15.415 Foundations of Modern Finance

Leonid Kogan and Jiang Wang
MIT Sloan School of Management

Lecture 8: Forwards and Futures

Key concepts

- Introduction: forward contracts
- Forward interest rates
- Pricing of forwards on financial assets
- Currency contracts
- Futures: introduction
- Commodity futures
- Swaps

Ancient derivatives

Financial contracts, with many features found in modern derivatives such as forwards, futures, and options, date back to early periods of human history.

In Ancient Mesopotamia, … Some types of contracts were arrangements on the future delivery of grain that stipulated for instance before planting that a seller would deliver a certain quantity of grain for a price paid at the time of contracting. Such types of contracts not only dealt with grain but also with all sorts of commodities. …

These types of contracts had the features of today's forwards and were used across borders. By about 1,400 BC, cuneiform script in the Babylonian language was even used in Egypt to record transactions with Crete, Cyprus, the Aegean Islands, Assyria and the Hittites.

Source: S. Kummer, 2012, "The History of Derivatives: A Few Milestones"

Forward contracts

- A **forward contract** is a commitment to buy (sell) at a future date a given amount of a commodity or an asset at a price agreed on today.

- The price fixed now for future exchange is the forward price.

- The buyer obtains a “long position” in the asset/commodity.

An example: forward contract

- A tofu manufacturer needs 100,000 bushels of soybeans in 3 months.
- Current price of soybeans is $12.50/bu but may go up.
- Wants to make sure that 100,000 bushels will be available.
- Enter 3-month forward contract for 100,000 bushels of soybeans at $13.50/bu.
- Long side buys 100,000 bushels from short side at $13.50/bu in 3 months.

Features of forward contracts

- Traded over the counter (not on exchanges);
- Custom tailored;
- No money changes hands until maturity.
- Advantages of forward contracts:
  - Full flexibility.
- Disadvantages of forward contracts:
  - Counterparty risk;
  - Illiquidity.

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Forward interest rates

- So far, we have focused on spot interest rates: rates for a transaction between today, $t = 0$, and a future date, t.
- Now, we study forward interest rates: rates for a transaction between two future dates, for instance, $t = 1$ and $t = 2$.
- For a forward transaction to borrow money in the future:
  - Terms of the transaction are agreed on today, $t = 0$;
  - Loan is received on a future date $t = 1$;
  - Repayment of the loan occurs on date $t = 2$.
- Note: Future spot rates are random, they can be different from current corresponding forward rates.

Example: forward interest rate

- As the CFO of a U.S. multinational, you expect to repatriate $10M from a foreign subsidiary in 1 year, which will be used to pay dividends 1 year later.

- Not knowing the interest rates in 1 year, you would like to lock into a lending rate one year from now for a period of one year.

- The current interest rates are as follows.

Time to maturity t (years)	1	2
Spot interest rate r_t	0.05	0.07

- What should you do?

Example: forward interest rate

- Strategy:
  - Borrow $9.524M now for one year at 5%;
  - Invest the proceeds $9.524M for two years at 7%.
- Outcome (in million dollars):

Year	0	1	2
1-yr borrowing	9.524	-10,000	0
2-yr borrowing	-9.524	0	10,904
Repatriation	0	10,000	0
Net	0	0	10.904

- The locked-in 1-year lending rate 1 year from now is 9.04%.

Forward interest rates vs spot rates

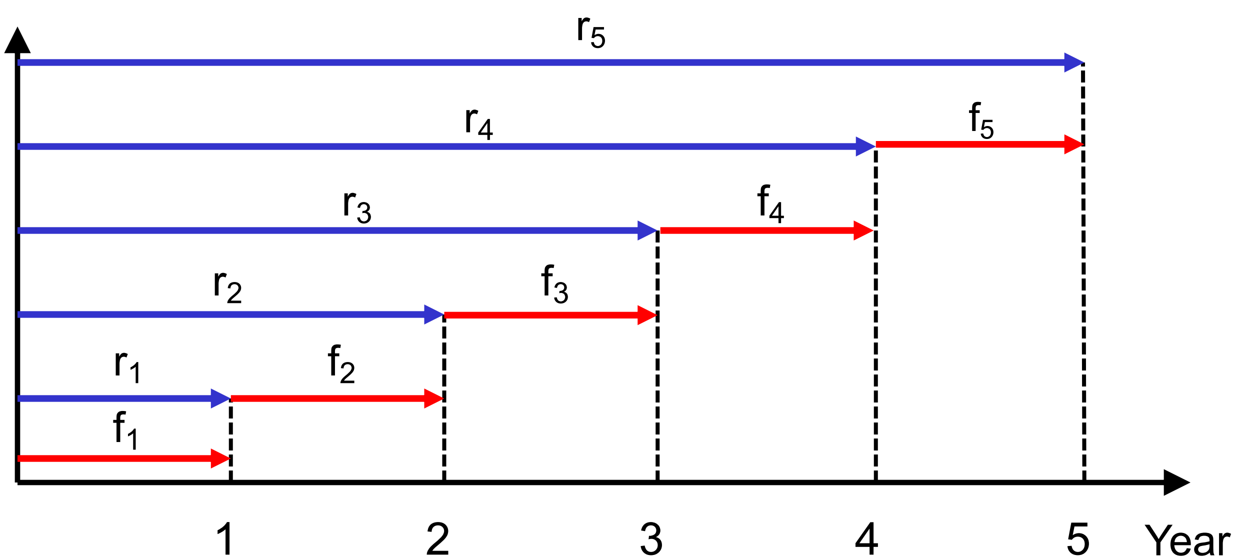
- The forward interest rate between time $t-1$ and t is:

$$(1+r_t)^t = (1+r_{t-1})^{t-1}(1+f_t)$$

or

$$f_t = \frac{B_{t-1}}{B_t} - 1 = \frac{(1+r_t)^t}{(1+r_{t-1})^{t-1}} - 1$$

Spot and Forward Rates



Example: forward interest rates

- Suppose that discount bond prices are as follows:

t	1	2	3	4
B_t	0.9524	0.8900	0.8278	0.7629
r_t	0.05	0.06	0.065	0.07

- A customer wants a forward contract to borrow \$20M for one year in three years from now. Can you (a bank) quote a rate?

- Answer: $f_4 = 8.51\%$.

Example: forward interest rates

- What should you do today to lock-in these cash flows?

1. Buy 20,000,000 of 3-year discount bonds, costing:

$$(\$20{,}000{,}000)(0.8278) = \$16{,}556{,}000$$

2. Finance this by selling 4-year discount bonds of amount:

$$\frac{\$16{,}556{,}000}{0.7629} = \$21{,}701{,}403$$

3. This creates a liability in year 4 in the amount $21,701,403.

Example: forward interest rates

- Cash flows from this strategy (in million dollars):

Year	0	1-2	3	4
Purchase of 3-year bonds	−16.556	0	20.000	0
Sale of 4-year bonds	16.556	0	0	−21.701
Total	0	0	20.000	−21.701

- The interest for this future investment is given by:

$$\frac{21{,}701{,}403}{20{,}000{,}000} - 1 = 8.51\%$$

Forward rates and the expectations hypothesis

- We can re-state the expectations hypothesis (EH) in terms of the relation between spot and forward rates.

- Under the EH, expected returns on all bonds are the same, and

$$\mathrm{E}_0[\tilde{r}_1(t)] = \underbrace{\frac{\left(1 + r_{t+1}(0)\right)^{t+1}}{\left(1 + r_t(0)\right)^{t}}}_{1+f_{t+1}} - 1 = f_{t+1}$$

- Under the EH, forward rates are unbiased predictors of future spot rates.

- Empirically, forwards rates over-predict future spot rates on average: forward rate reflects a risk premium in addition to the expectations of the future spot rates.

Forward rates and future spot rates

- Consider 2 strategies, both start with a $\$1$ initial investment.
- Strategy 1:
  - At $t = 0$, invest $\$1$ at the risk-free rate $r_T(0)$ up to time T.
  - At time T, re-invest $\left(1 + r_T(0)\right)^T$ for one more period at the spot rate $\tilde{r}_1(T)$.
- Strategy 2:
  - At $t = 0$, enter into a forward contract to invest the amount of $\left(1 + r_T(0)\right)^T$ at time T for one period.
  - At $t = 0$, invest $\$1$ at the risk-free rate up to time T.
  - At time T, re-invest $\left(1 + r_T(0)\right)^T$ for one more period at the forward rate f_{T+1}.

Forward rates and future spot rates

- Payoff of Strategy 1 at time $T+1$:

$$\left(1+r_T(0)\right)^T \times (1+\tilde{r}_1(T))$$

- Payoff of Strategy 2 at time $T+1$:

$$\left(1+r_T(0)\right)^T \times (1+f_{T+1})$$

- Both payoffs have the same PV at $t=0$: $1.
- Conclusion:

$$\mathrm{PV}_0[\tilde{r}_1(T) \text{ at } T+1] = \mathrm{PV}_0[f_{T+1} \text{ at } T+1]$$

- Recall that

$$\mathrm{E}_0[\tilde{r}_1(T) \text{ at } T+1] \neq f_{T+1}$$

because f_{T+1} is fixed while $\tilde{r}_1(T)$ is random, may earn a risk premium.

