Constant Elasticity of Substitution notes

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1 General

Define two goods as x_1 and x_2. Varian (1992, p. 112) defines a constant elasticity of substitution (CES) utility function as the following.

$$u(x_1, x_2) \equiv \left(x_1^\rho + x_2^\rho\right)^{\frac{1}{\rho}}, \quad x_1, x_2, \rho > 0 \tag{1}$$

The demand functions for x_1 and x_2 resulting from utility maximization or cost minimization of (1) given prices of each good p_1 and p_2 and a composite price P derived as the shadow price of composite consumption is the following.

$$x_1 = \left(\frac{p_1}{P}\right)^{-\frac{1}{1-\rho}} u(x_1, x_2) \tag{2}$$

$$x_2 = \left(\frac{p_2}{P}\right)^{-\frac{1}{1-\rho}} u(x_1, x_2) \tag{3}$$

Preceding Varian, Armington (1969, p. 167) proposes a specification motivated by modeling a trade economy with many goods. This is the most common form, that we follow in our model. Armington's is also a consumer utility function aggregator used to derive consumer demand functions, and it is equivalent to Varian's (1). With a minor change of variables to what Armington listed, the form is the following for N goods,

$$u\left(x_1, x_2, ...x_N\right) \equiv \left[\sum_{n=1}^{N} (\alpha_n)^{\frac{1}{\varepsilon}} (x_n)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \quad \text{for} \quad \varepsilon > 1 \tag{4}$$

where ε is the constant elasticity of substitution and α_n is a share parameter on the nth good such that $\sum_{n=1}^{N} \alpha_n = 1$.[1]

The demand functions for x_n resulting from utility maximization or cost minimization of (4) given prices of each good p_n and a composite price P derived as the shadow price of composite consumption is the following.

$$x_n = \alpha_n \left(\frac{p_n}{P}\right)^{-\varepsilon} u(\boldsymbol{x}) \tag{5}$$

[1] The mapping from (4) to what is shown in Armington (1969, p. 167) is $x_n = X_{i,m}$, $\varepsilon = \sigma_i$, $(\alpha_n)^{\frac{1}{\varepsilon}} = b_{i,1}$, and $\frac{\varepsilon-1}{\varepsilon} = -\rho_i$.

2 Jason's OG-USA Guide

In Jason's `OG-USA` Guide, the CES aggregator for assets between debt and equity is equation (2.14).

$$a_{j,s,t} \equiv \left[(\gamma_{a,s})^{-\frac{1}{\varepsilon_{a,s}}}\left(b_{j,s,t}\right)^{\frac{1+\varepsilon_{a,s}}{\varepsilon_{a,s}}} + (1-\gamma_{a,s})^{-\frac{1}{\varepsilon_{a,s}}}\left(e_{j,s,t}\right)^{\frac{1+\varepsilon_{a,s}}{\varepsilon_{a,s}}}\right]^{\frac{\varepsilon_{a,s}}{1+\varepsilon_{a,s}}} \quad \forall j,s,t \quad \text{and} \quad \varepsilon_{a,s} < 0 \tag{6}$$

If we assume that $\varepsilon_{a,s}$ in (6) is the negative of the elasticity of substitution, then (6) is equivalent to (4) and (1). That is, the $\varepsilon_{a,s}$ parameter in (6) must be negative $\varepsilon_{a,s} < 0$ for the demand functions to make sense.

However, the two demand functions for this functional form are incorrectly derived, as is the composite price function. I ran into this same mistake in my derivation, and I will demonstrate how to correct it. The incorrect demand functions from the OG-USA guide for private capital $b_{j,s,t}$ and equity $e_{j,s,t}$ (which same mistake I also made) are the following,

$$b_{j,s,t} = \gamma_{a,s}\left(\frac{\rho_{b,j,s,t}}{\rho_{e,j,s,t}}\right)^{\varepsilon_{a,s}} a_{j,s,t} \quad \forall j,s,t \tag{7}$$

$$e_{j,s,t} = (1-\gamma_{a,s})\left(\frac{\rho_{e,j,s,t}}{\rho_{b,j,s,t}}\right)^{\varepsilon_{a,s}} a_{j,s,t} \quad \forall j,s,t \tag{8}$$

where ρ_b and ρ_e are the respective after-tax returns to capital investment and equity investment.

