1 (a) $E(r_t) = E(\varepsilon_t + a\varepsilon_{t-1} + b) = b + E(\varepsilon_t) + aE(\varepsilon_{t-1}) = b$

(b) $\operatorname{var}(r_t) = E[(r_t - b)^2] = E[(\varepsilon_t + a\varepsilon_{t-1})^2]$

$= E[\varepsilon_t^2] + a^2 E[\varepsilon_{t-1}^2] = (a^2+1)\sigma^2$

$\sigma_{r_t} = \sigma\sqrt{a^2+1}$

(c) $\gamma_0 = \operatorname{cov}(r_t, r_t) = \operatorname{var}(r_t) = (a^2+1)\sigma^2$

$\gamma_1 = \operatorname{cov}(r_{t+1}, r_t) = E[(r_{t+1} - b)(r_t - b)]$

$= E[(\varepsilon_{t+1} + a\varepsilon_t)(\varepsilon_t + a\varepsilon_{t-1})]$

$= E(\varepsilon_{t+1}\varepsilon_t) + aE(\varepsilon_t^2) + a^2 E(\varepsilon_t\varepsilon_{t-1}) + aE(\varepsilon_{t+1}\varepsilon_{t-1})$

$= a\sigma^2$

for $k > 1$, $\gamma_k = \operatorname{cov}(r_{t+k}, r_t) = E[(r_{t+k} - b)(r_t - b)]$

$= E[(\varepsilon_{t+k} + a\varepsilon_{t+k-1})(\varepsilon_t + a\varepsilon_{t-1})]$

$= E(\varepsilon_{t+k}\varepsilon_t) + aE(\varepsilon_{t+k}\varepsilon_{t-1}) + aE(\varepsilon_{t+k-1}\varepsilon_t)$

$+ a^2 E[\varepsilon_{t+k-1}\varepsilon_{t-1}] = 0$

(d) $b > 0$ since on average (in expectation) stock price is going up over time (risk-free rate, inflation...)

$a = 0$: random walk model with drift b

$a \neq 0$: MA(1) model

2. (a) $X_t = \lambda Y_{t-1} + \sigma Z_{t-1} = \lambda(\lambda X_{t-2} + \sigma W_{t-2}) + \sigma Z_{t-1}$

$= \lambda^2 X_{t-2} + \lambda\sigma W_{t-2} + \sigma Z_{t-1}$

$E(X_t) = E(\lambda^2 X_{t-2} + \lambda\sigma W_{t-2} + \sigma Z_{t-1}) = \lambda^2 E(X_{t-2})$

given $E(X_t) = E(X_{t-2}) = \mu_X$ (stationary) $\Rightarrow$

$\mu_X = 0$ similarly, $\mu_Y = 0$

$\sigma_X^2 = E[X_t^2] = E[(\lambda X_{t-2} + \lambda\sigma W_{t-2} + \sigma Z_{t-1})^2] = \lambda^4 E[X_{t-2}^2] + \lambda^2\sigma^2 + \sigma^2$

$= \lambda^4 \sigma_X^2 + (\lambda^2+1)\sigma^2 \Rightarrow \sigma_X^2 = \frac{\sigma^2}{1-\lambda^2}$

similarly, $\sigma_Y^2 = \frac{\sigma^2}{1-\lambda^2}$

(b) $\mathrm{cov}(X_t, X_{t+1}) = E[X_t X_{t+1}] = E[X_t(\lambda Y_t + \sigma Z_t)] = \lambda E[X_t Y_t]$

$E(X_0 Y_0) = E(\sigma Z_0 \sigma W_t) = 0$; $E(X_1 Y_1) = E[(\lambda X_0 + \sigma Z_1)(\lambda Y_0 + \sigma W_1)]$

$= \lambda^2 E(X_0 Y_0) = 0 \ \cdots$

$\Rightarrow E(X_t Y_t) = 0 \Rightarrow \mathrm{cov}(X_t, X_{t+1}) = 0$ similarly, $\mathrm{cov}(Y_t, Y_{t+1}) = 0$

