0.1 E_6 singularity

The E_6 singulrity arises as a quotient of the binary tetrahedral group $G \subset \mathrm{SL}_2(\mathbb{C})$, which is of the form

$$G = \langle a,\, b,\, c \,|\, a^2 = b^2 = c^3 = -I_2,\, (ac)^3 = (bc)^3 = I_2 \rangle$$

This has two distinct representations in $GL_2(\mathbb{C})$ the first is given by

$$a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad c = \tfrac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$$

This is the standard embedding into $\mathrm{SL}_2(\mathbb{C})$. Consider σ_3 then using the previous embedding and considering $c \mapsto \sigma_3 c$ provides a different embedding of G in $\mathrm{GL}_2(\mathbb{C})$. We now consider the possible central extensions of G by a cyclic group μ_k. This splits into the two cases where this is a direct sum and where it is not. We start with the second case. This will have to be a map of the form $a,\, b,\, c \mapsto \sigma_i a,\, \sigma_j b,\, \sigma_k c$ where the $\sigma \in \mu_A$. Looking at where c is sent to, we see that clearly $3|A$ otherewise the extension would contain $-\sigma_k^3$ which would imply it contains σ_k factor out by the group generated by $-\sigma_k$ and get just an extension which changes a and b. Via similar logic we would have that this extension would have to be a power of 2. By looking at the second equation we would get it contains $\sigma_i^3,\, \sigma_j^3$ these generate μ_{2^B}. Hence this cannot occur. By this logic we see that $A = 3^n$. We now wish to show that fixing n gives rise to a unique extension. Clearly a, b need to be snet to a mutliple of $\sigma_{3^{n-1}}$, we still need that $a^2 = b^2 = c^3$ otherwise the group would not be finite. This fixes what c is sent to upto a factor of 3. Writing this out in explicit terms we take our earlier representation and consider $a \mapsto \sigma_{3^{n-1}}^{\frac{1}{2}} a,\, b \mapsto \sigma_{3^{n-1}}^{\frac{1}{2}} b,\, c \mapsto \sigma_{3^n} c$, where explicitly $\sigma_j = e^{\frac{2\pi i}{j}}$. This has equations

$$G_{3^n} = \langle a,\, b,\, c \,|\, a^2 = b^2 = c^3 = -\sigma_3^{n-1} I_2,\, (ac)^3 = (bc)^3 = \sigma_3^{n-1} I_2 \rangle$$

This has size $3^{n-1} \times 24$. Now we consider the group $G_{m,n} = \mu_m \oplus G_{3^n}$. Clearly m has no factors of 3, otherwise we could ignore the non direct extension. Now $G_{m,n} \cap \mathrm{GL}_2(\mathbb{C})$ is the binary dihedral group BD_{n-2}