

# PRACTICAL 1

## Aim of this practical:

In this first practical we are going to look at some simple models

1. A Gaussian model with simulated data
2. A Linear mixed model
3. A GLM model with random effects

we are going to learn:

- How to fit some commonly used models with `inlabru`
- How to explore the results
- How to get predictions for missing data points

## 1 Linear Model

In this practical we will:

- Simulate Gaussian data
- Learn how to fit a linear model with `inlabru`
- Generate predictions from the model

Start by loading useful libraries:

```
library(dplyr)
library(INLA)
library(ggplot2)
library(patchwork)
library(inlabru)
# load some libraries to generate nice plots
library(scico)
```

As our first example we consider a simple linear regression model with Gaussian observations

$$y_i \sim \mathcal{N}(\mu_i, \sigma^2), \quad i = 1, \dots, N$$

where  $\sigma^2$  is the observation error, and the mean parameter  $\mu_i$  is linked to the **linear predictor** ( $\eta_i$ ) through an identity function:

$$\eta_i = \mu_i = \beta_0 + \beta_1 x_i$$

where  $x_i$  is a covariate and  $\beta_0, \beta_1$  are parameters to be estimated. We assign  $\beta_0$  and  $\beta_1$  a vague Gaussian prior.

To finalize the Bayesian model we assign a  $\text{Gamma}(a, b)$  prior to the precision parameter  $\tau = 1/\sigma^2$  and two independent Gaussian priors with mean 0 and precision  $\tau_\beta$  to the regression parameters  $\beta_0$  and  $\beta_1$  (we will use the default prior settings in INLA for now).

## Question

What is the dimension of the hyperparameter vector and latent Gaussian field?

Answer

The hyperparameter vector has dimension 1,  $\theta = (\tau)$  while the latent Gaussian field  $\mathbf{u} = (\beta_0, \beta_1)$  has dimension 2, 0 mean, and sparse precision matrix:

$$\mathbf{Q} = \begin{bmatrix} \tau_{\beta_0} & 0 \\ 0 & \tau_{\beta_1} \end{bmatrix}$$

Note that, since  $\beta_0$  and  $\beta_1$  are fixed effects, the precision parameters  $\tau_{\beta_0}$  and  $\tau_{\beta_1}$  are fixed.

## i Note

We can write the linear predictor vector  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_N)$  as

$$\boldsymbol{\eta} = \mathbf{A}\mathbf{u} = \mathbf{A}_1\mathbf{u}_1 + \mathbf{A}_2\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \beta_0 + \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \beta_1$$

Our linear predictor consists then of two components: an intercept and a slope.

## 1 Simulate example data

First, we simulate data from the model

$$y_i \sim \mathcal{N}(\eta_i, 0.1^2), i = 1, \dots, 100$$

with

$$\eta_i = \beta_0 + \beta_1 x_i$$

where  $\beta_0 = 2$ ,  $\beta_1 = 0.5$  and the values of the covariate  $x$  are generated from an Uniform(0,1) distribution. The simulated response and covariate data are then saved in a `data.frame` object.

```
beta = c(2,0.5)
sd_error = 0.1

n = 100
x = rnorm(n)
y = beta[1] + beta[2] * x + rnorm(n, sd = sd_error)

df = data.frame(y = y, x = x)
```

## 1 Fitting a linear regression model with inlabru

## Defining model components

The model has two parameters to be estimated  $\beta_1$  and  $\beta_2$ . We need to define the two corresponding model components.

### Task

Define an object called `cmp` that includes and (i) intercept `beta_0` and (ii) a covariate `x` linear effect `beta_1`.

Take hint

The `cmp` object is here used to define model components. We can give them any useful names we like, in this case, `beta_0` and `beta_1`. You can remove the automatic intercept construction by adding a `-1` in the components

[Click here to see the solution](#)

```
cmp = ~ -1 + beta_0(1) + beta_1(x, model = "linear")
```

### Note

Note that we have excluded the default Intercept term in the model by typing `-1` in the model components. However, `inlabru` has automatic intercept that can be called by typing `Intercept()`, which is one of `inlabru` special names and it is used to define a global intercept, e.g.

```
cmp = ~ Intercept(1) + beta_1(x, model = "linear")
```

## Observation model construction

The next step is to construct the observation model by defining the model likelihood. The most important inputs here are the formula, the family and the data.

