1. Context

This is a first draft, context will be inserted

2. Standard notions and notation for quotient singularities

3. Singularities with small discrepancy

Recall from Section 2 our standard notation for quotient singularities. We consider the germ S of a cyclic quotient singularity appearing at a point P on a projective surface X. The minimal resolution of X is denoted $f\colon Y \longrightarrow X$. It contains a chain of exceptional (smooth, rational) curves $C_1, \ldots, C_n$, entirely determined by S itself, which are ordered so that the only intersections between these curves are $C_i \cap C_{i+1}$ which is a single transverse intersection for each $i = 1, \ldots, n-1$; in other words, C_1 and C_n are the two 'ends' of the chain. We also denote the discrepancies of each C_i (as curves in Y) by $d_i \in \mathbb{Q}$: thus

$$K_Y = f^*(K_X) + \sum_{i=1}^{n} d_i C_i.$$

We introduce a property of cyclic quotient singularities that is central to the rest of the chapter.

Definition 3.1. *Let S be a cyclic quotient singularity, and $C_1, \ldots, C_n$ the exceptional curves of the minimal resolution of S and $d_1, \ldots, d_n$ their discrepancies, as above. We say that S is a* singularity with small discrepancy *if $d_i \leq -\frac{1}{2}$ for all $i = 1, \ldots, n$.*

To simplify our calculations we introduce to the notation $e_i = d_i + 1$.

Proposition 3.2. *In the notation above, a singularity S has small discrepancy if and only if $C_1^2 \neq -2$ and $C_n^2 \neq -2$ and $S \ncong \frac{1}{3}(1,1)$.*

Proof. We use the fact that the discrepancy is a strictly decreasing sequence then a strictly increasing sequence. So it suffices to show this for C_1 and C_n and then apply this to show it for the intermediate values. We use the following formula for the discrepancy $e_i = \frac{e_{i-1}+e_{i+1}}{a_i}$. We note that if $a_1 \geq 4$, then as $e_0 = 1$ and $e_2 \leq 1$ we have $e_1 \leq \frac{2}{-4}$. This implies the inequality for small discrepancy. In the case where $a_1 = 3$ this results in the following, as $e_1 \geq e_2$ by substituting e_2 into $e_1 = \frac{1+e_2}{-3}$ we get $e_2 \leq \frac{1+e_2}{3}$ rearranges to $2e_2 - 1 < 0$. Hence $e_2 \leq \frac{1}{2}$. Substituting this back into the equation for e_1 we get $e_1 \leq \frac{1+\frac{1}{2}}{3} = \frac{1}{2}$. $\square$

Throughout the rest of this chapter we restrict the class of singularities we consider as follows:

Assumption 3.3. *Any singularity germ S that appears in this chapter is assumed to be a cyclic quotient singularity with small discrepancy.*

4. Log del Pezzo surfaces and small discrepancy

DEFINITION 4.1. *Let X be a surface. A -1 cycle $Z = \sum D_i \subset X$ is a rationally connected set of curves such that there is a sequence of maps*

$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n$$

Such that each map is a blowdown and the image of Z is a smooth point.

LEMMA 4.2. *Let X be a surface having cyclic quotient singularities of small discrepancy, and let $f\colon Y \to X$ be the minimal resolution of X. Let $C \subset X$ be a rational curve whose strict transform $\widetilde{C} \subset Y$ is smooth. Suppose in addition that $\widetilde{C}$ meets the exceptional locus of f with intersection multiplicity at least 2. Then if $\widetilde{C}^2 = -1$ then $-K_X \cdot C \leq 0$.*

In particular, $\widetilde{C}$ is smooth and C either meets at least two singularities of X or meets one singularity with at least branches or has a singular point of C at a singularity of X, then the hypotheses on C are satisfied.

PROOF. By the genus formula for $\widetilde{C} \subset Y$, as $\widetilde{C}$ and Y are both smooth, $K_Y \cdot \widetilde{C} = -1$. If $\widetilde{C}$ intersects two distinct exceptional curves E_i, E_j, with discrepancy d_i, d_j respectively, then $K_X \cdot C = f^*(K_X) \cdot \widetilde{C} \geq -1 - d_i - d_j \geq 0$, as X has only singularities with small discrepancy. If, on the other hand, $\widetilde{C}$ meets only one exceptional curve E_i, but with intersection multiplicity m_i, then $K_X \cdot C = f^*(K_X) \cdot \widetilde{C} \geq -1 - m_i d_i \geq 0$. □

We show next that in fact such rational curves cannot lie on a log del Pezzo. We need a preliminary lemma.

LEMMA 4.3. *Let X be a log del Pezzo and $f\colon Y \to X$ be the minimal resolution. Let $C \subset Y$ be a smooth rational curve. If $C^2 \leq -2$ then C is contracted by f to a point of X.*

PROOF. We proof this by contradiction. Assume there is a curve C that is not contracted, the $K_X \cdot f(C) = f^*(K_X) \cdot \widetilde{C} \geq K_Y \cdot \widetilde{C} \geq 0$, with the inequality following as there are no terminal surface singularities. □

PROPOSITION 4.4. *Let X is a log del Pezzo with singularities of small discrepancy and consider the following diagram*

$$\begin{tikzcd} & Y \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Z \end{tikzcd}$$

f is the minimal resolution of X and g is a birational morphism to a smooth surface Z. Let $E \subset Y$ be an f-exceptional curve. Then E is contracted to a point of Z by g, or $g(E)$ is a smooth curve and g_E is an isomorphism.

