Lecture 8

Time Series Variation in Volatility

What have we learned about the aggregate stock market?

- It yields a positive risk premium, but the risk premium is difficult to measure with precision because of
  - the “high” level of stock market volatility
  - and the limited length of the historical data.

- There is some evidence that expected returns are time varying. The autocorrelation of the aggregate stock returns is slightly positive for some horizons and the dividend-to-price ratio may have some predictability for future stock returns.

- Overall, only a small portion of future stock returns can be predicted (low R-squared’s), and much of the variation is unpredictable.

The volatility of the aggregate stock market

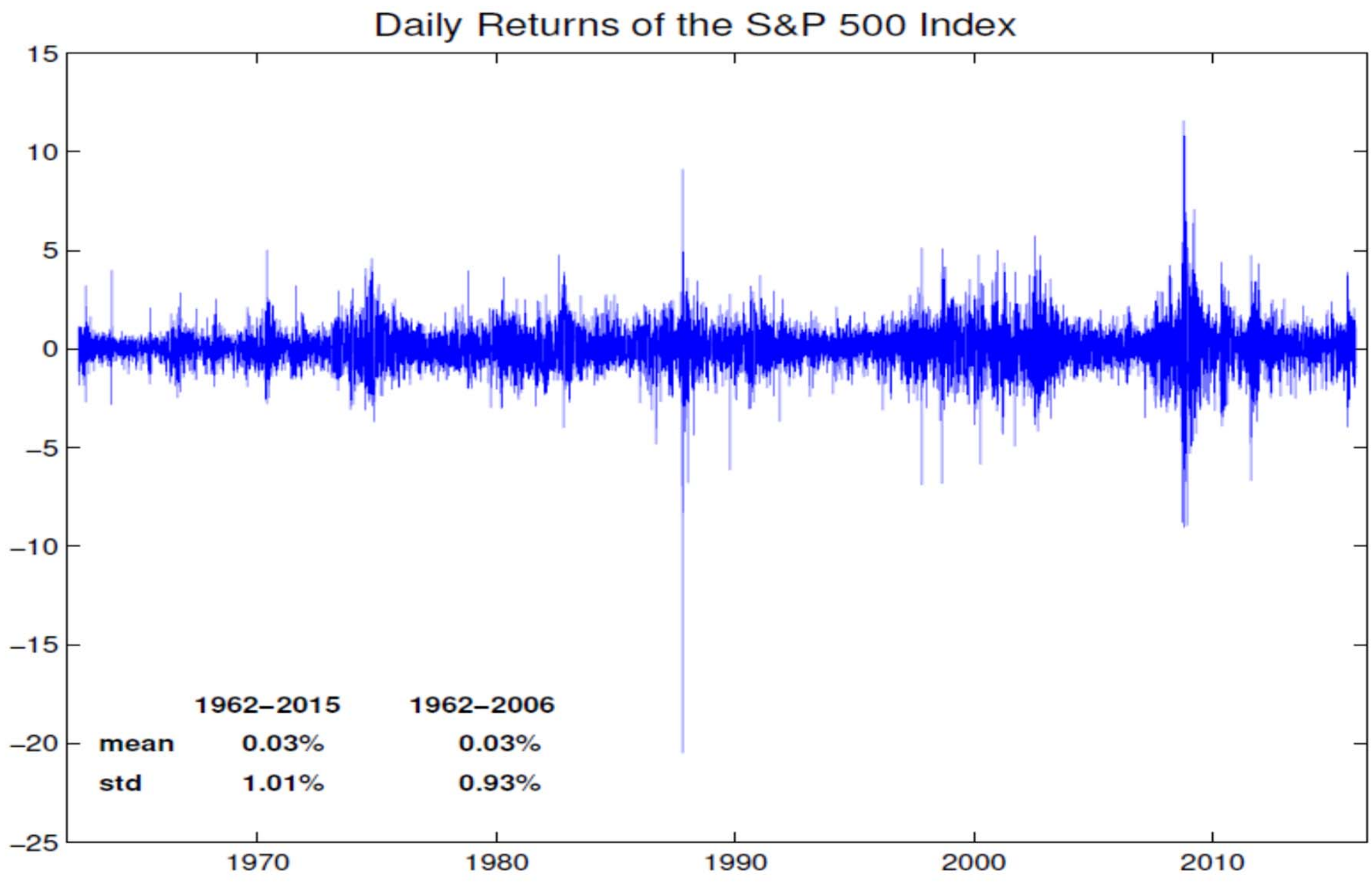
- Historical data can be used to measure volatility with much better precision
- In fact, we can learn about market volatility not only from the historical stock market data (backward looking), but also from derivatives prices (forward looking).
- We will study three volatility estimators:
  - SMA: simple moving average model
  - EWMA: exponentially weighted moving average model
  - ARCH and GARCH models

