Class Notes Topic 2

Duration and Convexity

and Related Risk Management Strategies

15.438 Fixed Income

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Topic 2: Conceptual Goals for This Topic

- Understand that **duration** and **convexity** are two fundamental concepts for measuring, hedging and speculating on price and reinvestment risk
  - We’ll see many applications throughout the course
- Become familiar with:
  - basic and more advanced definitions of duration and convexity
  - mathematical formulas and their interpretations
  - Graphical interpretations
  - Portfolio strategies
  - Hedging applications
    - The logic and uses of immunization strategies
    - The logic and uses of hedge ratios

Acquired skills

- *Basic:* Know how to compute **formula-based duration measures** (e.g., **Macaulay, modified,** and **dollar duration**) for a bond or bond portfolio
- *Advanced:* Know how to compute **model-based generalized duration measures** (e.g., **effective,** key rate) for fixed income securities and their derivatives
- Know how to compute basic and model-based measures of **convexity**
- Know how to **immunize** a stream of liabilities
- Know how to **delta** and **gamma hedge** a portfolio

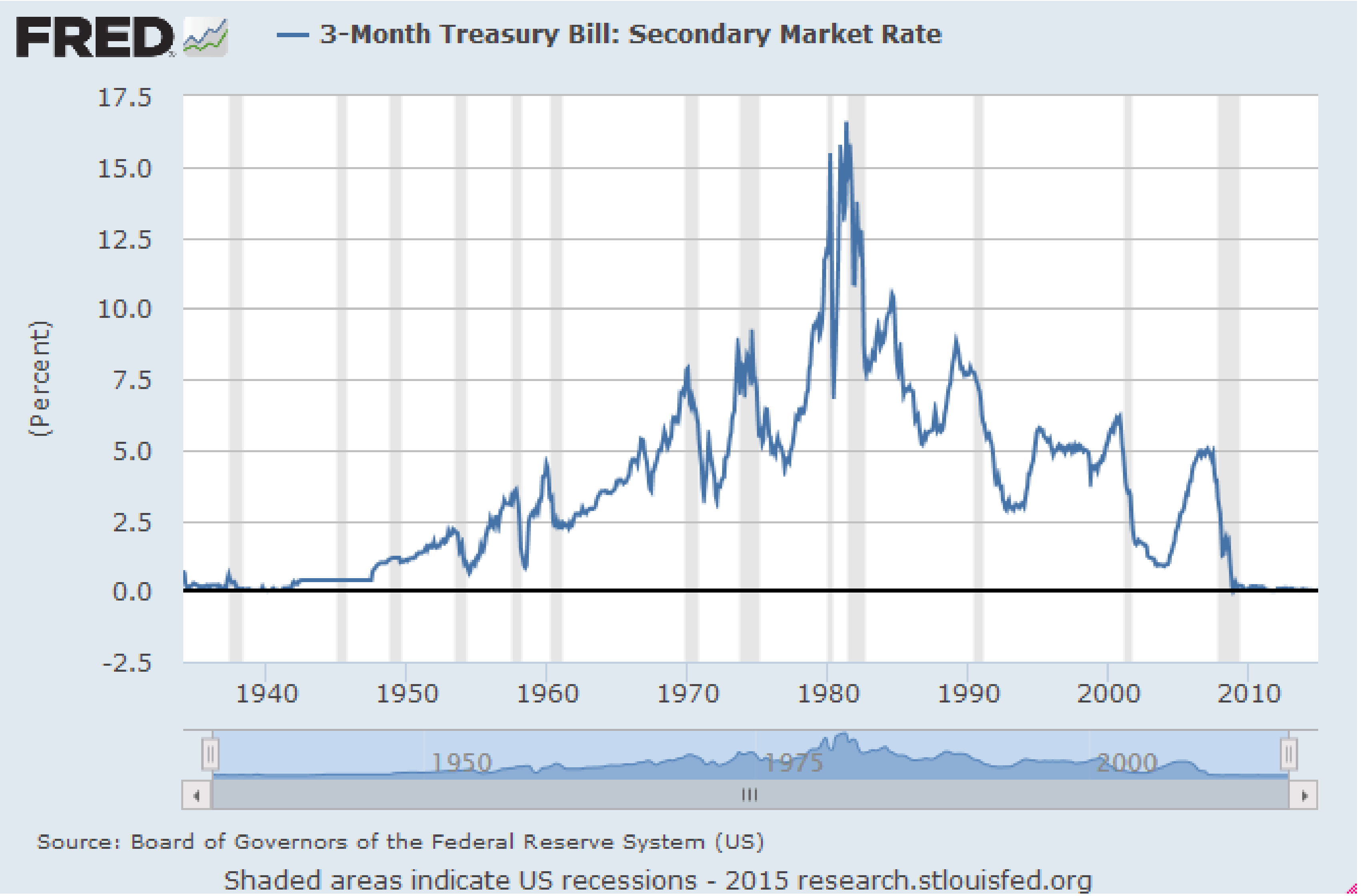
The big question: How much will prices (and future values) change for a small change in yields?

- A security's price "*p*" is function of its yield "*y*" and other factors:
  - Other factors include maturity, coupon, embedded options, default risk, market conditions, etc.

