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Growth of the South African Economy: An Empirical Evaluation

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Abstract

Abstract goes here.

1 Introduction

Intro goes here.

2 Mathematical Forms of the Energy-augmented Production Functions

In this paper, we assess three prominent energy-augmented production functions that appear in the literature: Cobb-Douglas (CD), Constant Elasticity of Substitution (CES), and LINear EXponential (LINEX). The following subsections describe each.

2.1 Energy-augmented Cobb-Douglas Production Function

The Cobb-Douglas production function without energy can be expressed as

$$y = \theta \mathrm{e}^{\lambda(t-t_0)} k^{\alpha} l^{\beta} \ , \tag{1}$$

where

- $y \equiv Y/Y_0$,
- $k \equiv K/K_0$,
- $l \equiv L/L_0$,
- Y economic output, measured by GDP,
- K capital stock, expressed in currency units,
- L labor, expressed in workers or work-hours/year,
- t time, measured in years,
- e the base of the natural logarithm, and

- 0 subscripts indicate values at an initial year. [1]

The capital-labor Cobb-Douglas production function shown in Equation 1 can be augmented to include an energy term:

$$y = \theta e^{\lambda(t-t_0)} k^{\alpha} l^{\beta} e^{\gamma} \ , \tag{2}$$

where $e \equiv E/E_0$, and E is in units of energy per time, typically TJ/year. The energy-augmented Cobb-Douglas production function is often assumed to have constant returns to scale for the three factors of production: $\alpha + \beta + \gamma = 1$.

2.2 Energy-augmented CES Production Function (CESe)

The `R` [R Core Team, 2012] package `micEconCES` [Henningsen and Henningsen, 2011] estimates CES production functions of the following forms (among others):

$$y = \theta\ A\ \left[\delta_1 k^{-\rho_1} + (1-\delta_1) l^{-\rho_1}\right]^{-1/\rho_1} ; A = e^{\lambda(t-t_0)} \tag{3}$$

$$y = \theta\ A\ \left\{\delta\left[\delta_1 k^{-\rho_1} + (1-\delta_1) l^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta) e^{-\rho}\right\}^{-1/\rho} ; A = e^{\lambda(t-t_0)} \tag{4}$$

Equation 3 is a CES production function with capital stock (k) and labor (l) factors of production. Equation 4 augments Equation 3 with energy using a $(kl)(e)$ nesting structure, as is typical in the literature. Equation 3 is a degenerate form of Equation 4 where $\delta \to 1$.

θ is a fitting parameter that represents productivity. The fitting parameters ρ_1 and ρ indicate elasticities of substitution (σ_1 and σ). The elasticity of substitution between capital (k) and labor (l) is given by $\sigma_1 = \frac{1}{1+\rho_1}$, and the elasticity of substitution between (kl) and (e) is given by $\sigma = \frac{1}{1+\rho}$. As $\rho_1 \to 0$, $\sigma_1 \to 1$, and the embedded CES production function for k and l degenerates to the Cobb-Douglas production function. Similarly, as $\rho \to 0$, $\sigma \to 1$, and the CES production function for (kl) and (e) degenerates to the Cobb-Douglas production function. As $\sigma \to \infty$ ($\rho \to -1$), (kl) and (e) are perfect substitutes. As $\sigma \to 0$ ($\rho \to \infty$), (kl) and (e) are perfect complements: no substitution is possible. Similarly, as $\sigma_1 \to 0$ ($\rho_1 \to \infty$), k and l are perfect complements. δ_1 describes the relative importance of capital (k) and labor (l), and δ describes the importance of (kl) relative to (e).

Constraints on the fitting parameters include $\delta_1 \in [0,1]$, $\delta \in [0,1]$, $\rho_1 \in [-1,0) \cup (0,\infty)$, and $\rho \in [-1,0) \cup (0,\infty)$.

