15.456 – Financial Engineering

Problem Set 3

1. A financial market consists of a riskless asset and a risky asset. Their initial values are \$1 and \$2, respectively. The riskless asset yields an interest rate of 50% in each period. The risky asset pays no dividend and its return is IID. Its price can double in each period (with probability 1/3) or drop by half (with probability 2/3).

   (a) Show that the market is dynamically complete.

   (b) Derive the state prices for each node/state.

   (c) Derive the risk-neutral measure in this market.

   (d) Derive the state price density (SPD) for this market.

   (e) Compute the risk premium for the risky asset. Does it change over time? Explain.

   (f) Consider a security that pays $\$3 - \max(0, \$3 - P_2)$, where P_2 is the price of the risky asset at $t = 2$. Use the risk-neutral pricing formula to compute its price at time 0.

2. In the market of the previous problem, consider a vanilla European call option on the risky asset with strike price $K = \$2$, expiring at $t = 3$.

   (a) Find the self-financing portfolio that replicates the payoff of the call option. Diagram the process, and indicate clearly the conditional values at each node for the quantity of shares held, the cash balance in the risk-free asset, and the market value of the replicating portfolio.

   (b) What is the value of the option at $t = 0$?

   (c) At times other than $t = 0$, what is the relationship between the option price and the value of the self-financing replicating portfolio?

3. Suppose that the option in the previous problem is mispriced on an exchange. Its market price at $t = 0$ is \$1 more than the value derived from risk-neutral pricing.

   (a) You decide to sell the option. What is the expected P/L and volatility of your trade?

   (b) Now suppose that you decide to sell the option in the market and simultaneously hedge the option using a dynamic, self-financing, replicating portfolio. What is the expected P/L and volatility of your trade? How does your answer depend on the market price of the option at intermediate times?

4. Use a binomial tree to analyze the contracts below under the following conditions: $S_0 = \$8$, $T = 3$, $R_u = 2$, $R_d = 1/2$. Find the price of each contract for two cases: (i) $R_f = 1$ and (ii) $R_f = 1.25$.

   (a) European call and put, strike price $K = \$10$, expiring at $T = 3$.

   (b) Lookback option which gives the right at $T = 3$ to buy the stock at the lowest price realized between $t = 0$ and $t = T$.

   (c) Barrier call option with strike price $K = S_0$ that is cancelled if the stock price reaches $\frac{1}{2}S_0$ at any time.

5. Write a function in R or Python to find the price of a European call and put option on a non-dividend paying stock using a binomial tree. The function `BTvalues(S, K, T, Ru, Rd, Rf)` should take as inputs the initial stock price, the strike price, the number of time steps, the up- and down-factors for each node, and the risk-free gross return for each time period. Use the function to find the $t = 0$ price of a European call and put when $S = \$100$, $K = \$105$, $T = 3$, $R_u = 2 = 1/R_d$, and $R_f = 1.25$. (*Optional, extra credit: have your function solve for the option delta as well.*