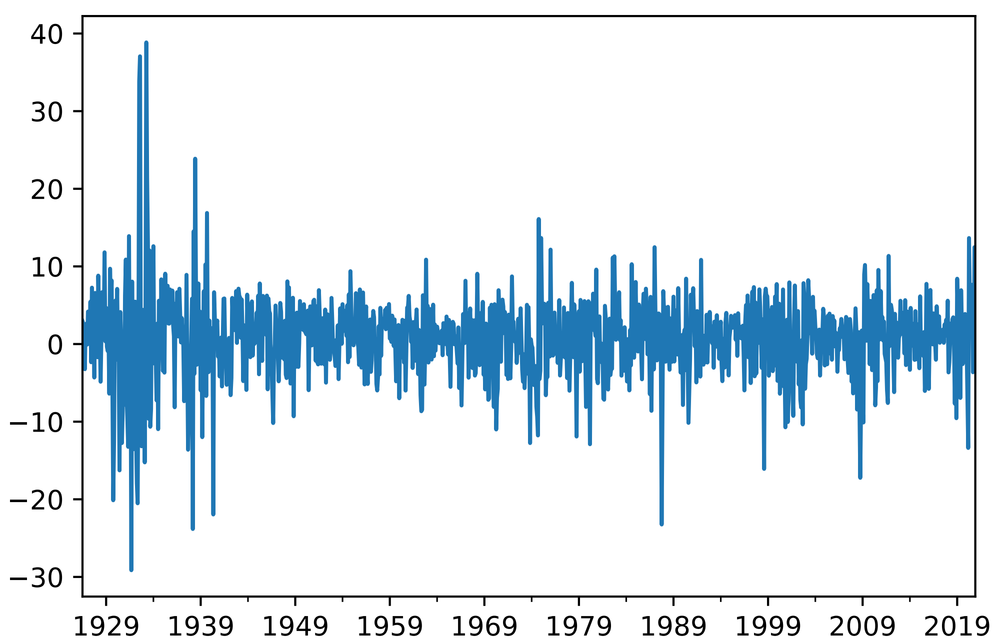
A

In [49]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
data = pd.read_csv('F-F_Research_Data_Factors.csv', skiprows = 2,index_col=0)
[:1134]
data.index = pd.to_datetime(data.index,format = '%Y%m')

for column in data.columns:
    data[column] = pd.to_numeric(data[column])
```

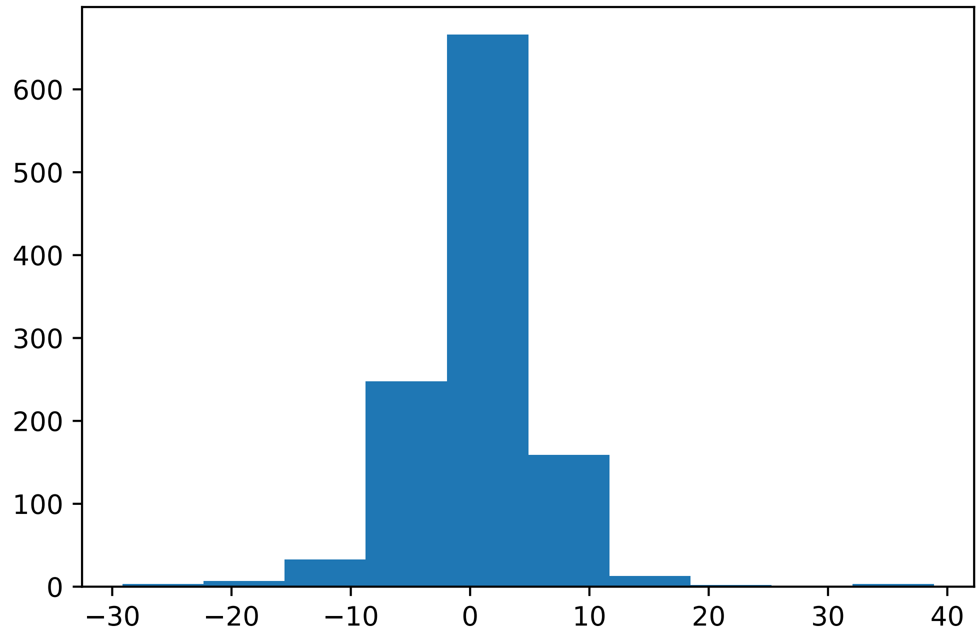
In [23]:
```
data['Mkt-RF'].plot()
```

Out[23]: <matplotlib.axes._subplots.AxesSubplot at 0x2266b0514c0>



In [30]:
```
plt.hist(data['Mkt-RF'])
```

Out[30]:
```
(array([  3.,   7.,  33., 248., 666., 159.,  13.,   2.,   0.,   3.]),
 array([-29.13 , -22.332, -15.534,  -8.736,  -1.938,   4.86 ,  11.658,
         18.456,  25.254,  32.052,  38.85 ]),
 <a list of 10 Patch objects>)
```



B

The hyperparameters that I choose are mean return and standard deviation from sample during 07/1926 to 12/2013

In [229]:
```
σ = 5.4
m0 = data['Mkt-RF'][data.index < pd.to_datetime(201401,format = '%Y%m')].mean
()
v0 = data['Mkt-RF'][data.index < pd.to_datetime(201401,format = '%Y%m')].std()

