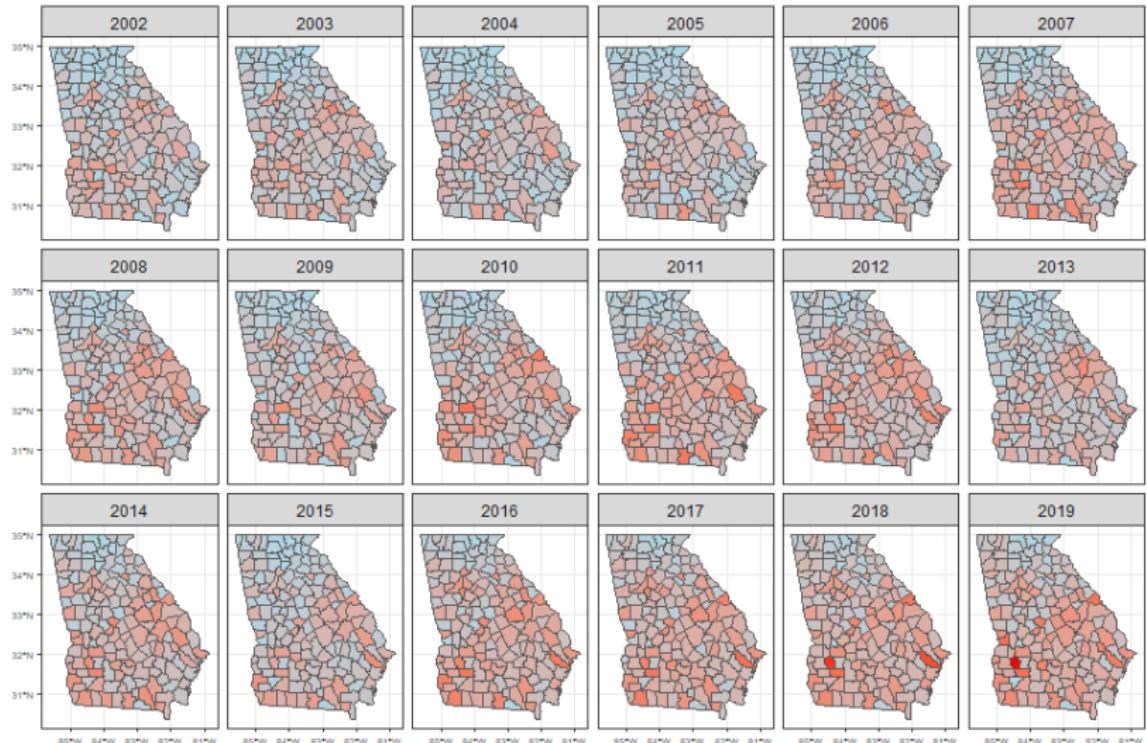


Lecture 8: Spatial-Temporal Models

July 11, 2023

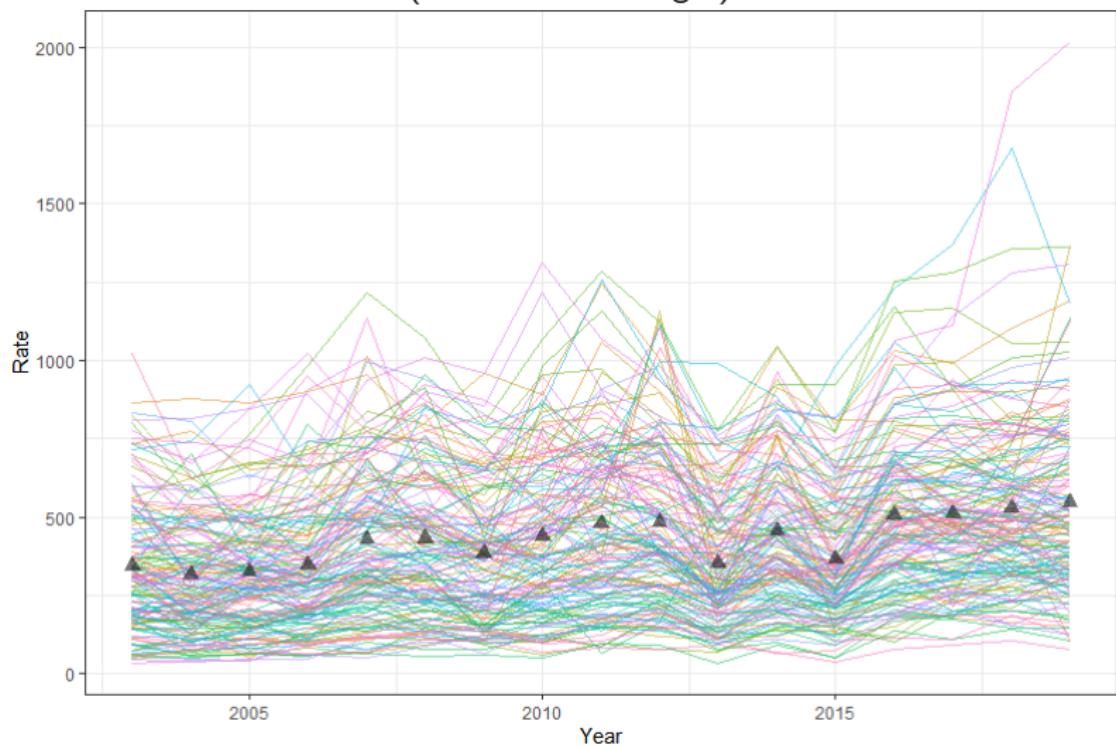
Annual Chlamydia Rate in Georgia

Rate per 100,000 population



County-specific Temporal Trends

Rate per 100,000 population
(▲ = annual averages)



Additive Models

Let

y_{st} = cases of chlamydia in county s in year t .

P_{st} = population in county s in year t .

x_t = year since 2002 (per 10 year),

$$y_{st} \sim \text{Pois} (P_{st} \times \mu_{st})$$

$$\log(\mu_{st}) = \beta_0 + \beta_1 x_t + \theta_{st}$$

In an *additive* model, we break the space-time residuals into a purely spatial component and a purely temporal component:

$$\theta_{st} = \theta_s + \theta_t .$$

- ▶ θ_s = overall county-specific residual log rate ratio .
- ▶ θ_t = overall year-specific residual log rate ratio.
- ▶ θ_s and θ_t are independent.

Additive Models

