

# Representing operational semantics with enriched Lawvere theories

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## Abstract

Many term calculi, like  $\lambda$ -calculus or  $\pi$ -calculus, involve binders for names, and the mathematics of bound variable names is subtle. Schönfinkel introduced the SKI combinator calculus in 1924 to clarify the role of quantified variables in intuitionistic logic by eliminating them. Yoshida demonstrated how to eliminate the bound names coming from the input prefix in the asynchronous  $\pi$ -calculus, but her combinators still depend on the `new` operator to bind names. Recently, Meredith and Stay showed how to modify Yoshida's combinators by replacing `new` and replication with reflective operators to provide the first combinator calculus with no bound names into which the asynchronous  $\pi$ -calculus has a faithful embedding. Here we provide an alternative set of combinators built from SKI plus reflection that also eliminates all nominal phenomena, yet provides a faithful embedding of a reflective higher-order pi calculus. We show that with the nominal features effectively eliminated as syntactic sugar, multisorted Lawvere theories enriched over graphs suffice to capture the operational semantics of the calculus.

**1998 ACM Subject Classification** F.1.2 Modes of computation, F.3.2 Semantics of Programming Languages, F.4 Mathematical Logic and Formal Languages, D.1.3 Concurrent Programming, D.3.1 Formal Definitions and Theory, D.3.3 Language Constructs and Features

**Keywords and phrases** Concurrent combinator, nominal, pi calculus

**Digital Object Identifier** 10.4230/LIPIcs.CVIT.2016.23

## 1 Introduction

Many term calculi, like  $\lambda$ -calculus or  $\pi$ -calculus, involve binders for names, and the mathematics of bound variable names is subtle. Schönfinkel introduced the SKI combinator calculus in 1924 to clarify the role of quantified variables in intuitionistic logic by eliminating them [14]. Yoshida demonstrated how to eliminate the bound names coming from the input prefix in the asynchronous  $\pi$ -calculus, but her combinators still depend on the `new` operator to bind names. Curry developed Schönfinkel's ideas much further. Recently, Meredith and Stay [11] showed how to modify Yoshida's combinators by replacing `new` and replication with reflective operators to provide the first combinator calculus with no bound names into which the asynchronous  $\pi$ -calculus has a faithful embedding of a reflective higher-order pi calculus. Here we provide an alternative set of combinators built from SKI plus reflection that also eliminates all nominal phenomena, yet provides a faithful embedding.

The recent work by Jamie Gabbay and Andrew Pitts [2] and others [1] on nominal set theory has put the study of bound names and substitution on a much nicer foundation, at the cost of an increase in the complexity of the semantic framework. It interprets nominal phenomena in terms of atoms in Fraenkl-Mostowski set theory. Clouston's work in particular



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42nd Conference on Very Important Topics (CVIT 2016).

Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:1–23:11

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

makes evident the additional machinery needed to interpret nominal phenomena as Lawvere theories. On the other hand, with the nominal features effectively eliminated as syntactic sugar, we show that multisorted Lawvere theories enriched over graphs suffice to capture the operational semantics of the calculus.

## 2 Previous work

There is a long history and an enormous body of work on modeling term rewriting and operational semantics with various notions of category enriched over category-like structures; we only have room here for a sampling. L  th and Ghani [8] use poset-enriched categories to study the modularity of strong normalization. One approach to nominality is the one we mentioned in the introduction. A different approach deals with nominal issues by allowing “functor types” in the signature: Seely [15] suggested using 2-categories for modeling the denotational semantics of lambda calculus in Scott domains to capture the adjunction between  $\beta$  reduction and  $\eta$  conversion; Hilken [4] expands Seely’s work by exploring the proof theory using categorical logic; and Hirschowitz [5] generalizes algebraic signatures to cartesian closed 2-signatures. A third approach is to model substitution explicitly: Stell [17] considered sesquicategories for term rewriting; in his system, objects are finite sets of variables, morphisms are substitutions, and 2-morphisms are roughly rewrite rules.

## 3 Gph-enriched categories

Here we review some standard definitions and results in enriched category theory; see [3], [13], [7], and [18] for more details.

