15.415x Foundations of Modern Finance

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Lecture 4: Fixed Income Securities

Key Concepts

- Introduction
- Yield Curve
- Discount versus Coupon Bonds
- Relative Bond Valuation
- Yield To Maturity
- Yield Curve Dynamics
- Interest Rate Risk and Bond Duration
- Bond Duration and Convexity
- Inflation Risk

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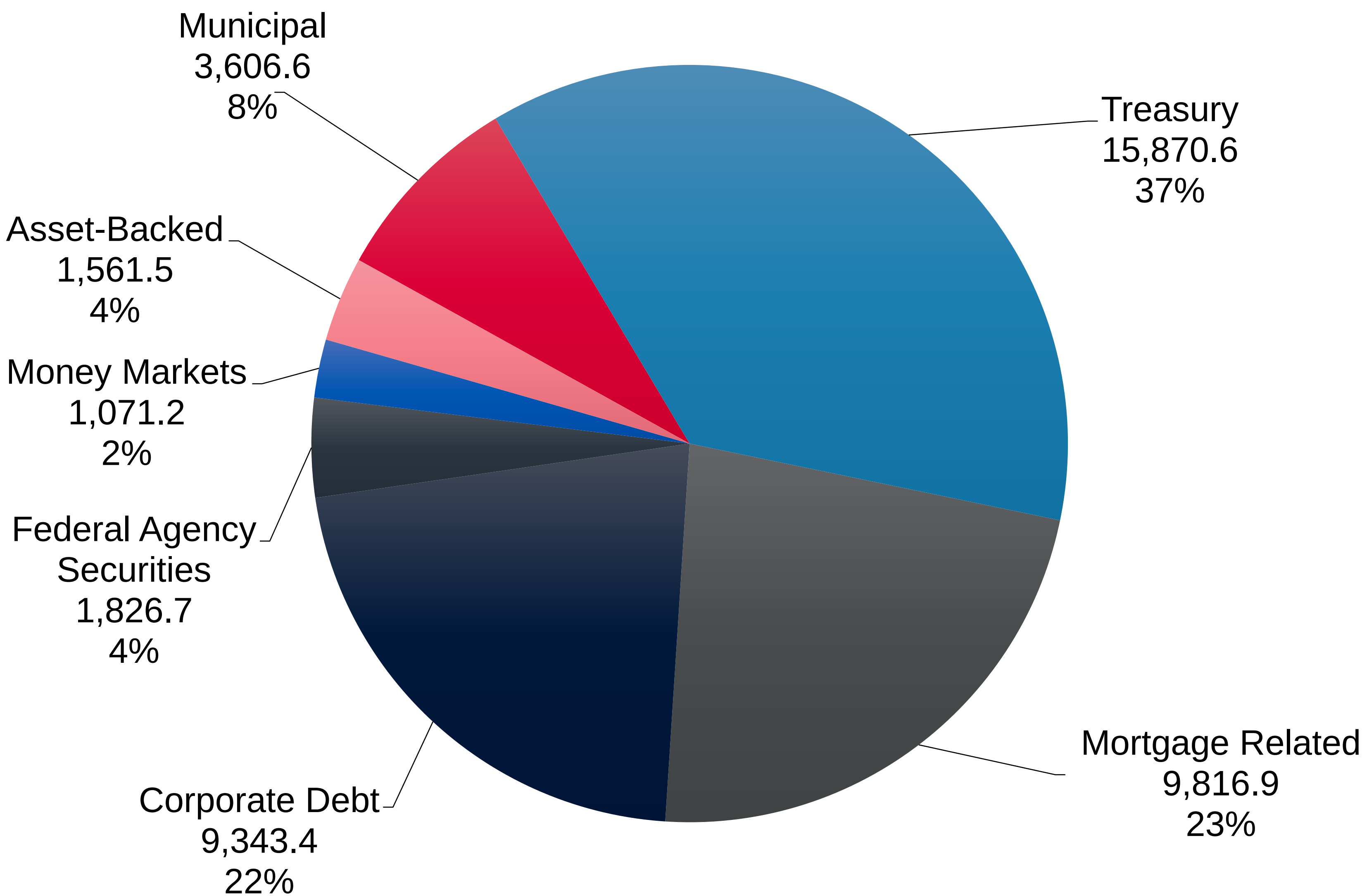
Fixed income securities

- Fixed income securities are financial claims with promised cash flows of fixed amount paid at fixed dates.
- Major classes of fixed-income securities:
  - Treasury: U.S. Treasuries, Bunds, JGBs, etc.;
  - Federal agency (U.S.): FNMA, FHLMC, etc.;
  - Municipal securities;
  - Corporate;
  - Mortgage backed and asset backed.

