Class Notes Topic 3

The Yield Curve

15.438 Fixed Income
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Spring 2021

Yield curve movie

Topic Outline

- Fitting spot yield curves
- Pricing with the yield curve
- Estimating implied forward rates
- Theories of the yield curve and related evidence
- Interpretations of the yield curve and relation to macro variables
- Linking international yield curves with interest rate parity

- Skills acquired:
  - Know how to **construct** yield curves and strip curves from bond market data at a point in time.
  - Know how to **use spot yield curve to price cash flows** and **look for arbitrage opportunities**.
  - Know how to **calculate and replicate by trading the implied forward rates** embedded in the yield curve.
  - Understand the **linkages of yield curves internationally** through interest rate parity conditions

- Knowledge acquired:
  - Basics about Treasury market
  - Leading theories of the yield curve
  - Relation between yield curves and the macro-economy

Approaches for fitting Treasury yield curves

- Generally there is no single set of spots rates that can explain all observed prices of Treasury securities at a point in time.
- Interpolation is required to find yields for dates on which no (or only illiquid) Treasury cash flows are promised.
- Estimation methods include:
  - Bootstrapping (often using “on-the-run” bonds)
  - Regressions (various specifications)
  - Non-line*ar* fits
    1. Quadratic and cubic splines
    2. Nelson Siegel model
    3. Less formal guess-timates

Constructing Spot Yield Curves

- The spot yield curve is the foundation for pricing fixed income securities.
  - It gives the precise yield for discounting cash flows at each horizon (corresponding to the discount function).
- Constructing a spot yield curve is easiest using the prices of pure discount bonds such as Treasury bills and/or Treasury strips.
- Example from Topic 1 (repeated): Here is a set of prices and the implied yields (on a bond equivalent basis):

Time To Maturity (in years)	**Price (P)** (per $100)	**Implied Yield** (b.e.b.)
1	$96	4.12%
2	$90	5.34%
3	$85	5.49%
4	$80	5.66%

These yields are found using the logic that:

$$P = \frac{100}{(1 + y_t/2)^{2t}} \quad =>$$

$$y_t = 2\left[\left(\frac{100}{P}\right)^{(1/2t)} - 1\right]$$

where t is the horizon in years.
y_t is the spot yield for cash flows arriving in year t, expressed on a bond equivalent basis.

Note that $y_t/2$ can be described as an effective 6-month rate.
To convert to an effective annual yield use $(1 + y_t/2)^2 - 1$.

Estimating spot yield curves from coupon bond prices by "**Bootstrapping**"

- Bootstrapping is a well-known method for **constructing a spot yield curve from coupon bond prices**.
  - Can be useful when pure discount instruments are unavailable or illiquid.
- <u>Step 1</u>: Gather price and coupon rate information.
- <u>Step 2</u>: Find effective spot yields sequentially, going from the shortest to longest maturity, using the formula:

$$P = \frac{C_1}{(1+Y_1)} + \frac{C_2}{(1+Y_2)^2} + \ldots + \frac{C_N}{(1+Y_N)^N}$$

  - C_i is the cash flow in the i^{th} period
  - Y_i is the effective spot yield for i periods

Bootstrapping

- <u>EXAMPLE 3-1</u>: **Finding Semi-Annual Spot Yields**

  The following information is available on Treasury bond prices

  (Prices here are decimal, not 32nds; s.a. = semiannual):

Maturity (months)	**Coupon Rate** (s.a. pmts)	**Price** (per $100 face)
6	7 1/2	99.473
12	11	102.068
18	8 3/4	99.410
24	10 1/8	101.019

$$99.473 = \frac{103.75}{(1+Y_1)}$$

so Y_1 = 4.3% (= **8.6% on a b.e.b.)**

$$102.068 = \frac{5.5}{(1+.043)} + \frac{105.5}{(1+Y_2)^2}$$

so Y_2 = 4.4% (= **8.8% on a b.e.b.)**

$$99.410 = \frac{4\frac{3}{8}}{(1+.043)} + \frac{4\frac{3}{8}}{(1+.044)^2} + \frac{104\frac{3}{8}}{(1+Y_3)^3}$$

so Y_3 = 4.6% (= **9.2% on a b.e. basis)**

Continuing in this manner generates a **yield curve** of:

$Y_{6\ mo}$ = 8.6% (b.e.b.)

$Y_{1\ yr}$ = 8.8% (b.e.b.)

$Y_{1.5\ yr}$ = 9.2% (b.e.b.)

$Y_{2\ yr}$ = 9.6% (b.e.b.)

You can use the same approach to construct spot yield curves based on other compounding periods (e.g., continuous, weekly, monthly, yearly).

<u>Example 3-2</u>: Using the Spot Yield Curve for Pricing Other Bonds

These yields can be used to estimate the value of other Treasury bonds, or any package of cash flows with similar characteristics.

What is the value of a 1-year, 9% coupon T-bond (semiannual pmts)?

$$P = \frac{4.5}{1.043} + \frac{104.5}{(1.044)^2}$$

$$= 100.192$$

A more general way to get bootstrapped rates involves linear algebra and a computer:

For example, the information in the preceding example can be summarized in matrix form as:

$$\begin{bmatrix} 103.75 & 0 & 0 & 0 \\ 5.5 & 105.5 & 0 & 0 \\ 4.375 & 4.375 & 104.375 & 0 \\ 5.0625 & 5.0625 & 5.0625 & 105.0625 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 99.473 \\ 102.068 \\ 99.410 \\ 101.019 \end{bmatrix}$$

where the semiannual discount factor is:

$d_i = \dfrac{1}{(1+Y_i)^i} = Z(0,i)$

This equation can be represented as:

$M\ d = P$

M is the matrix of payoffs. P is the vector of prices. The vector of discount factors is found by solving for d:

$d = M^{-1}\ P$

Bootstrapping

- The file "bootstrap.xls" has a spreadsheet programmed to do this, along with sample price information.
  - *Technical note:* using this method, a spot yield curve can be derived from any set of coupon bonds for which the matrix M is non-singular (i.e., no row is simply a multiple of another row).
    - For instance, you could use four 2-year, semiannual coupon bonds, each with a different coupon rate, to generate a 2-year yield curve at a semi-annual frequency.

Estimating Spot Yield Curves Using Regression Analysis

$$p^1 = b_1 C_1^{\ 1} + b_2 C_2^{\ 1} + \ldots + b_n C_n^{\ 1} + \varepsilon^1$$
$$p^2 = b_1 C_1^{\ 2} + b_2 C_2^{\ 2} + \ldots + b_n C_n^{\ 2} + \varepsilon^2$$
$$p^3 = b_1 C_1^{\ 3} + b_2 C_2^{\ 3} + \ldots + b_n C_n^{\ 3} + \varepsilon^3$$

.

As is needed for bootstrapping, imagine you have a set of bonds with coupons falling at regular intervals that are lined up in time. .

Run regression to estimate coefficients $b_1, b_2, \ldots b_n$. These represent a set of discount factors:

$$b_1 = \frac{1}{(1+y_1)}, \quad b_2 = \frac{1}{(1+y_2)^2}, \quad \text{etc.,}$$

where y_i is the i period spot yield. C_i^j is the cash flow in period i for security j.