r = data['Mkt-RF'][data.index == pd.to_datetime(201312,format = '%Y%m')]
m1 = (m0*(σ**2) + r*(v0**2))/(σ**2+v0**2)
v1 = np.sqrt((v0*σ)**2/(σ**2 + v0**2))

print('m0 = %.4f' %m0)
print('v0 = %.4f' %v0)

print('one-month ahead posterior mu is %.4f' %m1)
```

```
m0 = 0.6506
v0 = 5.4290
one-month ahead posterior mu is 1.7361
```

C

prior: $\mu \sim N(m_0, v_0)$, $p(\mu) = \frac{1}{\sqrt{2\pi v_0^2}} exp(-\frac{(\mu-m_0)^2}{2v_0^2})$

likelihood: $p(r|\mu) = p(r_1|\mu)p(r_2|\mu)\ldots p(r_T|\mu) = (\frac{1}{\sqrt{2\pi v_0^2}})^T exp(-\sum_{t=1}^{T}\frac{(r_t-\mu)^2}{2\sigma^2})$

posterior $p(\mu|r) \propto p(\mu)p(r|\mu) \propto exp(-\frac{(\mu-m_0)^2}{2v_0^2} - \sum_{t=1}^{T}\frac{(r_t-\mu)^2}{2\sigma^2}) \propto exp(-\frac{(\mu-m_T)^2}{2v_T^2})$

that is to say, $p(\mu|r) \sim N(m_T, v_T)$

where $m_T = \frac{m_0\sigma^2+\bar{r}Tv_0^2}{\sigma^2+Tv_0^2}$, $v_T = \frac{v_0^2\sigma^2}{\sigma^2+Tv_0^2}$

$p(r_{t+1}|r) = \int_\mu p(r_{t+1}|\mu)p(\mu|r)d\mu = \int_\mu \frac{1}{2\pi\sigma v_t}\exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right)\exp\left(-\frac{(\mu-m_t)^2}{2v_t^2}\right)\mathrm{d}\mu \propto= \exp$
$\left(-\frac{r^2-2m_t r}{2(v_t^2+\sigma^2)}\right)$

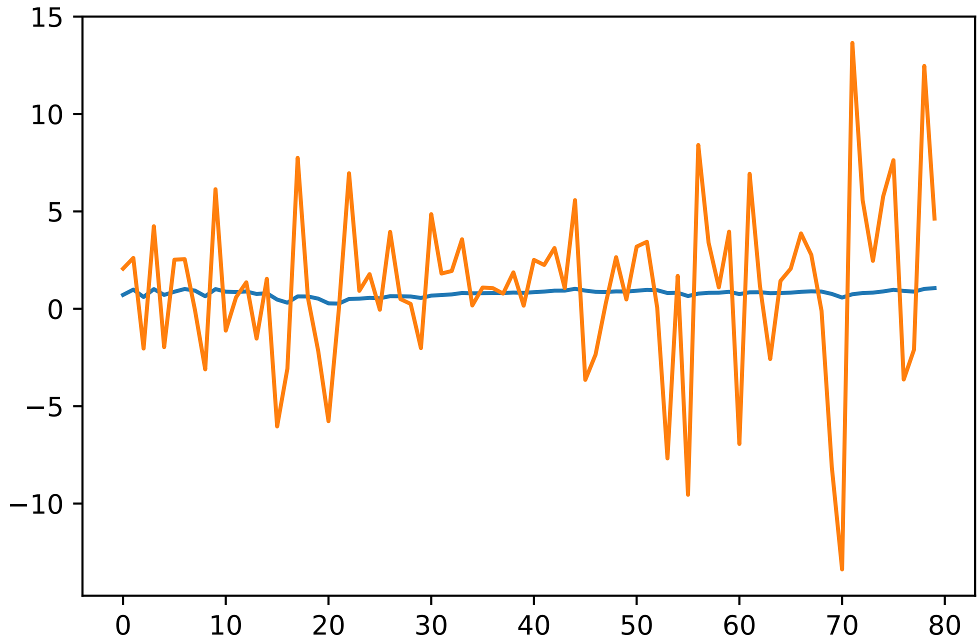
Thus, $r_{t+1}|r_t \sim N\left(\bar{\mu}_{r,t}, \bar{\sigma}_{r.t}^2\right)$, where $\bar{\mu}_{r,t} = m_T, \bar{\sigma}_{r.t}^2 = \sigma^2 + v_t^2$

D

In [234]:
```
ε = []
m = [m0]
v = [v0]
r = []
T = 0

for month in data.index[data.index >= pd.to_datetime(201401,format = '%Y%m')]:
    T += 1
    r.append(data['Mkt-RF'][month])
    ε.append(m[-1] - r[-1])
    r_bar = np.mean(r)

    m.append((m0*(σ**2) + r_bar*T*(v0**2))/(σ**2+T*(v0**2)) )
    v.append( v0*σ / (σ**2 + T*(v0**2))**0.5)
```

In [235]:
```
plt.plot(m[-80:])
plt.plot(r[-80:])
```

Out[235]:
```
[<matplotlib.lines.Line2D at 0x22672b4e3a0>]
```



In [282]:
```
MSE = np.square(ε).mean()
print("Out-of-sample MSE = %.3f" %MSE)
```

```
Out-of-sample MSE = 19.133
```

E

In [284]:
```
R = data['Mkt-RF'][data.index < pd.to_datetime(201401,format = '%Y%m')][:-1]
R_next = data['Mkt-RF'][data.index < pd.to_datetime(201401,format = '%Y%m')][1
:]
```

In [285]:
```
import statsmodels.api as sm
import numpy as np

x = sm.add_constant(R.values)
y = R_next.values

model = sm.OLS(y,x)

