1. The $\mathbb{Q}$G rigid singularities

We wish to study groups G that are finite subgroups $\mathrm{GL}_2(\mathbb{C})$ with the property $G \cap \mathrm{SL}_2(\mathbb{C}) = \mathrm{BD}_{4n}$. Consider the three elements

$$a = \begin{pmatrix} \zeta_m & 0 \\ 0 & \zeta_m \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad c_m = \begin{pmatrix} \sigma_{2n} & 0 \\ 0 & \sigma_{2n}^{-1} \end{pmatrix}$$

Here ζ We split this into two cases the case where m is odd or even. In the odd case $G = \langle a,\, b,\, c_m \rangle$, otherwise it it is $\langle c,\, b \circ a_{2m} \rangle$. Similar analysis can be done for the groups E_7 and E_8. We now discuss when their $\mathbb{Q}$g smoothings.

Case 1

Here m is odd so $G = \langle a,\, b,\, c \rangle$. Looking at the index one cover we have a $x^2 + y^2 z + z^{n+1}$. With $x = (u^{2n} - v^{2n})uv$, $y = u^{2n} + v^{2n}$, $z = u^2 v^2$. Looking at the derivatives we see that $\mathcal{T}^1$ is generated by $1, y, z, \cdots z^{n-1}$. As this is a rational surface singularity we see that $\mathcal{T}^2 = 0$, so these deformations are unobstructed. We see that c acts on x, y, z with weights $(2n+2, 2n, 4)$. This correspond to $\frac{1}{m}(n+1, n, 2)$ action. Our equation has weight $2n+2$. To check rigidity we first need $n \not\cong_m -2$. In this case we have y cannot be in the qG smoothing. Other than that we just need z^i cannot have weight $2n+2$. This corresponds to $2n+2 \not\cong_m 2i$ for i in $0 \ldots n-1$. These will be the only qG rigid singularities. In particular this means that if $m > 2n+2$ then it is rigid.

Case 2

Here $m = 2m'$ is even so $\langle a,\, b \circ c_{2m} \rangle$. So we have the same equation as above and we now get a $\frac{1}{2m'}(2n+m', 2n, 4)$, so to be rigid once again we need the y term not to be in the smoothing $2n+4 \not\cong_{2m'} 0$. Once again we need no z^i terms this means $4n+4 \not\cong_{2m'} 4i$ for i in $0 \ldots n-1$.

It is easy to see that if a partial smoothing exists then it either stay non cyclic quotient or, if you can deform equivariantly by y, it becomes a cyclic quotient singularity which is of the form $\frac{1}{m}(n+1, 2)$ or $\frac{1}{2m'}(2n_m, 4)$. From this it is easy to classify the $\mathbb{Q}g$ smoothable singularities. We note for a given n there exists only a finite amount of $\mathbb{Q}g$ smoothable singularities with the D_n singularity as its index one cover. This is in contrast to the A_n case.

2. The E_i Singularity

We start with the E_6 singularity as this behaves differently more similarly to the D_n singularities. Once again we aim to classify the $\mathbb{Q}$G rigid singularities. We split it into two cases again.

Case 1

This is a $\mathbb{Z}_n$ action with n coprime to 6. This will act with weights $(6, 4, 3)$ on the equation $x^2 + y^3 + z^4$. This lies in the eigenspace of degree 12. Once again by computing partial derivatives we get that the smoothing parameters are $1, y, z, z^2$. We wish for none of these two be invariant. This means that $12 \not\cong_n 0,\ 4,\ 3,\ 6$. This implies that $n \neq 2,\ 3,\ 4,\ 9,\ 12.$

However none of these cases can occur via the coprimality condition stated at the beginning.

Case 2
This is a $\mathbb{Z}_n$ action with $(n, 6) = 3$. Writing $n = 3n'$ we have the action with weights $(6 + n', 4, 3 + n')$. Once again this is non rigid if $n \neq 2, 3, 4, 9, 12$. Via the coprimality condition we get this is non rigid if $n = 3$ or $n = 9$.

The E_7 and E_8 cases are not divide by case. The E_7 has a single possibility of $\mathbb{Z}_n$ with $(n, 6) = 1$. This acts on $x^2 + y^3 + yz^3$ with the weights $(9, 6, 4)$. The partials give us a basis for the deformations of $1, y, y^2, z, z^2, yz$. This means that $18 \not\cong_n 0, 6, 12, 4, 8, 10$. All of these would correspond to n being even so there are no non rigid singularities.

3. Singularity content

We wish to generalise the notion of singularity content from cyclic quotients. We have that given a non cyclic quotient there exists a unique residual singularity. We wish to have other invariants indicating what it has smoothed from. The natural thing is to consider the invariant $h^2_{\text{top}}(X^0)$ where X^0 is $X - \{\text{singular locus}\}$. First considering the case where there is no smoothing in terms of y so the only terms we can put in are of the form z^i giving $x^2 + y^2 z + z^j \Pi_{k=1}^{p}(z^i - a_k)$ then it is clear that we have exactly by projecting $(x, y, z) \mapsto z$ we get a map to $\mathbb{C}$ with degenerate fibers of 2 curves transversely intersecting over the finitely many points corresponding to $\lfloor \frac{n+1}{p} \rfloor$. **Ignoring the central fiber over the origin**. We can apply Mayer-Vietoris repeatedly to get that $h^2_{\text{top}}(X^0)$ is a deformation invariant. In the case where we have y in the deformation we have