- ***Advantage is that can use the information in a large set of bonds.***
- ***Disadvantage is that if some of the bonds are illiquid or otherwise special, they may bias the estimates of discount rates.***

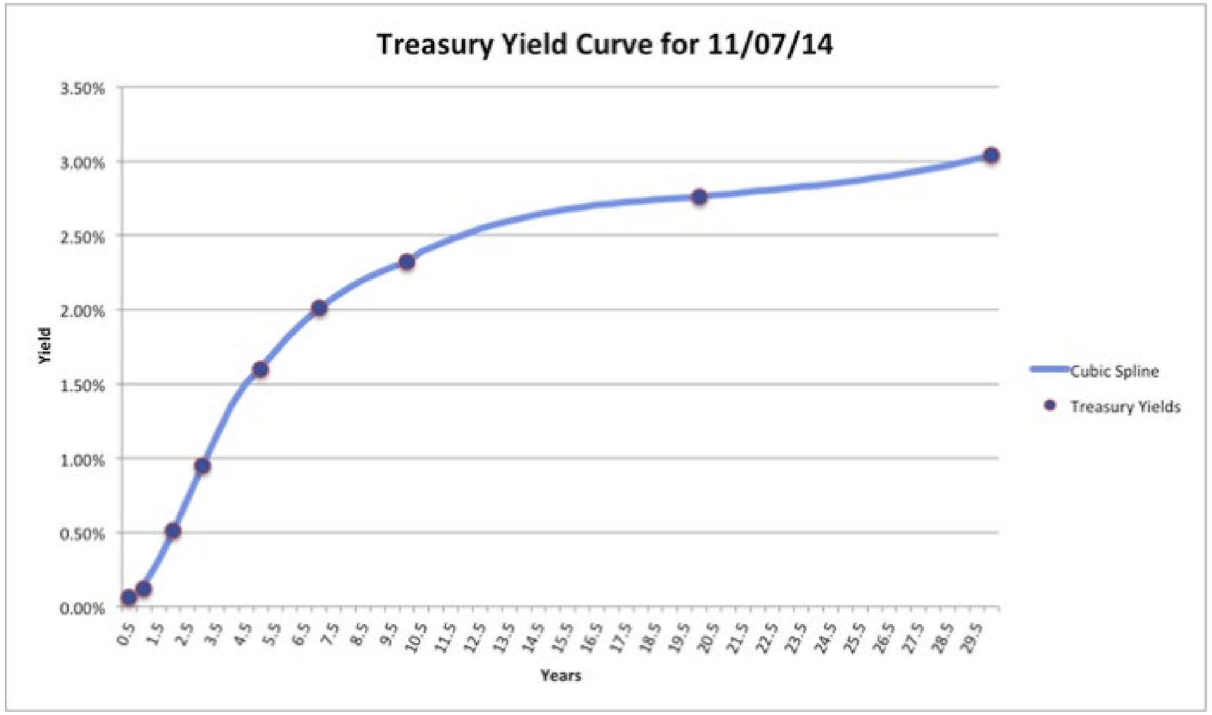
Estimating yield curves with splines

- Splines interpolate between data points with smooth functions (polynomials)
  - E.g., linear, quadratic, cubic
  - In cubic case:

$$\hat{r}(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$$

Cubic polynomial to fit to each segment of the yield curve.

  - With cubic spline, the fitted curve can be smoothed across data points by matching derivatives
    - Allows differentiation of yield curve to get smooth forward rate curves
  - Downside of splines include overfitting and spurious non-linear behavior with higher order splines



Graph of US Treasury yield curve in Excel.

R code to do cubic splines can be found at: https://puppyeconomics.wordpress.com/2014/11/11/yield-curve-interpolation-using-cubic-splines-in-excel-and-r/

Nelson Siegel Model

- Popular model (e.g., at Federal Reserve) for estimating a smooth curve that accommodates many shapes
- Assumes continuous compounding with discount function *Z(0,T)* = $e^{-r(0,T)T}$
- Spot rate curve given by:

$$r(0,T) = \theta_0 + (\theta_1 + \theta_2)\frac{1 - e^{-T/\lambda}}{T/\lambda} - \theta_2 e^{-T/\lambda}$$

- Fit to minimize squared error by choice of $\theta_0, \theta_1, \theta_2, and\ \lambda$

Link to spreadsheet

Using the Spot Yield Curve to Search for Arbitrage Opportunities

- An important use of the spot yield curve is to see whether bonds are correctly priced relative to other bonds.
- If not, the discrepancy represents an arbitrage opportunity (i.e., a risk-free profit opportunity).
  - *You're not likely to find many arbitrage opportunities in the Treasury market, but knowing how to search provides fundamental insights about how bond prices are tied together.*

<u>Example 3-3</u>: Looking for Arbitrage Profits

Based on the prices of three zero coupon risk-free bonds maturing in 1, 2 and 3 years respectively (e.g., Treasury "strips"), you derive a 3-year spot yield curve.

The effective annual rates are given by:

$$Y_1 = 9.9\%$$
$$Y_2 = 9.3\%$$
$$Y_3 = 9.1\%$$

Also available is a 3-year 11% risk-free annual coupon bond selling for $102 (per $100 face).

- ***Is there an arbitrage opportunity?***
- ***If so, how would you exploit it?***

The price of the coupon bond can be checked for consistency by discounting the promised cash flows using the yields derived from the discount bonds:

$$p = \frac{11}{1.099} + \frac{11}{(1.093)^2} + \frac{111}{(1.091)^3} = 104.69$$

- The bond at $102 is clearly under-priced!

- **How to profit?**
- Buy the under-priced coupon bond and sell a set of discount bonds whose payments mimic the cash flows of the bond you buy:
  - sell $11 face of the 1 yr discount bond,
  - sell $11 face of the 2 yr discount bond,
  - sell $111 face of the 3 yr discount bond.
- These sales generate:

$$\frac{11}{1.099}, \frac{11}{(1.093)^2}, and \frac{111}{(1.091)^3} \qquad totaling...104.69$$

Therefore you collect **$104.69** today.

At the same time, buy the under-priced bond for **$102.**
Profit today = **$2.69** per $100 face value transaction.

All future cash inflows and outflows cancel, so this is an arbitrage opportunity.

Remember that in practice, you also have to take into account the bid/ask spread.

A Question to Think about for Next Time

(PRACTICE PROBLEM 3.1) Is it a signal of a pricing inconsistency if two bonds with the same maturity but different coupon rates have different yields to maturity? (Recall that a bond's YTM is the constant yield that equates its price to the present value of its future payments.) **Give an example**.

Static vs. dynamic models of the yield curve

- Static models
  - These describe the term structure at a point in time
  - They do not predict how the term structure will move over time
  - We've been discussing static models

- Dynamic models
  - These predict the statistical distribution of yield curve shapes in the future
  - Main use is for pricing interest rate contingent claims
    - E.g., caps, floors, options on Treasury bonds
  - We will study dynamic models of yields when we turn to pricing options

Implied Forward Rates

- The returns over future time periods that can be locked in by buying and selling bonds today are called **implied forward rates**.
  - These forward rates are implicit in the spot yield curve.
  - They are also the key to **pricing** forward, future, and swap contracts through “no-arbitrage” conditions.
  - Implied forward rates can be **informative about the market’s consensus forecast** of future short-term interest rates, but they are a very noisy indicator.

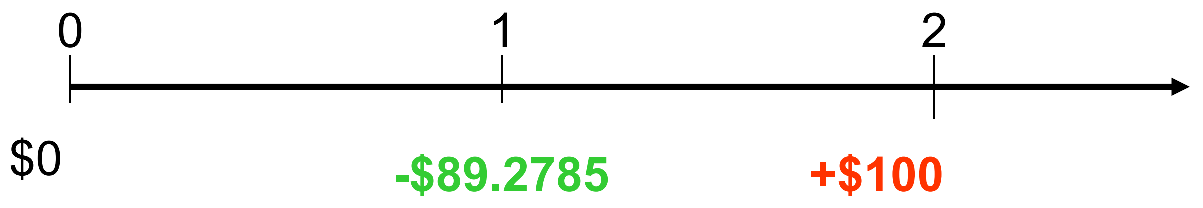
Example 3-4.a: The idea of an implied forward rate is easiest to see in a 2-period example

- Say we know that
  - the one period effective spot yield, $Y_1 = 10\%$
  - the two period effective spot yield $Y_2 = 11\%$
- **Definition**: the implied one period forward rate starting in one period, "f", solves: $(1 + Y_1)(1 + f) = (1 + Y_2)^2$
  - In this example, $(1.1)(1 + f) = (1.11)^2$ which implies **f = 12%**
- *Interpretation:*
  - The 1-period **implied forward rate** can be described as a breakeven rate:
    - It is the rate an investor would have to obtain on a 1-period zero purchased next period such that the return over 2 periods is the same from (1) buying and holding a 2-period zero, and (2) rolling over 1-period zeros.