A **directed multigraph with self loops**, hereafter **graph**, consists of a set  $E$  of edges, a set  $V$  of vertices, two functions  $s, t: E \rightarrow V$  picking out the source and target of each edge, and a function  $a: V \rightarrow E$  such that  $s \circ a$  and  $t \circ a$  are both the identity on  $V$ —that is,  $a$  equips each vertex in  $V$  with a chosen self loop. There are no constraints on  $E, V, s$ , or  $t$ , so a graph may have infinitely many vertices and infinitely many edges between any pair of vertices. A **graph homomorphism** from  $(E, V, s, t, a)$  to  $(E', V', s', t', a')$  is a pair of functions  $(\epsilon: E \rightarrow E', v: V \rightarrow V')$  such that  $v \circ s = s' \circ \epsilon$  and  $v \circ t = t' \circ \epsilon$ . **Gph** is the category of graphs and graph homomorphisms. Gph has finite products: the terminal graph is the graph with one vertex and one loop, while the product of two graphs  $(E, V, s, t, a) \times (E', V', s', t', a')$  is  $(E \times E', V \times V', s \times s', t \times t', a \times a')$ .

A **Gph-enriched category** consists of

- a set of objects;
- for each pair of objects  $x, y$ , a graph  $\text{hom}(x, y)$ ;
- for each triple of objects  $x, y, z$ , a composition graph homomorphism  $\circ: \text{hom}(y, z) \times \text{hom}(x, y) \rightarrow \text{hom}(x, z)$ ; and
- for each object  $x$ , a vertex of  $\text{hom}(x, x)$ , the identity on  $x$ .

such that composition is associative, and composition and the identity obey the unit laws. A Gph-enriched category has finite products if the underlying category does.

Any category is trivially Gph-enriched by taking the all the sets of edges to be empty. The category Gph is nontrivially Gph-enriched: Gph is a topos, and therefore cartesian closed, and therefore enriched over itself. Given two graph homomorphisms  $F, F': (E, V, s, t, a) \rightarrow (E', V', s', t', a')$ , a **graph transformation** assigns to each vertex  $v$  in  $V$  an edge  $e'$  in  $E'$  such that  $s'(e') = F(v)$  and  $t'(e') = F'(v)$ . Given any two graphs  $G$  and  $G'$ , there is an

exponential graph  $G'^G$  whose vertices are graph homomorphisms between them and whose edges are graph transformations.

A **Gph-enriched functor** between two Gph-enriched categories  $C, D$  is a functor between the underlying categories such that the graph structure on each hom set is preserved, *i.e.* the functions between hom sets are graph homomorphisms between the hom graphs.

Let  $S$  be a finite set,  $\text{FinSet}$  be a skeleton of the category of finite sets and functions between them, and  $\text{FinSet}/S$  be the category of functions into  $S$  and commuting triangles. A **multisorted Gph-enriched Lawvere theory**, hereafter **Gph-theory** is a Gph-enriched category with finite products  $\text{Th}$  equipped with a finite set  $S$  of **sorts** and a Gph-enriched functor  $\theta: \text{FinSet}^{\text{op}}/S \rightarrow \text{Th}$  that preserves products strictly. Any Gph-theory has an underlying multisorted Lawvere theory given by forgetting the edges of each hom graph.

A **model** of a Gph-theory  $\text{Th}$  is a Gph-enriched functor from  $\text{Th}$  to  $\text{Gph}$  that preserves products up to natural isomorphism. A **homomorphism of models** is a braided Gph-enriched natural transformation between the functors. Let  $\text{FPGphCat}$  be the 2-category of small Gph-enriched categories with finite products, product-preserving Gph-functors, and braided Gph-natural transformations. The forgetful functor  $U: \text{FPGphCat}[\text{Th}, \text{Gph}] \rightarrow \text{Gph}$  that picks out the underlying graph of a model has a left adjoint that picks out the free model on a graph.

