第九次作业

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题目 1. 设 (M,g) 为 Riemann 流形. (U,φ,x^i) 是以 q 为原点的法坐标图.

$$X_0=\xi^i\Big(\frac{\partial}{\partial x^i}\Big)_q,\quad Y_0=\eta^i\Big(\frac{\partial}{\partial x^i}\Big)_q$$

均为单位向量. $C\colon[0,r)\to C(s)$ 为在 $q=C(0)$ 点以 X_0 为切向量的测地线, $Y(s)$ 是将 Y_0 沿 C 平行移动而得的切向量. 证明:

(i) 在法坐标系的原点

$$\frac{\partial\Gamma_{ij}^k}{\partial x^l}=-\frac{1}{3}(\mathrm{R}_{ijl}^k+\mathrm{R}_{jil}^k),$$

(ii) 设 $Y(s)=\zeta^i\Big(\dfrac{\partial}{\partial x^i}\Big)_{C(s)}$, 则

$$\zeta^i(s)=\eta^i+\frac{1}{6}(\mathrm{R}_{jkl}^i)_q\xi^j\eta^k\xi^l s^2+o(s^3),$$

(iii) 若 $\langle X_0,Y_0\rangle=0$, 且令 $||Y(s)||_q^2=g_{ij}(q)\xi^i(s)\xi^j(s)$, 则

$$||Y(s)||_q=1+\frac{s^2}{6}\mathrm{R}(X_0,Y_0,X_0,Y_0)+o(s^3).$$

解答. 记 $e_i=\dfrac{\partial}{\partial x^i}$.

(i) 记 M 的维数为 n. 任取 $\mathbf{u}=(u^1,\cdots,u^n)\in T_qM$, 因为 $\exp_q(t\mathbf{u})=(tu^1,\cdots,tu^n)$ 是测地线, 所以由测地线方程,

$$\Gamma_{ij}^k(\exp_q(t\mathbf{u}))u^iu^j=0.$$

因此在这一点, Riemann 曲率张量为

$$\begin{aligned}\mathrm{R}_{jkl}^i(q)&=g^{im}\langle\mathrm{Rm}(e_k,e_l)e_j,e_m\rangle\\&=\frac{\partial\Gamma_{jl}^i}{\partial x^k}+\Gamma_{ks}^i\Gamma_{jl}^s-\frac{\partial\Gamma_{jk}^i}{\partial x^l}-\Gamma_{ls}^i\Gamma_{jk}^s\\&=\frac{\partial\Gamma_{jl}^i}{\partial x^k}-\frac{\partial\Gamma_{jk}^i}{\partial x^l}.\end{aligned}$$

由 u 的任意性, $\Gamma_{ij}^k(q)=0$. 对测地线方程关于 t 微分, 得到

$$\mathbf{u}(\Gamma_{ij}^k)u^iu^j=\frac{\partial\Gamma_{ij}^k}{\partial x^l}u^iu^ju^l=0.$$

取 $\mathbf{u}=u^ie_i$, 那么

$$\frac{\partial\Gamma_{ii}^k}{\partial x^i}u^iu^iu^i=0,$$

取 $\mathbf{u}=e_i+e_j$ 和 $\mathbf{u}=e_i-e_j$, 那么

$$\frac{\partial\Gamma_{ii}^k}{\partial x^j}+2\frac{\partial\Gamma_{ij}^k}{\partial x^i}+2\frac{\partial\Gamma_{ij}^k}{\partial x^j}+\frac{\partial\Gamma_{jj}^k}{\partial x^i}=0,$$

$$-\frac{\partial\Gamma_{ii}^k}{\partial x^j}-2\frac{\partial\Gamma_{ij}^k}{\partial x^i}+2\frac{\partial\Gamma_{ij}^k}{\partial x^j}+\frac{\partial\Gamma_{jj}^k}{\partial x^i}=0,$$

所以

$$\begin{aligned}0&=\frac{\partial\Gamma_{ii}^k}{\partial x^j}+2\frac{\partial\Gamma_{ij}^k}{\partial x^i}\\&=3\frac{\partial\Gamma_{ij}^k}{\partial x^i}+\mathrm{R}_{iji}^k\\&=3\frac{\partial\Gamma_{ii}^k}{\partial x^j}+2\mathrm{R}_{iij}^k\end{aligned}$$

