

# PRACTICAL 2

## Aim of this practical:

1. Set priors for different linear models
2. Compute and visualize posterior densities and summaries for marginal effects
3. Fit hierarchical flexible models

we are going to learn:

- How to change some of the R default priors in `inlabru`
- How to explore and visualize model parameters
- Fit different flexible models

## 0 Setting priors and model checking for Linear Models

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In this exercise we will:

- Learn how to set priors for linear effects  $\beta_0$  and  $\beta_1$
- Learn how to set the priors for the hyperparameter  $\tau = 1/\sigma^2$ .
- Visualize marginal posterior distributions

Start by loading useful libraries:

```
library(dplyr)
library(INLA)
library(ggplot2)
library(patchwork)
library(inlabru)
```

Recall a simple linear regression model with Gaussian observations

$$y_i \sim \mathcal{N}(\mu_i, \sigma^2), \quad i = 1, \dots, N$$

where  $\sigma^2$  is the observation error, and the mean parameter  $\mu_i$  is linked to the linear predictor through an identity function:

$$\eta_i = \mu_i = \beta_0 + \beta_1 x_i$$

where  $x_i$  is a covariate and  $\beta_0, \beta_1$  are parameters to be estimated. In INLA, we assume that the model is a latent Gaussian model, i.e., we have to assign  $\beta_0$  and  $\beta_1$  a Gaussian prior. For the precision hyperparameter  $\tau = 1/\sigma^2$  a typical prior choice is a  $\text{Gamma}(a, b)$  prior.

In R-INLA, the default choice of priors for each  $\beta$  is

$$\beta \sim \mathcal{N}(0, 10^3).$$

and the prior for the variance parameter in terms of the log precision is

$$\log(\tau) \sim \text{logGamma}(1, 5 \times 10^{-5})$$

**i Note**

If your model uses the default intercept construction (i.e., Intercept(1) in the linear predictor) INLA will assign a default  $\mathcal{N}(0, 0)$  prior to it.

Lets see how can we change the default priors using some simulated data

**0.1.0.1 Simulate example data** We simulate data from a simple linear regression model

```
beta = c(2,0.5)
sd_error = 0.1

n = 100
x = rnorm(n)
y = beta[1] + beta[2] * x + rnorm(n, sd = sd_error)

df = data.frame(y = y, x = x)
```

**0.1.0.2 Fitting the linear regression model with inlabru** Now we fit a simple linear regression model in inlabru by defining (1) the model components, (2) the linear predictor and (3) the likelihood.

```
# Model components
cmp = ~ -1 + beta_0(1) + beta_1(x, model = "linear")
# Linear predictor
formula = y ~ Intercept + beta_1
# Observational model likelihood
lik = bru_obs(formula = y ~.,
              family = "gaussian",
              data = df)
# Fit the Model
fit.lm = bru(cmp, lik)
```

### 0.1.1 Change the prior distributions

Until now, we have used the default priors for both the precision  $\tau$  and the fixed effects  $\beta_0$  and  $\beta_1$ . Let's see how to customize these.

To check which priors are used in a fitted model one can use the function `inla.prior.used()`

```
inla.prior.used(fit.lm)
```

```
section=[family]
  tag=[INLA.Data1] component=[gaussian]
    theta1:
      parameter=[log precision]
      prior=[loggamma]
      param=[1e+00, 5e-05]
section=[linear]
```

```

tag=[beta_0] component=[beta_0]
  beta:
    parameter=[beta_0]
    prior=[normal]
    param=[0.000, 0.001]
tag=[beta_1] component=[beta_1]
  beta:
    parameter=[beta_1]
    prior=[normal]
    param=[0.000, 0.001]

```

From the output we see that the precision for the observation  $\tau \sim \text{Gamma}(1e+00, 5e-05)$  while  $\beta_0$  and  $\beta_1$  have precision 0.001, that is variance  $1/0.001$ .

### Change the precision for the linear effects

The precision for linear effects is set in the component definition. For example, if we want to increase the precision to 0.01 for  $\beta_0$  we define the relative components as:

```
cmp1 = ~-1 + beta_0(1, prec.linear = 0.01) + beta_1(x, model = "linear")
```

#### Task

Run the model again using 0.1 as default precision for both the intercept and the slope parameter.

[Click here to see the solution](#)

```

cmp2 = ~ -1 +
  beta_0(1, prec.linear = 0.1) +
  beta_1(x, model = "linear", prec.linear = 0.1)

lm.fit2 = bru(cmp2, lik)

```

Note that we can use the same observation model as before since both the formula and the dataset are unchanged.