Outstanding U.S. bond market debt 2019 Q1 ($billions)



Examples of key market participants

Issuers	Intermediaries	Investors
Governments	Dealers: primary and other	Pension funds
Municipalities	Investment banks	Insurance companies
Corporations	Credit rating agencies	Mutual funds
		Hedge funds
		Banks
		Individuals

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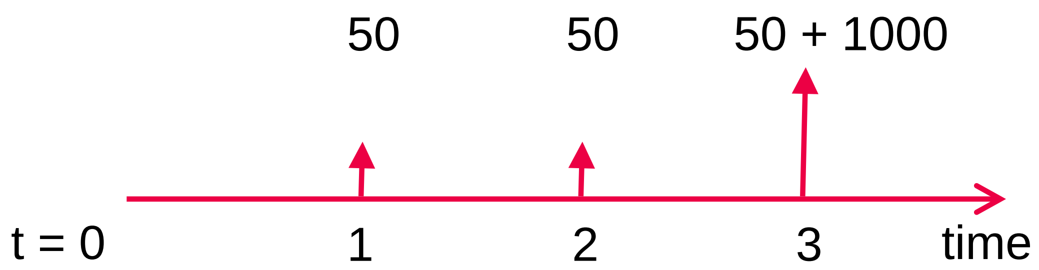
Cash flows of fixed income securities

A 3-year bond with principal of $1,000 and annual coupon payment of 5%



- Cash flow:
  - Maturity;
  - Principal;
  - Coupon.

Valuation of riskless cash flows

- Relative valuation, based on absence of arbitrage.
- Without risk, only time value of money is relevant.
- Prices of traded fixed-income securities provide information needed to value riskless cash flows at hand.

Market information: time value of money

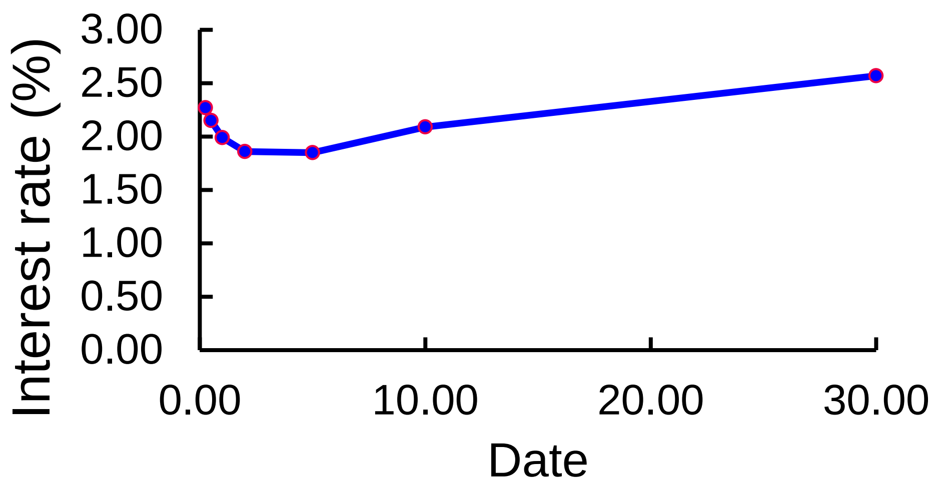
- In the market, the time value of money is captured in many different forms:
- Spot interest rates;
- Prices of discount bonds (zero-coupon bonds);
- Prices of coupon bonds.