We will start with conventional random intercept models that ignore spatial/temporal dependence.

Model	θ_s	θ_t	WAIC
1	$N(0, \sigma_s^2)$	-	45,615
2	$N(0, \sigma_s^2)$	$N(0, \sigma_t^2)$	41,862

Model	β_0	β_1	σ_s^2	σ_t^2
1	-5.86 (-5.95, -5.78)	0.28 (0.28, 0.29)	0.30	-
2	-5.86 (-6.00, -5.71)	0.27 (0.16, 0.37)	0.29	0.015

- ▶ Accounting for temporal clustering leads to larger interval estimates.
- ▶ Between-county variation is more important than between-year variation.

Model Parameter Interpretations

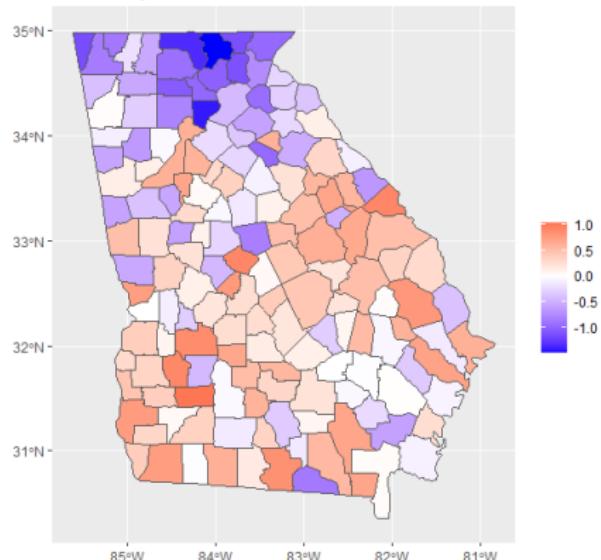
From Model 2:

- ▶ $e_0^\beta = 0.0028$ is the estimated rate of chlamydia in year 2002.
- ▶ $e_1^\beta = 1.32$ describes a 32% increase in rate of chlamydia for every 10 years.
- ▶ $\sigma_s = 0.54$ is the between-county standard deviation of log rate
- ▶ $\sigma_t = 0.12$ is the between-year standard deviation of log rate

