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Computing equilibrium in OLG models with stochastic production

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Abstract

In this paper we develop a projection algorithm to approximate equilibria in overlapping generations economies with a large number of generations and stochastic aggregate production. In these types of economies the state space includes the distribution of wealth across generations. We use Smolyak's algorithm to approximate policy functions, which map the current state into agents' optimal choices, by linear combinations of polynomials. This allows us to compute equilibria for models where agents live for 20–30 periods.

We also provide examples which demonstrate that approximating the cross-sectional wealth distribution only by its first moment (and thereby reducing the dimension of the state space to two continuous state variable, independently of the number of agents) often leads to very high relative errors in agents' Euler equations.

JEL classification: C63; C68; D58; D91; E32

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1. Introduction

Infinite horizon general equilibrium models with overlapping generations have many interesting implications which are absent in the standard Arrow–Debreu model. Under certainty, there are well established methods to approximate equilibria numerically and the model has been fruitfully applied in fields such as macroeconomics or public finance (see e.g. Auerbach and Kotlikoff, 1983, 1987).

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Under uncertainty, there do not exist steady state equilibria in these models, and even when the exogenous aggregate shock can take only finitely many values the equilibrium allocations in general do not have finite support. Using Malinvaud's (1973) approach it is possible to construct economies with *idiosyncratic* uncertainty where there is a continuum of ex ante identical agents and all uncertainty cancels out in the aggregate – in these economies there usually exists a steady state wealth distribution which can be computed fairly easily (see e.g. Imrohoroglu et al., 1995; Conesa and Krueger, 1999). With aggregate uncertainty, however, a steady state wealth distribution will not exist in general, as stochastic aggregate shocks affect everybody's return to physical and human capital, and by construction the effects do not cancel out in the aggregate, so that the distribution of wealth changes with the stochastic aggregate shock. This feature makes it difficult to approximate equilibria with many agents of different ages and aggregate uncertainty. It also may explain why relatively little work has been carried out using these types of models. [1]

In this paper we consider the simplest possible version of an overlapping generations model with stochastic production. There is a single agent per generation and the only asset available for trade is risky capital. For many applied policy question the interplay between aggregate uncertainty and household heterogeneity because of age is crucial. If, for example, returns to capital and labor are stochastic and imperfectly correlated, then, absent private insurance markets, a public insurance system between young households (whose income risks consist mostly of risky returns to labor) and old households (which face significant risks from stochastic returns to their assets) may provide beneficial intergenerational risk sharing. The reform of a social security system that provides this type of intergenerational risk sharing towards a system with individual retirement accounts (which does not provide such insurance) may be uncalled for, even if it has positive labor supply and capital accumulation effects. To study such a question, as we do in our companion paper, Krueger and Kubler (2003), one evidently needs to analyze a model which features both aggregate uncertainty as well as heterogeneity of households with respect to their age.

The purpose of this paper is to examine the *computational* problems which arise because of aggregate uncertainty in OLG models. We first develop a projection algorithm (as for example in Judd, 1992) to approximate equilibrium numerically. Since the distribution of wealth across generations has to be included in the endogenous state space, one faces a curse of dimensionality as the life span of agents (and therefore the number of agents active in markets) increases.

Methods which use classical tensor products to approximate the unknown policy functions are bound to be infeasible if the dimension of the state space is larger than 5 or 6. Judd and Gaspar (1997) suggest to use complete polynomials to reduce the number of unknown polynomial coefficients. However, their method does generally not reduce the number of points at which the approximated function needs to be evaluated. Therefore projection methods have previously been thought to be applicable only to models with relatively few agents. However, since in many economic applications the policy functions which have to be approximated numerically exhibit a high degree of

[1] Rios-Rull (1996) is a notable exception.

smoothness, high dimensional approximation of these functions can be done efficiently by using only relatively few function values, as we will demonstrate in this paper, using Smolyak's algorithm (see Barthelmann et al., 2000). With this method the number of unknown coefficients grows polynomially in the dimension of the problem (as in Judd and Gaspar, 1997) and since exact interpolation is used the number of interpolation points grows at the same rate. Our algorithm can approximate equilibria in models with 20–30 agents. While the original paper of Smolyak was published in 1963, this paper is, to the best of our knowledge, the first application of his method to the computation of dynamic equilibria in economic models.

In order to assess the quality of our algorithm we first consider a simple case (a variant of Huffman's (1987) model) for which an analytical solution of the model exists against which numerical approximations can be compared. We show that the allocations computed from the algorithm lie within 10^{-4} percent of the true equilibrium allocations. For (more realistic) models for which no such analytic solution can be obtained we resort to comparisons of the errors in Euler equations, as suggested by Judd (1992). The proposed projection algorithm can approximate equilibria for models with 20 agents in around 1.5 h. Maximal errors in Euler equations lie around 7×10^{-3}. For 30 agents running times increase to around 30 h and maximal errors in Euler equations lie around 10^{-2}. Models with more than 30 agents quickly become infeasible.

In the context of a model with infinitely lived agents, Krusell and Smith (1998) recognize and exploit that in solving his optimization problem the household only needs to forecast future interest rates, but not the future distribution of wealth. If interest rates can be forecasted with sufficient accuracy using a low-dimensional summary statistic of current endogenous variables (such as the current aggregate capital stock), one can simulate an approximate equilibrium allocation by just solving the individual problem given the approximate forecasting rule. Evidently, if a sufficiently accurate forecasting rule which only uses current aggregate capital exists, the dimension of the problem is independent of the number of agents in the economy.

Storesletten et al. (2001) and Gourinchas (2000) apply Krusell and Smith's idea to an overlapping generations model with aggregate uncertainty and examine if the aggregate capital stock alone provides such a sufficiently good approximation for the current endogenous state of the economy. While in their specifications of the model this often seems to be the case we argue that the scenarios where this simple approximation works sufficiently well to use it for welfare analysis are extremely special cases and that in general higher-dimensional approximations of the state are needed.

The paper is organized as follows. In Section 2 we introduce the economic model and define equilibrium. In Section 3 we develop a projection algorithm to approximate functional rational expectations equilibria numerically. In Section 4, we then evaluate our algorithm using a simple example for which an analytical solution can be obtained. In Section 5 we consider models without analytical solutions and show that our algorithm can handle models with up to 30 agents. In Section 6 we examine under which conditions the endogenous state can be approximated sufficiently well by the mean aggregate capital alone. Section 7 concludes.

2. The economic model

2.1. Uncertainty

Time is discrete and extends from $t = 0, \ldots, \infty$. Uncertainty is represented by an event tree. The root of the tree is given by some fixed event z_0. Each node of the tree is a history of exogenous aggregate shocks $z^t = (z_0, z_1, \ldots, z_t)$. The shocks are assumed to follow a Markov chain with finite support $\mathscr{Z}$ and with transition matrix Π.

2.2. Households

The economy is populated by overlapping generations of agents that live for N periods. At each date-event z^t a representative household is born. As mentioned in the introduction we focus on the case where there is no within generation heterogeneity and households only distinguish themselves by the date-event of their birth so that a household is fully characterized by z^t. To simplify notation, we collect all households which are alive at some node z^t in a set $\mathscr{I}_{z^t}$ and denote a typical household by $i \in \mathscr{I}_{z^t}$. When there is no ambiguity about the identity of households we will index households simply by their time of birth.

In each period i of her life, an agent born at node z^s has non-negative, possibly stochastic labor endowment $l^i(z_t)$ which depends on the agents' age, $i = t - s + 1$ and on the current shock z_t alone. The price of the consumption good at each date event is normalized to one and at each date event z^t the household supplies her labor endowment inelastically for a market wage $w(z^t)$.

Let c_t^s denote the consumption of an agent born at time s in period $t \geqslant s$.[2] Individuals value consumption according to

$$\mathrm{E}_s \left[\sum_{t=s}^{s+N-1} \beta^{t-s} u(c_t^s) \right], \tag{1}$$

where $u : \mathbb{R}_{++} \to \mathbb{R}$ is assumed to be smooth, strictly increasing, strictly concave and to satisfy the Inada condition $\lim_{c \to 0} u'(c) = \infty$.

