15.456 Financial Engineering

MIT Sloan School of Management
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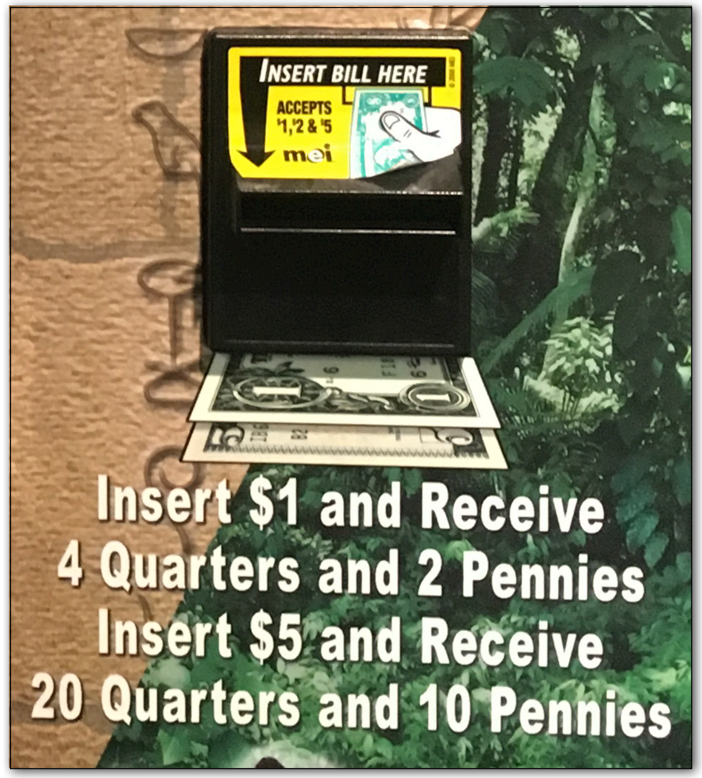
Fundamental theorem of asset pricing, approximate replication, and risk-neutral pricing
September 18, 2020

Agenda

- Announcements
  - Problem Set 1 solutions on Canvas
  - Problem Set 2 due Thursday, September 24 @ 11:59pm
  - Quiz alert
- Fundamental Theorem of Asset Pricing
- Approximate replication and hedging
- Risk-neutral probabilities and asset pricing
- Multi-period dynamics and the binomial tree



Summary and recap

- Law of one price:
  - Absence of arbitrage
  - Unique price (for securities in complete market)
  - Pricing bounds (from positive state prices in incomplete market
- Pricing is linear
  - Linear combination of assets
  - Linear functions and prices
  - Adjoint relates market and state prices

$$\text{If payoffs } A\mathbf{x}_1 = A\mathbf{x}_2, \text{ then } \mathbf{S}^*\mathbf{x}_1 = \mathbf{S}^*\mathbf{x}_2$$

$$\text{since } A(\mathbf{x}_1 - \mathbf{x}_2) = 0 \implies \mathbf{S}^*(\mathbf{x}_1 - \mathbf{x}_2) = 0$$

Arbitrage theorem

- Consider a market with *n* securities, *s* states of the world, payoff matrix

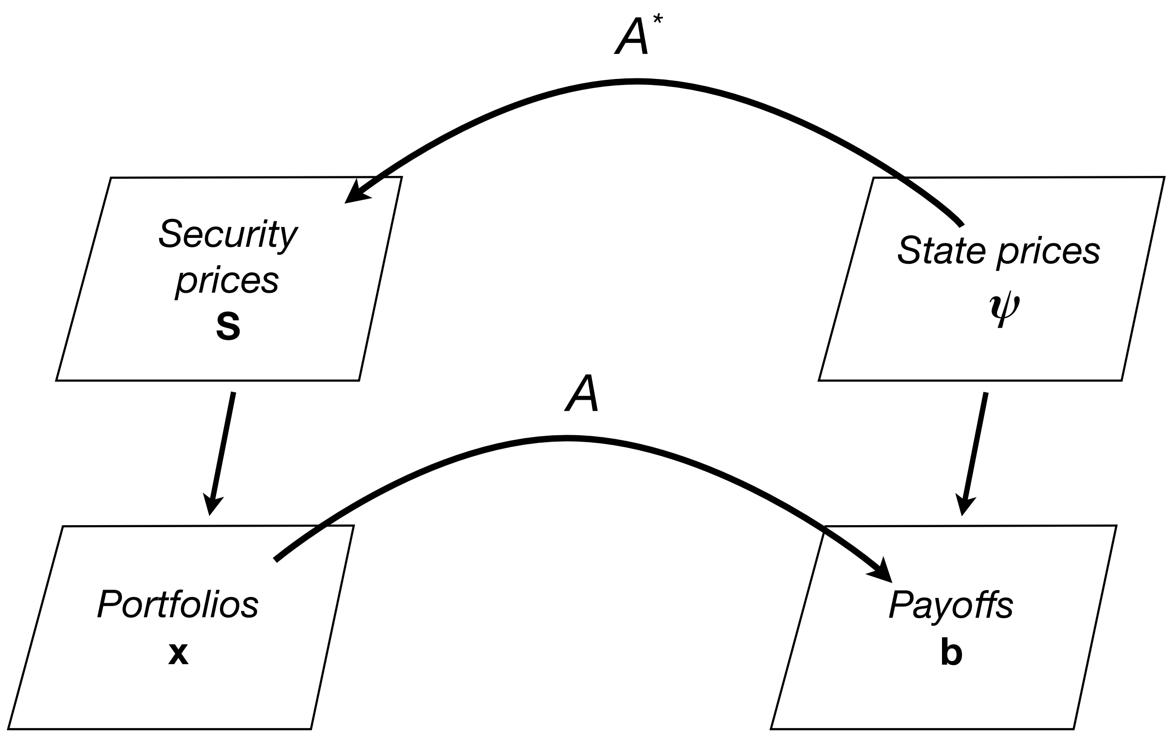
$$A \in \mathbb{R}^{s \times n}, \quad A : \mathbb{R}^n \to \mathbb{R}^s$$

- There is no arbitrage if and only if there exists a **strictly positive** state-price vector ψ **consistent with** the security-price vector,

$$\psi \in \mathbb{R}^s, \quad \mathbf{S} \in \mathbb{R}^n, \quad \mathbf{S} = A^* \psi$$

- Already shown this for complete markets. For **incomplete markets**, there can be (infinitely) multiple solutions ψ.
  - If there is **at least one solution** where $\psi > 0$, then no arbitrage.
  - If **none of the solutions** has $\psi > 0$, then there is arbitrage.

