第九次作业

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0.1 174 页习题 4

题目 1. 设 (M,g) 为 Riemann 流形. (U,φ,x^i) 是以 q 为原点的法坐标图.

$$X_0=\xi^i\Big(\frac{\partial}{\partial x^i}\Big)_q,\quad Y_0=\eta^i\Big(\frac{\partial}{\partial x^i}\Big)_q$$

均为单位向量. $C\colon[0,r)\to C(s)$ 为在 $q=C(0)$ 点以 X_0 为切向量的测地线, $Y(s)$ 是将 Y_0 沿 C 平行移动而得的切向量. 证明:

(i) 在法坐标系的原点

$$\frac{\partial\Gamma^k_{ij}}{\partial x^l}=-\frac{1}{3}(\mathrm{R}^k_{ijl}+\mathrm{R}^k_{jil}),$$

(ii) 设 $Y(s)=\zeta^i\Big(\dfrac{\partial}{\partial x^i}\Big)_{C(s)}$, 则

$$\zeta^i(s)=\eta^i+\frac{1}{6}(\mathrm{R}^i_{jkl})_q\xi^j\eta^k\xi^l s^2+o(s^3),$$

(iii) 若 $\langle X_0,Y_0\rangle=0$, 且令 $||Y(s)||_q^2=g_{ij}(q)\xi^i(s)\xi^j(s)$, 则

$$||Y(s)||_q=1+\frac{s^2}{6}\mathrm{R}(X_0,Y_0,X_0,Y_0)+o(s^3).$$

解答. 记 $e_i=\dfrac{\partial}{\partial x^i}$.

(i) 记 M 的维数为 n. 任取 $\mathbf{u}=(u^1,\cdots,u^n)\in T_qM$, 因为 $\exp_q(t\mathbf{u})=(tu^1,\cdots,tu^n)$ 是测地线, 所以由测地线方程,

$$\Gamma^k_{ij}(\exp_q(t\mathbf{u}))u^iu^j=0.$$

因此在这一点, Riemann 曲率张量为

$$\begin{aligned}\mathrm{R}^i_{jkl}(q)&=g^{im}\langle\mathrm{Rm}(e_k,e_l)e_j,e_m\rangle\\&=\frac{\partial\Gamma^i_{jl}}{\partial x^k}+\Gamma^i_{ks}\Gamma^s_{jl}-\frac{\partial\Gamma^i_{jk}}{\partial x^l}-\Gamma^i_{ls}\Gamma^s_{jk}\\&=\frac{\partial\Gamma^i_{jl}}{\partial x^k}-\frac{\partial\Gamma^i_{jk}}{\partial x^l}.\end{aligned}$$

由 u 的任意性, $\Gamma^k_{ij}(q)=0$. 对测地线方程关于 t 微分, 得到

$$\mathbf{u}(\Gamma^k_{ij})u^iu^j=\frac{\partial\Gamma^k_{ij}}{\partial x^l}u^iu^ju^l=0.$$

取 $\mathbf{u}=u^ie_i$, 那么

$$\frac{\partial\Gamma^k_{ii}}{\partial x^i}u^iu^iu^i=0,$$

取 $\mathbf{u}=e_i+e_j$ 和 $\mathbf{u}=e_i-e_j$, 那么

$$\frac{\partial\Gamma^k_{ii}}{\partial x^j}+2\frac{\partial\Gamma^k_{ij}}{\partial x^i}+2\frac{\partial\Gamma^k_{ij}}{\partial x^j}+\frac{\partial\Gamma^k_{jj}}{\partial x^i}=0,$$

$$-\frac{\partial\Gamma^k_{ii}}{\partial x^j}-2\frac{\partial\Gamma^k_{ij}}{\partial x^i}+2\frac{\partial\Gamma^k_{ij}}{\partial x^j}+\frac{\partial\Gamma^k_{jj}}{\partial x^i}=0,$$

所以

$$\begin{aligned}0&=\frac{\partial\Gamma^k_{ii}}{\partial x^j}+2\frac{\partial\Gamma^k_{ij}}{\partial x^i}\\&=3\frac{\partial\Gamma^k_{ij}}{\partial x^i}+\mathrm{R}^k_{iji}\\&=3\frac{\partial\Gamma^k_{ii}}{\partial x^j}+2\mathrm{R}^k_{iij}\end{aligned}$$

