

# A Sound and Complete Axiomatisation for Spatio-Temporal Specification Language

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**Abstract**—Specifying spatio-temporal aspects is one of the important areas in cyber-physical systems. Spatio-temporal logic with changes of truth value in discrete time and dense time has been researched, but a combination of spatial and temporal components with changes of spatial entities in dense time hasn't been well-done. The major problem is dense time and real-valued variables of the spatio-temporal properties of cyber-physical systems. In this paper, we propose a spatio-temporal specification language, named STSL, which integrates Signal Temporal Logic (STL) with a spatial logic  $S4_u$  to deal with the changes of real-values spatial entities in dense time. The combined language is divided into two formalisms,  $STSL_{PC}$  and  $STSL_{OC}$ , which is applied to interpret the Boolean semantics and quantitative semantics, respectively. The syntax of the two formalism and the corresponding semantics are provided. Besides, we present a Hilbert-style axiomatization for the proposed STSL and provide the soundness and completeness result by the spatio-temporal extension of maximal consistent set and canonical model.

**Index Terms**—Signal Temporal Logic (STL)  $S4_u$  Spatio-Temporal Specification Language (STSL)  $STSL_{PC}$   $STSL_{OC}$  Soundness Completeness Axiomatization System

## I. INTRODUCTION

It is a challenging work to model cyber-physical systems, not only because cyber-physical systems integrate cyber systems, physical environment and the interactive part of them, but also because cyber-physical systems combine temporal and spatial aspects, discrete and continuous behavior, and uncertainty. Describing spatio-temporal aspects is one of the important areas in cyber-physical systems. Many works have been done with hybrid [1] and stochastic behaviors of cyber-physical systems, but fewer researchers concentrate on spatio-temporal aspects. The major problem is multidimensional expressiveness and expensive verifiability for modeling and

analysis of the spatio-temporal behaviors of cyber-physical systems.

This work aims at building a spatio-temporal specification language (STSL) by solving spatio-temporal constraints concerning dense time and real-valued variables, as an intelligent object in physical environment is provided with changes in specified space and continuous time. More specifically, we confine ourselves to the combination of topological space and time constraints with real-valued interval, which may be an open, half-open and half-closed, half-closed and half-open, or closed interval in a flow of time. We adopt the modal spatial logic  $S4_u$  to express topological constraints, which is one of the most influential formalism and the most expressiveness for topological relations. As for signal temporal logic (STL) [2], [10], there are two approaches that can cope with signals, quantitative semantics and Boolean semantics. Quantitative semantics obtains real-valued signals from *satisfaction degree* of a trace in real-valued interval. While Boolean semantics evaluates Boolean signals from a trace which can be booleanized through a set of threshold predicates.

Combining spatio-temporal constraints from temporal logics and modal spatial logics is a very important problem. Given a spatio-temporal model  $\mathfrak{M}$  and a STSL formula  $\varphi$ , the satisfiability problem of the formula  $\varphi$  is to check if  $\varphi$  is satisfiable in model  $\mathfrak{M}$ .

Since the changes of spatial entities and the flows of time are not independent, the combination between modal spatial logics and temporal logics is divided into two formalism,  $STSL_{PC}$  and  $STSL_{OC}$ .  $STSL_{PC}$  means the changes of spatial propositions over time, while  $STSL_{OC}$  represents the changes or evolution of spatial objects over time. Each formalism is equipped with different expressiveness, so Boolean semantics and quantitative semantics need to be provided.

Many works have been done on the axiomatization and completeness of modal logics. Patrick Blackburn [4] present the completeness of normal modal logic through maximal consistent set and canonical model. J.M. Davoren [5] proposes topological semantics for intuitionistic tense logics and multi-modal logic and provide the Hilbert-style axiomatization and the completeness result. F.D. David [6] prove the absolute completeness of  $S4_u$  for its measure-theoretic semantics. In this paper, we present an axiomatization system for **STSL** and provide the soundness and completeness result of the axiomatization system.

In this work, there are three contributions:

- 1) We propose a spatio-temporal specification language **STSL**, based on STL and  $S4_u$ , to specify the changes in topological space and dense time,
- 2) We interpret the **STSL** language from two formalisms:  $STSL_{PC}$  and  $STSL_{OC}$ , and provide Boolean semantics and quantitative semantics for the language,
- 3) We present an axiomatization system and provide the completeness result for the proposed language **STSL**.

The next section introduces temporal logic STL and modal spatial logic  $S4_u$ . Section 3 presents the spatio-temporal specification language **STSL** and section 4 presents an axiomatization system for the language **STSL**. In section 5, we conclude the work and talk about the future work.

## II. SIGNAL TEMPORAL LOGIC AND $S4_u$

The section provides the background to the proposed spatio-temporal logic, including signal temporal logic (STL) and spatial logic  $S4_u$ .

