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Application of Bootstrap Resampling to REXS Data

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Abstract

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Keywords: economic growth, energy, Cobb-Douglas, CES, LINEX

1. Introduction

2. Coordinates of Analysis

This section describes the coordinates of analysis and briefly reviews literature related to each.

2.1. Mathematical Forms of the Energy-augmented Production Function

In this paper, we assess three prominent energy-augmented production functions that appear in the literature: Cobb-Douglas (CD), Constant Elasticity of Substitution (CES), and LINear EXponential (LINEX). The following subsections describe each.

2.1.1. Cobb-Douglas (CD) Production Function

This production function can be expressed as

$$y = \theta \mathrm{e}^{\lambda(t-t_0)} k^{\alpha} l^{\beta} \; , \tag{1}$$

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where $y \equiv Y/Y_0$, θ is a scale parameter, e is the base of the natural logarithm, λ is represents the pace of technological progress, t (time) is measured in years, $k \equiv K/K_0$, $l \equiv L/L_0$, Y is represented by GDP, K is expressed in currency units, L is expressed in workers or work-hours/year, and 0 subscripts indicate values at an initial year.[1]

The capital-labor Cobb-Douglas production function shown in Equation 1 can be augmented to include an energy term:

$$y = \theta \mathrm{e}^{\lambda(t-t_0)} k^{\alpha} l^{\beta} e^{\gamma} \ , \tag{2}$$

where $e \equiv E/E_0$, and E is in units of energy per time, typically TJ/year. The energy-augmented Cobb-Douglas production function is often assumed to have constant returns to scale for the three factors of production: $\alpha + \beta + \gamma = 1$.

2.1.2. Constant Elasticity of Substitution Production Function (CES)

Other energy economists use an energy-augmented Constant Elasticity of Substitution (CES) production function to describe economic growth. The `R` (R Core Team, 2012) package `micEconCES` (Henningsen and Henningsen, 2011) estimates CES production functions of the following forms (among others):

$$y = \gamma\, A\, \left[\delta_1 k^{-\rho_1} + (1-\delta_1) l^{-\rho_1}\right]^{-1/\rho_1} ; A = \mathrm{e}^{\lambda(t-t_0)} \tag{3}$$

$$y = \gamma\, A\, \left\{\delta \left[\delta_1 k^{-\rho_1} + (1-\delta_1) l^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta) e^{-\rho}\right\}^{-1/\rho} ; A = \mathrm{e}^{\lambda(t-t_0)} \tag{4}$$

Equation 3 is a CES production function with capital stock (k) and labor (l) factors of production. Equation 4 augments Equation 3 with energy using a $(kl)(e)$ nesting structure, as is typical in the literature. Equation 3 is a degenerate form of Equation 4 where $\delta \to 1$.

γ is a fitting parameter that represents productivity. The fitting parameters ρ_1 and ρ indicate elasticities of substitution (σ_1 and σ). The elasticity of substitution between capital (k) and labor (l) is given by $\sigma_1 = \frac{1}{1+\rho_1}$, and the elasticity of substitution between (kl) and (e) is given by $\sigma = \frac{1}{1+\rho}$. As $\rho_1 \to 0$, $\sigma_1 \to 1$, and the embedded CES production function for k and l degenerates to the Cobb-Douglas production function. Similarly, as $\rho \to 0$,

[1]Dimensionless, indexed quantities are represented by lower-case symbols (y, k, l, e, q, x, and u), and dimensional quantities are represented by upper-case symbols (Y, K, L, E, Q, X, and U). Model parameters are represented by Greek letters (α, β, λ, θ).

$\sigma \to 1$, and the CES production function for (kl) and (e) degenerates to the Cobb-Douglas production function. As $\sigma \to \infty$ ($\rho \to -1$), (kl) and (e) are perfect substitutes. As $\sigma \to 0$ ($\rho \to \infty$), (kl) and (e) are perfect complements: no substitution is possible. Similarly, as $\sigma_1 \to 0$ ($\rho_1 \to \infty$), k and l are perfect complements. δ_1 describes the relative importance of capital (k) and labor (l), and δ describes the importance of (kl) relative to (e).

Constraints on the fitting parameters include $\delta_1 \in [0, 1]$, $\delta \in [0, 1]$, $\rho_1 \in [-1, 0) \cup (0, \infty)$, and $\rho \in [-1, 0) \cup (0, \infty)$.

