Chapter 2
Static Consumption-Labor Framework

In our review of consumer theory, we have simply assumed that an individual has some given amount of "labor income," which we denoted by Y, to spend on consumption goods. Doing so allowed us to focus attention on the tools and principles of consumer theory.

Economics is at its core a set of theories about decision-making, and casual reflection reveals that individuals **do** have some control over how much **labor income** they earn. That is, at least to some degree, individuals "choose" how much income they earn just as they choose how much, say, good 1 and good 2 they consume. We now extend our model of consumer theory to incorporate this feature of individual decision-making. As we will see, the tools of analysis and general principles of this extended model are ones with which we are already familiar – simply the tools of indifference curves and budget constraints. To simplify our introduction to this topic, we will use a "one-shot" model in which the individual has no savings decision to make – that is, there is no future, so that the only economic decisions to be concerned with are the present. After we understand how the one-period consumption-leisure framework works and we later study the consumption-savings framework, we will bring the two analyses together to complete our analysis of macroeconomic consumer theory.

In addition to considering the structure of the **consumption-labor framework (alternatively and equivalently referred to as the consumption-leisure framework)** and to get our feet wet with government policy effects, we will embed within it from the start a consideration of government tax policy. We will have much more to say later about the role of macroeconomic tax policies. As we will see, one of the major schools of tax policy thought to have emerged in the past 30 years crucially hinges on the main features of the consumption-leisure model.

The Two "Goods": Consumption and Leisure

In our initial look at consumer theory, we supposed that there existed two broad categories of consumption goods, "good 1" and "good 2." We will now condense these two categories into just a single category called "consumption." That is, consumption is any and all "stuff" (goods and services) that individuals might purchase in order to obtain utility (happiness). Thus, consumption, which we will denote by c (without any subscripts), is an argument to individuals' utility functions.

Because we are interested in studying how consumers "choose" their income, we must specify how consumers in fact earn their incomes. One seemingly obvious way of proceeding is to suppose that consumers obtain their income by working. An individual can choose to work some number of hours (per day or per week or per month, etc – we will specify this more carefully below) for which he receives **pre-tax pay** of W dollars per hour. That is, W dollars per hour is the individual's gross wage rate, which in general is **not** what the individual actually gets to keep as the result of his efforts. In most countries, individuals are subject to a variety of government taxes – of the many kinds of taxes which exist, the most common type (and certainly the type to which the greatest number of people are subject) are income taxes. Income taxes in the U.S. and essentially all other countries are specified as some percentage of an individual's total earnings. For example, if the labor tax rate in the U.S. were 30 percent and an individual earned \$50,000 in a given year, the amount of tax he would have to pay that year is \$50,000 x 0.30 = \$15,000.[14] Thus, it is as if the hourly wage W is subject to a 30 percent tax, making the individual's **after-tax wage rate** $0.70W$ dollars per hour. More generally, if we denote the tax rate on labor income by t (where $0 \leq t \leq 1$), then the after-tax wage rate is $(1-t)W$ dollars per hour. Because it is ultimately disposable income that individuals care about, $(1-t)W$ is the relevant wage rate for an individual's decision-making.

Presumably, working is a "bad" for individuals – that is, individuals dislike working because it reduces their total utility. In order to fit our model into standard consumer theory, we can easily recast the "bad" of working into a "good" by defining **leisure** to be the total number of hours an individual has available to him during some relevant period of time minus the total number of hours he spends working during that period.

For example, suppose we were to consider each calendar week as a distinct period of time. If n is the number of hours in a week that an individual spends working, then, because there are 24 x 7 = 168 total hours in a week, the individual's hours spent in leisure, which we will denote by l, is $l = 168 - n$.[15]

Instead, suppose we think of a distinct period of time as one calendar month, which has roughly 30 days. Then, all of the hours spent in either work or leisure would = 720 hours. That is, $n + l$ = 720 (= 24 x 30) hours.

Alternatively, if we think in terms of one calendar year, then $n + l$ = 8,760 (= 24 x 365) hours.

You get the point.

Indeed, we could think of any of these timeframes as "one distinct period of time." So instead of writing 168 or 720 or 8,760 (which is itself quite cumbersome to write over and over), we will **normalize the hours available during a given time period to one.**

[14] The calculation of an individual's tax burden is not nearly so straightforward in reality due to a great many complicating features of tax laws. However, for our purposes this simple example will suffice.

[15] Notice that because of our definition, leisure should not be thought of as time spent "having fun." Rather, it is time spent **not working.** Thus, items like time spent sleeping, time spent watching TV, time spent cooking and cleaning at home, time spend taking care of children, and so on, all counts as "leisure." The American Time Use Survey (ATUS), a survey conducted by the U.S. Bureau of Labor Statistics (BLS) provides many more categories of how people spend their 24 hours per day (http://www.bls.gov/tus/).

This "one" unit of total hours is a stand-in for 168, 720, 8,760, or whatever frequency of "time" you consider best.

Thus, in our framework, the representative consumer makes optimal choices so that

$$n + l = 1$$

must be true. Here, *n* should be thought of as the **percentage** of the one unit of time the individual works, and hence in turn *l* should be thought of as the **percentage** of the one unit of time the individual spends in leisure. We will from here on colloquially refer to *n* to *l* as "hours spent working" and "hours spent in leisure," respectively.

The above is all about time accounting. Getting back to the framework, leisure is the opposite of working. Because working is a **bad,** leisure must be a **good.** We thus postulate leisure to be the second argument to the representative individual's utility function.

Indifference Map for Consumption and Leisure

The two objects in our model from which an individual obtains utility are thus consumption and leisure, giving rise to an abstract utility function $u(c,l)$. We will refer to both consumption and leisure from here on as "goods," even though clearly leisure is not a tangible object. Consumption and leisure as we have defined them are very broad categories of goods. As such, it is most useful to think of the general properties of the utility function $u(c,l)$ as being the same as those of the utility function $u(c_1, c_2)$ when we first studied consumer theory. Thus, we assume from now on the following properties:

1. Utility is always strictly increasing in consumption (that is , $\partial u / \partial c > 0$).

2. Utility is always strictly increasing in leisure (that is, $\partial u / \partial l > 0$).

3. Utility exhibits diminishing marginal utility in consumption (that is, $\partial^2 u / \partial c^2 < 0$).

4. Utility exhibits diminishing marginal utility in leisure (that is, $\partial^2 u / \partial l^2 < 0$).

Indeed, these are exactly the same properties of the utility function that we have already studied. With these assumptions, we can construct an indifference map over consumption and leisure, as illustrated in **Figure 9**. Each indifference curve has all the usual properties we initially encountered in our study of consumer theory. Specifically, each indifference curve is downward-sloping, is bowed-in towards the origin, and crosses no other indifference curve.[16]

Although our two goods consumption and leisure are not both "market goods" (that is, one cannot really "purchase leisure" in a market), there is still a well-defined notion of a **marginal rate of substitution (MRS)** between the two. Again as usual, the MRS measures how many units of one good the consumer is willing to give up to get one more unit of the other good. Graphically, the MRS is the slope of the indifference curve.

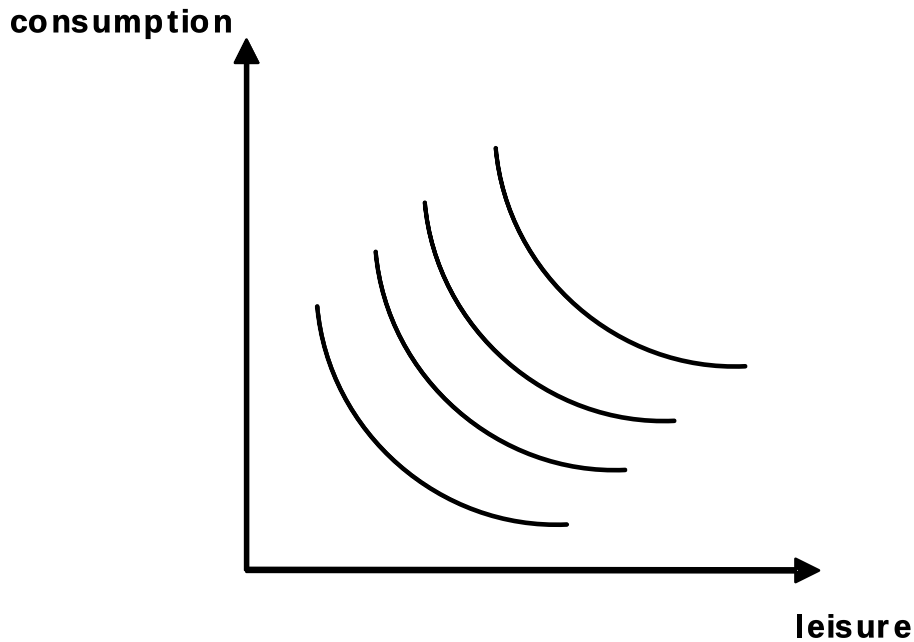


Figure 9. Indifference map defined over consumption and leisure.

Budget Constraint

Indifference maps alone are of course not enough to study an individual's optimal choice. To study optimal decision-making, we need to consider the individual's budget constraint, and it is here where our model of consumption and leisure most differs from the simple model of consumer theory we initially studied. In our simple model, a consumer simply had the income Y to spend on consumption (of good 1 and good 2). Here, the amount of income an individual has to spend on consumption (of the *one* market good) depends on how much he chooses to work. Let us now study formally the budget constraint in this model, reminding ourselves that the length of one period of time in our model is one unit (that is, recall that $n + l = 1$).

[16] At this point, this should all be review. Recall especially that these three properties of indifference curves arise precisely because of our four assumptions on the utility function.

We assume the individual can work as few or as many hours as he wants. Regardless of how many hours he works, he gets paid the pre-tax wage W dollars per hour.[17] As mentioned above, though, it is ultimately his after-tax income (his disposable income) that an individual cares about – his after-tax wage rate is $(1-t)W$ dollars per hour. Because he will choose to work n hours per week, his total disposable income is simply

$$Y = (1-t)\cdot W \cdot n .$$

Because $n = 1 - l$, we can write nominal disposable income as a function of leisure,

$$Y = (1-t)\cdot W \cdot (1-l) .$$

As in our earlier study, we make the simplifying assumption that the individual spends all of his income on consumption and saves nothing for the future. Each unit of consumption c can be purchased at the market price P (the individual is a price-taker). Thus, the individual's consumption each period (week) is

$$Pc = Y .$$

Combining the last two expressions yields the budget constraint in the consumption/leisure model,

$$Pc = (1-t)\cdot W \cdot (1-l) .$$

In this budget constraint, the consumer takes as given the nominal price P, the hourly nominal wage rate W, and the tax rate t,[18] and he chooses his level of consumption c and hours of leisure l.

A useful rearrangement of this budget constraint is

$$Pc + (1-t)Wl = (1-t)W .$$

In this version with both the consumption good and leisure on the left-hand side, we see that the "price" of leisure is the after-tax wage rate $(1-t)\cdot W$. Of course, leisure (time off from work) is not directly bought and sold in markets. But the wage is the opportunity cost of leisure – every hour spent in leisure is an hour that could have been spent working. Thus, from an economic point of view, where we explicitly take account of

[17] Clearly the assumption of being able to work as few or as many hours as one wants does not capture reality literally. Most workers have some semi-fixed schedule they must adhere to, at least in some relevant "short-run." In a "longer-run," workers are freer to move to jobs that better accommodate their lifestyles, etc. Thus, think of our consumption/leisure model as more of an attempt to capture this latter sense rather than the former sense.

[18] That is, the individual is a price-taker in both the consumption-good market as well as the labor market.

opportunity costs, the after-tax wage is the price of leisure because it is what is being given up for every extra hour of leisure taken.[19]

As always, a budget constraint describes the set of choices which are available to the consumer but does not tell us anything about which point in that set he will choose. To graph this budget constraint in a diagram like **Figure 9**, we can rearrange again to get

$$c = \frac{(1-t)\cdot W}{P} - \frac{(1-t)\cdot W}{P}\cdot l.$$

This budget constraint is a straight line, as shown in **Figure 10**, with vertical intercept $\frac{(1-t)W}{P}$ and slope $\left(\frac{-(1-t)W}{P}\right)$. By inserting the value $c = 0$ in the budget constraint, we find that the horizontal intercept is at $l = 1$, which simply states that if the individual wants no consumption, he can use all the hours in a week for leisure.

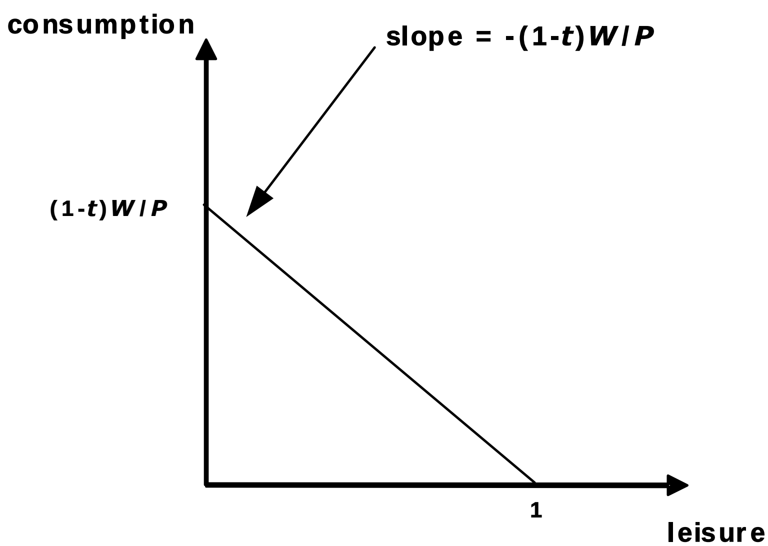


Figure 10. The budget line in the consumption-leisure model.

In our earlier analysis, in which consumers took their income Y as a constant, changes in income led to parallel shifts of the budget line. In the consumption-leisure framework, it is not income that individuals treat as a constant, but rather the after-tax wage rate $(1-t)W$. Notice how the after-tax wage rate enters the budget constraint here. Any change in the after-tax wage rate leads to a rotation of the budget constraint around the horizontal intercept (which recall is fixed at l = 1) because $(1-t)W$ appears in both the vertical intercept and the slope of the budget line.

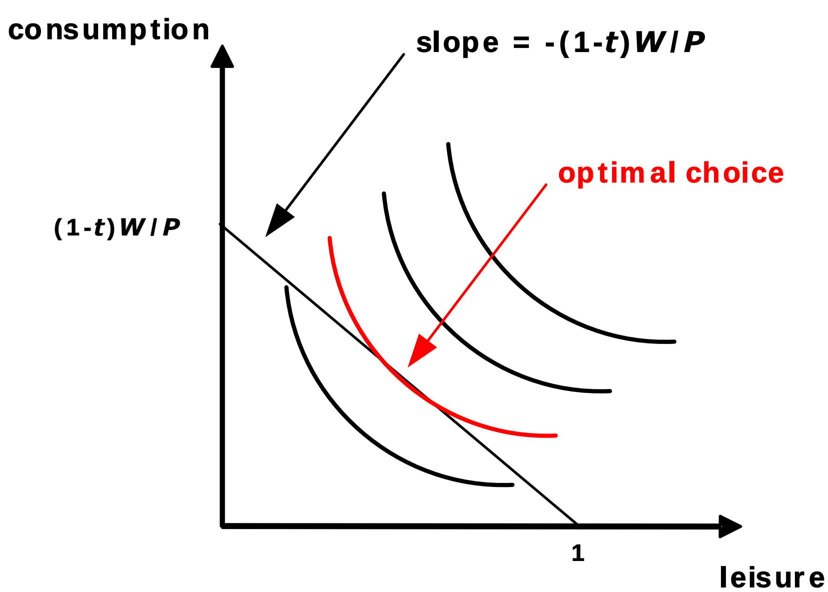
[19] This is a very general notion of a "price." A price is anything that must be given up in order to obtain something else.

Optimal Choice

As always, to consider optimal choice we must consider the interaction of the individual's preferences (indifference map) with his budget constraint. Superimposing the budget line and the indifference map, we have that the optimal choice of consumption and leisure is as shown in **Figure 11**.

Labor Supply Function

When the individual optimally chooses to spend l hours of his time in leisure, he is of course choosing to spend $n = 1 - l$ hours of his time working. He is thus supplying n hours of labor to the labor market. Clearly, the optimal choice of labor in **Figure 11** depends on the after-tax wage $(1-t)W$. A definition is in order before proceeding: the **real after-tax wage** is the after-tax wage in money terms divided by the price of consumption in money terms. With our notation, the real after-tax wage is simply the ratio $\frac{(1-t)W}{P}$,[20] and we see that it is only the real after-tax wage that matters for the vertical intercept and slope of the budget line. For the rest of this section, we will study how the optimal choice of labor varies as the real after-tax wage varies.