<u>Example 3-4.b</u>: Locking in the implied 1-period forward rate by trading in the spot market

- Say we know that:
  - 1-period effective spot yield, $Y_1 = 10\%$
  - 2-period effective spot yield $Y_2 = 11\%$

- Consider the following investment strategy:
  - Buy a 2-period zero coupon bond with F=$100
  - $P = \$100/(1.11)^2 = \$81.1622$
  - At the same time, sell (i.e., short) a 1-period zero with an equal price of $81.1622
  - $F = \$81.1622(1.10) = \$89.2785$
  - Cash flows locked in:



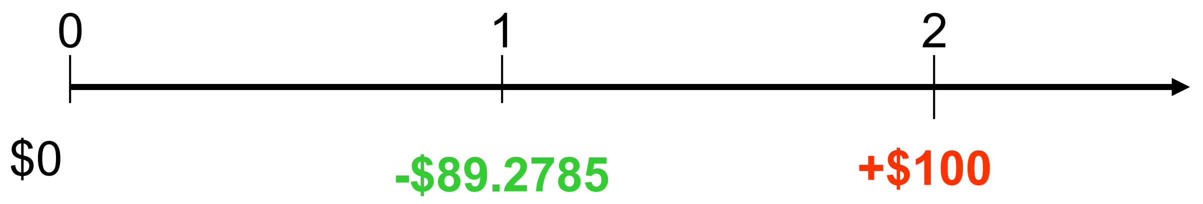
Example 3-4.b: continued

- Forward return locked in is:

$$\frac{100-89.2785}{89.2785}=.12$$

This is the forward rate in the yield curve, 12%! *(Note in terms of discount functions this can be written as Z(0,1)/Z(0,2) -1)*

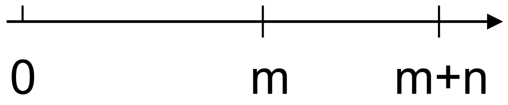
General lesson: implied forward rates can be locked in by taking equal value long/short positions



General relation between spot yields and forward rates
(on a per period basis)

- The forward rate today for a cash flow delivered in *m* periods and repaid in *m+n* periods, $f_1(0,m,m+n)$, can also be represented as a direct function of current spot yields
  - For simplicity, here m and n are numbers of periods
  - The subscript "1" in f_1 refers to having one compounding period per period

$$\left(1+f_1(0,m,m+n)\right)=\left[\frac{(1+Y_{m+n})^{m+n}}{(1+Y_m)^m}\right]^{1/n}$$



- **Cautions and additional notation:**
  - The only numbers you can plug into this formula are **effective rates per period**. For instance, for rates quoted on a b.e.b. the APR must be divided by 2 to get a 6-month effective rate, and then can be used for calculations based on 6-month periods.
  - Usually we will assume that the spot and forward rates are as of time 0 (i.e., based on the current yield curve).
  - Any reference to future spot rates requires additional notation. I will denote a 2-period spot rate starting at period 5 as *Y(5,2)*.

Please see Appendix A and B for the most general case, and for forward rates on a continuous basis.

<u>Example 3-5</u>. Find the implied 3-period forward rate starting in 2 periods, given the following information on effective spot yields:

$Y_1 = 2.0\%$
$Y_2 = 2.6\%$
$Y_3 = 3.0\%$
$Y_4 = 3.2\%$
$Y_5 = 3.4\%$

The implied forward rate solves:

$(1+Y_2)^2(1+f_1(0,2,5))^3 = (1+Y_5)^5$

$$(1+f_1(0,2,5)) = \left[\frac{(1+Y_5)^5}{(1+Y_2)^2}\right]^{1/3}$$

$= 3.93\%$

<u>Practice Problem 3.2</u>: Write down at least two other equations in forward rates that are equivalent to $(1+f_1(0,2,5))^3$.

Constructing spot yield curves from forward rates

It is also possible to construct a yield curve from a given a set of forward rates.
You can think of forward rates as the fundamental building blocks of longer-term spot yields:

$$(1+Y_n)^n = (1+f_1(0,0,1))(1+f_1(0,1,2))(1+f_1(0,2,3))\ldots(1+f_1(0,n-1,n))$$

This highlights the fact that long yields are geometric averages of short-term implied forward rates.

Practice Problem 3.3: Given the following set of one year forward rates, find the five year spot yield curve and plot the results.

$f_1(0,0,1) = 5.2$
$f_1(0,1,2) = 5.6$
$f_1(0,2,3) = 5.8$
$f_1(0,3,4) = 5.4$
$f_1(0,4,5) = 5.0$

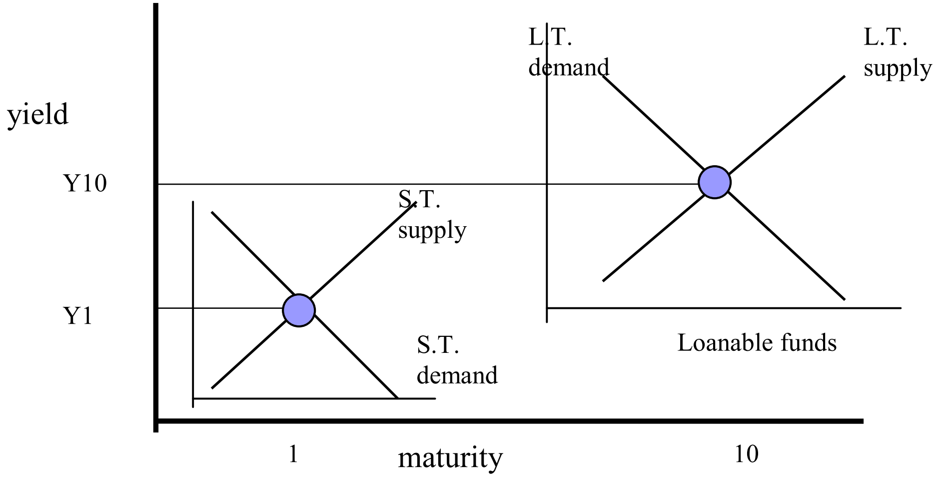
How does the slope of the yield curve change when forward rates increase? How does it change when they decrease? Why?

Interpreting the Yield Curve

- Theories of the yield curve help to explain:
  - The shape of the yield curve at a point in time
  - How the yield curve will change over time
  - What one can infer about future rates and economic conditions from the yield curve
- **Traditional Theories**
  - Market Segmentation / Preferred Habitat
  - Unbiased Expectations Hypothesis
  - Liquidity Preference
- There is also a (typically small) **convexity effect**
  - The volatility of interest rates pulls down on the long end of the curve
  - We will look into the math behind this fact in a few weeks

The Market Segmentation Theory

- Some investors/borrowers like long maturities (e.g., life insurers and pension funds)
- Others like short maturities (e.g., banks)
- The forces of supply and demand operate independently in these two essentially separate markets for loanable funds.



- The supply of loanable funds increases in the interest rate
- The demand for loanable funds decreases in the interest rate
- The interest rate at every maturity equates the supply and demand of loanable funds at that maturity.

Preferred Habitat Theory

- A milder version of the market segmentation theory
- It states that investors have preferred maturities in which to invest, but if expected return differentials become large across maturities they will switch habitats.

Is there support for segmented markets?

- The evidence is mixed…
- 30-year bond prices seemed to increase when that maturity was phased out in Oct. 2001
  - Explained as a scarcity effect
  - Reintroduced in 2006
- But other historical episodes suggest the effect is small (see box)
- These effects are probably sometimes relevant, particularly over short horizons

> Operation Twist: In 1971 Congress removed an interest rate ceiling on Treasury bonds.
>
> <u>Pre-1971</u> 4.5% interest rate ceiling on bonds.
> <u>Post-1971</u> ceiling lifted.
>
> Following the deregulation, the government rushed to issue new long-term bonds.
>
> What does the theory predict about the relative change in rates at different maturities?
>
> What happened? Not much…
>
> More recently, there has been much debate over how much "quantitative easing" by the Federal Reserve has affected the slope of the term structure.