res = model.fit()
print(res.summary())
```

```
                            OLS Regression Results
==============================================================================
=
Dep. Variable:                      y   R-squared:                       0.01
3
Model:                            OLS   Adj. R-squared:                  0.01
2
Method:                 Least Squares   F-statistic:                     13.6
8
Date:                Tue, 02 Mar 2021   Prob (F-statistic):           0.00022
8
Time:                        21:52:15   Log-Likelihood:               -3256.
2
No. Observations:                1049   AIC:                            651
6.
Df Residuals:                    1047   BIC:                            652
6.
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
=
                 coef    std err          t      P>|t|      [0.025      0.97
5]
------------------------------------------------------------------------------
-
const          0.5747      0.168      3.424      0.001       0.245       0.90
4
x1             0.1136      0.031      3.699      0.000       0.053       0.17
4
==============================================================================
=
Omnibus:                      164.022   Durbin-Watson:                   1.99
4
Prob(Omnibus):                  0.000   Jarque-Bera (JB):             2150.22
2
Skew:                           0.209   Prob(JB):                         0.0
0
Kurtosis:                      10.001   Cond. No.                         5.5
1
==============================================================================
=

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correc
tly specified.
```

In [288]:

```
print("a0 = %.3f, standard error for a0: %.3f" %(res.params[0],res.bse[0]))
print("a1 = %.3f,standard error for a1: %.3f" %(res.params[1],res.bse[1]))
```

```
a0 = 0.575, standard error for a0: 0.168
a1 = 0.114,standard error for a1: 0.031
```

F

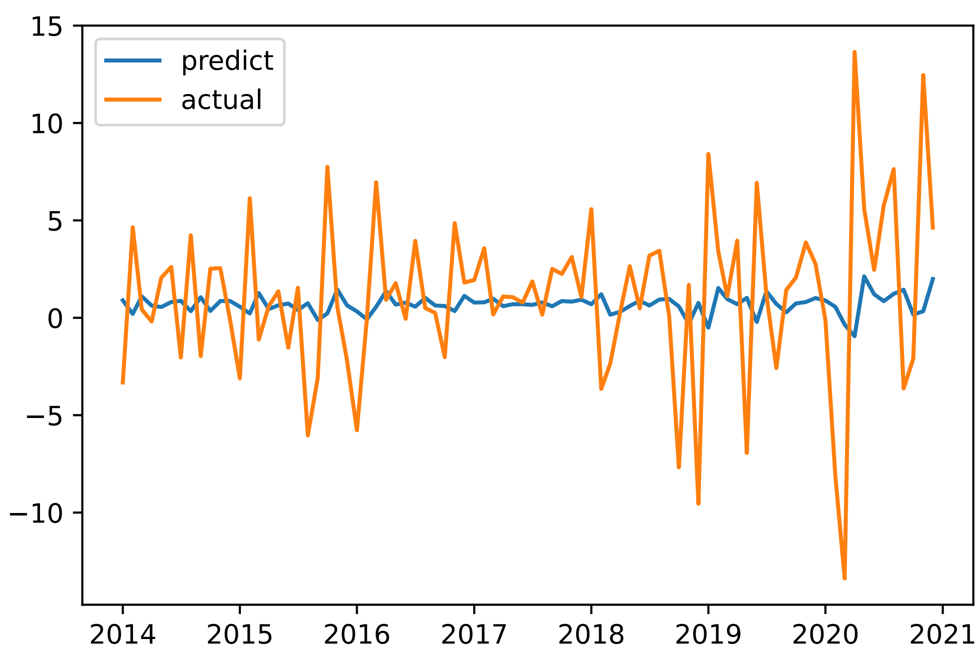
since $r^e_{t+1} = a_0 + a_1 r^e_t + \epsilon_{t+1}$;

a_0 and a_1 are known at the start of t+1, and $\epsilon_{t+1} \sim N(0, \sigma_e^2)$

So $r^e_{t+1} \sim N(a_0 + a_1 r^e_t, \sigma_e^2)$

In [292]:
```
a0 = res.params[0]
a1 = res.params[1]
σe = ((res.resid)**2).mean()**0.5

ε_F = []
r_F = [data['Mkt-RF'][data.index == pd.to_datetime(201312,format = '%Y%m')]]
r_predict=[]
for month in data.index[data.index >= pd.to_datetime(201401,format = '%Y%m')]:
    r_predict.append(a0 + a1*r_F[-1])
    r_realized = data['Mkt-RF'][month]
    ε_F.append(r_realized - r_predict[-1])
    r_F.append(r_realized)

