

第九次作业

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题目 1. 设 (M, g) 为 Riemann 流形. (U, φ, x^i) 是以 q 为原点的法坐标图.

$$X_0 = \xi^i \left(\frac{\partial}{\partial x^i} \right)_q, \quad Y_0 = \eta^i \left(\frac{\partial}{\partial x^i} \right)_q$$

均为单位向量. $C: [0, r) \rightarrow C(s)$ 为在 $q = C(0)$ 点以 X_0 为切向量的测地线, $Y(s)$ 是将 Y_0 沿 C 平行移动而得的切向量. 证明:

(i) 在法坐标系的原点

$$\frac{\partial \Gamma_{ij}^k}{\partial x^l} = -\frac{1}{3}(\mathbf{R}_{ijl}^k + \mathbf{R}_{jil}^k),$$

(ii) 设 $Y(s) = \zeta^i \left(\frac{\partial}{\partial x^i} \right)_{C(s)}$, 则

$$\zeta^i(s) = \eta^i + \frac{1}{6}(\mathbf{R}_{jkl}^i)_q \zeta^j \eta^k \xi^l s^2 + o(s^3),$$

(iii) 若 $\langle X_0, Y_0 \rangle = 0$, 且令 $\|Y(s)\|_q^2 = g_{ij}(q) \xi^i(s) \xi^j(s)$, 则

$$\|Y(s)\|_q = 1 + \frac{s^2}{6} \mathbf{R}(X_0, Y_0, X_0, Y_0) + o(s^3).$$

解答. 记 $e_i = \frac{\partial}{\partial x^i}$.

(i) 记 M 的维数为 n . 任取 $\mathbf{u} = (u^1, \dots, u^n) \in T_q M$, 因为 $\exp_q(t\mathbf{u}) = (tu^1, \dots, tu^n)$ 是测地线, 所以由测地线方程,

$$\Gamma_{ij}^k(\exp_q(t\mathbf{u})) u^i u^j = 0.$$

因此在这一点, Riemann 曲率张量为

$$\begin{aligned} \mathbf{R}_{jkl}^i(q) &= g^{im} \langle \mathbf{Rm}(e_k, e_l) e_j, e_m \rangle \\ &= \frac{\partial \Gamma_{jl}^i}{\partial x^k} + \Gamma_{ks}^i \Gamma_{jl}^s - \frac{\partial \Gamma_{jk}^i}{\partial x^l} - \Gamma_{ls}^i \Gamma_{jk}^s \\ &= \frac{\partial \Gamma_{jl}^i}{\partial x^k} - \frac{\partial \Gamma_{jk}^i}{\partial x^l}. \end{aligned}$$

由 \mathbf{u} 的任意性, $\Gamma_{ij}^k(q) = 0$. 对测地线方程关于 t 微分, 得到

$$\mathbf{u}(\Gamma_{ij}^k)u^i u^j = \frac{\partial \Gamma_{ij}^k}{\partial x^t} u^i u^j u^l = 0.$$

取 $\mathbf{u} = u^i e_i$, 那么

$$\frac{\partial \Gamma_{ii}^k}{\partial x^i} u^i u^i u^i = 0,$$

取 $\mathbf{u} = e_i + e_j$ 和 $\mathbf{u} = e_i - e_j$, 那么

$$\begin{aligned} \frac{\partial \Gamma_{ii}^k}{\partial x^j} + 2\frac{\partial \Gamma_{ij}^k}{\partial x^i} + 2\frac{\partial \Gamma_{ij}^k}{\partial x^j} + \frac{\partial \Gamma_{jj}^k}{\partial x^i} &= 0, \\ -\frac{\partial \Gamma_{ii}^k}{\partial x^j} - 2\frac{\partial \Gamma_{ij}^k}{\partial x^i} + 2\frac{\partial \Gamma_{ij}^k}{\partial x^j} + \frac{\partial \Gamma_{jj}^k}{\partial x^i} &= 0, \end{aligned}$$