Two other nestings of the factors of production (k, l, and e) are possible with the CES model.

$$y = \theta\ A\ \left\{\delta\left[\delta_1 l^{-\rho_1} + (1-\delta_1) e^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta) k^{-\rho}\right\}^{-1/\rho} ; A = e^{\lambda(t-t_0)} \tag{5}$$

$$y = \theta\ A\ \left\{\delta\left[\delta_1 e^{-\rho_1} + (1-\delta_1) k^{-\rho_1}\right]^{\rho/\rho_1} + (1-\delta) l^{-\rho}\right\}^{-1/\rho} ; A = e^{\lambda(t-t_0)} \tag{6}$$

[1] Dimensionless, indexed quantities are represented by lower-case symbols (y, k, l, e, q, x, and u), and dimensional quantities are represented by upper-case symbols (Y, K, L, E, Q, X, and U). Model parameters are represented by Greek letters (α, β, λ, θ).

Note that the ρ (σ) and δ parameters have different meanings depending upon the nesting of the factors of production.

**** Describe here how we implemented the constraints. ****

2.3 LINEX Production Function

A third production function, the LINear EXponential (LINEX) function

$$y = \theta \, \mathrm{e}^{a_0 \left[2\left(1-\frac{1}{\rho_k}\right)+c_t\left(\rho_l-1\right)\right]} \, e \,, \tag{7}$$

was derived from thermodynamic considerations [Kümmel, 1980, 1982, Kümmel et al., 1985, Kümmel, 1989, Kümmel et al., 2002, 2010]. In contrast to both the Cobb-Douglas model (Equations 1 and 2) and the CES model (Equations 3–6), the LINEX model (Equation 7) does not include a generic, time-dependent augmentation term (A). The LINEX models assumes energy (e) is the only factor of production.

The LINEX model does, however, include terms that represent efficiencies among capital, labor, and energy, in the form of the ratios ρ_k and ρ_l which are defined by[2]

$$\rho_k \equiv \frac{k}{(1/2)(l+e)} \tag{8}$$

and

$$\rho_l \equiv \frac{l}{e} \tag{9}$$

and represent (8) capital stock increase relative to the average of labor and energy and (9) labor increase relative to energy. The effect of energy (e) on production is enhanced if either

- capital stock (k) has increased more than the average of labor (l) and energy (e), i.e. if $\rho_k > 1$, or
- labor (l) has increased more than energy (e), i.e. if $\rho_l > 1$.

An economy that increases capital stock (k) and labor (l) at the same rate as energy will have capital and labor efficiency ratios (ρ_k and ρ_l, respectively) of unity. In that case, economic output (y) increases at the same rate as energy consumption (e). According to the LINEX model, an economy that increases capital stock (k) without a commensurate increase in labor (l) and energy (e) will experience an increase in output (y) in excess of its increase of energy consumption (e), because $\mathrm{e}^{a_0 \left[2\left(1-\frac{1}{\rho_k}\right)+c_t\left(\rho_l-1\right)\right]} > 1$. Similarly, an economy benefits by increasing labor (l) without a commensurate increase in energy (e). Thus, ρ_k is an efficiency of using additional labor and energy to make additional capital stock available to the economy, and ρ_l is an efficiency of using additional energy to make additional labor available to the economy. Increases in ρ_k and ρ_l provide upward pressure on economic output (y), as the only factor of production (e) is used more efficiently.

The parameter a_0 represents the overall importance of efficiencies ρ_k and ρ_l, and the parameter c_t represents the relative importance of the labor efficiency

[2]The notation for the LINEX model breaks from the conventions used in the Cobb-Douglas and CES models: a_0 and c_t are parameters and ρ_k and ρ_l are not.

term (ρ_l) compared to the capital stock efficiency term (ρ_k). Both a_0 and c_t are assumed constant with respect to time.

Kümmel et al. [2002] use the LINEX production function to describe economic output for the economies of the U.S., Japan, and Germany. Warr et al. [2010] compare the LINEX production function with the Cobb-Douglas production function to assess economic growth for Austria, the United Kingdom, the United States, and Japan. Ayres and Warr [2005] argued that introducing energy inputs in traditional production functions does not improve its explanatory power, but
2.1 → including useful work in a LINEX function does.