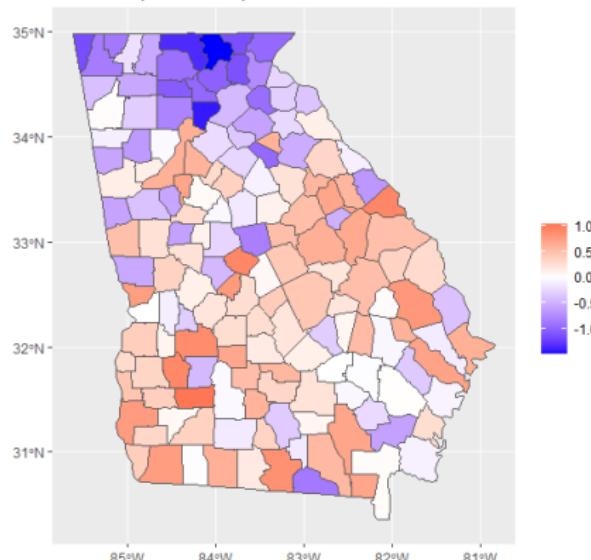
County-specific Residual Random Effects

Log rate ratio for chlamydia

Model 1: Spatial Random Effect

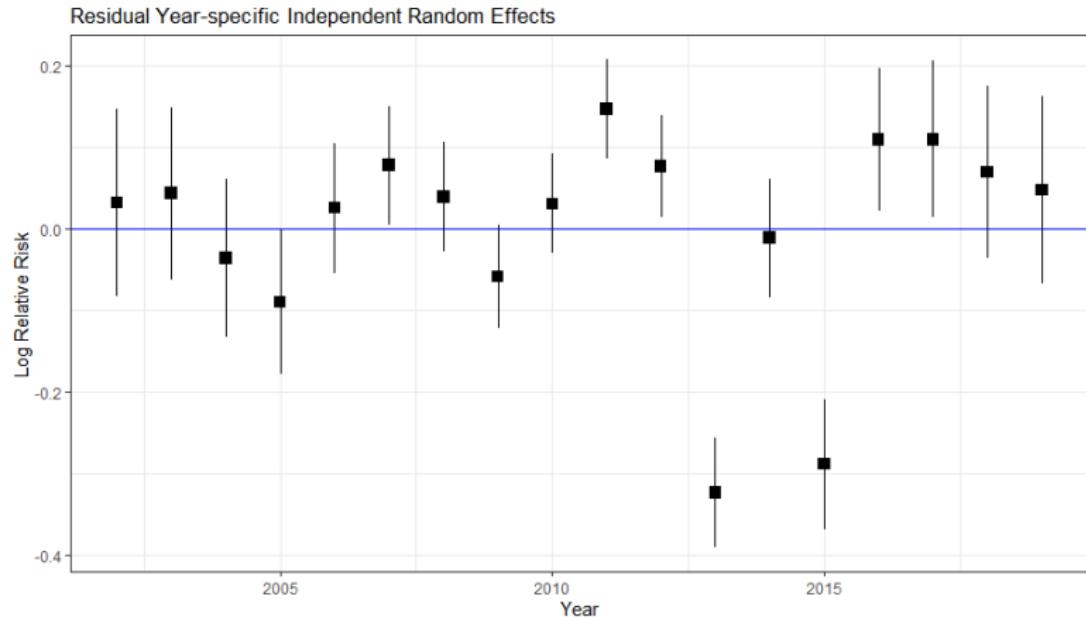


Model 2: Spatial + Temporal Random Effects



Evidence of spatial dependence?

Temporal Residual Random Effects



Evidence of temporal dependence?

Modeling Temporal Dependence

One common approach is to use a [random-walk](#) specification for equally-spaced covariate values.

Let $\Delta_i = f(x_i) - f(x_{i-1})$ for $i = 1, \dots, n - 1$. If we assume that the differences $\Delta_i \sim N(0, \sigma_x^2)$, the conditional distribution of $f(x)$ is

$$f(x_i) | f(x_{-i}) \sim N\left(\frac{1}{2}[f(x_{i-1}) + f(x_{i+1})], \frac{\sigma_x^2}{2}\right)$$

- This is the same model as an iCAR where the neighbors are defined as the first-order in time stamp.
- The function $f(x)$ will be smoother if we assume a 2nd-order structure.