Households have access to a storage technology: they can use one unit of the consumption good to obtain one unit of the capital good next period. We denote the investment of household s into this technology by a_t^s. We do not restrict $a_t^s \geqslant 0$, because we want to permit households to borrow against future labor income. One possible interpretation of this assumption is that there is a bank which acts as an intermediary and which stores the capital good for all households, and each individual household can then borrow from this bank. At time t the household sells its capital goods accumulated from last period, a_{t-1}^s, to the firm for a market price $r_t > 0$. The budget constraint of household s in period $t \geqslant s$ is

$$c_t^s + a_t^s = r_t a_{t-1}^s + l^{t-s} w_t. \tag{2}$$

[2] Whenever there is no ambiguity we will use the notation $c_t^s = c_t^s(z^t)$ to denote consumption of an agent born at s at node z^t of the event tree. The notation for all other variables is to be understood correspondingly.

We impose the restriction that in the last period of his life the agent is not allowed to borrow, $a^s_{s+N-1} \geqslant 0$, i.e. we rule out that households die in debt. Furthermore we assume that agents enter the economy without any assets, i.e. we assume that $a^s_{s-1} = 0$.

To start off the economy we assume that in period zero, there are N households of ages 1–N who enter the period with given capital holdings $a^0_{-1}, \ldots, a^{-N+1}_{-1}$, where by assumption $a^0_{-1} = 0$.

2.3. Firms

There is a single representative firm which in each period t uses labor and capital to produce the consumption good according to a constant returns to scale production function $f_t(K, L; z_t)$. Since firms make their decisions on how much capital to buy and how much labor to hire after the realization of the shock z_t they face no uncertainty and simply maximize current period profits. [3]

In the examples below we will always use the following parametric form for the production function.

$$f(K, L, z) = \eta(z)F(K, L) + K(1 - \delta(z)), \tag{3}$$

where $\eta(.)$ is the stochastic shock to productivity, where $\delta(.)$ can be interpreted as the (possibly) stochastic depreciation rate and where $F(.,.)$ is a Cobb–Douglas production function.

2.4. Markets

In this simple economy the only markets are spot markets for consumption, labor and capital, all of which are assumed to be perfectly competitive. It is not difficult to extend the model (but possibly difficult to compute its equilibrium) to include financial markets where J securities like bonds or options are traded. However, in order to focus on the main computational challenges we avoid unnecessary notation and additional prices and focus on the simplest possible asset structure.

2.5. Equilibrium

Definition 1. A competitive equilibrium, given initial conditions $z_0, (a^s_{-1})^0_{s=-N+1}$ is a collection of choices for households $(c^i_t, a^i_t)^{i \in \mathcal{I}_{z^t}}$ and for the representative firm $\{K_t, L_t\}$ as well as prices $\{r_t, w_t\}$ for all $t = 0, \ldots, \infty$. such that

1. For all $s = 0, \ldots$, given $\{r_t, w_t\}^\infty_{t=0}$, the choices $\{c^s_t, a^s_t\}^{s+N-1}_{t=s}$ maximize (1), subject to (2).

[3] We assume that households cannot convert capital goods back into consumption goods at the beginning of the period. This assumption is necessary to prevent households from consuming the capital at the beginning of the period instead of selling it to the firm in states where the net return to capital is negative. Alternatively one can assume that for all $z^t, r(z^t) \geqslant 1$ as is the case for our numerical examples below.

2. Given r_t, w_t the firm maximizes profits, i.e.

$$(K_t, L_t) \in \arg\max_{K_t, L_t \geqslant 0} f(K_t, L_t, z_t) - r_t K_t - w_t L_t. \tag{4}$$

3. All markets clear: For all t

$$L_t = \sum_{i \in \mathscr{I}_{z^t}} l_t^i,$$

$$K_t = \sum_{i \in \mathscr{I}_{z^t}} a_{t-1}^i.$$

Note that by Walras law market clearing in the labor and capital market imply market clearing in the consumption goods market. Note furthermore that the assumptions on the parametric form of the production function as well as concavity and differentiability of F imply that equilibrium prices satisfy

$$w(z^t) = \eta(z_t) F_L(K(z^t), L(z^t)),$$

$$r(z^t) = \eta(z_t) F_K(K(z^t), L(z^t)) + (1 - \delta(z_t)).$$

For future reference, the Euler equation for consumption for any given generation s in node z^t reads as

$$\begin{aligned} u'(c_t^s(z^t)) &= \beta \sum_{z_{t+1} \in \mathscr{Z}} \Pi(z_{t+1}|z_t) r(z^t, z_{t+1}) u'(c_{t+1}^s(z^t, z_{t+1})) \\ &= \beta \mathrm{E}_{z_t} u'(c_{t+1}^s(z^t, \tilde{z})) r(z^t, \tilde{z}), \end{aligned} \tag{5}$$

where E_{z_t} is the conditional expectation of $\tilde{z}$, conditional on z_t.

3. A projection algorithm to approximate equilibrium

Following Judd (1992) and Judd and Gaspar (1997) we use a projection method to approximate equilibria numerically. In order to do so we first need to describe equilibrium as a system of operator equations.

3.1. Functional rational expectation equilibrium

Our computational strategy searches for a recursive equilibrium where the distribution of capital holdings constitutes a sufficient endogenous state, where the endogenous state lies in a known compact set and where all transition functions are smooth. Following Spear (1988) we call this 'functional rational expectations equilibrium' [FREE]. In our specification, we require the compact endogenous state space to be a box. A natural endogenous state would be the vector of individual asset holdings. However, this turns out to be impractical in our setting. The lower bound of individuals' equilibrium asset holdings is not guaranteed to be positive. The sum of all lower bounds being negative results in a negative aggregate capital stock and thus returns to capital and wages

that are not well-defined. Instead we choose as an endogenous state the aggregate capital stock together with individual's savings as shares of aggregate capital, which assures that everywhere in the compact state space returns to capital and wages are well-defined.

The precise definition is as follows.

Definition 2. A FREE consists of an $N-1$ dimensional box $\mathscr{B} \subset \mathbb{R}^{N-1}$ and smooth asset demand functions $\theta_{zi} : \mathscr{B} \to \mathbb{R}$, $i = 1,\ldots,N-1$, $z \in \mathscr{Z}$ such that for all shocks $z \in \mathscr{Z}$ and all states $s = (s_1,\ldots,s_{N-1}) \in \mathscr{B}$

$$u'(\bar{c}_i(s,z)) = \beta \mathrm{E}_z r(s_+,\tilde{z}) u'(\bar{c}_{i+1}(s_+,\tilde{z})) \quad \text{for all } i = 1,\ldots,N-1, \tag{6}$$

where

$$s_+ = \left(\sum_{i=1}^{N-1} \theta_{iz}(s), \frac{\theta_{1z}(s)}{\sum_{i=1}^{N-1} \theta_{iz}(s)}, \ldots, \frac{\theta_{(N-2)z}(s)}{\sum_{i=1}^{N-1} \theta_{iz}(s)} \right) \in \mathscr{B}$$

is the state tomorrow and

$$r(s,z) = f_K\left(s_1, \sum_{i=1}^{N} l^i(z), z \right),$$

$$w(s,z) = f_L\left(s_1, \sum_{i=1}^{N} l^i(z), z \right),$$

$$\bar{c}_1(s,z) = l^1(z) w(s,z) - \theta_{iz}(s),$$

$$\bar{c}_i(s,z) = s_1 s_i \cdot r(s,z) + l^i(z) w(s,z) - \theta_{iz}(s) \quad \text{for } i = 2,\ldots,N-1,$$

$$\bar{c}_N(s,z) = s_1 \left(1 - \sum_{i=1}^{N-2} s_i \right) \cdot r(s,z) + l^i(z) w(s,z).$$

Note that in this definition the beginning of period asset holdings of agent i is given by his share of aggregate capital s_{i-1} multiplied by aggregate capital, s_1. Since all first order conditions are necessary and sufficient it is clear that any FREE induces a competitive equilibrium in the sense of Definition 1.

3.2. A projection algorithm

We approximate the unknown equilibrium asset demand functions θ_{iz}, $i=1,\ldots,N-1$, $z \in \mathscr{Z}$, by polynomial functions $\hat{\theta}_{iz}(\alpha)$ that are uniquely defined by finitely many coefficients α. Let the total number of unknown coefficients be M – evidently, this number will be an important determinant in the accuracy of the solution.