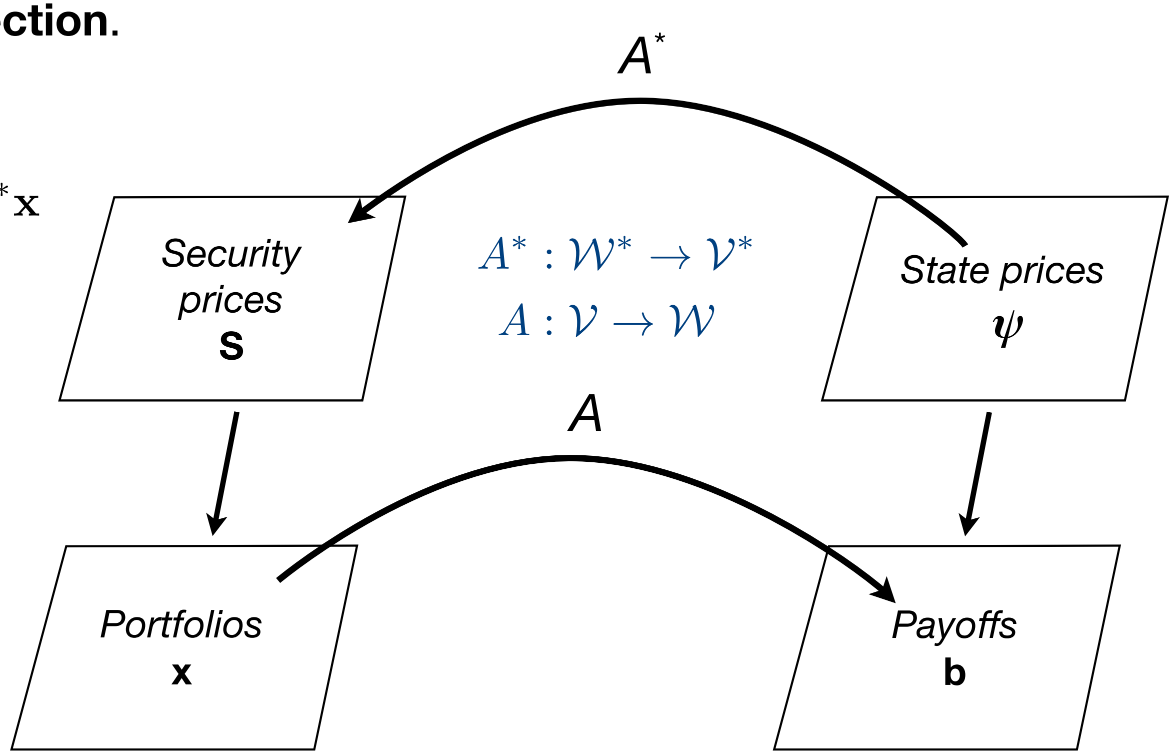
Dual spaces and arbitrage



Algebra of arbitrage

- The **adjoint transformation**, given by the **transpose** of a matrix, goes between dual spaces **in the opposite direction**.

$$S[\mathbf{x}] = \psi[\mathbf{b}] = \psi[A\mathbf{x}]$$
$$\mathbf{S}^*\mathbf{x} = \psi^* A\mathbf{x} = (A^*\psi)^*\mathbf{x}$$
$$\mathbf{S} = A^*\psi$$



Arbitrage pricing theorem

- Given payoffs *A,* prices ***S***, target asset with payoff **b**:

- Find all $\psi > 0 \text{ such that } A^*\psi = \mathbf{S}$
  - No solutions $\Rightarrow$ arbitrage
  - 1 solution $\Rightarrow$ complete market
  - Multiple solutions $\Rightarrow$ incomplete market

- Price asset using **all** solutions $\psi > 0 : \{\psi^*\mathbf{b}\}$
  - 1 solution $\Rightarrow$ redundant asset
  - Otherwise find full set of allowed non-arbitrage prices

Arbitrage pricing

- Example: incomplete market

$$\text{Suppose } A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} 1.5 \\ 0.5 \\ 0 \end{pmatrix}.$$

$$\text{Then } \psi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \implies A^* \psi = \mathbf{S} \text{ and } 0 < x < 1/2.$$

$$\begin{aligned} \text{Then price } S_b &= \psi^* \mathbf{b} = 0.5 + x(0.5) \\ &= \frac{1}{2}(1 + x), \\ \text{so } 1/2 < S_b &< 3/4. \end{aligned}$$

Asset pricing duality

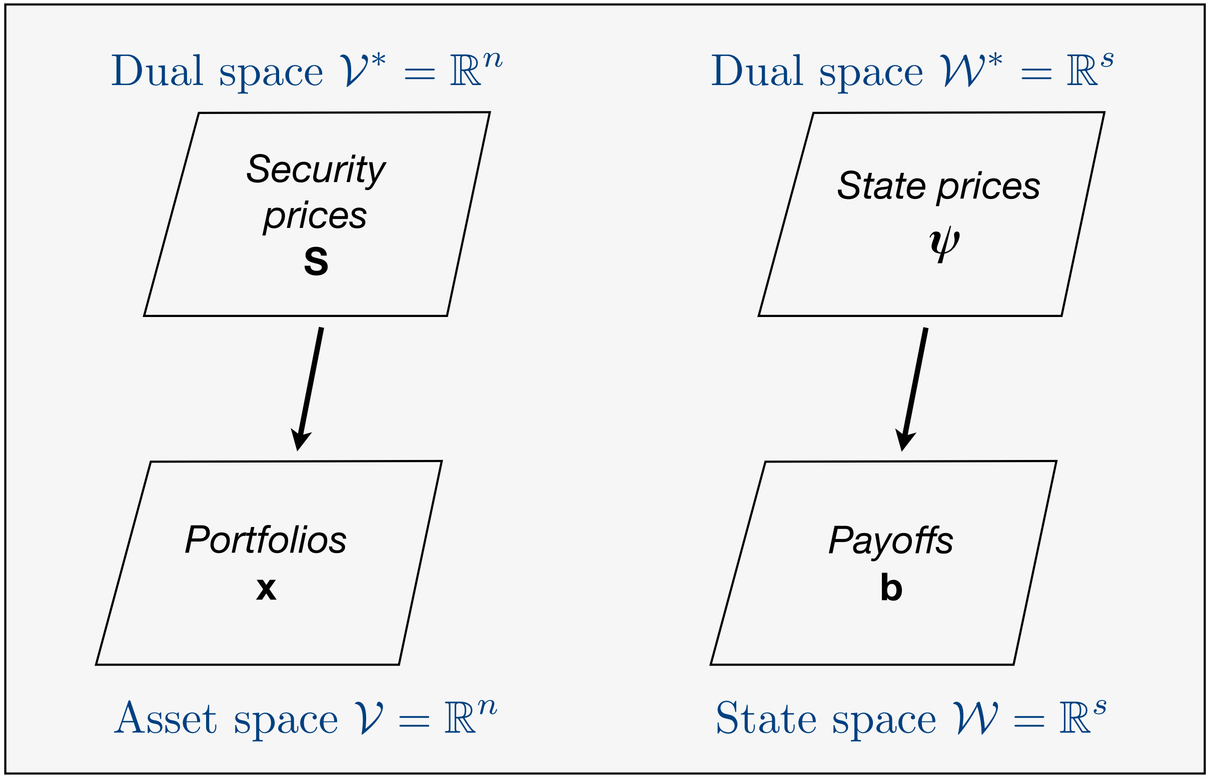
Algebra of arbitrage

- The **dual space** of a vector space consists of all **linear function(al)s** on vectors in the space.
  - Example: price of a portfolio