The Unbiased Expectations Hypothesis

- **Expectations Hypothesis:** Investors choose a bond portfolio with the highest expected return over their target holding period (all else equal).
- Therefore, due to the forces of supply and demand, the expected rate of return on any bond over a given holding period is the same, regardless of maturity (all else equal).
- **Example 3-6**: Consider an investor with a 2-year horizon. The yield curve is steeply upward-sloping.

  If the expectations hypothesis is true, which strategy has the highest expected return?

  - Buy \$1mm of a one-year bond and roll it over into another one-year bond;
  - Buy \$1mm of a two-year bond;
  - Buy \$1mm of a five-year bond and sell it after two years.

A key implication of the expectations hypothesis is the interpretation of implied forward rates:

- **Under the expectations hypothesis, the forward rates in the term structure are equal to the market's expectation of future spot rates over the periods covered by the forward rates.**
  - It follows that long-term yields are geometric averages of current and expected future short-term yields.
- The unbiased expectations theory relates current forward interest rates with expected future spot rates with the simple equation:
- $$f_1(0,m,m+n) = \mathrm{E}(Y(m,m+n))$$
  - $f_1(0,m,m+n)$ is the forward rate implied by the yield curve today for an n-period loan beginning in period m, as of time 0, on an effective per period basis
  - $Y(m,m+n)$ is the future spot rate for an n period loan beginning at period m, on an effective per period basis
  - E(Y) denotes the market's expectation of Y.

An application of the expectations hypothesis: "Arbitrage-free" Total Returns

Idea: Use the predictions of future rates embedded in the yield curve to estimate total returns.

<u>**Example 3-7**:</u> Computing Arbitrage Free Total Returns

Invest in a 3 year bond with a 7% coupon (semiannual payments) selling at $960.33 (per $1,000 face) to yield 8.53% (b.e.b.). You plan to sell the bond in two years. ***What is the expected arbitrage-free total return?***

t	Spot Yield Curve (effective 6-mo rates)	One Period Forward Rate at t-1
(1 period = 6 mos; all rates are on effective 6 mo. basis)		
1	3.25%	3.25%
2	3.50%	3.75%
3	3.70%	4.10%
4	4.00%	4.91%
5	4.20%	5.00%
6	4.30%	4.80%

Example 3.7 continued: "Arbitrage-free" Total Returns

1. Find the expected sales price in two years:

$$P_{year2} = \frac{35}{(1.05)} + \frac{1035}{(1.05)(1.048)} = 973.90$$

so your expected capital gain is:

$973.90 - $960.33 = $13.57

2. Find accumulated interest:

$35(1.0375)(1.0410)(1.0491) + $35(1.0410)(1.0491) + $35(1.0491) + $35 =
$149.60

3. Total expected dollar return =
$13.57 + $149.60 = $163.17

Average percentage return (on a b.e.b.) =

$$\left[\left[\frac{960.33 + 163.17}{960.33}\right]^{1/4} - 1\right] 2 = 8.00\%$$

Much less than the 8.53% yield to maturity! *Why?*

The expectations hypothesis and predicted business cycle conditions

- **Observation:** *The yield curve tends to slope up at the start of an expansion, and to slope down at the end of an expansion.*

- **Demand Side Story**
  - The demand for business investment is high during expansions. High expected demand for money implies high real interest rates.
  - If the economy is expected to slow, expected future rates fall since investment demand is expected to slacken.

- **Supply Side Story**
  - People like to smooth their consumption.
  - Therefore if they anticipate a recession they will want to save more, pushing down rates.

- *Also consistent with a nominal story: Inflation expected to be higher in expansions and lower in recessions.*

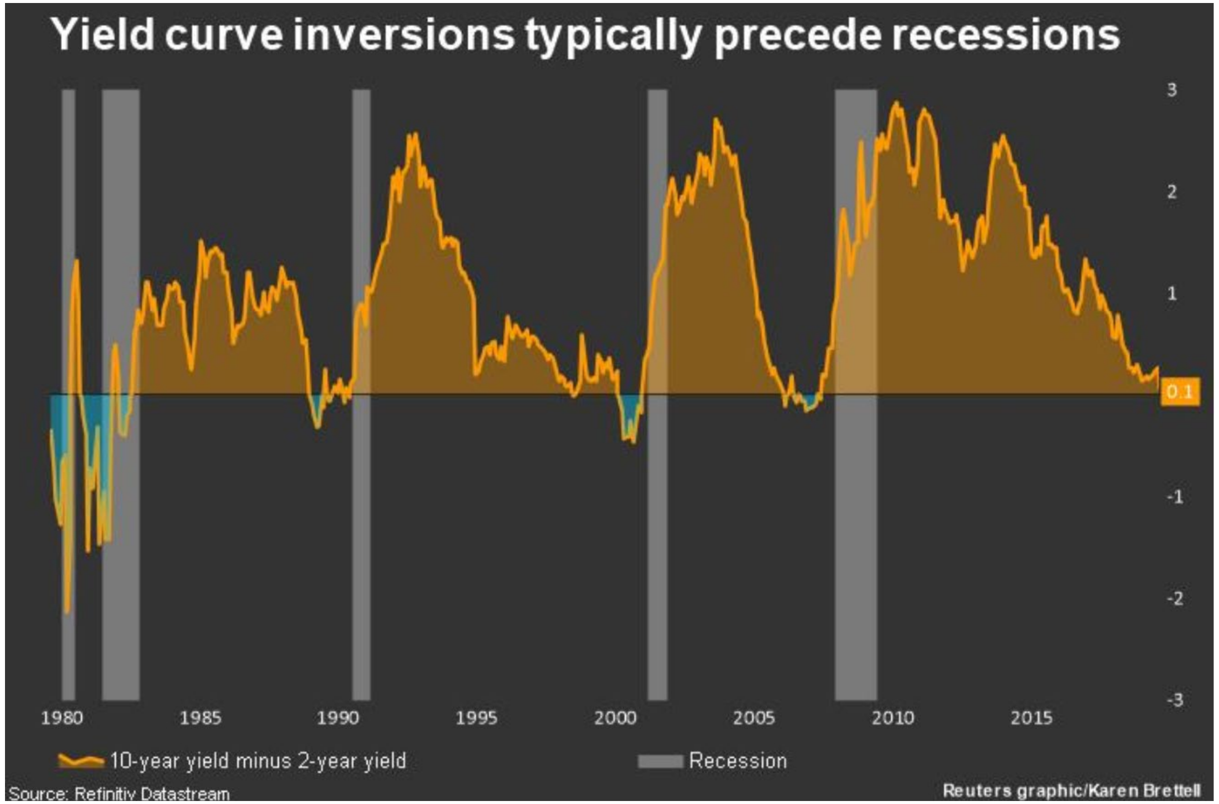
BUSINESS NEWS JANUARY 28, 2020 / 1:27 PM / 20 DAYS AGO

Countdown to recession: What an inverted yield curve means

Karen Brettell

5 MIN READ

NEW YORK - A dramatic rally in Treasuries this week led some key parts of the U.S. yield curve to reinvert, a signal that has traditionally been bearish for the U.S. economy.



Evidence on the expectations hypothesis and predicted business cycle conditions

- Based on *"The Term Structure and World Economic Growth" by* Campbell R. Harvey

- **Main finding:** The spread between long and short rates is a remarkably good predictor of GNP growth rates in many countries.

- Ran regression:
- $\ln(GNP_{t+5}) - \ln(GNP_t) = a + b(TS)_t + u_{t+5}$
  - TS = spread between 90 day bill and bond with maturity at least five years; data is quarterly.

- In the U.S. and Canada, this regression "explained" almost 50% of the growth in GNP.