[20] The term "real after-tax wage" comes from the units of measure associated with $(1-t)W/P$. Because the units of $(1-t)W$ is (\$ / hour of work) and the units of P is (\$ / unit of good), the units of $(1-t)W/P$ is (units of goods / hour of work), and hence the terminology: $(1-t)W/P$ measures the number of actual (real) goods a worker earns for each hour of work after he has paid his taxes. This is yet another example of how unit analysis helps us think about the relationship between economic variables.

Figure 11. At the optimal choice of consumption and leisure, the budget constraint is tangent to an indifference curve.

We begin our analysis by supposing that the initial real after-tax wage is quite low. Denote this initial wage by $((1-t)W/P)_1$. At this low initial real after-tax wage, the optimal choice is labeled point A in **Figure 12**. This initial optimal choice has associated with it n_1 hours of work (not shown of course because the axes contain c and l, not n). Now suppose that with the price P held constant, the nominal after-tax wage rises to $((1-t)W)_2$, so that the new after-tax real wage is $((1-t)W/P)_2$. **Notice that there are two ways the nominal after-tax wage rate can rise:** the gross wage W can rise while the tax rate remains constant or the tax rate can fall while the gross wage W remains constant. Regardless of which mechanism by which it occurs, the rise in the real after-tax wage causes the budget line to become steeper by pivoting around the horizontal intercept. With this higher real after-tax wage, the individual's optimal choice is point B. At point B, the individual has more consumption than at point A. The individual also enjoys less leisure at point B than at point A – which means that he now works n_2 hours, with $n_2 > n_1$. In other words, n has risen as the real after-tax wage has risen from $((1-t)W/P)_1$ to $((1-t)W/P)_2$.

An important note is in order here. You may be looking at **Figure 12** and wondering why the optimal choice under the higher real after-tax wage did not feature more consumption **and** more leisure. In other words, you may be wondering why the indifference map is not such that the new optimal choice lies to the northeast of the original optimal choice, rather than northwest as is drawn. The answer is not at all a theoretical one, but rather one due to evidence about the real world. Much microeconomic and behavioral research has shown that when individuals currently have a low real after-tax wage, an increase in the real after-tax wage induces them to work more (presumably because, for example, they simply need the earnings to meet basic expenses). This suggests the partial indifference map in **Figure 12**.

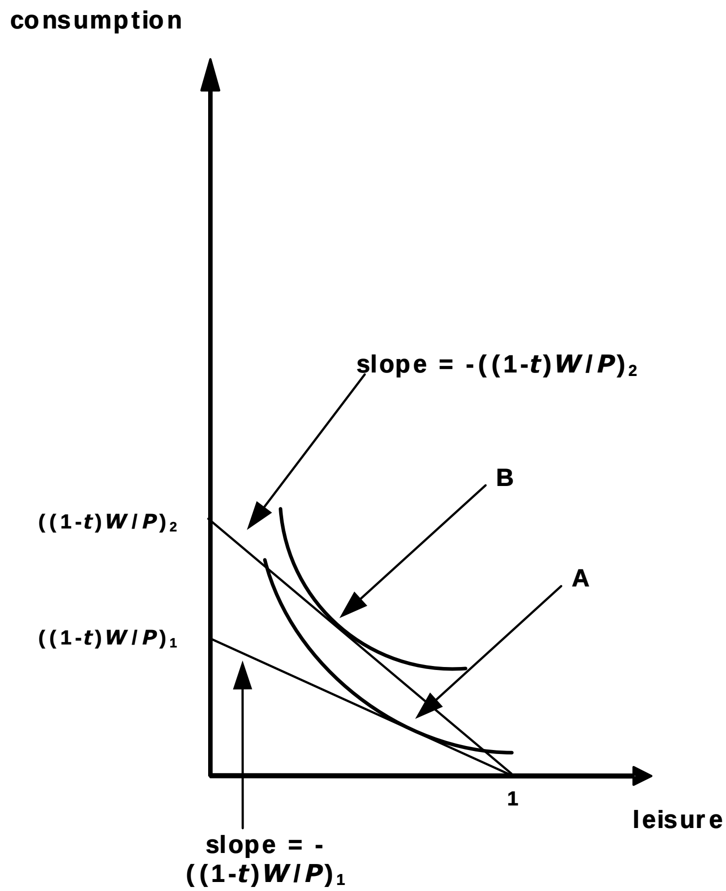


Figure 12. As the real after-tax wage rises from $((1\text{-t})W/P)_1$ to $((1\text{-t})W/P)_2$, the individual optimally chooses more consumption and less leisure – the latter implying that he chooses to work more.

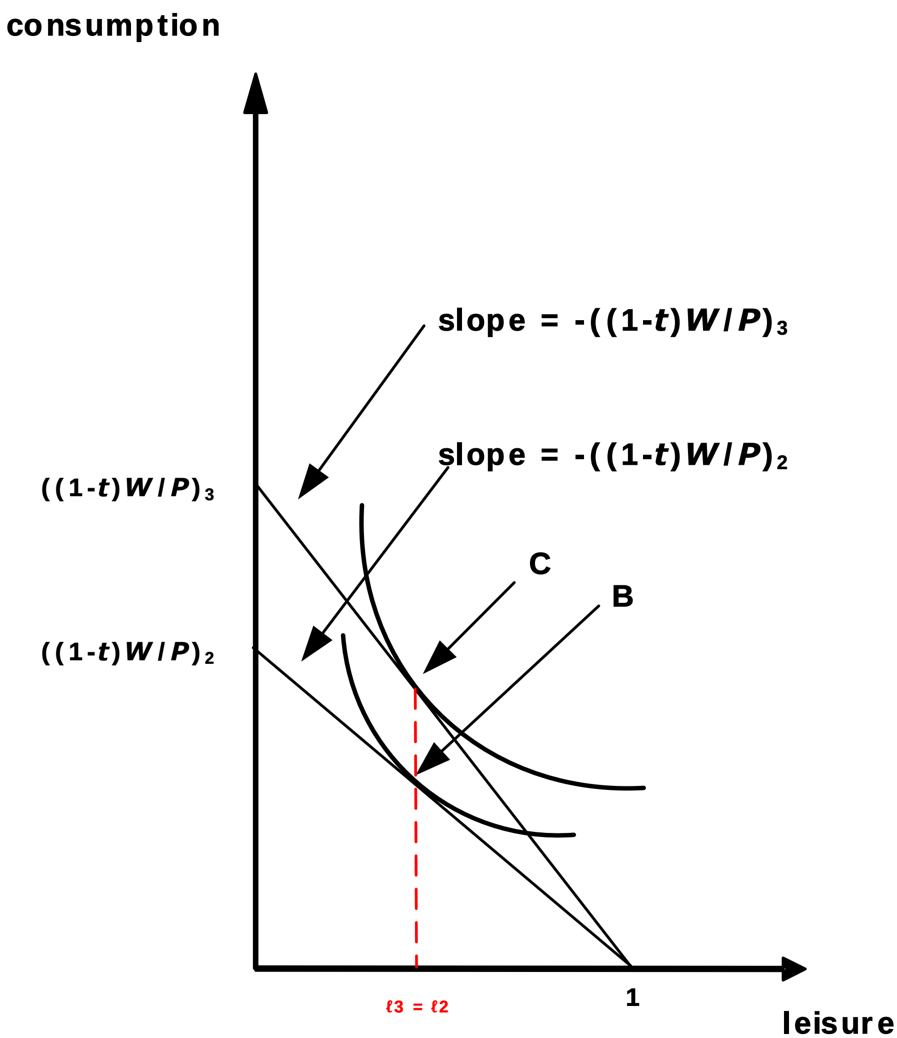


Figure 13. As the real after-tax wage rises from $((1\text{-}t)W/P)_2$ to $((1\text{-}t)W/P)_3$, the individual optimally chooses more consumption but an unchanged amount of leisure – the latter implying that he chooses to not adjust his hours worked.

Now suppose with the price P still held constant, the nominal after-tax wage rises again, to $((1-t)W)_3$. Thus, the real after-tax wage has now risen to $((1-t)W/P)_3$. The optimal choice at this new higher real after-tax wage is labeled point C in **Figure 13.** Comparing point C to point B, we see that the individual has chosen to not adjust the number of hours he works (and thus also not adjust the amount of leisure time he enjoys) when the real after-tax wage rose from $((1-t)W/P)_2$ to $((1-t)W/P)_3$. Thus, at this higher real after-tax wage, the individual is working n_3 hours, with $n_3 = n_2 > n_1$.

Consider yet another increase in the nominal after-tax wage, to $((1-t)W)_4$, which has associated with it the new real after-tax wage $((1-t)W/P)_4$. At this point, the real after-tax wage has gotten quite high and it may be that the individual simply does not need to spend more time working because his basic expenses (and perhaps even some luxuries) have already been met. At a very high after-tax wage, it may be reasonable to expect that the individual will now choose to spend **less** time working and spend more of his time in leisure. Such a situation is depicted in **Figure 14**, in which the rise in the real after-tax wage to $((1-t)W/P)_4$ induces the optimal choice to move from point C to point D. At point D, the individual is working fewer hours than at point C. That is, hours worked n_4 given the real after-tax wage $((1-t)W/P)_4$ is smaller than hours worked n_3 given the real after-tax wage $((1-t)W/P)_3$.

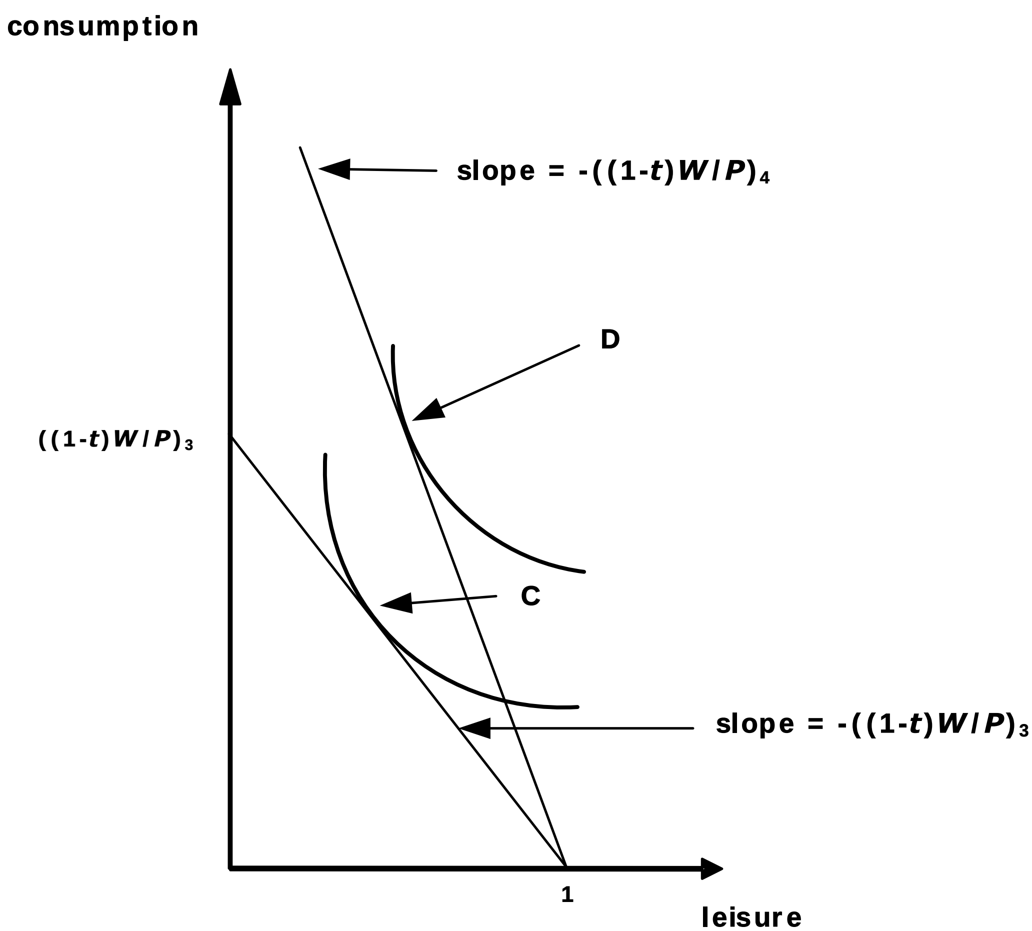


Figure 14. As the real wage rises $((1\text{-}t)W/P)_3$ to $((1\text{-}t)W/P)_4$, the individual optimally chooses more consumption and **more** leisure – the latter implying that he now chooses to work less.

To re-emphasize a point made above, notice there is no **theoretical** reason why the optimal choices of the individual should change in the way depicted in **Figure 12** through **Figure 14** when the real after-tax wage rate rises. Rather, such a description is justified on the grounds of evidence about how individuals do actually seem to respond to changes in their real after-tax wages.

Substitution Effect and Income Effect

We can decompose the effect of the change in the real after-tax wage on the optimal choice of leisure into two separate components: an effect called the substitution effect and an effect called the income effect. Both of these effects have very general meanings in economics and indeed can be applied to any optimal choice problem, not simply the consumption-leisure model. However, for our purposes, we will restrict our discussion of these effects to the consumption-leisure model.[21]

Simply put, in the context of our consumption-leisure model, the **substitution effect of a higher real after-tax wage leads an individual to take less leisure (and hence work more).** This is because as the real after-tax wage rises, the opportunity cost of leisure rises (because they are one and the same). As this "price" of leisure rises, an individual would tend to demand less leisure – simply because leisure has become more expensive!

Conversely, the **income effect of a higher real after-tax wage leads an individual to take more leisure (and hence work less).** This is due to the higher income that a higher real after-tax wage tends to bring.[22] **With a higher income, an individual would want to consume more of all normal goods.[23]** So long as leisure is a normal good, an increase in income would lead an individual to want to take more leisure, and thus spend less time working.

The substitution effect and income effect are both always present. From the preceding discussion, it should be clear that they have opposing effects on an individual's optimal choice of leisure (and hence opposing effects on an individual's optimal choice of labor). For any given real after-tax wage and subsequent rise in the real after-tax wage, then, one of two things must occur. Either the substitution effect dominates (is stronger than) the income effect and the rise in the real after-tax wage leads the individual to choose to work more (take less leisure), or the income effect dominates (is stronger than) the substitution effect and the rise in the real after-tax wage leads the individual to choose to work less (take more leisure).

With these notions of substitution and income effects, let us reconsider the events depicted in **Figure 12** through **Figure 14**. The rise in the real after-tax wage from $((1-t)W/P)_1$ to $((1-t)W/P)_2$ led the individual to work more (take less leisure), as illustrated in the move of the optimal choice from point A to point B. Thus, it must be that over this range of the real after-tax wage, the substitution effect dominates (is stronger than) the income effect.

When the real after-tax wage rose again from $((1-t)W/P)_2$ to $((1-t)W/P)_3$, the individual decided to not adjust the amount of time he spent working, as illustrated in the

[21] And defer a more general discussion of substitution effects and income effects to a more advanced course on microeconomic theory.
[22] Note the distinction between "wage" and "income." The wage is the hourly rate of pay, while income is the product of the wage and the actual number of hours worked.
[23] Recall that this is in fact the definition of a normal (as opposed to an inferior) good.

move of the optimal choice from point B to point C. Thus, it must be that over this range of the real after-tax wage, the substitution effect exactly cancels with the income effect.

When the real after-tax wage rose yet again from $((1-t)W/P)_3$ to $((1-t)W/P)_4$, the individual decided to work less (take more leisure), as illustrated in the move of the optimal choice from point C to point D. Thus, it must be that over this range of the real after-tax wage, the income effect dominates (is stronger than) the substitution effect.

The Backward-Bending Labor Supply Curve

Let us now graph this individual's choice of number of hours worked as a function of the nominal after-tax wage, with the price of consumption held constant at some value P. The resulting graph is the individual's labor supply curve. The following table summarizes the labor supply schedule we found above in **Figure 12** through **Figure 14**:

Nominal wage	**Number of hours worked**
$((1-t)W)_1$	n_1
$((1-t)W)_2$	n_2, with $n_2 > n_1$
$((1-t)W)_3$	n_3, with $n_3 = n_2$
$((1-t)W)_4$	n_4, with $n_4 < n_3$

Graphing this data and passing a smooth curve through the points gives the individual's labor supply curve in **Figure 15**. The labor supply curve is said to be "backward-bending" because at high levels of the after-tax wage, the amount of hours worked declines as the after-tax wage rises.

