Chapter 10: The Linear Search Theorem.

In which we introduce the simplest searching algorithm.

Suppose we are given an array f[0..N) of integer, where {$1 \le N$}, which is guaranteed to contain at least one occurrence of the value X, and we are asked to find the location of the leftmost X in f, i.e. the smallest index i where f.i = X.

We begin as usual with a problem specification.

$$\{\langle \exists j : 0 \le j < N : f.j = X \rangle\}$$

$$S$$

$$\{\langle \forall j : 0 \le j < n : f.j \ne X\rangle \wedge f.n = X\}$$

As usual, we begin by developing a model of the problem domain.

* (0) $C.n \quad \equiv \quad \langle \forall j : 0 \le j < n : f.j \ne X\rangle \quad , 0 \le n \le N$

Appealing to the empty range and associativity we get the following theorems

Consider.

$$\begin{array}{ll} & C.0 \\ = & \quad \{(0) \text{ in model }\} \\ & \langle \forall j : 0 \le j < 0: f.j \ne X\rangle \\ = & \quad \{ \text{ empty range }\} \\ & \text{true} \end{array}$$

- (1) $C.0 \quad \equiv \quad \text{true}$

Consider

$$\begin{array}{ll} & C.(n+1) \\ = & \quad \{(0) \text{ in model }\} \\ & \langle \forall j : 0 \le j < n+1: f.j \ne X\rangle \\ = & \quad \{ \text{ split off } j = n \text{ term}\} \\ & \langle \forall j : 0 \le j < n : f.j \ne X\rangle \wedge f.n \ne X \\ = & \quad \{(0) \text{ in model}\} \\ & C.n \wedge f.n \ne X \end{array}$$

- (2) $C.(n+1) \quad \equiv \quad C.n \wedge f.n \ne X \quad , 0 \le n < N$

Rewrite postcondition using the model.

Post : $C.n \wedge f.n = X$

Choose Invariants.

We choose as our invariants

P0: $C.n$
P1: $0 \le n \le N$

Establish Invariants.

Theorem (1) in our model shows us that we can establish P0 by the assignment

$n := 0$

This also establishes P1.

Termination.

We note that

$P0 \wedge P1 \wedge f.n = X \Rightarrow Post$

Guard

We choose our loop guard to be

B : $f.n \neq X$

Variant.

As our variant function we choose

$K - n$

Where $0 \le K < N$ and K is the (as yet unknown) index of the leftmost occurrence of X.

Calculate Loop body.

Decreasing the variant by the assignment n := n+1 is a standard step and maintains P1. Let us se what effect it has on P0

$$
\begin{array}{ll}
 & (n := n+1).\ P0 \\
= & \quad \{\text{textual substitution}\} \\
 & C.(n+1) \\
= & \quad \{(2)\ \text{above}\} \\
 & C.n \wedge f.n \neq X \\
= & \quad \{\ f.n \neq X \text{ at start of loop body}\} \\
 & C.i \\
= & \quad \{\ P0 \text{ is invariant }\} \\
 & \text{true}
\end{array}
$$

Finished program.

So our finished program is

```
  n := 0
; do f.n ≠ X →

            n := n+1

  od
{C.n ∧ f.n = X}
```

Generic Solution.

This problem is just one instance of a set of problems called the Linear Searches. We will now describe this family of problems and construct the generic solution.

Suppose we are given a finite, ordered domain, $f[\alpha..\beta)$ and a predicate Q defined on the elements of f. We are also given that Q holds true at at least one point in the domain. We are asked to find the lowest point at which this is the case. We begin with our specification.

$$\{\langle\ \exists\, j : \alpha \leq j < \beta : \neg Q.(f.j)\ \rangle\}$$

$$S$$

$$\{\langle\ \forall\, j : \alpha \leq j < n : \neg Q.(f.j)\ \rangle \wedge Q.(f.n)\}$$

As usual we develop our model

* (3) $C.n \quad \equiv \quad \langle \forall j : \alpha \leq j < n : f.j \neq X \rangle \qquad , \alpha \leq n \leq \beta$

Appealing to the empty range law and associativity we get the following theorems

Consider.

$$
\begin{array}{ll}
 & C.\alpha \\
= & \quad \{(0) \text{ in model }\} \\
 & \langle \forall j : \alpha \leq j < \alpha : f.j \neq X \rangle \\
= & \quad \{ \text{ empty range }\} \\
 & true
\end{array}
$$

- (1) $C.\alpha \quad \equiv \quad true$

Consider

$$
\begin{array}{ll}
 & C.(n+1) \\
= & \quad \{(0) \text{ in model }\} \\
 & \langle \forall j : \alpha \leq j < n+1 : f.j \neq X \rangle \\
= & \quad \{ \text{ split off } j = n \text{ term}\} \\
 & \langle \forall j : \alpha \leq j < n : f.j \neq X \rangle \wedge f.n \neq X \\
= & \quad \{(0) \text{ in model}\} \\
 & C.n \wedge f.n \neq X
\end{array}
$$

- (2) $C.(n+1) \quad \equiv \quad C.n \wedge f.n \neq X \qquad , \alpha \leq n < \beta$

We can now rewrite our postcondition as

Post : $C.n \wedge Q.(f.n)$

Choose Invariants.

We choose as our invariants

P0: $C.n$
P1: $\alpha \leq n \leq \beta$

Termination.

We note that

$$P0 \wedge P1 \wedge Q.(f.n) \ \Rightarrow Post$$

Establish Invariants.

Our model (4) shows us that we can establish P0 by the assignment

$$n := \alpha$$

This also establishes P1.

Guard.

We choose our loop guard to be

$$B : \neg Q.(f.n)$$

Variant.

As our variant function we choose

$$K - n$$

Where $0 \leq K < N$ and K is the (as yet unknown) index of the leftmost point where Q holds.

Calculate Loop body.

Decreasing the variant by the assignment n := n+1 is a standard step and maintains P1. Let us se what effect it has on P0

$$
\begin{array}{ll}
 & (n := n+1).\ P0 \\
= & \qquad \{\text{textual substitution}\} \\
 & C.(n+1) \\
= & \qquad \{(2)\ \text{above}\} \\
 & C.n \wedge \neg Q.(f.n) \\
= & \qquad \{\neg Q.(f.n)\ \text{at start of loop body}\} \\
 & C.n \\
= & \qquad \{\ P0\ \} \\
 & \text{true}
\end{array}
$$

Finished Program.

So our finished program is as follows

```
n := α
; do ¬Q.(f.n) →

            n := n+1

  od

{C.n ∧ Q.(f.n)}
```

This is called the Linear Search Theorem.