Class Notes Topic 5
(part 1)

Interest Rate Options

15.438 Fixed Income

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Topic 5 Part 1 Outline

- Introduction to interest rate dependent options
  - payoff diagrams
  - basic pricing using replicating trading strategy and no arbitrage
  - “risk-neutral” or arbitrage-free pricing
- Shortcomings of Black-Scholes-Merton for bond options
- Pricing bonds using binomial lattice models of short-term yields
  - Two simple binomial models
  - Estimating volatility and implementing it in binomial models
- Deriving long-term yields from models of short-term yields
  - Volatility and the shape of the yield curve
- Pricing options using binomial models
  - European bond options
  - American bond options
- Calibrating binomial models with market price data

Acquired Skills

- Know how to use binomial models of short rates to:
  - Derive term structure of interest rates and its time evolution
  - Price fixed income securities
  - Price simple and complex options on fixed income securities
- Know basics about how to calibrate binomial short rate models to match market data
- Understand
  - No-arbitrage logic as it applies to pricing bond options
  - How volatility affects shape of yield curve
  - Shortcomings of BSM for bond options

Recall the two basic types of options

- An **American option** can be exercised any time prior to expiration (maturity).

- A **European option** can only be exercised on the date it matures.

Common types options involving interest rates

- instruments with **embedded options**

  callable bonds
  puttable bonds
  prepayable mortgages

- caps, collars, and floors

- options on futures (e.g., Eurodollars and Treasury notes)

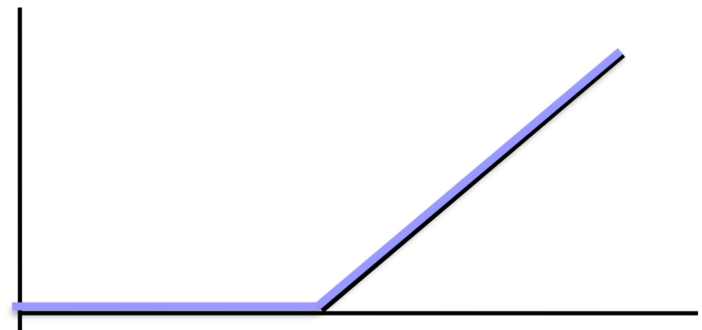
- swaptions

Payoff diagrams for bond options

A **Call Option** gives the buyer the **right to buy** a security at the specified strike price, in exchange for the premium paid up front.

(e.g., payoff diagram for a European call option on 20 year 10% bond with a strike price of 102 expiring in one year)

Option Value



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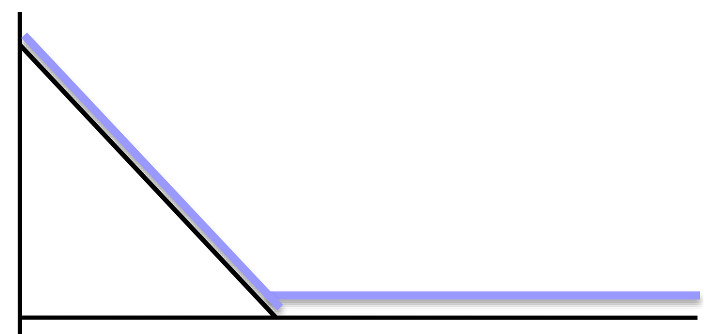
bond price

A **Put Option** gives the buyer the **right to sell** a security at the specified strike price, in exchange for the premium paid up front.

(e.g., payoff diagram for a European put option on 20 year 10% bond with a strike price of 102 expiring in one year)

Option Value



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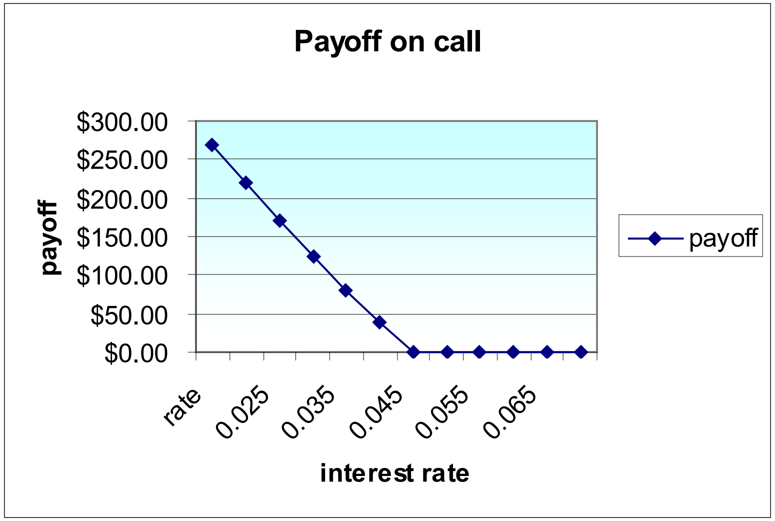
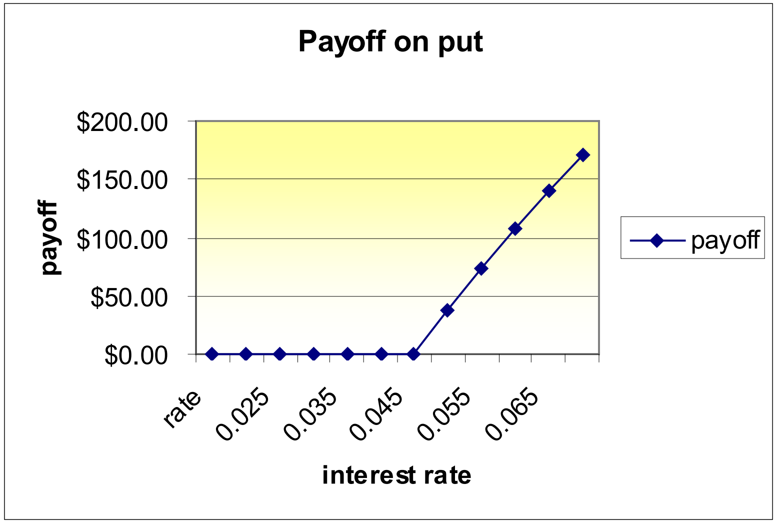
bond price

The seller is said to "write" the option.