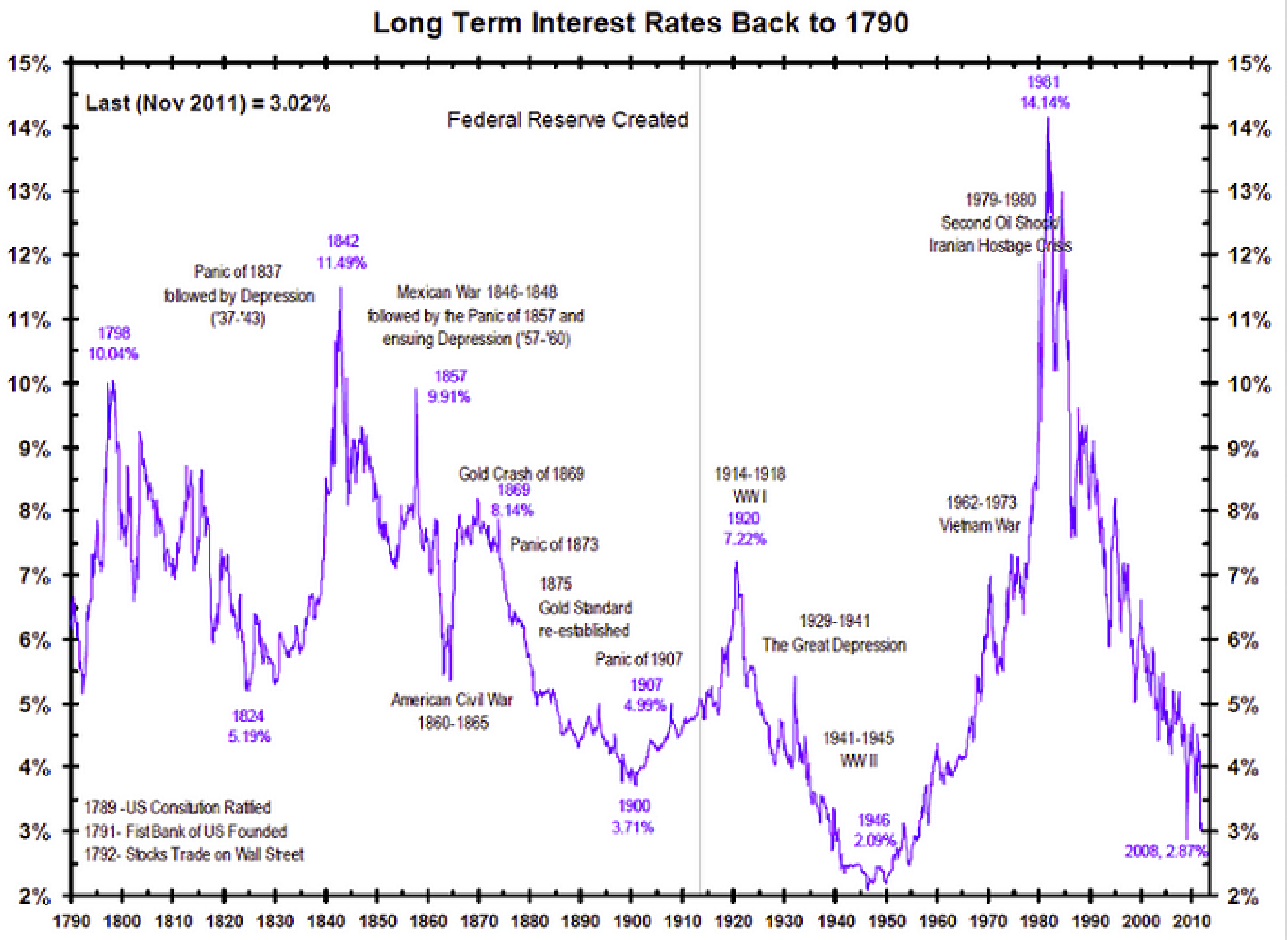
$$p = f(y; \text{other factors})$$

- Duration measures are related to the first partial derivative: $\partial f/\partial y$
- Convexity measures are related to the second derivative: $\partial^2 f/\partial y^2$

Short term yields have historically been volatile



Long term yields are also volatile, but less so than short-term yields



Three major factors affect a bond's price sensitivity to yield changes ($\partial P/\partial Y$):

- Remaining maturity
- Coupon rate
- Level of interest rates
- *Credit risk and attached options also affect $\partial P/\partial Y$. We'll study these effects later.*

Example 2-1:

The Effect of Maturity and Coupon on Price Sensitivity

Four bonds; each priced to yield 9% (b.e.b), semiannual payments. Two have 9% coupon, two have 5% coupon; two mature in 5 years, two mature in 20 years

[Prices at 9% yield are $100, $100, $84.175, and $63.1968]

Instantaneous Percentage Price Changes

New Yld (%)	BP change	9% 5 yr	9% 20 yr	5% 5 yr	5% 20 yr
6.00	-300	12.80	34.67	13.73	39.95
8.00	-100	4.06	9.90	4.35	11.26
8.90	-10	0.40	0.93	0.42	1.05
9.01	1	-0.04	-0.092	-0.042	-.104
9.50	50	-1.95	-4.44	-2.09	-5.01
10.00	100	-3.86	-8.58	-4.13	-9.64
12.00	300	-11.0	-22.6	-11.9	-25.1

Link to Excel calculations

Duration – Two Conceptual Definitions

- Primary: **Duration is the elasticity of bond prices with respect to changes in yield.** It answers the question, "approximately by what percentage will my bond price fall when its yield rises?"

- Secondary: **Duration is the holding period horizon at which price risk and reinvestment risk approximately cancel.**

- We'll look at its mathematical definition, its graphical and intuitive interpretation, and its use as a hedging tool.

- **We will discuss three traditional duration measures: *Macaulay*, *modified*, and *dollar*; and more generalized concepts of duration: *effective*, *partial*, and *key rate*.**

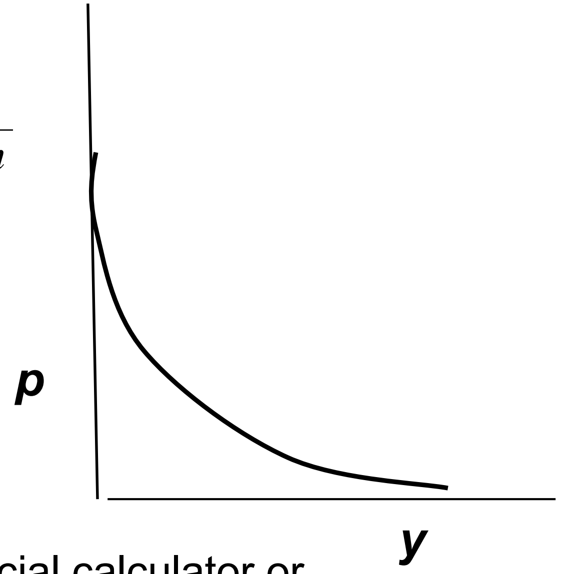
A reminder: Recall from last week the formula that relates price to yield, and that it abstracts from the term structure.

- **A bond's yield (YTM) answers the question: *"What is the constant rate of return that makes the bond price equal to the present value of promised future payments?"***

or

$$p = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \ldots + \frac{C_n}{(1+y)^n}$$

$$p = \sum_{i=1}^{n} \frac{Ci}{(1+y)^i}$$



- *p = price*
- *C_i = cash flow at end of period i*
- *y = yield to maturity (per period)*

- Given the other information, you can solve for the yield using a financial calculator or spreadsheet program.

- **However, prices are actually a function of *all* the relevant yields in the spot yield curve.**

Duration Basics

- **Duration** measures are commonly used to evaluate **bond price sensitivity to interest rate changes**.
  - Higher duration means higher price sensitivity to interest rate changes (more price volatility).

