The Longest Path Problem

Computational Complexity Analysis

Longest Path Problem

Input: Weighted graph $G = (V, E, w)$ where $w : E \to \mathbb{R}^+$

```
8 15 # Vertices, Edges
v2 v1 5 # vertex, vertex, weight
...
```

Output: A simple path P with maximum sum of edge weights

```
115
v5 v4 v7 v8 v3 v1 v6 v2
```

Optimization Version (NP-Hard): Given a weighted graph, find the simple path with maximum total weight.

Descision Version (NP-Complete): Given a weighted graph, does a simple path of weight $\geq k$ exist?

Shortest vs Longest Path

Why can't we use a shortest path algorithm like Bellman-Ford (Polynomial) to solve Longest Path?

Attempt #1 - Negate every weight:
Definitionally, Shortest Path is *not* required to be simple, Longest Path *is*.

- Bellman-Ford will stop if it encounters a negative weight cycle to prevent an infinite loop.
- Even in the presence of a positive weight cycle, verticies in Longest Path can not be repeated and so it will never loop.

Note: Allowing repeated veritices and positive weight cycle detection in Longest Path would make it polynomial. Conversely, requiring Shortest Path to be simple would make it NP-Hard.

Shortest vs Longest Path

Why can't we use a shortest path algorithm like Bellman-Ford (Polynomial) to solve Longest Path?

Attempt #2 - Use reciprocal weights:
This doesn't work either because the reciprocal of a sum is not equal to the sum of reciprocals (e.g. $\frac{1}{2+3} \neq \frac{1}{2} + \frac{1}{3}$)

Counter-Example

- Path A: $w_1 = 1, w_2 = 4$
- Path B: $w_3 = 2, w_2 = 2$

Longest Path: $(1 + 4) > (2 + 2) \therefore$ Longest Path: (w_1, w_2)
Shortest Path + Reciprocals: $(1 + \frac{1}{4}) > (\frac{1}{2} + \frac{1}{2}) \therefore$ Longest Path: (w_3, w_4)

NP (Certificate Verification)

Certificate: A sequence of vertices P = [v_1, v_2, ..., v_m] claimed to be a longest path

```
def verify_longest_path(G, certificate, k):
    # Check 1: O(|P|) - All vertices in certificate are unique (simple path)
    if len(certificate) != len(set(certificate)):
        return False

    # Check 2: O(|P|) - All consecutive pairs form edges in G
    total_weight = 0 # Accumulator
    for i in range(len(certificate) - 1):
        edge = G.get_edge(certificate[i], certificate[i+1])
        if edge is None: # Verify edge exists in graph
            return False

        total_weight += edge.weight

    # Check 3: O(1) - Total path weight ≥ k
    return total_weight >= k
```

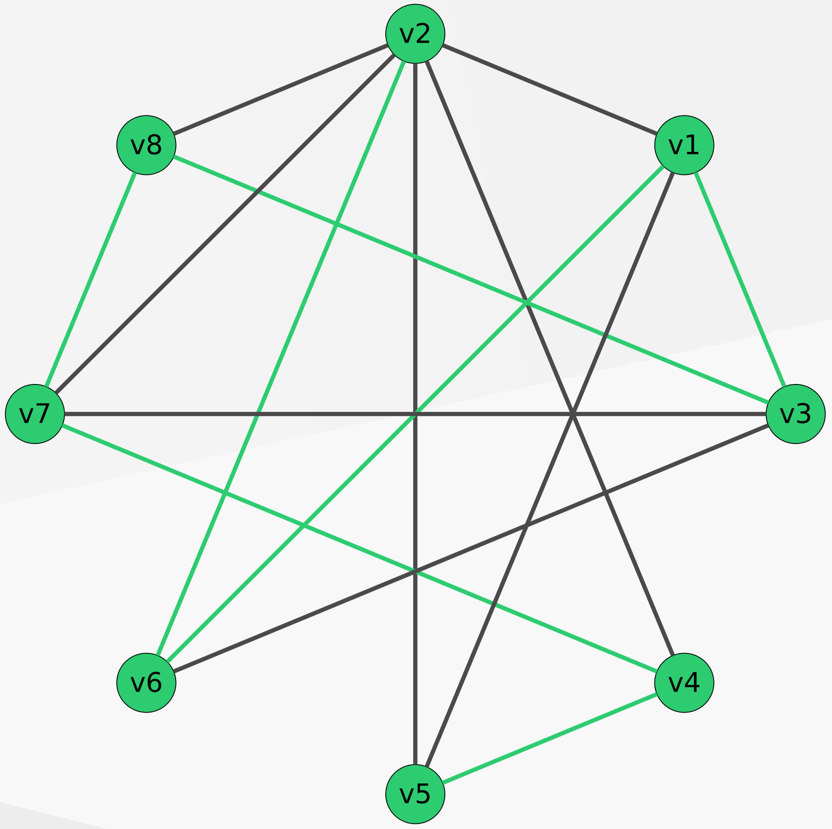
NP-Hardness

Hamiltonian Path Problem (known NP-Complete):

- Input: Unweighted Graph $G = (V, E)$
- Output: A path visiting every vertex exactly once (if it exists)