The importance of measuring market volatility

- Portfolio managers performing optimal asset allocation
- Risk managers assessing portfolio risk (e.g., Value-at-Risk)
- Derivatives investors trading non-linear contracts with values linked directly to market volatility
- Increasingly, the level of market volatility (e.g., VIX) has become a market indicator ("the fear gauge") watched closely by many institutional investors

Measuring Market Risk

- By early 1990s, increasing complexity in the financial instruments made the trading books of many investment banks too complex and diverse for executives to understand the overall risk of their firms
- Market risk management tools such as Value-at-Risk are ways to aggregate the firm-wide risk to a set of numbers that can be easily communicated to the chief executives. By the mid-1990s, most Wall Street firms have developed risk measurement into a firm-wide system
- Daily estimates of market volatility, along with correlations across financial assets, constitute the key inputs to Value-at-Risk. JP Morgan's RiskMetrics popularized exponentially weighted moving average (EWMA) model to estimate the volatilities and correlations of over 480 financial time series in order to construct a variance-covariance matrix of 480x480.


Daily Returns of the S&P 500 Index
15
10
5
0
−5
−10
−15
−20
−25
1970
1980
1990
2000
2010
1962−2015
1962−2006
mean
0.03%
0.03%
std
1.01%
0.93%

The Simple Moving Average Model

- Unlike expected returns, volatility can be measured with better precision using higher frequency data. So let's use daily data.
- Some have gone into higher frequency by using intra-day data. But micro-structure noises such as bid/ask bounce start to dominate in the intra-day domain.
- Suppose in month t, there are N trading days, with R_n denoting n-th day return. The simple moving average (SMA) model:

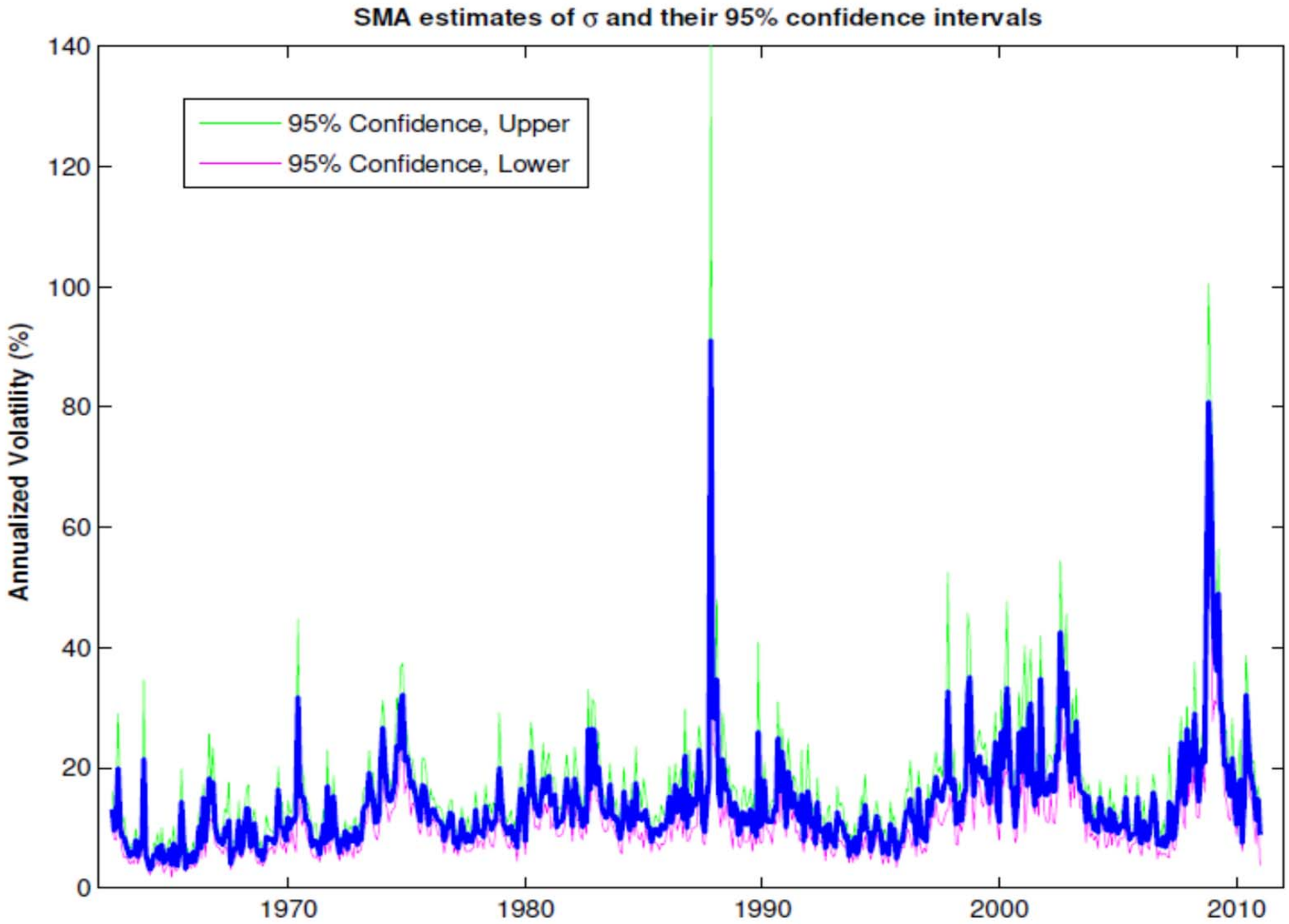
$$\sigma = \sqrt{\frac{1}{N}\sum_{n=1}^{N}(R_n)^2}$$

