

# Lecture 4

## Continuous and discrete-continuous decision problems

2021 Econometric Society Summer School in  
Dynamic Structural Econometrics

Fedor Iskhakov, Australian National University

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# Discrete and continuous choice?

In economics discrete and continuous choice co-exist

- How much to work + when to retire/become an entrepreneur
- How much to save + when to buy a house/car/durables
- Which car to buy + how much to drive

Often modeled separately using traditional solution methods:

- Discrete choice → optimization over finite set
- Continuous choice → first order conditions + concavity(?)
- Dynamic → dynamic programming (VFI, policy, time iterations)

# Discrete and continuous choice?

In discrete-continuous choice models:

- Intrinsic non-concavity
- First order conditions not sufficient
- Kinks in value functions + discontinuities in policy functions

Traditional methods are not ideal

- Need global optimizer in each point of the state space
- Need to locate and keep track of kinks and discontinuities
- Need special numerical procedures for non-smooth objects

⇒ **Endogenous grid point methods**

# Plan for the lecture

- 1 Original EGM for continuous choice **only**  
*Only for particular (yet interesting and important) models*  
*(stochastic growth models, consumption-savings (buffer stock) models)*
- 2 DC-EGM for discrete-continuous choice **without taste shocks**  
*For models with one continuous and additional discrete choices*  
*Nasty and scary*
- 3 DC-EGM for discrete-continuous choice **with taste shocks**  
*For models with one continuous and additional discrete choices*  
*Structural taste shocks or logit smoothing*  
*Much better, possible to work with*
- 4 Some words on multi-dimensional extensions and occasionally binding constraints

# What is EGM?

The **M**ethod of **E**ndogenous **G**ridpoints — fast method for solving dynamic stochastic consumption/savings problems

- 1 finite and infinite horizon
- 2 Strictly concave monotone and differentiable utility function
- 3 one continuous state variable (*wealth*) and one continuous choice (*consumption*)
- 4 particular structure of the law of motion for state variables (*intertemporal budget constraint*)
- 5 very well accommodate potentially binding borrowing constraints

# DC-EGM for Discrete-Continuous problems

Expand the class of problems to be solved:

- ① **A1.** Strictly concave monotone and differentiable utility function
- ② Continuous state  $M_t$  with a particular motion rule
- ③ Additional (discrete) state variables  $st_t$   
**A2.** Transition probabilities of  $st_t$  are independent of  $M_t$
- ④ One continuous ( $c_t$ ) and one\* discrete choice variable  $d_t$

Two flavors:

- ① **Without taste shocks:** DC-EGM iterates on value function and policy function, produces exact solutions for the optimal thresholds for discrete decisions (discrete policy)
- ② **With taste shocks:** DC-EGM iterates on **discrete choice specific** value and policy functions, produces choice probabilities for discrete alternatives

# Learning outcomes = points to remember

- 1 If your model has **one continuous** (consumption) choice and **additional discrete choices** → **Use DC-EGM**
- 2 In regular cases DC-EGM **avoids all root-finding** operations
- 3 If utility is separable in continuous and discrete choices, DC-EGM deals very easily with credit constraints
- 4 Extreme value taste shocks → solution is **much better behaved**
- 5 **Faster and more accurate than traditional approaches**

# EGM



# Simple consumption/savings model (Phelps)

$$V_t(M_t) = \max_{0 \leq c \leq M_t} \left[ u(c) + \beta EV_{t+1} \left( \tilde{R}(M_t - c) \right) \right]$$

$M_t$  cash-in-hand, all resources available at period  $t$   
 $A_t = M_t - c_t$  assets at the end of period  $t$  (savings)  
 $\tilde{R}$  *deterministic* or *stochastic* return on savings  
 $u(c)$  utility of current consumption

$$u(c) = \frac{c^{\rho} - 1}{\rho} \xrightarrow{\rho \rightarrow 0} \log(c)$$

# Analytic solution (Hakansson, 1970, Phelps, 1962)

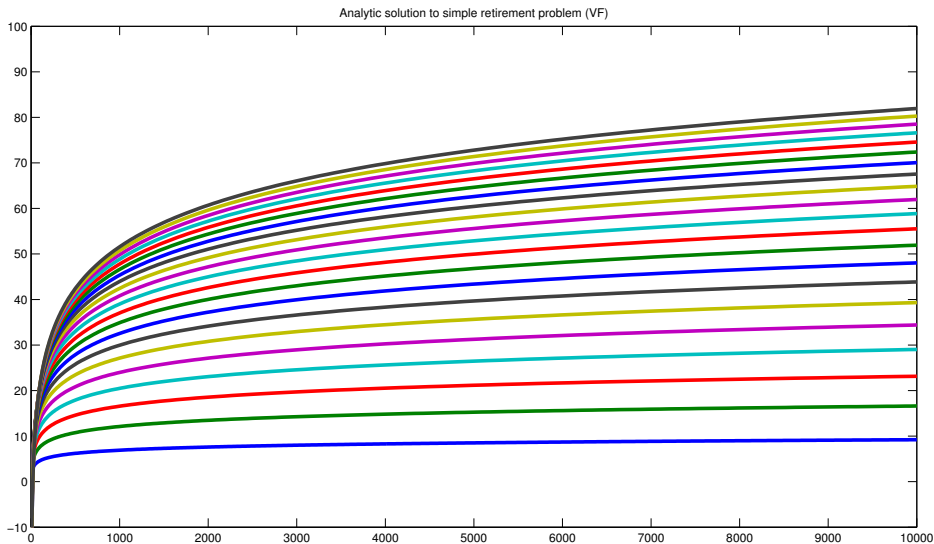
$$V_{T-t}(M) = \left[ \frac{M^\rho}{\rho} \right] \left( \sum_{i=0}^t K^i \right)^{(1-\rho)} - \frac{1}{\rho} \left( \sum_{i=0}^t \beta^i \right)$$

$$V_{T-t}(M) \xrightarrow{\rho \rightarrow 0} \log(M) \left( \sum_{i=0}^t \beta^i \right) + K_t$$

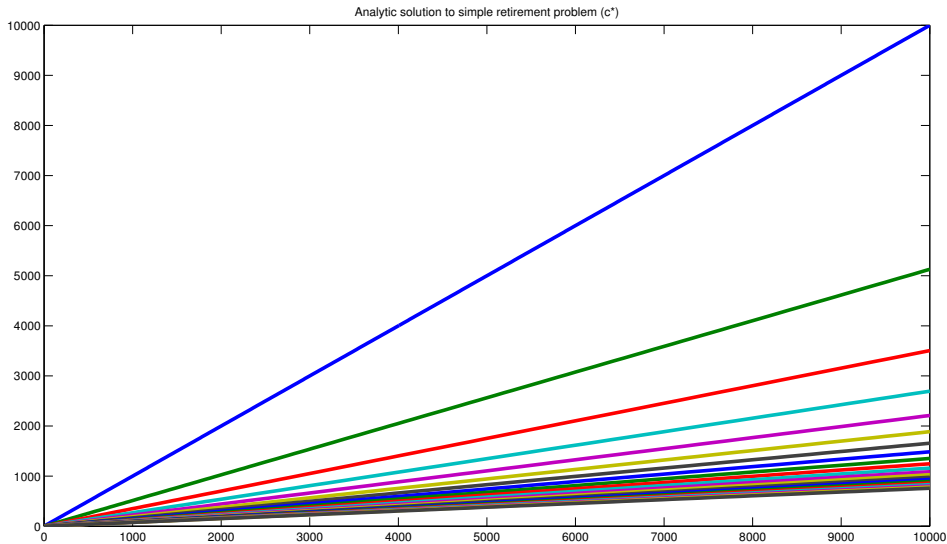
$$c_{T-t}(M) = M \left( \sum_{i=0}^t K^i \right)^{-1}$$

$K$  and  $K_t$  are functions of primitives,  $K \xrightarrow{\rho \rightarrow 0} \beta$

# Analytic solution : value functions



# Analytic solution : consumption rule



## Simple consumption/savings model (Deaton)

$$V_t(M_t) = \max_{0 \leq c \leq M_t} [u(c) + \beta EV_{t+1}(R(M_t - c) + \tilde{y})]$$

$M_t$	cash-in-hand, all resources available at period $t$
$A_t = M_t - c_t$	assets at the end of period $t$ (savings)
$R$	<i>deterministic</i> return on savings
$\tilde{y}$	<i>stochastic</i> income
$u(c)$	utility of current consumption

$$u(c) = \frac{c^\rho - 1}{\rho} \xrightarrow{\rho \rightarrow 0} \log(c)$$

No analytical solution!

# Traditional approach : value function iterations

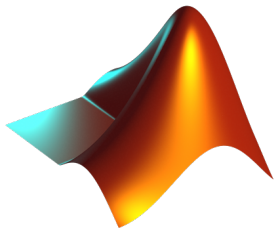
- 1 *Fix grid over  $M_t$ .* For every point on this grid:
- 2 In the terminal period calculate  

$$V_T(M_T) = \max_{0 \leq c_T \leq M_T} \{u(c_T)\}$$
 and  

$$c_T^* = \text{argmax}_{0 \leq c_T \leq M_T} \{u(c_T)\}$$
- 3 With  $t + 1$  value function at hand, proceed backward to period  $t$  and calculate  

$$V_t(M_t) = \max_{0 \leq c_t \leq M_t} \left\{ u(c_t) + \beta EV_{t+1} \left( \tilde{R}(M_t - c_t) \right) \right\}$$
 and  

$$c_t^* = \text{argmax}_{0 \leq c_t \leq M_t} \left\{ u(c_t) + \beta EV_{t+1} \left( \tilde{R}(M_t - c_t) \right) \right\}$$
 using Bellman equation



- 1 Introduction to the code
- 2 Phelps and Deaton models
- 3 Run VFI solver

# Euler equation

$$\text{Bellman equation: } V_t(M_t) = \max_{0 \leq c_t \leq M_t} \left[ u(c_t) + \beta E V_{t+1} \left( \tilde{R}(M_t - c_t) \right) \right]$$

$$\text{F.O.C. for Bellman equation: } u'(c_t) = \beta E \left[ \frac{\partial V_{t+1}(M_{t+1})}{\partial M_{t+1}} \tilde{R} \right]$$

Envelope theorem:

$$\begin{aligned} \frac{\partial V_t(M_t)}{\partial M_t} &= \beta E \left[ \frac{\partial V_{t+1}(M_{t+1})}{\partial M_{t+1}} \tilde{R} \right] \Rightarrow \frac{\partial V_t(M_t)}{\partial M_t} = u'(c_t) \Rightarrow \\ \Rightarrow \frac{\partial V_{t+1}(M_{t+1})}{\partial M_{t+1}} &= u'(c_{t+1}) \end{aligned}$$

$$\text{Euler equation to characterize the interior solutions: } u'(c_t) = \beta E \left[ u'(c_{t+1}) \tilde{R} \right]$$