### Change the prior for the precision of the observation error $\tau$

Priors on the hyperparameters of the observation model must be passed by defining argument `hyper` within `control.family` in the call to the `bru_obs()` function.

```

# First we define the logGamma (0.01,0.01) prior

prec.tau <- list(prec = list(prior = "loggamma", # prior name
                             param = c(0.01, 0.01))) # prior values

lik2 = bru_obs(formula = y ~.,
               family = "gaussian",
               data = df,
               control.family = list(hyper = prec.tau))

fit.lm2 = bru(cmp2, lik2)

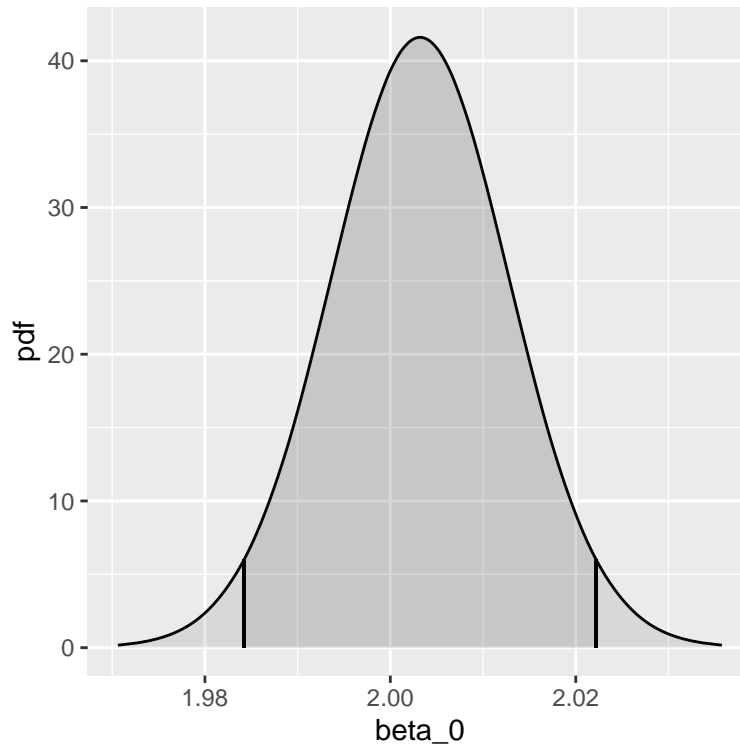
```

The names of the priors available in **R-INLA** can be seen with `names(inla.models())$prior`

### 0.1.2 Visualizing the posterior marginals

Posterior marginal distributions of the fixed effects parameters and the hyperparameters can be visualized using the `plot()` function by calling the name of the component. For example, if want to visualize the posterior density of the intercept  $\beta_0$  we can type:

```
plot(fit.lm, "beta_0")
```



#### Task

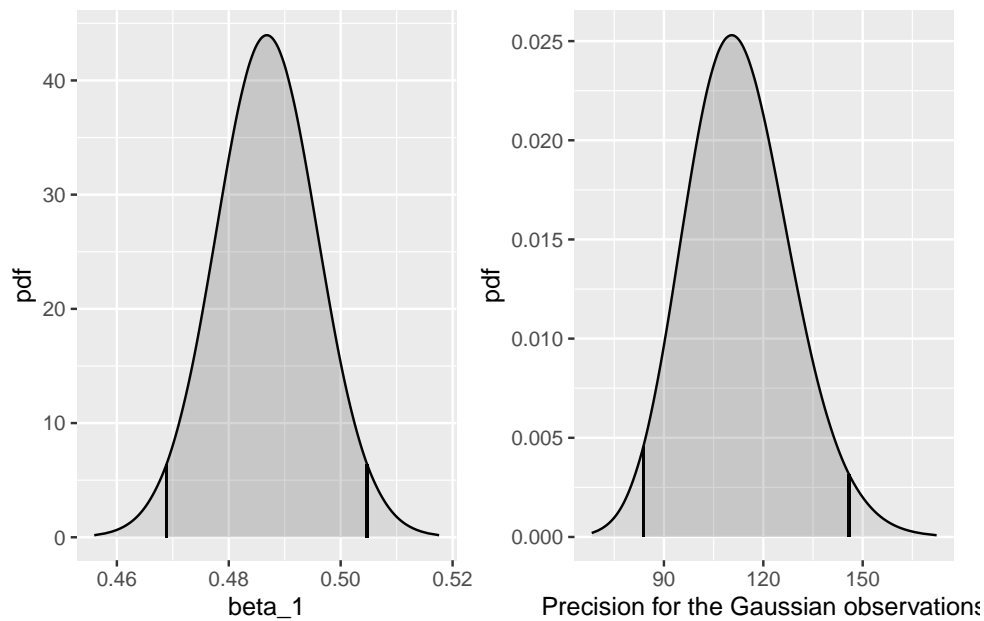
Plot the posterior marginals for  $\beta_1$  and for the precision of the observation error  $\pi(\tau|y)$

Take hint

See the `summary()` output to check the names for the different model components.

[Click here to see the solution](#)

```
plot(fit.lm, "beta_1") +  
plot(fit.lm, "Precision for the Gaussian observations")
```



## 0 Linear Mixed Model for fish weight-length relationship

In this exercise we will:

- Plot random effects of a LMM
- Compute posterior densities and summaries for the variance components

Libraries to load:

```
library(dplyr)
library(INLA)
library(ggplot2)
library(patchwork)
library(inlabru)
```

In this exercise, we will use a subset of the Pygmy Whitefish (*Prosopium coulterii*) dataset from the FSAdata R package, containing biological data collected in 2001 from Dina Lake, British Columbia.