Two other nestings of the factors of production (k, l, and e) are possible with the CES model.

$$y = \gamma\, A\, \left\{\delta\left[\delta_1 l^{-\rho_1} + (1-\delta_1)e^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta)k^{-\rho}\right\}^{-1/\rho} ; A = \mathrm{e}^{\lambda(t-t_0)} \tag{5}$$

$$y = \gamma\, A\, \left\{\delta\left[\delta_1 e^{-\rho_1} + (1-\delta_1)k^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta)l^{-\rho}\right\}^{-1/\rho} ; A = \mathrm{e}^{\lambda(t-t_0)} \tag{6}$$

Note that the ρ (σ) and δ parameters have different meanings that depend upon the nesting of the factors of production.

2.1.3. LINEX Production Function

A third production function, the LINear EXponential (LINEX) function

$$y = \theta\, \mathrm{e}^{a_0\left[2\left(1-\frac{1}{\rho_k}\right)+c_t\left(\rho_l-1\right)\right]}\, e\ , \tag{7}$$

was derived from thermodynamic considerations (Kümmel, 1980, 1982; Kümmel et al., 1985; Kümmel, 1989; Kümmel et al., 2002, 2010). In contrast to both the Cobb-Douglas model (Equations 1 and 2) and the CES model (Equations 3–6), the LINEX model (Equation 7) does not include a generic, time-dependent augmentation term (A). The LINEX models assumes energy (e) is the only factor of production.

The LINEX model does, however, include terms that represent efficiencies among capital, labor, and energy, in the form of the ratios ρ_k and ρ_l which are defined by[2]

$$\rho_k \equiv \frac{k}{(1/2)(l+e)} \tag{8}$$

[2]The notation for the LINEX model breaks from the conventions used in the Cobb-Douglas and CES models: a_0 and c_t are parameters and ρ_k and ρ_l are not.

and

$$\rho_l \equiv \frac{l}{e} \tag{9}$$

and represent (8) capital stock increase relative to the average of labor and energy and (9) labor increase relative to energy. The effect of energy (e) on production is enhanced if either

- capital stock (k) has increased more than the average of labor (l) and energy (e), i.e. if $\rho_k > 1$, or
- labor (l) has increased more than energy (e), i.e. if $\rho_l > 1$.

An economy that increases capital stock (k) and labor (l) at the same rate as energy will have capital and labor efficiency ratios (ρ_k and ρ_l, respectively) of unity. In that case, economic output (y) increases at the same rate as energy consumption (e). According to the LINEX model, an economy that increases capital stock (k) without a commensurate increase in labor (l) and energy (e) will experience an increase in output (y) in excess of its increase of energy consumption (e), because $\mathrm{e}^{a_0\left[2\left(1-\frac{1}{\rho_k}\right)+c_t\left(\rho_l-1\right)\right]} > 1$. Similarly, an economy benefits by increasing labor (l) without a commensurate increase in energy (e). Thus, ρ_k is an efficiency of using additional labor and energy to make additional capital stock available to the economy, and ρ_l is an efficiency of using additional energy to make additional labor available to the economy. Increases in ρ_k and ρ_l provide upward pressure on economic output (y), as the only factor of production (e) is used more efficiently.

The parameter a_0 represents the overall importance of efficiencies ρ_k and ρ_l, and the parameter c_t represents the relative importance of the labor efficiency term (ρ_l) compared to the capital stock efficiency term (ρ_k). Both a_0 and c_t are assumed constant with respect to time.

2.2. Economy Types

The REXS data set include US, UK, JP, and AT. (See Table 1.)

3. Sources of Data

REXS.

Table 1: REXS economies and years of evaluation.

	US	USA	
	UK	United Kingdom	1900–2000
	JP	Japan	
	AT	Austria	1950–2000

3.1. Historical Data

Figure ?? presents historical data, including indexed values for output (y), capital stock (k), labor (l), and energy (q) for 1900–2000 (US, UK, JP) and 1950–2000 (AT).

4. Parameter Estimation

4.1. Point Estimates

Each of the economic growth models discussed in Section 2.1 has parameters which must be estimated (fitted) using data for each economy (see Sections 2.2 and 3). We obtained parameter estimates by applying the method of least-squares to log-transformed data using the `R` (R Core Team, 2012) functions `lm` (for Cobb-Douglas and LINEX models) and `cesEst` (Henningsen and Henningsen, 2011) (for CES models).

