Econ 522 – Problem Set 2

Due — TBD

1. Let $\{Y_i, Z_i\}_{i=1}^n$ be an i.i.d. sample generated by: for some $\theta_0 \in \Theta \subset \mathbf{R}^k$,

$$Y = g(Z, \theta_0) + U \ , \tag{1}$$

where $E[U|Z] = 0$ and $g : \mathcal{Z} \times \Theta \to \mathbf{R}$ is a known map.

   (a) Define $Q_0(\theta) = E[(Y - g(Z, \theta))^2]$. Derive a condition under which $Q_0(\theta)$ has a unique minimizer $\theta_0 \in \Theta$.

   (b) Formalize conditions under which $\hat{\theta}_n \xrightarrow{p} \theta_0$ and provide a proof.

   (c) Formalize conditions under which $\hat{\theta}_n$ is asymptotically normal and provide a proof. You may directly assume $\hat{\theta}_n \xrightarrow{p} \theta_0$.

2. Consider the criterion function (2.29) in the lecture notes. The M-estimation theory implies that the MLE $\hat{\theta}_n$ is asymptotically normal under regularity conditions.

   (a) Derive the asymptotic variance of $\hat{\theta}_n$ that is robust to model misspecification.

   (b) Suppose that G is the standard logistic cdf. The odds ratio then equals $\exp\{z_0^\intercal \theta_0\}$ at a particular covariate z_0. Propose an estimator for it and derive the asymptotic variance of the estimator.

   (c) Propose an estimator for the asymptotic variance in part (b).

3. Let $\{Y_{1i}, Y_{2i}, Z_i\}_{i=1}^n$ be an i.i.d. sample generated by

$$Y_{1i} = \max\{Z_{1i}^\intercal \beta_0 + Y_{2i}\gamma_0 + U_i, 0\} \ , \tag{2}$$

$$Y_{2i} = Z_i^\intercal \pi_0 + V_i \ , \tag{3}$$

where Z_i is independent of (U_i, V_i) and contains Z_{1i} as a subvector, and $(U_i, V_i)^\intercal$ is bivariate normal with mean zero and variance

$$\begin{bmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{vu} & \sigma_v^2 \end{bmatrix} .$$

Thus, this is just the censored regression with an endogenous regressor Y_{2i}.

   (a) Define $\epsilon_i \equiv U_i - \sigma_{uv}\sigma_v^{-2} V_i$. Show that V_i and ϵ_i are independent.

   (b) Plugging $U_i = \epsilon_i + \sigma_{uv}\sigma_v^{-2} V_i$ into (2) to obtain

$$Y_{1i} = \max\{Z_{1i}^\intercal \beta_0 + Y_{2i}\gamma_0 + \sigma_{uv}\sigma_v^{-2} V_i + \epsilon_i, 0\} \ . \tag{4}$$

   Suppose for the moment that we observe V_i. How would you estimate β_0 and γ_0? Justify your proposal.

   (c) Now back to the reality where we actually do not observe V_i. How would you estimate β_0 and γ_0?

4. Do Exercise 26.12 in Hansen (2022).