### A. Signal Temporal Logic (STL)

An STL signal [2], [7] is defined on dense-time domain  $\mathbb{T}$ . A *signal function*  $\varepsilon: \mathbb{T} \rightarrow \mathbb{E}$  associates a set of time domain with a set of signals. Signals with  $\mathbb{E} = \mathbb{B} = \{0, 1\}$  are called Boolean signals, while those where  $\mathbb{E} = \mathbb{R}^+$  are called real-valued or quantitative signals. A Boolean signal, transformed from real-valued one, can be represented by Metric Temporal Logic [8], [9]. The comparison of some temporal logics is listed in Table I.

TABLE I  
THE COMPARISON OF LTL, MTL AND STL

Logic	Time domain	Variable
Linear Temporal Logic (LTL)	Discrete time	Boolean value
Metric Temporal Logic (MTL)	Dense-time	Boolean value
Signal Temporal Logic (STL)	Dense-time	Real-value

An execution trace  $w$  is a set of real-valued signals  $x_1^w, \dots, x_k^w$  bound in some interval  $I$  of  $\mathbb{R}^+$ , which is called the time domain of  $w$  [10]. Such an interval  $I \subseteq \mathbb{R}^+$  is open  $(t_1, t_2)$ , half-closed and half-open  $[t_1, t_2)$ , half-open and half-closed  $(t_1, t_2]$  or closed time interval  $[t_1, t_2]$ . For instance, a running car checks in and enters a highway at  $t_1$  and checks

out and leaves it at  $t_2$ . We say, the car keeps running on the highway in the interval  $[t_1, t_2]$ .

The syntax of STL is given by

$$\varphi ::= \top \mid x_i \geq 0 \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \mathcal{U}_I \varphi_2$$

where  $x_i \geq 0$  is an atomic predicate whose truth value is determined by the sign of an evaluation based on a signal  $x_i$ . The Boolean operators  $\neg$  and  $\wedge$  are negation and conjunction, respectively. The time *bounded until* operator  $\mathcal{U}_I$  is defined on the time interval  $I$ .

The formula  $\varphi_1 \vee \varphi_2$ ,  $\Diamond_I \varphi$  and  $\Box_I \varphi$  can be defined by:

$$\begin{aligned} \varphi_1 \vee \varphi_2 &= \neg(\neg\varphi_1 \wedge \neg\varphi_2) \\ \Diamond_I \varphi &= \top \mathcal{U}_I \varphi, \\ \Box_I \varphi &= \neg \Diamond_I \neg\varphi. \end{aligned}$$

Formula  $\Diamond_I \varphi$  indicates that at some time  $t \in I$ ,  $\varphi$  is eventually satisfiable, while  $\Box_I \varphi$  denotes that  $\varphi$  is always satisfiable at each time  $t \in I$ .

### B. An axiomatisation system for STL

The fundamental composition of the axiomatization system for STL contains all the tautologies like atomic proposition, boolean and quantitative operator in first-order logic. Temporal expressiveness and inference rules are shown as follows:

A0 All classical tautologies of first-order logic

A1  $\Box(\phi \rightarrow \varphi) \rightarrow (\Box\phi \rightarrow \Box\varphi)$

A2  $\neg \circ \phi \leftrightarrow \circ \neg \phi$

A3  $\circ(\phi \rightarrow \varphi) \rightarrow (\circ\phi \rightarrow \circ\varphi)$

A4  $\Box(\phi \rightarrow \circ\varphi) \rightarrow (\phi \rightarrow \Box\varphi)$

A5  $(\phi \mathcal{U} \varphi) \leftrightarrow \varphi \vee \circ(\phi \mathcal{U} \varphi)$

A6  $(\phi \mathcal{U} \varphi) \rightarrow \Diamond \varphi$

MP  $\frac{\phi \quad \phi \rightarrow \varphi}{\varphi}$

$N_{\Box} \frac{\phi}{\vdash \Box \phi}$

$N_{\circ} \frac{\phi}{\vdash \circ \phi}$

### C. Spatial Logic: $S4_u$

$S4$  [11] is a proposition modal logic and  $\tau$  are spatial terms under the topological space interpretation. In the absence of ambiguity, the terminology spatial terms denote spatial objects. According to an observation by [12],  $S4$  is a logic of topological spaces, and the propositional variable  $p$  is interpreted as a subset of the topological space. From the perspective of the topological space, propositional variables of  $S4$  will be understood as spatial variables [13]. The formula of  $S4$  is defined as follows:

$$\tau ::= p \mid \bar{\tau} \mid \tau_1 \sqcap \tau_2 \mid \mathbb{I}_{\tau}$$

where  $p$  is named as spatial variables and  $\bar{\tau}$  is the complementary of  $\tau$ ,  $\tau_1 \sqcap \tau_2$  the *intersection* operation of  $\tau_1$  and  $\tau_2$ .  $\mathbb{I}$  is an *interior* operator under the topological space interpretation. The *union* and *closure* operator can be defined by:

$$\tau_1 \sqcup \tau_2 = \overline{(\bar{\tau}_1 \sqcap \bar{\tau}_2)}, \quad \mathbb{C}_{\tau} = \bar{\bar{\tau}}$$

$\mathbb{C}_\tau$  refers to the *closure* of a spatial object  $\tau$ . For example, In order to ensure safety, the rear car  $c_{rear}$  can't reach the edge of the front car, the formula can be described by  $\overline{\mathbb{C}_{rear}} \sqcap \mathbb{C}_{front}$ .

