Nothing in this chapter is original work, and references are provided. Throughout this thesis we work only over the field $\mathbb{C}$.

1. Toric Geometry

We use the traditional language set up in [5]. In particular we use the fact that a normal toric variety X of dimension n can be associated with a (non unique) fan $\Sigma \subset N \cong \mathbb{Z}^n$. We consider the dual lattice to N, denoted M. Any $m \in M$ corresponds to a character of the torus which in turn corresponds to a monomial function in the function field of X. We denote the 1-skeleton of one dimension cones in Σ by Σ^1. We also use this to refer to the set of primitive vectors generating those rays; this is a small abuse of notation that is always clear in context. We say a fan is complete if every lattice point $u \in N$ lies inside some cone $\sigma \subset \Sigma$. The associated variety to a complete fan is complete.

1.1. Cox rings. Given a toric variety X, we wish to construct it as a GIT quotient. We follow the construction of [5]. Given a complete fan Σ with $\Sigma^1 = \{v_1, \dots, v_m\}$, we consider the toric variety given by a fan $\overline{\Sigma} \subset \mathbb{Z}^m$, with $\overline{\Sigma}^1 = \{e_i\}$, where e_i are the standard basis vectors. A set $\{e_i\}_{i \in S}$ spans a cone in $\overline{\Sigma}$ if and only if the set $\{v_i\}_{i \in S}$ spans a cone of Σ. The variety Y associated to $\overline{\Sigma}$ is a subset of K^m. By construction we have a well defined map of fans $\phi \colon \overline{\Sigma} \to \Sigma$ corresponding to a linear projection. This induces a map $\tilde{\phi} \colon Y \to X$ which can be seen as a GIT quotient with weights corresponding to the linear dependencies of Σ^1, and a finite group corresponding to the index of the sublattice of N generated by Σ^1.

1.2. Cyclic quotient singularities and singularity content. We also make frequent use of the following concepts introduce in [8] and [1]. Suppose given a cyclic quotient singularity $S = \frac{1}{r}(a, b)$ in two dimensions. Here S is the quotient of $\mathbb{C}^2$ by the group $G \cong \frac{\mathbb{Z}}{r\mathbb{Z}}$, with action defined by the matrix

$$\begin{pmatrix} \zeta^a & 0 \\ 0 & \zeta^b \end{pmatrix}$$

where $\zeta = e^{\frac{2\pi i}{r}}$. Without loss of generality a and b are coprime to r. This in turn implies that, by change of basis, we can write S as $\frac{1}{r}(1, u)$. The minimal resolution of this singularity is a chain of curves $C_1, \dots, C_n$ with self intersections equal to $[a_1, \dots, a_n]$, where these values a_i are equal to the coefficients of the Hirzebruch Jung continued fraction of $\frac{r}{u}$, as laid out in [8].

We are mainly interested in studying the restricted class of deformations know as $\mathbb{Q}$-Gorenstein as given in [10].

Definition 1.1. *For X a normal projective surface with quotient singularities, a $\mathbb{Q}$-Gorenstein smoothing is a one parameter flat family over $\mathcal{X} \to \mathcal{D}$ such that the total space is $\mathbb{Q}$-Gorenstein.*

Singularity content is a concept introduced in [1] as a $\mathbb{Q}$-Gorenstein deformation invariant of a surface. Given a surface singularity S we define the index one cover S_1 to be the quotient of $\mathbb{C}^2$ by the subgroup $H = G \cap SL_2(\mathbb{Z})$. This gives $\mathbb{C}^2 \to S_1 \to S$ where S_1 has a singularity of type A_n, and this has equation $xy = z^{n+1}$. The group G/H acts on S_1 with quotient S. That is, this group acts on $xy = z^{n+1}$ with some weight k; this means $G/H \cong \frac{\mathbb{Z}}{\frac{r}{n}\mathbb{Z}}$ acts naturally by some weights on the x, y, z and the equation has weight k. This gives us the $\mathbb{Q}$-Gorenstein deformations of S are the quotients of the equivariant deformations $xy = \sum a_i z^{k+i\frac{r}{n}}$. This is smooth if and only if $k = 0$. On the other hand if $k \neq 0$ the deformation has a residual singularity $\frac{1}{r'}(a', b')$. We call the pair $(n, \frac{1}{r'}(a', b'))$ the singularity content. If $n = 0$ we say the singularity is $\mathbb{Q}$-Gorenstein rigid. The value n can be seen to be equal to the topological Euler number of the $\mathbb{Q}$-Gorenstein smoothing with the singular point removed, although this is not used in this thesis.

Given a log del Pezzo surface X with only $\mathbb{Q}$-Gorenstein rigid singularities, we define the singularity content $(n, \{S_1, \ldots, S_n\})$ where S_i are the singularities of X and n is once again the topological Euler number of $X^0 = X - \{\text{Singular locus of } X\}$. In [?] we show how the value n fits into the language of affine manifolds.