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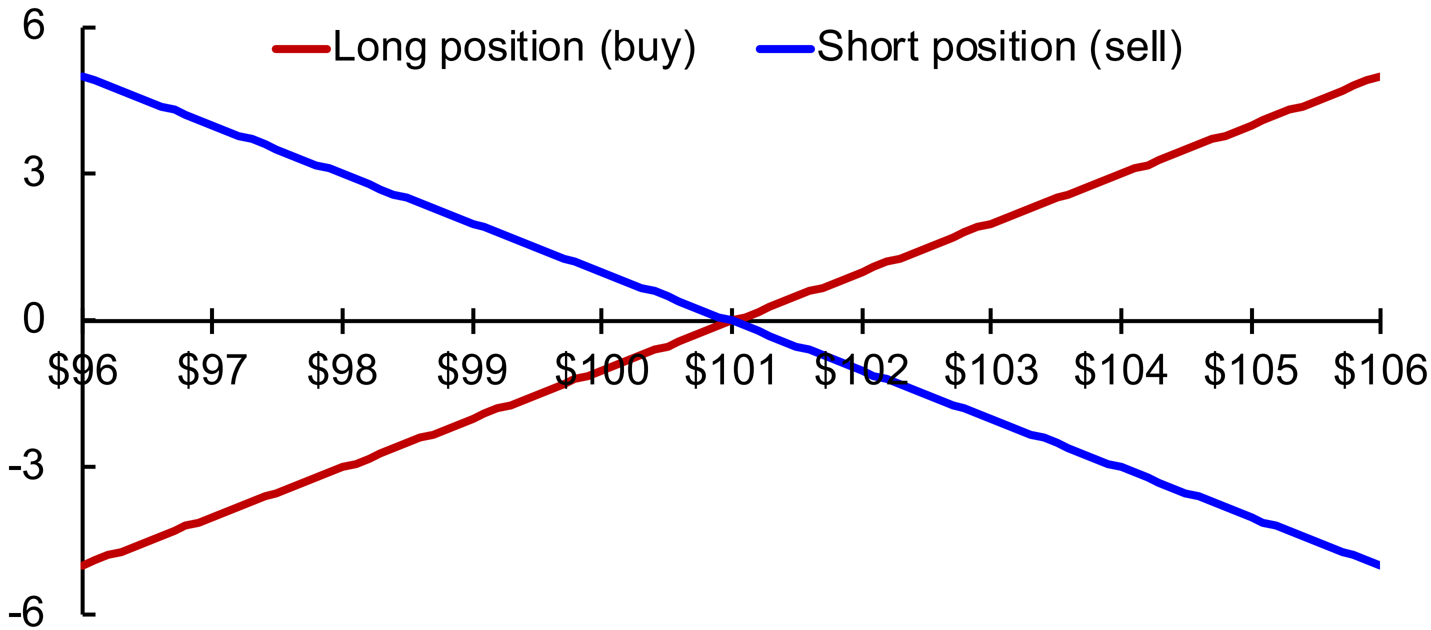
Payoff diagrams of forwards

- Forwards and futures are derivative securities.
  - Payoffs tied to prices of underlying assets/commodities;
  - Zero net supply (aggregate positions add to zero).
- Payoffs are linear in underlying asset price: $S(T) - F$.



Financial forwards

- For financial forwards, the underlying are financial assets.
  - Dividend or interest on the underlying asset.
- Stock index forwards, e.g., S&P, Nikkei,…
  - Underlying: baskets of stocks.
- Interest rate forwards.
  - Underlying: fixed income instruments (T-bonds,…).
- Currency forwards.

Forward prices

- Forward prices are linked to spot prices.
- Notation:

Contract	Spot now	Spot at T	Forward	Futures
Price	S	$S(T)$	F_T	H_T

- Two ways to buy the underlying asset for date-T delivery:

1. Buy a forward or futures contract with maturity date T,
2. Buy the underlying asset and hold it until T.

A model of payout

- Consider a financial asset paying dividends (more generally, net payout).
- We assume that the asset pays a continuous flow of dividends proportional to the asset price: dividend yield is constant, y.
- Reinvest dividends: number of units of the asset grows at rate y.
- Start with one share of the asset; by time T hold e^{yT} shares.
- To accumulate one share by T, need to start with e^{-yT} and continuously reinvest dividends.

Forward prices

- Let B_T denote the time-0 price of a discount bond paying $\$1$ at T.
- Suppose continuously-compounded interest rate is constant, r: $B_T = e^{-rT}$

Date	Forward Contract	Outright Asset Purchase
0	▪ Pay \$0 for contract with forward price F_T	▪ Borrow $e^{-yT}S$ ▪ Pay $e^{-yT}S$ for the asset
T	▪ Pay F_T ▪ Own asset	▪ Reinvest dividends ▪ Pay back $e^{-yT}S/B_T$ ▪ Own one share of the asset
Cash flow at T	$S_T - F_T$	$S_T - e^{-yT}S/B_T$

- Absence of arbitrage implies that

$$H_T \approx F_T = e^{(r-y)T}S$$

Forwards on financial assets

- Stock index forwards.
- The underlying asset (basket of stocks) pays dividends.
- Example: Prices on 2019.06.19 are
  - S&P500 closed at 2,926.46;
  - S&P forward with maturity in September is 2,929.00;
  - 3-month interest rate 2.16% (continuously compounded)

$$F = e^{0.25\times(r-y)}S$$

$$\Rightarrow y = r - 4\ln\left(\frac{F}{S}\right) = 2.16\% - 4\ln\left(\frac{2{,}929.00}{2{,}926.46}\right) = 1.81\%$$

  - The annual dividend yield $y = 1.81\%$.