Although the details vary from model to model, each of our three models has the general form

$$y = f(t, k, l, e; \theta) + \text{error} ,$$

where θ is a vector of parameters for the function f. Parameter estimates are chosen to minimize

$$sse = \sum_i (y_i - f(t, k, l, e; \theta))^2 \tag{10}$$

within constraints. For the optimal parameter values $\hat{\theta}$, we define the fitted response by

$$\hat{y}_i = f(t_i, k_i, l_i, e_i; \hat{\theta})$$

and the residuals (r_i) as the difference between the observed response and the fitted response

$$r_i = y_i - \hat{y}_i = y_i - f(t, k, l, e; \hat{\theta}) \tag{11}$$

The details for each of the models are discussed in Sections 4.3–4.5 below.

4.2. Resampling Methods

Bootstrapping is a statistical resampling technique for assigning measures of accuracy and precision to sample estimates by measuring properties of an estimator when sampling from a resampling distribution. Resampling distributions can be formed in a number of ways in accordance with the type of data, experimental design, and modeling assumptions involved. In each case, many new resamples are created, each of which is a randomized version of the original sample data to which the desired analysis method can be applied. By investigating, for example, the variability of a parameter estimate from one resample to another, one can learn about the precision of the estimation method.

In the context of linear models (regression), resamples are generally created by residual resampling. In our case, we formed resamples by adding to the fitted response ($\hat{y}$) the product of a residual from the original model fit and random sign (-1 or 1, each with probability 0.5). For the case of Cobb-Douglas and LINEX models, this residual resampling occurs on the log-transformed data.

Intuitively, this method assumes that the residuals are indicative of the variability (from many potential sources) inherent in the data such that it would be unsurprising if the residual from any particular year had been observed in a different year. Thus, a resampled response $\tilde{Y}'$ can be computed as

$$\tilde{Y}_i' = \hat{y}_i \pm r_j$$

where

$$r_i = y_i - \hat{y}_i$$

both the sign ($\pm$) and the index of the residual (j, typically different from i) are chosen at random (with replacement). We repeated the resampling process 1000 times for each combination of growth model and country, both with and without energy as appropriate.

The coefficients from the fit to a resampled time series (the "resample coefficients") will be different from the coefficients obtained from the fit to historical data (the "base coefficients"). When these resample coefficients are highly variable, it is an indication that the data do not determine the parameter estimates very precisely. Lack of precision can stem from a number of factors, most obviously a poor model fit, low model sensitivity to one or more parameters, or correlations between parameter estimates.

It is important to note that large residuals can arise from either (a) poor quality historical time series data, or (b) a mathematical model that does not describe the underlying phenomena well. It is also important to note that (c) even when the residuals are small and the model produces fitted values that track the observed data closely, it may yet be difficult to estimate some or all of the model parameters precisely. The resampling method employed herein reflects all three of these potential sources of uncertainty in parameter estimates.

4.3. Cobb-Douglas Models

The Cobb-Douglas model without energy is given by Equation 1. Equation 1 was reparameterized as

$$y = \theta \mathrm{e}^{\lambda(t-t_0)} k^{\alpha} l^{1-\alpha} \tag{12}$$

to ensure $\alpha + \beta = 1$ for constant returns to scale. θ, λ, and α were estimated by the `R` (R Core Team, 2012) function `lm` in log-transform space. If the estimated value for α was found outside the interval $[0, 1]$, we set α to its boundary value and re-estimated λ. The value of β was found with $\beta = 1 - \alpha$.

To estimate the parameters θ, λ, α, β, and γ in the energy-augmented Cobb-Douglas model, Equation 2 was reparameterized as

$$y = \theta \mathrm{e}^{\lambda(t-t_0)} k^{\alpha} l^{\beta} e^{1-\alpha-\beta} \tag{13}$$

ensuring that $\alpha + \beta + \gamma = 1$, thereby providing constant returns to scale. The `R` (R Core Team, 2012) function `lm` was used to estimate values of λ, θ, α, and β in log-transform space. If the fitted value for α or β fell outside the interval $[0, 1]$, we fit along all boundaries ($\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\alpha = 0$, $\beta = 0$, and $\gamma = 0$) and chose the boundary fit with minimum sse (Equation 10) as the winner. γ was recovered with $\gamma = 1 - \alpha - \beta$.

Figure **??** shows ternary plots for α, β, and γ parameter values for Cobb-Douglas (without and with energy) resample models. Each reample is shown as a gray dot on the graph. The fit to historical data is shown as a gray crosshairs.

Figure **??** compares predictions from the Cobb-Douglas models (without and with energy) to historical data. Historical data are shown as a black line. The fit to historical data is shown as a white line. The gray band encompasses 95% of the fits to resampled data.