The data set contains the following information:

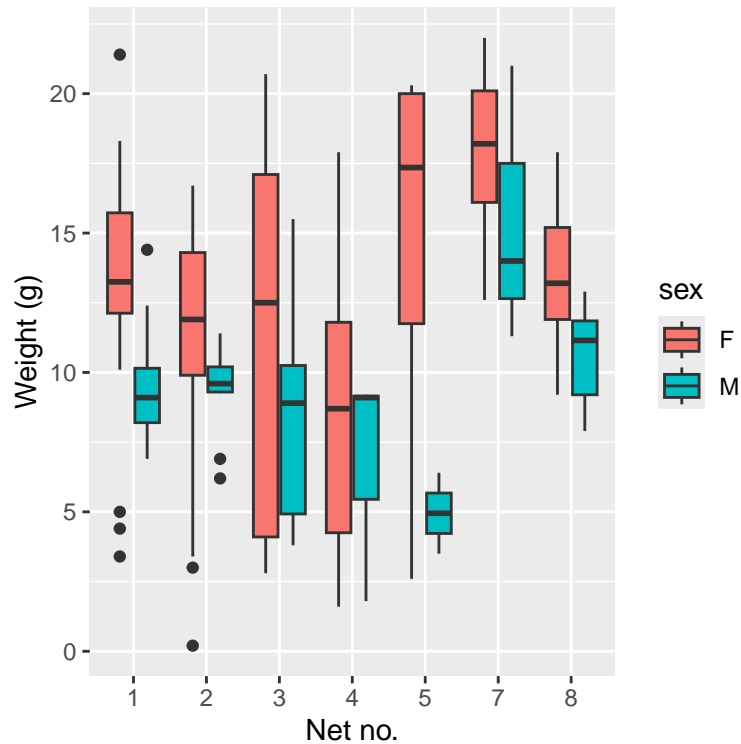
- net\_no Unique net identification number
- wt Fish weight (g)
- t1 Total fish length (cm)
- sex Sex code (F=Female, M = Male)

We can visualize the distribution of the response (weight) across the nets split by sex as follows:

```
PygmyWFBC <- read.csv("datasets/PygmyWFBC.csv")

ggplot(PygmyWFBC, aes(x = factor(net_no), y = wt, fill = sex)) +
  geom_boxplot() +
```

```
labs(y="Weight (g)", x = "Net no.")
```



Suppose we are interested in modelling the weight-length relationship for captured fish. The exploratory plot suggest some important variability in this relationship, potentially attributable to differences among sampling nets deployed across various sites in the Dina Lake.

To account for this between-net variability, we model net as a random effect using the following linear mixed model:

$$\begin{aligned}
 y_{ij} &\sim \mathcal{N}(\mu_{ij}, \sigma_e^2), & i = 1, \dots, a & \quad j = 1, \dots, n \\
 \mu_{ij} &= \beta_0 + \beta_1 \times \text{length}_{ij} + \beta_2 \times \mathbb{I}(\text{Sex}_{ij} = \text{M}) + u_i \\
 u_i &\sim \mathcal{N}(0, \sigma_u^2)
 \end{aligned}$$

where:

- $y_{ij}$  is the weight of the  $j$ -th fish from net  $i$
- $\text{length}_{ij}$  is the corresponding fish length
- $\mathbb{I}(\text{Sex}_{ij} = \text{M})$  is an indicator/dummy such that for the  $i$ th net

$$\mathbb{I}(\text{Sex}_{ij}) = \begin{cases} 1 & \text{if the } j\text{th fish is Male} \\ 0 & \text{otherwise} \end{cases}$$

- $u_i$  represents the random intercept for net  $i$
- $\sigma_u^2$  and  $\sigma_e^2$  are the between-net and residual variances, respectively

To run this model in `inllabru` we first need to create our sex dummy variable :

```
PygmyWFBC$sex_M <- ifelse(PygmyWFBC$sex=="F",0,1)
```

inlabru will treat 0 as the reference category (i.e., the intercept  $\beta_0$  will represent the baseline weight for females ). Now we can define the model component, the likelihood and fit the model.

```
cmp = ~ -1 + sex_M + beta_0(1) + beta_1(tl, model = "linear") + net_eff(net_no, model = "linear")

lik = bru_obs(formula = wt ~ .,
              family = "gaussian",
              data = PygmyWFBC)

fit = bru(cmp, lik)

summary(fit)
```

inlabru version: 2.13.0

INLA version: 25.08.21-1

Components:

sex\_M: main = linear(sex\_M), group = exchangeable(1L), replicate = iid(1L), NULL

beta\_0: main = linear(1), group = exchangeable(1L), replicate = iid(1L), NULL

beta\_1: main = linear(tl), group = exchangeable(1L), replicate = iid(1L), NULL

net\_eff: main = iid(net\_no), group = exchangeable(1L), replicate = iid(1L), NULL

Observation models:

Family: 'gaussian'

Tag: <No tag>

Data class: 'data.frame'

Response class: 'numeric'

Predictor: wt ~ .

