Demonstration of Effect of Indexing on Cobb-Douglas and CES Aggregate Production Functions

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1 Models

For demonstration purposes, we focus on models that assume capital and labor factors of production only.

1.1 Raw historical data

The Cobb-Douglas model for using raw historical data is given by

$$Y = \theta \, \mathrm{e}^{\lambda t} \, K^{\alpha_k} \, L^{\alpha_l} \ , \tag{1}$$

where Y, K, and L are the raw (non-indexed) time series. Time (t) is indexed by difference to the intial year (1960).

The CES model with raw historical data is given by

$$Y = \gamma \, \mathrm{e}^{\lambda t} \left[\delta K^{-\rho} + (1 - \delta) L^{-\rho} \right]^{-1/\rho} \ . \tag{2}$$

1.2 Indexed factors of production

The Cobb-Douglas model indexed historical data is given by

$$y = \theta \, \mathrm{e}^{\lambda t} \, k^{\alpha_k} \, l^{\alpha_l} \ , \tag{3}$$

where y, k, and l are indexed by ratio to the initial year (1960). For example,

$$y = Y / Y_{1960} \ . \tag{4}$$

The CES model with indexed historical data is given by

$$y = \gamma \, \mathrm{e}^{\lambda t} \left[\delta k^{-\rho} + (1 - \delta) l^{-\rho} \right]^{-1/\rho} \ . \tag{5}$$

2 Data

All data are from the Penn World Table. Figure 1 below shows historical data as dots.

```
UKdata <- read.csv(file.path("data", "IndexingDemoRawData.csv"))
```

3 Fitting

Next, we fit both the Cobb-Douglas and CES aggregate production functions to both indexed and non-indexed data. Figure 1 below shows historical data as lines.

```
cd_nonindexed <- cdModel(GDP ~ K + L + iYear, data = UKdata)
cd_indexed <- cdModel(iGDP ~ iK + iL + iYear, data = UKdata)
ces_nonindexed <- cesModel(GDP ~ K + L + iYear, data = UKdata)
ces_indexed <- cesModel(iGDP ~ iK + iL + iYear, data = UKdata)
```

We graph each type of fitted model against historical data. Note that all models provide excellent fits.

```
ggplot(data = quadplotdata, mapping = aes_string(x = "Year")) +
  geom_point(mapping = aes_string(y = "y_historical"),
             size = 0.2) +
  geom_line(mapping = aes_string(y = "y_hat")) +
  facet_grid(indexed ~ apf, scales = "free") +
  xlab(NULL) +
  ylab("GDP [raw: millions 2005$, indexed: 1960 = 1]") +
  xy_theme()
```

4 Coefficients

4.1 Cobb-Douglas

Here are the coefficients for the Cobb-Douglas models.

```
cd_nonindexed

## Generalized least squares fit by REML
##   Model: log(GDP) - log(L) ~ I(log(K) - log(L)) + iYear
##   Data: sdata
##   Log-restricted-likelihood: 107.8529
##
```

