```
(*To do:
  (1) Find approximation using Gompertz function for mortality rate part of l_x,
that is, mortality rate = z exp(b t) instead of just z
   (2) Find approximation using Van's approach, compare with my apporximation
  (3) Substitute Briere type functions and generate predictions for density M
  (4) Compare with Parham's model
*)
```

(*For b_x modeled using Gamma distribution*)

```
(*f = E^((-(Log[x-a]-k)^2)/s)(*Log-normal distribution*)*)
(*Plot[f/.{k→1,s→2,a→0},{x,0,50}(*check  function*)*)
(*f = (x-a)^(k-1) E^(-(x-a)/(k+1))/(Gamma[k] (k+1)^k )
(*Simplified gamma distribution with θ = k+1*)*)
(*f = (x-a)^(k-1) E^(-(x-a)/θ)(*The reduced gamma distribution*)*)
```

f = FullSimplify$\left[e^{-1+k+\frac{a-x}{\theta}}\,(-a+x)^{-1+k}\,\left((-1+k)\,\theta\right)^{1-k}\right]$

(*/.θ→(x_{pk}-a)/(k-1):The reduced gamma distribution expressed in terms of x_{pk}*)

$$e^{-1+k+\frac{a-x}{\theta}}\,(-a+x)^{-1+k}\,\left((-1+k)\,\theta\right)^{1-k}$$

D[f, x]

$$e^{-1+k+\frac{a-x}{\theta}}\,(-1+k)\,(-a+x)^{-2+k}\,\left((-1+k)\,\theta\right)^{1-k}-\frac{e^{-1+k+\frac{a-x}{\theta}}\,(-a+x)^{-1+k}\,\left((-1+k)\,\theta\right)^{1-k}}{\theta}$$

Simplify[Solve[D[f, x] == 0, θ]](*To find maximum (mode) for normalization*)

$$\left\{\left\{\theta\to\frac{-a+x}{-1+k}\right\}\right\}$$

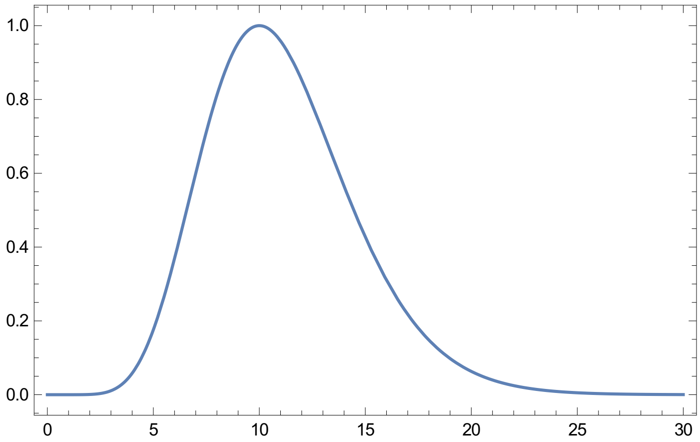
b_x = b_{pk} Simplify [f / (f /. x → a + (-1 + k) θ)](*To normalize it by mode*)

$$e^{-1+k+\frac{a-x}{\theta}}\,(-a+x)^{-1+k}\,\left((-1+k)\,\theta\right)^{1-k}\,b_{pk}$$

(*b_x = Simplify$\left[e^{-1+k+\frac{a-x}{((x_{pk}-a)/(k-1))}}\,(-a+x)^{-1+k}\,\left((-1+k)\,((x_{pk}-a)/(k-1))\right)^{1-k}\right]$
(*express it in terms of x_{pk}*)*)

$$\texttt{Plot}\left[e^{k-1-\frac{x-a}{\left((x_{pk}-a)/(k-1)\right)}}\left((x-a)\Big/\left(((x_{pk}-a)/(k-1))\ (k-1)\right)\right)^{-1+k}\ \texttt{/.}\ \{a\to 0,\ k\to 10,\ x_{pk}\to 10\},\ \{x,\ 0,\ 30\},\ \texttt{Frame}\to\texttt{True},\ \texttt{PlotRange}\to\texttt{All}\right]$$

(*check normalized function*)



(*with exponential survivorship l_x *)

$$\texttt{FullSimplify}\Big[\texttt{Integrate}\left[E\,\hat{}\,(-r_{max}\ x)\ E\,\hat{}\left(-\left(zbar\ a+z\ (x-a)\right)\right)\ b_{pk}\ e^{k-1-\frac{x-a}{(\theta)}}\left((x-a)\Big/\left((\theta)\ (k-1)\right)\right)^{-1+k},\ \{x,\ a,\ \infty\}\right]\ \texttt{/.}\ \theta\to(x_{pk}-a)\Big/(k-1)\Big]$$

$$\texttt{ConditionalExpression}\left[e^{-1+k-a\,zbar-a\,r_{max}}\ \texttt{Gamma}[k]\ b_{pk}\left(\frac{1}{-a+x_{pk}}\right)^{-1+k}\left(z+r_{max}+\frac{-1+k}{-a+x_{pk}}\right)^{-k},\ \texttt{Re}[k]>0\ \&\&\ \texttt{Re}\left[z+r_{max}+\frac{-1+k}{-a+x_{pk}}\right]>0\ \&\&\ a>0\right]$$

