15.456 Financial Engineering

MIT Sloan School of Management
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Fundamental theorem of asset pricing, approximate replication, and risk-neutral pricing
September 18, 2020

Agenda

- Announcements
  - Problem Set 2 due Thursday, September 24 @ 11:59pm
  - Quiz alert
- Risk-neutral probabilities and asset pricing
- Multi-period dynamics and the binomial tree
- Measures and martingales
- Dynamic arbitrage theorem

Risk-neutral probabilities and pricing

An improbable trick

- Whenever there is a bilinear sum (or integral), and all the coefficients are positive, we can interpret it **as if** probabilities were involved.

- Define these fictitious probabilities by dividing each coefficient by their sum

$$s(\mathbf{y}) = \sum_i c_i y_i, \quad \text{define } q_i \equiv \frac{c_i}{C}, \quad \text{where } C = \sum_i c_i$$

- Then,

$$s(\mathbf{y}) = C \sum_i q_i y_i = C\,\mathrm{E}[\mathbf{y}] = \mathrm{E}\left[C\mathbf{y}\right]$$

where the (fictitious) expectation is taken with respect to these “probabilities.”

Risk-neutral probabilities

- There is an interesting interpretation of the state price vector: if we multiply by a constant so that the sum of the components is unity, those components can be interpreted **as if** they were probabilities for the occurrence of each state.

In components, write $\mathbf{S} = A^* \psi$ as $S_i = \sum_{\nu=1}^{s} A_{\nu i} \psi_\nu$, where $\nu = 1, 2, \ldots, s$ labels the states, and $i = 1, 2, \ldots, n$ labels the assets.

- Define

$$
\begin{aligned}
R_{\nu i} &= A_{\nu i}/S_i = \text{return of asset } i \text{ in state } \nu, \\
R_f &= A_{\nu 1}/S_1 = \text{risk-free rate, independent of state } \nu, \\
q_\nu &= R_f \psi_\nu = \text{risk-neutral probability of state } \nu.
\end{aligned}
$$

- Then

$$
\sum_\nu q_\nu = \sum_\nu R_f \psi_\nu = \frac{\sum A_{\nu 1} \psi_\nu}{S_1} = 1
$$

Risk-neutral pricing

- Now re-express the pricing equation to get two interesting forms.

$$S_i = \sum A_{\nu i} \psi_\nu = \sum \frac{A_{\nu i}}{R_f} q_\nu = \mathrm{E}\left[\frac{A_{\nu i}}{R_f}\right]$$

- This says that the security price of risky assets is the **expectation** of the discounted present value of the **payoff** under the **risk-neutral measure**. Divide by price and multiply by the risk-free rate to get

$$\sum R_{\nu i} q_\nu = \mathrm{E}\left[R_{\nu i}\right] = R_f$$

- This says that the **expected return** of every risky asset is just the **risk-free rate** under the risk-neutral measure