Shortcomings of the expectations hypothesis

- **Theoretically, it requires several strong assumptions that do not hold in practice**:
  - Investors maximize expected returns, with no consideration of risk
  - Investors view securities with different maturities as perfect substitutes for one another
    - E.g., liquidity is not a concern
  - There are no transactions costs
  - Expectations are held with absolute certainty.
    - It can be shown that if there is uncertainty about rates, then mathematically the theory must be false due to Jensen's inequality. This effect need not be large.
    - Short-term expectations hypothesis recognizes the problem by assuming that only the instantaneous return over the next instant is equated across securities
- **More disturbingly, it appears to be seriously violated in historical data.**
- **Still, most experts agree that it is helpful in interpreting the shape of the yield curve.**

The Liquidity Preference Theory

- The liquidity preference theory states that **investors require a premium for investing in longer-term debt**. The required premium is called a ***"liquidity premium"*** or “***term premium***” or “***risk premium***”

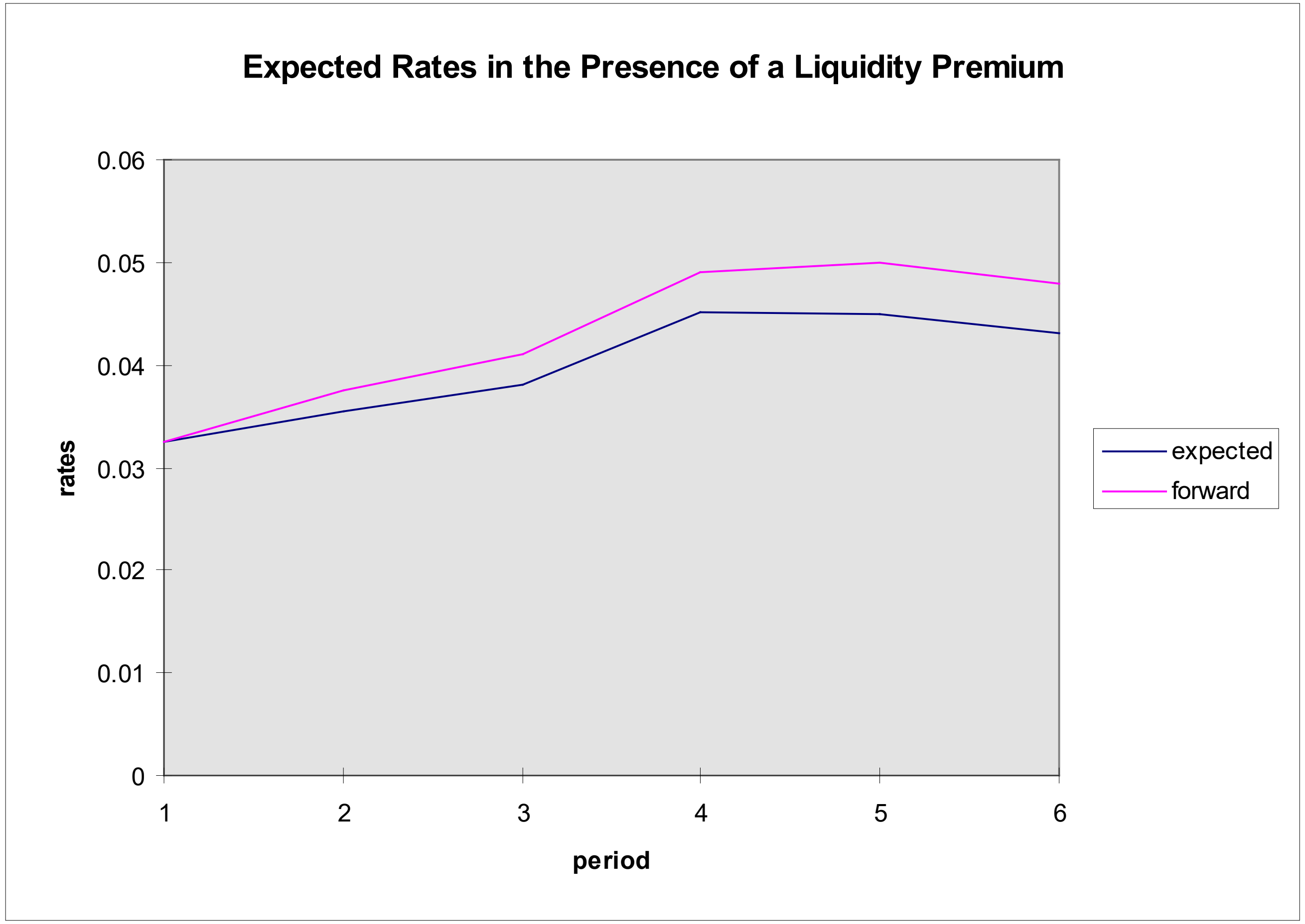
- This suggests modifying our interpretation of implied forward rates:

$$\mathbf{f_1(0,m,m+n) = E[Y(m,m+n)] + L_1(0,m,m+n)}$$

  - $\mathbf{f_1(0,m,m+n)}$ is the forward rate for an n period loan beginning in period m (as of time 0) on an effective per period basis;
  - $\mathbf{L_1(0,m,m+n)}$ is the liquidity premium on an n period loan beginning in period m (as of time 0), on an effective per period basis;
  - **E[Y(m,m+n)]** is the expected future spot rate (or yield) for an n period loan beginning in period m (as of time 0) on an effective per period basis.

- ***Interpreting forward rates as the sum of the expected future spot rate and a liquidity premium is called the “biased expectations hypothesis.”***

Figure 3.1: Interpreting the yield curve in the presence of a liquidity premium.



How to read this graph: The x-axis shows the number of periods spanned by a particular forward rate. All forward rates are for periods starting at a fixed future time t. The rates are implied by the current spot yield curve.

Evidence on the Liquidity Premium

- Forward rates tend to be higher than estimates of expected spot rates, supporting the existence of a liquidity premium.
  - On average the yield curve is upward sloping, even though on average interest rates don't increase over time.
- The measured premium is thought to increase with maturity over short maturities, and level off for long maturities.
- Attempts to estimate liquidity premiums suggest that they vary significantly over time.
- Intrinsically difficult to estimate since it requires identifying expectations of future rates.
- Statistical analyses suggest that the size of the premium ranges from a few basis points to over 1%.

Synthesis of Traditional Theories

- The "**modified expectations hypothesis**" provides a coherent story to explain the yield curve:
  - It says implied forward rates have three main components.
    - The first represents **expectations of future short-term rates**, as in the unbiased expectations hypothesis.
    - In longer rates, a second component is a **liquidity premium**.
    - In long-term rates there is also a **convexity effect** pulling down on the yield curve (we'll cover this later on).
  - The presence of a liquidity premium means that implied forward rates generally will be higher than expected future spot rates.
  - Also, at times, shocks to supply or demand conditions for different maturities may affect the slope through a market segmentation effect. This is usually a transient effect.

Considerations for forecasting interest rates

- Choice between using only interest rate data, or also using macro variables
  - Interest rate only models use current and lagged rates, and possibly term structure
  - Macro models also include variables such as inflation, unemployment, exchange rates and deficits
  - Example of Deutsche Bank model in case (exhibit 3)
- Real or nominal rates?
- How to forecast expected inflation?
- Challenges for statistical models
  - Regime shifts in central bank policies
  - Zero (or slightly negative) lower bound on nominal rates
  - Data is backward looking, rates are forward looking
  - None work too well

Is the Deutsche Bank model a pure interest rate or a macro model?

Exhibit 3 Deutsche Bank's Zero-Coupon Yield Model

- **Key variables:** Short rate, slope, and long rate (or short rate, output gap, and inflation)
- **Model specified by a system of equations (in Q measure)**
  - Long rate mean reverts slowly (possibly to nonzero mean)

    $$dX_t = (\mu_X - k_X X_t)dt + \sigma_X dW_t^X$$

  - Slope mean reverts faster (to zero)

    $$dY_t = -k_Y Y_t dt + \sigma_Y dW_t^Y$$

  - In equilibrium short rate, r_t, follows the target X_t+Y_t (an analogue of the Taylor rule)

    $$X_t + Y_t - r_t = 0$$

  - Short rate mean reverts fast in order to restore the equilibrium

    $$dr_t = k_r(X_t + Y_t - r_t)dt + \sigma_r dW_t^r$$

Source: Adapted by casewriter from "Quantitative Models for Fixed Income," Deutsche Bank presentation, October 2003.

