```r
fileName <- "data/USData.txt"

# Read the data file as a table with a header.
## dataTable is a poor name.  Name it after contents, unless this is the
## start of a script where this will be provided at the command line.
dataTable <- read.table(fileName, header = TRUE)

# Identifies the header names associated with dataTable
names(dataTable)
```

```
 [1] "Year"
 [2] "GDP.Millionsofreal2005USdollars."
 [3] "Labour.Millionsofhoursworked."
 [4] "CapitalStock.Millionsofreal2005USdollars."
 [5] "Thermalenergy.TJ."
 [6] "Exergy.TJ."
 [7] "UsefulWork.TJ."
 [8] "iYear"
 [9] "iGDP"
[10] "iLabor"
[11] "iCapStk"
[12] "iQ"
[13] "iX"
[14] "iU"
```

1 Cobb-Douglas Without Energy

```r
# Establish guess values for alpha and lambda.
lambdaGuess <- 0.0 # guessing lambda = 0 means there is no technological progress.
alphaGuess <- 0.3 # a typical value for alpha, the coefficient on capital stock

# Runs a non-linear least squares fit to the data. We've replaced beta with 1-alpha for simplicity.
modelCD <- nls(iGDP ~ exp(lambda*iYear) * iCapStk^alpha * iLabor^(1 - alpha),
               start=(list(lambda=lambdaGuess,alpha=alphaGuess)),
               data=dataTable)

# Checks validity of the model. AIC stands for Akaike's Information Criterion.
aicCD  <- AIC(modelCD, k=2); aicCD
```

```
[1] -163
```

```r
summaryCD <- summary(modelCD) # Gives the nls summary table.
print(summaryCD)
```

```

Formula: iGDP ~ exp(lambda * iYear) * iCapStk^alpha * iLabor^(1 - alpha)

Parameters:
       Estimate Std. Error t value Pr(>|t|)
lambda 0.010256   0.000682   15.03  1.7e-15
alpha  0.270030   0.028311    9.54  1.4e-10

Residual standard error: 0.0178 on 30 degrees of freedom

Number of iterations to convergence: 4
Achieved convergence tolerance: 3.81e-07
```

```r
ciCD <- confint(modelCD, level = 0.95); ciCD # Displays confidence intervals for the CD model.
```

```
Waiting for profiling to be done...

            2.5%   97.5%
lambda 0.008862 0.01164
alpha  0.212405 0.32785


# Calculate beta and its confidence interval and report it.
alpha <- as.numeric(coef(modelCD)["alpha"])
beta <- 1.0 - alpha
beta.est <- deltaMethod(modelCD, "1 - alpha"); beta.est # Estimates beta and its standard error (SE).

          Estimate      SE
1 - alpha     0.73 0.02831

# Now calculate a confidence interval on beta
dofCD <- summaryCD$df[2]; dofCD # Gives the degrees of freedom for the model.

[1] 30

tvalCD <- qt(0.975, df = dofCD); tvalCD

[1] 2.042

betaCICD <- with(beta.est, Estimate + c(-1.0, 1.0) * tvalCD * SE); betaCICD # Gives the confidence interval on

[1] 0.6722 0.7878


coef(modelCD)

 lambda   alpha
0.01026 0.27003


# Combine all estimates and their confidence intervals into data frames with intelligent row names
estCD <- data.frame(lambda = coef(modelCD)["lambda"], alpha = coef(modelCD)["alpha"], beta = beta, gamma = NA)
row.names(estCD) <- "Cobb-Douglas"
#row.names(estCD) <- "Cobb-Douglas: $y = e^{\\lambda t}k^{\\alpha}l^{\\beta}$"
 # The [1] subscripts pick off the lower confidence interval
lowerCD <- data.frame(lambda = ciCD["lambda","2.5%"], alpha = ciCD["alpha", "2.5%"], beta = betaCICD[1], gamma
row.names(lowerCD) <- "- 95% CI"
 # The [2] subscripts pick off the lower confidence interval
upperCD <- data.frame(lambda = ciCD["lambda","97.5%"], alpha = ciCD["alpha", "97.5%"], beta = betaCICD[2], gamm
row.names(upperCD) <- "+ 95% CI"

# Now create the data for a table.
dataCD <- rbind(lowerCD, estCD, upperCD); dataCD

               lambda  alpha   beta gamma
- 95% CI     0.008862 0.2124 0.6722    NA
Cobb-Douglas 0.010256 0.2700 0.7300    NA
+ 95% CI     0.011645 0.3279 0.7878    NA

colnames(dataCD)  <- c("$\\lambda$", "$\\alpha$", "$\\beta$", "$\\gamma$")
tableCD  <- xtable(dataCD)

Warning:  provided 3 variables to replace 1 variables
```

```
print(xtable(dataCD), floating=FALSE)
```

