Effect of quality-adjusted production function inputs

João dos Santos[a], Paul Brockway[b], Matthew Kuperus Heun[c,*], Marco Sekai[b], Tiago Domingos[a], Julia Steinberger[b]

[a] *IST, Lisbon, Portugal*
[b] *Leeds University, Leeds, UK*
[c] *Engineering Department, Calvin College, Grand Rapids, MI 49546, USA*

Abstract

**** Add abstract ****

Keywords: economic growth, exergy, energy, Cobb-Douglas, CES

1. Introduction

This is a study of the effects of quality-adjusted production function inputs on the importance of energy in the production function.

2. Coordinates of Analysis

This section describes the coordinates of analysis and briefly reviews literature related to each.

2.1. Mathematical Forms of the Energy-augmented Production Function

In this paper, we assess two prominent energy-augmented production functions that appear in the literature: Cobb-Douglas (CD) and Constant Elasticity of Substitution (CES). These production functions are assessed relative to a model of exponential growth only. The following subsections describe each.

*Corresponding author
Email address: `mkh2@calvin.edu` (Matthew Kuperus Heun)

2.1.1. Cobb-Douglas Production Function

The Cobb-Douglas production function can be expressed as

$$y = \theta A k^{\alpha_1} l^{\alpha_2} \; ; \; A \equiv \mathrm{e}^{\lambda(t-t_0)} \; , \tag{1}$$

where $y \equiv Y/Y_0$, θ is a scale parameter, e is the base of the natural logarithm, λ is represents the pace of technological progress, t (time) is measured in years, $k \equiv K/K_0$, $l \equiv L/L_0$, Y (economic output) is represented by GDP, K (capital) is expressed in currency units, L (labor) is expressed in workers or work-hours/year, and the 0 subscript indicates values at an initial year.[1] Constant returns to scale are represented by the constraint $\alpha_1 + \alpha_2 = 1$.

The capital-labor Cobb-Douglas production function shown in Equation 1 can be augmented to include an energy term:

$$y = \theta A k^{\alpha_1} l^{\alpha_2} e^{\alpha_3} \; ; \; A \equiv \mathrm{e}^{\lambda(t-t_0)} \; , \tag{2}$$

where $e \equiv E/E_0$, and E is in units of energy per time, typically TJ/year. The energy-augmented Cobb-Douglas production function is often assumed to have constant returns to scale for the three factors of production: $\alpha_1 + \alpha_2 + \alpha_3 = 1$. The term A is known as total factor productivity, and λ is the Solow residual.

2.1.2. Constant Elasticity of Substitution Production Function (CES)

Other energy economists use an energy-augmented Constant Elasticity of Substitution (CES) production function to describe economic growth. The `R` package `micEconCES` estimates CES production functions of the following form

$$y = \gamma \, A \, \left\{ \delta \left[\delta_1 x_1^{-\rho_1} + (1-\delta_1) x_2^{-\rho_1} \right]^{\rho/\rho_1} + (1-\delta) x_3^{-\rho} \right\}^{-1/\rho} ; A \equiv \mathrm{e}^{\lambda(t-t_0)}, \tag{3}$$

where x_1, x_2, and x_3 are factors of production and permutations of capital (k), labor (l), and energy (e).

The CES model without energy is given in Equation 4.

$$y = \gamma \, A \, \left[\delta_1 k^{-\rho_1} + (1-\delta_1) l^{-\rho_1} \right]^{-1/\rho_1} ; A \equiv \mathrm{e}^{\lambda(t-t_0)}. \tag{4}$$

[1] Dimensionless, indexed quantities are represented by lower-case symbols (y, k, l, e, q, x, and u), and dimensional quantities are represented by upper-case symbols (Y, K, L, E, Q, X, and U). Model parameters are represented by Greek letters (α_1, α_2, λ, θ).

Equation 5 augments Equation 4 with energy using a $(kl)(e)$ nesting structure, as is typical in the literature. Equation 4 is a degenerate form of Equation 5 where $\delta \to 1$.

$$y = \gamma\, A\, \left\{\delta \left[\delta_1 k^{-\rho_1} + (1-\delta_1) l^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta) e^{-\rho}\right\}^{-1/\rho} ; A \equiv \mathrm{e}^{\lambda(t-t_0)}. \quad (5)$$

In the CES production function, γ is a fitting parameter that accounts for an atypical first year. The fitting parameters ρ_1 and ρ indicate elasticities of substitution (σ_1 and σ). The elasticity of substitution between capital (k) and labor (l) is given by $\sigma_1 = \frac{1}{1+\rho_1}$, and the elasticity of substitution between (kl) and (e) is given by $\sigma = \frac{1}{1+\rho}$. As $\rho_1 \to 0$, $\sigma_1 \to 1$, and the embedded CES production function for k and l degenerates to the Cobb-Douglas production function. Similarly, as $\rho \to 0$, $\sigma \to 1$, and the CES production function for (kl) and (e) degenerates to the Cobb-Douglas production function. As $\sigma \to \infty$ ($\rho \to -1$), (kl) and (e) are perfect substitutes. As $\sigma \to 0$ ($\rho \to \infty$), (kl) and (e) are perfect complements: no substitution is possible. Similarly, as $\sigma_1 \to 0$ ($\rho_1 \to \infty$), k and l are perfect complements. δ_1 describes the relative importance of capital (k) and labor (l), and δ describes the importance of (kl) relative to (e).

Constraints on the fitting parameters include $\delta_1 \in [0, 1]$, $\delta \in [0, 1]$, $\rho_1 \in [-1, 0) \cup (0, \infty)$, and $\rho \in [-1, 0) \cup (0, \infty)$.

Two other nestings of the factors of production (k, l, and e) are possible with the CES model.

$$y = \gamma\, A\, \left\{\delta \left[\delta_1 l^{-\rho_1} + (1-\delta_1) e^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta) k^{-\rho}\right\}^{-1/\rho} ; A \equiv \mathrm{e}^{\lambda(t-t_0)} \quad (6)$$

$$y = \gamma\, A\, \left\{\delta \left[\delta_1 e^{-\rho_1} + (1-\delta_1) k^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta) l^{-\rho}\right\}^{-1/\rho} ; A \equiv \mathrm{e}^{\lambda(t-t_0)} \quad (7)$$

Note that ρ (σ), ρ_1 (σ_1), δ, and δ_1 have different meanings depending upon the nesting of the factors of production.

2.1.3. Exponential Production Function

We define an exponential-only reference model for economic growth in Equation 8.

$$y = \theta A\ ;\ A \equiv \mathrm{e}^{\lambda(t-t_0)}\ . \quad (8)$$

The reference model is a degenerate case of the Cobb-Douglas production function wherein all factor shares are zero ($\alpha_1 = \alpha_2 = \alpha_3 = 0$) and the constant returns to scale constraint is not respected.

We expect that the reference model will have a larger fitted Solow residual term than the Cobb-Douglas and CES models, because no factors of production are included in the reference model to drive growth. Indeed, in the reference model, all growth is attributed to the Solow residual.

In contrast, it is not necessarily true that Cobb-Douglas and CES models will exhibit lower *mse* than the reference model shown in Equation 8. The Cobb-Douglas and CES models have more fitting parameters, but they incorporate the factors of production at constant returns to scale. The reference model has $\alpha_1 + \alpha_2 + \alpha_3 = 0$, whereas our implementations of both the Cobb-Douglas and CES models requires constant returns to scale, $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Thus, if the factors of production (α_1, α_2, and α_3) are poorly correlated to output (y), the constant returns to scale constraint may cause higher *mse* for the Cobb-Douglas or CES models relative to the reference model.

In the sections that follow, we assess the Cobb-Douglas and CES models relative to the reference model in terms of goodness of both goodness of fit (*mse*) and Solow residual (λ).

2.2. Economies

Discuss economies here.

UK and Portugal.

3. Sources of Data

Discuss data sources here.

3.1. Historical Data

Historical data are stored in the `IST` and `Leeds` data sets.

For unadjusted variables, we use the time series shown in Table 1.

For quality-adjusted variables, we use the time series shown in Table 2.

Unadjusted and quality-adjusted data for both Portugal and the United Kingdom are shown in Figure 1.

Table 1: Unadjusted time-series variables.

Variable	Portugal	UK
Output	iGDP	iGDP
Capital	iKstkS.L	iKstkO.WwithRD
Labor	iL	iL
Energy	iXpMP	iXp

Table 2: Quality-adjusted time series variables.

Variable	Portugal	UK
Output	iGDP	iGDP
Capital	iKservS.L	iKservO.WwithRD
Labor	ihLest	ihLest
Energy	iUMP	iU

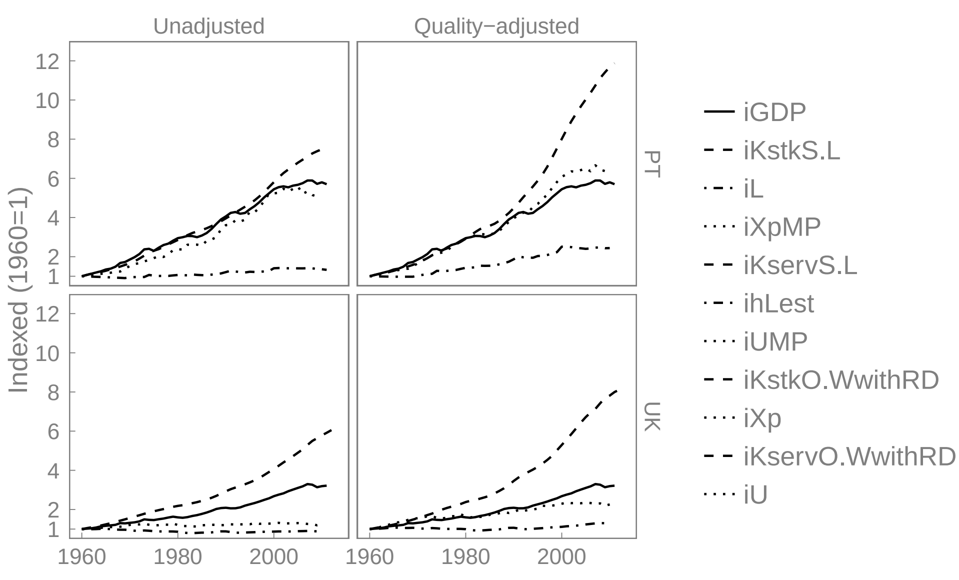


Figure 1: Historical data.

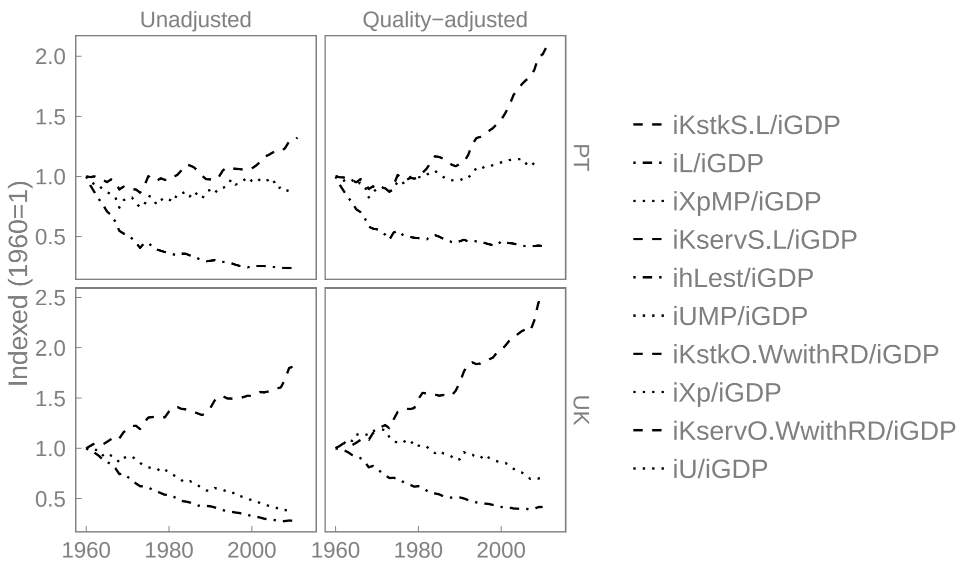


Figure 2: Pre-econometric data.

4. Parameter Estimation (Without Cost-share Theorem)

The models being fit can be described by the algebraic form of the model (Cobb-Douglas, CES, etc.) and a formula that enumerates which data variables are being used in which roles to fit the model. A formula of the form

$$y \sim x_1 + x_2 + x_3 + t, \tag{9}$$

or

$$y \sim x_1 + x_2 + t, \tag{10}$$

describes the economic output variable (y, usually `iGDP`, indexed GDP) and the factors of production (x_1, x_2, and x_3, which will be some measure of capital, labor, and energy, but perhaps not in that order), and a time variable (t, usually `iYear`, the number of years since the beginning of data collection). All of the models assume an error term that is additive on the logarithmic scale and are fit by the method of least squares. Model fitting provides estimates for all parameters in the model.

Table 3: Equations for α_1, α_2, and α_3 (factor shares for capital, labor, and energy, respectively) for the various CES nestings, provided that the `formula` is specified as `y ~ k + l + e + time`.

Nesting	`nest`	α_1	α_2	α_3
$(kl) + ()$	`c(1,2)`	δ_1	$1 - \delta_1$	0
$(kl) + (e)$	`c(1,2,3)`	$\delta\delta_1$	$\delta(1 - \delta_1)$	$1 - \delta$
$(le) + (k)$	`c(2,3,1)`	$1 - \delta$	$\delta\delta_1$	$\delta(1 - \delta_1)$
$(ek) + (l)$	`c(3,1,2)`	$\delta(1 - \delta_1)$	$1 - \delta$	$\delta\delta_1$

For the CES model, values of α_1, α_2, and α_3 are calculated by $\alpha_1 = \delta_1$, $\alpha_2 = 1 - \delta_1$, and $\alpha_3 = 0$ for the CES model with two factors of production and by $\alpha_1 = \delta\delta_1$, $\alpha_2 = \delta(1 - \delta_1)$, and $\alpha_3 = 1 - \delta$ for the CES model with three factors of production and the $(kl)e$ nesting. The values of α_1, α_2, and α_3 are interpreted as factor shares for capital, labor, and energy, respectively, as shown in Table 3, assuming factors of production are specified in the fitting `formula` as `y ~ k + l + e + time`.

```
Error in saveRDS(models, file = fileName):  object 'fileName' not
found
```

5. Results

5.1. Fits to historical data

Both historical GDP and fitted GDP are shown in Figures 3–6.

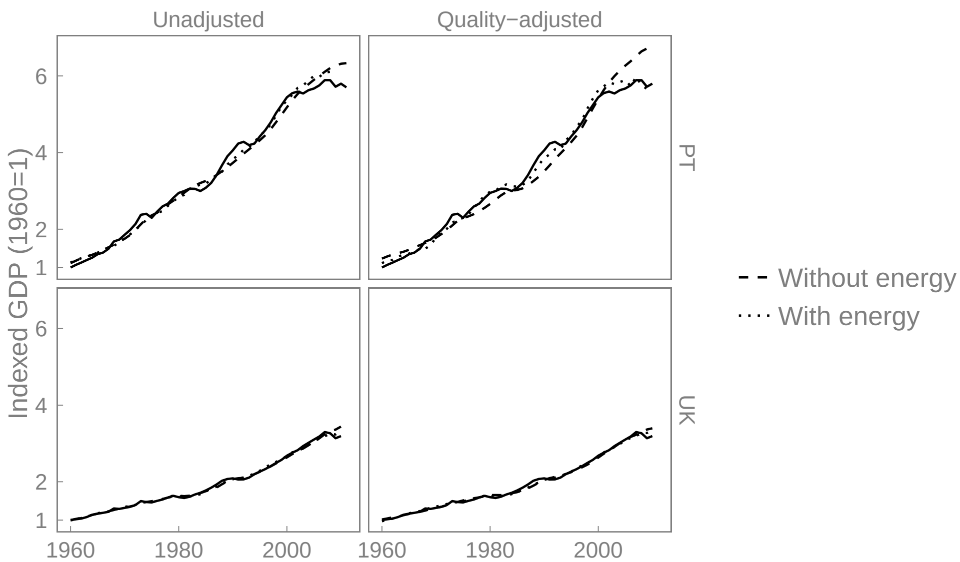


Figure 3: Cobb-Douglas models that reject the cost-share theorem. Solid line is historical GDP.

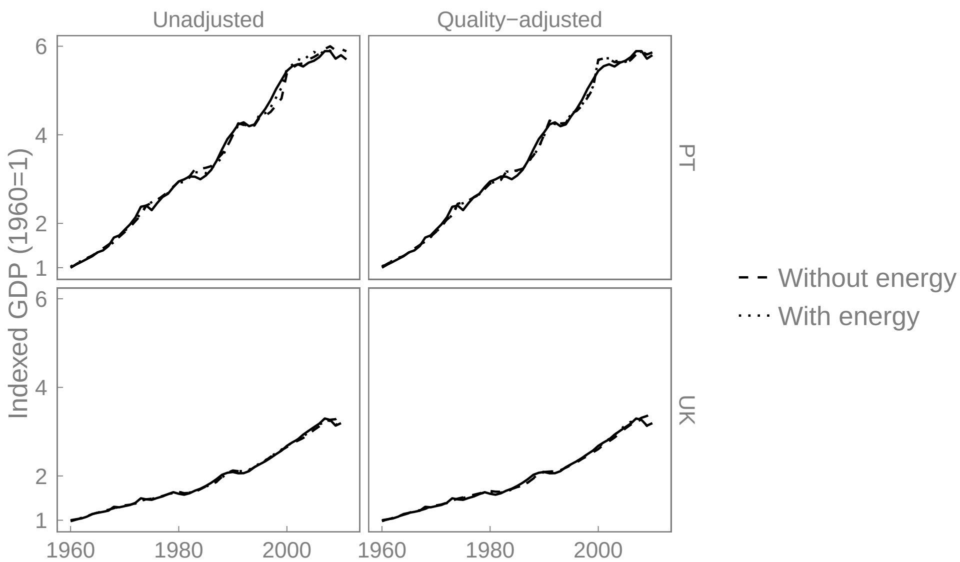


Figure 4: CES models with $(kl)e$ nesting. Solid line is historical GDP.

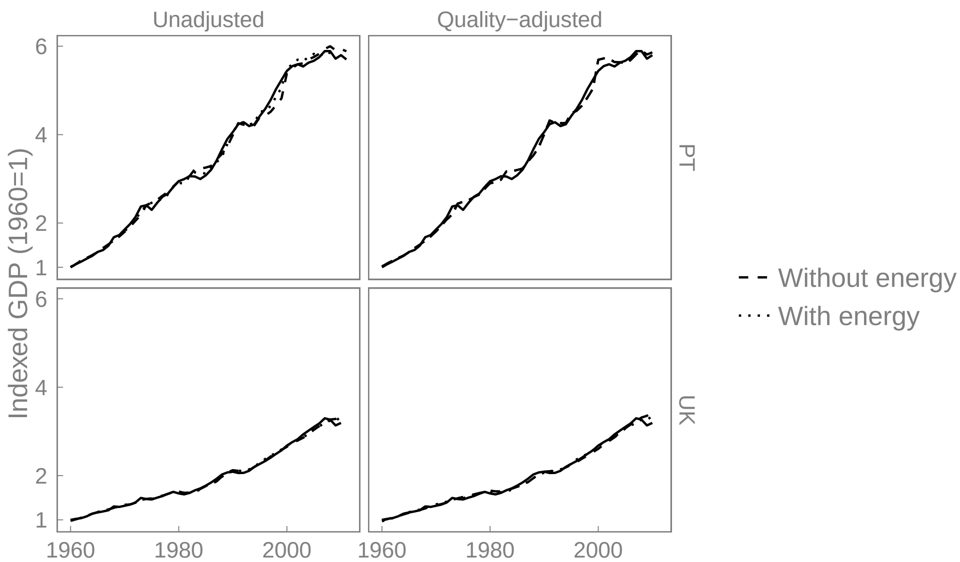


Figure 5: CES models with $(le)k$ nesting. Solid line is historical GDP.

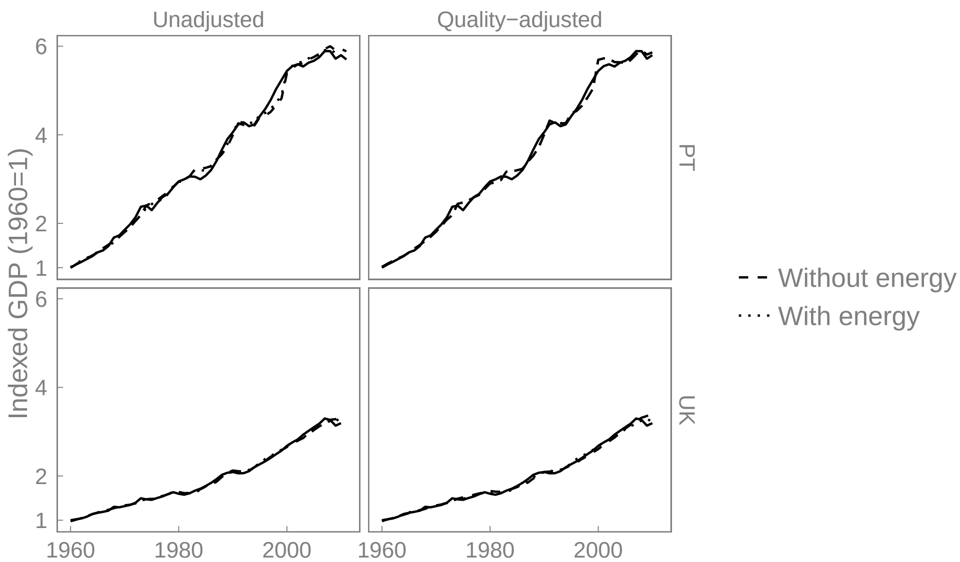


Figure 6: CES models with $(ek)l$ nesting. Solid line is historical GDP.

5.2. Fitting residuals

Because we fit in log-space, fitting residuals (r_i) are defined as

$$r_i \equiv \ln(y_i) - \ln(\hat{y}_i) = \ln\left(\frac{y_i}{\hat{y}_i}\right), \tag{11}$$

where r_i will be zero when there is agreement between historical (y_i) and fitted $(\hat{y}_i)$ economic output.

The mean squared error (mse) for any fitted model can be calculated by

$$mse \equiv \frac{1}{N}\sum_{i=1}^{N} r_i^2. \tag{12}$$

Figures 7–10 show fitting residuals for all models.

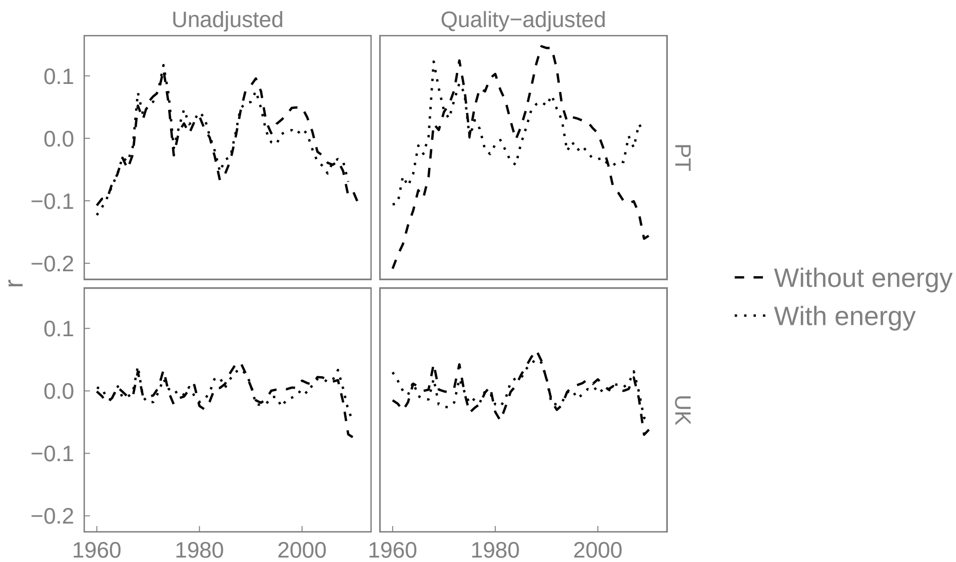


Figure 7: Fitting residuals for Cobb-Douglas models.

