0.1 context

This is a first draft, context will be inserted

0.2 Standard notions and notation for quotient singularities

0.3 Result

Recall from Section 0.2 our standard notation for quotient singularities. We consider the germ S of a cyclic quotient singularity appearing at a point P on a projective surface X. The minimal resolution of X is denoted $f \colon Y \longrightarrow X$. It contains a chain of exceptional (smooth, rational) curves $C_1, \ldots, C_n$, entirely determined by S itself, which are ordered so that the only intersections between these curves are $C_i \cap C_{i+1}$ which is a single transverse intersection for each $i = 1, \ldots, n-1$; in other words, C_1 and C_n are the two 'ends' of the chain. We also denote the discrepancies of each C_i (as curves in Y) by $d_i \in \mathbb{Q}$: thus

$$K_Y = f^*(K_X) + \sum_{i=1}^{n} d_i C_i.$$

We introduce a property of cyclic quotient singularities that is central to the rest of the chapter.

Definition 0.3.1. Let S be a cyclic quotient singularity, and $C_1, \ldots, C_n$ the exceptional curves of the minimal resolution of S and $d_1, \ldots, d_n$ their discrepancies, as above. We say that S is a **singularity with small discrepancy** if $d_i \leq -\frac{1}{2}$ for all $i = 1, \ldots, n$.

Proposition 0.3.2. In the notation above, a singularity S has small discrepancy if and only if $C_1^2 \neq -2$ and $C_n^2 \neq -2$.

Proof. We use the fact that the discrepancy is a strictly decreasing sequence then a strictly increasing sequence. So it suffices to show this for C_1 and C_n. We only care about the case of where the square is 3. Without loss of generality we can assume $d_1 \geq d_2$ so $d_2 \leq -1 - \frac{2+d_2}{-3}$ rearranges to $2d_2 + 1 < 0$. Substituting this back into the equation for d_1 we get $d_1 \leq \frac{-1}{2}$. $\qquad\square$

Throughout the rest of this chapter we restrict the class of singularities we consider as follows:

Assumption 0.3.3. Any singularity germ S that appears in this chapter is assumed to be a cyclic quotient singularity with small discrepancy.

Lemma 0.3.4. Let X be a surface having cyclic quotient singularities of small discrepancy, and let $f \colon Y \to X$ be the minimal resolution of X. Let $C \subset X$ be a rational curve whose

strict transform $\widetilde{C} \subset Y$ is smooth. Suppose in addition that $\widetilde{C}$ meets the exceptional locus of f with intersection multiplicity at least 2. Then if $\widetilde{C}^2 = -1$ then $-K_X \cdot C \leq 0$.

In particular, $\widetilde{C}$ is smooth and C either meets at least two singularities of X or meets one singularity with at least branches or has a singular point of C at a singularity of X, then the hypotheses on C are satisfied.

Proof. By the genus formula for $\widetilde{C} \subset Y$, as $\widetilde{C}$ and Y are both smooth, $K_Y \cdot \widetilde{C} = -1$. If $\widetilde{C}$ intersects two distinct exceptional curves E_i, E_j, with discrepancy d_i, d_j respectively, then $K_X \cdot C = f^*(K_X) \cdot \widetilde{C} \geq -1 - d_i - d_j \geq 0$, as X has only singularities with small discrepancy. If, on the other hand, $\widetilde{C}$ meets only one exceptional curve E_i, but with intersection multiplicity m_i, then $K_X \cdot C = f^*(K_X) \cdot \widetilde{C} \geq -1 - m_i d_i \geq 0$. $\qquad\square$

We show next that in fact such rational curves cannot lie on a log del pezzo. We need a preliminary lemma.

Lemma 0.3.5. Let X be a log del pezzo and $f\colon Y \to X$ be the minimal resolution. Let $C \subset Y$ be a smooth rational curve. If $C^2 \leq -2$ then C is contracted by f to a point of X.

Proof. We proof this by contradiction. Assume there is a curve C that is not contracted, the $K_X \cdot f(C) = f^*(K_X) \cdot \widetilde{C} \geq K_Y \cdot \widetilde{C} \geq 0$, with the inequality following as there are no terminal surface singularities. $\qquad\square$

Lemma 0.3.6. Let X is a log del pezzo with singularities of small discrepancy and consider the following diagram

$$
\begin{array}{ccc}
 & Y & \\
{}^{f}\swarrow & & g\searrow \\
X & & Z
\end{array}
$$

f is the minimal resolution of X and g is a birational morphism to a smooth surface Z. Let $E \subset Y$ be an f-exceptional curve. Then E is contracted to a point of Z by g, or $g(E)$ is a smooth curve and g_E is an isomorphism.

