第九次作业

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题目 1. 设 (M, g) 为 Riemann 流形. (U, φ, x^i) 是以 q 为原点的法坐标图.

$$X_0 = \xi^i\Big(\frac{\partial}{\partial x^i}\Big)_q, \quad Y_0 = \eta^i\Big(\frac{\partial}{\partial x^i}\Big)_q$$

均为单位向量. $C\colon [0, r) \to C(s)$ 为在 $q = C(0)$ 点以 X_0 为切向量的测地线, $Y(s)$ 是将 Y_0 沿 C 平行移动而得的切向量. 证明:

(i) 在法坐标系的原点

$$\frac{\partial \Gamma^k_{ij}}{\partial x^l} = -\frac{1}{3}(\mathrm{R}^k_{ijl} + \mathrm{R}^k_{jil}),$$

(ii) 设 $Y(s) = \zeta^i\Big(\dfrac{\partial}{\partial x^i}\Big)_{C(s)}$, 则

$$\zeta^i(s) = \eta^i + \frac{1}{6}(\mathrm{R}^i_{jkl})_q \xi^j \eta^k \xi^l s^2 + o(s^3),$$

(iii) 若 $\langle X_0, Y_0\rangle = 0$, 且令 $||Y(s)||^2_q = g_{ij}(q)\xi^i(s)\xi^j(s)$, 则

$$||Y(s)||_q = 1 + \frac{s^2}{6}\mathrm{R}(X_0, Y_0, X_0, Y_0) + o(s^3).$$

解答. 记 $e_i = \dfrac{\partial}{\partial x^i}$.

(i) 记 M 的维数为 n. 任取 $\mathbf{u} = (u^1, \cdots, u^n) \in T_qM$, 因为 $\exp_q(t\mathbf{u}) = (tu^1, \cdots, tu^n)$ 是测地线, 所以由测地线方程,

$$\Gamma^k_{ij}(\exp_q(t\mathbf{u}))u^iu^j = 0.$$

因此在这一点, Riemann 曲率张量为

$$\begin{aligned}
\mathrm{R}^i_{jkl}(q) &= g^{im}\langle \mathrm{Rm}(e_k, e_l)e_j, e_m\rangle \\
&= \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^i_{ks}\Gamma^s_{jl} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \Gamma^i_{ls}\Gamma^s_{jk} \\
&= \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l}.
\end{aligned}$$

由 u 的任意性, $\Gamma_{ij}^k(q)=0$. 对测地线方程关于 t 微分, 得到

$$\mathbf{u}(\Gamma_{ij}^k)u^iu^j=\frac{\partial\Gamma_{ij}^k}{\partial x^l}u^iu^ju^l=0.$$

取 $\mathbf{u}=u^ie_i$, 那么

$$\frac{\partial\Gamma_{ii}^k}{\partial x^i}u^iu^iu^i=0,$$

取 $\mathbf{u}=e_i+e_j$ 和 $\mathbf{u}=e_i-e_j$, 那么

$$\frac{\partial\Gamma_{ii}^k}{\partial x^j}+2\frac{\partial\Gamma_{ij}^k}{\partial x^i}+2\frac{\partial\Gamma_{ij}^k}{\partial x^j}+\frac{\partial\Gamma_{jj}^k}{\partial x^i}=0,$$
$$-\frac{\partial\Gamma_{ii}^k}{\partial x^j}-2\frac{\partial\Gamma_{ij}^k}{\partial x^i}+2\frac{\partial\Gamma_{ij}^k}{\partial x^j}+\frac{\partial\Gamma_{jj}^k}{\partial x^i}=0,$$

所以

$$\begin{aligned}0&=\frac{\partial\Gamma_{ii}^k}{\partial x^j}+2\frac{\partial\Gamma_{ij}^k}{\partial x^i}\\&=3\frac{\partial\Gamma_{ij}^k}{\partial x^i}+\mathrm{R}_{iji}^k\\&=3\frac{\partial\Gamma_{ii}^k}{\partial x^j}+2\mathrm{R}_{iij}^k\end{aligned}$$