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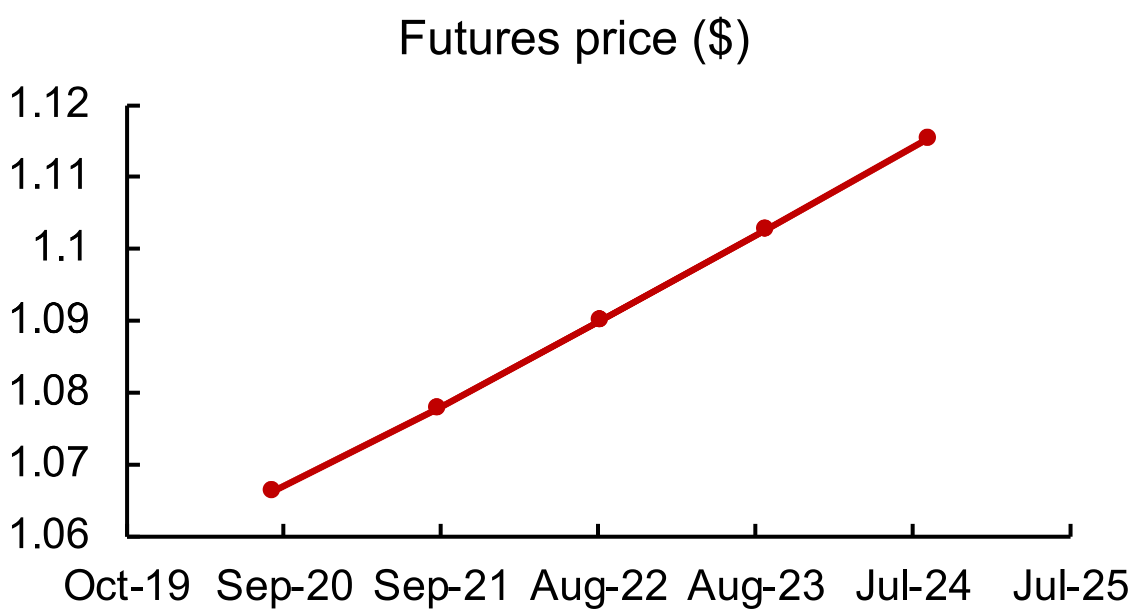
Currency forwards

- A forward contract to exchange a unit of one currency for a specified number of units of another currency.
- Example: a forward contract to exchange one Swiss Franc for 1.10 US Dollars in September of 2023.



Data source: CME Globex, Swiss Franc Futures Quotes, 07/20/2020

Pricing of currency forwards

- Suppose the forward price is F USD for 1 CHF; contract matures at time T.
- Let X_t denote the spot exchange rate: the price of 1 CHF in USD.
- Let r_{USD} and r_{CHF} denote continuously-compounded spot interest rates for tenor T, in US Dollars and Swiss Francs, respectively.
- The payoff of the forward at time T, is

$$₣1 - \$F = \$(X_T - F)$$

Pricing of currency forwards

- The payoff of the forward at time T, is

$$₣1 - \$F = \$(X_T - F)$$

- The present value (in USD) of the payoff from the forward must be zero:

$$\mathrm{PV}_0[X_T] = e^{-r_{CHF}T} X_0 = e^{-r_{USD}T} F$$

  - Note that the PV_0 of ₣1 $@T$ is ₣$e^{-r_{CHF}T}$, which is valued at $\$e^{-r_{CHF}T} X_0$.

- Find the forward rate:

$$\mathrm{F} = X_0 \times e^{(r_{USD} - r_{CHF})T}$$

Replication of currency forwards

- At $t = 0$:
  - Borrow $\$\mathrm{F}e^{-r_{USD}T} = \$X_0 \times e^{-r_{CHF}T}$ at the interest rate r_{USD};
  - Convert the borrowed amount into ₣$e^{-r_{CHF}T}$;
  - Invest the proceeds at the interest rate r_{CHF} (buy a CHF-denominated discount bond).
- At time $t = T$:
  - The payoff is identical to the forward:

$$₣1 - \$F$$

- The relation $F = X_0 e^{(r_{USD} - r_{CHF})T}$ is called the covered interest rate parity – it is a no-arbitrage condition.

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Futures contracts: main characteristics

- A futures contract is an exchange-traded, standardized, forward-like contract that is marked to the market daily.
- Futures contract can be used to establish a long (or short) position in the underlying commodity/asset.
- Standardized contracts:
  - Underlying commodity or asset,
  - Quantity,
  - Maturity.

Trading of futures contracts

- Traded on exchanges:
  - Chicago Board Options Exchange, CME Group, Eurex Exchange, London Metal Exchange, China Financial Futures Exchange, Tokyo Commodity Exchange, etc.
- Guaranteed by the clearing house – little counter-party risk.