4.4. CES Models

The CES model without and with energy is given by Equations 3–6. The `R` (R Core Team, 2012) package `micEconCES` (Henningsen and Henningsen, 2011) was used to estimate parameters λ, γ, δ, δ_1, ρ, and ρ_1. The `cesEst` function of `micEconCES` provides several algorithm options for parameter estimation. The default algorithm (Levenberg-Marquardt) does not respect parameter constraints (see Section 2.1.2) and, in our testing, nearly always violated them, often returning negative values for elasticity of substitution (σ) parameters. Thus, we used the two fitting algorithms available in `cesEst` that respect coefficient constraints: `PORT` and `L-BFGS-B`.

Our CES parameter estimation algorithm starts with an eleven-value grid search in ρ and ρ_1 (9, 2, 1, 0.43, 0.25, 0.1, -0.1, -0.5, -0.75, -0.9, -0.99), which corresponds to σ and σ_1 values of 0.1, 0.33, 0.5, 0.7, 0.8, 0.9, 1.11, 2, 4, 10, and 100, respectively. During the grid search, values of ρ and ρ_1 are fixed, and values of λ, γ, δ, and δ_1 are estimated by gradient search with the `PORT` and `L-BFGS-B` algorithms. In all, 121 gradient searches in λ, γ, δ, and δ_1 at grid points representing all combinations of ρ and ρ_1 are attempted. During the grid search portion of our algorithm, starting values for the free parameters are $\lambda = 0.015/\text{year}$, $\delta = 0.5$, $\delta_1 = 0.5$, and γ is set to a value such that the mean of the residuals is zero by the `cesEst` function.

Next, a gradient search (using both `PORT` and `L-BFGS-B`) is attempted wherein all fitting parameters (λ, γ, δ, δ_1, ρ, and ρ_1) are allowed to float. The start values for fitting parameters are taken from the grid search fit that provided the lowest sse (Equation 10).

If resampled data are being fitted, a prior fit to historical data is available. In the final step of our algorithm, a gradient search (using both `PORT` and `L-BFGS-B`) uses coefficients from the best fit to historical data as its starting point. In this final gradient search, all model parameters (λ, γ, δ, δ_1, ρ, and ρ_1) are considered free parameters.

The fit with lowest sse of all above trials is deemed the winner, and its fitting parameters are used as the model.

Henningsen and Henningsen (2011), in their detailed analysis of Kemfert (1998), found that

> …the Levenberg-Marquardt and the `PORT` algorithms are—at least in this study—most likely to find the coefficients that give the best fit to the model, where the `PORT` algorithm can be used to restrict the estimates to the economically meaningful region.

In our testing, we found that to be mostly true. `PORT` nearly always provided lower *sse* than `L-BFGS-B`, despite the fact that `L-BFGS-B` often reports convergence and `PORT` does not for the same data (i.e., for the same y, k, l, and e data).

Coefficients for each resampled CES model for all countries and all nests are shown in Figures 1–3. The white crosshair on each graph shows the location of the fit to historical data. In Figures 2 and 3, the bottom of the vertical axis represents $\sigma = 0$. The top of the vertical axis represents $\sigma = \infty$.

Note that as δ or δ_1 approaches 0 (1) in Equations 3–6, the corresponding value of σ or σ_1 becomes irrelevant, and resample points along the left (right) edge of Figures 2 and 3 are equivalent. That is, all points along the left (right) edge of Figures 2 and 3 have the same *sse*. As a consequence, the `cesEst` fitting algorithm has no way to discriminate among points along the left (right) edge of Figures 2 and 3. Thus, resample point estimates where δ or δ_1 approaches 0 (1) tend to cluster at the locations of the initial grid search in σ or σ_1 (0.1, 0.33, 0.5, 0.7, 0.8, 0.9, 1.11, 2, 4, 10, and 100).