因此

$$\frac{\partial\Gamma_{ij}^k}{\partial x^i}=-\frac{1}{3}\mathrm{R}_{iji}^k=-\frac{1}{3}(\mathrm{R}_{iji}^k+\mathrm{R}^{jii}),$$
$$\frac{\partial\Gamma_{ii}^k}{\partial x^j}=-\frac{2}{3}\mathrm{R}_{iij}^k.$$

取 $\mathbf{u}=u^ie_i+u^je_j+u^ke_k$, 由上面的计算, 如果求和中 i,j,k 只选到一个或两个下标, 那么这部分求和项为 0. 所以

$$\begin{aligned}0&=\frac{\partial\Gamma_{ij}^k}{\partial x^l}+\frac{\partial\Gamma_{jl}^k}{\partial x^i}+\frac{\partial\Gamma_{il}^k}{\partial x^j}\\&=3\frac{\partial\Gamma_{ij}^k}{\partial x^l}+\big(\frac{\partial\Gamma_{jl}^k}{\partial x^i}-\frac{\partial\Gamma_{ij}^k}{\partial x^l}\big)+\big(\frac{\partial\Gamma_{il}^k}{\partial x^j}-\frac{\partial\Gamma_{ij}^k}{\partial x^l}\big)\\&=3\frac{\partial\Gamma_{ij}^k}{\partial x^l}+\mathrm{R}_{jil}^k+\mathrm{R}_{ijl}^k.\end{aligned}$$

因此

$$\frac{\partial\Gamma_{ij}^k}{\partial x^l}=-\frac{1}{3}(\mathrm{R}_{jil}^k+\mathrm{R}_{ijl}^k).$$

(ii) $Y(s)$ 满足平行移动方程

$$\frac{\partial\zeta^i(s)}{\partial s}+\Gamma_{jk}^i\zeta^j(s)\xi^k(s)=0,$$

所以在 q 点

$$\frac{\partial\zeta^i(s)}{\partial s}=-\Gamma_{jk}^i\eta^j\xi^k=0,$$

并且利用 $\Gamma_{jk}^i(q)=0$, 在 q 点二阶导数为

$$\frac{\partial^2\zeta^i(s)}{\partial s^2}=-\frac{\partial\Gamma_{jk}^i}{\partial s}(q)\eta^j\xi^k.$$

利用 (i) 的结论,

$$\frac{\partial\Gamma_{ij}^k}{\partial s}(q)=-\frac{1}{3}(\mathrm{R}_{jil}^k+\mathrm{R}_{ijl}^k)\xi^l,$$

所以

$$\begin{aligned}\zeta^i(s) &= \eta^i + \frac{\partial \zeta^i(s)}{\partial s}s + \frac{1}{2}\frac{\partial^2\zeta^i(s)}{\partial s^2}s^2 + O(s^3)\\ &= \eta^i + \frac{1}{6}(\mathrm{R}^i_{jkl} + \mathrm{R}^i_{kjl})\eta^j\xi^k\xi^l s^2 + O(s^3).\end{aligned}$$

因为 R^i_{jkl} 交换 kl 会变号, 所以 $\mathrm{R}^i_{jkl}\eta^j\xi^k\xi^l = 0$. 因此

$$\zeta^i(s) = \eta^i + \frac{1}{6}\mathrm{R}^i_{kjl}\xi^j\eta^k\xi^l s^2 + O(s^3).$$

(iii) $\langle X_0, Y_0\rangle = 0$, 则 $\sum_i \eta^i\xi^i = 0$. 所以

$$\begin{aligned}||Y(s)||_q^2 &= g_{ij}(q)\xi^i(s)\xi^j(s)\\ &= |\eta^i|^2 + \sum_i \frac{1}{6}\mathrm{R}_{ijkl}\eta^i\xi^j\eta^k\xi^l s^2 + O(s^3)\\ &= 1 + \frac{1}{6}R(X_0, Y_0, X_0, Y_0)s^2 + O(s^3).\end{aligned}$$

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题目 2. 设 m 维 Riemann 流形 (M, g) 在测地极坐标系 $(r, \theta^1, \cdots, \theta^{m-1})$ 下具有度量形式

$$\mathrm{d}s^2 = (\mathrm{d}r)^2 + (f(r))^2 h_{ij}(\theta)\,\mathrm{d}\theta^i\theta^j,$$

