Temperature dependence

Equations from Smallwood CMEE MSc thesis:

[Eq. 11] $\frac{dM}{dT} = \frac{\partial M}{\partial b_{pk}}\frac{db_{pk}}{dT} + \frac{\partial M}{\partial v}\frac{dv}{dT} + \frac{\partial M}{\partial a}\frac{da}{dT} + \frac{\partial M}{\partial z}\frac{dz}{dT} + \frac{\partial M}{\partial \bar{z}}\frac{d\bar{z}}{dT} + \frac{\partial M}{\partial k}\frac{dk}{dT}$

[Eq. 8] $M_t = \frac{M_0 K e^{r_{max} t}}{K + M_0(e^{r_{max} t} - 1)}$

[Eq. 7] $r_{max} = \left(log\left(\frac{\Gamma(k) b_{pk} v}{(zv+k-1)^k}\right) - a\bar{z} + k - 1\right)\left(\frac{zv+k-1}{azv-a+k(v+a)}\right)$

This (Eq. 7) is an approximate derivation, per Smallwood, but treat it as though exact

Question to consider: What is the biological meaning of some of these expressions? For example, $-a\bar{z}$ is deducting the juvenile mortality during the whole juvenile development period (a is that period, $\bar{z}$ is the mortality rate per unit time).

Observe that $azv - a + k(v+a) = a(zv+k-1) + kv$. Would that be a neater representation of the denominator of the right hand term in r_{max} (Eq. 7)? Derivative results below use original form. Does an alternative representation help clarify biological meaning?

$$\left(\frac{zv+k-1}{azv-a+k(v+a)}\right) = \left(\frac{zv+k-1}{a(zv+k-1)+kv}\right) = \left(\frac{1}{a + \frac{kv}{(zv+k-1)}}\right)$$

State variables, parameters and symbols (see Smallwood thesis Table 2)

T	Temperature
M_t	Number of mosquitoes at time *t*
r_{max}	Population growth rate
M_0	Initial population size
K	Carrying capacity (adult mosquitoes)
k	A scaling parameter (not temperature dependent), fecundity schedule shape parameter, simulated value constant 2
a	Total juvenile development time
$\bar{z}$	Mean juvenile mortality rate across all juvenile stages
z	Adult mortality rate
b_{pk}	Peak fecundity
v	Time between maturation and peak reproduction
M_{pk}	(relative?) peak population density
T_{pk}	Temperature at which a trait is maximized

Approach

By the chain rule, for each parameter / trait x (i.e. in turn x represents b_{pk}, …, $\bar{z}$, … etc),

$$\frac{\partial M}{\partial x} = \frac{\partial M}{\partial r_{max}}\frac{\partial r_{max}}{\partial x}$$

so work in terms of r_{max}.

Results

For converting results in terms of r_{max} into results in terms of M,

$$\frac{\partial M}{\partial r_{max}} = \frac{M_0 K(K - M_0) t e^{r_{max} t}}{\left(K + M_0(e^{r_{max} t} - 1)\right)^2}$$

An alternative formulation (in terms of M)

$$\frac{\partial M}{\partial r_{max}} = tM\left(1 - \frac{M}{K}\right)$$

Individual partial derivative components of form $\frac{\partial r_{max}}{\partial x}$ are as follows.

$$\frac{\partial r_{max}}{\partial b_{pk}} = \left(\frac{zv + k - 1}{azv - a + k(v + a)}\right)\frac{1}{b_{pk}}$$

$$\frac{\partial r_{max}}{\partial v} = \frac{1}{azv - a + k(v + a)}\left[\frac{zv + k - 1 + kzv}{v} - \frac{k(k-1)}{\left(azv - a + k(v + a)\right)}\left(\log\left(\frac{\Gamma(k) b_{pk} v}{(zv + k - 1)^k}\right) - a\bar{z} + k - 1\right)\right]$$

$$= \frac{1}{azv - a + k(v + a)}\left[\frac{zv + k - 1 + kzv}{v} - \frac{k(k-1)}{zv + k - 1} r_{max}\right]$$

alternative formulation (substituting in for r_{max})

$$\frac{\partial r_{max}}{\partial a} = -\left(\frac{zv + k - 1}{azv - a + k(v + a)}\right)\left[\left(\log\left(\frac{\Gamma(k) b_{pk} v}{(zv + k - 1)^k}\right) - a\bar{z} + k - 1\right)\left(\frac{zv + k - 1}{azv - a + k(v + a)}\right) - \bar{z}\right]$$

$$= \left(\frac{zv + k - 1}{azv - a + k(v + a)}\right)(\bar{z} - r_{max})$$

alternative formulation (substituting in for r_{max})

$$\frac{\partial r_{max}}{\partial z} = \frac{kv}{azv - a + k(v + a)}\left[\left(\frac{v}{azv - a + k(v + a)}\right)\left(\log\left(\frac{\Gamma(k) b_{pk} v}{(zv + k - 1)^k}\right) - a\bar{z} + k - 1\right) - 1\right]$$

$$= \left(\frac{kv}{azv - a + k(v + a)}\right)\left(\frac{v}{zv + k - 1} r_{max} - 1\right)$$

alternative formulation (substituting in for r_{max})

$$\frac{\partial r_{max}}{\partial \bar{z}} = -a\left(\frac{zv + k - 1}{azv - a + k(v + a)}\right)$$

$$\frac{\partial r_{max}}{\partial k} = \frac{1}{\big(azv - a + k(v + a)\big)}\Bigg[\left(\frac{v(1 - zv)}{azv - a + k(v + a)}\right)\left(\log\left(\frac{\Gamma(k) b_{pk} v}{(zv + k - 1)^k}\right) - a\bar{z} + k - 1\right) + (v + k - 1)\left(\Psi(k) - \frac{k}{(zv + k - 1)} + \log(zv + k - 1) + 1\right)\Bigg]$$

where $\Psi(k)$ is the digamma function (logarithmic derivative of the gamma function)

$$= \frac{1}{\big(azv - a + k(v + a)\big)}\Bigg[\left(\frac{v(1 - zv)}{(zv + k - 1)}\right) r_{max} + (v + k - 1)\left(\Psi(k) - \frac{k}{(zv + k - 1)} + \log(zv + k - 1) + 1\right)\Bigg]$$

alternative formulation (substituting in for r_{max})