第三次作业

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0.1 132 页习题 4

题目 1. 设 M 为光滑流形, ∇ 为对称仿射联络. 设 $\{e_i\}$ 是局部基向量场, $\{\omega^i\}$ 和 $\{\omega_j^i\}$ 分别是对偶基和联络 1–形式, 证明:

$$\nabla_X \omega^i = -\omega_j^i(X)\omega^j, \quad \forall X \in \mathscr{X}(M).$$

解答. 取向量场 $Y \in \mathscr{X}(M)$, 设 $Y = Y^i e_i$.

$$\begin{aligned}
(\nabla_X \omega^i)(Y) =& X(\omega^i(Y)) - \omega^i(\nabla_X Y) \\
=& X(Y^i) - \omega^i(X(Y^j)e_j + Y^j \omega_j^k(X) e_k) \\
=& X(Y^i) - X(Y^i) - Y^j \omega_j^i(X) \\
=& -\omega_j^i(X)\omega^j(Y).
\end{aligned}$$

因此 $\nabla_X \omega^i = -\omega_j^i(X)\omega^j$.

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题目 2. 设 (M^n, g) 为 Riemann 流形, $\omega = \omega_{i_1\cdots i_m} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_m}$ 为 m–形式, 其中

$$\begin{aligned}
\omega_{i_1\cdots i_m} =& \sqrt{G}\varepsilon_{i_1\cdots i_m}, \\
\varepsilon_{i_1\cdots i_m} =& \begin{cases} 0 & (i_1, \cdots, i_m)\text{中有相同时}, \\ 1 & (i_1, \cdots, i_m)\text{为偶置换}, \\ -1 & (i_1, \cdots, i_m)\text{为奇置换}. \end{cases}
\end{aligned}$$

证明: $\omega_{i_1\cdots i_m, k} = 0$, 即 ω 是平行的.

解答. 如果按照题目, $\omega \equiv 0$, 所以此题应该是说明体积元关于 Riemann 联络是平行的.

由于体积 $\mathrm{vol} = \sqrt{G}\,\mathrm{d}x^1 \wedge \cdots \mathrm{d}x^m$ 是张量, 其协变导数也是张量. 任取 $\{v_1, \cdots, v_n\} \subset \mathscr{X}(M)$, $v_i = \alpha_i^j e_j$, 则

此时

$$\begin{aligned}\nabla_{e_k}\mathrm{vol}(v_1,\cdots,v_m)=&e_k(\mathrm{vol}(v_1,\cdots,v_m))-\sum_{i=1}^m\mathrm{vol}(v_1,\cdots,\nabla_{e_k}v_i,\cdots,v_m)\\=&e_k(\mathrm{vol}(v_1,\cdots,v_m))-\sum_{i=1}^m\mathrm{vol}(v_1,\cdots,e_k(\alpha_i^l)e_l+\alpha_i^j\omega_j^l(e_k)e_l,\cdots,v_m).\end{aligned}$$

记矩阵 $A=(\alpha_i^j)$, 以及记其伴随阵为 $A^\star=(A_i^j)$, 满足 $\sum_i\alpha_l^iA_j^i=\delta_{jl}\det A$, $\sum_i\alpha_i^jA_i^l=\delta_{jl}\det A$. 则利用这些记号, 以及

$$e_k(\det A)=\sum_{i=1}^m\sum_{j=1}^m e_k(\alpha_i^j)A_i^j,$$

和

$$e_k(\sqrt{G})=\frac{e_k(G)}{2\sqrt{G}}=\sqrt{G}\sum_{i,j=1}^m\frac{1}{2}(\langle\nabla_{e_k}e_i,e_j\rangle+\langle e_i,\nabla_{e_k}e_j\rangle)g^{ij},$$

可以得到

$$\begin{aligned}\nabla_{e_k}\mathrm{vol}(v_1,\cdots,v_m)=&e_k(\mathrm{vol}(v_1,\cdots,v_m))-\sum_{i=1}^m\mathrm{vol}(v_1,\cdots,\left(e_k(\alpha_i^l)+\alpha_i^j\omega_j^l(e_k)\right)e_l,\cdots,v_m)\\=&e_k(\sqrt{G}\det A)-\sqrt{G}\sum_{i=1}^m\sum_{l=1}^m\left(e_k(\alpha_i^l)+\alpha_i^j\omega_j^l(e_k)\right)\cdot A_i^l\\=&e_k(\sqrt{G}\det A)-\sqrt{G}\sum_{i=1}^m\sum_{l=1}^m e_k(\alpha_i^l)A_i^l-\sqrt{G}\sum_{l=1}^m\omega_j^l(e_k)\left(\sum_{i=1}^m\alpha_i^j\cdot A_i^l\right)\\=&e_k(\sqrt{G}\det A)-\sqrt{G}e_k(\det A)-\sqrt{G}\sum_{j,l=1}^m\omega_j^l(e_k)\delta_{jl}\det A\\=&e_k(\sqrt{G}\det A)-\sqrt{G}e_k(\det A)-\sqrt{G}\sum_{j=1}^m\omega_j^j(e_k)\det A.\end{aligned}$$