所以

$$\begin{aligned} 0 &= \frac{\partial \Gamma_{ii}^k}{\partial x^j} + 2\frac{\partial \Gamma_{ij}^k}{\partial x^i} \\ &= 3\frac{\partial \Gamma_{ij}^k}{\partial x^i} + R_{iji}^k \\ &= 3\frac{\partial \Gamma_{ii}^k}{\partial x^j} + 2R_{iij}^k \end{aligned}$$

因此

$$\begin{aligned} \frac{\partial \Gamma_{ij}^k}{\partial x^i} &= -\frac{1}{3}R_{iji}^k = -\frac{1}{3}(R_{iji}^k + R^{jii}), \\ \frac{\partial \Gamma_{ii}^k}{\partial x^j} &= -\frac{2}{3}R_{iij}^k. \end{aligned}$$

取 $\mathbf{u} = u^i e_i + u^j e_j + u^k e_k$, 由上面的计算, 如果求和中 i, j, k 只选到一个或两个下标, 那么这部分求和项为 0. 所以

$$\begin{aligned} 0 &= \frac{\partial \Gamma_{ij}^k}{\partial x^l} + \frac{\partial \Gamma_{jl}^k}{\partial x^i} + \frac{\partial \Gamma_{il}^k}{\partial x^j} \\ &= 3\frac{\partial \Gamma_{ij}^k}{\partial x^l} + \left(\frac{\partial \Gamma_{jl}^k}{\partial x^i} - \frac{\partial \Gamma_{ij}^k}{\partial x^l}\right) + \left(\frac{\partial \Gamma_{il}^k}{\partial x^j} - \frac{\partial \Gamma_{ij}^k}{\partial x^l}\right) \\ &= 3\frac{\partial \Gamma_{ij}^k}{\partial x^l} + R_{jil}^k + R_{ijl}^k. \end{aligned}$$

因此

$$\frac{\partial \Gamma_{ij}^k}{\partial x^l} = -\frac{1}{3}(R_{jil}^k + R_{ijl}^k).$$

(ii) $Y(s)$ 满足平行移动方程

$$\frac{\partial \zeta^i(s)}{\partial s} + \Gamma_{jk}^i \zeta^j(s) \zeta^k(s) = 0,$$

所以在 q 点

$$\frac{\partial \zeta^i(s)}{\partial s} = -\Gamma_{jk}^i \eta^j \zeta^k = 0,$$

并且利用 $\Gamma_{jk}^i(q) = 0$, 在 q 点二阶导数为

$$\frac{\partial^2 \zeta^i(s)}{\partial s^2} = -\frac{\partial \Gamma_{jk}^i}{\partial s}(q) \eta^j \zeta^k.$$

利用 (i) 的结论,

$$\frac{\partial \Gamma_{ij}^k}{\partial s}(q) = -\frac{1}{3}(R_{jil}^k + R_{ijl}^k) \zeta^l,$$

所以

$$\begin{aligned}\zeta^i(s) &= \eta^i + \frac{\partial \zeta^i(s)}{\partial s} s + \frac{1}{2} \frac{\partial^2 \zeta^i(s)}{\partial s^2} s^2 + O(s^3) \\ &= \eta^i + \frac{1}{6} (R_{jkl}^i + R_{kjl}^i) \eta^j \xi^k \xi^l s^2 + O(s^3).\end{aligned}$$

因为 R_{jkl}^i 交换 kl 会变号, 所以 $R_{jkl}^i \eta^j \xi^k \xi^l = 0$. 因此

$$\zeta^i(s) = \eta^i + \frac{1}{6} R_{kjl}^i \xi^j \eta^k \xi^l s^2 + O(s^3).$$

(iii) $\langle X_0, Y_0 \rangle = 0$, 则 $\sum_i \eta^i \xi^i = 0$. 所以

$$\begin{aligned}\|Y(s)\|_q^2 &= g_{ij}(q) \xi^i(s) \xi^j(s) \\ &= |\eta^i|^2 + \sum_i \frac{1}{6} R_{ijkl} \eta^i \xi^j \eta^k \xi^l s^2 + O(s^3) \\ &= 1 + \frac{1}{6} R(X_0, Y_0, X_0, Y_0) s^2 + O(s^3).\end{aligned}$$

