15.415 Foundations of Modern Finance

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Lecture 9: Options, Part 1

Key concepts

- Introduction: option types
- Payoffs of European options
- Option strategies
- Corporate securities as options
- Pricing bounds for European options
- Put-Call parity for European options
- Early exercise of American options
- Binomial model: single period
- Binomial model: multiple periods

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Introduction to option types

- Option types:
  - Call: The right to buy an asset (the underlying asset) for a given price (exercise price) on or before a given date (expiration date).
  - Put: The right to sell an asset for a given price on or before the expiration date.
- Exercise styles:
  - European: Owner can exercise the option only on expiration date.
  - American: Owner can exercise the option on or before expiration date.

Introduction to option types

- Key elements in defining an option:
  - Underlying asset and its price S,
  - Exercise price (strike price) K,
  - Expiration date (maturity date) T (today is 0),
  - European or American.

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Example: a European call option

- A European call option on IBM with exercise price $100.
- It gives the owner (buyer) of the option the right (not the obligation) to buy one share of IBM at $100 on the expiration date.
- The option's payoff depends on the share price of IBM on the expiration date.

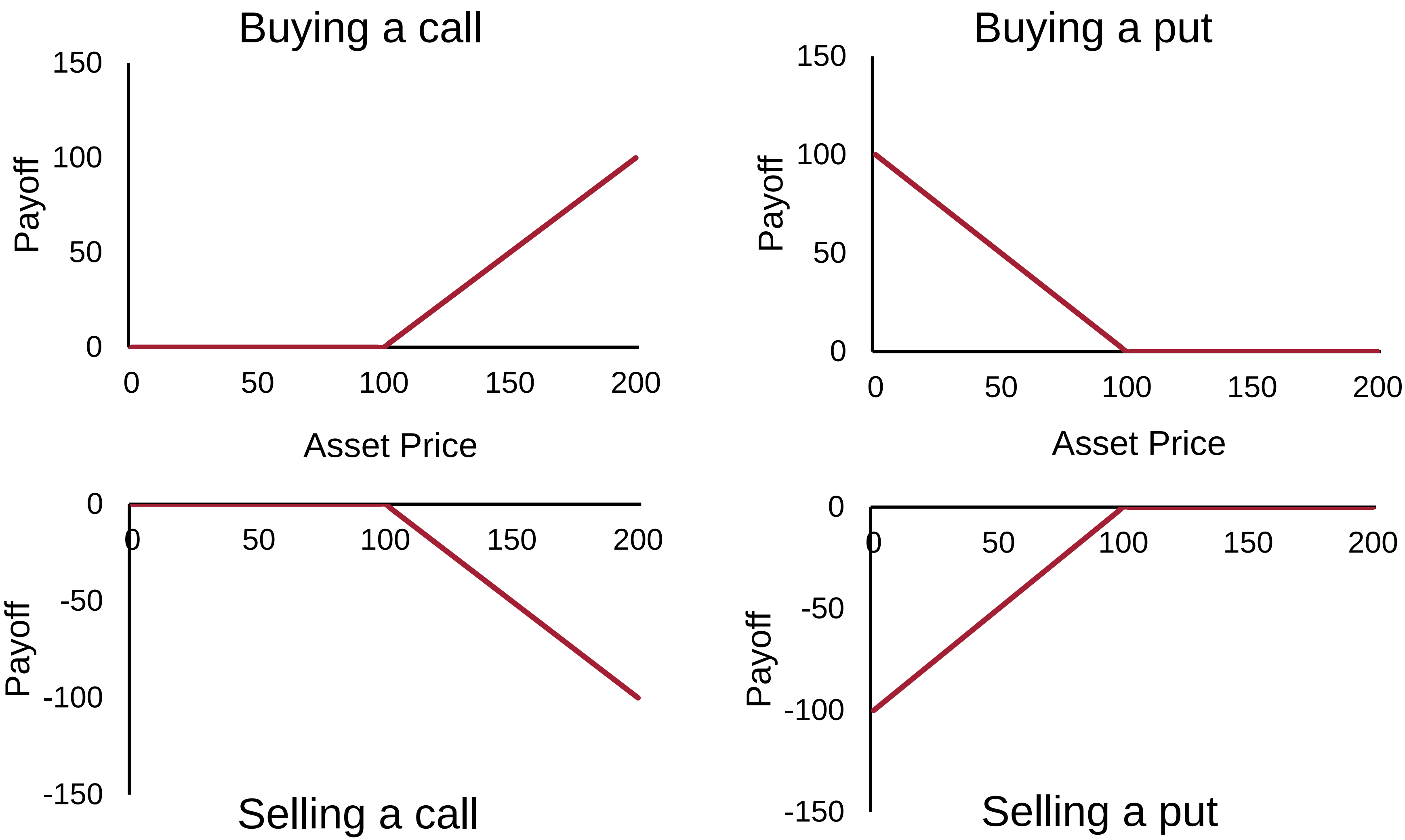
Example: a European call option

IBM Price at T	Action	Payoff
< 80	Not Exercise	0
80	Not Exercise	0
90	Not Exercise	0
100	Not Exercise	0
110	Exercise	10
120	Exercise	20
130	Exercise	30
S_T	Exercise	$S_T - 100$

- Observations:
  - The payoff of an option is never negative; sometimes, it is positive.
  - Actual payoff depends on the price of the underlying asset:

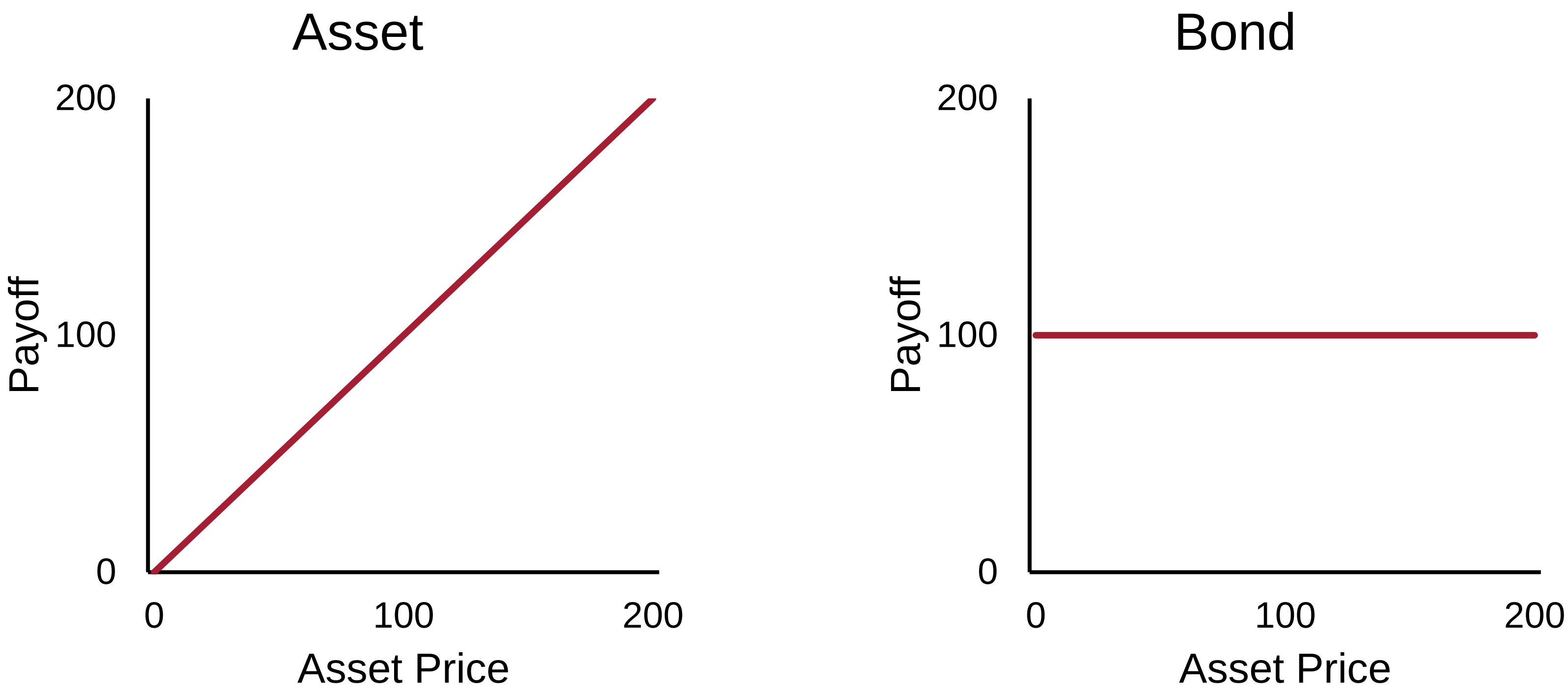
$$CF_T(\text{call}) = \max[S_T - K, 0]$$

Payoff diagrams



Payoff diagrams

- The underlying asset and the bond (with face value $100) have the following payoff diagrams

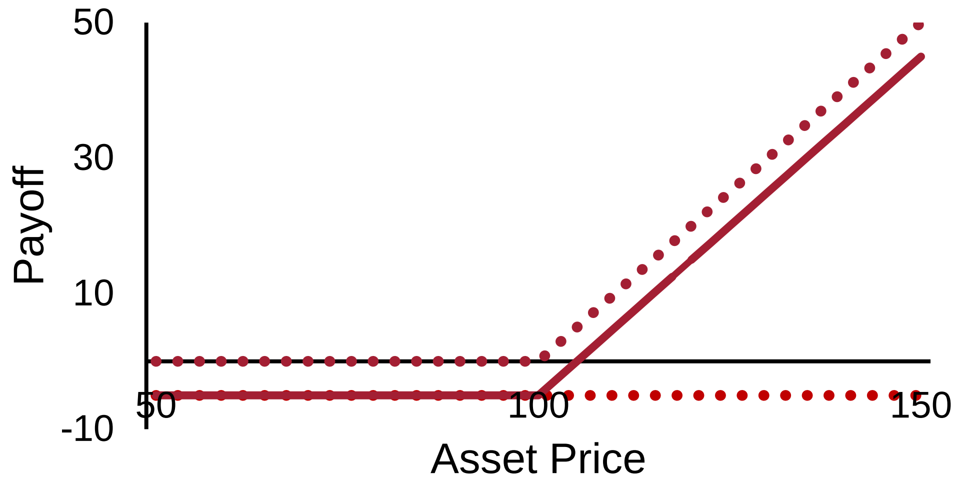


Net option payoff

- The net payoff of an option must include its cost.
- Example: a European call on IBM shares with an exercise price of \$100 and maturity of three months is trading at \$5.
- The 3-month interest rate, not annualized, is 0.5%.
- At maturity, the call's net payoff is as follows.

IBM Price	Action	Payoff	Net payoff
< 80	Not Exercise	0	−5.025
80	Not Exercise	0	−5.025
90	Not Exercise	0	−5.025
100	Not Exercise	0	−5.025
110	Exercise	10	4.975
120	Exercise	20	14.975
S_T	Exercise	$S_T - 100$	$S_T - 100$ −5.025