A topological model is a pair of the form  $\mathfrak{M} = (\mathfrak{L}, \mathfrak{V}(p))$ , where  $\mathfrak{L} = (U, \mathbb{I})$  is a topological space.  $U$  is a nonempty set denoting the universe of the space, and  $\mathbb{I}$  is the interior operator on  $U$ .  $\mathfrak{V}(p)$ , as a set of valuations on spatial variables, is a subset of  $U$ . Therefore we get the valuation of other spatial formulas as follows:

$$\begin{aligned}\mathfrak{V}(\bar{\tau}) &= U - \mathfrak{V}(\tau), \mathfrak{V}(\tau_1 \sqcup \tau_2) = \mathfrak{V}(\tau_1) \cup \mathfrak{V}(\tau_2), \\ \mathfrak{V}(\mathbb{I}\tau) &= \mathbb{I}\mathfrak{V}(\tau), \mathfrak{V}(\tau_1 \sqcap \tau_2) = \mathfrak{V}(\tau_1) \cap \mathfrak{V}(\tau_2), \\ \mathfrak{V}(\mathbb{C}\tau) &= \mathbb{C}\mathfrak{V}(\tau).\end{aligned}$$

A spatial logic is a formal language interpreted over a class of structures featuring geometrical entities and relations. Among the well-known spatial logics such as RCC-8 [14], [15], BRCC-8 and  $\mathcal{S4}_u$ , the most expressive spatial formalism is  $\mathcal{S4}_u$  [16].  $\mathcal{S4}_u$  extends  $\mathcal{S4}$  with the *universal* and *existential* quantifiers  $\boxtimes$ ,  $\boxdot$  based on a spatial term  $\tau$ .  $\boxdot\tau$  refers to that there is at least one element in space  $\tau$ ;  $\boxtimes\tau$  means that all elements in the space belong to  $\tau$ . The formula  $\varphi$  is defined in the form of BNF:

$$\varphi ::= \boxtimes\tau \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2$$

where,  $\neg\varphi$  is the negation of  $\varphi$  and  $\varphi_1 \wedge \varphi_2$  the conjunction of  $\varphi_1$  and  $\varphi_2$ . Correspondingly, the *disjunction* and *existential* operator can be defined by:

$$\varphi_1 \vee \varphi_2 = \neg(\neg\varphi_1 \wedge \neg\varphi_2), \boxdot\tau = \neg\boxtimes\bar{\tau}$$

The axiomatization system for  $\mathcal{S4}_u$  includes the classical propositional logic in topology  $\mathfrak{L}$ , the modal logics K, T, 4 and  $\neg$  and inference rules.

CP axioms of classical propositional logic in  $\mathfrak{L}$

$$\Box K \quad \Box(\phi \rightarrow \varphi) \rightarrow (\Box\phi \rightarrow \Box\varphi)$$

$$\Box T \quad \Box\phi \rightarrow \phi$$

$$\Box 4 \quad \Box\phi \rightarrow \Box\Box\phi$$

$$\neg \quad \neg\Box\phi \leftrightarrow \Box\neg\phi$$

and the inference rules

$$\text{MP} \quad \frac{\phi \quad \phi \rightarrow \varphi}{\varphi}$$

$$N_{\Box} \quad \frac{\phi}{\Box\phi}$$

$$N_{\Box} \quad \frac{\phi}{\Box\phi}$$

### III. SPATIO-TEMPORAL SPECIFICATION LANGUAGE

STL provides an approach that combines the truth value and quantitative value of general signals. But it is inadequate to represent the changes of a spatial entity and the binary relation between spatial entities and temporal aspects. We propose the spatio-temporal specification language that combines STL and  $\mathcal{S4}_u$  to describe the evolution in spatial and temporal domain.

#### A. Spatio-temporal signal

A spatio-temporal signal is defined with continuous time and topological space [17], [18]. The time domain  $\mathbb{T}$  will

usually be a real-valued interval  $[0, t]$ , where  $t \in \mathbb{R}_{\geq 0}$ . The *signal function* is extended to spatial-temporal domain with  $\varepsilon: \mathbb{T} \times \mathbb{L} \rightarrow \mathbb{B}$ , where  $\mathbb{L}$  denotes the topological space. The domain of Boolean and quantitative signals extends the domain of STL signals to topological space.

A spatio-temporal trace  $w(t, l)$  provides a notation about execution sequence of time  $t$  and space  $l$ . For a spatio-temporal trace, there are two different interpretations:

- A trace represents a sequence of spatial objects and time point and each point in the trace evaluates a pair of spatial objects and time.
- Another interpretation means that a spatio-temporal trace takes spatial objects as the basic entities and spatio-temporal primitive relations could be obtained by the changes of ontology of space over time.

In this work, we treat the spatio-temporal trace as the second interpretation. The changes of spatial objects are influenced by the flow of time.