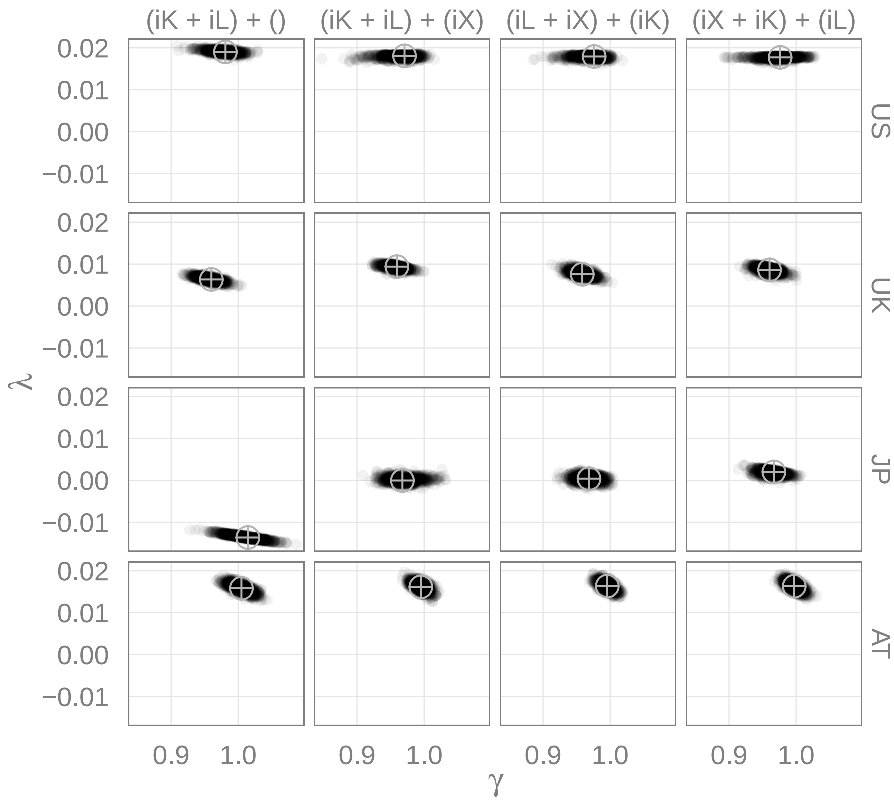

Figure 1: λ and γ values for CES resample models with exergy (x).

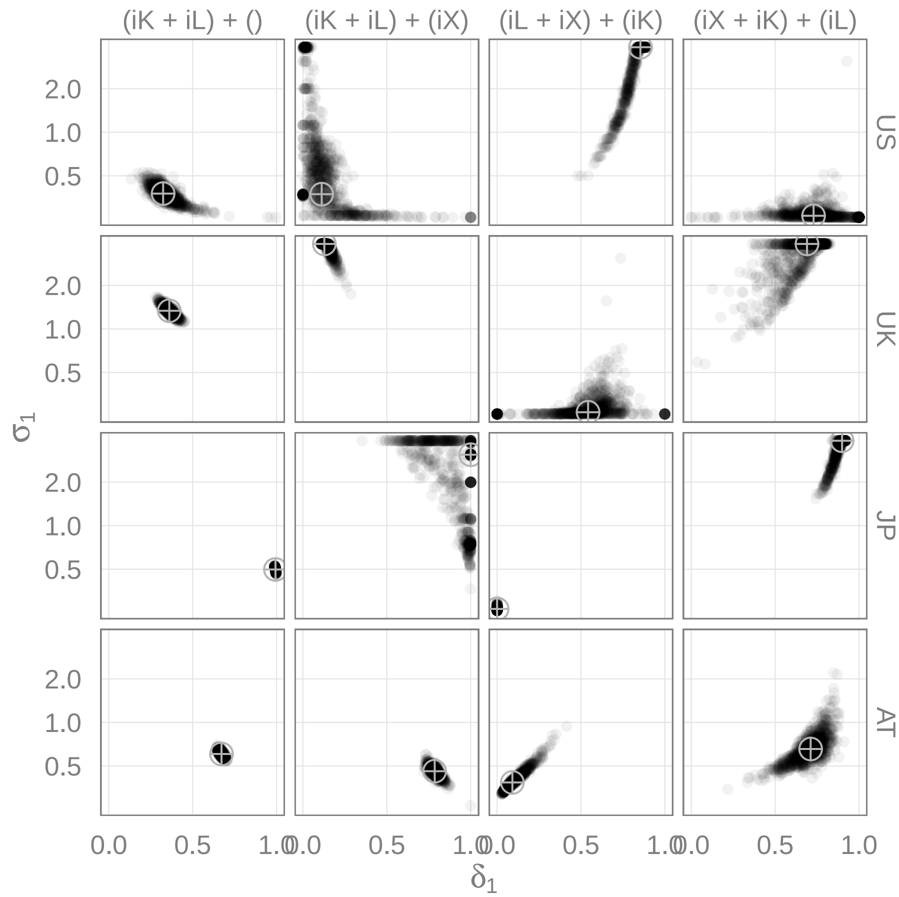

Figure 2: σ_1 and δ_1 values for CES resample models with exergy (x).

