OBVP作业推导

- 针对末状态部分自由的线性双积分系统求解最优控制量及状态轨线
- 由于原题中机器人可独立分析三个轴向的运动，此处仅分析单个轴向

优化模型

$$minJ = \frac{1}{T}\int_0^T j(t)^2 dt$$

$$s.t. \frac{ds}{dt} = As + Bu$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$s = (p, v, a)^T, s(0) = (p_0, v_0, a_0)^T, s_1(T) = p_T$$

推导

构造Hamilton方程:

$$H(s, u, \lambda) = \frac{1}{T}j^2 + \lambda^T\frac{ds}{dt} = \frac{1}{T}j^2 + \lambda_1 v + \lambda_2 a + \lambda_3 j$$

由正则方程:

$$\dot{\lambda} = -\frac{\partial H}{\partial s} = (0, -\lambda_1, -\lambda_2)^T$$

$$\dot{s} = f(s, u) = As + Bu$$

解得

$$\lambda(t) = \frac{1}{T}\begin{bmatrix} -2\alpha \\ 2\alpha t + 2\beta \\ -\alpha t^2 - 2\beta - 2\gamma \end{bmatrix}$$

由横截条件:

$$s_1(T) = p_T$$

$$\lambda_2(T) = \lambda_3(T) = 0$$

得约束条件

$$2\alpha t + 2\beta = 0$$

$$-\alpha t^2 - 2\beta - 2\gamma = 0$$

由庞特里亚金极小值原理

$$u^*(t)=\underset{j(t)}{\operatorname{argmax}}\, H(s,u,\lambda)=\frac{1}{2}\alpha t^2+\beta t+\gamma$$

代入状态方程

$$s^*(t)=\begin{bmatrix}\frac{\alpha}{120}t^5+\frac{\beta}{24}t^4+\frac{\gamma}{6}t^3+\frac{a_0}{2}t^2+v_0t+p_0\\ \frac{\alpha}{24}t^4+\frac{\beta}{6}t^3+\frac{\gamma}{2}t^2+a_0t+v_0\\ \frac{\alpha}{6}t^3+\frac{\beta}{2}t^2+\gamma t+a_0\end{bmatrix}$$

结合末端位置约束条件

$$\begin{bmatrix}\frac{1}{120}T^5 & \frac{1}{24}T^4 & \frac{1}{6}T^3\\ 2T & 2 & 0\\ T^2 & 2T & 2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=\begin{bmatrix}\Delta p\\ 0\\ 0\end{bmatrix}$$

$$\Delta p=p_f-p_0-v_0T-\frac{1}{2}a_0T^2$$

解得

$$\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=\begin{bmatrix}\frac{20\Delta p}{T^5}\\ -\frac{20\Delta p}{T^4}\\ \frac{10\Delta p}{T^3}\end{bmatrix}$$

则最优控制输入为

$$u^*(t)=-\frac{20\Delta p}{T^4}t+\frac{10\Delta p}{T^3}$$

最优状态轨线为

$$s^*(t)=\begin{bmatrix}\frac{\Delta p}{6T^5}t^5-\frac{5\Delta p}{6T^4}t^4+\frac{5\Delta p}{3T^3}t^3+\frac{a_0}{2}t^2+v_0t+p_0\\ \frac{5\Delta p}{6T^5}t^4-\frac{10\Delta p}{3T^4}t^3+\frac{5\Delta p}{T^3}t^2+a_0t+v_0\\ \frac{10\Delta p}{3T^5}t^3-\frac{10\Delta p}{T^4}t^2+\frac{10\Delta p}{T^3}t+a_0\end{bmatrix}$$