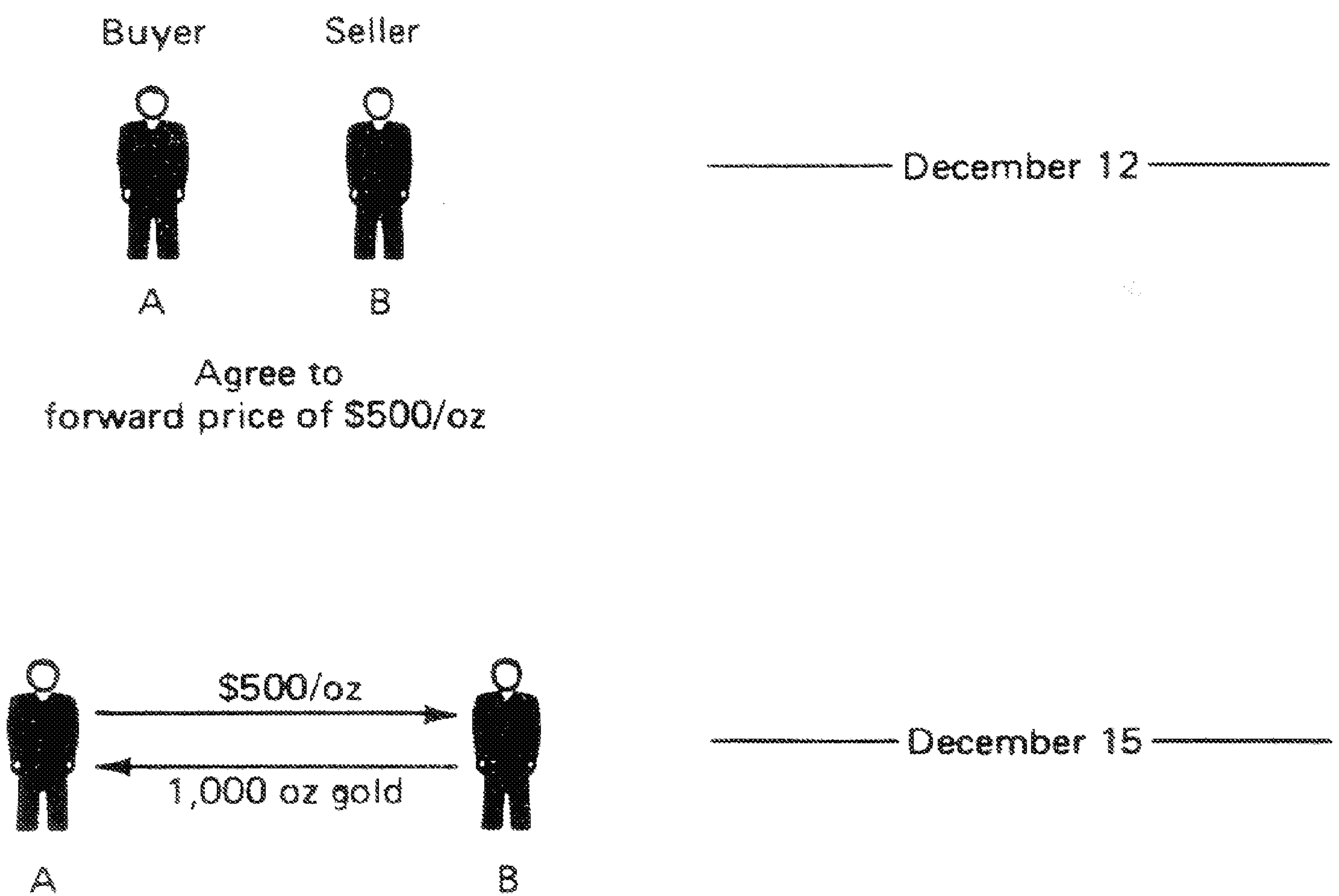
- Gains/losses settled daily – marked to market.
- Margin account required as collateral to cover losses.

Margin account

- Example: NYMEX crude oil (light sweet) futures with delivery in Oct. 2019 were traded at a price of $58.83/barrel on July 1, 2019.
- Each contract is for 1,000 barrels.
- Tick size: $0.01 per barrel, or $10 per contract.
- Initial margin: $3,960.
- Maintenance margin: $3,600.
- No cash changes hands today (contract price is $0).
- Buyer has a “long” position (wins if prices go up).
- Seller has a “short” position (wins if prices go down).