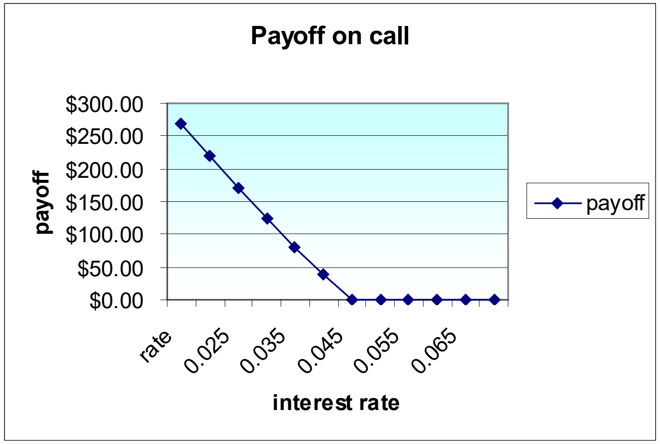
The payoff diagrams flip and become slightly curved when option value is plotted against interest rates

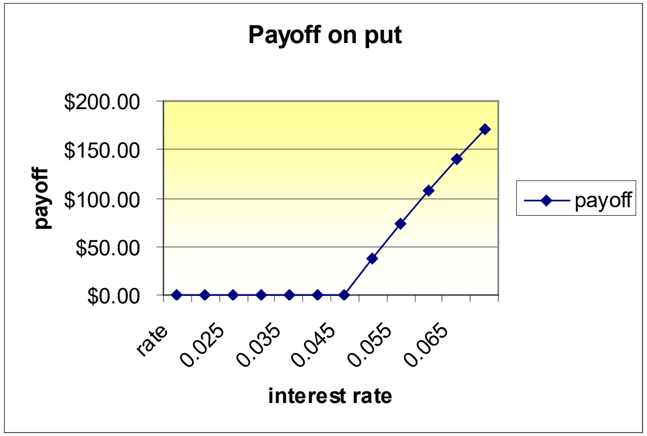


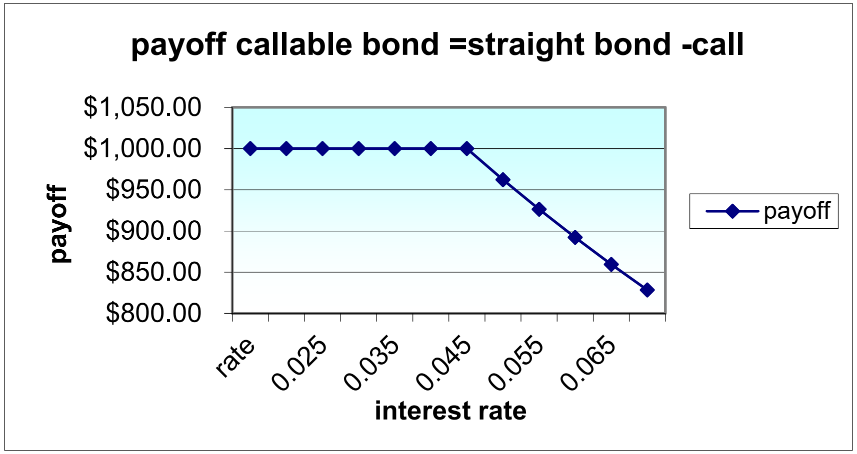


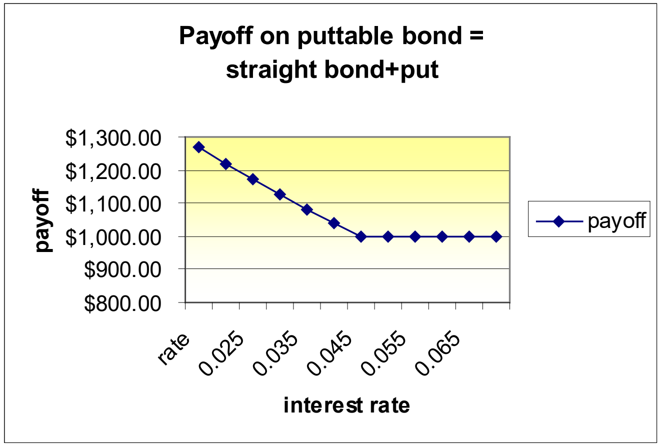
Link to spreadsheet

This explains the negative convexity of callable bonds, and more positive convexity of puttable bonds









Pricing options on bonds with no-arbitrage logic

- Recall that the price of stock options can be derived using a replicating portfolio of stocks and a money market fund, and no arbitrage conditions

- Bond options can be priced similarly, using a replicating portfolio of bonds and a money market fund

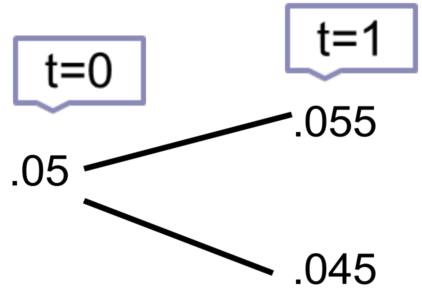
- The intuition is simplest to explain with a binomial tree

- It also illustrates why “risk neutral” pricing is possible

Example 5.1: You have a European call option on a two period pure discount bond that allows you to purchase it in one period for $95 per $100 face value.

Also available are a one period pure discount bond, and the underlying two period pure discount bond. The two-period bond is currently priced at $90.5304 per $100 face value.

The interest rate tree with the time path of 1-period rates relevant for pricing all of these securities is:



Consistent with this, the current price of the one-period bond is $100/1.05 = 95.238, and the two-period bond will be worth either $100/1.055 = $94.787 or $100/1.045 = $95.694 next year.

This implies that the value of the call is either $0 or $.694 at expiration.

To price the call, create a **"replicating portfolio"**

To do this, invest in "X" 1-year bonds, and "Y" 2-year bonds, to replicate the option payoff:

$$X(100) + \Delta(94.787) = 0$$
$$X(100) + \Delta(95.694) = .694$$

Two equations in two unknowns is solved for X and Δ;
$X = -.725272$, $\Delta = .76516$.

- The option price must be $X(95.238) + \Delta(90.5304) = .197$, since by buying Y of the 2-period bond and selling X of the 1-period bond, the option's payoff is replicated in every future state.

- If the option price were different, there would be an arbitrage opportunity by taking a long position in the underpriced securities and a short position in the overpriced securities.

Risk-neutral probabilities

Notice that no probabilities were associated with the above tree for the evolution of short rates. Hence we can use the trick of risk-neutral pricing to find the options price also.