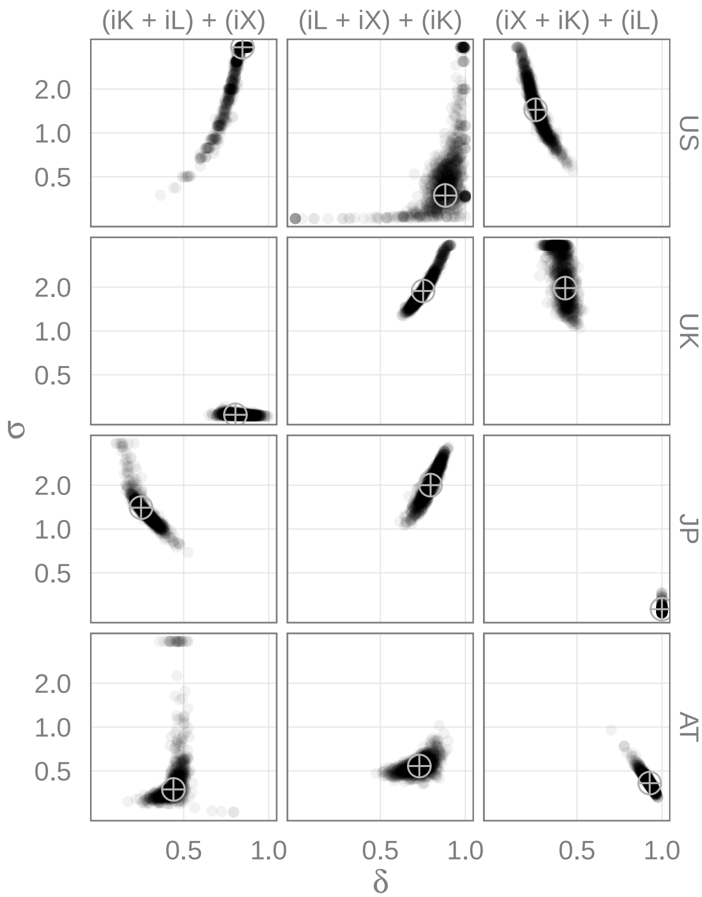

Figure 3: σ and δ values for CES resample models using exergy (x).

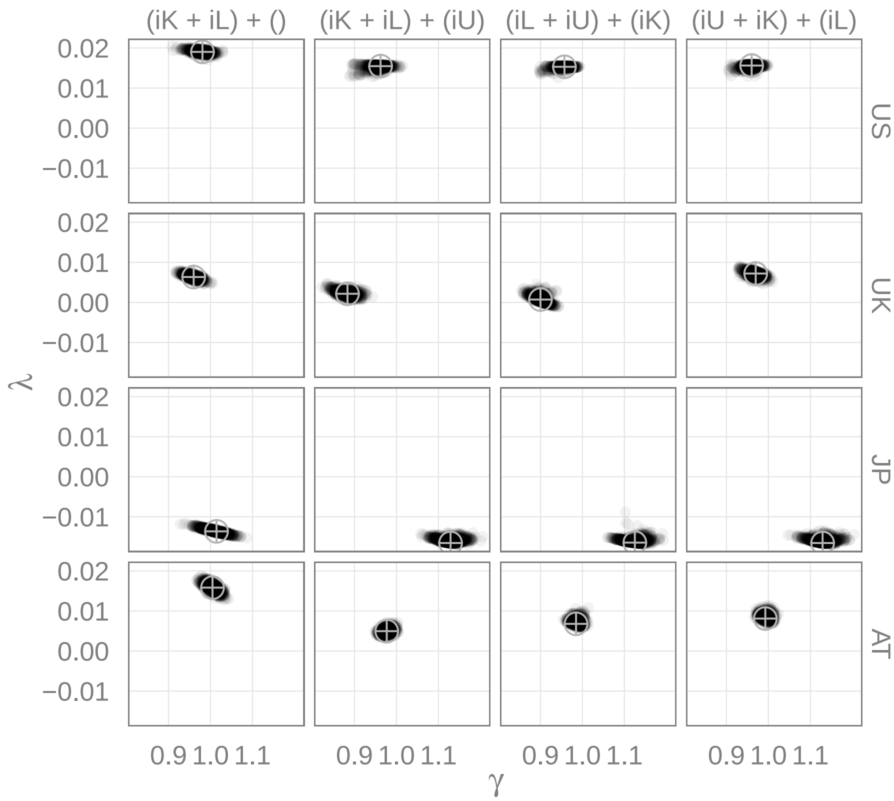

Figure 4: λ and γ values for CES resample models with useful work (u).

