15.456 Financial Engineering

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Arbitrage pricing in dynamic markets
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Agenda

- Announcements
  - Mid-term exam: Monday, October 19
- Measures and martingales
- Dynamic arbitrage theorem
- Arbitrage pricing applications
  - Option pricing
  - Bond pricing

Of Measures and Martingales

Measuring up

- We've explored two kinds of measure and their application
  - Risk neutral: no-arbitrage pricing
  - Objective: expected utility, portfolio optimization

- We can bridge the two by relating two different ways to compute expectations. For instance, a risk-neutral expectation can also be written as an objective expectation, though of a different random variable

$$\mathbb{E}^Q[X] = \sum_i q_i X_i = \sum_i p_i \left(\frac{q_i}{p_i}\right) X_i = \mathbb{E}[mX]$$

- The original RV gets multiplied times a correction factor known as the **change of measure**

$$m_i = \frac{q_i}{p_i}$$

Martingales

- A **martingale** is a stochastic process for which the expectation at any point in time of its future value is equal to the current value

$$\mathbb{E}_t[X_{t'}] = X_t, \quad t < t'$$

- Martingales will be the natural way that our single-period results extend to a dynamic, multiperiod world. In one period, we had the pricing relation for any risky asset

$$\mathbb{E}^Q[R_i] = R_f$$

- For a single asset, this gives $\mathbb{E}_t^Q[R_t] = R_{ft} = \mathbb{E}_t^Q\left[\frac{S_{t+1}}{S_t}\right] = \frac{1}{S_t}\mathbb{E}_t^Q[S_{t+1}]$

$$S_t = \frac{1}{R_{ft}}\mathbb{E}_t^Q[S_{t+1}]$$

Martingales: examples and counterexamples

- Examples

- Random walk $X_t = X_{t-1} + \epsilon_t, \quad e.g., \epsilon_t \sim \mathcal{N}(0, \sigma^2)$
  - Require zero-mean increments
  - Re-write for applications by shifting time ahead (forecasted) instead of behind (recursion)

$$X_{t+1} = X_t + \epsilon_{t+1}, \quad \mathbb{E}_t[\epsilon_{t+1}] = 0$$
$$\mathbb{E}_t[X_{t+1}] = X_t$$

- Random walk with drift

$$X_{t+1} = X_t + \mu + \epsilon_{t+1}, \quad \mathbb{E}_t[\epsilon_{t+1}] = 0$$
$$\mathbb{E}_t[X_{t+1}] = X_t + \mu \neq X_t$$
$$r_{t+1} = X_{t+1} - X_t$$

Time to iterate

- Allow risk-free rate to be time-varying. Can be moved in or out of expectations with respect to which it is (conditionally) constant.
  - Conventions: time index of risk-free rate identifies **start** of period for which rate applies

$$
\begin{aligned}
S_{T-1} &= \frac{1}{R_{f,T-1}} \mathbb{E}^Q_{T-1}[S_T] \\
S_{T-2} &= \frac{1}{R_{f,T-2}} \mathbb{E}^Q_{T-2}\left[\frac{1}{R_{f,T-1}} \mathbb{E}^Q_{T-1}[S_T]\right] \\
&= \mathbb{E}^Q_{T-2}\left[\mathbb{E}^Q_{T-1}\left[\frac{S_T}{R_{f,T-2}R_{f,T-1}}\right]\right]
\end{aligned}
$$

Time to iterate

- Continuing backward in time to any earlier time,

$$S_t = \mathbb{E}_t^Q \left[\mathbb{E}_{t+1}^Q \left[\cdots \mathbb{E}_{T-1}^Q \left[\frac{S_T}{R_{f,t} R_{f,t+1} \cdots R_{f,T-1}} \right] \right] \right]$$

- Denominator is compound return over period. Simplify by defining

$$\beta_t \equiv R_{f,0} R_{f,1} \cdots R_{f,t-1}, \quad \beta_0 \equiv 1$$

$$\prod_{\tau=t}^{T-1} R_{f\tau} = \frac{\prod_{\tau=0}^{T-1} R_{f\tau}}{\prod_{\tau=0}^{t-1} R_{f\tau}} = \frac{\beta_T}{\beta_t}$$

$$S_t = \mathbb{E}_t^Q \left[\frac{S_T}{\beta_T / \beta_t} \right] = \beta_t \mathbb{E}_t^Q \left[\frac{S_T}{\beta_T} \right]$$