Additive Models with ST Dependence

Let's now consider spatial/temporally-dependent random effects

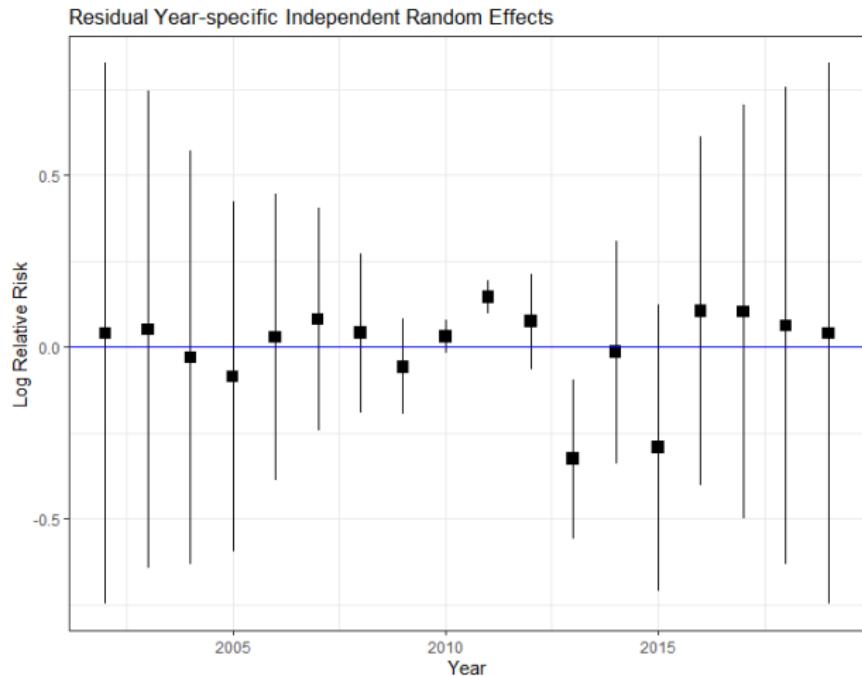
Model	θ_s	θ_t	β_1	WAIC
3	iCAR (τ_s^2)	RW-1 (τ_t^2)	0.28 (-0.69, 1.25)	41,861
4	Conv (τ_s^2, σ_s^2)	RW-1 (τ_t^2)	0.28 (-0.60, 1.16)	41,861
5	iCAR (τ_s^2)	RW-2 (τ_t^2)	0.09 (-51.0, 51.0)	41,862
6	Conv (τ_s^2, σ_s^2)	RW-2 (τ_t^2)	0.09 (-51.0, 51.0)	41.862

Conv (τ_s^2, σ_s^2) denotes the convolution (BYM) model: $\theta_s = \theta_{1,s} + \theta_{2,s}$ with $\theta_{1,s} \sim \text{iCAR}(\theta_s^2)$ and $\theta_{2,s} \sim N(0, \sigma_s^2)$.

All the models have similar WAIC. We shall go with the simplest Model (i.e. Model 3).

What happened to our time trend β_1 estimate?

Temporal Residual Random Effects



The annual estimates are very noisy, especially at the ends. Too much smoothing?

Non-additive Models

Additive models are quite restrictive because they may not capture how the spatial map **evolves** over time.

Let's consider the following model

$$y_{st} \sim \text{Pois} (P_{st} \times \mu_{st})$$

$$\log(\mu_{st}) = \beta_0 + \beta_1 x_t + \theta_s + \theta_t + \theta_{st}$$

$$\theta_s \sim \text{iCAR}(\sigma_s^2) \quad \theta_t \sim N(0, \sigma_t^2) \quad \theta_{st} \sim N(0, \nu^2) .$$

We can think of θ_{st} as space-time interaction that is independent across space and time.

The improvement in WAIC is huge!

Model	θ_s	θ_t	β_1	WAIC
7	iCAR	iid	0.33 (0.23, 0.35)	23,021

$$\sigma_s^2 = 0.91 \quad \sigma_t^2 = 0.04 \quad \nu^2 = 0.11$$

Space-time Interaction Model

There are several ways we can build in spatially-correlated residual errors. One common approach is to model time-specific **random maps** using **separable** covariance functions.

Let Σ_S be the covariance matrix between counties defined by an iCAR model in any year. And let Σ_T be the correlation matrix between time for any county.

If we stack all the space-time random effects by time, then space:

$$\boldsymbol{\theta} = (\boldsymbol{\theta}_{t=1}, \boldsymbol{\theta}_{t=2}, \dots, \boldsymbol{\theta}_T)',$$

a separable model entails a covariance for the entire vector as

$$\text{Cov}(\boldsymbol{\theta}) = \Sigma_T \otimes \Sigma_S .$$

where \otimes is the Kronecker product.