Additive/Linear: TRUE/TRUE

Used components: effects[sex\_M, beta\_0, beta\_1, net\_eff], latent[]

Time used:

Pre = 0.323, Running = 0.293, Post = 0.218, Total = 0.834

Fixed effects:

	mean	sd	0.025quant	0.5quant	0.975quant	mode	kld
sex_M	-1.106	0.218	-1.534	-1.106	-0.678	-1.106	0
beta_0	-15.817	0.870	-17.516	-15.820	-14.100	-15.820	0
beta_1	2.555	0.072	2.414	2.555	2.696	2.555	0

Random effects:

Name Model

net\_eff IID model

Model hyperparameters:

	mean	sd	0.025quant	0.5quant	0.975quant	mode
Precision for the Gaussian observations	0.475	0.044	0.392	0.473		
Precision for net_eff	2.145	1.312	0.562	1.839		
Precision for the Gaussian observations	0.567	0.47				
Precision for net_eff	5.518	1.32				

Marginal log-Likelihood: -467.54

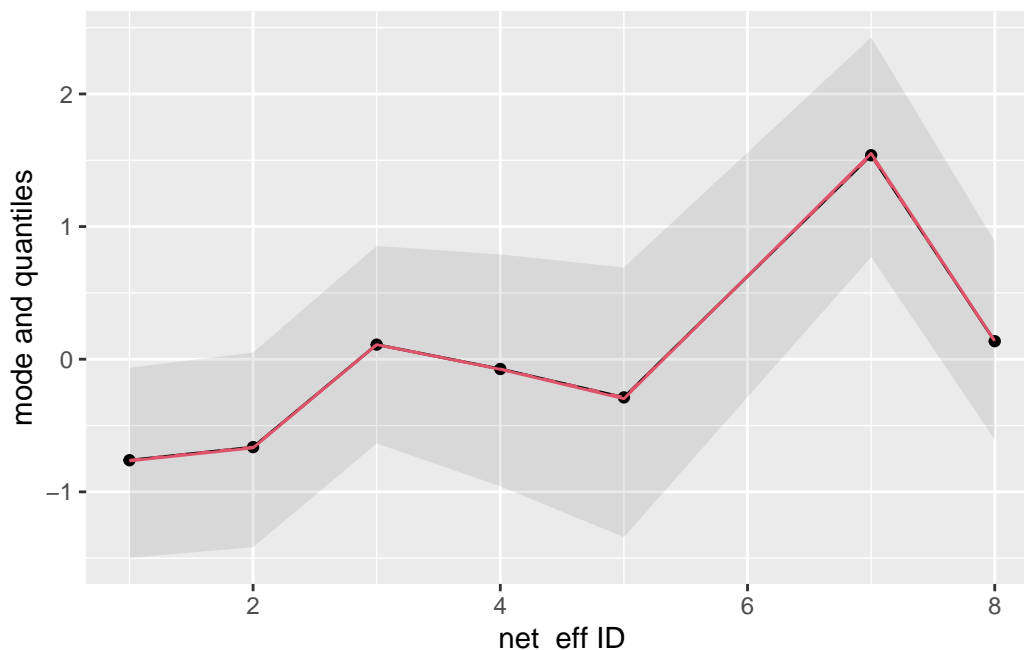
is computed

Posterior summaries for the linear predictor and the fitted values are computed (Posterior marginals needs also 'control.compute=list(return.marginals.predictor=TRUE)')

For interpretability, we could have centered the predictors, but our primary focus here is on estimating the variance components of the mixed model.

We can plot the posterior density of the nets random intercept as follows:

```
plot(fit,"net_eff")
```



For theoretical and computational purposes, INLA works with the precision which is the inverse of the variance. To obtain the posterior summaries on the SDs scale we can sample from the posterior distribution for the precision while back-transforming the samples and then computing the summary statistics. Transforming the samples is necessary because some quantities such as the mean and mode are not invariant to monotone transformation; alternatively we can use some of the in-built R-INLA functions to achieve this (see supplementary note).

We use the `inla.hyperpar.sample` function to draw samples from the approximated joint posterior for the hyperparameters, then invert them to get variances and lastly compute the mean, std. dev., quantiles, etc.

```
sampvars <- 1/inla.hyperpar.sample(1000,fit,improve.marginals = T)
colnames(sampvars) <- c("Error variance","Between-net Variance")
apply(sampvars,2,
      function(x) c("mean"=mean(x),
                    "std.dev" = sd(x),
                    quantile(x,c(0.025,0.5,0.975)))))
```

	Error variance	Between-net Variance
mean	2.1251137	0.6516507
std.dev	0.1982475	0.4172313



2.5%	1.7674637	0.1838441
50%	2.1143470	0.5428131
97.5%	2.5420159	1.7529186

### Task

Another useful quantity we can compute is the intraclass correlation coefficient (ICC) which help us determine how much the response varies within groups compared to between groups. The intraclass correlation coefficient is defined as:

$$\text{ICC} = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_e^2}$$

Compute the median, and quantiles for the ICC using the posterior samples we draw for  $\sigma_e^2$  and  $\sigma_u^2$ .