Proof. THINK: TO DO $\qquad\square$

Lemma 0.3.7. Let X be a log del pezzo with only singularities of small discrepancy, and let $f\colon Y \to X$ be the minimal resolution. We suppose $X \neq \mathrm{P}^2$, so that $\rho_Y \geq 2$. The resolution Y admits a morphism $\pi\colon Y \to \mathbb{F}_l$ for some $l \geq 0$: this is the minimal model of Y, unless that minimal model is P^2, in which case there is a factorisation $Y \to \mathrm{F}_1 \to \mathrm{P}^2$.

For a germ S of a singularity of X, denote by $E_i^S \subset Y$ the exceptional curves in the resolution of S. For each singularity S on X:

1. Every exceptional curve E_i^S is either contracted to a point of F_l by π, or the pushdown $\pi_* E_i^S \subset \mathrm{F}_l$ is a smooth rational curve with self intersection one of $-l$, 0, l, $l+2$.

2

2. In the case $l \geq 2$, there is always some curve E_j^S not contracted by π.

Proof. Let $E \subset Y$ be any one of the exceptional curves E_i^S; in particular, E is a smooth rational curve with $E^2 \leq -2$. We first show that if $\pi_* E \subset F_l$ is a curve, then it must be a smooth curve.

For contradiction, suppose $\pi_* E$ is a curve with a singular point P. Let $C_1, \ldots, C_s \subset Y$ be the curves that contract to P under π. As these curves are contracted, $C_i^2 \leq -1$. Notice that if $C_i^2 \leq -2$, then $f(C_i)$ is a point of X by Lemma 0.3.5. There are two cases to consider: set-theoretically, either $\pi^{-1}(P)$ meets E in a single point or in more than one point.

In the case of more than one intersection point, since $\pi^{-1}(\pi_*(E))$ is connected, among the curves C_i there must be a shortest chain $C_1 \cup \cdots \cup C_r$ with $C_k \cdot E = 0$ for $k = 2, \ldots, r-1$, and $(\sum_{i=1}^r C_i) \cdot E = 2$. At least one of the curves $A = C_k$ of the cycle must have $A^2 = -1$, otherwise the whole cycle is contracted to a point R of X, but then $R \in X$ would not be a rational singularity, and so in particular not a cyclic quotient singularity. And of course A cannot meet another -1-curve C_j with $\pi(C_j) = P$. Thus A must lie in one of the following configurations:

1. A meets two distinct π-exceptional curves, C_j and $C_{j'}$, both of which have self-intersection ≤ -2.

2. A meets E in one point and a distinct π-exceptional curves C_j with $C_j^2 \leq -2$.

3. A meets E in two distinct points.

In each of these situations, $C = f_*(A) \subset X$ would be a curve on which K_X is nef, by Lemma 0.3.4, which contracts X being log del pezzo. Indeed $A = \widetilde{C}$ meets the f-exceptional locus with multiplicity at least 2 in each case.

The argument in the nodal case follows similarly, up to the case division of configurations at which there is an additional case:

4. A meets E in one point with intersection multiplicity ≥ 2.

In the case that $\pi^{-1}(P) \cap E$ is supported on a single point $Q \in E$, we claim there is a unique $C_j \subset Y$ with $C_j^2 = -1$ and $Q \in C_j \cap E$. In particular, the local intersection number $I_Q(C_j, E) \geq 2$. THINK: TO DO

and there is some $1 \leq j \leq s$ for which $C_j^2 = -1$. Moreover, among such -1-curves, at least one of them either meets two distinct other C_i (that is to say, it lies within a chain of exceptional curves), or

Clearly C_j intersects either two curves with self intersection less than -1, or it could intersect E twice. By the above lemma neither case could appear on the minimal resolutionof

3

a log del pezzo.

To show that not all the curves E_j^S can be contracted to a point, we go for a proof by contradiction. Assume they are all contracted to a point $P \in \mathbb{F}_l$. Then P lies on a fiber F which intersects the curve B. We have to blow up P to get hence we get the following curve configuration.

$$B, -1, -1$$

This configuration is locally toric. To recover the curve E_i^S we start with the toric blowups. Assuming we have done at least one blowup this results in a curve configuration with at least one -1 curve joining together two curves with self intersection less than -1. If this curve stayed a -1 curve after the non toric blowups then we would have a -1 curve joining together two singularities. This contradicts the previous lemma. Hence the -1 curve has to be blown up in a non toric way. This would result in it being a $-a$ curve with $a > 1$. Hence it is exceptional and contracted. This results in our curves all being connected to via a chain of curves with self intersection less than -1 to B. This means there exists i such that $B = E_i^S$ contradicting our assumptions. $\qquad\square$

Remark. In the case where the length, n, of the singularity is 1 or 2, this lemma follows via easy toric geometry as any curve joining two singularities is a locally toric configuration. This corresponds to the associated fan being non convex.