<u>Example 5.1</u> (continued): We find the risk neutral probabilities based on the 2-period bond price today and next period, and the 1-period risk free rate:

$$\frac{q94.787+(1-q)95.694}{1.05}=90.5304$$

Solving gives q = .7024

The price of the option is:
[.7024(0) + (1-.7024)(.694)]/1.05 = .197

<u>Key Concept</u>: In most binomial (or "lattice") models used in fixed income to price options, the probabilities of up and down moves represent **risk-neutral probabilities,** selected to make the model match observed bond prices.

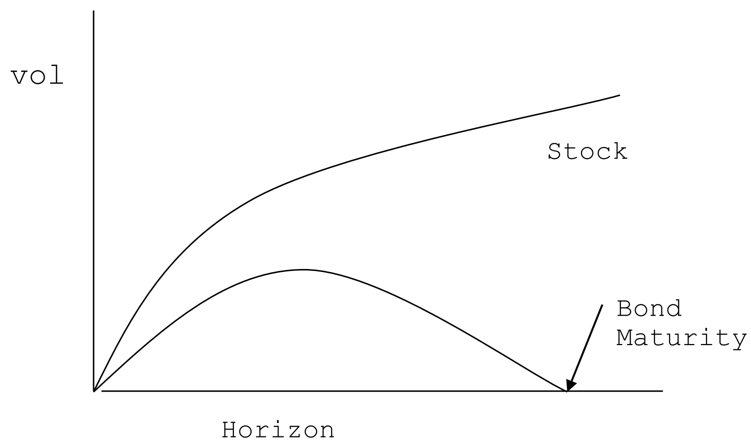
Example 5.2: **Shortcomings of Black-Scholes-Merton for Pricing Bond Options**

What is the price of a 3-year European call option on a 3-year zero coupon bond with exercise price $110 (per $100 face)?

The answer is obviously zero.

Under the assumption of r = 10% and 4% annual bond price volatility, the Black-Scholes formula gives a price of 7.78!

Figure 5.1: **Price Volatility vs. Horizon -- Stocks vs. Bonds**



Note: When stock prices are lognormal, stock price volatility at horizon t is $\sigma\sqrt{t}$, where σ is a constant

Problems with Black-Scholes-Merton for pricing bond options

The assumptions underlying the traditional Black-Scholes option pricing model do not hold for bond prices:

- constant return volatility parameter for the underlying security
- positive probability of any future price
- constant short-term rate assumed over life of option
- many bond options are European and early exercise can be optimal

The problems with binomial models in <u>bond prices</u> are similar

The fix: Price options on fixed income securities using stochastic models of the yield curve, not of bond prices

Modeling the Evolution of Interest Rates

- **Payoffs on fixed income derivatives depend on the term structure of interest rates in the future**
- Pricing derivatives requires a model of the stochastic evolution of the term structure that answers:
  - how are spot and forward rates expected to move over time?
  - what is the volatility of those movements?

- First we will work with with **simple binomial models,** to learn to price interest rate derivatives and understand how to calibrate the models to match market prices.

- Later we will study **Monte Carlo simulation techniques** for generating interest rate paths.
- We will also consider some of the mathematically more complex **continuous time models** of stochastic yield curves.

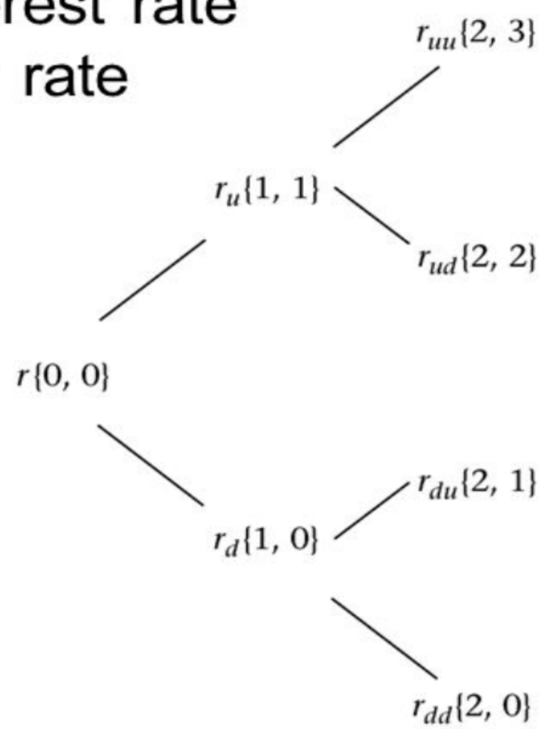
Simple Lattice Models

- A "lattice model" describes the evolution of the term structure over time when rates are stochastic
  - Usually **binomial**
    - sometimes trinomial
  - Usual starting point is evolution of a one-period rate (short rate).
  - Tree is usually restricted to be recombining
  - The entire yield curve and its evolution follows from the model of short rates and the imposition of no-arbitrage conditions

- Calibration
  - **Nodes, step size and probabilities are chosen to match**:
    - **Current bond prices** (hence implied rate trends)
    - **Volatility of bond prices** (can reflect a term structure of vol)
    - Models may also be calibrated with options data
  - Probabilities are typically risk-neutral, not physical

A generic binomial tree in short rates

Three-period interest rate tree of the 1-year rate



- Probabilities of up or down can be physical or risk neutral
- Probabilities at each node can be constant or can vary
- Infinite choices for how to calibrate; danger of over-fitting
- Different dynamic models restrict the choices in different ways

The choice variables are: (1) the short rate at each node, and (2) the probability that rates go up or down at each node.

For tractability, the pattern of rate changes and probabilities is usually restricted to a simple form.

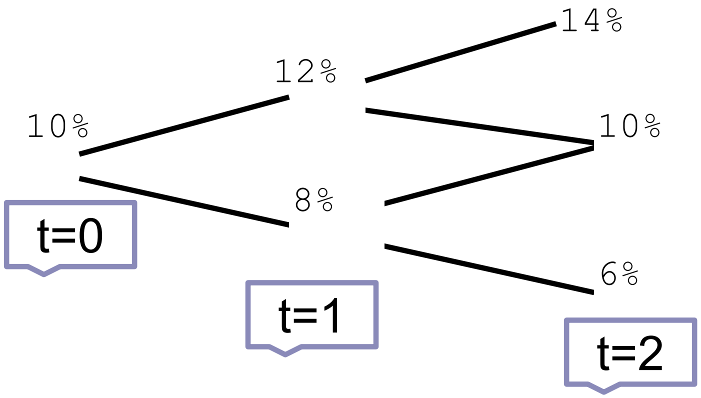
The following two examples illustrate the simplest possible assumptions. They are no longer used in practice.