	λ	α	β	γ
- 95% CI	0.01	0.21	0.67	
Cobb-Douglas	0.01	0.27	0.73	
+ 95% CI	0.01	0.33	0.79	

```
# According to http://cran.r-project.org/web/packages/xtable/vignettes/xtableGallery.pdf, Section 3.1, I shoul
# be able to use the "sanitize.text.function" parameter to allow markup in column headers. But this next
# line is not working at the present time. --MKH, 18 Jan 2012.
# print(tableCD, sanitize.text.function = function(x){x})
```

2 Cobb-Douglas With Q

```
# Establish guess values for alpha, beta, and lambda.
# lambdaGuess <- 0.0 # guessing lambda = 0 means there is no technological progress.
# alphaGuess <- 0.2 # a typical value for alpha
# betaGuess <- 0.6 # a typical value for beta

# Runs a non-linear least squares fit to the data.
# modelCDQ <- nls(iGDP ~ exp(lambda*iYear) * iCapStk^alpha * iLabor^beta * iQ^(1.0 - alpha - beta),
#                 start=(list(lambda=lambdaGuess,alpha=alphaGuess,beta=betaGuess)),
#                 data=dataTable)

# Reparameterize to ensure that we meet the constraint that alpha + beta + gamma = 1.0.
# 0 < a < 1
# 0 < b < 1
# alpha = min(a, b)
# beta = b - a
# gamma = 1 - max(a, b)

lambdaGuess <- 0.0 # guessing lambda = 0 means there is no technological progress.
alphaGuess <- 0.2 # a typical value for alpha
betaGuess <- 0.8 # a typical value for beta
modelCDq <- nls(iGDP ~ exp(lambda*iYear) *
                           iCapStk^min(a,b) * iLabor^abs(b-a) *
                           iQ^(1.0 - max(a,b)),
  algorithm = "port",
  start = list(lambda=lambdaGuess, a=alphaGuess, b=alphaGuess+betaGuess),
lower = list(lambda=-Inf, a=0, b=0),
upper = list(lambda=Inf, a=1, b=1),
  data = dataTable)

aicCDq <- AIC(modelCDq, k=2); aicCDq # Checks validity of the model. AIC stands for Akaike's Information Criter

[1] -161.1


summaryCDq <- summary(modelCDq); summaryCDq # Gives the nls summary table


Formula: iGDP ~ exp(lambda * iYear) * iCapStk^min(a, b) * iLabor^abs(b -
    a) * iQ^(1 - max(a, b))

Parameters:
       Estimate Std. Error t value Pr(>|t|)
lambda  0.01049    0.00108    9.67  1.4e-10
a       0.26322    0.03795    6.94  1.3e-07
b       0.97890    0.07647   12.80  1.9e-13

Residual standard error: 0.0181 on 29 degrees of freedom
```

```
Algorithm "port", convergence message: relative convergence (4)

# Provides confidence intervals on lambda, a, and b. But, we need CIs on alpha and beta.
ciCDq <- confint(modelCDq, level = 0.95); ciCDq

Waiting for profiling to be done...

          2.5%  97.5%
lambda 0.00885 0.0127
a      0.18580 0.3283
b      0.82175     NA

a <- as.numeric(coef(modelCDq)["a"])
b <- as.numeric(coef(modelCDq)["b"])
lambda <- as.numeric(coef(modelCDq)["lambda"])
alpha <- a
beta <- b - a
gamma <- 1.0 - alpha - beta

# Report results with SE
beta.est <- deltaMethod(modelCDq, "b-a"); beta.est # Reports results for beta, because beta = b - a.

      Estimate      SE