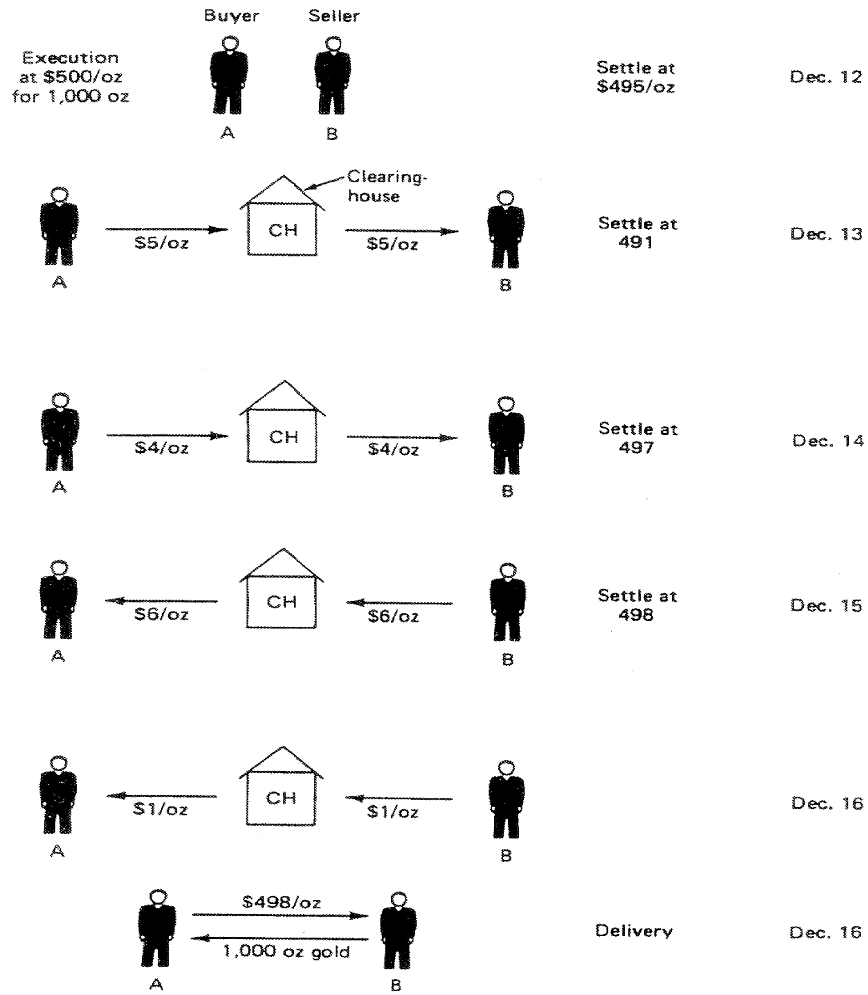
Mark to market

A forward contract



Mark to market

A futures contract



Forwards prices vs futures prices

- Both forward and futures prices are linked to spot prices.
- Differences have to do with the mark-to-market process for futures.

Contract	Spot now	Spot at T	Forward	Futures
Price	S	$S(T)$	F_T	H_T

- Ignore differences between forward and futures prices for now:

$$F_T \approx H_T$$

- Futures are different from forwards under stochastic interest rates.

Role of financial futures

- Since the underlying asset is a portfolio in the case of stock index futures, trading in the futures market is easier than trading in cash market when trading portfolios.

- Futures prices may react more quickly to macro-economic news than the index itself.

- Index futures are very useful to market makers, investment bankers, stock portfolio managers:

  - Creating synthetic index fund,

  - Implementing portfolio insurance,

  - Managing market risk in block purchases & underwriting.

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Commodity forward and futures prices

- Can price commodity futures using the same arbitrage argument as for financial futures.
- When holding commodities, net payout must reflect storage costs and any effective convenience yield from holding the physical commodity.
- Continuous compounding, assume storage costs flow in proportion to commodity value:

$$Cost_t = cS_t$$

- Valuation formula is

$$H_T \approx F_T = e^{(r-\hat{y})T} S$$

where $\hat{y}$ denotes the net convenience yield

$$\hat{y} = y - c$$

Example: gold futures

- Gold is often held for long-run investments.
- Easy to store — negligible cost of storage.
- No dividends or benefits.
- Two ways to buy gold at T :
  - Buy now for S and hold until T;
  - Buy forward, pay F and take delivery at T.
- No-arbitrage requires that:

$$H \approx F = Se^{rT}$$

Example: gold futures

$$H \approx F = Se^{rT}$$

- Prices on 2019.07.01:
  - Spot price of Gold: \$1,387.45/oz.
  - 2019 October futures (CME): \$1,397.80/oz.
- Implied continuously-compounded 3-month interest rate is $r = 2.23\%$, relative to the 3-month T-bill rate of 2.05%.

Example: oily futures

- Unlike gold, held for future use and not for long-term investment.
- Costly to store.
- Additional benefits (convenience yield) for holding physical commodity (over holding futures).
- Valuation equation:

$$H_T \approx F_T = e^{(r-\hat{y})T} S$$

Example: oily futures

$$H_T \approx F_T = e^{(r-\hat{y})T} S$$

- Prices on 2019.07.19:
  - Spot oil price 55.68/barrel (light sweet);
  - Oct. oil futures price 55.83/barrel (NYMEX);
  - 3-month continuously-compounded interest rate is 2.3%,
- 3.5 months to expiration.
- Annualized net convenience yield: $\hat{y} = 2\%$.