其中 $m-1$ 维度量 $\mathrm{d}\sigma^2 = h_{ij}(\theta)\,\mathrm{d}\theta^i\theta^j$ 具有常数截面曲率 1. 求证 $\mathrm{d}s^2$ 具有常数截面曲率 c 的充要条件是

$$f(r) = \begin{cases} \sin\left(\sqrt{cr^2}\right)/\sqrt{c} & c > 0,\\ r & c = 0,\\ \sinh\left(\sqrt{-cr^2}\right)/\sqrt{-c} & c < 0.\end{cases}$$

解答. 第一步要计算 Christoffel 记号.,

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right).$$

1. 若三个指标都是 r:

$$\Gamma^r_{rr} = 0.$$

2. 若两个指标是 r, 一个是 θ^i:

$$\Gamma^r_{ri} = \frac{1}{2}g^{rr}\left(\frac{\partial g_{rr}}{\partial \theta^i} + \frac{\partial g_{ri}}{\partial r} - \frac{\partial g_{ri}}{\partial r}\right) = 0,$$

$$\Gamma^i_{rr} = \frac{1}{2}g^{ij}\left(\frac{\partial g_{rj}}{\partial r} + \frac{\partial g_{jr}}{\partial r} - \frac{\partial g_{rr}}{\partial \theta^j}\right) = 0.$$

3. 若一个指标是 r, 其余两个是 θ^i, θ^j:

$$\Gamma^r_{ij} = \frac{1}{2}g^{rr}\left(\frac{\partial g_{ir}}{\partial \theta^j} + \frac{\partial g_{rj}}{\partial \theta^i} - \frac{\partial g_{ij}}{\partial r}\right) = -f(r)f'(r)h_{ij}(\theta),$$

$$\Gamma^i_{rj} = \frac{1}{2}g^{ik}\left(\frac{\partial g_{rk}}{\partial \theta^j} + \frac{\partial g_{jk}}{\partial r} - \frac{\partial g_{rj}}{\partial \theta^k}\right) = \frac{f'(r)}{f(r)}\delta^i_j.$$

4. 若所有指标都不为 r: 设 $\bar{\Gamma}$ 表示 $\mathrm{d}\sigma^2$ 的 Christoffel 记号,

$$\Gamma^i_{jk} = \bar{\Gamma}^i_{jk}.$$

那么 Riemann 曲率张量

$$\mathrm{R}^\alpha_{\beta\delta\eta} = \frac{\partial \Gamma^\alpha_{\beta\eta}}{\partial x^\delta} + \Gamma^\alpha_{\delta\xi}\Gamma^\xi_{\beta\eta} - \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\eta} - \Gamma^\alpha_{\eta\xi}\Gamma^\xi_{\beta\delta},$$

所以,

$$\begin{aligned}
\mathrm{R}_{riri} &= g_{rr}\mathrm{R}^r_{iri} = \mathrm{R}^r_{iri} \\
&= \frac{\partial \Gamma^r_{ii}}{\partial r} + \Gamma^r_{r\xi}\Gamma^\xi_{ii} - \frac{\partial \Gamma^r_{ir}}{\partial \theta^i} - \Gamma^r_{i\xi}\Gamma^\xi_{ir} \\
&= -(f(r))^2 h_{ii}(\theta) - f(r)f''(r)h_{ii}(\theta) - \sum_j \left(-f(r)f' h_{ij}(\theta)\cdot \frac{f'(r)}{f(r)}\delta^i_j\right) \\
&= -f(r)f''(r)h_{ii}(\theta). \\
\mathrm{R}_{ijij} &= g_{ik}\mathrm{R}^k_{jij} \\
&= g_{ik}\left(\frac{\partial \Gamma^k_{jj}}{\partial \theta^i} + \Gamma^k_{i\xi}\Gamma^\xi_{jj} - \frac{\partial \Gamma^k_{ji}}{\partial \theta^j} - \Gamma^k_{j\xi}\Gamma^\xi_{ji}\right) \\
&= g_{ik}\left(\frac{\partial \Gamma^k_{jj}}{\partial \theta^i} + \Gamma^k_{il}\Gamma^l_{jj} - \frac{\partial \Gamma^k_{ji}}{\partial \theta^j} - \Gamma^k_{jl}\Gamma^l_{ji}\right) + g_{ik}\left(\Gamma^k_{ir}\Gamma^r_{jj} - \Gamma^k_{jr}\Gamma^r_{ji}\right)
\end{aligned}$$