b - a   0.7157 0.05916

gamma.est <- deltaMethod(modelCDq, "1-b"); gamma.est # Reports results for gamma, because gamma = 1 - b.

      Estimate      SE
1 - b   0.0211 0.07647

# Now calculate confidence intervals.
dofCDq <- summaryCDq$df[2]; dofCDq # Gives the degrees of freedom for the model.

[1] 29

tvalCDq <- qt(0.975, df = dofCDq); tvalCDq

[1] 2.045

betaCICDq <- with(beta.est, Estimate + c(-1.0, 1.0) * tvalCDq * SE); betaCICDq # Gives the confidence interval

[1] 0.5947 0.8367

gammaCICDq <- with(gamma.est, Estimate + c(-1.0, 1.0) * tvalCDq * SE); gammaCICDq # Gives the confidence inter

[1] -0.1353  0.1775

# Combine all estimates and their confidence intervals into data frames with intelligent row names
estCDq <- data.frame(lambda = lambda, alpha = alpha, beta = beta, gamma = gamma); estCDq

   lambda  alpha   beta  gamma
1 0.01049 0.2632 0.7157 0.0211

#row.names(estCDq) <- "Cobb-Douglas with q: $y = e^{\\lambda t}k^{\\alpha}l^{\\beta}q^{\\gamma}$"
row.names(estCDq) <- "CobbDouglas with q"
# The [1] subscripts pick off the lower confidence interval
lowerCDq <- data.frame(lambda = ciCDq["lambda","2.5%"], alpha = ciCDq["a", "2.5%"], beta = betaCICDq[1], gamma
row.names(lowerCDq) <- "- 95% CI"
# The [2] subscripts pick off the lower confidence interval
upperCDq <- data.frame(lambda = ciCDq["lambda","97.5%"], alpha = ciCDq["a", "97.5%"], beta = betaCICDq[2], gam
row.names(upperCDq) <- "+ 95% CI"

# Now create the data for a table.
dataCDq <- rbind(lowerCDq, estCDq, upperCDq); dataCDq
```

```
                     lambda  alpha   beta   gamma
- 95% CI            0.00885 0.1858 0.5947 -0.1353
CobbDouglas with q  0.01049 0.2632 0.7157  0.0211
+ 95% CI            0.01270 0.3283 0.8367  0.1775
```

```
colnames(dataCDq)  <- c("$\\lambda$", "$\\alpha$", "$\\beta$", "$\\gamma$")
tableCDq <- xtable(dataCDq)

dataAll <- rbind(dataCD, dataCDq); dataAll
```

```
                   $\\lambda$ $\\alpha$ $\\beta$ $\\gamma$
- 95% CI             0.008862    0.2124   0.6722        NA
Cobb-Douglas         0.010256    0.2700   0.7300        NA
+ 95% CI             0.011645    0.3279   0.7878        NA
- 95% CI1            0.008850    0.1858   0.5947   -0.1353
CobbDouglas with q   0.010486    0.2632   0.7157    0.0211
+ 95% CI1            0.012700    0.3283   0.8367    0.1775
```

```
print(xtable(dataCDq), floating=FALSE)
```

	λ	α	β	γ
- 95% CI	0.01	0.19	0.59	-0.14
CobbDouglas with q	0.01	0.26	0.72	0.02
+ 95% CI	0.01	0.33	0.84	0.18

```
# According to http://cran.r-project.org/web/packages/xtable/vignettes/xtableGallery.pdf, Section 3.1, I shoul
# be able to use the "sanitize.text.function" parameter to allow markup in column headers. But this next
# line is not working at the present time. --MKH, 18 Jan 2012.
# print(tableCDq, sanitize.text.function = function(x){x})

print(xtable(dataAll, floating=FALSE))
```