**Definition III.1** (Spatio-temporal signal). A *spatio-temporal signal*  $\varepsilon(t, l)$  is an evaluation of spatial entities in a trace  $w$  over time. A *Boolean signal*  $\mu(t, l)$  is an evaluation of an atomic proposition transferred from quantitative signals  $\mathbf{x}(t, l)$  by atomic predicate  $\mu(t, l) = (\mathbf{x}(t, l) \geq 0)$  in a trace  $w$ .

$$\varepsilon(t, l) := \begin{cases} \mu(t, l) & \text{if } \mathbb{T} \times \mathbb{L} \rightarrow \mathbb{B} \\ \mathbf{x}(t, l) & \text{if } \mathbb{T} \times \mathbb{L} \rightarrow \mathbb{R}_{\geq 0} \end{cases}$$

where boolean function  $\mu: \mathbb{T} \times \mathbb{L} \rightarrow \mathbb{B}$  gives rise to the Boolean signals  $\varepsilon(t, l) = \mu(t, l)$ , while quantitative signals are obtained as the real-valued function  $\mathbf{x}: \mathbb{T} \times \mathbb{L} \rightarrow \mathbb{R}_{\geq 0}$ , with  $\varepsilon(t, l) = \mathbf{x}(t, l)$ .

#### B. The interpretation of the combined logic

It is essential that a combined spatio-temporal formalism should be provided with enough expressiveness to contain the three parameters [13]:

- 1) the expressiveness of the spatial component;
- 2) the expressiveness of the temporal component;
- 3) the interaction between the two components allowed in the combined logic.

The interaction between the spatial and temporal components should comply with the principle of *PC* and *OC*, which is used to evaluate the interaction:

- 1)  $STSL_{PC}$ : the language should be able to express changes over time of the truth-values of purely spatial propositions.
- 2)  $STSL_{OC}$ : the language should be able to express changes or evolution of spatial objects over time.

$STSL_{PC}$  expresses the change of truth-value of proposition and it is the elementary requirement for a combined spatio-temporal logic. For instance:

- Eiffel Tower is located in Paris.
- Two cars running on the road never occupy the same location simultaneously.

For  $STSL_{OC}$ , spatio-temporal properties are described about the changes of spatial objects over time. We use spatial objects in topological space to interpret the principle such as

- One wave gradually formed in the sea and disappeared eventually on the shore.
- If train A will pass the location in one hour that train B occupies now on the railway, train B must run within one hour.
- A package from a courier service company will eventually be delivered to its destination through several transfer stations.

The spatio-temporal signals are divided into Boolean and quantitative signals. According to the category of spatio-temporal signals, we will present syntax and semantics for the proposed spatio-temporal specification language from two sides:

- The Boolean semantics returns true if the trace of spatio-temporal model satisfies the properties described by  $STSL_{PC}$  formulas.
- The quantitative semantics returns a real value in different time that can be interpreted as an evaluation of satisfaction of the  $STSL_{OC}$  formulas.

The Boolean semantics of the spatio-temporal specification language interprets that a formula of **STSL** over spatio-temporal traces returns true or false, so it is able to express changes of the truth-values concerning purely spatial propositions of  $PC$ . Meanwhile, the  $\mathcal{U}_I$  and  $\Box_I$  operators of the quantitative semantics are able to express changes or evolution of spatial objects over some fixed finite periods and the whole duration of time of  $OC$ , respectively. So, the expressiveness power of the combined spatio-temporal logic is enough to describe the spatio-temporal behavior.

### C. The syntax of $STSL_{PC}$

We extend the real-value interval into spatio-temporal domain. Formally, we define the spatial temporal interval  $I$  as  $[(t, l), (t', l')]$ ,  $\forall t, t' \in \mathbb{T}$  and  $t < t'$ . The  $STSL_{PC}$  is defined on spatio-temporal terms  $\tau$  over the spatio-temporal interval  $I$ , combining temporal logic STL and modal spatial logic  $S4_u$ . The syntax of  $STSL_{PC}$  is given by:

$$\begin{aligned} \tau &::= p \mid \bar{\tau} \mid \tau_1 \sqcap \tau_2 \mid \mathbb{I}\tau \\ \varphi &::= \Box\tau \mid x_i \geq 0 \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \mathcal{U}_I \varphi_2 \end{aligned}$$

- $\tau$  is a spatio-temporal term,
- $p$  is a spatio-temporal variable,
- $\bar{\tau}$  is the complementary of  $\tau$ ,
- $\tau_1 \sqcap \tau_2$  is the intersection of  $\tau_1$  and  $\tau_2$ ,
- $\mathbb{I}$  is the *interior* operator under the topological space interpretation. Moreover, the dual operator of  $\mathbb{I}$  is the *closure* operator  $\mathbb{C}$ , which means possible or consistent,
- $x_i \geq 0$  is an atomic predicate,
- $\neg, \vee$  and  $\wedge$  are the Boolean operators,
- $\mathcal{U}_I$  is the *until* operator.