- To get an annualized number: $\sigma \times \sqrt{252}$. (252 trading days per year).

Whether or not to take out μ?

- The industry convention is such that $(R_t - \mu)^2$ is replaced by R_t^2 in the volatility calculation.
- The reason is that, at daily frequency, μ^2 is too small compared with $E(R^2)$, μ is several basis points while σ is close to 1%.
- So instead of going through the trouble of doing $E(R^2) - \mu^2$, people just do $E(R^2)$.

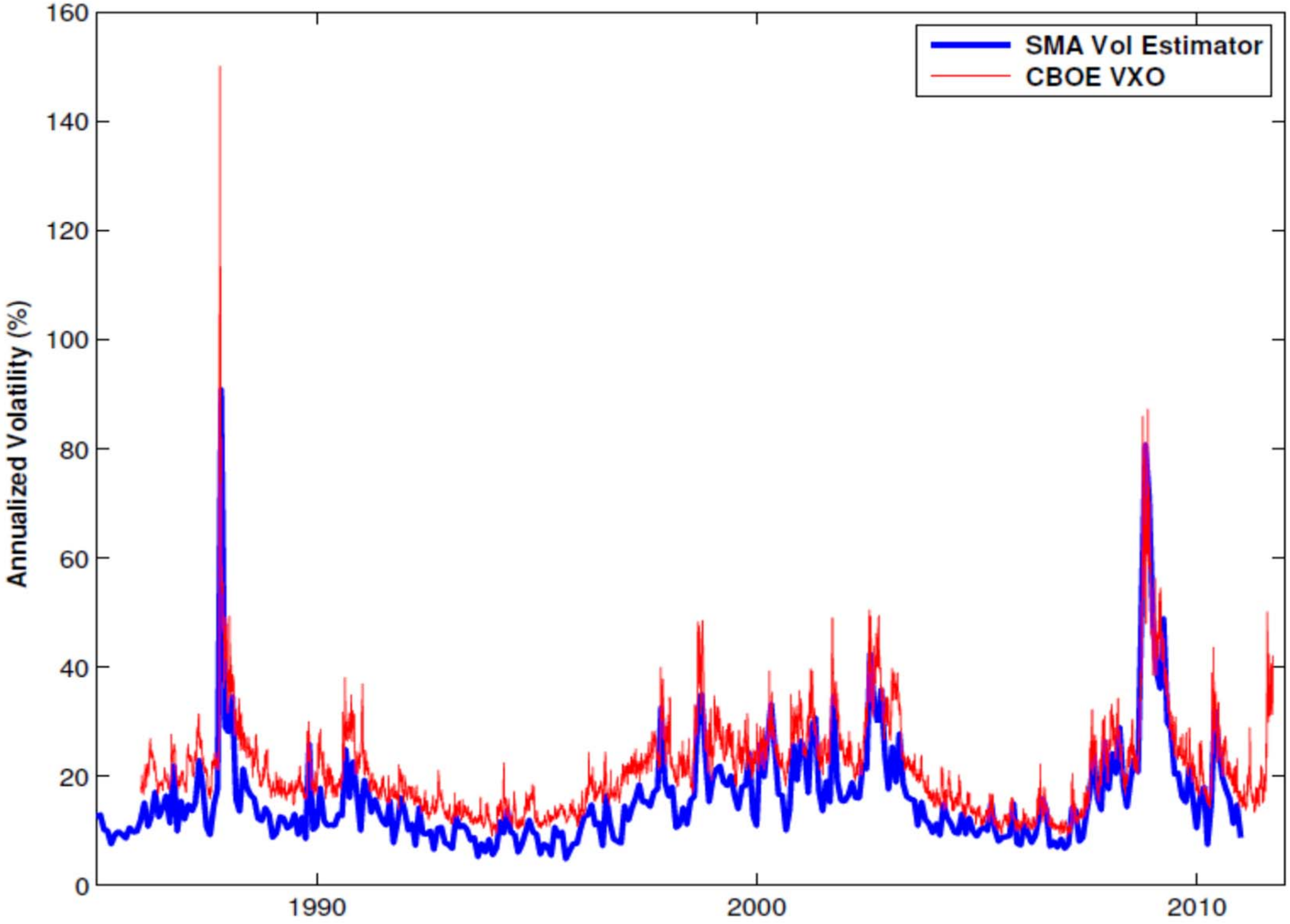
How precise are SMA volatility estimates?



