frac{1}{2}X(\rho_i) = A(\nabla_X \tilde{e}_i, \tilde{e}_i) = X(\rho_i).$$
所以 $X(\rho_i) \equiv 0$. 由 X 任意性, ρ_i 在 M 上均为常数.
2. 正交性: $\rho_h\langle \tilde{e}_h, \tilde{e}_k\rangle = A(\tilde{e}_h, \tilde{e}_k) = \rho_k\langle \tilde{e}_h, \tilde{e}_k\rangle$, 因为 $\rho_h \neq \rho_k$, 所以 $\langle \tilde{e}_h, \tilde{e}_k\rangle = 0$.
$\langle \nabla_{\tilde{e}_i}\tilde{e}_h, \tilde{e}_k\rangle = 0$ 利用内积为 0 和 $\nabla A = 0$, 不妨设 $\rho_h \neq 0$:
$$\begin{aligned} \langle \nabla_{\tilde{e}_i}\tilde{e}_h, \tilde{e}_k\rangle =& -\langle \tilde{e}_h, \nabla_{\tilde{e}_i}\tilde{e}_k\rangle \\ =& -\frac{1}{\rho_h}A(\tilde{e}_h, \nabla_{\tilde{e}_i}\tilde{e}_k) \\ =& -\frac{1}{\rho_h}\tilde{e}_i(A(\tilde{e}_h, \tilde{e}_k)) + \frac{1}{\rho_h}A(\nabla_{\tilde{e}_i}\tilde{e}_h, \tilde{e}_k) \\ =& \frac{1}{\rho_h}A(\nabla_{\tilde{e}_i}\tilde{e}_h, \tilde{e}_k) \\ =& \frac{\rho_k}{\rho_h}\langle \nabla_{\tilde{e}_i}\tilde{e}_h, \tilde{e}_k\rangle, \end{aligned}$$
由于系数不为 1, 所以 $\langle \nabla_{\tilde{e}_i}\tilde{e}_h, \tilde{e}_k\rangle = 0$.
3. 我们依然取 $\{\tilde{e}_i\}$ 为单位正交的, 因为这不影响分布的生成. 利用分布 Frobenius 定理, 分布完全可积当且仅当其对合, 即 $s, t > r$ 时 $[\tilde{e}_s, \tilde{e}_t]$ 可由 $\tilde{e}_{r+1}, \cdots, \tilde{e}_m$ 表示.

设 $[\tilde{e}_s, \tilde{e}_t] = a^p \tilde{e}_p$. 则与 $\tilde{e}_1, \cdots, \tilde{e}_r$ 内积得

$$\langle [\tilde{e}_s, \tilde{e}_t], \tilde{e}_i \rangle = \sum_{p=1}^{r} a^p \delta_{ip} = a^i.$$

而根据第二问的结论

$$\langle [\tilde{e}_s, \tilde{e}_t], \tilde{e}_i \rangle = \langle \nabla_{\tilde{e}_s} \tilde{e}_t, \tilde{e}_i \rangle - \langle \nabla_{\tilde{e}_t} \tilde{e}_s, \tilde{e}_i \rangle = 0.$$

所以 $a^i = 0$, 即分布是对合的. 因此 $\mathscr{D}$ 是完全可积的.

0.3 题目 B

题目 3. 证明 Ricci 恒等式: 假设 ∇ 是对称联络, ϕ 是 (r,s)-型张量场, $X, Y \in \mathscr{X}(M)$, 则

$$\begin{aligned}
&\nabla^2 \phi(\theta^1, \cdots, \theta^r, X_1, \cdots, X_s; X, Y) - \nabla^2 \phi(\theta^1, \cdots, \theta^r, X_1, \cdots, X_s; Y, X) \\
=& - \mathrm{Rm}(X,Y)(\phi(\theta^1, \cdots, \theta^r, X_1, \cdots, X_s)) \\
&+ \sum_{a=1}^{r} \phi(\theta^1, \cdots, \theta^{a-1}, \mathrm{Rm}(X,Y)\theta^a, \theta^{a+1}, \cdots, \theta^r, X_1, \cdots, X_s) \\
&+ \sum_{b=1}^{r} \phi(\theta^1, \cdots, \theta^r, X_1, \cdots, X_{b-1}, \mathrm{Rm}(X,Y)X_b, X_{b+1}, \cdots, X_s)
\end{aligned}$$