Proof. Let $E \subset Y$ be any one of the exceptional curves E_i^S; in particular, E is a smooth rational curve with $E^2 \leq -2$. We first show that if $g_*E \subset Z$ is a curve, then it must be a smooth curve.

For contradiction, suppose g_*E is a curve with a singular point P. Let $C_1, \ldots, C_s \subset Y$ be the curves that contract to P under g. As these curves are contracted, $C_i^2 \leq -1$. Notice that if $C_i^2 \leq -2$, then $f(C_i)$ is a point of X by Lemma 4.3. There are two cases to consider: set-theoretically, either $g^{-1}(P)$ meets E in a single point or in more than one point.

In the case of more than one intersection point, since $g^{-1}(g_*(E))$ is connected, among the curves C_i there must be a shortest chain $C_1 \cup \cdots \cup C_r$ with $C_k \cdot E = 0$ for $k = 2, \ldots, r-1$, and $(\sum_{i=1}^r C_i) \cdot E = 2$. At least one of the curves $A = C_k$ of the cycle must have $A^2 = -1$, otherwise the whole cycle is contracted to a point R of X, but then $R \in X$ would not be a rational singularity, and so in particular not a cyclic quotient singularity. And of course A cannot meet another -1-curve C_j with $g(C_j) = P$. Thus A must lie in one of the following configurations:

(1) A meets two distinct π-exceptional curves, C_j and $C_{j'}$, both of which have self-intersection ≤ -2.

(2) A meets E in one point and a distinct π-exceptional curves C_j with $C_j^2 \leq -2$.

(3) A meets E in two distinct points.

In each of these situations, $C = f_*(A) \subset X$ would be a curve on which K_X is nef, by Lemma 4.2, which contradicts X being a log del Pezzosurface. Indeed $A = \widetilde{C}$ meets the f-exceptional locus with multiplicity at least 2 in each case.

The argument in the nodal case follows similarly, up to the case division of configurations at which there is an additional case:

We note that if $g^{-1}P$ contains two curve C_i, C_j with $I_Q(C_i, C_j) \geq 2$ then either one of them is a -1-curve or it cannot occur on the minimal resolution of a log del Pezzo surface. This is because every curve in $\pi^{-1}P$ has negative self intersection, if its intersection is less than -1 then it would have to be contracted on the map down to X, resulting in a noncyclic quotient singularity. Hence one of them is -1 curve, and this cannot occur as it would contradict Lemma 4.2. Hence this cannot occur, so we have to blow up the point P enough times such that all the intersections are transverse. At this point we have a curve A such that A intersects transversely at least three other curves, E, C_1, $C_2 \ldots$ with $C_i \in \pi^{-1}P$. In addition C is the only -1 curve in $\pi^{-1}P$. As Y is constructed from further blowups we split into configurations

(1) The strict transform of A, denoted $\widetilde{A}$ has $\widetilde{A}^2 = -1$.

(2) The strict transform of A, denoted $\widetilde{A}$ has $\widetilde{A}^2 \leq -2$.

In the first case Lemma 4.2 this cannot occur on the minimal resolution of a log del Pezzo surface due to the curves $\widetilde{E}$, $\widetilde{C}_1$, $\widetilde{C}_2$. In the second case, if none of the intersection points

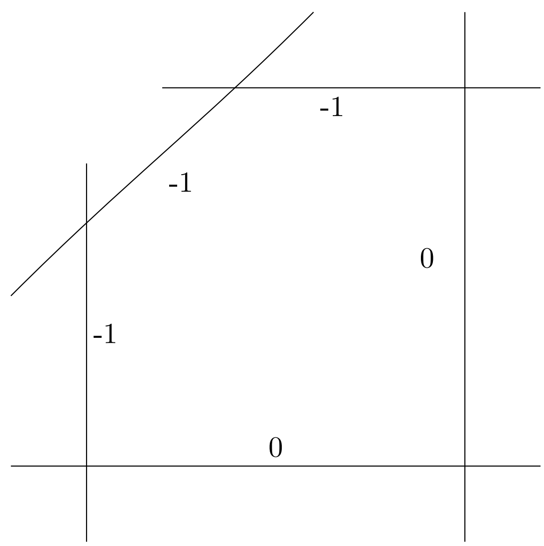
$A \cap E$, $A \cap C_1$, $A \cap C_2$ have been blown up then we are left with a noncyclic quotient singularity. Hence one of these points has to be blown up. This results in a -1-curve intersecting $\widetilde{A}$ and another negative curve hence we have a contradiction to Lemma 4.2.

For a completely general curve singularity it follows by a combination of the above arguments. $\square$

PROPOSITION 4.5. *There is a unique family of singular log del Pezzo surfaces S_p, indexed by $p \in \mathbb{N}$, such that given the minimal resolution Y of S_p, Y does not admit a map to $\mathbb{F}_i$ with $i \geq 2$. Here S_p has one $\frac{1}{p}(1,1)$ singularity.*

PROOF. The first case is that Y only admits a map to $\mathbb{F}_0$. Then Y must be F_0, since a blow up of any point of $\mathbb{F}_0$ also permits a map to $\mathbb{F}_1$; but then $X = Y$ is smooth, contradicting the assumption.

