Project 2: 2-period-lived agent overlapping generations model

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This is a description of a two-period overlapping generations model and its steady-state equilibrium.

1 Individuals

Assume that a unit measure of individual agents are born each period t and live for two periods, indexed by $s = \{1, 2\}$.[1] Table 1 shows the structure of this economy, with the economy beginning in period $t = 1$ and continuing in perpetuity. This asymmetric structure of time is important for many of the welfare properties that are unique to the OG framework.

Table 1: Two-period-lived OG structure

	Period					
Birthday	...	t	$t+1$	$t+2$	$t+3$	...
born $t-1$	...	$c_{2,t}$				
born t		$c_{1,t}$	$c_{2,t+1}$			
born $t+1$			$c_{1,t+1}$	$c_{2,t+2}$		
born $t+2$				$c_{1,t+2}$	$c_{2,t+3}$	
					$\vdots$	...

The rows represent the lifetime of a given cohort, and the columns represent a cross section of the population alive at time t.

Assume that each individual agent supplies labor $n_{s,t}$ exogenously each period

[1] Individuals face a zero probability of death between ages $s = 1$ and $s = 2$, and have a 100-percent probability of death at the end of age $s = 2$.

such that the following conditions hold.

$$n_{1,t} = n_1 > 0 \quad \forall t \tag{1}$$
$$n_{2,t} = n_2 \geq 0 \quad \forall t \tag{2}$$
$$\tag{3}$$

The young age $s = 1$ and the old age $s = 2$ budget constraints for the individual are the following.

$$c_{1,t} + b_{2,t+1} = w_t n_1 \tag{4}$$
$$c_{2,t+1} = (1 + r_{t+1})b_{2,t+1} + w_{t+1}n_2 \tag{5}$$

Note that this assumes that individuals are born with no wealth $b_{1,t} = 0$ for all t and that individuals save nothing when they are old $b_{3,t+2} = 0$ for all t.

Agents choose consumption $c_{s,t}$ and savings $b_{s+1,t+1}$ each period in each period to maximize lifetime utility.

$$\max_{c_{1,t},c_{2,t+1},b_{2,t+1}} u\big(c_{1,t}\big) + \beta u\big(c_{2,t+1}\big) \quad \forall t \tag{6}$$
$$\text{s.t.} \quad c_{1,t} = w_t n_1 - b_{2,t+1} \tag{4}$$
$$\text{and} \quad c_{2,t+1} = (1 + r_{t+1})b_{2,t+1} + w_{t+1}n_2 \tag{5}$$

Assume that the period utility function $u(c)$ is constant relative risk aversion (CRRA),

$$u(c) \equiv \frac{(c)^{1-\gamma} - 1}{1-\gamma} \quad \forall c, \gamma > 0 \tag{7}$$

with the properties that the $\gamma = 1$ case reduces to log utility in the limit,

$$\lim_{\gamma \to 1} \frac{c^{1-\gamma} - 1}{1-\gamma} = \ln(c) \quad \forall c > 0 \tag{8}$$

and that the marginal utility of consumption is simply $c^{-\gamma}$.

$$\frac{\partial}{\partial c}\left(\frac{c^{1-\gamma} - 1}{1-\gamma}\right) = c^{-\gamma} \quad \forall c, \gamma > 0 \tag{9}$$

If we substitute the period budget constraints (4) and (5) into the lifetime utility function (6), the individual reduces to a maximization problem in one variable.

$$\max_{b_{2,t+1}} u\big(w_t n_1 - b_{2,t+1}\big) + \beta u\Big([1 + r_{t+1}]b_{2,t+1} + w_{t+1}n_2\Big) \quad \forall t \tag{10}$$

The solution to the individual's problem is characterized by taking the derivative of the life time utility subject to the budget constraints (10) with respect to savings $b_{2,t+1}$, which results in the following Euler equation.

$$\big(w_t n_1 - b_{2,t+1}\big)^{-\gamma} = \beta(1 + r_{t+1})\Big([1 + r_{t+1}]b_{2,t+1} + w_{t+1}n_2\Big)^{-\gamma} \quad \forall t \tag{11}$$