In order to solve for the unknown coefficients, we require that the functional equation (6) holds exactly at M collocation points $s \in \mathscr{H} \subset \mathscr{B}$. We therefore transform the

infinite dimensional functional equation into the following finite dimensional (non-linear) system of equations in the M unknown coefficients α.

$$u'(\hat{c}_i(s,z;\alpha)) = \beta \mathrm{E}_z r(\hat{s}_+,\tilde{z}) u'(\hat{c}_{i+1}(\hat{s}_+,\tilde{z};\alpha)) \quad \text{for all } i=1,\ldots,N-1,$$
$$z \in \mathscr{Z}, \quad s \in \mathscr{H}, \tag{7}$$

where

$$\hat{s}_+ = \left(\sum_{i=1}^{N-1} \hat{\theta}_{iz}(s;\alpha), \frac{\hat{\theta}_{1z}(s;\alpha)}{\sum_{i=1}^{N-1} \hat{\theta}_{iz}(s;\alpha)}, \ldots, \frac{\hat{\theta}_{(N-2)z}(s;\alpha)}{\sum_{i=1}^{N-1} \hat{\theta}_{iz}(s;\alpha)} \right),$$

$$\hat{c}_1(s,z;\alpha) = l^1(z) w(s,z) - \hat{\theta}_{iz}(s;\alpha),$$

$$\hat{c}_i(s,z;\alpha) = s_1 s_i \cdot r(s,z) + l^i(z) w(s,z) - \hat{\theta}_{iz}(s;\alpha) \quad \text{for } i=2,\ldots,N-1,$$

$$\hat{c}_N(s,z;\alpha) = \bar{c}_N(s,z).$$

The main computational challenges are caused by the fact that the endogenous state space has dimension $N-1$. In a model where agents live for 30 periods and where one therefore has to approximate 29 dimensional functions. As mentioned in the introduction, we will use Smolyak's algorithm to render this problem feasible.

3.3. Smolyak's algorithm for high dimensional interpolation

Suppose we want to approximate $f : [-1,1]^d \to \mathbb{R}$ by interpolating it at points $(x^i, y^i = f(x^i))$ with $x^i \in \mathscr{H} \subset [-1,1]^d$. In order to obtain a good approximation, both the choice of $\mathscr{H}$ and the interpolating function $\hat{f}$ are crucial.

The simplest way is to take $\mathscr{H}$ to be a grid and interpolate the points by the tensor product[4] of univariate interpolating polynomials. Obviously this is not feasible for our problem where the dimension d is large. Judd and Gaspar (1997) suggest to use complete polynomials instead of tensor products. For $\mathbf{i} \in \mathbb{N}^{\mathbf{d}}$ define $|\mathbf{i}| = i_1 + \cdots + i_d$. The complete set of polynomials of total degree k in d variables is given by the set $\{h \mid h(x) = (x_1)^{i_1} \cdot (x_2)^{i_2} \cdot \ldots \cdot (x_d)^{i_d} \text{ with } |\mathbf{i}| \leqslant k\}$. Judd (1998) gives an intuitive explanation for why this scheme yields approximation errors which are not much worse than tensor products – however, he does not provide a general method to interpolate function-values using complete polynomials.

In this paper we use Smolyak's (1963) method (also known as Boolean methods, see Delvos, 1982, and sometimes referred to as 'sparse grid methods') which provides a general principle to construct good approximations for d-dimensional problems, based on approximations for the univariate case. Barthelmann et al. (2000) show how to use the method for high dimensional interpolation on sparse grids. For completeness, we repeat the main steps of their construction.

Define $m_1 = 1$ and $m_i = 2^{i-1} + 1$ for $i = 2, \ldots$. The total degree of the interpolating polynomials will be $m_i - 1$ – by construction, only polynomials of total degree 2,4,8 etc. can therefore be used for interpolation.

[4] If A and B are sets of functions their tensor product is $A \otimes B = \{\phi(x)\psi(y) \mid \phi \in A, \psi \in B\}$.

Define $\mathscr{G}^i = \{\zeta_1^i, \ldots, \zeta_{m_i}^i\} \subset [-1,1]$ as the set of the extrema of the Chebyshev polynomials

$$\zeta_j^i = -\cos\left(\frac{\pi(j-1)}{m_i - 1}\right), \quad j = 1, \ldots, m_i \tag{8}$$

with $\mathscr{G}^1 = \{0\}$. These are also known as Gauss–Lobotto nodes and it can be shown (see e.g. Quarteroni et al., 2000) that univariate interpolation over these nodes has similar optimality properties as interpolation over Chebyshev zeros (as used in Judd, 1992). The crucial advantage of using the extrema as opposed to Chebyshev zeros lies in the fact that $\mathscr{G}^i \subset \mathscr{G}^{i+1}$ for all $i = 1, 2 \ldots$.

For an integer $q > d$ we can then define a *sparse* grid of points in $[-1,1]^d$ as follows:

$$\mathscr{H}(q,d) = \bigcup_{q-d+1 \leqslant |\mathbf{i}| \leqslant q} (\mathscr{G}^{i_1} \times \cdots \times \mathscr{G}^{i_d}). \tag{9}$$

Defining the set of points over which we interpolate in terms of q simplifies the notation. Note that by the definition of m_i, the maximal number of points in a given set $\mathscr{G}^i$ will be equal to $2^{q-d}+1$. The total degree of the interpolating complete polynomial is therefore always 2^{q-d}.

Fig. 1 shows the grids for $d = 2$ and for $q = 3$, 4, 5 and 6. For example for $q = 4$, the grid is the union of $\mathscr{G}^1 \times \mathscr{G}^3$, $\mathscr{G}^2 \times \mathscr{G}^2$ and $\mathscr{G}^3 \times \mathscr{G}^1$ (and the redundant $\mathscr{G}^1 \times \mathscr{G}^2$

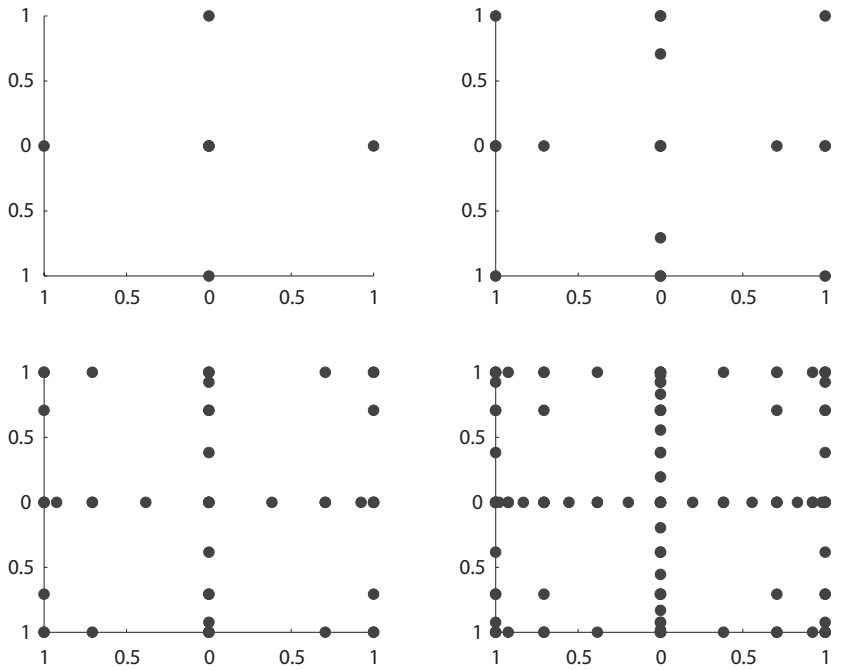

Fig. 1. The Grids $\mathscr{H}(3,2)$, $\mathscr{H}(4,2)$, $\mathscr{H}(5,2)$ and $\mathscr{H}(6,2)$.

Table 1
Number of points

d	q	Number of points in $\mathscr{H}(q,d)$
2	3	5
2	4	13
2	5	29
5	6	11
5	7	61
5	8	241
10	11	21
10	12	221
10	13	1581

and $\mathscr{G}^2 \times \mathscr{G}^1$). If the sets $\mathscr{G}^i$ were not nested the total number of points would be 18, but since the 3 points in $\mathscr{G}^2$ are contained in $\mathscr{G}^3$ the number of points reduces to 14. For $q=5$ exactly 8 points are added at the corners and 4 points are added along each dimension in the middle – these come from the sets $\mathscr{G}^4 \times \mathscr{G}^1$, $\mathscr{G}^1 \times \mathscr{G}^4$ and from the sets $\mathscr{G}^3 \times \mathscr{G}^2$ and $\mathscr{G}^2 \times \mathscr{G}^3$.

