Edward Ryan

18th May, 2022

Topics in Theoretical Physics

Question 1: Soap film Bubble

- General Form
    - $y = C_1\, cosh\frac{x-C_2}{C_1}$

Part a.) Investigate the Goldschmidt discontinuous solution (when the soap film can no longer stretch between the rings and breaks) for the boundary conditions of $(x, y) = (\pm x_0, 1)$.

In elementary calculus, the derivative of any function at the minimum and maximum value will give a zero. In calculus variation the function at the minimum and maximum will be zero as shown in the following function.

$$dS = \mathrm{S}(r + dr) - S(r) = 0$$

The quantity dr represents a small variation within the function of r in order to satisfy the boundary conditions of

$$dr(-L) = dr(L) = 0$$

We should note that $dS\ =\ 0$ does not mean that the extremum is a minimum. If we take the surface as

$$S = \int 2\pi\, dS = 2\pi \int_{-L}^{+L} \mathrm{r}\sqrt{1 + r'^2}\mathrm{dz}$$

We can compute dS when setting it equal to 0

$$dS = 2\Pi \int\limits_{-L}^{+L} d\left(r\sqrt{1 + r'^2}\right) \mathrm{dz} = 0$$

This will solve to

$$\frac{d}{d\mathrm{z}}\left(\frac{rr^1}{\sqrt{1+r'^2}}\right) = \sqrt{1 + r'^2} \text{ or } \mathrm{rr''} = 1 + r'^2$$

When solving this we can use basic calculus by using

$$\left(1 + r'^2\right)' = 2r'r''$$

This will produce

$$r = \mathrm{K}\sqrt{1 + r'^2}$$

We will proceed with a hyberbolic trigonometry

$$r' = \frac{dr}{dz} = \frac{\sqrt{r^2 - \mathrm{K}^2}}{\mathrm{K}}$$

To solve this, we will first solve for z which will give us

$$z = \mathrm{K}\, cosh^{-1}\left(\frac{r}{\mathrm{K}}\right) + k$$

Then solve for r

$$r = \mathrm{K}\, cosh\left(\frac{z - \mathrm{k}}{\mathrm{K}}\right)$$

If x = ± x_0 we get

$$k\ cosh\left(\frac{L - k}{\mathrm{K}}\right) = k\ cosh\left(\frac{\text{-}\ L - k}{\mathrm{K}}\right)$$

It should be noted that k and K are different values. Since cosh is an even function, we have $L - k =$ - $L - k$ or $L - k =$ - (- $L - k$). The first case implies that L = -L and the second case implies that k = 0. Thus, our solution to the radius is

$$r(z) = \mathrm{K}\, cosh\left(\frac{z}{\mathrm{K}}\right)$$

This is the general form.

Part b.) Find the boundary conditions for

$$y = a\left[cosh\frac{2x}{\sqrt{3}a} + {}^{1}\!/_{2}\, sin\, h\frac{2x}{\sqrt{3}a}\right]$$

We know that a = 1.

So, the angle formed from the catenary curves is 120 degrees. This means when solving we can find y by finding x. If we use

$$120 = cosh\frac{2x}{\sqrt{3}} + \frac{1}{z} sin\, h\frac{2x}{\sqrt{3}}$$

We know cos(120 degrees) = -.5, sin(120 degrees) = .5 from there we use the

$cosh = \frac{e^{x}+e^{-x}}{2}$ and $sin\, h = \frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}$

So, we get x = 3.5696

This gives us a y of ±46.2575

From here we solve for x.