The Euler equation (11) implicitly characterizes the individual's optimal choice of $b_{2,t+1}$ as a function of prices over the lifetime of the individual.[2]

$$b_{2,t+1} = \psi\big(w_t, w_{t+1}, r_{t+1}\big) \quad \forall w_t, w_{t+1}, r_{t+1} > 0 \tag{12}$$

2 Firms

The economy also includes a unit measure of identical, perfectly competitive firms that rent investment capital from individuals for real return r_t and hire labor for real wage w_t. Firms use their total capital K_t and labor L_t to produce output Y_t every period according to a Cobb-Douglas production technology.

$$Y_t = F(K_t, L_t) \equiv AK_t^\alpha L_t^{1-\alpha} \quad \text{where} \quad \alpha \in (0,1) \quad \text{and} \quad A > 0 \tag{13}$$

We assume that the price of the output in every period $P_t = 1$.[3] The representative firm chooses how much capital to rent and how much labor to hire to maximize profits,

$$\max_{K_t, L_t} AK_t^\alpha L_t^{1-\alpha} - (r_t + \delta)K_t - w_t L_t \tag{14}$$

where $\delta \in [0,1]$ is the rate of capital depreciation.[4] The two first order conditions that characterize firm optimization are the following.

$$r_t = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta \tag{15}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{16}$$

3 Market clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

$$L_t = n_1 + n_2 \quad \forall t \tag{17}$$

$$K_t = b_{2,t} \quad \forall t \tag{18}$$

$$\begin{aligned} &Y_t = C_t + I_t \quad \forall t \\ &\quad \text{where} \quad I_t \equiv K_{t+1} - (1-\delta)K_t \\ &\quad \text{and} \quad C_t \equiv c_{1,t} + c_{2,t} \end{aligned} \tag{19}$$

[2] It is true that we can solve (11) analytically for $b_{2,t+1}$. However, in the steady-state equilibrium, both the wage w_t and the interest rate r_{t+1} will be functions of $b_{2,t+1}$. So we will just leave the policy function $b_{2,t+1} = \psi(w_t, w_{t+1}, r_{t+1})$ implicitly defined.

[3] This is just a cheap way to assume no monetary policy. Relaxing this assumption is important in many applications for which price fluctuation is important.

[4] Note that it is equivalent whether we put depreciation on the firms' side as in equation (14) or on the household side making the return on capital savings $1 + r_t - \delta$. Depreciation must be in one place or the other, not both. We choose to put depreciation on the firm's side here because the tax model we are building up to includes taxes and subsidies to firms for depreciation expenses.

The goods market clearing equation (19) is redundant by Walras' Law.

4 Steady-state equilibrium

The steady-state equilibrium is a long run concept. It is the equilibrium in which all the endogenous variables have "settled down" and are constant over time. The t subscripts go away. In the steady-state, generic variable x_t satisfies the following property for all periods t and after.

$$\text{steady-state} \quad x_t = x_{t+1} = \bar{x} \tag{20}$$

So for example, in the steady-state, $w_t = \bar{w}$ and $b_{2,t+1} = \bar{b}_2$ for all t.

Definition 1 (**Steady-state equilibrium**). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with 2-period lived agents is defined as constant allocations of consumption $\{\bar{c}_1, \bar{c}_2\}$, savings $\bar{b}_2$, and prices $\bar{w}$ and $\bar{r}$ such that:

1. households optimize according to (11), (4), and (5),
2. firms optimize according to (15) and (16),
3. markets clear according to (17) and (18).

This means that the steady-state equilibrium is characterized by the following five steady-state equations and the following five unknowns $(\bar{b}_2, \bar{w}, \bar{r}, \bar{K}, \bar{L})$.

$$\big(\bar{w}n_1 - \bar{b}_2\big)^{-\gamma} = \beta(1+\bar{r})\Big([1+\bar{r}]\bar{b}_2 + \bar{w}n_2\Big)^{-\gamma} \tag{21}$$

$$\bar{r} = \alpha A\left(\frac{\bar{L}}{\bar{K}}\right)^{1-\alpha} - \delta \tag{22}$$

$$\bar{w} = (1-\alpha)A\left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha} \tag{23}$$

$$\bar{L} = n_1 + n_2 \tag{24}$$

$$\bar{K} = \bar{b}_2 \tag{25}$$

Note that the steady-state equations for $\bar{c}_1$ and $\bar{c}_2$ are in (21). For reference, the steady-state equations for consumption are the following.