It can be verified that the number of points in $\mathscr{H}(q,d)$ is $1+2d$ for $q=d+1$, $1+4d+4\ d(d-1)/2$ for $q=d+2$ and $1+8d+12\ d(d-1)/2+8\ d(d-1)(d-2)/6$ for $q=d+3$. The nestedness of the Gauss–Lobotto nodes implies, in general, that the number of points in $\mathscr{H}(q,d)$ grows polynomially in d if q is taken to be $q=d+n$ for some fixed n. However, given d it might grow very fast in q. Table 1 illustrates this.

The table shows how the number of points grows in the dimension d and with q. In interpreting this table one should keep in mind that the total degree of the interpolating polynomial will be 2^{q-d}. For a fixed degree of the polynomial increasing the number of dimensions therefore comes at a very low cost, while increasing the degree of the polynomial for fairly high dimension is computationally very costly.

Smolyak's method interpolates functions at nodes in $\mathscr{H}(q,d)$ using complete polynomials of total degree $m_{q-d+1}-1=2^{q-d}$. We denote by p^i a univariate polynomial of degree m_i. For any $d>1$ denote the d dimensional tensor product of polynomials of degree $(i_1,\ldots,i_d)$ by

$$p^{\mathbf{i}}(x) = p^{i_1}(x_1)\ldots p^{i_d}(x_d). \tag{10}$$

The coefficients of the polynomials are chosen to ensure that $p^{\mathbf{i}}$ interpolates the points (x^i, y^i) with $x^i \in \mathscr{G}^{i_1} \times \cdots \times \mathscr{G}^{i_d}$.

For $q>d$, the Smolyak 'algorithm'[5] is given by the weighted sum of low dimensional tensor products. At a point $x \in [-1,1]^d$ we therefore approximate $f(x)$ by

$$\hat{f}^{q,d}(x) = \sum_{q-d+1 \leqslant |\mathbf{i}| \leqslant q} (-1)^{q-|\mathbf{i}|} \begin{pmatrix} d-1 \\ q-|\mathbf{i}| \end{pmatrix} p^{\mathbf{i}}(x). \tag{11}$$

[5] The approximation formula is usually referred to as Smolyak's algorithm because its relatively complicated and nested form can be interpreted as an algorithm.

It can be easily checked that $\hat{f}^{d+n,d}$ reproduces the complete polynomials of total degree $m_{n+1}-1$. Barthelmann et al. (2000) show that Smolyak's algorithm interpolates functions on $\mathscr{H}(q,d)$ and that the error bounds of the method are almost optimal (that is, optimal, up to a logarithmic factor in the number of points) for different classes of functions with bounded mixed derivative (see Theorem 2 and Remark 4 of their paper for details).

3.3.1. Numerical evaluation of Smolyak polynomials

Since our algorithm requires the evaluation of the polynomial approximating functions at thousands of points it is crucial to implement the evaluation efficiently. We use weighted sums of Chebyshev polynomials since they can be evaluated relatively efficiently and since they can be computed quite accurately.

Let the function $T_j : [-1,1] \to \mathbb{R}$, defined as

$$T_j(x) = \cos(j \cos^{-1} x) \tag{12}$$

denote the Chebyshev polynomial of degree j. A function $f : [-1,1] \to \mathbb{R}$ at points $\zeta_j \in \mathscr{G}^i$ is then interpolated by

$$\hat{f}(x) = p^i_f(x) = \sum_{k=0}^{m_{i-1}} \xi_k(f) T_k(x), \tag{13}$$

where the coefficients are given by

$$\xi_k(f) = \frac{2}{m_i d_k} \sum_{j=1}^{m_i} \frac{1}{d_j} T_k(\zeta_j) f(\zeta_j) \tag{14}$$

with $d_1 = d_{m_i} = 2$ and $d_j = 1$ for $j = 1,\ldots,m_i - 1$ (see Quarteroni et al., 2000) for a derivation).

In our economic model, the state generally does not lie in $[-1,1]^d$; however, it lies in the box $\mathscr{B} = [\underline{s}_1, \bar{s}_1] \times \cdots \times [\underline{s}_{N-1}, \bar{s}_{N-1}]$. Given an endogenous state $s \in \mathscr{B}$ we therefore define, for $n = 1,\ldots,N-1$,

$$x_n(s_n) = 2 \frac{s_n - \underline{s}_n}{\bar{s}_n - \underline{s}_n} - 1 \in [-1,1]. \tag{15}$$

We implemented Smolyak's method for the interpolation of the asset demand functions $\hat{\theta}_{iz}$ for $q = d+1$, $d+2$ and $d+3$. Table 1 shows that the computational burden increases polynomially with the number of state variables, and not exponentially.

3.4. Solving for the polynomial coefficients

Even when there are only four exogenous shocks and agents only live for nine periods, the approximation of the policy functions by Smolyak's method with degree $q = d+2$ leads to a system with 4640 unknown coefficients (for each agent and each shock we have 145 coefficients, there are eight (active) agents and four shocks). This makes a simple Newton method unfeasible. Therefore we use a time-iteration algorithm to solve for Chebyshev coefficients in Eq. (7). This method can be interpreted as solving finite horizon approximations to our infinite horizon model.

Given a grid $\mathscr{H}(q,d)$ and a stopping tolerance $\tau > 0$, the algorithm proceeds as follows:

1. Guess coefficients α^0 for a starting $\hat{\theta}^0 = (\hat{\theta}^0_{iz})$.
2. Given coefficients α^n and thus the function $\hat{\theta}^n$, for each $z \in \mathscr{Z}$, $s \in \mathscr{H}$ solve the following $N-1$ non-linear equations for the $N-1$ unknown choices today, θ_{iz}, $i = 1,\ldots,N-1$.

$$u'(c_i(\theta_{iz};s,z)) = \beta \mathrm{E}_z r(s_+,\tilde{z}) u'(\hat{c}_{i+1}(s_+,\tilde{z};\alpha^n)) \quad \text{for all } i = 1,\ldots,N-1, \tag{16}$$

with

$$s_+ = \left(\sum_{i=1}^{N-1} \theta_{iz}, \frac{\theta_{1z}}{\sum_{i=1}^{N-1} \theta_{iz}}, \ldots, \frac{\theta_{(N-2)z}}{\sum_{i=1}^{N-1} \theta_{iz}} \right),$$

$$c_1 = l^1(z) w(s,z) - \theta_{iz},$$

$$c_i = s_1 s_i \cdot r(s,z) + l^i(z) w(s,z) - \theta_{iz} \quad \text{for } i = 2,\ldots,N-1,$$

$$c_N = s_1 \left(1 - \sum_{i=1}^{N-2} s_i \right) \cdot r(s,z) + l^i(z) w(s,z).$$

3. Compute the new coefficients α^{n+1} by choosing α^{n+1} so that the function $\hat{\theta}^{n+1}_{iz}$ interpolates the solutions from step 2., θ_{iz}, on $\mathscr{H}$.
4. Check stopping criterion: If $\sup_{z,s\in\mathscr{H}} |\hat{\theta}^{n+1} - \hat{\theta}^n| < \tau$ terminate – otherwise go to Step 2.

Conceptually our algorithm is similar to a basic policy function iteration as proposed, e.g., by Coleman (1990). Note that, as we solve for optimal policy functions and equilibrium prices jointly, an iterative procedure is required to do so. Alternatively one could hold prices fixed, solve for optimal policies by backward induction and then check market clearing and adjust prices.

3.4.1. Finding interval bounds

Part of our definition of a FREE is the existence of an ergodic set which is a subset of some box $\mathscr{B} = [\underline{s}_1, \bar{s}_1] \times \cdots \times [\underline{s}_{N-1}, \bar{s}_{N-1}]$. It is essential for our computational strategy that such bounds exist.

We approximate these bounds by solving for the deterministic steady state for all values of the exogenous shock z. We then take 70 percent of the lowest value and 170 percent of the highest value as our interval bounds (because of precautionary savings, the ergodic set of possible capital holdings usually lies above the deterministic steady states). We also check, after the optimal policies have been computed, that the constraints $s_+ \in \mathscr{B}$ are never binding.