Question 2: Lagrangian Mechanics

$$\ddot{\theta} + \frac{\mathrm{g}}{\mathrm{l}} sin\, \theta = 0$$

Use Lagrange equation to find the dynamic equation of motion

Part a.) The Rotating pendulum

- Mass = m, length = b which is attached to the edge of a disk with radius = a, angular velocity = ω in a constant gravitational field
- Show that the Lagrange equation for θ is

$$\ddot{\theta} + \frac{g}{b} sin\,\theta - \frac{a}{b}\omega^2\, cos(\theta - \omega t) = 0$$

- We start with the pivot points which are simply $x = acos(\omega t)$ and $y = asin(\omega t)$
- Next, we need the coordinates of the mass $x = x + bsin(\theta)$ and $y = y\text{-}bcos(\theta)$
- This will produce time derivatives for both x and y as

$$\frac{dx}{dt} = -a\omega\, sin(\omega t) + b\, cos(\theta)\frac{d\theta}{dt}$$

$$\frac{dy}{dt} = a\omega \cos(\omega t) + b \sin(\theta)\frac{d\theta}{dt}$$

- We know the Kinetic Energy is $T = \frac{1}{2}mv^2$ which can be written as $T = \frac{1}{2}m((\frac{dx}{dt})^2 + (\frac{dy}{dt})^2)$
- When plugging in the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ along with using the identity $sin^2\,\theta + cos^2\,\theta = 1$ we see that the Kinetic Energy is

$$T = \frac{1}{2}m\left[a^2\omega^2 + b^2\left(\frac{d\theta}{dt}\right)^2 + 2a\omega b\left(\frac{d\theta}{\partial t}\right)(-\cos\theta\sin\omega t + \cos\omega t \sin\theta)\right]$$

- Now using the identity $\sin(a\text{-}b) = \sin a \cos b - \cos a \sin b$

$$T = \frac{1}{2}m\left[a^2\omega^2 + b^2\left(\frac{d\theta}{dt}\right)^2 + 2a\omega b\left(\frac{\partial\theta}{\partial t}\right)\sin(\theta\text{-}\omega t)\right]$$

- Now that we have the Kinetic energy, we need the Potential energy which is defined as

$$V = mgy$$

$$V = mg(asin\,\omega t - b\cos\theta)$$

- Using Lagrange (L = T -V)
- The Equation of motion is

$$\frac{d}{dt}\left(\frac{dL}{d\left({}^{d\theta}/_{\partial t}\right)}\right) - \frac{dL}{d\theta} = 0$$

$$\frac{dL}{d\dot{\theta}} = mb^2\left(\frac{d\theta}{dt}\right) + ma\omega b\, sin(\theta - \omega t)$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}}\right) = mb^2\left(\frac{d^2\theta}{dt^2}\right) + ma\omega b\cos(\theta - \omega t)\left(\frac{\partial\theta}{dt} - \omega\right)$$

$$\frac{dL}{d\theta} = ma\omega b\left(\frac{d\theta}{\partial t}\right) cos(g - \omega t) - mgb\sin(\theta)$$

- Using all of these we will get

$$\left(\frac{d^2\theta}{dt^2}\right) - \frac{a}{b}\omega^2 \cos(\theta - \omega t) + \frac{\mathrm{g}}{b}\sin\theta = 0$$

- This is our desired result

Part b.) Pendulum with oscillating fulcrum

- Mass = m, Length l attached to massless block, spring with constant k
- Show that the Lagrange Equation for x and θ are

$$m\ddot{x} + kx = \mathrm{ml}\left(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta\right)$$

$$\ddot{\theta} + \frac{\mathrm{g}}{\mathrm{l}} \sin\theta = -\frac{\ddot{x}}{1}\cos\theta$$

- We will have

$$x_1 = a + x + \mathrm{l}\sin\theta, \mathrm{y}_1 = \text{-}\mathrm{l}\cos\theta$$

$$\frac{dx_1}{dt} = \dot{x} + \mathrm{l}sin^{\theta}\left(\frac{\mathrm{d}\theta}{dt}\right), \frac{d\mathrm{y}_1}{dt} = \mathrm{l}sin^{\theta}\left(\frac{\mathrm{d}\theta}{dt}\right)$$

- Kinetic energy

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mq^2\left(\frac{\partial\theta}{dt}\right)^2 + \mathrm{ml}\cos\theta\,\dot{x}\left(\frac{\partial\theta}{\partial t}\right)$$