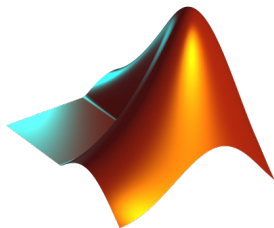


# Traditional approach : solving Euler equation

- 1 *Fix grid over  $M_t$ . For every point on this grid:*
- 2 In the terminal period calculate  

$$c_T^* = \text{argmax}_{0 \leq c_T \leq M_T} \{u(c_T)\}$$
- 3 With  $t + 1$  optimal consumption rule  $c_{t+1}^*(M_{t+1})$  at hand, proceed backward to period  $t$  and calculate  $c_t$  from *equation*  

$$u'(c_t) = \beta E \left[ u' \left( c_{t+1}^* \left( \tilde{R}(M_t - c_t) \right) \right) \tilde{R} \right]$$
 to recover  $c_t^*(M_t)$
- 4 When  $M_t$  is small enough so credit constraint binds, the Euler equation does not hold, and special provisions are necessary



### Exercise 1:

- 1 Code up the Euler equation solver
- 2 Verify solution against VFI solver

# What if no root-finding is necessary?

## With numerical optimization

- Relatively slow: iterative numerical optimization in each point of state space!
- Hard to find global optimum in non-convex problems
- Loss of accuracy due to the absence of the point where credit constraint starts to bind on the fixed grid

Even when using state-of-the-art solvers!

## Without numerical optimization

- Much faster: no iterative methods in each point of the state space
- More accurate: using analytical structure of the problem

# Endogenous gridpoint method (EGM)



Carroll 2006 *Economics Letters*

The method of endogenous gridpoints for solving dynamic stochastic optimization problems.

## Idea

- Instead of searching for optimal decision in each point of the state space (traditional approaches)
- Look for the state variable (level of assets) where arbitrary chosen decision (consumption  $\rightarrow$  savings) would be optimal (EGM)

# EGM algorithm

Start with  $c_T^* = M_T$ . In each period  $t = T, T - 1, \dots, 1$ :

## EGM step

- 1 Take a guess  $A$  = current period savings ( $= M_t - c_t$ )  
(from fixed or adaptive list/grid)
- 2 Intertemporal budget constraint:  $A \rightarrow M_{t+1}$   
 $M_{t+1} = \tilde{R}(M_t - c_t) = \tilde{R} \cdot A$
- 3 Policy function at period  $t + 1$ :  $M_{t+1} \rightarrow c_{t+1}$   
 $c_{t+1} = c_{t+1}^*(M_{t+1})$
- 4 Inverted Euler equation:  $c_{t+1} \rightarrow c_t$   
 $c_t = (u')^{-1} \left( \beta E \left[ \tilde{R} \cdot u' (c_{t+1}^*(M_{t+1})) | A \right] \right)$
- 5 Intratemporal budget constraint:  $c_t + A = M_t \rightarrow c_t(M_t)$   
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# Matlab implementation (minimal.m)

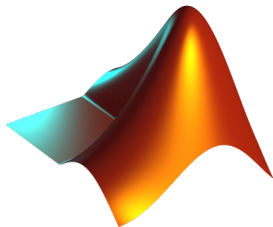
```

o [quadp quadw]=quadpoints(EXPN,0,1);
quadstnorm=norminv(quadp,0,1);
sgrid=linspace(0,MMAX,NM);
policy{TBAR}.w=[0 MMAX];
policy{TBAR}.c=[0 MMAX];
5 for it=TBAR-1:-1:1
    w1=Y+exp(quadstnorm*SIGMA)*(1+R)*sgrid;
    c1=interp1(policy{it+1}.w,policy{it+1}.c,w1,'linear';
    rhs=quadw.*(1./c1);
    policy{it}.c=[0 1./(DF*(1+R)*rhs)];
10 policy{it}.w=[0 sgrid+policy{it}.c(2:end)];
end

```

# Accuracy and speed of EGM

	Traditional Euler	EGM
Running time	37 sec.	0.11 sec.
Max abs error, $c_t^*$	5e-9	4e-14
Mean abs error, $c_t^*$	1.4e-12	1.5e-14
Max abs error, $V_t(M, \mathbb{R})$	39.466	15.163
Mean abs error, $V_t(M, \mathbb{R})$	2.5e-02	3.2e-02



- 1 Compare speed of VFI and EGM solvers
- 2 Look at the simulator code
- 3 Simulate flat consumption path using VFI and EGM solutions

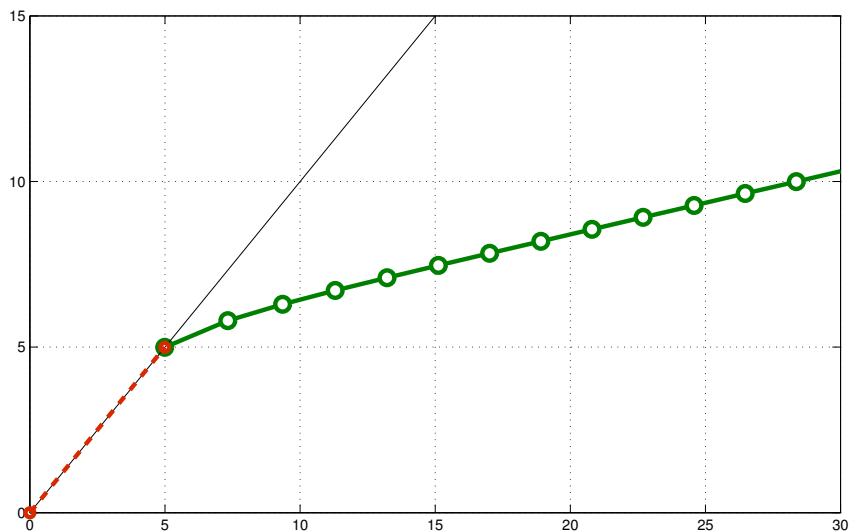
# EGM and credit constraint

## Theorem: Monotonicity of savings

Monotone and concave utility function  $\Rightarrow$   
end-of-period assets  $A_t = M_t - c_t$  are non-decreasing in  $M_t$

- With  $A = 0$  the EGM loop recovers the value of cash-in-hand  $M_t^{cc}$  that bounds the credit constrained region
- For all  $M_t < M_t^{cc}$  credit constrained binds  $\Rightarrow c_t = M_t$
- Consumption rule in the credit constrained region is 45° line between  $(0, 0)$  and  $(M_t^{cc}, M_t^{cc})$
- As simple as “connect the dots”  $(0, 0)$  and  $(M_t^{cc}, M_t^{cc})$

# EGM and credit constraint



# Credit constraints and value function

- Credit constraints are handled so well by EGM because it is never necessary to compute utility of nearly zero consumption
- Inevitable when value functions have to be computed, but..

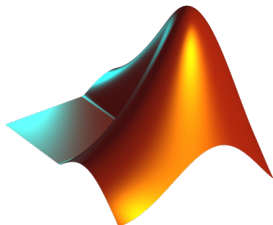
$$M_t < M_t^{cc}$$

$$V_t(M) = u(M) + \beta EV_{t+1}(0)$$

$EV_{t+1}(0)$  — expected value of ending period  $t$  with  $A_t = 0$

- Value function has analytic form for  $M_t < M_t^{cc}$ !





- 1 How to evaluate value function with EGM

## Exercise 2:

- 1 Code up an EGM solver for infinite horizon problem

# DC-EGM

# Generalization of EGM



Iskhakov, Jørgensen, Rust, Schjerning, QE forthcoming  
The Endogenous Grid Method for Discrete-Continuous Dynamic Choice Models with (or without) Taste Shocks

- The DC-EGM paper
- Two flavors: with and without EV taste shocks
- Solution method made for empirical applications



Giulio Fella, RED 2014  
A Generalized Endogenous Grid Method for Non-Smooth and Non-Concave Problems

- Identify the regions of the problem where Euler equation is not sufficient for optimality
- Use global optimization methods inside (VFI) and EGM outside
- Similar to DC-EGM without taste shocks

# Simple retirement model

$$V_t(M_t, \mathbb{W}) = \max \left\{ \begin{array}{l} \max_{0 \leq c \leq M_t} u(c, \mathbb{R}) + \beta EV_{t+1} \left( \tilde{R}(M_t - c), \mathbb{R} \right) \\ \max_{0 \leq c \leq M_t} u(c, \mathbb{W}) + \beta EV_{t+1} \left( \tilde{R}(M_t + y - c), \mathbb{W} \right) \end{array} \right\}$$

$$V_t(M_t, \mathbb{R}) = \max_{0 \leq c \leq M_t} \left[ u(c, \mathbb{R}) + \beta EV_{t+1} \left( \tilde{R}(M_t - c), \mathbb{R} \right) \right]$$

- $\mathbb{R}, \mathbb{W}$  retirement and working **states**  $st_t$  that evolve according to **discrete choices**  $d \in \{\mathbb{R}, \mathbb{W}\}$   
 $y$  deterministic wage income

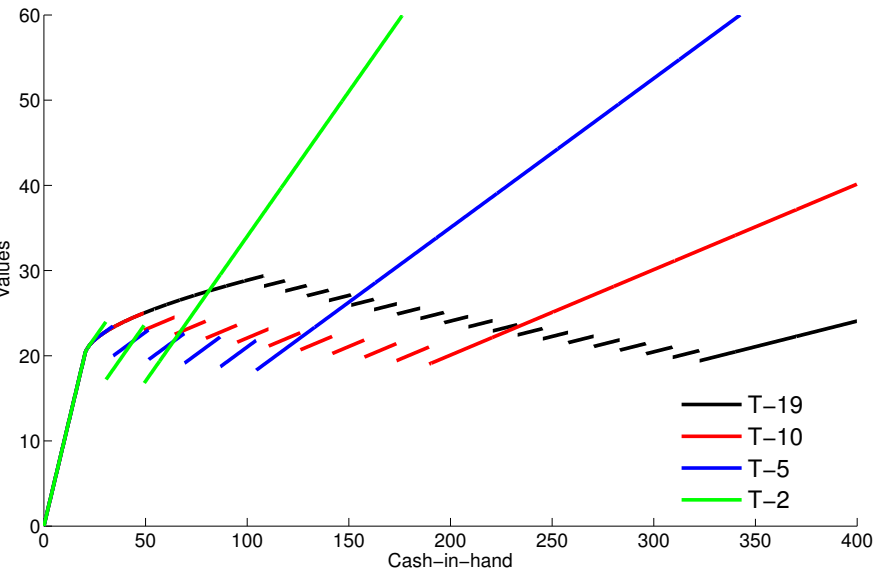
$$u(c, d) = \frac{c^\rho - 1}{\rho} - 1(d = \mathbb{W}) \xrightarrow{\rho \rightarrow 0} \log(c) - 1(d = \mathbb{W})$$