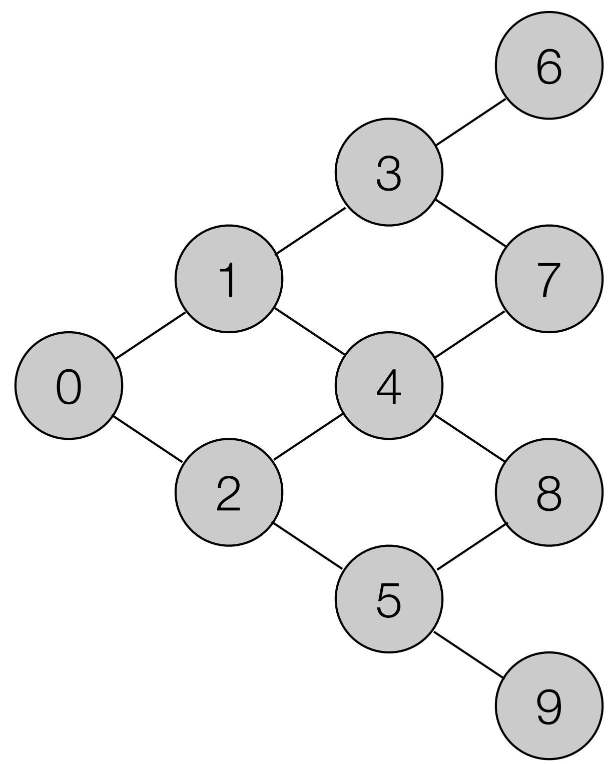
Risk-neutral pricing

- Are risk-neutral probabilities really fictitious?
- Inferred from prices, sentiment in market
- Compare the questions
  - How **likely** is this event?
  - How much does it **cost** to insure against this event?
- Does a disagreement between objective and risk-neutral probabilities imply arbitrage?
- No. On the contrary, arbitrage is absent as long as risk-neutral probabilities exist.

Dynamic markets

Next-to-simplest model

- Let's go from one period to many.
- Discrete time: 0,1,2,...,T,....
- Discrete states, distinct **at each time step**.
- Start with two basis assets
  - Bond – same payoff, regardless of state, between any two fixed times
  - Stock – uncertain payoff
    - State-dependence
    - Path-dependence



Static replication

- Example: *T*=3, stock moves only up or down.

$$A = \begin{pmatrix} R_f^3 & S_0 R_u^3 \\ R_f^3 & S_0 R_u^2 R_d \\ R_f^3 & S_0 R_u R_d^2 \\ R_f^3 & S_0 R_d^3 \end{pmatrix}$$

- Suppose there is an option with strike price *K*

$$S_1 > S_2 > K > S_3 > S_4$$

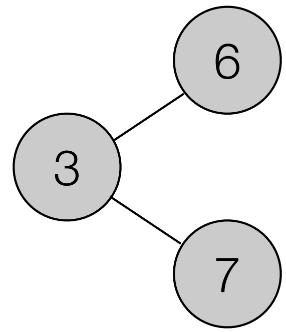
- Payoff on a call or a put is (in terms of terminal prices)

$$C = \begin{pmatrix} S_1 - K \\ S_2 - K \\ 0 \\ 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 \\ 0 \\ K - S_3 \\ K - S_4 \end{pmatrix}$$

- Cannot replicate either with stock and bond – an **incomplete** market.

Dynamic hedging

- Instead, let's consider replication one time-step at a time
- Focus on each node and the possible conditional outcomes
- Example: payoff at *t=3*, given that one is already in a specific state at *t=2*



$$\mathbf{b} = \begin{pmatrix} S_1 - K \\ S_2 - K \end{pmatrix} = \begin{pmatrix} C_u \\ C_d \end{pmatrix}$$

$$A = \begin{pmatrix} R_f & S_0 R_u^3 \\ R_f & S_0 R_u^2 R_d \end{pmatrix} = \begin{pmatrix} R_f & S_u \\ R_f & S_d \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 \\ S(t=2) = S_0 R_u^2 \end{pmatrix}$$

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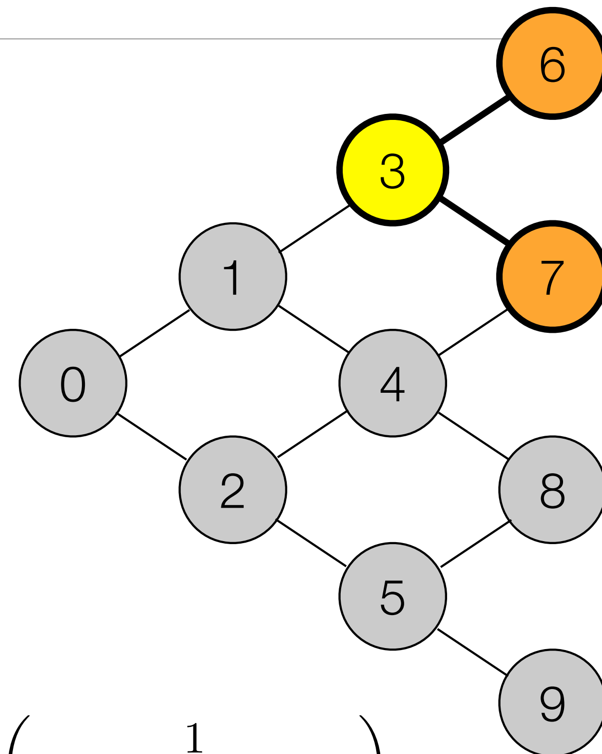