- The Potential energy

$$V = \frac{1}{2}kx^2 + mgy$$

$$V = \frac{1}{2}kx^2 - mg\mathrm{l}\cos\theta$$

- The Lagrange is L = T -V
- Taking the momenta for x and θ

$$\mathrm{p}_x = \frac{\partial L}{\partial \dot{x}}$$

$$\mathrm{p}_\theta = \frac{\partial L}{\partial \dot{\theta}}$$

- This means the force are

$$F_x = \frac{\partial L}{\partial x}$$

$$F_\theta = \frac{\partial L}{\partial \theta}$$

- So, the equations of motion after plugging in all we have found we get the desired results of

$$m\ddot{x} + kx = m\mathrm{l}\left(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta\right)$$

$$\ddot{\theta} + \frac{\mathrm{g}}{\mathrm{l}}\sin\theta = -\frac{\ddot{x}}{1}\cos\theta$$

Question 3: Determine the value of the following definite integral via three routes

$$\int_0^\infty \frac{x^2}{(x^2-4)(x^2+9)}dx$$

1st route, look up answer in tables of integrals, online source or textbook. (https://www.integral-calculator.com/)

- First step is to factor denominator

$$\int_0^\infty \frac{x^2}{(x-2)(x+2)(x^2+9)}dx$$

- Preform partial fraction decomposition

$$\int_0^\infty \left(\frac{9}{13(x^2+9)} - \frac{1}{13(x+2)} + \frac{1}{13(x-2)}\right)dx$$

- Apply linearity

$$\frac{9}{13}\int_0^\infty \frac{1}{x^2+9}dx - \frac{1}{13}\int_0^\infty \frac{1}{x+2}dx + \frac{1}{13}\int_0^\infty \frac{1}{x-2}dx$$

- Now solving the first term

$$\int_0^\infty \frac{1}{x^2+9}dx$$

- Preform u substitute using $u = \frac{x}{3} \Rightarrow \frac{du}{dx} = \frac{1}{3} \Rightarrow dx = 3\,du$

$$\int_0^\infty \frac{3}{9u^2+9}du$$

- Simplify

$$\frac{1}{3}\int_0^\infty \frac{1}{u^2+1}du$$

- Solving $\int_0^\infty \frac{1}{U^2+1}du$ which is equal to $arctan(u)$
- When resubstituting x in for u and plugging in the constant before the function we end up with

$$\frac{3\,arctan\left(\frac{x}{3}\right)}{13}$$

- Now solving $\int_0^\infty \frac{1}{x+2}dx$ and $\int_0^\infty \frac{1}{x-2}dx$
- For the $\int_0^\infty \frac{1}{x+2}dx$ we preform u substitution using $u = x + 2 \Rightarrow du = dx$ and for $\int_0^\infty \frac{1}{x-2}dx$ we preform u substitution using $u = x - 2 \Rightarrow du = dx$
- Which gives us $\int_0^\infty \frac{1}{u}du$ for both equations
- When preforming the integral, we get the result $ln(u)$
- Undoing the u substitution, we get $ln(x + 2)$ and $ln(x - 2)$
- Reintroducing the constant before the function we end up with $\frac{ln(x+2)}{13}$ and $\frac{ln(x-2)}{13}$
- Putting all of this together we get

$$\frac{-\,ln(x+2) + 3\,arctan\left(\frac{x}{3}\right) + ln(x-2)}{13}$$

- When applying Cauchy principal value, we receive

$$\frac{3\pi}{26}$$

2nd route, evaluate analytically, by extending it to the complex plane and solving an appropriate integral involving a complex variable, z = x + iy.