# Analytic solution

$$u(c) = \log(c), R = 1 \Rightarrow c_{T-t}^*(M, \mathbb{W}) =$$

$$\left\{ \begin{array}{ll} M & \text{if } M \leq y/\beta \\ (y + M)/(1 + \beta) & \text{if } y/\beta \leq M \leq \overline{M}_{T-t}^1 \\ (2y + M)/(1 + \beta + \beta^2) & \text{if } \overline{M}_{T-t}^1 \leq M \leq \overline{M}_{T-t}^2 \\ \dots & \dots \\ ((t-1)y + M) \left( \sum_{i=0}^{t-1} \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{l_{t-2}} \leq M \leq \overline{M}_{T-t}^{l_{t-1}} \\ (ty + M) \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{l_{t-1}} \leq M \leq \overline{M}_{T-t}^{r_1} \\ [(t-1)y + M] \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{r_1} \leq M \leq \overline{M}_{T-t}^{r_2} \\ \dots & \dots \\ (2y + M) \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{r_{t-2}} \leq M \leq \overline{M}_{T-t}^{r_{t-1}} \\ (y + M) \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{r_{t-1}} \leq M \leq \overline{M}_{T-t} \\ M \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t} < M \end{array} \right.$$

# Analytic solution



# How to approach discrete/continuous choice

The goal:

- Avoid root finding
- Keep efficient treatment of credit constraints

## DC-EGM ver. 1.0

- 1 EGM step for each discrete choice  $d$  and every state  $st$
- 2 Compute  $d$ -specific value functions and consumption rules
- 3 Compare the  $d$ -specific value functions to find optimal switching points (compute upper envelope)
- 4 Reconstruct overall consumption rule and value function from optimal switching points

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# How to approach discrete/continuous choice

## DC-EGM ver. 1.0

- ➊ EGM step for each discrete choice  $d$  and every state  $st$
  - ➋ Compute  $d$ -specific value functions and consumption rules
  - ➌ Compare the  $d$ -specific value functions to find optimal switching points (compute upper envelope)
  - ➍ Reconstruct overall consumption rule and value function from optimal switching points
- No root finding!
  - Efficient treatment of credit constraints (to be shown)
  - Need to compute value functions
  - Need to compute upper envelope

# Is Euler equation still a necessary condition?

## DC-EGM ver. 1.0

- 1 EGM step for each discrete choice  $d$  and every state  $st$
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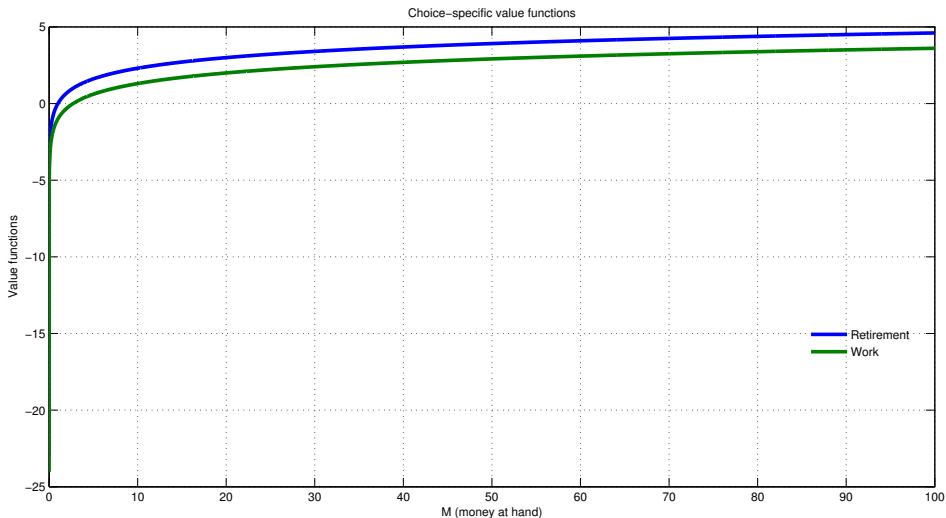


Clausen & Strub, 2010-2013

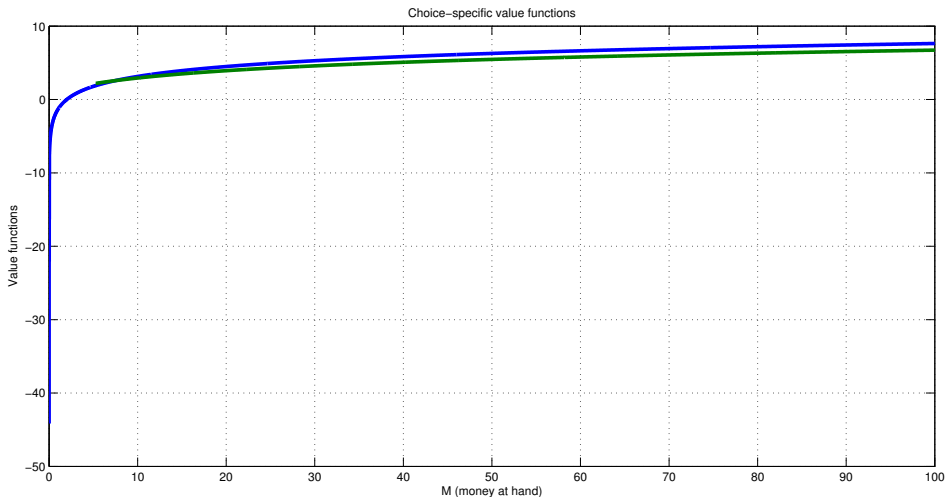
A General and Intuitive Envelope Theorem.

Show that Euler equation remains a necessary condition for the optimal continuous consumption.

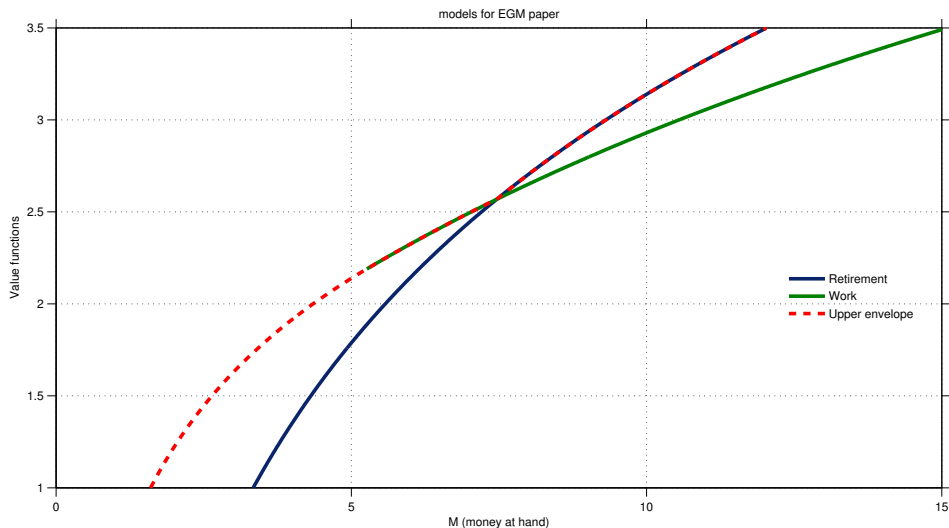
# Period $T$ : choice specific value functions