I. Simple Additive Binomial Model:

$r_{t+1,H} = r_t + d$ and $r_{t+1,L} = r_t - d$; each with equal probability

e.g., $d = .02$; $r_0 = 10\%$, $pr_{up} = pr_{down} = .5$

r_t = one period rate starting at time t

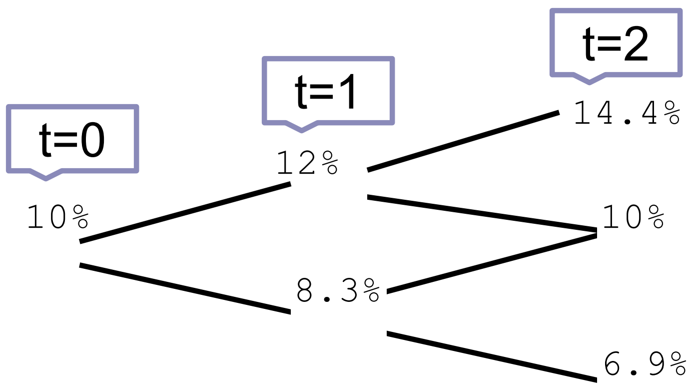


- Shortcoming is that very negative rates have a positive probability

II. Simple Multiplicative Binomial Model:

$r_{t+1,H} = r_t(1 + d)$ and $r_{t+1,L} = r_t/(1 + d)$ with equal probability

e.g., $d=.2$; $r_0 = 10\%$; $pr_{up} = pr_{down} = .5$



•Shortcoming is that short rates on average always increase

Volatility

- Key input for all stochastic interest rate models.
- Ties down vertical distance between nodes in a tree (physical and risk neutral)
  - E.g., determines “d” in the above binomial examples.

Estimating Volatility from Historical Data

Procedure:

a) collect recent sample of yields (e.g., daily data on 1-year rates)

b) calculate sample std. dev. of the yields
  - here we calculate proportional changes
  - but in some models risk is measured in levels

c) annualize by multiplying by $\sqrt{365}$ (or $\sqrt{250}$)
  - or convert to volatility over t-day period by multiplying by $\sqrt{t}$
  - note: this assumes rate changes are uncorrelated over time

$$\text{variance} = \sum_{s=1}^{N} \frac{(X_s - E[X])^2}{N-1}$$

X_s = percentage yield change from previous day
N = number of observations
$E(X)$ = average percentage yield change

Example 5.3: Estimating Short Yield Volatility

Date	Observed Yield	Proportional Change
7/16	0.0388	
7/17	0.039	0.00515
7/20	0.0391	0.00256
7/21	0.0393	0.00512
7/22	0.039	-0.00763
7/23	0.0383	-0.01795
7/24	0.0385	0.00522
7/27	0.0385	0.00000
7/28	0.0381	-0.01039
7/29	0.0383	0.00525
7/30	0.0386	0.00783
	Mean	**-0.00048**
	std dev (daily)	**0.00857**
	std dev (annual)	**0.16379**

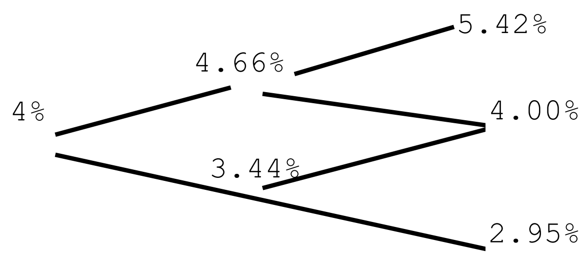
e.g. [(.039-.0388)/.0388] =.00515

(here the annual standard deviation was found by multiplying by $\sqrt{365}$ *.)*

Implementing Volatility in Multiplicative Model

Say r_0 = .04 = current 1-year rate
Estimate of **σ =** .164 = annual volatility
Assume probability of rates up or down = .5

Then in a tree with each step representing one year, setting d = **σ** makes the model volatility match observed volatility.



<u>Proof that d = σ:</u>

σ = std dev of $\dfrac{r_{t+1} - r_t}{r_t}$ ≅ std dev of [ln(r_{t+1}) – ln(r_t)].

Since r_{t+1} will equal (1+d)r_t or r_t/(1+d), then ln(r_{t+1}) – ln(r_t) = ln(1+d) or –ln(1+d), which is approximately equal to d or –d, for small d.

Then E(ln(r_{t+1}) – ln(r_t)) = .5(d) + .5(-d) = 0,

and std. dev. of [ln(r_{t+1}) – ln(r_t)]= $[.5(d)^2 + .5(-d)^2]^{1/2}$ = d.

Note: Some models take the vertical distance between two nodes at a point in time to be two standard deviations in levels

The Multiplicative Model on a Continuous Basis

Recall that we wrote the simple multiplicative model as:

$r_{t+1} = r_t(1 + \sigma_m)$ or $r_t/(1 + \sigma_m)$,
each with equal probability

Alternatively, if volatility is given on a continuous basis:

$r_{t+1} = r_t e^{\sigma}$ or $r_t e^{-\sigma}$
each with equal probability

(Recall that when σ is small, $(1 + \sigma) \cong e^{\sigma}$.)

- The simple multiplicative model states volatility σ_m on a “simple per period basis”
- The exponential representation states volatility σ on a “continuous basis”

More sophisticated models incorporate time-varying volatility.

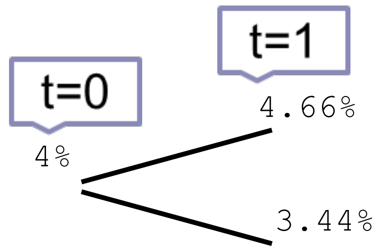
Finding the implied yield curve from a binomial model in 1-period yields

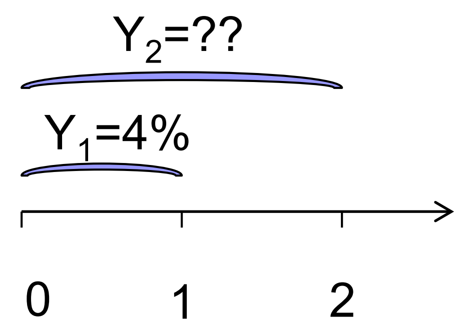
- Any bond can be priced working backwards on a risk-neutral tree
- The plan:
  - The rate at the t=0 node is the 1-period spot yield, $\mathbf{Y_1}$.
  - To find $\mathbf{Y_2}$, the 2-period spot yield, we find the **yield on a 2-period zero coupon bond** implied by the tree, and so forth.
  - We do this working backwards along the tree to price a two-period zero coupon bond with F=100.
  - Given the price of the two-period zero coupon bond, we can find the 2-period spot yield using the usual formula:

$$P_0 = \frac{100}{(1+y_2)^2}$$

Going From Short Rates to Long Rates

Example 5.4: Derive the 2 period yield curve based on the following short rates (assume risk-neutral $pr_{up} = pr_{down} = .5$)





Consider a 2 period, risk-free zero coupon bond that pays $100 in two periods.