Take hint

The `rowSums` function can be used to compute  $\sigma_{u,s}^2 + \sigma_{e,s}^2$  for the  $s$ th posterior draw. Click here to see the solution

```
sampicc <- sampvars[,2]/(rowSums(sampvars))
quantile(sampicc, c(0.025,0.5,0.975))
```

2.5%	50%	97.5%
0.07743354	0.20456838	0.46701249

### Supplementary Material

The marginal densities for the hyper parameters can be also found by calling `inlabru_model$marginals.hyperpar`. We can then apply a transformation using the `inla.tmarginal` function to transform the precision posterior distributions.

```
var_e <- fit$marginals.hyperpar$`Precision for the Gaussian observations` %>%
  inla.tmarginal(function(x) 1/x,.)

var_u <- fit$marginals.hyperpar$`Precision for net_eff` %>%
  inla.tmarginal(function(x) 1/x,.)
```

The marginal densities for the hyper parameters can be found with `inlabru_model$marginals.hyperpar`, then we can apply a transformation using the `inla.tmarginal` function to transform the precision posterior distributions. Then, we can compute posterior summaries using `inla.zmarginal` function as follows:

```
post_var_summaries <- cbind( inla.zmarginal(var_e,silent = T),
                             inla.zmarginal(var_u,silent = T))
colnames(post_var_summaries) <- c("sigma_e","sigma_u")
post_var_summaries
```

	sigma_e	sigma_u
mean	2.124695	0.6509834
sd	0.1981598	0.4175519
quant0.025	1.765464	0.1816892
quant0.25	1.985473	0.3686155

```
quant0.5    2.113604  0.5411288  
quant0.75    2.252083  0.8073278  
quant0.975    2.543166  1.759271
```

## 0 Hierarchical generalised additive mixed models with `inlabru`

In this exercise we will:

- Fit an hierarchical generalised additive mixed models
- Fit a model with a global smooth term
- Fit a model with global and group-level smooth terms

Libraries to load:

```
library(dplyr)  
library(INLA)  
library(ggplot2)  
library(patchwork)  
library(inlabru)
```

The oceans represent Earth's largest habitat, with life distributed unevenly across depths primarily due to variations in light, temperature, and pressure. Biomass generally decreases with depth, though complex factors like water density layers create non-linear patterns. A significant portion of deep-sea organisms exhibit bioluminescence, which scientists measure using specialized equipment like free-fall camera systems to profile vertical distribution.

In this exercise, we analyze the ISIT dataset, which contains bioluminescence measurements from the northeast Atlantic Ocean. This dataset was previously examined in Zuur et al. (2009) and Gillibrand et al. (2007) and consists of observations collected across a depth gradient (0–4,800 m) during spring and summer cruises in 2001–2002 using an ISIT free-fall profiler.

The focus of this exercise will be on characterizing seasonal variation in the relationship between bioluminescent source density (sources  $\text{m}^2$ ) and depth. We begin by exploring distribution patterns of pelagic bioluminescence through source-depth profiles, with each profile representing measurements from an individual sampling station. These profiles will be grouped by month to examine temporal patterns in the water column's bioluminescent structure.