MSE_F = np.square(ε_F).mean()
print("Out-of-sample MSE = %.3f" %MSE_F)
print("Bayesian Model is better")

plt.plot(data.index[data.index >= pd.to_datetime(201401,format = '%Y%m')],r_pr
edict, label = 'predict')
plt.plot(data.index[data.index >= pd.to_datetime(201401,format = '%Y%m')],r_F[
1:], label = 'actual')
plt.legend();plt.show()
```

```
Out-of-sample MSE = 19.254
Bayesian Model is better
```



G

$R_p = \omega\,(1 + r_{t+1}) + (1 - \omega)(1 + r_{f,t})$

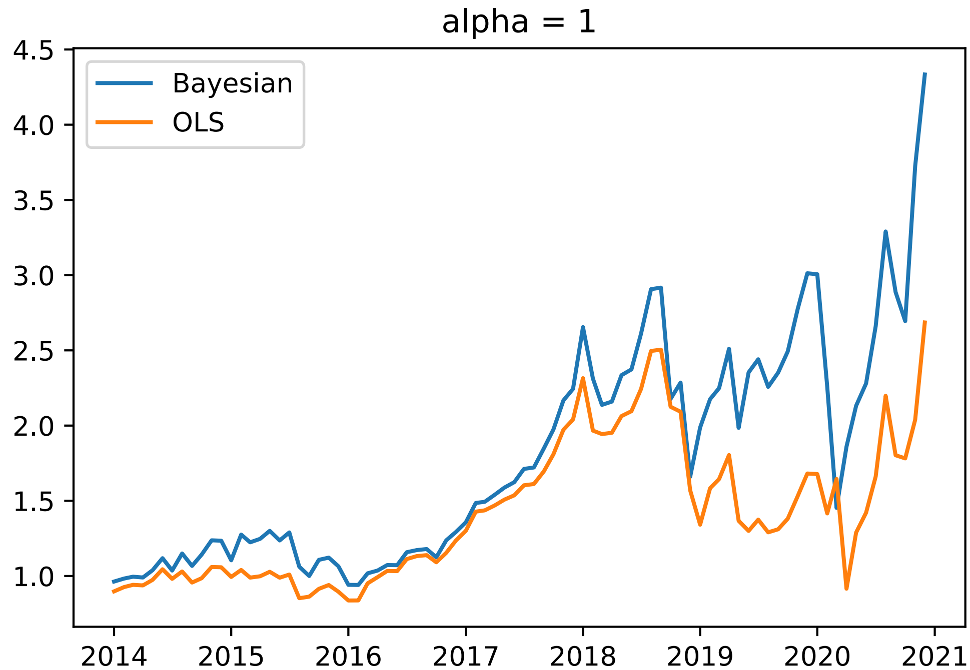