- Basic duration measures take into account information about yield, coupon, and maturity.

$$\boxed{D = \left(\frac{-dP}{dY} \cdot \frac{(1 + Y/k)}{P} \right)}$$

"*Y*" is APR

  - For Macaulay duration, *dP/dY* is based on the standard formula relating bond price *to promised* cash flows

(Note: *k* is the number of compounding periods per year.
For continuous compounding *k*=∞)

- Duration is a property of a security or a portfolio at a point in time.
- It changes over time.

Deriving Macaulay Duration

Macaulay Duration is derived as follows:

$$\text{Bond price } P_B = \sum_{t=1}^{T} \frac{C_t}{(1+\frac{y}{k})^t}\ .$$

C_t is the promised cash flow in period t, T is the total number of periods (= k times maturity in years), y is the annual percentage rate, and k is the number of compounding periods in a year.

To find change in bond price for a small change in yield differentiate:

$$\frac{dP_B}{dy} = \sum_{t=1}^{T} (\frac{-t}{k}) C_t (1+\frac{y}{k})^{-t-1}$$

This gives the dollar price change per small change in annual yield.

To get the percent price change, divide by $P_{B.}$

To get the Macaulay duration (measured in years) multiply the percentage price change by *-(1+y/k)*:

$$D = \sum_{t=1}^{T} (\frac{t}{k}) C_t (1+\frac{y}{k})^{-t} / P_B$$

Our derivation implies that the formula for Macauley duration can be written as:

$$D = \sum_{t=1}^{T} \frac{\dfrac{C_t}{(1+\frac{y}{k})^t}}{P_B} \times \frac{t}{k}$$

D = Macauley duration
C_t = period t cash flow
T = total number of periods
y = yield as an APR
k = assumed compounding periods in a year
P_B = bond price (or present value of cash flows)

That formula implies that Macaulay duration is a **weighted average arrival time of cash flows**, where the weights are the fraction of present value represented by that cash flow:

$$D = \frac{\left(\frac{1}{k}\right) \times PVCF_1}{PVTCF} + \frac{\left(\frac{2}{k}\right) \times PVCF_2}{PVTCF} + \ldots + \frac{\left(\frac{T}{k}\right) \times PVCF_T}{PVTCF}$$

$PVCF_i$ = present value of i^{th} cash flow
PVTCF = present value of total cash flows
(Note in this representation, duration can be described in terms of years although it is intrinsically unitless)

Example 2-2: Calculating Macaulay duration as p.v. weighted average arrival time of cash flows

5 year, 6% bond, yielding 9% (b.e.b.)

Period	Cash Flow	PV $1 @ 4.5%	PV of CF	t x PVCF
1	3.00	.9569	2.871	2.871
2	3.00	.9157	2.747	5.494
3	3.00	.8763	2.629	7.887
4	3.00	.8386	2.516	10.063
5	3.00	.8025	2.407	12.037
6	3.00	.7679	2.304	13.822
7	3.00	.7348	2.204	15.431
8	3.00	.7032	2.109	16.876
9	3.00	.6729	2.019	18.168
10	103.00	.6439	66.325	663.246
		Total	88.131	765.895

Macaulay duration = (765.89/88.13)/2 = 4.35 (in years; here k = 2)

You can calculate using duration and convexity calculator

Macauley Duration vs. Maturity for Bonds in Example 2-1

Coupon	Maturity	Macaulay Duration
9%	5	4.13
9%	20	9.61
5%	5	4.43
5%	20	10.87

Check using calculator!

Properties of Macauley Duration

$$D = \frac{\left(\frac{1}{k}\right) \times PVCF_1}{PVTCF} + \frac{\left(\frac{2}{k}\right) \times PVCF_2}{PVTCF} + \ldots + \frac{\left(\frac{T}{k}\right) \times PVCF_T}{PVTCF}$$

- Duration of an option-free coupon bond is less than or equal to its time to maturity
- Duration of a zero coupon option-free bond is equal to its time to maturity
- The higher the coupon rate the shorter the duration
- As market yield increases, duration decreases

Using Duration to Estimate Price Volatility and Defining Modified Duration

Recall that $\boxed{D = \left(\frac{-dP_B}{dY} \cdot \frac{(1+Y/k)}{P_B} \right)}$ Rearrange to get:

$$\frac{dP_B}{P_B} = -D \frac{dy}{(1+\frac{y}{k})}$$

where D is Macauley duration, y is an APR, and k is the number of compounding periods in a year. Rewrite this as:

$$\boldsymbol{\frac{dP_B}{P_B} = -D_M \times dy}$$

<u>**Definition**</u>: D_M is called the "**modified duration**." It is defined as:

$$D_M = \frac{D}{1+\frac{y}{k}}$$

In words, the percentage price change is approximated by:

(modified duration) times (the change in the yield).

<u>Example 2-3</u>: Using Modified Duration to Approximate Price Changes

20 yr, 5% coupon bond (semiannual payments), P=63.1968, to yield 9% (b.e.b.).

D = 10.87 years (see table above)