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题目 2. 设 m 维 Riemann 流形 (M, g) 在测地极坐标系 $(r, \theta^1, \dots, \theta^{m-1})$ 下具有度量形式

$$ds^2 = (dr)^2 + (f(r))^2 h_{ij}(\theta) d\theta^i d\theta^j,$$

其中 $m-1$ 维度量 $d\sigma^2 = h_{ij}(\theta) d\theta^i d\theta^j$ 具有常数截面曲率 1. 求证 ds^2 具有常数截面曲率 c 的充要条件是

$$f(r) = \begin{cases} \sin(\sqrt{cr^2})/\sqrt{c} & c > 0, \\ r & c = 0, \\ \sinh(\sqrt{-cr^2})/\sqrt{-c} & c < 0. \end{cases}$$

解答. 第一步要计算 Christoffel 记号.,

$$\Gamma_{jk}^i = \frac{1}{2} g^{il} \left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right).$$

1. 若三个指标都是 r :

$$\Gamma_{rr}^r = 0.$$

2. 若两个指标是 r , 一个是 θ^i :

$$\Gamma_{ri}^r = \frac{1}{2} g^{rr} \left(\frac{\partial g_{rr}}{\partial \theta^i} + \frac{\partial g_{ri}}{\partial r} - \frac{\partial g_{ri}}{\partial r} \right) = 0,$$

$$\Gamma_{rr}^i = \frac{1}{2} g^{ij} \left(\frac{\partial g_{rj}}{\partial r} + \frac{\partial g_{jr}}{\partial r} - \frac{\partial g_{rr}}{\partial \theta^j} \right) = 0.$$

3. 若一个指标是 r , 其余两个是 θ^i, θ^j :

$$\Gamma_{ij}^r = \frac{1}{2} g^{rr} \left(\frac{\partial g_{ir}}{\partial \theta^j} + \frac{\partial g_{rj}}{\partial \theta^i} - \frac{\partial g_{ij}}{\partial r} \right) = -f(r) f'(r) h_{ij}(\theta),$$

$$\Gamma_{rj}^i = \frac{1}{2} g^{ik} \left(\frac{\partial g_{rk}}{\partial \theta^j} + \frac{\partial g_{jk}}{\partial r} - \frac{\partial g_{rj}}{\partial \theta^k} \right) = \frac{f'(r)}{f(r)} \delta_j^i.$$

4. 若所有指标都不为 r : 设 $\bar{\Gamma}$ 表示 $d\sigma^2$ 的 Christoffel 记号,

$$\Gamma_{jk}^i = \bar{\Gamma}_{jk}^i.$$

那么 Riemann 曲率张量

$$R_{\beta\delta\eta}^\alpha = \frac{\partial \Gamma_{\beta\eta}^\alpha}{\partial x^\delta} + \Gamma_{\delta\xi}^\alpha \Gamma_{\beta\eta}^\xi - \frac{\partial \Gamma_{\beta\delta}^\alpha}{\partial x^\eta} - \Gamma_{\eta\xi}^\alpha \Gamma_{\beta\delta}^\xi,$$

所以,

$$\begin{aligned} R_{riri} &= g_{rr} R_{iri}^r = R_{iri}^r \\ &= \frac{\partial \Gamma_{ii}^r}{\partial r} + \Gamma_{r\xi}^r \Gamma_{ii}^\xi - \frac{\partial \Gamma_{ir}^r}{\partial \theta^i} - \Gamma_{i\xi}^r \Gamma_{ir}^\xi \\ &= - (f(r))^2 h_{ii}(\theta) - f(r) f''(r) h_{ii}(\theta) - \sum_j \left(-f(r) f' h_{ij}(\theta) \cdot \frac{f'(r)}{f(r)} \delta_j^i \right) \\ &= -f(r) f''(r) h_{ii}(\theta). \end{aligned}$$

$$\begin{aligned} R_{ijij} &= g_{ik} R_{jij}^k \\ &= g_{ik} \left(\frac{\partial \Gamma_{jj}^k}{\partial \theta^i} + \Gamma_{i\xi}^k \Gamma_{jj}^\xi - \frac{\partial \Gamma_{ji}^k}{\partial \theta^j} - \Gamma_{j\xi}^k \Gamma_{ji}^\xi \right) \\ &= g_{ik} \left(\frac{\partial \Gamma_{jj}^k}{\partial \theta^i} + \Gamma_{il}^k \Gamma_{jj}^l - \frac{\partial \Gamma_{ji}^k}{\partial \theta^j} - \Gamma_{jl}^k \Gamma_{ji}^l \right) + g_{ik} (\Gamma_{ir}^k \Gamma_{jj}^r - \Gamma_{jr}^k \Gamma_{ji}^r) \end{aligned}$$