Note that (7) implies that demand for saving with firms $b_{j,s,t}$ decreases if the relative return on saving with firms ρ_b increases because $\varepsilon_{a,s}$ is negative. Similarly, (8) implies that demand for saving with equity $e_{j,s,t}$ decreases if the relative return on equity saving ρ_e increases. Furthermore, the composite price for this CES aggregation function is incorrectly specified as the following.

$$\rho_{a,j,s,t} = \left[(\gamma_{a,s})(\rho_{b,j,s,t})^{\varepsilon_{a,s}} + (1-\gamma_{a,s})(\rho_{e,j,s,t})^{\varepsilon_{a,s}}\right]^{\frac{1}{1+\varepsilon_{a,s}}} \quad \forall j,s,t \tag{9}$$

It should be the following.

$$\rho_{a,j,s,t} = \left[(\gamma_{a,s})(\rho_{b,j,s,t})^{-(1+\varepsilon_{a,s})} + (1-\gamma_{a,s})(\rho_{e,j,s,t})^{-(1+\varepsilon_{a,s})}\right]^{-\frac{1}{1+\varepsilon_{a,s}}} \quad \forall j,s,t \tag{10}$$

3 Corrected demand and composite price functions

The place where we made a mistake was in setting up the cost minimization problem for this portfolio problem. We treated the returns as prices, whereas the prices are actually the inverse of the return. The correct cost minimization problem is the following Lagrangian.

$$\mathcal{L} = \frac{1}{\rho_b}b + \frac{1}{\rho_e}e + \lambda\left(a - \left[(\gamma)^{-\frac{1}{\varepsilon}}b^{\frac{1+\varepsilon}{\varepsilon}} + (1-\gamma)^{-\frac{1}{\varepsilon}}e^{\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{1+\varepsilon}}\right) \tag{11}$$

The multiplier λ is the shadow price of an extra unit of composite asset a, so we can rename it the composite price and denote it similarly as the inverse of the composite return $1/\rho_a$.

$$\mathcal{L} = \frac{1}{\rho_b}b + \frac{1}{\rho_e}e + \frac{1}{\rho_a}\left(a - \left[(\gamma)^{-\frac{1}{\varepsilon}}b^{\frac{1+\varepsilon}{\varepsilon}} + (1-\gamma)^{-\frac{1}{\varepsilon}}e^{\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{1+\varepsilon}}\right) \tag{12}$$

Taking derivatives with respect to b, e, and $1/\rho_a$ and setting them equal to zero gives the following three first order conditions.

$$b = \gamma\left(\frac{\rho_b}{\rho_a}\right)^{-\varepsilon}a \tag{13}$$

$$e = (1-\gamma)\left(\frac{\rho_e}{\rho_a}\right)^{-\varepsilon}a \tag{14}$$

$$a = \left[(\gamma)^{-\frac{1}{\varepsilon}}b^{\frac{1+\varepsilon}{\varepsilon}} + (1-\gamma)^{-\frac{1}{\varepsilon}}e^{\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{1+\varepsilon}} \tag{6}$$

Note first that equations (13) and (14) are different from the Euler equations listed in the OG-USA guide (7) and (8). The denominators on the right-hand-side of (7) and (8) are not correct. Further, note that (13) and (14) imply that b increases when its return ρ_b increases (because $\varepsilon < 0$) and e increases when its return ρ_e increases.

Finally, we derive the composite return ρ_a on the composite asset a by substituting equations (13) and (14) into (6).

$$\begin{aligned}
a &= \left[(\gamma)^{-\frac{1}{\varepsilon}}b^{\frac{1+\varepsilon}{\varepsilon}} + (1-\gamma)^{-\frac{1}{\varepsilon}}e^{\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{1+\varepsilon}} \\
\Rightarrow\quad a &= \left[(\gamma)^{-\frac{1}{\varepsilon}}(\gamma)^{\frac{1+\varepsilon}{\varepsilon}}\left(\frac{\rho_b}{\rho_a}\right)^{-(1+\varepsilon)}a^{\frac{1+\varepsilon}{\varepsilon}} + (1-\gamma)^{-\frac{1}{\varepsilon}}(1-\gamma)^{\frac{1+\varepsilon}{\varepsilon}}\left(\frac{\rho_e}{\rho_a}\right)^{-(1+\varepsilon)}a^{\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{1+\varepsilon}} \\
\Rightarrow\quad a &= a\rho_a^{\varepsilon}\left[(\gamma)(\rho_b)^{-(1+\varepsilon)} + (1-\gamma)(\rho_e)^{-(1+\varepsilon)}\right]^{\frac{\varepsilon}{1+\varepsilon}} \\
\Rightarrow\quad \rho_a^{-\varepsilon} &= \left[(\gamma)(\rho_b)^{-(1+\varepsilon)} + (1-\gamma)(\rho_e)^{-(1+\varepsilon)}\right]^{\frac{\varepsilon}{1+\varepsilon}} \\
\Rightarrow\quad \rho_a &= \left[(\gamma)(\rho_b)^{-(1+\varepsilon)} + (1-\gamma)(\rho_e)^{-(1+\varepsilon)}\right]^{-\frac{1}{1+\varepsilon}}
\end{aligned} \tag{10}$$

That should do it.

References

Armington, Paul S., "A Theory of Demand for Products Distinguished by Place of Production," *Staff Papers, International Monetary Fund*, March 1969, *16* (1), 159–178.

Varian, Hal R., *Microeconomic Analysis*, third edition ed., W. W. Norton & Company, 1992.