$P_1(4.66\%) = 100/1.0466 = 95.547$
$P_1(3.44\%) = 100/1.0344 = 96.674$

$$P_0(4.00\%) = \frac{.5(95.547)+.5(96.674)}{1.04} = 92.414$$

$92.414 = 100/(1+Y_2)^2 \Rightarrow Y_2 = 4.02\%$

The effect of volatility on the shape of the yield curve

- Even when future short rates are expected to equal the current short rate, if yields are uncertain then the yield curve will be slightly downward sloping.

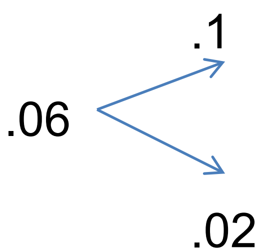
- This helps explain the empirical observation that the spot yield curve almost always curves down at very long maturities.

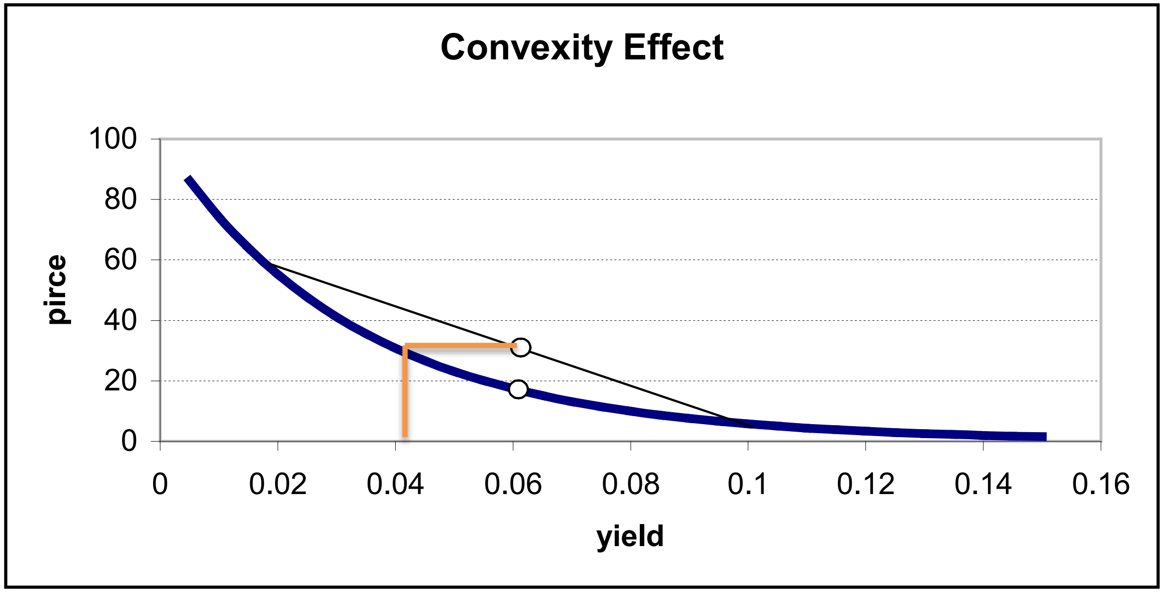
- This is effect can be illustrated using either the additive or multiplicative model.

Implied multiplicative yield curve

Link to implied linear yield curve

The fact that the curvature of the yield curve varies with rate volatility is another convexity effect…





Note: This example is based on a 30-yr zero coupon bond, F=100

prob	yield	price(yield)	
0.5	0.02	55.20709	$=100/(1.02)^{30}$
0.5	0.1	5.730855	

Expected yield = .5(.1 + .02) = .06;

avg price 30.46897 yld at avg price
0.040411 $=(100/30.46897)^{(1/30)}$

E(P(Y)) > P(E(Y))

The yield implied by the expected price will be lower than expected yield.

Using binomial trees for bond pricing, the spot yield curve is based on expected bond prices.

Valuing a callable or puttable bond on a binomial tree

Value Callable Bond = Value Non-Callable Bond - Value Call Option

Value Puttable Bond = Value Non-Puttable Bond + Value Put Option

Strategy for pricing callable bond:

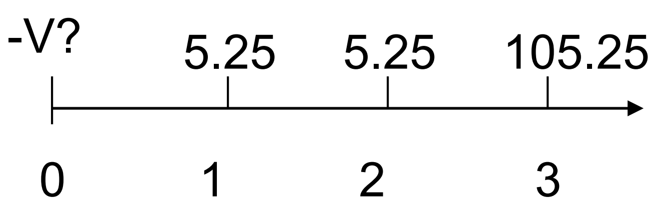
- Use interest rate model to price non-callable bond.
- Use same interest rate model to price embedded call option.