Expected Inflation

- $(1+r_{nominal}) = (1+r_{real})(1+E(\pi))$
- $E(\pi) \approx r_{nominal} - r_{real}$

- How to infer expected inflation?
  - Statistical models
    - e.g., regress realized inflation on macro and market variables and use fitted model calibrated to expected macro/market conditions
  - Ask people (survey data)
  - Look at TIPS vs. nominal Treasuries

What can we learn about inflation from TIPS?

- Treasury Inflation Protected Securities (TIPS)
  - Principal is indexed to CPI (slightly lagged)
  - Quoted rate is a real rate
  - About 10% of Treasury issues are TIPS
  - Performed phenomenally well for investors for many years following introduction in the late 1990s
  - Issuance motivated by helping people protect against inflation, and allowing Fed to measure inflation expectations directly

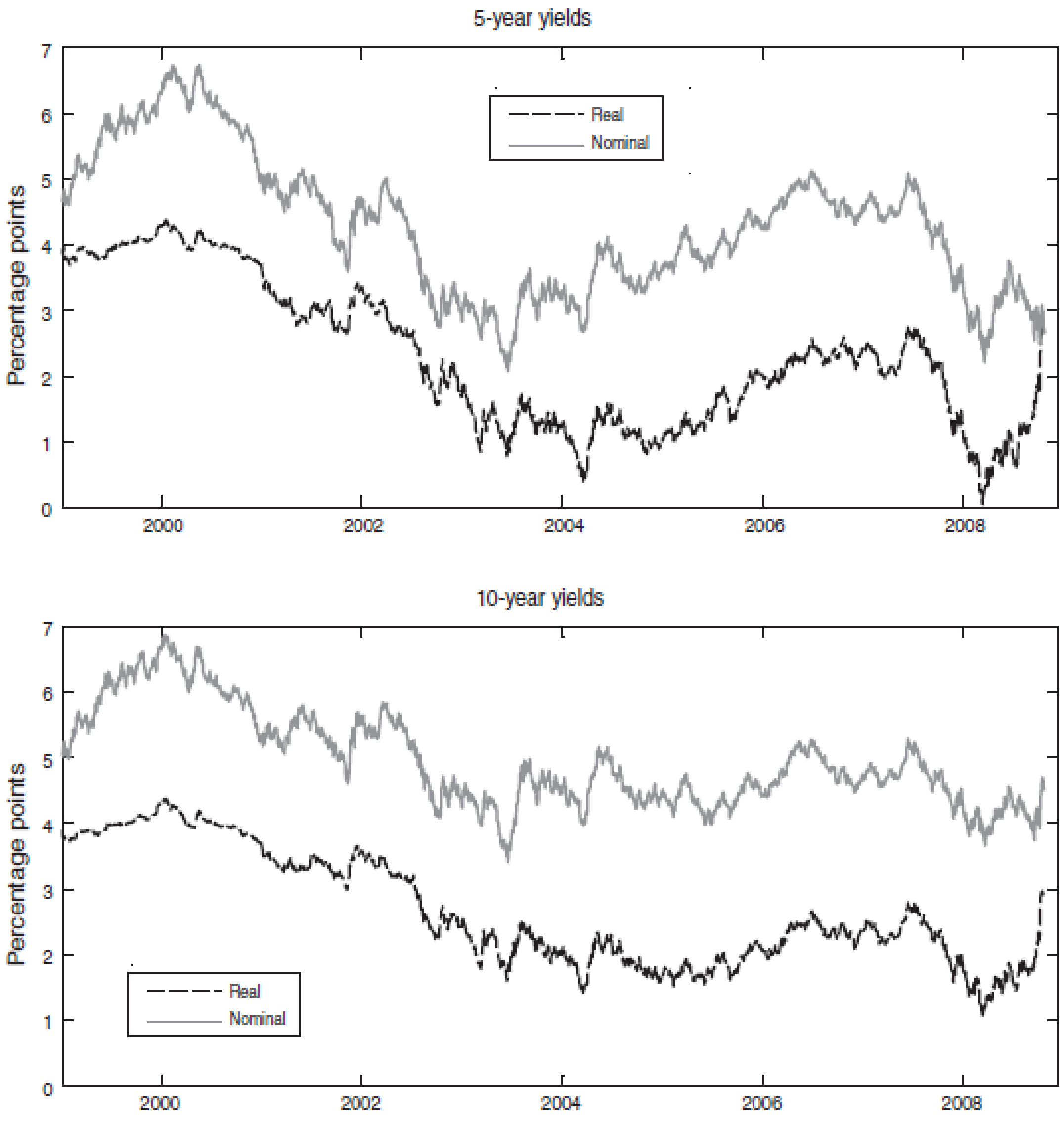


Figure 5. Zero-Coupon Yields: TIPS and Nominal

Source: Gurkaynak, Sack and Wright, "The TIPS Yield Curve and Inflation Compensation"

Volatile gap between TIPS and survey data attributed to:
1) Liquidity premium
2) Inflation risk premium

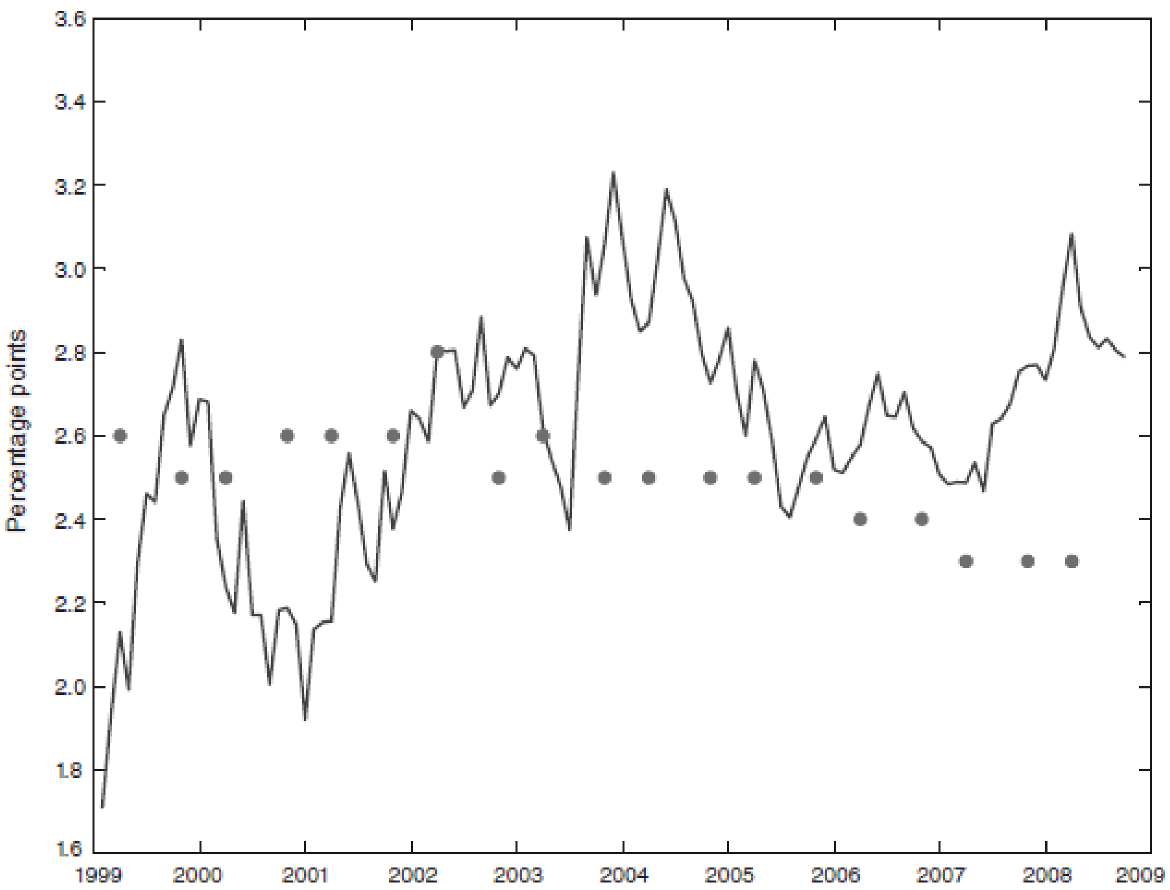
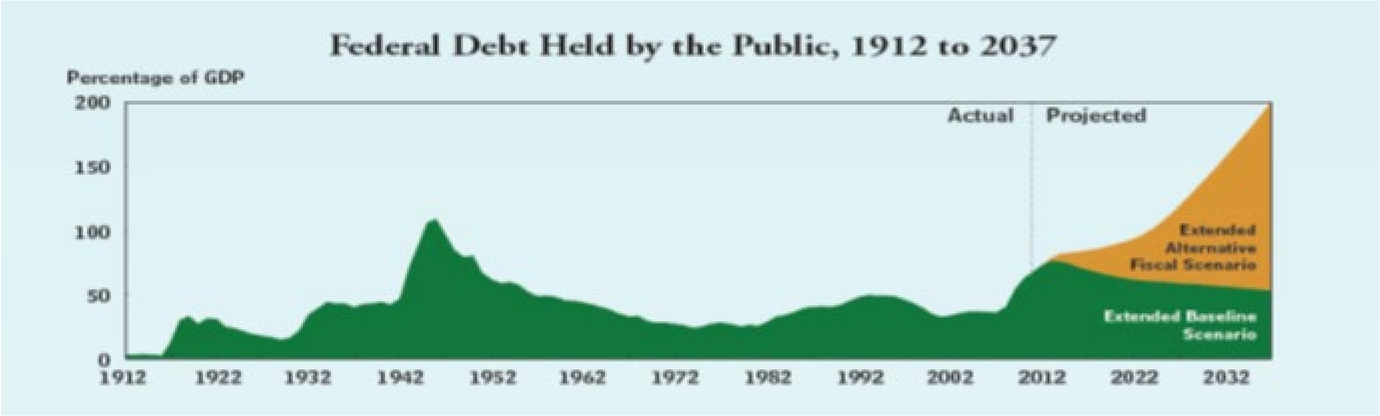