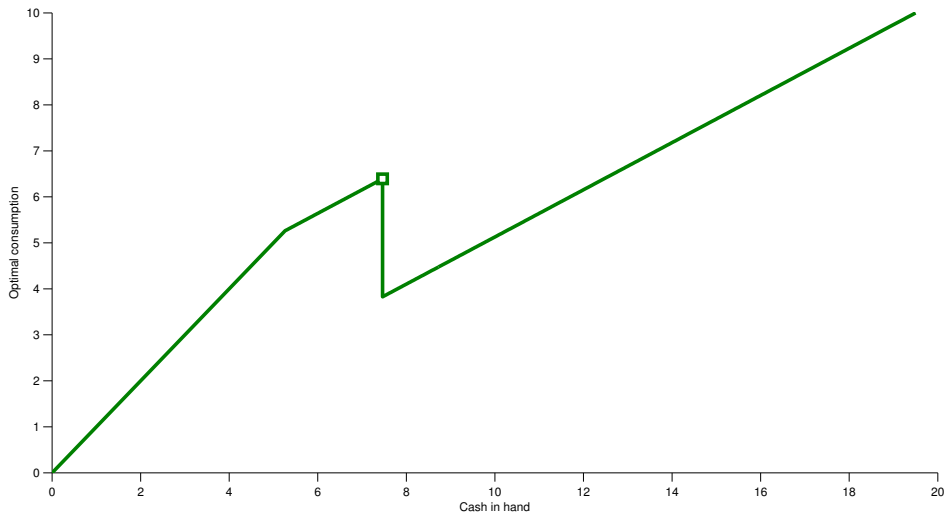
# Period $T - 1$ : Choice specific VF



# Period $T - 1$ : Choice specific VF



# Period $T - 1$ : Optimal consumption

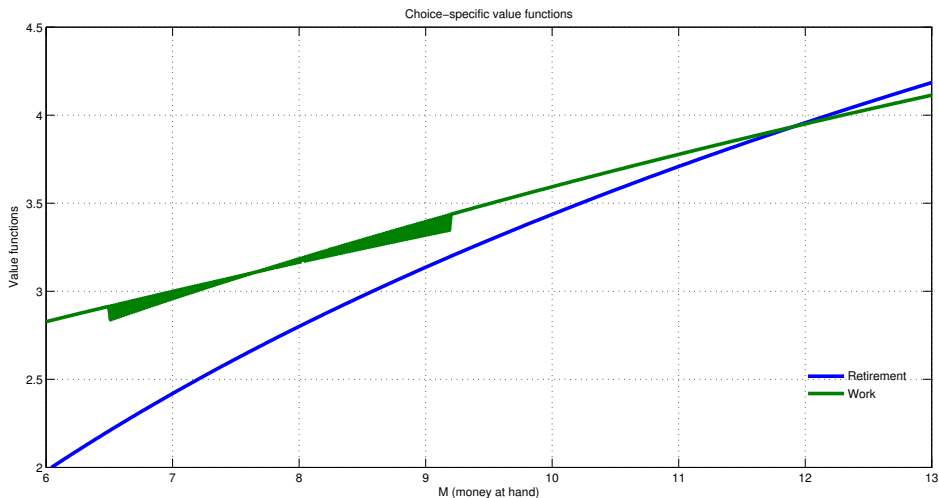




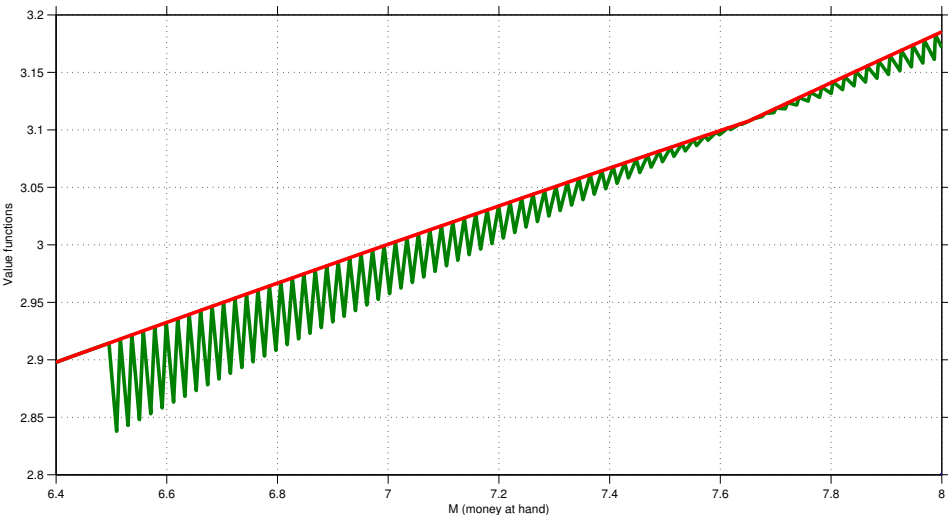
# So, what is going on

- 1  $d$ -specific value functions intersect  
(due to trade-off between income and disutility of work)  
↓
- 2 The **upper envelope** of the value functions has a kink  
and combined consumption function has a discontinuity

# Period $T - 2$ : Choice specific VF



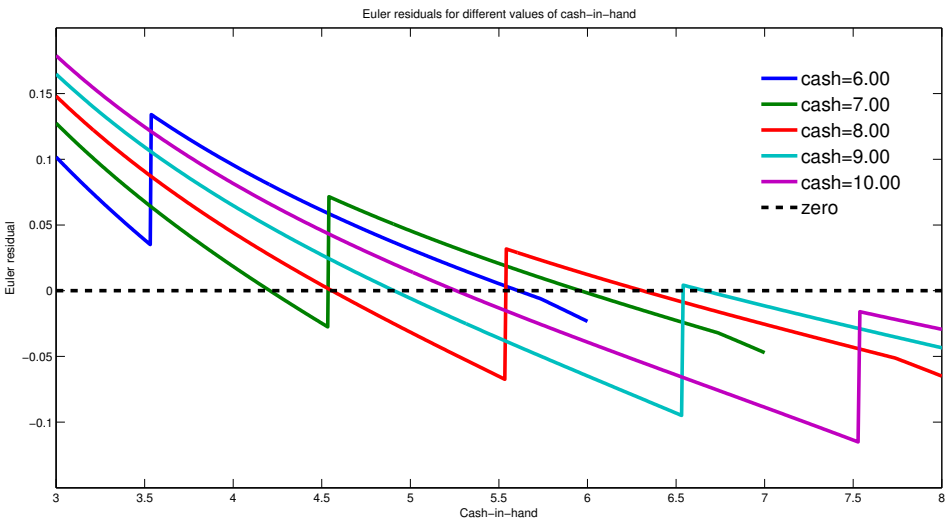
## Period $T - 2$ : Secondary upper envelope



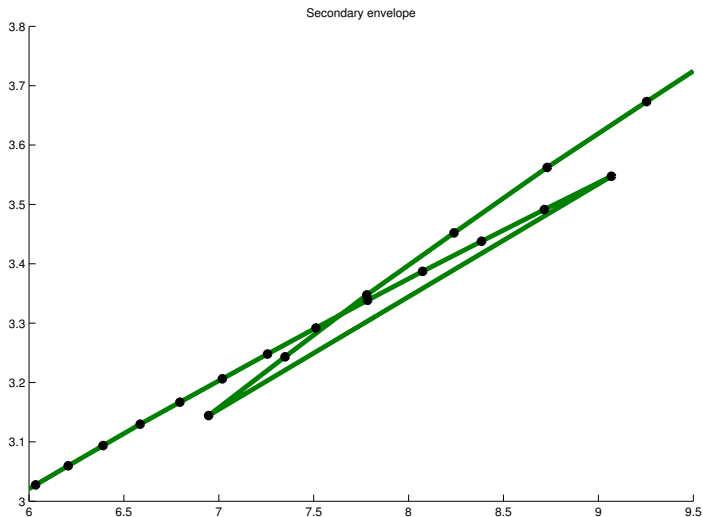
# So, what is going on

- ①  $d$ -specific value functions intersect  
(due to trade-off between income and disutility of work)  
⇓
- ② The **upper envelope** of the value functions has a kink  
and combined consumption function has a discontinuity  
⇓
- ③ Derivative of the value function has a discontinuity  
at the kink  
⇓
- ④ For some values of wealth (on endogenous grid) Euler equation has  
two solutions!  
If endogenous grid points are sorted → **zigzag region**

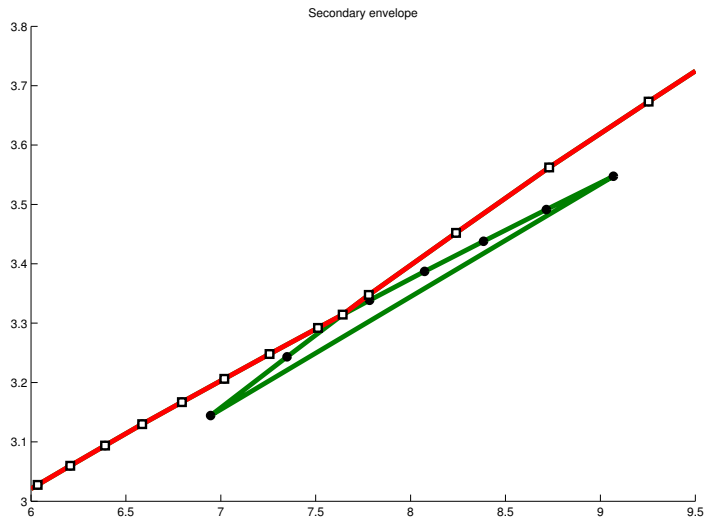
# Multiple zeros of Euler residuals



## Period $T - 2$ : Secondary upper envelope: detect



# Period $T - 2$ : Secondary upper envelope: result



# How to algorithmically detect “zigzag” regions?

## Theorem: monotonicity

Under weak regularity conditions on the utility function and intertemporal budget constraint, savings function is weakly increasing.

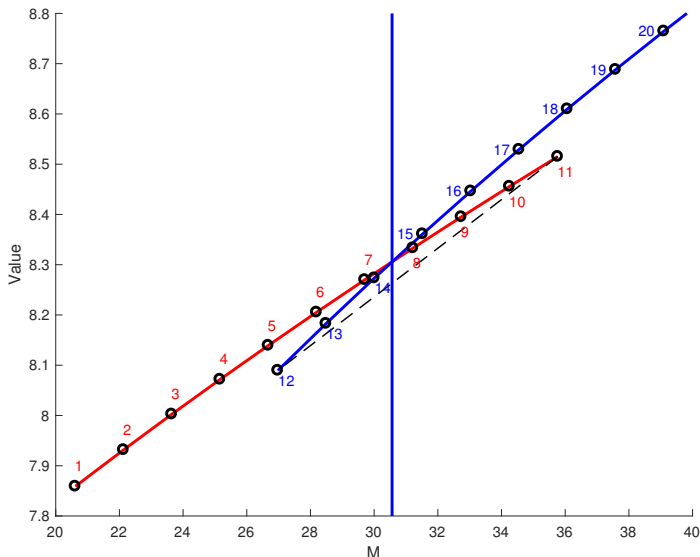
$A_t(M'_t) \geq A_t(M''_t)$  for every  $M'_t \geq M''_t$  for all  $t$ .

Note: savings function may still have “upward” jumps

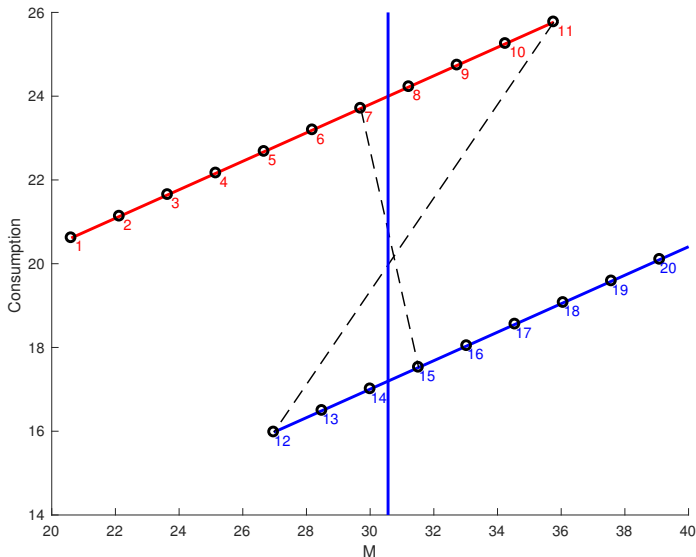
- 1 Sort the exogenous grid over  $A$  in **ascending order**
- 2 Then the sequence of endogenous grid points over  $M$  has to be in **ascending order as well** as long as Euler equation is sufficient
- 3 Every time the endogenous grid “**bends back**” the endogenous grid is separated into subsets of points
- 4 Calculate the **Upper envelope** on the segments over the subsets
- 5 **Delete suboptimal endogenous points**
- 6 Find and add a kink point to the endogenous grid



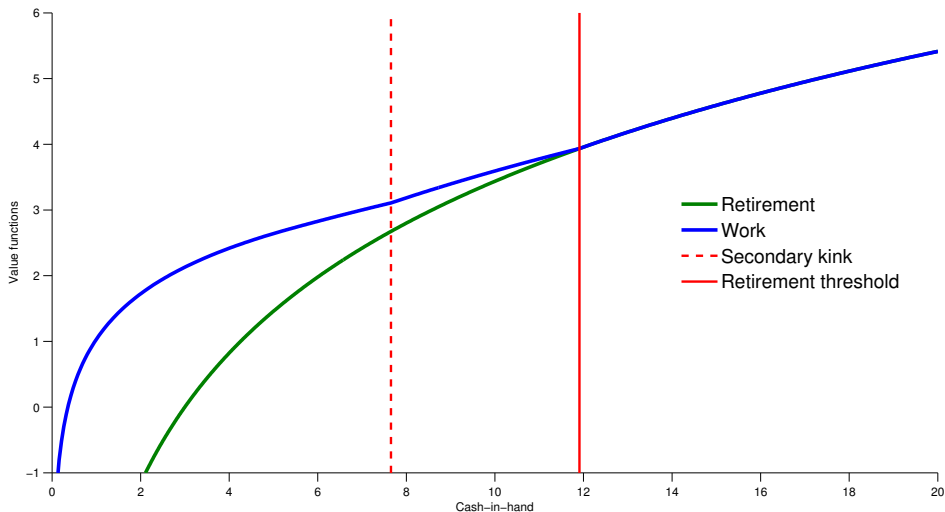
# What happens to optimal consumption?