$$S[\mathbf{x}] = \mathbf{S}^* \mathbf{x}$$

  - Example: state price of a payoff

$$\psi[\mathbf{b}] = \psi^* \mathbf{b}$$



Algebra of arbitrage

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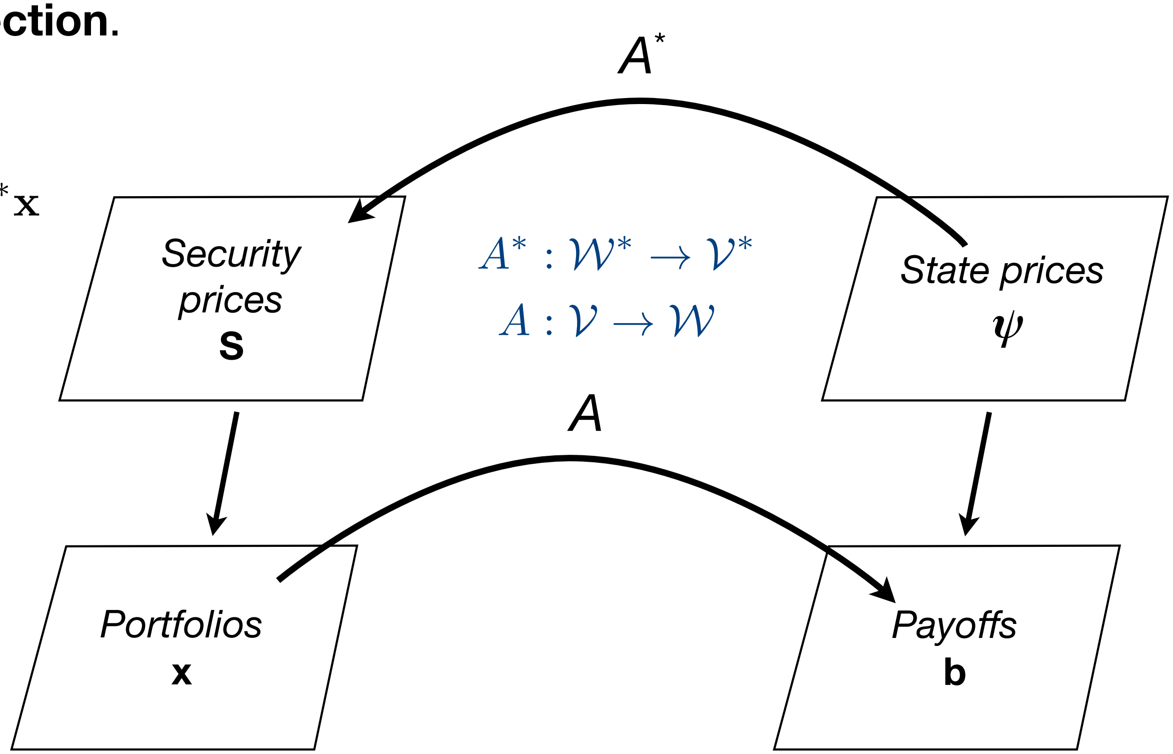
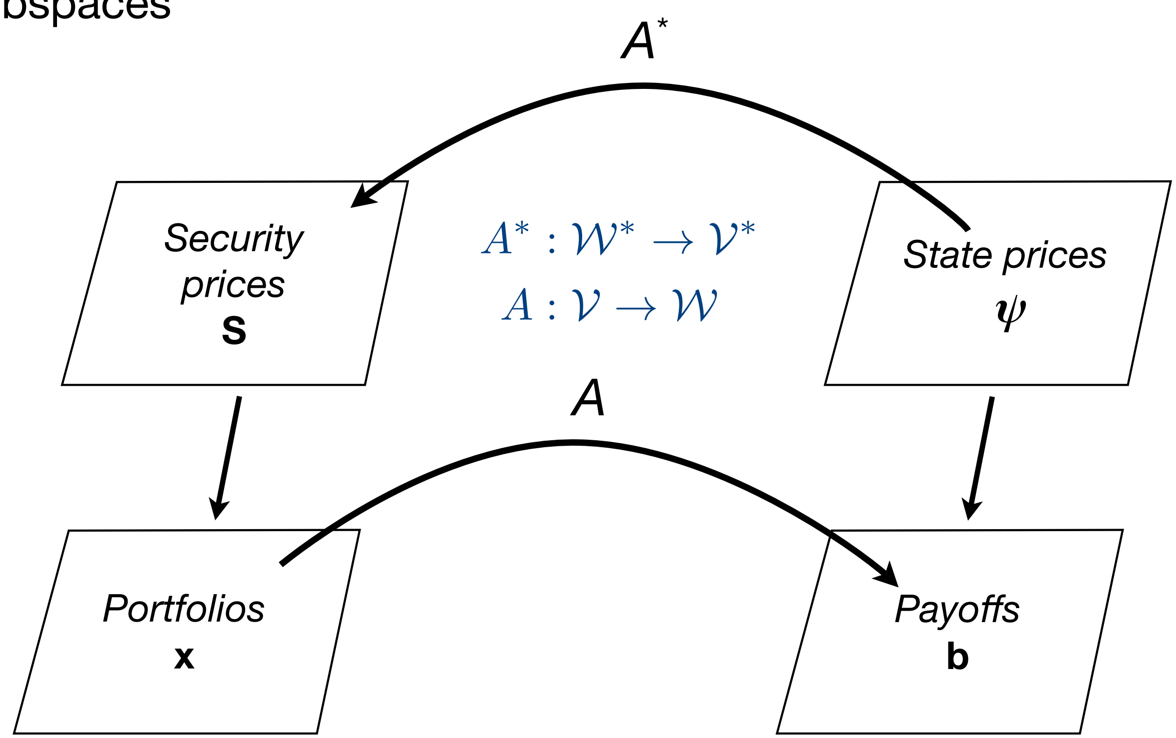


Algebra of arbitrage

- The operators have special relationships among their subspaces

$$\operatorname{Ker} A^* \perp \operatorname{Im} A$$
$$\operatorname{Im} A^* \perp \operatorname{Ker} A$$