Figure 7. Five-to-Ten Year Forward Inflation Compensation and Blue Chip Forecasts

Notes: The solid line gives forward par inflation compensation. The dots are the Blue Chip survey inflation expectations.

Source: Gurkaynak, Sack and Wright, "The TIPS Yield Curve and Inflation Compensation"

Can the yield curve and fiscal policy be reconciled?



The puzzle:

- Current fiscal policy is unsustainable in the U.S. and many other places
  - high projected spending, insufficient taxes to cover spending
- Debt projected to grow to unprecedented levels
- Why doesn't this increase long-term interest rates?

Quote from Peter Fisher (formerly of BlackRock):
"Do not wait for the bond market to help us out; that is not its job."

Linking International Yield Curves with Covered Interest Rate Parity

- S = Spot Exchange Rate (domestic currency/foreign currency)
- F = Forward Exchange Rate (domestic currency/foreign currency)
- R = domestic interest rate
- R^*= comparable foreign interest rate

- Assume the domestic currency is the US dollar.
- If you invest $1 in the US market you will have (1+R) dollars at year end.
- What would be the return in dollars from investing in a foreign country? *(assume both investments have no risk)*

Deriving Covered Interest Parity

Investing Abroad

- Step 1:

  Convert the dollar into foreign currency using the spot market. You will obtain 1/S units of foreign currency.

- Step 2:

  Invest the resulting foreign currency in the foreign market. You will have $(1+R^*)/S$ at year end.

- Step 3:

  Arrange to convert the foreign currency you will receive in the end of the investment period into dollars at the forward exchange rate, F. You will have $F(1+R^*)/S$.

Covered Interest Parity – The Definition

Investing in the U.S. and abroad has to yield the same return, otherwise there is an arbitrage opportunity that can be exploited.

$$(1+R) = F(1+R^{*})/S \qquad \text{(eq. 3.1)}$$

Example 3.8: Looking for international arbitrage opportunities

Imagine the following contracts are available (U.S. Dollar vs. Foreign Currency):

- 180 - day Commercial Paper (USD): 8% per annum (simple, 360 day basis)
- 180 - day Commercial Paper (FC): 12.0% per annum (simple, 360 day basis)
- Spot Exchange Rate: FC1.6360/$1
- 180-day Forward Exchange Rate: FC1.6741/$1

Example 3.8: (continued)

- *One option is just to invest in the U.S.*

- The dollar interest rate for a 180 days period is:

$$0.08 \times (180/360) = 4\%$$

- Investing \$10,000 in the U.S. yields:

$$\$10{,}000 \times 1.04 = \mathbf{\$10{,}400}$$

- The FC rate is

$$0.12 \times (180/360) = 6\%$$

<u>Example 3.8</u>: (continued)

- An alternative to get a dollar return is to invest in the foreign currency and at the same time enter into a forward contract locking in the forward exchange rate to convert back to dollars:
- Step 1: buy FC spot

  $10,000xFC1.6360/$ = 16,360.FC
- Step 2: invest the FC in its home market

  FC16,360x(1.06) = FC17,341.6
- Step 3: Convert into USD at forward rate

  FC17,341.6/(FC1.6741/$) = **$10,358.8**

- *It is better to invest in the U.S. This means one could construct an arbitrage opportunity by borrowing abroad (i.e., taking a short position in the foreign currency CD) and investing in the U.S.*

Implications for Yield Curves

- International yield curves are linked by the covered interest rate parity condition for convertible currencies.
- In some countries there is only short-term gov’t debt, or gov’t debt is illiquid.
- There may be a more active forward currency market or currency swap market. If so, one can find the “implied yield curve” from the currency forward rates and U.S. dollar rates using equation 4.1.

Violations of covered interest rate parity

- Historically the relation held tightly in the data
- Since 2008 there have been significant and persistent violations
  - See “Deviations from Covered Interest Rate Parity” by Wenxin Du, Alexander Tepper, and Adrien Verdelhan