«««< Updated upstream Gph-enriched categories are part of a spectrum of 2-category-like structures. A strict 2-category is a category enriched over  $\text{Cat}$  with its usual product. Sesquicategories are categories enriched over  $\text{Cat}$  with the “funny” tensor product [6]; a sesquicategory can be thought of as a 2-category where the interchange law doesn’t hold. A Gph-enriched category can be thought of as a sesquicategory where 2-morphisms (now edges) can’t be composed. Any strict 2-category has an underlying sesquicategory, and any sesquicategory has an underlying Gph-enriched category. These forgetful functors have left adjoints. ===== Gph-enriched categories are part of a spectrum of 2-category-like structures. A strict 2-category is a category enriched over  $\text{Cat}$  with its usual product. Sesquicategories are categories enriched over  $\text{Cat}$  with the “funny” tensor product [6]; a sesquicategory can be thought of as a 2-category where the interchange law does not hold. A Gph-enriched category can be thought of as a sesquicategory where 2-morphisms (now edges) cannot be composed. Any strict 2-category has an underlying sesquicategory, and any sesquicategory has an underlying Gph-enriched category, and these forgetful functors have left adjoints. »»»> Stashed changes

## 4 Gph-theories as models of computation

Lawvere theories and their generalizations are categories with infinitely many objects and morphisms, but most theories of interest are finitely generated. A presentation of the underlying multisorted Lawvere theory of a finitely-generated Gph-theory is a signature for a term calculus, consisting of a set of sorts, a set of term constructors, and a set of equations, while the edges in the hom graphs of the theory encode the reduction relation.

Here is a presentation of the SKI combinator calculus as a Gph-theory:

- one sort  $T$ , for terms
- term constructors

$$\begin{aligned} S &: 1 \rightarrow T \\ K &: 1 \rightarrow T \\ I &: 1 \rightarrow T \\ (-\ -) &: T^2 \rightarrow T \end{aligned}$$

- structural congruence (no equations)
- rewrites

$$\begin{aligned}\sigma & : (((S\ x)\ y)\ z) \Rightarrow ((x\ z)\ (y\ z)) \\ \kappa & : ((K\ y)\ z) \Rightarrow y \\ \iota & : (I\ z) \Rightarrow z\end{aligned}$$

where in the rewrites we have used expressions like  $((K\ y)\ z)$  as shorthand for

$$T \times T \xrightarrow{\text{left}^{-1}} 1 \times T \times T \xrightarrow{K \times T \times T} T \times T \times T \xrightarrow{(-) \times T} T \times T \xrightarrow{(-)} T.$$

A model  $M$  of this Gph-theory in Gph picks out a graph  $M(T)$  of terms and rewrites. It picks out three special vertices  $S, K$ , and  $I$  of  $M(T)$ ; it equips  $M(T)$  with a graph homomorphism from  $M(T)^2$  to  $M(T)$  that says for every pair of vertices  $(u, v)$ , there is a vertex  $(u\ v)$ , and similarly for edges; and it equips  $M(T)$  with graph transformations asserting the existence of an edge out of a reducible expression to the term it reduces to.

That this Gph-theory captures the operational semantics of the SKI calculus is almost definitional: there is an edge between distinct vertices in the free model on the empty graph if and only if the source vertex is reducible to the target vertex in a single step.

It is straightforward to verify that Gph-theories suffice to capture the operational semantics of any calculus where every context is a reduction context. This restriction on reduction contexts is a consequence of the fact that models map term constructors to graph homomorphisms: given a model  $M$ , a graph homomorphism  $F: M(T) \rightarrow M(T)$ , and an edge  $e: t_1 \rightarrow t_2$ , there is necessarily an edge  $F(e): F(t_1) \rightarrow F(t_2)$ .

## 5 Gph-theory for SKI with the weak head normal form evaluation strategy

In modern programming languages, many contexts are *not* reduction contexts. In Haskell, for instance, there are no reductions under a lambda abstraction: even if  $t_1$  reduces to  $t_2$  as a program, the term  $\lambda x \rightarrow t_1$  does not reduce to  $\lambda x \rightarrow t_2$ .