Yield curve

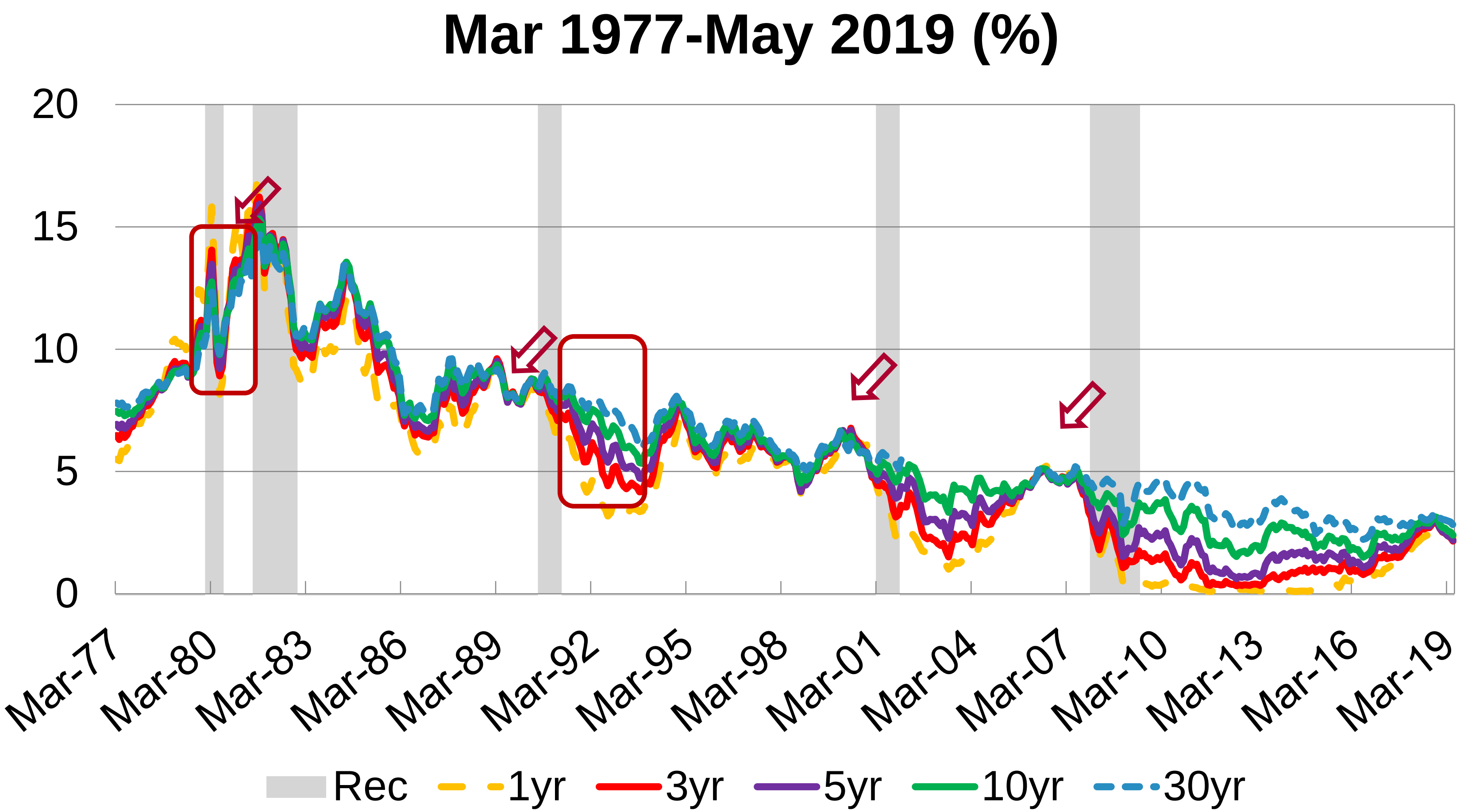
- **Spot interest rate** is the current (annualized) interest rate (r_t) for maturity date t:
  - r_t is for payments only on date t;
  - r_t is different for each date t.
- Spot interest rates on 2019/06/08

Maturity (year)	0.25	0.5	1	2	5	10	30
Interest Rate (%)	2.27	2.15	1.99	1.86	1.85	2.09	2.57

Yield curve (term structure of interest rates): the set of spot interest rates for different maturities.



Historical U.S. Treasury rates



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Bond prices and interest rates

- Let B_t denote the current price (time 0) of a discount bond maturing at t.
- Prices of discount bonds provide information about spot interest rates:

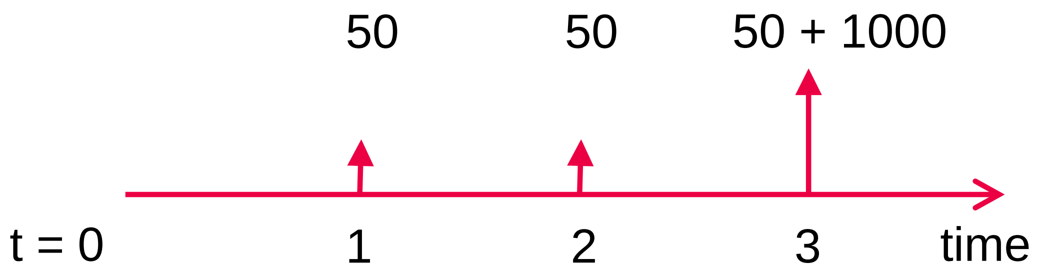
$$B_t = \frac{1}{(1+r_t)^t} \quad \text{or} \quad r_t = \frac{1}{B_t^{1/t}} - 1$$

Coupon bonds vs discount bonds

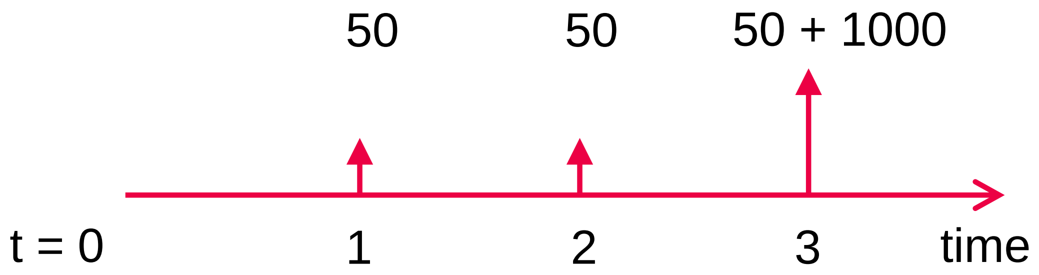
- A coupon bond pays a stream of regular coupon payments and a principal at maturity.
- A coupon bond is a portfolio of discount bonds.
- Relative pricing: establish the value of a coupon bond relative to discount bonds, and vice versa.