Discounted price process

- Grouping factors by their time references, the **discounted price process is a martingale**:

$$\frac{S_t}{\beta_t} = \mathbb{E}_t^Q \left[\frac{S_T}{\beta_T} \right]$$

- Nothing special about terminal point

$$\text{If } X_t = \frac{S_t}{\beta_t}, \quad \text{then } \mathbb{E}_{t_1}^Q \left[X_{t_2} \right] = \mathbb{E}_{t_1}^Q \left[\mathbb{E}_{t_2}^Q \left[X_T \right] \right]$$
$$= \mathbb{E}_{t_1}^Q \left[X_T \right]$$
$$= X_{t_1}$$

Martingale propositions

- If a process equals its expectation one time-step ahead, then it is a martingale

$$\text{Suppose } X_t = \mathbb{E}_t[X_{t+1}], \text{ for } t = 0, 1, \ldots, T-1.$$
$$X_t = \mathbb{E}_t[X_{t+1}] = \mathbb{E}_t\left[\mathbb{E}_{t+1}[X_{t+2}]\right] = \mathbb{E}_t[X_{t+2}],$$
$$X_t = \mathbb{E}_t[X_{t+1}] = \mathbb{E}_t\left[\mathbb{E}_{t+1}\left[\mathbb{E}_{t+2}[X_{t+3}]\right]\right] = \mathbb{E}_t[X_{t+3}], \textit{etc.}$$

- If a process is defined as a sequence of expectations of a fixed random variable, then it is a martingale

$$\text{Suppose } X_t = \mathbb{E}_t[Y], \text{ for } t = 0, 1, \ldots, T-1.$$
$$\text{Then } \mathbb{E}_t[X_{t'}] = \mathbb{E}_t\left[\mathbb{E}_{t'}[Y]\right] = \mathbb{E}_t[Y] = X_t.$$

Self-financing trading strategies

- What is the cost of all the re-hedging in the binomial tree model? Zero.
  - Portfolio is risk-free at each time step
  - Can liquidate to cash and re-establish new re-hedged position.
- Example: suppose you buy a **mispriced** call option and hedge; this is equivalent to selling (the payoff of) a correctly-priced option and buying an underpriced one.
  - P/L on initial trade date equals mispricing minus (or plus) half-spread
  - Mark-to-market vs. mark-to-model
  - P/L on subsequent days expected to be zero for each day