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Dynamic hedging

- Solve for the replicating portfolio for the node:

$$A\mathbf{x} = \mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b}$$

$$A = \begin{pmatrix} R_f & S_u \\ R_f & S_d \end{pmatrix}, \quad A^{-1} = \frac{1}{R_f(S_d - S_u)} \begin{pmatrix} S_d & -S_u \\ -R_f & R_f \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} B \\ \Delta \end{pmatrix} = A^{-1} \begin{pmatrix} C_u \\ C_d \end{pmatrix} = \frac{1}{R_f(S_d - S_u)} \begin{pmatrix} S_d C_u - S_u C_d \\ -R_f C_u + R_f C_d \end{pmatrix} = \begin{pmatrix} \frac{S_u C_d - S_d C_u}{R_f(S_u - S_d)} \\ \frac{C_u - C_d}{S_u - S_d} \end{pmatrix}$$

- Note that

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \begin{cases} 1 & \text{if both states in the money since } C = S - K, \\ 0 & \text{if both states out of the money, } C_u = C_d = 0 \end{cases}$$

Dynamic hedging

- Now that we have a replicating portfolio for this node (and **any** node),

$$\mathbf{S}^*\mathbf{x} = B + S\Delta$$

- State prices are coefficients of the payoff, or more directly,

$$\begin{aligned}
\psi &= (A^*)^{-1}\mathbf{S} = (A^{-1})^*\mathbf{S} \\
&= \frac{-1}{R_f(S_u - S_d)} \begin{pmatrix} S_d & -R_f \\ -S_u & R_f \end{pmatrix} \begin{pmatrix} 1 \\ S \end{pmatrix} = \frac{-1}{R_f(S_u - S_d)} \begin{pmatrix} S_d - R_f S \\ -S_u + R_f S \end{pmatrix} \\
&= \frac{+1}{R_f(R_u - R_d)} \begin{pmatrix} R_f - R_d \\ R_u - R_f \end{pmatrix} = \frac{1}{R_f}\mathbf{q}
\end{aligned}$$

- No arbitrage requires $R_u > R_f > R_d$

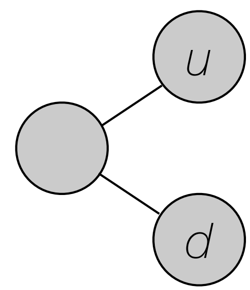
Dynamic hedging

- The state prices and risk-neutral probabilities depend only on the return parameters

$$\mathbf{q} = \begin{pmatrix} \frac{R_f - R_d}{R_u - R_d} \\ \frac{R_u - R_f}{R_u - R_d} \end{pmatrix}$$

- Therefore at any node, call value given in terms of **next period** state values as

$$\begin{aligned} C &= \psi^* \mathbf{b} = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix} \begin{pmatrix} C_u \\ C_d \end{pmatrix} \\ &= q_u \frac{C_u}{R_f} + q_d \frac{C_d}{R_f} \end{aligned}$$