For $\mathbb{F}_1$ other cases arise. Clearly if we blow up a point on the -1-curve we get a map to $\mathbb{F}_2$. So the only option is a blowup at a smooth point. At this point you get the following toric variety



This results in three adjacent -1-curves. We now split into cases;

Case 1: Our next blow up is a blow up at a smooth point. This results in DP_6. We note that on any surface Z which occurs as a blowup at general points of DP_6 has the property that for every -1 curve C there is a map $\pi_C \colon Z \to \mathbb{F}_1$ sending C to the negative section B. Hence, in order for X to be non singular this involves blowing up a point on a -1 curve.

Hence we get the following diagram:

$$\begin{tikzcd}
C \subset Z \arrow[d] & Z' \arrow[l, "\pi_1"'] \arrow[d] \\
B \subset \mathbb{F}_1 & T \arrow[l, "\pi_2"'] \arrow[d] \\
 & \mathbb{F}_2
\end{tikzcd}$$

Where the π_i are blow ups of a point on a -1 curve. Hence this case cannot occur.

Case 2: Blowing up a point on the curve which is the pullback of the section B of $\mathbb{F}_1$. Once again this results in an obvious map to $\mathbb{F}_2$. We note that our surface admits two maps to $\mathbb{F}_1$ via the symmetry of the surface.

Case 3: Blowing up a general point of the -1 curve which occurred as the strict transform of the fiber. By blowing up $p-1$ distinct points. This results in an infinite family of log del Pezzo's with a single $\frac{1}{p}(1,1)$ singularity and potentially A_n singularities. Via the previous arguments any subsequent blowups not on this curve would induce a map to $\mathbb{F}_2$. $\square$

LEMMA 4.6. *Let X be a log del Pezzo with only singularities of small discrepancy, and let $f\colon Y \to X$ be the minimal resolution. We suppose that Y admits a map π to $\mathbb{F}_l$ where $l \geq 2$.*

For a germ S of a singularity of X, denote by $E_i^S \subset Y$ the exceptional curves in the resolution of S. For each singularity S on X:

(1) Every exceptional curve E_i^S is either contracted to a point of $\mathbb{F}_l$ by π, or the pushdown $\pi_* E_i^S \subset \mathrm{F}_l$ is a smooth rational curve with self intersection one of $-l$, 0, l, $l+2, 4l$.

(2) In addition there is always a curve E_j^S not contracted by π for all singularities S.

PROOF. To prove the first statement note that $\pi_* E_i^S$ cannot be a singular curve by Proposition 4.4, hence it is a smooth rational curve. The only smooth rational curves on a Hirzebruch Surface $\mathbb{F}_l$ are the curves B, with $B^2 = -l$, F with $F^2 = 0$ and the curves lieing inside the linear systems $|lF+B|$, $|(l+1)F+B|$, $|2F|$, $|2(lF+B)|$ and finally $|(l+2)F+B|$. We note that the final case could not arise on $\mathbb{F}_l$ when $l \geq 2$. In this case the curve B is also the image of an exceptional curve from a singularity. Hence any curve in $|(l+2)F+B$ would intersect B, when counting multiplicities, 2 times. This would be a contradiction to Lemma 4.2. A similar argument occurs with $2F$ which is meeting the curve B at a single point with multiplicity 2.

To show that not all the curves E_j^S can be contracted to a point if $l \geq 2$, we go for a proof by contradiction. Assume $l \geq 2$ and every exceptional curve in a singularity S is contracted to a point $P \in \mathbb{F}_l$. Then P lies on a fiber F which intersects the curve B. First we consider

$P \notin B$. We have $E_i^S \in \pi^{-1}P$ for all i. Hence we have to blow up several times. However the strict transform of the fiber F, denoted $\widetilde{F}$ now has $\widetilde{F}^2 \leq -1$. If $\widetilde{F}^2 \leq -2$ then it has to be contracted, meaning $\widetilde{F}$, $B \in \{E_i^S\}$ which would be curves not contracted to a point. If $\widetilde{F}^2 = -1$, then the only -1 curves in $\pi^{-1}P$ cannot intersect $\widetilde{F}$. This is because after the first blowup we have an exceptional curve E and the fiber $\widetilde{F}$. These both have square -1. If we blow up the intersection point of $\widetilde{F}$ and E then $\widetilde{F}^2 \leq -2$, hence we can only blowup general points on E. At this point we have non e of the -1-curves intersecting E. If we blowup no points on E then clearly we are not introducing a singularity so this does not occur. Now finally we note that our curve configuration would contradict Lemma 4.2.

□

REMARK. *In the case where the length, n, of the singularity is 1 or 2, Lemma 4.2 follows via easy toric geometry as any curve joining two singularities is a locally toric configuration. This corresponds to the associated fan being non convex.*

Now we can classify these log del Pezzos in a straightforwards way.

THEOREM 4.7. *Let X be a non-smooth log del Pezzo with only singularities of small discrepancy. Then*

(1) X has either one singularity or two singularities, each of type $\frac{1}{p}(1,1)$ for some, possibly different, p.

(2) If X admits no floating -1-curves then X admits a toric degeneration.