The labor supply curve in **Figure 15** is for a single individual. Every individual in an economy makes a similar labor supply decision, so in principle we have backward-bending labor supply curves for each individual. If the positions of every individual's labor supply curve is the same, then summing these individual labor supply curves horizontally yields an economy's labor supply curve (called the aggregate labor supply curve), which will also be backward-bending.

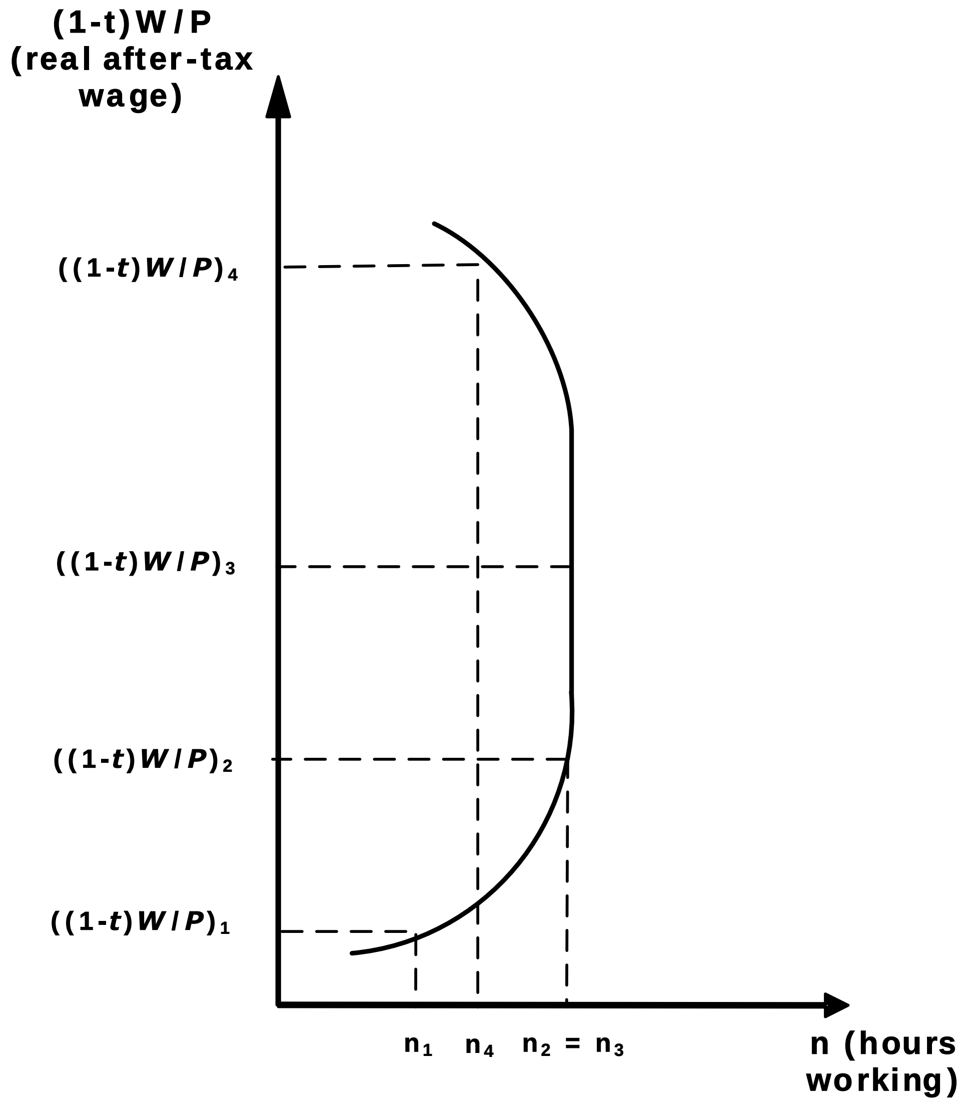


Figure 15. The backward-bending labor supply curve. In this diagram, the price of consumption is held constant at some value P. With this, at very low levels of the nominal after-tax wage, the substitution effect outweighs the income effect and thus the labor supply curve has a positive slope. At very high levels of the nominal after-tax wage, the income effect outweighs the substitution effect and thus the labor supply curve has a negative ("backward-bending") slope. At intermediate levels of the nominal after-tax wage, the substitution effect roughly cancels out against the income effect, giving the labor supply curve its vertical region.

Aggregate Labor Supply Curve

Even if every individual an economy has a backward-bending labor supply curve, though, the *aggregate* labor supply curve actually need not be backward-bending. This can occur if the exact positions of each individual's labor supply functions are *not* identical. More precisely, if we are interested only in some "usual" range of macroeconomic outcomes *and* we wish to model events using the representative agent framework, then our representative agent should *not* have a labor supply curve that is backward-bending. This means that our analysis in **Figure 12** through **Figure 14** must be modified: the successive optimal choices traced out in the progression of these diagrams must feature always decreasing quantities of leisure, which equivalently means always increasing quantities of labor supply. In the terminology of substitution and income effects, we require indifference curves (more fundamentally, a utility function) that features a substitution effect that is always stronger than the income effect with regards to leisure.[24]

Because in macroeconomic data, there is no evidence of a "backward-bending" labor supply curve, the utility functions used in macroeconomic analysis feature just such a property. The particular functional forms for utility we encounter throughout our studies will exhibit this property.

Consumption Demand Function

Regardless of whether the aggregate labor supply curve is backward-bending or not, we derived it by considering how the optimal labor choice varied as the real after-tax wage varied. We can use the same analysis in **Figure 12** through **Figure 14** to consider how the optimal choice of consumption varies as the price P varies, **holding the nominal wage W and the tax rate t constant.** After all, a change in the real after-tax wage rate can be initiated by any one of P, W, or t changing with the other two held constant. In this section, we will suppose that it is the price of consumption which varies and examine how the optimal consumption demand varies.

Begin again at point A in **Figure 12**, and suppose that the price of consumption falls. This means that the budget line rotates to become steeper, just as shown in **Figure 12**. Point B then shows the new optimal choice of consumption, clearly larger than at point A. Turning to **Figure 13**, we see that if the price of consumption falls yet again, the budget line again becomes steeper, leading to a yet higher consumption choice at point C. If the price falls yet again, the budget line becomes even steeper and the optimal consumption choice increases again, as at point D in **Figure 14**. The point should by now be clear: a fall in the price leads to rise in optimal consumption, all else being held constant. Indeed, this is simply the law of demand that you learned in basic microeconomics. This analysis yields the downward sloping aggregate consumption function in **Figure 16** (its lineararity is for illustrative purposes).

[24] You should be able to trace out for yourself the analogs of **Figure 12** through **Figure 14** for the labor supply curve to not be backward-bending.

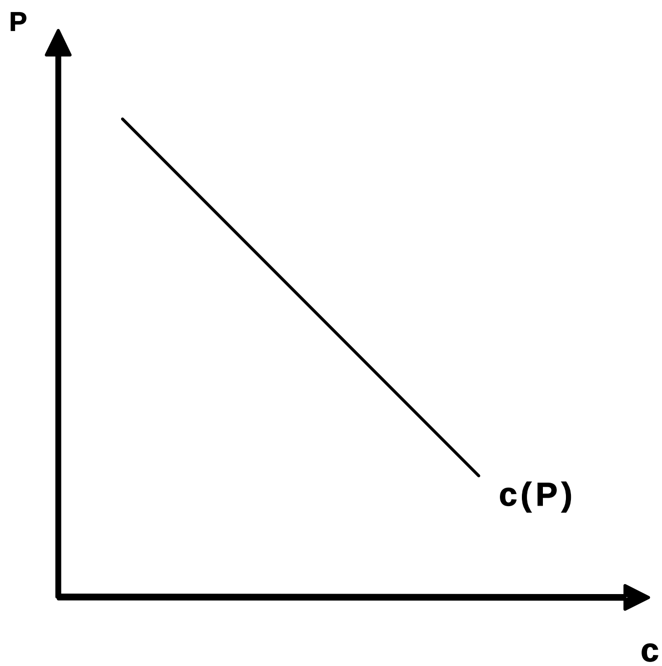


Figure 16. The downward-sloping aggregate consumption function, which is derived holding the labor tax rate t fixed and the nominal wage rate W fixed.

Lagrange Characterization – the Consumption-Leisure Optimality Condition

Let's now return to the decision problem of the representative agent (that is, before we aggregated things up to an aggregate labor supply function and an aggregate consumption demand function) and study the optimization problem our Lagrange tools.

To cast the problem into Lagrange form, we must first identify the objective function (i.e., the function that the consume seeks to maximize) – that is simply the utility function $u(c, l)$. Then, we must identify the constraint(s) on the maximization problem. The only constraint is the budget constraint; to cast it in our general Lagrange form, we write it as $g(c, l) = (1\text{-}t)W - Pc - (1\text{-}t)Wl = 0$. Proceeding as we have a couple of times already now, having identified the objective function and the constraint function(s), we now must construct the Lagrange function; in our problem here, the Lagrange function is

$$L(c,l,\lambda) = u(c,l) + \lambda\left[(1-t)W - Pc - (1-t)Wl\right].$$

The first-order conditions we thus require are those with respect to c, l, and λ; respectively, they are:

$$\frac{\partial u}{\partial c} - \lambda P = 0$$
$$\frac{\partial u}{\partial l} - \lambda(1-t)W = 0$$
$$(1-t)W - Pc - (1-t)Wl = 0$$

The usual second step of the Lagrange method is to eliminate the Lagrange multiplier between the first-order conditions on the main variables of economic interest – here, c and l. Solve the first expression above for the multiplier gives us $\lambda = \frac{\partial u / \partial c}{P}$. Inserting this in the second expression above gives us $\frac{\partial u}{\partial l} = \frac{\partial u}{\partial c}\frac{(1-t)W}{P}$. Rearranging in one more step gives us

$$\frac{\partial u / \partial l}{\partial u / \partial c} = \frac{(1-t)W}{P},$$

which is the representative consumer's **consumption-leisure optimality condition.** It states that when consumers are making their utility-maximizing consumption-leisure choices, they choose consumption and leisure such that their MRS between consumption and leisure (the left-hand-side of the above expression – recall that the ratio of marginal utilities **is** the MRS) is equated to the slope of the relevant budget constraint. As we saw graphically above, the slope of the relevant budget line here is the after-tax real wage $(1-t)W/P$.

Unemployment?

Let's zoom out. In the consumption-leisure framework, is the representative individual employed? Or unemployed?

Given our analysis above of the optimal choice of **hours supplied,** it should be apparent that our representative individual is **employed with 100% certainty.** Keeping in mind the strict representative-agent framework, this means that **every individual is employed with 100% certainty.**

Thus, there is **no unemployment** in this framework, which seems like a major shortcoming.

Clearly, in reality, there are people who would like to work – that is, would like to supply hours – but cannot find a job and hence are **unemployed.** Broadly, the U.S. Bureau of Labor Statistics (BLS) categorizes individuals into the three groups shown in Figure 17.

As the diagram indicates, the red-outlined pools of individuals that are unemployed but actively seeking work and those individuals that are outside the labor force are grouped together into “leisure” in the consumption-leisure framework.

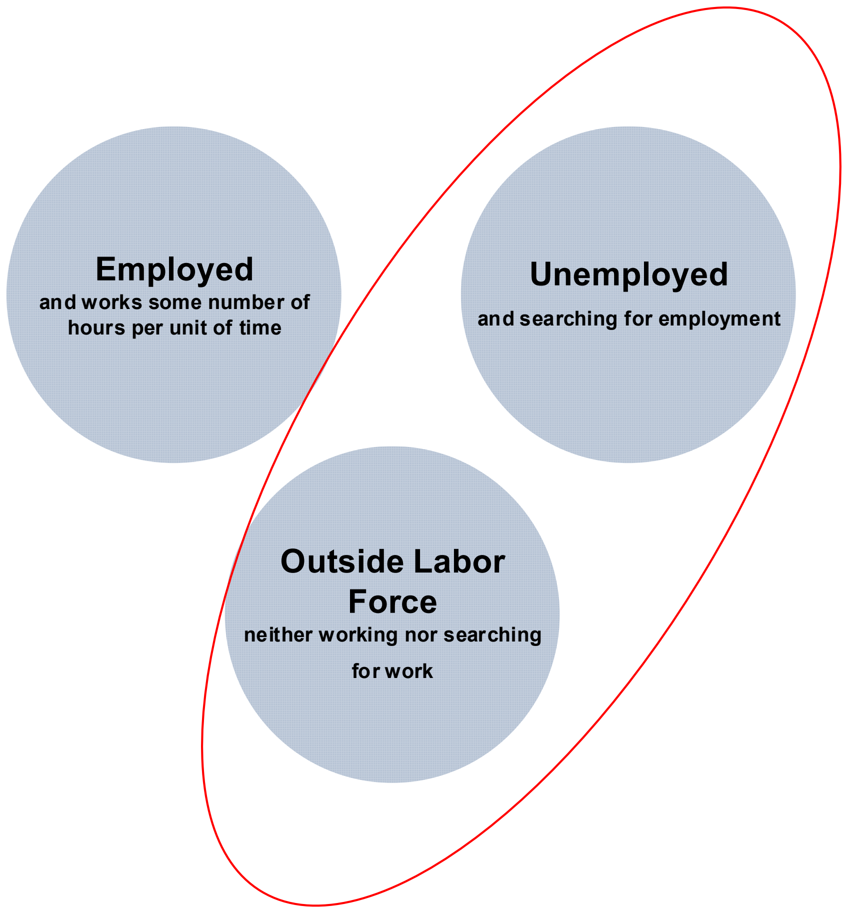


Figure 17. The three widely-recognized categories of an individual’s labor-market status are employment, unemployment but actively seeking a job, and neither working nor searching for a job. The consumption-leisure framework bundles the latter two categories together into "leisure."

Thus, not only is there no notion of “unemployment” in the framework just presented, but there is also no notion of “actively searching” for work. We will later construct an extended version of the consumption-leisure framework that explicitly incorporates time spent searching for work; in that extension, it will **not be the case** that each unit of search activity leads to employment with 100% certainty.

Chapter 3
Dynamic Consumption-Savings Framework

We just studied the consumption-leisure model as a "one-shot" model in which individuals had no regard for the future: they simply worked to earn income, all of which they then spent on consumption right away, putting away none of it for the future.

Individuals do, of course, consider their future prospects when making economic decisions about the present. When an individual makes his or her optimal choices about consumption and leisure in the current period, he/she usually recognizes that he/she will make a similar consumption-leisure choice in the future. In effect, then, it seems there are multiple consumption-leisure choices an individual makes over the course of his/her lifetime.

However, these choices are not independent of each other because **consumers can save for the future or borrow against future income** (borrowing is simply negative savings, also known as **"dissaving"**). That is, current choices affect future choices, and, conversely, (expectations of) future events and choices affect current choices.

In this section, we will focus on the study of **intertemporal** (literally, "across time") choices of individuals. The easiest way to understand the basics of intertemporal choice theory is by first ignoring leisure and labor altogether. That is, we will revert to our assumption that an individual has no control over his or her income.

Rather, we will enrich our model of consumer theory by now supposing that each individual plans economic events for two time periods – the "present" period and the "future" period. We will designate the present period as "period 1" and the future period as "period 2." There is no "period 3" in the economic planning horizon, and every individual knows there is no period 3.[25] This stark division of all time into just two periods will serve to illustrate the basic principles of (macro)economic events unfolding as a sequence over time; after mastering the basics of ***dynamic macroeconomics*** by using the two-period model, we will eventually extend to consideration of an infinite-period model, which arguably may be more realistic because, after all, when does time "end?" But let's build that up slowly.

In the two-period model, our stylized (that is, representative) individual will receive "labor income" (over which he/she has no control) in each of the two periods, and has to make a choice about consumption in each of the two periods. Savings or borrowings are allowed during period 1. The notation we will use here, indeed the entire method of analysis, should remind you of our initial study of consumer theory.

[25] Think of this as meaning that the world (and hence the economy) ends with certainty after two periods.

A Simple Intertemporal Utility Function

As always, in order to study consumer choice, we need to first specify the individual's utility function. In our present intertemporal context, the two arguments to the utility function are consumption in period 1 and consumption in period 2, which we will denote by c_1 and c_2, respectively.[26] We will assume all the usual properties of utility functions: utility is always strictly increasing in both arguments and always displays diminishing marginal utility in both arguments. In abstract form, we (again!) will write this utility function as $u(c_1, c_2)$, and the utility function can be represented by an indifference map featuring downward-sloping indifference curves that are bowed in towards the origin.