Gph-theories can still capture the operational semantics of calculi with restrictions on reduction contexts by introducing term constructors that explicitly mark the reduction contexts. For example, suppose that we want an evaluation strategy for the SKI calculus that only reduces the leftmost combinator when it has been applied to sufficiently many arguments, *i.e.* we want the *weak head normal form*; we can accomplish this by introducing a term constructor  $R: T \rightarrow T$  that explicitly marks the reduction contexts. We then add a structural congruence rule for propagating the context and modify the existing reduction rules to apply only to marked contexts.

- one sort  $T$ , for terms
- term constructors

$$\begin{aligned}S & : 1 \rightarrow T \\ K & : 1 \rightarrow T \\ I & : 1 \rightarrow T \\ (-) & : T^2 \rightarrow T \\ R & : T \rightarrow T\end{aligned}$$

- structural congruence

$$R(x\ y) = (R\ x\ y)$$

■ rewrites

$$\begin{aligned}\sigma & : (((RS\ x)\ y)\ z) \Rightarrow ((Rx\ z)\ (y\ z)) \\ \kappa & : ((RK\ y)\ z) \Rightarrow Ry \\ \iota & : (RI\ z) \Rightarrow Rz\end{aligned}$$

► **Theorem 1.** *Let  $t$  be a term in which  $R$  does not appear. Then  $Rt$  reduces to  $Rt'$ , where  $t'$  is the weak head normal form of  $t$ .*

**Proof.** If we form the term  $Rt$  where  $t$  contains no uses of  $R$ , no reductions will ever take place in the right-hand argument of an application: the structural congruence and rewrite rules enforce that the  $R$  context can only move to the left term in an application, never the right. The result follows by induction on the number of steps to reach  $t'$ . ◀

## 6 Explicit reduction contexts as gas

The Ethereum [19] and RChain [10] projects are building virtual machines on the blockchain. Both use the concept of a linear resource called “gas” (as in gasoline) that is consumed as the virtual machine executes. Gph-theories can capture the operational semantics of a calculus where reduction contexts are consumable, and thus play a role similar to that of gas [16].

■ one sort  $T$ , for terms

■ term constructors

$$\begin{aligned}S & : 1 \rightarrow T \\ K & : 1 \rightarrow T \\ I & : 1 \rightarrow T \\ (-\ -) & : T^2 \rightarrow T \\ R & : T \rightarrow T\end{aligned}$$

■ structural congruence

$$R(x\ y) = (Rx\ y)$$

■ rewrites

$$\begin{aligned}\sigma & : (((RS\ x)\ y)\ z) \Rightarrow ((x\ z)\ (y\ z)) \\ \kappa & : ((RK\ y)\ z) \Rightarrow y \\ \iota & : (RI\ z) \Rightarrow z\end{aligned}$$

► **Theorem 2.** *Let  $t$  be a term in which  $R$  does not appear; let  $t'$  be the weak head normal form of  $t$ ; let  $m$  be the number of steps by which  $Rt$  reduces to  $Rt'$  in the calculus of section 5; and let  $n \geq m$ . Then in this calculus,  $R^n t$  reduces to  $R^{n-m} t'$  in  $m$  steps.*

**Proof.** As before, if we form the term  $Rt$  where  $t$  contains no uses of  $R$ , no reductions will ever take place in the right-hand argument of an application. Each application of the reduction rules reduces the number of  $R$ s by one, and structural equivalence preserves the number of  $R$ s. The result follows by induction on the number of steps to reach  $t'$ . ◀

## 7 Gph-theory for a pi calculus variant

Gph-theories can capture the operational semantics of concurrent calculi as well as serial calculi like SKI above.

Meredith and Radestock [9] describe a reflective higher-order variant of pi calculus we call the RHO calculus. Rather than the usual replication and **new** operators, they have quoting and unquoting operators. Quoting turns a process into a name and unquoting does the opposite; freshness of names is obtained using a type discipline. They prove that there is a faithful embedding of the monadic asynchronous pi calculus into the RHO calculus.