设 $d\sigma^2$ 的曲率张量为 \bar{R} , 则有

$$\bar{R}_{jij}^i = \frac{\partial \Gamma_{jj}^i}{\partial \theta^i} + \Gamma_{ik}^i \Gamma_{jj}^k - \frac{\partial \Gamma_{ji}^i}{\partial \theta^j} - \Gamma_{jk}^i \Gamma_{ji}^k$$

所以

$$\begin{aligned} R_{ijij} &= (f(r))^2 \bar{R}_{ijij} + g_{ik} \left(\frac{f'(r)}{f(r)} \delta_i^k \cdot -f(r) f'(r) h_{jj}(\theta) - \frac{f'(r)}{f(r)} \delta_j^k \cdot -f(r) f'(r) h_{ij}(\theta) \right) \\ &= (f(r))^2 \bar{R}_{ijij} - (f(r))^2 f'(r)^2 (h_{ii}(\theta) h_{jj}(\theta) - h_{ij}(\theta)^2). \end{aligned}$$

因此截面曲率为

$$\begin{aligned} K(\partial r, \partial \theta^i) &= \frac{R_{riri}}{g_{rr} g_{ii} - g_{ri}^2} = -\frac{f''(r)}{f(r)}, \\ K(\partial \theta^i, \partial \theta^j) &= \frac{R_{ijij}}{g_{ii} g_{jj} - g_{ij}^2} = \frac{1}{(f(r))^2} - \frac{(f(r))^2 (f'(r))^2 (h_{ii}(\theta) h_{jj}(\theta) - h_{ij}(\theta)^2)}{f(r)^4 (h_{ii}(\theta) h_{jj}(\theta) - h_{ij}(\theta)^2)} \\ &= \frac{1}{(f(r))^2} - \frac{(f'(r))^2}{(f(r))^2}. \end{aligned} \tag{1}$$

因此如果是常截面曲率流形, 那么

$$-\frac{f''(r)}{f(r)} = \frac{1}{(f(r))^2} - \frac{(f'(r))^2}{(f(r))^2} = c.$$

即

$$f''(r) = -c f(r), \quad 1 - (f'(r))^2 = c (f(r))^2.$$

所以

1. 如果 $c > 0$, 那么 $f(r) = A \sin(\sqrt{c}r) + B \cos(\sqrt{c}r)$, 利用第二个方程,

$$1 = c(f(r))^2 + (f'(r))^2 = cB^2 + cA^2,$$

此外, 计算 $|\partial B(r)|$ 的面积为

$$A(r) = \int_{\partial B(r)} (f(r))^{2(m-1)} \det(h_{ij}(\theta)) d\theta,$$

当 $r \rightarrow 0$ 时, $A(r)$ 应当趋向于 0. 所以 $\lim_{r \rightarrow 0} f(r) = 0$. 所以 $A = \frac{1}{\sqrt{c}}$, $B = 0$. 即

$$f(r) = \sin(r\sqrt{c})/\sqrt{c}.$$

2. 如果 $c = 0$, 那么 $f(r) = Ax + B$. 结合上面所说, 有

$$1 = c(Ax + B)^2 + A^2, \quad \lim_{r \rightarrow 0} f(r) = 0,$$

所以 $A = 1$, $B = 0$. 即

$$f(r) = r.$$

3. 如果 $c < 0$, 那么 $f(r) = Ae^{r\sqrt{-c}} + Be^{-r\sqrt{-c}}$. 所以有

$$1 = c(f(r))^2 + (f'(r))^2 = 4cAB, \quad \lim_{r \rightarrow 0} f(r) = A + B = 0.$$