其中 $\theta^i \in A^1(M)$, $X_i \in \mathscr{X}(M)$,

$$\mathrm{Rm}(X,Y)\psi := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\psi,$$

这里 ψ 是任何 (p,q)-型张量场.

解答. 类似张量的联络导数, 记

$$\begin{aligned}
&\mathrm{Rm}(X,Y)\phi(\theta^1, \cdots, \theta^r, X_1, \cdots, X_s) \\
:=&\mathrm{Rm}(X,Y)(\phi(\theta^1, \cdots, \theta^r, X_1, \cdots, X_s)) \\
&- \sum_{a=1}^{r} \phi(\theta^1, \cdots, \theta^{a-1}, \mathrm{Rm}(X,Y)\theta^a, \theta^{a+1}, \cdots, \theta^r, X_1, \cdots, X_s) \\
&- \sum_{b=1}^{r} \phi(\theta^1, \cdots, \theta^r, X_1, \cdots, X_{b-1}, \mathrm{Rm}(X,Y)X_b, X_{b+1}, \cdots, X_s).
\end{aligned}$$

则 Ricci 恒等式为 $\nabla^2 \phi(\cdots; X, Y) - \nabla^2 \phi(\cdots; Y, X) = -\mathrm{Rm}(X,Y)\phi(\cdots)$.

对称联络说明, $\nabla_X Y - \nabla_Y X - [X,Y] = 0$. 1. 对函数 $f \in C^\infty(M)$,

$$\begin{aligned}
&\nabla^2 f(X,Y) - \nabla^2 f(Y,X) \\
=&Y(X(f)) - \nabla_Y X(f) - X(Y(f)) + \nabla_X Y(f) \\
=&Y(X(f)) - X(Y(f)) - [Y,X](f) \\
=& - \mathrm{Rm}(X,Y)f = 0.
\end{aligned}$$

2. 对向量场 $Z \in \mathscr{X}(M)$,

$$
\begin{aligned}
&\nabla^2 Z(\theta; X, Y) - \nabla^2 Z(\theta; Y, X) \\
=&Y(X(\theta(Z))) - Y(\nabla_X \theta(Z)) - X(\nabla_Y \theta(Z)) + \nabla_X \nabla_Y \theta(Z) - \nabla_Y X(\theta(Z)) + \nabla_{\nabla_Y X} \theta(Z) \\
-&X(Y(\theta(Z))) + X(\nabla_Y \theta(Z)) + Y(\nabla_X \theta(Z)) - \nabla_Y \nabla_X \theta(Z) + \nabla_X Y(\theta(Z)) - \nabla_{\nabla_X Y} \theta(Z) \\
=&(YX - XY - [Y, X])(\theta(Z)) + (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\theta(Z) \\
=&-\mathrm{Rm}(X, Y)(Z(\theta)) + Z(\mathrm{Rm}(X, Y)\theta).
\end{aligned}
$$

3. 对 1-形式 $\theta \in \Gamma(T^*M)$,

$$
\begin{aligned}
&\nabla^2 \theta(Z; X, Y) - \nabla^2 \theta(Z; Y, X) \\
=&Y(X(\theta(Z))) - Y(\theta(\nabla_X Z)) - X(\theta(\nabla_Y Z)) + \theta(\nabla_X \nabla_Y Z) - \nabla_Y X(\theta(Z)) + \theta(\nabla_{\nabla_Y X} Z) \\
-&X(Y(\theta(Z))) + X(\theta(\nabla_Y Z)) + Y(\theta(\nabla_X Z)) - \theta(\nabla_Y \nabla_X Z) + \nabla_X Y(\theta(Z)) - \theta(\nabla_{\nabla_X Y} Z) \\
=&(YX - XY - [Y, X])(\theta(Z)) + \theta((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z) \\
=&-\mathrm{Rm}(X, Y)(\theta(Z)) + \theta(\mathrm{Rm}(X, Y)Z).
\end{aligned}
$$