	λ	α	β	γ
- 95% CI	0.01	0.21	0.67	
Cobb-Douglas	0.01	0.27	0.73	
+ 95% CI	0.01	0.33	0.79	
- 95% CI1	0.01	0.19	0.59	-0.14
CobbDouglas with q	0.01	0.26	0.72	0.02
+ 95% CI1	0.01	0.33	0.84	0.18

3 Cobb-Douglas With X

```
# Establish guess values for alpha, beta, and lambda.
lambdaGuess <- 0.0 # guessing lambda = 0 means there is no technological progress.
alphaGuess <- 0.2 # a typical value for alpha
betaGuess <- 0.6 # a typical value for beta

# Runs a non-linear least squares fit to the data.
modelCDX <- nls(iGDP ~ exp(lambda*iYear) * iCapStk^alpha * iLabor^beta * iX^(1.0 - alpha - beta),
                start=(list(lambda=lambdaGuess,alpha=alphaGuess,beta=betaGuess)),
                data=dataTable)

# Gives the nls summary table
summary(modelCDX)
```

```
Formula: iGDP ~ exp(lambda * iYear) * iCapStk^alpha * iLabor^beta * iX^(1 -
    alpha - beta)

Parameters:
       Estimate Std. Error t value Pr(>|t|)
lambda  0.01071    0.00105   10.16  4.6e-11
alpha   0.25689    0.03666    7.01  1.0e-07
beta    0.70011    0.05916   11.83  1.3e-12

Residual standard error: 0.018 on 29 degrees of freedom

Number of iterations to convergence: 4
Achieved convergence tolerance: 2.88e-06
```

```
confint(modelCDX, level = 0.95)
```

```
Waiting for profiling to be done...
```

```
           2.5%   97.5%
lambda 0.008559 0.01287
alpha  0.182137 0.33181
beta   0.578423 0.82111
```

```
# Calculate gamma and report it.
alpha <- coef(modelCDX)["alpha"]
beta <- coef(modelCDX)["beta"]
gamma <- as.numeric(1.0 - alpha - beta)
c(coef(modelCDX),gamma=gamma)
```

```
 lambda   alpha    beta   gamma
0.01071 0.25689 0.70011 0.04300
```

```
# Checks validity of the model. AIC stands for Akaike's Information Criterion
AIC(modelCDX, k=2)
```

```
[1] -161.4
```

4 Cobb-Douglas With U

An issue arrises in this example because contraining α and β is not sufficient to guarantee that γ is properly constrained.

```
# Establish guess values for alpha, beta, and lambda.
lambdaGuess <- 0.0 # guessing lambda = 0 means there is no technological progress.
alphaGuess <- 0.2 # a typical value for alpha
betaGuess <- 0.6 # a typical value for beta
gammaGuess <- 0.01 # a nice low value

# Runs a non-linear least squares fit to the data with constraints
modelCDU <- nls(iGDP ~ exp(lambda*iYear) * iCapStk^alpha * iLabor^beta * iU^(1.0 - alpha - beta),
  algorithm="port",
start = list(lambda=lambdaGuess, alpha=alphaGuess, beta=betaGuess),
lower = list(lambda=-Inf, alpha=0, beta=0),
upper = list(lambda=Inf, alpha=1, beta=1),
  data=dataTable)

# Gives the nls summary table
summary(modelCDU)
```

```
Formula: iGDP ~ exp(lambda * iYear) * iCapStk^alpha * iLabor^beta * iU^(1 -
    alpha - beta)

Parameters:
       Estimate Std. Error t value Pr(>|t|)
lambda  0.00920    0.00129    7.11  1.2e-06
alpha   0.32389    0.07187    4.51  0.00027
beta    0.71425    0.07641    9.35  2.5e-08

Residual standard error: 0.00994 on 18 degrees of freedom

Algorithm "port", convergence message: relative convergence (4)
  (11 observations deleted due to missingness)

confint(modelCDU, level = 0.95)

Waiting for profiling to be done...

          2.5%   97.5%
lambda 0.00649 0.01192
alpha  0.17289 0.47457
beta   0.55373 0.87481
```

```
# Checks validity of the model. AIC stands for Akaike's Information Criterion
AIC(modelCDU, k=2)

[1] -129.3
```