PROOF. Given a log del Pezzo X_0 we start by contracting all floating -1 curves. This gives rise to a log del Pezzo X_1; note that X_1 is not P^2 since the contraction map is an isomorphism in the neighbourhood of any singularity of X_0. Let $\sigma\colon Y \to X_1$ be the minimal resolution of X_1. We know that there is a map $\pi\colon Y \to \mathbb{F}_l$, and we may suppose l is maximal. There is a curve $B \subset \mathbb{F}_l$ with $B^2 = -l$. If $l \geq 2$ then B has to be the image of a σ-exceptional curve E_i inside Y.

We first show that π cannot contract a curve to a point on B. If on the contrary there is a curve contracted to B, then without loss of generality we may assume that it is the exceptional curve of the final blowdown $Y \to Y_2 \to \mathrm{F}_l$. In that case, there two curves C_1, C_2 on Y_2, both -1 curves, with C_2 being the strict transform of 0 fiber. But then we could instead contract C_2 from Y_2 and get a map to $\mathbb{F}_{l+1}$, contradicting maximality of l. Hence π is indeed an isomorphism in a neighbourhood of B.

We note that $l \leq 1$ has been classified in Proposition 4.5. So we restrict to $l \geq 2$. Now there is a singularity S such that $B \in \{\pi_* E_i^S\}$. Assume first that S is not a $\frac{1}{p}(1,1)$ singularity. Note that there is a curve E_j^S such that $\pi_* E_j^S$ is B. The adjacent (one or two) exceptional curves cannot be contracted (by the argument of the previous paragraph). We suppose there are two adjacent curves $E_{j\pm 1}^S$; the case where E_j^S is at the end of a chain of blowups with only one adjacent exceptional curve works in exactly the same way. Thus each of $\pi_* E^S_{j\pm1}$ is either a 0 curve (a fiber) or an $l+2$ curve on F_l (by the classification of smooth rational curves on F_l in Lemma **??**). Denote these two adjacent curves by C_1 and C_2 respectively. Assume there was another singularity with exceptional curves $\{E^{S'}_i\}_{i=0}^{m_{S'}}$ on Y. Then by Lemma 4.6 there would be a curve $E^{S'}_j$ such that $\pi_* E^{S'}_j$ is a curve with self intersection 0, l, $l+2$. However these curves would necessarily intersect C_1 and C_2 meaning either S' is not distinct from S or there is a -1-curve in Y connecting two of their curves in the minimal resolution. Hence X has precisely one singularity.

To complete the analysis of this step, suppose S is a $\frac{1}{p}(1,1)$ singularity and that its unique exceptional curve is mapped to the negative section B. Then consider the possibility of there being another singularity S' on X. By Lemma 4.6, there is a curve $E^{S'}_j$ such that $A = \pi_* E^{S'}_j$ has self intersection l or $4l$; it cannot be 0 or $l+2$ as it must not meet B. If S' is not a $\frac{1}{p}(1,1)$ then there is at least one exceptional curve among the $E^{S'}_k$ that is contracted to a point on $A \subset \mathrm{F}_l$. However each blowup of a point $Q \in A$ introduces a -1-curve D which is joined to curve B by another -1-curve, the birational transform of the fiber through Q. Hence none of these curves $E^{S'}_k$ can be mapped to D, as otherwise it would be adjoined to B by a -1-curve, contradicting Lemma 4.2. Thus any other singularity on X is also of type $\frac{1}{p}(1,1)$ (though possibly for a different p).

Suppose now that there was a third singularity of type $\frac{1}{p}(1,1)$. Once again, its exceptional curve would have to be sent to a 0, l, $l+2$. Any smooth rational curve on F_l with one of these intersection numbers intersects the curve A. Thus on Y it must either meet the birational transform of A or meet some curve that contacts to A. Once again in the second case it will result in two singularities connected by a -1-curve. This is a contradiction to small discrepancy.

Thus X has exactly one or two singularities of type $\frac{1}{p}(1,1)$, and part (1) is complete in the case $l \geq 2$.

For part (2), we first observe that neither of the adjacent curves $E^S_{j\pm1}$ can map to an $l+2$ curve, since in that case X will have a floating -1-curve. This is because $l+1$ points in general position on the $l+2$ curve can be cut out as the intersection of the $l+2$ curve with an l curve.

Because of this we see that the only possibilities for $\pi_*(E^S_{j\pm1})$ are two different 0 curves. (Again we suppose there are two adjacent curves; the case of one adjacent curve is the same.) We can then proceed to construct the configuration of all exceptional curves inductively. This means that when a surface of this form is able to be constructed we can obtain it by doing two weighted blowups at a general point of a Hirzebruch surface and then doing a series of non toric blowups on the boundary. The following surface is one example, arising from blowing up two general points of a Hirzebruch surface with weight $(1,i)$ and $(1,n-i)$. This is the picture where the map to the Hirzebruch surface is an isomorphism on an exceptional curve E_i, where $1 < i < n$. Here the red curve s indicate -1-curves and the blue