Net option payoff



The break even point is given by:

$$\begin{aligned}\text{Net payoff} &= \max[S_T - K, 0] - C(1+r)^T \\ &= S_T - 100 - (5)(1+0.005) \\ &= 0\end{aligned}$$

or

$$S_T = \$105.025$$

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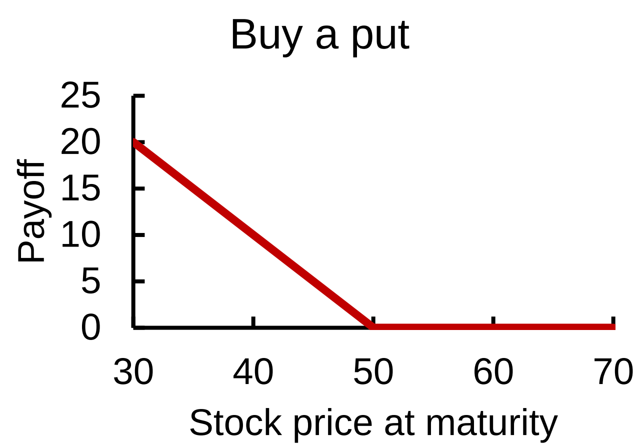
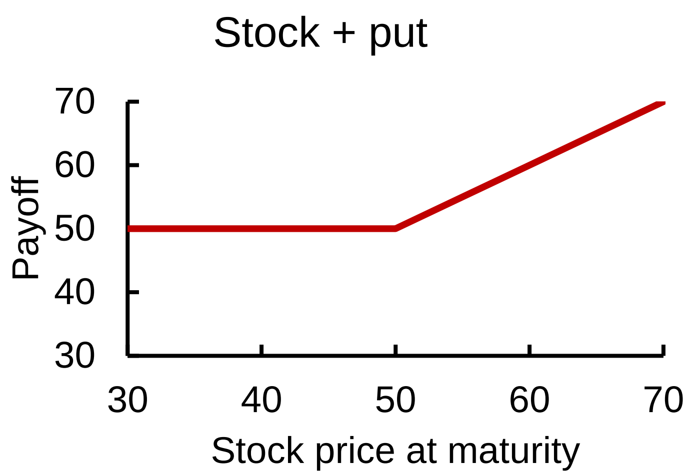
Option strategies: protective put

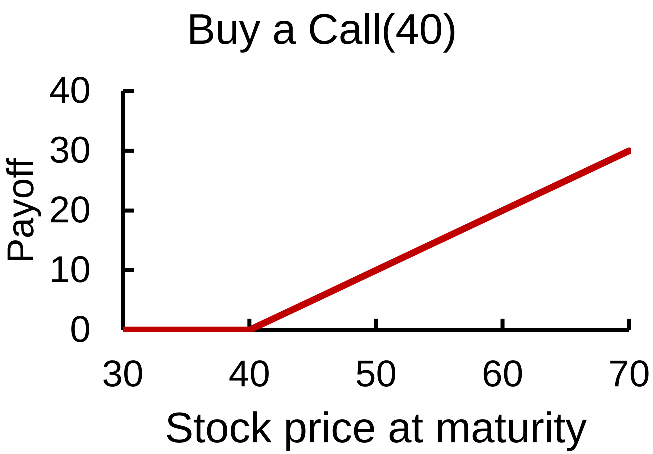
Buy the underlying stock, and buy a put with a strike price of $50:

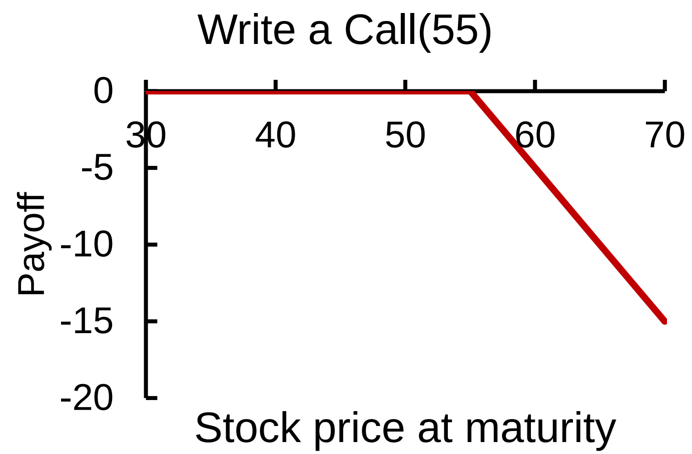


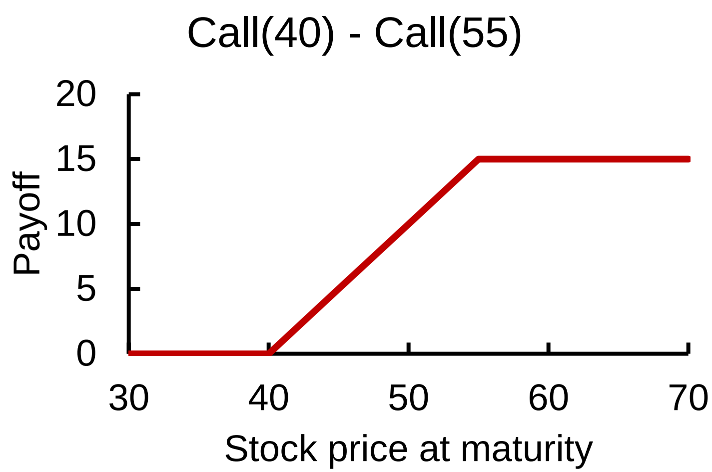


Option strategies: bull spread spread

Buy a call with a strike price of $45, and write a call with a strike price of $55:

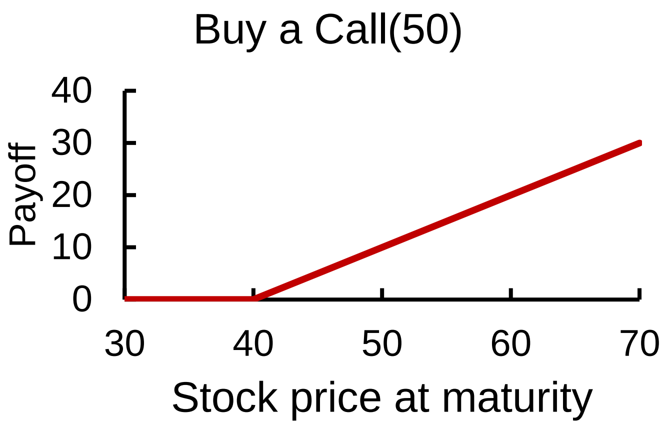


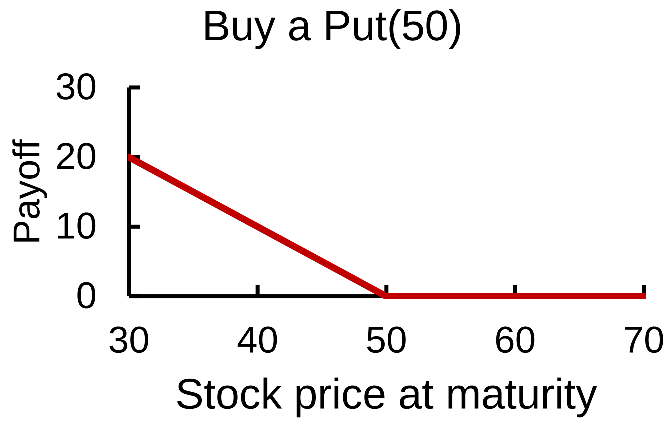


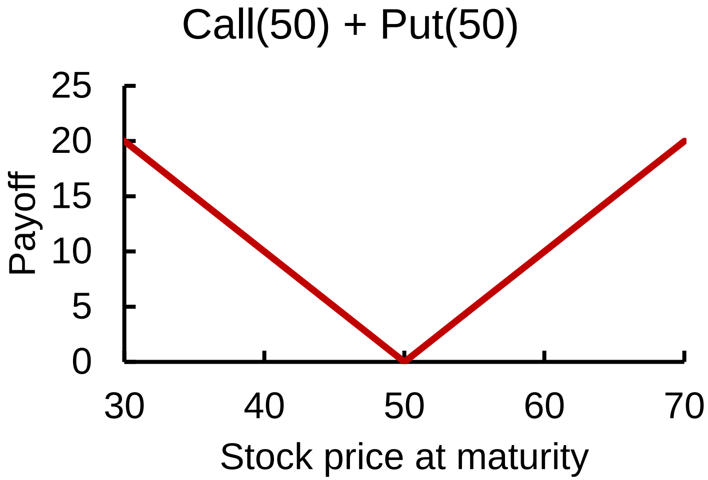


Option strategies: straddle

Buy a call with a strike price of $50, and buy a put with a strike price of $50:







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Corporate securities as options

- Consider two firms, A and B, with identical assets but different capital structures (in market value terms).

Balance sheet of A			
Assets	\$30	\$0	Bond
		\$30	Equity
Total	\$30	\$30	

Balance sheet of B			
Assets	\$30	\$25	Bond
		\$5	Equity
Total	\$30	\$30	

- Firm B's bond has a face value of \$50.
- Default is likely: If the firm's assets are worth less than \$50 when the bond matures, the firm will be unable to afford its debt.
- In that event, the assets are turned over to the bondholders, and the equity is worth zero.

Corporate securities as options

- Consider the value of stock A, stock B, and a call on the underlying assets of firm B with an exercise price of $50:

Asset value	Value of stock A	Value of stock B	Value of call on assets of B
⋮	⋮	⋮	⋮
$20	$20	$0	$0
$40	$40	$0	$0
$50	$50	$0	$0
$60	$60	$10	$10
$80	$80	$30	$30
$100	$100	$50	$50
⋮	⋮	⋮	⋮

- Stock B gives the same payoff as a call option written on its assets;
- Thus B’s common stocks really are call options on firm’s assets.

Corporate securities as options

- Many corporate securities can be viewed as options.
- Equity (E): A call option on the firm's assets (A) with the exercise price equal to its bond's redemption value.
- Debt (D): A portfolio combining the firm's assets (A) and a short position in the call with the exercise price equal to its bond's redemption value:

$$E \equiv \max[0, A - B]$$

$$D \equiv \min[A, B] = A - \max[0, A - B]$$

$$A = D + E$$

Corporate securities as options

- Warrant: Call options on the firm's stock.
- Convertible bond: A portfolio combining straight bonds and a call on the firm's stock with the exercise price related to the conversion ratio;
- Callable bond: A portfolio combining straight bonds and a call written on the bonds.

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Preliminaries

- For convenience, we refer to the underlying asset as stock. It could also be a bond, foreign currency or some other asset.
- Notation:
  - S : Price of stock now;
  - S_T : Price of stock at T;
  - B : Price of discount bond of par \$1 and maturity T $(B \leq 1)$;
  - C : Price of a European call with strike K and maturity T;
  - P : Price of a European put with strike K and maturity T.
- For our discussion:
  - Consider only European options (no early exercise);
  - Assume no dividends (option cash flow occurs only at maturity).

Basic properties of options

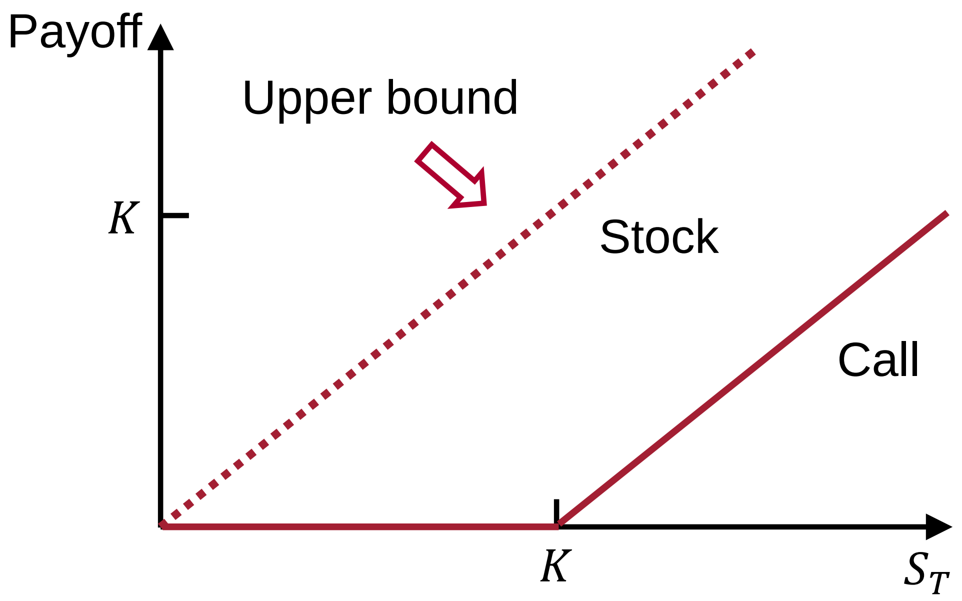
- If an option is exercised now, the resulting cash flow is called its exercise value.
  - For a call, its exercise value is $S - K$, where S is the current stock price;
  - For a put, its exercise value is $K - S$.