3.5. Implementation

We implemented the algorithm in Fortran 90. In each iteration, given a function $\hat{\theta}$ from the previous iteration, for each collocation point Θ we solve system (16) for optimal saving choices today using a simple Newton method. In models where agents live for 20–30 periods Newton's method often fails to converge when there is no good starting point known from previous iterations. In this case we use a homotopy solver (see Garcia and Zangwill, 1981 for an introduction to homotopy methods and Judd et al., 2003 for a description of the use of homotopy methods in time iteration algorithms).

In all the examples reported we start with a $\hat{\theta}^0 \equiv 0$ – while this is by no means a 'good starting value', it allows us to compare running times of the algorithm across different specifications of the model.

4. Evaluating the algorithm I: an example with an analytic solution

In this section we evaluate the performance of the algorithm for an economy in which the exact equilibrium is known and where we can compare the computed allocation to the true equilibrium allocation.

The example is an adapted version of Huffman's (1987) model and a special case of the model described above. Agents all have log-utility, $u(c) = \log(c)$, and they only receive non-zero labor endowments in the first period of their lives, i.e. $l^i(z^t) = 0$ for all $i > 1$. For simplicity we assume that $l^1(z^t) \equiv 1$, although the existence of an analytical solution does not hinge on the first period labor endowment being non-stochastic and constant.

4.1. The analytical solution

In this economy we can characterize the equilibrium analytically because the consumer's maximization problem has the following simple solution. Optimal savings for a typical generation that is born at node z^t are given by

$$
\begin{aligned}
a_t^t(z^t) &= \left(\frac{\beta(1-\beta^{N-1})}{1-\beta^N}\right) w_t(z^t),\\
a_{t+1}^t(z^{t+1}) &= \left(\frac{\beta(1-\beta^{N-2})}{1-\beta^{N-1}}\right) r(z^{t+1}) a_t^t(z^t),\\
&= \left(\frac{\beta^2-\beta^N}{1-\beta^N}\right) r(z^{t+1}) w_t(z^t),\\
&\vdots\\
a_{t+N-2}^t(z^{t+N-2}) &= \left(\frac{(\beta^{N-1}-\beta^N)}{1-\beta^N}\right) r(z^{t+N-2})\ldots r(z^{t+1}) w(z^t). \qquad (17)
\end{aligned}
$$

In order to solve for the FREE one needs to determine the law of motion for the aggregate capital stock K since it determines, together with the stochastic shock, the gross return to capital $r()$. By the market clearing condition from the asset market we find that

$$K(z^t) = \sum_{i=t-N+1}^{t-1} a_{t-1}^i(z^{t-1}). \tag{18}$$

Since $r(z^t) = f_K(K(z^t), 1, z_t)$ and $w(z^t) = f_L(K(z^t), 1, z_t)$, K will simply follow an $(N-1)$th order stochastic difference equation. Define

$$\gamma_i = \left(\frac{\beta^i - \beta^N}{1-\beta^N}\right). \tag{19}$$

Then we find, denoting $K_t = K(z^t)$ and using (17)–(19), the analytical law of motion for this economy to be

$$\begin{aligned} K_t = {} & \gamma_1 * f_L(K_{t-1}, 1, z_{t-1}) + \gamma_2 * f_L(K_{t-2}, 1, z_{t-2}) * f_K(K_{t-1}, 1, z_{t-1}) \\ & + \cdots + \gamma_{N-1} f_L(K_{t-N+1}, 1, z_{t-N+1}) * f_K(K_{t-1}, 1, z_{t-1}) \\ & * \cdots * f_K(K_{t-N+2}, 1, z_{t-N+2}). \end{aligned} \tag{20}$$

For a given set of initial conditions $\{K_{-1}, \ldots, K_{-N+1}\}$ and a given simulated path of exogenous shocks $\{z_t\}_{t=0}^T$ we can use (20), (17) and the budget constraints (2) to generate the true equilibrium allocations $\{K(z^t), a_t^{t-s}(z^t), c_t^{t-s}(z^t)\}$.

4.2. Performance of our algorithm

Following our numerical algorithm described above we now approximate policy functions $\hat{\theta}$. Once these are obtained we choose some[6] initial conditions $(a_{-1}^0, \ldots, a_{-1}^{-N+1})$ and use these, together with a simulated path of shocks $\{z_t\}_{t=0}^T$ to generate approximated equilibrium allocations.

We use a sequence of N simulated aggregate capital stocks and the same simulated path of shocks $\{z_t\}_{t=N+1}^T$ to generate true equilibrium allocations, for initial conditions that are consistent with the initial conditions for the approximated equilibrium. We then compare these allocations to the true equilibrium allocations. For the evaluation we discard the first 1000 of our 15,000 simulated observations, to avoid the influence of initial conditions on our results.

We parametrize the economy as follows. We set $\beta = 0.95^{60/N}$. We consider a Cobb-Douglas production function, $F(K,L) = K^\alpha L^{1-\alpha}$ and pick a capital share of $\alpha = 0.3$. The stochastic production shock is assumed to be *iid* across time and can take four values, i.e. $\mathscr{Z} = \{z_1, z_2, z_3, z_4\}$ with equal probability of $\frac{1}{4}$. The *iid* assumption is mainly made for expositional simplicity (the model has an analytical solution with serially correlated aggregate shocks as well). Experiments with positively correlated technology shocks deliver results that are similar to those reported in the text.

[6] In practice we use the steady state asset holdings for some deterministic steady state in which the aggregate shock is fixed permanently at an intermediate level.

Table 2
Specifications for shocks

	State 1	State 2	State 3	State 4
Case 1: η	0.95	1.05	0.95	1.05
Case 1: δ	0.7	0.7	0.7	0.7
Case 2: η	0.85	1.15	0.85	1.15
Case 2: δ	0.7	0.7	0.7	0.7
Case 3: η	0.95	1.05	0.95	1.05
Case 3: δ	0.5	0.5	0.9	0.9
Case 4: η	0.85	1.15	0.85	1.15
Case 4: δ	0.5	0.5	0.9	0.9

Table 3
Maximal errors in allocations and average running times

ρ, N	Case 1	Case 2	Case 3	Case 4	Avg. run. (min) time (s)
2,3	6.7 (−4)	2.2 (−3)	8.2 (−3)	1.9 (−2)	0, 0.1
4,3	1.1 (−5)	1.0 (−4)	3.4 (−4)	5.1 (−4)	0, 0.2
8,3	4.2 (−6)	1.2 (−5)	4.9 (−5)	1.1 (−4)	0, 0.3
2,6	6.3 (−2)	9.1 (−2)	1.2 (−1)	2.8 (−1)	0, 0.7
4,6	3.9 (−4)	9.8 (−4)	2.3 (−3)	3.2 (−3)	0, 3.6
8,6	7.4 (−5)	1.0 (−4)	3.5 (−4)	7.7 (−4)	1, 26.3
2,9	9.8 (−2)	2.4 (−1)	5.1 (−1)	6.9 (−1)	1, 50
4,9	1.1 (−3)	7.3 (−3)	2.0 (−2)	3.8 (−2)	6, 30
8,9	6.7 (−4)	9.9 (−4)	3.1 (−3)	7.2 (−3)	42, 20

In order to shed some light on whether and to what extent the size of aggregate shocks matters and whether it is shocks to the return to capital or shocks to returns to labor that matter more for the quality of the approximation we consider four different cases for the values the productivity and depreciation shocks.

In Table 2 we list the parameters for the four cases. While it is obvious that for realistic calibrations the magnitude of the shocks has to depend on the length of a period (and therefore on the number of agents), we keep these four cases fixed throughout the paper. Our goal in this paper is to evaluate the proposed algorithm for a variety of different shocks.

In Table 3 we show how the number of generations N, the degree of the interpolating polynomial ρ (recall that the dimension of the state space is $N-1$ and that in the notation of Section 3 $\rho = 2^{q-d}$) as well as the magnitude of productivity and depreciation shocks influences the accuracy of our algorithm, as compared to the true

equilibrium allocations (recall that the dimension d of the problem equals $d = N-1$). We implemented the algorithm for 2nd degree complete polynomials ($q = d + 1$), 4th ($q = d + 2$) and 8th ($q = d + 3$) degree polynomials. We compute equilibria for $N = 3$, 6 and 9. For all examples we set the stopping tolerance of the time iteration method to $\tau = 10^{-6}$. As it turns out for this model specification, the results are numerically highly unstable for large N. This is caused by the fact, that agents have zero endowments for all but one period of their lives. We therefore do not consider $N > 10$ in this section.