Separable Space-Time Interaction Model

Here are a few commonly used models.

A. Independent.

When $\Sigma_T = \mathbf{I}$, we the spatial random effects are independent maps across time

$$\boldsymbol{\theta}_t \stackrel{iid}{\sim} iCAR .$$

B. Exchangeable.

When $\Sigma_T = \rho \mathbf{1}^T \mathbf{1} + (1 - \rho) \mathbf{I}$ for $\rho < 1$, random effects in the same county by at any two points in time are correlated by ρ :

$$\text{Cov}(\theta_{st}, \theta_{s',t'}) = \rho \Sigma_{T,(s,s')} .$$

C. Dynamic (Auto-regressive).

Assume $\boldsymbol{\theta}_t$ evolves in time depending on the previous value (AR-1).

$$\boldsymbol{\theta}_t = \alpha \boldsymbol{\theta}_{t-1} + \boldsymbol{\gamma}_t \sim iCAR .$$

Separable Space-Time Interaction Model

Model	β_0	β_1	ρ/α	WAIC
8 (Ind)	-5.9 (-6.0, -5.8)	0.33 (0.22, 0.43)		23,736
9 (Exch)	-5.9 (-6.0, -5.9)	0.33 (0.23, 0.44)	0.86 (0.83, 0.86)	23,274
10 (Dynamic)	-5.9 (-6.0, -5.8)	0.33 (0.23, 0.43)	0.96 (0.95, 0.97)	22,961

- ▶ Very robust β_0 and β_1 estimates.
- ▶ ρ and α being close to 1 indicate very strong temporal similarity in residual maps.
- ▶ WAIC prefers the dynamic model and suggests the residual has a temporal (ordered) dependency.

Case Study: Estimating Unrecognized COVID-19 Deaths in the US

- ▶ COVID-19 deaths reported from death certificates is likely an underestimate.
- ▶ It is important to estimate the full burden of infectious disease.
- ▶ One approach is to use infection activity proxies (e.g., weekly percent test positive) to estimate **counterfactual** baseline mortality rates.
- ▶ Need to handle spatial-temporal correlation, sparse data situations and over-dispersion in mortality counts.
- ▶ This analysis is only for ages ≥ 85 from March 2020 to April 2021.

Zhang Y, Chang HH, Iuliano AD, Reed C (2022). Application of Bayesian spatial-temporal models for estimating unrecognized COVID-19 deaths in the United States. *Spatial statistics* 50:100584.

Spatial-Temporal Binomial Model for Positive Tests

Let N_{st} = number of positive SARS-CoV-2 test results for state s during week t .

$$N_{st} \sim \text{Binom}(D_{st}, p_{st})$$

$$\log\left(\frac{p_{st}}{1 - p_{st}}\right) = \alpha_0 + \theta_{s,t}$$

- ▶ D_{st} = number of tests conducted
- ▶ p_{st} percent positive test results

For θ_{st} ,

- ▶ Exchangeable: $\theta_s \sim N(0, \sigma^2)$ and $\theta_t \sim \text{RW-1}$.
- ▶ Spatial: $\theta_s \sim iCAR$ and $\theta_t \sim \text{RW-1}$.
- ▶ Dynamic (AR-1)

Spatial-Temporal Negative-Binomial Model for Mortality

Let Y_{st} denote the number of all-cause deaths with COVID-19 coded deaths removed.

$$Y_{st} \sim \text{Pois}(\lambda_{st}) \quad \lambda_{st} \sim \text{Gamma}(\xi, e^{\eta_{st}})$$

$$\log \text{eta}_{st} = \log \text{Pop}_{st} + \beta_0 + \beta_1 p_{s,t-1} + \beta_2 p_{s,t-2} + \delta_{s,t} .$$

- ▶ Similar space-time options for δ_{st}
- ▶ Unrecognized deaths are estimated as