The problem here is that $\hat{\gamma} < 0$.

```
# Calculate gamma and report it.
alpha <- coef(modelCDU)["alpha"]
beta <- coef(modelCDU)["beta"]
gamma <- as.numeric(1.0 - alpha - beta)
c(coef(modelCDU), gamma=gamma)

   lambda     alpha      beta     gamma
 0.009203  0.323887  0.714248 -0.038135
```

Our $\hat{\gamma}$ is not much below 0. Let's compute the standard error.

```
require(car)  # the methods in alr3 have been deprecated.  Use car instead.
gamma.est <- deltaMethod(modelCDU, "1-alpha-beta"); gamma.est

                 Estimate      SE
1 - alpha - beta -0.03813 0.03055

# crude CI (using 2 as a rough estimate for critical value)
with(gamma.est, Estimate + c(-1,1) * 2 * SE)

[1] -0.09924  0.02297
```

So no real cause for concern: our data don't convince us that the real γ is different from 0.

4.1 Forcing $\gamma \geq 0$

We can force α, β, and γ to be in $[0, 1]$ by a reparameterization:

$$a \in [0,1], b \in [0,1], \alpha = \min(a,b), \beta = |b-a|, \gamma = 1 - \max(a,b)$$

```
modelCDUforced <- nls(iGDP ~ exp(lambda*iYear) *
                             iCapStk^min(a,b) * iLabor^abs(b-a) *
                             iU^(1.0 - max(a,b)),
  algorithm="port",
start = list(lambda=lambdaGuess, a=alphaGuess, b=alphaGuess + betaGuess),
lower = list(lambda=-Inf, a=0, b=0),
upper = list(lambda=Inf, a=1, b=1),
  data=dataTable)

coef(summary(modelCDUforced))

       Estimate Std. Error t value  Pr(>|t|)
lambda 0.009299   0.001349   6.893 1.908e-06
a      0.318805   0.074953   4.253 4.780e-04
b      1.000000   0.031938  31.310 3.765e-17

with( as.data.frame(t(coef(modelCDUforced))), c(alpha=min(a,b), beta=abs(b-a), gamma=1-max(a,b)))

 alpha   beta  gamma
0.3188 0.6812 0.0000
```

But the naive delta method to calculate significance information fails because R doesn't know how to calculate the derivatives for the minimum and maximum functions. So we need to be clever, using the fact that we know now that $a < b$:

```
# alpha = a
deltaMethod( modelCDUforced, "a")

  Estimate      SE
a   0.3188 0.07495

# beta = b - a
deltaMethod( modelCDUforced, "b-a")

      Estimate      SE
b - a   0.6812 0.07972

# gamma = 1-b
deltaMethod( modelCDUforced, "1-b")

      Estimate      SE
1 - b        0 0.03194
```

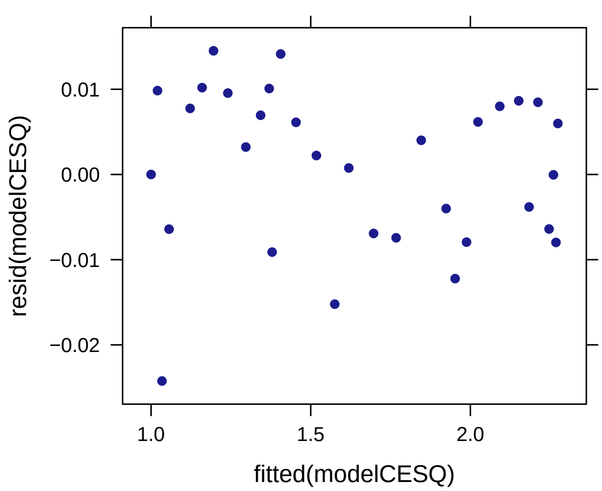
So this seems to give us what we want for this case: We have parameter estimates and standard errors subject to all of our constraints.