Reduction Strategy: Hamiltonian Path $\leq_p$ Longest Path

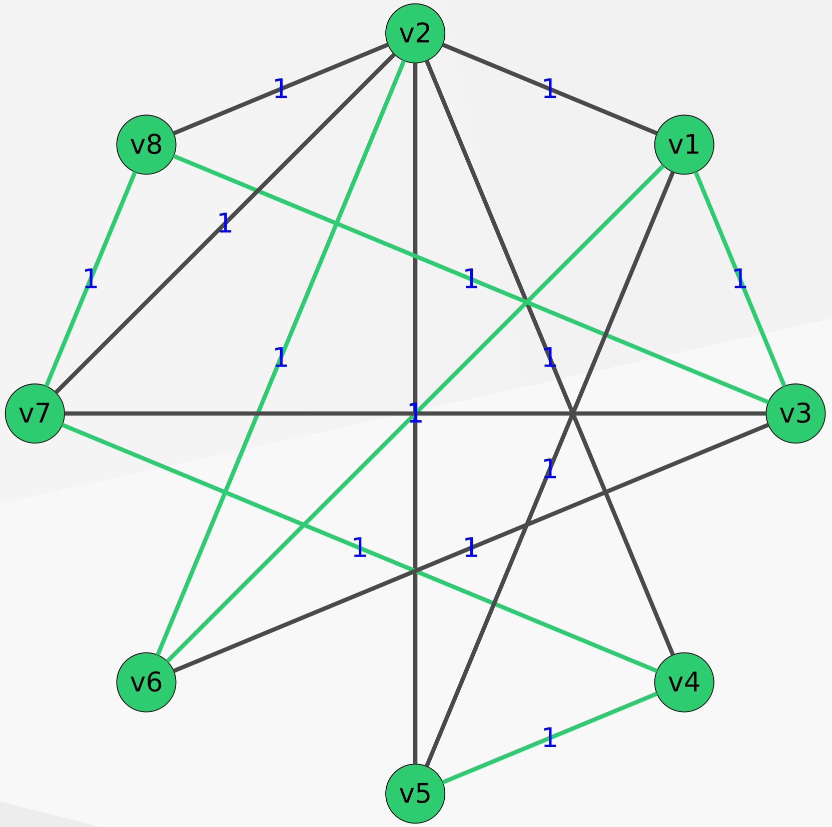
If the Hamiltonian Path Poblem can be reduced to Longest Path in polynomial time, then the Longest Path problem is also NP-Hard.



Reduction

Reduce Hamiltonian Path → Longest Path in polynomial time:

Assign every edge in the graph a weight of 1.
$G = (V, E) \to G = (V, E, w)$ where $w : E \to 1.$
$O(|E|)$



Reframe decision:

Does a path of total weight $\geq |V| - 1$ exist?

- If YES, then the path found by Longest Path is a Hamiltonian path.
- If NO, then no Hamiltonian Path exists.

Reduction Correctness

Forward (⇒): If G has Hamiltonian path P:

- P visits all vertices exactly once
- P uses exactly $|V| - 1$ edges.
- All edges have weight 1 so the total weight of P is $|V| - 1$
- Thus there exists a simple path of total weight $\geq |V| - 1$, so Longest Path answers YES ✓

Reduction Correctness

Backward (⇐): If Longest Path returns path P of weight $\geq |V| - 1$:

- P is simple (no repeated vertices)
- P uses exactly $|V|$ vertices in a graph with $|V|$ total vertices
- Therefore P visits every vertex exactly once
- P is a Hamiltonian path ✓

Conclusion: Reduction runs in polynomial time, therefore, because Hamiltonian Path is NP-Complete, Longest Path is NP-Hard

Implementation (Simplified)

```python
# graph: dict[int, dict[int, float]]

def recurse(
    start: int, weight: float, visited: list[bool]
) -> tuple[list[int], float]:
    # ...

    for adj in graph[start].keys():
        if visited[adj]:
            continue
        # DOMINANT OPERATION: Recursive exploration
        new_path, new_weight = recurse(adj, weight + graph[start][adj], visited)
        # ...

        if new_weight > best_weight:
            best_weight = new_weight
            new_path.insert(0, start)
            best_path = new_path

    return (best_path, best_weight)
```

Analytical Runtime Analysis

Worst Case (Complete Graph) Analysis:

- Start from each of n vertices: $O(n)$
- At each vertex, try extending to each unvisited neighbor
- At depth d, we have d visited vertices, $(n-d)$ unvisited
- Total simple paths explored: $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n!$
- Total: $O(n! \cdot n) = O(n!) = O(|V|!)$

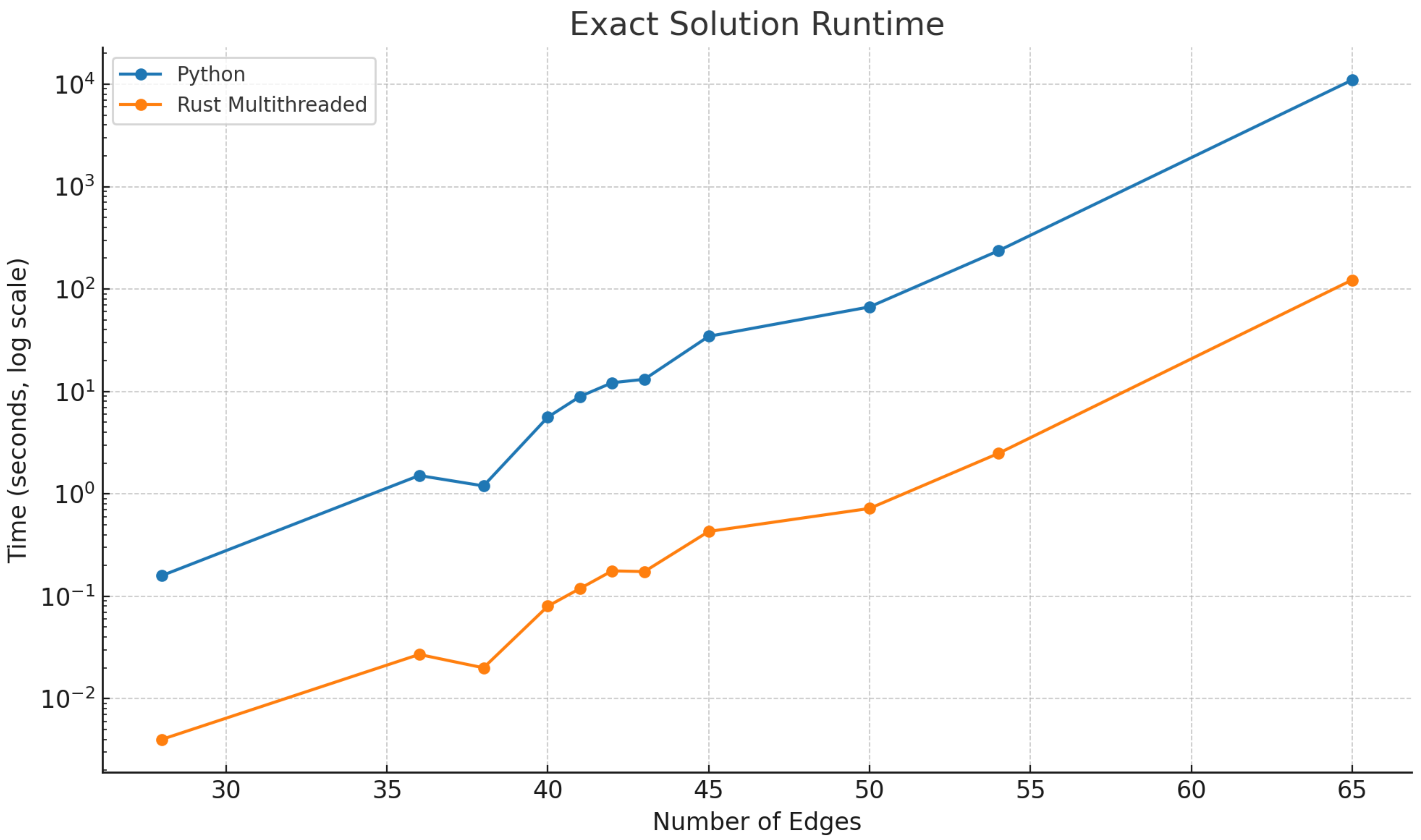
Dominant Operation: Recursive call exploring all possible paths