We can define equivalence of operators as syntactic abbreviations:

$$\begin{aligned} \mathbb{C}\tau &= \mathbb{U} - \mathbb{I}\tau \\ \tau_1 \sqcup \tau_2 &= \overline{\tau_1 \sqcap \tau_2} \\ \varphi_1 \vee \varphi_2 &= \neg(\neg\varphi_1 \wedge \neg\varphi_2) \\ \bigcirc_I \varphi &= \perp \mathcal{U}_I \varphi \\ \Diamond_I \varphi &= \top \mathcal{U}_I \varphi \\ \Box_I \varphi &= \neg \Diamond_I \neg \varphi. \end{aligned}$$

Atomic predicates, Boolean operators, and the spatio-temporal *bounded until* operator  $\mathcal{U}_I$  are from STL. The new spatial operators are the *interior* operator  $\mathbb{I}$ , the *closure* operator  $\mathbb{C}$  with reference to  $S4_u$ . The  $\bigcirc_I$ ,  $\Diamond_I$  and  $\Box_I$  operators are derived unary operators.  $\Box_I \varphi$  denotes that  $\varphi$  holds within the whole trace of spatio-temporal interval  $I$ , and  $\Diamond_I \varphi$  means that  $\varphi$  holds in at least one time point of the spatio-temporal interval  $I$ .

### D. The semantics of $STSL_{PC}$

A spatio-temporal model is defined on topological space and temporal model. Formally, a spatio-temporal model  $\mathfrak{M} = (\mathfrak{T}, \mathfrak{L}, \mathfrak{V})$ , where

- $\mathfrak{T}$  is a pair  $(\mathbb{T}, <)$ , where  $\mathbb{T}$  is a set of time point and  $<$  R an irreflexive, transitive and asymmetric relation on  $\mathbb{T}$  with a linear strict time flow,
- $\mathfrak{L}$  is a topological space domain with the definition of  $\mathbb{U}, \mathbb{I}$  in which  $\mathbb{U}$  is a nonempty set, the universe of the space, and  $\mathbb{I}$  is the interior operator on  $\mathbb{U}$  satisfying the standard Kuratowski axioms:  $\forall X, Y \subseteq \mathbb{U}, \mathbb{I}(X \cap Y) = \mathbb{I}X \cap \mathbb{I}Y, \mathbb{I}X \subseteq \mathbb{I}\mathbb{I}X$  and  $\mathbb{I}(\mathbb{U}) = \mathbb{U}$ ,
- $\mathfrak{V}$  is a valuation on the time point set  $\mathfrak{T}$  and the spatial variable set  $\mathbb{P}$ , i.e.,  $\forall p \in \mathbb{P}$ , and  $t \in \mathfrak{T}$  and  $\mathfrak{U}(p, t)$  which means the space occupied by  $p$  at time point  $t$ . As for the spatial term  $\tau$ , the valuation can be defined as:  $\mathfrak{V}(\bar{\tau}, t) = \mathbb{U} - \mathfrak{V}(\tau, t)$ ,  $\mathfrak{V}(\tau_1 \sqcap \tau_2, t) = \mathfrak{V}(\tau_1, t) \cap \mathfrak{V}(\tau_2, t)$ ,  $\mathfrak{V}(\mathbb{I}\tau, t) = \mathbb{I}\mathfrak{V}(\tau, t)$ .

We define a spatio-temporal trace  $w$  as the changes of spatial objects over time. In spatio-temporal model  $\mathfrak{M}$ ,  $\mathfrak{V}(p)$  evaluates the spatial object  $p$ . Further, the spatio-temporal trace  $w$  represents  $\mathfrak{V}(p)$  changes over time  $t$ , where  $t$  belongs to the domain  $\mathbb{T}$ .

The satisfaction relation for a  $STSL_{PC}$  formula  $\varphi$  over a spatio-temporal model  $\mathfrak{M}$  is given by:

- $(\mathfrak{M}, t) \models \Box\tau \Leftrightarrow \mathfrak{V}(\Box\tau, t) = \top$
- $(\mathfrak{M}, t) \models x_i \geq 0 \Leftrightarrow \mathfrak{V}(x_i \geq 0, t) = x_i$
- $(\mathfrak{M}, t) \models \neg\varphi \Leftrightarrow (\mathfrak{M}, t) \not\models \varphi$
- $(\mathfrak{M}, t) \models \varphi_1 \wedge \varphi_2 \Leftrightarrow (\mathfrak{M}, t) \models \varphi_1$  and  $(\mathfrak{M}, t) \models \varphi_2$
- $(\mathfrak{M}, t) \models \varphi_1 \mathcal{U}_I \varphi_2 \Leftrightarrow \exists t' \in t + I$  s.t.  $(\mathfrak{M}, t') \models \varphi_2$  and  $\forall t'' \in [t, t']$ ,  $(\mathfrak{M}, t'') \models \varphi_1$

A model  $\mathfrak{M}$  satisfies  $\varphi$  in  $t$ , denoted by  $(\mathfrak{M}, t) \models \varphi$ .