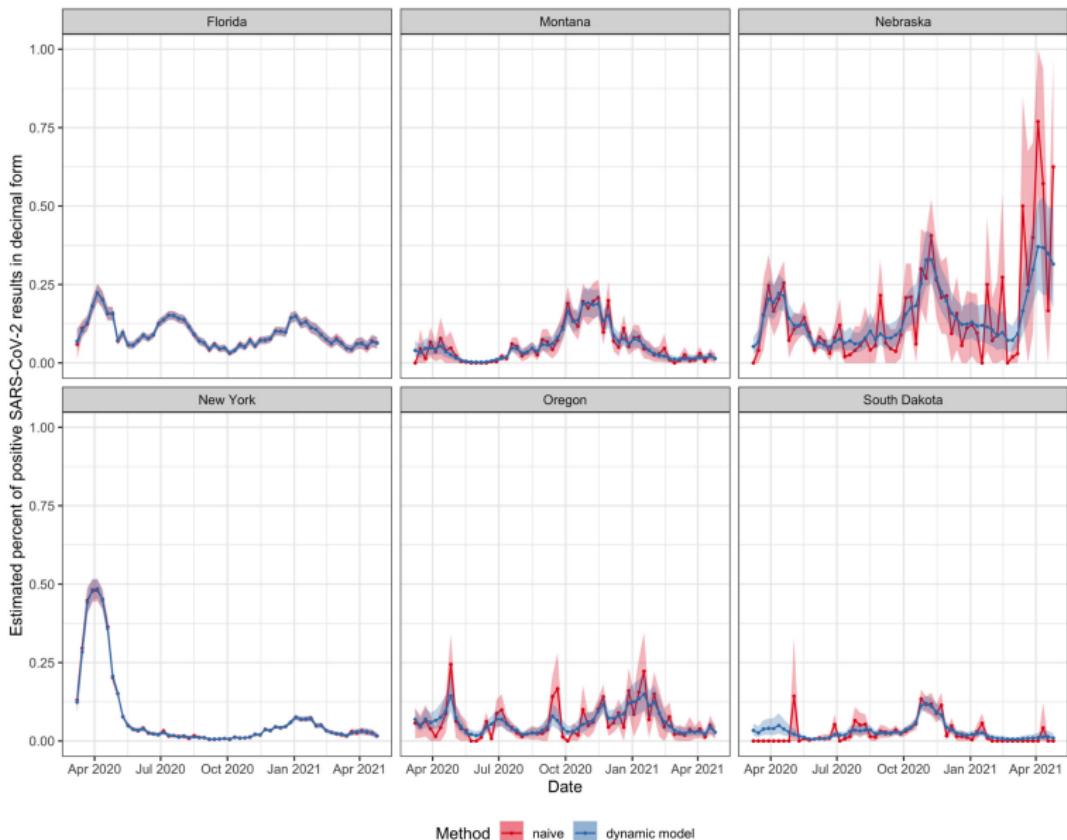
$$\xi \left(e^{\eta_{st}} - e^{\log \text{Pop}_{st} + \beta_0 + \delta_{s,t}} \right)$$

Model Selection

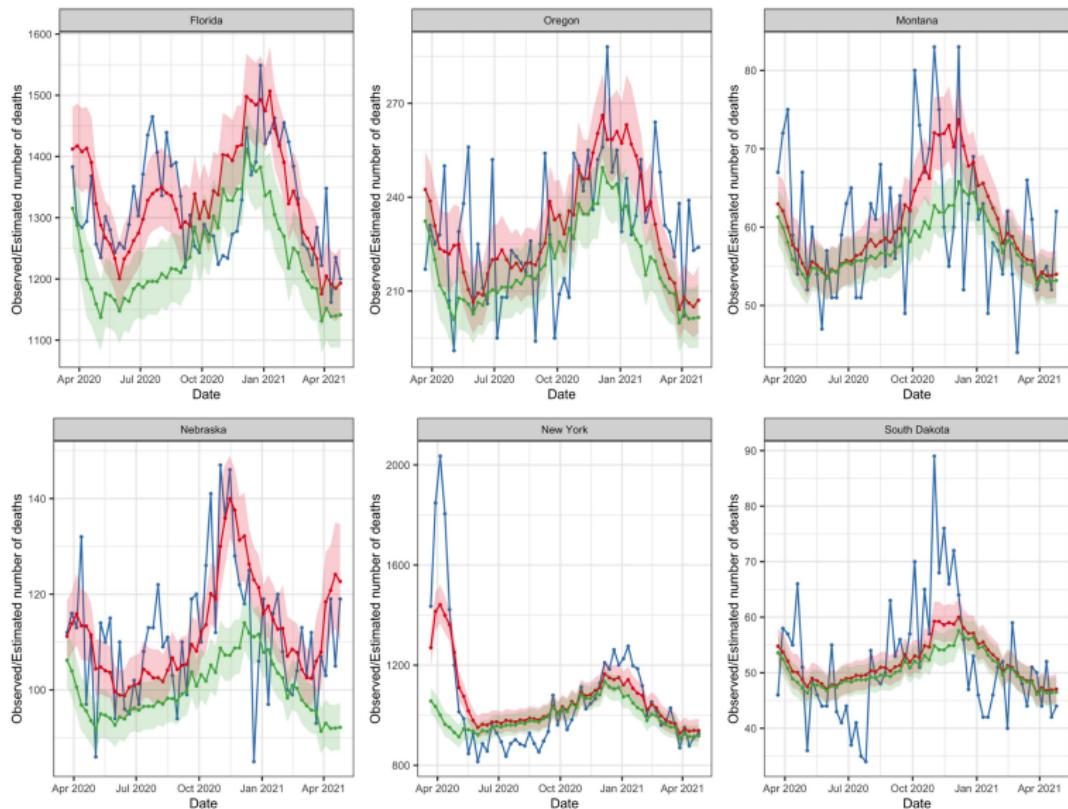
Model	Exchangeable	Spatial	Dynamic
Binomial (% Positive)	59,699	59,619	18,271
Neg-Binom (Mortality)			
Imputation 1	26,015	26,040	26,943
Imputation 2	25,949	25,895	27,873
Imputation 3	25,801	25,929	27,719
Imputation 4	25,887	25,877	27,427
Imputation 5	25,665	25,577	27,957

