Production Function Primer

Abstract

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Keywords: production function, economic growth, energy, Cobb-Douglas, CES, LinEx

1. Introduction

2. Differential Economic Growth

The differential equation for economic growth (y) with capital (k), labor (l), and energy (e) factors of production, i.e., $y = y(k, l, e; t)$ is

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = \lambda\frac{1}{t-t_0} + \alpha\frac{1}{k}\frac{\mathrm{d}k}{\mathrm{d}t} + \beta\frac{1}{l}\frac{\mathrm{d}l}{\mathrm{d}t} + \gamma\frac{1}{e}\frac{\mathrm{d}e}{\mathrm{d}t} \ , \tag{1}$$

where $y \equiv Y/Y_0$, t (time) is measured in years, λ represents the pace of technological progress, $k \equiv K/K_0$, $l \equiv L/L_0$, $e \equiv E/E_0$ Y (economic output) is measured by GDP in currency units, K (capital) is expressed in currency units, L (labor) is expressed in workers or work-hours/year, and the 0 subscript indicates values at an initial year. Constant returns to scale are represented by the constraint $\alpha + \beta + \gamma = 1$.

The output elasticities α, β, and γ are defined as

$$\alpha \equiv \frac{k}{y}\frac{\partial k/\partial t}{\partial y/\partial t} \ , \ \beta \equiv \frac{l}{y}\frac{\partial k/\partial t}{\partial y/\partial t} \ , \ \gamma \equiv \frac{e}{y}\frac{\partial e/\partial t}{\partial y/\partial t} \ . \tag{2}$$

2.1. Cobb-Douglas Production Function

Integrating Equation 1 with respect to time gives

$$y = \theta \mathrm{e}^{\lambda(t-t_0)} k^\alpha l^\beta \ , \tag{3}$$

The capital-labor Cobb-Douglas production function shown in Equation 3 can be augmented to include an energy term:

$$y = \theta \mathrm{e}^{\lambda(t-t_0)} k^\alpha l^\beta e^\gamma \ , \tag{4}$$

where $e \equiv E/E_0$, and E is in units of energy per time, typically TJ/year. The energy-augmented Cobb-Douglas production function is often assumed to have constant returns to scale for the three factors of production: $\alpha + \beta + \gamma = 1$.

The term $\mathrm{e}^{\lambda(t-t_0)}$ is known as total factor productivity (A), and λ is the Solow residual. Assuming that parameters θ, α, and β are known from parameter estimation, we can estimate the value of λ as follows. First, assume constant returns to scale such that $\gamma = 1 - \alpha - \beta$. Then, take the natural logarithm (ln) of Equation 3 to obtain

$$\ln y = \ln \theta + \ln A + \alpha \ln k + \beta \ln l + (1 - \alpha - \beta) \ln e \; . \tag{5}$$

By taking the derivative of Equation 5 with respect to time (t) and noting that model parameters θ, α, and β are constant with respect to time, we obtain

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} + \alpha\frac{1}{k}\frac{\mathrm{d}k}{\mathrm{d}t} + \beta\frac{1}{l}\frac{\mathrm{d}l}{\mathrm{d}t} + (1 - \alpha - \beta)\frac{1}{e}\frac{\mathrm{d}e}{\mathrm{d}t} \; . \tag{6}$$

Solving for the Solow Residual term gives

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} - \left[\alpha\frac{1}{k}\frac{\mathrm{d}k}{\mathrm{d}t} + \beta\frac{1}{l}\frac{\mathrm{d}l}{\mathrm{d}t} + (1 - \alpha - \beta)\frac{1}{e}\frac{\mathrm{d}e}{\mathrm{d}t}\right] \; . \tag{7}$$

Recognizing that $A \equiv \mathrm{e}^{\lambda(t-t_0)}$ gives

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{\mathrm{e}^{\lambda(t-t_0)}}\;\lambda\mathrm{e}^{\lambda(t-t_0)} = \lambda \; , \tag{8}$$

which can be substituted into Equation 7 to find

$$\lambda = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} - \left[\alpha\frac{1}{k}\frac{\mathrm{d}k}{\mathrm{d}t} + \beta\frac{1}{l}\frac{\mathrm{d}l}{\mathrm{d}t} + (1 - \alpha - \beta)\frac{1}{e}\frac{\mathrm{d}e}{\mathrm{d}t}\right] \; . \tag{9}$$

Equation 9 applies for any instant in time.