### Task

Define a linear predictor `eta` using the component labels you have defined on the previous task.

Take hint

The `eta` object defines how the components should be combined in order to define the model predictor.

[Click here to see the solution](#)

```
eta = y ~ beta_0 + beta_1
```

The likelihood for the observational model is defined using the `bru_obs()` function.

### Task

Define the observational model likelihood in an object called `lik` using the `bru_obs()` function.

Take hint

The `bru_obs` is expecting three arguments:

- The linear predictor `eta` we defined in the previous task
- The data likelihood (this can be specified by setting `family = "gaussian"`)
- The data set `df`

[Click here to see the solution](#)

```
lik = bru_obs(formula = eta,
              family = "gaussian",
              data = df)
```

## Fit the model

We fit the model using the `bru()` functions which takes as input the components and the observation model:

```
fit.lm = bru(cmp, lik)
```

## Extract results

The `summary()` function will give access to some basic information about model fit and estimates

```
summary(fit.lm)
## inlabru version: 2.13.0.9011
## INLA version: 25.09.19
## Components:
## Latent components:
## beta_0: main = linear(1)
## beta_1: main = linear(x)
## Observation models:
##   Family: 'gaussian'
##   Tag: <No tag>
##   Data class: 'data.frame'
##   Response class: 'numeric'
##   Predictor: y ~ beta_0 + beta_1
##   Additive/Linear: TRUE/TRUE
##   Used components: effects[beta_0, beta_1], latent[]
## Time used:
##   Pre = 1.07, Running = 0.184, Post = 0.0185, Total = 1.27
## Fixed effects:
##      mean    sd 0.025quant 0.5quant 0.975quant  mode kld
## beta_0 2.001 0.01      1.982    2.001      2.02 2.001  0
## beta_1 0.491 0.01      0.472    0.491      0.51 0.491  0
##
## Model hyperparameters:
##                                mean    sd 0.025quant 0.5quant
## Precision for the Gaussian observations 107.34 15.18      79.70  106.62
##                                0.975quant  mode
## Precision for the Gaussian observations   139.09 105.19
##
## Marginal log-Likelihood: 69.44
## is computed
## Posterior summaries for the linear predictor and the fitted values are computed
## (Posterior marginals needs also 'control.compute=list(return.marginals.predictor=TRUE)')
```

We can see that both the intercept and slope and the error precision are correctly estimated.

## 1 Generate model predictions

Now we can take the fitted `bru` object and use the `predict` function to produce predictions for  $\mu$  given a new set of values for the model covariates or the original values used for the model fit

```
new_data = data.frame(x = c(df$x, runif(10)),
                      y = c(df$y, rep(NA, 10)))
pred = predict(fit.lm, new_data, ~ beta_0 + beta_1,
              n.samples = 1000)
```

The `predict` function generate samples from the fitted model. In this case we set the number of samples to 1000.