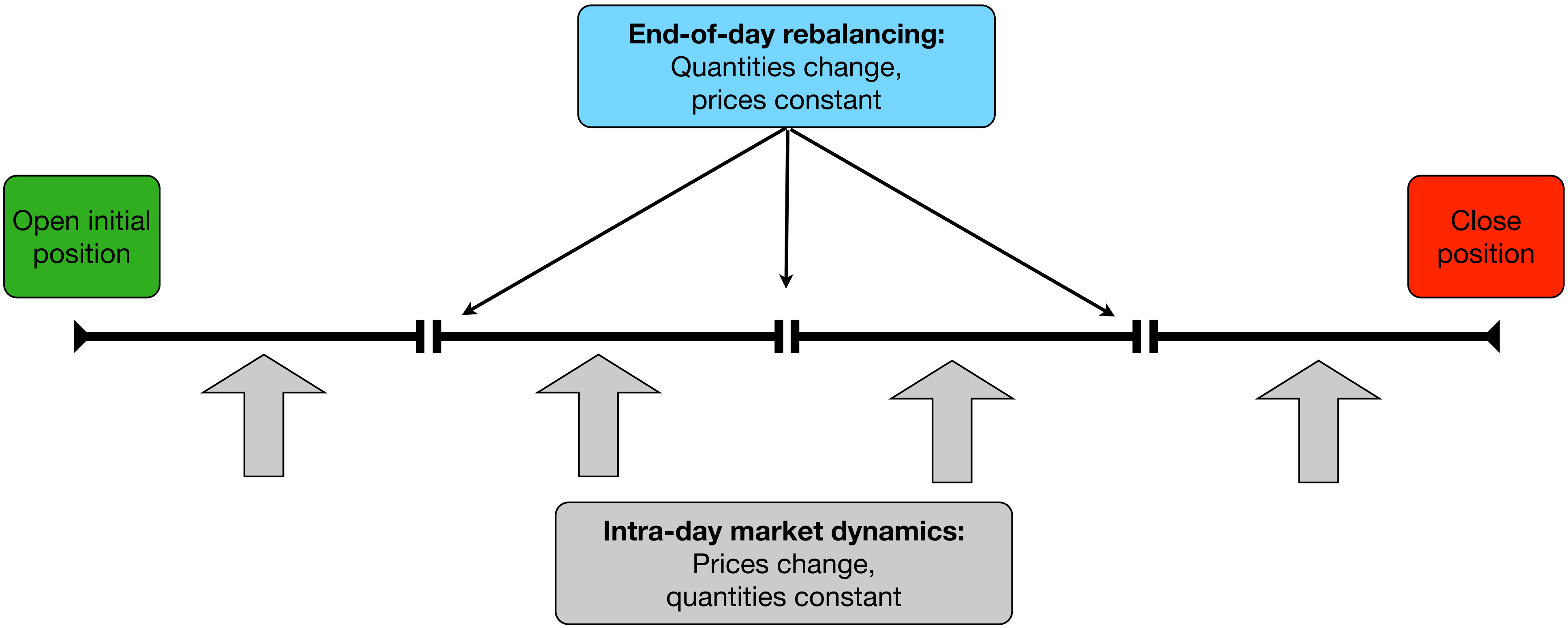
Self-financing trading strategies

- More generally, suppose a **dynamic, self-financing trading strategy** has portfolio value, share price, and share quantity equal to V_t, S_t, θ_t

- Recursive value equation
  - Cash decreased by purchase price of shares
  - Over one day, cash grows at risk-free rate
  - Over one day, stock changed price while quantity is constant

$$
\begin{aligned}
V_{t+1} &= R_{f,t} V_t + \theta_t S_t (R_{t+1} - R_{f,t}), \quad R_{t+1} = S_{t+1}/S_t \\
&= \underbrace{R_{f,t}(V_t - \theta_t S_t)}_{\text{Cash minus purchase price}} + \underbrace{\theta_t S_{t+1}}_{\text{Market value of shares one day later}}
\end{aligned}
$$

Discrete hedging and portfolio rebalancing



Self-financing trading strategies

- Taking the risk-neutral expectation,

$$\begin{aligned}\mathbb{E}_t^Q\left[V_{t+1}\right] &= R_{f,t}(V_t - \theta_t S_t) + \theta_t \mathbb{E}_t^Q[S_{t+1}] \\ &= R_{f,t}(V_t - \theta_t S_t) + \theta_t(R_{f,t} S_t) = R_{f,t} V_t\end{aligned}$$

- Therefore the discounted price process of the self-financing **strategy** is a martingale:

$$V_t = \mathbb{E}_t^Q\left[\frac{V_{t+1}}{R_{f,t}}\right] \implies \frac{V_t}{\beta_t} = \mathbb{E}_t^Q\left[\frac{V_T}{\beta_T}\right], \quad \text{for } t \leq T$$

Dynamic arbitrage

- Define two types of **dynamic arbitrage**, analogous to Type I and II static arbitrage

- Type I: there exists a (no-lose) self-financing strategy with

$$V_0 \leq 0, \quad V_T \geq 0, \text{ and } \operatorname{Prob}(V_T > 0) \neq 0$$

- Type II: there exists a (riskless) self-financing strategy with

$$V_0 < 0, \quad V_T = 0$$

Dynamic arbitrage theorem

- There is no dynamic arbitrage if and only if
  - There exists a positive measure Q
  - Discounted price process of strategies are martingales

- Multiperiod model has no arbitrage if and only if each 1-period model is arbitrage-free.
  - The "if" direction is true by construction since multiperiod model is built up using 1-period models.