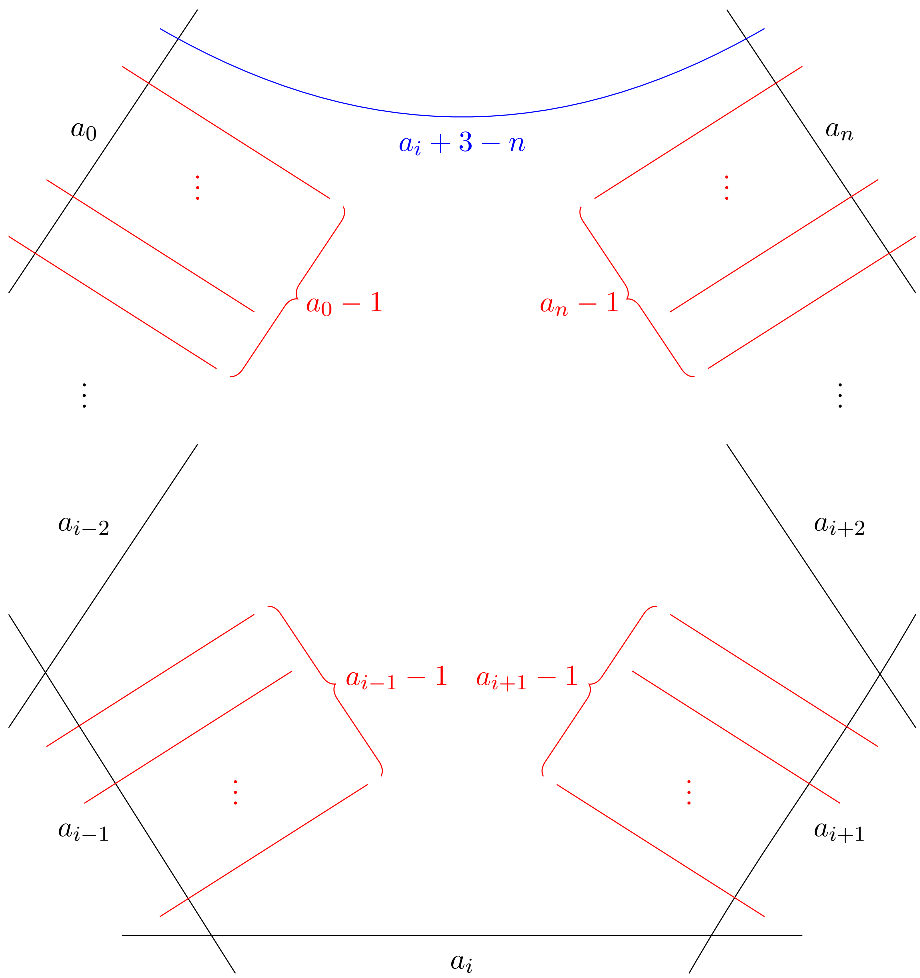


FIGURE 1. Example of surface.

curves indicate curves with positive self intersection. The blue curve has self intersection $a_i + 3 - n$. This value is dependent on $n \geq 3$ and the map to the Hirzebruch surface being an isomorphism on a curve E_i with $1 < i < n$. This is because on the Hirzebruch surface $\mathbb{F}_{a_i}$ this curve had self intersection a_i. The $n - 3$ of our exceptional curves are mapped to points and hence it has self intersection $a_i - (n - 3)$. In the case that our map is an isomorphism on the curve E_1 or E_n then we have a similar looking configuration except with positive curve now having self intersection $a_i + 4 - n$ as there is an extra point being blown up however the image of the curve in the Hirzebruch surface now is in the linear system $|(l + 1)F + B|$. Hence its self intersection is now $l - (n - 3) - 1 = l - n + 4$.

The toric degeneration property now follows. By construction all these surfaces X are Looijenga pairs, and so admit a toric degeneration by [**?**, Theorem ??]. $\square$

This leads to the following corollary in which we classify all log del Pezzo surfaces with singularities of small discrepancy, each of which is resolved by a single exceptional curve.

5. Examples

COROLLARY 5.1. *Let X be a log del Pezzo surface with small discrepancy and basket $\{\{\frac{1}{p_1}(1,1), \dots \frac{1}{p_n}(1,1)\}, m\}$ for $n \geq 0$ and $m \geq 0$. Then $n \leq 2$ and moreover*

(1) if $n \leq 1$ then either X is a smooth del Pezzo surface or lies in a cascade over $\mathrm{P}(1,1,k)$ (see [**?**, Table ??]*);*

(2) if $n = 2$ then X is isomorphic to a quasismooth weighted hypersurface $X_{p+q} \subset \mathbb{P}(1,\, 1,\, p,\, q)$. Conversely any such hypersurface with $p, q \geq 4$ is a log del Pezzo surface with small discrepancy.

In particular, in the case of two singularities there is no cascade.

The small discrepancy condition is equivalent to the condition that $p_i \geq 4$ for each $i = 1, \dots, n$. For the sake of completeness, we outline the classification result of [**?**] that describes part 1, which also follows independently from MY PROOF ? LEMMA ??

PROOF. With these constrictions on singularities, it fits the criterion for the above theorem. The explicit classification was done in the proof of the theorem. The case of one singularity was done in CP. The only examples of these surfaces with more than one singularity are constructed by blowing up a Hirzebruch surface in several points along a line and then contracting the two curves. Denote this surface by X. Then X admits a toric degeneration to $(-p_1, -1)$, $(0, 1)$, $(p_2, 1)$. Note X admits a $\mathbb{C}^*$ action and the degeneration is equivariant with respect to the torus action. We have $-K_X^2 = \frac{4}{p_1} + \frac{4}{p_2}$. Even in cases where $-K_X^2 > 1$ we see that X cannot be blown up while preserving $-K_X$ ample. If X admitted a blow up at a general point P then there is a fiber F such that $P \in F$. Then $\widetilde{F}$ is a -1-curve on the minimal resolution connecting the $-p_1$ curve with the $-p_2$ curve. This is a contradiction. Hence there is only one element in the cascade.