4. 利用 $\nabla_X(\phi \otimes \psi) = \nabla_X \phi \otimes \psi + \phi \otimes \nabla_X \psi$,

$$
\begin{aligned}
&\nabla^2(\phi \otimes \psi)(\cdots; X, Y) - \nabla^2(\phi \otimes \psi)(\cdots; Y, X) \\
=&\nabla_Y(\nabla(\phi \otimes \psi))(\cdots; X) - \nabla_X(\nabla(\phi \otimes \psi))(\cdots; X) \\
=&\nabla_Y(\nabla_X(\phi \otimes \psi)) - \nabla_{\nabla_Y X}(\phi \otimes \psi) - \nabla_X(\nabla_Y(\phi \otimes \psi)) + \nabla_{\nabla_X Y}(\phi \otimes \psi) \\
=&\nabla_Y(\nabla_X \phi \otimes \psi) + \nabla_Y(\phi \otimes \nabla_X \psi) - \nabla_{\nabla_Y X} \phi \otimes \psi - \phi \otimes \nabla_{\nabla_Y X} \psi \\
-&\nabla_X(\nabla_Y \phi \otimes \psi) - \nabla_X(\phi \otimes \nabla_Y \psi) + \nabla_{\nabla_X Y} \phi \otimes \psi + \phi \otimes \nabla_{\nabla_X Y} \psi \\
=&\nabla_Y \nabla_X \phi \otimes \psi + \nabla_X \phi \otimes \nabla_Y \psi + \nabla_Y \phi \otimes \nabla_X \psi + \phi \otimes \nabla_Y \nabla_X \psi - \nabla_{\nabla_Y X} \phi \otimes \psi - \phi \otimes \nabla_{\nabla_Y X} \psi \\
-&\nabla_X \nabla_Y \phi \otimes \psi - \nabla_Y \phi \otimes \nabla_X \psi - \nabla_X \phi \otimes \nabla_Y \psi - \phi \otimes \nabla_X \nabla_Y \psi + \nabla_{\nabla_X Y} \phi \otimes \psi + \phi \otimes \nabla_{\nabla_X Y} \psi \\
=&(\nabla_Y \nabla_X - \nabla_X \nabla_Y - \nabla_{[Y,X]})\phi \otimes \psi + \phi \otimes (\nabla_Y \nabla_X - \nabla_X \nabla_Y - \nabla_{[Y,X]})\psi \\
=&\nabla^2 \phi(\cdots; X, Y) \otimes \psi(\cdots) + \phi(\cdots) \otimes \nabla^2 \psi(\cdots; X, Y).
\end{aligned}
$$

5. 最后利用归纳法, 假设 $r + s < k$ 时, (r, s)-型张量场都满足 Ricci 恒等式. 则对于 $r + s = k$ 的张量场 ϕ, 其总可以分解为若干 (r, s) 型张量单项式 (即由 X_i 和 θ^j 张量得到的). 则我们只需证明 Ricci 恒等式对单项式成立. 对某单项式, 其可以表示为两个指标和低于 k 的张量的张量积 $\phi \otimes \psi$. 则根据第 4 部分和归纳假设,