Why would volatility change over time?

- If the rate of information arrival is time-varying, so is the rate of price adjustment, causing volatility to change over time.
- The time-varying volatility of the market return is related to the time-varying volatility of a variety of economic variables, including inflation, unemployment rate, money growth and industrial production.
- Stock market volatility increases with financial leverage: a decrease in stock price causes an increase in financial leverage, causing volatility to increase.
- Investors' sudden changes of risk attitudes, changes in market liquidity, and temporary imbalance of supply and demand could all cause market volatility to change over time.

SMA vs. Option-Implied



Exponentially weighted moving average model

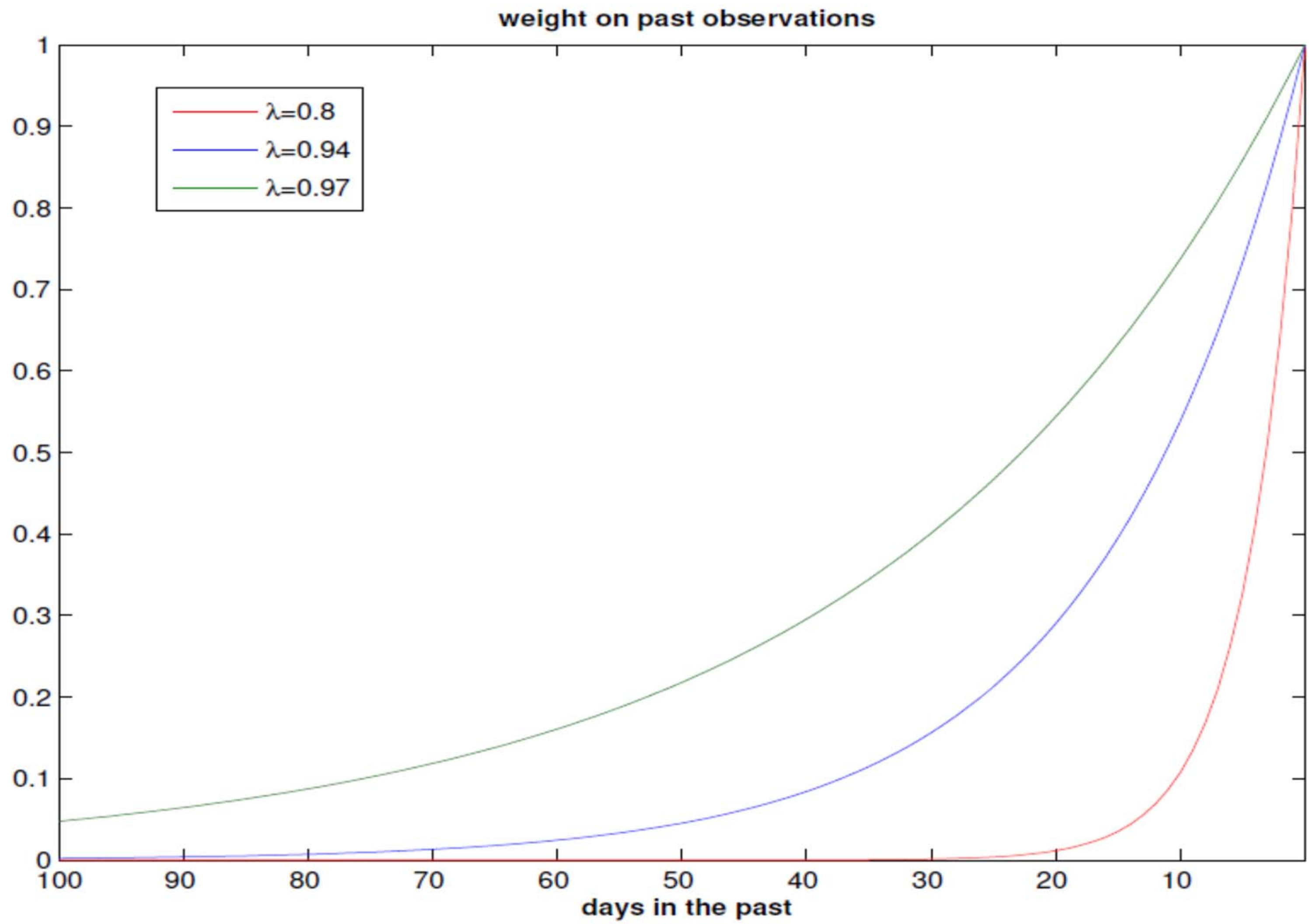
- The simple moving average (SMA) model fixes a time window and applies equal weight to all observations within the window.
- In the exponentially weighted moving average (EWMA) model, the more recent observation carries a higher weight in the volatility estimate.
- The relative weight is controlled by a decay factor λ.
- Suppose R_t is today's realized return, R_{t-1} is yesterday's, and R_{t-n} is the daily return realized n days ago. Volatility estimate σ:

Equally Weighted

$$\sqrt{\frac{1}{N}\sum_{n=0}^{N-1}\left(R_{t-n}\right)^2}$$

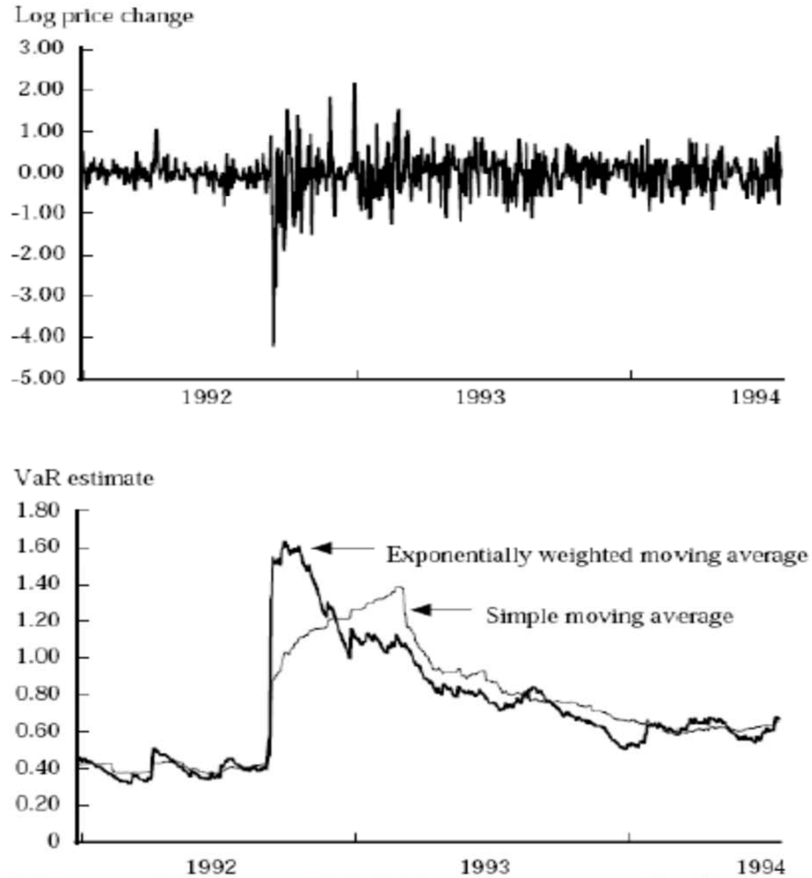
Exponentially Weighted

$$\sqrt{(1-\lambda)\sum_{n=0}^{N-1}\lambda^n\left(R_{t-n}\right)^2}$$


weight on past observations
λ=0.8
λ=0.94
λ=0.97
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
100
90
80
70
60
50
40
30
20
10
days in the past

SMA and EWMA Estimates after a Crash

Chart 5.2
Log price changes in GBP/DEM and VaR estimates (1.65σ)



Source: J.P.Morgan/Reuters RiskMetrics — Technical Document, 1996

Computing EWMA recursively

- One attractive feature of the exponentially weighted estimator is that it can be computed recursively.
- You will appreciate this convenience if you have to compute the EWMA volatility estimator in Excel.
- Let σ_t be the EWMA volatility estimator using all the information available on day $t-1$ for the purpose of forecasting the volatility on day t.
- Moving one day forward, it's now day t. After the day is over, we observe the realized return R_t.
