Documentation of R Programs for the COVID-adjusted Laubach and Williams Model,

in “Measuring the Natural Rate of Interest After COVID-19”*

Kathryn Holston Thomas Laubach

John C. Williams

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This note documents the R code used for the estimation of the natural rate of interest, natural rate of output, and its trend growth rate for the United States using the COVID-adjusted Laubach and Williams model, presented in “Measuring the Natural Rate of Interest After COVID-19” (Holston, Laubach, Williams 2023; henceforth HLW). The COVID-adjusted LW model presented here builds on the model in Laubach and Williams (2003). The code documented here corresponds to the FRBNY Staff Report published in June 2023.[1]

# 1 Code Layout and Directory Structure

There is one main R file, *run.lw.R*, which does the following:

1. Reads in pre-processed data to be used in the LW estimation;
2. Defines the sample period, constraints, and variables to be used throughout the estimation;
3. Runs the three-stage LW estimation;
4. Saves output.

This file calls multiple R functions and files, each of which are described in this guide. To run the code without modification, use the following structure:

1. A subdirectory titled “inputData” should contain the provided input data;
2. An empty subdirectory titled “output” should be created and will populate with estimates and model output.
3. Optional: create a subdirectory titled “Rpackages” to use as the library location for downloaded packages.

*Holston: Harvard University, kathrynaholston@gmail.com.

Williams: Federal Reserve Bank of New York.

[1]https://www.newyorkfed.org/medialibrary/media/research/staff_reports/sr1063.pdf

**Input data for the United States will be published quarterly with the current LW estimates on the FRBNY website. Please save this Excel file in your *inputData* directory.** For reference, we are using R Version 4.2.1 at the time of release.

# 2 Raw Data

See Laubach and Williams (2003) for a description of data used in the estimation. In the estimation after COVID-19, we add an indicator variable described below, which we provide with the release of pre-processed data and estimates each quarter.

## 2.1 COVID-19 Indicator Variable

We use a COVID indicator variable equal to the quarterly average of the COVID-19 Stringency Index from the Oxford COVID-19 Government Response Tracker (OxCGRT) for the United States (Hale et al., 2021). We use the national weighted average of the stringency indices for vaccinated and unvaccinated populations. As the OxCGRT project suspended data collection at the end of 2022, we assume each indicator variable declines linearly beginning in 2023:Q1, reaching zero in 2024:Q4. The COVID indicator is set equal to zero up to and including 2019:Q4.

# 3 Basic Functions used Throughout LW Programs

In the accompanying set of code, these functions are stored in *utilities.R*.

**Function:** *shiftQuarter*
**Description:** This function takes in a (year, quarter) date in time series format and a shift number, and returns the (year, quarter) date corresponding to the shift. Positive values of shift produce leads and negative values of shift produce lags. For example, entering 2014q1 with a shift of -1 would return 2013q4. Entering 2014q1 with a shift of 1 would return 2014q2. In each case, the first argument of the function must be entered as a two-element vector, where the first element corresponds to the year and the second element corresponds to the quarter. For example, 2014q1 must be entered as "c(2014, 1)".

**Function**: *shiftMonth*
**Description**: This function takes in a (year, month) date in time series format and a shift number, and returns the (year, month) date corresponding to the shift. Positive values of shift produce leads and negative values of shift produce lags. For example, entering 2014m1 with a shift of -1 would return 2013m12. Entering 2014m1 with a shift of 1 would return 2014m2. In each case, the first argument of the function must be entered as a two-element vector, where the first element corresponds to the year and the second element corresponds to the month. This function is analogous to shiftQuarter().

**Function**: *gradient*
**Description**: This function computes the gradient of a function f given a vector input x.

# 4 R Packages

The "tis" package is used to manage time series data. The "mFilter" package contains the hpfilter() function. We use the "nloptr" package for optimization. The "openxlsx" package is used to read from and write to Excel.

# 5 Estimation

The results reported in Laubach and Williams (2003) are based on our estimation method described in Section 2 of the paper. The estimation proceeds in sequential steps through three stages, each of which is implemented in an R program. These and all other R programs are described in this section.

## 5.1 Main Estimation Program: *run.lw.R*

The program *run.lw.R* defines and trims the sample for the key variables, included in the pre-processed data: log output, inflation, the real and nominal short-term interest rates, oil and import price inflation, and the COVID indicator variable. Constraints on $a_r$ and $b_y$, specifications for the COVID-adjusted model, and initialization of the state vector and covariance matrix are defined in this file. It calls the programs *rstar.stageX.R* to run the three stages of the LW estimation. Additionally, it calls the programs *median.unbiased.estimator.stageX.R* to obtain the signal-to-noise ratios $\lambda_g$ and $\lambda_z$.