# 1 Plot

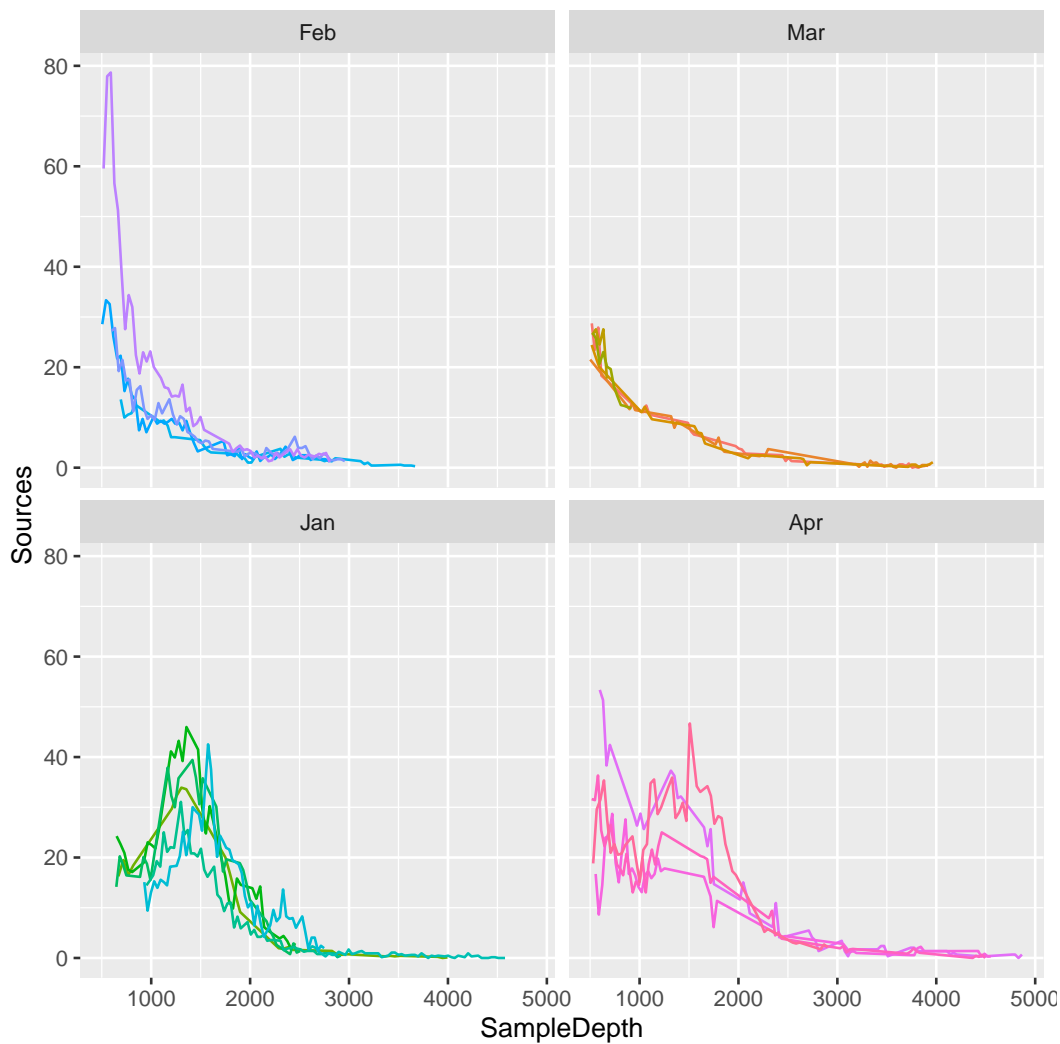


Figure 1: Source–depth profiles per month. Each line represents a station.

## 2 R-Code

```

icit <- read.csv("datasets/ISIT.csv")

icit$Month <- as.factor(icit$Month)
levels(icit$Month) <- month.abb[unique(icit$Month)]

ggplot(icit, aes(x=SampleDepth, y= Sources,
                 group=as.factor(Station),
                 colour=as.factor(Station)))+
  geom_line()+
  facet_wrap(~Month)+
  theme(legend.position = "none")

```

As expected, there seems to be a non-linear depth effect with some important variability across months.

## 2.0.1 Fitting a global smoother

We could begin analysing these data with a global smoother and a random intercept for each month. Thus, a possible model is of the form:

$$S_{is} = \beta_0 + f(\text{Depth})_s + \text{Month}_i + \epsilon_{is} \text{ such that } \epsilon \sim \mathcal{N}(0, \sigma_e^2); \text{Month} \sim \mathcal{N}(0, \sigma_m^2).$$

where the source during month  $i$  at depth  $s$ ,  $S_{is}$ , are modelled as smoothing function of depth and a month effect. The model has one smoothing curve for all months and can be fitted in inlabru as follows:

```
icit$Month_id <- as.numeric(icit$Month) # numeric index for the i-th month

cmp_g = ~ -1+ beta_0(1) +
  smooth_g(SampleDepth, model = "rw1") +
  month_reff(Month_id, model = "iid")

lik = bru_obs(formula = Sources ~ .,
              family = "gaussian",
              data = icit)

fit_g = bru(cmp_g, lik)

summary(fit_g)
```

inlabru version: 2.13.0

INLA version: 25.08.21-1

Components:

beta\_0: main = linear(1), group = exchangeable(1L), replicate = iid(1L), NULL

smooth\_g: main = rw1(SampleDepth), group = exchangeable(1L), replicate = iid(1L), NULL

month\_reff: main = iid(Month\_id), group = exchangeable(1L), replicate = iid(1L), NULL

Observation models:

Family: 'gaussian'

Tag: <No tag>

Data class: 'data.frame'

Response class: 'numeric'

Predictor: Sources ~ .

Additive/Linear: TRUE/TRUE

Used components: effects[beta\_0, smooth\_g, month\_reff], latent[]

Time used:

Pre = 0.346, Running = 0.696, Post = 0.188, Total = 1.23

Fixed effects:

	mean	sd	0.025quant	0.5quant	0.975quant	mode	kld
beta_0	10.016	1.629	6.656	10.025	13.323	10.023	0

Random effects:

Name Model

smooth\_g RW1 model

month\_reff IID model

Model hyperparameters:

	mean	sd	0.025quant	0.5quant
Precision for the Gaussian observations	0.024	0.001	0.021	0.024
Precision for smooth_g	21.178	5.412	12.463	20.533
Precision for month_reff	0.139	0.096	0.028	0.115

	0.975quant	mode
Precision for the Gaussian observations	0.026	0.024
Precision for smooth_g	33.623	19.315
Precision for month_reff	0.387	0.073