## 2 Plot

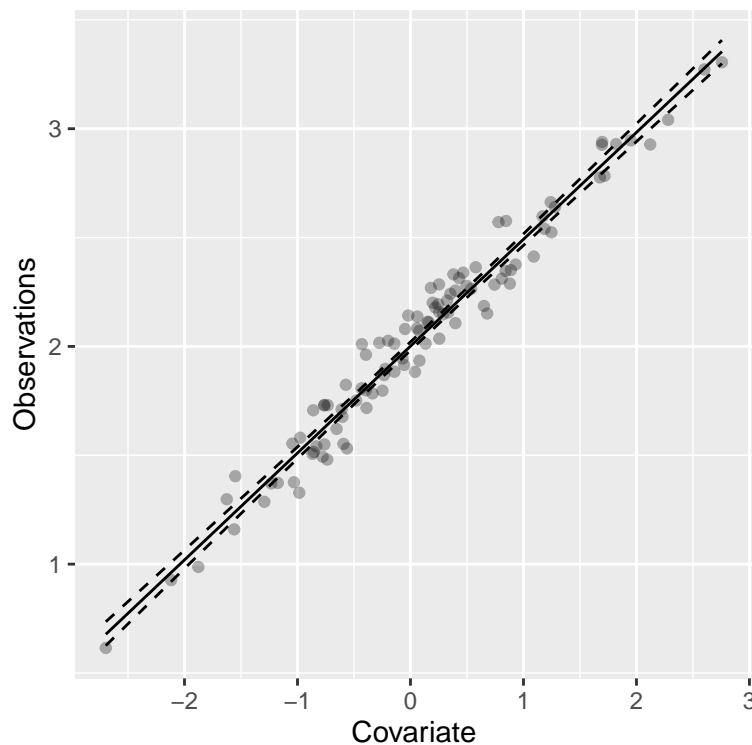


Figure 1: Data and 95% credible intervals

## 3 R Code

```
pred %>% ggplot() +
  geom_point(aes(x,y), alpha = 0.3) +
  geom_line(aes(x,mean)) +
  geom_line(aes(x, q0.025), linetype = "dashed")+
  geom_line(aes(x, q0.975), linetype = "dashed")+
  xlab("Covariate") + ylab("Observations")
```

## Task

Generate predictions for a new observation with  $x_0 = 0.45$

Take hint

You can create a new data frame containing the new observation  $x_0$  and then use the `predict` function.

[Click here to see the solution](#)

```
new_data = data.frame(x = 0.45)
pred = predict(fit.lm, new_data, ~ beta_0 + beta_1,
               n.samples = 1000)
```

## 4 Linear Mixed Model

In this practical we will:

- Understand the basic structure of a Linear Mixed Model (LLM)
- Simulate data from a LMM
- Learn how to fit a LMM with `inlabru` and predict from the model.

Consider the a simple linear regression model except with the addition that the data that comes in groups. Suppose that we want to include a random effect for each group  $j$  (equivalent to adding a group random intercept). The model is then:

$$y_{ij} = \beta_0 + \beta_1 x_i + u_j + \epsilon_{ij} \quad \text{for } i = 1, \dots, N \text{ and } j = 1, \dots, m.$$

Here the random group effect is given by the variable  $u_j \sim \mathcal{N}(0, \tau_u^{-1})$  with  $\tau_u = 1/\sigma_u^2$  describing the variability between groups (i.e., how much the group means differ from the overall mean). Then,  $\epsilon_j \sim \mathcal{N}(0, \tau_\epsilon^{-1})$  denotes the residuals of the model and  $\tau_\epsilon = 1/\sigma_\epsilon^2$  captures how much individual observations deviate from their group mean (i.e., variability within group).

The model design matrix for the random effect has one row for each observation (this is equivalent to a random intercept model). The row of the design matrix associated with the  $ij$ -th observation consists of zeros except for the element associated with  $u_j$ , which has a one.

$$\boldsymbol{\eta} = \mathbf{A}\mathbf{u} = \mathbf{A}_1\mathbf{u}_1 + \mathbf{A}_2\mathbf{u}_2 + \mathbf{A}_3\mathbf{u}_3$$

### Supplementary material: LMM as a LGM

In matrix form, the linear mixed model for the  $j$ -th group can be written as:

$$\underset{N \times 1}{\widehat{\mathbf{y}}_j} = \underset{N \times 2}{\widehat{\mathbf{X}}_j} \underset{1 \times 1}{\widehat{\boldsymbol{\beta}}} + \underset{n_j \times 1}{\widehat{\mathbf{Z}}_j} \underset{1 \times 1}{u_j} + \underset{n_j \times 1}{\widehat{\boldsymbol{\epsilon}}_j},$$

In a latent Gaussian model (LGM) formulation the mixed model predictor for the  $i$ -th observation can be written as :

$$\eta_i = \beta_0 + \beta_1 x_i + \sum_k^K f_k(u_j)$$

where  $f_k(u_j) = u_j$  since there's only one random effect per group (i.e., a random intercept for group  $j$ ). The fixed effects  $(\beta_0, \beta_1)$  are assigned Gaussian priors (e.g.,  $\beta \sim \mathcal{N}(0, \tau_\beta^{-1})$ ). The random effects  $\mathbf{u} = (u_1, \dots, u_m)^T$  follow a Gaussian density  $\mathcal{N}(0, \mathbf{Q}_u^{-1})$  where  $\mathbf{Q}_u = \tau_u \mathbf{I}_m$  is the precision matrix for the random intercepts. Then, the components for the LGM are the following:

- Latent field given by

$$\begin{bmatrix} \beta \\ \mathbf{u} \end{bmatrix} \sim \mathcal{N} \left( \mathbf{0}, \begin{bmatrix} \tau_\beta^{-1} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \tau_u^{-1} \mathbf{I}_m \end{bmatrix} \right)$$

- Likelihood:

$$y_i \sim \mathcal{N}(\eta_i, \tau_\epsilon^{-1})$$

- Hyperparameters:

- $\tau_u \sim \text{Gamma}(a, b)$
- $\tau_\epsilon \sim \text{Gamma}(c, d)$

## 4 Simulate example data

```
set.seed(12)
beta = c(1.5, 1)
sd_error = 1
tau_group = 1

n = 100
n.groups = 5
x = rnorm(n)
v = rnorm(n.groups, sd = tau_group^{-1/2})
y = beta[1] + beta[2] * x + rnorm(n, sd = sd_error) +
  rep(v, each = 20)

df = data.frame(y = y, x = x, j = rep(1:5, each = 20))
```

Note that `inlabru` expects an integer indexing variable to label the groups.

```
ggplot(df) +
  geom_point(aes(x = x, colour = factor(j), y = y)) +
  theme_classic() +
  scale_colour_discrete("Group")
```



Figure 2: Data for the linear mixed model example with 5 groups

## 4 Fitting a LMM in inlabru

### Defining model components and observational model

In order to specify this model we must use the group argument to tell inlabru which variable indexes the groups. The model = "iid" tells INLA that the groups are independent from one another.

```
# Define model components
cmp = ~ -1 + beta_0(1) + beta_1(x, model = "linear") +
  u(j, model = "iid")
```

The group variable is indexed by column j in the dataset. We have chosen to name this component v() to connect with the mathematical notation that we used above.

```
# Construct likelihood
lik = bru_obs(formula = y ~.,
  family = "gaussian",
  data = df)
```

### Fitting the model

The model can be fitted exactly as in the previous examples by using the bru function with the components and likelihood objects.

```
fit = bru(cmp, lik)
summary(fit)
## inlabru version: 2.13.0.9011
```



```
## INLA version: 25.09.19
## Components:
## Latent components:
## beta_0: main = linear(1)
## beta_1: main = linear(x)
## u: main = iid(j)
## Observation models:
##   Family: 'gaussian'
##   Tag: <No tag>
##   Data class: 'data.frame'
##   Response class: 'numeric'
##   Predictor: y ~ .
##   Additive/Linear: TRUE/TRUE
##   Used components: effects[beta_0, beta_1, u], latent[]
## Time used:
##   Pre = 1.04, Running = 0.21, Post = 0.0446, Total = 1.3
## Fixed effects:
##      mean      sd 0.025quant 0.5quant 0.975quant  mode kld
## beta_0 2.108 0.438      1.229      2.108      2.986 2.108  0
## beta_1 1.172 0.120      0.936      1.172      1.407 1.172  0
##
## Random effects:
##   Name      Model
##   u IID model
##
## Model hyperparameters:
##
##      mean      sd 0.025quant 0.5quant
## Precision for the Gaussian observations 0.995 0.144      0.738      0.986
## Precision for u                        1.613 1.060      0.369      1.356
##
##      0.975quant  mode
## Precision for the Gaussian observations      1.30 0.971
## Precision for u                        4.35 0.918
##
## Marginal log-Likelihood: -179.93
## is computed
## Posterior summaries for the linear predictor and the fitted values are computed
## (Posterior marginals needs also 'control.compute=list(return.marginals.predictor=TRUE)')
```