2.2 → Abramovitz [1956] wrote:

> This result [that "technological progress" explained 175% of the growth of the US economy since 1870] is surprising in the lopsided importance which it appears to give to productivity increase, and it should be, in a sense, sobering, if not discouraging, to students of economic growth. Since we know little about the causes of productivity increase, the indicated importance of this element may be taken to be some sort of measure of our ignorance about the causes of economic growth in the United States and some sort of indication of where we need to concentrate our attention.

3 Data

4 Results

4.1 Cobb-Douglas Model

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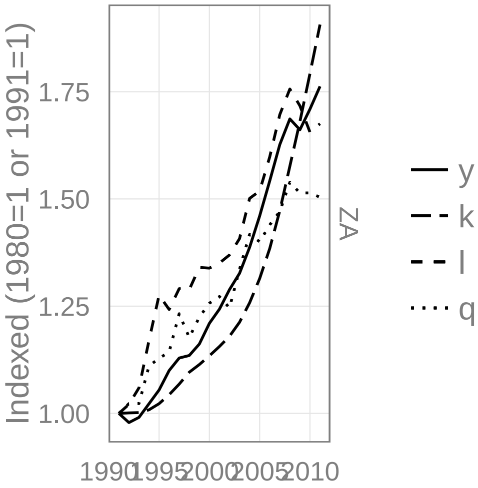

Figure 1: Historical data: indexed GDP (y), capital stock (k), labor (l), and thermal energy (q).

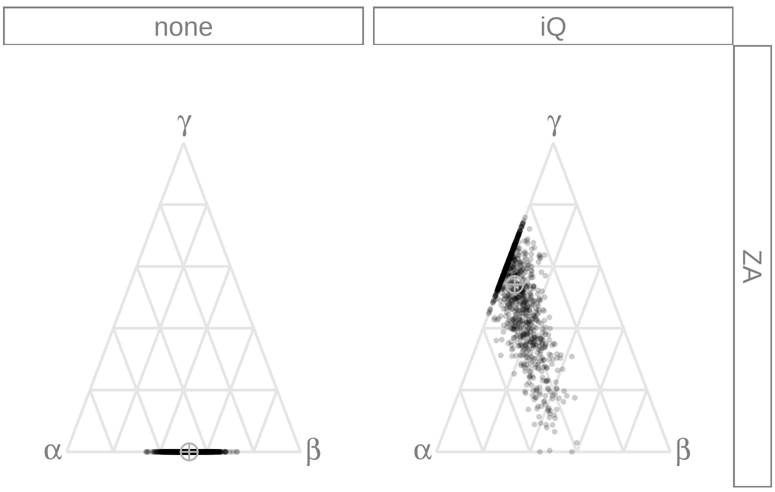

Figure 2: Cobb-Douglas resample model parameters.

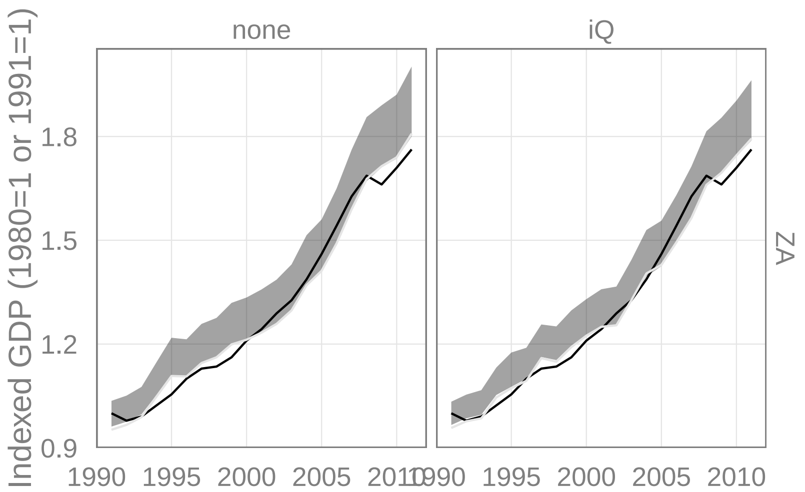

Figure 3: Cobb-Douglas fitted models.