The programs *unpack.parameters.stageX.R* set up coefficient matrices for the corresponding state-space models for the given parameter vectors. In all stages, we impose the constraint $b_y \geq 0.025$. In stages 2 and 3, we impose $a_r \leq -0.0025$. These constraints are labeled as a.r.constraint and b.y.constraint, respectively, in the code.

## 5.2 The Stage 1, 2, and 3 COVID-adjusted State-Space Models

This section presents the COVID-adjusted state-space models; that is, the version of the LW models that are estimated in HLW (2023). See Section 3 of HLW (2023) for a description of COVID-related modifications to the LW (2003) model and Appendix A4 for additional technical changes. For reference, the next section presents the LW (2003) state-space models. The following section documents the corresponding R programs. Notation matches that of Hamilton (1994) and is also used in the R programs. All of the state-space models can be cast in the form:

$$\mathbf{y}_t = \mathbf{A}' \cdot \mathbf{x}_t + \mathbf{H}' \cdot \xi_t + \epsilon_t \tag{1}$$

$$\xi_t = \mathbf{F} \cdot \xi_{t-1} + \mathbf{c} + \eta_t \tag{2}$$

Here, $\mathbf{y}_t$ is a vector of contemporaneous endogenous variables, while $\mathbf{x}_t$ is a vector of exogenous and lagged exogenous variables. $\xi_t$ is vector of unobserved states. In the LW (2003) model, the vectors of stochastic disturbances $\epsilon_t$ and $\eta_t$ are assumed to be Gaussian and mututally uncorrelated, with mean zero and covariance matrices $\mathbf{R}$ and $\mathbf{Q}$, respectively. The covariance matrix $\mathbf{R}$ is always assumed to be diagonal. In the COVID-adjusted LW model, the covariance matrix $\mathbf{R}_t$ is time-varying. $\mathbf{c}$ is 0 in Stages 2 and 3.

For each model, there is a corresponding vector of parameters to be estimated by maximum likelihood. Because maximum likelihood estimates of the innovations to $g$ and $z$, $\sigma_g$ and $\sigma_z$, are likely to be biased towards zero (see Section 2 of LW for explanation), we use Stock and Watson's (1998) mediun unbiased estimator to obtain estimates of two ratios, $\lambda_g \equiv \frac{\sigma_g}{\sigma_{y^*}}$ and $\lambda_z \equiv \frac{a_r \sigma_z}{\sigma_{\tilde{y}}}$. We impose these ratios when estimating the remaining model parameters by maximum likelihood. Departures from the LW (2003) model related to the COVID pandemic are highlighted in red, while other modifications are highlighted in blue.

## 5.3 The COVID-Adjusted State-Space Models

### 5.3.1 The COVID-adjusted Stage 1 Model

The first-stage model, which corresponds to the *rstar.stage1.R* program, can be represented by the following matrices:

$$\begin{aligned}
\mathbf{y}_t &= [y_t, \pi_t]' \\
\mathbf{x}_t &= \left[y_{t-1}, y_{t-2}, \pi_{t-1}, \pi_{t-2,4}, \pi_{t-5,8}, \pi^0_{t-1} - \pi_{t-1}, \pi^m_t - \pi_t, d_t, d_{t-1}, d_{t-2}\right]' \\
\xi_t &= \left[y^*_t, y^*_{t-1}, y^*_{t-2}\right]'
\end{aligned}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_1 & -a_2 \\ 0 & -b_3 & 0 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & \phi & -\phi a_1 & -\phi a_2 \\ b_3 & 0 & b_1 & b_2 & 1-b_1-b_2 & b_4 & b_5 & 0 & -\phi b_3 & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R}_t = \begin{bmatrix} (\kappa_t \sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t \sigma_\pi)^2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_1 = [a_1, a_2, b_1, b_2, b_3, b_4, b_5, g, \sigma_1, \sigma_2, \sigma_4, \phi, \kappa_{2020Q2-Q4}, \kappa_{2021}, \kappa_{2022}]$$

#### 5.3.2 The COVID-adjusted Stage 2 Model

The second-stage model, which corresponds to the *rstar.stage2.R* program, can be represented by the following matrices:

$$\begin{aligned} \mathbf{y}_t &= [y_t, \pi_t]' \\ \mathbf{x}_t &= [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, \pi_{t-5,8}, \pi_{t-1}^0 - \pi_{t-1}, \pi_t^m - \pi_t, 1, d_t, d_{t-1}, d_{t-2}]' \\ \xi_t &= [y_t^*, y_{t-1}^*, y_{t-2}^*, g_t, g_{t-1}, g_{t-2}]' \end{aligned}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_1 & -a_2 & 0 & \frac{a_5}{2} & \frac{a_5}{2} \\ 0 & -b_3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} a_1 & a_2 & \frac{a_3}{2} & \frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 & a_4 & \phi & -\phi a_1 & -\phi a_2 \\ b_3 & 0 & 0 & 0 & b_1 & b_2 & 1-b_1-b_2 & b_4 & b_5 & 0 & 0 & -\phi b_3 & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R}_t = \begin{bmatrix} (\kappa_t \sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t \sigma_\pi)^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_2 = [a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5, \sigma_1, \sigma_2, \sigma_4, \phi, \kappa_{2020Q2-Q4}, \kappa_{2021}, \kappa_{2022}]$$