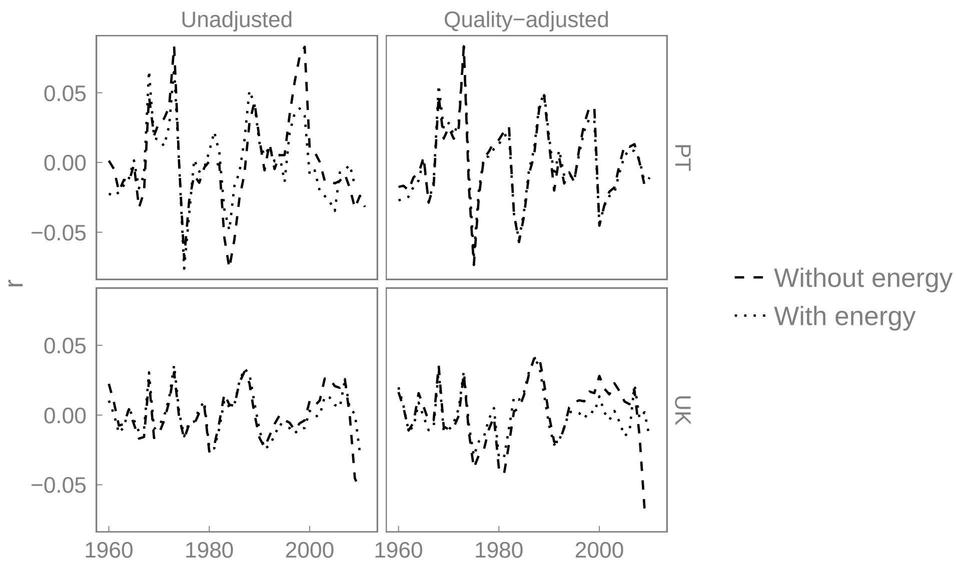


Figure 8: Fitting residuals for CES models with $(kl)e$ nesting.

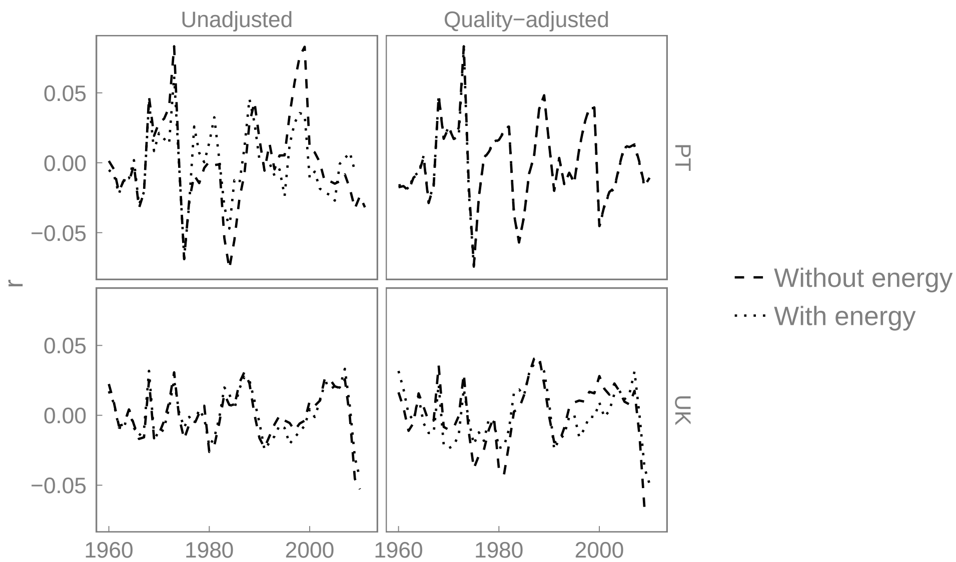


Figure 9: Fitting residuals for CES models with $(le)k$ nesting.

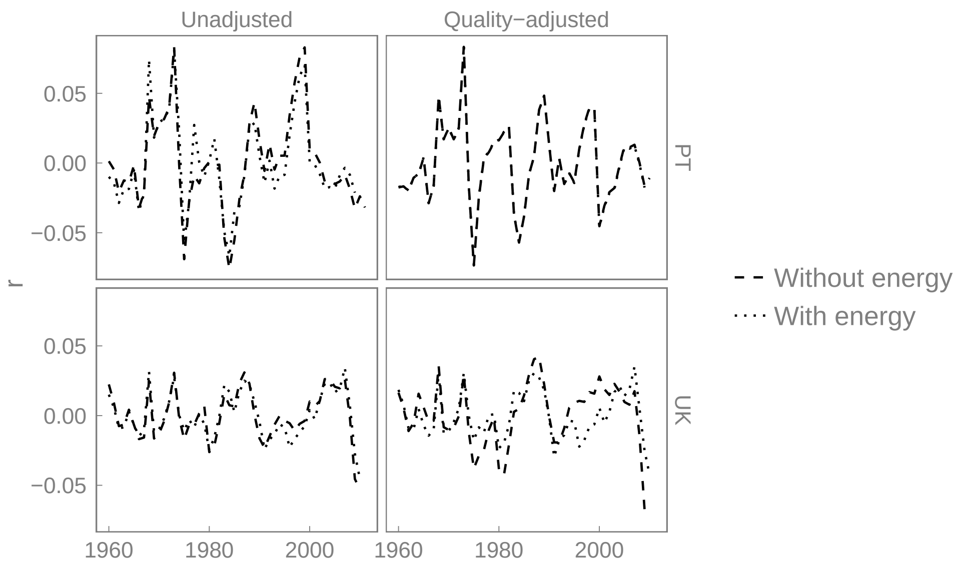


Figure 10: Fitting residuals for CES models with $(ek)l$ nesting.

6. Analysis

For now, I've organized the analysis around questions that we would like to answer.

6.1. Does rejecting the cost-share theorem decrease the mean squared error and/or the Solow residual?

Our hypothesis is that rejecting the cost-share theorem (CST) will decrease both mean squared error (mse) and the Solow residual (λ). The mean squared error (mse) is expected to decrease because rejecting the CST adds one or more fitting parameters to the model. The Solow residual (λ) is expected to decrease because the factor shares for capital (α_1) and labor ($\alpha_2 = 1 - \alpha_1$) are free to float, allowing each factor of production to contribute optimally toward production and bringing the model's prediction for economic output closer to historical output. In so doing, we expect less "unexplained" economic growth and a decreased Solow residual (λ).

Table 4 shows the effect of the CST on fitted parameters for the Cobb-Douglas model and the exponential-only model. The Cobb-Douglas models show lower mse than the exponential-only models. The Cobb-Douglas models also reduce solow residual (λ) relative to the exponential-only case.

Table 4: Model parameters for Cobb-Douglas models with unadjusted factors of production, without energy.

Country	model	cst	scale	lambda	alpha_1	alpha_2	mse
PT	exp	Reject CST	1.283537	0.034507	0.000000	0.000000	0.013359
PT	CD	Adhere to CST	1.328501	0.016080	0.300000	0.700000	0.011724
PT	CD	Reject CST	1.113489	−0.005539	1.000000	0.000000	0.003190
UK	exp	Reject CST	1.021833	0.023565	0.000000	0.000000	0.001212
UK	CD	Adhere to CST	1.032978	0.015510	0.300000	0.700000	0.000775
UK	CD	Reject CST	1.000925	0.006654	0.544956	0.455044	0.000494

Table 5 shows the effect of the CST on fitted parameters for the CES model. The CES models show lower mse than the exponential-only models. The CES models also reduce solow residual (λ) relative to the exponential-only case.

Table 5: Model parameters for CES models with unadjusted factors of production, without energy.

Country	model	cst	gamma	lambda	alpha_1	alpha_2	sigma_1	mse
PT	exp	Reject CST		0.034507	0.000000	0.000000		0.013359
PT	CES	Adhere to CST	1.318284	0.018825	0.300000	0.700000	0.665163	0.011450
PT	CES	Reject CST	0.998729	0.012404	0.983352	0.016648	0.173623	0.001113
UK	exp	Reject CST		0.023565	0.000000	0.000000		0.001212
UK	CES	Adhere to CST	1.006734	0.020914	0.300000	0.700000	0.509524	0.000491
UK	CES	Reject CST	0.977872	0.017904	0.436298	0.563702	0.500000	0.000326

Both Cobb-Douglas and CES models do a better job of fitting historical data than an exponential-only model as evidenced by decreased mse values. Both Cobb-Douglas and CES models exhibit reduced Solow residual (λ compared to the exponential-only model. For both Cobb-Douglas and CES, rejecting the Cost Share Theorem yields decreased mean squared error (mse) and smaller Solow residual (λ, on an absolute-value basis), as expected.

****

Note that in Table 5, we see $\sigma_1 = 0.5$ for UK when the cost share theorem is rejected. This even-numbered result is coming from the grid search in ρ_1. For an unknown reason, the gradient search from the best grid search point fails. I will investigate.

This result means that we have found a set of fitting coefficients for this situation that is *close* to providing the minimum possible mse, but it may not be the *exact* minimum. Regardless, the point above (that mse and λ always decrease when rejecting the cost share theorem) remains valid. Investigating and fixing this problem will only serve to *further* reduce mse and λ from the values reported in Table 5.

****

6.2. *Does quality-adjusting the factors of production decrease the mean squared error and/or the Solow residual?*

We can test our hypothesis that quality-adjusting the factors of production and including energy will both reduce the Solow residuals (λ) and improve the fit to historical data (thereby reducing the fitting residuals, r_i, and mean squared error, mse) by calculating $\Delta\lambda$ and Δmse, where

$$\Delta\lambda \equiv |\lambda| - |\lambda_{Unadjusted,\,Without\,energy,\,CST}| \tag{13}$$

and

$$\Delta mse \equiv mse - mse_{Unadjusted,\,Without\,energy,\,CST}. \tag{14}$$

where N is the number of years of data. On a percentage basis,

$$\Delta\lambda\,[\%] = \frac{100\,\Delta\lambda}{|\lambda_{Unadjusted,\,Without\,energy,\,CST}|} = \frac{100\,|\lambda|}{|\lambda_{Unadjusted,\,Without\,energy,\,CST}|} - 1 \tag{15}$$

When $\Delta\lambda$, $\Delta\lambda\,[\%]$, or Δmse are negative, we observe reduction in the Solow residual (λ) or the fitting residuals (mse) relative to the unadjusted, no-energy, CST case.

To calculate $\Delta\lambda$ and Δmse, we need to fit all combinations of country, model, flavor, energy, nest, and CST. Table 6 shows coefficients for all fitted Cobb-Douglas models and the reference model (exponential-only).

Table 6: Model parameters for all Cobb-Douglas models.

Country	model	flavor	energy	cst	scale	lambda	alpha_1	alpha_2	alpha_3	mse
PT	exp		Without energy	Reject CST	1.283537	0.034507	0.000000	0.000000	0.000000	0.013359
PT	CD	Unadjusted	Without energy	Adhere to CST	1.328501	0.016080	0.300000	0.700000	0.000000	0.011724
PT	CD	Unadjusted	Without energy	Reject CST	1.113489	−0.005539	1.000000	0.000000	0.000000	0.003190
PT	CD	Unadjusted	With energy	Reject CST	1.130790	−0.004439	0.755978	0.000000	0.244022	0.002628
PT	CD	Quality-adjusted	Without energy	Adhere to CST	1.364226	0.004183	0.300000	0.700000	0.000000	0.011340
PT	CD	Quality-adjusted	Without energy	Reject CST	1.231989	−0.015079	1.000000	0.000000	0.000000	0.008937
PT	CD	Quality-adjusted	With energy	Reject CST	1.111839	−0.004569	0.000000	0.000000	1.000000	0.002268
UK	exp		Without energy	Reject CST	1.021833	0.023565	0.000000	0.000000	0.000000	0.001212
UK	CD	Unadjusted	Without energy	Adhere to CST	1.032978	0.015510	0.300000	0.700000	0.000000	0.000775
UK	CD	Unadjusted	Without energy	Reject CST	1.000925	0.006654	0.544956	0.455044	0.000000	0.000494
UK	CD	Unadjusted	With energy	Reject CST	0.995030	0.018134	0.162500	0.460325	0.377175	0.000331
UK	CD	Quality-adjusted	Without energy	Adhere to CST	1.038384	0.008725	0.300000	0.700000	0.000000	0.001564
UK	CD	Quality-adjusted	Without energy	Reject CST	1.015146	−0.008149	0.751058	0.248942	0.000000	0.000720
UK	CD	Quality-adjusted	With energy	Reject CST	0.970614	0.001981	0.403433	0.330248	0.266319	0.000453

Tables 7 and 8 show coefficients for all fitted CES models and the reference model (exponential-only).

Table 9 shows $\Delta\lambda$ and Δmse results for the Cobb-Douglas models.

Table 7: Model parameters for all CES models.

Country	model	flavor	energy	nest	cst	gamma	lambda	alpha_1	alpha_2	alpha_3
PT	exp		Without energy		Reject CST		0.034507	0.000000	0.000000	0.000000
PT	CES	Unadjusted	Without energy	kl	Adhere to CST	1.318284	0.018825	0.300000	0.700000	0.000000
PT	CES	Unadjusted	Without energy	kl	Reject CST	0.998729	0.012404	0.983352	0.016648	0.000000
PT	CES	Unadjusted	With energy	kle	Reject CST	1.022631	0.009738	0.727922	0.000064	0.272014
PT	CES	Unadjusted	With energy	lek	Reject CST	1.005212	0.008318	1.000000	0.000000	0.000000
PT	CES	Unadjusted	With energy	ekl	Reject CST	1.009966	0.010742	0.983292	0.016708	0.000000
PT	CES	Quality-adjusted	Without energy	kl	Adhere to CST	1.335143	0.009131	0.300000	0.700000	0.000000
PT	CES	Quality-adjusted	Without energy	kl	Reject CST	1.017672	0.005739	1.000000	0.000000	0.000000
PT	CES	Quality-adjusted	With energy	kle	Reject CST	1.027582	0.004508	0.902835	0.000000	0.097165
PT	CES	Quality-adjusted	With energy	lek	Reject CST	1.016454	0.005790	1.000000	0.000000	0.000000
PT	CES	Quality-adjusted	With energy	ekl	Reject CST	1.017327	0.005789	1.000000	0.000000	0.000000
UK	exp		Without energy		Reject CST		0.023565	0.000000	0.000000	0.000000
UK	CES	Unadjusted	Without energy	kl	Adhere to CST	1.006734	0.020914	0.300000	0.700000	0.000000
UK	CES	Unadjusted	Without energy	kl	Reject CST	0.977872	0.017904	0.436298	0.563702	0.000000
UK	CES	Unadjusted	With energy	kle	Reject CST	0.989712	0.013857	0.445243	0.554757	0.000000
UK	CES	Unadjusted	With energy	lek	Reject CST	0.982919	0.020230	0.570049	0.134387	0.295564
UK	CES	Unadjusted	With energy	ekl	Reject CST	0.985088	0.022682	0.254298	0.738243	0.007460
UK	CES	Quality-adjusted	Without energy	kl	Adhere to CST	1.008638	0.013592	0.300000	0.700000	0.000000
UK	CES	Quality-adjusted	Without energy	kl	Reject CST	0.983725	0.001956	0.628728	0.371272	0.000000
UK	CES	Quality-adjusted	With energy	kle	Reject CST	0.980315	0.007103	0.426684	0.421804	0.151512
UK	CES	Quality-adjusted	With energy	lek	Reject CST	0.968799	0.007115	0.469167	0.144017	0.386817
UK	CES	Quality-adjusted	With energy	ekl	Reject CST	0.981695	0.011338	0.305749	0.642022	0.052229

Tables 10–12 show $\Delta\lambda$ and Δmse results for CES models with various nestings.

Figures 11 and 12 summarize $\Delta\lambda$ and Δmse results for the Cobb-Douglas model.

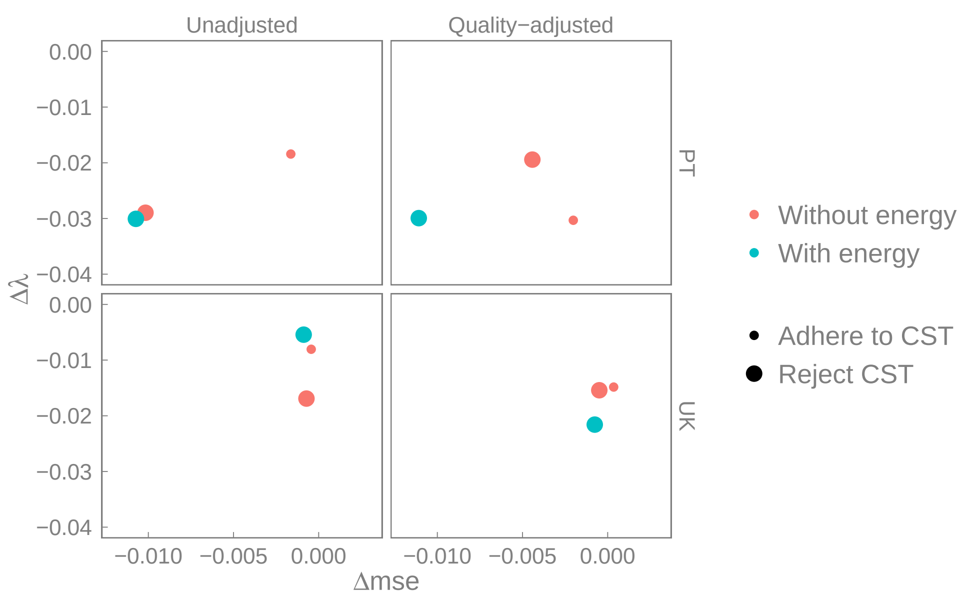


Figure 11: Change in Solow Residuals ($\Delta\lambda$) and mean squared error (Δmse) for the Cobb-Douglas model relative to the exponential-only model.

Table 8: Model parameters for all CES models.

Country	model	flavor	energy	nest	cst	sigma_1	sigma	mse
PT	exp		Without energy		Reject CST			0.013359
PT	CES	Unadjusted	Without energy	kl	Adhere to CST	0.665163		0.011450
PT	CES	Unadjusted	Without energy	kl	Reject CST	0.173623		0.001113
PT	CES	Unadjusted	With energy	kle	Reject CST	0.078512	1.747120	0.000746
PT	CES	Unadjusted	With energy	lek	Reject CST	0.398887	0.013645	0.000588
PT	CES	Unadjusted	With energy	ekl	Reject CST	0.005193	0.186281	0.000985
PT	CES	Quality-adjusted	Without energy	kl	Adhere to CST	0.399944		0.009694
PT	CES	Quality-adjusted	Without energy	kl	Reject CST	0.017511		0.000804
PT	CES	Quality-adjusted	With energy	kle	Reject CST	0.017991	Inf	0.000792
PT	CES	Quality-adjusted	With energy	lek	Reject CST	0.017595	0.005358	0.000812
PT	CES	Quality-adjusted	With energy	ekl	Reject CST	0.017013		0.000817
UK	exp		Without energy		Reject CST			0.001212
UK	CES	Unadjusted	Without energy	kl	Adhere to CST	0.509524		0.000491
UK	CES	Unadjusted	Without energy	kl	Reject CST	0.500000		0.000326
UK	CES	Unadjusted	With energy	kle	Reject CST	0.720815	0.009391	0.000223
UK	CES	Unadjusted	With energy	lek	Reject CST	Inf	0.465188	0.000289
UK	CES	Unadjusted	With energy	ekl	Reject CST	0.103338	1.340508	0.000281
UK	CES	Quality-adjusted	Without energy	kl	Adhere to CST	0.545219		0.001207
UK	CES	Quality-adjusted	Without energy	kl	Reject CST	0.607601		0.000571
UK	CES	Quality-adjusted	With energy	kle	Reject CST	0.611818	0.014357	0.000267
UK	CES	Quality-adjusted	With energy	lek	Reject CST	Inf	0.611994	0.000380
UK	CES	Quality-adjusted	With energy	ekl	Reject CST	0.190138	1.247783	0.000301

Table 9: $\Delta\lambda$ and Δmse for Cobb-Douglas models.

Country	model	flavor	energy	cst	Dlambda	Dmse
PT	CD	Unadjusted	Without energy	Adhere to CST	−0.018427	−0.001635
PT	CD	Unadjusted	Without energy	Reject CST	−0.028968	−0.010168
PT	CD	Unadjusted	With energy	Reject CST	−0.030068	−0.010730
PT	CD	Quality-adjusted	Without energy	Adhere to CST	−0.030324	−0.002019
PT	CD	Quality-adjusted	Without energy	Reject CST	−0.019428	−0.004422
PT	CD	Quality-adjusted	With energy	Reject CST	−0.029938	−0.011091
UK	CD	Unadjusted	Without energy	Adhere to CST	−0.008055	−0.000437
UK	CD	Unadjusted	Without energy	Reject CST	−0.016911	−0.000717
UK	CD	Unadjusted	With energy	Reject CST	−0.005432	−0.000880
UK	CD	Quality-adjusted	Without energy	Adhere to CST	−0.014840	0.000352
UK	CD	Quality-adjusted	Without energy	Reject CST	−0.015416	−0.000492
UK	CD	Quality-adjusted	With energy	Reject CST	−0.021584	−0.000758

Table 10: $\Delta\lambda$ and Δmse for CES models with $(kl)e$ nesting.

Country	model	flavor	energy	nest	cst	Dlambda	Dmse
PT	exp		Without energy		Reject CST	0.000000	0.000000
PT	CES	Unadjusted	Without energy	kl	Adhere to CST	−0.015683	−0.001908
PT	CES	Unadjusted	Without energy	kl	Reject CST	−0.022103	−0.012246
PT	CES	Unadjusted	With energy	kle	Reject CST	−0.024769	−0.012613
PT	CES	Quality-adjusted	Without energy	kl	Adhere to CST	−0.025377	−0.003665
PT	CES	Quality-adjusted	Without energy	kl	Reject CST	−0.028768	−0.012555
PT	CES	Quality-adjusted	With energy	kle	Reject CST	−0.029999	−0.012566
UK	exp		Without energy		Reject CST	0.000000	0.000000
UK	CES	Unadjusted	Without energy	kl	Adhere to CST	−0.002651	−0.000721
UK	CES	Unadjusted	Without energy	kl	Reject CST	−0.005661	−0.000886
UK	CES	Unadjusted	With energy	kle	Reject CST	−0.009708	−0.000989
UK	CES	Quality-adjusted	Without energy	kl	Adhere to CST	−0.009973	−0.000004
UK	CES	Quality-adjusted	Without energy	kl	Reject CST	−0.021609	−0.000641
UK	CES	Quality-adjusted	With energy	kle	Reject CST	−0.016463	−0.000944

Table 11: $\Delta\lambda$ and Δmse for CES models with $(le)k$ nesting.