$$\bar{c}_1 = \bar{w}n_1 - \bar{b}_2 \tag{26}$$

$$\bar{c}_2 = \big(1+\bar{r}\big)\bar{b}_2 + \bar{w}n_2 \tag{27}$$

This system of five equations and five unknowns can be reduced to one equation and one unknown. First, substitute the steady-state market clearing conditions (24) and (25) into the firm's steady-state first order conditions (22) and (23). This shows that steady-state interest rate $\bar{r}$ and steady-state wage $\bar{w}$ are only functions of the household savings $\bar{b}_2$.

$$\bar{r}\big(b_2\big) = \alpha A\left(\frac{n_1+n_2}{\bar{b}_2}\right)^{1-\alpha} - \delta \tag{28}$$

$$\bar{w}\big(b_2\big) = (1-\alpha)A\left(\frac{\bar{b}_2}{n_1+n_2}\right)^{\alpha} \tag{29}$$

Now substitute those two functions for $\bar{r}(\bar{b}_2)$ and $\bar{w}(\bar{b}_2)$ (28) and (29) into the steady-state Euler equation to get one big nonlinear equation and one unknown $\bar{b}_2$.

$$\Big(\bar{w}\big(\bar{b}_2\big)n_1 - \bar{b}_2\Big)^{-\gamma} = \beta\Big[1+\bar{r}\big(\bar{b}_2\big)\Big]\bigg(\Big[1+\bar{r}\big(\bar{b}_2\big)\Big]\bar{b}_2 + \bar{w}\big(\bar{b}_2\big)n_2\bigg)^{-\gamma} \tag{30}$$

5 Group Git and GitHub Exercise

5.1 Stage 1: Create the basic functions

Exercise 1. Make the appropriate modifications to the `FirmMC.py` file in the `nyupredocs/githubtutorial/Projects/Project2` folder to create a Python module that has the following four functions that generate $\bar{r}$, $\bar{w}$, $\bar{K}$, and $\bar{L}$ according to steady-state equations (24), (25), (28), and (29). Follow the instructions in the `FirmMC.py` file. Test your functions using either iPython or Jupyter notebook to make sure they work properly. Submit your changes to the repository via pull request.

Exercise 2. Make the appropriate modifications to the `household.py` file in the `nyupredocs/githubtutorial/Projects/Project2` folder to create a Python module that has the following three functions that generate $\bar{c}_1$, $\bar{c}_2$, and the marginal utility of consumption $c^{-\gamma}$ according to steady-state equations (26), (26), and (9). Follow the instructions in the `household.py` file. Test your functions using either iPython or Jupyter notebook to make sure they work properly. Submit your changes to the repository via pull request.

Exercise 3. Make the appropriate modifications to the `euler.py` file in the `nyupredocs/githubtutorial/Projects/Project2` folder to create a Python module that has the following two functions that generate a steady-state Euler error for a given value of $\bar{b}_2$ and that generate the optimal value of $\bar{b}_2$ according to steady-state equations (30). Follow the instructions in the `euler.py` file. Test your functions using either iPython or Jupyter notebook to make sure they work properly. Submit your changes to the repository via pull request.

5.2 Stage 2: Put functions together to solve for SS

Exercise 4. Make the appropriate modifications to the `execute.py` file in the `nyupredocs/githubtutorial/Projects/Project2` folder to create a Python script that declares parameter values, solves for and prints the steady-state savings $\bar{b}_2$ and solves for and prints the other steady-state variables $\bar{c}_1$, $\bar{c}_2$, $\bar{r}$, $\bar{w}$, $\bar{K}$, $\bar{L}$, $\bar{Y}$, $\bar{C}$, and $\bar{I}$. Follow the instructions in the `execute.py` file. Test your script using either iPython or Jupyter notebook to make sure they work properly. Submit your changes to the repository via pull request.