This factorial runtime is the dominant term and confirms the exponential nature of the problem

Empirical Runtime Analysis

Test Setup: Test cases timed with zsh time built-in

Test Case	Vertices	Edges	Time - Python	Time - Rust	Test Case	Vertices	Edges	Time - Python	Time - Rust
lp_172.txt	15	28	0:00.159	0:00.004	lp_136.txt	19	43	0:13.131	0:00.174
lp_083.txt	11	36	0:01.509	0:00.027	lp_175.txt	17	45	0:34.522	0:00.429
lp_156.txt	20	38	0:01.195	0:00.020	lp_135.txt	18	50	1:06.896	0:00.720
lp_100.txt	12	40	0:05.613	0:00.080	lp_151.txt	21	54	3:55.820	0:02.478
lp_152.txt	16	41	0:08.921	0:00.119	lp_108.txt	21	65	DNF (>20min)	2:02.490
lp_101.txt	16	42	0:12.165	0:00.177					


Exact Solution Runtime
Python
Rust Multithreaded
Time (seconds, log scale)
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
30
35
40
45
50
55
60
65
Number of Edges

High-Level Approximation Strategy

The algorithm combines **two major ideas**:

1. Greedy
2. Random Exploration

Greedy Construction

```
def GREEDY_STEP(G, current, visited):
    # All neighbors of current that are not yet visited
    candidates = [v for v in G[current] if v not in visited]
    if not candidates:
        return None

    # Score(v): degree + number of unvisited neighbors
    def SCORE(v):
        degree = len(G[v])
        unseen = sum(1 for u in G[v] if u not in visited)
        return (degree, unseen)

    # Return the candidate with maximum score
    return max(candidates, key=SCORE)
```

Random Exploration

```
def RANDOM_EXPLORE(G, current, visited, JUMP_PROB):
    r = random.random()      # number in [0, 1)

    # With probability JUMP_PROB, explore randomly
    if r < JUMP_PROB:
        candidates = [v for v in G[current] if v not in visited]
        if not candidates:
            return None
        return random.choice(candidates)

    # Otherwise, follow greedy rule
    return GREEDY_STEP(G, current, visited)
```

Runtime Analysis of `sample_path`

We analyze the worst-case runtime of one greedy + random-jump path.

At each step:

- Scan all neighbors of the current vertex → **O(degree(current))**
- Filter unvisited neighbors
- Pick best-weight or random-choice → **O(1)**

Each vertex becomes `current` at most once,
so each edge is scanned at most **twice**.

Overall Worst-Case Runtime

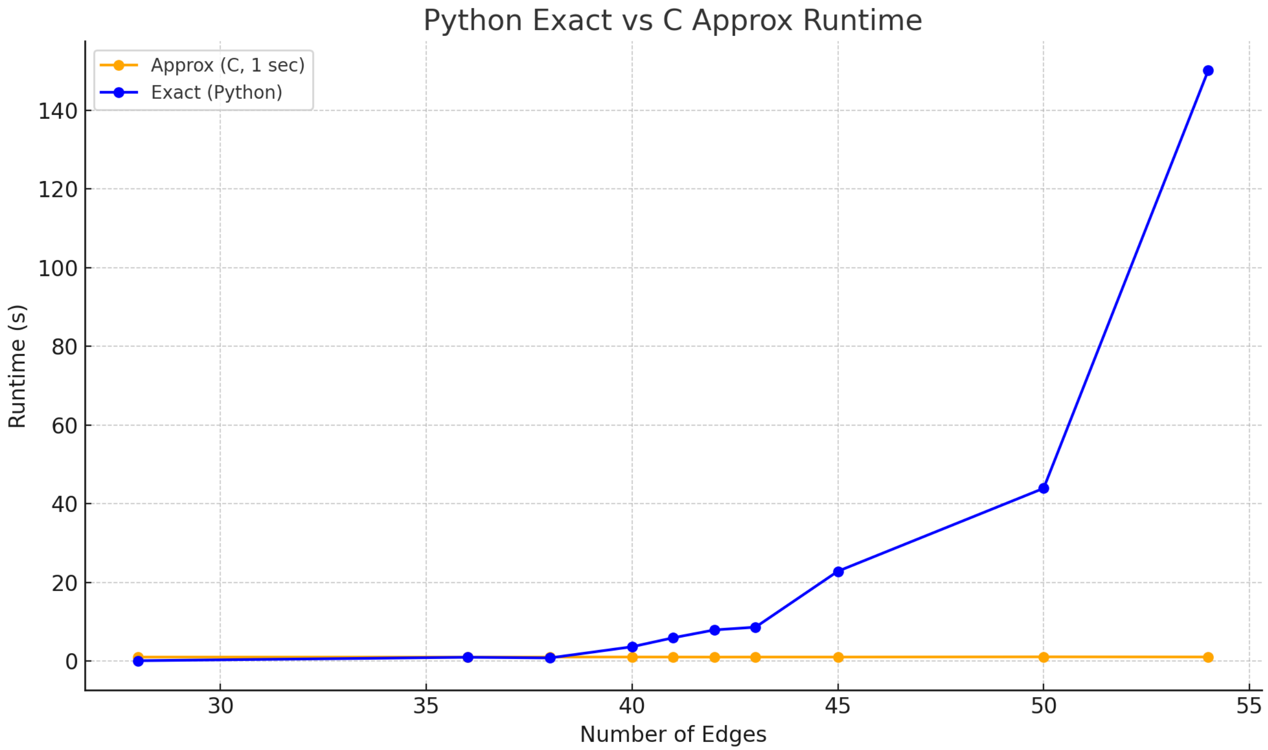
Total neighbor scanning:

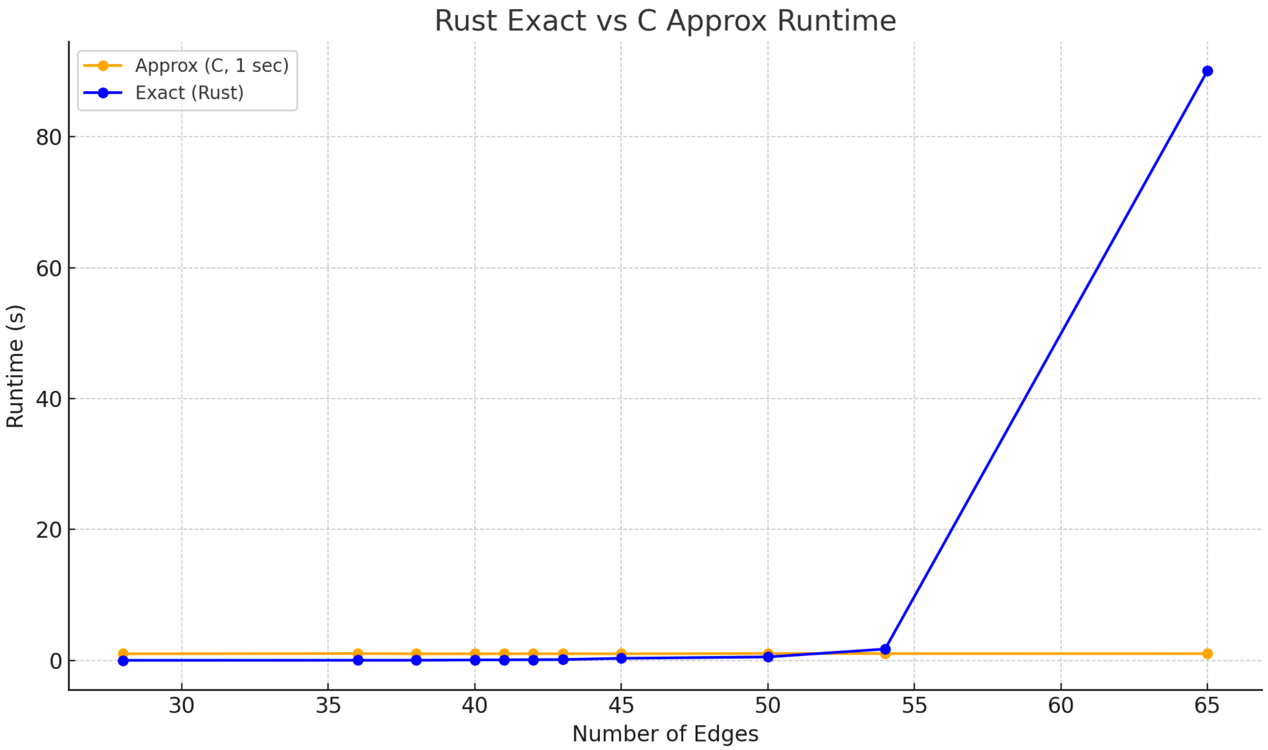
$\text{degree}(v_1) + \text{degree}(v_2) + \ldots + \text{degree}(v_n)$
= **$2E = O(E)$**

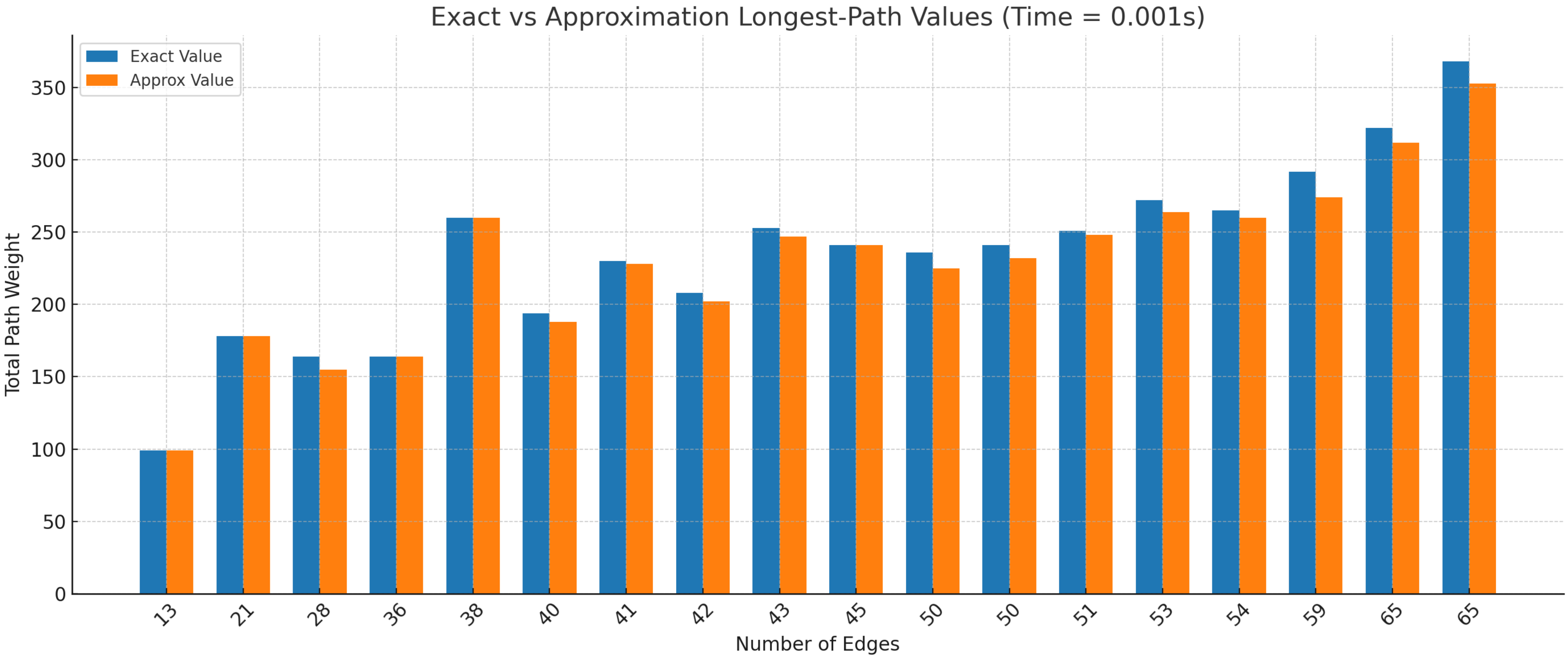
Thus:

T_sample(E) = O(E)

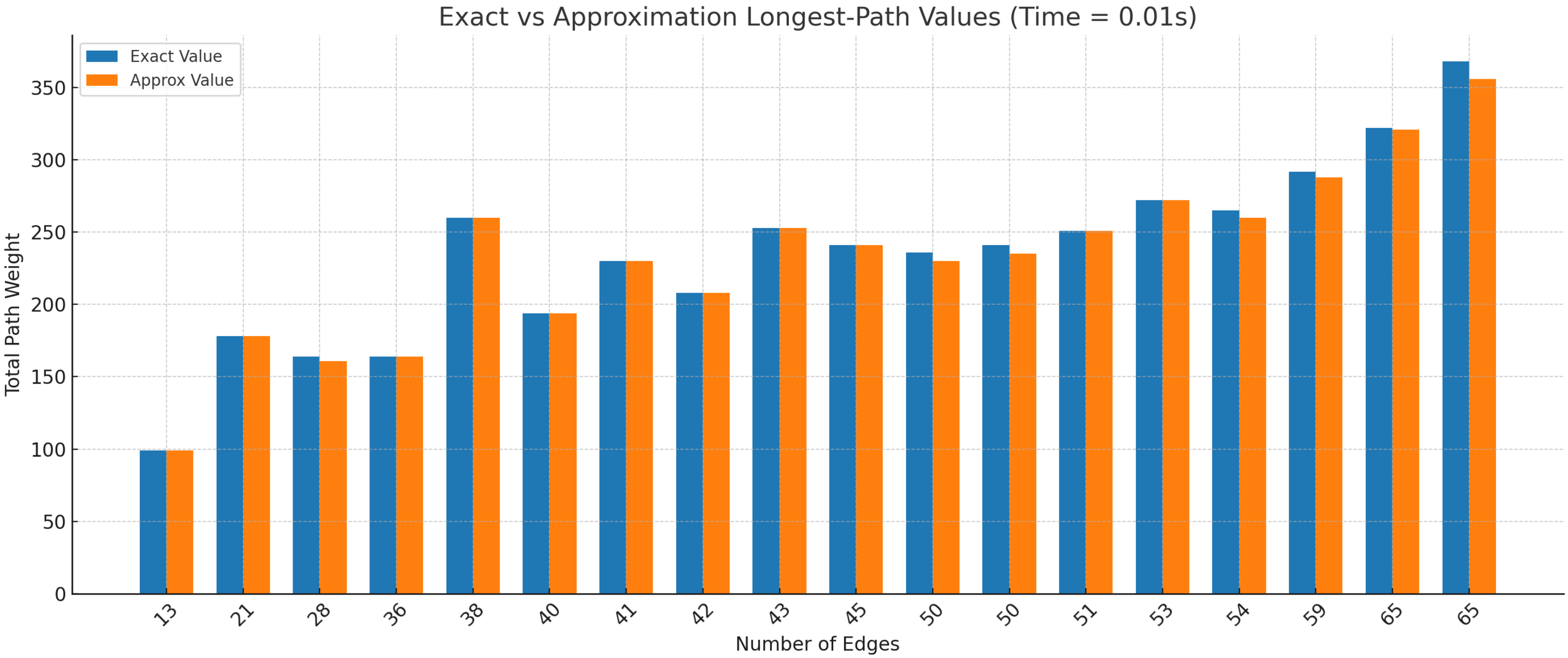
Each sampled path runs in
linear time in the number of edges.

