Pset 5

~~1~~ $\cancel{dF = \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}dt + \frac{\partial F}{\partial S}dS}$

$\cancel{= \frac{\partial F}{\partial S}}$

$\cancel{= aS^{\alpha}\frac{\partial F}{\partial x} + b^2 S}$

1 $dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}b^2 S^{2\beta}\frac{\partial^2 F}{\partial X^2} + aS^{\alpha}\frac{\partial F}{\partial X}\right)dt + bS^{\beta}\frac{\partial F}{\partial X}dB$

$\Rightarrow bS^{\beta}\frac{\partial F}{\partial S} = 6$ since F is a function of S only

$\Rightarrow dF = \frac{6}{b}S^{-\beta}dB$

$F = \int \frac{6}{b}S^{-\beta}dS = \frac{6}{(1-\beta)b}S^{1-\beta}$

2. $F = XY$, $dF = YdX + XdY + (dX)(dY)$

$F = X^{\alpha}Y^{\beta}$ if $\alpha, \beta \neq 1, 0$

$dF = \frac{\partial F}{\partial X}dX + \frac{\partial F}{\partial Y}dY + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}(dY)^2 + \frac{\partial^2 F}{(dX)(dY)}$

$= \alpha X^{\alpha-1}Y^{\beta}dX + \beta X^{\alpha}Y^{\beta-1}dY + \frac{1}{2}\alpha(\alpha-1)X^{\alpha-2}Y^{\beta}(dX)^2 + \frac{1}{2}\beta(\beta-1)X^{\alpha}Y^{\beta-2}(dY)^2$

$+ \alpha\beta X^{\alpha-1}Y^{\beta-1}(dX)(dY)$

if $\alpha, \beta = 0$ or 1, corresponding item will have parameter of 0, so this equation fits all situations

$F = X/Y$ $dF = \frac{1}{Y}dX - \frac{X}{Y^2}dY + \frac{X}{Y^3}(dY)^2 - \frac{1}{Y^2}(dX)(dY)$

$F = e^{\lambda X}$ $dF = \frac{\partial F}{\partial X}dX + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2$

$= \lambda e^{\lambda X}dX + \frac{1}{2}\lambda^2 e^{\lambda X}(dX)^2$

$F = e^{\lambda t}X$ $dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2$

$= \lambda e^{\lambda t}Xdt + e^{\lambda t}dX$

$F = X\exp(\int_0^t \lambda(s)ds)$ $\frac{\partial F}{\partial X} = \exp(\int_0^t \lambda(s)ds)$ $\frac{\partial^2 F}{\partial X^2} = 0$

~~dF~~

$$\frac{\partial F}{\partial t} = \cancel{\ }\cdot \lambda(t)\cdot \exp\left(\int_0^t \lambda(s)ds\right)\cdot X$$

$$dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial X} dX$$

$$= X\lambda(t)\exp\left(\int_0^t \lambda(s)ds\right)dt + \exp\left(\int_0^t \lambda(s)ds\right)dX$$

they are all Itô processes

4. since $T \to \infty$, time decay of options $\cancel{\ }=0$, $\frac{\partial V}{\partial t}=0$

for B-S equation $\frac{\partial V}{\partial t} + \frac{(\sigma S)^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$

$\Rightarrow \frac{(\sigma S)^2}{2}\frac{dV^2}{dS^2} + (r-\delta)S\frac{dV}{dS} - rV = 0$

which is a ~~homogeneous~~ ODE

~~solve characteristic equation~~ $\cancel{\frac{(\sigma S)^2}{2}x + (r-\delta)}$)

general solution: $V(S) = A_1 S^{\lambda_1} + A_2 S^{\lambda_2}$

$A_1, \lambda_1, A_2, \lambda_2$ are given based on boundary conditions

Q3

Monte-Carlo, from R studio:

```r
t <- 1;
S0 <- 100
K <- 100
r = 0.06
sigma = 0.15

dt <- 1/10000;
Nt <- t / dt;
Np <- 10000;
return <- matrix(rnorm(Nt*Np, mean = r * dt, sd = sigma * sqrt(dt)), nrow = Nt)


s <- matrix (0, Nt + 1, Np)

for (k in 1:Nt) {
  s[k+1, ] <- s[k,] + return[k,]
}

S <- exp(s[Nt+1,])* S0
V <- rep(0,Np)

for (i in 1:Np) {
  if (S[i] < K){
    V[i] = 0;
  }
  else {
    V[i] = S[i] - K;
  }
}

C <- exp(-r*t) * mean(V)
```

C = 9.8155

Black-Scholes, from Python codes:

```python
S = 100
sigma = 0.15
r = 0.06
K = 100
T = 1

x1 = (math.log(S/K) + T * r)/(sigma * math.sqrt(T)) + 0.5 * sigma * math.sqrt(T)
x2 = (math.log(S/K) + T * r)/(sigma * math.sqrt(T)) - 0.5 * sigma * math.sqrt(T)
delta = stat.norm.cdf(x1)
B = - K * math.exp(-r * T) * stat.norm.cdf(x2)

C = S * delta + B
```

C = 9.173453198408012