Coupon bonds vs discount bonds

A 3-year bond of $1,000 par and 5% annual coupon



Portfolio of discount bonds



Pricing a coupon bond

- Law of one price: price of the coupon bond must equal the price of the replicating portfolio of discount bonds:

$$B = \sum_{t=1}^{T} (C_t \times B_t) + (P \times B_T) = \frac{C_1}{1+r_1} + \cdots + \frac{C_{T-1}}{(1+r_{T-1})^{T-1}} + \frac{C_T + P}{(1+r_T)^T}$$

- Suppose that discount bond prices are as follows:

t	1	2	3	4	5
B_t	0.952	0.898	0.863	0.807	0.757

- Coupon bond price is

$$(50)(0.952) + (50)(0.898) + (1{,}050)(0.863) = 998.65$$

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Relative valuation of bonds

- There are 1-year, 2-year, and 3-year bonds (all with face values of $100) traded in the market.
- The coupons are paid in annual installments.

Maturity and type	Coupon rate	Price
A: 1-year discount	--	$96.00
B: 2-year coupon bond	5.0%	$99.30
C: 3-year coupon bond	10.0%	$108.80

- Consider a 3-year discount bond D with the face value of $100, traded at $84.00. Is this bond fairly priced?
  - If not, how can we take advantage of mispricing?

Relative valuation of bonds

- First, visualize bond cash flows as a payoff matrix.

Bond	$t = 1$	$t = 2$	$t = 3$	Price
A	100	0	0	96
B	5	105	0	99.3
C	10	10	110	108.8

- Denote the prices of discount bonds with maturities of 1, 2, and 3 years respectively and face value of \$1 as $P_1, P_2,$ and P_3.

- Prices of bonds A, B, and C are related to prices of zero-coupon bonds.

- Consider bond B:

$$5 \times P_1 + 105 \times P_2 + 0 \times P_3 = 99.30$$

Relative valuation of bonds

- Collect all three pricing relations into a system of linear equations:

$$\begin{bmatrix} 100 & 0 & 0 \\ 5 & 105 & 0 \\ 10 & 10 & 110 \end{bmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} 96 \\ 99.3 \\ 108.8 \end{pmatrix} \quad \begin{matrix} (A) \\ (B) \\ (C) \end{matrix}$$

- We find a unique solution:

$$P_1 = 0.96, P_2 = 0.90, P_3 = 0.82$$

- The price of bond D is inconsistent with the prices of A, B, and C: the implied price of D is $\$82 < \$84$.

Arbitrage strategy

- We now construct an arbitrage strategy to take advantage of mispricing:
  - This strategy is self-financing: it requires no infusion of capital;
  - It has no risk of losing money;
  - It produces positive profits with a positive probability.
- Construct a portfolio with x_A, x_B, x_C, and x_D shares of bonds A, B, and C, and D respectively.
- We require our portfolio to produce a cash flow of \$1 at time 0, and nothing in periods 1, 2, and 3.
  - This portfolio delivers arbitrage profits.

Arbitrage strategy

- Collect conditions on cash flows at $t = 0, 1, 2, 3$ into a system of linear equations:

$$\begin{bmatrix} -96 & -99.3 & -108.8 & -84 \\ 100 & 5 & 10 & 0 \\ 0 & 105 & 10 & 0 \\ 0 & 0 & 110 & 100 \end{bmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \\ x_D \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} (t = 0) \\ (t = 1) \\ (t = 2) \\ (t = 3) \end{matrix}$$

- Solving the equations, we find

$$x_A = -0.0433$$

$$x_B = -0.0433$$

$$x_C = 0.4545$$

$$x_D = -0.5000$$

Conclusion

- With a rich collection of fixed-income assets traded in the market, absence of arbitrage imposes strong restrictions on prices of securities relative to each other.
- Prices of coupon bonds, in particular, contain information about the yield curve (interest rates).
- So far we explore implications of arbitrage restrictions in a static framework – trade only at time 0.
    - Arbitrage-based pricing methods can be used in a dynamic setting to describe variation in bond prices over time.

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Yield to maturity

- Yield-to-maturity (YTM) of a bond, denoted by y, solves

$$B = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} + \frac{P}{(1+y)^T}$$

- Yield to maturity is a convention for quoting prices: in general, YTM does not represent the expected return on a bond.

- YTM is a function of interest rates of various maturities.