If we approximate derivatives in Equation 9 with forward differences between times i and j, we find

$$\lambda_{i,j} \approx \frac{1}{y_i}\frac{y_j - y_i}{t_j - t_i} - \left[\alpha\frac{1}{k_i}\frac{k_j - k_i}{t_j - t_i} + \beta\frac{1}{l_i}\frac{l_j - l_i}{t_j - t_i} + (1 - \alpha - \beta)\frac{1}{e_i}\frac{e_j - e_i}{t_j - t_i}\right] \; , \tag{10}$$

where $\lambda_{i,j}$ approximates the true, instantaneous value of λ between times i and j.

Interestingly,

$$\lambda_{1,n} \neq \sum_{i=2}^{n} \lambda_{i,i-1} \ . \tag{11}$$

Rather,

$$\lambda_{1,n} \approx \frac{1}{y_1}\frac{y_n - y_1}{t_n - t_1} - \left[\alpha \frac{1}{k_1}\frac{k_n - k_1}{t_n - t_1} + \beta \frac{1}{l_1}\frac{l_n - l_1}{t_n - t_1} + (1-\alpha-\beta)\frac{1}{e_1}\frac{e_n - e_1}{t_n - t_1}\right] \ , \tag{12}$$

which will become increasingly inaccurate over large time spans, because y_1, k_1, l_1, and e_1 will be less representative of the average value of y, k, l, and e in the time span, respectively. This suggests that an averaging approach such as

$$\lambda_{1,n} \approx \frac{\sum_{i=2}^{n} \lambda_{i,i-1}}{n-1} \tag{13}$$

is a better approximation of Equation 12, which is, itself, an approximation of the true value of λ given by Equation 9.

2.2. Constant Elasticity of Substitution Production Function (CES)

Other energy economists use an energy-augmented Constant Elasticity of Substitution (CES) production function to describe economic growth. The `R` (?) package `micEconCES` (?) estimates CES production functions of the following forms (among others):

$$y = \gamma \, A \, \left[\delta_1 k^{-\rho_1} + (1-\delta_1) l^{-\rho_1}\right]^{-1/\rho_1} ; A = \mathrm{e}^{\lambda(t-t_0)} \tag{14}$$

$$y = \gamma \, A \, \left\{\delta \left[\delta_1 k^{-\rho_1} + (1-\delta_1) l^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta) e^{-\rho}\right\}^{-1/\rho} ; A = \mathrm{e}^{\lambda(t-t_0)} \tag{15}$$

Equation 14 is a CES production function with capital stock (k) and labor (l) factors of production. Equation 15 augments Equation 14 with energy using a $(kl)(e)$ nesting structure, as is typical in the literature. Equation 14 is a degenerate form of Equation 15 where $\delta \to 1$.

γ is a fitting parameter that represents productivity. The fitting parameters ρ_1 and ρ indicate elasticities of substitution (σ_1 and σ). The elasticity of substitution between capital (k) and labor (l) is given by $\sigma_1 = \frac{1}{1+\rho_1}$, and the elasticity of substitution between (kl) and (e) is given by $\sigma = \frac{1}{1+\rho}$. As $\rho_1 \to 0$, $\sigma_1 \to 1$, and the embedded CES production function for k and l degenerates to the Cobb-Douglas production function. Similarly, as $\rho \to 0$,

$\sigma \to 1$, and the CES production function for (kl) and (e) degenerates to the Cobb-Douglas production function. As $\sigma \to \infty$ ($\rho \to -1$), (kl) and (e) are perfect substitutes. As $\sigma \to 0$ ($\rho \to \infty$), (kl) and (e) are perfect complements: no substitution is possible. Similarly, as $\sigma_1 \to 0$ ($\rho_1 \to \infty$), k and l are perfect complements. δ_1 describes the relative importance of capital (k) and labor (l), and δ describes the importance of (kl) relative to (e).