2. Log del Pezzo background

2.1. Definitions. We here relate some basic definitions and facts about surfaces.

Given a normal surface singularity S and minimal resolution $\pi\colon \widetilde{S} \to S$ then we have

$$K_{\widetilde{S}} = \pi^*(K_S) + \sum a_i E_i$$

Definition 2.1. *Throughout this thesis a log del Pezzo surface is a normal complex projective surface with log terminal singularities and $-K_X$ ample.*

Where we say a singularity is

- terminal singularities if $a_i > 0$
- canonical singularities if $a_i \geq 0$
- log terminal singularities if $a_i \geq 0$
- log canonical singularities is $a_i \geq 1$

A surface singuarity is log terminal if and only if it can be constructed as a quotient of $\mathbb{C}^2$ by a, not necessarily cyclic, group action [9]. The classification of smooth log del Pezzo surfaces have been classified as the blowups of $\mathbb{P}^2$ at less than 9 general points.

Given an orbifold log del Pezzo surface we frequently use the invariants $-K_X^2$ and $h^0(-K_X)$. These can be via orbifold Riemann Roch as set out in [3]. For a rough sketch of how we do these calculations, given a singular X, with minimal resolution Y. Then $-K_Y^2$ and $h^0(-K_Y)$. To account for these the contractions, there is a correction term which we calculate via toric geometry, in the case of case of $-K_X^2$ this corresponds to the area of lattice cones contained in N corresponding to the singularities and in the case of $h^0(-K_X)$ this corresponds to a count of lattice points in the dual of the cone inside the lattice M. These are invariant under $\mathbb{Q}$-Gorenstein deformation.

2.2. Hirzebruch Surfaces. We briefly state some basic results about Hirzebruch surfaces [7]. A Hirzebruch surface is a rational scroll defined as the quotient of $\mathbb{C}^4$ by $(\mathbb{C}^*)^2$ with weights $(1, -1, 0, 0)$ and $(n, 0, 1, 1)$. Alternatively it is the minimal resolution of $\mathbb{P}(11n)$. From this we see that we have the picard group generated by B and F, where $B^2 = -n$ and F is a fiber of the map to $\mathbb{P}^1$. From this it is straight forwards to see the possible smooth rational curves on a Hirzebruch surface, let $A = B + nF$ then every smooth reduced rational curve lies in one of the linear equivalence classes $|A|$, $|2A|$, $|A + F|$, $|A + 2F|$, $|B|$, $|F|$ and $|2F|$.

2.3. Basic Surfaces. We finish with a very brief overview of [?], [?] and [?] as some of the methods we employ are similar. Respectively these paper classify log del Pezzos with singularities with minimal resolution [3] in [?], $[3, 2]$ and [3] in [?], and finally one singularity with resolution $[n]$ in [?]. The structure is similar, classify the possible surfaces X which admit no Mori contractions to another surface which could arise from these choices of singularities. These are called basic surfaces. Then study their blowups and their birational relations, often in the context of cascades as introduced by [?]. Via these explicit classifications they have been able to give explicit coordinate contstructions and their toric degenerations (when they exist). In Chapter 2 we classify log del Pezzo surfaces with singularities of the form $[a_1, \ldots, a_n]$ with both a_1 and a_n greater than two via similar although modified methods.

3. Gross Siebert

In this section we do not use or even refer to the full power of the Gross-Siebert program, we are mainly using results referencing how certain SYZ fibrations are constructed and how these give rise to toric degenerations. Our main reference throughout is [6], mainly chapters 1.4 and 1.5. We have made a variety of small changes to notation, namely our fans lie inside N instead of M to be consistent with the rest of the notation within the thesis and we do not consider fans on non compact affine manifolds.

We start with the definition of a tropical affine manifold

Definition 3.1 ([6] 1.22). *A tropical affine manifold is a real topological manifold (possibly with boundary) with an atlas of coordinate charts such that every transition function lies in $N_{\mathbb{R}} \rtimes GL(N)$.*

and then we allow singularities on a tropical manifold is a manifold B by taking a codimension two or more locus Δ such that $B - \Delta$ is a tropical affine manifold.

We finish by extending the definition of a fan to lie on tropical affine manifolds with singularities. We say a collection of cones Σ on a tropical affine manifold B is a fan, if it satisfies the usual condition of a fan in a lattice N and the condition that locally around every cone $\sigma \in \Sigma$ there is embedding of σ into a lattice. For the full technical definition see [6][1.4]. Now given a surface X such that the minimal resolution Y can be constructed from a toric variety Z_Σ by blowing up points on the boundary we show how this related to a fan on a tropical affine manifold. Denote by C_i a curve in the boundary of Z_Σ, and $\rho_i \in N$ the corresponding ray in Σ. If we blow up C_i k times then we introduce k singular points along the ray ρ_i each with monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in a a basis u_i, v_i, with u_i being the primitive generator of ρ_i. This gives a fan Σ' on a singular integral tropical manifold, such that when we look at the self intersection of the divisors D_i corresponding to rays we have exactly the same values as on the boundary of Y.

We now sketch out the discrete Legendre transform. In [6] this is defined with respect to a polarisation. For the sake of brevity we ignore this technicality, however for a complete definition see [6] section 1.6. We define the discrete Legendre transform of a fan Σ on a tropical affine manifold with singularities to be the integral tropical affine manifold with boundary such that the normal fan to each face is Σ.

We finish by providing the following example: the del Pezzo surface of degree five. We can consider this by blowing up $\mathbb{P}^2$ twice torically giving a boundary of $[-1, -1, -1, 0, 0]$ and then blowing up a non toric point on both the 0-curves. This gives the following fan on an affine manifold:

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