## 4 Model predictions

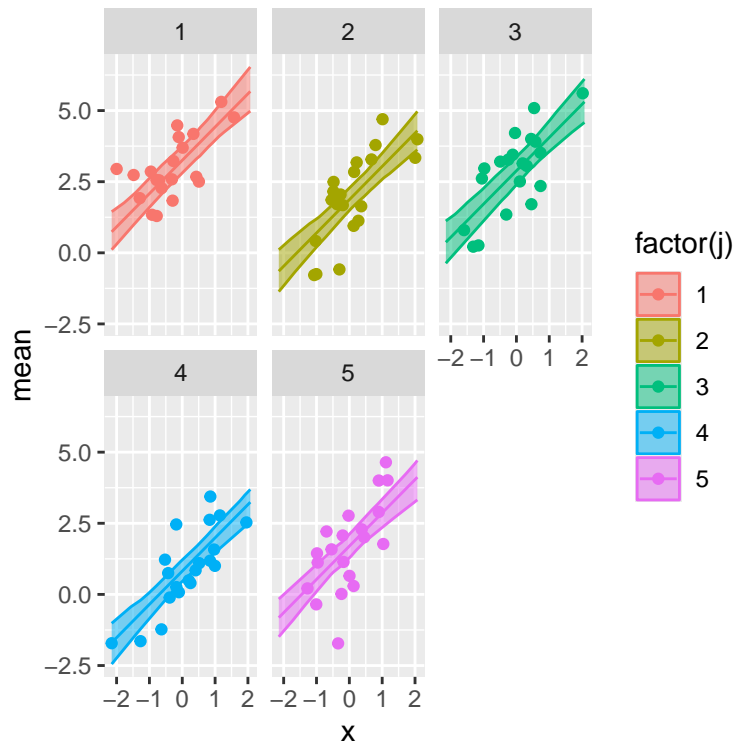
To compute model predictions we can create a `data.frame` containing a range of values of covariate where we want the response to be predicted for each group. Then we simply call the `predict` function while specifying the model components.

```
# New data
xpred = seq(range(x)[1], range(x)[2], length.out = 100)
j = 1:n.groups
pred_data = expand.grid(x = xpred, j = j)
pred = predict(fit, pred_data, formula = ~ beta_0 + beta_1 + u)
```

```

pred %>%
  ggplot(aes(x=x,y=mean,color=factor(j)))+
  geom_line()+
  geom_ribbon(aes(x,ymin = q0.025, ymax= q0.975,fill=factor(j)), alpha = 0.5) +
  geom_point(data=df,aes(x=x,y=y,colour=factor(j)))+
  facet_wrap(~j)

```



### Question

Suppose that we are also interested in including random slopes into our model. Assuming intercept and slopes are independent, can you write down the linear predictor and the components of this model as a LGM?

Give me a hint

In general, the mixed model predictor can be decomposed as:

$$\boldsymbol{\eta} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{u}$$

Where  $\boldsymbol{X}$  is a  $n \times p$  design matrix and  $\boldsymbol{\beta}$  the corresponding  $p$ -dimensional vector of fixed effects. Then  $\boldsymbol{Z}$  is a  $n \times q_J$  design matrix for the  $q_J$  random effects and  $J$  groups;  $\boldsymbol{u}$  is then a  $q_J \times 1$  vector of  $q$  random effects for the  $J$  groups. In a latent Gaussian model (LGM) formulation this can be written as:

$$\eta_i = \beta_0 + \sum \beta_j x_{ij} + \sum_k f(k)(u_{ij})$$

See Solution

- The linear predictor is given by

$$\eta_i = \beta_0 + \beta_1 x_i + u_{0j} + u_{1j} x_i$$

- Latent field defined by:

- $\beta \sim \mathcal{N}(0, \tau_\beta^{-1})$
- $\mathbf{u}_j = \begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix}, \mathbf{u}_j \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_u^{-1})$  where the precision matrix is a block-diagonal matrix with entries  $\mathbf{Q}_u = \begin{bmatrix} \tau_{u_0} & 0 \\ 0 & \tau_{u_1} \end{bmatrix}$

• The hyperparameters are then:

- $\tau_{u_0}, \tau_{u_1}$  and  $\tau_\epsilon$

To fit this model in `inlabru` we can simply modify the model components as follows:

```
cmp = ~ -1 + beta_0(1) + beta_1(x, model = "linear") +
      u0(j, model = "iid") + u1(j,x, model = "iid")
```

## 5 Generalized Linear Model

In this practical we will:

- Simulate non-Gaussian data
- Learn how to fit a generalised linear model with `inlabru`
- Generate predictions from the model

A generalised linear model allows for the data likelihood to be non-Gaussian. In this example we have a discrete response variable which we model using a Poisson distribution. Thus, we assume that our data

$$y_i \sim \text{Poisson}(\lambda_i)$$

with rate parameter  $\lambda_i$  which, using a log link, has associated predictor

$$\eta_i = \log \lambda_i = \beta_0 + \beta_1 x_i$$

with parameters  $\beta_0$  and  $\beta_1$ , and covariate  $x$ .