- Convert the function into $\oint f(z)\,dz$ by using $z = x \Rightarrow dz = dx$
- This gives us our function

$$f = \frac{z^2\,dz}{(z^2-4)(z^2+9)}$$

- Since f(z) contains a singularity at a z_0 we can write $f(z) = \frac{g(t)}{z-z_0}$ where g(t) is the analytic function
- We find the singularities for the function to be z = ± 2 and z = ± 3i
- To solve Residues, we need to use the function $R_f(z_0) = \lim_{z\to z_0}(z - z_0)f(z)$
- This gives us

$$R_f(-2) = \lim_{z\to -2}\frac{(z+2)z^2}{(z-2)(z+2)(z^2+9)} = \frac{z^2}{(z-2)(2^2+9)} = \frac{4}{(-4)(13)} = \frac{-1}{13}$$

$$R_f(2) = \lim_{z\to 2}\frac{(z-2)z^2}{(z-2)(z+2)(z^2+9)} = \frac{z^2}{(z+2)(2^2+9)} = \frac{4}{(4)(13)} = \frac{1}{13}$$

$$R_f(-3i) = \lim_{z\to -3i}\frac{(z+3i)z^2}{(z^2-4)(z-3i)(z+3i)} = \frac{z^2}{(z^2-4)(z-3i)} = \frac{-9}{(-3i-2)(-3i+2)(-6i)}$$
$$= \frac{-9}{-13(-6i)} = \frac{-9}{78i} = \frac{3i}{26}$$

$$R_f(3i) = \lim_{z\to 3i}\frac{(z-3i)z^2}{(z^2-4)(z-3i)(z+3i)} = \frac{z^2}{(z^2-4)(z+3i)} = \frac{-9}{(3i-2)(3i+2)(6i)}$$
$$= \frac{-9}{-13(6i)} = \frac{9}{78i} = \frac{-3i}{26}$$

- In order to further solve this integral we need Cauchy Integral Formula which is

$$\oint f(z)\,dz = 2\pi i \sum_{1=1}^{N} R_f^i(z_i)$$

- If assuming that the integral encloses all poles, we get $2\pi i\left[-\frac{1}{13}+\frac{1}{13}+\frac{3i}{26}-\frac{3i}{26}\right]=0$
- However, since the integral has an undefined determinate at z = 2 and is from [0, ∞) the only pole enclosed by the integral is at 3i. This will give us

$$2\pi i\left[-\frac{3i}{26}\right]=\frac{6\pi}{26}$$

But this is twice the value we expect, in order to resolve this issue, we need to realize that the equation $\oint f(z)\,dz = 2\pi i \sum_{1=1}^{N} R_f^i(z_i)$ is when the integral is from (∞, ∞) so the fact that the integral is from [0, ∞) we can dive the equation in half. This will then give us

$$\pi i\left[-\frac{3i}{26}\right]=\frac{3\pi}{26}$$

- This is the exact desired result we expected

3rd route, Explain how to write a computer program (pseudocode) to evaluate it numerically.

sympy.integrate method	Rectangle method
Program starts	Program starts
Import the necessary modules (math, sympy)	Define a function that's purpose is to return the integrated equation
Define a function that's purpose is to return the integrated equation	Next all upper and lower conditions, number of steps as well as 2 variables need to be initialized - The 2 variables will contain the 2 returned values at each step in the integration
Next, you need to initialize the bounds of the integral	Define a variable that is equal to $\left(\frac{upper-lower}{steps}\right)$
Since we have not initialized the variable x within the previous defined function, we need to initialize the Symbol "x"	Start a for loop that will call the function that was defined at the beginning of the code
Preform the integral twice in the intervals of [0, 1.9999] and [2.0001, ∞)	Preform the integral twice in the intervals of [0, 1.9165] and [2.0835, 100,000,000) taking steps 1,000,000 for each integral
Add the two integrations and print the results to the screen	Add the two integrations and print the results to the screen

- The integration ranges were selected for the two methods as the closest approximation to the Cauchy Principal Value ($\frac{3\pi}{26}$) given by the function $\int_0^\infty \frac{x^2}{(x^2-4)(x^2+9)}dx$. The sympy.integrate method will give a result of $\frac{3\pi}{26}$ + ($-6.805*10^{-6}$) and the rectangle method will produce a result of (0.3621236). Since $\frac{3\pi}{26} = 0.3624915$ the rectangle method has a percent error of 0.101%.