设 $\mathrm{d}\sigma^2$ 的曲率张量为 $\bar{R}$, 则有

$$\bar{\mathrm{R}}^i_{jij} = \frac{\partial \Gamma^i_{jj}}{\partial \theta^i} + \Gamma^i_{ik}\Gamma^k_{jj} - \frac{\partial \Gamma^i_{ji}}{\partial \theta^j} - \Gamma^i_{jk}\Gamma^k_{ji}$$

所以

$$\begin{aligned}
\mathrm{R}_{ijij} &= (f(r))^2\bar{\mathrm{R}}_{ijij} + g_{ik}\left(\frac{f'(r)}{f(r)}\delta^k_i \cdot -f(r)f'(r)h_{jj}(\theta) - \frac{f'(r)}{f(r)}\delta^k_j \cdot -f(r)f'(r)h_{ij}(\theta)\right) \\
&= (f(r))^2\bar{\mathrm{R}}_{ijij} - (f(r))^2 f'(r)^2\big(h_{ii}(\theta)h_{jj}(\theta) - h_{ij}(\theta)^2\big).
\end{aligned}$$

因此截面曲率为

$$\begin{aligned}
\mathrm{K}(\partial r, \partial\theta^i) &= \frac{\mathrm{R}_{riri}}{g_{rr}g_{ii} - g_{ri}^2} = -\frac{f''(r)}{f(r)}, \\
\mathrm{K}(\partial\theta^i, \partial\theta^j) &= \frac{\mathrm{R}_{ijij}}{g_{ii}g_{jj} - gij^2} = \frac{1}{(f(r))^2} - \frac{(f(r))^2(f'(r))^2\big(h_{ii}(\theta)h_{jj}(\theta) - h_{ij}(\theta)^2\big)}{f(r)^4\big(h_{ii}(\theta)h_{jj}(\theta) - h_{ij}(\theta)^2\big)} \\
&= \frac{1}{(f(r))^2} - \frac{(f'(r))^2}{(f(r))^2}.
\end{aligned} \tag{1}$$

因此如果是常截面曲率流形, 那么

$$-\frac{f''(r)}{f(r)} = \frac{1}{(f(r))^2} - \frac{(f'(r))^2}{(f(r))^2} = c.$$

即

$$f''(r) = -cf(r), \qquad 1 - (f'(r))^2 = c(f(r))^2.$$

所以

1. 如果 $c>0$, 那么 $f(r)=A\sin(\sqrt{c}r)+B\cos(\sqrt{c}r)$, 利用第二个方程,

$$1=c(f(r))^2+(f'(r))^2=cB^2+cA^2,$$

此外, 计算 $|\partial B(r)|$ 的面积为

$$A(r)=\int_{\partial B(r)}(f(r))^{2(m-1)}\det(h_{ij}(\theta))\,\mathrm{d}\theta,$$

当 $r\to 0$ 时, $A(r)$ 应当趋向于 0. 所以 $\lim_{r\to 0}f(r)=0$. 所以 $A=\dfrac{1}{\sqrt{c}}$, $B=0$. 即

$$f(r)=\sin\bigl(r\sqrt{c}\bigr)/\sqrt{c}.$$

2. 如果 $c=0$, 那么 $f(r)=Ax+B$. 结合上面所说, 有

$$1=c(Ax+B)^2+A^2,\qquad \lim_{r\to 0}f(r)=0,$$

所以 $A=1$, $B=0$. 即

$$f(r)=r.$$

3. 如果 $c<0$, 那么 $f(r)=Ae^{r\sqrt{-c}}+Be^{-r\sqrt{-c}}$. 所以有

$$1=c(f(r))^2+(f'(r))^2=4cAB,\qquad \lim_{r\to 0}f(r)=A+B=0.$$

所以

$$f(r)=e^{r\sqrt{-c}}/(2\sqrt{-c})+e^{-r\sqrt{-c}}/(2\sqrt{-c})=\sinh\bigl(r\sqrt{-c}\bigr)/\sqrt{-c}.$$