因此

$$\begin{aligned}\frac{\partial\Gamma^k_{ij}}{\partial x^i}&=-\frac{1}{3}\mathrm{R}^k_{iji}=-\frac{1}{3}(\mathrm{R}^k_{iji}+\mathrm{R}^{jii}),\\\frac{\partial\Gamma^k_{ii}}{\partial x^j}&=-\frac{2}{3}\mathrm{R}^k_{iij}.\end{aligned}$$

取 $\mathbf{u}=u^ie_i+u^je_j+u^ke_k$, 由上面的计算, 如果求和中 i,j,k 只选到一个或两个下标, 那么这部分求和项为 0. 所以

$$\begin{aligned}0&=\frac{\partial\Gamma^k_{ij}}{\partial x^l}+\frac{\partial\Gamma^k_{jl}}{\partial x^i}+\frac{\partial\Gamma^k_{il}}{\partial x^j}\\&=3\frac{\partial\Gamma^k_{ij}}{\partial x^l}+\Big(\frac{\partial\Gamma^k_{jl}}{\partial x^i}-\frac{\partial\Gamma^k_{ij}}{\partial x^l}\Big)+\Big(\frac{\partial\Gamma^k_{il}}{\partial x^j}-\frac{\partial\Gamma^k_{ij}}{\partial x^l}\Big)\\&=3\frac{\partial\Gamma^k_{ij}}{\partial x^l}+\mathrm{R}^k_{jil}+\mathrm{R}^k_{ijl}.\end{aligned}$$

因此

$$\frac{\partial\Gamma^k_{ij}}{\partial x^l}=-\frac{1}{3}(\mathrm{R}^k_{jil}+\mathrm{R}^k_{ijl}).$$

(ii) $Y(s)$ 满足平行移动方程

$$\frac{\partial\zeta^i(s)}{\partial s}+\Gamma^i_{jk}\zeta^j(s)\xi^k(s)=0,$$

所以在 q 点

$$\frac{\partial\zeta^i(s)}{\partial s}=-\Gamma^i_{jk}\eta^j\xi^k=0,$$

并且利用 $\Gamma^i_{jk}(q)=0$, 在 q 点二阶导数为

$$\frac{\partial^2\zeta^i(s)}{\partial s^2}=-\frac{\partial\Gamma^i_{jk}}{\partial s}(q)\eta^j\xi^k.$$

利用 (i) 的结论,

$$\frac{\partial\Gamma^k_{ij}}{\partial s}(q)=-\frac{1}{3}(\mathrm{R}^k_{jil}+\mathrm{R}^k_{ijl})\xi^l,$$

所以

$$
\begin{aligned}
\zeta^i(s) &= \eta^i + \frac{\partial \zeta^i(s)}{\partial s}s + \frac{1}{2}\frac{\partial^2 \zeta^i(s)}{\partial s^2}s^2 + O(s^3) \\
&= \eta^i + \frac{1}{6}(\mathrm{R}^i_{jkl} + \mathrm{R}^i_{kjl})\eta^j\xi^k\xi^l s^2 + O(s^3).
\end{aligned}
$$

因为 R^i_{jkl} 交换 kl 会变号, 所以 $\mathrm{R}^i_{jkl}\eta^j\xi^k\xi^l = 0$. 因此

$$
\zeta^i(s) = \eta^i + \frac{1}{6}\mathrm{R}^i_{kjl}\xi^j\eta^k\xi^l s^2 + O(s^3).
$$

(iii) $\langle X_0, Y_0\rangle = 0$, 则 $\sum_i \eta^i\xi^i = 0$. 所以

$$
\begin{aligned}
||Y(s)||^2_q &= g_{ij}(q)\xi^i(s)\xi^j(s) \\
&= |\eta^i|^2 + \sum_i \frac{1}{6}\mathrm{R}_{ijkl}\eta^i\xi^j\eta^k\xi^l s^2 + O(s^3) \\
&= 1 + \frac{1}{6}R(X_0, Y_0, X_0, Y_0)s^2 + O(s^3).
\end{aligned}
$$