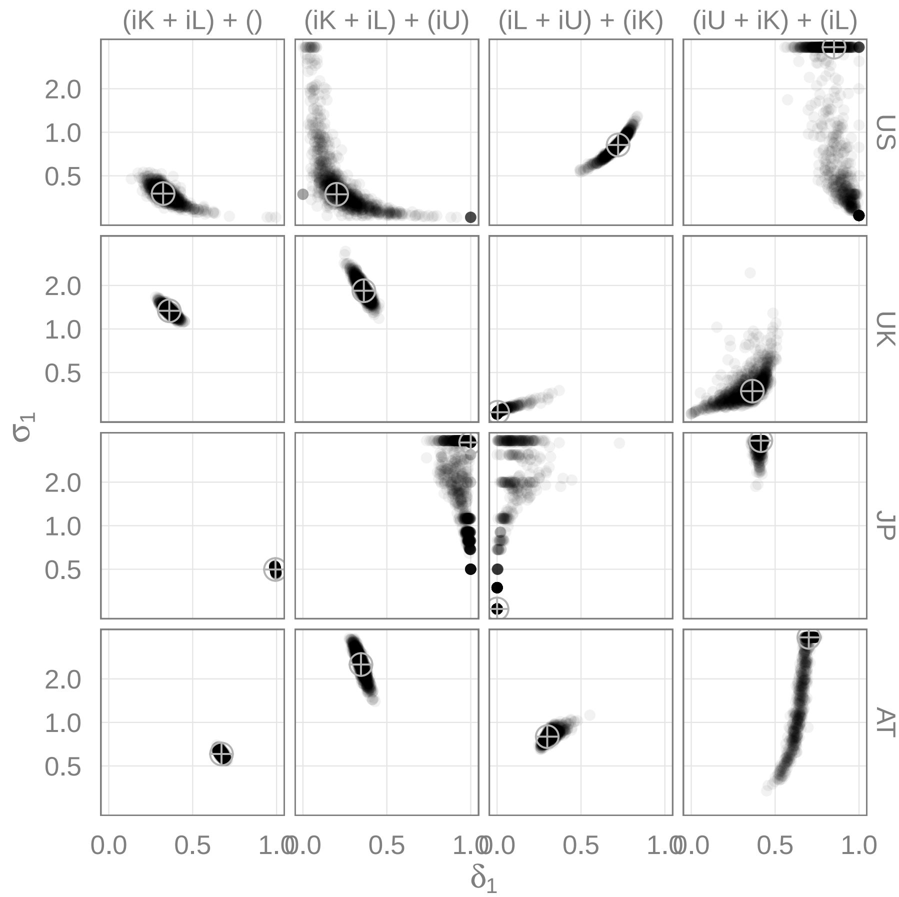

Figure 5: σ_1 and δ_1 values for CES resample models with useful work (u).

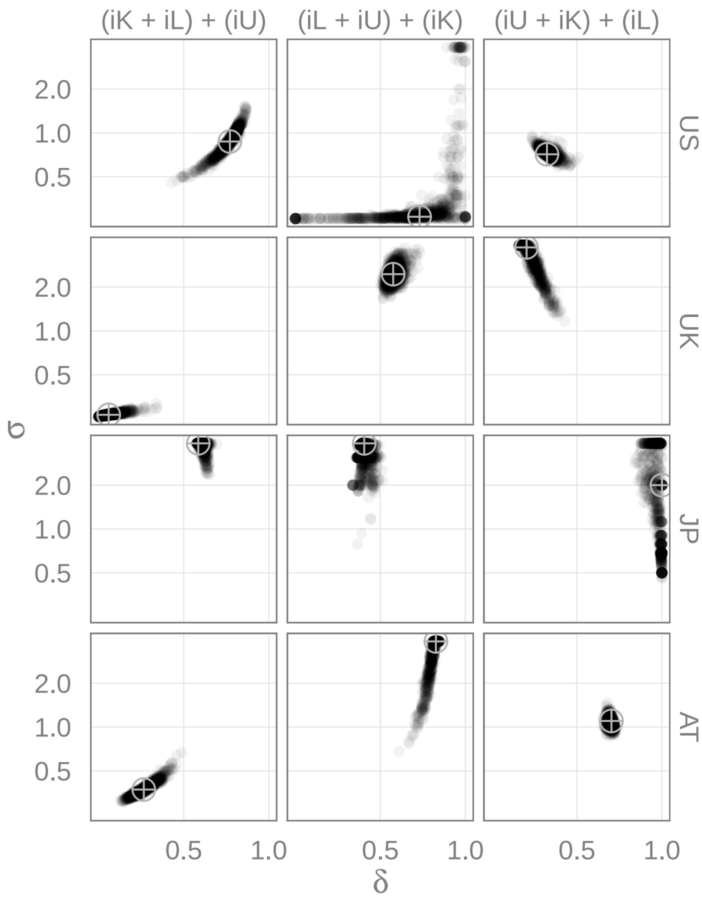

Figure 6: σ and δ values for CES resample models with useful work (u).