The results are based on simulations of the economy for 14,000 periods (after discharging the initial 1000 observations). What we report is the maximal error $\max_t|(\hat{K}_t - K_t)/K_t|$ in the aggregate capital stock $\hat{K}_t$ computed with our numerical approximation, as compared to the true analytical solution K_t. We also report running times in minutes, seconds.

We see that in almost all cases where $N = 3$ the quality of the approximation is excellent – a maximal error of 10^{-4} (in the table written as $1.0(-4)$) implies that if the true capital stock were always equal to 1 the approximated capital stock is never smaller than 0.9999 or larger than 1.0001. In general the quality of the approximation (but also obviously the running time of the algorithm) improves with the number of points. It is generally not advisable to only use polynomials of order 2 for the approximations.

It is interesting to note that the quality of the approximation decreases with the size of the technology shocks (compare errors for cases 1 and 2, and for cases 3 and 4); this is due to the fact that with a larger support of technology shocks the deterministic steady states lie further apart and hence the approximation occurs over a larger state space, which leads to poorer quality of the approximation. More importantly from a quantitative point of view, however, is that the approximation becomes significantly worse for economies for which stochastic returns to labor and to capital are imperfectly correlated (cases 3 and 4). In these cases the decision rules are more 'curved' in the own asset holdings and thus cannot be approximated as well with Chebyshev polynomials as for the case with non-stochastic depreciation.

As the number of agents, N, increases, the quality of the approximation decreases quite rapidly. In Section 5 below we will examine the performance of the algorithm for models with many agents in more detail. In this section we restrict ourselves to cases where agents live for a maximal number of nine periods.

In the table also we report running times of our algorithm for economies with different numbers of generations and varying degrees of polynomials (and correspondingly, collocation points). The running times were very similar across the different cases we therefore only report the averages across the cases. All running times reported refer to an implementation on a 450 MHz Pentium II.

The table shows that running times only increase moderately with d when one uses polynomials of degree 4 or smaller. For $q = d + 3$ (i.e. polynomials of degree 8) running times increase dramatically in d – we will come back to this point in Section 5 below.

4.3. Errors in Euler equations

As mentioned above, we are mainly be interested in economies where the numerically approximated equilibrium cannot be compared to a true solution. Judd (1992) proposes

Table 4
Maximal Euler equation error

ρ, N	Case 1	Case 2	Case 3	Case 4
4,3	8.3 (−5)	2.5 (−4)	9.2 (−4)	2.3 (−3)
8,3	9.2 (−6)	4.2 (−5)	9.9 (−5)	2.5 (−4)
4,6	8.2 (−4)	3.3 (−3)	4.9 (−3)	7.7 (−3)
8,6	1.5 (−4)	2.4 (−4)	8.8 (−4)	1.5 (−3)
4,9	3.9 (−3)	2.1 (−2)	4.8 (−2)	5.9 (−2)
8,9	1.7 (−3)	2.9 (−3)	6.1 (−3)	2.2 (−2)

the following criterion to assess the quality of an approximation:

$$e^s(z^t) = 1 - \frac{u'^{-1}(\beta \sum_{z_{t+1} \in \mathscr{Z}} \Pi(z_{t+1}|z_t)\hat{r}(z^{t+1})u'(\hat{c}^s_{t+1}(z^{t+1})))}{\hat{c}^s_t(z^t)}. \tag{21}$$

The quantity $e^s(z^t)$ is the relative Euler equation error for generation s at node z^t. In this expression $\hat{c}$ and $\hat{r}$ are the approximated consumption allocations and interest rates as delivered by a particular numerical algorithm. As Judd points out this measure has the practical advantage that it is unit free, i.e. it is invariant to a scaling of the utility function. It has the intuitive appeal in that it describes the magnitude of agents' optimization errors, holding next period consumption fixed.

Throughout this paper we will report maximal optimization errors and average optimization errors along a simulated path. The rationale for reporting both errors is that low average errors could be caused by zero errors for long periods of time followed by huge errors over a few periods. Such an approximate equilibrium would seem very unsatisfactory. On the other hand, each agent could make the same error over and over again, which would lead to relatively low maximal errors but high average errors.

For a FREE, since we know the box $\mathscr{B}$, we could compute maximal errors over a grid which covers $\mathscr{B}$. However, when the dimension of the box is large, this turns out to be impractical. In order to obtain 'maximal errors' we therefore simulate the economy for 15,000 periods and report maximal errors along the simulated path. We compared this error with maximal errors on a grid for low dimensional problems ($N \leqslant 10$) and find that they always lie within the same order of magnitude.

Even though for the present model we know the true solution, it is useful to report errors in Euler equations in order to obtain an indication as to how errors in the allocations translate into errors in Euler equations.

Table 4 shows the results. As with the errors in allocations, increasing the number of collocation points reduces the approximation error, when measured by the relative Euler equation errors. In addition, expanding the state space, due to the inclusion of more generations into the model, and keeping the number of collocation points in each dimension fixed reduces the accuracy of the solutions quite quickly, with relative errors in Euler equations increasing by a factor 10 when increasing the number of generations by 3 (for Case 1).

5. Models without an analytic solution

In order to demonstrate how our algorithm performs for more realistic specifications of endowments, preferences and technology, we now consider cases where agents have positive labor endowments in more than the first period and where utility is not restricted to be logarithmic. No analytic solution is available for these economies and the evaluation of the algorithm has to be based on relative errors in Euler equations. Throughout we consider the same specification as above for production functions and aggregate shocks (see Table 2) but we assume that agents receive endowments throughout their lives. Furthermore we assume that their utility exhibits constant relative risk aversion with a coefficient of relative risk aversion of 2.

We consider the following life cycle-profile for labor incomes

$$l_n = 1 \qquad \text{for } n \leqslant 7/10,$$

$$l_n = 0.25 \quad \text{otherwise.}$$

For $N = 10$ (i.e. $d = 9$) we compare the performance of our algorithm for different degrees of the approximating polynomial. We want to examine the trade-off between running times and approximation error in some more detail. We focus on Case 4 for the specification of the aggregate shock.

As above, we set the stopping tolerance $\tau = 10^{-6}$. Smaller values do not decrease the maximal error in Euler equations.

Table 5 shows that for the computation of models with more than 10 agents it is infeasible to use polynomials of degree 8 (recall that we denote the polynomial degree by ρ). On the other hand, the increase in precision when going from fourth to eighth order approximations is significant and in some applications Euler equation errors of 10^{-2} may be unacceptably high.

The reason for the high running times in the case $q = d + 3$ lies in the fact that the evaluation of the approximating polynomial at a single point involves $d(d-1)(d-2)/6$ evaluations of three-dimensional tensor products of polynomials of degree two as well as $d(d-1)$ evaluations of two-dimensional tensor products of polynomials of degree two times degree four. Since our algorithm needs to solve a non-linear system of equations at each collocation point, and since it uses Newton's method and finite differences to compute the Jacobian of the system, in each iteration the polynomials have to be evaluated at least $M \cdot d^2/2$ times (where M is the number of collocation points) for each exogenous shock.

Table 5
Running times and errors

ρ, N	Max. Err.	Avg. Err.	Run. Time
(2,10)	8.9 (−1)	4.5 (−2)	0 min, 6 s
(4,10)	1.2 (−2)	2.7 (−3)	3 min, 16 s
(8,10)	3.6 (−3)	9.6 (−4)	109 min, 52 s

Table 6
Errors and running times

	Case 1	Case 2	Case 3	Case 4
$N = 20$, Max. Err.	6.9 (−3)	7.8 (−3)	8.1 (−3)	9.0 (−3)
$N = 20$, Avg. Err.	1.4 (−3)	4.4 (−3)	2.8 (−3)	4.2 (−3)
running time	1 min, 16 s	1 min, 17 s	1 min, 32 s	1 min, 32 s
$N = 30$, Max. Err.	1.1 (−2)	2.1 (−2)	3.0 (−2)	4.2 (−2)
$N = 30$, Avg. Err.	2.3 (−3)	4.3 (−3)	5.1 (−3)	5.5 (−3)
running time	27 min, 12 s	27 min, 49 s	28 min, 1 s	28 min, 2 s

Running times can be reduced effectively, if a solution is computed using a 4th order approximation and then one time iteration is carried out to compute 8th order approximations. In the example above (for $q = 11$ and $N = 10$) maximal errors can be reduced to 7.6(−3) while running times remain around 6 minutes.