Yield to maturity: example

- Current spot rates

1 year	2 years
5%	6%

- 2-year Treasury coupon bond with a par value of $100 and a coupon rate of 6%, annual coupon payments:

$$Price = \frac{6}{1+0.05} + \frac{106}{(1+0.06)^2} = 100.0539$$

- Yield to maturity is 5.9706%:

$$100.0539 = \frac{6}{1+0.059706} + \frac{106}{(1+0.059706)^2}$$

YTM vs coupon rate

- Bond price is inversely related to YTM.
- A bond sells at par only if its coupon rate equals the YTM. Let $P = 1$, coupon rate is c:

$$B = \sum_{t=1}^{T} \frac{c}{(1+y)^t} + \frac{1}{(1+y)^T} = \frac{c}{y} + \frac{1}{(1+y)^T}\left(1 - \frac{c}{y}\right)$$

$$c = y \quad \text{iff} \quad B = 1$$

- A bond sells at a discount if its coupon rate is below the YTM, $c < y$.
- A bond sells at a premium if its coupon rate is above the YTM, $c > y$.

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Expectations hypothesis

- Consider two alternative investment strategies:

  1. Invest \$1 in a 10-year discount bond;
  2. Invest \$1 in a 9-year discount bond + re-invest for 1 more year at the prevailing spot rate.

- Assume that bond returns do not carry a risk premium, the two strategies should produce the same return in expectation.

Expectations hypothesis

- Notation: $r_s(t)$ denotes s-period spot rate at time t. By default, $t = 0$.
- Two strategies should produce the same return in expectation:

$$\mathrm{E}_0\left[\left(1 + r_9(0)\right)^9\left(1 + \tilde{r}_1(9)\right)\right] = \left(1 + r_{10}(0)\right)^{10}$$

  - $r_9(0)$and $r_{10}(0)$ are spot rates, known at time $t = 0$;
  - $\tilde{r}_1(9)$ is random, spot rate at time $t = 9$.
- Without risk premium,

$$\mathrm{E}_0[\tilde{r}_1(9)] = \frac{\left(1+r_{10}(0)\right)^{10}}{\left(1+r_9(0)\right)^9} - 1$$

Expectations hypothesis

- Under the expectations hypothesis, the slope of the yield curve predicts future spot rates.

- Suppose the 10-year spot rate is above the 9-year rate. Then

$$\mathrm{E}_0[\tilde{r}_1(9)] = \frac{\left(1 + r_{10}(0)\right)^9}{\left(1 + r_9(0)\right)^9}\left(1 + r_{10}(0)\right) - 1 > r_{10}(0) > r_9(0)$$

- Suppose the 10-year spot rate is equal to the 9-year rate. Then

$$\mathrm{E}_0[\tilde{r}_1(9)] = \frac{\left(1+r_{10}(0)\right)^9}{\left(1+r_9(0)\right)^9}\left(1 + r_{10}(0)\right) - 1 = r_{10}(0) = r_9(0)$$

Expectations hypothesis

- The slope of the term structure reflects the market's expectations of future short-term interest rates:

$$\mathrm{E}_0[\tilde{r}_1(t)] = \frac{\left(1 + r_{t+1}(0)\right)^{t+1}}{\left(1 + r_t(0)\right)^{t}} - 1$$

Liquidity preference hypothesis

- Investors regard long bonds as riskier than short bonds, earn a premium λ_t -- “risk premium”, or “liquidity premium”

$$\mathrm{E}_0[\tilde{r}_1(t)] + \lambda_t = \left\{ \frac{\left(1 + r_{t+1}(0)\right)^{t+1}}{\left(1 + r_t(0)\right)^{t}} - 1 \right\}$$

- Implications:
  - Long-term bonds on average receive higher returns than short-term bonds.
  - Long-term interest rates “over-predict” future short-term rates.
- Term structure reflects expectations of future interest rates + risk (or “liquidity”) premium demanded by investors in long bonds.

Hypotheses on interest rates

Long-term	Bills
5.9%	3.4%

Average Rates of Return on Treasuries, 1926 - 2018

(Source: Ibbotson Associates, 2019 Yearbook)

- Why long-term bonds may earn a positive premium?
  - Short-term bonds are more money-like: hold value better in the short run;
  - Nominal bonds are exposed to inflation risk – lose value when inflation spikes.

Models of interest rates

- What determines the term structure of interest rates?
  - Expected future spot rates;
  - Risk of long bonds.

- Models of interest rates:
  - Expectations Hypothesis;
  - Liquidity Preference Hypothesis;
  - Dynamic Models: Vasicek, Cox-Ingersoll-Ross, etc.