因此

$$\frac{\partial\Gamma_{ij}^k}{\partial x^i}=-\frac{1}{3}\mathrm{R}_{iji}^k=-\frac{1}{3}(\mathrm{R}_{iji}^k+\mathrm{R}^{jii}),$$

$$\frac{\partial\Gamma_{ii}^k}{\partial x^j}=-\frac{2}{3}\mathrm{R}_{iij}^k.$$

取 $\mathbf{u}=u^ie_i+u^je_j+u^ke_k$, 由上面的计算, 如果求和中 i,j,k 只选到一个或两个下标, 那么这部分求和项为 0. 所以

$$\begin{aligned}0&=\frac{\partial\Gamma_{ij}^k}{\partial x^l}+\frac{\partial\Gamma_{jl}^k}{\partial x^i}+\frac{\partial\Gamma_{il}^k}{\partial x^j}\\&=3\frac{\partial\Gamma_{ij}^k}{\partial x^l}+\Big(\frac{\partial\Gamma_{jl}^k}{\partial x^i}-\frac{\partial\Gamma_{ij}^k}{\partial x^l}\Big)+\Big(\frac{\partial\Gamma_{il}^k}{\partial x^j}-\frac{\partial\Gamma_{ij}^k}{\partial x^l}\Big)\\&=3\frac{\partial\Gamma_{ij}^k}{\partial x^l}+\mathrm{R}_{jil}^k+\mathrm{R}_{ijl}^k.\end{aligned}$$

因此

$$\frac{\partial\Gamma_{ij}^k}{\partial x^l}=-\frac{1}{3}(\mathrm{R}_{jil}^k+\mathrm{R}_{ijl}^k).$$

(ii) $Y(s)$ 满足平行移动方程

$$\frac{\partial\zeta^i(s)}{\partial s}+\Gamma_{jk}^i\zeta^j(s)\xi^k(s)=0,$$

所以在 q 点

$$\frac{\partial\zeta^i(s)}{\partial s}=-\Gamma_{jk}^i\eta^j\xi^k=0,$$

并且利用 $\Gamma_{jk}^i(q)=0$, 在 q 点二阶导数为

$$\frac{\partial^2\zeta^i(s)}{\partial s^2}=-\frac{\partial\Gamma_{jk}^i}{\partial s}(q)\eta^j\xi^k.$$

利用 (i) 的结论,

$$\frac{\partial\Gamma_{ij}^k}{\partial s}(q)=-\frac{1}{3}(\mathrm{R}_{jil}^k+\mathrm{R}_{ijl}^k)\xi^l,$$

所以

$$\begin{aligned}\zeta^i(s) &= \eta^i + \frac{\partial \zeta^i(s)}{\partial s}s + \frac{1}{2}\frac{\partial^2\zeta^i(s)}{\partial s^2}s^2 + O(s^3)\\ &= \eta^i + \frac{1}{6}(\mathrm{R}^i_{jkl} + \mathrm{R}^i_{kjl})\eta^j\xi^k\xi^l s^2 + O(s^3).\end{aligned}$$

因为 R^i_{jkl} 交换 kl 会变号, 所以 $\mathrm{R}^i_{jkl}\eta^j\xi^k\xi^l = 0$. 因此

$$\zeta^i(s) = \eta^i + \frac{1}{6}\mathrm{R}^i_{kjl}\xi^j\eta^k\xi^l s^2 + O(s^3).$$

(iii) $\langle X_0, Y_0\rangle = 0$, 则 $\sum_i \eta^i\xi^i = 0$. 所以

$$\begin{aligned}||Y(s)||_q^2 &= g_{ij}(q)\xi^i(s)\xi^j(s)\\ &= |\eta^i|^2 + \sum_i \frac{1}{6}\mathrm{R}_{ijkl}\eta^i\xi^j\eta^k\xi^l s^2 + O(s^3)\\ &= 1 + \frac{1}{6}R(X_0, Y_0, X_0, Y_0)s^2 + O(s^3).\end{aligned}$$

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题目 2. 设 m 维 Riemann 流形 (M, g) 在测地极坐标系 $(r, \theta^1, \cdots, \theta^{m-1})$ 下具有度量形式

$$\mathrm{d}s^2 = (\mathrm{d}r)^2 + (f(r))^2 h_{ij}(\theta)\,\mathrm{d}\theta^i\theta^j,$$

其中 $m-1$ 维度量 $\mathrm{d}\sigma^2 = h_{ij}(\theta)\,\mathrm{d}\theta^i\theta^j$ 具有常数截面曲率 1. 求证 $\mathrm{d}s^2$ 具有常数截面曲率 c 的充要条件是

$$f(r) = \begin{cases}\sin\left(\sqrt{cr^2}\right)/\sqrt{c} & c > 0,\\ r & c = 0,\\ \sinh\left(\sqrt{-cr^2}\right)/\sqrt{-c} & c < 0.\end{cases}$$