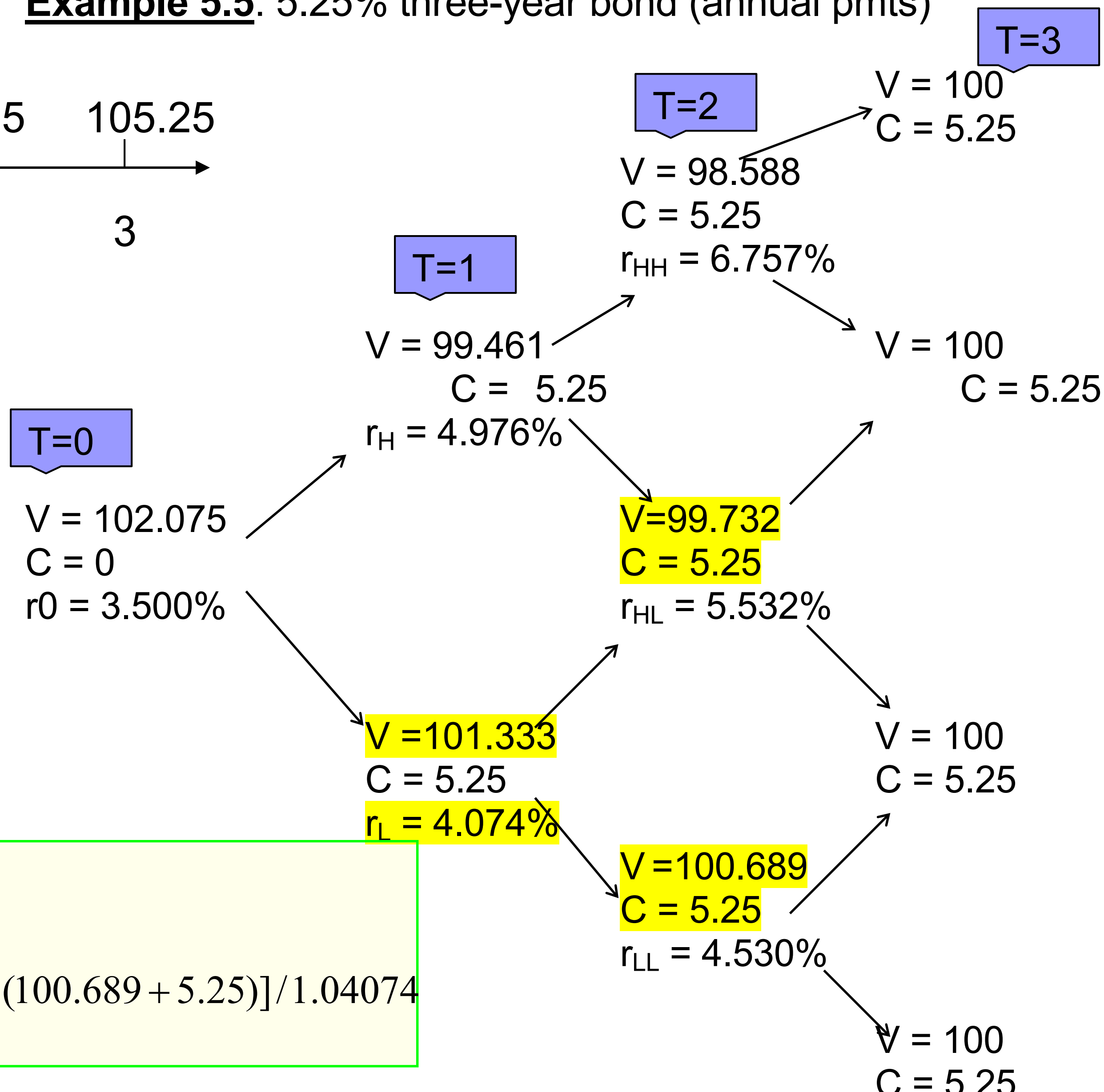
Assume the following annual binomial tree is correct for risk-neutral pricing of bonds and bond options: **(pr(up) = pr(down) = .5)**

$$
\begin{tikzcd}
 & & r_{HH} = 6.757\% \\
 & r_H = 4.976\% \arrow[ur] \arrow[dr] & \\
r_0 = 3.500\% \arrow[ur] \arrow[dr] & & r_{HL} = 5.532\% \\
 & r_L = 4.074\% \arrow[ur] \arrow[dr] & \\
 & & r_{LL} = 4.530\%
\end{tikzcd}
$$

The tree can be used to price option-free bonds:

Example 5.5: 5.25% three-year bond (annual pmts)





e.g.,

$$101.333 = .5[(99.732 + 5.25) + (100.689 + 5.25)]/1.04074$$

Pricing a European Call Option

Example 5.6: Assume that the 5.25% bond is callable at the end of two years for $99.50.

What is the value of the call option? What is the value of the callable bond?

e.g., .110=.5(.232 + 0)/1.04976

T=0

V_{call}= 0.383

r0 = 3.500%

T=1

V_{call} = 0.110

r_H = 4.976%

V_{call} =0.683

r_L = 4.074%

V_{call} = 0

r_{HH} = 6.757%

V_{call}=99.732-99.5=.232

r_{HL} = 5.532%

V_{call}=100.689-99.5=1.19

r_{LL} = 4.530%

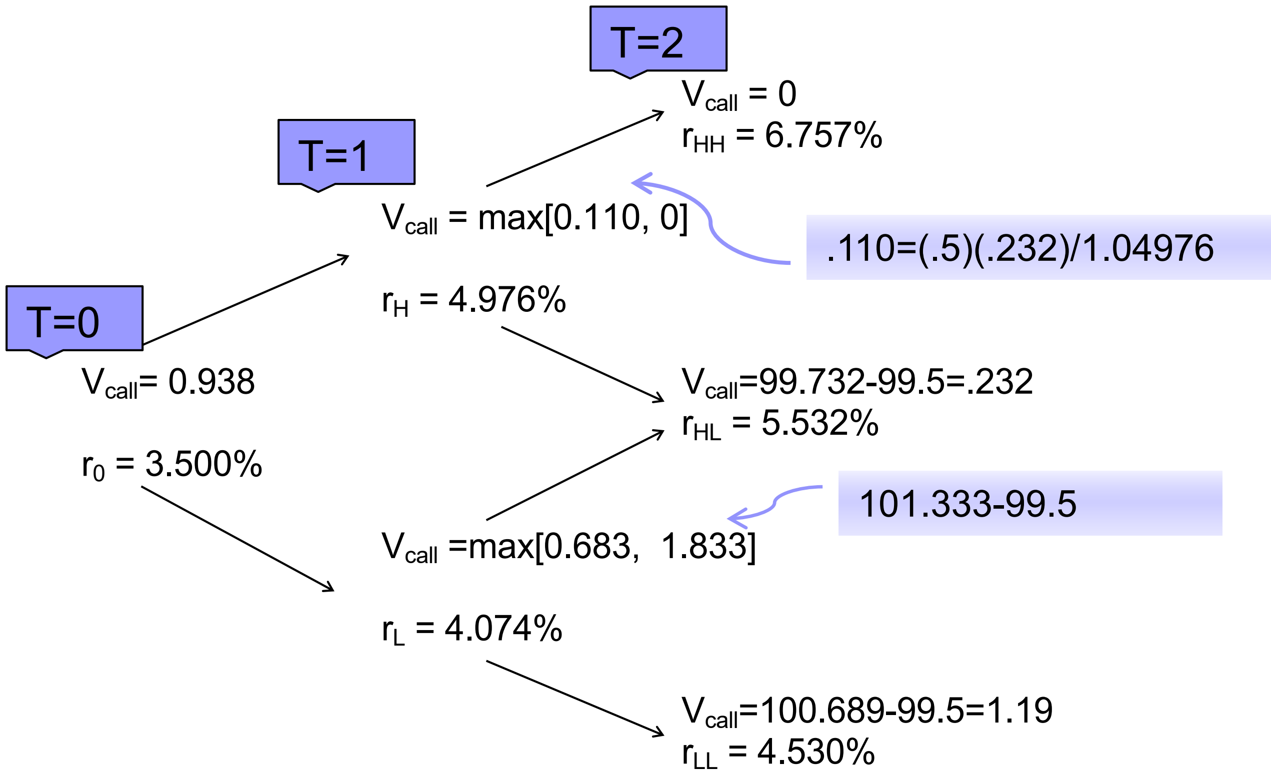
The call is worth $0.383.