- Key idea: martingale rule formalizes notion of one-period no-arbitrage implying

$$\mathbb{E}^Q[R] = R_f$$

Dynamic arbitrage theorem

- The unconditional measure is the product of positive conditional measures.

- Given a set of martingale price processes, show there is a contradiction if dynamic arbitrage exists.

- Type I: $\mathbb{E}_t^Q\left[\frac{V_T}{\beta_T}\right] > 0$ since Q is positive and $V_T > 0$.
  - But this also equals

$$\mathbb{E}_t^Q\left[\frac{V_T}{\beta_T}\right] = \frac{V_0}{\beta_0} = V_0 \leq 0, \text{ which is a contradiction.}$$

- Type II:

$$0 = \mathbb{E}_t^Q\left[\frac{V_T}{\beta_T}\right] = \frac{V_0}{\beta_0} = V_0 < 0, \text{ which is a contradiction.}$$

Applications of arbitrage pricing

Applications of arbitrage pricing

- Our basic result from no-arbitrage analysis of dynamic prices, that discounted price processes are martingales, is a powerful computational tool.

- We will consider a few examples now, and compare later with results obtained using the tools and techniques of stochastic calculus.

- Basic approach:
  - Determine Q-measure
  - Determine relationship between Q-measure and P-measure (optional)
  - Compute the discounted expectations to find values of interest

Application: option pricing

- Options are derivative contracts giving the holder the right, but not the obligation, to buy or sell an underlying security.

- Once the underlying is used to determine the proper measure, many contracts whose payoffs are functions of the underlying can be computed directly as integrals – no PDE's required.

- Examples:
  - Vanilla European call option
  - Digital call option
  - Power option

Application: option pricing

- Suppose we are given that stock returns are log-normally distributed under *P*. Then between any two points, say *0* and *t,*

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma\sqrt{t}Z}, \quad Z \sim \mathcal{N}(0,1)$$

$$\begin{aligned}\mathbb{E}\left[e^{a+bZ}\right] &= \frac{e^a}{\sqrt{2\pi}} \int e^{-z^2/2+bz} \mathrm{d}z \\ &= \frac{e^{a+b^2/2}}{\sqrt{2\pi}} \int e^{-(z-b)^2/2} \mathrm{d}z \\ &= e^{a+b^2/2}.\end{aligned}$$

- *S* is **not** a martingale under this measure since

$$\begin{aligned}\mathbb{E}\left[S_t\right] &= S_0 e^{\mu t}, \\ \mathbb{E}\left[\frac{S_t}{\beta_t}\right] &= e^{-rt}\mathbb{E}\left[S_t\right] = S_0 e^{(\mu - r)t} \neq \frac{S_0}{\beta_0}\end{aligned}$$

- Notice that it **would be** a martingale if *Z* were shifted by a constant: $Z \to Z + \left(\frac{r-\mu}{\sigma}\right)\sqrt{t}$

Application: option pricing

- Under *Q,* the distribution of stock prices is $S_t = S_0 e^{(r-\sigma^2/2)t+\sigma\sqrt{t}Z^{\mathbb{Q}}}, \quad Z^{\mathbb{Q}} \sim \mathcal{N}(0,1)$
- With respect to *Q*-measure, the market satisfies no-arbitrage since
  - The discounted price process is a martingale
  - The return under *Q*-measure is the risk-free rate

$$\mathbb{E}^{\mathbb{Q}}\left[S_t\right] = S_0 e^{rt},$$
$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t}{\beta_t}\right] = e^{-rt}\mathbb{E}\left[S_t\right] = \frac{S_0}{\beta_0},$$
$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t}{S_0}\right] = e^{rt}$$

- The measures are connected by the SPD $\pi_t = e^{-(r+\eta^2/2)t-\eta\sqrt{t}Z}, \quad \eta \equiv \frac{\mu - r}{\sigma}$

