Chapter 18 : The longest all zero segment.

In which we continue to develop our calculational style.

Given f[0..N) of int, $\{0 \le N\}$, we are asked to construct a program to determine the length of the longest all zero segment in f.

{f[0..N) of int contains values}

S

$\{ r = \langle \uparrow i,j : 0 \le i \le j \le N \wedge Az.i.j : j\text{-}i \rangle \}$

Domain model.

As usual we begin by building a model of the domain.

* (0) $Az.i.j \equiv \langle \forall k : i \le k < j : f.k = 0 \rangle$ $\quad ,0 \le i \le j \le N$

Exploiting the empty range and associativity gives us

\- (1) $Az.i.i \equiv true$ $\quad ,0 \le i \le N$

\- (2) $Az.i.(j+1) \equiv Az.i.j \wedge (f.j = 0)$ $\quad ,0 \le i \le j < N$

We name the quantified expresion in our postcondition.

* (3) $C.n = \langle \uparrow i,j : 0 \le i \le j \le n \wedge Az.i.j : j\text{-}i \rangle$ $\quad ,0 \le n \le N$

Appealing to the "1 point" rule and (1) gives us

\- (4) $C.0 = 0$

In an effort to exploit associativity we calculate as follows

$C.(n+1)$

= {(3)}

$\langle \uparrow i,j : 0 \le i \le j \le n+1 \wedge Az.i.j : j\text{-}i \rangle$

= {split off j = n+1 term}

$\langle \uparrow i,j : 0 \le i \le j \le n \wedge Az.i.j : j\text{-}i \rangle \uparrow \langle \uparrow i : 0 \le i \le n+1 \wedge Az.i.(n+1) : (n+1)\text{-}i \rangle$

= {(3)}

$C.n \uparrow \langle \uparrow i : 0 \le i \le n+1 \wedge Az.i.(n+1) : (n+1)\text{-}i \rangle$

= {name the new expression (6)}

$C.n \uparrow D.(n+1)$

Which gives us

- (5) $C.(n+1) \quad = \quad C.n \uparrow D.(n+1) \qquad ,0 \le n < N$

* (6) $D.n \quad = \quad \langle \uparrow i : 0 \le i \le n \wedge Az.i.n : n\text{-}i \rangle \qquad ,0 \le n \le N$

An appeal to the "1 point rule" and (1) gives us

- (7) $D.0 \quad = \quad 0$

Seeking to exploit associativity, we calculate as follows

$$
\begin{array}{ll}
 & D.(n+1) \\
= & \quad \{(6)\} \\
 & \langle \uparrow i : 0 \le i \le n+1 \wedge Az.i.(n+1) : (n+1)\text{-}i \rangle \\
= & \quad \{\text{split off } i = n+1 \text{ term}\} \\
 & \langle \uparrow i : 0 \le i \le n \wedge Az.i.(n+1) : (n+1)\text{-}i \rangle \uparrow (n+1)\text{-}(n+1) \\
= & \quad \{\text{arithmetic}\} \\
 & \langle \uparrow i : 0 \le i \le n \wedge Az.i.(n+1) : (n+1)\text{-}i \rangle \uparrow 0 \\
= & \quad \{(2)\} \\
 & \langle \uparrow i : 0 \le i \le n \wedge Az.i.n \wedge (f.n{=}0) : (n+1)\text{-}i \rangle \uparrow 0 \\
 & \quad \{\text{case analysis, } f.n{=}0,\ \text{true} \equiv \text{Id}\wedge\} \\
 & \langle \uparrow i : 0 \le i \le n \wedge Az.i.n \ : (n+1)\text{-}i \rangle \uparrow 0 \\
= & \quad \{+/\uparrow \text{ for non-empty ranges}\} \\
 & (1 + \langle \uparrow i : 0 \le i \le n \wedge Az.i.n \ : n\text{-}i \rangle) \uparrow 0 \\
= & \quad \{(6)\} \\
 & (1 + D.n) \uparrow 0
\end{array}
$$