The callable bond is worth $102.075 - $.383 = $101.692.

Pricing an American Call Option

<u>Example 5.7</u>: Assume that the 5.25% bond is callable in years 1 and 2 at $99.50?

What is the value of the call option? What is the value of the callable bond?



The call is worth $0.938.
The callable bond is worth $102.075 - $.938 = $101.137.

Option Value in Terms of the Spread

The cost of the option can be represented in terms of the change in the quoted yield.

In the last examples, the yield to maturity of the option-free bond solves:

$$102.075 = \frac{5.25}{(1+y)} + \frac{5.25}{(1+y)^2} + \frac{105.25}{(1+y)^3}; \quad y = 4.495\%$$

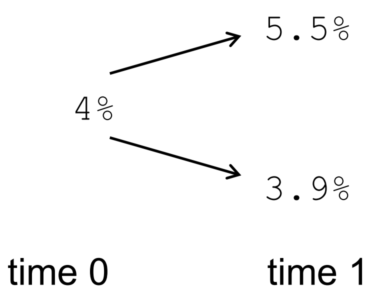
The yield to maturity of the bond with an American call option solves:

$$101.137 = \frac{5.25}{(1+y^*)} + \frac{5.25}{(1+y^*)^2} + \frac{105.25}{(1+y^*)^3}; \quad y^* = 4.834\%$$

The borrower pays about 34 bps each year for the option.

- The **OAS (options adjusted spread)** is different! *It is defined as the difference between the yield on the bond and the risk-free rate, after accounting for embedded options.*

<u>Practice Problem 5.1</u>: Suppose that you have estimated the following binomial model for one-year interest rates, where the probability of rates rising or falling equals .5:

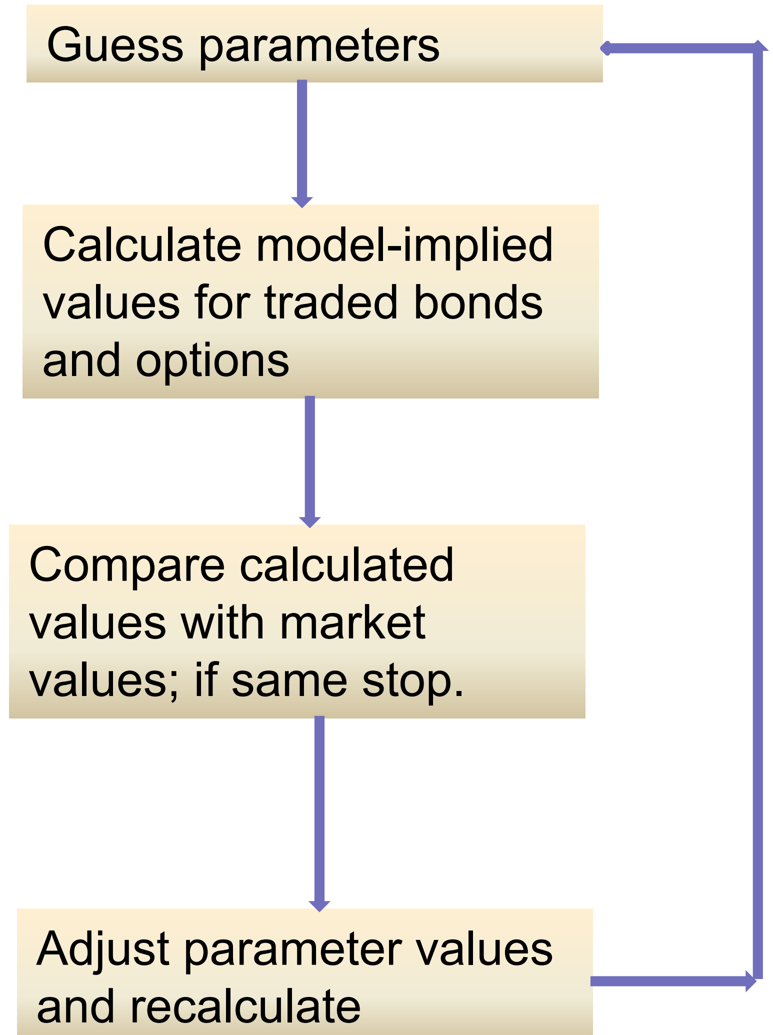


(a) Using this model, what is the theoretical price of a two year 6% coupon bond with no options attached? Assume the coupon payments are paid annually and the face value is $100.

(b) Now consider the same bond as in part (a), but with a call option that allows the issuer to call the bond at the end of the first year for $101. What is the value of the call option? What is the theoretical value of the callable bond?

Calibrating Lattice Models

General iterative procedure:



Say you observe the current term structure (annual rates):

Year	Spot Rate	Implied One-Year Forward (t-1)
1	3.500%	3.500%
2	4.010%	4.523%
3	4.531%	5.580%

Price of a two year, 4% coupon bond (annual payments)

$$\frac{4}{1.035} + \frac{104}{(1.0401)^2} = 100$$

Price of a three year, 4.5% coupon bond (annual payments)

$$\frac{4.5}{1.035} + \frac{4.5}{(1.0401)^2} + \frac{104.5}{(1.04531)^3} = 100$$

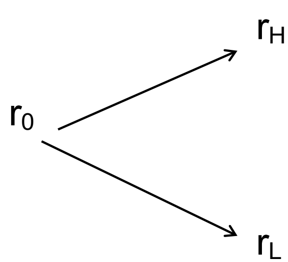
Goal: <u>Construct binomial lattice model for evolution of one year rate that correctly prices these bonds</u>.