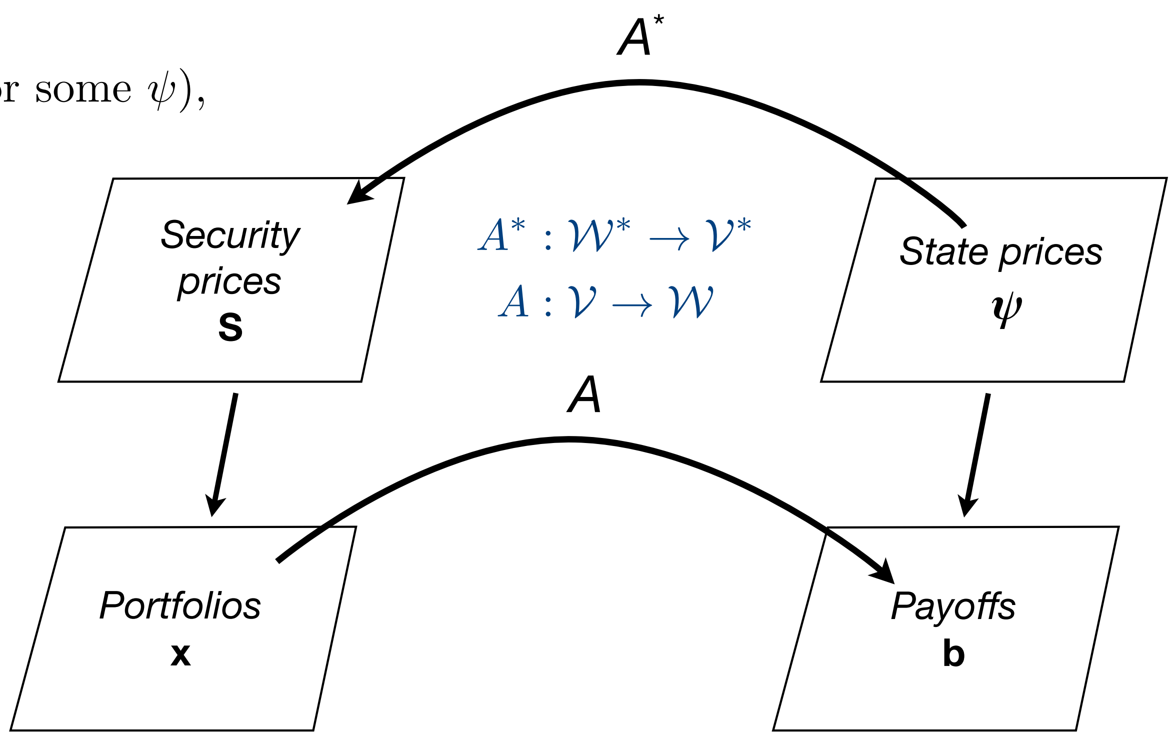
Algebra of arbitrage

- For arbitrage portfolios,

$$\text{If } \mathbf{x} \in \operatorname{Ker} A \quad (i.e., A\mathbf{x} = 0)$$

$$\text{and } \mathbf{S} \in \operatorname{Im} A^* \ (i.e., \mathbf{S} = A^*\psi \text{ for some } \psi),$$

$$\begin{aligned} \text{then } \mathbf{S}^*\mathbf{x} &= (A^*\psi)^*\mathbf{x} \\ &= (\psi^* A)\mathbf{x} \\ &= \psi^*(A\mathbf{x}) \\ &= 0. \end{aligned}$$



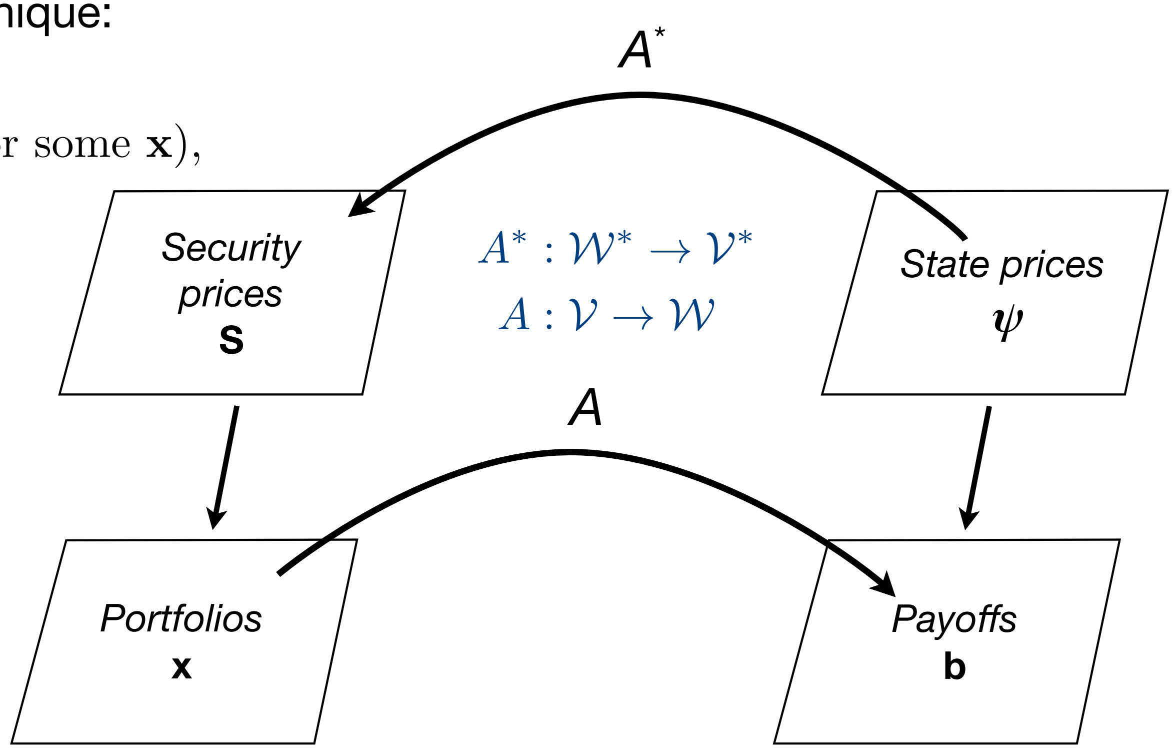
Algebra of arbitrage

- For incomplete markets portfolios, where state prices are not unique:

  If $\psi \in \mathrm{Ker}\, A^* \;\; (i.e., A^*\psi = 0)$

$$\text{and } \mathbf{b} \in \mathrm{Im}\, A \quad (i.e., \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x}),$$

$$\begin{aligned} \text{then } \psi^*\mathbf{b} &= \psi^*(A\mathbf{x}) \\ &= (\psi^* A)\mathbf{x} \\ &= (A^*\psi)^*\mathbf{x} \\ &= 0. \end{aligned}$$



Algebra of arbitrage

- Example: incomplete market $\mathrm{Im}\, A \subset \mathbb{R}^s, \ \mathrm{rank}(A) < s$

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix},$$

$$A^* = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}, \quad (\mathrm{Ker}\, A^*) = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

If $\psi = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = A\mathbf{x}$ for some portfolio $\mathbf{x}$,

then $\psi^*\mathbf{b} = \psi^*(A\mathbf{x}) = (\psi^* A)\mathbf{x} = (A^*\psi)^*\mathbf{x} = 0.$

Asset pricing duality

- We have two approaches to pricing, and they are not just equivalent, they are **dual:**
- Given *A*, **S**, and payoff **b** for "focus" asset:
- **State pricing**: compute price(s) of focus asset by applying all allowed $\boldsymbol{\psi}$
  - The price is uniquely determined if **b** is the payoff of a redundant asset

$$S_b = \psi[\mathbf{b}] = \{\psi^* \mathbf{b} : A^* \psi = \mathbf{S}, \psi \in \mathcal{W}^*, \psi > 0\}$$

- **Replication pricing** (no state prices involved): compute price(s) of replicating portfolio
  - The price is uniquely determined if **b** is the payoff of a redundant asset

$$S_b = \mathbf{S}[\mathbf{x}] = \{\mathbf{S}^* \mathbf{x} : \mathbf{x} \in \mathcal{V}, A\mathbf{x} = \mathbf{b}\}$$

Asset pricing duality

- **Replication pricing** (cont.): for non-redundant assets, can frame allowed prices as a **bound** on asset prices between $S_{\min} < S_b < S_{\max}$ where
  - most expensive sub-replicating portfolio $S_{\min} = \max\{\mathbf{S}^*\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$,
  - least expensive super-replicating portfolio $S_{\max} = \min\{\mathbf{S}^*\mathbf{x} : A\mathbf{x} \geq \mathbf{b}\}$

- Application: in this form, bounds identify (potential) arbitrage trade to execute *if* bound were ever violated (temporarily, for example).