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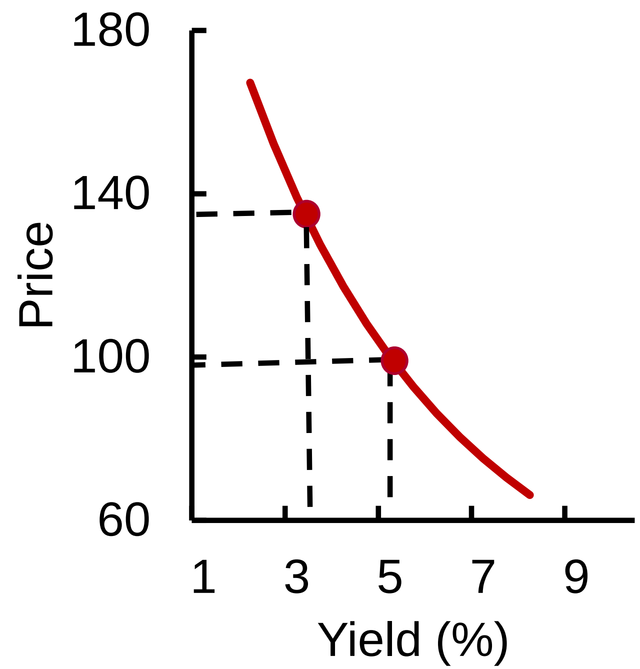
Interest rate risk

- As interest rates change (stochastically) over time, bond prices also change.
- The value of a bond is subject to interest rate risk.
- Bond prices and bond yields are inversely related:
  - As bond yield rises, bond price falls.



Bond Duration

- Assume a flat term structure at $r_t = y$.
- Measure bond's interest rate risk by its relative price change with respect to a unit change in yield (with a negative sign):
  - Suppose bond price is \$90 at $y = 0.05$.
  - As yield changes to 0.04, bond price rises to \$91.8.
  - Relative price change is $(91.8 - 90)/90 = 0.02$.
  - Normalizing by the change in the yield, risk measure is $\frac{0.02}{0.05-0.04} = 2$.
- We consider infinitesimal changes in bond yield, and use derivatives to define bond risk.

Bond Duration

- Measure of bond risk:

$$-\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}y}$$

- This is called Modified Duration (MD).
- For a discount bond, $\mathrm{B}_t = (1+y)^{-t}$, hence

$$MD(B_t) = -\frac{1}{B_t}\frac{\mathrm{d}B_t}{\mathrm{d}y} = \frac{t}{1+y}$$

- Modified duration is closely related to physical timing of cash flows:

$$(1+y\,)MD(B_t) = t$$

Macaulay duration

- Consider general streams of cash flows, CF_t (e.g., coupons).
- Macaulay duration is the weighted average term to maturity.

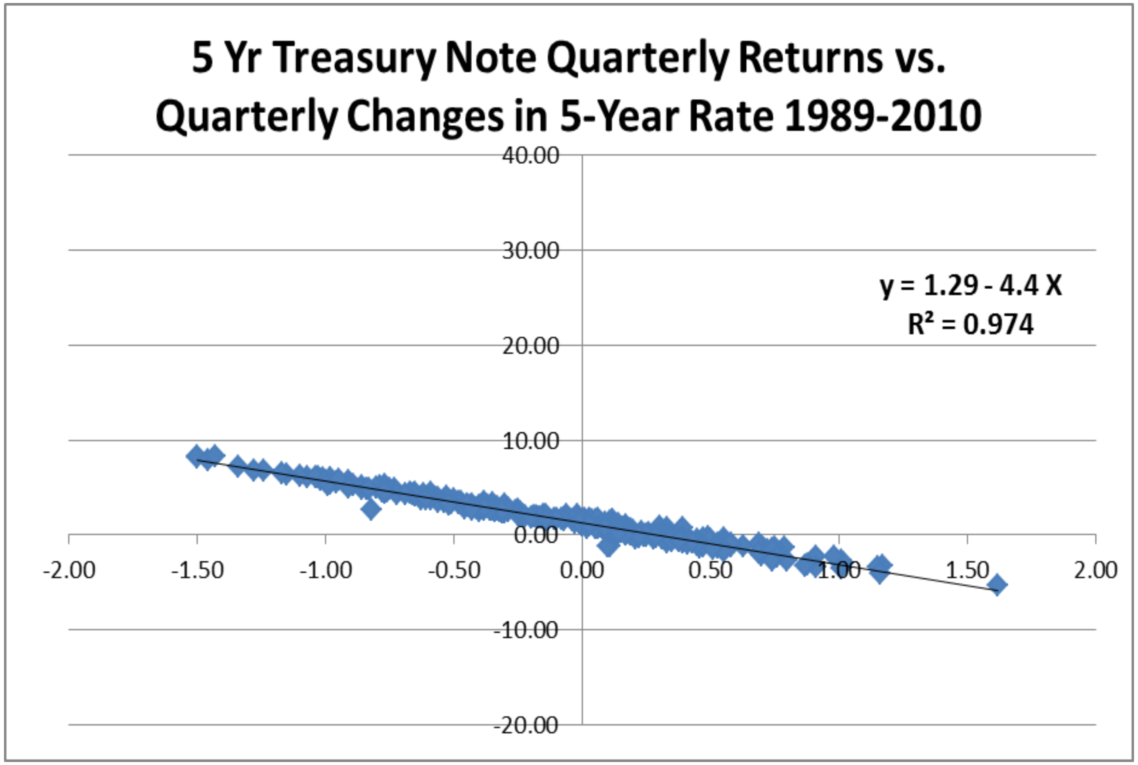
$$D = \sum_{t=1}^{T} \left(\frac{PV(CF_t)}{B} \right) t = \frac{1}{B} \sum_{t=1}^{T} \left(\frac{CF_t}{(1+y)^t} \right) t$$