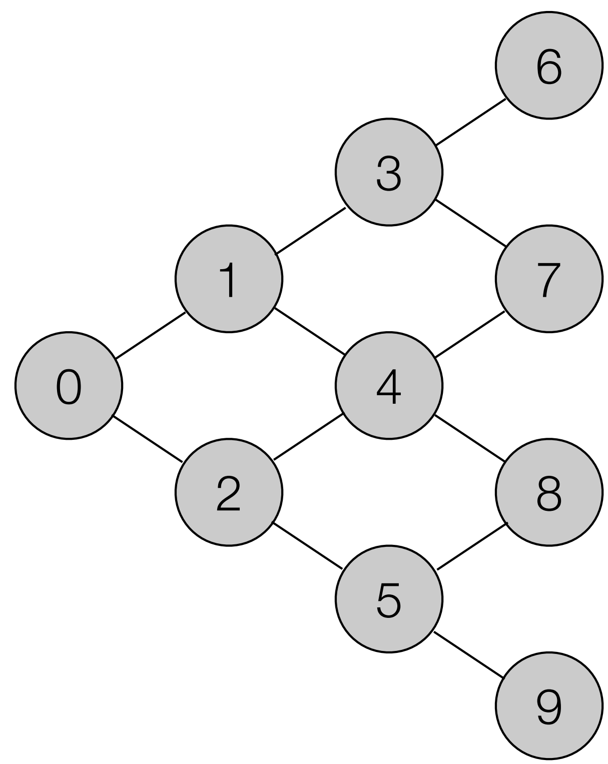


- This is the expected discounted payoff, under the risk-neutral measure, at the node.

Pricing: work backward from terminal values

- Omitting time subscripts for clarity,

$$
\begin{aligned}
t = 2: \quad & C_3 = \frac{1}{R_f}\left(q_u C_6 + q_d C_7\right) \\
& C_4 = \frac{1}{R_f}\left(q_u C_7 + q_d C_8\right) \\
& C_5 = \frac{1}{R_f}\left(q_u C_8 + q_d C_9\right) \\
t = 1: \quad & C_1 = \frac{1}{R_f}\left(q_u C_3 + q_d C_4\right) \\
& C_2 = \frac{1}{R_f}\left(q_u C_4 + q_d C_5\right) \\
t = 0: \quad & C_0 = \frac{1}{R_f}\left(q_u C_1 + q_d C_2\right)
\end{aligned}
$$



Pricing: work backward from terminal values

- Each step of recursion is an expectation

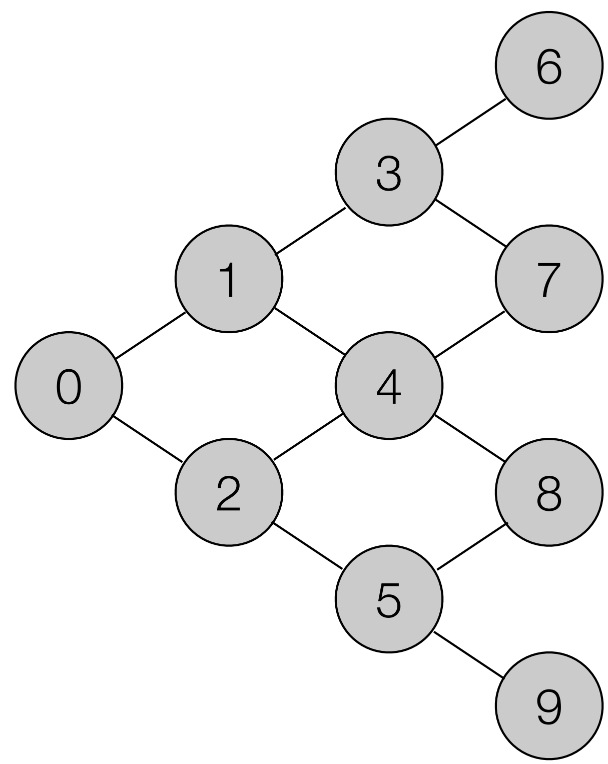
- In general, write at each node

$$C_t = \mathrm{E}_t\left[\frac{C_{t+1}}{R_f}\right]$$

where the expectation is conditional on the given time. For example, for *t=2* at node 5, the only states that enter on the right-hand side are the *t=3* states: {8,9}

- Then combining steps, the *t=0* price is

$$C_{t=0} = \frac{1}{R_f^3}\mathrm{E}\left[\mathrm{E}_1\left[\mathrm{E}_2\left[C_3\right]\right]\right] = \frac{1}{R_f^3}\mathrm{E}\left[C_3\right]$$