Fundamental theorem of asset pricing (FTAP)

- There is no arbitrage if and only if there exists a **strictly positive** state-price vector ψ **consistent with** the security-price vector,

$$\mathbf{S} = A^* \psi, \quad \psi > 0, \qquad \text{where } \psi \in \mathcal{W}^*,\ \mathbf{S} \in \mathcal{V}^*,\ A^* : \mathcal{W}^* \to \mathcal{V}^*$$

- Absence of type-I arbitrage:

$$\text{If } A\mathbf{x} = \mathbf{b} \geq 0, \text{ then } \mathbf{S}^*\mathbf{x} > 0.$$
$$\textit{Proof:} \quad \mathbf{S}^*\mathbf{x} = (\psi^* A)\mathbf{x} = \psi^*(A\mathbf{x}) = \psi^*\mathbf{b} > 0.$$

- Absence of type-II arbitrage:

$$\text{If } A\mathbf{x} = 0, \text{ then } \mathbf{S}^*\mathbf{x} = 0.$$
$$\textit{Proof:} \quad \mathbf{S}^*\mathbf{x} = \psi^*(A\mathbf{x}) = 0.$$

Fundamental theorem of asset pricing (FTAP)

- The opposite direction of the FTAP follows from Farkas' lemma, which dictates strong alternatives:

- Farkas' Lemma:

$$\begin{aligned}&\text{Either}\\&\qquad (a)\ A^*\psi = \mathbf{S}, \quad \psi \geq 0 \text{ has a solution, or}\\&\qquad (b)\ A\mathbf{x} \geq 0, \quad \mathbf{S}^*\mathbf{x} < 0 \text{ has a solution}\\&\text{but not both.}\end{aligned}$$

- Geometric interpretation: the lemma says that a vector either lies within a convex cone (defined by A^*) or is separated from it by a hyperplane. (See Boyd & Vandenberghe (2004) Ch. 5 for proof and applications in linear optimization.)

Approximate hedge

Approximate hedge

- If the market is incomplete, can we find an **approximate** solution?
  - One idea: minimize least square error:

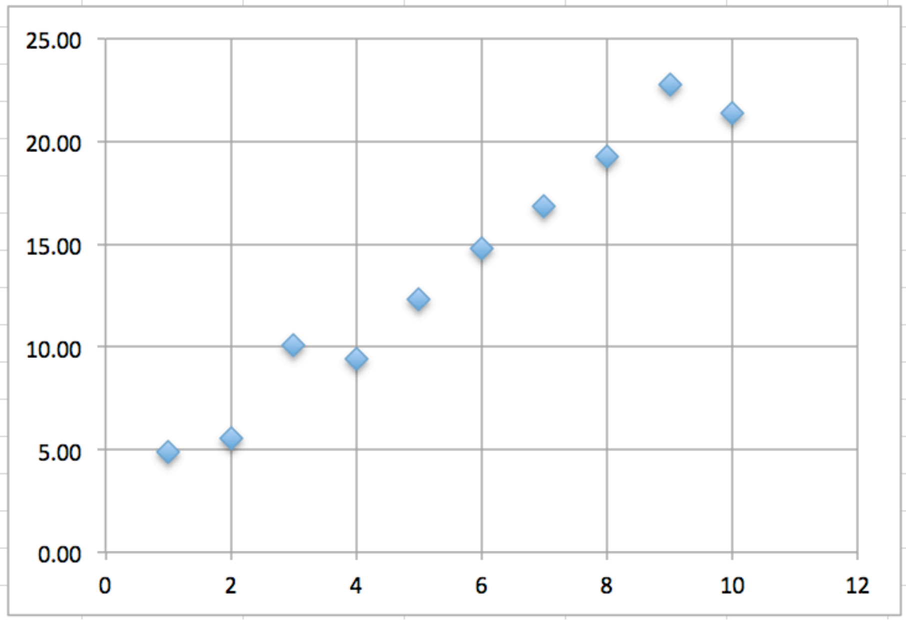
$$
\begin{aligned}
\mathbf{e} &= A\mathbf{x} - \mathbf{b} \neq 0 \\
f(\mathbf{x}) &= ||\mathbf{e}||^2 = (A\mathbf{x} - \mathbf{b})^*(A\mathbf{x} - \mathbf{b}) \\
\partial f / \partial x_i = 0 &\implies 2A^*(A\mathbf{x} - \mathbf{b}) = 0 \\
A^*A\mathbf{x} &= A^*\mathbf{b} \\
\mathbf{x} &= (A^*A)^{-1}A^*\mathbf{b}
\end{aligned}
$$

- How should we measure quality or closeness of solution?

- When does solution exist?

Recall least squares solution from statistics

	x	y
	1	4.91
	2	5.55
	3	10.10
	4	9.42
	5	12.32
	6	14.78
	7	16.82
	8	19.23
	9	22.79
	10	21.35
Mean	5.50	13.73
Std. Dev.	3.03	6.32
Median	5.50	13.55



Least squares solution

- Relationship structure

$$y_i = \alpha + \beta x_i + \epsilon_i,$$

$$\mathbf{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad A = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix}, \quad \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \end{pmatrix}$$

- Minimize square error by finding "**pseudoinverse**"

$$||\epsilon||^2 = ||A\mathbf{v} - \mathbf{b}||^2,$$

$$\mathbf{v} = (A^* A)^{-1} A^* \mathbf{b}$$

Least squares solution

```
> A <- matrix(cbind(ones,x),ncol=2); A
      [,1] [,2]
 [1,]    1    1
 [2,]    1    2
 [3,]    1    3
 [4,]    1    4
 [5,]    1    5
 [6,]    1    6
 [7,]    1    7
 [8,]    1    8
 [9,]    1    9
[10,]    1   10

> b <- y; b
       [,1]
 [1,]  4.91
 [2,]  5.55
 [3,] 10.10
 [4,]  9.42
 [5,] 12.32
 [6,] 14.78
 [7,] 16.82
 [8,] 19.23
 [9,] 22.79
[10,] 21.35
```

```
> Apinv <- solve(t(A) %*% A) %*% t(A)
> Apinv %*% A
             [,1]         [,2]
[1,] 1.000000e+00 1.332268e-15
[2,] 1.387779e-17 1.000000e+00

> round(Apinv %*% A,2)
     [,1] [,2]
[1,]    1    0
[2,]    0    1

> Apinv %*% y
         [,1]
[1,] 2.428667
[2,] 2.054242

> lm(y ~ x)

Call:
lm(formula = y ~ x)

Coefficients:
(Intercept)            x
      2.429        2.054
```

Least squares solution

```
A =

     1     1
     1     2
     1     3
     1     4
     1     5
     1     6
     1     7
     1     8
     1     9
     1    10

b =

    4.9100
    5.5500
   10.1000
    9.4200
   12.3200
   14.7800
   16.8200
   19.2300
   22.7900
   21.3500
>> M=inv(A'*A)*A';
```

```
>> M*A

ans =

    1.0000         0
    0.0000    1.0000


>> v=M*b

v =

    2.4287
    2.0542

>> R2 = corr(b,A(:,2))^2

R2 =

    0.9684
```