Marginal log-Likelihood: -2217.23

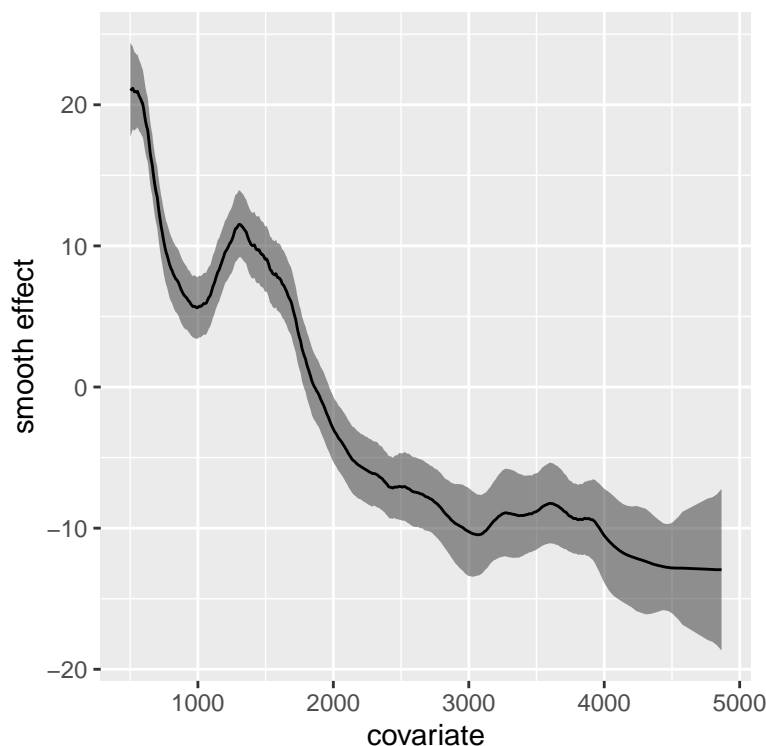
is computed

Posterior summaries for the linear predictor and the fitted values are computed

(Posterior marginals needs also 'control.compute=list(return.marginals.predictor=TRUE)')

We can plot the smoother marginal effect as follows:

```
data.frame(fit_g$summary.random$smooth_g) %>%
  ggplot() +
  geom_ribbon(aes(ID,ymin = X0.025quant, ymax= X0.975quant), alpha = 0.5) +
  geom_line(aes(ID,mean)) +
  xlab("covariate") + ylab("smooth effect")
```



You might want to have a smoother function by placing a RW2 prior. Unfortunately, this assumes that all the knots are regularly spaced and some depth values are too close to be used for building the RW2 priors. For the case, it is possible to use function `inla.group()` to bin data into groups according to the values of the covariate:

```
icit$depth_grouped <- inla.group(icit$SampleDepth,n=50)
```

## Task

Re-run the global smoother model using a RW2 prior for the depth smoother and compare your results with the RW1 model.

Take hint

Use the `depth_grouped` covariate to define the smoother.

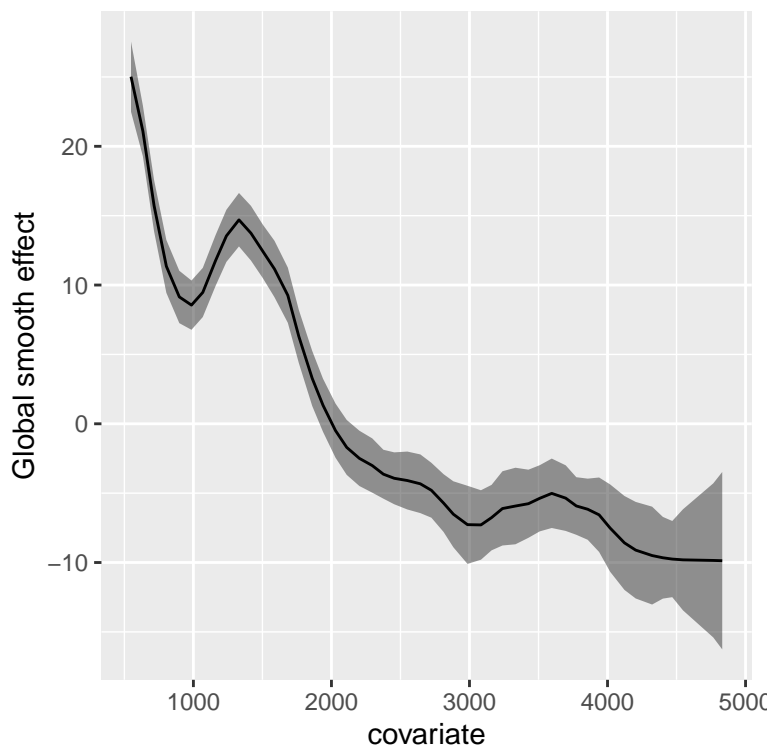
[Click here to see the solution](#)

```
cmp_rw2 = ~ -1+ beta_0(1) +
  smooth_g(depth_grouped, model = "rw2") +
  month_reff(Month_id, model = "iid")

lik = bru_obs(formula = Sources ~.,
              family = "gaussian",
              data = icit)

fit_rw2 = bru(cmp_rw2, lik)

data.frame(fit_rw2$summary.random$smooth_g) %>%
  ggplot() +
  geom_ribbon(aes(ID, ymin = X0.025quant, ymax= X0.975quant), alpha = 0.5) +
  geom_line(aes(ID, mean)) +
  xlab("covariate") + ylab("Global smooth effect")
```