D_M = 10.87/(1.045) = 10.40

Say yields increase from 9% to 9.10%. ***What is the predicted price change?***

-10.40(.0010) = -1.04%
The actual change in price is -1.03%

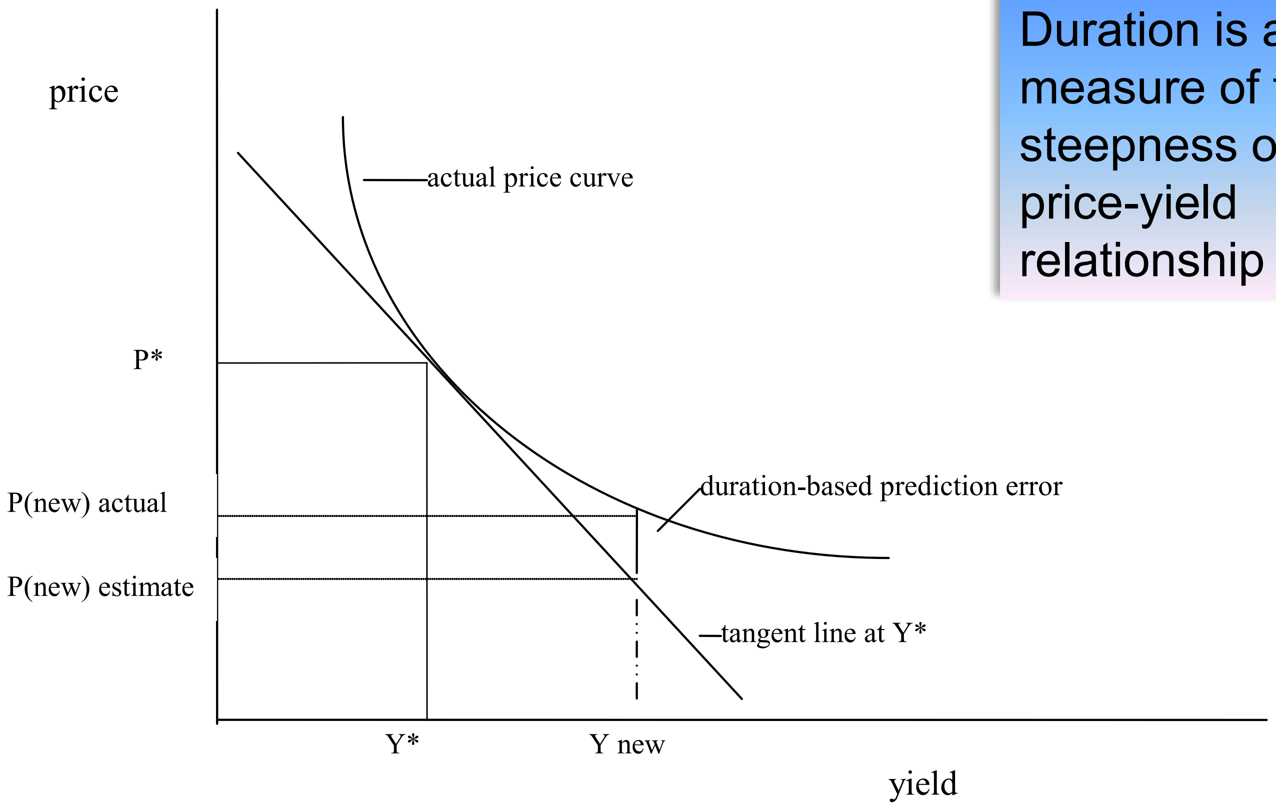
EXCEL example of actual change.

Say yields increase from 9% to 11%. ***What is the predicted price change?***

-10.40(.020) = -20.80%
The actual change in price is -17.94%

A general conclusion is that duration only gives an accurate estimate of price change for small yield changes. It is like "δ"in a derivatives context.

Graphical Interpretation of Duration



Duration is a measure of the steepness of the price-yield relationship

Other Duration Measures

1. Dollar Duration

We have been using duration to estimate percentage price changes. It also can be used to estimate dollar price changes:

$$dP_B = -D_M \times P_B \times dy$$

Definition: D_d is dollar duration. It is defined as:

$$\boldsymbol{D_d = D_M \times P_B}$$

- Dollar duration is useful in hedging strategies and for understanding risk of zero NPV portfolios.

Portfolio Duration

One can prove that

(1) The **modified duration of a bond portfolio** is the **value-weighted modified duration of bonds in the portfolio**

(2) The **dollar duration of a portfolio** is the **sum of the dollar duration of the bonds in the portfolio**

- Added flexibility in duration-targeting comes from the fact that a short position contributes negative duration.
- For a zero-value portfolio, only dollar duration is defined.

If the bonds have different yields, this means that the duration of each bond in the portfolio will be based on a different yield.

Example 2-4: Two-bond portfolio. $P(1) = \$8{,}000$, $D_M(1) = 4.3$ years. $P(2) = \$12{,}000$, $D_M(2) = 3.6$ years.

$$D_M(\text{portfolio}) = (8/20)(4.3) + (12/20)(3.6) = 3.88$$

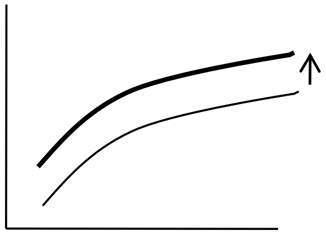
Later we will see that portfolio duration can be changed using forward, futures and swap contracts.

Limitations of Macauley and Modified Duration Measures for Estimating Price Sensitivity to Interest Rates

- Assumes a flat yield curve and parallel shifts

- Bonds are assumed to be default and option-free

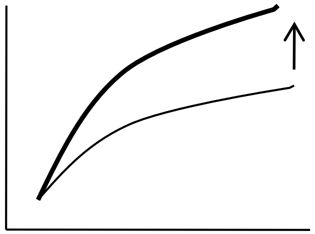
Fortunately, non-parallel shifts are infrequent:



<u>Types of Yield Curve Changes</u>

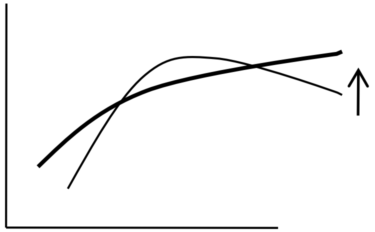
Level or Parallel Shifts

85% of Treasury yield curve changes



Steepness Shifts

5% of Treasury yield curve changes



Butterfly or Curvature Shifts

3-4% of Treasury yield curve changes

Generalized Measures of Duration

Traditional duration measures price sensitivity to small changes in the general level of interest rates.

But other factors also influence bond prices.

Generalized measures of duration describe the total % change in bond price as the sum of the partial effects of multiple factors in a linear model:

$$\frac{dP}{P} = \frac{1}{P}\left[\frac{\partial P}{\partial f_1}\Delta f_1 + \frac{\partial P}{\partial f_2}\Delta f_2 + \ldots + \frac{\partial P}{\partial f_n}\Delta f_n\right]$$

where f_i is the i^{th} factor that influences price.

