15.456 Financial Engineering Homework 2

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Question 1.

(a)

$$A^* = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 3 & 3 & 3 & 3 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} 1/3 \\ 1/2 \\ 2 \end{pmatrix}$$

$$A^*\psi = \mathbf{S} \Rightarrow \psi = \begin{pmatrix} \psi_1 \\ 1/6 \\ 1/6 \\ 1/3 - \psi_1 \end{pmatrix}$$

where $0 < \psi_1 < 1/3$

(b)
the market is not complete since there are multiple state prices

(c)
The third security (third column) in matrix A is a risk-free bond
$R_f = 3/2 = 1.5$

(d)

$$q_R = R_f \psi = \begin{pmatrix} 1.5\psi_1 \\ 1/4 \\ 1/4 \\ 1/2 - 1.5\psi_1 \end{pmatrix}$$

(e)

$$p_{\mathbf{b}} = \psi * \mathbf{b} = 0.5$$

The only possible no-arbitrage price for this contract is 0.5

Question 2.

(a) payoff matrix:

$$\begin{pmatrix} 1.2 & 1.2 & 1.05 \\ 1 & 1 & 1.05 \\ 1.2 & 1 & 1.05 \\ 1 & 1.2 & 1.05 \end{pmatrix}$$

(b)

$$A^*\psi = \mathbf{S} \Rightarrow \psi = \begin{pmatrix} 5/21 - \psi_4 \\ 5/7 - \psi_4 \\ \psi_4 \\ \psi_4 \end{pmatrix}$$

$0 < \psi_4 < 5/21$, there is no arbitrage in the market

$$R_f = 1.05 q_R = R_f \psi = \begin{pmatrix} 0.25 - 1.05\psi_4 \\ 0.75 - 1.05\psi_4 \\ 1.05\psi_4 \\ 1.05\psi_4 \end{pmatrix}$$

(c)
call for stock X:

$$b_X = \begin{pmatrix} 0.1 \\ 0 \\ 0.1 \\ 0 \end{pmatrix}, c_X = \psi^* b_X = 1/42$$

call for stock Y:

$$b_Y = \begin{pmatrix} 0.1 \\ 0 \\ 0 \\ 0.1 \end{pmatrix}, c_Y = \psi^* b_Y = 1/42$$

(d)

$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, c = \psi^* b = (10/21 - \psi_4) \in [5/21, 10/21]$$

Question 3.

The optimal hedging portfolio is existent and unique only when both (a) and (b) hold, where the market is complete and without redundant securities.

(c) and (d) lead to redundant securities; (e) leads to incomplete market

Question 4.

(a)

$$S_t = 2^x * (0.5)^{t-x} * S_0 = 2^{2x-t} S_0, x \in \{0, 1, ..., t\}$$
$$P = C_t^x p^x (1-p)^{t-x}$$

(b)

$$S_3 = 1, C_3 = 0, prob = (1-p)^3$$
$$S_3 = 4, C_3 = 0, prob = 3p(1-p)^2$$
$$S_3 = 16, C_3 = 8, prob = 3p^2(1-p)$$
$$S_3 = 64, C_3 = 56, prob = p^3$$

if $p = 1/2$, $E_{t=0}[C_{t=3}] = 10$; if $p = 2/3$, $E_{t=0}[C_{t=3}] = 20.15$

(c) say the replicating portfolio is x_1 risk-free bond and x_2 stock

$$E(SSE) = p_1(x_1 + x_2)^2 + p_2(x_1 + 4x_2)^2 + (x_1 + 16x_2)^2 + p3(x_1 + 16x_2 - 8)^2 + p4(x_1 + 64x_2 - 56)^2$$

$$\partial E(SSE)/\partial x1 = 0, \partial E(SSE)/\partial x2 = 0 \Rightarrow$$

$$B\mathbf{y}x = \mathbf{y}, B = \begin{pmatrix} p_1 + p_2 + p_3 + p_4 & p_1 + 4p_2 + 16p_3 + 64p_4 \\ p1 + 4p_2 + 16p_3 + 64p_4 & p_1 + 16_p 2 + 256p_3 + 4096p_4 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 8p_3 + 56p_4 \\ 128p_3 + 3584p_4 \end{pmatrix}$$

when p = 1/2, $x = \begin{pmatrix} -4.38 \\ 0.918 \end{pmatrix}$ when p = 2/3, $x = \begin{pmatrix} -5.63 \\ 0.955 \end{pmatrix}$

(d)
cost = $x1 + 8x2 = 3.00$ when p = 1/2
cost = $x1 + 8x2 = 2.01$ when p = 2/3