### 7.1 The RHO calculus

#### 7.1.1 Syntax

$P, Q$	$::= 0$	the stopped process
	$\text{for}(y \leftarrow x)P$	input guarded process
	$x!P$	output process
	$P \mid Q$	parallel composition
	$*x$	deference
$x, y$	$::= \&P$	quotation

Note that in the original rho-calculus papers the notation was somewhat different. The quotation and dereference constructions were originally written,  $\ulcorner P \urcorner$  and  $\lrcorner x \lrcorner$ , respectively. Here we have adopted a more programmer friendly style employing the  $\&$  and  $*$  of the C programming language for reference (quotation) and dereference, respectively. Input guards which were written with a whimper  $?$  in more traditional process calculi style are now written in for-comprehension style as adopted in languages like Scala; e.g.  $x?(y)P$  is written here  $\text{for}(y \leftarrow x)P$ .

#### 7.1.2 Free and bound names

$$\begin{array}{ll}
 FN(0) &= \emptyset \\
 FN(\text{for}(y \leftarrow x)P) &= \{x\} \cup (FN(P) \setminus \{y\}) \\
 FN(x!P) &= \{x\} \cup FN(P)
 \end{array}
 \qquad
 \begin{array}{ll}
 FN(P \mid Q) &= FN(P) \cup FN(Q) \\
 FN(*x) &= \{x\}
 \end{array}$$

#### 7.1.3 Structural congruence

Structural (process) congruence is the smallest congruence  $\equiv$  containing  $\alpha$ -equivalence and making  $(\mid, 0)$  into a commutative monoid.

#### 7.1.4 Name equivalence

Name equivalence is the smallest equivalence relation  $\equiv_N$  on names such that

$$\frac{}{\& * x \equiv_N x} \text{ and } \frac{P \equiv Q}{\&P \equiv_N \&Q} .$$

### 7.1.5 Substitution

Syntactic substitution:

$$\begin{aligned}
 (0)\{\&Q/\&P\} &= 0 \\
 (\text{for}(y \leftarrow x)R)\{\&Q/\&P\} &= \text{for}(z \leftarrow (x\{\&Q/\&P\}))(R\{z/y\}\{\&Q/\&P\}) \\
 (x!R)\{\&Q/\&P\} &= (x\{\&Q/\&P\})!(R\{\&Q/\&P\}) \\
 (R|S)\{\&Q/\&P\} &= (R\{\&Q/\&P\}) | (S\{\&Q/\&P\}) \\
 (*x)\{\&Q/\&P\} &= \begin{cases} *\&Q & \text{when } x \equiv_N \&Q \\ *x & \text{otherwise,} \end{cases}
 \end{aligned}$$

where

$$x\{\&Q/\&P\} = \begin{cases} \&Q & \text{if } x \equiv_N \&P \\ x & \text{otherwise} \end{cases}$$

and, in the rule for input,  $z$  is chosen to be distinct from  $\&P, \&Q$ , the free names in  $Q$ , and all the names in  $R$ .

Semantic substitution, for use in  $\alpha$ -equivalence:

$$(*x)\{\&Q/\&P\} = \begin{cases} Q & \text{when } x \equiv_N \&Q \\ *x & \text{otherwise} \end{cases}$$

### 7.1.6 Reduction rules

We use  $\rightarrow$  to denote single-step reduction.

$$\begin{array}{c}
 \frac{x_0 \equiv_N x_1}{\text{for}(y \leftarrow x_1)P \mid x_0!Q \rightarrow P\{\&Q/y\}} \\
 \frac{P \rightarrow P'}{P \mid Q \rightarrow P' \mid Q} \\
 \frac{P \equiv P' \quad P' \rightarrow Q' \quad Q' \equiv Q}{P \rightarrow Q}
 \end{array}$$

## 7.2 RHO combinators

We can define an embedding  $\llbracket - \rrbracket$  of closed RHO calculus terms into a set of RHO combinators. We follow Milner [12] in thinking of an input-prefixed process  $\text{for}(x \leftarrow y)P$  as consisting of two parts: the first names the channel  $y$  on which the process is listening, while the second describes the continuation  $\lambda x.P$  in terms of an abstracted name. The right hand side of the communication rule, in effect, applies the continuation to the name to be substituted. Since the only bound names in the RHO calculus come from input prefixing, we can completely eliminate bound names by using abstraction elimination on the continuation. Like the weak head normal form SKI calculus above, this combinator calculus uses a linear resource  $C$  to reify reduction contexts.