- The step-size in the rectangular method was used as to keep the potential code run time low. Even increasing the steps by a factor of 10 will increase the runtime of any code.
- Dealing with the poles within the equation was done by setting all real poles within the integration range as bounds for integration, since 2 was the only real pole within the range $[0, \infty)$ so the integration range only needed to be split into two parts where the range is $[0,2)$ and $(2,\infty)$.

Question 4

Part a.) Define Fourier Transform and Laplace Transforms and highlight their differences.

Definitions

- The Fourier transform of a function f(t) can be represented by an exponential function of form $e^{i\omega t}$ using a continuous summation. The Fourier transform cannot be used to analyze unstable systems. The Fourier transform does not have any convergence factor. The Fourier transform is equivalent to the Laplace transform evaluated along the imaginary axis of the s-plane.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

- The Laplace transform of function f(t) can be represented by a continuous summation of complex exponential of damped waves of form e^{st}. The Laplace transform can be used to analyze unstable system. The Laplace transform has a convergence factor making it more general than the Fourier transform. The Laplace transform of a signal f(t) is equivalent to the Fourier transform of the signal f(t) $e^{-\sigma t}$.

$$\mathcal{L}(f(t)) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

Part b.) Find the Fourier Transform of g(ω) of the function $f(t) = \cos(\omega_0 t)$. Compare answer with Laplace Transform

- The Delta function is defined as zero for every value except for the point in which t = a.
- Making use of the the definition of the delta function below

$$2\pi\delta(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}dt$$

- Find the Fourier Transform g(ω) of the function $f(t) = \cos(\omega_0 t)$
- Using the forward Fourier transform written out in part a.): $\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ we obtain:

$$\mathcal{F}\{f(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos(\omega_0 t)e^{-i\omega t}dt$$

- The identity $\cos(\omega_0 t) = \frac{e^{i\omega t}+e^{-i\omega t}}{2}$ we can rewrite the equation above as

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\frac{e^{i\omega t}+e^{-i\omega t}}{2})e^{-i\omega t}dt$$

- Expanding the integral, we find

$$=\frac{1}{2}\int_{-\infty}^{\infty}e^{i(\omega_{0+}\omega)t}+e^{i(\omega-\omega_0)t}dt\cdot\frac{1}{\sqrt{2\pi}}$$

- Using the definition of the delta function we can the above separated integrals become

$$=\frac{\pi}{\sqrt{2\pi}}[\delta(\omega_0+\omega)+\delta(\omega-\omega_0)]$$

- The Laplace transform $\mathcal{L}(f(t)) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$ leads to the function becoming

$$=\int_{-\infty}^{\infty}e^{st}\cos(\omega_0 t)dt$$

- Once again using the identity of cos

$$=\frac{1}{2}\int_{-\infty}^{\infty}e^{-(s-i\omega_0)t}+e^{-(s+i\omega_0)t}dt$$

- Use the standard integral identity $\int_{-\infty}^{\infty} e^{ax}dx = \frac{1}{a}$ in order to simplify the equation

$$\frac{1}{2}\left[\frac{-1}{s-i\omega}+\frac{-1}{s+i\omega}\right]$$

- We next need to multiple the two-inside fraction by the other denominator in order to create a single fraction within the brackets

$$\frac{1}{2}\left[\frac{(s-i\omega)+(s+i\omega)}{(s-i\omega)(s+i\omega)}\right]$$

- Now simplify this

$$\frac{s}{s^2+\omega^2}$$

- This value will represent the Laplace transform for the original function $f(t) = \cos{(\omega_0 t)}$
- Now we can evaluate the integral

$$\int_{-\infty}^{\infty}\frac{g(\omega)\sin{(\omega)}}{\omega}d\omega$$

- g(ω) represents the Fourier transform of the original function which will allow the integral to be written as

$$\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty}\frac{[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]\sin\omega}{\omega}d\omega$$

- We separate the terms and use u substitution $\mathrm{u} = \omega-\omega_0 \Rightarrow d_{\mathrm{u}} = d\omega$

$$\int_{\infty}^{-\infty} \frac{\delta(u)\sin(u+\omega_0)}{u+\omega_0}\text{du}$$