In everything that follows, we will continue to write $u(c_1, c_2)$ to stand for the intertemporal, or lifetime, utility function. To dip our feet a bit into macroeconomics, though, a commonly-used intertemporal utility function is

$$u(c_1, c_2) = \ln c_1 + \ln c_2,$$

in which "ln" stands for the natural logarithm. The indifferences curves are plotted in three-dimensional space in Figure 18 and in two-dimensional coordinates in Figure 19. Both Figure 18 and Figure 19 should remind you of basic micro concepts.

[26] With this choice of notation, you can already start to see the parallels between the intertemporal consumption model and our initial study of consumer theory. Keep in mind the different interpretation here though, that of intertemporal choice.

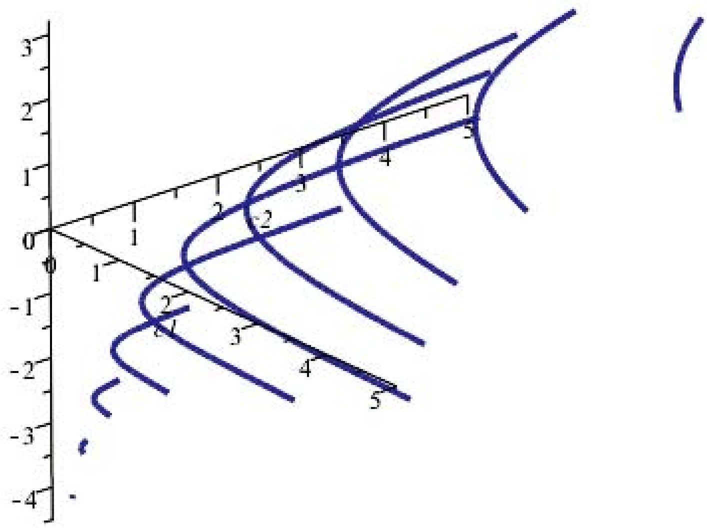


Figure 18. An indifference map of the utility function $u(c_1, c_2) = \ln(c_1) + \ln(c_2)$, where each solid curve represents a given (positive or negative) height above the c_1-c_2 plane and hence a particular level of utility. The three axes are the c_1 axis, the c_2 axis, and the utility axis.

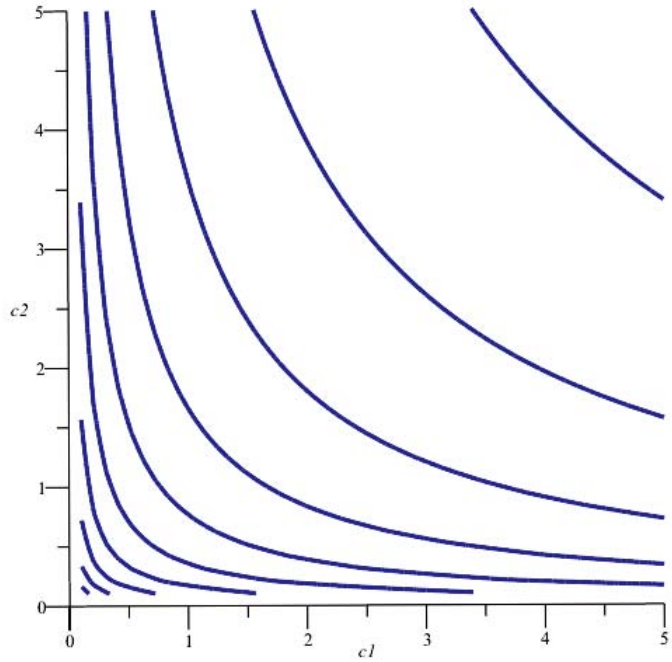


Figure 19. The contours of the utility function $u(c_1, c_2) = \ln(c_1) + \ln(c_2)$ viewed in the two-dimensional c_1-c_2 plane. The utility axis is coming perpendicularly out of the page at you. Each contour is called an indifference curve. Indifference curves further to the northeast are associated with higher levels of utility.

Budget Constraints

The most important way in which the intertemporal consumption model differs from our model of consumer theory heretofore is in the budget constraint(s). Before describing the model further, we need to distinguish between income and wealth, two conceptually different economic ideas.

Income Vs. Wealth

Income is a receipt of money by an individual during some period of time – the most common forms of income are **labor income** (money earned by working) and **interest income** (money earned on assets). On the other hand, an individual's wealth is the level of assets (cash, checking accounts, savings accounts, stock, bonds, etc.) an individual has in store. An individual's wealth may be negative, for example if he is overdrawn on his checking account or otherwise is in debt.

A simple example will illustrate the point. If you currently have \$1,000 in your savings account, an economist would say that you have \$1,000 in wealth. Say your savings account pays three percent interest per year. If you leave your funds in your savings account alone for the next one year (making neither deposits nor withdrawals), at the end of one year you will have $(1+0.03)\cdot\$1,000=\$1,030$ in your account. This amount can be decomposed into \$1,000 of wealth and \$30 of interest income. Suppose during that year you also earned \$10,000 by working – this amount, not surprisingly, we would call your labor income. Thus, your total income during the year is the sum of your labor income and interest income, in this case \$10,030. The \$1,000 still in your savings account is **not** part of your income, although it was the basis of your \$30 of interest income.

Period-by-Period Budget Constraints

Returning to the description of the two-period model: individuals receive labor income twice in their lives – once in period one and again in period two. As we said above, for now, the amounts of labor income are outside the control of the individual. Soon, we will relax this assumption and allow the individual to have some control over how much labor income he earns. In describing the sequence of economic events, we will need to introduce several elements of notation. The individual receives labor income Y_1 dollars at the beginning of period 1. In addition, the individual begins period 1 with some initial wealth (which may be negative), which we denote by A_0 — we make no assertion about where this initial wealth came from (perhaps it was bequeathed to him by his ancestors). Regardless of where this initial wealth (or initial debt if A_0 is negative) came from, in period 1 it becomes available to the individual along with some nominal interest income.

He chooses consumption c_1 in period 1, each unit of which costs P_1 dollars. He also decides how much wealth to carry into period 2. Denote this level of wealth A_1.

To emphasize, A_1 is chosen in period 1 and is the amount of dollars the individual carries with him (in a savings account, say) from period 1 into period 2. **Notice that A_1 may be negative,** just as A_0 may be negative. A negative A_1 means that the individual is in debt at the beginning of period 2. With this notation, we can write down the **period-1 budget constraint** of the individual

$$P_1c_1 + A_1 = (1+i)A_0 + Y_1, \tag{6}$$

where i denotes the nominal interest rate (we will say more about this shortly). An **equivalent** version of the period-1 budget constraint is obtained by subtracting P_1c_1 from both sides, which gives

$$A_1 = (1+i)A_0 + Y_1 - P_1c_1$$

This equivalent expression of the period-1 budget constraint emphasizes that out of all the resources that were available for the first period, A_1, that were **NOT** spent on period-1 consumption and thus carries over to the next period.

At the beginning of period 2, the individual receives nominal income Y_2. If he chose to carry positive wealth A_1 from period 1 into period 2, he receives back (from his bank account, say) the full amount A_1 plus interest earned on that amount. Denote this **nominal interest rate** by i, where $0 \le i \le 1$. For our purposes, t**he nominal interest rate is the return on each dollar kept in a bank account from one period to the next.**

We need to be very clear about the events occurring here, so to re-emphasize: if the individual chose to carry a positive amount A_1 dollars from period 1 into period 2, he receives at the beginning of period 2 his original A_1 dollars plus another iA_1 dollars in interest. On the other hand, if the individual chose to carry a negative A_1 into period 2 (that is, the individual is in debt at the beginning of period 2), he must repay A_1 (to, say, the bank to whom he is in debt) with an interest rate of i – that is, he would repay $A_1 + iA_1$.[27] This nominal interest rate i is the same interest rate that appears in the period-1 budget constraint in expression (6).

[27] For simplicity we are supposing that the interest rate at which the individual can save is the same as the interest rate at which the individual can borrow. In general, this need not and usually is not the case. More generally, we can say that there is an interest rate i_s which the individual would receive if he had a positive level of wealth and a different interest rate i_b which the individual would face if he had a negative level of wealth.

After settling his accounts, the individual then chooses consumption c_2 in period 2, each unit of which costs P_2 dollars. He also decides how much wealth to carry into period 3. Denote this level of wealth by A_2. But the economy ends at the end of period 2 and every individual knows the economy ends at the end of period 2! Thus, there is no period 3 to save for, and no rational bank would allow anyone to die in debt to it – so we must have that $A_2 = 0$.

With this notation, we can write down the **period-2 budget constraint** of the individual:

$$P_2 c_2 + A_2 = (1+i)A_1 + Y_2, \tag{7}$$

where, as we just said, we must have $A_2 = 0$, and A_1 may be positive or negative.

This timing of events is depicted by the timeline in **Figure 20**, which is crucial to understand.

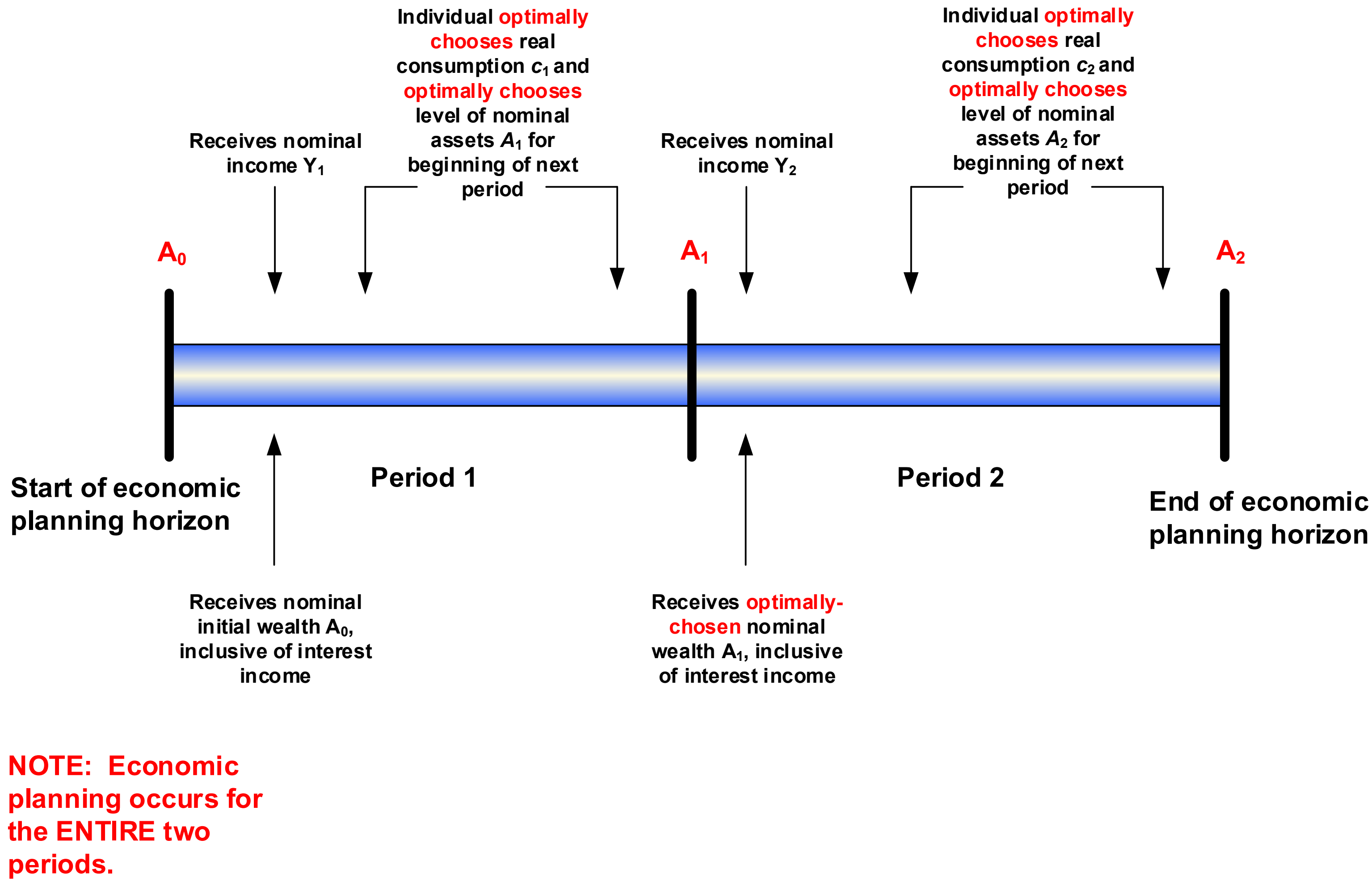


Figure 20. Timing of events in the two-period consumption framework, stated in nominal units.

Before making our next point, we introduce important new terminology. We define an individual's **private savings in a given time period as the difference between his total income in that period and his total expenditures in that period.** The two main categories of expenditures for individuals in any economy are consumption and taxes. We have not yet discussed taxes, but we will soon. Examining the period-1 budget constraint (6) above, we see that the individual's total income in period 1 is $iA_0 + Y_1$ (the sum of his labor income and interest income), and his total expenditure on consumption in period 1 is P_1c_1. Thus, we have that the individual's private savings in period 1 is

$$S_1^{priv} = iA_0 + Y_1 - P_1c_1, \tag{8}$$

where the "priv" superscript indicates that this is the savings of the private individual.[28] If we rearrange expression (6) a bit, we get that

$$A_1 - A_0 = iA_0 + Y_1 - P_1c_1. \tag{9}$$

Comparing expressions (8) and (9), we see that $S_1^{priv} = A_1 - A_0$.

Thus, the private individual's savings in period 1 is equal to the change in his wealth during period 1. This is a second useful way of computing an individual's private savings – as the change in wealth. To re-emphasize, this is a CRITICAL idea to understand, as it is pervasive throughout macroeconomic analysis. At the end of the chapter, we will emphasize this point again with rigorous definitions of "stock variables" and "flow variables."

To continue the savings account example from above, starting from an initial balance of $1,000 if you withdrew $400 from your savings account during the course of one year (and made no deposits), your savings during the course of the year would be $600-$1000 = -$400. That is, you would have dissaved during the year.

Similarly, the private individual's savings in period 2 is $S_2^{priv} = iA_1 + Y_2 - P_2c_2$, which, using the period-2-budget constraint, can also be expressed as $S_2^{priv} = A_2 - A_1$.

Lifetime Budget Constraint

Examining the period-1 budget constraint and the period-2 budget constraint, we see that they are linked by wealth at the beginning of period 2, A_1. Mathematically, this is the only term that appears in both expressions. **The economic interpretation, an important one, is that an individual's wealth position is what links economic decisions of the past with economic decisions of the future.** Again continuing the savings account

[28] Later, we will also have something called "public savings," in which the government engages – we will denote this by S^{gov}.

example from above, the \$1,000 in your savings account somehow reflects your past income and consumption decisions. Obviously, just knowing that you currently have \$1,000 in your savings account does not allow anyone to know exactly what or how much "stuff" you bought in the past or how much income you earned in the past. Nonetheless, it is essentially a summary of your past income and consumption behavior, albeit a condensed one. The fact that you have \$1,000 in your account now implies some level of interest income for you in the upcoming year, income which is available for your consumption needs over the next year. Thus, that \$1,000 is a reflection of your past economic behavior and represents part of your future economic opportunities.

Thus, economic decisions over time are linked by wealth. A useful first approximation to actual economic behavior is to suppose that individuals are completely rational over the course of their lifetimes in the sense that they save and/or borrow appropriately during their whole lifetimes. In the context of our two-period model here, such an assumption amounts to an individual deciding on his consumption and savings for his whole life (i.e., both period 1 and period 2) at the beginning of period 1. This latter point is an important one for the analysis of the two-period model: **all of our analysis of the two-period model proceeds from the point of view of the very beginning of period one.** That is, we will consider the very beginning of period one as the "moment in time" in which our (and the consumer's) analysis is conducted; hence, in our (and the consumer's) analysis of the two-period world, the entire two periods will always be yet to unfold.

Proceeding, then: armed with the assumption of rationality on the part of consumers and the perspective of economic events from the very beginning of period one, it is neither the period-1 budget constraint alone nor the period-2 budget constraint alone that is the relevant one for decision-making, but rather a combination of both of them.[29] The way to combine the budget constraints (6) and (7) is to exploit the observation that A_1 is the only term that appears in both. The mathematical strategy to employ is to solve for A_1 from one of the constraints and then substitute the resulting expression into the other constraint. Doing this will yield the individual's **lifetime budget constraint** – which we will abbreviate **LBC** for short.