A Gph-theory for the operational semantics of these combinators has:

- one sort  $T$ , for terms
- term constructors

$$\begin{array}{ll}
 C & : 1 \rightarrow T \\
 0 & : 1 \rightarrow T \\
 | & : 1 \rightarrow T \\
 \text{for} & : 1 \rightarrow T \\
 ! & : 1 \rightarrow T \\
 \& & : 1 \rightarrow T
 \end{array}
 \qquad
 \begin{array}{ll}
 * & : 1 \rightarrow T \\
 S & : 1 \rightarrow T \\
 K & : 1 \rightarrow T \\
 I & : 1 \rightarrow T \\
 () & : T \times T \rightarrow T
 \end{array}$$

■ structural congruence rules

$$\begin{aligned}
((| 0) P) &= P && \text{unit law} \\
((| ((| P) Q)) R) &= ((| P) ((| Q) R)) && \text{associativity} \\
((| P) Q) &= ((| Q) P) && \text{commutativity}
\end{aligned}$$

■ reduction rules

$$\begin{aligned}
\sigma: (((S P) Q) R) &\Rightarrow ((P R) (Q R)) && \text{action of } S \\
\kappa: ((K P) Q) &\Rightarrow P && \text{action of } K \\
\iota: (I P) &\Rightarrow P && \text{action of } I \\
\xi: ((| C) ((| ((\text{for } (\& P)) Q)) ((\text{! } (\& P)) R))) &\Rightarrow ((| C) (Q (\& R))) && \text{communication} \\
\epsilon: ((| C) (* (\& P))) &\Rightarrow ((| C) P) && \text{evaluation}
\end{aligned}$$

### 7.3 Embeddings

We define an interpretation function  $\llbracket - \rrbracket$  from RHO calculus terms into RHO combinators by

$$\begin{aligned}
\llbracket 0 \rrbracket &= 0 \\
\llbracket \text{for}(x \leftarrow \&P)Q \rrbracket &= ((\text{for } (\& \llbracket P \rrbracket)) \llbracket Q \rrbracket_x) \\
\llbracket \&P!Q \rrbracket &= ((\text{! } (\& \llbracket P \rrbracket)) \llbracket Q \rrbracket) \\
\llbracket P|Q \rrbracket &= ((| \llbracket P \rrbracket) \llbracket Q \rrbracket) \\
\llbracket * \&P \rrbracket &= (* (\& \llbracket P \rrbracket))
\end{aligned}$$

where  $\llbracket - \rrbracket_x$  eliminates the free name  $x$  :

$$\begin{aligned}
\llbracket P \rrbracket_x &= (K \llbracket P \rrbracket) \text{ where } x \text{ is not free in } P \\
\llbracket \text{for}(y \leftarrow \&P)Q \rrbracket_x &= ((S ((S (K \text{for})) ((S (K \&)) \llbracket P \rrbracket_x))) \llbracket Q \rrbracket_{y,x}) \\
\llbracket \&P!Q \rrbracket_x &= ((S ((S (K \text{!})) ((S (K \&)) \llbracket P \rrbracket_x))) \llbracket Q \rrbracket_x) \\
\llbracket P|Q \rrbracket_x &= ((S ((S (K |)) \llbracket P \rrbracket_x)) \llbracket Q \rrbracket_x) \\
\llbracket * \&P \rrbracket &= \begin{cases} ((S (K *)) I) & \text{when } \&P \equiv_N x \\ ((S (K *)) ((S (K \&)) \llbracket P \rrbracket_x)) & \text{otherwise.} \end{cases}
\end{aligned}$$