Application: option pricing

- Pricing a vanilla call on a non-dividend-paying stock:

$$
\begin{aligned}
C &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[C_T] = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)] \\
&= e^{-rT}\int_{z^*}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} e^{-z^2/2}\left[S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}z} - K\right]
\end{aligned}
$$

- The lower limit of corresponds to the value of *z* where the payoff becomes zero,

$$
S_T = K \implies z^* = -\left[\frac{\log(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]
$$

Application: option pricing

- The integral has two terms,

$$C = S_0 \int_{z^*}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} e^{-(z-\sigma\sqrt{T})^2/2} - Ke^{-rT} \int_{z^*}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} e^{-z^2/2}$$

- Shifting the integration variable in the first term, both integrals can be written in terms of the normal CDF,

$$C = S_0 \Phi\left(-z^* + \sigma\sqrt{T}\right) - Ke^{-rT}\Phi(-z^*), \quad \Phi(x) \equiv \int_{-\infty}^{x} \frac{\mathrm{d}z}{\sqrt{2\pi}} e^{-z^2/2}$$

- Substituting for *z**, this is the Black-Scholes formula...obtained without the Black-Scholes equation.

Application: option pricing

- Pricing a digital call option that pays off $1 if *S* > *K* at expiry and zero otherwise:
  - Integrand is even simpler!
  - Integration limit at z^* is the same

$$
\begin{aligned}
V &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[V_T] \\
&= e^{-rT}\int_{z^*}^{\infty}\frac{\mathrm{d}z}{\sqrt{2\pi}}e^{-z^2/2} = e^{-rT}\Phi(-z^*)
\end{aligned}
$$

Application: option pricing

- Pricing a so-called power option, whose payoff $V_T = S_T^n$
- For example if *n=3*,

$$\begin{aligned} V &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T^3] \\ &= S_0^3 e^{-rT}\int_{z^*}^{\infty}\frac{\mathrm{d}z}{\sqrt{2\pi}}e^{-z^2/2}e^{3(r-\sigma^2/2)T+3\sigma\sqrt{T}z} \\ &= S_0^3 e^{2rT+3\sigma^2 T} \end{aligned}$$

Application: bond pricing

- Firms finance their activities using equity and debt
- Arbitrage relationships related pricing of debt instruments and their behavior under events such as bankruptcy or credit default
- The risk of credit default is an essential feature of debt pricing
  - Model default as occurring when the firm value drops below a given level
  - In simple default model, creditors recover remaining asset value; shareholders wiped out
  - What are the payoffs for bondholders and shareholders?

Application: bond pricing

- A firm that has issued both equity and zero-coupon bonds of par value *F* has total assets valued at *A,* where *A* evolves with a random component according to

$$A_t = A_0 e^{(\mu - \sigma^2/2)t + \sigma\sqrt{t}Z}, \quad Z \sim \mathcal{N}(0,1)$$

- What is the **probability** of default? What is the **value** of the equity and the debt?
  - At maturity, bondholders receive *F* – assuming the firm is still solvent – and the shareholder get the remainder of the firm value.
  - In the event of default, equity holders receive nothing and debt holders receive the remaining asset value. In summary,

$$\begin{aligned} A_t &= E_t + D_t \\ D_T &= \min(F, A_T) = F - \max(F - A_T, 0) \\ E_T &= \max(A_T - F, 0) \end{aligned}$$

Credit risk

- The debt holders have effectively written a put option with strike *F*. Therefore

$$
\begin{aligned}
D_t &= e^{-r(T-t)}\mathbb{E}^Q[D_T] = Fe^{-r(T-t)} - P_{\mathrm{BS}}(A_t; K, T-t, r, \sigma) \\
E_t &= C_{\mathrm{BS}}(A_t; K, T-t, r, \sigma)
\end{aligned}
$$

- What happens in credit event at $t < T$? Suppose the debt holders can put their bonds as soon as an intermediate credit boundary is hit, e.g.,

$$A_t = X \leq Fe^{-r(T-t)}$$

- When the asset value *A* first hits the boundary *X* from above, the option is essentially cancelled. In this case, the equity holders' option is effectively a down-and-out barrier call option.