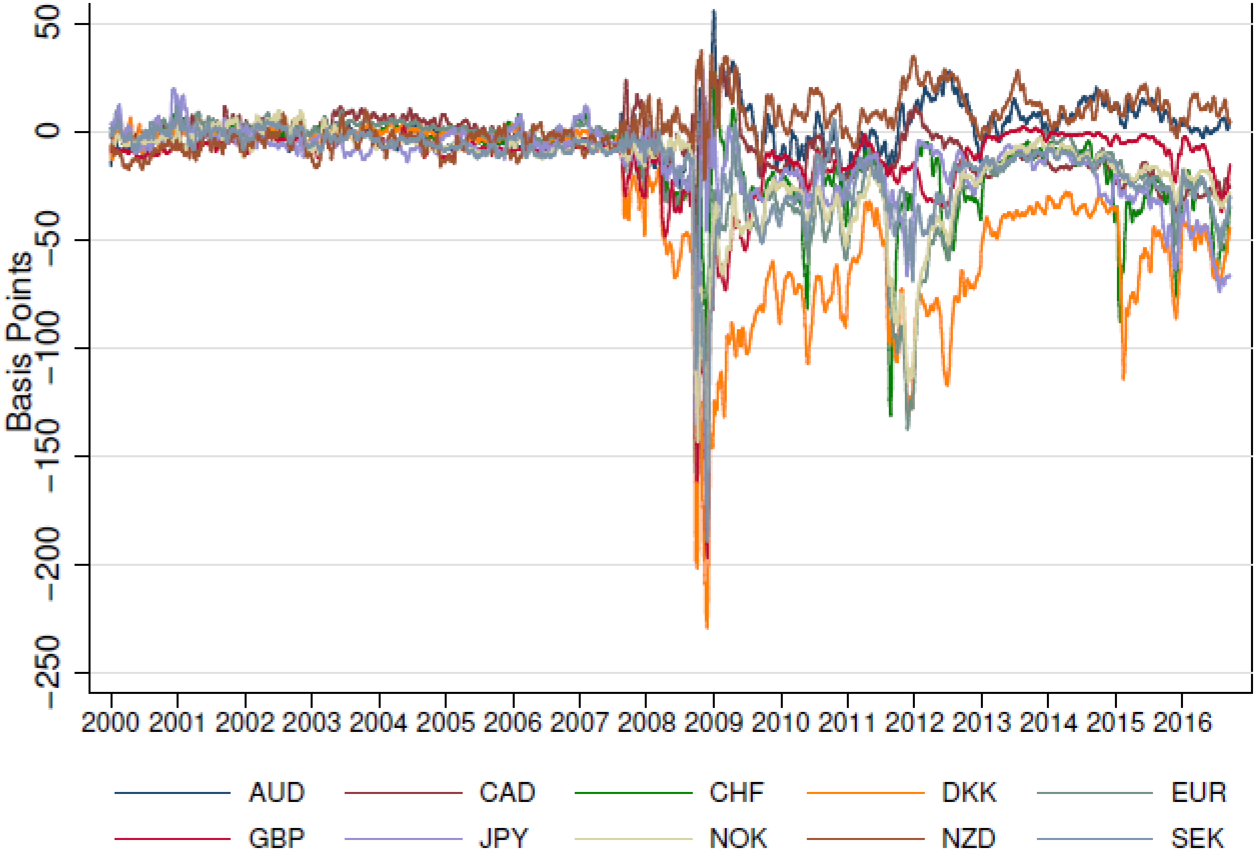


Figure 2: **Short-Term Libor-Based Deviations from Covered Interest Rate Parity:** This figure plots the 10-day moving averages of the three-month Libor cross-currency basis, measured in basis points, for G10 currencies. The covered interest rate parity implies that the basis should be zero. One-hundred basis points equal one percent. The Libor basis is

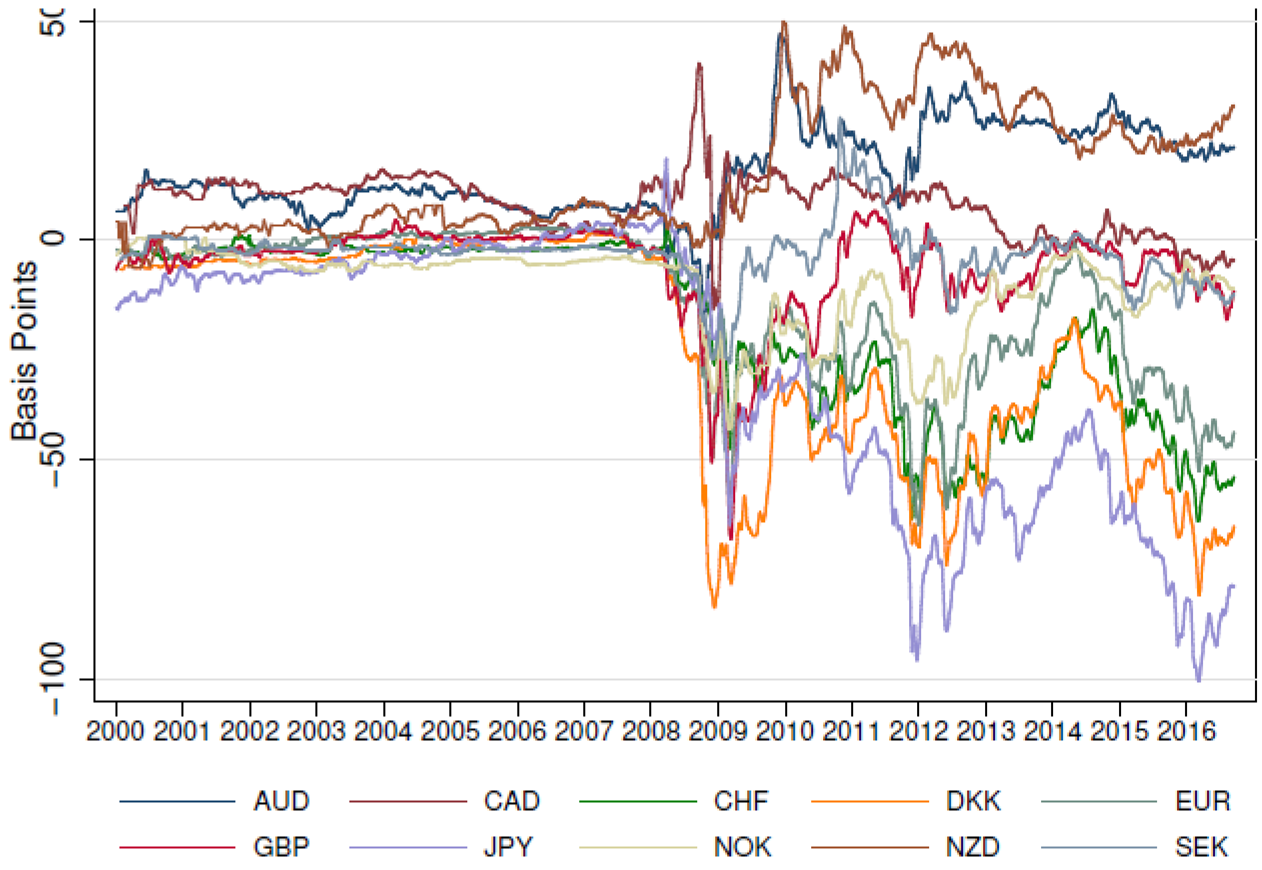


Figure 5: **Long-Term Libor-Based Deviations from Covered Interest Rate Parity**: This figure plots the 10-day moving averages of the five-year Libor cross-currency basis, measured in basis points, for G10 currencies. The covered interest rate parity implies that the basis should be zero. One-hundred basis points equal one percent.

Uncovered Interest Rate Parity

- **Covered interest rate parity** says that the risk-free returns you can *lock in* today must be the same everywhere.

$$(1+R_t) = F_0(1+R_t^*)/S_0$$

- **Uncovered interest rate parity** says that investors *expect* to realize the same real return whatever currency they invest in.

$$(1+R_t) = E(S_t)(1+R_t^*)/S_0$$

  - Here currency risk is not hedged with a forward contract

- ***Uncovered interest rate parity can be thought of as the expectations hypothesis applied to international returns.***
  - Like the expectations hypothesis, it appears to be often violated in the data. The violation is usually attributed to a risk premium.

Appendix A:
More on generalized representations of forward rates

- Implied forward rates can be found for any future time period. This can be expressed in terms of discount functions, and implies some additional notation:
  - *$F(t,T_1,T_2) = Z(t,T_2)/Z(t,T_1)$* gives the **forward discount factor**.
  - **Used to find time T_1 value of \$1 delivered at time T_2, as of time t.**

$$F(t,T_1,T_2) \qquad \$1$$
$$t \qquad\qquad T_1 \qquad T_2$$

- Going from discount factors to **forward rates** (stated here as APRs):
- $F(t,T_1 T_2) = \frac{1}{\left(1+\frac{f_n(t,T_1 T_2)}{n}\right)^{n\times(T_2-T_1)}}$ or $\mathbf{1+f_n(t,T_1\,T_2)} = \left(\frac{1}{F(t,T_1,T_2)}\right)^{\left(\frac{1}{n\times(T_2-T_1)}\right)}$
- Forward rates are denoted by a small "*f*" and forward discount factors by a capital "*F*."
- In this notation, time is in years. There are an assumed "n" compounding periods per year. The rates are APRs.

Appendix B: Relation between spot yields and forward rates on a continuous basis

Many advanced treatments of fixed income mathematics represent all rates assuming continuous compounding.

Recall that the relationship between the t-year yield on a continuous basis, r_t, and the t-year spot yield on an effective annual basis, y_t, is given by:

$$1/Z(0,t) = (1+y_t)^t = e^{r_t t}$$

Under this representation, the implied forward rate (as of time 0) on a continuous basis between time T_1 and time T_2, satisfies:

$$e^{r_{T_1} T_1} e^{f(0,T_1,T_2)(T_2-T_1)} = e^{r_{T_2} T_2}$$

Rearranging and taking the natural logs yields the simple formula:

$$f(0,T_1,T_2) = \frac{r_{T_2} T_2 - r_{T_1} T_1}{T_2 - T_1}$$