- We now need to update our EWMA volatility estimator σ_{t+1} using the newly arrived information (i.e. R_t). It turns out that we can do so by

$$\sigma_{t+1}^2 = \lambda\,\sigma_t^2 + (1-\lambda)\,R_t^2$$

What about the first observation?

- The recursive formula has to start from the beginning:

$$\sigma_2^2 = \lambda\,\sigma_1^2 + (1-\lambda)\,R_1^2$$

  So what to use for σ_1?

- In practice, the choice of σ_1 does not matter in any significant way after running the iterative process long enough:

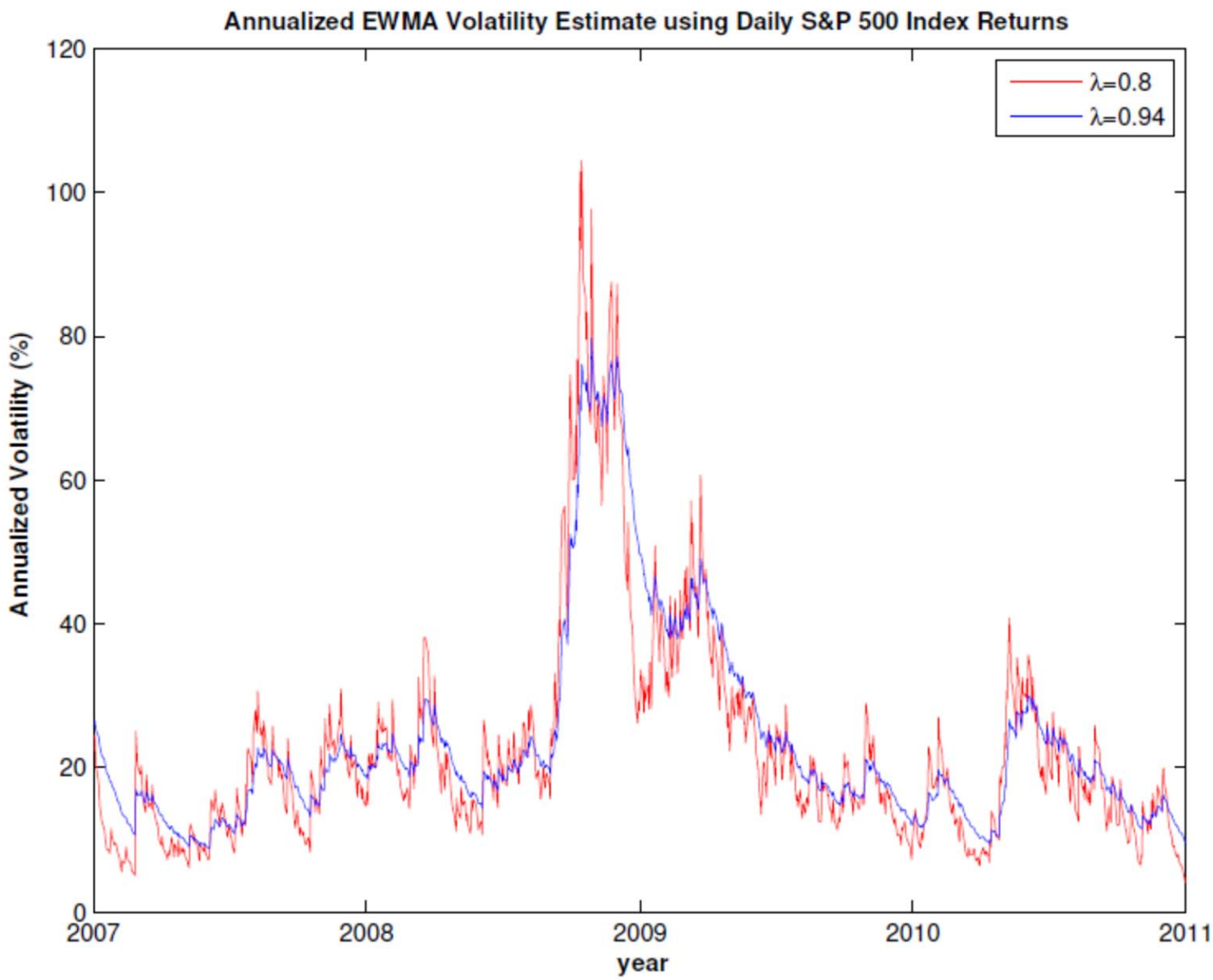
$$\begin{aligned}
\sigma_3^2 &= \lambda\,\sigma_2^2 + (1-\lambda)\,R_2^2 \\
&= \lambda^2\,\sigma_1^2 + (1-\lambda)\left(\lambda R_1^2 + R_2^2\right) \\
\sigma_4^2 &= \lambda\,\sigma_3^2 + (1-\lambda)\,R_3^2 \\
&= \lambda^3\,\sigma_1^2 + (1-\lambda)\left(\lambda^2 R_1^2 + \lambda R_2^2 + R_3^2\right) \\
\ldots \\
\sigma_t^2 &= \lambda^{t-1}\,\sigma_1^2 + (1-\lambda)\left(\lambda^{t-2} R_1^2 + \ldots + R_{t-1}^2\right)
\end{aligned}$$