所以

$$f(r) = e^{r\sqrt{-c}}/(2\sqrt{-c}) + e^{-r\sqrt{-c}}/(2\sqrt{-c}) = \sinh(r\sqrt{-c})/\sqrt{-c}.$$

综上, 如果 M 是常截面曲率流形, 且截面曲率为 c , 则

$$f(r) = \begin{cases} \sin(r\sqrt{c})/\sqrt{c} & c > 0, \\ r & c = 0, \\ \sinh(r\sqrt{-c})/\sqrt{-c} & c < 0. \end{cases}$$

反过来, 如果 f 表现如上, 那么根据 (1) 式, 可以计算截面曲率为 c . 因此题目得证.

0.3 问题 1.2

题目 3. 假设 (\mathcal{M}, g) 是 m 维 Riemann 流形, $p \in \mathcal{M}$ 是流形上任意一点. 在 p 的法坐标 (\mathcal{U}_p, x_i) 中证明:

$$(i) \quad g_{ij}(x) = \delta_{ij} - \frac{1}{3}R_{ikjl}(0)x_kx_l + O(|x|^3);$$

$$(ii) \quad \det(g_{ij}) = 1 - \frac{1}{3}\text{Ric}_{kl}(0)x_kx_l + O(|x|^3).$$

解答.

1. 核心是计算 g_{ij} 沿由 p 出发到 x 的测地线的导数. 设 $x = (x^1, \dots, x^n)$, 同时其也 p 到 x 指数映射的切向量. 设 $\gamma(t) = \exp_p(tx)$, 那么 $x = \gamma(1)$. 所以一阶导数为

$$\frac{d}{dt}g_{ij}(\gamma(t)) = x^k \partial_k (g_{ij}(\gamma(t))).$$

而 $\partial_k(g_{ij}) = \langle \nabla_{\partial_k} \partial_i, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_k} \partial_j \rangle = g_{jl}\Gamma_{ik}^l + g_{il}\Gamma_{jk}^l$. 因为在法坐标的原点, Christoffel 记号为 0, 所以

$$\frac{d}{dt}g_{ij}(\gamma(t)) = 0.$$

二阶导数为

$$\begin{aligned} \frac{d^2}{dt^2}g_{ij}(\gamma(t)) &= \frac{d}{dt} [x^k (g_{jl}\Gamma_{ik}^l + g_{il}\Gamma_{jk}^l)] \\ &= x^k \left(g_{jl} \frac{d\Gamma_{ik}^l}{dt} + g_{il} \frac{d\Gamma_{jk}^l}{dt} \right) \\ &= x^k \left(g_{jl} x^m \frac{\partial \Gamma_{ik}^l}{\partial x^m} + g_{il} x^m \frac{\partial \Gamma_{jk}^l}{\partial x^m} \right). \end{aligned}$$

在 0 点, 利用第 0.1 节的 (i), 遂有

$$\begin{aligned} \frac{d^2}{dt^2}g_{ij}(0) &= x^k \left(x^m g_{jl} \frac{\partial \Gamma_{ik}^l}{\partial x^m}(0) + x^m g_{il} \frac{\partial \Gamma_{jk}^l}{\partial x^m}(0) \right) \\ &= -\frac{1}{3} x^k x^m g_{jl}(0) (R_{ikm}^l(0) + R_{kim}^l(0)) - \frac{1}{3} x^k x^m g_{il}(0) (R_{jkm}^l(0) + R_{kjm}^l(0)) \\ &= -\frac{1}{3} x^k x^m (R_{jikm}(0) + R_{jkim}(0) + R_{ijkm}(0) + R_{ikjm}(0)) \\ &= -\frac{2}{3} x^k x^l R_{ijkl}(0), \end{aligned}$$

代入 Taylor 展式, 得到

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ijkl}(0) x^k x^l + O(|x|^3).$$

2. 依然要计算导数. 记 $G = (g_{ij})_{n \times n}$,

$$\frac{d}{dt} \det(g_{ij})(\gamma(t)) = \det G \operatorname{tr} G^{-1} G' = \det(g_{ij})(\gamma(t)) (g^{ki} g'_{ik})(\gamma(t)),$$