Descriptive statistics

	x	*y*
	1	4.91
	2	5.55
	3	10.10
	4	9.42
	5	12.32
	6	14.78
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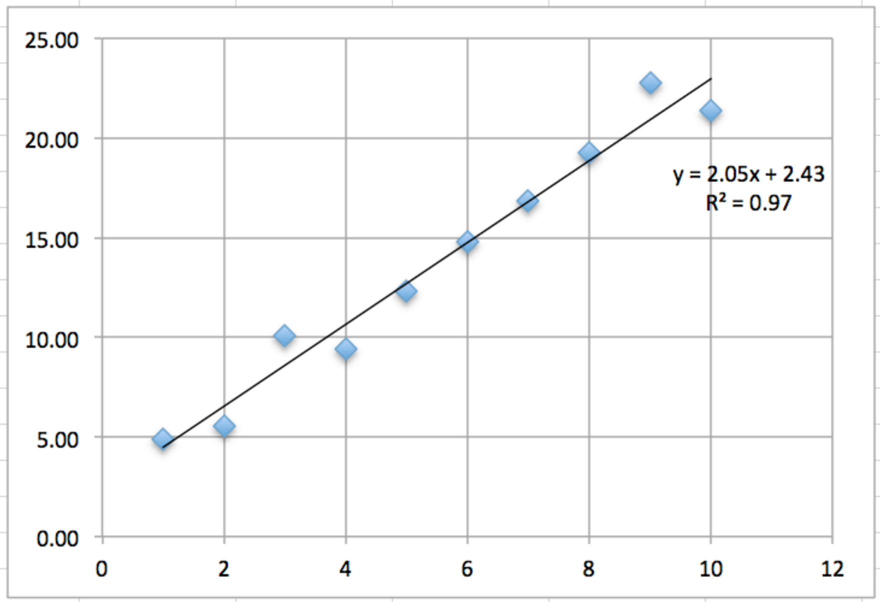


Risk-neutral probabilities and pricing

An improbable trick

- Whenever there is a bilinear sum (or integral), and all the coefficients are positive, we can interpret it **as if** probabilities were involved.

- Define these fictitious probabilities by dividing each coefficient by their sum

$$s(\mathbf{y}) = \sum_i c_i y_i, \quad \text{define } q_i \equiv \frac{c_i}{C}, \quad \text{where } C = \sum_i c_i$$

- Then,

$$s(\mathbf{y}) = C \sum_i q_i y_i = C\,\mathrm{E}[\mathbf{y}] = \mathrm{E}\left[C\mathbf{y}\right]$$

where the (fictitious) expectation is taken with respect to these "probabilities."

Risk-neutral probabilities

- There is an interesting interpretation of the state price vector: if we multiply by a constant so that the sum of the components is unity, those components can be interpreted **as if** they were probabilities for the occurrence of each state.

In components, write $\mathbf{S} = A^*\psi$ as $S_i = \sum_{\nu=1}^{s} A_{\nu i}\psi_\nu$, where $\nu = 1, 2, \ldots, s$ labels the states, and $i = 1, 2, \ldots, n$ labels the assets.

- Define

$$\begin{aligned} R_{\nu i} &= A_{\nu i}/S_i = \text{return of asset } i \text{ in state } \nu, \\ R_f &= A_{\nu 1}/S_1 = \text{risk-free rate, independent of state } \nu, \\ q_\nu &= R_f \psi_\nu = \text{risk-neutral probability of state } \nu. \end{aligned}$$

- Then

$$\sum_\nu q_\nu = \sum_\nu R_f \psi_\nu = \frac{\sum A_{\nu 1}\psi_\nu}{S_1} = 1$$

Risk-neutral pricing

- Now re-express the pricing equation to get two interesting forms.

$$S_i = \sum A_{\nu i}\psi_\nu = \sum \frac{A_{\nu i}}{R_f} q_\nu = \mathrm{E}\left[\frac{A_{\nu i}}{R_f}\right]$$

- This says that the security price of risky assets is the **expectation** of the discounted present value of the **payoff** under the **risk-neutral measure**. Divide by price and multiply by the risk-free rate to get

$$\sum R_{\nu i} q_\nu = \mathrm{E}\left[R_{\nu i}\right] = R_f$$

- This says that the **expected return** of every risky asset is just the **risk-free rate** under the risk-neutral measure

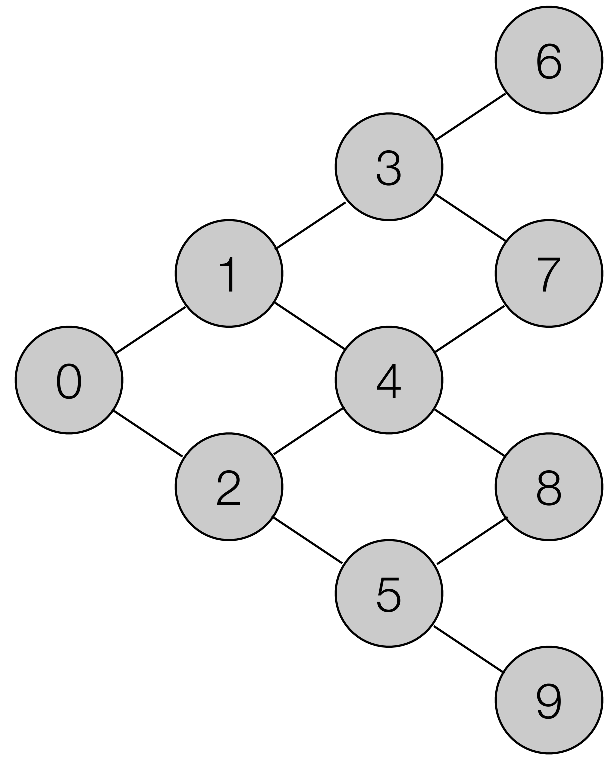
Risk-neutral pricing

- Are risk-neutral probabilities really fictitious?
- Inferred from prices, sentiment in market
- Compare the questions
  - How **likely** is this event?
  - How much does it **cost** to insure against this event?
- Does a disagreement between objective and risk-neutral probabilities imply arbitrage?
- No. On the contrary, arbitrage is absent as long as risk-neutral probabilities exist.

Dynamic markets

Next-to-simplest model

- Let's go from one period to many.
- Discrete time: 0,1,2,...,T,....
- Discrete states, distinct **at each time step**.
- Start with two basis assets
  - Bond – same payoff, regardless of state, between any two fixed times
  - Stock – uncertain payoff
    - State-dependence
    - Path-dependence



Static replication

- Example: *T*=3, stock moves only up or down.

$$A = \begin{pmatrix} R_f^3 & S_0 R_u^3 \\ R_f^3 & S_0 R_u^2 R_d \\ R_f^3 & S_0 R_u R_d^2 \\ R_f^3 & S_0 R_d^3 \end{pmatrix}$$

- Suppose there is an option with strike price *K*

$$S_1 > S_2 > K > S_3 > S_4$$

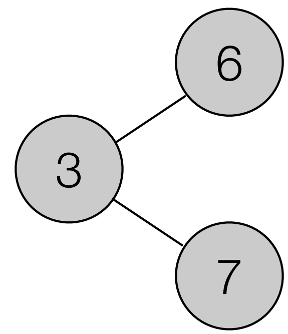
- Payoff on a call or a put is (in terms of terminal prices)

$$C = \begin{pmatrix} S_1 - K \\ S_2 - K \\ 0 \\ 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 \\ 0 \\ K - S_3 \\ K - S_4 \end{pmatrix}$$

- Cannot replicate either with stock and bond – an **incomplete** market.