Country	model	flavor	energy	nest	cst	Dlambda	Dmse
PT	exp		Without energy		Reject CST	0.000000	0.000000
PT	CES	Unadjusted	Without energy	kl	Adhere to CST	−0.015683	−0.001908
PT	CES	Unadjusted	Without energy	kl	Reject CST	−0.022103	−0.012246
PT	CES	Unadjusted	With energy	lek	Reject CST	−0.026189	−0.012771
PT	CES	Quality-adjusted	Without energy	kl	Adhere to CST	−0.025377	−0.003665
PT	CES	Quality-adjusted	Without energy	kl	Reject CST	−0.028768	−0.012555
PT	CES	Quality-adjusted	With energy	lek	Reject CST	−0.028717	−0.012546
UK	exp		Without energy		Reject CST	0.000000	0.000000
UK	CES	Unadjusted	Without energy	kl	Adhere to CST	−0.002651	−0.000721
UK	CES	Unadjusted	Without energy	kl	Reject CST	−0.005661	−0.000886
UK	CES	Unadjusted	With energy	lek	Reject CST	−0.003335	−0.000923
UK	CES	Quality-adjusted	Without energy	kl	Adhere to CST	−0.009973	−0.000004
UK	CES	Quality-adjusted	Without energy	kl	Reject CST	−0.021609	−0.000641
UK	CES	Quality-adjusted	With energy	lek	Reject CST	−0.016450	−0.000831

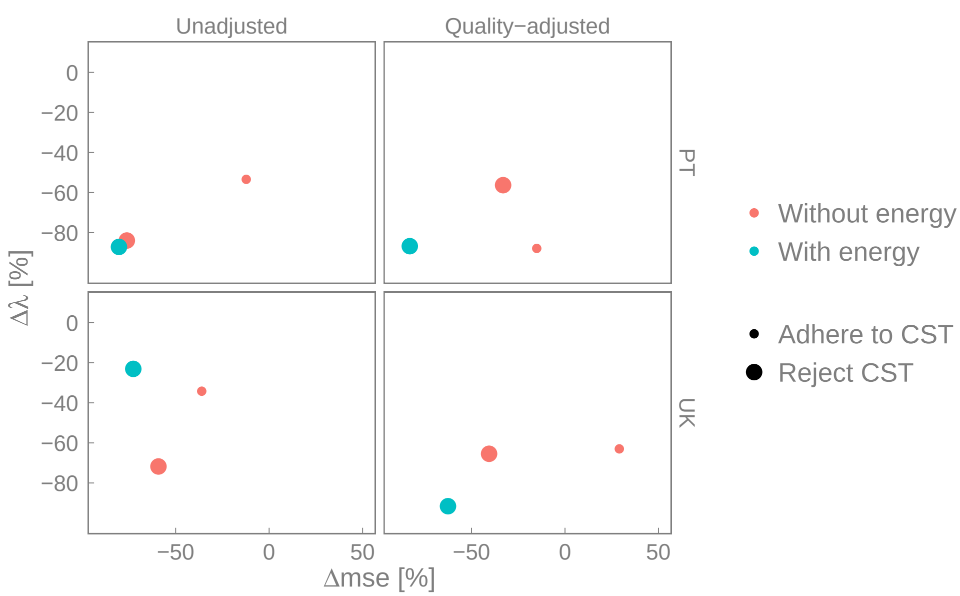


Figure 12: Percentage change in Solow Residuals ($\Delta\lambda\,[\%]$) and mean squared error ($\Delta mse\,[\%]$) for the Cobb-Douglas model relative to exponential-only models.

Table 12: $\Delta\lambda$ and Δmse for CES models with $(ek)l$ nesting.

Country	model	flavor	energy	nest	cst	Dlambda	Dmse
PT	exp		Without energy		Reject CST	0.000000	0.000000
PT	CES	Unadjusted	Without energy	kl	Adhere to CST	−0.015683	−0.001908
PT	CES	Unadjusted	Without energy	kl	Reject CST	−0.022103	−0.012246
PT	CES	Unadjusted	With energy	ekl	Reject CST	−0.023766	−0.012374
PT	CES	Quality-adjusted	Without energy	kl	Adhere to CST	−0.025377	−0.003665
PT	CES	Quality-adjusted	Without energy	kl	Reject CST	−0.028768	−0.012555
PT	CES	Quality-adjusted	With energy	ekl	Reject CST	−0.028718	−0.012542
UK	exp		Without energy		Reject CST	0.000000	0.000000
UK	CES	Unadjusted	Without energy	kl	Adhere to CST	−0.002651	−0.000721
UK	CES	Unadjusted	Without energy	kl	Reject CST	−0.005661	−0.000886
UK	CES	Unadjusted	With energy	ekl	Reject CST	−0.000883	−0.000930
UK	CES	Quality-adjusted	Without energy	kl	Adhere to CST	−0.009973	−0.000004
UK	CES	Quality-adjusted	Without energy	kl	Reject CST	−0.021609	−0.000641
UK	CES	Quality-adjusted	With energy	ekl	Reject CST	−0.012227	−0.000910

Figures 13 and 14 summarize $\Delta\lambda$ and Δmse results for CES models.

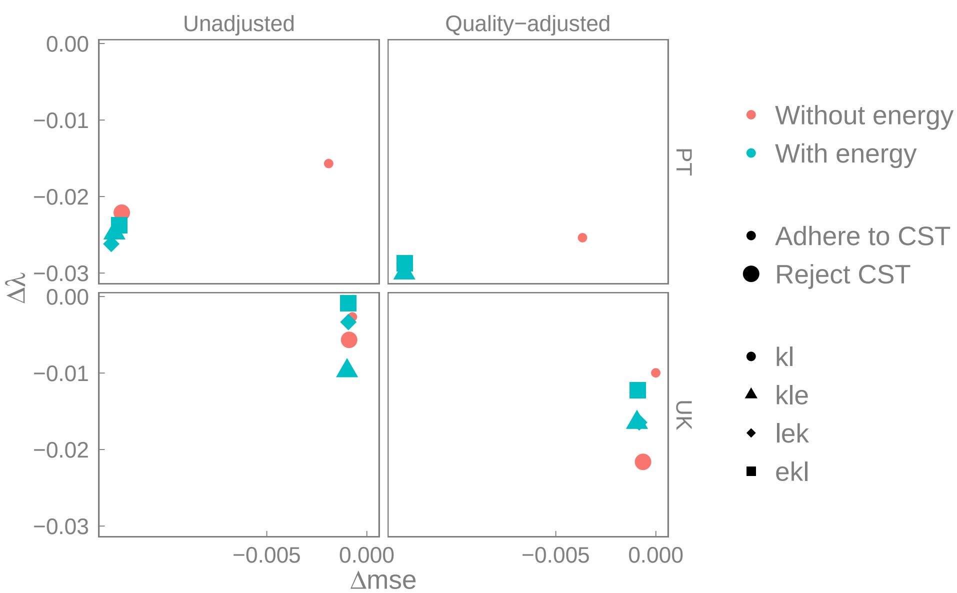


Figure 13: Change in Solow Residuals ($\Delta\lambda$) and mean squared error (Δmse) for the CES modles relative to the exponential-only model.

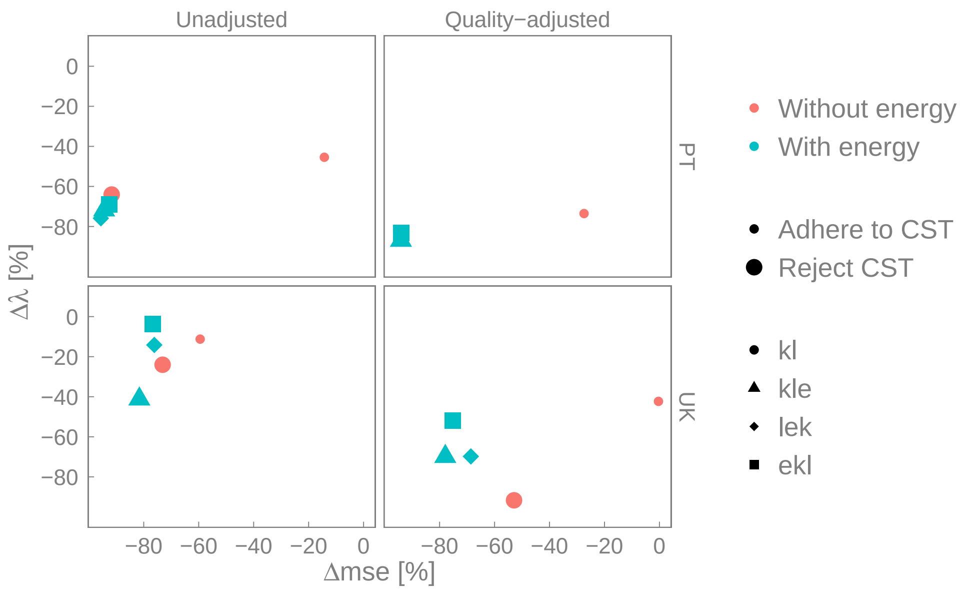


Figure 14: Percentage change in Solow Residuals ($\Delta\lambda$ [%]) and mean squared error (Δmse [%]) for the CES models relative to the exponential-only model.

6.3. Are the means of the fitting residuals significantly different from zero?

The set of fitting residual values (r_i) for each combination of country (PT or UK), model (Cobb-Douglas or CES), flavor (Unadjusted or Quality-adjusted), energy (with or without), and nest [(kl), $(kl)e$, $(le)k$, or $(ek)l$, for the CES models] can be assessed for statistical significance compared to zero by a t-test. The null hypothesis for the t-test is that the mean of each group is equal to zero, and the alternative hypothesis is that the mean is not equal to zero. Smaller p-values indicate increasing confidence that the mean is different from zero. We expect that p-values will be large (close to unity), because the mean of the residuals is expected to be small (close to zero). Indeed, that is the case. Table 13 summarizes the mean and associated p-values values for the fitting residuals (r_i).

6.4. Does quality-adjusting the factors of production or including energy decrease the fitting residuals in a statistically-significant manner?

The set of annual Δr values for each combination of country (PT or UK), model (Cobb-Douglas or CES), flavor (Unadjusted or Quality-adjusted),

Table 13: Means and p-values for fitting residuals (r_i).

Country	model	flavor	energy	nest	Residuals.mean	Residuals.p.value
PT	CD	Unadjusted	Without energy		−0.000000	1.000000
PT	CD	Unadjusted	With energy		−0.000000	1.000000
PT	CD	Quality-adjusted	Without energy		0.000000	1.000000
PT	CD	Quality-adjusted	With energy		0.000000	1.000000
PT	CES	Unadjusted	Without energy	kl	0.000000	0.999999
PT	CES	Unadjusted	With energy	kle	−0.000000	0.999999
PT	CES	Unadjusted	With energy	lek	−0.000162	0.962887
PT	CES	Unadjusted	With energy	ekl	−0.000080	0.985890
PT	CES	Quality-adjusted	Without energy	kl	−0.000025	0.997238
PT	CES	Quality-adjusted	With energy	kle	−0.000083	0.983608
PT	CES	Quality-adjusted	With energy	lek	−0.000007	0.998657
PT	CES	Quality-adjusted	With energy	ekl	0.000001	0.999715
PT	exp		Without energy		−0.000000	1.000000
UK	CD	Unadjusted	Without energy		0.000000	1.000000
UK	CD	Unadjusted	With energy		−0.000000	1.000000
UK	CD	Quality-adjusted	Without energy		0.000000	1.000000
UK	CD	Quality-adjusted	With energy		−0.000000	1.000000
UK	CES	Unadjusted	Without energy	kl	−0.000000	1.000000
UK	CES	Unadjusted	With energy	kle	−0.000037	0.986099
UK	CES	Unadjusted	With energy	lek	−0.000000	1.000000
UK	CES	Unadjusted	With energy	ekl	0.000000	0.999997
UK	CES	Quality-adjusted	Without energy	kl	0.000000	0.999999
UK	CES	Quality-adjusted	With energy	kle	−0.000000	0.999998
UK	CES	Quality-adjusted	With energy	lek	0.000000	1.000000
UK	CES	Quality-adjusted	With energy	ekl	0.000000	0.999998
UK	exp		Without energy		−0.000000	1.000000

energy (with or without), and nest [(kl), $(kl)e$, $(le)k$, or $(ek)l$, for the CES models] can be assessed for statistical significance compared to zero by a t-test. The null hypothesis for the t-test is that the mean of each group is equal to zero, and the alternative hypothesis is that the mean is less than zero. Smaller p-values indicate increasing confidence that the mean is actually less than zero. Table 14 summarizes the mean and associated p-values values for Δr.

6.5. What trends exist in factors shares?

Figures 15 and 16 show factor shares (α values) for CES models for Portugal and the UK, respectively. See Table 3 for details.

Table 14: Means and p-values for Δr.

Country	model	flavor	energy	nest	Dr.mean	Dr.p.value
PT	CD	Unadjusted	Without energy		−0.025733	0.000000
PT	CD	Unadjusted	With energy		−0.048181	0.000000
PT	CD	Quality-adjusted	Without energy		−0.009587	0.003533
PT	CD	Quality-adjusted	With energy		−0.051804	0.000000
PT	CES	Unadjusted	Without energy	kl	−0.038049	0.000000
PT	CES	Unadjusted	With energy	kle	−0.069223	0.000000
PT	CES	Unadjusted	With energy	lek	−0.071483	0.000000
PT	CES	Unadjusted	With energy	ekl	−0.065928	0.000000
PT	CES	Quality-adjusted	Without energy	kl	−0.042697	0.000000
PT	CES	Quality-adjusted	With energy	kle	−0.067761	0.000000
PT	CES	Quality-adjusted	With energy	lek	−0.067419	0.000000
PT	CES	Quality-adjusted	With energy	ekl	−0.067335	0.000000
PT	exp		Without energy		0.000000	
UK	CD	Unadjusted	Without energy		−0.011254	0.000003
UK	CD	Unadjusted	With energy		−0.014450	0.000000
UK	CD	Quality-adjusted	Without energy		−0.004538	0.055557
UK	CD	Quality-adjusted	With energy		−0.012454	0.000072
UK	CES	Unadjusted	Without energy	kl	−0.013655	0.000000
UK	CES	Unadjusted	With energy	kle	−0.017349	0.000000
UK	CES	Unadjusted	With energy	lek	−0.015158	0.000000
UK	CES	Unadjusted	With energy	ekl	−0.015557	0.000000
UK	CES	Quality-adjusted	Without energy	kl	−0.007222	0.003270
UK	CES	Quality-adjusted	With energy	kle	−0.016547	0.000000
UK	CES	Quality-adjusted	With energy	lek	−0.013366	0.000007
UK	CES	Quality-adjusted	With energy	ekl	−0.014973	0.000000
UK	exp		Without energy		0.000000	

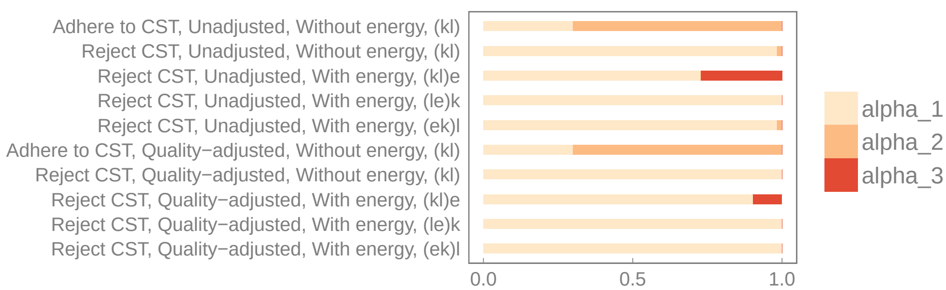


Figure 15: Factor shares (α values) for CES models for Portugal.

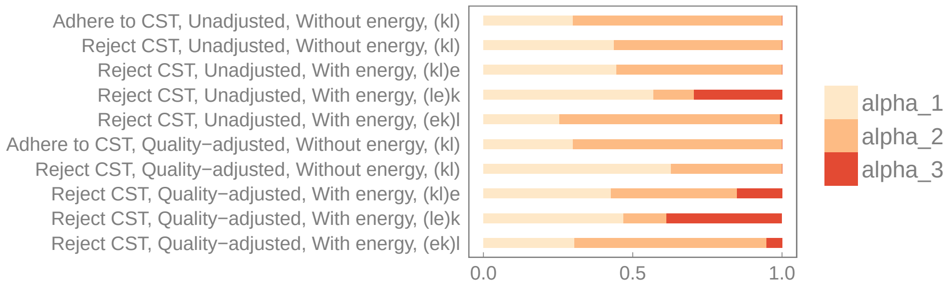


Figure 16: Factor shares (α values) for CES models for the UK.

7. Conclusion

8. Future Work

Acknowledgements

References

Appendix A. Derivation of dynamic Solow residual for the CD equation

Assuming that parameters θ, α, and β in Equation 2 are known from parameter estimation, we can estimate the value of λ as follows. First, assume constant returns to scale such that $\alpha+\beta+\gamma=1$, and calculate $\gamma = 1-\alpha-\beta$. Then, take the natural logarithm (ln) of Equation 2 to obtain

$$\ln y = \ln\theta + \ln A + \alpha \ln k + \beta \ln l + (1-\alpha-\beta)\ln e \; . \tag{A.1}$$

By taking the derivative of Equation A.1 with respect to time (t) and noting that model parameters θ, α, and β are constant with respect to time, we obtain

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} + \alpha\frac{1}{k}\frac{\mathrm{d}k}{\mathrm{d}t} + \beta\frac{1}{l}\frac{\mathrm{d}l}{\mathrm{d}t} + (1-\alpha-\beta)\frac{1}{e}\frac{\mathrm{d}e}{\mathrm{d}t} \; . \tag{A.2}$$

Solving for the total factor productivity term gives

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} - \left[\alpha\frac{1}{k}\frac{\mathrm{d}k}{\mathrm{d}t} + \beta\frac{1}{l}\frac{\mathrm{d}l}{\mathrm{d}t} + (1-\alpha-\beta)\frac{1}{e}\frac{\mathrm{d}e}{\mathrm{d}t}\right] \; . \tag{A.3}$$

Recognizing that $A \equiv \mathrm{e}^{\lambda(t-t_0)}$ gives

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{\mathrm{e}^{\lambda(t-t_0)}}\ \lambda \mathrm{e}^{\lambda(t-t_0)} = \lambda\ , \tag{A.4}$$

which can be substituted into Equation A.3 to find

$$\lambda = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} - \left[\alpha\frac{1}{k}\frac{\mathrm{d}k}{\mathrm{d}t} + \beta\frac{1}{l}\frac{\mathrm{d}l}{\mathrm{d}t} + (1-\alpha-\beta)\frac{1}{e}\frac{\mathrm{d}e}{\mathrm{d}t}\right]\ . \tag{A.5}$$

Equation A.5 applies for any instant in time.

If we approximate derivatives in Equation A.5 with forward differences between times i and j, we find

$$\lambda_{i,j} = \frac{1}{y_i}\frac{y_j - y_i}{t_j - t_i} - \left[\alpha\frac{1}{k_i}\frac{k_j - k_i}{t_j - t_i} + \beta\frac{1}{l_i}\frac{l_j - l_i}{t_j - t_i} + (1-\alpha-\beta)\frac{1}{e_i}\frac{e_j - e_i}{t_j - t_i}\right]\ , \tag{A.6}$$

where $\lambda_{i,j}$ approximates the true, instantaneous value of λ from Equation A.5 between times i and j.

Interestingly,

$$\lambda_{1,n} \neq \sum_{i=2}^{n} \lambda_{i,i-1}\ . \tag{A.7}$$

Rather,

$$\lambda_{1,n} = \frac{1}{y_1}\frac{y_n - y_1}{t_n - t_1} - \left[\alpha\frac{1}{k_1}\frac{k_n - k_1}{t_n - t_1} + \beta\frac{1}{l_1}\frac{l_n - l_1}{t_n - t_1} + (1-\alpha-\beta)\frac{1}{e_1}\frac{e_n - e_1}{t_n - t_1}\right]\ , \tag{A.8}$$

which will become increasingly inaccurate over large time spans, because y_1, k_1, l_1, and e_1 will be less representative of the average value of y, k, l, and e in the time span, respectively. This suggests that an averaging approach such as

$$\lambda_{1,n} = \frac{\sum_{i=2}^{n} \lambda_{i,i-1}}{n-1} \tag{A.9}$$

is a better approximation of Equation A.8, which is, itself, an approximation of the true value of λ given by Equation A.5.

The fraction (f) of GDP growth explained by the Solow residual for the CD model can be given as

$$f \equiv \frac{\lambda}{\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t}} = 1 - \frac{\alpha\frac{1}{k}\frac{\mathrm{d}k}{\mathrm{d}t} + \beta\frac{1}{l}\frac{\mathrm{d}l}{\mathrm{d}t} + (1-\alpha-\beta)\frac{1}{e}\frac{\mathrm{d}e}{\mathrm{d}t}}{\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t}}\ . \tag{A.10}$$

Using forward differences to estimate f yields

$$f_{i,j} = \frac{\lambda_{i,j}}{\frac{1}{y_i}\frac{y_j-y_i}{t_j-t_i}} = 1 - \frac{\alpha\frac{1}{k_i}\frac{k_j-k_i}{t_j-t_i} + \beta\frac{1}{l_i}\frac{l_j-l_i}{t_j-t_i} + (1-\alpha-\beta)\frac{1}{e_i}\frac{e_j-e_i}{t_j-t_i}}{\frac{1}{y_i}\frac{y_j-y_i}{t_j-t_i}} \ . \tag{A.11}$$

Cancelling $t_j - t_i$ terms and simplifying gives

$$f_{i,j} = 1 - \frac{\alpha\left(\frac{k_j}{k_i} - 1\right) + \beta\left(\frac{l_j}{l_i} - 1\right) + (1-\alpha-\beta)\left(\frac{e_j}{e_i} - 1\right)}{\left(\frac{y_j}{y_i} - 1\right)} \ , \tag{A.12}$$

which is an estimate of the instantaneous value of f given by Equation A.10.

Appendix B. Derivation of dynamic Solow residual for the CES equation

In this section, we derive the dynamic Solow residual (λ) for the CES equation with three generic factors of production, namely x_1, x_2, and x_3.

We can define

$$a \equiv \delta b^{\rho/\rho_1} + (1-\delta)x_3^{-\rho} \tag{B.1}$$

and

$$b \equiv \delta_1 x_1^{-\rho_1} + (1-\delta_1)x_2^{-\rho_1} \ , \tag{B.2}$$

such that Equation 3 can be restated as

$$y = \gamma A a^{-1/\rho} \ . \tag{B.3}$$

Taking the natural logarithm of Equation B.3 and realizing that γ is not a function of time, we find

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} + \left(-\frac{1}{\rho}\right)\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \ . \tag{B.4}$$

By Equation A.4, and after rearranging, the dynamic Solow residual for the CES equation can be stated as

$$\lambda = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{1}{a}\frac{1}{\rho}\frac{\mathrm{d}a}{\mathrm{d}t} \ . \tag{B.5}$$

To find $\frac{\mathrm{d}a}{\mathrm{d}t}$ and $\frac{\mathrm{d}b}{\mathrm{d}t}$, we take the time derivatives of Equations B.1 and B.2 and rearrange slightly to obtain

$$\frac{1}{\rho}\frac{\mathrm{d}a}{\mathrm{d}t} = \delta b^{(\rho/\rho_1 - 1)}\frac{1}{\rho_1}\frac{\mathrm{d}b}{\mathrm{d}t} - (1-\delta)x_3^{-(\rho+1)}\frac{\mathrm{d}x_3}{\mathrm{d}t} \tag{B.6}$$

and

$$\frac{1}{\rho_1}\frac{\mathrm{d}b}{\mathrm{d}t} = -\delta_1 x_1^{-(\rho_1+1)}\frac{\mathrm{d}x_1}{\mathrm{d}t} - (1-\delta_1)x_2^{-(\rho_1+1)}\frac{\mathrm{d}x_2}{\mathrm{d}t} \; . \tag{B.7}$$

Equations B.5–B.7 can be used to calculate a dynamic time series for λ given fitted parameters from the CES model (ρ, ρ_1, δ, and δ_1) and time series for economic output (y) and factors of production (x_1, x_2, and x_3). The time derivatives $\left(\frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}x_1}{\mathrm{d}t}, \frac{\mathrm{d}x_2}{\mathrm{d}t}, \text{and } \frac{\mathrm{d}x_3}{\mathrm{d}t}\right)$ can be approximated from historical data by forward finite differences in a manner similar to Equation A.6.