解答. 第一步要计算 Christoffel 记号.,

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right).$$

1. 若三个指标都是 r:

$$\Gamma^r_{rr} = 0.$$

2. 若两个指标是 r, 一个是 θ^i:

$$\Gamma^r_{ri} = \frac{1}{2}g^{rr}\left(\frac{\partial g_{rr}}{\partial \theta^i} + \frac{\partial g_{ri}}{\partial r} - \frac{\partial g_{ri}}{\partial r}\right) = 0,$$

$$\Gamma^i_{rr} = \frac{1}{2}g^{ij}\left(\frac{\partial g_{rj}}{\partial r} + \frac{\partial g_{jr}}{\partial r} - \frac{\partial g_{rr}}{\partial \theta^j}\right) = 0.$$

3. 若一个指标是 r, 其余两个是 θ^i, θ^j:

$$\Gamma^r_{ij} = \frac{1}{2}g^{rr}\left(\frac{\partial g_{ir}}{\partial \theta^j} + \frac{\partial g_{rj}}{\partial \theta^i} - \frac{\partial g_{ij}}{\partial r}\right) = -f(r)f'(r)h_{ij}(\theta),$$

$$\Gamma^i_{rj} = \frac{1}{2}g^{ik}\left(\frac{\partial g_{rk}}{\partial \theta^j} + \frac{\partial g_{jk}}{\partial r} - \frac{\partial g_{rj}}{\partial \theta^k}\right) = \frac{f'(r)}{f(r)}\delta^i_j.$$

4. 若所有指标都不为 r: 设 $\bar{\Gamma}$ 表示 $\mathrm{d}\sigma^2$ 的 Christoffel 记号,

$$\Gamma^i_{jk} = \bar{\Gamma}^i_{jk}.$$

那么 Riemann 曲率张量

$$\mathrm{R}^\alpha_{\beta\delta\eta} = \frac{\partial \Gamma^\alpha_{\beta\eta}}{\partial x^\delta} + \Gamma^\alpha_{\delta\xi}\Gamma^\xi_{\beta\eta} - \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\eta} - \Gamma^\alpha_{\eta\xi}\Gamma^\xi_{\beta\delta},$$

所以,

$$\begin{aligned}
\mathrm{R}_{riri} &= g_{rr}\mathrm{R}^r_{iri} = \mathrm{R}^r_{iri} \\
&= \frac{\partial \Gamma^r_{ii}}{\partial r} + \Gamma^r_{r\xi}\Gamma^\xi_{ii} - \frac{\partial \Gamma^r_{ir}}{\partial \theta^i} - \Gamma^r_{i\xi}\Gamma^\xi_{ir} \\
&= -(f(r))^2 h_{ii}(\theta) - f(r)f''(r)h_{ii}(\theta) - \sum_j \left(-f(r)f' h_{ij}(\theta)\cdot \frac{f'(r)}{f(r)}\delta^i_j\right) \\
&= -f(r)f''(r)h_{ii}(\theta). \\
\mathrm{R}_{ijij} &= g_{ik}\mathrm{R}^k_{jij} \\
&= g_{ik}\left(\frac{\partial \Gamma^k_{jj}}{\partial \theta^i} + \Gamma^k_{i\xi}\Gamma^\xi_{jj} - \frac{\partial \Gamma^k_{ji}}{\partial \theta^j} - \Gamma^k_{j\xi}\Gamma^\xi_{ji}\right) \\
&= g_{ik}\left(\frac{\partial \Gamma^k_{jj}}{\partial \theta^i} + \Gamma^k_{il}\Gamma^l_{jj} - \frac{\partial \Gamma^k_{ji}}{\partial \theta^j} - \Gamma^k_{jl}\Gamma^l_{ji}\right) + g_{ik}\left(\Gamma^k_{ir}\Gamma^r_{jj} - \Gamma^k_{jr}\Gamma^r_{ji}\right)
\end{aligned}$$

设 $\mathrm{d}\sigma^2$ 的曲率张量为 $\bar{R}$, 则有

$$\bar{\mathrm{R}}^i_{jij} = \frac{\partial \Gamma^i_{jj}}{\partial \theta^i} + \Gamma^i_{ik}\Gamma^k_{jj} - \frac{\partial \Gamma^i_{ji}}{\partial \theta^j} - \Gamma^i_{jk}\Gamma^k_{ji}$$

所以

$$\begin{aligned}
\mathrm{R}_{ijij} &= (f(r))^2\bar{\mathrm{R}}_{ijij} + g_{ik}\left(\frac{f'(r)}{f(r)}\delta^k_i \cdot -f(r)f'(r)h_{jj}(\theta) - \frac{f'(r)}{f(r)}\delta^k_j \cdot -f(r)f'(r)h_{ij}(\theta)\right) \\
&= (f(r))^2\bar{\mathrm{R}}_{ijij} - (f(r))^2 f'(r)^2\big(h_{ii}(\theta)h_{jj}(\theta) - h_{ij}(\theta)^2\big).
\end{aligned}$$