We can develop pseudo α, β, and γ parameters for the CES model, based on values of δ and δ_1. Table 2 shows the permuted equations for each CES nesting.

Figure **??** shows α, β, and γ parameters for the CES model for all nests and countries. Figure **??** is comparable to the ternary graph for the Cobb-Douglas models, Figure **??**.

Figure **??** compares CES model predictions to historical data for all nestings. As with the Cobb-Douglas results, historical data are shown as a black line, the fit to historical data is shown as a white line, and the gray band encompasses 95% of the fits to resampled data.

Table 2: Equations for α, β, and γ for the various CES nestings.

Nesting	Equation	α	β	γ
(kl)	3	δ_1	$1-\delta_1$	0
$(kl)e$	4	$\delta\delta_1$	$\delta(1-\delta_1)$	$1-\delta$
$(le)k$	5	$1-\delta$	$\delta\delta_1$	$\delta(1-\delta_1)$
$(ek)l$	6	$\delta(1-\delta_1)$	$1-\delta$	$\delta\delta_1$

Figure **??** shows details of the resample fits for China with the (kl)e nesting. Again, historical data are shown as a black line. The fit to historical data is shown as a white line. The gray band encompasses 95% of the fits to resampled data.

4.5. LINEX Models

The LINEX model is given in Equation 7. If the fitting algorithm drives $a_0 \to 0$, any changes to c_t have no effect on sse. Therefore, we transformed the LINEX model into

$$y = \theta A e; A = \mathrm{e}^{\left[2a_0\left(1-\frac{1}{\rho_k}\right)+a_1\left(\rho_l-1\right)\right]}, \tag{14}$$

where $a_1 \equiv a_0 c_t$. We used the `lm` function in `R` (R Core Team, 2012) to estimate least-squares values for a_0 and a_1 in Equation 14 in log-transformed space. We recovered a value for c_t with $c_t = a_1/a_0$. Figure **??** shows the coefficients estimated from resampled data.

Figure **??** compares LINEX model predictions to historical data. Again, historical data are shown as a black line, the fit to historical data is shown as a white linem and the gray band encompasses 95% of the fits to resampled data.

The LINEX model coefficients a_0 and c_t imply time-dependent values of α, β, and γ as given by the following equations (Warr and Ayres, 2012):

$$\alpha = \frac{k}{y}\frac{\partial y}{\partial k} = a_0 \left(\frac{l+e}{k}\right) \ , \tag{15}$$

$$\beta = \frac{l}{y}\frac{\partial y}{\partial l} = a_0 \left[c_t \left(\frac{l}{e}\right) - \frac{l}{k}\right] \ , \tag{16}$$

and

$$\gamma = \frac{e}{y}\frac{\partial y}{\partial e} = 1 - a_0 \left[\frac{e}{k} + c_t \frac{l}{e}\right] = 1 - \alpha - \beta \ . \tag{17}$$

Thus, each fitted model implies a trajectory of α, β, and γ values as time proceeds. If the trajectory stays within the triangle, the constraints on α, β, and γ are respected. Conversely, if the trajectory goes outside the triangle, one of the constraints has been violated.

Figures **??** and **??** show trajectories for China and Iran, respectively. **** Can we make a figure that has all 9 graphs in it? –MKH ****

In our testing, Iran is the only country for which the trajectory stays within the constraint triangle for the entire time period. Thus, in our fitted models, the LINEX model is nearly always violating constraints on α, β, and γ. If the LINEX model were made to respect the constraints, it would fit worse than it already does for those countries where the constraints are violated (that is, for every country except Iran). We did nothing to correct this situation, because it is not clear what should be done. Warr and Ayres (2012) constrained the values of a_0 and c_t, while still holding them constant for the entire time series, such that the constraints on α, β, and γ were respected. But, Warr's approach necessarily leads to larger *sse* values than unconstrained fitting. Because unconstrained LINEX models already fit worse than constrained Cobb-Douglas and CES models, we chose to report the unconstrained LINEX model fits in this paper.

4.6. Parameter Estimation Summary

5. Discussion of Results

6. Conclusion

7. Future Work

Acknowledgements

Reproducible Research

Appendix A. Appendix

Include data tables here?

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