Theorem 0.0.1. Let S be a cyclic quotient singularity. Let $\widetilde{S}$ be minimal resolution . Let E be an exceptional divisor. Let $\widetilde{X}$ be the variety obtained by blowing up a general point k times on E. Then X is obtained as the blowdown of the strict transform of the E_i. Then we can extend this contraction to the deformation family and its associated toric degenerations.

Given the assumptions above we can assume that S is a $\frac{1}{r}(1,s)$ singularity and has a fan Σ with rays (a,b) and $(c-kd,-d)$ with $a,b,c,d>0$. We can also assume that the ray $(1,0)$ corresponds to E in the minimal resolution. We also note that r is equal to the determinant of the rays, $ad-b(c-kd)$.

We note that X has a torus acting on it. This means that it can be written as a polyhedral divisor $\mathcal{S}$.

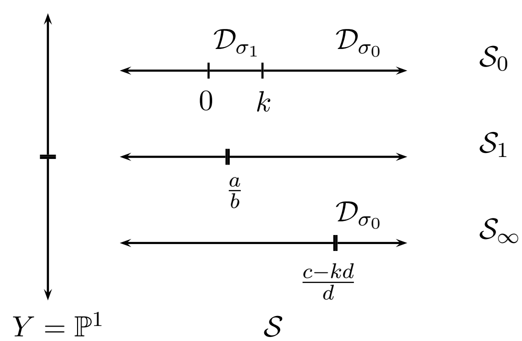

Figure 1: Divisorial fan associated to X.

Here the tail fan is $\mathbb{Z}_{\geq 0}$. When constructing the equivariant toric degeneration, we take the Minkowski sum of $\frac{c-kd}{d}$ and the cone $[0,k]$. After summation this is the toric variety with rays (a,b), $(c,-d)$, $(c-kd,-d)$. Call this X_Σ. The cone $(c,-d)$, $(c-kd,-d)$ is a T-singularity. The cone (a,b), $(c,-d)$ is a $\frac{1}{t}(1,u)$ singularity with once again $t=bc+ad$. Labeling these three rays v_1,v_2,v_3 we get the relation $kd^2v_1-rv_2+tv_3=0$. This implies that the Cox Ring $\mathcal{R}(X_\Sigma)$ is $\mathbb{A}^3_{<x,y,z>}$ with a quotient of $\mathbb{C}^*$ with weights $(kd^2,-r,t)$ and a finite group action μ. It is easy to show that $|\mu|=$ hcf(b,d) $=e$ as (x,ey) is clearly a sub lattice which contains all the vertices and $(1,0)$ is in the lattice as b and d are coprime. This is a well defined quotient except along $x=z=0$. To construct the deformation, we take the d-fold veronese embedding getting

$$\frac{\mathbb{C}[y_1,y_2,y_3,y_4]}{y_2^d-y_3y_4} \text{ with weights } (kd,kb,-r,t)$$

Here the b occurs as $t - r = kdb$. We can now construct the deformation family by the following equation

$$\frac{\mathbb{C}[y_1, y_2, y_3, y_4]}{\lambda y_1^b + \mu y_2^d - y_3 y_4} \text{ with weights } (kd, kb, -r, t)$$

At $\lambda = 0$ this is our original variety. At $\mu = 0$ we get the variety corresponding to the Minkowski sum of the cones $\frac{a}{b}$ and the cone $[0, 1]$. This is the toric variety with rays $(c - d, -d)$, $(a + b, b)$, (a, b). Note that the index of the sublattice generated by these rays is still equal to $\mathrm{hcf}(b, d)$. All that remains is to show that this is the desired complexity one variety. From the above polyhedral divisor we get that $\mathcal{R}(X) = \frac{\mathbb{C}[y_1,y_2,y_3,y_4]}{\lambda y_1^b+\mu y_2^d-y_3y_4}$. We now have to calculate the weights of the torus action on it. The Weil divisors on X satisfy the relations given by the rows of the following matrix.

$$\begin{pmatrix} b & 0 & -1 & -1 \\ 0 & c & -1 & -1 \\ a & d & 0 & k \end{pmatrix}$$

To calculate the grading is equivalent to finding a relation between the columns of the matrix. It is easy to verify that the weights $kd, kb, -r, t$ give the corresponding column sum of 0. To verify that the same finite group acts on the fiber we see that if we do Gaussian elimination. We get a row $(d, -b, 0, 0)$ if these two numbers are not coprime this means that we have torsion in the class of size $\mathrm{hcf}(b, d)$. So μ action extends across the family.

We now classify cyclic log fano extractions on surfaces.

Lemma 0.0.2. Let $f : X \to S$ be a cyclic log fano extarction. Then S has to be a cyclic singularity and f has to be the map as described above. This results in potentially two cyclic quotient singularities. One of them is a general $\frac{1}{r}(1, a)$ and the other is of type A_n.