因此截面曲率为

$$\begin{aligned}
\mathrm{K}(\partial r, \partial\theta^i) &= \frac{\mathrm{R}_{riri}}{g_{rr}g_{ii} - g_{ri}^2} = -\frac{f''(r)}{f(r)}, \\
\mathrm{K}(\partial\theta^i, \partial\theta^j) &= \frac{\mathrm{R}_{ijij}}{g_{ii}g_{jj} - gij^2} = \frac{1}{(f(r))^2} - \frac{(f(r))^2(f'(r))^2\big(h_{ii}(\theta)h_{jj}(\theta) - h_{ij}(\theta)^2\big)}{f(r)^4\big(h_{ii}(\theta)h_{jj}(\theta) - h_{ij}(\theta)^2\big)} \\
&= \frac{1}{(f(r))^2} - \frac{(f'(r))^2}{(f(r))^2}.
\end{aligned} \tag{1}$$

因此如果是常截面曲率流形, 那么

$$-\frac{f''(r)}{f(r)} = \frac{1}{(f(r))^2} - \frac{(f'(r))^2}{(f(r))^2} = c.$$

即

$$f''(r) = -cf(r), \qquad 1 - (f'(r))^2 = c(f(r))^2.$$

所以

1. 如果 $c>0$, 那么 $f(r)=A\sin(\sqrt{c}r)+B\cos(\sqrt{c}r)$, 利用第二个方程,

$$1=c(f(r))^2+(f'(r))^2=cB^2+cA^2,$$

此外, 计算 $|\partial B(r)|$ 的面积为

$$A(r)=\int_{\partial B(r)}(f(r))^{2(m-1)}\det(h_{ij}(\theta))\,\mathrm{d}\theta,$$

当 $r\to 0$ 时, $A(r)$ 应当趋向于 0. 所以 $\lim_{r\to 0}f(r)=0$. 所以 $A=\dfrac{1}{\sqrt{c}}$, $B=0$. 即

$$f(r)=\sin\bigl(r\sqrt{c}\bigr)/\sqrt{c}.$$

2. 如果 $c=0$, 那么 $f(r)=Ax+B$. 结合上面所说, 有

$$1=c(Ax+B)^2+A^2,\qquad \lim_{r\to 0}f(r)=0,$$

所以 $A=1$, $B=0$. 即

$$f(r)=r.$$

3. 如果 $c<0$, 那么 $f(r)=Ae^{r\sqrt{-c}}+Be^{-r\sqrt{-c}}$. 所以有

$$1=c(f(r))^2+(f'(r))^2=4cAB,\qquad \lim_{r\to 0}f(r)=A+B=0.$$

所以

$$f(r)=e^{r\sqrt{-c}}/(2\sqrt{-c})+e^{-r\sqrt{-c}}/(2\sqrt{-c})=\sinh\bigl(r\sqrt{-c}\bigr)/\sqrt{-c}.$$

综上, 如果 M 是常截面曲率流形, 且截面曲率为 c, 则

$$f(r)=\begin{cases}\sin\bigl(r\sqrt{c}\bigr)/\sqrt{c} & c>0,\\ r & c=0,\\ \sinh\bigl(r\sqrt{-c}\bigr)/\sqrt{-c} & c<0.\end{cases}$$

反过来, 如果 f 表现如上, 那么根据 (1) 式, 可以计算截面曲率为 c. 因此题目得证.

0.3 问题 1.2

题目 3. 假设 (M,g) 是 m 维 Riemann 流形, $p\in M$ 是流形上任意一点. 在 p 的法坐标 $(\mathcal{U}_p,x_i)$ 中证明:

(i) $g_{ij}(x)=\delta_{ij}-\frac{1}{3}\mathrm{R}_{ikjl}(0)x_kx_l+O(|x|^3)$;

(ii) $\det(g_{ij})=1-\frac{1}{3}\mathrm{Ric}_{kl}(0)x_kx_l+O(|x|^3)$.

解答.

1. 核心是计算 g_{ij} 沿由 p 出发到 x 的测地线的导数. 设 $x=su$, 其中 $u=(u^1,\cdots,u^n)\in T_p\mathcal{U}_p$ 是 p 点的单位切向量, 同时其也对应 p 到 x 测地线上某点的法坐标. 设 $\gamma(t)=\exp_p(tu)$, 那么 $x=\gamma(s)$.

$$\frac{\mathrm{d}}{\mathrm{d}s}g_{ij}(\gamma(s))$$