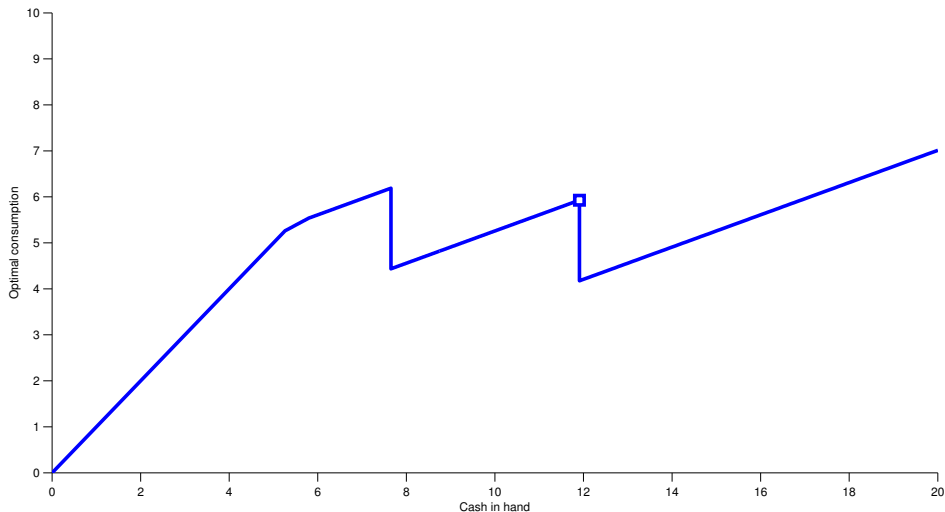
# What happens to optimal consumption?



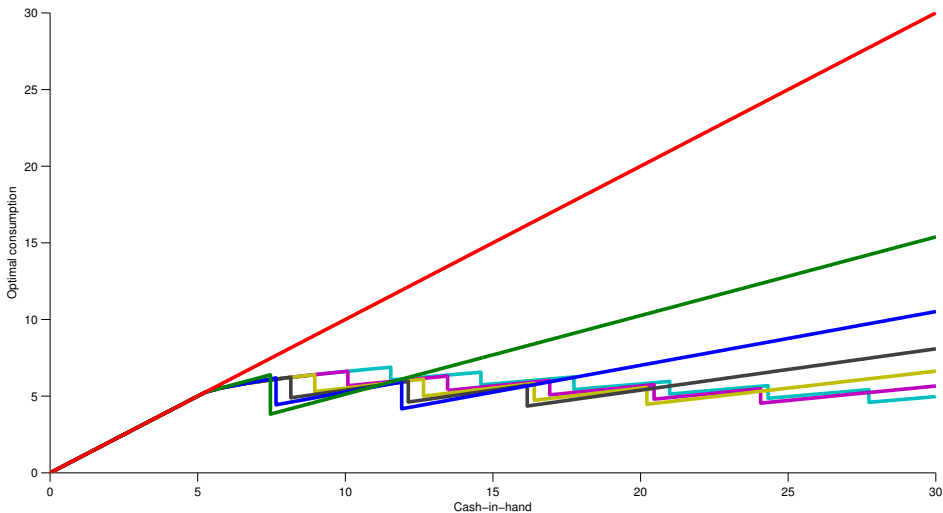
## Period $T - 2$ : VF, primary and secondary kinks



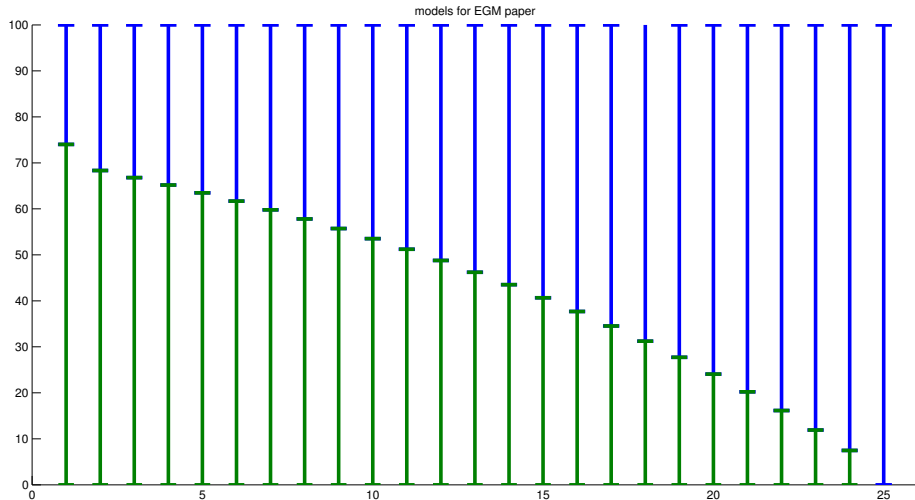
# Period $T - 2$ : Optimal consumption



# Optimal consumption (many periods)



# Optimal retirement (many periods)



# DC-EGM full algorithm

## DC-EGM ver. 2.0

- 1 Start from terminal period, compute optimal consumption rule and value function. Loop backwards over time:
- 2 EGM step for each discrete choice  $d$  and every state  $st$
- 3 Compute  $d$ -specific value functions and consumption rules
- 4 Compute the “secondary” upper envelope over the “zig-zag” regions of the  $d$ -specific value functions and update the corresponding consumption rules
- 5 Compare the  $d$ -specific value functions to find optimal switching points (compute upper envelope)
- 6 Reconstruct overall consumption rule and value function from optimal switching points

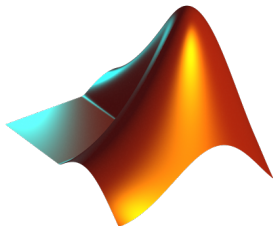
# Properties of the full solution

- 1 Value functions are non-concave and have **kinks**
- 2 Consumption functions have **discontinuities**
- 3 Discontinuities/kinks **propagate** through time and **accumulate**

This properties are attributes of the model itself.  
Any solution method has to deal with these complexities.

**DC-EGM matches the analytical solution perfectly!**





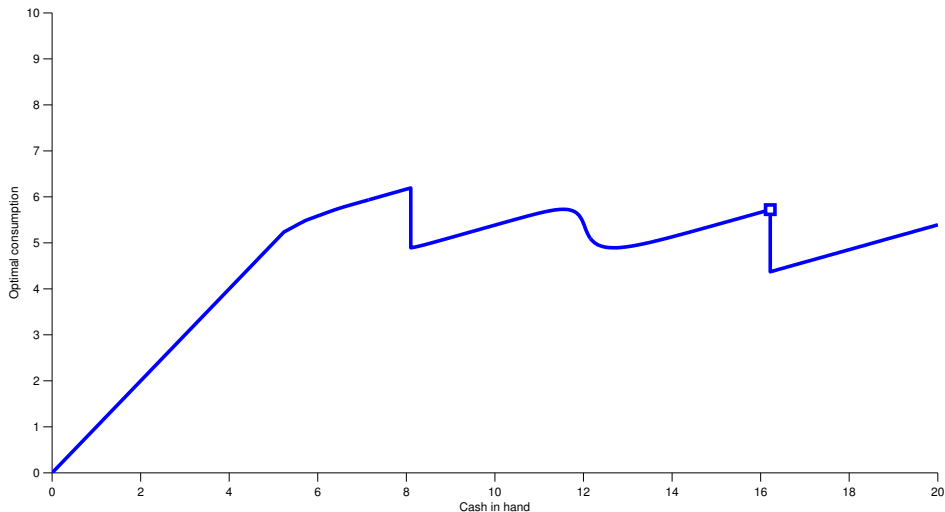
- 1 Replicate the solution using `model_retirement.m`
- 2 Simulate the consumption path for  $\beta R = 1$  and discuss the accuracy of the solutions

# Random returns $\tilde{R}$

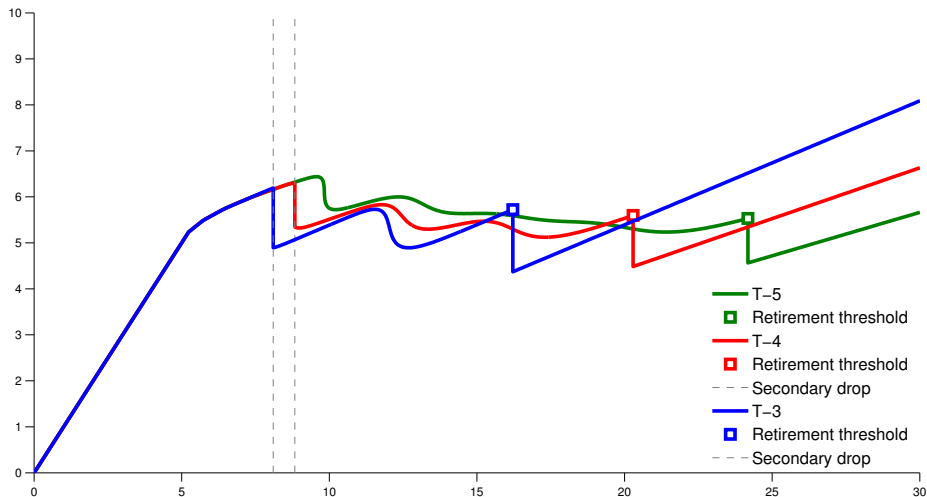
Random shocks do help, however:

- Smooth out **secondary** kinks **only**
- Primary kinks (switching between discrete options) **remain**
- May not smooth out all kinks: continuous but sharp declines in optimal consumption at  $t$  may lead to a discontinuity/kink at  $t - 1$
- Expectations in Euler equation have to be taken over discontinuous functions
  - More kinks/discontinuities from sloppy computation
  - Need to integrate over “continuous” intervals separately