- An option is deemed to be:
  - In the money (ITM) if its exercise value is positive;
  - At the money (ATM) if its exercise value is zero;
  - Out of the money (OTM) if its exercise value is negative.

Price bounds

- European options on a non-dividend paying stock.

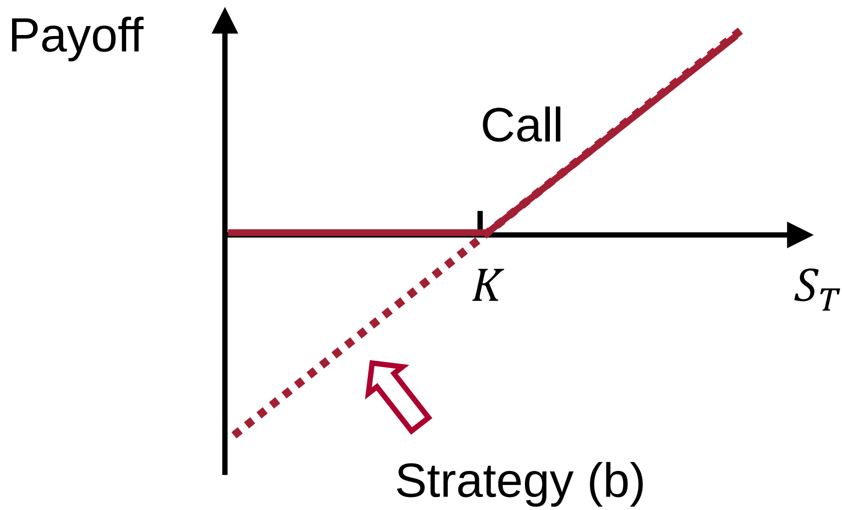
1. The payoff of call can never be negative: $C \geq 0$.
2. The payoff of stock dominates that of call: $C \leq S$.



Price bounds (cont'd)

3. Lower bound: $C \geq S - K\,B$

   - Strategy (a): Buy a call;
   - Strategy (b): Buy a share of stock by borrowing *K B.*
   - The payoff of strategy (a) dominates that of strategy (b):



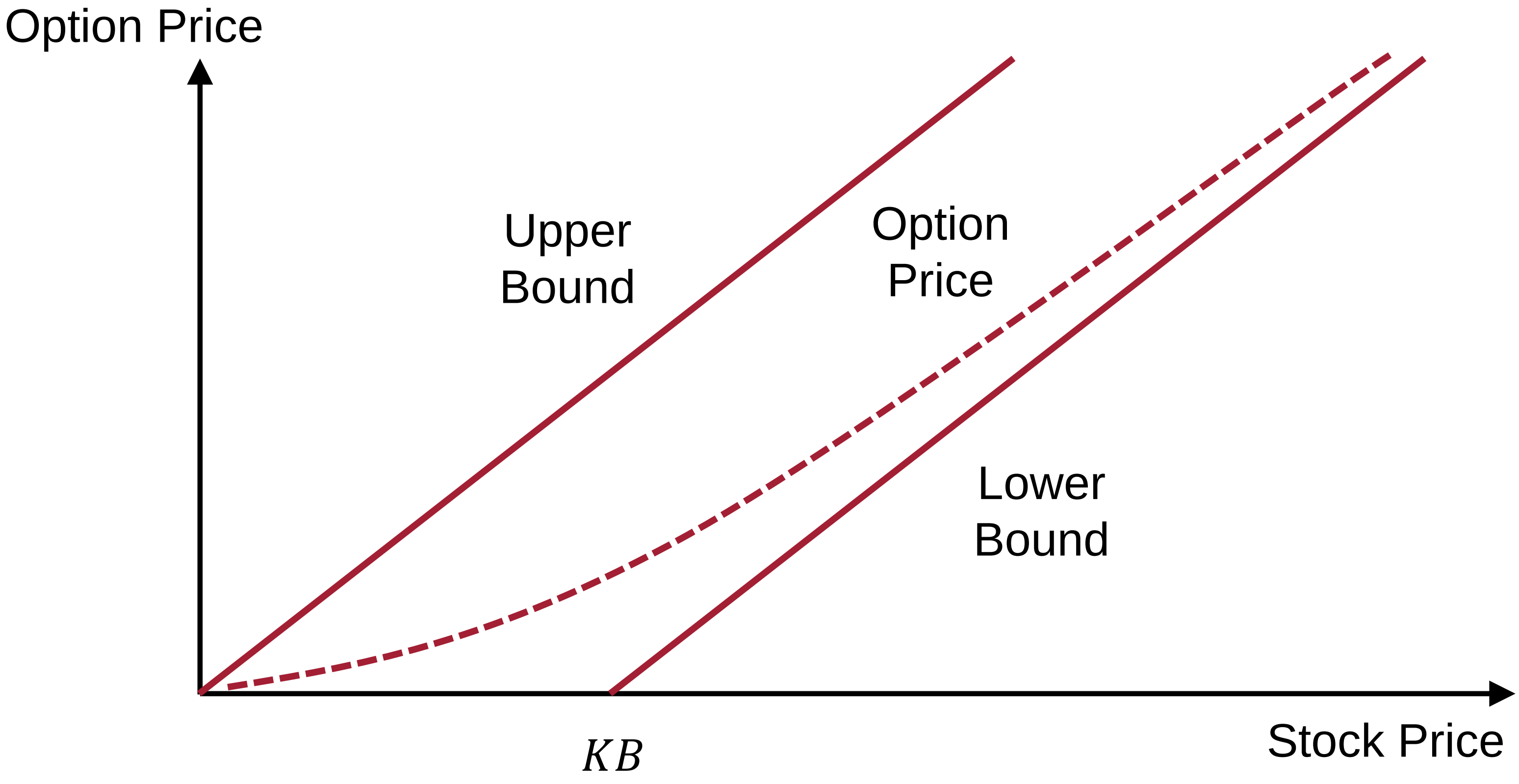
- Since $C \geq 0$, we have:

$$C \geq \max[S - KB, 0]$$

Price bounds (cont'd)