## 2.0.2 Fitting group-level smoothers

Here we fit a model where each month is allowed to have its own smoother for depth, i.e.,  $f_i(\text{Depth})_s$ . The model structure is given by:

$$S_{is} = \beta_0 + f_i(\text{Depth})_s + \text{Month}_i + \epsilon_{is}.$$

Notice the only different between the global smoother model (Model G) and the group level model (Model GS) is the indexing of the smooth function for depth. We can fit a group-level smoother using the group argument within the model component as follows:

```
cmp_gs = ~ -1+ beta_0(1) +
  smooth_g(SampleDepth, model = "rw1") +
  month_reff(Month_id, model = "iid")+
  smooth_loc(SampleDepth, model = "rw1", group = Month_id)
```

Then, we simply run the model (since the observational model has not changed -only the model components have):

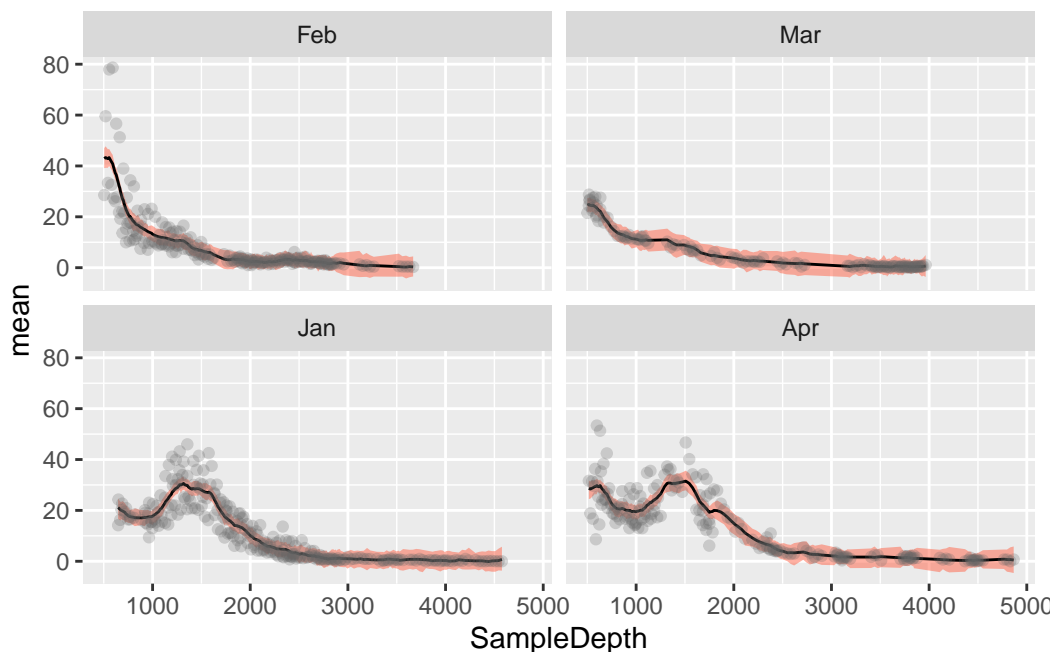
```
fit_gs = bru(cmp_gs, lik)
```

Lastly, we can generate model predictions using the predict function.

```
pred_gs = predict(fit_gs, icit, ~ (beta_0 + smooth_g+month_reff+smooth_loc))
```

Then, we plot the predicted mean values with their corresponding 95% Crls.

```
ggplot(pred_gs, aes(y=mean, x=SampleDepth))+
  geom_ribbon(aes(SampleDepth, ymin = q0.025, ymax= q0.975), alpha = 0.5, fill="tomato") +
  geom_line()+
  geom_point(aes(x=SampleDepth, y=Sources ), alpha=0.25, col="grey40")+
  facet_wrap(~Month)
```



#### Task

Re-fit the model GS without the global smoother. By omitting the global smoother, we do not longer force group-level smooths to follow a shared pattern, which is useful when groups may differ substantially from a common trend.

Take hint

You only need to modify the model components in `cmp_gs`

Add hint details here...

[Click here to see the solution](#)

```
cmp_s = ~ -1+ beta_0(1) +
  month_reff(Month_id, model = "iid")+
  smooth_loc(SampleDepth, model = "rw1", group = Month_id)

fit_s = bru(cmp_s, lik)

pred_s = predict(fit_s, icit, ~ (beta_0 +month_reff+smooth_loc))

ggplot(pred_s,aes(y=mean,x=SampleDepth))+
  geom_ribbon(aes(SampleDepth,ymin = q0.025, ymax= q0.975), alpha = 0.5,fill="tomato") +
  geom_line()+
  geom_point(aes(x=SampleDepth,y=Sources ),alpha=0.25,col="grey40")+
  facet_wrap(~Month)
```