### 5.3.3 The COVID-adjusted Stage 3 Model

The third-stage model, which corresponds to the *rstar.stage3.R* program, can be represented by the following matrices:

$$\begin{aligned}
\mathbf{y}_t &= [y_t, \pi_t]' \\
\mathbf{x}_t &= \left[y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, \pi_{t-5,8}, \pi^0_{t-1} - \pi_{t-1}, \pi^m_t - \pi_t, d_t, d_{t-1}, d_{t-2}\right]' \\
\xi_t &= \left[y^*_t, y^*_{t-1}, y^*_{t-2}, g_t, g_{t-1}, g_{t-2}, z_t, z_{t-1}, z_{t-2}\right]'
\end{aligned}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_1 & -a_2 & 0 & -c\frac{a_3}{2} & -c\frac{a_3}{2} & 0 & \frac{-a_3}{2} & \frac{-a_3}{2} \\ 0 & -b_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} a_1 & a_2 & \frac{a_3}{2} & \frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 & \phi & -\phi a_1 & -\phi a_2 \\ b_3 & 0 & 0 & 0 & b_1 & b_2 & 1-b_1-b_2 & b_4 & b_5 & 0 & -\phi b_3 & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma^2_{y^*} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{\lambda_z \sigma_{\tilde{y}}}{a_r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{R}_t = \begin{bmatrix} (\kappa_t \sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t \sigma_\pi)^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_3 = [a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, c, \sigma_1, \sigma_2, \sigma_4, \phi, \kappa_{2020Q2-Q4}, \kappa_{2021}, \kappa_{2022}]$$

The law of motion for the natural rate of interest is $r^*_t = c \cdot g_t + z_t$, as in LW (2003).

### 5.4 The Standard Stage 1, 2, and 3 State-Space Models

This section includes the standard LW model presented in our paper, without the COVID-19 adjustment, for reference.

#### 5.4.1 The Stage 1 Model

The first-stage model, which corresponds to the *rstar.stage1.R* program, can be represented by the following matrices:

$$
\begin{aligned}
\mathbf{y}_t &= [y_t, \pi_t]' \\
\mathbf{x}_t &= \left[y_{t-1}, y_{t-2}, \pi_{t-1}, \pi_{t-2,4}, \pi_{t-5,8}, \pi^0_{t-1} - \pi_{t-1}, \pi^m_t - \pi_t\right]' \\
\xi_t &= \left[y^*_t, y^*_{t-1}, y^*_{t-2}\right]'
\end{aligned}
$$

$$
\mathbf{H}' = \begin{bmatrix} 1 & -a_1 & -a_2 \\ 0 & -b_3 & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 & 0 \\ b_3 & 0 & b_1 & b_2 & 1-b_1-b_2 & b_4 & b_5 \end{bmatrix}
$$

$$
\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma^2_{y^*} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix}
$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$
\theta_1 = [a_1, a_2, b_1, b_2, b_3, b_4, b_5, g, \sigma_1, \sigma_2, \sigma_4]
$$

#### 5.4.2 The Stage 2 Model

The second-stage model, which corresponds to the *rstar.stage2.R* program, can be represented by the following matrices:

$$
\begin{aligned}
\mathbf{y}_t &= [y_t, \pi_t]' \\
\mathbf{x}_t &= \left[y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, \pi_{t-5,8}, \pi^0_{t-1} - \pi_{t-1}, \pi^m_t - \pi_t, 1\right]' \\
\xi_t &= \left[y^*_t, y^*_{t-1}, y^*_{t-2}, g_{t-1}\right]'
\end{aligned}
$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_1 & -a_2 & a_5 \\ 0 & -b_3 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_1 & a_2 & \frac{a_3}{2} & \frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 & a_4 \\ b_3 & 0 & 0 & 0 & b_1 & b_2 & 1-b_1-b_2 & b_4 & b_5 & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_2 = [a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5, \sigma_1, \sigma_2, \sigma_4]$$