Some parameters are fixed by assumption (which ones depend on the model chosen):

$p = 1/2$ (equal probability up or down move)

$\sigma = .1$ (volatility of one-year rate estimated from data)

A standard implementation of volatility is:



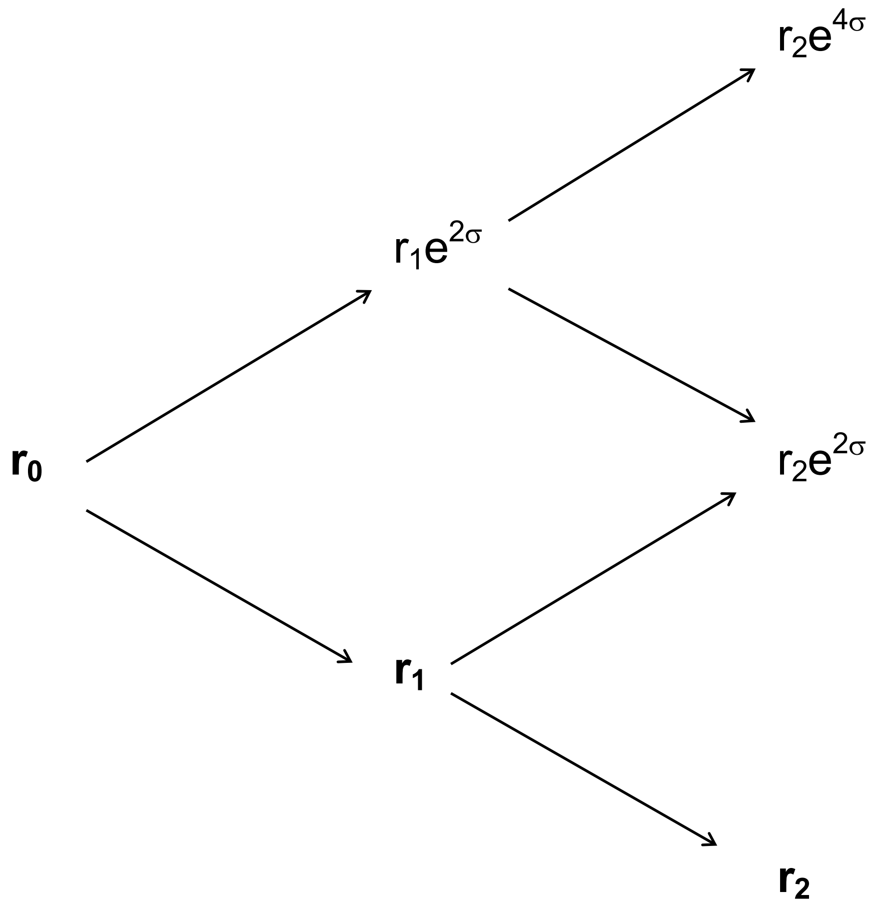
$r_H = r_L\, e^{2\sigma}$ where $e = 2.71828...$

For instance,

$r_L = 4.074\%$ implies $r_H = 4.074\% \times e^{2\times.1} = 4.976\%$.

Notice that $e^{2\sigma} \cong 1 + 2\sigma$ for 2σ small. Then with an equal probability of an up or down, the variance is $.5(r(1+ 2\sigma) - r(1+\sigma))^2 +.5(r - r(1+\sigma))^2 = (r\sigma)^2$.

Binomial Interest Rate Tree



Each period there is one free parameter: the one year rate along the lowest path, r_t.

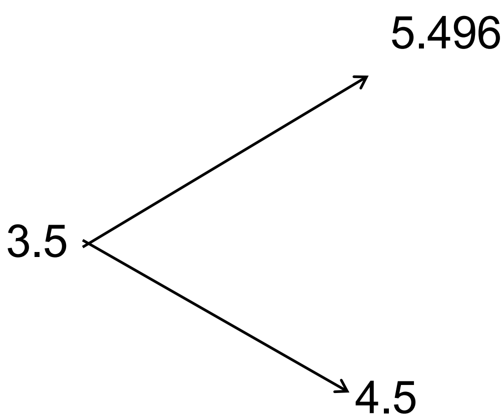
Lets take two period case.

r_0 = 3.5% (from current term structure)

For now assume r_1 = 4.5% (= r_L).

Then $r_H = 4.5\% \times e^{2\times .1} = 5.496\%$

Then lattice for evolution of one-year rates is:



Pricing Two Period Coupon Bond on Lattice

Method:

Find bond value at each node, working backwards from final period.

Discount expected one-step-ahead payoff at one-year rate at that node.

V = 100
C = 4.00

V = 98.582
C = 4.00
r_H = 5.496%

V = 99.567
C = 0.00
r_0 = 3.500%

V = 100
C = 4.00

V = 99.522
C = 4.00
r_L = 4.500%

V = 100
C = 4.00

Notes:
98.582 = 104/1.05496

$$99.567 = \frac{1}{2}\left[\frac{98.582+4}{1.035} + \frac{99.522+4}{1.035}\right]$$

What’s wrong with this model?

It misprices the two-year bond!

How can we fix It?

The minimum one period rate starting in one year, r_1, was chosen arbitrarily.

Pick a new rate and repeat process.

Should the new rate be higher or lower?

Repeat process of picking rate, filling out lattice, and pricing bond.

Iterate until bond price is correct.

Practice Problem 5.2: Verify that the model prices the bond correctly at r_L=4.074

To extend the tree out to three years, price the three year bond, adjusting the guess of the short rate at the bottom of the tree…

Also see the spreadsheet **"tree-fitter"** for a more general implementation of this model that you can experiment with:

- It allows the volatility to vary in future periods.
- It also illustrates how a model can be calibrated using implied volatilities.

<u>Practice Problem 5.3</u>: on Tree_Fitter.xls:

Set the model's input parameters to correspond to the example that we have just been working on, and verify that the resulting interest rate tree is the same.

Typical bond option pricing model

Inputs:

1. Current bond price
2. Strike price of option
3. Time to expiration
4. Coupon rate
5. Expected interest rate volatility over life of option

Option Pricing Model

Output:
Theoretical option price

Implied Volatility

Idea is to infer the market's estimate of volatility from options prices and an options pricing model. To obtain implied volatility:

Inputs:

1. Current bond price
2. Strike price of option
3. Time to expiration
4. Coupon rate
5. Observed option price

Option Pricing Model

Output:
Implied interest rate volatility