For a given formula  $\varphi$  and execution trace  $w$ , we define the *satisfaction* signal  $\chi(\varphi, w, t, l)$  over a trace  $w(t, l)$ :

$$\forall t \in I, \chi(\varphi, w, t, l) := \begin{cases} \top & \text{if } (\mathfrak{M}, t) \models \varphi \\ \perp & \text{otherwise} \end{cases} \quad (1)$$

In order to compute the *satisfaction* of a formula  $\varphi$ , we divide the formula  $\varphi$  into each subformula  $\phi_i$  until atomic

formula so that formula  $\varphi$  can be computed through the subformula and atomic formulas instead of the entire *satisfaction* signal  $\chi(\varphi, w, t, l)$ . The procedure can be treated as a hierarchical structure from the full formula  $\varphi$  down to each atomic formula.

#### E. STSL<sub>OC</sub>

The difference between  $STSL_{PC}$  and  $STSL_{OC}$  is that  $STSL_{PC}$  involves in the change of truth-values of propositions, while  $STSL_{OC}$  describes the change of extensions of predicates. The syntax of  $STSL_{OC}$  is given by:

$$\begin{aligned}\tau &::= p \mid \bar{\tau} \mid \tau_1 \sqcap \tau_2 \mid \mathbb{I}\tau \mid \tau_1 \mathcal{U}_I \tau_2 \\ \varphi &::= \boxtimes \tau \mid x_i \geq 0 \mid \neg \varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \mathcal{U}_I \varphi_2\end{aligned}$$

The change exists in that the operator  $\mathcal{U}_I$  can be applied to spatial terms  $\tau$ .  $\tau_1 \mathcal{U}_I \tau_2$  refers to  $\tau_1$  holds until  $\tau_2$  holds within the interval  $I$ . Similar to the atomic formulas, the  $\square_I$ ,  $\square_I$  and  $\diamond_I$  of spatio-temporal term can also be derived from the operator  $\mathcal{U}_I$ . The unary operators  $\square_I$  and  $\diamond_I$  in  $STSL_{OC}$  formulas fuse the  $\square_I$  and  $\diamond_I$  operators of STL with that of  $\boxtimes$  and  $\diamond$  operators of  $S4_u$ .

A spatio-temporal trace  $w$  is a sequence over signals  $\varepsilon$ . The Boolean satisfaction relation and *satisfaction degree* for a  $STSL_{OC}$  formula  $\varphi$  over a spatio-temporal trace  $w$  is similar to that of  $STSL_{PC}$  formula.

We define  $\rho$  to quantify the *satisfaction degree* of the property  $\varphi$  over the trace  $w(t, l)$ , and it returns a real number  $\rho(\varphi, w, t, l)$ . The quantitative satisfaction relation for a formula  $\varphi$  over a spatio-temporal trace  $w$  is given by:

- $\rho(\boxtimes \tau, w, t, l) = \top$
- $\rho(x_i \geq 0, w, t, l) = f(w(t, l))$  where  $\mu \equiv (f \geq 0)$
- $\rho(\neg \varphi, w, t, l) = -\rho(\varphi, w, t, l)$
- $\rho(\varphi_1 \wedge \varphi_2, w, t, l) = \min\{\rho(\varphi_1, w, t, l), \rho(\varphi_2, w, t, l)\}$
- $\rho(\varphi_1 \vee \varphi_2, w, t, l) = \max\{\rho(\varphi_1, w, t, l), \rho(\varphi_2, w, t, l)\}$
- $\rho(\square_I \varphi, w, t, l) = \inf_{t' \in t+I} \{\rho(\varphi, w, t', l')\}$
- $\rho(\diamond_I \varphi, w, t, l) = \sup_{t' \in t+I} \{\rho(\varphi, w, t', l')\}$
- $\rho(\varphi_1 \mathcal{U}_I \varphi_2, w, t, l) = \sup_{t' \in t+I} \{\min\{\rho(\varphi_2, w, t', l'), \inf_{t'' \in [t, t']} \{\rho(\varphi_1, w, t'', l'')\}\}\}$

The intuitive meaning for  $\rho(\square_I \varphi, w, t, l)$  refers to that we achieve the infimum of  $\rho(\varphi, w, t', l'), \forall t' \in t + I$  over the trace. Similar to  $\rho(\square_I \varphi, w, t, l)$ ,  $\rho(\diamond_I \varphi, w, t, l)$  returns the supremum of  $\rho(\varphi, w, t', l'), \forall t' \in t + I$ .

Especially, if  $x_i \geq 0$ , the *satisfaction* signal will be

$$\forall t \in I, \chi(x_i \geq 0, w, t, l) := \begin{cases} \top & \text{if } x_i \geq 0 \\ \perp & \text{otherwise} \end{cases} \quad (2)$$

The connection is built between Boolean signals and quantitative signals by the way of predicate  $x_i \geq 0$  and obtain the satisfaction signal  $\chi(x_i \geq 0, w, t, l)$ . In the quantitative semantics, however, atomic predicates  $x_i \geq 0$  do not evaluate to  $\top$  or  $\perp$  but give a real value of the quantitative signals  $x_i$  by *satisfaction degree*  $\rho(\varphi, w, t, l)$  representing the distance to satisfaction or not.