$\frac{\partial P}{\partial f_i}\Delta f_i$ is the sensitivity of price to the i^{th} factor times a unit change in the i^{th} factor. This is sometimes called a **"partial duration."**

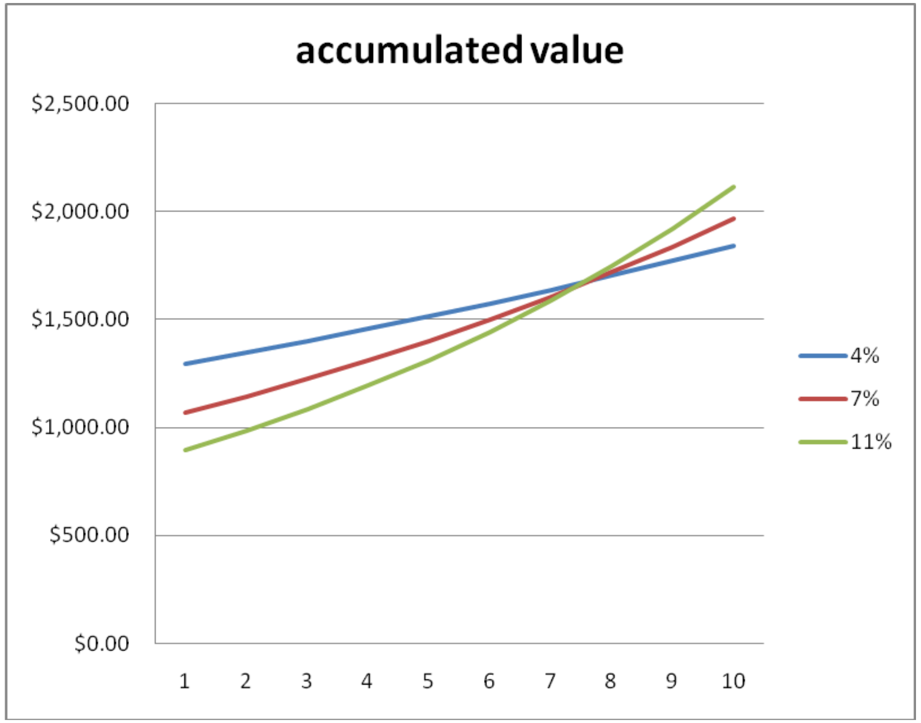
When the factors of interest are rates along the yield curve, the resulting partial durations are called **"key rate durations."**

What is the value of these generalized measured?

- Value is that they increase the accuracy of sensitivity estimates, which in turn makes hedging strategies more robust
  - For example, partial durations are inputs into statistical analyses, such as value-at-risk

- **The most important generalized measure is called "Effective Duration"**
- For securities with uncertain cash flows (like MBS or securities with credit risk), **effective duration is defined as the true price sensitivity of the security to changes in yield**
- It may be based on a theoretical model, or it may be estimated empirically. It is not measured using a standard duration formula because the cash flows are highly uncertain
- We will come back to this important measure later in the course

Interpretation of duration as a break-even point for future value



Example 2.9

This chart shows the accumulated value (bond value + coupon payments + interest on coupon payments) from investing $1000 in a 7%, 10-year, bond at time 0. The bond has annual payments.

In general the accumulated value will depend on the interest rate. The picture assumes a one-time change in interest rates at time 0.

The duration of the bond is 7.5 years. Notice that the accumulated value is about the same for each interest rate at 7.5 years.

Algebra for Break-even Example:

Value at year 7.5 if rates stay at 7%:

$70(1.07)^{6.5} + 70(1.07)^{5.5} + \ldots + 70(1.07)^{.5} + 70/(1.07)^{.5} + 70/(1.07)^{1.5} + 1070/(1.07)^{2.5}$
$= 626.6 + 1034.4 = \mathbf{1661}$

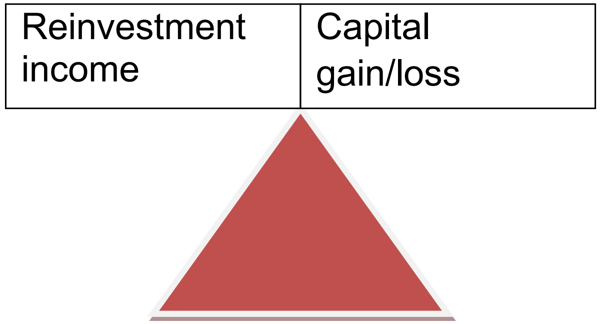
Value at year 7.5 if rates fall to 4% just after time 0:

$70(1.04)^{6.5} + 70(1.04)^{5.5} + \ldots + 70(1.04)^{.5} + 70/(1.04)^{.5} + 70/(1.04)^{1.5} + 1070/(1.04)^{2.5}$
$= 563.8 + 1104.7 = \mathbf{1668.5}$

Value at year 7.5 if rates rise to 10% just after time 0:

$70(1.10)^{6.5} + 70(1.10)^{5.5} + \ldots + 70(1.10)^{.5} + 70/(1.10)^{.5} + 70/(1.10)^{1.5} + 1070/(1.10)^{2.5}$
$= 696.5 + 970.6 = \mathbf{1667.1}$

Practice Problem 2.2: Refer back to practice problem 1.1. For what holding period do you expect the horizon return to be invariant to a change in interest rates? Verify your guess with a calculation like the one here.



Convexity and its properties

- Convexity measures the degree of inward curvature of the price-yield relationship. It is based on the second derivative of a security price with respect to yield.
- It is used to improve upon duration-based approximations and hedging strategies. (Like "gamma" in derivatives pricing.)
- A long position in non-callable bonds always has positive convexity.
- Positive convexity is a desirable property for a long position.
  - It means that duration underestimates the price increase resulting from a drop in yields, and overestimates the price decrease from an increase in yields.
- For bonds with positive convexity, convexity decreases with yield.

Calculating Convexity

Convexity is found by taking the second derivative of the bond price function and then dividing by the price. An explicit expression for convexity of an option-free bond is:

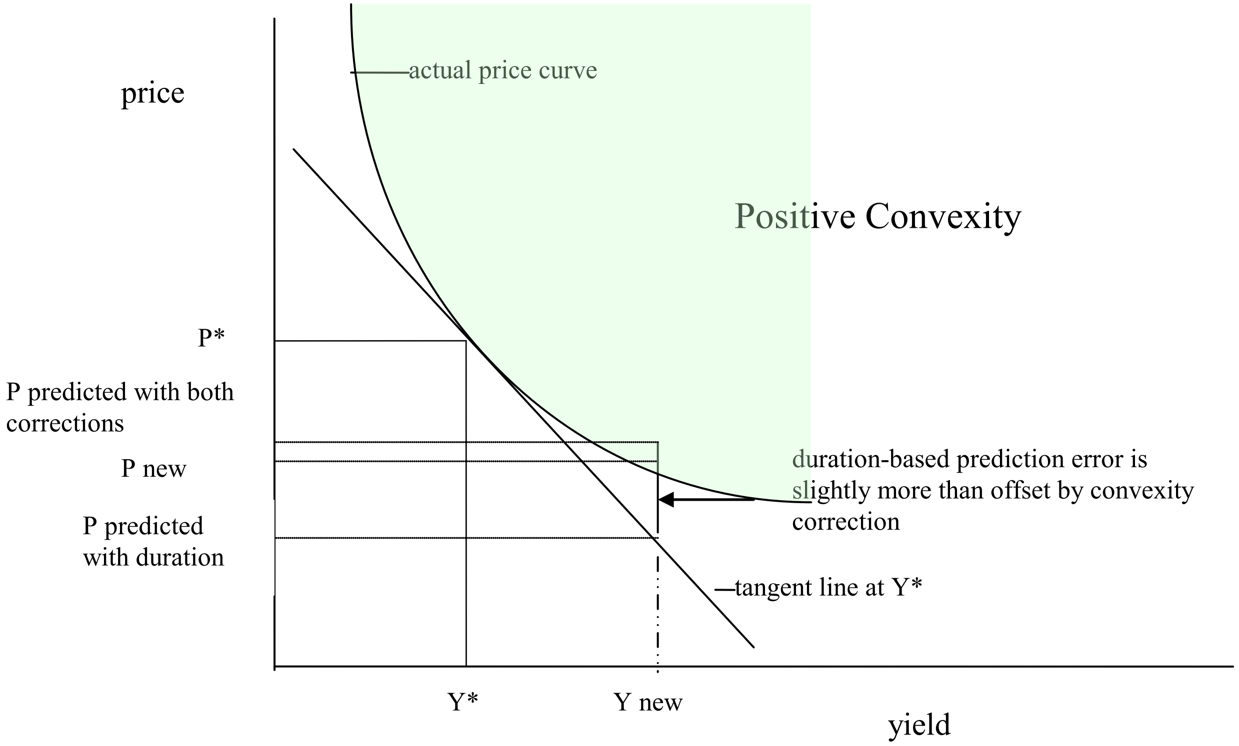
$$C_0 = \sum_{t=1}^{T} \frac{t(t+1)X_t}{(1+y/k)^{t+2}k^2} \,/\, P_B$$

T = number of periods (= maturity in years $\times$ k)
P_B = bond price (present value of cash flows)
X_t = time t cash flow
y = quoted annual percentage rate (so y/k = effective yield per period)
k = number of compounding periods per year

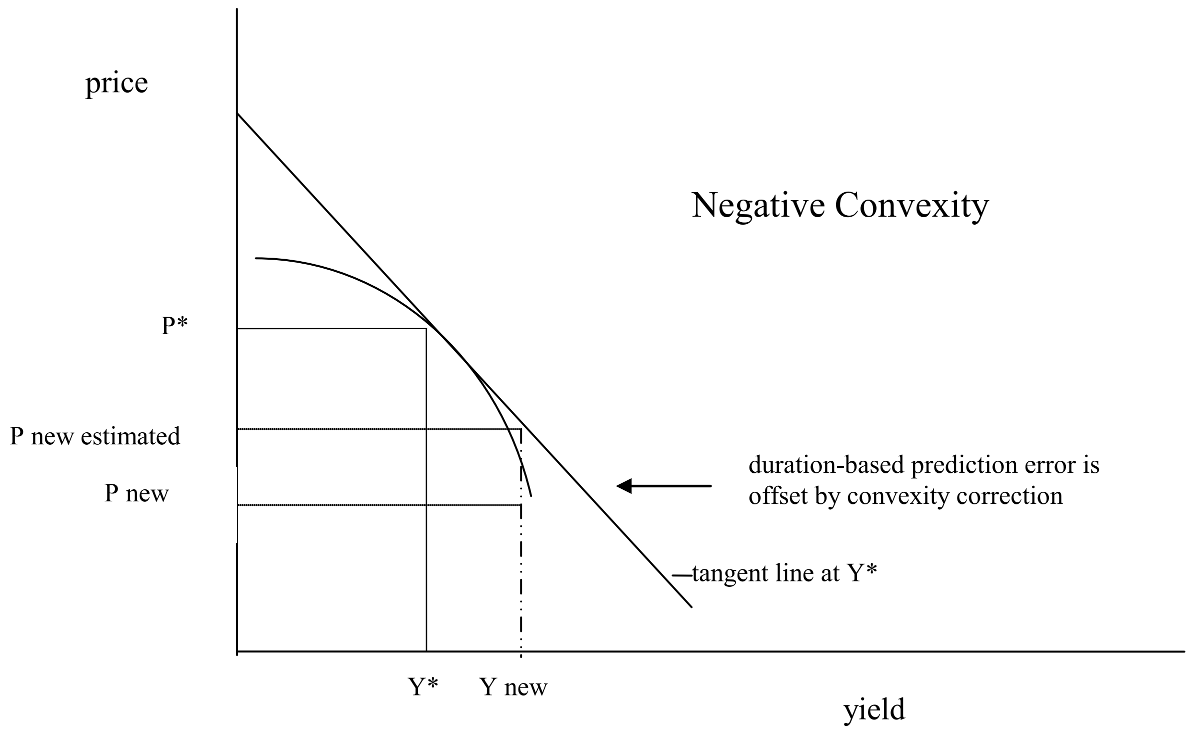
Note that convexity is measured in terms of years squared. The units have no intuitive interpretation.

Dollar Convexity = C_0P_B

Graphical Interpretation of Convexity



Graphical Interpretation of Negative Convexity



Example 2-5: Convexity Calculation

Five year, 9% bond, semi-annual payments, priced at 100.

Period	Cashflow	$1/(1.045)^{t+2}$	t(t+1)CF	col(4)* col(3)
1	4.50	.8763	9	7.9
2	4.50	.8386	27	22.6
3	4.50	.8025	54	43.3
4	4.50	.7679	90	69.1
5	4.50	.7348	135	99.1
6	4.50	.7032	189	132.9
7	4.50	.6729	252	169.6
8	4.50	.6439	324	208.6
9	4.50	.6162	405	249.6
10	104.50	.5897	11,495	6,778.0
		Total	12,980	7,781.0

convexity = 7781/(100x4) = 19.45

dollar convexity = 19.45(100) = 1,945

Check using calculator!

Using Convexity to Improve Price Sensitivity Estimates

The prediction equation with duration and convexity is:

$$\frac{dP_B}{P_B} = -D_M(dy) + \tfrac{1}{2}C_0(dy)^2$$

(This is a 2nd order Taylor's Series expansion of the price-yield function.)

Example 2-6: Approximating price change with duration and convexity

Estimate the percent price change of a 6% coupon bond with 25 years to maturity, selling to yield 9% if there is a 200 basis point increase in required yield.

D_M = 10.62
C_0 = 182.92

$$\frac{dP_B}{P_B} = -10.62(.02) + \tfrac{1}{2}182.92(.02)^2 = -21.24\% + 3.66\% = -17.58\%$$

Contrast to the actual change = -18.03%.