Appendix C. Thoughts on the "dynamic" Solow residual

Appendix A and Appendix B derived expressions for a "dynamic" Solow residual for Cobb-Douglas and CES production functions, respectively. However, these approaches conflate stochastic error terms and the time-dependent contribution of "technology" to economic growth in one variable ($\lambda_{i,j}$). In this appendix, we let the Solow residual (expressed as time-independent λ) stand on its own and try to separate the stochastic and time-dependent aspects of the fitting residuals (r_i). Note: this section is speculative, and we need feedback from Marco and Randy before moving ahead with a presentation or publication based on this work.

We can re-state the models (in general) as

$$y_i = \hat{y}_i \epsilon_i \tag{C.1}$$

where ϵ_i is a multiplicative error term, $\hat{y}_i$ is the predicted economic output, and $\hat{y}_i \equiv f(\theta, \lambda, \alpha, \beta, \gamma;\ t_i)$. (For the Cobb-Douglas model, $f = \theta A k^\alpha l^\beta e^\gamma$.)

Taking the natural logarithm (ln) of Equation C.1 yields

$$\ln y_i = \ln \hat{y}_i + \ln \epsilon_i \; , \tag{C.2}$$

which can be rearranged to show

$$\ln y_i - \ln \hat{y}_i = \ln \epsilon_i \; . \tag{C.3}$$

Using Equation 11, we can write

$$r_i = \ln \epsilon_i \ . \tag{C.4}$$

Our concern is that the residuals (r_i) may have a time-dependent component in addition to a stochastic component. To decompose, we can define

$$\epsilon_i = \mathrm{e}^{m(t_i - t_0)} \mathrm{e}^{\varepsilon_i^*} \ . \tag{C.5}$$

Substituting Equation C.5 into C.4 yields

$$r_i = m(t_i - t_0) + \varepsilon_i^* \ . \tag{C.6}$$

where m is a slope and captures the linear (in natural logarithmic space), time-dependent component of r_i. ε_i^* is a variable that captures a stochastic component of ϵ_i. Equation C.6 can be fitted via linear regression in time (t_i) as

$$r_i = m(t_i - t_0) + b + \varepsilon_i^* \ . \tag{C.7}$$

In Equation C.7, b is the y-intercept of the regression and captures a time-independent offset of r_i. Recall that ϵ_i is related to the fitting residuals (r_i) by Equation C.4. Thus, obtaining estimates of m and b and values for ε_i^* is equivalent to estimating the time-dependent, time-independent, and stochastic components, respectively, of the multiplicative errors (ϵ_i). This regression is equivalent to performing a linear fit to data in Figures 7–10.

For PT, Cobb-Douglas, unadjusted, without energy, we obtain the following:

```
mod <- models$PT$unadjusted$noE$CD
r_regression <- lm(formula = resid(mod) ~ mod$data$iYear)
print(summary(r_regression))


Call:
lm(formula = resid(mod) ~ mod$data$iYear)

Residuals:
      Min        1Q    Median        3Q       Max
-0.107498 -0.040098  0.009557  0.044471  0.109683
```

```
Coefficients:
                 Estimate Std. Error t value Pr(>|t|)
(Intercept)    -1.366e-17  1.575e-02       0        1
mod$data$iYear  3.847e-19  5.322e-04       0        1

Residual standard error: 0.0576 on 50 degrees of freedom
Multiple R-squared:  5.069e-32,Adjusted R-squared:  -0.02
F-statistic: 2.535e-30 on 1 and 50 DF,  p-value: 1
```

Note that the estimate for the coefficient in front of **`mod$data$iYear`** (m in Equation C.6) is incredibly small relative to the scale of the residuals (r_i), $3.8468693 \times 10^{-19}$, indicating that there is little linear time-dependent information in the residuals (r_i). This result is the same as saying that the slope of a line fitted through the PT, Unadjusted, Without energy data in Figure 7 is (effectively) zero. There is (effectively) no linear time trend in the residuals.

Furthermore, the intercept (b in Equation C.6) is very small relative to the scale of the residuals (r_i), $-1.3658522 \times 10^{-17}$, indicating that there is little time-independent information in the residuals (r_i). This result is the same as saying that the line fitted to the PT, Unadjusted, Without energy data in Figure 7 is nearly coincident with the x-axis.

Almost all of the information in the residuals (r_i) is stochastic and captured by the ε_i^* term.

Perhaps Marco can help us interpret the rest of the output here (`F-statistic`, `p-value`, etc.). (An aside: what about fitting a quadratic curve, instead of a line, through the r_i values in Figure 7?)

We can compare r_i and ε_i^*, graphically, but the result is uninteresting. We would expect r_i and ε_i^* to be equal to each other by Equation C.7, because (a) $m(t_i - t_0) \ll \max(r_i)$ for any value of t_i and (b) $b \ll \max(r_i)$. Indeed, Figure C.17 shows this is the case.

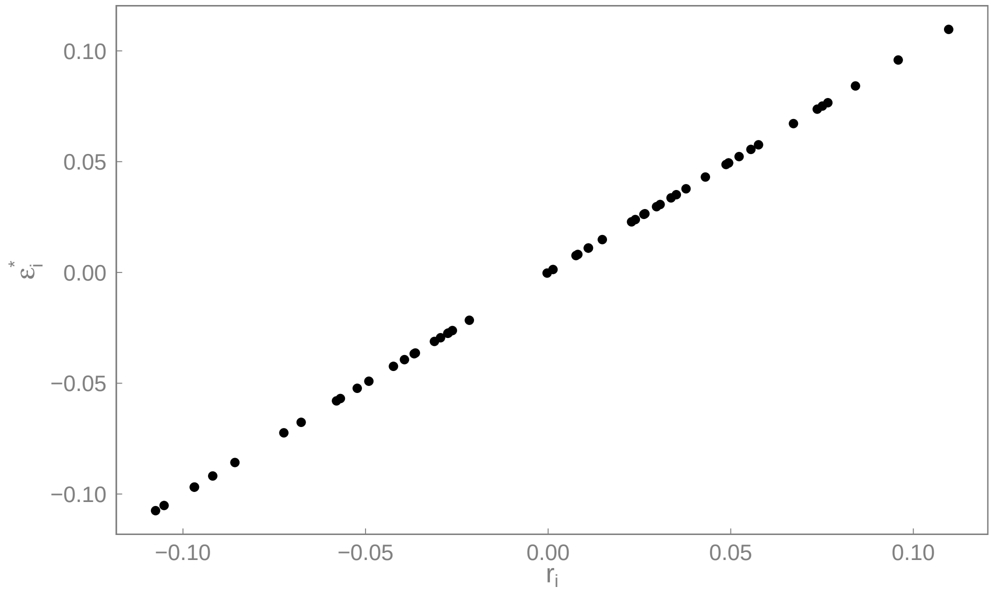


Figure C.17: Comparison between ε_i^* and r_i

I'm not sure where this leaves us. Marco and I agreed that I would re-derive the "dynamic" Solow residual equation while properly including the multiplicative error term (ϵ_i). I think I have done that, but the results aren't terribly interesting. There is little (linear) time-dependent (m) or time-independent (b) information in the residuals (r_i), as evidenced by Figures 7–10.

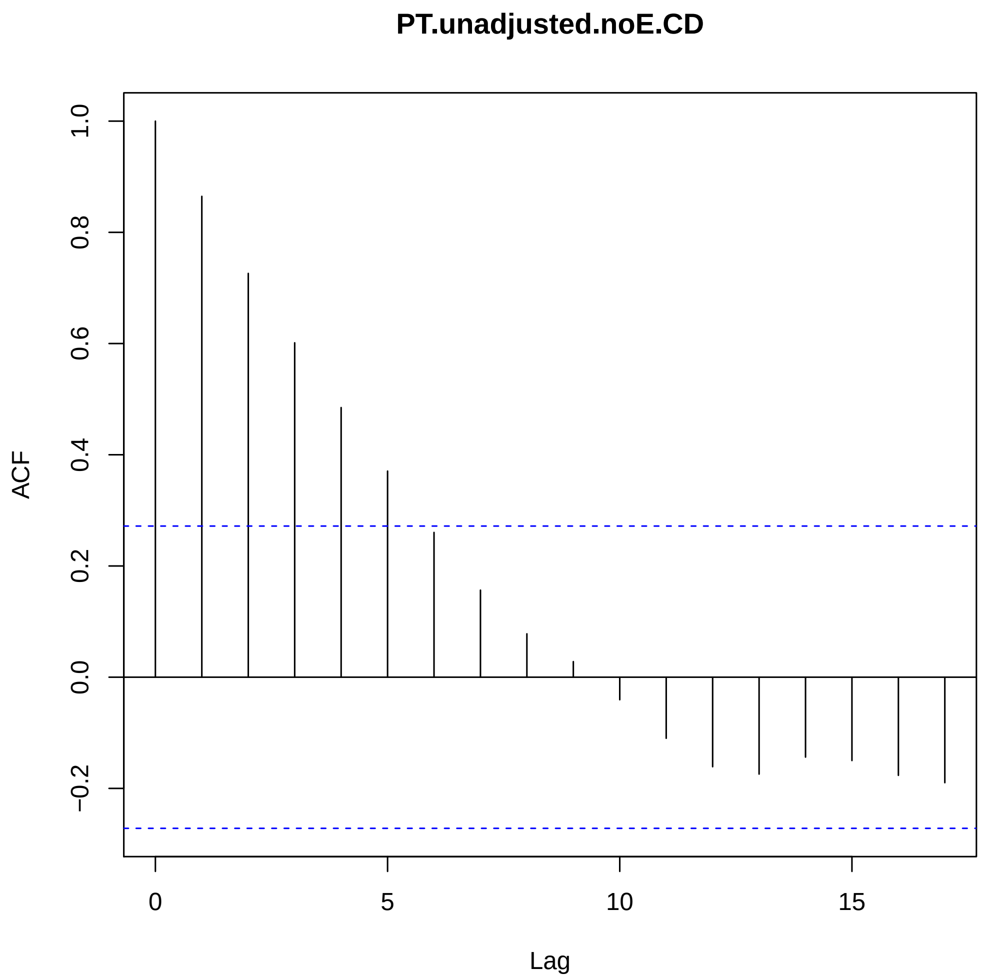
Perhaps there is another direction in which I should go with this analysis?

Appendix D. Correlograms

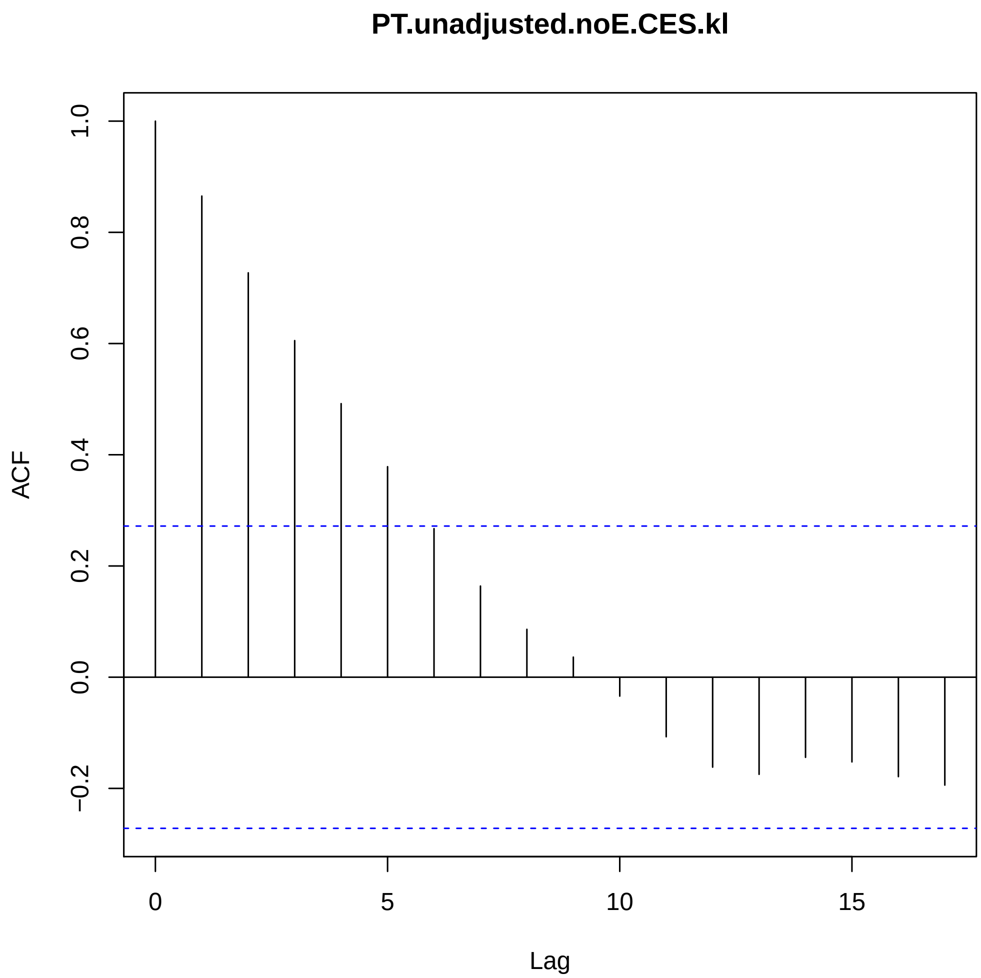
This appendix contains correlograms for all models.

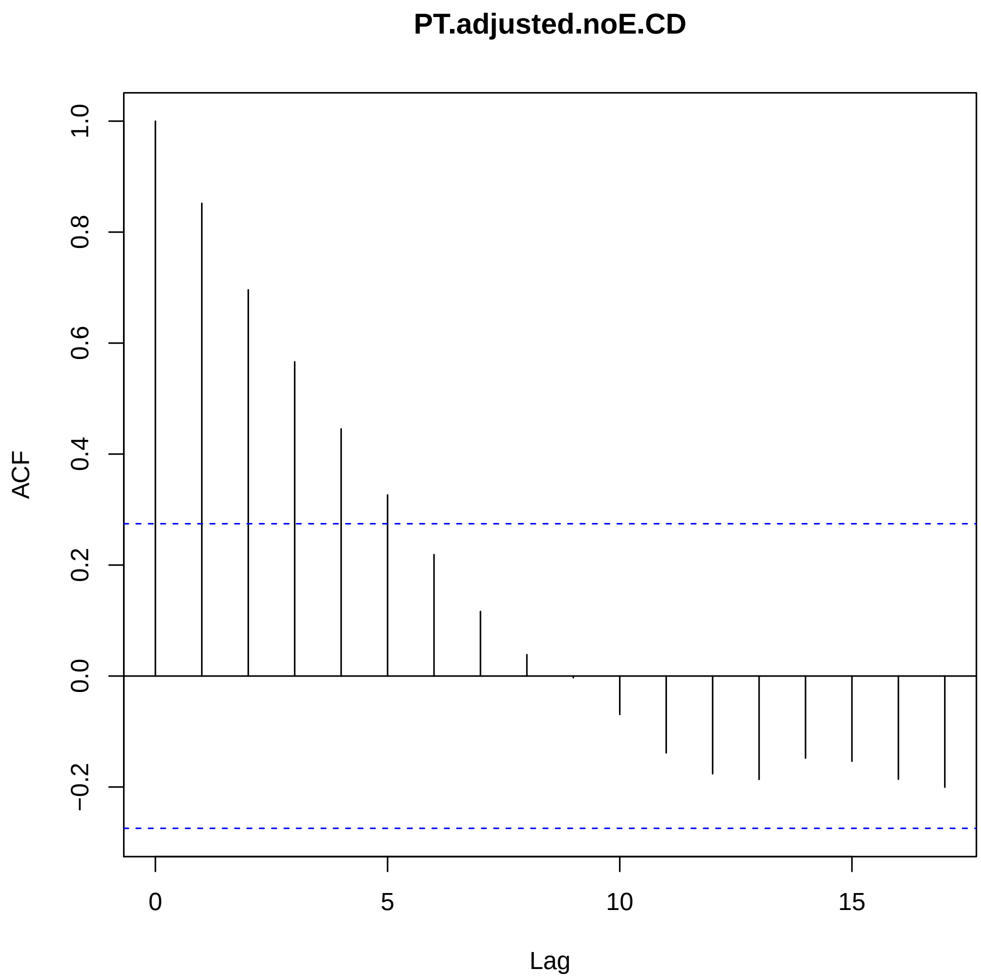
Appendix D.1. Correlograms for models respecting the cost-share theorem

In this section we present correlograms for all models that respect the cost-share theorem.

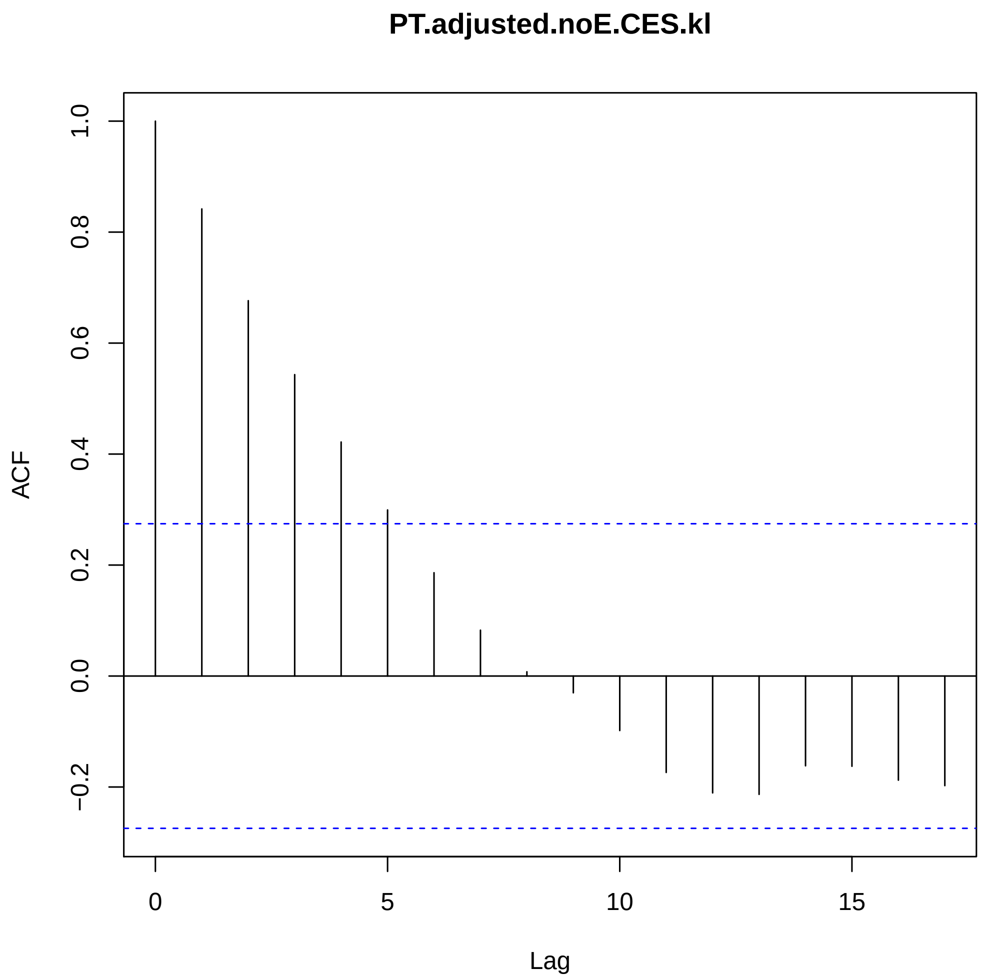


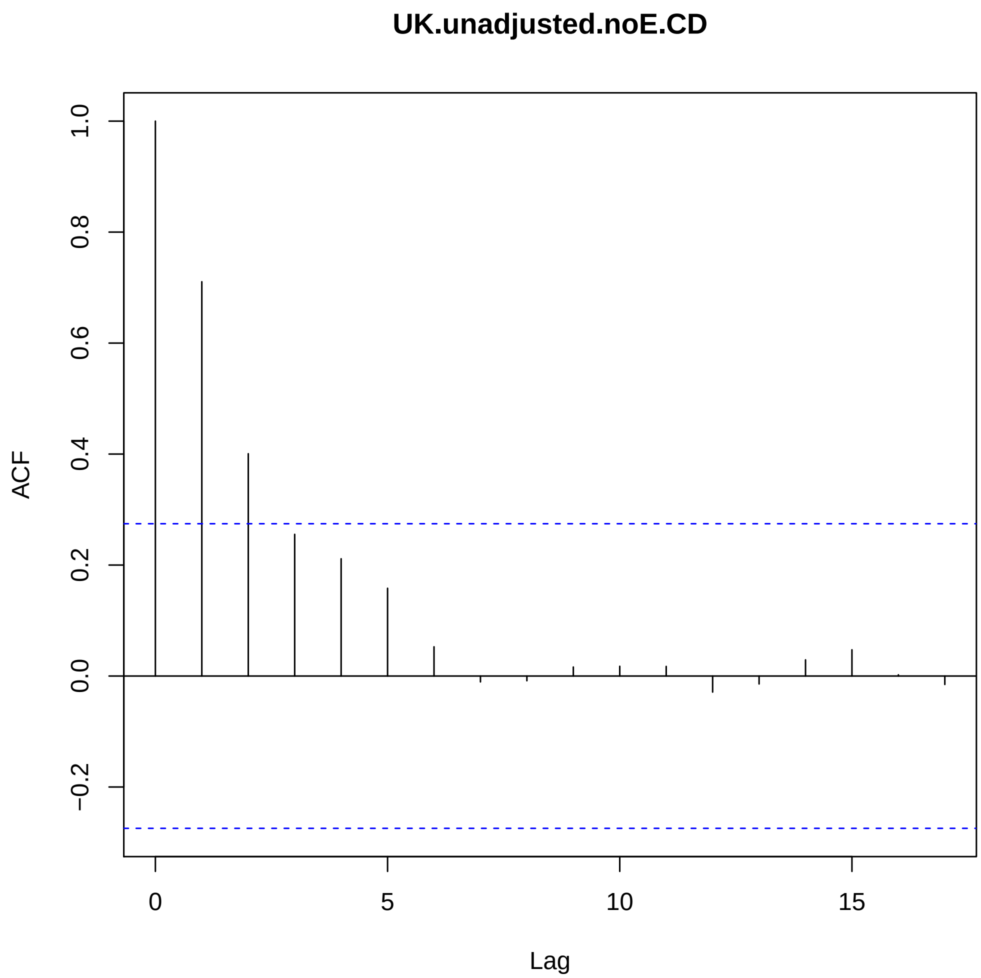
PT.unadjusted.noE.CD
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag

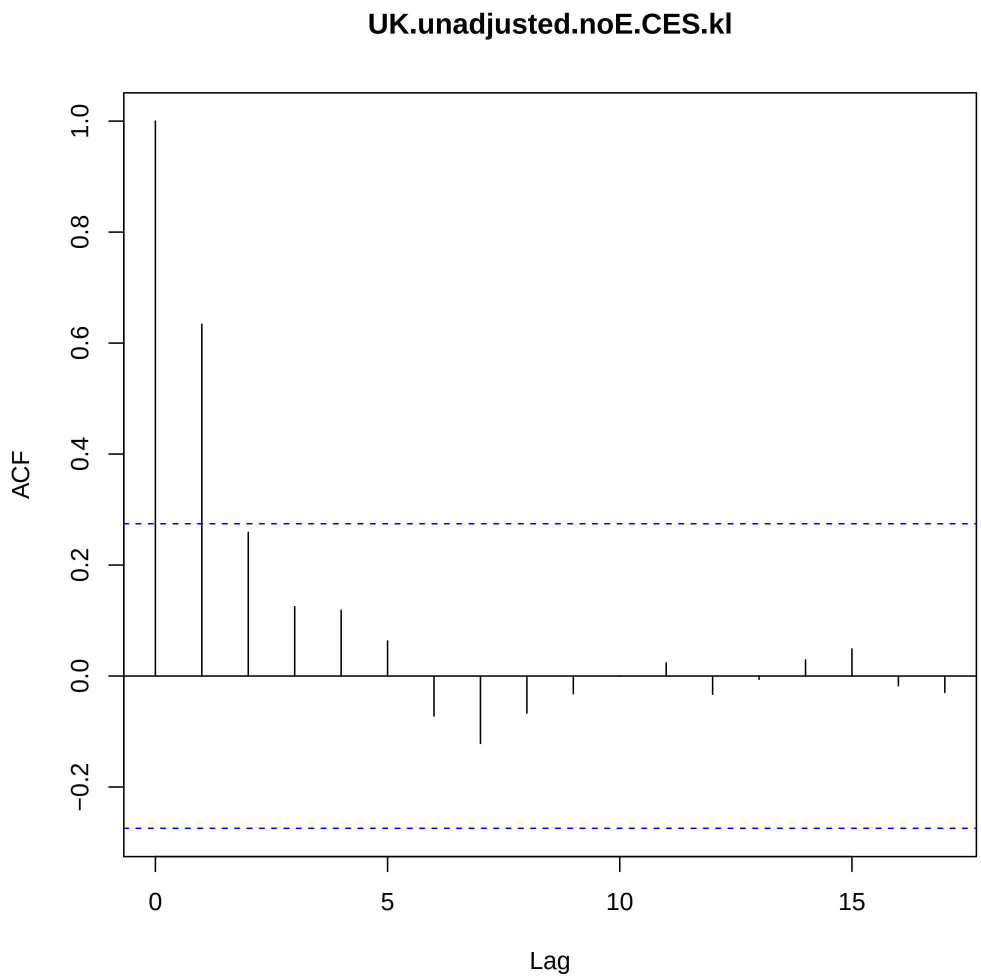

PT.unadjusted.noE.CES.kl
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag

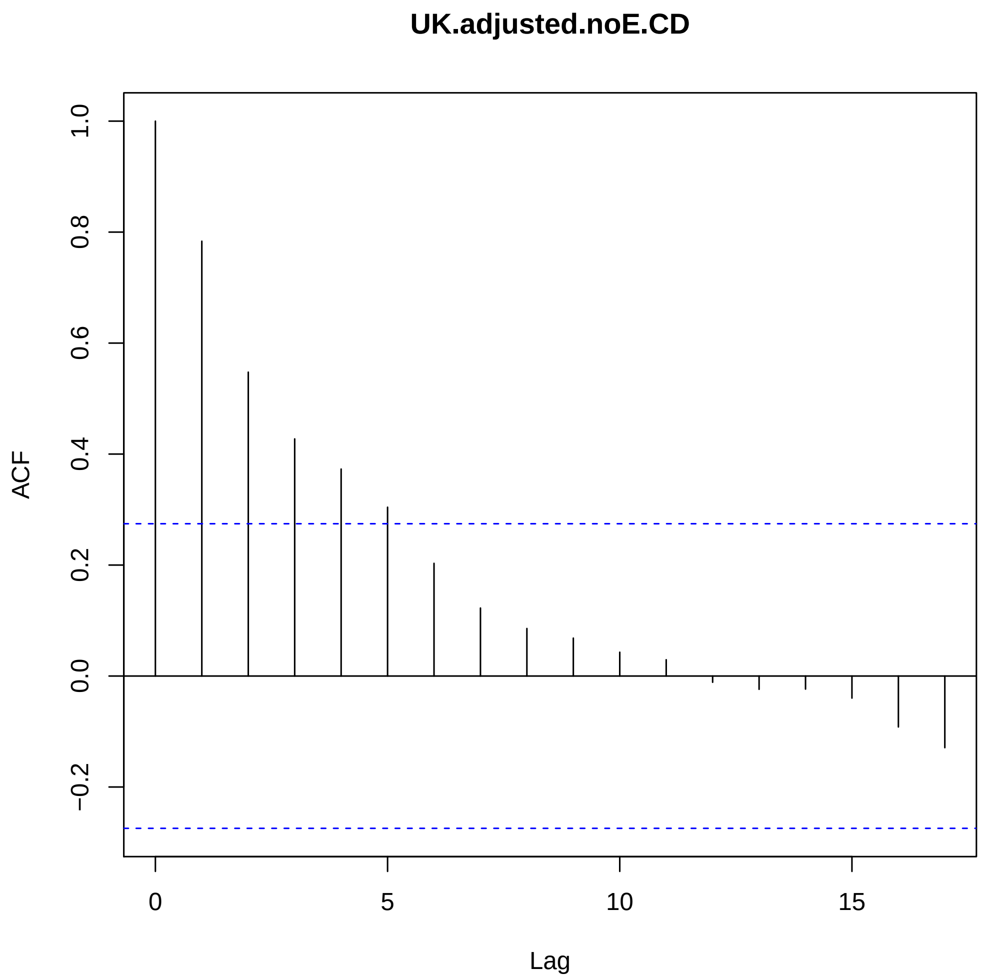


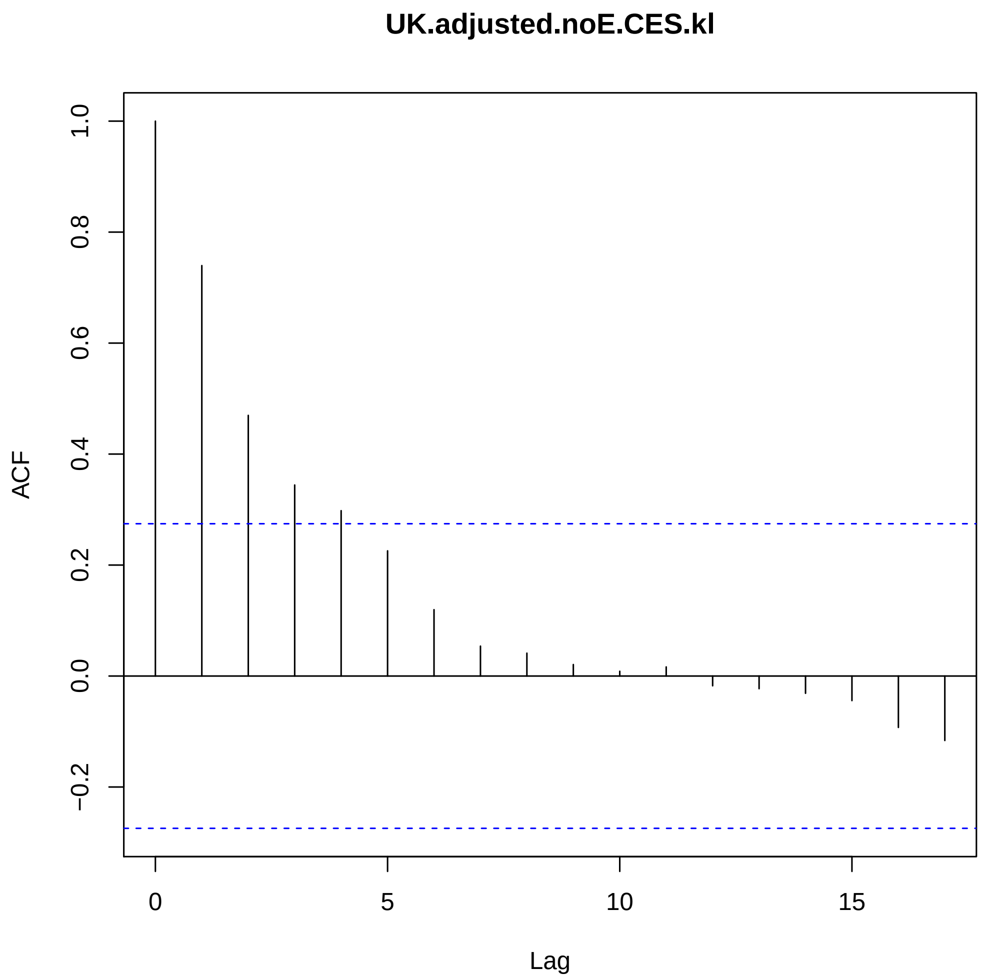
PT.adjusted.noE.CD
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag


PT.adjusted.noE.CES.kl
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag


UK.unadjusted.noE.CD
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag


UK.unadjusted.noE.CES.kl
ACF
Lag
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15


UK.adjusted.noE.CD
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag



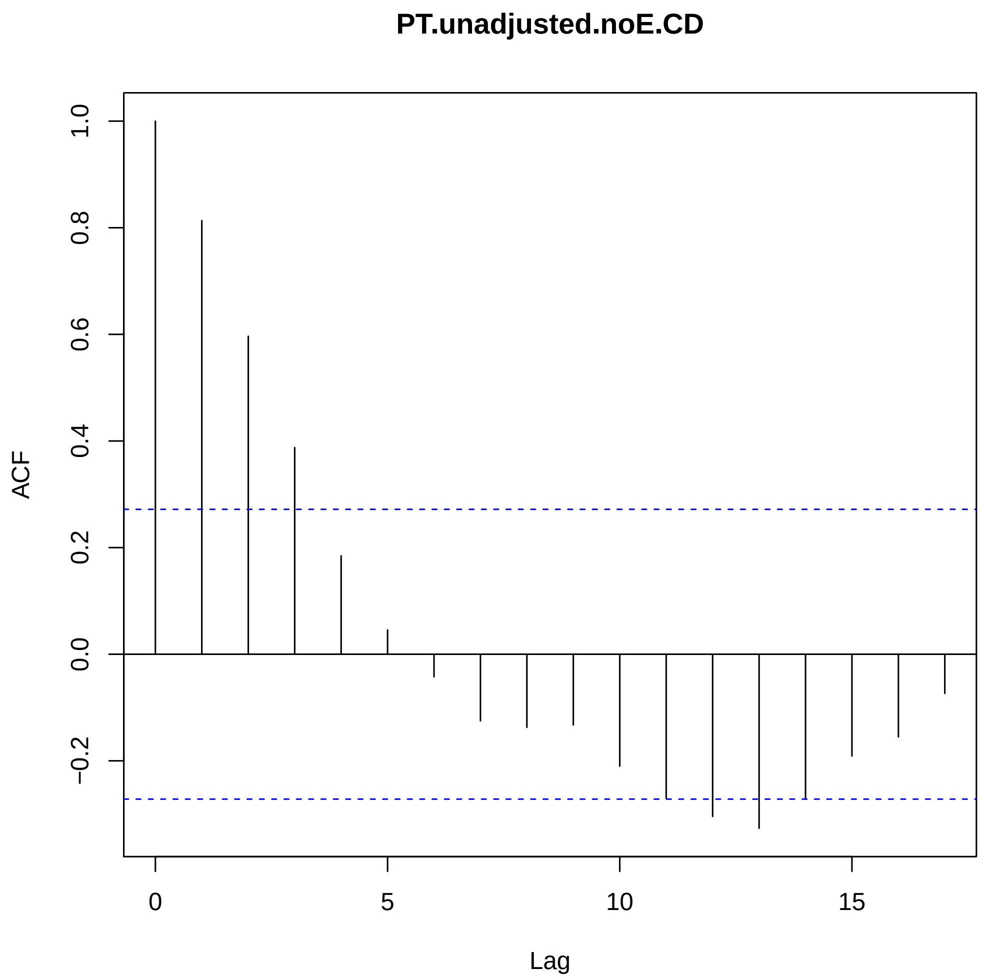

```
$PT.unadjusted.noE.CD

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.865  0.726  0.601  0.485  0.371  0.260  0.157  0.078  0.028
    10     11     12     13     14     15     16     17
-0.041 -0.110 -0.161 -0.174 -0.144 -0.150 -0.177 -0.190
```

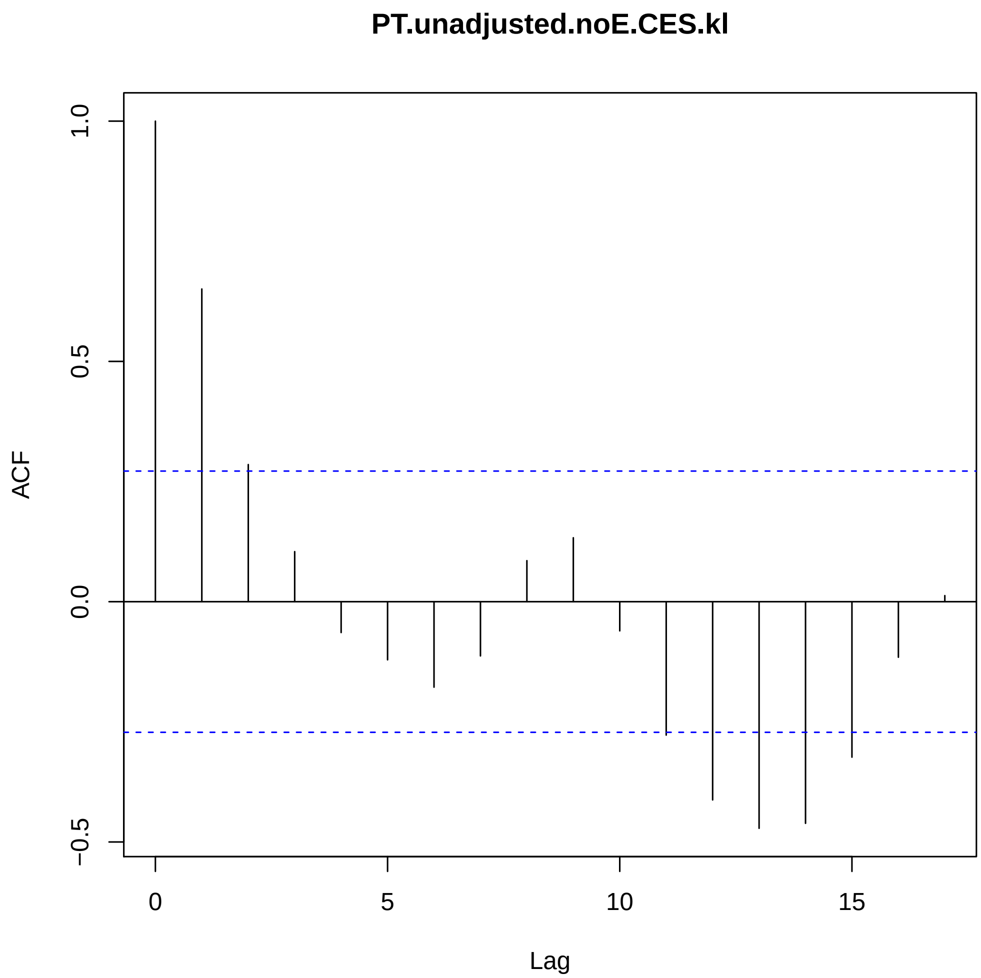
```
$PT.unadjusted.noE.CES.kl

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.865  0.727  0.605  0.492  0.379  0.267  0.164  0.086  0.036
    10     11     12     13     14     15     16     17
-0.034 -0.107 -0.162 -0.175 -0.144 -0.152 -0.179 -0.194

$PT.adjusted.noE.CD

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.852  0.696  0.566  0.445  0.326  0.219  0.116  0.039 -0.003
    10     11     12     13     14     15     16     17
-0.069 -0.139 -0.176 -0.186 -0.148 -0.154 -0.186 -0.200

$PT.adjusted.noE.CES.kl

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.842  0.676  0.543  0.422  0.299  0.186  0.083  0.008 -0.030
    10     11     12     13     14     15     16     17
-0.098 -0.174 -0.211 -0.213 -0.162 -0.163 -0.188 -0.198

$UK.unadjusted.noE.CD

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.711  0.401  0.255  0.211  0.158  0.053 -0.011 -0.009  0.016
    10     11     12     13     14     15     16     17
 0.018  0.017 -0.029 -0.014  0.029  0.047  0.002 -0.015

$UK.unadjusted.noE.CES.kl
```

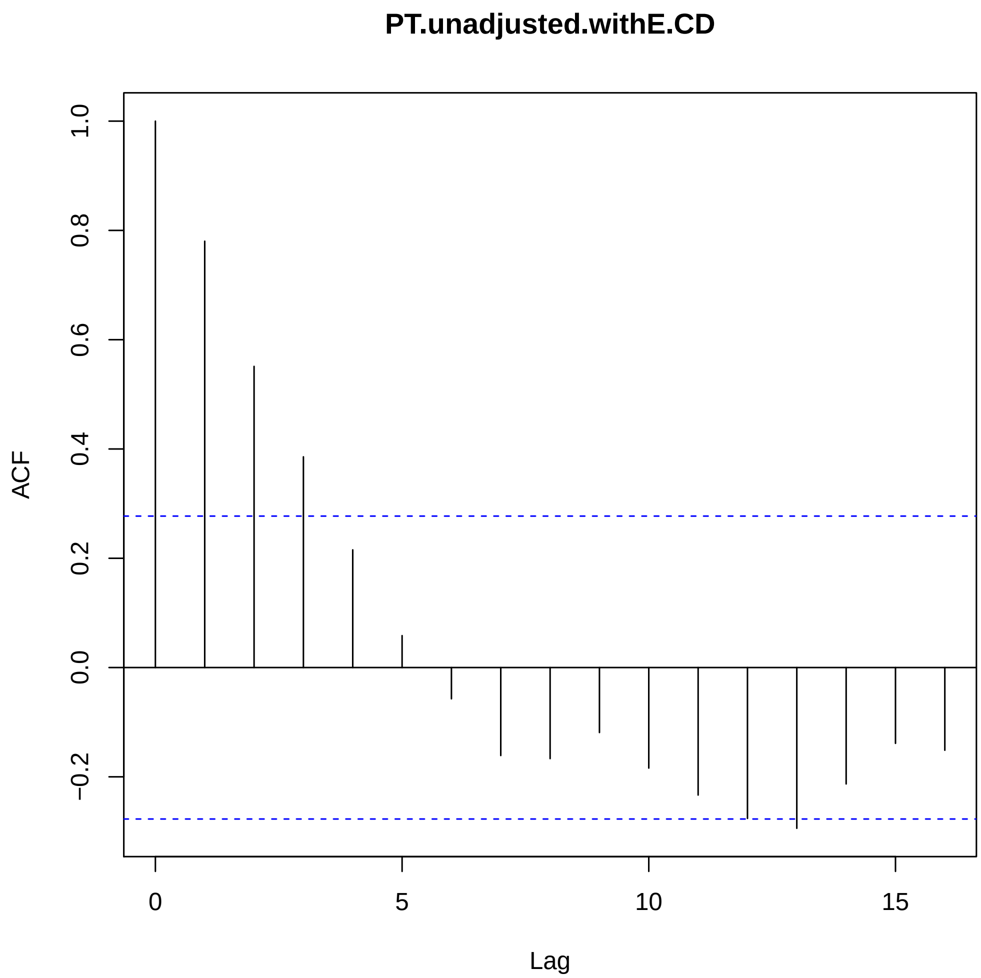
```
Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.634  0.259  0.125  0.119  0.063 -0.072 -0.121 -0.067 -0.032
    10     11     12     13     14     15     16     17
 0.000  0.024 -0.033 -0.006  0.029  0.049 -0.018 -0.030

$UK.adjusted.noE.CD

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.784  0.548  0.427  0.373  0.304  0.203  0.123  0.086  0.068
    10     11     12     13     14     15     16     17
 0.043  0.029 -0.011 -0.024 -0.024 -0.040 -0.092 -0.129

$UK.adjusted.noE.CES.kl

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.740  0.470  0.344  0.298  0.226  0.120  0.054  0.041  0.021
    10     11     12     13     14     15     16     17
 0.009  0.016 -0.018 -0.023 -0.031 -0.044 -0.093 -0.116
```

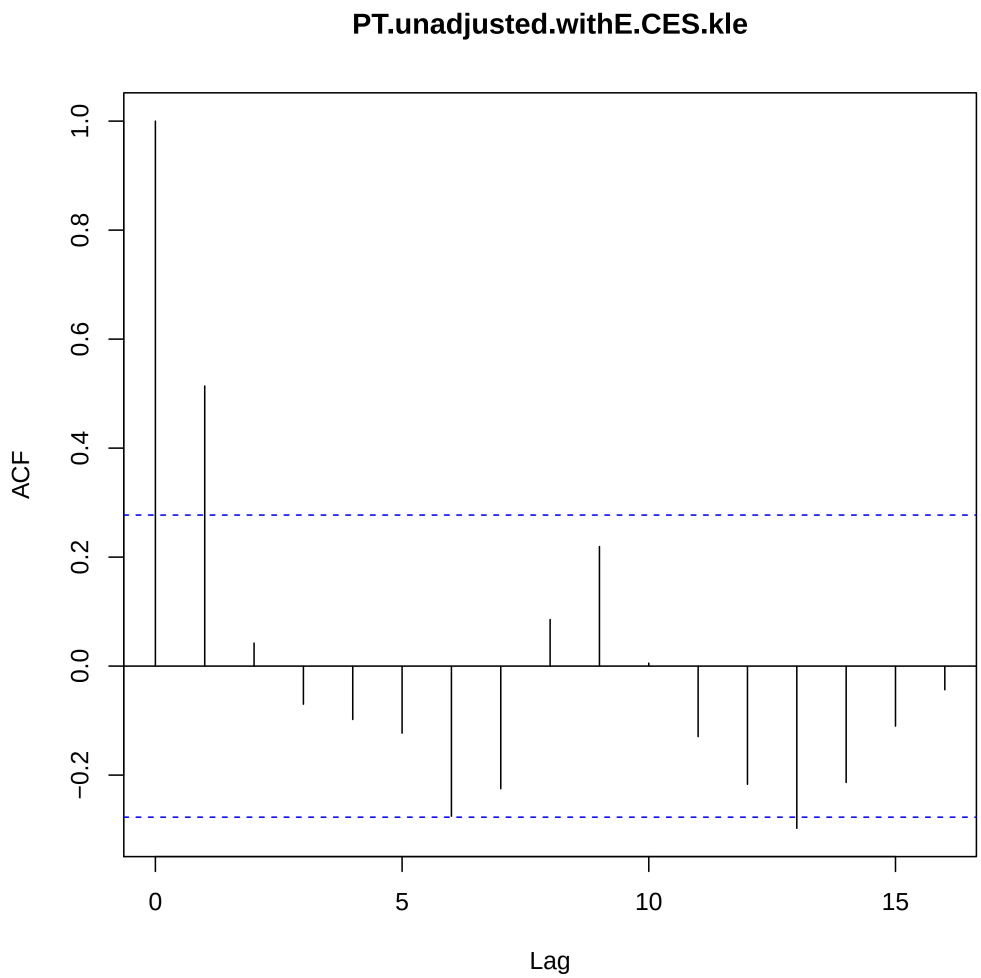
Appendix D.2. Correlograms for models reject the cost-share theorem

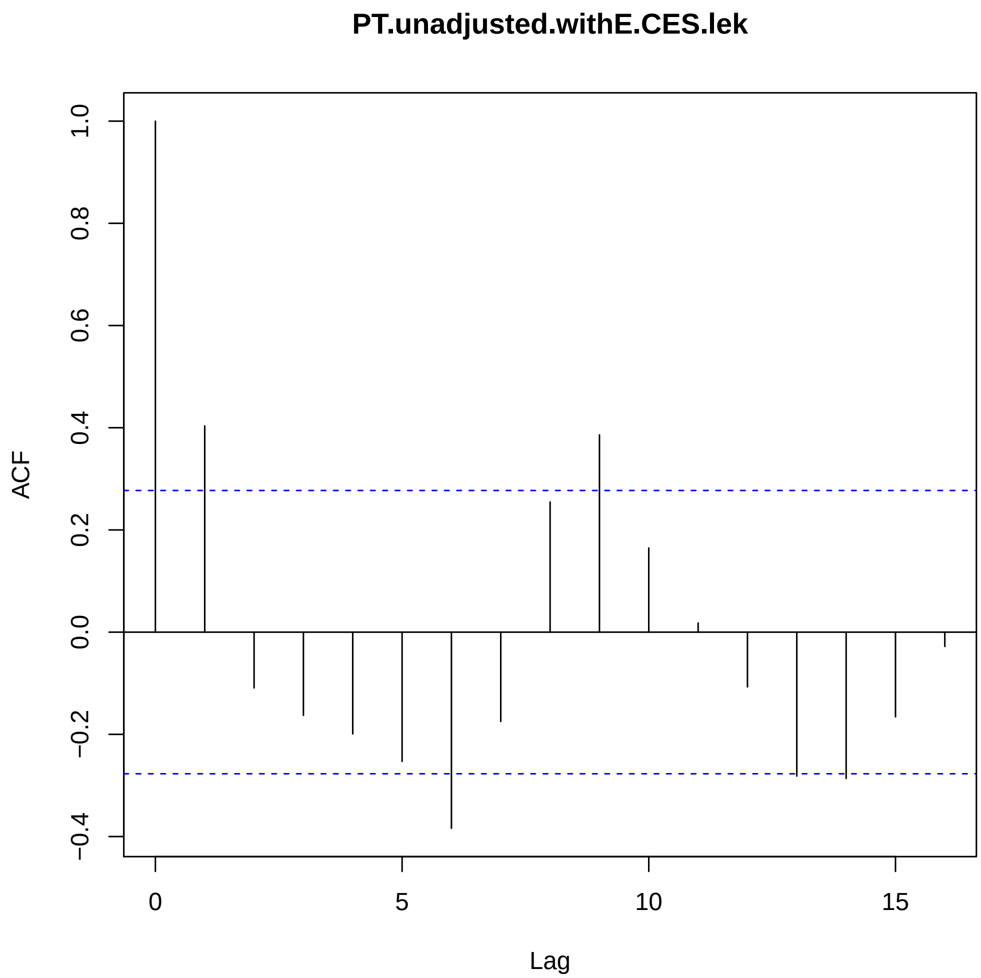
In this section we present correlograms for all models that do not adhere to the cost-share theorem.