Conditional and iterated expectations

- Basic result on **iterated** expectations: the expectation of an expectation is an expectation!
  - Important because hedging and pricing are conditioned on information at a given time.
  - Filtrations: information revealed over time

- Conditional probability and expectation:

$$\mathrm{Prob}(A = a, B = b) = \mathrm{Prob}(A = a|B = b)\,\mathrm{Prob}(B = b)$$

$$\mathrm{E}\left[A|B = b\right] = \sum_a a\,\mathrm{Prob}(A = a|B = b) = f(b)$$

- The expectation of *A* conditional on *B* = *b* is a function of *b.* Taking **its** expectation...

Conditional and iterated expectations

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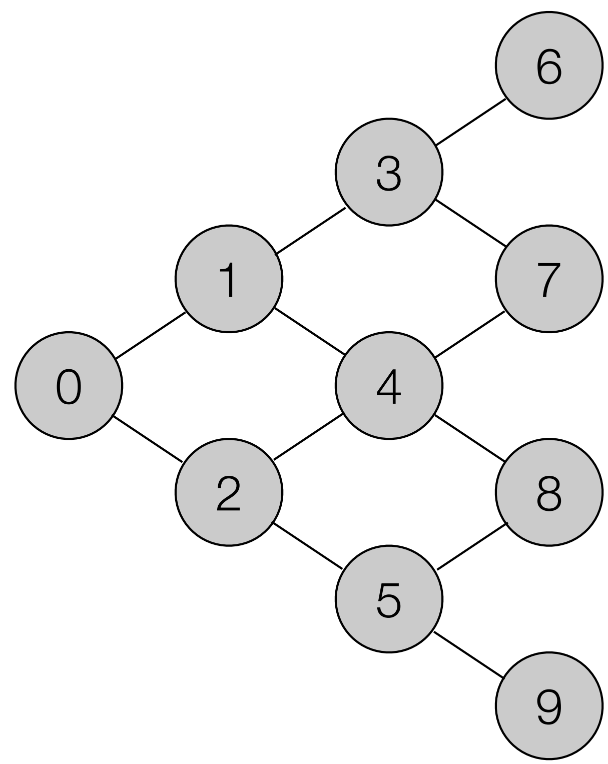
$$
\begin{aligned}
\mathrm{E}\left[\mathrm{E}\left[A|B\right]\right] = \mathrm{E}\left[f(b)\right] &= \sum_b \mathrm{Prob}(B=b) f(b) \\
&= \sum_b \mathrm{Prob}(B=b) \left[\sum_a a \,\mathrm{Prob}(A=a|B=b)\right] \\
&= \sum_{a,b} a \left[\mathrm{Prob}(A=a|B=b)\,\mathrm{Prob}(B=b)\right] \\
&= \sum_{a,b} a \,\mathrm{Prob}(A=a, B=b) \\
&= \sum_a a \,\mathrm{Prob}(A=a) \\
&= \mathrm{E}\left[A\right]
\end{aligned}
$$

Pricing: work backward from terminal values

- This works for **any function** of the stock price on the terminal nodes
  - Calls of any strike
  - Puts of any strike
  - Forwards
  - Etc.

- Key ingredients
  - Bond: captures time value of money
  - Stock: captures uncertainty over time
  - Delta: solves for unique dynamic hedging ratio
  - Risk-neutral probabilities: consistently assigned



Calibration

- Identification of binomial model parameters
  - Returns (up, down)
  - Probabilities (up, down)
  - Average return per period
  - Volatility of return

$$
\begin{aligned}
\overline{R} = \mathrm{E}\left[R\right] &= pR_u + (1-p)R_d \\
&= R_d + p(R_u - R_d) \\
\sigma^2 &= \mathrm{E}\left[(R-\overline{R})^2\right] = \mathrm{E}\left[R^2\right] - \overline{R}^2 \\
&= \left[pR_u^2 + (1-p)R_d^2\right] - \left[pR_u + (1-p)R_d\right]^2 \\
&= p(1-p)(R_u - R_d)^2
\end{aligned}
$$