Now we can classify these log del Pezzos in a straightforwards way.

Theorem 0.3.8. Let X be a log del pezzowith only singularities of small discrepancy. Then X has either one singularity or two $\frac{1}{p}(1,1)$ singularities or it does not occur. If X admits no floating -1 curves then X admits a toric degeneration.

Proof. Given a log del pezzoX_0 we start by contracting all floating -1 curves. This gives rise to a log del pezzoX_1, let Y be the minimal resolution of X_1. We know that there is a map $\pi :: Y \to \mathbb{F}_l$. We start by considering the case $l > 1$. There is a curve $B \subset \mathbb{F}_l$ with $B^2 = -l$. Assume there is no $l' > l$ such that $Y \to \mathbb{F}_{l'}$. Then B has to be the image of an exceptional curve E_i inside Y. Assume our map π involves blowing up a point on B. Without loss of generality we can assume that this is the first blowup, so we have curves C_1, C_2 which are both -1 curve, with C_2 being the strict transform of 0 fiber. We could then instead contract C_2 and get a larger value of l. Hence this does not occur.

We will deal with the case of $l = 0$ and $l = 1$ after this. Now there is a singularity S such that $B \in \{\pi_* E_i^S\}$. Assume S is not a $\frac{1}{p}(1,1)$ singularity. Note that there is a curve E_j^S such that $\pi_* E_j^S$ is a B, hence $E_{j\pm 1}^S$ is a 0 curve or a $l+2$ curve, as we are assuming l is the largest possible value of l and hence B could not be blown up. Denote these two curves by C_1 and C_2. Assume there was another singularity giving rise to exceptional curves $\{E_i^{S'}\}_0^{m_{S'}}$. Then by the previous lemma there would be a curve $E_j^{S'}$ such that $\pi_* E_j^{S'}$ is a curve with self intersection 0, l, $l+2$. However these curves would intersect C_1 and C_2

contradicting this being a distinct singularity. Hence there is only one singularity. If S is a $\frac{1}{p}(1,1)$ singularity. Then consider the possibility of there being another singularity S'. There is a curve $E_j^{S'}$ such that $\pi_* E_j^{S'}$ has self intersection l, not 0 or $l+2$ too avoid it meeting B. Denote this image by A. If S' is not a $\frac{1}{p}(1,1)$ then we would have to blow up this curve several times. However each blowup introduces a -1 which is joined to curve B by another -1 curve. Hence this curve cannot be blown up further, by the argument in the above lemma. If there was a third $\frac{1}{p}(1,1)$ singularity, then its exceptional curve would have to be sent to a 0, l, $l+2$, however all of these would intersect the curve A or intersect the curves introduced by blowing up points on A. In the first case it contradicts it being a new singularity and in the second it contradicts the singularities not being joined by a -1 curve.

Dealing with the case of $l = 0$ first. Assume that Y is such that Y only admits a map down to $\mathbb{F}_0$. However a blow up of any point of $\mathbb{F}_0$ results in a map to $\mathbb{F}_1$. So the only possibility is $\mathbb{F}_0$ itself. For $\mathbb{F}_1$ other cases arise. Clearly if we blow up a point on the -1 curve we get a map to $\mathbb{F}_2$. So the only option is a blowup at a smooth point. This results in three adjacent -1 curves. If we blowup a point on either of the two end curves we could get a map to $\mathbb{F}_2$. So the only option is blowing up the middle curve arbitrarily many times. This results in an infinite family of log del pezzo's with a single $\frac{1}{p}(1,1)$ singularity.

We finish by discussing the condition that there are no floating minus one curves. We note that in the case where there is a curve E_i^S such that $\pi_* E_i^S$ is an $l+2$ curve then the blowup introduces floating -1 curves corresponding to the l curve that goes through $l+1$ of the points blown up. Hence this surface is not minimal. $\square$

This leads to the following corollary.
Corollary 0.3.9. Let X be a surface such that the basket is $\{\{\frac{1}{p_1}(1,1), \ldots \frac{1}{p_n}(1,1)\}, n\}$, with the condition that either $p_i \geq 5$. Then there are at most two singularities. The case of one singularity was classified by CP. In the case of two singularities there is no cascade.

Proof. With these constrictions on singularities, it fits the criterion for the above theorem. The explicit classification was done in the case fo the theorem. We note that the cascade is of length 3 and all the surfaces admit toric degenerations. Let X be the surface with no floating curves. Then X admits a toric degeneration to $(-p_1, -1)$, $(0,1)$, $(p_2, 1)$. Here the smoothing is equivariant with respect to the torus action. We have $-K_X^2 = \frac{4}{p_1} + \frac{4}{p_2}$. Even in cases where $-K_X^2 > 1$ we see that X cannot be blown up while preserving $-K_X$ ample. If X admitted a blow up at a general point P then there is a fiber F such that $P \in F$. Then $\widetilde{F}$ is a -1 curve on the minimal resolution connecting the $-p_1$ curve with the $-p_2$ curve. This is a contradiction.

