Autoregressive-distributed lag models

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Contents

Introduction

- Multivariate time series applications consider temporal and cross-sectional dependence
- Provide insight into the dynamic relationship between variables
- May be used to improve the forecasting accuracy
- Could be used in policy analysis purposes or for making specific inference
- Early multivariate time series models include distributed lag models
- Examples of these models include the polynomial and geometric distributed lag models
- Autoregressive distributed lag (ARDL) models continue to used in a number of current studies

Polynomial distributed lag model

- General specification for DL model is:

$$
\begin{aligned}
y_t &= \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \cdots + \beta_q x_{t-q} + \varepsilon_t \\
&= \alpha + \beta(L) x_t + \varepsilon_t
\end{aligned}
$$

- where ε_t is assumed to be white noise, while $\beta(L)$ is a lag polynomial and L is the lag operator

Rational distributed lag model

- Modification to the above model would provide the rational distributed lag model:

$$y_t = \alpha + \frac{\beta(L)}{\lambda(L)} x_t + v_t$$

- where $\beta(L)$ as defined above and

$$\lambda(L) = \lambda_0 + \lambda_1 L + \ldots + \lambda_p L^p$$

Autoregressive-distributed lag model

- Above model may be extended to incorporate a number of autoregressive terms
- Gives rise to the ARDL (p, q) model that may be specified as

$$\begin{aligned} y_t + \lambda_1 y_{t-1} + \lambda_2 y_{t-2} + \ldots + \lambda_p y_{t-p} = \ldots \\ \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \ldots + \beta_q x_{t-q} + \varepsilon_t \end{aligned}$$

or

$$\lambda(L) y_t = \alpha + \beta(L) x_t + \varepsilon_t$$

- Rational DL model can also be written in the form of an ARDL $(p,\ q)$ model with MA errors

$$\lambda(L) y_t = \alpha \lambda(1) + \beta(L) x_t + \lambda(L) \varepsilon_t,$$

- Represents previous model except that the error term is now given by

$$\varepsilon_t = \lambda(L) \upsilon_t$$

Autoregressive-distributed lag model

- Of the three models the ARDL (p, q) specification is easier to work
- By selecting relatively large values for p and q, one can approximate the rational DL
- Deterministic trends, or seasonal dummies, can be incorporated in the ARDL model

$$y_t + \lambda_1 y_{t-1} + \lambda_2 y_{t-2} + \ldots + \lambda_p y_{t-p} = \zeta_0 + \zeta_1 \cdot t + \beta_0 x_t + \beta_1 x_{t-1} + \ldots + \beta_q x_{t-q} + \varepsilon_t$$

- Model may be extended to k regressors, each with a specific number of lags

Autoregressive-distributed lag model

- Consider an ARDL $(p, q_1, q_2, \ldots, q_k)$ model, which may be expressed as

$$\lambda(L, p) y_t = \sum_{j=1}^{k} \beta_j \left(L, q_j\right) x_{t,j} + \varepsilon_t$$

- where

$$\begin{aligned} \lambda(L, p) &= 1 + \lambda_1 L + \lambda_2 L^2 + \ldots + \lambda_p L^p \\ \beta_j(L, q_j) &= \beta_{j,0} + \beta_{j,1} L + \ldots + \beta_{j,q_j} L^{q_j}, \quad j = 1, 2, \ldots, k \end{aligned}$$

Conclusion

- ARDL models make use of a straightforward extension of the univariate AR model
- Include additional explanatory variables and their lagged values
- Framework is used in several settings and may allow for a distinction for both long-run and short-run dynamics