4. Combining the above, we have:

$$\max[S - KB, 0] \leq C \leq S$$



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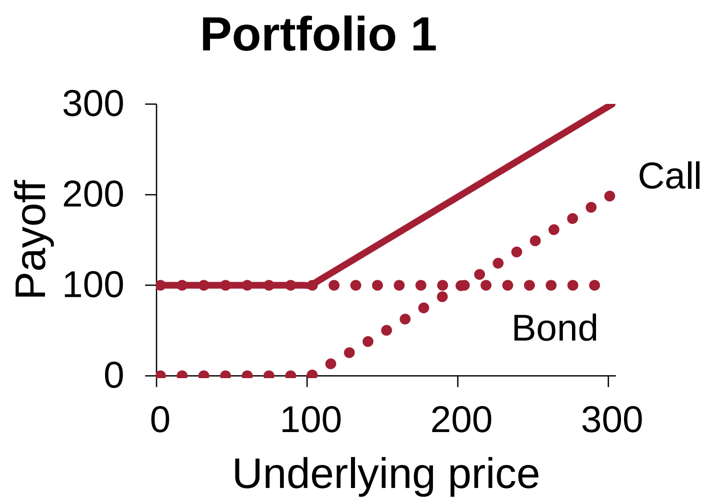
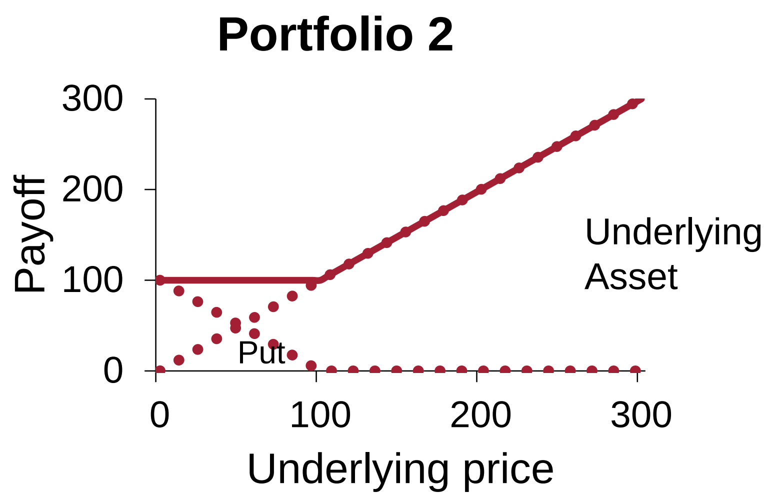
Put-call parity

Portfolio 1: A call with strike \$100 and a bond with par \$100;

Portfolio 2: A put with strike \$100 and a share of the underlying asset.





$$C + \frac{K}{(1+r)^T} = P + S$$

Their payoffs are identical, so must be their prices:

This is called the put-call parity.

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American options, early exercise

- American options are worth more than their European counterparts.
- Without dividends, never exercise American call early.
- If exercise at $t < T$, collect $S_t - K$
- If sell the option at instead, collect at least the price of a European call, which is

$$C(S_t, K, T-t) \geq \max[0, S_t - KB_t] \geq \max[0, S_t - K]$$

- Better to sell than exercise, thus early exercise is never optimal!
- By the law of one price:

$$c(S_t, K, T-t) = C(S_t, K, T-t)$$

American options, early exercise

- Without dividends, it can be optimal to exercise an American put early.
- Example. A put with a strike of $10 on a stock with price of zero.
  - Exercise now gives $10 today;
  - Exercise later gives $10 later.
- Better to exercise now (assuming positive interest rate).

Effect of dividends

- With dividends:

$$\max[0, S - KB - PV(D)] \leq C \leq S$$

- Dividends make early exercise more likely for American calls and less likely for American puts.

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Underlying volatility affects option value

- Option value increases with the volatility of underlying asset.
- Example: two firms, A and B, with the same current price of \$100.
- B has higher volatility of future prices.
- Consider call options written on A and B, respectively, with the same exercise price \$100.

	Good state	**Bad state**
	Probability = p	Probability = $1 - p$
Stock A	120	80
Stock B	150	50
Call on A	20	0
Call on B	50	0

- Clearly, call on stock B should be more valuable.

Binomial option pricing model

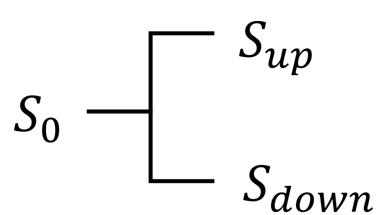
0 ——————————————— 1

$S_0, B_T;\ C_0 = ?$ $\qquad$ $S_1, 1; C_1 = \text{Max}\,[S_1 - K, 0]$

- Determinants of option value:
  1. price of underlying asset S,
  2. strike price K,
  3. time to maturity T,
  4. interest rate r,
  5. volatility of underlying asset σ.

Binomial option pricing model

- In order to have a complete option pricing model, we need to make additional assumptions about:
  - Price process of the underlying asset (stock),
  - Other factors.
- We will assume, in particular, that:
- Prices do not allow arbitrage,
- Prices are “reasonable”.
- A benchmark model – price follows a binomial process.



