HW3

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1. (a) For positivity, assume (i) $K \geq 0$. For the integration, assume (ii) K is even ($K(-v) = K(v)$) and (iii) $\int K(v)dv = 1$, so that

$$\int \frac{1}{h_n} K(\frac{x-u}{h_n})du = \int \frac{1}{h_n} K(\frac{u-x}{h_n})du = \int K(v)dv = 1.$$

(b) We will need some regularity conditions. Assume

(iv) $\int u^2 K(u)du = \mu_2 < \infty$,

(v) f admits the second order Taylor expansion,

(vi) f and K are bounded[1].

Then,

$$\begin{aligned}
\mathbb{E}[\hat{f}_n(x)] &= \int \frac{1}{nh_n} \sum_i K(\frac{x-X_i}{h_n}) dP_{X_1,\ldots,X_n} \\
&= \int \frac{1}{h_n} K(\frac{x-X_1}{h_n}) dP_{X_1} \\
&= \int \frac{1}{h_n} K(\frac{x-v}{h_n}) f_0(v)dv \\
&= \int K(-u) f_0(x+uh_n)du \\
&= \int K(u) \left\{ f_0(x) + f_0'(x)uh_n + \frac{1}{2} f_0''(x)u^2 h_n^2 + o(h_n^2) \right\} du \\
&= f_0(x) + 0 + \frac{1}{2} f_0''(x) h_n^2 \underbrace{\int u^2 K(u)du}_{\mu_2} + o(h_n^2).
\end{aligned}$$

The middle term is 0 because K is even.

[1] I use this assumption only for DCT, so it would be stronger than necessary.

(c)

(d) Biased. Does not vanish. At 0, we don't have observations from the left, so the choice of the kernel could be inappropriate. Maybe we can split the case 0,(0,1), and 1 and use different kernels in each case.

2. (a) Large. The bias should be small regardless of h_n because it is just a function of U_i.

   (b) Consider (3.15).

$$\mathrm{Var}(\hat{\theta}_{LL}|Z) = \frac{v_0^2(z)}{nh_n}\{1+o_p(1)\} = \frac{\kappa_2\sigma_0^2}{nh_n f_z(z)}\{1+o_p(1)\}.$$

   If $\sigma_0^2(z) = \mathbb{E}[U^2|z]$ is high, the variance should be high, so it makes sense. If $f_z(z)$ is high, there will be more observations near z and we have more information about the location, so it makes sense.

3. $K(\cdot) = K(0)$ is a constant and cancels out in the optimization. Manipulation of the objective function completes the proof.

$$Y_i - b_0 - b_1(Z_i - z) = Y_i - (b_0 - b_1 z) - b_1 Z_i = Y_i - a_0 - a_1 Z_i,$$

   so that

$$\hat{\alpha}_0 + \hat{\alpha}_1 z = (\hat{\beta}_0 - \hat{\beta}_1 z) + \hat{\beta}_1 z = \hat{\beta}_0 = \hat{\theta}.$$

4. Submitted via GitHub.