Now let us calculate using the other case. We observe

$$
\begin{array}{ll}
 & D.(n+1) \\
= & \quad \{(6)\} \\
 & \langle \uparrow i : 0 \le i \le n+1 \wedge Az.i.(n+1) : (n+1)\text{-}i \rangle \\
= & \quad \{\text{split off } i = n+1 \text{ term}\} \\
 & \langle \uparrow i : 0 \le i \le n \wedge Az.i.(n+1) : (n+1)\text{-}i \rangle \uparrow (n+1)\text{-}(n+1) \\
= & \quad \{\text{arithmetic}\} \\
 & \langle \uparrow i : 0 \le i \le n \wedge Az.i.(n+1) : (n+1)\text{-}i \rangle \uparrow 0 \\
= & \quad \{(2)\} \\
 & \langle \uparrow i : 0 \le i \le n \wedge Az.i.n \wedge (f.n{=}0) : (n+1)\text{-}i \rangle \uparrow 0 \\
= & \quad \{\text{case analysis, } f.n{\neq}0,\ P \wedge \text{false} \equiv \text{false}\} \\
 & \langle \uparrow i : \text{false} \ : (n+1)\text{-}i \rangle \uparrow 0
\end{array}
$$

= {empty range}

$Id\uparrow \uparrow 0$

= {defn. of Id}

0

So we have

- (8) $D.(n+1) = (1+D.n)\uparrow 0 \quad \Leftarrow f.n=0 \quad ,0 \le n < N$

- (9) $D.(n+1) = 0 \quad \Leftarrow f.n\neq 0 \quad ,0 \le n < N$

Rewrite and strengthen the postcondition.

Post : $r = C.N$

Choose invariants.

We choose the following invariants

P0 : $r = C.n \wedge d = D.n$

P1 : $0 \le n \le N$

Establish invariants.

From our model, in particular laws (4) and (7), we can see that the following assignment establishes the invariants

$n, r, d := 0, 0, 0$

Guard.

$n\neq N$

Variant.

n

Calculate the loop body.

We achieve our standard decrease of vf by increasing n by 1. We calculate the loop body using this.

$(n, r, d := n+1, E, E').P0$

= {textual substitution}

$E = C.(n+1) \wedge E' = D.(n+1)$

= {(5)}
$E = C.n \uparrow D.(n+1) \wedge E' = D.(n+1)$
= {case analysis, f.n=0, (8) twice}
$E = C.n \uparrow (1+D.n)\uparrow 0 \wedge E' = (1+D.n)\uparrow 0$
= {P0}
$E = r \uparrow (1+d)\uparrow 0 \wedge E' = (1+d)\uparrow 0$

This gives us

If f.n=0 → n. r. d := n+1, $r \uparrow (1+d)\uparrow 0$, $(1+d)\uparrow 0$

We now consider the other case

(n, r, d := n+1, E, E').P0
= {textual substitution}
$E = C.(n+1) \wedge E' = D.(n+1)$
= {(5)}
$E = C.n \uparrow D.(n+1) \wedge E' = D.(n+1)$
= {case analysis, f.n≠0, () twice}
$E = C.n \uparrow 0 \uparrow 0 \wedge E' = 0\uparrow 0$
= {P0 and ↑ idempotent}
$E = r\uparrow 0 \wedge E' = 0$

This gives us

If f.n≠0 → n. r. d := n+1, $r\uparrow 0$, 0

Finished program.

n, r, d := 0, 0, 0 {P0 ∧ P1}
;do n≠N → {P0 ∧ P1 ∧ n≠N}

If f.n=0 → n. r. d := n+1, $r \uparrow (1+d)\uparrow 0$, $(1+d)\uparrow 0$
[] f.n≠0 → n. r. d := n+1, $r\uparrow 0$, 0
fi

{P0 ∧ P1}
od
{r = C.N}

Note.

Our final observation is that as both D.n and C.n are natural values they cannot be negative. Therefore we can simplify some of the expressions. We note that

$r \uparrow 0 = r$

$d \uparrow 0 = d$

Using this we can rewrite our finished program like this.

```
n, r, d := 0, 0, 0 {P0 ∧ P1}
;do n≠N →      {P0 ∧ P1 ∧ n≠N}

               If f.n=0 → n. r. d := n+1, r ↑ (1+d), (1+d)
               [] f.n≠0 → n. r. d := n+1, r, 0
               fi

               {P0 ∧ P1}
od
{r = C.N}
```