Let us proceed by first solving for A_1 in expression (7). After a couple of steps of algebra, we get

$$A_1 = \frac{P_2 c_2}{(1+i)} - \frac{Y_2}{(1+i)}, \tag{10}$$

where we have used the fact that $A_2 = 0$ from above.[30] Inserting this resulting expression for A_1 into the period-1 budget constraint in (6) above yields

[29] Keep this point in mind when we later formulate, two different types of Lagrange problems to analyze the two-period framework.
[30] It is a good idea for you to verify these algebraic manipulations and the ones that follow for yourself.

$$P_1c_1 + \frac{P_2c_2}{(1+i)} = Y_1 + \frac{Y_2}{(1+i)} + (1+i)A_0, \tag{11}$$

which is the LBC. The LBC has very important economic meaning. The right-hand-side of expression (11) represents the **present discounted value of lifetime resources,** which takes into account both initial wealth as well as all lifetime labor income.[31] The left-hand-side of expression (11) represents the **present discounted value of lifetime consumption,** which takes into account consumption in all periods of the individual's life (here, only two periods). Thus, over the course of his lifetime, the individual spends all his lifetime resources on lifetime consumption, leaving nothing behind when he dies (and indeed why should he because, after all, the world ends with certainty at the end of period 2). It is this LBC that our perfectly rational individual uses in making his choices over time. As such, in order to proceed graphically, we need to represent this LBC in $c_1 - c_2$ space.

Before graphing the LBC, we make one simplifying assumption, that $A_0 = 0$, which means the individual begins his economic life with zero initial wealth (and zero initial debts). None of the qualitative results change if we do not make this assumption – it simply makes the graphical analysis to follow more straightforward.

To graph the LBC with c_2 on the vertical axis and c_1 on the horizontal axis, we need to solve expression (11) for c_2, which gives us, after a few lines of algebra,

$$c_2 = -\left(\frac{P_1(1+i)}{P_2}\right)c_1 + \left(\frac{1+i}{P_2}\right)Y_1 + \frac{Y_2}{P_2}. \tag{12}$$

Thus, the vertical intercept is the entire term $\left(\frac{1+i}{P_2}\right)Y_1 + \frac{Y_2}{P_2}$, and the slope is the term $-\left(\frac{P_1(1+i)}{P_2}\right)$. The graph of the LBC is in **Figure 21**.

[31] You should be familiar with the notion of present discounted value from introductory economics – if your recollection is a bit hazy on this point, now is the time to refresh yourself because we will use the concept repeatedly.

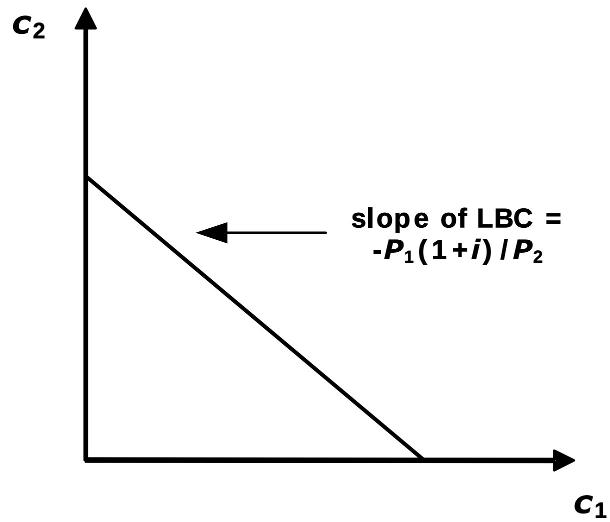


Figure 21. The lifetime budget constraint (LBC) of the individual, with the simplifying assumption that $A_0 = 0$.

Optimal Intertemporal Choice – Consumption and Savings

As in all of consumer theory, the individual's actual optimal choice is determined by the interaction of his budget constraint and his indifference map (i.e., his utility function) – the former represents all of the choices available to him and the latter represents his own personal preferences. **Figure 22** depicts an example, in which the individual's optimal choice is c_1^* in period 1 and c_2^* in period 2.

Also shown in **Figure 22** are the individual's labor incomes in both period 1 and period 2. Actually, what are shown are Y_1 / P_1 and Y_2 / P_2, which represent **real labor income** in the two periods, respectively. We will soon discuss exactly what is meant by this term, but for now just think of it as the labor income we have been discussing all along in this two-period model. We see in **Figure 22** that consumption c_1^* in period 1 is higher than real labor income in period 1 Y_1 / P_1. This individual is spending more in period 1 than he earns, which means that the individual must be decumulating wealth (i.e., borrowing) during period 1. We can see this mathematically by looking at the period 1 budget constraint in expression (6) (and recall our simplifying assumption that $A_0 = 0$). Rearranging that expression a bit gives

$$c_1 - \frac{Y_1}{P_1} = -\frac{A_1}{P_1}. \tag{13}$$

So for the individual in **Figure 22**, the left-hand-side of expression (13) is positive, which must mean that A_1 for this individual is negative. This individual is in debt at the end of period 1. By similar logic and using the period 2 budget constraint in expression (7) we have that

$$c_2 - \frac{Y_2}{P_2} = \frac{(1+i)A_1}{P_2}. \tag{14}$$

We already know that A_1 is negative, implying the left-hand-side of expression (14) must be negative, which is in fact the case looking at **Figure 22**. The reason why consumption is smaller than income in period 2 is because the individual has to repay the loan obligations he took on during period 1. Thus, consumption higher than labor income in one period has to be balanced with consumption lower than income in another period, a result which should strike you as not surprising.

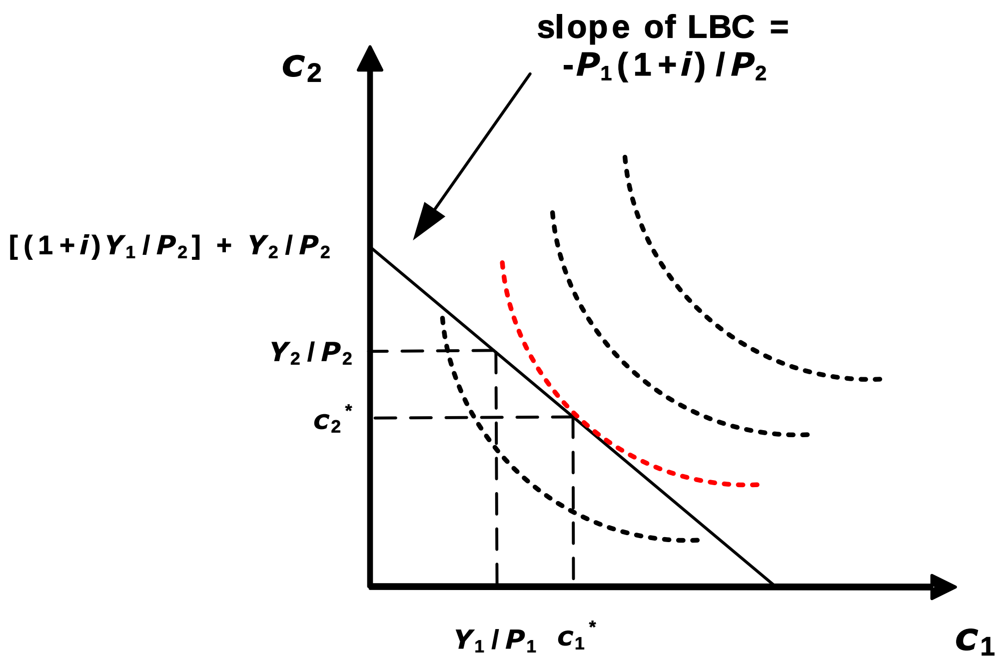


Figure 22. The interaction of the individual's LBC and his/her preferences (represented by the indifference map) determine the individual's optimal consumption over time, here c_1* in period 1 and c_2* in period 2.

One final point regarding the example in **Figure 22**: notice that no mention was made of interest income, only labor income – despite the careful distinction we made earlier between labor income and interest income. The reason for this is that when considering the **lifetime** choices he makes and as long as asset markets are perfectly functioning (we will discuss in more depth the content of this qualifier), the individual can completely disregard interest income because the only reason for the existence of non-zero wealth at the end of any period is simply to transfer resources across time.

When explicitly considering the lifetime decisions of an individual, as we are here, **those "intermediate" wealth positions appear to "completely cancel out."** Specifically, notice that **from a *mathematical* point of view,** A_1 does not appear at all in the LBC in expression (12), and the only relevant income for the individual is that which he receives in period 1 and period 2. **However, from an *economic* perspective, the A_1 net wealth term that links activities across time periods is still present. This is a *critical* point to understand about multi-period economic frameworks – there is some "state of economic conditions" that occurred in the past and have implications for current and future outcomes.**

Stocks vs. Flows

Understanding the two-period model (and as Figure 20 portrays) requires understanding a **critical** conceptual difference between two different types of variables: **stock variables** and **flow variables.** This conceptual difference arises **entirely because of the dynamic nature of the two-period framework.**

Stock Variables (alternative terminology: Accumulation Variables)

Quantity variables whose natural measurement occurs at a particular moment in time
Examples:
- **Checking account balance**
- **Credit card indebtedness**
- **Mortgage loan payoff**
- **College loan balance**

In our two-period model so far, and as displayed in Figure 20, the three stock variables (aka accumulation variables) are A_0, A_1, and A_2.

Flow Variables

Quantity variables whose natural measurement occurs during the course of a given interval of time
Examples:
- **Income**
- **Consumption**
- **Savings**

In our two-period model so far, and as displayed in Figure 20, the six flow variables are c_1, c_2, Y_1, Y_2, S_1, and S_2.

Chapter 4
Inflation and Interest Rates in the Consumption-Savings Framework

The lifetime budget constraint (LBC) from the two-period consumption-savings model is a useful vehicle for introducing and analyzing the important macroeconomic relationships between inflation, nominal interest rates, real interest rates, savings, and debt. Before doing so, we present definitions of these terms and a basic relationship among them.

The Fisher Equation

Inflation is a general rise in an economy's price level over time. Formally, an economy's rate of inflation is defined as the percentage increase in the price level from one period of time to another period of time. In any period t, the inflation rate relative to period $t-1$ is defined as

$$\pi_t = \frac{P_t - P_{t-1}}{P_{t-1}},$$

where π denotes the inflation rate.[32] As a matter of terminology, a **deflation** (negative inflation) occurs when $\pi < 0$, and a **disinflation** occurs when π decreases over time (but is still positive at every point in time). For example, if in four consecutive years, inflation was 20%, 15%, 10%, and 5%, we say that disinflation is occurring – even though the price level increased in each of the four years.

In our consideration of the consumption-savings model, we defined the nominal interest rate as the return on each dollar kept in a bank account from one period to the next. For example, if your savings account (in which you keep dollars) pays you \$3 per year for every \$100 you have on balance, the nominal interest rate on your savings account is three percent.

Because of inflation, however, a dollar right now is not the same thing as a dollar one year from now because a dollar one year from now will buy you less (generally) than a dollar right now. That is, the purchasing power of a dollar changes over time due to inflation. Because it is goods (i.e., consumption) that individuals ultimately care about and not the dollars in their pockets or bank accounts, it is extremely useful to define another kind of interest rate, the **real interest rate.** **A real interest rate is a return that is measured in terms of goods rather than in terms of dollars.** Understanding the

[32] Not to be confused with profits, which is what π often represents in microeconomics. The usage is almost always clear from the context.

difference between a nominal interest rate and a real interest rate is important. An example will help illustrate the issue.

Example:
Consider an economy in which there is only one good – macroeconomics textbooks, say. In the year 2012, the price of a textbook is \$100. Wishing to purchase 5 textbooks (because macroeconomics texts are so much fun to read), but having no money with which to buy them, you borrow \$500 from a bank. The terms of the loan contract are that you must pay back the principal plus 10% interest in one year – in other words, you must pay back \$550 in one year. After one year has passed, you repay the bank \$550. If there has been zero inflation during the intervening one year, then the purchasing power of that \$550 is 5.5 textbooks, because the price of one textbook is still \$100. Rather than thinking about the loan and repayment in terms of dollars, however, we can think about it in terms of real goods (textbooks). In 2012, you borrowed 5 textbooks (what \$500 in 2012 could be used to purchase) and in 2013, you paid back 5.5 textbooks (what \$550 in 2013 could be used to purchase). Thus, in terms of textbooks, you paid back 10% more than you borrowed.

However, consider the situation if there **had** been inflation during the course of the intervening year. Say in the year 2013 that the price of a textbook had risen to \$110, meaning that there had been 10% inflation during the year. In this case, the \$550 repayment can be used to purchase only 5 textbooks, rather than 5.5 textbooks. So we can think about this case as if you had borrowed 5 textbooks and repaid 5 textbooks – that is, you did not pay back any additional textbooks, *even though* you repaid more dollars than you had borrowed.

In the zero-inflation case in the above example, the nominal interest rate is 10% and the real interest rate is 10%. In the 10%-inflation case, however, the nominal interest rate was still 10% but the real interest rate (the extra textbooks you had to pay back) was zero percent. This relationship between the nominal interest rate, the real interest rate, and the inflation rate is captured by the **Fisher equation,**

$$r_t = i_t - \pi_t \tag{15}$$

where r is the real interest rate, i is the nominal interest rate, and π is the inflation rate. Although almost all interest rates in economic transactions are specified in nominal terms, we will see that it is actually the real interest rate that determines much of macroeconomic activity.

Actually, however, the Fisher equation as stated in expression (15) is a bit of a simplification. The **exact Fisher equation** is

$$(1+i_t) = (1+r_t)(1+\pi_t)\,, \tag{16}$$

the details of which we will not describe here. This more accurate form of the Fisher equation turns out to be more convenient than its simplification in thinking about our two-period consumption-savings model. Before we analyze the topics of inflation, nominal interest rates, and real interest rates in the consumption-savings model, let's quickly see why expression (15) is in fact an approximation of expression (16). Multiplying out the terms on the right-hand-side of expression (16), we get

$$1+i_t = 1+\pi_t + r_t + r_t\pi_t . \tag{17}$$

If both r and π are small, which they usually are in developed economies (e.g., the U.S., Europe, Japan, etc.), then the term $r\pi$ is very close to zero. For example, if $r = 0.02$ and $\pi = 0.02$, then $r\pi = 0.0004$, which is essentially zero. So we may as well ignore this term. Dropping this term and then canceling the ones on both sides of expression (17) immediately yields the "casual" Fisher Equation of expression (15). The simplified Fisher equation of (15) is useful for quick analysis, but for our consumption-savings model it will almost always be more useful to think in terms of the exact Fisher equation (16).

For the two-period analysis below, the only economically meaningful inflation rate is that occurs between period 1 and period 2. According to our definition of inflation above, the inflation rate between period 1 and period 2 is

$$\pi_2 = \frac{P_2 - P_1}{P_1} . \tag{18}$$

So π_2 measures the percentage change in the price level (here, the nominal price of the consumption basket) between period 1 and period 2. For use below, it is helpful to re-arrange expression (18). First, separate the two terms on the right-hand side to get

$$\pi_2 = \frac{P_2}{P_1} - 1;$$

next, add 1 to both sides, which gives

$$1+\pi_2 = \frac{P_2}{P_1}.$$

Finally, taking the inverses of both sides leads to

$$\frac{1}{1+\pi_2} = \frac{P_1}{P_2} . \tag{19}$$

Consumption-Savings Model in Real Units

Recall the **nominal LBC** of the two-period model,

$$P_1c_1 + \frac{P_2c_2}{1+i} = Y_1 + \frac{Y_2}{1+i} + (1+i)A_0, \tag{20}$$

where the notation is exactly as we have already developed. Each term is in **nominal** units in this expression. As shown in the diagram, we can recast the framework into purely **real (goods-denominated) units** and re-do the entire analysis.

Dividing the nominal LBC by P_1 is the first step in re-casting the analysis in real units:

$$c_1 + \frac{P_2}{P_1}\cdot\frac{c_2}{1+i} = \frac{Y_1}{P_1} + \frac{Y_2}{P_1\cdot(1+i)} + (1+i)\cdot\frac{A_0}{P_1}.$$

The "labor income" terms Y_1 and Y_2 are nominal income. Define **real income** in period 1 and period 2, respectively, as

$$y_1 \equiv \frac{Y_1}{P_1} \tag{21}$$

and

$$y_2 \equiv \frac{Y_2}{P_2}. \tag{22}$$

Notice now we have to be careful in distinguishing upper-case Y from lower-case y!