A European call on a stock

- Current stock price is \$50;
- There is one period to go;
- Stock price will either go up to \$75 or go down to \$25;
- There are no cash dividends;
- The strike price is \$50;
- One period borrowing and lending rate is 10%.

A European call on a stock

- The stock and bond present two investment opportunities:

Stock	Bond
$50 \begin{cases} 75 \\ 25 \end{cases}$	$1 \begin{cases} 1.1 \\ 1.1 \end{cases}$

- The option's payoff at expiration is:

$$C_0 \begin{cases} 25 \\ 0 \end{cases}$$

- What is C_0, the value of the option today?

Replicating portfolio

- Form a portfolio of stock and bond that replicates the call's payoff:
- a shares of the stock;
- b dollars in the riskless bond.

such that:

$$\begin{aligned} 75a + 1.1b &= 25 \\ 25a + 1.1b &= 0 \end{aligned}$$

- Unique solution: $a = 0.5$ and $b = -11.36$.

Replicating portfolio

- Replication strategy:
  - buy half a share of stock and sell \$11.36 worth of bond;
  - payoff of this portfolio is identical to that of the call;
  - present value of the call must equal the current cost of this “replicating portfolio” which

$$(50)(0.5) - 11.36 = 13.64$$

- Number of shares needed to replicate one call option is called the option’s hedge ratio or delta.
- In the above problem, the option’s delta is $a = 0.5$.

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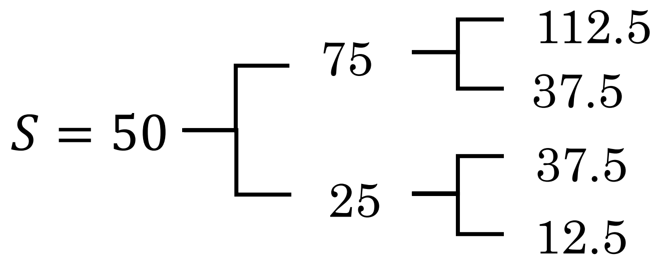
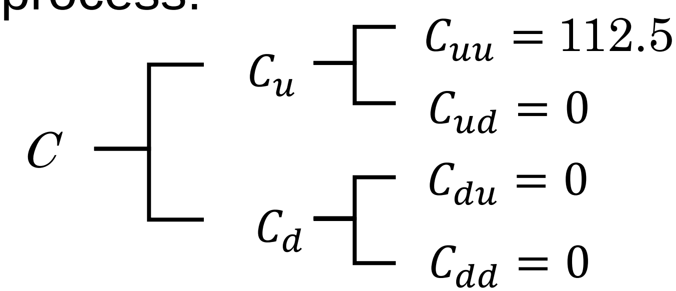
Binomial option pricing model

- Multiple periods:



- European call price process:



- The terminal value of the call is known

- C_u and C_d denote the option value next period when the stock price goes up and goes down, respectively.

- Compute the initial value working backwards: first C_u and C_d and then C.

Period 1, "up" node

Start with Period 1:

- Suppose the stock price goes up to \$75 in period 1.
- Construct the replicating portfolio at node ($t = 1$, up):

$$
\begin{aligned}
112.5a + 1.1b &= 62.5 \\
37.5a + 1.1b &= 0
\end{aligned}
$$

- Unique solution: $a = 0.833, b = -28.4$.
- The cost of this portfolio: $(0.833)(75) - 28.4 = 34.075$.
- Thus, $C_u = 34.075$.

Period 1, "down" node

- Suppose the stock price goes down to \$25 in period 1. Repeat the above for node ($t = 1$, down):

$$
\begin{aligned}
112.5a + 1.1b &= 0 \\
37.5a + 1.1b &= 0
\end{aligned}
$$

- The replicating portfolio: $a = 0$, $b = 0$.

- The call value at the lower node next period is $C_d = 0$.

Period 0

- Now go back one period, to period 0:
- The option's value next period is either 34.075 or 0:

$$
C_0 \begin{cases} C_u = 34.075 \\ C_d = 0 \end{cases}
$$

- If we can construct a portfolio of the stock and bond to replicate the value of the option next period, then the cost of this replicating portfolio must equal the option's present value.

Period 0

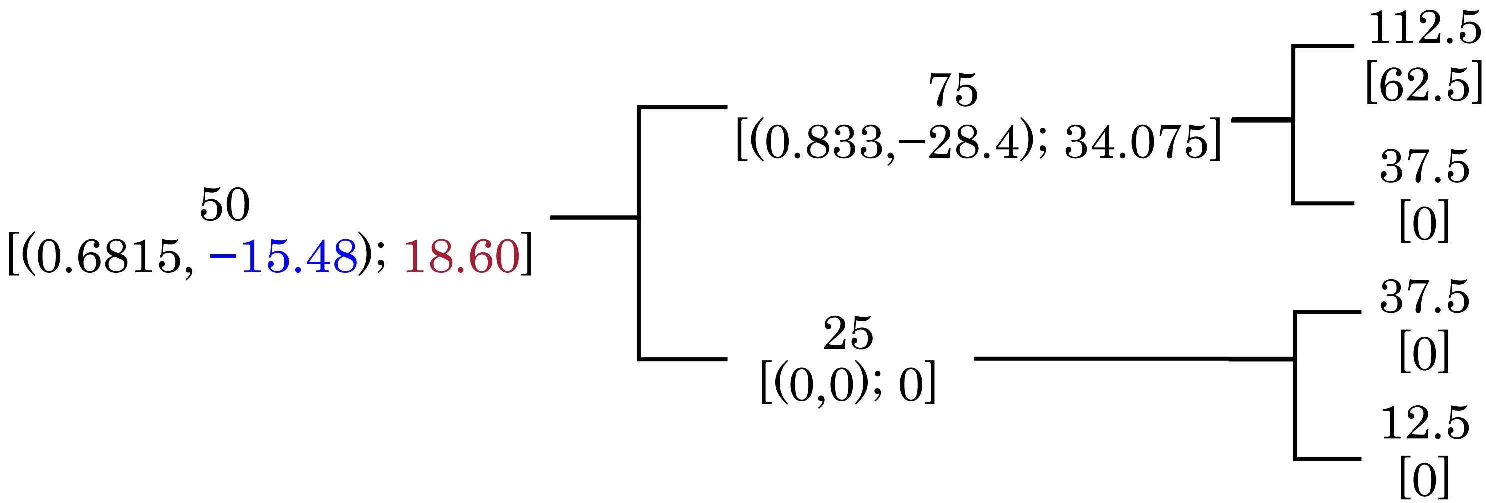
$$C_0 \quad \begin{cases} C_u = 34.075 \\ C_d = 0 \end{cases}$$