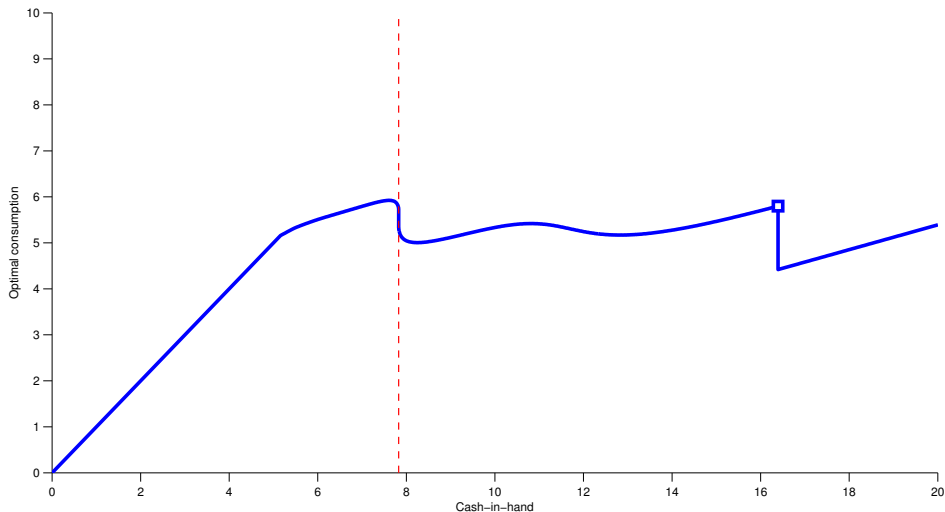
# Period $T - 3$ : Optimal consumption with $\sigma = .1$



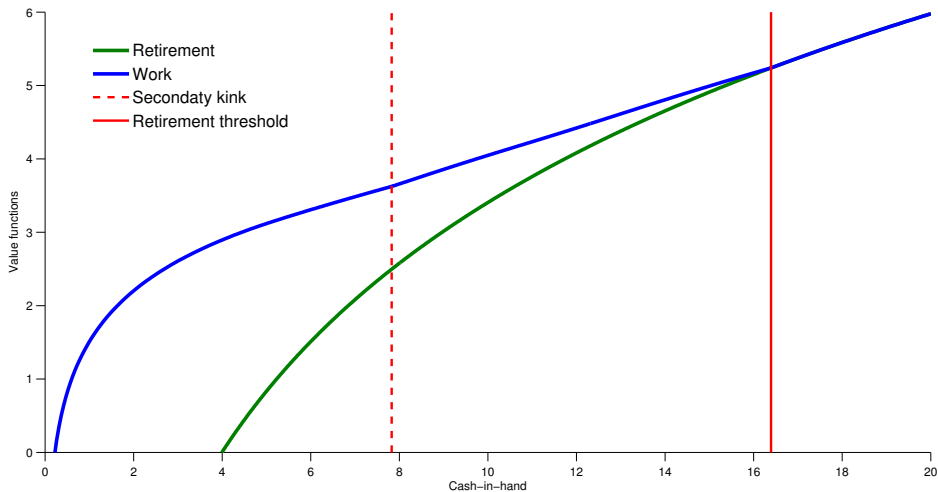
Before  $T - 3$  : Optimal consumption with  $\sigma = .1$



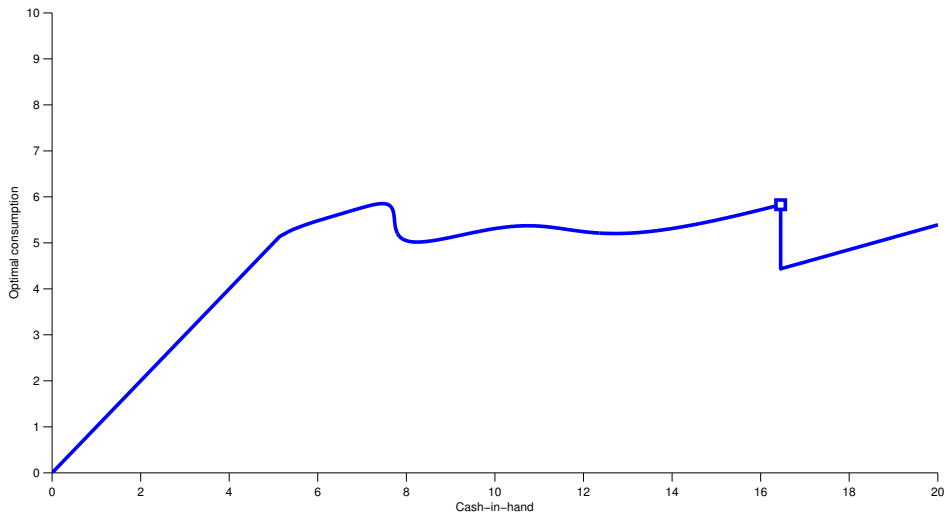
## Period $T - 3$ : Optimal consumption with $\sigma = .2$



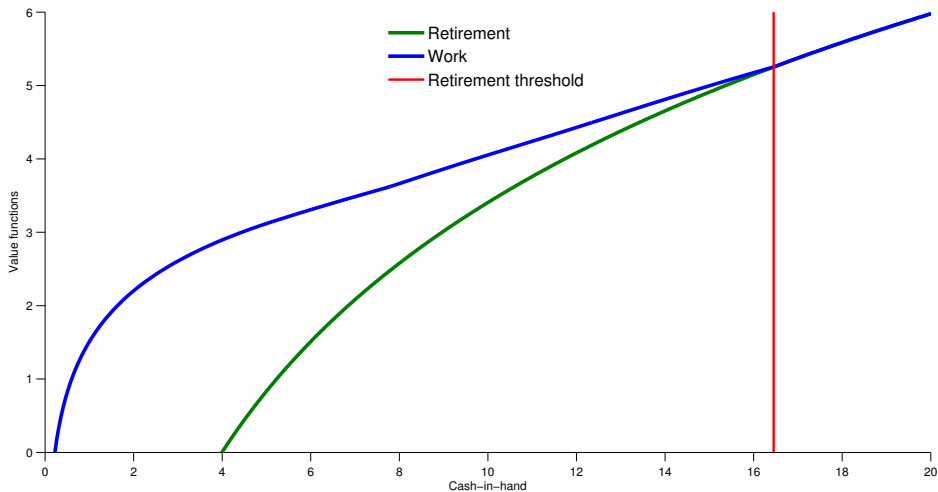
Period  $T - 3$  : VF with  $\sigma = .2$



## Period $T - 3$ : Optimal consumption with $\sigma = .22$



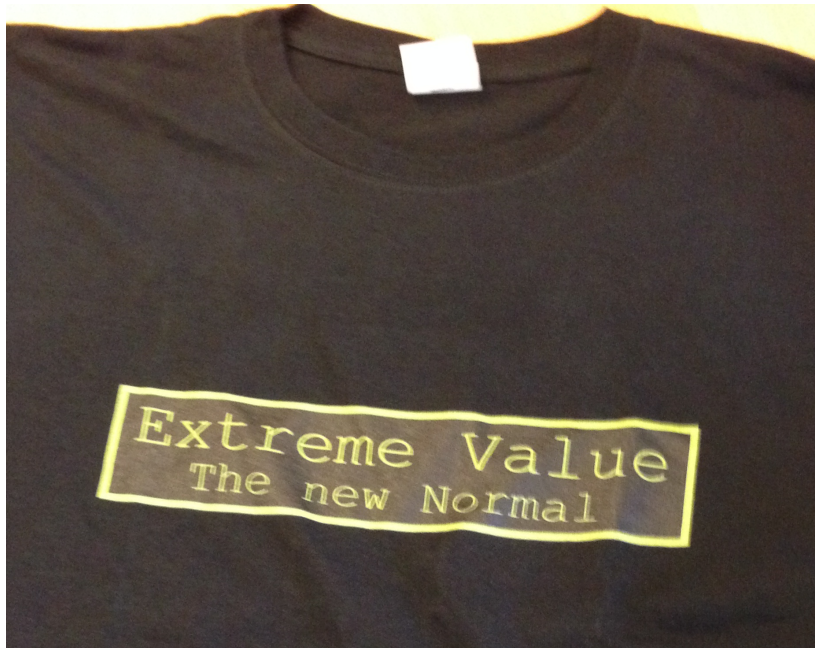
Period  $T - 3$  : VF with  $\sigma = .22$





# Extreme value distributed taste shocks

- Smooth out **primary kinks**
- Extreme value distribution – closed form expectations and standard in empirical applications
- Two interchangeable interpretations
  - Structural: unobserved state variables
  - Logit smoothing: to streamline the solution
- Work together with other shocks in the model
  - EV taste shocks smooth out primary kinks
  - Random returns smooth out secondary kinks
- Complete smoothing is not guaranteed in general: secondary kinks may persist



# Retirement problem with taste shocks

Re-formulate in terms of choice specific value functions

$$V_t(M_t, \mathbb{W}) = \max \left\{ \begin{array}{l} v_t(M_t, \mathbb{W}, \mathbb{R}) + \sigma \epsilon_{\mathbb{R}} \\ v_t(M_t, \mathbb{W}, \mathbb{W}) + \sigma \epsilon_{\mathbb{W}} \end{array} \right\}$$

$$v_t(M_t, \mathbb{W}, \mathbb{W}) = \max_{0 \leq c \leq M_t} \left[ u(c) + \beta EV_{t+1} \left( \tilde{R}(M_t + y - c), \mathbb{W} \right) \right]$$

$$v_t(M_t, \mathbb{W}, \mathbb{R}) = \max_{0 \leq c \leq M_t} \left[ u(c) + \beta EV_{t+1} \left( \tilde{R}(M_t - c), \mathbb{R} \right) \right]$$

$$EV_{t+1}(x, \mathbb{W}) = \sigma \log \left[ \exp \frac{v_{t+1}(x, \mathbb{W}, \mathbb{W})}{\sigma} + \exp \frac{v_{t+1}(x, \mathbb{W}, \mathbb{R})}{\sigma} \right]$$

$$V_t(M_t, \mathbb{R}) = \max_{0 \leq c \leq M_t} \left[ u(c) + \beta EV_{t+1} \left( \tilde{R}(M_t - c), \mathbb{R} \right) \right]$$

# Smoothed Euler equation

Without taste shocks – “discontinuous” Euler equation:

$$u'(c_t) = \beta E \left[ u'(c_{t+1}(\mathbb{W}/\mathbb{R})) \tilde{R} \right]$$