Substituting y_1 into the LBC gives

$$c_1 + \frac{P_2}{P_1}\cdot\frac{c_2}{1+i} = y_1 + \frac{Y_2}{P_1\cdot(1+i)} + (1+i)\cdot\frac{A_0}{P_1}.$$

To substitute y_2, observe that we can multiply and divide the second term on the right-hand side by P_2, which gives

$$c_1 + \frac{P_2}{P_1}\cdot\frac{c_2}{1+i} = y_1 + \frac{Y_2}{P_2}\cdot\frac{P_2}{P_1}\cdot\frac{1}{1+i} + (1+i)\cdot\frac{A_0}{P_1}$$

(all we have done is multiply by "1," which is always a valid mathematical operation). Now using the definition y_2, we have

$$c_1 + \frac{P_2}{P_1}\cdot\frac{c_2}{1+i} = y_1 + y_2\cdot\frac{P_2}{P_1}\cdot\frac{1}{1+i} + (1+i)\cdot\frac{A_0}{P_1}.$$

The definition of inflation allows us to replace the $\frac{P_2}{P_1}$ terms to obtain

$$c_1 + c_2\cdot\left(\frac{1+\pi_2}{1+i}\right) = y_1 + y_2\cdot\left(\frac{1+\pi_2}{1+i}\right) + (1+i)\cdot\frac{A_0}{P_1}$$

Next, using the exact Fisher expression $\frac{1+i}{1+\pi_2} = 1+r$, rewrite the LBC once again as

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} + (1+i)\cdot\frac{A_0}{P_1}.$$

What's left to deal with is the seemingly complicated term at the far right-hand side. In terms of economics, it represents the **nominal receipts from the A_0** wealth with which the consumer began period 1, **stated in terms of period-1 purchasing power,** hence the appearance of P_1 in the denominator.

Using the same procedure as before, let's multiply and divide this term by P_0 (the nominal price level in period zero, or more generally stated, the nominal price level "in the past"), which gives us

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} + (1+i)\cdot\frac{P_0}{P_1}\cdot\frac{A_0}{P_0}$$

Using the definition of inflation allows us to rewrite this as

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} + \frac{1+i}{1+\pi_1}\cdot\frac{A_0}{P_0}.$$

Two steps remain. First, invoke the exact Fisher relationship. Second, define $a_0 \equiv \frac{A_0}{P_0}$ as the **real net wealth** of the consumer at the very end of period 0 and hence, equivalently and as shown in the timeline, at the very start of period 1. Finally, **the LBC in real terms is**

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} + (1+r)\cdot a_0,$$

which is highly analogous to the LBC in nominal terms. Indeed, the two describe the same exact budget restriction on consumer optimization.

The real form of the LBC emphasizes that consumption (which is a real variable! Nobody eats dollar bills or sits down in front of a dollar bill to watch a baseball game!) decisions over time are ultimately dependent on real factors of the economy: the real interest rate and real ("labor") income.

It is true that in modern economies with developed monetary exchange and financial markets, dollar prices and nominal interest rates are the objects people seem to think in terms of when making consumption and savings decisions. This facet of reality is indeed why our analysis so far has been framed in nominal units

But we can boil these dollar prices and nominal interest rates down to real interest rates and describe much of consumer theory solely in terms of real factors.

None of this is to say, though, that consideration of currencies, dollar prices, and nominal interest rates are unimportant or uninteresting topics. Indeed, the whole field of "monetary economics" is primarily concerned with these issues, and we will have a lot to say later about monetary economics. Depending in which issues we are analyzing, we will use either the LBC in real terms or the LBC in nominal terms. If we are considering issues of inflation, for example, then the nominal LBC will typically be more appropriate.

We proceed now with the nominal LBC. For diagrammatical purposes, it will be, just as before, easier to assume that $a_0 = 0$ (that is, the individual has no initial wealth). Rearranging the real LBC into the ready-to-be-graphed "slope-intercept" form, we have

$$c_2 = -(1+r)c_1 + (1+r)y_1 + y_2. \tag{23}$$

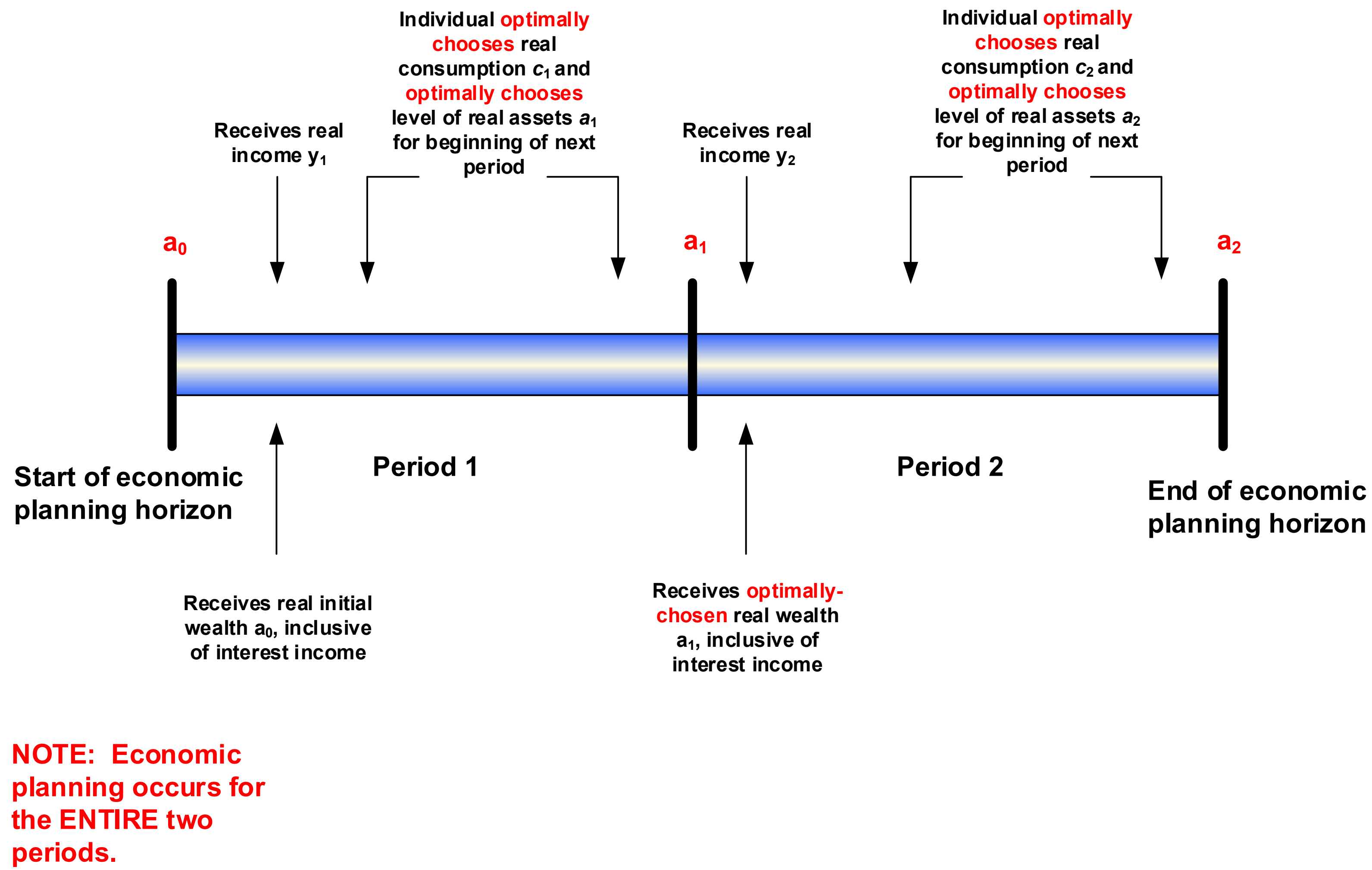


Figure 23. Timing of events in two-period consumption-savings framework, stated in real units.

The utility function $u(c_1, c_2)$ is unaffected by all of these manipulations of the LBC, meaning the indifference map is unaffected – as it must be, since budget constraints and indifference curves are two completely independent concepts.

Graphically, then, an example of an individual's optimal choice is shown in Figure 24 (which takes as given $a_0 = 0$). In this example, the individual consumes more than his real income in period 1, leading him to be in debt at the end of period 1; in period 2, he must repay the debt with interest and therefore consume less than his period-2 income. The definition of **real private savings** during the course of period 1 can be stated as

$$s_1^{priv} = ra_0 + y_1 - c_1,$$

which is quite analogous to one statement of **nominal private savings** during the course of period 1 (which, recall, was $S_1^{priv} = iA_0 + Y_1 - P_1c_1$

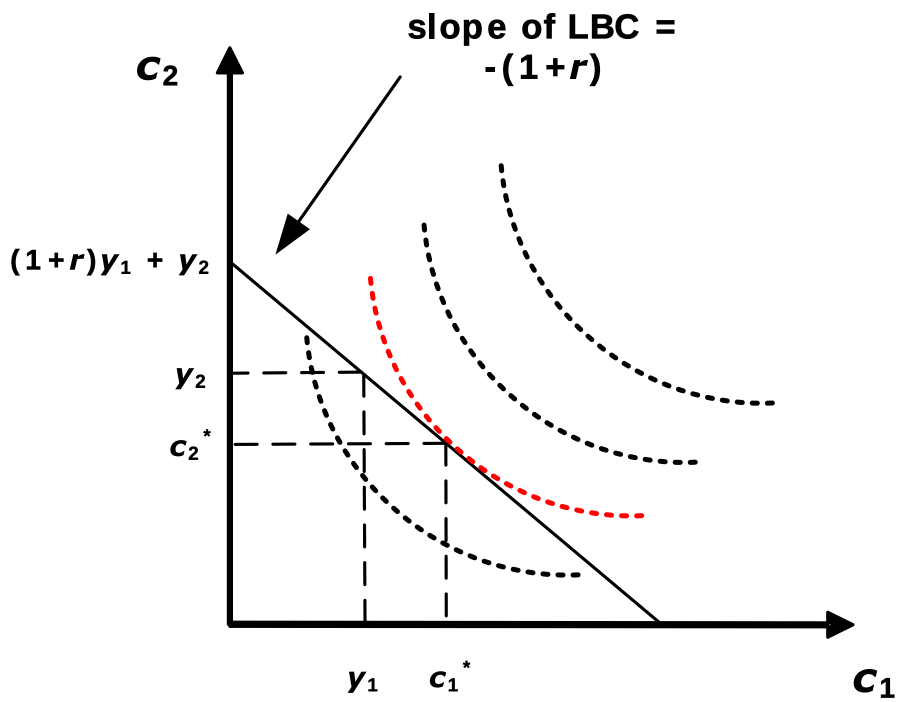


Figure 24. The interaction of the individual's LBC (here presented in real terms) and his preferences (represented by the indifference map) determine the individual's optimal consumption over time, here c_1* in period 1 and c_2* in period 2. The individual begins period 1 with $a_0 = 0$.

The Aggregate Private Savings Function

With the aid of Figure 24, we will now consider how changes in the real interest rate affect savings decisions of individuals. In our two-period model, there is only one time that the individual actually makes a decision about saving/borrowing: in period 1, when he must decide how of his period-1 labor income to save for period 2 or how much to borrow so that he can consume more than his period-1 labor income. As such, what we are exactly interested in is how S_1^{priv} (the same notation as before – private savings in period 1) is affected by r. Put more mathematically, what we are interested in is what the **private savings function** looks like.

Let us begin by supposing that the initial situation is as shown in Figure 24, in which the individual is a debtor at the end of period 1. Consider what happens to his optimal choice if the real interest rate r rises, while his real labor income y_1 and y_2 both remain constant. Such a rise in the real interest rate causes the LBC to both become steeper and have a higher vertical intercept, which we can see by analyzing the LBC (23). **In fact, the new LBC must still go through the point (y_1, y_2) because that is still a possible consumption choice for the individual.** That is, regardless of what the real interest rate is, it is always possible for the individual to simply not borrow or save in period 1 and simply consume his real labor income in each period. Because this is always possible, the point (y_1, y_2) must always lie on the LBC. Thus, the new LBC at the higher real interest rate is as shown in Figure 25. Also shown in Figure 25 are the new optimal consumption choices of the individual at the new higher interest rate. Specifically, notice that consumption in period 1 has decreased.

Because labor income in period 1 is unchanged, this means that his savings in period 1 has risen. Recall that private savings in period 1 is

$$S_1^{priv} = Y_1 - P_1 c_1 \tag{24}$$

in nominal terms. We can divide this expression through by P_1 to get savings in real terms,

$$s_1^{priv} = y_1 - c_1, \tag{25}$$

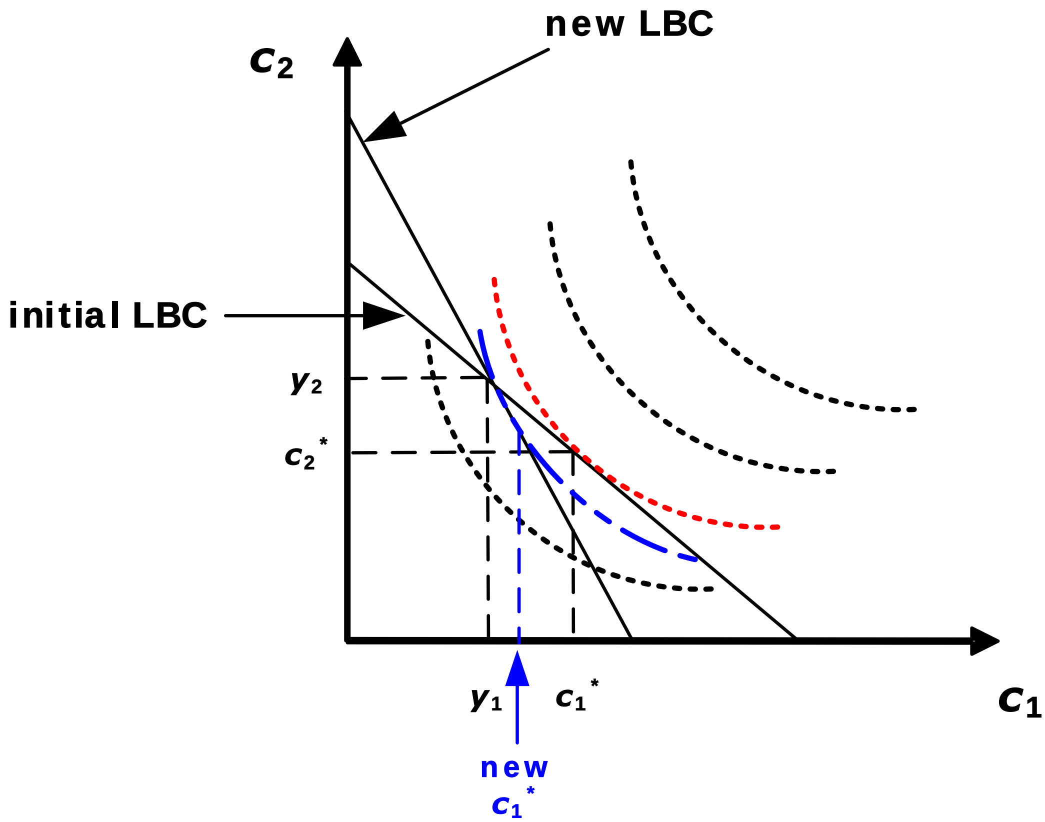


Figure 25. If at the initial real interest rate the individual chose to be a debtor at the end of period 1, then a rise in the real interest rate necessarily lowers consumption in period 1, implying that savings during period 1 has increased (or, equivalently, as shown, dissaving has decreased).

Notice the distinction between lower-case s_1^{priv}, which denotes real savings, and our earlier upper-case S_1^{priv}, which denotes nominal savings. The relationship is simply that $s_1^{priv} = S_1^{priv} / P_1$.[33] Thus, with unchanged y_1 and a decreased c_1^*, s_1^{priv} has increased. Actually, in Figure 25, savings is still negative after the rise in the real interest rate – but it is less negative, so indeed private savings has increased.

The preceding analysis seems to suggest that there is a positive relationship between the real interest rate and private savings. However, the conclusion is not so straightforward because we need to consider a different possible initial situation. Rather than the initial situation depicted in Figure 24, suppose instead that Figure 26 depicted the initial situation of the individual. In Figure 26, the optimal choice of the individual is such that

[33] By now, you should be noticing how to convert any nominal variable into its corresponding real variable – simply divide by the price level. The one slight exception is the nominal interest rate – to convert to the real interest rate requires use of the inflation rate (which itself depends on price levels, so the idea is still the same).

he consumes less in period 1 than his labor income in period 1, allowing him to accumulate positive wealth for period 2. That is, he saves during period 1.