We note that this surface can be see as a hypersurface of degree $p + q$ inside $\mathbb{P}(1,\, 1,\, p,\, q)$.

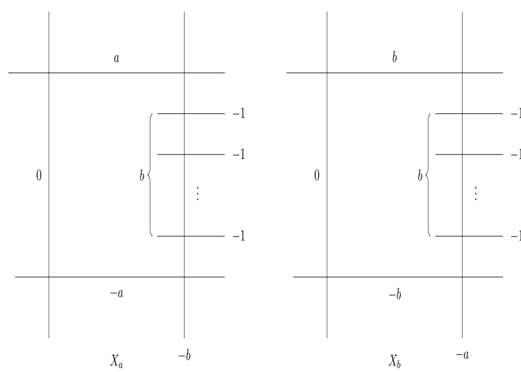
$\square$

We now do a more difficult example by classifying the log del Pezzo's with singularities $S_{a,b}$ with resolution E_1, E_2 with $E_1^2 = -a$, $E_2^2 = -b$. To make sure that this obeys they conditions on the theorem we insist $a, b \neq 2$. We note that the case of $S_{3,3}$ does satisfy the conditions for the theorem. However we are interested in $\mathbb{Q}$g smoothings and $S_{3,3}$ is not $\mathbb{Q}$g rigid and admits a partial smoothing to $\frac{1}{6}(1,1)$ singularity. These were classified above. This is the only one of these singularities which is not $\mathbb{Q}$Gorenstein rigid. This is a more complicated example of how the above theorem can be used.

Corollary 5.2. *Let X be a surface such that the basket is $(\{S_{a_1,b_1}, \ldots, S_{a_m,b_m}\}, n)$, with the condition that a_i, $b_i \geq 3$ and we exclude the case $a_i = b_i = 3$. Then there is at most one singularities. They fall into a cascade of the form*

$$\begin{tikzcd}
X_a^0 & X_a^1 \arrow[l, "\phi_a^0"] & \cdots \arrow[l, "\phi_a^1"] & X_a^{a-1} \arrow[l, "\phi_a^{a-2}"] & & & \\
& & & & X_1 \arrow[ul, "\phi_a^{a-1}"] \arrow[dl, "\phi_b^{b-1}"'] & X_2 \arrow[l, "\Phi_1"'] & X_3 \arrow[l, "\Phi_2"'] \\
X_b^0 & X_b^1 \arrow[l, "\phi_b^0"'] & \cdots \arrow[l, "\phi_b^1"'] & X_b^{b-1} \arrow[l, "\phi_b^{b-2}"'] & & &
\end{tikzcd}$$

Proof. Once again by Theorem 4.7 there are two heads of the cascade given by the following two surfaces. These correspond to surfaces constructed by blowing up $\mathbb{F}_a$ in b points and $\mathbb{F}_b$ in a points, then contracting the negative curves. We call these surfaces X_a and X_b respectively.



These admits a toric degeneration to $\mathbb{P}(1, b, ab-1)$ and $\mathbb{P}(1, a, ab-1)$ respectively. We only consider the case of X_a as X_b is completely symmetric. We see that the we can smooth it by taking the bth Veronese embedding and getting $\mathbb{P}_{u,v,w,t}(1, 1, ab-1, a)$ with the relation $uw = t^b$. This admits a smoothing giving us the surface lies as $X_{ab} \subset \mathbb{P}(1,\, 1,\, ab-1,\, a)$. We get the following formula for the anticanonical degree of X_a:

$$-K_{X_a}^2 = 8 - b + a\left(1 - \frac{b+1}{ab-1}\right)^2 + b\left(1 - \frac{a+1}{ab-1}\right)^2 - 2\left(1 - \frac{a+1}{ab-1}\right)\left(1 - \frac{b+1}{ab-1}\right)$$

We will show that this admits a cascade of length $a+2$. The first a terms are easy to describe as we see that these admit a toric degeneration to X_Σ with Σ being the fan with rays $(-1, b)$, $(-1, 0)$, $(a, -1)$, $(a-u, -1)$, where u is the number of blowups. This has an

A_{b-1} singularity and an A_{u-1} singularity. Via Cox rings this can be viewed as $\mathbb{C}^4_{\{}x, y, z, t\}$ with a quotient

$$\begin{array}{cccc} x & y & z & t \\ \end{array}$$
$$\begin{pmatrix} u & 0 & bu-(ab-1) & ab-1 \\ 1 & ab-1 & b & 0 \end{pmatrix}$$

Taking the Veronese embedding of degree u in the variables x, z, t gets us the coordinates z^u, t^u, zt. And to smooth the A_{b-1} singularity we take the b Veronese embedding of the variables x^b, y^b, xy. Once again we can smooth this out. This gives us the surface as complete intersection with weights $\begin{matrix} b & b^2u \\ ab & u \end{matrix}$ inside the toric variety with weights

$$\begin{array}{cccccc} x^b & y^b & z^u & xy & t^u & t^{bu}z \end{array}$$
$$\begin{pmatrix} b & 0 & b(bu-(ab-1)) & 1 & ab-1 & b^2 \\ 1 & ab-1 & u & a & 0 & 1 \end{pmatrix}$$