$\mathrm{cov}(X_t, X_{t+2}) = E[X_t(\lambda^2 X_{t-2} + \lambda\sigma W_{t-2}) + \sigma Z_{t-1}] = \lambda^2 E[X_t^2] = \frac{\lambda^2\sigma^2}{1-\lambda^2}$

similarly, $\mathrm{cov}(Y_t, Y_{t+2}) = \frac{\lambda^2\sigma^2}{1-\lambda^2}$

if $l \geq 3$, $\mathrm{cov}(X_t, X_{t+l}) = E[X_t(\lambda^2 X_{t+l-2} + \lambda\sigma W_{t+l-2} + \sigma Z_{t-2})]$

$= \lambda^2 E(X_t + X_{t+l-2})$

if $l = 2k$, $k \in \mathbb{N}$, $\mathrm{cov}(X_t, X_{t+l}) = \lambda^l \frac{\sigma^2}{1-\lambda^2}$

if $l = 2k+1$, $k \in \mathbb{N}$, $\mathrm{cov}(X_t, X_{t+l}) = \lambda^{l-1} E(X_t X_{t+1}) = 0$

similarly, if $l = 2k$, $k \in \mathbb{N}$, $\mathrm{cov}(Y_t, Y_{t+l}) = \lambda^l \frac{\sigma^2}{1-\lambda^2}$

$l = 2k+1$, $k \in \mathbb{N}$, $\mathrm{cov}(Y_t, Y_{t+l}) = 0$

2(c) $\operatorname{cov}(x_t, y_{t-l}) = E[x_t y_{t-l}] = E[(\lambda y_{t-1} + \sigma z_t) y_{t-l}]$

$= \lambda E(y_{t-1} y_{t-l})$

if $l = 2k$, $k \in \mathbb{N}$, $\operatorname{cov}(x_t, y_{t-l}) = \lambda \operatorname{cov}(y_{t-1}, y_{t-l}) = 0$

if $l = 2k+1$, $k \in \mathbb{N}$, $\operatorname{cov}(x_t, y_{t-l}) = \lambda \cdot \lambda^{l-1} \frac{\sigma^2}{1-\lambda^2} = \frac{\lambda^l \sigma^2}{1-\lambda^2}$

3(a) $R_t = \mu - \lambda(R_{t-1} - \mu) + \sigma z_t$

$E[R_t] = \mu - \lambda E[R_{t-1}] + \lambda\mu \Rightarrow (\lambda+1) E[R_t] = (\lambda+1)\mu$

since $\lambda + 1 \neq 0$, $E[R_t] = \mu$

$E[(R_t - \mu)^2] = \lambda^2 E[(R_{t-1} - \mu)^2] + \sigma^2 \Rightarrow \operatorname{var}(R_t) = \frac{\sigma^2}{1-\lambda^2}$, $0 < \lambda < 1$

$E[\pi_t] = E(-c R_{t-1} R_t) = -c E[R_{t-1} R_t]$

$= -c E[(-\lambda(R_{t-1} - \mu) + \sigma z_t + \mu) R_{t-1}]$

$= c\lambda E(R_{t-1}^2) - c\lambda\mu E[R_{t-1}] - c\sigma E[z_t R_{t-1}] - c\mu E[R_{t-1}]$

$= c\left(\frac{\lambda\sigma^2}{1-\lambda^2} - \mu^2\right) > 0$ to be positive

(b) $E[\pi_t] = E[-c R_{t-1} R_t] = -c E[R_{t-1} R_t]$

$= -c E[(\mu + \sigma z_{t-1} + \phi z_{t-2})(\mu + \sigma z_t + \phi z_{t-1})]$

$= -c[\mu^2 + \sigma\phi E[z_{t-1}^2]] = -c(\mu^2 + \sigma\phi)$

so if $\sigma, \phi > 0$, if we set $c < 0$ we can generate positive expected return without further requirement, which is more attractive

A better way could be: since $R_t = \mu + \sigma z_t + \phi z_{t-1}$,

$E_{t-1}(R_t) = E(\mu + \sigma z_t + \phi z_{t-1} \mid z_{t-1}, z_{t-2}, \dots)$

$= \mu + \phi z_{t-1} + \sigma E_{t-1}(z_t) = \mu + \phi z_{t-1}$

we can loose the requirement that $\mu, \phi > 0$

so essentially we can set $I_t = c(\mu + \phi z_t)$ after observing z_t, $E(\pi_t) = E(I_{t-1} R_t) = E[c(\mu + \phi z_{t-1})(\mu + \sigma z_t + \phi z_{t-1})] = c(\mu^2 + \phi^2)$