- Find a and b so that:

$$75a + 1.1b = 34.075$$
$$25a + 1.1b = 0$$

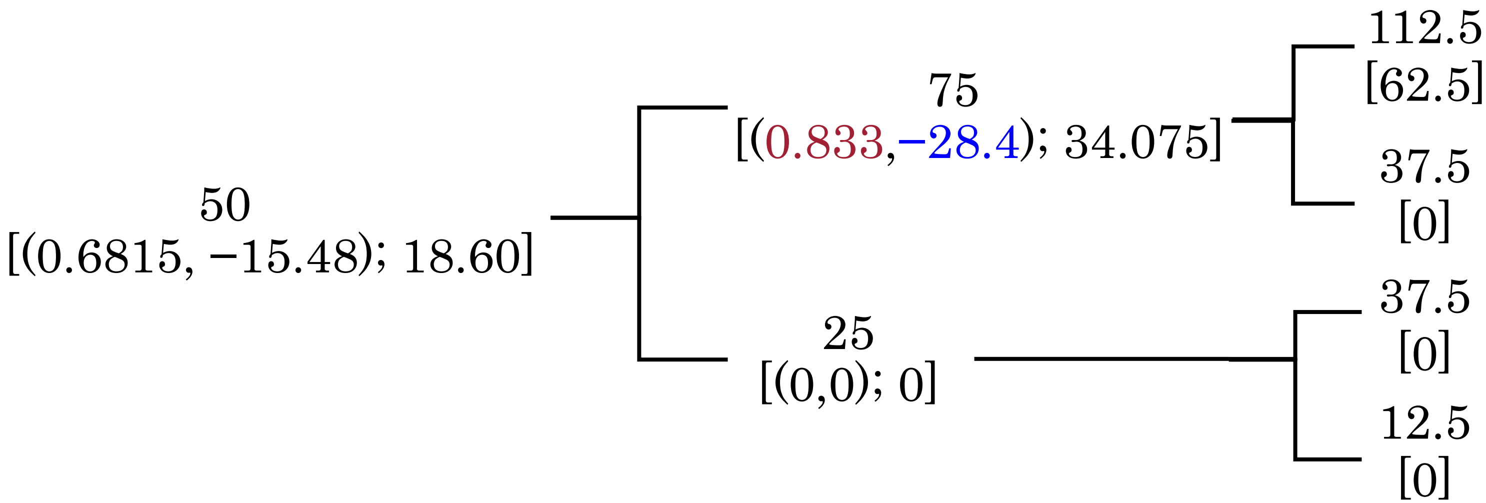
- Unique solution: $a = 0.6815, b = -15.48$.
- The cost of this portfolio: $(0.6815)(50)$ – $15.48 = 18.60$.
- The present value of the option must be $C_0 = 18.60$.
- It is greater than the exercise value 0 (thus no early exercise).

Option replication



- Period 0: Spend \$18.60 and borrow \$15.48 at 10% interest rate to buy 0.6815 shares of the stock.

Option replication



- Period 1, “up node”: the portfolio value is 34.075. Re-balance the portfolio to include 0.833 stock shares, financed by borrowing 28.4 at 10%.
  - One period later, the payoff of this portfolio exactly matches that of the call.
- Period 1, “down node”: the portfolio becomes worthless. Close out the position.
  - The portfolio payoff one period later is zero.

Binomial option pricing model

- Bottom line:
  - Replicating strategy gives payoffs identical to those of the call.
  - Initial cost of the replicating strategy must equal the call price.

Binomial option pricing model

- What we have used to calculate option's value:
  - current stock price,
  - magnitude of possible future changes of stock price — volatility,
  - interest rate,
  - strike price,
  - time to maturity.

Binomial option pricing model

- What we have not used:
  - probabilities of upward and downward movements,
  - investor's attitude towards risk.
- Questions on the Binomial Model:
  - What is the length of a period?
  - Price can take more than two possible values.
  - Trading takes place continuously.
- Response: The length of a period can be arbitrarily small.

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Appendix: Additional results

Put-call parity with a forward

Portfolio 1: A call with strike K and a bond with par K;

Portfolio 2: A put with strike K, long forward contract (forward price F), long bond with fact value F.

$$C + \frac{K}{(1+r)^T} = P + \frac{F}{(1+r)^T}$$

	Portfolio 2	
	PV_0	Payoff
Put	P	Max(0, $K - S_T$)
Forward	0	$S_T - F$
Bond	$\frac{F}{(1+r)^T}$	F
Total	$P + \frac{F}{(1+r)^T}$	Max(0, $K - S_T$) $+S_T$= Max(S_T, K)

Put-call parity with a forward

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Portfolio 2: A put with strike K, long forward contract (forward price F), long bond with fact value F.

$$C + \frac{K}{(1+r)^T} = P + \frac{F}{(1+r)^T}$$

	Portfolio 1	
	PV_0	Payoff
Call	C	Max(0, $S_T - K$)
Bond	$\frac{K}{(1+r)^T}$	K
Total	$C + \frac{K}{(1+r)^T}$	Max(0, $S_T - K$) + K = Max(S_T, K)