### 5 Simulate example data

This code generates 100 samples of covariate  $x$  and data  $y$ .

```
set.seed(123)
n = 100
beta = c(1,1)
x = rnorm(n)
lambda = exp(beta[1] + beta[2] * x)
y = rpois(n, lambda = lambda)
df = data.frame(y = y, x = x)
```

## 5 Fitting a GLM in inlabru

### Define model components

The predictor here only contains only 2 components (Intercept and Slope).

#### Task

Define an object called `cmp` that includes and (i) intercept `beta_0` and (ii) a covariate `x` linear effect `beta_1`.

[Click here to see the solution](#)

```
cmp = ~ -1 + beta_0(1) + beta_1(x, model = "linear")
```

### Define linear predictor

#### Task

Define a linear predictor `eta` using the component labels you have defined on the previous task.

[Click here to see the solution](#)

```
eta = y ~ beta_0 + beta_1
```

### Build observational model

When building the observation model likelihood we must now specify the Poisson likelihood using the `family` argument (the default link function for this family is the log link).

```
lik = bru_obs(formula = eta,
              family = "poisson",
              data = df)
```

### Fit the model

Once the likelihood object is constructed, fitting the model is exactly the same process, we just need to specify the model components and the observational model, and pass this on to the `bru` function:

```
fit_glm = bru(cmp, lik)
```

And model summaries can be viewed using

```
summary(fit_glm)
```

```
inlabru version: 2.13.0.9011
```

```
INLA version: 25.09.19
```

```
Components:
```

```
Latent components:
```

```
beta_0: main = linear(1)
```

```
beta_1: main = linear(x)
```

```
Observation models:
```

```

Family: 'poisson'
Tag: <No tag>
Data class: 'data.frame'
Response class: 'integer'
Predictor: y ~ beta_0 + beta_1
Additive/Linear: TRUE/TRUE
Used components: effects[beta_0, beta_1], latent[]
Time used:
  Pre = 0.875, Running = 0.19, Post = 0.00572, Total = 1.07
Fixed effects:
      mean      sd 0.025quant 0.5quant 0.975quant  mode kld
beta_0 0.915 0.071      0.775   0.915      1.054 0.915  0
beta_1 1.048 0.056      0.938   1.048      1.157 1.048  0

Marginal log-Likelihood: -204.02
is computed
Posterior summaries for the linear predictor and the fitted values are computed
(Posterior marginals needs also 'control.compute=list(return.marginals.predictor=TRUE)')

```

## 5 Generate model predictions

To generate new predictions we must provide a data frame that contains the covariate values for  $x$  at which we want to predict.

This code block generates predictions for the data we used to fit the model (contained in `df$x`) as well as 10 new covariate values sampled from a uniform distribution `runif(10)`.

```

# Define new data, set to NA the values for prediction

new_data = data.frame(x = c(df$x, runif(10)),
                      y = c(df$y, rep(NA,10)))

# Define predictor formula
pred_fml <- ~ exp(beta_0 + beta_1)

# Generate predictions
pred_glm <- predict(fit_glm, new_data, pred_fml)

```

Since we used a log link (which is the default for `family = "poisson"`), we want to predict the exponential of the predictor. We specify this using a general R expression using the formula syntax.

### **i** Note

Note that the `predict` function will call the component names (i.e. the “labels”) that were decided when defining the model.

Since the component definition is looking for a covariate named  $x$ , all we need to provide is a data frame that contains one, and the software does the rest.

