0.1 The $\mathbb{Q}$g rigid singularities

We wish to study groups G that are finite subgroups $\mathrm{GL}_2(\mathbb{C})$ with the property $G \cap \mathrm{SL}_2(\mathbb{C}) = \mathrm{BD}_{4n}$. Consider the three elements

$$a = \begin{pmatrix} \zeta_m & 0 \\ 0 & \zeta_m \end{pmatrix} \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad c_m = \begin{pmatrix} \sigma_n & 0 \\ 0 & \sigma_n^{-1} \end{pmatrix}$$

We split this into two cases the case where m is odd or even. In the odd case $G = \langle a,\, b,\, c_m \rangle$, otherwise it it is $\langle a,\, b \circ c_{2m} \rangle$. Similar analysis can be done for the groups E_7 and E_8. We now discuss when their $\mathbb{Q}$g smoothings.

Case 1

Here m is odd so $G = \langle a,\, b,\, c \rangle$. Looking at the index one cover we have a $x^2 + y^2 z + z^{n+1}$. With $x = (u^{2n} - v^{2n})uv$, $y = u^{2n} + v^{2n}$, $z = u^2 v^2$. Looking at the derivatives we see that $\mathcal{T}^1$ is generated by $1, y, z, \cdots z^{n-1}$. As this is a rational surface singularity we see that $\mathcal{T}^2 = 0$, so these deformations are unobstructed. We see that c acts on x, y, z with weights $(2n+2, 2n, 4)$. This correspond to $\frac{1}{m}(n+1, n, 2)$ action. Our equation has weight $2n+2$. To check rigidity we first need $n \not\cong_m -2$. In this case we have y cannot be in the qG smoothing. Other than that we just need z^i cannot have weight $2n+2$. This corresponds to $2n+2 \not\cong_m 2i$ for i in $0 \dots n-1$. These will be the only qG rigid singularities. In particular this means that if $m > 2n+2$ then it is rigid.

Case 2

Here $m = 2m'$ is even so $\langle a,\, b \circ c_{2m} \rangle$. So we have the same equation as above and we now get a $\frac{1}{2m'}(2n+m', 2n, 4)$, so to be rigid once again we need the y term not to be in the smoothing $2n+4 \not\cong_{2m'} 0$. Once again we need no z^i terms this means $4n+4 \not\cong_{2m'} 4i$ for i in $0 \dots n-1$.

It is easy to see that if a partial smoothing exists then it either stay non cyclic quotient or, if you can deform equivariantly by y, it becomes a cyclic quotient singularity which is of the form $\frac{1}{m}(n+1, 2)$ or $\frac{1}{2m'}(2n_m, 4)$. From this it is easy to classify the $\mathbb{Q}g$ smoothable singularities. We note for a given n there exists only a finite amount of $\mathbb{Q}g$ smoothable singularities with the D_n singularity as its index one cover. This is in contrast to the A_n case.

0.2 Singularity content

We wish to generalise the notion of singularity content from cyclic quotients. We have that given a non cyclic quotient there exists a unique residual singularity. We wish to have other invariants indicating what it has smoothed from. The natural thing is to consider the invariant $h^2_{\mathrm{top}}(X^0)$ where X^0 is $X - \{singularlocus\}$. First considering the case where there is no smoothing in terms of y so the only terms we can put in are of the form z^i giving

$x^2 + y^2 z + z^j \Pi_{k=1}^{p} (z^i - a_k)$ then it is clear that we have exactly by projecting $(x, y, z) \mapsto z$ we get a map to $\mathbb{C}$ with degenerate fibers of 2 curves transversely intersecting over the finitely many points corresponding to $\lfloor \frac{n+1}{p} \rfloor$. **Ignoring the central fiber over the origin**. We can apply Mayer-Vietoris repeatedly to get that $h^2_{\text{top}}(X^0)$ is a deformation invariant. In the case where we have y in the deformation we have