Consider the following sorting on RHO combinators:

$$\begin{array}{ll}
C : W & \& : W \Rightarrow N \\
0 : W & * : N \Rightarrow W \\
| : W \Rightarrow W \Rightarrow W & S : \forall X, Y, Z. (Z \Rightarrow Y \Rightarrow X) \Rightarrow (Z \Rightarrow Y) \Rightarrow Z \Rightarrow X \\
\text{for} : N \Rightarrow (N \Rightarrow W) \Rightarrow W & K : \forall X, Y. X \Rightarrow Y \Rightarrow X \\
\text{!} : N \Rightarrow W \Rightarrow W & I : \forall X. X \Rightarrow X
\end{array}$$

The left- and right-hand sides of each of the structural congruence and rewrite rules have the sort  $W$ , the interpretation of any RHO calculus term has the sort  $W$ , and the result of eliminating an abstraction has the sort  $N \Rightarrow W$ .

We define an interpretation function  $\{\!\{-\}\!\}$  from  $W$ -sorted RHO combinators not containing  $C$  into the RHO calculus by

$$\begin{aligned}
\{\!\{0\}\!\} &= 0 \\
\{\!\{P | Q\}\!\} &= \{\!\{P\}\!\} | \{\!\{Q\}\!\} \\
\{\!\{((\text{for } (\& P)) Q)\}\!\} &= \text{for}(\&\{\!\{P\}\!\} \leftarrow \&\{\!\{P\}\!\})\{\!\{(Q (\& R))\}\!\} \\
\{\!\{((\text{! } (\& P)) Q)\}\!\} &= \&\{\!\{P\}\!\}!\{\!\{Q\}\!\} \\
\{\!\{(* (\& P))\}\!\} &= * \&\{\!\{P\}\!\} \\
\{\!\{(((S P) Q) R)\}\!\} &= \{\!\{((P R) (Q R))\}\!\} \\
\{\!\{((K P) Q)\}\!\} &= \{\!\{P\}\!\} \\
\{\!\{(I P)\}\!\} &= \{\!\{P\}\!\}
\end{aligned}$$



where  $R$  is any  $W$ -sorted RHO combinator.

Some simple calculation shows that

► **Theorem 3.**  $\llbracket \{P\} \rrbracket = P$  and  $\{\llbracket Q \rrbracket\}$  is  $\alpha$ -equivalent to  $Q$ .

See the appendix for more details.

## 7.4 Barbed bisimilarity

An **observation relation**  $\downarrow_{\mathcal{N}}$  over a set of names  $\mathcal{N}$  is the smallest relation satisfying

$$\frac{y \in \mathcal{N} \quad x \equiv_N y}{x!P \downarrow_{\mathcal{N}} x} \quad \text{and} \quad \frac{P \downarrow_{\mathcal{N}} x \text{ or } Q \downarrow_{\mathcal{N}} x}{P \mid Q \downarrow_{\mathcal{N}} x}$$

for the RHO calculus or

$$\frac{y \in \mathcal{N} \quad x \equiv_N y}{((!x)P) \downarrow_{\mathcal{N}} x} \quad \text{and} \quad \frac{P \downarrow_{\mathcal{N}} x \text{ or } Q \downarrow_{\mathcal{N}} x}{((\mid P)Q) \downarrow_{\mathcal{N}} x}.$$

for the RHO combinators.

We denote eventual reduction by  $\rightarrow^*$  and write  $P \downarrow_{\mathcal{N}}^* x$  if there exists a process  $Q$  such that  $P \rightarrow^* Q$  and  $Q \downarrow_{\mathcal{N}} x$ .

An  $\mathcal{N}$ -**barbed bisimulation** over a set of names  $\mathcal{N}$  is a symmetric binary relation  $S_{\mathcal{N}}$  between agents such that  $P S_{\mathcal{N}} Q$  implies

1. if  $P \rightarrow P'$  then  $Q \rightarrow^* Q'$  and  $P' S_{\mathcal{N}} Q'$ , and
2. if  $P \downarrow_{\mathcal{N}} x$ , then  $Q \downarrow_{\mathcal{N}}^* x$ .

$P$  is  $\mathcal{N}$ -barbed bisimilar to  $Q$ , written  $P \approx Q$ , if  $P S_{\mathcal{N}} Q$  for some  $\mathcal{N}$ -barbed bisimulation  $S_{\mathcal{N}}$ .