Dynamic hedging

- Instead, let's consider replication one time-step at a time
- Focus on each node and the possible conditional outcomes
- Example: payoff at *t=3*, given that one is already in a specific state at *t=2*



$$\mathbf{b} = \begin{pmatrix} S_1 - K \\ S_2 - K \end{pmatrix} = \begin{pmatrix} C_u \\ C_d \end{pmatrix}$$

$$A = \begin{pmatrix} R_f & S_0 R_u^3 \\ R_f & S_0 R_u^2 R_d \end{pmatrix} = \begin{pmatrix} R_f & S_u \\ R_f & S_d \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 \\ S(t=2) = S_0 R_u^2 \end{pmatrix}$$

Dynamic hedging

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- Focus on each node and the possible conditional outcomes
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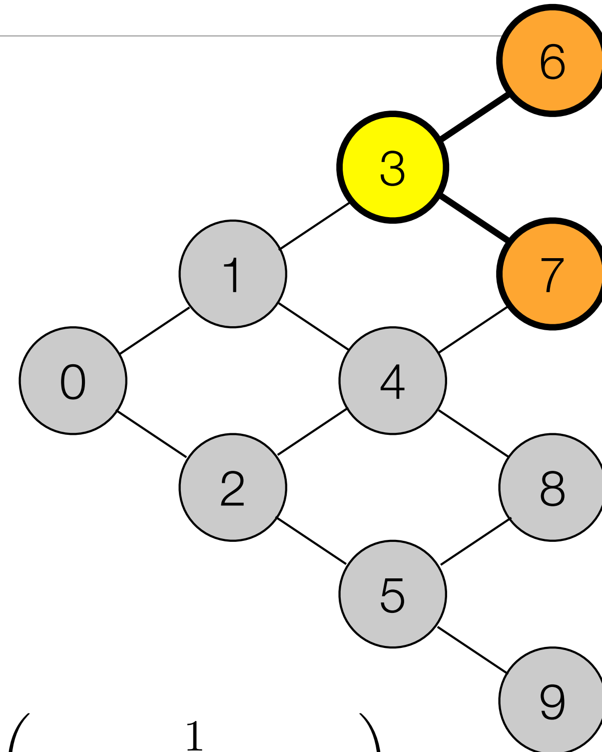

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Dynamic hedging

- Solve for the replicating portfolio for the node:

$$A\mathbf{x} = \mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b}$$

$$A = \begin{pmatrix} R_f & S_u \\ R_f & S_d \end{pmatrix}, \quad A^{-1} = \frac{1}{R_f(S_d - S_u)} \begin{pmatrix} S_d & -S_u \\ -R_f & R_f \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} B \\ \Delta \end{pmatrix} = A^{-1} \begin{pmatrix} C_u \\ C_d \end{pmatrix} = \frac{1}{R_f(S_d - S_u)} \begin{pmatrix} S_d C_u - S_u C_d \\ -R_f C_u + R_f C_d \end{pmatrix} = \begin{pmatrix} \frac{S_u C_d - S_d C_u}{R_f(S_u - S_d)} \\ \frac{C_u - C_d}{S_u - S_d} \end{pmatrix}$$

- Note that

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \begin{cases} 1 & \text{if both states in the money since } C = S - K, \\ 0 & \text{if both states out of the money, } C_u = C_d = 0 \end{cases}$$

Dynamic hedging

- Now that we have a replicating portfolio for this node (and **any** node),

$$\mathbf{S}^*\mathbf{x} = B + S\Delta$$

- State prices are coefficients of the payoff, or more directly,

$$\begin{aligned}
\psi &= (A^*)^{-1}\mathbf{S} = (A^{-1})^*\mathbf{S} \\
&= \frac{-1}{R_f(S_u - S_d)} \begin{pmatrix} S_d & -R_f \\ -S_u & R_f \end{pmatrix} \begin{pmatrix} 1 \\ S \end{pmatrix} = \frac{-1}{R_f(S_u - S_d)} \begin{pmatrix} S_d - R_f S \\ -S_u + R_f S \end{pmatrix} \\
&= \frac{+1}{R_f(R_u - R_d)} \begin{pmatrix} R_f - R_d \\ R_u - R_f \end{pmatrix} = \frac{1}{R_f}\mathbf{q}
\end{aligned}$$

- No arbitrage requires $R_u > R_f > R_d$

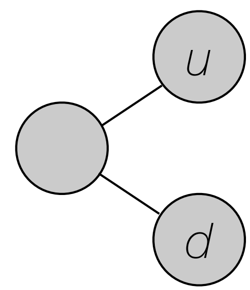
Dynamic hedging

- The state prices and risk-neutral probabilities depend only on the return parameters

$$\mathbf{q} = \begin{pmatrix} \frac{R_f - R_d}{R_u - R_d} \\ \frac{R_u - R_f}{R_u - R_d} \end{pmatrix}$$

- Therefore at any node, call value given in terms of **next period** state values as

$$C = \psi^* \mathbf{b} = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix} \begin{pmatrix} C_u \\ C_d \end{pmatrix}$$
$$= q_u \frac{C_u}{R_f} + q_d \frac{C_d}{R_f}$$

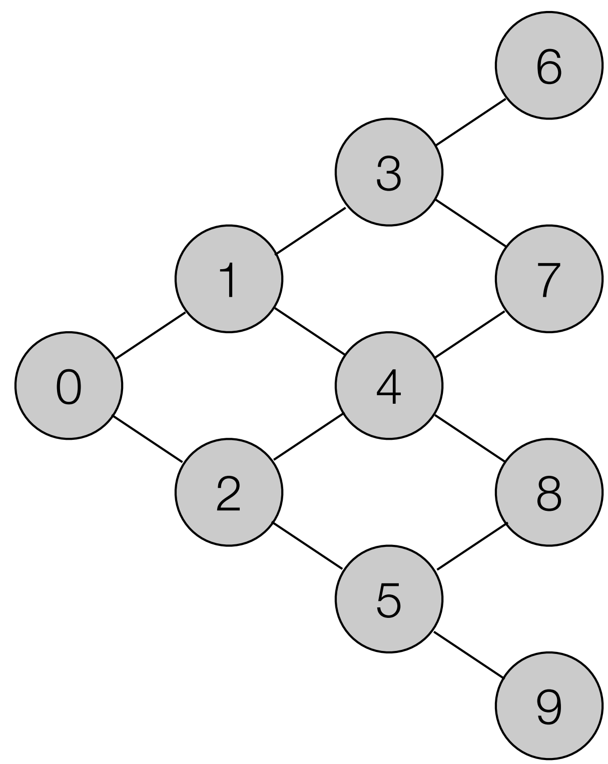


- This is the expected discounted payoff, under the risk-neutral measure, at the node.