所以 $\frac{d}{dt} \det(g_{ij})(0) = 0$. 继续计算, 二阶导数为

$$\frac{d^2}{dt^2} \det(g_{ij})(\gamma(t)) = \frac{d}{dt} \sum_{j,k} [g^{jk} (g^{ki} g'_{ij})](\gamma(t)),$$

因为 $0 = (g^{ij} g_{jk})' = (g^{ij})' g_{jk} + g^{ij} (g_{jk})'$, 在 0 点处, $0 = (g^{ij})' g_{jk}$. 所以 $(g^{ij})'(0) = 0$. 因此上式中求导只会求在 g' 上. 于是利用 (1) 的结果,

$$\frac{d^2}{dt^2} \det(g_{ij})(0) = -\frac{2}{3} \sum_{j,k} [g^{jk} (g^{ki} x^k x^l R_{ijkl}(0))] = -\frac{2}{3} x^k x^l \operatorname{Ric}_{kl}(0),$$

代入 Taylor 展式, 得到

$$\det(g_{ij})(x) = 1 - \frac{1}{3} \operatorname{Ric}_{kl}(0) x^k x^l + O(|x|^3).$$

0.4 问题 1.3

题目 4. 假设 (\mathcal{M}, g) 是 m 维 Riemann 流形. 假设 $r > 0$ 足够小使得 $\exp_p: D_r(0) \rightarrow B_r(p)$ 是微分同胚, 其中 $D_r(0) = \{v \in T_p \mathcal{M} : |v| < r\}$, $B_r(p) = \exp_p D_r(0)$. 记 $S_r(p) = \partial B_r(p)$. 证明:

1. $\text{vol}(B_r(p)) = \omega_m r^m \left(1 - \frac{\text{scal}(p)}{6(m+2)} r^2 + O(r^3) \right)$, 其中 ω_m 是 \mathbb{R}^m 的单位球的欧氏体积;
2. $\text{area}(S_r(p)) = m\omega_m r^{m-1} - \frac{1}{6} \text{scal}(p) \omega_m r^{m+1} + O(r^{m+2})$.

解答.

1. $\text{vol}(B_r(p))$ 相当于在欧氏空间 \bar{B}_r 上对 \mathcal{M} 的体积元 dvol 积分. 而 $\text{dvol} = \sqrt{\det(g_{ij})} \text{d}m_{\mathbb{R}^m}$. 设

$$\sqrt{\det(g_{ij})}(x) = 1 + a_k x^k + b_{kl} x^k x^l + O(|x|^3),$$

那么平方可得

$$a_k = 0, \quad 2b_{kl} = -\frac{1}{3} \text{Ric}_{kl}(p),$$

所以

$$\sqrt{\det(g_{ij})}(x) = 1 - \frac{1}{6} \text{Ric}_{kl}(p) x^k x^l + O(|x|^3).$$

因此积分得

$$\begin{aligned} \text{vol}(B_r(p)) &= \int_{\bar{B}_r} 1 - \frac{1}{6} \text{Ric}_{kl}(p) x^k x^l + O(|x|^3) \text{d}x \\ &= \text{vol}(\bar{B}_r) - \frac{1}{6} \int_{\bar{B}_r} \text{Ric}_{kl}(p) x^k x^l \text{d}x + \text{vol}(\bar{B}_r) O(r^3) \end{aligned}$$

考虑 p 点附近的测地极坐标 $(r, \theta_1, \dots, \theta_{m-1})$. 则

$$\begin{cases} x_1 = r \cos \theta_1 \\ x_2 = r \sin \theta_1 \cos \theta_2 \\ \vdots \\ x_{m-1} = r \sin \theta_1 \cdots \sin \theta_{m-2} \cos \theta_{m-1} \\ x_m = r \sin \theta_1 \cdots \sin \theta_{m-2} \sin \theta_{m-1} \end{cases}$$

其中 $1 \leq i \leq m-2$ 时, $\theta_i \in [0, \pi]$; 而 $\theta_{m-1} \in [0, 2\pi]$. 观察对 θ_{m-1} 的积分, 有以下结论:

引理. 若 $k \neq l$, 则

$$\int_{\bar{B}_r} x^k x^l \text{d}x = 0.$$

引理的证明. 由于 $k \neq l$, 所以 θ_{m-1} 的项只可能有三种情况: $\cos \theta_{m-1}$, $\sin \theta_{m-1}$, $\sin \theta_{m-1} \cos \theta_{m-1} = \sin(2\theta_{m-1})/2$. 而这三者在 $[0, 2\pi]$ 上的积分都是 0. 因此引理成立. \square

有此结论, 则

$$\int_{\bar{B}_r} \text{Ric}_{kl}(p) x^k x^l \text{d}x = \int_{\bar{B}_r} \sum_k \text{Ric}_{kk}(p) (x^k)^2 \text{d}x.$$

由对称性, 不同 k 的 $\int_{\bar{B}_r} (x^k)^2 dx$ 都相等, 并且其均等于

$$\frac{1}{m} \int_{\bar{B}_r} \sum_{k=1}^m (x^k)^2 dx = \frac{1}{m} \int_{\bar{B}_r} |x|^2 dx = \omega_m \frac{r^{m+2}}{m+2},$$

故

$$\int_{\bar{B}_r} \text{Ric}_{kl}(p) x^k x^l dx = m \omega_m \frac{\text{scal}(p)}{m+2} r^{m+2}.$$

因此

$$\text{vol}(B_r(p)) = \omega_m r^m - \frac{1}{6} m \omega_m \frac{\text{scal}(p)}{m+2} r^{m+2} + \omega_m r^m O(r^3) = \omega_m r^m \left(1 - \frac{\text{scal}(p)}{6(m+2)} r^2 + O(r^3)\right).$$

2. 计算 $\text{area}(S_r(p))$, 只需要对 $\text{vol}(B_r(p))$ 关于 r 求导即可.

0.5 问题 1.4

题目 5. 假设 (\mathcal{M}, g) 是 m 维 Riemann 流形, $p \in M$, $\Pi_p \subset T_p \mathcal{M}$ 是 2 维截面. 令 $C_r^0 = \{v \in \Pi_p : |v| = r\}$, $C_r = \exp_p C_r^0$. 用 L_r 记曲线 C_r 的长度. 证明:

$$K(\Pi_p) = \frac{3}{\pi} \lim_{r \rightarrow 0} \frac{2\pi r - L_r}{r^3},$$

其中 $K(\Pi_p)$ 表示 p 点处关于 Π_p 的截面曲率.

解答. 对 $\exp_p \Pi_p$ 应用问题 1.3 的结论. 这是一个 2 维的子流形, 其截面曲率就是 $K(\Pi_p)$. 因此

$$L_r = \text{area}(C_r) = 2\pi r - \frac{2K(\Pi_p)}{6} \pi r^3 + O(r^4).$$

故

$$\lim_{r \rightarrow 0} \frac{2\pi r - L_r}{r^3} = \lim_{r \rightarrow 0} \frac{\frac{2K(\Pi_p)}{6} \pi r^3 + O(r^4)}{r^3} = \frac{\pi}{3} K(\Pi_p),$$

问题得证.

0.6 问题 1.5

题目 6. 假设 (\mathcal{M}, g) 是 m 维 Riemann 流形, $\gamma_k: [0, 1] \rightarrow \mathcal{M}$, $k = 1, 2$ 是两条正规测地线 (速度为 1 的测地线), 且满足 $\gamma_1(0) = \gamma_2(0) = p$. 用 $\theta \in (0, \pi)$ 记 $v = \dot{\gamma}_1(0)$ 与 $w = \dot{\gamma}_2(0)$ 的夹角, $\Pi_p = \text{span}\{v, w\}$ 是 $T_p \mathcal{M}$ 的 2 维截面. 证明:

$$d(\gamma_1(t), \gamma_2(t)) = \sqrt{2(1 - \cos \theta)} \left[1 - \frac{1}{6} K(\Pi_p) \cos^2(\theta/2) t^2 + O(t^3)\right].$$