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From 2020/10/20 Notes

$$\frac{d}{dx} LG5(x) = \frac{r(K-L)\frac{1}{\theta}\exp(-r(x-d))}{\left(1+\frac{1}{\theta}\exp(-r(x-d))\right)^2} = \frac{r(K-L)}{\theta} \frac{f(x)}{g(x)}$$

Inflection point occurs @ x s.t. $\frac{d^2}{dx^2} LG5(x) = 0$

$$\frac{d^2}{dx^2} LG5(x) = \frac{r(K-L)}{\theta} \frac{d}{dx}\frac{f(x)}{g(x)} = \underbrace{\frac{r(K-L)}{\theta}\left(\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\right)}_{\text{quotient rule}}$$

thus

$\frac{d^2}{dx^2} LG5(x) = 0$ when $f'(x)g(x) = f(x)g'(x)$

① $$f'(x)g(x) = \frac{1}{\theta}\exp(-r(x-d))(-r)\left(1+\frac{1}{\theta}\exp(-r(x-d))\right)^2$$

$$= \cancel{-}\frac{\cancel{r}}{\cancel{\theta}}\cancel{\exp(-r(x-d))}\left(1+\frac{1}{\theta}\exp(-r(x-d))\right)^{\cancel{2}}$$

② $$f(x)g'(x) = \frac{1}{\theta}\exp(-r(x-d))\,2\left(1+\frac{1}{\theta}\exp(-r(x-d))\right)\frac{1}{\theta}\exp(-r(x-d))(-$$

$$= \cancel{-}\frac{2\cancel{r}}{\theta^{\cancel{2}}}\left(\exp(-r(x-d))\right)^{\cancel{2}}\cancel{\left(1+\frac{1}{\theta}\exp(-r(x-d))\right)}$$

thus

$f'(x)g(x) = f(x)g'(x)$

$1+\frac{1}{\theta}\exp(-r(x-d)) = \frac{2}{\theta}\exp(-r(x-d)) \Rightarrow$

$1 = \frac{1}{\theta}\exp(-r(x-d)) \Rightarrow \log(\theta) = -r(x-d) \Rightarrow$

$\boxed{-\frac{\log(\theta)}{r} + d = x}$ is true inflection point NOT $x = d$

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Max growth rate = $\frac{d}{dx}$ LG5(x) evaluated @

$x = -\frac{\log(\theta)}{r} + d$

$$\boxed{\text{Since } \exp(-r(x-d))\Big|_{x=-\frac{\log(\theta)}{r}+d} = \exp\left(-r\left(-\frac{\log(\theta)}{r}+d-d\right)\right) = \exp\left(-r\left(-\frac{\log(\theta)}{r}\right)\right) = \theta}$$

then

$$\frac{d}{dx}LG5(x)\Big|_{x=-\frac{\log(\theta)}{r}+d} = \frac{r(K-L)}{\cancel{\theta}}\frac{\cancel{\theta}}{(1+\cancel{\frac{\theta}{\theta}})^2} = \frac{r(K-L)}{4}$$

$$\boxed{\boxed{\frac{r(K-L)}{4} \text{ is the max growth rate}}}$$

Previously we stated max growth was

$\frac{K-L}{(1+\frac{1}{\theta})^2}\frac{r}{\theta}$, but this was tested empirically

for $\theta = 1 \Rightarrow \frac{K-L}{(1+\frac{1}{1})^2}\frac{r}{1} = \frac{r(K-L)}{4}$

which is consistent with result.