Using convexity gets much closer than the estimate using duration alone!

Practice Problem 2.1:

a. Calculate the modified duration and convexity for a 7 year 9% coupon bond with semiannual coupon payments, priced to yield 8% on a b.e.b.

b. Compare the approximate price change (using duration and convexity) with the actual price change if the yield increases to 8.5%

Application of Duration and Convexity to Portfolio Strategies

Holding duration constant, and all else equal, convexity is often considered a valuable portfolio attribute. How to achieve it?

Barbells vs. Bullets

Bullet payment = single payment

Barbell payments = cash flows widely spaced in time

> **For a given yield and modified duration**, the lower the coupon rate the smaller the convexity.
>
> Related to this is that bullets have a smaller convexity than barbells
>
> This leads to the portfolio strategy of picking barbells over bullets, holding duration constant and all else equal.

Example 2-7: Bullet vs. Barbell Strategy

The following T-bonds are available, each selling at par:

Bond	Coupon (%)	Maturity (yrs)	Yields	D_m	C_0
A	8.5	5	8.5	4.00	19.81
B	9.5	20	9.5	8.88	124.2
C	9.25	10	9.25	6.43	55.45

Consider two alternative portfolio investment strategies:

(a) Invest only in C (bullet strategy)

(b) Invest in A and B in proportions so that the dollar duration is the same as for C.

This implies 50.2% in A and 49.8% in B since:

$$.502(4.00) + .498(8.88) = 6.43$$

The portfolio manager expects to do better if rates change with the barbell portfolio.

Hence the manager may be willing to give up some yield for an increase in convexity.

Here, the yield on the bullet is 9.25%.

Taking the yield on the barbell as approximately the weighted average yield of the two bonds, the yield on the barbell is:

$$.502(8.5\%) + .498(9.5\%) = 8.998\%$$

In this case the manager would have to give up yield for convexity.

Example 2-8: Barbells and bullets when yield curve shifts are not parallel

This Table shows the relative performance of bullets vs. barbells in the last example under different yield curve shift assumptions. **Notice that the barbell does not always do better!**

(Table reports dollar bullet returns minus dollar barbell returns over 6 month holding period)

% yield change	level shift	flattening	Steepening
-5.0	-7.19	-10.69	-3.89
-4.0	-4.00	-6.88	-1.27
-3.5	-2.82	-5.44	-0.35
-2.0	-0.59	-2.55	1.25
-1.0	0.06	-1.54	1.57
0.0	0.25	-1.06	1.48
2.0	-0.31	-1.18	0.49
2.75	-0.73	-1.46	-0.05
3.00	-0.88	-1.58	-0.24
3.75	-1.39	-1.98	-0.85

Flattening means 5 yr yield increased by 25 basis pts more than col. 1 and 20 yr yield increased by 25 basis pts less than col. 1.

Steepening means 5 yr yield increased by 25 basis pts less than col. 1 and 20 yr yield increased by 25 basis pts more than col. 1.

Common Duration-Based Hedging Strategies

- Immunization
- Delta Hedging
- Gamma Hedging

Portfolio Strategies Involving Duration and Convexity

1. Immunization

- The goal of immunization is to be able to meet a set of liabilities (i.e., a target future value) out of the proceeds of a dedicated investment portfolio, without having to add any capital to the portfolio after the initial investment.

- The simplest way to do this is “cash matching”

  (i.e., invest in a security with a maturity equal to the maturity of the liability, and a face value equal to the face value of the liability)

- Immunization allows the same goal to be accomplished without so drastic a limitation on investment choices.

Major Users of Immunization Policies

- Pension funds
- Life insurance companies
- Banks (managing “duration gap”)

The Immunization Procedure:

(1) Find the modified duration of the liability.

(2) Choose a portfolio such that the **modified duration of the portfolio equals the modified duration of the liability**.

(3) Select amounts to invest in each security so that the **present value of the portfolio equals the present value of the liability**, discounting at the rate implied by the immunization portfolio.

(4) Rebalance the investment portfolio as interest rates change and liabilities are paid off.

By matching the modified duration of assets and liabilities, both have the same price sensitivity to changes in interest rates.

Thus the value of assets and liabilities should move together over time.

Example 2-10: **Immunizing a single payment liability**.

Suppose that you have a liability where you must make a single payment of $1931 in 10 years. The yield curve is flat at 10%.

Present value of liability = $\$1931/(1.1)^{10} = \$745$

Macaulay Duration of the liability = 10 years
Modified Duration of liability = 10/(1.1) years

To immunize this liability, you invest in a bond (or portfolio of bonds) that has an equal modified duration. You need to buy enough so that the present values are also equated.

Considering the various alternatives you come up with a bond that fulfills these requirements:

A 20-year bond with face value of 1000 and a 7% coupon rate (annual payments) costs $745 and has a Macaulay duration of approximately 10 years, and a modified duration of 10/(1.1) years.

If the yield curve shifts immediately after the initial investment, this is what happens to the present value of both the liability and the bond:

Yield	**Bond Value**	**Liability Value**
4%	$1409	$1305
6%	1115	1078
8%	902	895
10%	745	745
12%	627	622
14%	536	521
16%	466	438

Here any permanent one-time shift in interest rates results in the bond value exceeding the liability value. Unfortunately, this won't always be the case.

Can you think of a case (based on this example), which has the opposite result?

You can also see what happens when the yield curve shifts in this example by considering terminal values of the bond and the liability:

	Rates Stay at 10%	Rates Fall to 4%	Rates Rise to 16%
Accum. value of interest received and reinvested at indicated rates	$\$70(1.1)^9 = \$165$	$\$70(1.04)^9 = \$100$	$\$70(1.16)^9 = \$256$
	$\$70(1.1)^8 = \$150$	$\$70(1.04)^8 = \$\ 96$	$\$70(1.16)^8 = \$229$
	$\$70(1.1)^7 = \$136$	$\$70(1.04)^7 = \$\ 92$	$\$70(1.16)^7 = \$198$
	$\$70(1.1)^6 = \$126$	$\$70(1.04)^6 = \$\ 89$	$\$70(1.16)^6 = \$171$
	...	...	...
	$ 70	$ 70	$ 70
TOTAL	$1115	$842	$1492
Market value of bond in the 10th year at indicated interest rate	$816	$1243	$565
Grand Total	$1931	$2085	$2057
Less required payment	$1931	$1931	$1931
Surplus	$ 0	$ 154	$ 126

Now imagine that instead of a one-time shift in interest rates, the following happens:

a) interest rates fall immediately to 4% and stay there for 9 years.

b) in year 9.5 interest rates rise to 16%.