- If we set u = 0 we get

$$\sqrt{\frac{\pi}{2}}\left[\frac{\sin\omega_0}{\omega_0}+\frac{\sin\omega_0}{\omega_0}\right]$$

- Which will give us a result of

$$\sqrt{\frac{\pi}{2}}\left[\frac{2\sin\omega_0}{\omega_0}\right]=\frac{\sqrt{2\pi}\sin\omega_0}{\omega_0}$$

Part c.) Solve the following simultaneous differential equations. Boundary conditions y(0) = -1, z(0) =1, using Laplace

- Given equations

$$\frac{dy}{dt}+z=2\cos t$$

$$\frac{dz}{dt}-y=1$$

- The First Equation: If we take a derivation with respect to time we can use the dot notation, then taking the Laplace transform we can get

$$\mathcal{L}\{\dot{y}+z=2\cos t\}$$

- Separated out

$$\mathcal{L}\{\dot{y}\}+\mathcal{L}\{z\}=\mathcal{L}\{2\cos t\}$$

- We can write the above Laplace as

$$sY(s)+Y(0)+Z(s)=\frac{2s}{s^2+1}$$

- The Second Equation: Using the same methods as we did for the first equation. Due to words limitation, I am using $\dot{x}$ to represent zDot, however I will revert back to z for the remainder of the problem.

$$\mathcal{L}\{\dot{\text{x}}-y=s\}$$

- zDot worked this time, so we are switching back to zDot notation, but separating out

$$\mathcal{L}\{\dot{\text{z}}\}+\mathcal{L}\{\text{y}\}=\mathcal{L}\{\text{s}\}$$

- Rewrite as

$$sZ(s)-Z(0)-Y(s)=\frac{1}{s}$$

- Plug in our boundary conditions as from before y(0)=-1 and z(0)=1
- The first equation becomes

$$sY(s) - (-1) + Z(s) = \frac{2s}{s^2 + 1}$$

- The second equation becomes

$$sZ(s) - 1 - Y(s) = \frac{1}{s}$$

- Now we want to move all constants to the RHS

$$sY(s) + Z(s) = \frac{2s}{s^2 + 1} - 1 = \frac{-s^2 + 2s + 1}{s^2 + 1}$$

- And for the second equation

$$sZ(s) - Y(s) = \frac{1}{s} + 1 = \frac{\mathrm{s} + 1}{s}$$

- If we take the second equation and multiply by s through the entire equation, we can then take the first equation (which will need to have Z(s) on the RHS by way of subtraction) and place it into the second equation

$$(s^2 + 1)\mathrm{Z}(s) + \frac{s^2 - 2s + 1}{s^2 + 1} = s + 1$$

- When multiplying this by s^2+1 we get

$$(s^4 + 2\mathrm{s}^2 + 1)\mathrm{Z}(s) + \mathrm{s}^2 - 2\mathrm{s} + 1 = s^3 + s^2 + \mathrm{s} + 1$$

- Move all individual components to the RHS. Then divide by $(s^4 + 2\mathrm{s}^2 + 1)$ to get

$$\mathrm{Z}(s) = \frac{s^3 + 3s}{s^4 + 2s^2 + 1}$$

- From here take the inverse Laplace transform of Z(s) to get $z = t \sin t + \cos t$
- Now plug in Z(s) to equation 1 to get:

$$s\left(\frac{s^3 + 3s}{(s^4 + 2s^2 + 1)}\right) - Y(s) = \frac{s + 1}{s}$$

- Now rearranging and doing algebra we get

$$\frac{-s^4 + s^3 - 2s^2 - s - 1}{s(s^2 + 1)(s^2 + 1)} = Y(s)$$

- Find the inverse Laplace Transform of Y(s) To find Y(t)

$$y(t) = t\cos(t) - 1$$

- So,

$$z(t) = t \sin t + \cos t$$

$$y(t) = t\cos(t) - 1.$$