The $(a+1)$st blowup admits a toric degeneration to $(-1, b)$, $(-1, -1)$, $(a, -1)$. The toric degeneration of the $a+2$'nd blowup is $(-1, 0)$, $(-a, -1-a)$, $(-1, -1-a)$, $(b, ab-1)$. We note that this surface still has a boundary of a curve $C \in -K_X$. This strict transform $\widetilde{C} \subset Y$ has self intersection 0 as it started of as an $a+2$ curve and has been blown up $a+2$ times. Hence X admits a degeneration as it is a Looijenga pair. To see the cascade result we note that if you blow up the surface X_a times at points $P_1 \dots P_a$. To each of these points there is a unique fiber going through it F_i. The strict transform of these fibers after blowing up is a -1-curve going through the $-a$ curve. Hence after blowing X_a and X_b respectively a and b times we get a surface which has as a boundary three curves with self intersection $0, -a, -b$ and in both cases you have your a and your b curves have a and b minus one curves intersecting them respectively. Hence they are isomorphic.

We note that we have made in the above calculations no effort to show that the elements in the cascade are log del Pezzo surfaces. However it is not hard to show assume we are blowing up $a+2$ points giving surface X_3. This has minimal resolution Y_3. The class group of Y_3 is generated by the curves D_1, D_2, D_3, D_4, $E^0_1, \dots E^0_b$, $E^1_1, \dots E^1_{a+2}$. Here the D_i for a cycle such that $\sum D_i \in |-K_{Y_3}|$. These have self intersections $-a$, $-b$, -1, -1 respectively. Here D_3 was a curve of degree a on $\mathbb{F}_a$ blown up $a+1$ times and D_4 was a fiber on which a point has been blown up. The E^0_i are -1-curves intersecting the $-b$ curve. The E^1_i are floating -1-curves. We wish to show $-K_{X_3}$ is ample. We note that showing $-K_{X_3} \cdot C > 0$ for all C generating the class group would suffice. We note that the curves D_1, D_2 are contracted when sent to X_3. We note that $-K_{X_3} \cdot E^0_i = -K_{X_a} \cdot E^0_i$ as we are blowing up points not on these curves. Then $-K_{X_3} \cdot E^0_i = 1$ as these are floating -1-curves. Finally to see that $-K_{X_3} \cdot D_3 > 0$ we note that, when pushed forwards to X_3, it only goes through the one singularity on X_3 with multiplicity -1. This is because on Y_3 it is only intersecting the $-a$ curve transversely. Hence $-K_{X_3} \cdot D_3 = 1 + d_a$ where d_a is the discrepancy of the $-a$ curve. Via log terminality we have $d_a > -1$. Hence the product is greater than 0. The argument for the curve is exactly the same with d_a replaced with d_b. From this we see X_3 is a log del Pezzo, hence every surface in the cascade is a log del Pezzo surface. $\square$

This structure of the cascade can be put in more general terms.

6. Cascades

THEOREM 6.1. *Given a singularity with small discrepancy such that the minimal resolution is $a_1, \dots a_n$, there are a finite number of basic surfaces classified in the previous theorem. Let X_i be the surface constructed from X_{a_i}. If $i \neq 1, n$ then this admits a cascade of length $a_i + 3 - n$, if $i = 1, n$ then the cascade is of length $a_i + 4 - n$.*

In addition we can describe the shape of the cascade. In the case when the singularities are of the form $\frac{1}{p}(1,1)$ the cascades have been classified by [?] and the above example. We explained the case of the singularity of length 2 above. In the case of the singularity having length 3, then the cascade looks like:

$$\begin{tikzcd}
X_1^0 & X_1^1 \arrow[l, "\phi_1^1"'] & \cdots \arrow[l, "\phi_1^2"'] & X_1^{a_1-1} \arrow[l, "\phi_1^{a_1-2}"'] & & & \\
X_2^0 & X_2^1 \arrow[l, "\phi_2^1"'] & \cdots \arrow[l, "\phi_2^2"'] & X_2^{a_2-1} \arrow[l, "\phi_2^{a_2-1}"'] & X_1 \arrow[l, "\phi_2^{a_2}"] \arrow[ul, "\phi_1^{a_1-1}"'] \arrow[dl, "\phi_3^{a_3-1}"] & X_2 \arrow[l, "\Phi_1"'] & X_3 \arrow[l, "\Phi_2"'] \\
X_3^0 & X_3^1 \arrow[l, "\phi_3^1"'] & \cdots \arrow[l, "\phi_3^2"'] & X_2^{a_3-1} \arrow[l, "\phi_3^{a_3-2}"'] & & &
\end{tikzcd}$$

For any singularity of length greater than 3, then a basic surface either falls into a cascade of the above form or it lies in a straight series.