5.1. Many agents

In this section we evaluate the performance of the algorithm for examples where agents live for more than 10 periods. We consider $N = 20$ and 30. In all examples, we use fourth order polynomials to calculate a solution and 8th order polynomials for the final approximation.

Table 6 shows how running times and errors increase with the dimension of the state space. For models with 20 agents the algorithm still provides good approximations to equilibria in reasonable running times. For models with 30 periods errors become larger and at the same time running times increase dramatically. For models with more than 30 periods the algorithm quickly becomes infeasible. If one uses lower degree approximations (i.e. $q = d + 1$), running times remain acceptable for models with 40-50 periods but errors in Euler equations exceed 0.5 and the quality of the approximation becomes unacceptable.

In summary, the projection algorithm we propose can only be applied effectively for models with less than 30 agents. If the number of agents is larger than this, one has to resort to methods which approximate the endogenous state itself.

6. Low-dimensional approximation of the state space

In a sequence of influential papers Krusell and Smith (1997, 1998) propose a method to compute approximate equilibria in economies with a large number of heterogeneous agents, aggregate uncertainty and incomplete asset market structure. Storesletten et al. (2001) and Gourinchas (2000) use the same approach to approximate equilibria in stochastic OLG models.

Agents, in order to solve their dynamic programming problem, have to forecast future wages and interest rates, which are stochastic due to aggregate productivity

shocks. To optimally forecast these factor prices, agents have to form forecasts about the future aggregate capital stock. Knowledge of the entire current wealth distribution is required to form exact forecasts. The key insight of Krusell and Smith is that in many specifications of the model very accurate forecasts can be obtained using a low dimensional summary statistic of the wealth distribution. By restricting the endogenous state space to a small number of moments of the current wealth distribution their algorithm breaks the curse of dimensionality and is applicable in a wide range of models, with a large number of agents that are heterogeneous along several dimensions. As such, it is a prime candidate for the approximation of equilibria in stochastic OLG economies with a large number of generations.

Applying their algorithm to a model with a continuum of infinitely lived agents (no heterogeneity with respect to age or preferences), borrowing constraints and uninsurable idiosyncratic labor income uncertainty Krusell and Smith (1998) find that the aggregate capital stock is sufficient for private agents to forecast future returns to labor and capital with high accuracy. Thus, although exact aggregation of saving behavior across households fails, 'quasi-aggregation' is obtained, in the sense that agents do not make large forecasting errors and do not suffer large welfare losses when behaving as if exact aggregation prevails.

In this section we will compare the projection algorithm developed in the last sections to a simple version of Krusell and Smith's algorithm which conjectures that quasi-aggregation obtains in the OLG model, i.e. that the law of motion of aggregate capital can be described with high accuracy using only current aggregate capital and the exogenous shock. Our objective in comparing our algorithm with theirs is not to provide a critique to their algorithm, but rather to identify when and under which circumstances quasi-aggregation fails; i.e. under which circumstances more information about the wealth distribution, besides the first and possibly higher moments, are needed for agents to form accurate forecasts when judged by Euler equations residuals.[7] We also aim at investigating whether and when their measures of goodness of fit of price forecasts are appropriate in assessing the accuracy of the numerical approximation of equilibrium.

In order to compare results obtained with their algorithm to results from our algorithm it is useful to briefly describe theirs in the context of our model. We restrict the discussion to the case where only the first moment of the wealth distribution is used in order to form forecasts.

1. Guess on a functional form for the forecasting rules

$$\begin{aligned}\log K' &= \Gamma(K, z; \phi)\\ &= \phi_{0z} + \phi_{1z} \log K\end{aligned}$$

for all $z \in \mathscr{Z}$. Here K' is the aggregate capital stock tomorrow and ϕ is a parameter vector parameterizing the function Γ.

[7] Our algorithm and our findings in this paper suggest that it would be particularly fruitful to add age-specific mean capital holdings to the state space (rather than, say, the variance of the asset distribution). Of course, in the limit as all age-specific asset holdings are added both approaches coincide.

2. Guess $\phi = \phi^0$.
3. For $\phi = \phi^j$ solve the consumers problem. We approximate the policy function over the two-dimensional endogenous 'state-space' by tensor products of Chebyshev polynomials of degree 5.[8] The policy function $a' = a^i(a,z,K)$ solves

$$u'(r(z,K)a + l^i w(z,K) - a') = \beta \mathrm{E}_z r(K',z')u'(r(z',K')a' + l^{i+1}w(z',K') - a''),$$

$$K' = \Gamma(K,z;\phi),$$

where $a'' = a^{i+1}(a',z',K')$ and $a^N \equiv 0$.
4. Simulate the economy using the policy functions from step 3 to obtain $\{z_t, K_t\}_{t=0}^T$. Use these data to obtain a new estimate of ϕ^{j+1} in the following way. First sort the data $\{z_t, K_t\}_{t=0}^T$ according to the aggregate shock z_t: for all $z \in \mathscr{Z}$, if $z_t = z$, then $y_\tau^z = \log(K_{t+1})$ and $x_\tau^z = \log(K_t)$. For all $z \in \mathscr{Z}$, obtain ϕ_{iz}^{j+1} as OLS estimates of the regression

$$y_\tau^z = \phi_{0z}^{j+1} + \phi_{1z}^{j+1} x_\tau^z + \varepsilon_\tau^z, \tag{22}$$

where ε_τ^z is the error term of the regression. Also compute measures of fit of this regression (standard error and the R^2 of the regression).
5. Repeat steps 3 and 4 until convergence of the ϕ^j is achieved.[9]

6.1. Sufficient conditions for quasi aggregation

The key to understanding why in the paper of Krusell and Smith the algorithm leads to good approximations even if only the first moment is used, is to realize that, apart from the (small number of) agents right at the borrowing constraint all agents in their model have approximately the same marginal propensity to save out of current wealth, so that quasi-aggregation is obtained. However, when agents differ with respect to their age as in our model, in the absence of operative bequest motives their propensity to save will vary greatly by age and thus quasi-aggregation may fail.

However, as we will see, there are many specifications of the model, where quasi-aggregation still obtains. Only if there is sufficient *variation* in the shares of the aggregate capital stock held by different agents in equilibrium the aggregate law of motion will not be characterized sufficiently well by just the current aggregate capital stock.

In order to make this concrete it is useful to return to the model from Section 4 and investigate under which conditions quasi-aggregation holds in that model. For $N = 3$, a Cobb–Douglas production function and total labor normalized to one, Eq. (20) reduces to

$$K_t = \gamma_1(1-\alpha)\eta_{t-1}K_{t-1}^\alpha + \gamma_2\alpha(1-\alpha)\eta_{t-2}K_{t-2}^\alpha(\eta_{t-1}K_{t-1}^{\alpha-1} + 1 - \delta_{t-1}).$$

In order to explore under which conditions the aggregate capital alone forms an approximately sufficient state, we set $\alpha = 0.3$, $\beta = 0.7$ and we simulate this equation for 15,000 periods and compute the R^2 of the regression described in Eq. (22) above for

[8] In the examples reported below a higher-dimensional approximation of the policy functions did not improve the maximal errors in Euler equations substantially, but increased running times considerably.

[9] In most examples considered below relaxation has to be used to ensure convergence of the algorithm.

Table 7
Minimal R^2

δ, η	(0.95, 1.05)	(0.9, 1.1)	(0.85, 1.15)	(0.8, 1.2)
(0.9, 0.5)	0.662	0.842	0.913	0.944
(0.7, 0.5)	0.852	0.932	0.967	0.979
(0.9, 0.7)	0.857	0.951	0.976	0.982
(0.5, 0.5)	0.984	0.984	0.984	0.985
(0.7, 0.7)	0.995	0.995	0.995	0.995
(0.9, 0.9)	0.999	0.999	0.999	0.999

different specifications of the productivity shock and depreciation. If and only if exact aggregation obtains, this value is equal to 1. A value close to one implies that on average, agents forecast next period's aggregate capital stock with high precision.

Table 7 shows the results. The rows correspond to different specifications of the depreciation shock δ, and the columns to different specifications of the TFP shock η. There are many cases where it is sufficient to describe the evolution of the capital stock as a function of last period's aggregate capital stock alone. In particular, if depreciation is non-stochastic the OLS coefficient estimates describe the evolution of the aggregate capital stock well, independently of the magnitude of the productivity shock. The reason for this result is that in this case the distribution of wealth does not change much along the equilibrium path. A bad productivity shock influences both the income of the young and the income of the middle aged. With stochastic depreciation this result disappears. It is worth mentioning that we obtain better measures of fit R^2 for TFP specifications that allow for larger-size technology shocks.