You can see that you'll come out way behind.

Does this prove that immunization doesn't work?

Certainly not. Once rates shift, the portfolio must be reimmunized.

After the rate shift the new price and new duration of the bond is:

$$P = \frac{70}{1.04} + \frac{70}{(1.04)^2} + \ldots + \frac{1070}{(1.04)^{20}} = \$1408$$

$$D_{\text{mod}} = \left[\frac{70}{1.04} + \frac{70(2)}{(1.04)^2} + \ldots + \frac{1070(20)}{(1.04)^{20}}\right] \times \frac{1}{1408(1.04)} = 12.71/1.04 = 12.22$$

To reimmunize:

(1) sell this bond

(2) rebalance the portfolio to have a modified duration of 10/1.04=9.61 years

Related Questions

- How often to reimmunize?
- What to look for besides duration?
- How well does immunization work in practice?

2. Using Duration and Convexity to Structure a Hedge – Delta and Gamma Hedging

How much does a bond price change with a change in interest rates?

This is approximated by:

$$dP = -D_m(P)dy + .5(P)C_0(dy)^2$$

<u>Definitions</u>

D_m = modified duration
P = price
C_0 = convexity
Hedge Ratio = $-D_m(P)$ (= "dollar duration")
Gamma = $(P)C_0$ (= "dollar convexity")

A delta neutral portfolio equates the hedge ratio of assets and liabilities:

$$P_{asset}D_{m,asset} = P_{liability}D_{m,liability}$$

A gamma neutral portfolio is delta neutral, and also equates the gammas of assets and liabilities.

Example 2-11: Hedging Dealer Portfolio Risk

A dealer in corporate bonds finds herself with an inventory of \$1mm in a 5 year 6.9% bonds (semiannual payments) at the end of the trading day, priced at par. The bonds are illiquid, so selling them would entail a loss. Holding them overnight is risky, since their price might fall if rates rise.

An alternative to selling the corporate bonds is to short more liquid Treasury bonds. The following bonds are available:

1. 10 yr 8% Treasury, p = \$1,109.0 per \$1,000 face
2. 3 yr 6.3% Treasury, p = \$1,008.1 per \$1,000 face

a. How much of the 10 year bond would she need to short to hedge? How much of the 3 year bond?

b. If yields rise by 1% overnight on all the bonds, show the result of the transactions the next day when the short position is closed out.

c. If she wanted to short \$1 mm of the Treasury bonds, what combination of the 10 year and 3 year bonds would she need to use?

Plan to answer (a):

1. Find modified duration of the bond to be hedged
2. Find modified duration of the bonds to be shorted
3. Use this to find hedge ratios.

For 5 year 6.9% bond, y = 6.9% b.e.b., D_m = 4.1688

For 10 year 8% bond, y=6.5% b.e.b, D_m=7.005

For 3 year 6.3% bond y = 6.00% b.e.b., D_m = 2.700

Amount of 10 year bond to sell, x, solves: x(7.005) = \$1mm(4.1688). x = \$593,861.5

Amount of 3 year bond to sell, y, solves: y(2.7) = \$1mm(4.1688) y = \$1.54072 mm

Part b: If yields rise by 1% overnight on all the bonds, show the result of the transactions the next day when the short position is closed out.

For 5 yr, yield to 7.9%, => price to \$959.344/\$1000. Loss on long position = \$1mm(1-.959344) = \$40,656

For 10 yr, yield to 7.5% => price to 1034.74/\$1000. 1034.74/1109 = .933. (1-.933)(593,861.5) = \$39,765.7 gain.

For 3 yr, yield to 7% => price to 981.35/\$1000. 981.35/1,008.1 = .97346. (1-.97346)(1,540,720) = \$40,891 gain.

Part c: If she wanted to short \$1 mm of the Treasury bonds, what combination of the 10 year and 3 year bonds would she need to use?

For equal present values to constitute a hedge, the duration of the asset and liability must be equal. Solve for x in:

x(7.005) + (1-x)(2.7) = 4.1688 to find the fraction of \$1mm to short of the 10 year bond...

Note: You can also find price change by first finding face value and then rediscounting cash flows at new rate.

Example 2-11 extended to gamma hedging…

- "What if the dealer wants the added protection of doing a gamma neutral hedge?"
  - Let bond 1 be the bond to be hedged (5 year 6.9% corporate)
  - Let bond 2 be the 10 year 8% Treasury
  - Let bond 3 be the 3 year 6.3% Treasury
- Investment must be both delta neutral and gamma neutral. (Note that unlike for immunization, there is no restriction on matching the PV of the long and short positions.)
- This requires matching deltas and gammas, and requires investments in both bonds.
  - P1 = $1 million by assumption. D1 = 4.1688, C1 = 21.038
  - P2 = ?, D2 = 7.005, C2 = 62.98
  - P3 = ?, D3 = 2.700, C3 = 8.939
- Match hedge ratios:
  - $1m(4.1688) = P2(7.005) + P3(2.700)
- Match gamma:
  - $1m(21.038) = P2(62.98) + P3(8.939)
- 2 linear equations in two unknowns. Solve for P2 and P3. (Full solution is posted on the class web page.)

- <u>Exercise</u>: Redo calculation of what happens when rates move (as in example in notes), and verify that hedge is better than with delta hedge.

Appendix:

Two More Sensitivity Measures

1. <u>Price Value of a Basis Point (PV01)</u>

Definition: *The price value of a basis point is the change in the price of a bond if the required yield to maturity changes by one basis point (1/100th of 1%).*

Example A2-1: 5 year 9% coupon bond (semiannual payments), selling for $100.

The yield increases 1 basis point to 9.01%

New price =

$4.5/(1.04505) + ... + 104.5/(1.04505)^{10} = 99.9604$

The price value of a basis point =

$100 - $99.9604 = $0.0396
(per $100 face, or $.396 per $1000 face)

2. <u>Yield Value of a Price Change</u>

Definition: *The yield value of a price change is the change in the yield of a bond if the required price changes by a specified dollar amount (often 1/32 of \$1).*

<u>Example A2-2</u>: 5 year 9% coupon bond (semiannual payments), priced at \$100 to yield 9% (b.e.b.).

The price increases by 1/32 to 100 1/32.

New yield solves:
$100\ 1/32 = 4.5/(1+y/2) + ... + 104.5/(1+y/2)^{10}$

The yield value of a price change =
.09 - .08992 = .00008,
or .008%