PROOF. Given a singularity S of small discrepancy with length $m > 1$. Then a basic surface X with singularity S has minimal resolution Y. The surface Y is constructed by taking a Hirzebruch surface $\mathbb{F}_{a_i}$ picking two points P_1, P_2 and blowing them up k_1 and k_2 times, this gives rise to an intermediate surface Z and then X is constructed by doing further blow ups. We can assume $k_1 \leq k_2$ and this gives the relations that either $k_1 = k_2 = 0$ and $min\{1, 2, 3\}$ or $k_1 = 0$ and $k_1 = m - 2$ or $k_1 + k_2 = m - 3$. These cases arise by considering the case where the strict transform of both/ one/ none of the fibers are exceptional curves.

We note that k_1 and k_2 should not be viewed as invariant of the surface X but an invariant of given map to $\mathbb{F}_{a_i}$:

$$X \longleftarrow Y \xrightarrow{\pi} \mathbb{F}_{a_i}$$

We now note that by construction on Y with self intersection $a_i - k_1 - k_2$. This implies that cascade of X is of length $L = a_i - k_1 - k_2$. If $L < 0$ then this surface is not a log del Pezzo surface. On X we have two sets of curves which are mapped to fibers on the Hirzebruch surface. Denote these by $A_1 \dots A_n$ and by $B_1, \dots B_m$. We have three rational curves U, V, W. We now assume that both A_1, A_n and B_1, B_m have been blown up non torically giving rise to curves C_1^A, C_n^A, C_1^B, C_m^B. The curve U intersects with C_i. □

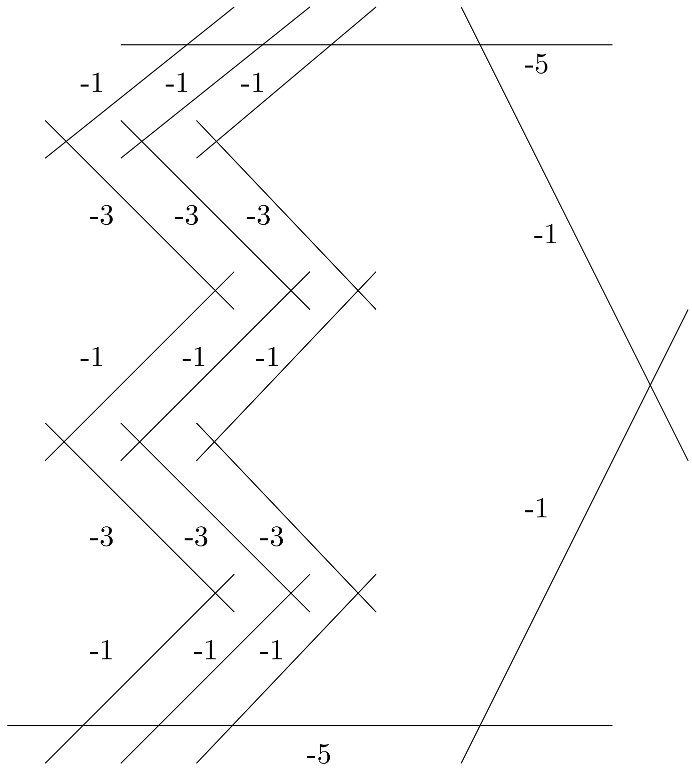
Corollary 6.2. *Let S be a singularity with small discrepancy, $-a_1, \dots, -a_n$ be the self intersection of the resolutions. Then if $n \geq \max(a_i) + 5$. Then there exists no log del Pezzo with only singularities of type S.*

Remark. *It is fully possible for both X_1 and X_2 to exist but one of the $X_{a_i}^0$ to not exist. For example consider a singularity with resolution -3, -8, -2, -2, -2, -2, -2, -2, -2, -3. There will be a surface X such that the resolution will have a map to $\mathbb{F}_8$ but there will be no surface with a map from its resolution to $\mathbb{F}_3$.*

7. Outside of the small discrepancy

If you consider singularities of the type $\frac{1}{p}(1,1)$ we note that if $p \geq 7$ then a $\frac{1}{p}(1,1)$ singularity cannot be joined to any other $\frac{1}{p}(1,1)$ singularity by a -1 curve. Hence a similar analysis to Theorem 4.7 gives us the bound that there cannot be a surface X with singularities $\frac{1}{p_1}(1,1)$, $\dots$, $\frac{1}{p_n}(1,1)$ and $p_1 \geq 7$, then if X is a log del Pezzo surface then $n = 2$.

However when we enter the case where $p_1 < 7$ you can get surfaces with many more singularities. For instance consider the surface X with the following minimal resolution:



This has $h^0(-K_X) \neq 0$, $-K_X^2 = \frac{3}{5}$, six $\frac{1}{3}(1,1)$ singularities and two $\frac{1}{5}(1,1)$ singularities. In addition it admits no normal toric degeneration, as it is a complexity one surface and hence any degeneration would have to be equivariant. We can construct this as a toric complete intersection via cox rings and we see that it lies as a complete intersection in the toric variety given by the GIT quotient

$$\begin{array}{c} \begin{array}{ccccccccccc} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9 & T_{10} & T_{11} \end{array} \\ \begin{pmatrix} 0 & 3 & -6 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -4 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 2 & -4 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 7 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

with the equations

$$T_1T_2^2T_3 + T_4T_5^2T_6 + T_7T_8^2T_9$$
$$T_1T_2^2T_3 + T_4T_5^2T_6 + \lambda T_{10}T_{11}$$

We note that there are also surfaces which admit toric degenerations with the same numerics.