0.1 context

This is a first draft, context will be inserted

0.2 Result

Throughout the rest of this chapter we limit ourselves to cyclic quotient singularities S with the following property - let $C_1, \dots C_n$ be the minimal resolution of S, let the values $a_1 \dots a_n$ be the respective discrepancies, we insist that $a_i \leq \frac{1}{2}$. We call this a singularity with small discrepancy. This can be compared to other peoples work, in the following lemma

Lemma 0.2.1. Let S be a cyclic quotient singularity, $C_1 \dots C_n$ the minimal resolution. If $C_i^2 \leq -4$ then S has small discrepancy.

Proof. This is easy to see as S can be described as a toric singularity with rays (u_0, v_0), (u_{n+1}, v_{n+1}). Where $v_0 = v_{n+1} = h$ is the gorenstein index of S. A given curve C_i in the minimal resolution corresponds to a ray (u_i, v_i) inside the above cone, so $v_i \leq h \forall i$. Now $(u_i, v_i) = \frac{(u_{i-1}, v_{n-1}) + (u_{i+1}, v_{i+1})}{C_i^2}$. So $v_i \leq \frac{\max(v_{i-1}, v_{i+1})}{2} \leq \frac{h}{2}$. The discrepancy of the curve C_i is equal to $\frac{v_i - h}{h}$ which is clearly less than $\frac{-1}{2}$. $\square$

We no explain why this makes classification so easy

Lemma 0.2.2. Let X be a surface and $f : Y \to X$ be the minimal resolution of X. Let $C \subset X$ be a curve that such that C intersects two singularities (potentially the same) with small discrepancy. Consider the curve $\widetilde{C} \subset Y$ the strict transform. Then if $C^2 = -1$ then $-K_X \cdot C \leq= 0$.

Proof. Let $f : Y \to X$ be the minimal resolution of X, $\widetilde{C} \subset Y$ the strict transform of C. As C is a smooth curve on a smooth surface $-K_Y \cdot \widetilde{C} = 1$. We know that C intersects at least two exceptional curve E_i, E_j, with discrepancy a, b. Via ??? we see that $-K_X \cdot C = f^*(-K_X) \cdot \widetilde{C} \leq 1 - a - b \leq 0$. $\square$

Hence this curve configuration cannot lie on a log del pezzo . We also make the quick remark that in the case where the length n of the singularity is 1 or 2, this remark follows via easy toric geometry as any curve joining two singularities is a locally toric configuration. This corresponds to the asscoiated fan being non convex.

Lemma 0.2.3. Let X be a log del pezzo with only singularities of small discrepancy. As above let $f : Y \to X$ be the minimal resolution then we consider the map $\pi : Y \to \mathbb{F}_l$. Consider E exceptional curves in the minimal resolution. Then $\pi_* E$ is one of the following

- A smooth curve with positive self intersection

- A smooth curve with negative self intersection
- A point

Proof. This just amounts to it being impossible for $\pi_* E$ to be a non smooth curve. Hence assume that it has a singular point P. In order for us to get a cyclic quotient singularity, E needs to be smooth. Hence there is a collection of curves $C_i \subset Y$ which blowdown to P. As all these curves are contracted $C_i^2 \leq= 0$ and $E^2 \leq 0$, and there is a curve C_j with $C_j^2 = -1$. Clearly C_j intersects either two curves with self intersection less than -1, or it could intersect E twice. By the above lemma neither case could appear on the minimal resolution of a log del pezzo . $\square$