$u = \mathrm{E}_t\,[\omega\,(1 + r_{t+1}) + (1 - \omega)\,(1 + r_{f,t})] - \frac{\alpha}{2}\mathrm{Var}_t[\omega\,(1 + r_{t+1}) + (1 - \omega)\,(1 + r_{f,t})]$

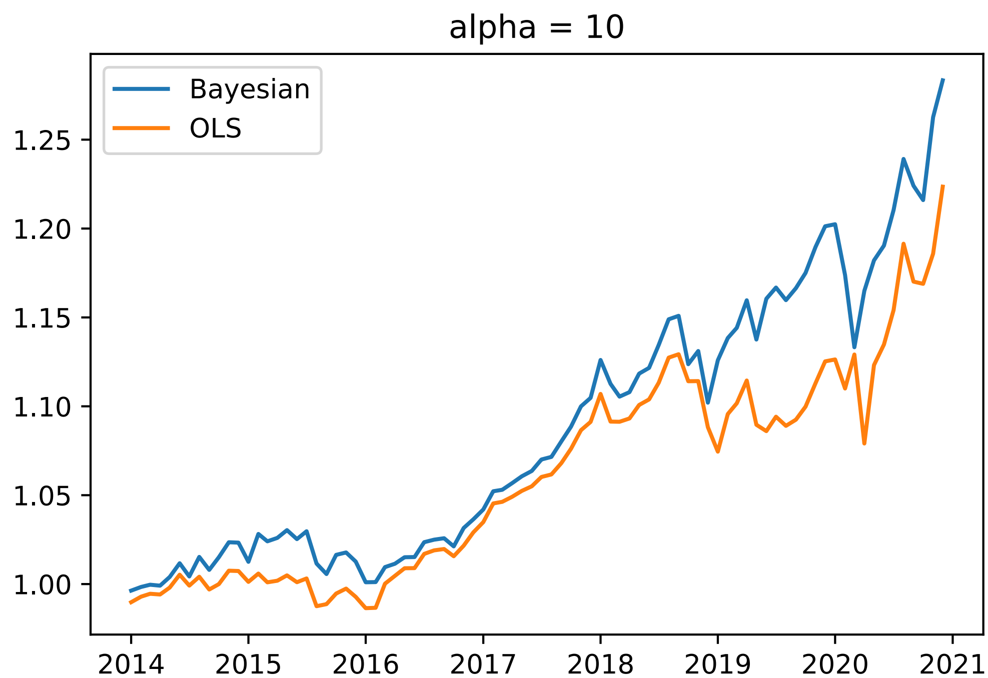
FOC: $0 = du/d\omega = \mathrm{E}_t[r_{t+1} - r_{f,t}] - \alpha\mathrm{E}_t[\omega(R_p - \mathrm{E}_t[R_p])^2] = \mathrm{E}_t[r^e_{t+1}]$
$\quad - \alpha\omega\mathrm{E}_t[(r^e_{t+1}) - \mathrm{E}_t(r^e_{t+1})] = \mu_{r,t} - \alpha\omega\,\mathrm{Var}_t[r^e_{t+1}] = \mu_{r,t} - \alpha\omega\sigma^2_{r,t}$

$\Rightarrow \omega_t = \frac{\mu_{r,t}}{\alpha\sigma^2_{r,t}}$

F

In [280]:
```
alphas = [1,10] # 10
a0 = res.params[0]
a1 = res.params[1]