Prices of commodity futures

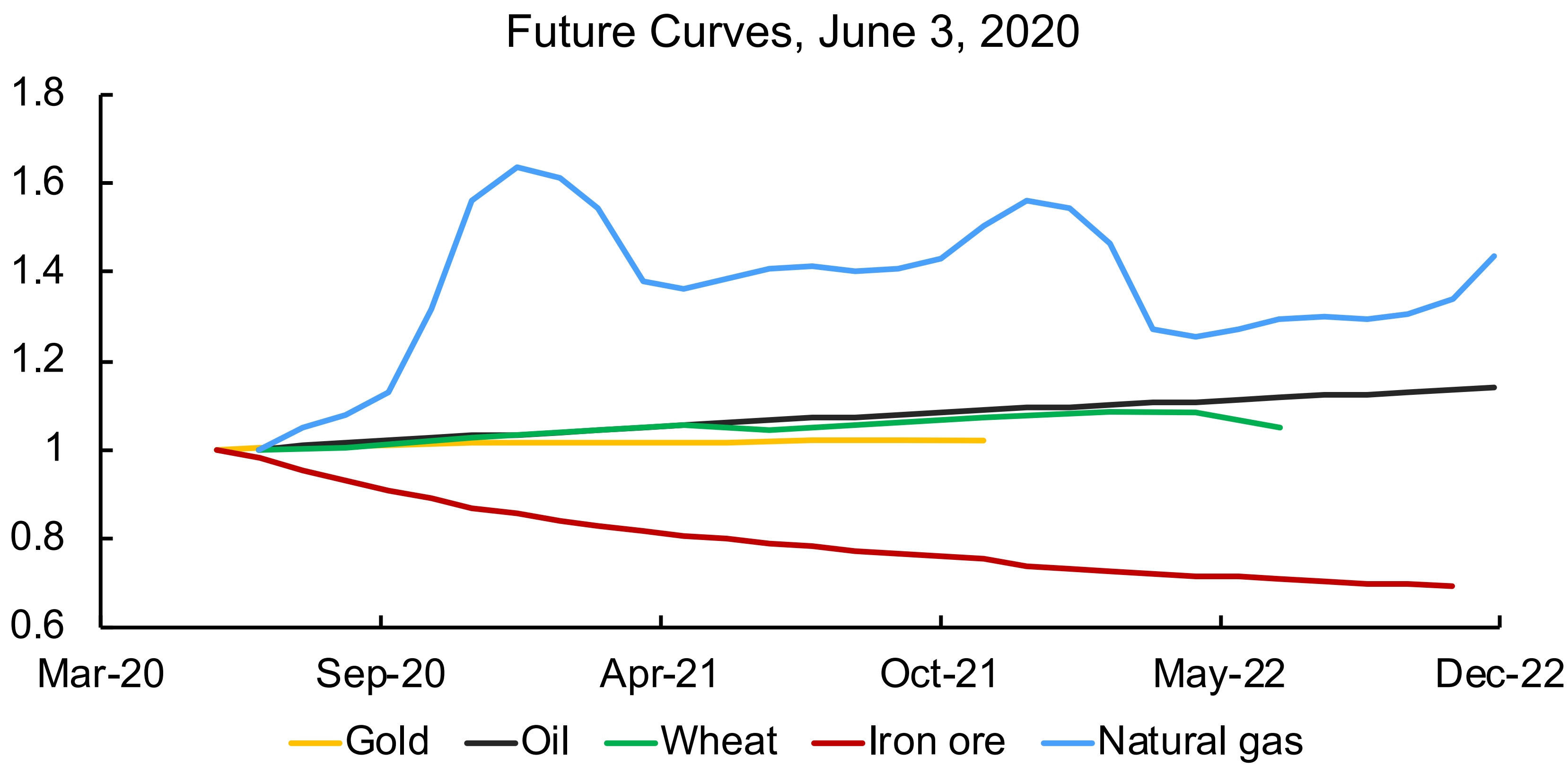
- For commodity futures:
  - Contango: Futures prices increase with maturity;
  - Backwardation: Futures prices decrease with maturity.

- Another definition adjusts for the time-value of money:
  - Contango: $H > Se^{rT}$;
  - Backwardation: $H < Se^{rT}$.

Various shapes of commodity forward curves

Backwardation occurs if net convenience yield exceeds the interest rate:

$$\hat{y} - r = y - c - r > 0$$



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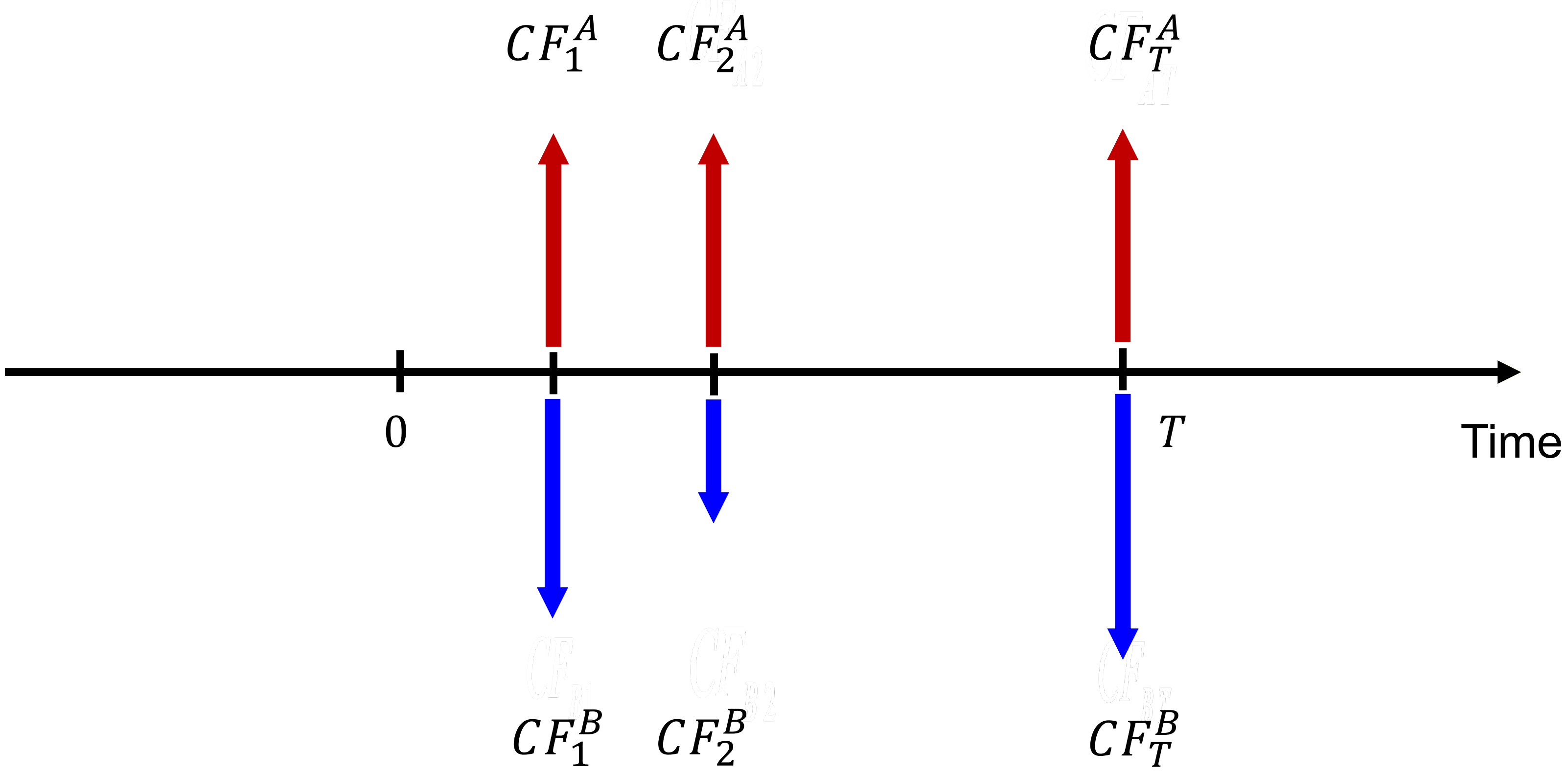
Swaps