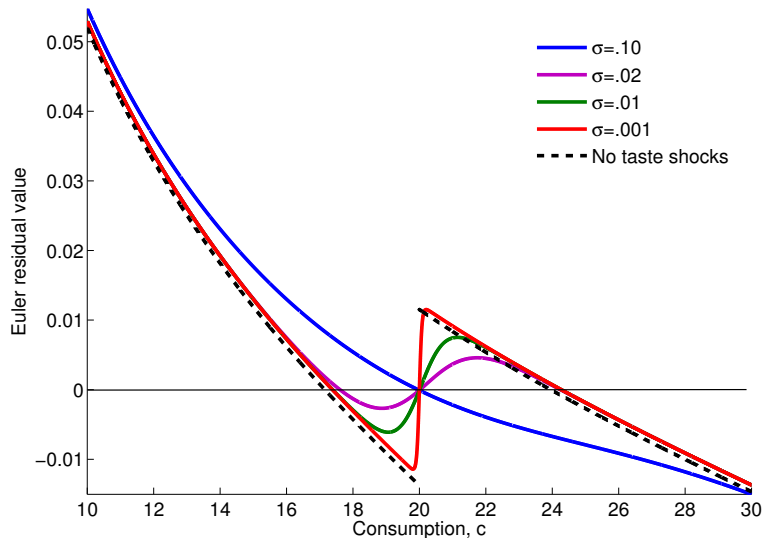
With EV taste shocks – smoothed Euler equation:

$$u'(c_t) = \beta E \left[ P_{t+1}(\mathbb{W}) u'(c_{t+1}(\mathbb{W})) \tilde{R} + P_{t+1}(\mathbb{R}) u'(c_{t+1}(\mathbb{R})) \tilde{R} \right]$$

Choice probability

$$P_{t+1}(\mathbb{W}) = \frac{\exp \frac{v_{t+1}(M_{t+1}, \mathbb{W}, \mathbb{W})}{\sigma}}{\exp \frac{v_{t+1}(M_{t+1}, \mathbb{W}, \mathbb{W})}{\sigma} + \exp \frac{v_{t+1}(M_{t+1}, \mathbb{W}, \mathbb{R})}{\sigma}}$$

# Smoothed Euler equation



# DC-EGM with taste shocks

## DC-EGM ver. 3.0

- 1 Start from terminal period, compute optimal consumption rule and value function. Loop backwards over time:
- 2 EGM step for each discrete choice  $d$  and every state  $st$
- 3 Compute  $d$ -specific value functions and consumption rules
- 4 Compute the “secondary” upper envelope over the “zig-zag” regions of the  $d$ -specific value functions and update the corresponding consumption rules
- 5 ~~Compare the  $d$ -specific value functions to find optimal switching points (compute upper envelope)~~
- 6 ~~Reconstruct overall consumption rule and value function from optimal switching points~~

# DC-EGM with taste shocks

- 1 With EV taste shocks DC-EGM becomes **simpler**
- 2 The problem is re-formulated in terms of **choice specific value functions**
- 3 Calculation of *primary* upper envelope is replaced by calculation of **logsum**
- 4 Easier computation of expectations (due to less discontinuities)
- 5 More memory is required to store choice specific value functions

# Extreme value Homotopy

## Theorem: approximation with logit smoother

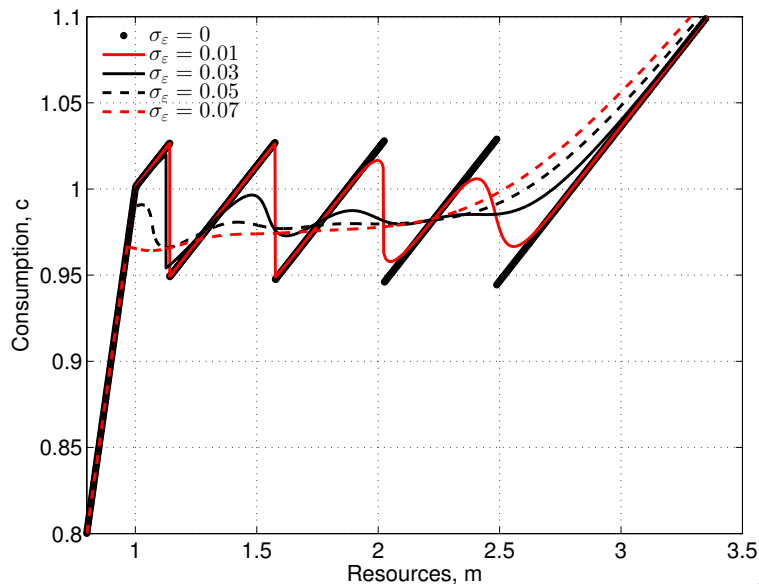
Let  $\sigma$  index scale of Type 1 extreme value taste shocks for the discrete choices in a DC-DP problem with  $D$  choices. Then we have the following bound

$$|EV_{\sigma,t}(s) - V_t(s)| \leq \sigma \left[ \sum_{j=0}^{T-t} \beta^j \right] \log(D)$$

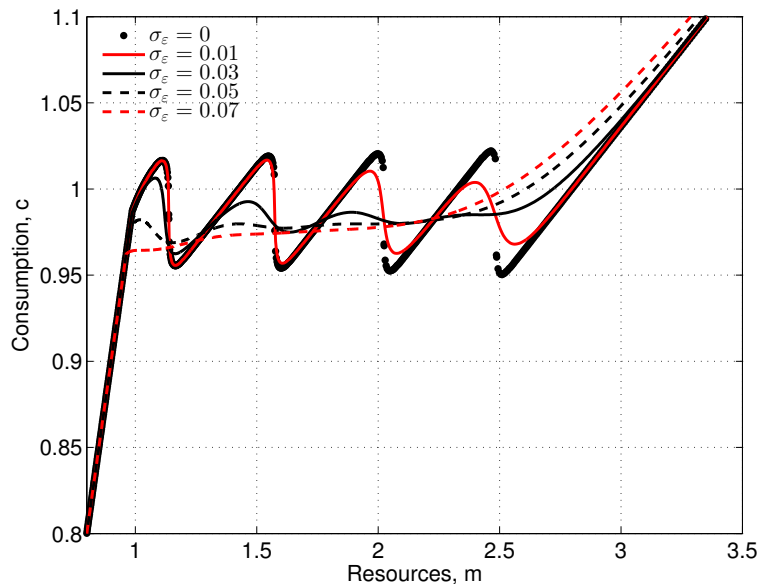
This implies that the extreme-value perturbed policy functions  $c_{\sigma,t}(s, \epsilon)$  and  $\delta_{\sigma,t}(s, \epsilon)$  converge pointwise to  $c_t(s)$  and  $\delta_t(s)$ , the optimal continuous and discrete decision rules to a DP problem without any taste shocks as  $\sigma \rightarrow 0$ .



# Optimal consumption with taste shocks only



# Optimal consumption with random returns



# Credit constraints

- Credit constraints are handled so well by EGM because it is never necessary to compute utility of nearly zero consumption
- Instead we “connect the dots”  $(0, 0)$  and  $(M_t^{cc}, M_t^{cc})$

$M_t^{cc}$  — level of wealth corresponding to  $A_t = 0$

- Inevitable when value functions have to be computed
- If utility is additively separable in consumption and discrete choices (AS), the problem can be avoided entirely!

# Credit constraints

## Dealing with credit constraints

- 1 For each  $d_t$  compute  $M_{t,d_t}^{cc}$  correspond to zero savings  
EGM loop can be started from  $A = 0$

$$M_{t,d_t}^{cc} : \forall M < M_{t,d_t}^{cc} \quad c_t^* = M$$

- 2 Value function for  $M < M_{t,d_t}^{cc}$  has analytic form

$$V_t^{d_t}(M) = u(M, d_t) + \beta EV_{t+1}^0(d_t)$$

$EV_{t+1}^0(d_t)$  — expected value of ending period  $t$  with  $A_t = 0$

- 3 (AS)  $\Rightarrow V_t^{d_t}(M) = u(M) + v(d_t) + \beta EV_{t+1}^0(d_t)$

- 4  $V_t^{d_t}(M)$  do not intersect when  $M < \min_{d_t} \{M_{t,d_t}^{cc}\}$   
 $\Rightarrow$  No need to compute utility of nearly zero consumption

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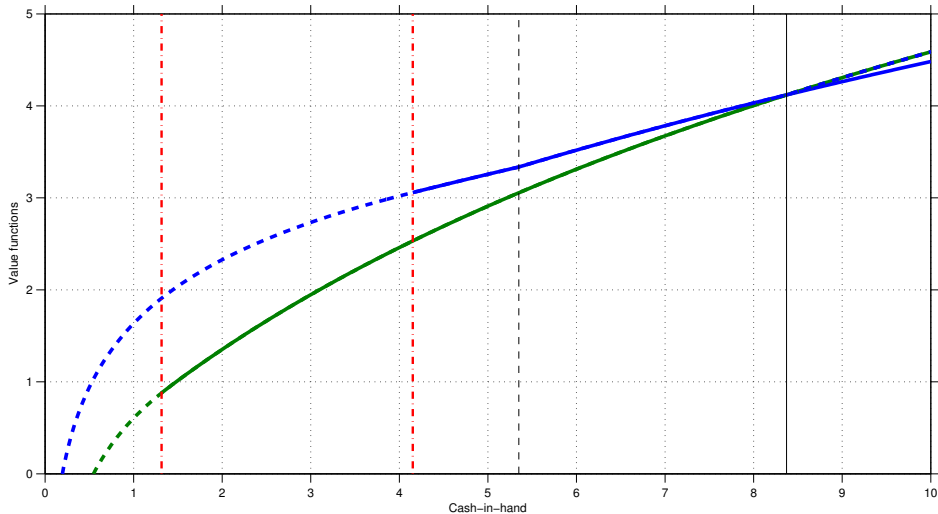
$$V_t^{d_t}(M) = u(M, d_t) + \beta EV_{t+1}^0(d_t)$$

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 $\Rightarrow$  No need to compute utility of nearly zero consumption

# Pension benefit .25y





# Multi-dimensional generalizations

# EGM + VFI



Barillas & Fernandez-Villaverde, JEDC 2007

A Generalization of the Endogenous Grid Method

- 1 Run EGM w.r.t. *one* choice keeping other controls fixed
- 2 Perform a VFI w.r.t. the rest of decision variables