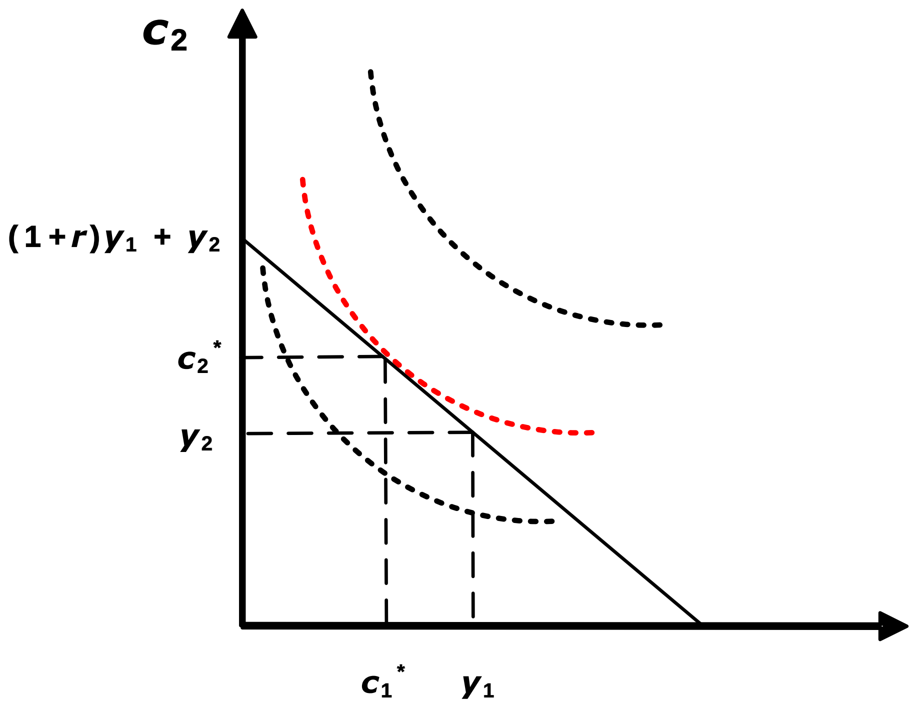


Figure 26. At the initial real interest rate, the individual's optimal choice may be such that he is **not** a debtor at the end of period 1 but rather a saver. This is because he chooses to consume less in period 1 than his labor income in period 1, which allows him to consume more in period 2 than his labor income in period 2.

Now suppose the real interest rate rises, with labor income y_1 and y_2 both held constant. The budget line again becomes steeper by pivoting around the point (y_1, y_2), as shown in both Figure 27 and Figure 28. However, depending on the exact shapes of the individual's indifference curves, the individual's consumption in period 1 may fall (shown in Figure 27) or rise (shown in Figure 28). In terms of his savings in period 1, then, a rise in the real interest rate may induce either a rise in savings (shown in Figure 27) or a fall in savings (shown in Figure 28).

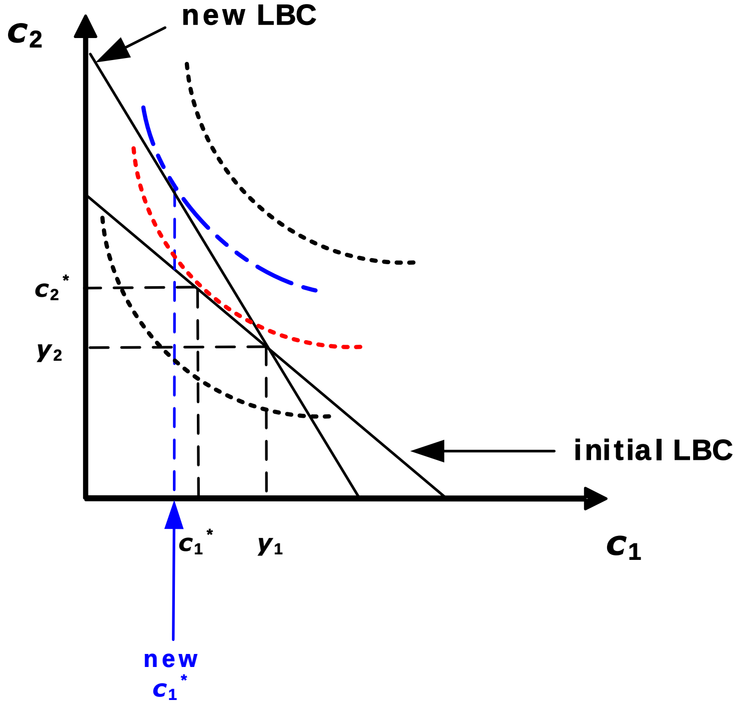


Figure 27. If the initial situation is such that the individual optimally chose to be a saver at the end of period 1, then a rise in the real interest rate may cause his savings in period 1 to increase…..

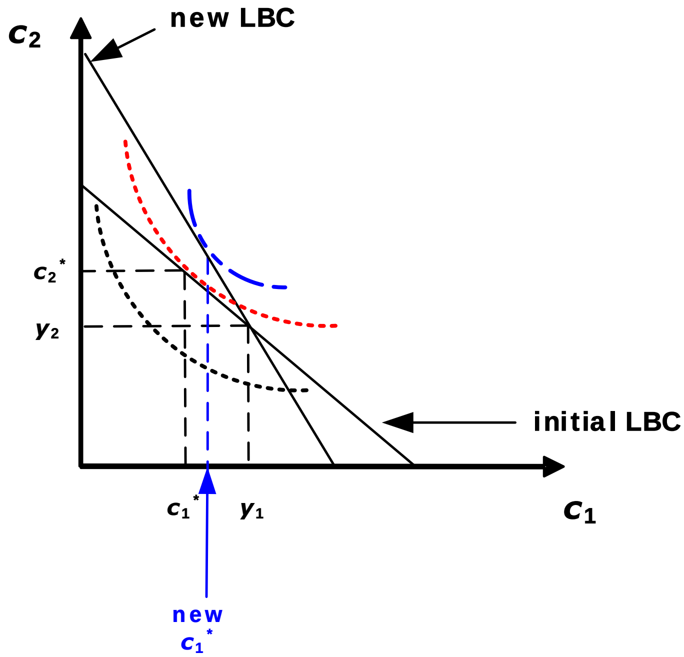


Figure 28. … or decrease, depending on the shape of his indifference map (i.e., depending on exactly what functional form his utility function has). Thus, for an individual who optimally initially chooses to be a

saver during period 1, it is impossible to determine theoretically in which direction his savings changes if the real interest rate rises.

Where does this leave us in terms of our ultimate conclusion about how private savings reacts to a rise in the real interest rate? Not very far theoretically, unfortunately. The summary of the above analysis is as follows. If an individual is initially a debtor at the end of period 1, then a rise in the real interest rate necessarily increases his savings during period 1. On the other hand, if an individual is initially a saver at the end of period 1, then a rise in the real interest rate may increase or decrease his savings during period 1. Overall, then, theory cannot guide us as to how private savings at the macroeconomic level responds to a rise in the real interest rate!

Where theory fails, we can turn to data. Many empirical studies conclude that the real interest rate in fact has a very weak effect, if any effect at all, on private savings behavior. The studies that do show that real interest rates do influence savings almost always conclude that a rise in the real interest rate leads to a rise in savings. The interpretation of such an effect seems straightforward: if all of a sudden the interest rate on your savings account rises (and inflation is held constant), then you may be tempted to put more money in your savings account in order to earn more interest income in the future.

We will adopt the (somewhat weak) empirical conclusion that the real interest rate has a positive effect on private savings – thus we will proceed with our macroeconomic models as if Figure 25 and Figure 27 are correct and Figure 28 is incorrect.[34]

This leads us to graph the upward-sloping aggregate private savings function in Figure 29.

Stocks vs. Flows

Let's return to the critical difference between stock variables and flow variables. Stated in terms of real goods (and as Figure 23 displays), the **stock (or, equivalently, accumulation) variables** are a_0, a_1, and a_2; and the **flow variables** are c_1, c_2, y_1, y_2; s_1, and s_2.

It is hard to emphasize how much the distinction between **stock variables** and **flow variables matters for all of macroeconomic analysis!** As our multi-period frameworks soon begin to include more and more time periods, the critical concepts of stocks vs. flows will continue to help us think about various economic events play out. So you are highly encouraged to understand the difference right away.

[34] Though debate amongst macroeconomists over this issue is not yet settled, this seems to be the most commonly-accepted interpretation of the results.

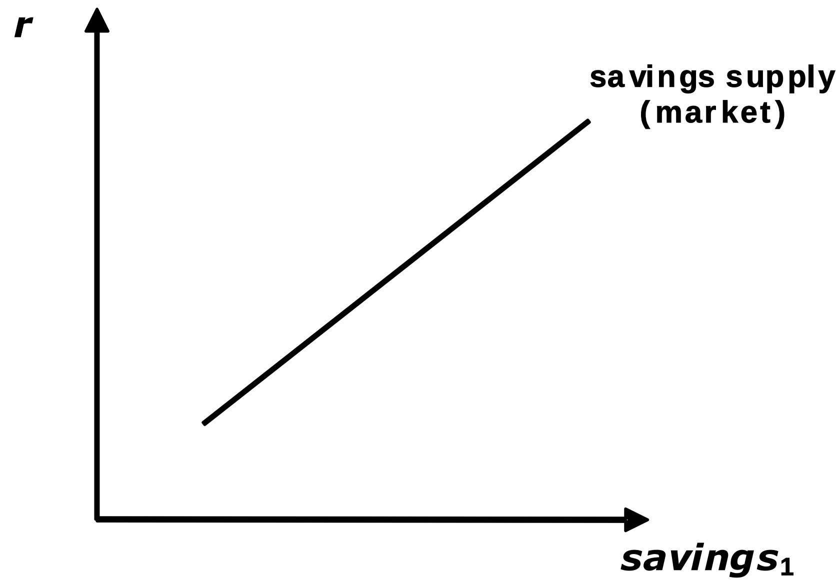


Figure 29. The upward-sloping aggregate private savings function**.**

Lagrange Characterization – the Consumption-Savings Optimality Condition

As we did with the consumption-leisure model, it is useful to work through the mechanics of analyzing the two-period model using our Lagrange tools. In analyzing multi-period models using Lagrangians, it turns out we have two alternative and distinctly useful ways of proceeding: an approach we will refer to as a **lifetime Lagrange formulation** and an approach we will refer to as a **sequential Lagrange formulation.**

These ideas will hopefully become clear as we describe how to pursue these two different Lagrange approaches, but the advantages and disadvantages of the two approaches can be summarized as follows. For a simple two-period model, the lifetime Lagrange formulation is essentially nothing more than a formal mathematical statement of the graphical analysis we have already conducted. It emphasizes, as the terminology suggests, that consumers can be viewed as making *lifetime* choices. The sequential Lagrange formulation, on the other hand, emphasizes the unfolding of economic events and choices over time, rather than starting from an explicitly lifetime view. In the end, the sequential approach will bring us to exactly the same conclusion(s) as the lifetime approach; the sequential approach will thus seem like a more circuitous mode of analysis.

We introduce the sequential Lagrangian approach, however, for two reasons. One reason is that when we soon extend things to an infinite-period model, in which graphical analysis becomes quite infeasible, the lifetime Lagrangian formulation (which, as just stated, is really just a mathematical formulation of analysis that can otherwise be carried out purely graphically) inhererently becomes a bit less interesting.

A second, and quite related, reason that sequential Lagrangian analysis is of interest is that it will allow us to explicitly track the dynamics of *asset prices* over time as macroeconomic events unfold over time. In the lifetime view of the two-period model, we effectively end up removing from our analysis the "intermediate asset position" A_1. In the richer infinite-period models to come, we will offer quite specific various interpretations of what A_1 "is," and we will naturally end up being concerned with "its price." Here, we have been loosely speaking of A as the "amount of money in the bank." This is a fine enough interpretation for now, but we will develop the concept of "A" much further in the chapters ahead, and the sequential Lagrangian approach will prove extremely useful in thinking about specific instantiations of A.

In what follows, we will formulate both the lifetime and sequential Lagrangians in nominal terms, but one could easily pursue either in real terms, as well – a useful exercise for you to try yourself.

Lifetime Lagrangian Formulation

To construct the lifetime Lagrangian for the two-period model, the general strategy is just as we have seen several times already: sum the objective function together with the constraint function (with a Lagrange multiplier attached to it) to form the Lagrangian, compute first-order conditions, and then conduct relevant analysis using the first-order conditions. The objective function to be maximized is obviously the consumer's lifetime utility function $u(c_1, c_2)$. The relevant constraint – recall we are pursuing the *lifetime* Lagrangian here – is the consumer's LBC. Associating the multiplier λ with the LBC, the lifetime Lagrangian for the two-period model is

$$u(c_1, c_2) + \lambda\left[Y_1 + \frac{Y_2}{1+i} - P_1 c_1 - \frac{P_2 c_2}{1+i}\right].$$

Note for simplicity we have dropped any initial assets, just as we did in our graphical analysis, by assuming $A_0 = 0$; none of the subsequent analysis depends on this simplifying assumption.

It should be clear by now that, apart from the first-order condition on the Lagrange multiplier, the two relevant first-order conditions that we need to compute are those with respect to c_1 and c_2. Indeed, these are the formal objects we need to compute. However, before simply proceeding to the mathematics, let's remind ourselves of what it means conceptually when we construct these objects. A first-order condition with respect to any particular variable (think in terms of basic calculus here) mathematically describes *how* a maximum is achieved by optimally setting/choosing that particular variable, taking as given the settings/choices for all other variables. In terms of the economics of our model, the consumer is *optimally* choosing *both* c_1 *and* c_2 (in order to maximize utility), which, from the formal mathematical perspective, requires computing first-order conditions of the Lagrangian with respect to both c_1 and c_2. Keep this discussion in mind when we consider the sequential Lagrangian.

The first-order conditions with respect to c_1 and c_2 (we'll neglect here the first-order condition with respect to λ, which, as should be obvious by now, simply returns to us the LBC) thus are:

$$\frac{\partial u}{\partial c_1} - \lambda P_1 = 0$$

$$\frac{\partial u}{\partial c_2} - \lambda \frac{P_2}{1+i} = 0$$

The next step, as usual, is to eliminate λ from these two conditions. From the first expression, we have $\lambda = \frac{\partial u / \partial c_1}{P_1}$; inserting this into the second expression gives us $\frac{\partial u}{\partial c_2} = \frac{\partial u}{\partial c_1} \frac{P_2}{P_1(1+i)}$. From earlier, we know that $\frac{P_2}{P_1(1+i)} = \frac{1+\pi_2}{1+i}$, which in turn, from the exact Fisher equation, we know is equal to $\frac{1}{1+r}$. Slightly rearranging the resulting expression $\frac{\partial u}{\partial c_2} = \frac{\partial u}{\partial c_1} \frac{1}{1+r}$ gives us

$$\frac{\partial u / \partial c_1}{\partial u / \partial c_2} = 1 + r,$$

which is our two-period model's **consumption-savings optimality condition**. The consumption-savings optimality condition describes what we saw graphically in **Figure 24**: when the representative consumer is making his optimal intertemporal choices, he chooses c_1 and c_2 in such a way as to equate his MRS between period-1 consumption and period-2 consumption (the left-hand-side of the above expression) to (one plus) the real interest rate (the right-hand-side of the above expression). The real interest rate (again, more precisely, one plus the real interest rate) is simply the slope of the consumer's LBC. The two-period model's consumption-savings optimality condition will be present in the richer infinite-period model we will build soon.

Sequential Lagrangian Formulation

We can alternatively cast the representative consumer's choice problem in the two-period world on a period-by-period basis. That is, rather than take the lifetime view of the consumer's decision-making process, we can take a more explicitly sequential view of events. A bit more precisely, we can think of the consumer as making optimal decisions for period 1 and *then* making optimal decisions for period 2. If there were more than just two periods, we could think of the consumer as *then* making optimal decisions for period 3, and *then* making optimal decisions for period 4, and *then* making optimal decisions for period 5, and so on.

In this explicitly sequential view of events, the consumer, in a given period, *chooses consumption for that period along with an asset position to carry into the subsequent period.* That is, in period t (where, in the two-period model, either $t = 1$ or $t = 2$), the consumer chooses consumption c_t and asset position A_t; note well the time-subscripts here. Also, crucially, note that in the sequential formulation, we are thinking explicitly of the consumer as making an optimal choice with regard to intermediate asset positions; in the lifetime formulations of the two-period model, whether graphical or Lagrangian, we effectively removed intermediate asset positions from the analysis, as we have noted a couple of times. In the sequential formulation, we do not remove intermediate asset positions from the analysis; think of this as the consumer deciding how much to put in (or borrow from) the bank.