Pricing: work backward from terminal values

- Omitting time subscripts for clarity,

$$
\begin{aligned}
t=2: \quad & C_3 = \frac{1}{R_f}\left(q_u C_6 + q_d C_7\right) \\
& C_4 = \frac{1}{R_f}\left(q_u C_7 + q_d C_8\right) \\
& C_5 = \frac{1}{R_f}\left(q_u C_8 + q_d C_9\right) \\
t=1: \quad & C_1 = \frac{1}{R_f}\left(q_u C_3 + q_d C_4\right) \\
& C_2 = \frac{1}{R_f}\left(q_u C_4 + q_d C_5\right) \\
t=0: \quad & C_0 = \frac{1}{R_f}\left(q_u C_1 + q_d C_2\right)
\end{aligned}
$$



Pricing: work backward from terminal values

- Each step of recursion is an expectation

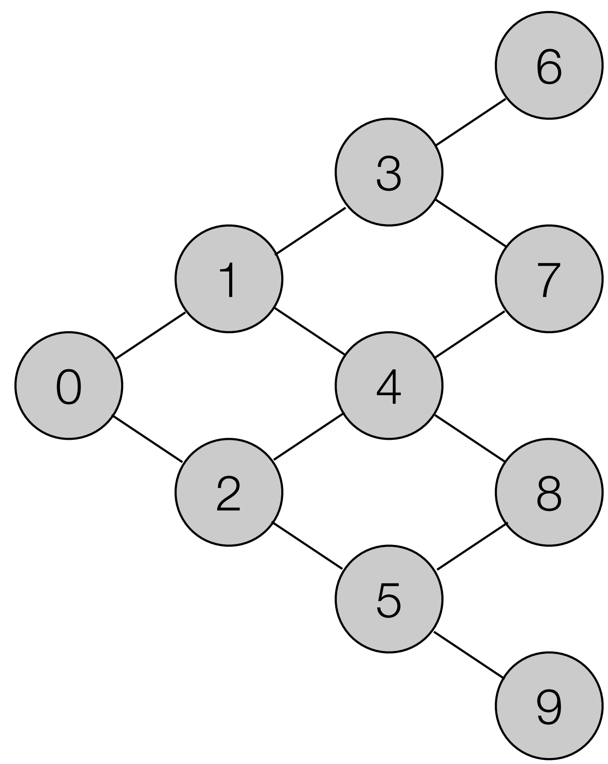
- In general, write at each node

$$C_t = \mathrm{E}_t\left[\frac{C_{t+1}}{R_f}\right]$$

  where the expectation is conditional on the given time. For example, for *t=2* at node 5, the only states that enter on the right-hand side are the *t=3* states: {8,9}

- Then combining steps, the *t=0* price is

$$C_{t=0} = \frac{1}{R_f^3}\mathrm{E}\left[\mathrm{E}_1\left[\mathrm{E}_2\left[C_3\right]\right]\right] = \frac{1}{R_f^3}\mathrm{E}\left[C_3\right]$$



Conditional and iterated expectations

- Basic result on **iterated** expectations: the expectation of an expectation is an expectation!
  - Important because hedging and pricing are conditioned on information at a given time.
  - Filtrations: information revealed over time

- Conditional probability and expectation:

$$\mathrm{Prob}(A = a, B = b) = \mathrm{Prob}(A = a | B = b)\,\mathrm{Prob}(B = b)$$
$$\mathrm{E}\left[A | B = b\right] = \sum_a a\,\mathrm{Prob}(A = a | B = b) = f(b)$$

- The expectation of *A* conditional on *B* = *b* is a function of *b*. Taking **its** expectation...

Conditional and iterated expectations

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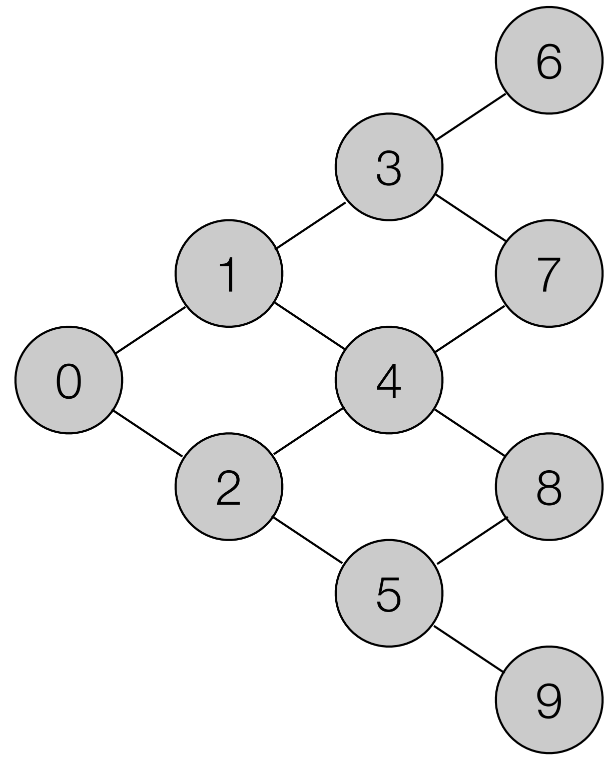
$$
\begin{aligned}
\mathrm{E}\left[\mathrm{E}\left[A|B\right]\right] = \mathrm{E}\left[f(b)\right] &= \sum_b \mathrm{Prob}(B=b) f(b) \\
&= \sum_b \mathrm{Prob}(B=b) \left[\sum_a a \,\mathrm{Prob}(A=a|B=b)\right] \\
&= \sum_{a,b} a \left[\mathrm{Prob}(A=a|B=b)\,\mathrm{Prob}(B=b)\right] \\
&= \sum_{a,b} a \,\mathrm{Prob}(A=a, B=b) \\
&= \sum_a a \,\mathrm{Prob}(A=a) \\
&= \mathrm{E}\left[A\right]
\end{aligned}
$$

Pricing: work backward from terminal values

- This works for **any function** of the stock price on the terminal nodes
  - Calls of any strike
  - Puts of any strike
  - Forwards
  - Etc.

- Key ingredients
  - Bond: captures time value of money
  - Stock: captures uncertainty over time
  - Delta: solves for unique dynamic hedging ratio
  - Risk-neutral probabilities: consistently assigned



Calibration

- Identification of binomial model parameters
  - Returns (up, down)
  - Probabilities (up, down)
  - Average return per period
  - Volatility of return

$$\begin{aligned}\overline{R} = \mathrm{E}[R] &= pR_u + (1-p)R_d \\ &= R_d + p(R_u - R_d)\end{aligned}$$

$$\begin{aligned}\sigma^2 &= \mathrm{E}\left[(R-\overline{R})^2\right] = \mathrm{E}\left[R^2\right] - \overline{R}^2 \\ &= \left[pR_u^2 + (1-p)R_d^2\right] - \left[pR_u + (1-p)R_d\right]^2 \\ &= p(1-p)(R_u - R_d)^2\end{aligned}$$

- Example:

$$\begin{aligned}\overline{R} &= 1.009 = 1 + 0.9\% \\ \sigma &= 4.4\% \\ p &= 1/2 \\ R_u &= 1.053 \\ R_d &= 0.965\end{aligned}$$

$$R_u = \overline{R} + \sigma\sqrt{\frac{1-p}{p}},$$

$$R_d = \overline{R} - \sigma\sqrt{\frac{p}{1-p}},$$