It may be that we can avoid using min and max and just use a, $b - a$ and $1 - b$. If that works, then this generalizes fairly easily to any number of parameters that must be bounded by 0 and 1 and sum to 1. (Else we have to "sort" the dummy parameters first, which is OK but makes the coding a bit uglier.)

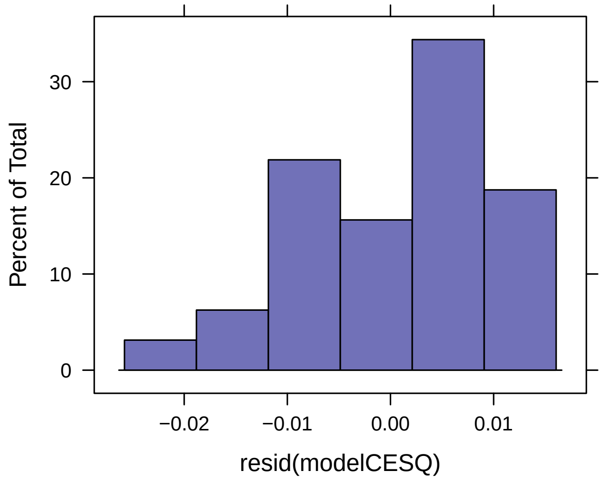
5 CES With Q

```
# Establish guess values for alpha, beta, and lambda.
phiGuess <- -20
betaGuess <- 0.5 # a typical value for beta (exponent on labor)
zetaGuess <- 0.0004 # a small value
lambda_LGuess <- 0.007 #assuming no technical progress on the labor-capital portion of the function
lambda_EGuess <- 0.008 #assuming no technical progress on the energy portion of the function

# Runs a non-linear least squares fit to the data with constraints
modelCESQ <- nls(iGDP ~ ((1-zeta) * (exp(lambda_L*iYear) * iLabor^beta * iCapStk^(1-beta))^phi
```

```
                          + zeta*(exp(lambda_E*iYear) * iQ)^phi)^(1/phi),
  algorithm = "port",
control = nls.control(maxiter = 500, tol = 1e-06, minFactor = 1/1024,
                        printEval = FALSE, warnOnly = FALSE),
start = list(phi=phiGuess, beta=betaGuess, zeta=zetaGuess, lambda_L=lambda_LGuess,
               lambda_E=lambda_EGuess),
lower = list(phi=-Inf, beta=0, zeta=0, lambda_L=-Inf, lambda_E=-Inf),
upper = list(phi=0, beta=1, zeta=1, lambda_L=Inf, lambda_E=Inf),
  data=dataTable)

# Gives the nls summary table
summary(modelCESQ)


Formula: iGDP ~ ((1 - zeta) * (exp(lambda_L * iYear) * iLabor^beta * iCapStk^(1 -
    beta))^phi + zeta * (exp(lambda_E * iYear) * iQ)^phi)^(1/phi)

Parameters:
          Estimate Std. Error t value Pr(>|t|)
phi      -2.22e+01   1.50e+01   -1.48   0.1512
beta      5.82e-01   5.20e-02   11.19  1.2e-11
zeta      3.51e-04   1.38e-03    0.25   0.8014
lambda_L  7.62e-03   8.30e-04    9.18  8.5e-10
lambda_E  8.05e-03   2.84e-03    2.84   0.0085

Residual standard error: 0.00993 on 27 degrees of freedom

Algorithm "port", convergence message: relative convergence (4)

xyplot( resid(modelCESQ) ~ fitted(modelCESQ) )
```




```
histogram( ~resid(modelCESQ) )
```




```
qqmath( ~resid(modelCESQ) )
```