- Swap: A contract in which two counterparties agree to exchange specified amounts of assets (e.g., cash, financial assets or commodities) at a set of given future dates.



Example: LIBOR swap

- A fixed rate of interest is exchanged for a reference floating rate.
- For example, the London Interbank Offered Rate (LIBOR) could be used as the reference floating rate.
- Payments are made periodically, e.g., at the end of each 6-month sub-period.
- Example: assume the current 1-month LIBOR is 0.2%. You enter into a 5-year fixed-for-floating swap with a fixed rate of 0.35%.
  - If LIBOR rises in the future, you receive higher payments on the floating leg, continue making fixed payments on the fixed leg.
  - The swap represents a bet on higher future values of LIBOR.

Valuation of an interest rate swap

- Suppose that at the end of each period t, the floating arm of the swap pays the spot risk-free rate over that period, $\tilde{r}_1(t-1)$.
- The fixed arm pays a fixed rate, r_S.
- The fixed rate is chosen so that the swap contract is at par initially: no money changes hands.
- The swap is over T periods.
- What is the swap rate r_S?

Valuation of an interest rate swap

- Recall B_t denotes the time-0 price of a discount bond paying $\$1$ at time t.
- The present value of the fixed arm of the swap is $r_S \times \sum_{t=1}^{T} B_t$.
- The present value of the floating arm of the swap is $\sum_{t=1}^{T} \mathrm{PV}_0[\tilde{r}_1(t-1) \text{ at } t]$
- We impose that no money should change hands initially:

$$r_S \times \sum_{u=1}^{T} B_u = \sum_{t=1}^{T} \mathrm{PV}_0[\tilde{r}_1(t-1) \text{ at } t] = \sum_{t=1}^{T} \mathrm{PV}_0[f_t \text{ at } t] = \sum_{t=1}^{T} B_t f_t$$

  - Recall that $\mathrm{PV}_0[\tilde{r}_1(t-1) \text{ at } t] = \mathrm{PV}_0[f_t \text{ at } t] = B_t f_t$.
- Conclude that the swap rate is a weighted average of forward rates:

$$r_S = \frac{\sum_{t=1}^{T} B_t f_t}{\sum_{u=1}^{T} B_u} = \sum_{t=1}^{T} w_t \times f_t, \qquad \text{where} \quad w_t = \frac{B_t}{\sum_{u=1}^{T} B_u}$$

Valuation of an interest rate swap

- Start with $r_S = \frac{\sum_{t=1}^{T} B_t f_t}{\sum_{u=1}^{T} B_u}$.
- Recall that $f_t = \frac{B_{t-1}}{B_t} - 1$ to obtain an alternative expression:

$$r_S = \frac{\sum_{t=1}^{T} B_t \left(\frac{B_{t-1}}{B_t} - 1\right)}{\sum_{u=1}^{T} B_u} = \frac{1 - B_T}{\sum_{u=1}^{T} B_u}$$

- Suppose that the bond with coupon rate c trades at par. Then,

$$\sum_{u=1}^{T} B_u\, c + B_T = 1 \;\;\Rightarrow\;\; c = \frac{1 - B_T}{\sum_{u=1}^{T} B_u}$$

- We conclude that the swap rate equals the coupon rate on the coupon bond trading at par.

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Short instructional videos from the CME

- Futures vs forwards: https://www.cmegroup.com/education/courses/introduction-to-futures/futures-contracts-compared-to-forwards.html
- Margin: https://www.cmegroup.com/education/courses/introduction-to-futures/margin-know-what-is-needed.html
- Mark-to-market: https://www.cmegroup.com/education/courses/introduction-to-futures/mark-to-market.html

Scratch pad