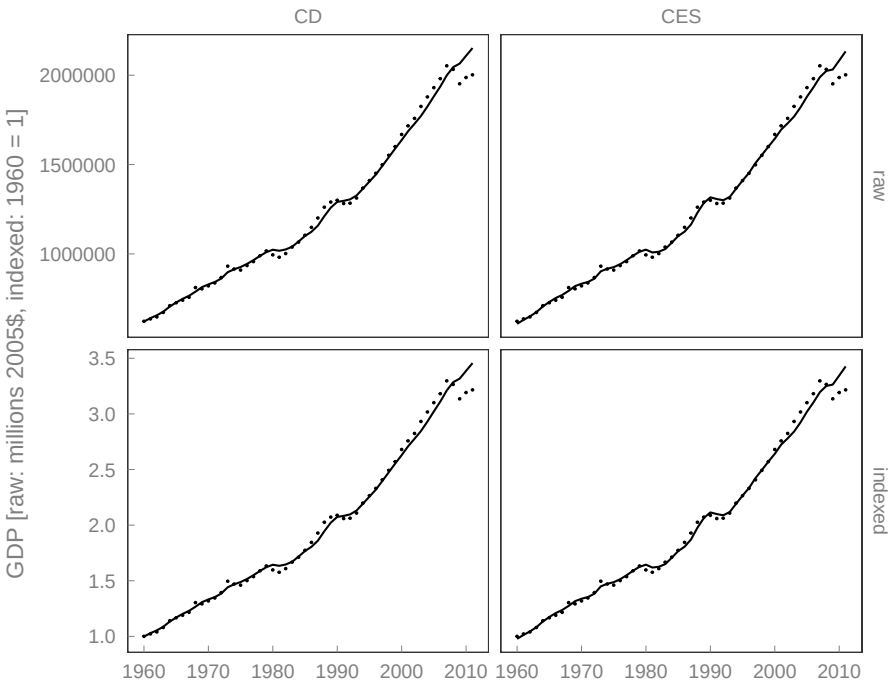

Figure 1: Fits to historical data. Dots are historical data. Lines are fits.

```
## Coefficients:
##    logscale     alpha_1      lambda
## -5.37466764  0.56178381  0.01019556
##
## Degrees of freedom: 52 total; 49 residual
## Residual standard error: 0.02369495
```

```
cd_indexed
```

```
## Generalized least squares fit by REML
##   Model: log(iGDP) - log(iL) ~ I(log(iK) - log(iL)) + iYear
##   Data: sdata
##   Log-restricted-likelihood: 107.8529
##
## Coefficients:
##     logscale      alpha_1       lambda
## -0.002331844  0.561783810  0.010195560
##
## Degrees of freedom: 52 total; 49 residual
## Residual standard error: 0.02369495
```

Note that the fitted parameters output elasticity `alpha_1` (α_k) and Solow residual growth rate `lambda` are identical, regardless of whether the historical data are indexed or not.

The pre-multiplier (in this case `logscale`) accounts for the fact that historical data have been indexed. We can convert `logscale` to θ as follows:

```
exp(cd_nonindexed$coefficients[["logscale"]])
```

```
## [1] 0.004632458
```

```
exp(cd_indexed$coefficients[["logscale"]])
```

```
## [1] 0.9976709
```

Note that θ for the indexed model is near 1.0, as expected.

4.2 CES

The following shows parameters for CES models fitted to non-indexed and indexed historical data.

```
ces_nonindexed
```

```
## Estimated CES function
##
## Call:
```

```
## cesEst(yName = "GDP", xNames = c("K", "L"), data = data, tName = "iYear",
##     method = "PORT", start = c(0.0078125108214483, 0.00583042323510128,
##     0.484469079677766, 0.1), lower = NULL, upper = NULL, multErr = TRUE,
##     control = list(iter.max = 2000, eval.max = 2000))
##
## Coefficients:
##     gamma    lambda     delta       rho
## 2.047e-05 1.784e-02 2.855e-06 1.200e+00
##
## Elasticity of Substitution: 0.4546

ces_indexed

## Estimated CES function
##
## Call:
## cesEst(yName = "iGDP", xNames = c("iK", "iL"), data = data, tName = "iYear",
##     method = "PORT", start = c(0.980366854292426, 0.0169427187219337,
##     0.509982506947314, 1), lower = NULL, upper = NULL, multErr = TRUE,
##     control = list(iter.max = 2000, eval.max = 2000))
##
## Coefficients:
##   gamma  lambda   delta     rho
## 0.97753 0.01784 0.50893 1.19979
##
## Elasticity of Substitution: 0.4546
```

As expected, and similar to the Cobb-Douglas model, the growth rate of the Solow residual (`gamma`) is identical, regardless of whether the historical data are indexed or not. Again, similar to the Cobb-Douglas model, the pre-multiplier term (in this case `gamma`) is different depending upon whether the historical data were indexed or not. We note that the value of `gamma` is close to 1, when fitting to indexed data, as expected. Furthermore, the elasticity of substitution parameters (`rho`) are identical regardless of whether the historical data are indexed or not.

However, the share parameter (`delta`) varies significantly, depending upon whether the historical data are indexed. Any inferences about the economy that depend upon the share parameter (`delta`) will be significantly different depending upon whether the historical data are indexed prior to fitting.