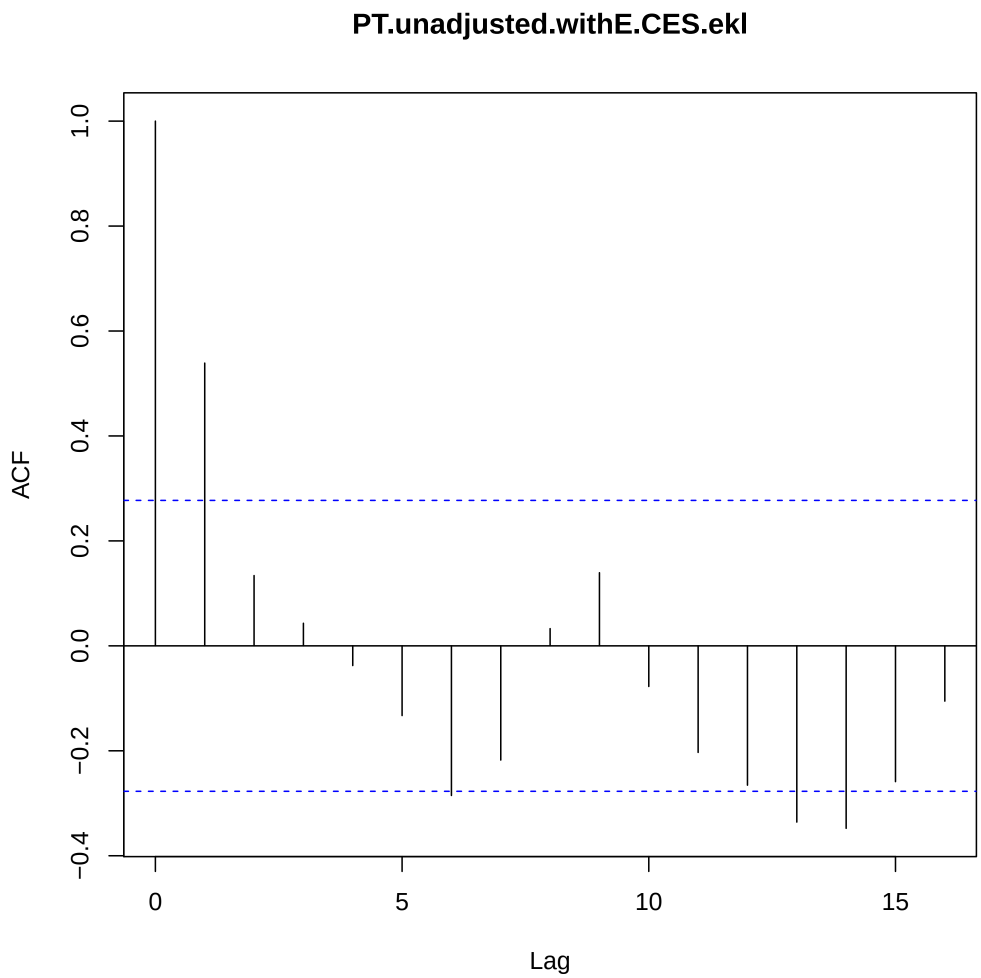

PT.unadjusted.noE.CD
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag

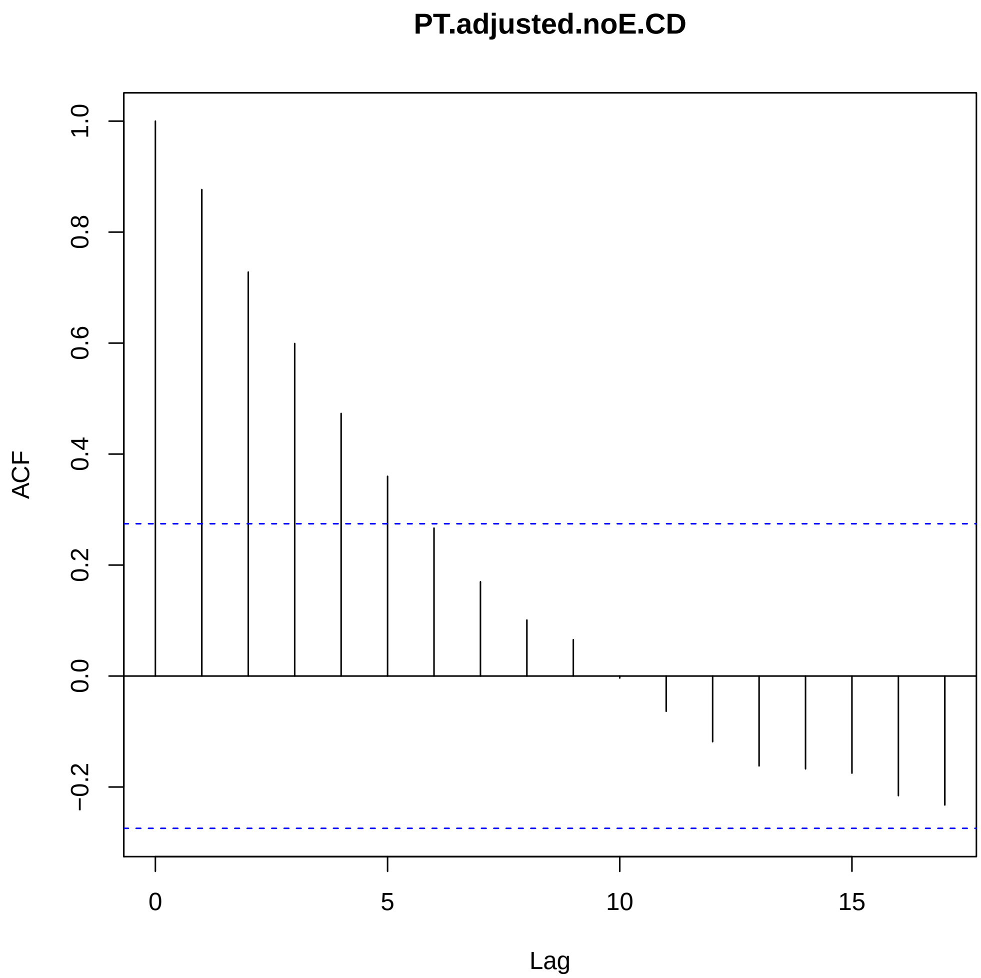

PT.unadjusted.noE.CES.kl
ACF
1.0
0.5
0.0
−0.5
0
5
10
15
Lag


PT.unadjusted.withE.CD
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag

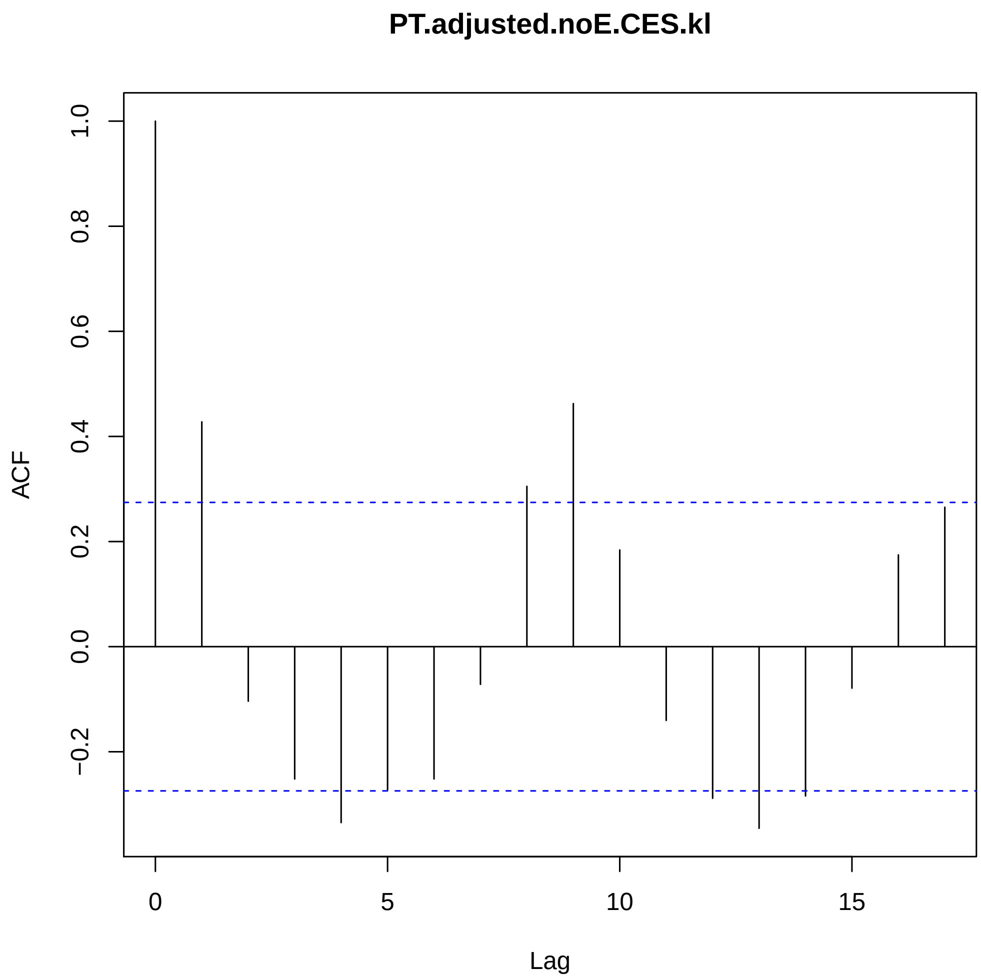

PT.unadjusted.withE.CES.kle
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag

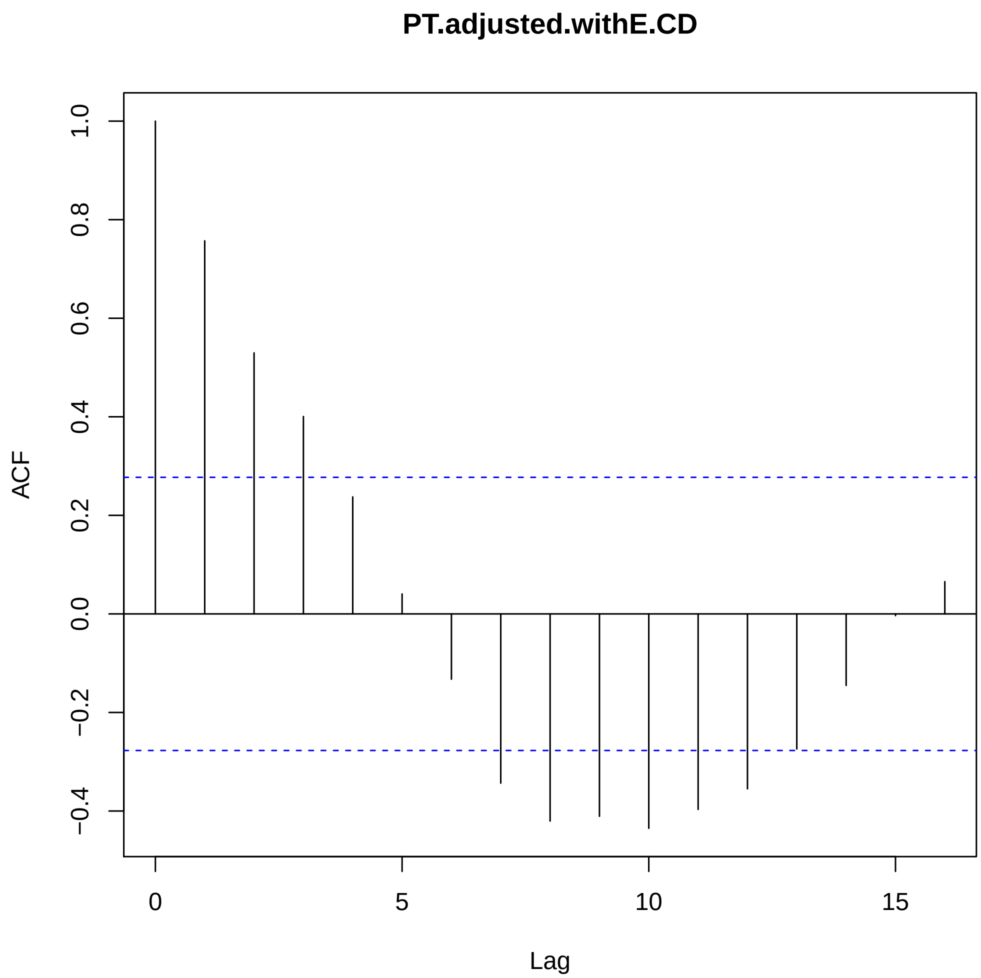

PT.unadjusted.withE.CES.lek
ACF
Lag
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0
5
10
15

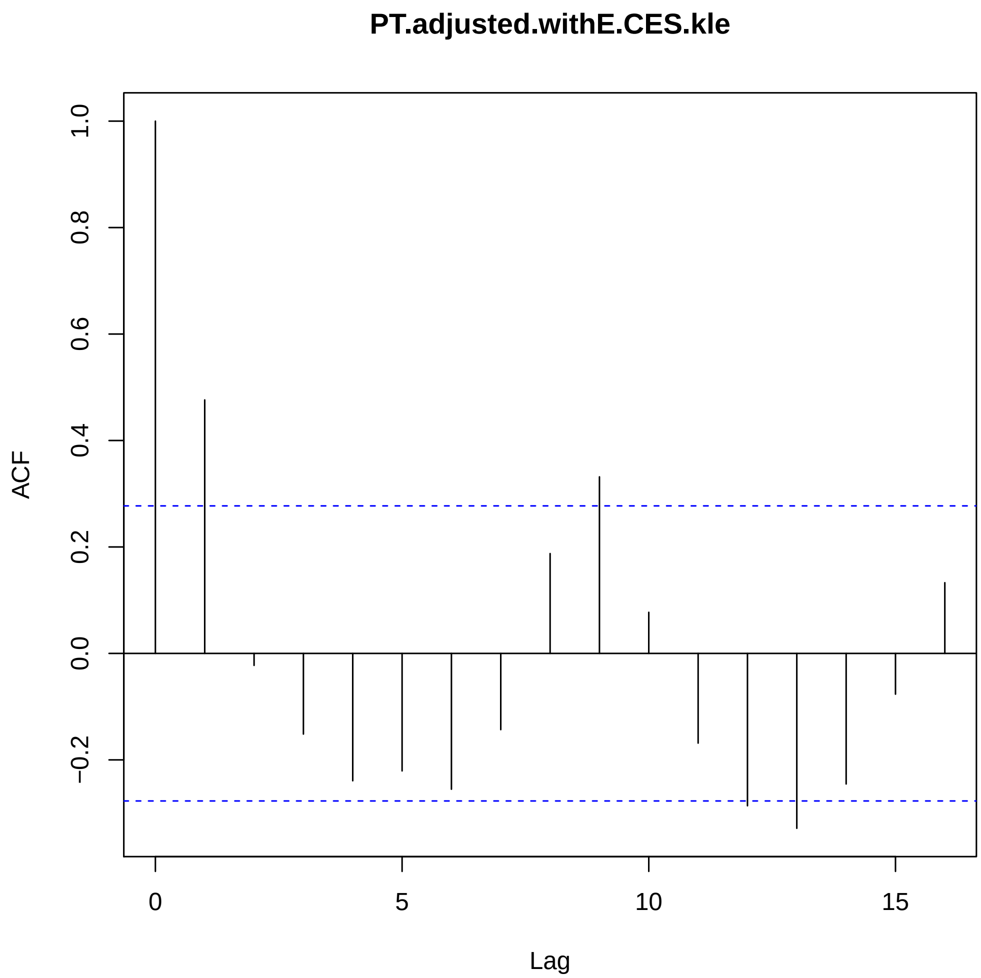

PT.unadjusted.withE.CES.ekl
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0
5
10
15
Lag



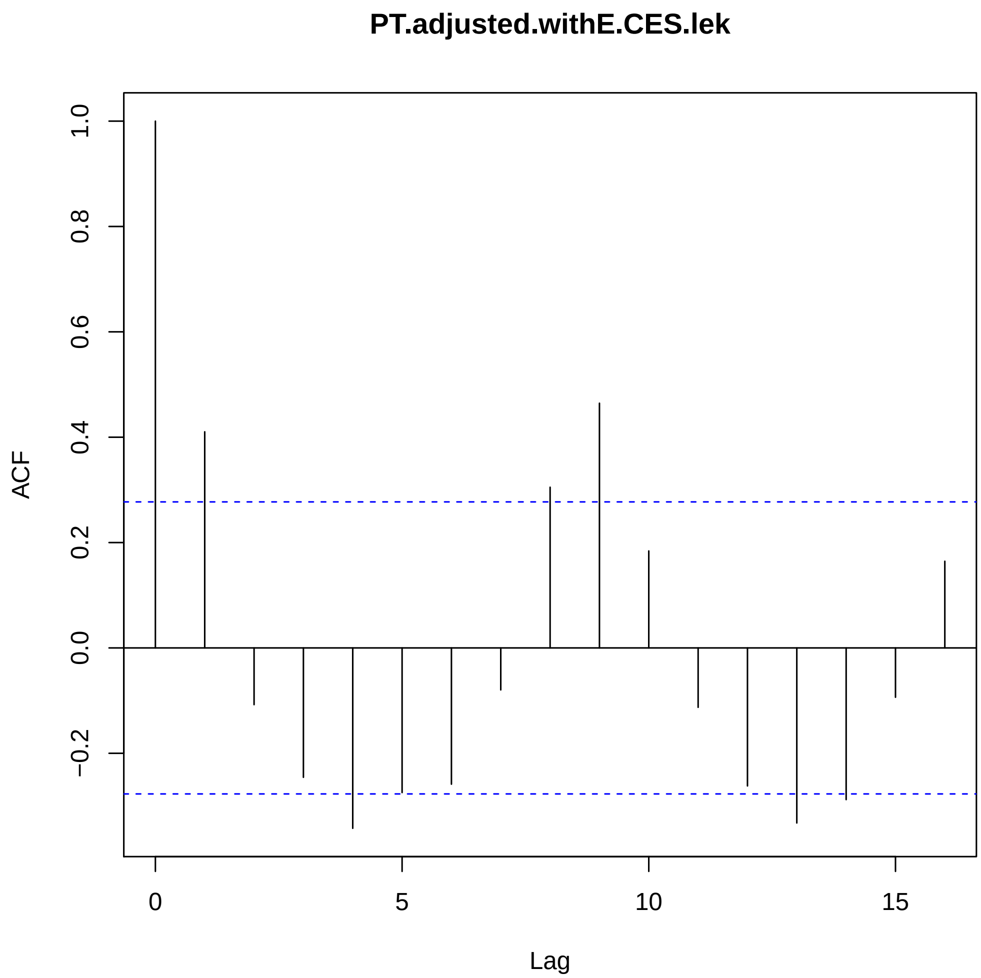
PT.adjusted.noE.CD
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag

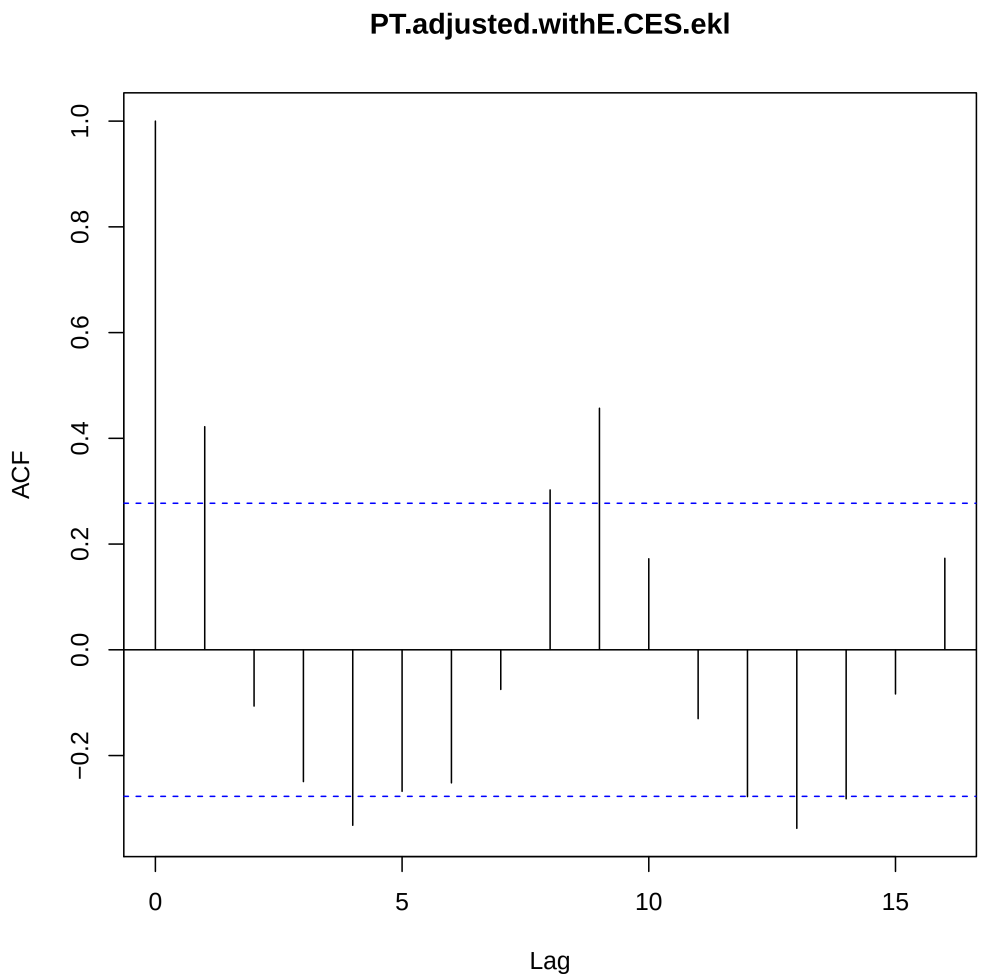

PT.adjusted.noE.CES.kl
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag

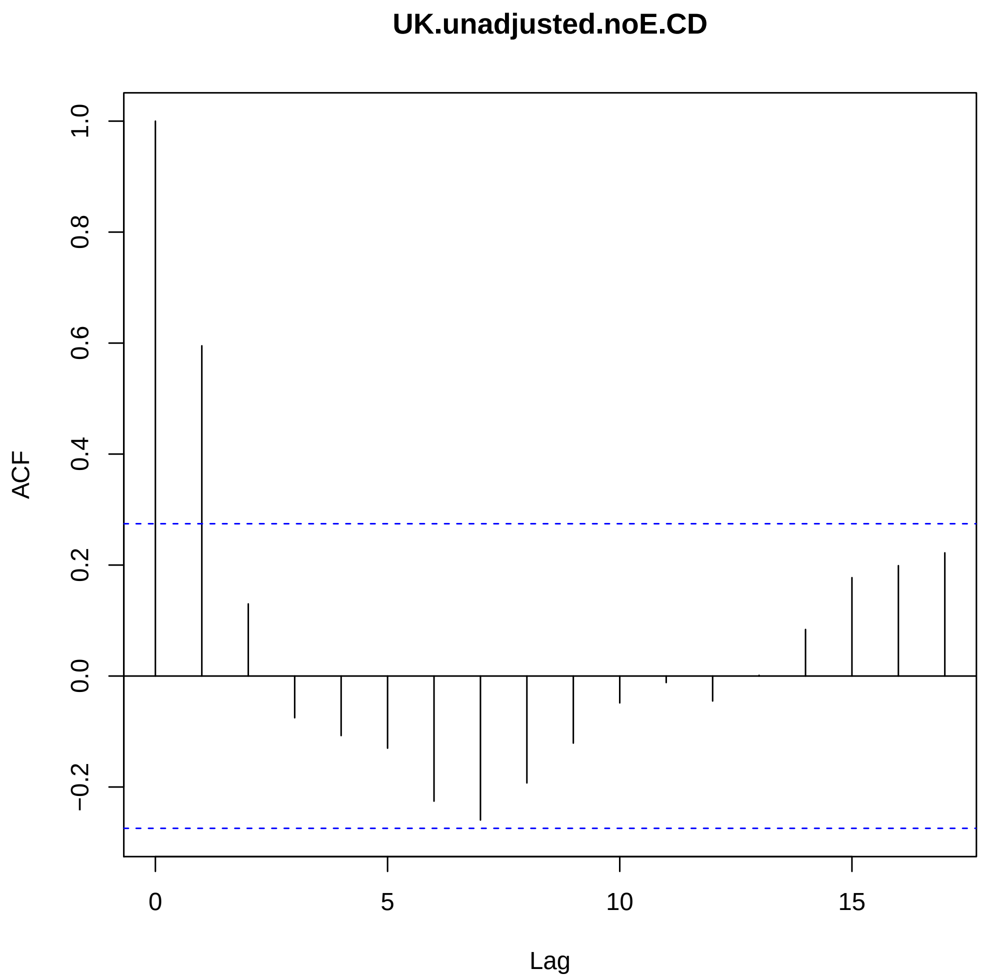

PT.adjusted.withE.CD
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0
5
10
15
Lag



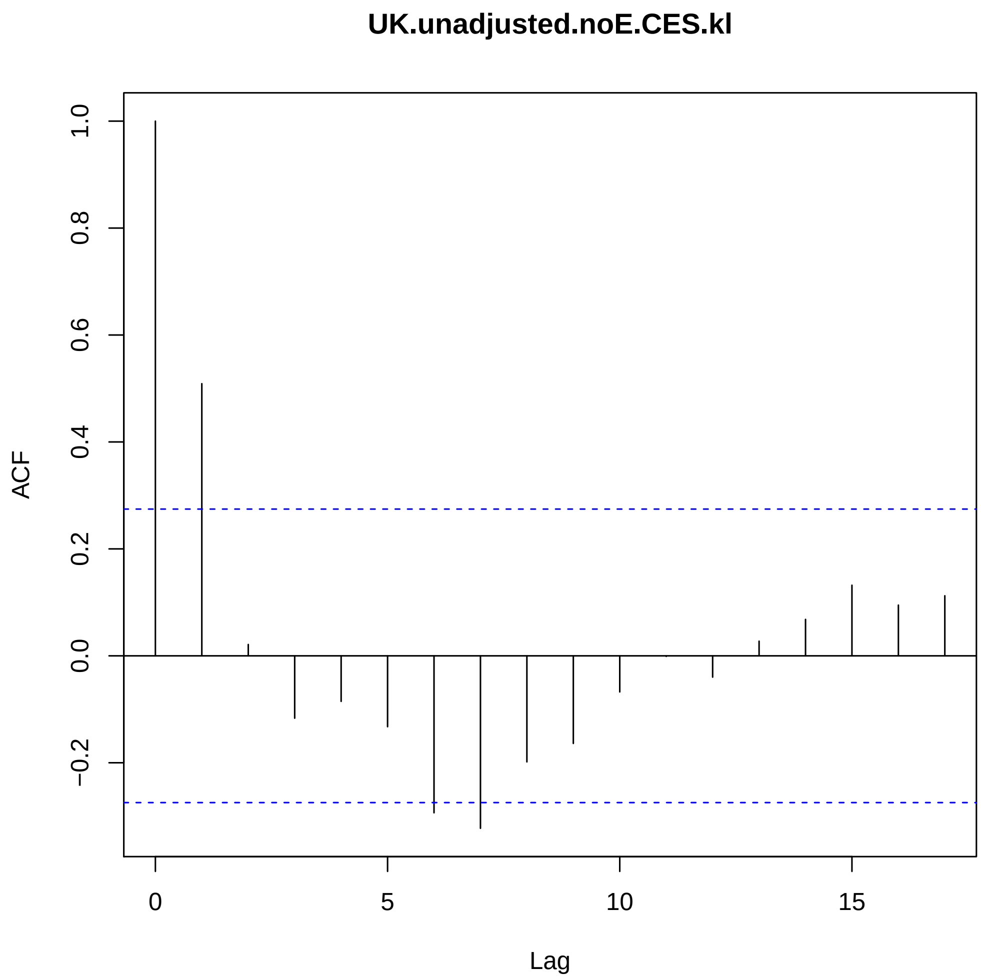
PT.adjusted.withE.CES.kle
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag

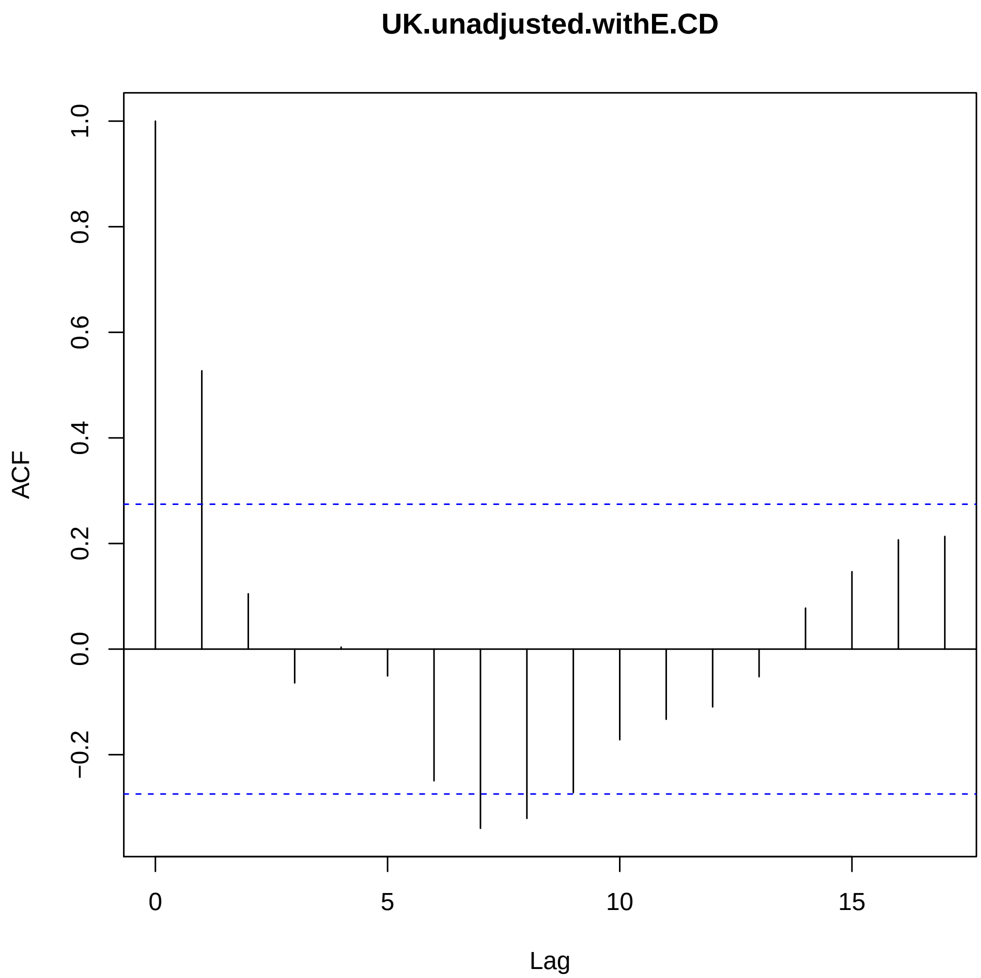

PT.adjusted.withE.CES.lek
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag

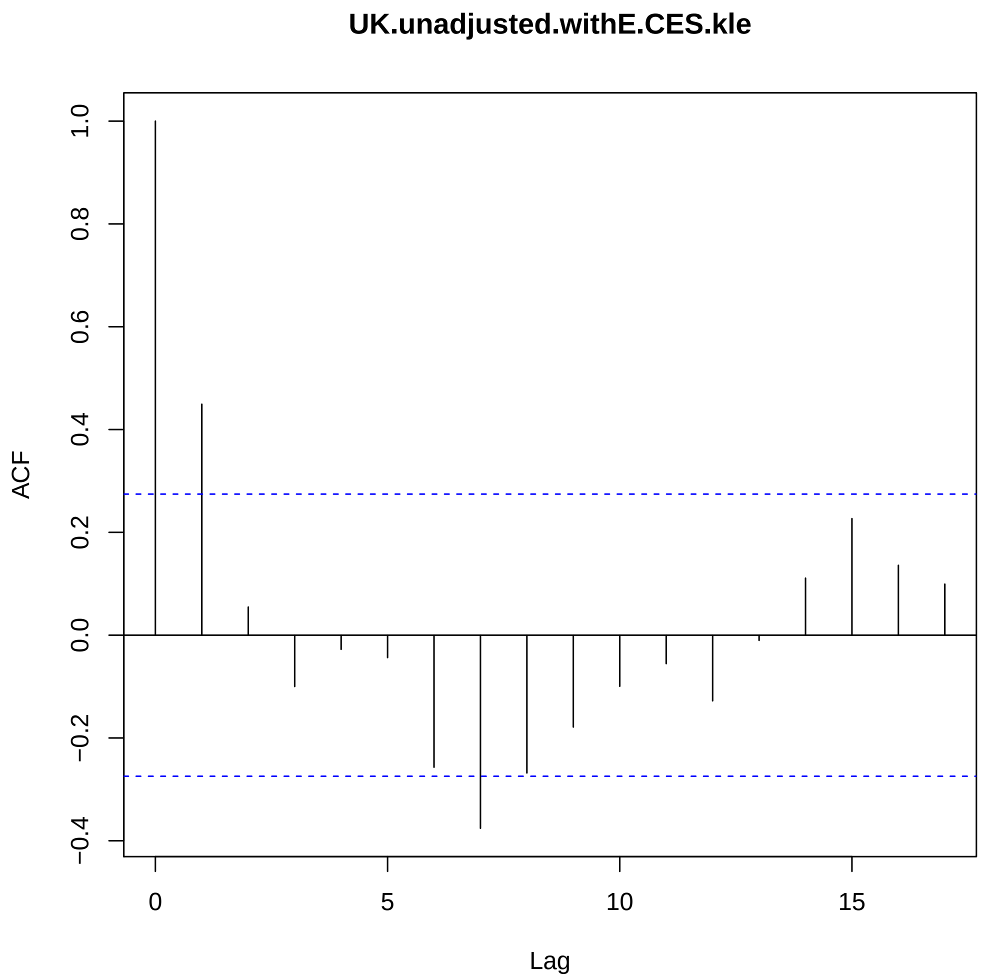

PT.adjusted.withE.CES.ekl
ACF
Lag
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15

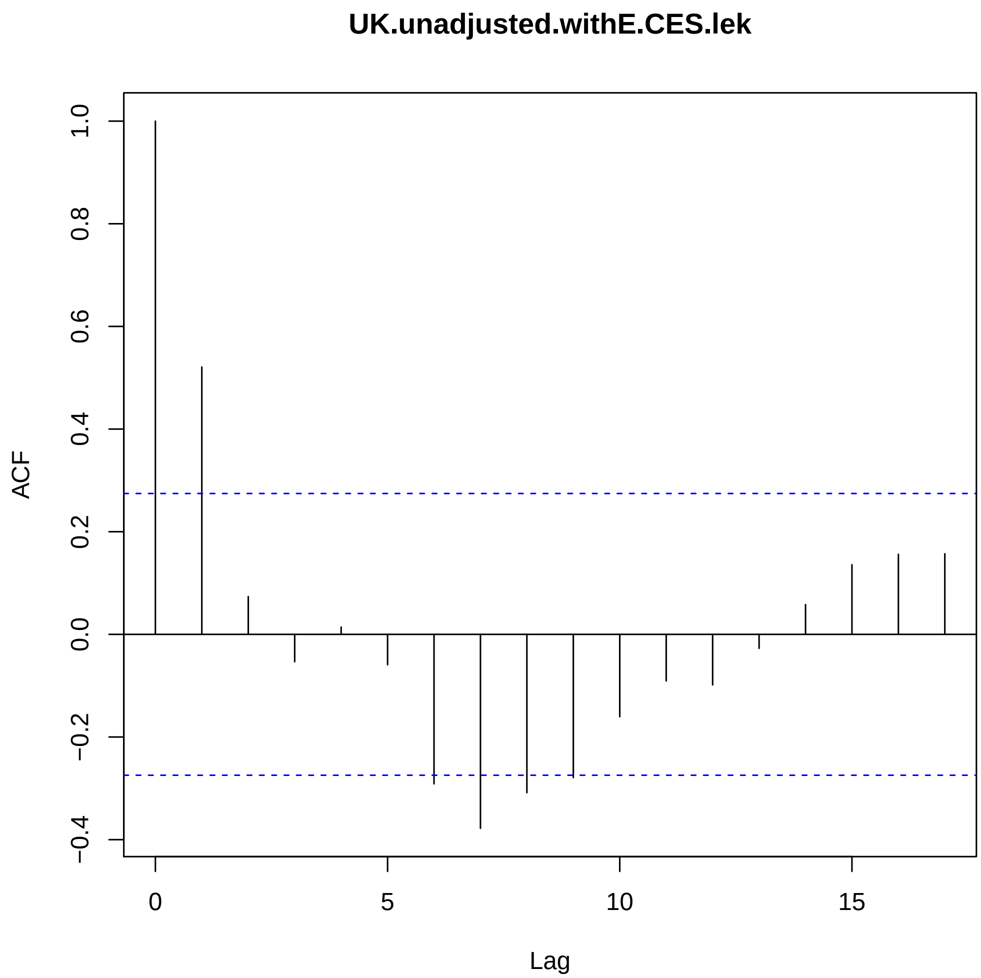


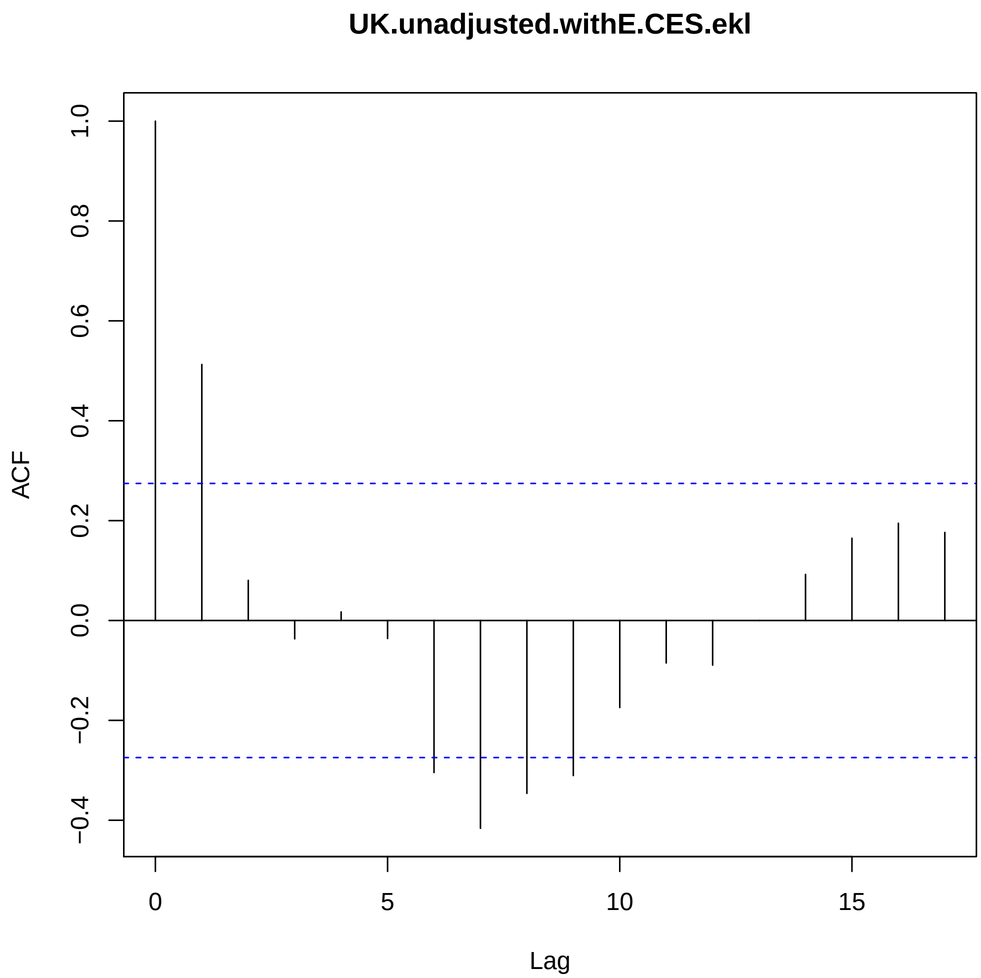
UK.unadjusted.noE.CD
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag

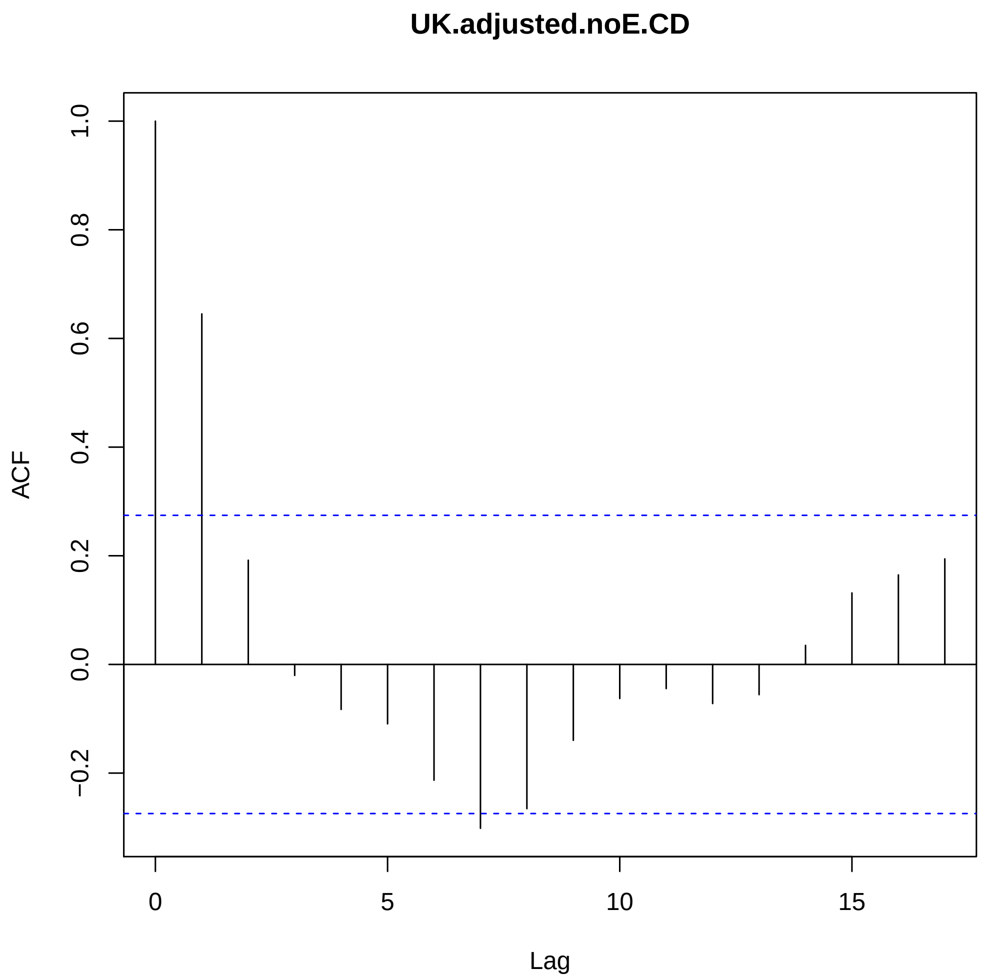

UK.unadjusted.noE.CES.kl
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag

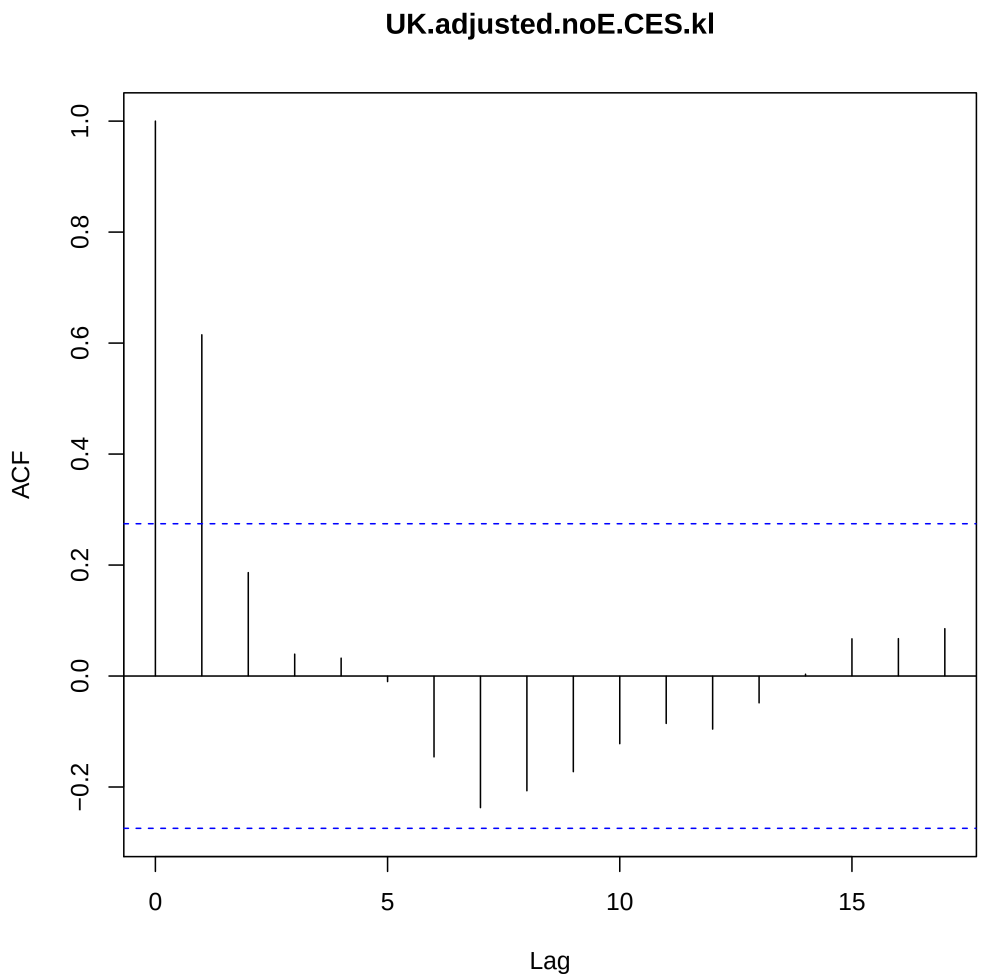

UK.unadjusted.withE.CD
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag

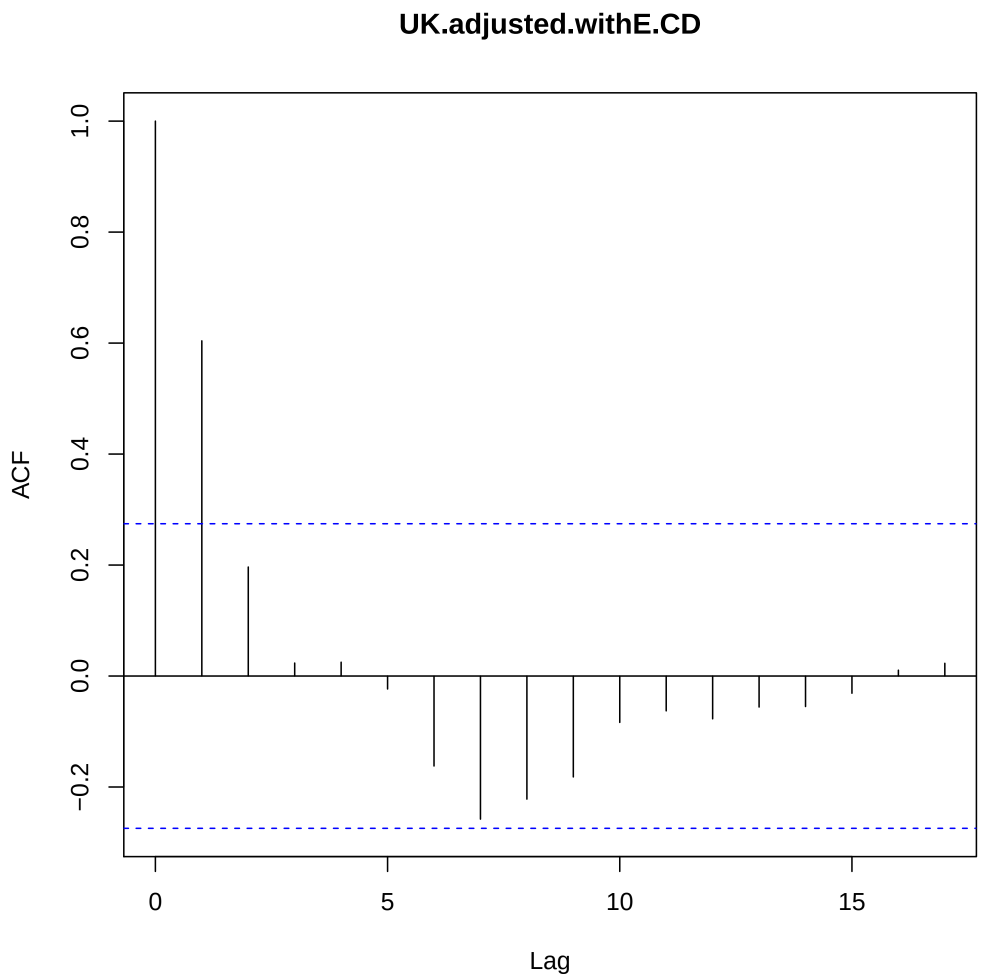

UK.unadjusted.withE.CES.kle
ACF
Lag
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0
5
10
15

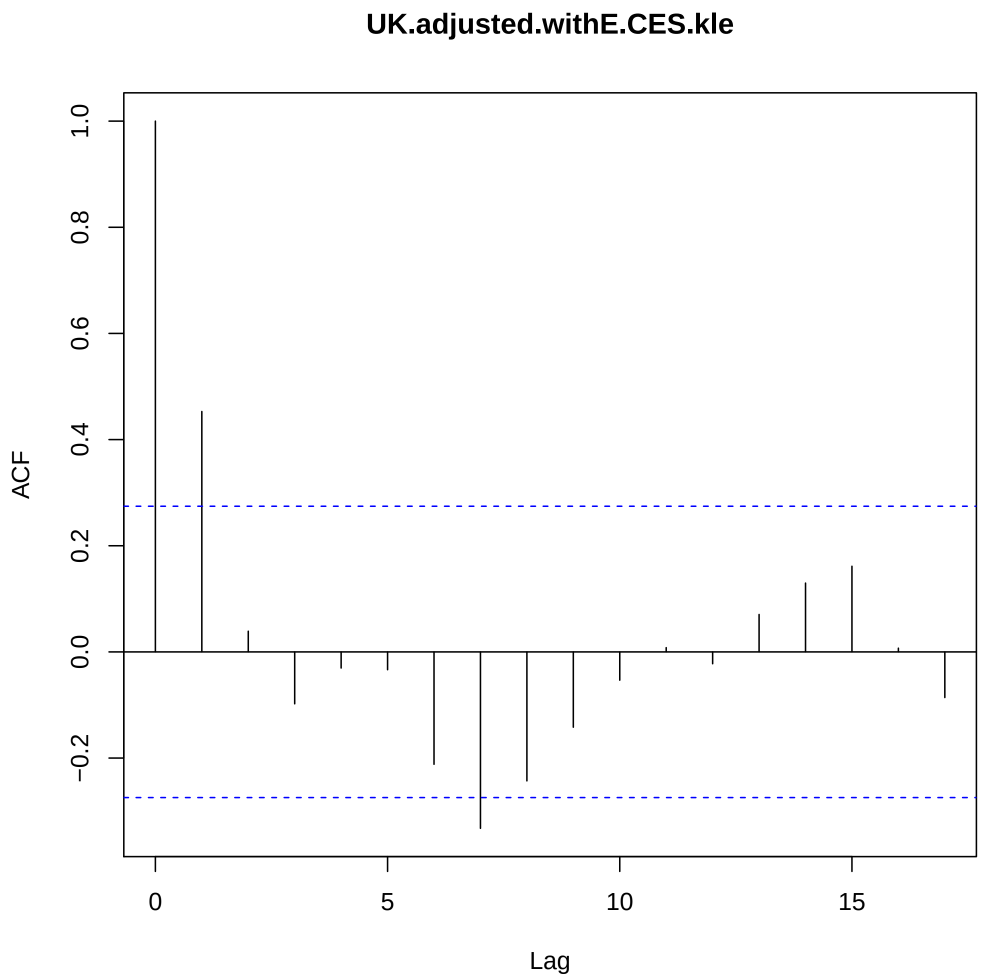

UK.unadjusted.withE.CES.lek
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0
5
10
15
Lag

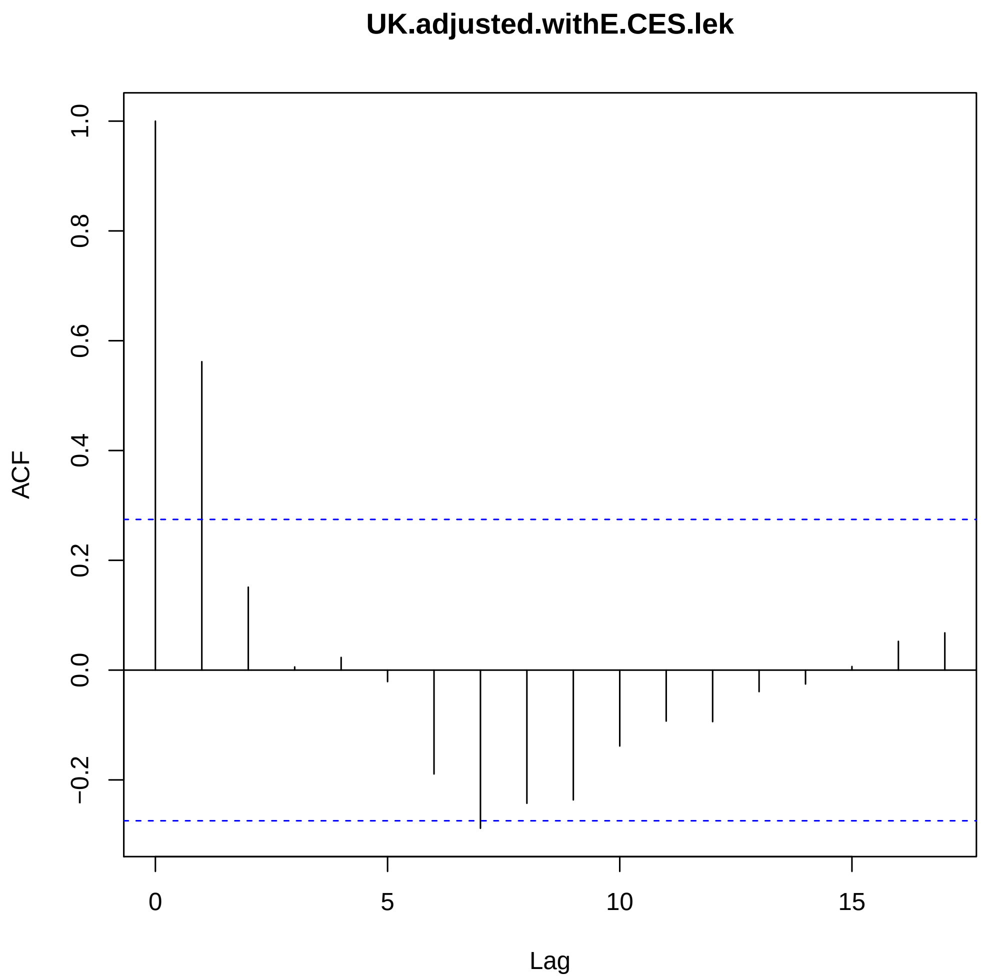

UK.unadjusted.withE.CES.ekl
ACF
Lag
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0
5
10
15


UK.adjusted.noE.CD
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag

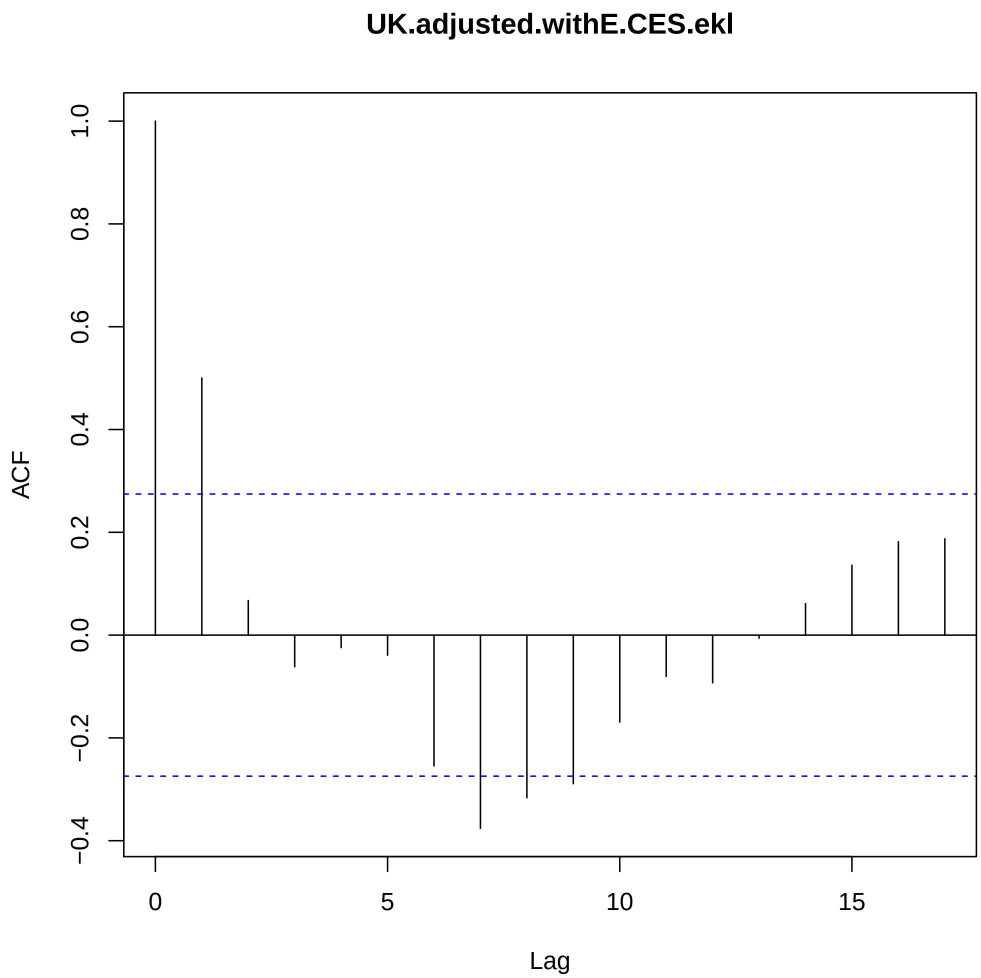

UK.adjusted.noE.CES.kl
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag


UK.adjusted.withE.CD
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag


UK.adjusted.withE.CES.kle
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag



UK.adjusted.withE.CES.lek
ACF
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
Lag




```
$PT.unadjusted.noE.CD

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.813  0.597  0.388  0.185  0.045 -0.042 -0.125 -0.137 -0.133
    10     11     12     13     14     15     16     17
-0.210 -0.270 -0.304 -0.327 -0.270 -0.191 -0.155 -0.073
```

```
$PT.unadjusted.noE.CES.kl

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9 
 1.000  0.651  0.285  0.104 -0.064 -0.121 -0.178 -0.113  0.085  0.133 
    10     11     12     13     14     15     16     17 
-0.061 -0.278 -0.413 -0.472 -0.461 -0.324 -0.116  0.013 

$PT.unadjusted.withE.CD

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9 
 1.000  0.780  0.551  0.386  0.215  0.058 -0.057 -0.161 -0.167 -0.119 
    10     11     12     13     14     15     16 
-0.184 -0.233 -0.276 -0.294 -0.213 -0.139 -0.151 

$PT.unadjusted.withE.CES.kle

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9 
 1.000  0.514  0.042 -0.070 -0.098 -0.123 -0.275 -0.225  0.085  0.219 
    10     11     12     13     14     15     16 
 0.005 -0.129 -0.217 -0.298 -0.213 -0.110 -0.043 

$PT.unadjusted.withE.CES.lek

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9 
 1.000  0.404 -0.109 -0.163 -0.199 -0.253 -0.384 -0.175  0.255  0.386 
    10     11     12     13     14     15     16 
 0.165  0.018 -0.107 -0.282 -0.286 -0.166 -0.028 

$PT.unadjusted.withE.CES.ekl
```

```
Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9 
 1.000  0.539  0.134  0.043 -0.038 -0.133 -0.285 -0.218  0.033  0.139 
    10     11     12     13     14     15     16 
-0.077 -0.203 -0.265 -0.336 -0.348 -0.259 -0.105 

$PT.adjusted.noE.CD

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9 
 1.000  0.877  0.728  0.599  0.473  0.360  0.267  0.170  0.101  0.066 
    10     11     12     13     14     15     16     17 
-0.004 -0.064 -0.118 -0.162 -0.167 -0.175 -0.216 -0.232 

$PT.adjusted.noE.CES.kl

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9 
 1.000  0.428 -0.104 -0.252 -0.335 -0.273 -0.252 -0.072  0.305  0.463 
    10     11     12     13     14     15     16     17 
 0.184 -0.140 -0.289 -0.346 -0.284 -0.079  0.175  0.265 

$PT.adjusted.withE.CD

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9 
 1.000  0.757  0.530  0.401  0.237  0.040 -0.133 -0.343 -0.420 -0.411 
    10     11     12     13     14     15     16 
-0.435 -0.397 -0.355 -0.274 -0.145 -0.004  0.065 

$PT.adjusted.withE.CES.kle

Autocorrelations of series 'resid(model)', by lag
```

```
    0      1      2      3      4      5      6      7      8      9
 1.000  0.476 -0.023 -0.152 -0.239 -0.221 -0.255 -0.143  0.188  0.332
    10     11     12     13     14     15     16
 0.077 -0.169 -0.286 -0.329 -0.245 -0.077  0.133

$PT.adjusted.withE.CES.lek

Autocorrelations of series 'resid(model)', by lag

    0      1      2      3      4      5      6      7      8      9
 1.000  0.410 -0.108 -0.246 -0.342 -0.275 -0.259 -0.080  0.305  0.465
    10     11     12     13     14     15     16
 0.184 -0.113 -0.262 -0.332 -0.288 -0.094  0.165

$PT.adjusted.withE.CES.ekl

Autocorrelations of series 'resid(model)', by lag

    0      1      2      3      4      5      6      7      8      9
 1.000  0.422 -0.106 -0.249 -0.332 -0.268 -0.252 -0.075  0.302  0.457
    10     11     12     13     14     15     16
 0.172 -0.130 -0.278 -0.338 -0.282 -0.084  0.173

$UK.unadjusted.noE.CD

Autocorrelations of series 'resid(model)', by lag

    0      1      2      3      4      5      6      7      8      9
 1.000  0.595  0.130 -0.075 -0.107 -0.130 -0.225 -0.260 -0.193 -0.121
    10     11     12     13     14     15     16     17
-0.048 -0.012 -0.045  0.001  0.084  0.177  0.199  0.222

$UK.unadjusted.noE.CES.kl

Autocorrelations of series 'resid(model)', by lag

    0      1      2      3      4      5      6      7      8      9
```

```
 1.000  0.509  0.021 -0.117 -0.085 -0.133 -0.294 -0.323 -0.198 -0.164
    10     11     12     13     14     15     16     17
-0.068 -0.001 -0.040  0.027  0.068  0.132  0.095  0.112

$UK.unadjusted.withE.CD

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.527  0.105 -0.064  0.004 -0.051 -0.249 -0.339 -0.321 -0.272
    10     11     12     13     14     15     16     17
-0.172 -0.133 -0.109 -0.052  0.078  0.147  0.207  0.213

$UK.unadjusted.withE.CES.kle

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.449  0.055 -0.100 -0.028 -0.044 -0.257 -0.376 -0.268 -0.179
    10     11     12     13     14     15     16     17
-0.099 -0.055 -0.128 -0.010  0.111  0.227  0.136  0.099

$UK.unadjusted.withE.CES.lek

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.521  0.074 -0.053  0.014 -0.059 -0.291 -0.378 -0.309 -0.279
    10     11     12     13     14     15     16     17
-0.161 -0.091 -0.099 -0.027  0.058  0.136  0.156  0.157

$UK.unadjusted.withE.CES.ekl

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.513  0.080 -0.037  0.017 -0.036 -0.304 -0.416 -0.346 -0.311
```

```
    10     11     12     13     14     15     16     17 
-0.174 -0.085 -0.089  0.000  0.092  0.165  0.195  0.176 

$UK.adjusted.noE.CD

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9 
 1.000  0.645  0.192 -0.020 -0.083 -0.109 -0.213 -0.302 -0.266 -0.140 
    10     11     12     13     14     15     16     17 
-0.063 -0.045 -0.072 -0.056  0.035  0.132  0.165  0.194 

$UK.adjusted.noE.CES.kl

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9 
 1.000  0.615  0.186  0.039  0.032 -0.010 -0.146 -0.237 -0.207 -0.172 
    10     11     12     13     14     15     16     17 
-0.122 -0.085 -0.096 -0.048  0.003  0.067  0.067  0.085 

$UK.adjusted.withE.CD

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9 
 1.000  0.604  0.196  0.023  0.025 -0.023 -0.162 -0.258 -0.222 -0.182 
    10     11     12     13     14     15     16     17 
-0.084 -0.063 -0.077 -0.056 -0.055 -0.031  0.011  0.023 

$UK.adjusted.withE.CES.kle

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9 
 1.000  0.453  0.039 -0.098 -0.030 -0.034 -0.212 -0.332 -0.243 -0.142 
    10     11     12     13     14     15     16     17 
```

```
-0.053  0.008 -0.022  0.071  0.130  0.161  0.007 -0.086

$UK.adjusted.withE.CES.lek

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.562  0.151  0.006  0.023 -0.021 -0.189 -0.288 -0.243 -0.236
    10     11     12     13     14     15     16     17
-0.138 -0.093 -0.094 -0.039 -0.025  0.007  0.052  0.068

$UK.adjusted.withE.CES.ekl

Autocorrelations of series 'resid(model)', by lag

     0      1      2      3      4      5      6      7      8      9
 1.000  0.500  0.067 -0.062 -0.024 -0.039 -0.255 -0.376 -0.316 -0.289
    10     11     12     13     14     15     16     17
-0.169 -0.080 -0.093 -0.006  0.061  0.136  0.182  0.188
```

Appendix E. Statistical details on all models

In this section, we present statistical details of all models.

At the moment, this is a simple example of two models. I hope to develop a table later, after I figure out some things. Both models are CES models with energy fitted to Quality-adjusted data using the $(kl)(e)$ nesting. In Table E.15, Model 1 is for Portugal, and Model 2 is for the UK.

Table E.15: Example output from `texreg`.

	Model 1	Model 2
gamma	1.027582^{***}	0.980315^{***}
	(0.011759)	(0.012946)
lambda	0.004508^{***}	0.007103^{***}
	(0.001169)	(0.001737)
delta_1	1.000000^{***}	0.502875^{***}
	(0.000000)	(0.036155)
delta	0.902835^{***}	0.848488
	(0.161689)	(0.661918)
rho_1	54.581844	0.634472^{***}
	(119.786669)	(0.138784)
rho	-1.000000	68.650078
	(6.852668)	(115.484907)
R^2	0.997138	0.997823
Num. obs.	50	51

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$