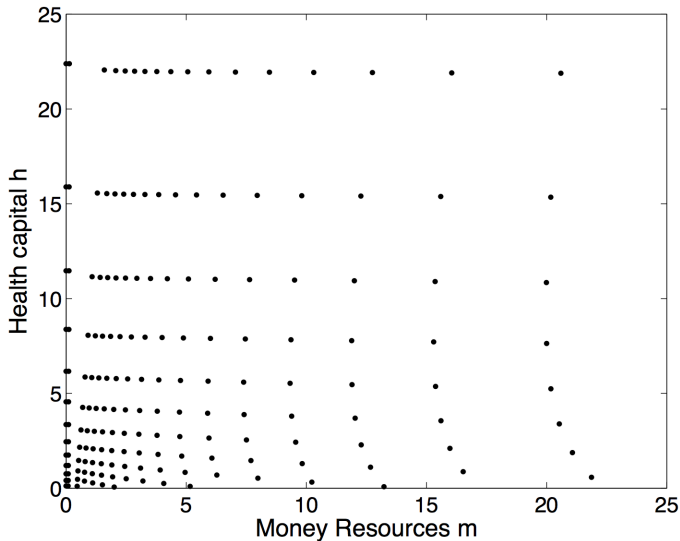


Ludwig & Schön, Computational Economics, 2018

Endogenous Grids in Higher Dimensions: Delaunay Interpolation and Hybrid Methods

- 1 Solve the model of human capital investment + consumption/savings
- 2 Compare three approaches which differ by the interpolation method
- 3 Need to interpolate on irregular multidimensional grid

# Multidimensional endogenous grid



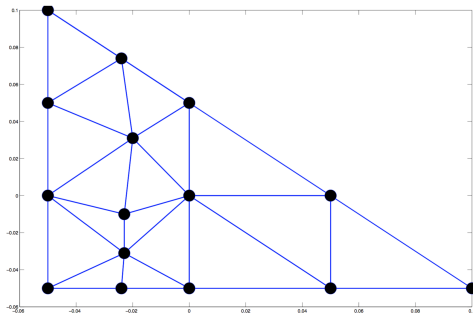
# Interpolation on the irregular grid



Johannes Brumm, Michael Grill, JEDC 2014

Computing equilibria in dynamic models with occasionally binding constraints

- Delaunay triangulation based interpolation



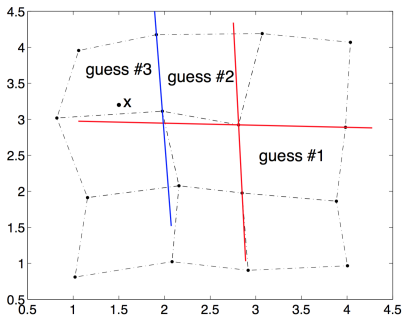
# Interpolation on the irregular grid



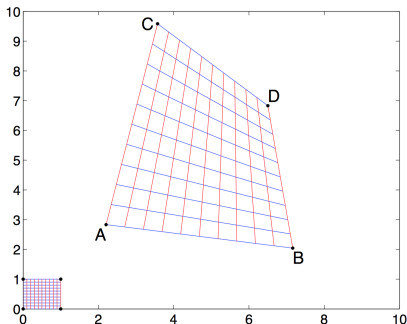
Matthew White, JEDC 2015

The Method of Endogenous Gridpoints in Theory and Practice

- Focus on general theory of multidimensional EGM
- Map non-linear rectangles into regular ones



(a) Identifying the sector by visibility walk



(b) Identifying relative coordinates

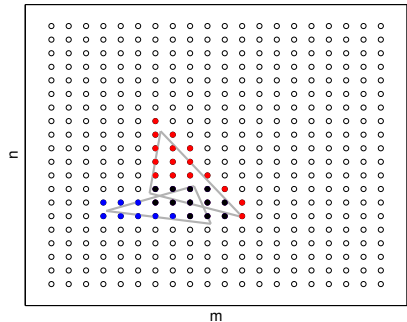
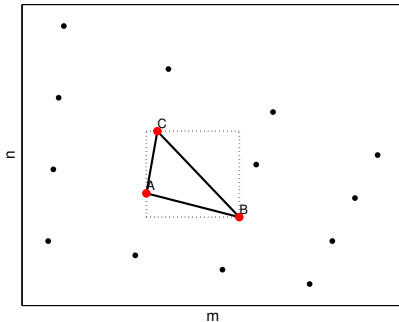
# Interpolation on the irregular grid



Jeppe Druedahl, Thomas Jørgensen, JEDC 2017

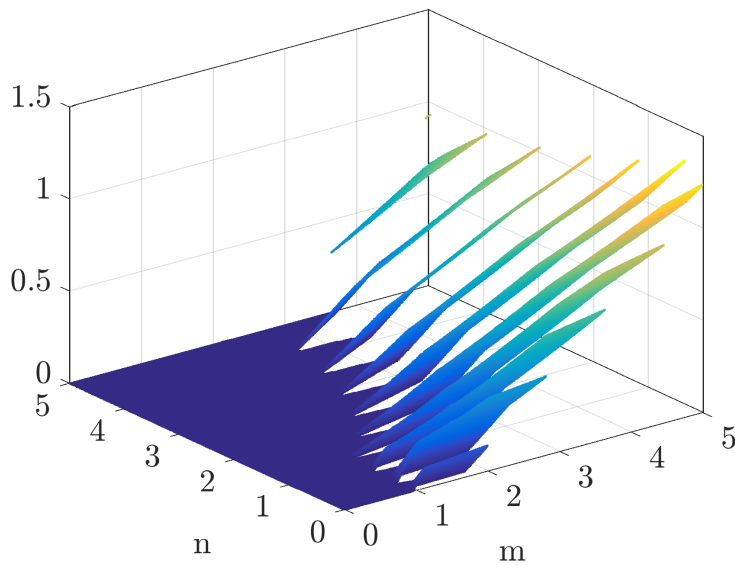
A General Endogenous Grid Method for Multi-Dimensional Models with Non-Convexities and Constraints

- Add occasionally binding constraints and allow for non-convexities
- Re-interpolate on regular grid while performing upper envelope



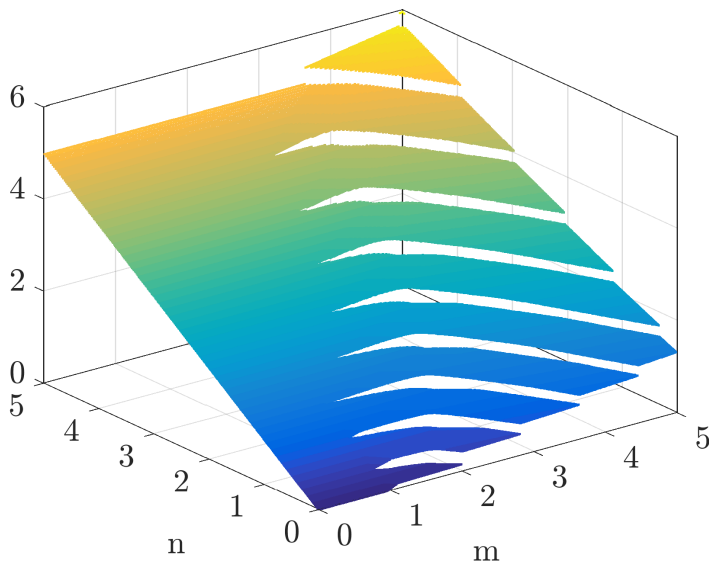
# Consumption + pension contributions model

## Pension fund contributions policy function



# Consumption + pension contributions model

Next period pension wealth  $n$





# General theory on multidimensional EGM



Matthew White, JEDC 2015

The Method of Endogenous Gridpoints in Theory and Practice

- **Invertibility condition** for the system of non-linear equations



Jeppe Druedahl, Thomas Jørgensen, JEDC 2017

A General Endogenous Grid Method for Multi-Dimensional Models with Non-Convexities and Constraints

- Formulate the **sufficient condition**,  
i.e. particular mapping has to be an **injection**



Iskhakov, Econ Letters 2015

Multidimensional endogenous gridpoint method: solving triangular dynamic stochastic optimization problems without root-finding operations + Corrigendum

- Focus on **analytical invertibility** to avoid root-finding operations

# Sufficient conditions for EGM to be applicable

- 1 Concave utility function
- 2 Post decision states ( $A_t$ ) form a set of sufficient statistics for the states and decisions in period  $t$
- 3 State variables can be analytically computed from post decision states ( $M_t = A_t + c_t$ )
- 4 The Hessian of the utility function can be converted to lower-triangular by permuting its rows and relabeling the variables

Then the dynamic problem can be solved (for interior solution) without root-finding operations by multidimensional EGM

# Estimating life cycle models using endogenous gridpoint methods

# What to do with EGM methods

We can solve **many** problems of this type  $\Rightarrow$

- ➊ Fast solver for important problems with discrete/continuous choice  
 $\rightarrow$ 
  - calibration
  - structural estimation with your favourite method
  - NFXP: inner loop to solve the model, outer loop to optimize the objective function
- ➋ Use the solver repeatedly in some “outer loop”  $\rightarrow$ 
  - individual heterogeneity : solve the model for each individual in the sample
  - unobserved heterogeneity : random effects
  - flexibility of distributional assumptions

# EGM vs. MPEC



Jørgensen, 2012 *Economics Letters*

Structural Estimation of Continuous Choice Models: Evaluating EGM and MPEC.

Deaton consumption/savings model in infinite horizon, MC experiment with ML on synthetic data, 1 structural parameter

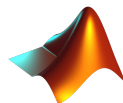
$\beta$		RMSE	Time
.70	EGM	0.002	0.1 sec.
	MPEC	0.049	112.4 sec.
.95	EGM	0.006	1.9 sec.
	MPEC	0.009	93.7 sec.
.99	EGM	0.000	5.0 sec.
	MPEC	0.000	30.9 sec.

## Points to take home

- 1 EGM and DC-EGM is fast and accurate solution methods
- 2 No root-finding operations in regular case
- 3 Efficient with credit constraint
- 4 Deterministic discrete-continuous problems are hard:
- 5 Kinks in value functions, discontinuous policy functions
- 6 Snowball effect in the accumulation of kinks over time
- 7 With EV taste shocks the problem is alleviated
- 8 EV taste shocks can be structural or added for smoothing
- 9 Facilitate estimation using discrete choice data

[github.com/fediskhakov/dcegm](https://github.com/fediskhakov/dcegm)

# Exercises



- 1 Code up the Euler equation solver in `model_phelps.m` and `model_deaton.m` and verify solution against FVI solver
- 2 Code up an EGM solver for infinite horizon version of the problems in `model_phelps.m` and `model_deaton.m`
- 3 (a) Replicate the solution to consumption/retirement problem using `model_retirement.m`;  
(b) Investigate how the variance of shocks to income and scale of the taste shock effect the solution;  
(c) Simulate the consumption path for  $\beta R = 1$  and discuss the accuracy of the solutions
- 4 Add education to the retirement model so that wage incomes varies by education, discuss the differences in labor supply decisions
- 5 Add part time work decision in the retirement model and simulate the case of phased retirement