Constraints on the fitting parameters include $\delta_1 \in [0,1]$, $\delta \in [0,1]$, $\rho_1 \in [-1,0) \cup (0,\infty)$, and $\rho \in [-1,0) \cup (0,\infty)$.

Two other nestings of the factors of production (k, l, and e) are possible with the CES model.

$$y = \gamma\, A\, \left\{\delta \left[\delta_1 l^{-\rho_1} + (1-\delta_1) e^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta) k^{-\rho}\right\}^{-1/\rho} ; A = \mathrm{e}^{\lambda(t-t_0)} \quad (16)$$

$$y = \gamma\, A\, \left\{\delta \left[\delta_1 e^{-\rho_1} + (1-\delta_1) k^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta) l^{-\rho}\right\}^{-1/\rho} ; A = \mathrm{e}^{\lambda(t-t_0)} \quad (17)$$

Note that the ρ (σ) and δ parameters have different meanings that depend upon the nesting of the factors of production.

2.2.1. LINEX Production Function

A third production function, the LINear EXponential (LINEX) function

$$y = \theta\, \mathrm{e}^{a_0\left[2\left(1-\frac{1}{\rho_k}\right)+c_t\left(\rho_l-1\right)\right]}\, e\,, \quad (18)$$

was derived from thermodynamic considerations (**??????**). In contrast to both the Cobb-Douglas model (Equations 3 and 4) and the CES model (Equations 14–17), the LINEX model (Equation 18) does not include a generic, time-dependent augmentation term (A). The LINEX model assumes energy (e) is the only factor of production.

The LINEX model does, however, include terms that represent efficiencies among capital, labor, and energy, in the form of the ratios ρ_k and ρ_l which are defined by[1]

$$\rho_k \equiv \frac{k}{(1/2)(l+e)} \quad (19)$$

and

$$\rho_l \equiv \frac{l}{e} \quad (20)$$

[1]The notation for the LINEX model breaks from the conventions used in the Cobb-Douglas and CES models: a_0 and c_t are parameters and ρ_k and ρ_l are not.

and represent (19) capital stock increase relative to the average of labor and energy (capital deepening) and (20) labor increase relative to energy. The effect of energy (e) on output (y) is enhanced if either

- capital stock (k) has increased more than the average of labor (l) and energy (e), i.e. if $\rho_k > 1$, or
- labor (l) has increased more than energy (e), i.e. if $\rho_l > 1$.

An economy that increases capital stock (k) and labor (l) at the same rate as energy will have capital and labor efficiency ratios (ρ_k and ρ_l, respectively) of unity. In that case, economic output (y) increases at the same rate as energy consumption (e). According to the LINEX model, an economy that increases capital stock (k) without a commensurate increase in labor (l) and energy (e) will experience an increase in output (y) in excess of its increase of energy consumption (e), because $\mathrm{e}^{a_0\left[2\left(1-\frac{1}{\rho_k}\right)+c_t\left(\rho_l-1\right)\right]} > 1$. Similarly, an economy benefits by increasing labor (l) without a commensurate increase in energy (e). Thus, ρ_k is an efficiency of using additional labor and energy to make additional capital stock available to the economy, and ρ_l is an efficiency of using additional energy to make additional labor available to the economy. Increases in ρ_k and ρ_l provide upward pressure on economic output (y), as the only factor of production (e) is used more efficiently.

The parameter a_0 represents the overall importance of efficiencies ρ_k and ρ_l, and the parameter c_t represents the relative importance of the labor efficiency term (ρ_l) compared to the capital stock efficiency term (ρ_k). Both a_0 and c_t are assumed constant with respect to time.

3. Conclusion

4. Future Work

Acknowledgements

References

Appendix A. Appendix

Include data tables here?