## 7.5 Faithfulness

► **Theorem 4.**  $P \approx_{\text{calc}} Q \iff ((\mid C) \llbracket P \rrbracket) \approx_{\text{comb}} ((\mid C) \llbracket Q \rrbracket)$ .

**Proof.** The only occurrence of  $C$  on the right is at the topmost context and the rewrite rules preserve the location of  $C$ , so the only reduction context is the topmost one. The rest follows from the two interpretation functions and theorem 3. In particular, while the only reduction rule in the RHO calculus is synchronizing on a name, there are extra reduction rules for the RHO combinators; however, these extra reduction rules never send or receive on a name and never prevent sending or receiving on a name. Therefore, each synchronization in the evaluation of a RHO calculus term corresponds to a synchronization in the corresponding RHO combinator term and some number of reductions of  $S, K, I$ , or evaluating a quoted process. ◀

## 8 Conclusion and future work

This paper is part of a pair of papers demonstrating that reflection provides a powerful technique for treating nominal phenomena as syntactic sugar, thus paving the way for simpler semantic treatments of richly featured calculi, such as the  $\pi$ -calculus and other calculi of concurrency. We illustrated the point by providing faithful semantics of both the  $\lambda$ -calculus and the  $\pi$ -calculus in terms of graph-enriched Lawvere theories. This work may be considered preparatory for a more elaborate study of logics for concurrency in which the nominal phenomena have logical status, but may be treated in a technically simpler fashion.

## 9 Appendix: abstraction elimination calculations

$$\begin{aligned}
& ((K \llbracket P \rrbracket) x) \\
= & \llbracket P \rrbracket \\
& (((S ((S (K \text{ for})) ((S (K \&)) \llbracket P \rrbracket_x))) \llbracket \llbracket Q \rrbracket_y \rrbracket_x) x) \\
= & (((((S (K \text{ for})) ((S (K \&)) \llbracket P \rrbracket_x) x) (\llbracket \llbracket Q \rrbracket_y \rrbracket_x x)) \\
= & (((((K \text{ for}) x) (((S (K \&)) \llbracket P \rrbracket_x) x)) \llbracket Q \rrbracket_y) \\
= & ((\text{for} (((K \&) x) (\llbracket P \rrbracket_x x))) \llbracket Q \rrbracket_y) \\
= & ((\text{for} (\& \llbracket P \rrbracket)) \llbracket Q \rrbracket_y) \\
& (((S ((S (K !)) ((S (K \&)) \llbracket P \rrbracket_x))) \llbracket Q \rrbracket_x) x) \\
= & ((((((S (K !)) ((S (K \&)) \llbracket P \rrbracket_x) x) (\llbracket Q \rrbracket_x x)) \\
= & (((((K !) x) (((S (K \&)) \llbracket P \rrbracket_x) x)) \llbracket Q \rrbracket) \\
= & ((! (((K \&) x) (\llbracket P \rrbracket_x x))) \llbracket Q \rrbracket) \\
= & (! (\& \llbracket P \rrbracket)) \llbracket Q \rrbracket) \\
& (((S ((S (K |)) \llbracket P \rrbracket_x)) \llbracket Q \rrbracket_x) x) \\
= & (((((S (K |)) \llbracket P \rrbracket_x) x) (\llbracket Q \rrbracket_x x)) \\
= & (((((K |) x) (\llbracket P \rrbracket_x x)) \llbracket Q \rrbracket) \\
= & (| \llbracket P \rrbracket) \llbracket Q \rrbracket) \\
& (((S (K *)) I) x) \\
= & (((K *) x) (I x)) \\
= & (* x) \\
& (((S (K *)) ((S (K \&)) \llbracket P \rrbracket_x) x) \\
= & (((K *) x) (((S (K \&)) \llbracket P \rrbracket_x) x)) \\
= & (* (((K \&) x) (\llbracket P \rrbracket_x x))) \\
= & (* (\& \llbracket P \rrbracket))
\end{aligned}$$

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