综上, 如果 M 是常截面曲率流形, 且截面曲率为 c, 则

$$f(r)=\begin{cases}\sin\bigl(r\sqrt{c}\bigr)/\sqrt{c} & c>0,\\ r & c=0,\\ \sinh\bigl(r\sqrt{-c}\bigr)/\sqrt{-c} & c<0.\end{cases}$$

反过来, 如果 f 表现如上, 那么根据 (1) 式, 可以计算截面曲率为 c. 因此题目得证.

0.3 问题 1.2

题目 3. 假设 $(\mathcal{M},g)$ 是 m 维 Riemann 流形, $p\in\mathcal{M}$ 是流形上任意一点. 在 p 的法坐标 $(\mathcal{U}_p,x_i)$ 中证明:

(i) $g_{ij}(x)=\delta_{ij}-\frac{1}{3}\mathrm{R}_{ikjl}(0)x_kx_l+O(|x|^3)$;

(ii) $\det(g_{ij})=1-\frac{1}{3}\mathrm{Ric}_{kl}(0)x_kx_l+O(|x|^3)$.

解答.

1. 核心是计算 g_{ij} 沿由 p 出发到 x 的测地线的导数. 设 $x=(x^1,\cdots,x^n)$, 同时其也 p 到 x 指数映射的切向量. 设 $\gamma(t)=\exp_p(tx)$, 那么 $x=\gamma(1)$. 所以一阶导数为

$$\frac{\mathrm{d}}{\mathrm{d}t}g_{ij}(\gamma(t))=x^k\partial_k\bigl(g_{ij}(\gamma(t))\bigr).$$

而 $\partial_k(g_{ij}) = \langle\nabla_{\partial_k}\partial_i, \partial_j\rangle + \langle\partial_i, \nabla_{\partial_k}\partial_j\rangle = g_{jl}\Gamma^l_{ik} + g_{il}\Gamma^l_{jk}$. 因为在法坐标的原点, Christoffel 记号为 0, 所以

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{ij}(\gamma(t)) = 0.$$

二阶导数为

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_{ij}(\gamma(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[x^k\left(g_{jl}\Gamma^l_{ik} + g_{il}\Gamma^l_{jk}\right)\right] \\
&= x^k\left(g_{jl}\frac{\mathrm{d}\Gamma^l_{ik}}{\mathrm{d}t} + g_{il}\frac{\mathrm{d}\Gamma^l_{jk}}{\mathrm{d}t}\right) \\
&= x^k\left(g_{jl}x^m\frac{\partial\Gamma^l_{ik}}{\partial x^m} + g_{il}x^m\frac{\partial\Gamma^l_{jk}}{\partial x^m}\right).
\end{aligned}$$

在 0 点, 利用第 0.1 节的 ((i)), 遂有

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_{ij}(0) &= x^k\left(x^m g_{jl}\frac{\partial\Gamma^l_{ik}}{\partial x^m}(0) + x^m g_{il}\frac{\partial\Gamma^i_{jk}}{\partial x^m}(0)\right) \\
&= -\frac{1}{3}x^k x^m g_{jl}(0)(\mathrm{R}^l_{ikm}(0) + \mathrm{R}^l_{kim}(0)) - \frac{1}{3}x^k x^m g_{il}(0)(\mathrm{R}^l_{jkm}(0) + \mathrm{R}^l_{kjm}(0)) \\
&= -\frac{1}{3}x^k x^m(\mathrm{R}_{jikm}(0) + \mathrm{R}_{jkim}(0) + \mathrm{R}_{ijkm}(0) + \mathrm{R}_{ikjm}(0)) \\
&= -\frac{2}{3}x^k x^l \mathrm{R}_{ijkl}(0),
\end{aligned}$$