Formally, in order to construct the sequential Lagangian, we must, as always, determine what the relevant objective function and constraint(s) are. The objective function, as usual, is simply the representative consumer's utility function. In terms of constraints, in the sequential formulation we will impose **all of the period-by-period budget constraints,** rather than the LBC. In our two-period model, we obviously have only two budget constraints, one describing choice sets in period 1 and one describing choice sets in period 2.

Almost all of our Lagrangian analyses thus far have used only one constraint function. But recall from our review of basic mathematics that it is straightforward to extend the Lagrangian method to handle optimization problems with multiple constraints. All we need to do, once we have identified the appropriate constraints, is associate *distinct* Lagrange multipliers with each constraint and then proceed as usual.

To construct the sequential Lagrangian, then, associate the multiplier λ_1 with the period-1 budget constraint and the multiplier λ_2 with the period-2 budget constraint – note that λ_1 and λ_2 are distinct multipliers, which in principle have nothing to do with each other. The sequential Lagrangian is thus

$$u(c_1, c_2) + \lambda_1[Y_1 - P_1 c_1 - A_1] + \lambda_2[Y_2 + (1+i)A_1 - P_2 c_2].$$

In writing this Lagrangian, we have used our assumption that $A_0 = 0$ and our result that $A_2 = 0$. The sequential analysis then proceeds as follows. Compute the first-order conditions for the consumer's choice problem in period 1: recall from our discussion above that in period 1, the consumer optimally chooses c_1 and A_1. Mathematically, this requires us to compute the first-order conditions of the Lagrangian with respect to these two variables; they are

$$\frac{\partial u}{\partial c_1} - \lambda_1 P_1 = 0$$
$$-\lambda_1 + \lambda_2(1+i) = 0$$

Next, compute the first-order conditions for the consumer's choice problem in period 2: in period 2, the consumer optimally chooses c_2 and A_2. Mathematically, this requires us to compute the first-order conditions of the Lagrangian with respect to these two variables. Of course, in the two-period model, we have that $A_2 = 0$, so due solely to the artifice of the two-period model, we actually do not need to compute the first-order condition with respect to A_2; only if we had more than two periods in our model would we need to compute it. Thus, all we need from the period-2 optimization is the first-order condition with respect to c_2, which is

$$\frac{\partial u}{\partial c_2} - \lambda_2 P_2 = 0.$$

Let's proceed to eliminate multipliers from the three first-order conditions we just obtained (and note that we'll skip considering the first-order conditions with respect to the two multipliers – as should be obvious by now, they simply deliver back to us the period-1 budget constraint and the period-2 budget constraint). Note that we now have *two* multipliers to deal with. From the first-order condition on A_1, we have $\lambda_1 = \lambda_2(1+i)$. We'll have much more to say about this type of relationship between multipliers – this expression that links multipliers across time periods – when we study the infinite-period model; for now, let's just exploit the mathematics it provides. Take this expression for λ_1 and insert it in the first-order condition on c_1, yielding $\frac{\partial u}{\partial c_1} = \lambda_2(1+i)P_1$. We've gotten rid of the multiplier λ_1 but are still left with λ_2. Fortunately, we can use the first-order condition on c_2 to obtain an expression for the period-2 multiplier: $\lambda_2 = \frac{\partial u / \partial c_2}{P_2}$. Now, insert this expression into the previously-obtained condition to get

$$\frac{\partial u}{\partial c_1} = \frac{\partial u}{\partial c_2}\frac{(1+i)P_1}{P_2},$$

in which we finally have eliminated all multipliers. Rearranging this expression a bit,

$$\frac{\partial u / \partial c_1}{\partial u / \partial c_2} = \frac{(1+i)P_1}{P_2}.$$

We have seen the right-hand-side of this expression a couple of times already, and we know that we can transform it (using the definition of inflation and the Fisher relationship) into 1 + *r*. Thus, the last expression becomes

$$\frac{\partial u / \partial c_1}{\partial u / \partial c_2} = 1 + r,$$

which clearly is simply the consumption-savings optimality condition we derived above in the lifetime formulation of the problem. Because we have already derived and discussed it, there of course is no reason to discuss the economics of it again.

The idea to really understand and appreciate here is that, whether we pursue the lifetime Lagrangian approach or the sequential Lagrangian approach, we arrive at exactly the same prediction regarding how consumers optimally allocate their intertemporal consumption choices: they do so in such a way as to equate the MRS between period-1 consumption and period-2 consumption to (one plus) the real interest rate.

The mathematical difference between the two approaches is that in the sequential approach we had to proceed by explicitly considering the first-order condition on the intermediate asset position A_1, which generated a relationship between Lagrange multipliers over time. Through the optimal decision on A_1, the consumer **does** take into account future period events, even though the mathematics may not make it seem apparent. In the lifetime approach, no such relationship had to formally be considered because there was, by construction, only one multiplier.

In the end, we should not be surprised that we reached the same conclusion using either approach – indeed, they are simply *alternative* approaches to the *same* problem, the problem being the representative consumer's utility maximization problem over time.

Optimal Numerical Choice

Regardless of a lifetime or sequential analysis, the same exact consumption-savings optimality condition arises: $\frac{\partial u / \partial c_1}{\partial u / \partial c_2} = 1 + r$. This expression is part of the heart of macroeconomic analysis.

However, if we actually wanted to solve for numerical values of the optimal choices of period-1 and period-2 consumption, the consumption-savings optimality condition is not enough. Why? **Because the consumption-savings optimality condition is one equation in two unknown variables.** A simple way to see this is to take the case of $u(c_1,c_2)=\ln c_1+\ln c_2$. The consumption-savings optimality condition is thus $c_2/c_1 = 1+r$ (which at this point you should be able to obtain yourself). Even though the market real interest rate r is taken as given, it is clearly impossible to solve for **both** c_2 and c_1 from this **one** equation.

This might be obvious by this point (especially given all of the indifference-curve/budget constraint diagrams in Figure 25, Figure 26, Figure 27, and Figure 28!), but to complete the numerical solution of the two-period framework requires us to use **both the consumption-savings optimality condition and the budget constraint** to pin down the optimal numerical choices of consumption across time. In other words, there are two equations in the two unknowns, period-1 consumption and period-2 consumption. The ensuing example takes us step-by-step though the analysis, and it also raises an important economic interpretation of the optimal consumption choices across time that arise.

Consumption Smoothing

The concept of **"consumption smoothing"** is an important underlying theme of the results that emerge from multi-period representative consumer utility maximization. This powerful and intuitive economic result arises not just in the two-period framework, but also in the progressively richer models we will construct later.

An example using the two-period model sheds light on the idea of consumption smoothing.

Consumption-Smoothing Example

Suppose the lifetime utility function is $u(c_1,c_2)=\ln c_1+\ln c_2$. And also assume that P_1 = 1, P_2 = 1, A_0 = 0, r = 0.10.

Case 1: Suppose the lifetime stream of nominal income is concentrated in the "later" period of the consumer's economic planning horizon – for example, Y_1 = 2 and Y_2 = 11.

To solve for the optimal numerical values of c_1 and c_2 requires use of the pair of expressions

$$\left(\frac{\partial u/\partial c_1}{\partial u/\partial c_2}=\right)\frac{c_2}{c_1}=1+r$$

and

$$P_1c_1 + \frac{P_2c_2}{1+i} = Y_1 + \frac{Y_2}{1+i}.$$

The few steps of algebra are left for you to go through (which is good reinforcement of basics). The numerical values of the optimal choices of consumption across time turn out to be

$$c_1^* = 6, \quad c_2^* = 6.6.$$

Case 2: Suppose instead the lifetime stream of nominal income is more evenly spread through the "early" period and the "later" period of the consumer's economic planning horizon – for example, $Y_1 = 7$ and $Y_2 = 5.5$. Once again, the optimal numerical values of consumption are determined by the consumption-savings optimality condition and the budget constraint. And also once again leaving the few steps of algebra for you to verify, optimal choices of consumption across time turn out to be

$$c_1^* = 6, \quad c_2^* = 6.6.$$

Clearly, the lifetime path of optimal consumption is the same, despite the large difference between the Case 1 lifetime income path ($Y_1 = 2$, $Y_2 = 11$) and the Case 2 lifetime income path ($Y_1 = 7$, $Y_2 = 5.5$).

This example demonstrates the **two different facets of consumption smoothing.** The first aspect is that individuals prefer their consumption across time to not vary very much. This result arises due to strictly increasing and strictly concave lifetime utility, which is part of the **preference** side of the framework.

The second aspect arises from the **constraint** side of the framework. Despite the two very different income scenarios in the example, optimal c_1 and c_2 are identical. The identical optimal consumption streams, despite the very different income streams, is due to the ability of the individual to borrow (in Case 1) as much as he or she wants during period one, and hence be in debt at the very beginning of period two. This is highlighted in the **negative value of the** A_1 term that arises in Case 1:

$$\begin{aligned} A_1 &= Y_1 - P_1c_1 + (1+i)A_0 \\ &= 2 - 6 + 0 \\ &= -4 \end{aligned}$$

If, counter to the example, the individual faced **another** constraint, in addition to the budget constraints, that allowed no borrowing at all during period one, the Case 1 consumption outcomes would be quite different: we would have $c_1 = 2$ and $c_2 = 11$ as the **"credit-constrained"** optimal choices for Case 1.

Without the credit constraint, the Case 1 individual is borrowing (that is, **dissaving**) during period one, and repaying the accumulated debt, inclusive of interest payments, in period two. In Case 2, the individual is saving during period one, and using the accumulated wealth (inclusive of interest earnings) for consumption in period two. Using all of the terminology and definitions of the two-period consumption-savings framework, you should be able to verify all of this for yourself.

Chapter 5
Dynamic Consumption-Labor Framework

We have now studied the consumption-leisure model as a "one-shot" model in which individuals had no regard for the future: they simply worked to earn income, all of which they then spent on consumption right away, socking away none of it for the future. Individuals do, of course, consider their future prospects when making economic decisions about the present. We saw this idea in our study of the two-period consumption-savings model. It should not strike you as unusual, then, that when an individual makes his optimal choice about consumption and leisure in the current period, he recognizes that he will make a similar consumption-leisure choice in the future. In effect, then, it seems there are multiple consumption-leisure choices an individual makes over the course of his lifetime. However, these choices are not independent of each other because consumers can save for the future or borrow against future income.

In this section we will bring the consumption-leisure model together with the consumption-savings model. As we will see, doing so in effect is just "gluing" the two models together. The main benefit is that it allows consideration of a broader range of consequences of macroeconomic policies – in particular it allows us to see that economic policies have their consequences not just in the time period in which they are implemented but also other periods.

Individual's Preferences

With two periods, in each of which the individual makes a consumption-leisure choice, there are four objects which determine the individual's lifetime utility: consumption in period 1, leisure in period 1, consumption in period 2, and leisure in period 2. Denote these, respectively, by c_1, l_1, c_2, and l_2, and let the lifetime utility function be $v(c_1, l_1, c_2, l_2)$. We will assume that this lifetime utility function is **additively-separable** across time in the following way:

$$v(c_1, l_1, c_2, l_2) = u(c_1, l_1) + u(c_2, l_2).$$

The function $v(c_1, l_1, c_2, l_2)$ is the **lifetime utility function** and the function u is the **sub-utility function** which measures utility over consumption and leisure in each of the two periods. Note especially that the function u is the same function in each of the two periods, meaning that the indifference map over c_1 and l_1 is identical to the indifference map over c_2 and l_2. Furthermore, because consumption in two different periods appears in the lifetime utility function, an indifference map over c_1 and c_2 exists, just as in the two-period model we have already consider.

Lifetime Budget Constraint

The more complicated object to describe in this model is the individual's lifetime budget constraint (LBC). Just as in the simpler two-period model, a budget constraint exists for period 1

$$P_1c_1 + A_1 = (1+i)A_0 + (1-t_1)W_1(1-l_1)$$

as well as for period 2

$$P_2c_2 + A_2 = (1+i)A_1 + (1-t_2)W_2(1-l_2),$$

in which W_1 denotes the hourly wage in period 1, W_2 denotes the hourly wage in period 2, t_1 denotes the labor tax rate in period 1, and t_2 denotes the labor tax rate in period 2. All of the other notation is the same as in our simple consumption-savings model and our simple consumption-leisure model. The interpretation of these period-by-period budget constraints is the same as before – in each period the individual has some wealth (which may be negative) and some labor income at his disposal, and he must decide how much to consume and how much to save for the future. The difference here versus the simple consumption-leisure model is that the individual decides how much labor income he earns.[35]

Because the rational individual consider his entire (two-period) lifetime when making his decisions, the relevant budget constraint is a lifetime budget constraint, which we derive using the two period-by-period budget constraints above. First, note that because there is no period 3, it must be that $A_2 = 0$, just as before, because there is no reason to save for after the end of the world. Then, we can solve the period-2 budget constraint to get

$$A_1 = \frac{P_2c_2}{(1+i)} - \frac{(1-t_2)W_2(1-l_2)}{(1+i)},$$

which we can in turn substitute into the period-1 budget constraint. After a few steps of algebra, we have

[35] If this brief description of these budget constraints, as well as the derivation of the LBC to follow, seems unfamiliar, it is a good idea to review the simple consumption-savings model and the simple consumption-leisure model at this point.

$$P_1c_1 + \frac{P_2c_2}{(1+i)} = (1-t_1)W_1(1-l_1) + \frac{(1-t_2)W_2(1-l_2)}{(1+i)} + (1+i)A_0.$$

Finally, as in the consumption-leisure model, we can expand the terms on the right-hand-side and then move the terms involving leisure to the left-hand-side to get

$$P_1c_1 + \frac{P_2c_2}{(1+i)} + (1-t_1)W_1l_1 + \frac{(1-t_2)W_2l_2}{(1+i)} = \left((1-t_1)W_1 + \frac{(1-t_2)W_2}{(1+i)}\right) + (1+i)A_0.$$

As always, it is a good idea for you to verify these algebraic manipulations for yourself.

We will now graph the LBC in expression in three different graphs: in $c_1 - c_2$ space, in $c_1 - l_1$ space, and in $c_2 - l_2$ space. As in the simple consumption-savings model, we will assume for graphical simplicity that $A_0 = 0$, but the results that follow in no way depend on this assumption. Solving the previous expression for c_2 gives

$$c_2 = -\left(\frac{P_1(1+i)}{P_2}\right)c_1 + \frac{(1+i)(1-t_1)W_1}{P_2}(1-l_1) + \frac{(1-t_2)W_2}{P_2}(1-l_2).$$

This equation can be usefully viewed in one of two ways: either c_2 as a function of l_2 (in which case we are thinking of the consumption-leisure decision in period 2) or c_2 as a function of c_1 (in which case we are thinking of the consumption-savings decision that spans period 1 and period 2).

First let's consider graphing the c_2= -…. equation with c_2 on the vertical axis and c_1 on the horizontal axis. The slope of this function is $-(P_1(1+i)/P_2)$. If $c_1 = 0$, then $c_2 = \frac{(1+i)(1-t_1)W_1}{P_2}(1-l_1) + \frac{(1-t_2)W_2}{P_2}(1-l_2)$, while if $c_2 = 0$, $c_1 = \frac{(1-t_1)W_1}{P_1(1+i)}(1-l_1) + \frac{(1-t_2)W_2}{P_1(1+i)}(1-l_2)$ – so we now have the intercepts of this function. Notice that these intercepts depend on the choices of leisure in the two periods, l_1 and l_2.

Alternatively, if we graph c_2= -…. equation with c_2 on the vertical axis and l_2 on the horizontal axis, we see that the slope is $-(1-t_2)W_2/P_2$, just as in our simple consumption-leisure model. If $l_2 = 0$, then $c_2 = -\left(\frac{P_1(1+i)}{P_2}\right)c_1 + \frac{(1+i)(1-t_1)W_1}{P_2}(1-l_1) + \frac{(1-t_2)W_2}{P_2}$, while if $c_2 = 0$, then

$l_2 = 1 - \frac{P_1(1+i)c_1}{(1-t_2)W_2} + \frac{(1+i)(1-t_1)W_1}{(1-t_2)W_2}(1-l_1)$ – so now we have the intercepts of this function. Notice that these intercepts depend on the choice of consumption in period 1 and leisure in period 1.

The main point which emerges from the preceding discussion is that all four choices (of consumption in the two periods as well as leisure in the two periods) are interdependent. Essentially, we need a five-dimensional graph (which obviously is impossible) in order to visualize the solution to this model. So use of graphical tools here is complicated.