for α in alphas:
    # model †
    m = m0
    v = v0
    r = [data['Mkt-RF'][data.index == pd.to_datetime(201312,format = '%Y%m')]]
    portfolio_value_1 = [1];portfolio_value_2 = [1]

    T = 0
    months = data.index[data.index >= pd.to_datetime(201401,format = '%Y%m')]

    for month in months:
        T += 1
        ω1 = 100 * m/(α*(v**2+σ**2))
        r_predict = a0 + a1*r[-1]
        ω2 = 100*r_predict/(α*σe**2)

        r.append(data['Mkt-RF'][month])
        portfolio_value_1.append((1 + ω1*r[-1]/100 + data['RF'][month]/100)*po
rtfolio_value_1[-1])
        portfolio_value_2.append((1 + ω2*r[-1]/100 + data['RF'][month]/100)*po
rtfolio_value_2[-1])

        r_bar = np.mean(r)
        m = (m0*(σ**2) + r_bar*T*(v0**2))/(σ**2+T*v0**2)
        v = np.sqrt((v0*σ)**2/(σ**2 + T*v0**2))


    plt.plot(months,portfolio_value_1[1:],label = 'Bayesian')
    plt.plot(months, portfolio_value_2[1:],label = 'OLS')
    plt.legend()

    plt.title('alpha = ' + str(α))
    plt.show()
```





Bayesian model is better since it procues higher return, and the portfolio value is above OLS method for the most of time during 01/2014 to 12/2020