- A good idea is to have the iterative process run for a while (say a few months) before recording volatility estimates.

- (The industry practice:) It is typical to set $\sigma_2^2 = R_1^2$ and start the recursive process from σ_3. The rationale is that σ_1 is unknowable and the only data we have at time 1 is R_1. So R_1^2 is our best estimate for σ_2^2. This approach is adopted by most of the practitioners.

Decay factor, Strong or Weak?

- A strong decay factor (that is, small λ) underweights the far away events more strongly, making the effective sample size smaller.
- A strong decay factor improves on the timeliness of the volatility estimate, but that estimate could be noisy and suffers in precision.
- On the other hand, a weak decay factor improves on the smoothness and precision, but that estimate could be sluggish and slow in response to changing market conditions.
- So there is a tradeoff.



Annualized EWMA Volatility Estimate using Daily S&P 500 Index Returns
λ=0.8
λ=0.94
Annualized Volatility (%)
0
20
40
60
80
100
120
2007
2008
2009
2010
2011
year

Picking the optimal decay factor based on volatility forecast

- RiskMetrics sets $\lambda = 0.94$ in estimating volatility and correlation. One of their key criteria is to minimize the forecast error.
- We form σ_{t+1} on day t in order to forecast the next-day volatility. So after observing R_{t+1}, we can check how good σ_{t+1} is in doing its job.
- This leads to the daily root mean squared prediction error

$$RMSE = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(R_{t+1}^2 - \sigma_{t+1}^2\right)^2}$$

- The deciding factor of RMSE is our choice of λ.