$$
\begin{aligned}
&\nabla^2(\phi \otimes \psi)(\cdots; X, Y) - \nabla^2(\phi \otimes \psi)(\cdots; Y, X) \\
=&\nabla^2 \phi(\cdots; X, Y) \otimes \psi(\cdots) + \phi(\cdots) \otimes \nabla^2 \psi(\cdots; X, Y) \\
=&-\mathrm{Rm}(X, Y)\phi(\cdots) \otimes \psi(\cdots) - \phi(\cdots) \otimes \mathrm{Rm}(X, Y)\psi(\cdots) \\
=&-\mathrm{Rm}(X, Y)(\phi(\cdots))\psi(\cdots) + \sum_i \phi(\cdots, \mathrm{Rm}(X, Y)u_i, \cdots)\psi(\cdots) \\
-&\phi(\cdots)\mathrm{Rm}(X, Y)\psi(\cdots) + \phi(\cdots) \sum_j \psi(\cdots, \mathrm{Rm}(X, Y)v_j, \cdots) \\
=&-\mathrm{Rm}(X, Y)(\phi \otimes \psi(\cdots)) + \sum_k \phi \otimes \psi(\cdots, \mathrm{Rm}(X, Y)w_k, \cdots) \\
=&-\mathrm{Rm}(X, Y)(\phi \otimes \psi)(\cdots).
\end{aligned}
$$

因此 Ricci 恒等式对任意张量成立.

0.4 习题 C

题目 4. 证明局部标架的 Ricci 恒等式

$$\phi^{i_1\cdots i_r}_{j_1\cdots j_s,kl}-\phi^{i_1\cdots i_r}_{j_1\cdots j_s,lk}=\sum_{\alpha=1}^{s}\phi^{i_1\cdots i_r}_{j_1\cdots j_{\alpha-1}hj_{\alpha+1}\cdots j_s}\mathrm{R}^h_{j_\alpha kl}-\sum_{\beta=1}^{s}\phi^{i_1\cdots i_{\beta-1}hi_{\beta+1}\cdots i_r}_{j_1\cdots j_s}\mathrm{R}^{i_\beta}_{hkl}.$$

解答. 首先

$$\mathrm{Rm}(e_k,e_l)e_j=\mathrm{R}^h_{jkl}e_h,$$

以及

$$\mathrm{Rm}(e_k,e_l)\omega^i(e_s)=-\omega^i(\mathrm{Rm}(e_k,e_l)e_s)=-\omega^i(\mathrm{R}^h_{skl}e_h)=-\mathrm{R}^i_{skl}.$$

所以

$$\mathrm{Rm}(e_k,e_l)\omega^i=-\mathrm{R}^i_{hkl}\omega^h.$$

因此利用习题 B 证明的 Ricci 恒等式,

$$\begin{aligned}
&\phi^{i_1\cdots i_r}_{j_1\cdots j_s,kl}-\phi^{i_1\cdots i_r}_{j_1\cdots j_s,lk}\\
=&\nabla^2\phi(\omega^1,\cdots,\omega^r,e_1,\cdots,e_s;e_k,e_l)-\nabla^2\phi(\omega^1,\cdots,\omega^r,e_1,\cdots,e_s;e_l,e_k)\\
=&-\mathrm{Rm}(e_k,e_l)(\phi(\omega^1,\cdots,\omega^r,e_1,\cdots,e_s))\\
&+\sum_{\alpha=1}^{r}\phi(\omega^1,\cdots,\omega^{\alpha-1},\mathrm{Rm}(e_k,e_l)\omega^\alpha,\omega^{\alpha+1},\cdots,\omega^r,e_1,\cdots,e_s)\\
&+\sum_{b=1}^{r}\phi(\omega^1,\cdots,\omega^r,e_1,\cdots,e_{\beta-1},\mathrm{Rm}(e_k,e_l)e_\beta,e_{\beta+1},\cdots,e_s)\\
=&\sum_{\alpha=1}^{s}\phi^{i_1\cdots i_r}_{j_1\cdots j_{\alpha-1}hj_{\alpha+1}\cdots j_s}\mathrm{R}^h_{j_\alpha kl}-\sum_{\beta=1}^{s}\phi^{i_1\cdots i_{\beta-1}hi_{\beta+1}\cdots i_r}_{j_1\cdots j_s}\mathrm{R}^{i_\beta}_{hkl}.
\end{aligned}$$