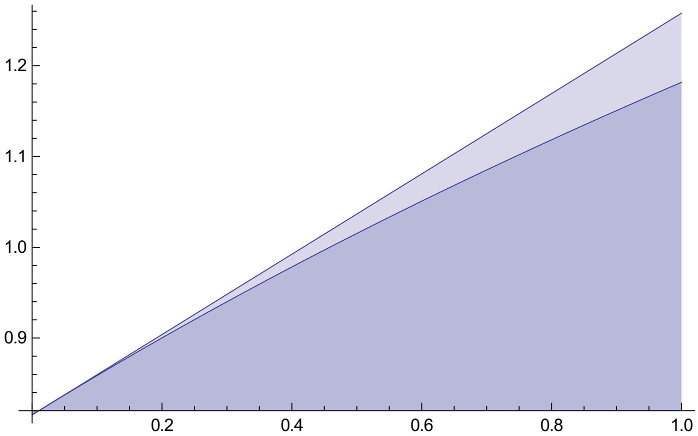
$$\texttt{Log}\left[e^{-1+k-a\,zbar-a\,r_{max}}\ \texttt{Gamma}[k]\ b_{pk}\left(\frac{1}{-a+x_{pk}}\right)^{-1+k}\left(z+r_{max}+\frac{-1+k}{-a+x_{pk}}\right)^{-k}\right]$$

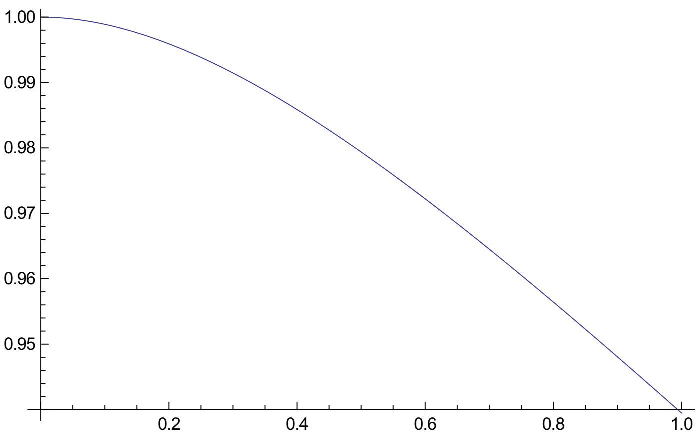
$$\texttt{Log}\left[e^{-1+k-a\,zbar-a\,r_{max}}\ \texttt{Gamma}[k]\ b_{pk}\left(\frac{1}{-a+x_{pk}}\right)^{-1+k}\left(z+r_{max}+\frac{-1+k}{-a+x_{pk}}\right)^{-k}\right]$$

$$\texttt{Simplify}\left[\texttt{Series}\left[\texttt{Log}\left[z+\left((k-1)\Big/(x_{pk}-a)\right)+r_{max}\right],\ \{r_{max},\ 0,\ 2\}\right]\right]\ \texttt{(*approximation*)}$$

$$\texttt{Log}\left[z+\frac{-1+k}{-a+x_{pk}}\right]+\frac{r_{max}}{z+\frac{-1+k}{-a+x_{pk}}}-\frac{r_{max}^2}{2\left(z+\frac{-1+k}{-a+x_{pk}}\right)^2}+O[r_{max}]^3$$

```
Plot[{Log[z + ((k - 1) / (xpk - a)) + rmax], Log[z + (-1 + k)/(-a + xpk)] + rmax/(z + (-1+k)/(-a+xpk))} /.
   {z → .01, a → 1, k → 10, xpk → 5},
  {rmax, 0, 1}, Filling → Axis] (*check aproximation*)
Plot[{Log[z + ((k - 1) / (xpk - a)) + rmax] / (Log[z + (-1 + k)/(-a + xpk)] + rmax/(z + (-1+k)/(-a+xpk)))} /.
   {z → .01, a → 1, k → 10, xpk → 5}, {rmax, 0, 1}]
(*Approximation as percent of true value*)
```





```
FullSimplify[
  (Log[Gamma[k] bpk] - 1 + k - a zbar + (k - 1) Log[ (1/((-1 + k) θ))] - k Log[z + 1/θ]) /
   (a + k/(z + 1/θ)) /. {k → k, θ → θ}]
```

$$\frac{-1+k-a\,\mathrm{zbar}-k\,\mathrm{Log}\left[z+\frac{1}{\theta}\right]+(-1+k)\,\mathrm{Log}\left[\frac{1}{(-1+k)\,\theta}\right]+\mathrm{Log}[\mathrm{Gamma}[k]\,b_{pk}]}{a+\frac{k\,\theta}{1+z\,\theta}}$$

(*with gompertz survivorship and b_x defined according to Amarasekare and Savage*)

$$\text{FullSimplify}\left[\text{Integrate}\left[E \wedge (-r_{max}\, x)\; E \wedge \left(-\left(\text{zbar}\; a + (z/\gamma)\; E \wedge (\gamma\,(x-a)) - 1\right)\right)\; b_{pk}\; e^{k-1-\frac{x-a}{\theta}} \left((x-a) \big/ \left(\theta\, (k-1)\right)\right)^{-1+k},\; \{x, a, \infty\}\right] \;/.\; \left\{\theta \to (x_{pk}-a) \big/ (k-1),\; \gamma \to 1\right\}\right]$$

$$\int_a^\infty e^{k-e^{-a+x}\, z - a\, \text{zbar} - x\, r_{max} + \frac{(-1+k)\,(a-x)}{-a+x_{pk}}}\, b_{pk} \left(\frac{a-x}{a-x_{pk}}\right)^{-1+k} dx$$