S&P 500 index returns 2007-2010,

λ	0.80	0.9075*	0.94	0.97
RMSE	8.1844	8.0124	8.0544	8.2444

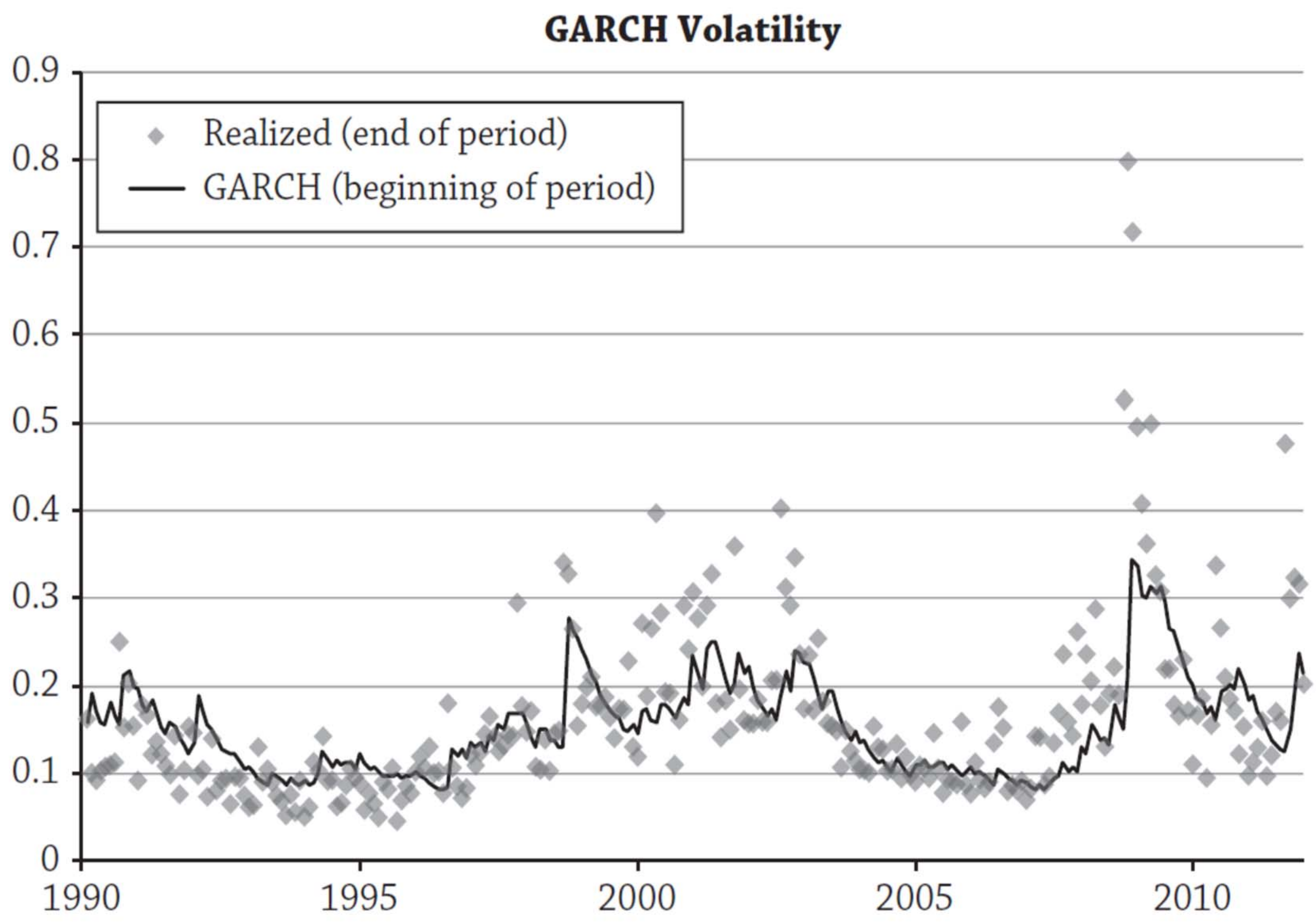
(MLE Estimate of lambda with same data is .932)

The ARCH and GARCH models

- The ARCH model, autoregressive conditional heteroskedasticity, was proposed by Professor Robert Engle in 1982. The GARCH model is a generalized version of ARCH.
- ARCH and GARCH are statistical models that capture the time-varying volatility:

$$\sigma_{t+1}^2 = a_0 + a_1 \, R_t^2 + a_2 \, \sigma_t^2$$

- As you can see, it is very similar to the EWMA model. In fact, if we set $a_0 = 0$, $a_2 = \lambda$, and $a_1 = 1 - \lambda$, we are doing the EWMA model.
- So what's the value added? This model has three parameters while the EWMA has only one. So it offers more flexibility (e.g., allows for mean reversion and better captures volatility clustering).


GARCH Volatility
Realized (end of period)
GARCH (beginning of period)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1990
1995
2000
2005
2010