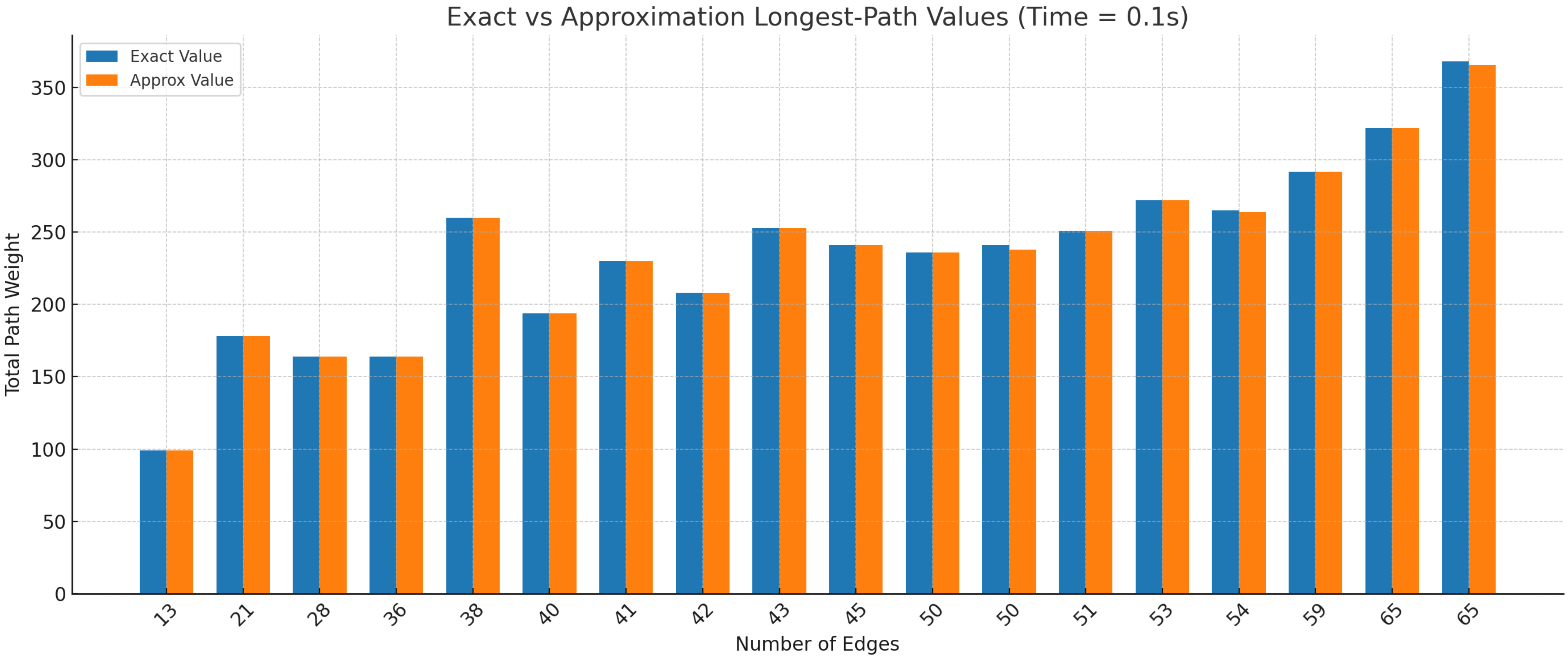

Python Exact vs C Approx Runtime
Approx (C, 1 sec)
Exact (Python)
Runtime (s)
Number of Edges
0
20
40
60
80
100
120
140
30
35
40
45
50
55

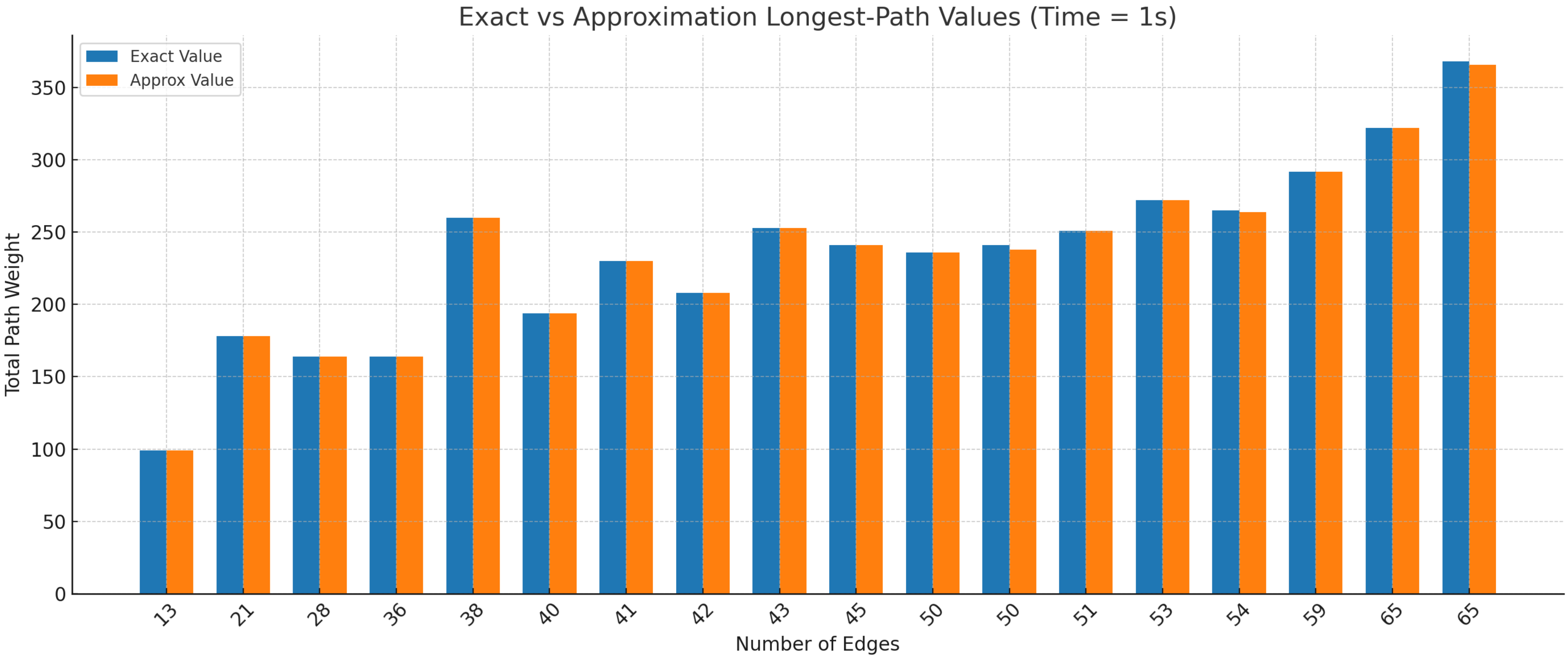

Rust Exact vs C Approx Runtime
Approx (C, 1 sec)
Exact (Rust)
Runtime (s)
Number of Edges
0
20
40
60
80
30
35
40
45
50
55
60
65