- Example:

$$
\begin{aligned}
\overline{R} &= 1.009 = 1 + 0.9\% \\
\sigma &= 4.4\% \\
p &= 1/2 \\
R_u &= 1.053 \\
R_d &= 0.965
\end{aligned}
$$

$$
\begin{aligned}
R_u &= \overline{R} + \sigma\sqrt{\frac{1-p}{p}}, \\
R_d &= \overline{R} - \sigma\sqrt{\frac{p}{1-p}},
\end{aligned}
$$

Of Measures and Martingales

Measuring up

- We've explored two kinds of measure and their application
  - Risk neutral: no-arbitrage pricing
  - Objective: expected utility, portfolio optimization

- We can bridge the two by relating two different ways to compute expectations. For instance, a risk-neutral expectation can also be written as an objective expectation, though of a different random variable

$$\mathbb{E}^Q[X] = \sum_i q_i X_i = \sum_i p_i \left(\frac{q_i}{p_i} \right) X_i = \mathbb{E}[mX]$$

- The original RV gets multiplied times a correction factor known as the **change of measure**

$$m_i = \frac{q_i}{p_i}$$

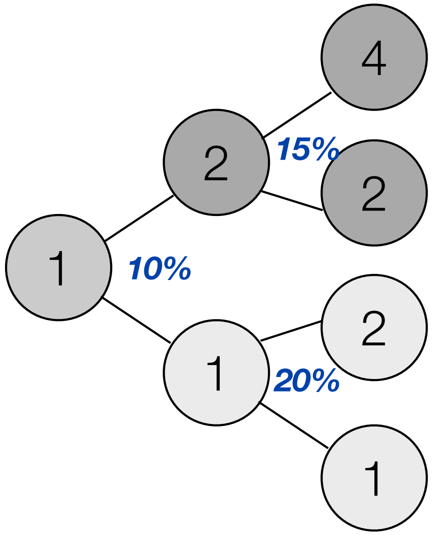
Conditional change

- Let's generalize and let probabilities change along different paths.

- Example: (very special numerical choice!) $\begin{aligned} R_u &= 2 \\ R_d &= 1 \end{aligned} \implies q_u = \dfrac{R_f - R_d}{R_u - R_d} = R_f - 1$

- Time-varying conditional risk-free rates

$$R_{f,0} = 1.1, \quad R_{f,1} = \begin{cases} 1.15 & \text{if stock goes up,} \\ 1.20 & \text{if stock goes down.} \end{cases}$$



- Note:
  - Objective probabilities are constant
  - Risk-neutral probabilities are path-dependent

Martingales

- A **martingale** is a stochastic process for which the expectation at any point in time of its future value is equal to the current value

$$\mathbb{E}_t[X_{t'}] = X_t, \quad t < t'$$

- Martingales will be the natural way that our single-period results extend to a dynamic, multiperiod world. In one period, we had the pricing relation for any risky asset

$$\mathbb{E}^Q[R_i] = R_f$$

- For a single asset, this gives $\mathbb{E}_t^Q[R_t] = R_{ft} = \mathbb{E}_t^Q\left[\frac{S_{t+1}}{S_t}\right] = \frac{1}{S_t}\mathbb{E}_t^Q[S_{t+1}]$

$$S_t = \frac{1}{R_{ft}}\mathbb{E}_t^Q[S_{t+1}]$$

Martingales: examples and counterexamples

- Examples

- Random walk $X_t = X_{t-1} + \epsilon_t, \quad e.g., \epsilon_t \sim \mathcal{N}(0, \sigma^2)$
  - Require zero-mean increments
  - Re-write for applications by shifting time ahead (forecasted) instead of behind (recursion)

$$X_{t+1} = X_t + \epsilon_{t+1}, \quad \mathbb{E}_t[\epsilon_{t+1}] = 0$$
$$\mathbb{E}_t[X_{t+1}] = X_t$$