We note that the total family can be see as a hypersurface of degree $p+q$ inside $\mathbb{P}(1, 1, p, q)$.

□

We now do a more difficult example by classifying the log del pezzo's with singularities $S_{a,b}$ with resolution E_1, E_2 with $E_1^2 = -a$, $E_2^2 = -b$. To make sure that this obeys they conditions on the theorem we insist a, $b \neq 2$. We note that the case of $S_{3,3}$ does not satisfy the conditions for the theorem. However we are interested in $\mathbb{Q}$g smoothings and $S_{3,3}$ is not $\mathbb{Q}$g rigid and admits admits a partial smoothing to $\frac{1}{6}(1,1)$ singularity. These were classified above. This is a more complicated example of how the above theorem can be used.

There are two heads of the cascade given by the following four surfaces. These correspond to surfaces constructed by blowing up $\mathbb{F}_a$ in b points and $\mathbb{F}_b$ in a points, then contracting the negative curves. We call these surfaces X_a and X_b respectively. These admits a toric degeneration to $\mathbb{P}(1, b, ab-1)$ and $\mathbb{P}(1, a, ab-1)$ respectively. We only consider the case of X_a as X_b is completely symmetric. We see that the we can smooth it by taking the b'th veronese embedding and getting $\mathbb{P}_{u,v,w,t}(1, 1, ab-1, a)$ with the relation $uw = t^b$. This admits a smoothing giving us the surface lies as $X_{ab} \subset \mathbb{P}(1, 1, ab-1, a)$. This gives us $-K_X^2 = \frac{a^2(b+1)^2}{b(ab-1)}$. We note that this admits a cascade of length $a+2$. The first a terms are easy to describe as we see that these admit a toric degeneration to X_Σ with Σ being the fan with rays $(-1, b)$, $(-1, 0)$, $(a, -1)$, $(a-u, -1)$, where u is the number of blowups. The $(a+1)$'st blowup admits a toric degeneration to $(-1, b)$, $(-1, -1)$, $(a, -1)$. The toric degeneration of the $a+2$'nd blowup is a bit more finicky and goes on a case by case analysis. However we see it admits a degeneration as it is a Looijenga pair . We note that there is a birational relationship between X_a and X_b in that the a'th blowup of X_a is isomorphic to the b'th blowup of X_b. This generalises, in the following way.

Theorem 0.3.10. Given a singularity with small discrepancy such that the minimal resolution is $a_1, \ldots a_n$, this has at most n basic surfaces form the previous theorem, possibly less via symmetry. Let X_i be the surface constructed from X_{a_i}, if $i \neq 1$, n then this admits a cascade of length $a_i + 3 - n$, if $i = 1$, n then the cascade is of length $a_i + 4 - n$. After a_i blowups X_i is isomorphic to X_j blownup a_j times.

Proof. Assume the length of the singularity is not of length one. We start by noting that Y_i, the minimal resolution of X_i, has a cycle $C_1 \ldots C_n \in -K_{Y_i}$ with intersections $-a_1, \cdots -a_n, a_i + 4 - n$. If $a_i + 4 - n \leq -2$ then this is clearly not the minimal resolution of a log del pezzowith cyclic quotients. If $a_i + 4 - n = -1$ then we have a -1 curve intersecting the singularity twice.

So now assume $a_i + 4 - n > -1$, then if we blowup the surface $a_i + 4 - n$ times we can assume that all the points lie on this curve. Clearly if we blow up $a_i + 5 - n$ times we have $-1-$ curve intersecting the singularity twice. If we blowup point $P_1, \ldots, P_k$, with $k \leq a_i + 4 - n$, to obtain a surface x'. We wish to show $-K_{X'}$ is ample, note $-K_{X'} \cdot C = -K_X - \sum E_i \cdot C$. We note that the Weil group of X is generated by the toric boundary and the curves which

are occurring as blowups on the boundary. In the latter case clearly our product is unchanged and hence it is still greater than zero. As our singularity has length greater than 2, there are at most only two components to the toric boundary. The first is the strict transform of the l curve. $\qquad\square$

Corollary 0.3.11. Let S be a singularity with small discrepancy, $-a_1, \ldots, -a_n$ be the self intersection of the resolutions. Then if $n \geq \max(a_i) + 5$. Then there exists no log del pezzowith only singularities of type S.