#### IV. AN AXIOMATIZATION SYSTEM FOR STSL

**STSL** is a logical system. A proof in **STSL** is a sequence of finite formula:  $A_0, A_1, \dots, A_n$ , where each of them is an axiom,

or there exists  $j, k < i$ , such that  $A_i$  is the conclusion derived from  $A_j$  and  $A_k$  using *MP* inference rule. The last term  $A_n$  is a theorem in **STSL**, using the sign  $\vdash A_n$ , where  $n$  is the length of proof.

The notions of *deducibility* and *consistency* [4], [19] is fundamental to deduce the logic system **STSL**. A formula  $A$  is *deducible* from a set of formulas  $\Gamma$  in a system  $\mathcal{ST}$ , written  $\Gamma \Vdash_{\mathcal{ST}} A$ , if and only if  $\mathcal{ST}$  contains a theorem of the form  $(A_1 \wedge \dots \wedge A_n) \rightarrow A$ , where the conjuncts  $A_i (i = 1, \dots, n)$  of the antecedent are formulas in  $\Gamma$ . A set of formulas  $\Gamma$  is consistent in  $\mathcal{ST}$ , written  $Con_{\mathcal{ST}} \Gamma$ , just in case the formula  $\perp$  is not  $\mathcal{ST}$ -deducible from  $\Gamma$ .

**Definition IV.1** ( $\mathcal{ST}$ -MCS). *A set of formulas  $\Gamma$  is maximal  $\mathcal{ST}$ -consistent iff*

- (i)  $\Gamma$  is  $\mathcal{ST}$ -consistent, and
- (ii) for every formula  $A$ , if  $\Gamma \cup \{A\}$  is  $\mathcal{ST}$ -consistent, then  $A \in \Gamma$ .

If  $\Gamma$  is a maximal  $\mathcal{ST}$ -consistent set of formulas then we say it is an  $\mathcal{ST}$ -MCS. The (ii) condition refers to that any set of formulas properly containing  $\Gamma$  is  $\mathcal{ST}$ -inconsistent.

The canonical model defined by [19] to induce the soundness and completeness of modal logics. We extend the notation of *canonical model* to spatio-temporal systems for completeness of **STSL**.

**Definition IV.2** ( $\mathcal{ST}$ -canonical Model). *The  $\mathcal{ST}$ -canonical model  $\mathfrak{M}^\Gamma$  for a spatio-temporal logic is a triple  $(W^\Gamma, R^\Gamma, V^\Gamma)$  where:*

- (i)  $W^\Gamma$  is the set of all  $\Gamma$ -MCSs;
- (ii)  $R^\Gamma$  is a topological relation on topological space over a quasi-order on time. It is the canonical binary relation on  $W^\Gamma$  defined by  $sR_i^\Gamma s'$  over state  $s$  and  $s'$  if for all formulas  $\phi$ ,  $\phi \in s$  implies  $\phi \in s'$ .
- (iii)  $V^\Gamma$  is the valuation defined by  $V^\Gamma(p) = \{s \in W^\Gamma \mid p \in s\}$ .  $V^\Gamma$  is called the canonical valuation.

**Lemma IV.1** (Truth Lemma). *Let  $\mathcal{ST}$ -canonical model be a class of tt-model. For all  $\phi \in \mathcal{ST}$ -MCS,  $\mathcal{ST} \Vdash \phi$  iff  $\phi \in \mathcal{ST}$ -MCS.*

*Proof.* The proof is by induction on the structure of  $\phi$ .

*Base case:* Suppose  $\phi$  is a spatial formula  $\boxtimes \tau$  or an atomic predicate  $x_i \geq 0$ .

$$\begin{aligned}(\mathcal{ST}, s) \Vdash \boxtimes \tau &\Leftrightarrow V^\Gamma(\boxtimes \tau, s) = \top \Leftrightarrow \boxtimes \tau \in s \\ (\mathcal{ST}, s) \Vdash x_i \geq 0 &\Leftrightarrow V^\Gamma(x_i \geq 0, s) = x_i \Leftrightarrow x_i \geq 0 \in s\end{aligned}$$

*Inductive step:* Suppose  $\phi$  is an atomic predicate  $\neg \phi$ ,  $\phi_1 \wedge \phi_2$ ,  $\phi_1 \vee \phi_2$ ,  $\square \phi$ ,  $\diamond \phi$ ,  $\phi \mathcal{U} \varphi$ . We show the proof of the case  $\square \phi$ , and leave the others to reader. We have  $(\mathcal{ST}, s) \Vdash \square \phi \Leftrightarrow \square \phi \in s$  (assuming the inductive hypothesis).

$$\begin{aligned}(\mathcal{ST}, s) \Vdash \square \phi &\Leftrightarrow \forall s', sR^\Gamma s' \Rightarrow \mathcal{ST}, s' \Vdash \phi \\ &\Leftrightarrow \forall s', sR^\Gamma s' \Rightarrow \phi \in s'\end{aligned}$$

we need to show that  $\square \phi \in s \Leftrightarrow \forall s', sR^\Gamma s' \Rightarrow \phi \in s'$ .  
 $\Rightarrow$  follows immediately from the definition IV.2.