代入 Taylor 展式, 得到

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}\mathrm{R}_{ijkl}(0)x^k x^l + O(|x|^3).$$

2. 依然要计算导数. 记 $G = (g_{ij})_{n\times n}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(g_{ij})(\gamma(t)) = \det G \operatorname{tr} G^{-1}G' = \det(g_{ij})(\gamma(t))(g^{ki}g'_{ik})(\gamma(t)),$$

所以 $\dfrac{\mathrm{d}}{\mathrm{d}t}\det(g_{ij})(0) = 0$. 继续计算, 二阶导数为

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\det(g_{ij})(\gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j,k}\left[g^{jk}(g^{ki}g'_{ij})\right](\gamma(t)),$$

因为 $0 = (g^{ij}g_{jk})' = (g^{ij})'g_{jk} + g^{ij}(g_{jk})'$, 在 0 点处, $0 = (g^{ij})'g_{jk}$. 所以 $(g^{ij})'(0) = 0$. 因此上式中求导只会求在 g' 上. 于是利用 (1) 的结果,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\det(g_{ij})(0) = -\frac{2}{3}\sum_{j,k}\left[g^{jk}(g^{ki}x^k x^l \mathrm{R}_{ijkl}(0))\right] = -\frac{2}{3}x^k x^l \mathrm{Ric}_{kl}(0),$$

代入 Taylor 展式, 得到

$$\det(g_{ij})(x) = 1 - \frac{1}{3}\mathrm{Ric}_{kl}(0)x^k x^l + O(|x|^3).$$

0.4 问题 1.3

题目 4. 假设 $(\mathcal{M}, g)$ 是 m 维 Riemann 流形. 假设 $r > 0$ 足够小使得 $\exp_p\colon D_r(0) \to B_r(p)$ 是微分同胚, 其中 $D_r(0) = v \in T_p\mathcal{M} : |v| < \varepsilon$, $B_r(p) = \exp_p D_r(0)$. 记 $S_r(p) = \partial B_r(p)$. 证明:

1. $\operatorname{vol}(B_r(p)) = \omega_m r^m \left(1 - \dfrac{\operatorname{scal}(p)}{6(m+2)} r^2 + O(r^3)\right)$, 其中 ω_m 是 $\mathbb{R}^m$ 的单位球的欧氏体积;
2. $\operatorname{area}(S_r(p)) = m\omega_m r^{m-1} - \dfrac{1}{6}\operatorname{scal}(p)\omega_m r^{m+1} + O(r^{m+2})$.

解答.

1. $\operatorname{vol}(B_r(p))$ 相当于在欧氏空间 $\bar{B}_r$ 上对 $\mathcal{M}$ 的体积元 dvol 积分. 而 $\mathrm{dvol} = \sqrt{\det(g_{ij})}\,\mathrm{d}m_{\mathbb{R}^m}$. 设

$$\sqrt{\det(g_{ij})}(x) = 1 + a_k x^k + b_{kl} x^k x^l + O(|x|^3),$$

那么平方可得

$$a_k = 0, \qquad 2b_{kl} = -\frac{1}{3}\operatorname{Ric}_{kl}(p),$$

所以

$$\sqrt{\det(g_{ij})}(x) = 1 - \frac{1}{6}\operatorname{Ric}_{kl}(p) x^k x^l + O(|x|^3).$$

因此积分得

$$\begin{aligned}\operatorname{vol}(B_r(p)) &= \int_{\bar{B}_r} 1 - \frac{1}{6}\operatorname{Ric}_{kl}(p) x^k x^l + O(|x|^3)\,\mathrm{d}x \\ &= \operatorname{vol}(\bar{B}_r) - \frac{1}{6}\int_{\bar{B}_r} \operatorname{Ric}_{kl}(p) x^k x^l\,\mathrm{d}x + \operatorname{vol}(\bar{B}_r) O(r^3)\end{aligned}$$