- ▶ The dynamic spatial model is favored for modeling percent positives.
- ▶ Across imputations the exchangeable and spatial models perform similarly for mortality counts.

Estimated Percent Positivity (2020-03 to 2021-04)



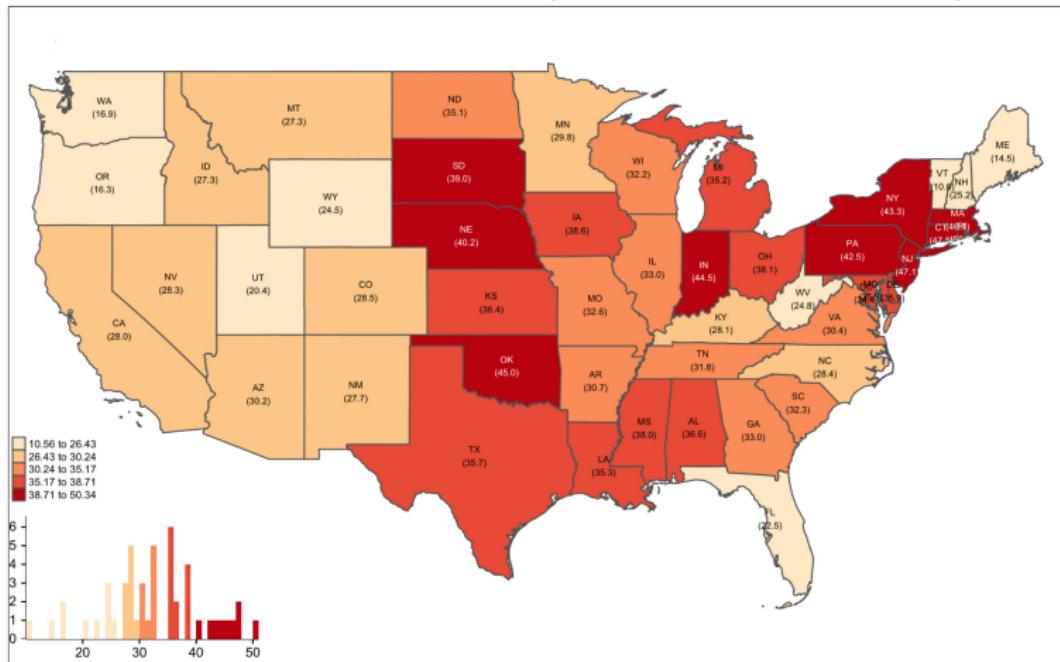
Weekly Mortality Counts (2020-03 to 2021-04)



Death counts — observed — predicted — predicted: no COVID-19

Total COVID-19 Deaths (2020-03 to 2021-04)

Total Reported = 162,085 versus
Total Estimated = 220,285 (95% PI: 213,385 226,985)



Percent Unrecognized COVID-19 Deaths (2020-03 to 2021-04)