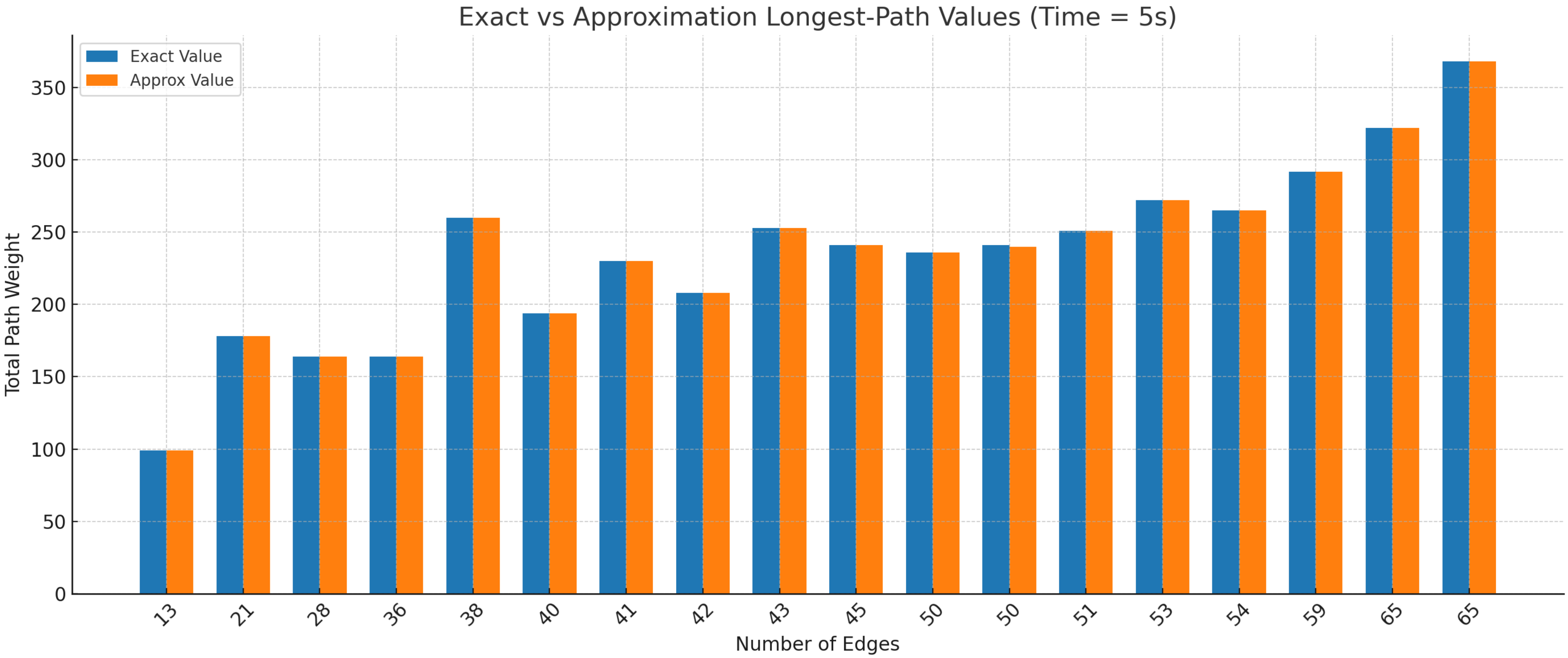


Exact vs Approximation Longest-Path Values (Time = 0.001s)
Exact Value
Approx Value
Total Path Weight
0
50
100
150
200
250
300
350
13
21
28
36
38
40
41
42
43
45
50
50
51
53
54
59
65
65
Number of Edges


Exact vs Approximation Longest-Path Values (Time = 0.01s)
Exact Value
Approx Value
Total Path Weight
0
50
100
150
200
250
300
350
13
21
28
36
38
40
41
42
43
45
50
50
51
53
54
59
65
65
Number of Edges


Exact vs Approximation Longest-Path Values (Time = 0.1s)
Exact Value
Approx Value
Total Path Weight
0
50
100
150
200
250
300
350
13
21
28
36
38
40
41
42
43
45
50
50
51
53
54
59
65
65
Number of Edges


Exact vs Approximation Longest-Path Values (Time = 1s)
Exact Value
Approx Value
Total Path Weight
0
50
100
150
200
250
300
350
13
21
28
36
38
40
41
42
43
45
50
50
51
53
54
59
65
65
Number of Edges


Exact vs Approximation Longest-Path Values (Time = 5s)
Exact Value
Approx Value
Total Path Weight
0
50
100
150
200
250
300
350
13
21
28
36
38
40
41
42
43
45
50
50
51
53
54
59
65
65
Number of Edges

Anytime Approximation Strategy

The algorithm has **two main components**:

1. Greedy Top-k Selection
2. Random Edge Selection

Greedy Top-k Selection (Pseudocode)

```
# Only consider unvisited neighbors
candidates = [(neighbor, weight)
                for neighbor, weight in graph[current_vertex].items()
                if neighbor not in visited_vertices]

if not candidates:
            break

# pick randomly from top-k heaviest using heapq.nlargest
top_k_candidates = heapq.nlargest(k, candidates, key=lambda x: x[1])
neighbor, weight = random.choice(top_k_candidates)

visited_vertices.add(neighbor)
current_path.append(neighbor)
current_weight += weight
current_vertex = neighbor
```

Random Edge Selection (Pseudocode)

```
# Only consider unvisited neighbors
candidates = [(neighbor, weight)
                for neighbor, weight in graph[current_vertex].items()
                if neighbor not in visited_vertices]

if not candidates:
    break

# probability set to 0.3 by default
if random.random() <= probability:
    neighbor, weight = random.choice(candidates)
else:
    # pick heaviest neighbor in linear time using max
    neighbor, weight = max(candidates, key=lambda x: x[1])

visited_vertices.add(neighbor)
current_path.append(neighbor)
current_weight += weight
current_vertex = neighbor
```

Runtime Analysis of the Algorithm

Let:

n = number of vertices

E = number of edges

Δ = max degree of a vertex

Top-K Selection Runtime

- **Per vertex iteration**:
  - Scan neighbors: O(Δ) (Δ = degree of current vertex)
  - Pick top-k using heapq.nlargest: O(Δ)
  - Random choice from top-k: O(1)
- **Total Runtime**:
  - Worst-case O(Δ) = **O(E)**

Random Edge Selection Runtime

- **Per vertex iteration**:
  - Scan neighbors: O(Δ)
  - Random pick: O(1) or max by weight: O(Δ)
- **Total Runtime**:
  - Worst-case O(Δ) = **O(E)**