### 5.4.3 The Stage 3 Model

The third-stage model, which corresponds to the *rstar.stage3.R* program, can be represented by the following matrices:

$$\begin{aligned} \mathbf{y}_t &= [y_t, \pi_t]' \\ \mathbf{x}_t &= [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, \pi_{t-5,8}, \pi_{t-1}^0 - \pi_{t-1}, \pi_t^m - \pi_t]' \\ \xi_t &= [y_t^*, y_{t-1}^*, y_{t-2}^*, g_{t-1}, g_{t-2}, z_{t-1}, z_{t-2}]' \end{aligned}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_1 & -a_2 & -c\frac{a_3}{2} & -c\frac{a_3}{2} & \frac{-a_3}{2} & \frac{-a_3}{2} \\ 0 & -b_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_1 & a_2 & \frac{a_3}{2} & \frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & b_1 & b_2 & 1-b_1-b_2 & b_4 & b_5 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \left(1+\lambda_g^2\right)\sigma_{y^*}^2 & 0 & 0 & (\lambda_g\sigma_{y^*})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\lambda_g\sigma_{y^*})^2 & 0 & 0 & (\lambda_g\sigma_{y^*})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{\lambda_z\sigma_{\tilde{y}}}{a_r}\right)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_3 = [a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, c, \sigma_1, \sigma_2, \sigma_4]$$

## 5.5 R Programs to Run the State-Space Models

The programs *rstar.stageX.R* run the models in stages 1-3 of the LW estimation.

## 5.6 R Programs for Median Unbiased Estimators

The function *median.unbiased.estimator.stage1.R* computes the exponential Wald statistic of Andrews and Ploberger (1994) for a structural break with unknown break date from the first difference of the preliminary estimate of the natural rate of output from the stage 1 model to obtain the median unbiased estimate of $\lambda_g$.

The function *median.unbiased.estimator.stage2.R* applies the exponential Wald test for an intercept shift in the IS equation at an unknown date to obtain the median unbiased estimate of $\lambda_z$, taking as input estimates from the stage 2 model.

## 5.7 Kalman Filter Programs

Within the program *kalman.states.R*, the function kalman.states() calls kalman.states.filtered() and kalman.states.smoothed() to apply the Kalman filter and smoother. It takes as input the coefficient matrices for the given state-space model as well as the conditional expectation and covariance matrix of the initial state, xi.tm1tm1 $\left(\xi_{t-1|t-1}\right)$ and P.tm1tm1 $\left(P_{t-1|t-1}\right)$, respectively. *kalman.states.wrapper.R* is a wrapper function for *kalman.states.R* that specifies inputs based on the estimation stage.

## 5.8 Log Likelihood Programs

The function *kalman.log.likelihood.R* takes as input the coefficient matrices of the given state-space model and the conditional expectation and covariance matrix of the initial state and returns the log likelihood value and a vector with the log likelihood at each time $t$. *log.likelihood.wrapper.R* is a wrapper function for *kalman.log.likelihood.R* that specifies inputs based on the estimation stage.

## 5.9 Standard Error Program

The function *kalman.standard.errors.R* computes confidence intervals and corresponding standard errors for the estimates of the states using Hamilton's (1986) Monte Carlo procedure that accounts for both filter and parameter uncertainty. See footnote 7 in HLW (2017).

## 5.10 Miscellaneous Programs

The function *calculate.covariance.R* calculates the covariance matrix of the initial state from the gradients of the likelihood function. The function *format.output.R* generates a dataframe to be written to a CSV containing one-sided estimates, parameter values, standard errors, and other statistics of interest.

## 6 Specifications

The file *run.lw.R* reads in the provided data, runs the LW estimation, and saves a spreadsheet of output.

The following variables defined in *run.lw.R* determine the model specification:

- **Sample dates:** Set *sample.start* and *sample.end* corresponding to the input data.
- **Initialization of state vector:** Set *xi.00.stageX* to a vector of values for the initial states, or set as *NA* (default) to initializing following the HLW (2017) procedure. One can also set the initial covariance matrix for each stage, *P.00.stageX*.
- **Constraints:** Set *a.r.constraint* and *b.y.constraint* to impose bounds on the slopes of the IS and Phillips curve equations, respectively.
- **COVID indicator parameter ($\phi$):** Set *fix.phi* equal to a numeric value in order to fix the parameter $\phi$ (e.g. at $\phi = 0$ to impose no role for the COVID supply shock). Set the value to *NA* (default) to estimate $\phi$. Note that while the code is set up to accommodate any numeric value in *fix.phi*, the procedure to obtain the initial guess of the parameter vector for maximum likelihood estimation implicitly assumes $\phi = 0$ when $\phi$ is not estimated.
- **Time-varying volatility specification:** Set the flag *use.kappa* to *TRUE* when introducing time-varying volatility. Set *kappa.inputs* corresponding to the instructions in *run.lw.R* to specify the variance scale parameters.

# 7 References


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