Fig. 2 shows a scatter plot from simulated aggregate capital stock data, plotting tomorrow's capital stock (contingent on the exogenous shock being 3) against today's capital stock (in logs) for the parameterization of technology and depreciation shocks $\delta \in \{0.9, 0.5\}$ and $\eta \in \{0.95, 1.05\}$. The figure shows that the true law of motion is nowhere close to the one obtained by Krusell and Smith's algorithm, when restricting attention to aggregate laws of motion that contain only the first moment of the cross-sectional wealth distribution. The R^2 of an OLS regression for this case is 0.662. For this specification clearly quasi-aggregation fails; if one were to simulate the economy by using the approximated aggregate law of motion to generate a time series for the aggregate capital stock and then use this time series and the individual policy functions to compute individual allocations and welfare, potentially large mistakes are committed.

It is interesting to note that for this special case with log utility and non-zero labor endowments only in the first period of an agent's life (Huffman economy) the perception of a wrong aggregate law of motion does not lead to wrong consumption and savings decisions and therefore does not result in welfare losses from poor forecasting of future interest rates. In particular, if one were to compute relative errors in Euler equations, these would be only non-zero because the agents' policy function have to be approximated, but not because predicted and realized interest rates differ. Agents do make substantial prediction errors about future interest rates, but since, in

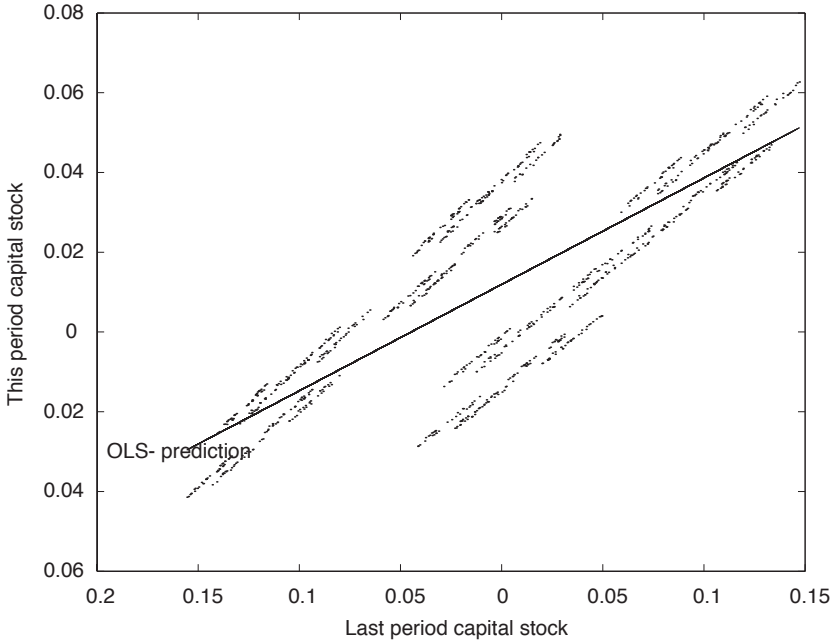

Fig. 2. Scatter plot of aggregate capital as function of lagged capital.

the simple example, decisions do not depend on interest rates, prediction errors do not lead to optimization errors. The simple example economy is only useful to illustrate that approximating the true law of motion by a log-linear function of last periods' aggregate capital stock and the current exogenous shock can, in some cases, lead to serious errors if one uses the aggregate law of motion in order to generate time series for the aggregate capital stock. It also illustrates that a low R^2 alone is by no means sufficient to conclude that private agents make large optimization errors in their private decisions.

6.2. Models without an analytical solution

In order to examine examples in which a wrong forecasting rule for prices does lead to welfare losses we now reconsider the examples from Section 5 above. The Euler equation errors reported in this section use as equilibrium price sequence the sequence obtained from correct aggregation of individual decisions and not from the estimated aggregate law of motion.

Table 8 shows the errors in Euler equations for different specifications of the aggregate shock and N. There are two things to note: For Cases 1 and 2 (i.e. cases without stochastic depreciation) the errors are generally very low. In fact the errors in these cases are comparable to the errors obtained in the projection approach. For these cases the quality of the approximation does decrease slightly, as N increases. However, the errors remain fairly low even if N is increased to 40 and beyond.

Table 8
Euler equation errors

	Case 1	Case 2	Case 3	Case 4
$N = 10$, Max. Err.	6.7 (−3)	1.2 (−2)	6.77 (−2)	1.1 (−1)
$N = 10$, Avg. Err.	3.5 (−3)	8.7 (−3)	4.2 (−2)	6.4 (−2)
$N = 10$, R^2	0.999	0.994	0.890	0.941
$N = 20$, Max. Err.	1.1 (−2)	3.8 (−2)	1.1 (−1)	2.9 (−1)
$N = 20$, Avg. Err.	7.3 (−3)	2.2 (−2)	6.8 (−2)	9.1 (−2)
$N = 20$, R^2	0.994	0.993	0.792	0.932
$N = 30$, Max. Err.	2.0 (−2)	4.2 (−2)	2.1 (−1)	4.1 (−1)
$N = 30$, Avg. Err.	1.1 (−2)	2.2 (−2)	8.7 (−2)	1.1 (−1)
$N = 30$, R^2	0.989	0.985	0.712	0.913

Table 9
Selected running times

N	Case 1	Case 2	Case 3	Case 4
10	1 min, 5 s	1 min, 9 s	1 min, 41 s	1 min, 59 s
20	1 min, 12 s	1 min, 21 s	1 min, 53 s	2 min, 5 s
30	1 min, 15 s	1 min, 31 s	2 min, 1 s	2 min, 12 s

However, for Cases 3 and 4 the errors are unacceptably high throughout: for example, for $N = 20$ in Case 4 agents make optimization errors of up to 29 percent and on average errors of 9.1 percent! This is consistent with the intuition developed above. Disentangling returns to capital and returns to labor leads to a variation of the wealth distribution across time and quasi-aggregation fails.

Also note, that there is not a one to one correlation between R^2 and maximal errors in Euler equations. As in the analytical example above, high depreciation shocks with small TFP shocks lead to a very low R^2 while errors in Euler equations are highest with the high TFP shock.

6.2.1. Running times

In terms of running times, for $N = 10$ the projection algorithm is about 10–20 times faster than Krusell and Smith's algorithm. In Table 9 we report running times for the case where we simulate the economy for 10,000 periods in each iteration.

We do not want to stress long running times as one can expedite their algorithm by reducing the number of simulation periods for each update of the aggregate law of motion, which is the single most time-consuming step in their procedure. Furthermore, for models with more than $N = 20$ generations the Krusell–Smith algorithm quickly becomes competitive in running times, compared to ours. This is to expected since our algorithm does not break the curse of dimensionality, whereas theirs does, at the cost of making agents ignore (not necessarily, but potentially) valuable information when

forming forecasts about future prices. Our algorithm precisely computes decision rules even at points of the state space that are never visited along equilibrium paths. Krusell and Smith's ingenious insight is to reduce the state space to a lower dimension, to ignore a large number of points of the full state space that are never visited and to conjecture that not many points that would be visited along equilibrium paths, had they been in the state space, are thrown out.

To us this suggests that the choice of the algorithm depends heavily on the application: when a model with large number of generations and heterogeneity within generations is desired, Krusell and Smith's approach appears to be the only feasible way to proceed, whereas for fairly large OLG models without intragenerational heterogeneity we would argue to have provided a viable contender which maintains full rationality of agents.

7. Conclusion

In this paper we developed a projection algorithm to compute equilibria in OLG economies with multiple generations and aggregate uncertainty. We compared its accuracy with those of alternative methods to compute recursive equilibria for models with a large state space, both for an example economy for which the equilibrium can be computed analytically and for more realistic examples for which it cannot.

While this paper provides an evaluation of several algorithms for stochastic OLG models with simple asset market structure, it remains to be explored whether a projection algorithm along the lines proposed in this paper can be successfully applied for other classes of stochastic dynamic general equilibrium model with a large number of agents and richer financial markets. We leave this to future research.

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