- Intuitive interpretation – center of gravity of payment tenors.
- Macaulay duration is proportional to Modified duration

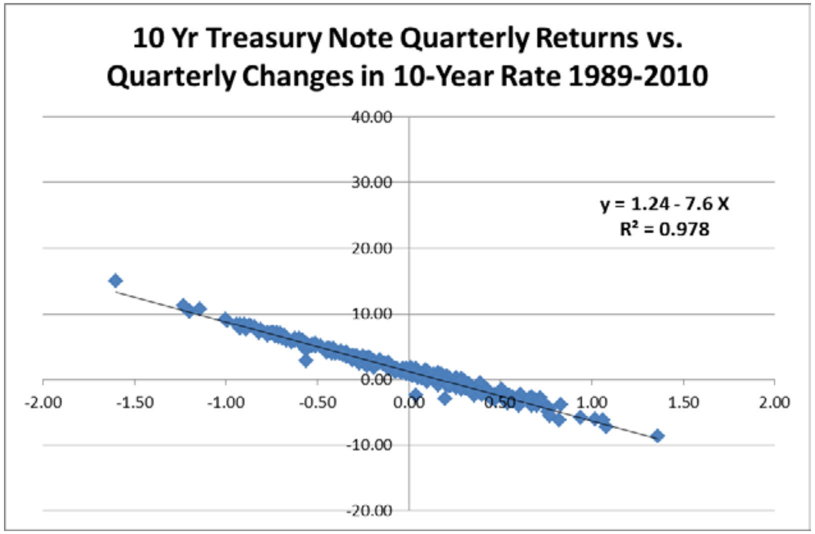
$$MD = \frac{D}{1+y}$$

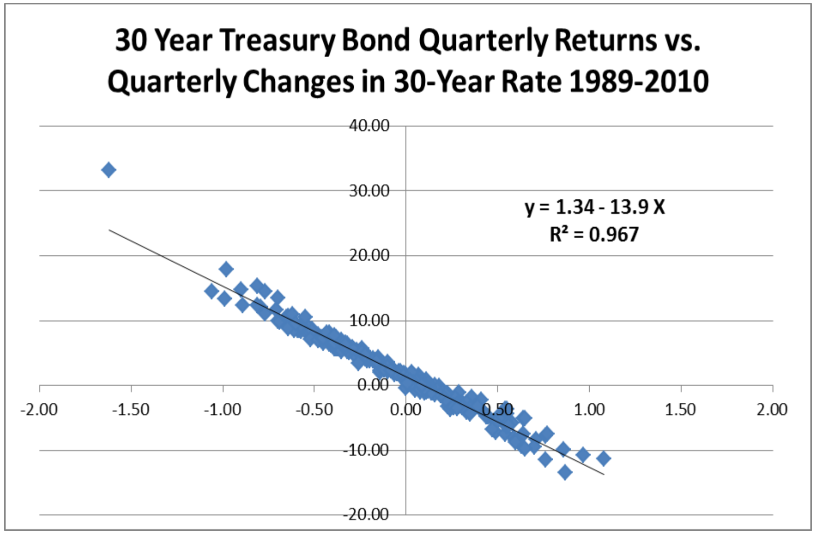
Empirical example

- Regress returns of T-year bonds on changes in T-year interest rate.
- Consider 5, 10, and 30-year Treasury bonds.
- Returns capture price changes in response to interest rate changes: higher slope coefficient indicates longer bond duration.



Empirical example, continued





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Example

- Consider a 4-year T-note: face value $100 and 7% coupon, selling at $103.50.

t	CF	PV	$t \times PV(CF)$
1	3.5	3.40	3.40
2	3.5	3.30	6.60
3	3.5	3.20	9.60
4	3.5	3.11	12.44
5	3.5	3.02	15.10
6	3.5	2.93	17.60
7	3.5	2.85	19.95
8	103.5	81.70	653.60
		103.50	738.28

- For T-notes, coupons are paid semi-annually.
- Use 1/2 year (6 months) as time unit.

Example, continued

- The bond yield (semi-annual) is 3% -- recall that 1 period corresponds to 6 months.
- Macaulay duration: $D = 738.28/103.50 = 7.13$ in $1/2$ year units.
- Modified duration: $MD = D/(1+y) = 7.13/1.03 = 6.92$.
- If the semi-annual yield moves up by 0.1%, the bond price decreases roughly by 0.692%.
- What if interest rates move by a larger amount – would duration still capture changes in the bond price accurately?