- Random walk with drift

$$X_{t+1} = X_t + \mu + \epsilon_{t+1}, \quad \mathbb{E}_t[\epsilon_{t+1}] = 0$$
$$\mathbb{E}_t[X_{t+1}] = X_t + \mu \neq X_t$$
$$r_{t+1} = X_{t+1} - X_t$$

Time to iterate

- Allow risk-free rate to be time-varying. Can be moved in or out of expectations with respect to which it is (conditionally) constant.
  - Conventions: time index of risk-free rate identifies **start** of period for which rate applies

$$
\begin{aligned}
S_{T-1} &= \frac{1}{R_{f,T-1}} \mathbb{E}^{Q}_{T-1}[S_T] \\
S_{T-2} &= \frac{1}{R_{f,T-2}} \mathbb{E}^{Q}_{T-2}\left[\frac{1}{R_{f,T-1}} \mathbb{E}^{Q}_{T-1}[S_T]\right] \\
&= \mathbb{E}^{Q}_{T-2}\left[\mathbb{E}^{Q}_{T-1}\left[\frac{S_T}{R_{f,T-2}R_{f,T-1}}\right]\right]
\end{aligned}
$$

Time to iterate

- Continuing backward in time to any earlier time,

$$S_t = \mathbb{E}_t^Q \left[\mathbb{E}_{t+1}^Q \left[\cdots \mathbb{E}_{T-1}^Q \left[\frac{S_T}{R_{f,t} R_{f,t+1} \cdots R_{f,T-1}} \right] \right] \right]$$

- Denominator is compound return over period. Simplify by defining

$$\beta_t \equiv R_{f,0} R_{f,1} \cdots R_{f,t-1}, \quad \beta_0 \equiv 1$$

$$\prod_{\tau=t}^{T-1} R_{f\tau} = \frac{\prod_{\tau=0}^{T-1} R_{f\tau}}{\prod_{\tau=0}^{t-1} R_{f\tau}} = \frac{\beta_T}{\beta_t}$$

$$S_t = \mathbb{E}_t^Q \left[\frac{S_T}{\beta_T / \beta_t} \right] = \beta_t \mathbb{E}_t^Q \left[\frac{S_T}{\beta_T} \right]$$

Discounted price process

- Grouping factors by their time references, the **discounted price process is a martingale**:

$$\frac{S_t}{\beta_t} = \mathbb{E}_t^Q \left[\frac{S_T}{\beta_T} \right]$$

- Nothing special about terminal point

$$\begin{aligned} \text{If } X_t = \frac{S_t}{\beta_t}, \quad \text{then } \mathbb{E}_{t_1}^Q \left[X_{t_2} \right] &= \mathbb{E}_{t_1}^Q \left[\mathbb{E}_{t_2}^Q \left[X_T \right] \right] \\ &= \mathbb{E}_{t_1}^Q \left[X_T \right] \\ &= X_{t_1} \end{aligned}$$

Martingale propositions

- If a process equals its expectation one time-step ahead, then it is a martingale

$$\begin{aligned} \text{Suppose } X_t &= \mathbb{E}_t[X_{t+1}], \text{ for } t = 0, 1, \ldots, T-1. \\ X_t &= \mathbb{E}_t[X_{t+1}] = \mathbb{E}_t\left[\mathbb{E}_{t+1}[X_{t+2}]\right] = \mathbb{E}_t[X_{t+2}], \\ X_t &= \mathbb{E}_t[X_{t+1}] = \mathbb{E}_t\left[\mathbb{E}_{t+1}\left[\mathbb{E}_{t+2}[X_{t+3}]\right]\right] = \mathbb{E}_t[X_{t+3}], \textit{etc.} \end{aligned}$$

- If a process is defined as a sequence of expectations of a fixed random variable, then it is a martingale

$$\text{Suppose } X_t = \mathbb{E}_t[Y], \text{ for } t = 0, 1, \ldots, T-1.$$
$$\text{Then } \mathbb{E}_t[X_{t'}] = \mathbb{E}_t\left[\mathbb{E}_{t'}[Y]\right] = \mathbb{E}_t[Y] = X_t.$$

Self-financing trading strategies

- What is the cost of all the re-hedging in the binomial tree model? Zero.
  - Portfolio is risk-free at each time step
  - Can liquidate to cash and re-establish new re-hedged position.