As for  $\Leftarrow$ : suppose  $\Box\phi \notin s$ . We need to show

$$\begin{aligned} & \exists s', sR^\Gamma s' \text{ and } \phi \notin s' \\ & \Leftrightarrow \exists s', sR^\Gamma s' \text{ and } \neg\phi \in s' \\ & \Leftrightarrow \exists s', \{\varphi \mid \Box\varphi \in s\} \subseteq s' \text{ and } \neg\phi \in s' \\ & \Leftrightarrow \exists s', \{\varphi \mid \Box\varphi \in s\} \cup \{\neg\phi\} \subseteq s' \end{aligned}$$

It is easy to show that  $\{\varphi \mid \Box\varphi \in s\} \cup \{\neg\phi\}$  is  $\mathcal{ST}$ -consistent. Suppose not, i.e.,  $\{\varphi \mid \Box\varphi \in s\} \cup \{\neg\phi\}$  is  $\mathcal{ST}$ -inconsistent. Then  $\vdash_{\mathcal{ST}} (\varphi_1 \wedge \dots \wedge \varphi_n) \rightarrow \phi$  for some  $\{\Box\varphi_1, \dots, \Box\varphi_n\} \subseteq s$ . But  $\mathcal{ST}$  is canonical and  $s$  is  $\mathcal{ST}$ -MCS, so  $s$  must contain  $(\Box\varphi_1 \wedge \dots \wedge \Box\varphi_n) \rightarrow \Box\phi$ . From  $\Box\varphi_i \in s$ , it follows  $\Box\phi \in s$ . This contradicts the hypothesis that  $\Box\phi \notin s$ .  $\square$

**Definition IV.3.** *The Hilbert-style proof system for the logic STSL has the following axiom schemes:*

- A0 All classical tautologies of first-order logic
- A1  $\neg \circ \phi \leftrightarrow \circ \neg \phi$
- A2  $\neg \Diamond \phi \leftrightarrow \Box \neg \phi$
- A3  $\circ(\phi \rightarrow \varphi) \rightarrow (\circ\phi \rightarrow \circ\varphi)$
- A4  $\Box(\phi \rightarrow \circ\varphi) \rightarrow (\phi \rightarrow \Box\varphi)$
- A5  $(\phi U \varphi) \leftrightarrow \varphi \vee \circ(\phi U \varphi)$
- A6  $(\phi U \varphi) \rightarrow \Diamond\varphi$
- $\Box K$   $\Box(\phi \rightarrow \varphi) \rightarrow (\Box\phi \rightarrow \Box\varphi)$
- $\Box T$   $\Box\phi \rightarrow \phi$
- $\Box 4$   $\Box\phi \rightarrow \Box\Box\phi$

And the inference rules:

$$\begin{aligned} \text{MP} & \frac{\phi \quad \phi \rightarrow \varphi}{\varphi} \\ N_\Box & \frac{\phi}{\vdash \Box\phi} \\ N_\Diamond & \frac{\phi}{\vdash \Diamond\phi} \end{aligned}$$

Soundness refers to that all the theorems in **STSL** are logically valid. Equivalently, A spatio-temporal logic is *sound* with respect to  $tt$ -model if for all the formula  $\phi$ ,  $\vdash_{\mathcal{ST}} \phi$  implies  $\models \phi$ . Let  $\mathcal{ST}$  be a class of  $tt$ -model, A spatio-temporal logic is *strongly complete* in  $\mathcal{ST}$  if for any set of formulas  $\Gamma \cup \{\phi\}$ , if  $\Gamma \models_{\mathcal{ST}} \phi$  then  $\Gamma \vdash_{\mathcal{ST}} \phi$ . If the semantics of  $\Gamma$  satisfies  $\phi$  on  $\mathcal{ST}$  then  $\phi$  is deducible from  $\Gamma$ .

**Theorem IV.2.** *The above axiomatization is sound for  $tt$ -model.*

*Proof.* This follows from the fact that all axioms are valid and all rules preserve validity.  $\square$

**Theorem IV.3.** *The system for STSL is weakly complete with respect to  $tt$ -model. i.e., for every STSL formula,  $\models_{\mathcal{ST}} \phi$  implies  $\vdash_{\Gamma} \phi$*

The full proof can be seen in

## V. CONCLUSION AND FUTURE WORK

In this paper, we build a spatio-temporal specification language, combining STL with spatial logic  $S4_u$ , specifically containing dense time and topological space. We provide the syntax and semantics of the spatio-temporal language,

and guarantee the seamless integration of spatial logic with temporal aspect from the perspective of the changes of purely spatial proposition  $STSL_{PC}$  and spatial objects  $STSL_{OC}$  over time. A Hilbert-style proof axiomatization system and the soundness and completeness result is present for the language.

The proposed **STSL** has a powerful expressiveness. It will be interesting to find a strongly complete fragment of **STSL**. For that, a more constrained axiomatization system need to be present, and more restricted inference and proof should be provided.

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