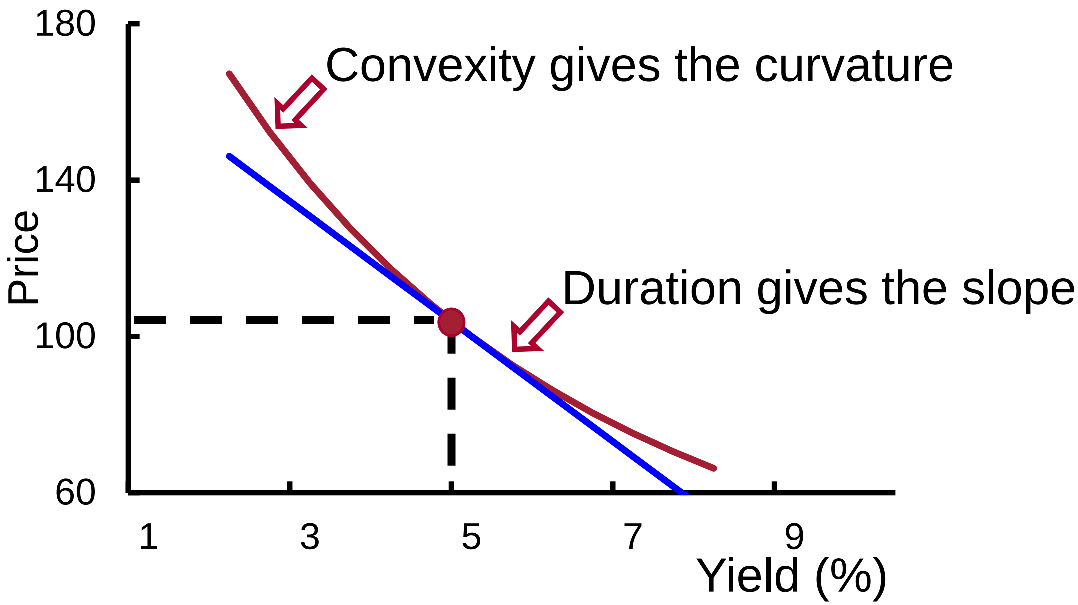
Example, continued

- For small yield changes, pricing by MD is accurate.
- For large yield changes, pricing by MD is inaccurate.

Yield per 6-month period

Yield	Price	Using MD	Difference
0.040	96.63	96.35	0.29
0.035	100.00	99.93	0.07
0.031	102.80	102.79	0.00
0.030	103.50	–	–
0.029	104.23	104.23	0.00
0.025	107.17	107.09	0.08
0.020	110.99	110.67	0.32

Duration and convexity



Convexity, definition

- Bond price is not a linear function of the yield. For larger yield changes, the effect of curvature (i.e., nonlinearity) becomes important.

$$
\begin{aligned}
(\Delta B) &= \frac{\mathrm{d}B}{\mathrm{d}y}(\Delta y) + \frac{1}{2}\frac{\mathrm{d}^2 B}{\mathrm{d}y^2}(\Delta y)^2 + \cdots \\
&\approx [-MD \times (\Delta y) + CX \times (\Delta y)^2] \times B
\end{aligned}
$$

- Convexity (CX) measures the curvature of the bond price as a function of the yield:

$$
CX = \frac{1}{2}\frac{1}{B}\frac{\mathrm{d}^2 B}{\mathrm{d}y^2}
$$

Example

- 10-year bond with 10% coupon and 10% flat yield curve (semi-annual periods: coupon of 5% and yield of 5% per 6-month period).
- Modified Duration: $MD = 6.23$.

6-month yield change is 1/2 of annual yield change

	Annual yield changes (bps)								
	-400	-300	-200	-100	0	100	200	300	400
Price changes	6-month yield change is 150 bps								
Total	29.8	21.3	13.6	6.5	0.0	-6.0	-11.5	-16.5	-21.2
Due to duration	24.9	18.7	12.5	6.2	0.0	-6.2	-12.5	-18.7	-24.9
Due to convexity	4.2	2.4	1.1	0.3	0.0	0.3	1.1	2.4	4.2
Residual	0.6	0.2	0.1	0.0	0.0	-0.0	-0.1	-0.2	-0.5

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Inflation risk

- Most bonds deliver nominal payoffs.
- In the presence of inflation risk, real payoffs are risky even when nominal payoffs are safe.
    - Inflation is the rate of change of the nominal price level.
    - As the price level rises, the real value of a dollar falls – purchasing power declines.
    - Real returns of bonds, all else equal, are inversely related to changes in the price level.

Example

- Suppose that inflation next year is uncertain ex ante, with equally possible rate of 10%, 8% and 6%.
- The real interest rate is 2%.
- The 1-year nominal interest rate will be approximately 10%.
- Real return from investing in a 1-year Treasury security depends on realized inflation.

Year 0 value	Inflation rate (%)	Year 1 nominal payoff	Year 1 real payoff
1,000	0.10	1,100	1,000
1,000	0.08	1,100	1,019
1,000	0.06	1,100	1,038

Valuation of riskless cash flows

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