考虑 p 点附近的测地极坐标 $(r, \theta_1, \cdots, \theta_{m-1})$. 则

$$\begin{cases} x_1 = r\cos\theta_1 \\ x_2 = r\sin\theta_1\cos\theta_2 \\ \quad\vdots \\ x_{m-1} = r\sin\theta_1\cdots\sin\theta_{m-2}\cos\theta_{m-1} \\ x_m = r\sin\theta_1\cdots\sin\theta_{m-2}\sin\theta_{m-1} \end{cases}$$

其中 $1 \leqslant i \leqslant m-2$ 时, $\theta_i \in [0, \pi]$; 而 $\theta_{m-1} \in [0, 2\pi]$. 观察对 θ_{m-1} 的积分, 有以下结论:

引理. *若 $k \neq l$, 则*

$$\int_{\bar{B}_r} x^k x^l\,\mathrm{d}x = 0.$$

引理的证明. 由于 $k \neq l$, 所以 θ_{m-1} 的项只可能有三种情况: $\cos\theta_{m-1}$, $\sin\theta_{m-1}$, $\sin\theta_{m-1}\cos\theta_{m-1} = \sin(2\theta_{m-1})/2$. 而这三者在 $[0, 2\pi]$ 上的积分都是 0. 因此引理成立. □

有此结论, 则

$$\int_{\bar{B}_r} \operatorname{Ric}_{kl}(p) x^k x^l\,\mathrm{d}x = \int_{\bar{B}_r} \sum_k \operatorname{Ric}_{kk}(p)(x^k)^2\,\mathrm{d}x.$$

由对称性, 不同 k 的 $\int_{\bar{B}_r}(x^k)^2\,\mathrm{d}x$ 都相等, 并且其均等于

$$\frac{1}{m}\int_{\bar{B}_r}\sum_{k=1}^m (x^k)^2\,\mathrm{d}x = \frac{1}{m}\int_{\bar{B}_r}|x|^2\,\mathrm{d}x = \omega_m\frac{r^{m+2}}{m+2},$$

故

$$\int_{\bar{B}_r}\mathrm{Ric}_{kl}(p)x^kx^l\,\mathrm{d}x = m\omega_m\frac{\mathrm{scal}(p)}{m+2}r^{m+2}.$$

因此

$$\mathrm{vol}(B_r(p)) = \omega_m r^m - \frac{1}{6}m\omega_m\frac{\mathrm{scal}(p)}{m+2}r^{m+2} + \omega_m r^m O(r^3) = \omega_m r^m(1-\frac{\mathrm{scal}(p)}{6(m+2)}r^2 + O(r^3)).$$

2. $\mathrm{area}(S_r(p))$ 只需要对 $\mathrm{vol}(B_r(p))$ 关于 r 求导即可.

0.5 问题 1.4

题目 5. 假设 $(\mathcal{M},g)$ 是 m 维 Riemann 流形, $p\in M$, $\Pi_p\subset T_p\mathcal{M}$ 是 2 维截面. 令 $C_r^0=\{v\in\Pi_p : |v|=r\}$, $C_r=\exp_p C_r^0$. 用 L_r 记曲线 C_r 的长度. 证明:

$$\mathrm{K}(\Pi_p) = \frac{3}{\pi}\lim_{r\to 0}\frac{2\pi r - L_r}{r^3},$$

其中 $\mathrm{K}(\Pi_p)$ 表示 p 点处关于 Π_p 的截面曲率.

0.6 问题 1.5

题目 6. 假设 $(\mathcal{M},g)$ 是 m 维 Riemann 流形, $\gamma_k\colon[0,1]\to\mathcal{M}$, $k=1,2$ 是两条正规测地线 (速度为 1 的测地线), 且满足 $\gamma_1(0)=\gamma_2(0)=p$. 用 $\theta\in(0,\pi)$ 记 $v=\dot\gamma_1(0)$ 与 $w=\dot\gamma_2(0)$ 的夹角, $\Pi_p=\mathrm{span}\{v,w\}$ 是 $T_p\mathcal{M}$ 的 2 维截面. 证明:

$$d(\gamma_1(t),\gamma_2(t)) = \sqrt{2(1-\cos\theta)}\big[1-\frac{1}{6}\mathrm{K}(\Pi_p)\cos^2(\theta/2)t^2 + O(t^3)\big].$$