Anytime Algorithm Total Runtime

- Top-k Selection **O(E)** + Random Edge Selection **O(E)** = **O(E)**
- Algorithm repeats these iterations until time limit is reached,
- **Total Runtime Per Iteration:** O(E)
- **Total Runtime:** unbounded (anytime nature)
    - Can run as long as needed, producing progressively better solutions

Calculating an Upper Bound

Use a **Maximum Spanning Tree**:

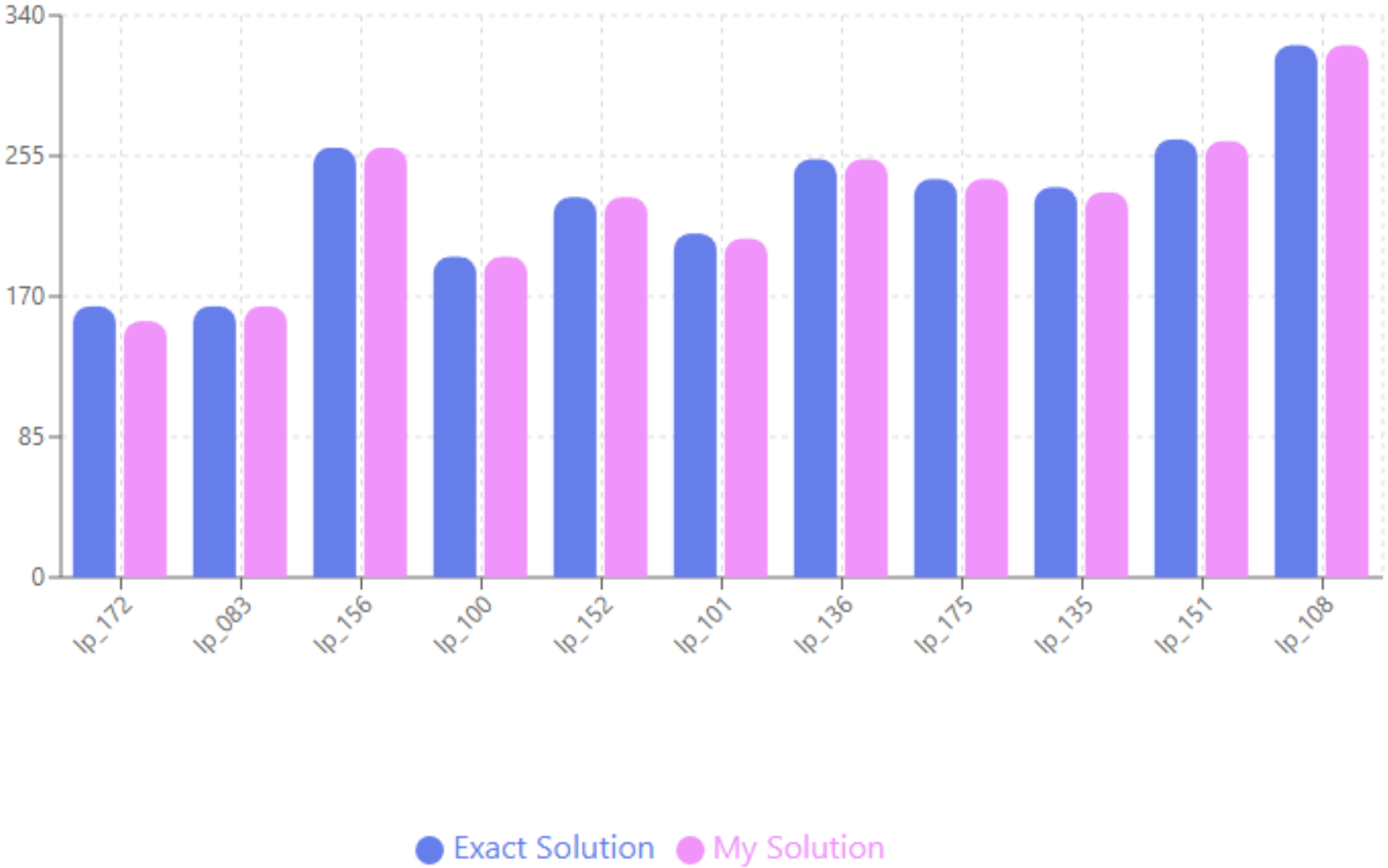
- A spanning tree connects all n vertices using exactly n-1 edges
- An MST is the spanning tree with maximum total weight among all trees
- The **Longest Simple Path Weight <= Maximum Spanning Tree Weight**

Patrick's Runtime VS Alston's Runtime

FILE	UPPER BOUND	1 SECOND	2.5 SECONDS	10 SECONDS	30 SECONDS	60 SECONDS
lp_151	315	260	264	265	265	265
lp_184	686	609	609	613	616	616
lp_208	702	633	619	622	635	635
lp_251	2128	1856	1848	1864	1882	1883
lp_324	1044	852	861	860	871	882

FILE	UPPER BOUND	1 SECOND	2.5 SECONDS	10 SECONDS	30 SECONDS	60 SECONDS
lp_151	315	264	265	265	265	265
lp_184	686	625	629	628	636	637
lp_208	702	635	632	635	640	635
lp_251	2128	1957	1941	1956	1950	1960
lp_324	1044	879	885	893	890	894

- Calculated upper bound using **Maximum Spanning Tree**
- Randomness can cause worse results even with more runtime
- Less improvements after 30 seconds


340
255
170
85
0
lp_172
lp_083
lp_156
lp_100
lp_152
lp_101
lp_136
lp_175
lp_135
lp_151
lp_108
Exact Solution
My Solution

DETAILED RESULTS

lp_172
My solution: 155
Exact: 164
Diff: 9

lp_083
My solution: 164
Exact: 164

lp_156
My solution: 260
Exact: 260

lp_100
My solution: 194
Exact: 194

lp_152
My solution: 230
Exact: 230

lp_101
My solution: 205
Exact: 208
Diff: 3

lp_136
My solution: 253
Exact: 253

lp_175
My solution: 241
Exact: 241

lp_135
My solution: 233
Exact: 236
Diff: 3

lp_151
My solution: 264
Exact: 265
Diff: 1

lp_108
My solution: 322
Exact: 322