- Example: suppose you buy a **mispriced** call option and hedge; this is equivalent to selling (the payoff of) a correctly-priced option and buying an underpriced one.
  - P/L on initial trade date equals mispricing minus (or plus) half-spread
  - Mark-to-market vs. mark-to-model
  - P/L on subsequent days expected to be zero for each day

Self-financing trading strategies

- More generally, suppose a **dynamic, self-financing trading strategy** has portfolio value, share price, and share quantity equal to V_t, S_t, θ_t

- Recursive value equation
  - Cash decreased by purchase price of shares
  - Over one day, cash grows at risk-free rate
  - Over one day, stock changed price while quantity is constant

$$\begin{aligned} V_{t+1} &= R_{f,t} V_t + \theta_t S_t (R_{t+1} - R_{f,t}), \quad R_{t+1} = S_{t+1}/S_t \\ &= R_{f,t}(V_t - \theta_t S_t) + \theta_t S_{t+1} \end{aligned}$$

Cash minus purchase price

Market value of shares one day later

Self-financing trading strategies

- Taking the risk-neutral expectation,

$$\begin{aligned}\mathbb{E}_t^Q\left[V_{t+1}\right] &= R_{f,t}(V_t - \theta_t S_t) + \theta_t \mathbb{E}_t^Q[S_{t+1}] \\ &= R_{f,t}(V_t - \theta_t S_t) + \theta_t (R_{f,t} S_t) = R_{f,t} V_t\end{aligned}$$

- Therefore the discounted price process of the self-financing **strategy** is a martingale:

$$V_t = \mathbb{E}_t^Q\left[\frac{V_{t+1}}{R_{f,t}}\right] \implies \frac{V_t}{\beta_t} = \mathbb{E}_t^Q\left[\frac{V_T}{\beta_T}\right], \quad \text{for } t \leq T$$

Dynamic arbitrage

- Define two types of **dynamic arbitrage**, analogous to Type I and II static arbitrage
- Type I: there exists a (no-lose) self-financing strategy with

$$V_0 \leq 0, \quad V_T \geq 0, \text{ and } \mathrm{Prob}(V_T > 0) \neq 0$$

- Type II: there exists a (riskless) self-financing strategy with

$$V_0 < 0, \quad V_T = 0$$

Dynamic arbitrage theorem

- There is no dynamic arbitrage if and only if
  - There exists a positive measure Q
  - Discounted price process of strategies are martingales

- Multiperiod model has no arbitrage if and only if each 1-period model is arbitrage-free.
  - The "if" direction is true by construction since multiperiod model is built up using 1-period models.

- Key idea: martingale rule formalizes notion of one-period no-arbitrage implying

$$\mathbb{E}^Q[R] = R_f$$

Dynamic arbitrage theorem

- The unconditional measure is the product of positive conditional measures.

- Given a set of martingale price processes, show there is a contradiction if dynamic arbitrage exists.

- Type I: $\mathbb{E}_t^Q\left[\frac{V_T}{\beta_T}\right] > 0$ since Q is positive and $V_T > 0$.
  - But this also equals

$$\mathbb{E}_t^Q\left[\frac{V_T}{\beta_T}\right] = \frac{V_0}{\beta_0} = V_0 \leq 0, \text{ which is a contradiction.}$$

- Type II:

$$0 = \mathbb{E}_t^Q\left[\frac{V_T}{\beta_T}\right] = \frac{V_0}{\beta_0} = V_0 < 0, \text{ which is a contradiction.}$$