## 6 Plot

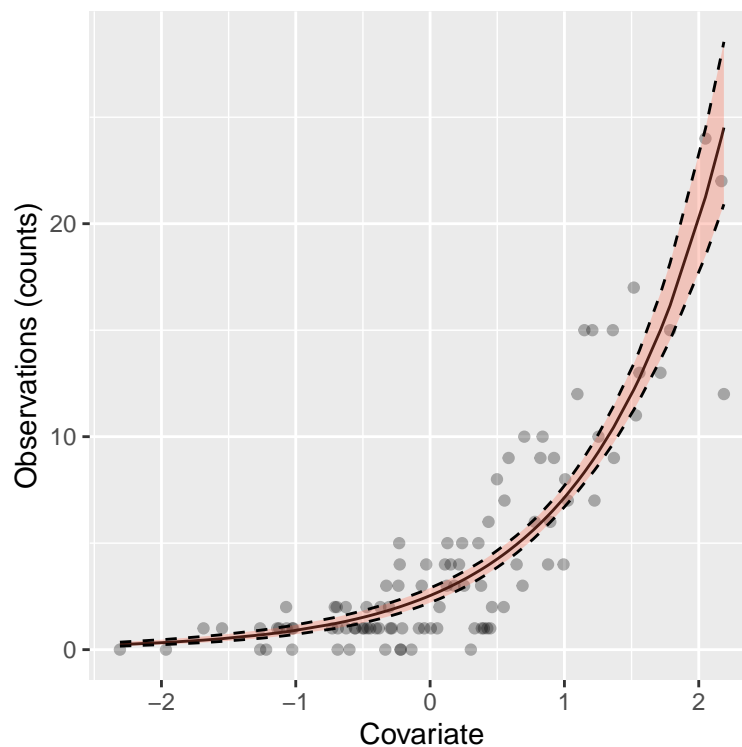


Figure 3: Data and 95% credible intervals

## 7 R Code

```
pred_glm %>% ggplot() +
  geom_point(aes(x,y), alpha = 0.3) +
  geom_line(aes(x,mean)) +
  geom_ribbon(aes(x = x, ymax = q0.975, ymin = q0.025), fill = "tomato", alpha = 0.3)+
  xlab("Covariate") + ylab("Observations (counts)")
```

### Task

Suppose a binary response such that

$$y_i \sim \text{Bernoulli}(\psi_i)$$

$$\eta_i = \text{logit}(\psi_i) = \alpha_0 + \alpha_1 \times w_i$$

Using the following simulated data, use `inlabru` to fit the logistic regression above. Then, plot the predictions for the data used to fit the model along with 10 new covariate values

```

set.seed(123)
n = 100
alpha = c(0.5,1.5)
w = rnorm(n)
psi = plogis(alpha[1] + alpha[2] * w)
y = rbinom(n = n, size = 1, prob = psi) # set size = 1 to draw binary observations
df_logis = data.frame(y = y, w = w)

```

Here we use the logit link function  $\text{logit}(x) = \log\left(\frac{x}{1-x}\right)$  (plogis() function in R) to link the linear predictor to the probabilities  $\psi$ .

Take hint

You can set family = "binomial" for binary responses and the plogis() function for computing the predicted values.

### Note

The Bernoulli distribution is equivalent to a Binomial(1,  $\psi$ ) pmf. If you have proportional data (e.g. no. successes/no. trials) you can specify the number of events as your response and then the number of trials via the Ntrials = n argument of the bru\_obs function (where n is the known vector of trials in your data set).

[Click here to see the solution](#)

```

# Model components
cmp_logis = ~ -1 + alpha_0(1) + alpha_1(w, model = "linear")
# Model likelihood
lik_logis = bru_obs(formula = y ~.,
                    family = "binomial",
                    data = df_logis)
# fit the model
fit_logis <- bru(cmp_logis,lik_logis)

# Define data for prediction
new_data = data.frame(w = c(df_logis$w, runif(10)),
                      y = c(df_logis$y, rep(NA,10)))
# Define predictor formula
pred_fml <- ~ plogis(alpha_0 + alpha_1)

# Generate predictions
pred_logis <- predict(fit_logis, new_data, pred_fml)

# Plot predictions
pred_logis %>% ggplot() +
  geom_point(aes(w,y), alpha = 0.3) +
  geom_line(aes(w,mean)) +
  geom_ribbon(aes(x = w, ymax = q0.975, ymin = q0.025),fill = "tomato", alpha = 0.3)+
  xlab("Covariate") + ylab("Observations")

```

