Lecture 16. Learning with expert advice

COMP90051 Statistical Machine Learning

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This lecture

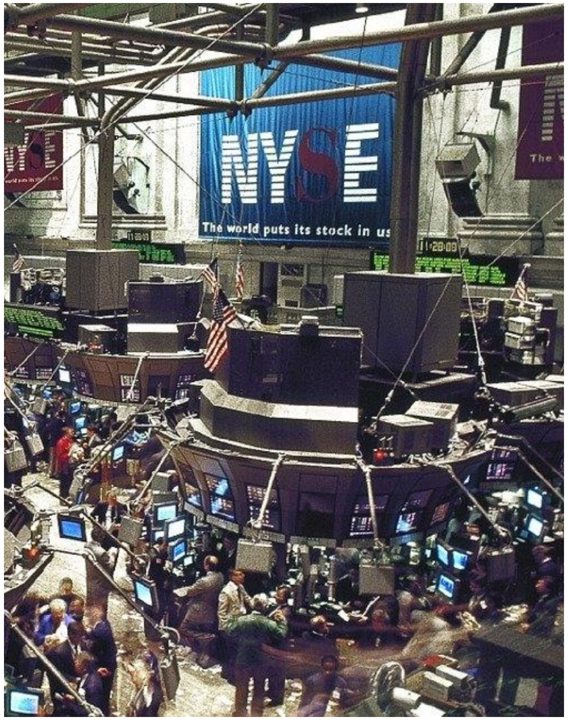
- Learning from expert advice / multiplicative weights
  * Learner listens to some/all experts making predictions
  * True outcomes are ADVERSARIAL!
  * Learner updates weights over experts based on their losses
  * Algorithms all forms of “multiplicative weights”
  * Nice clean bounds on total mistakes/loss: by “potential function” technique
- Infallible expert (one always perfect)
  * Majority Vote Algorithm
- Imperfect experts (none guaranteed perfect) – increasingly better
  * Weighted Majority Vote Algorithm by Halving
  * Weighted Majority Voting by General Multiplicative Weights
  * Probabilistic Experts Algorithm

An infallible expert and the Majority Algorithm

Warming up example

Warm-up: Case of the infallible expert

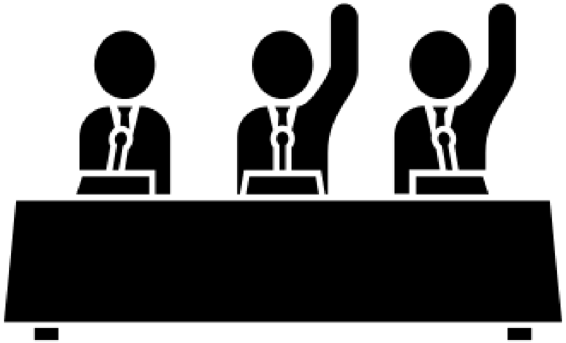
- Experts $E_1, \dots, E_n$ predict the stock market daily
  - Each expert prediction is binary: stocks will go up/down
- Learner's game, daily:
  - Observe predictions of all experts
  - Make a prediction of its own
  - Observe outcome (could be anything!)
  - Goal: minimise number total mistakes
- Infallible expert assumption:
  - 1 or more experts makes no mistakes



NYSE. CCA: skeeze

Infallible Expert Algorithm: Majority Vote

1. Initialise set of experts who haven't made mistakes $E = \{1, \ldots, n\}$
2. Repeat per round
   a) Observe predictions E_i for all $i \in \{1, \ldots, n\}$
   b) Make majority prediction $\arg\max_{y \in \{-1,1\}} \sum_{i \in E} 1[E_i = y]$
   c) Observe correct outcome
   d) Remove mistaken experts from E



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Mistake Bound for Majority Vote

Proposition: Under infallible expert assumption, majority vote makes total mistakes $M \leq \log_2 n$

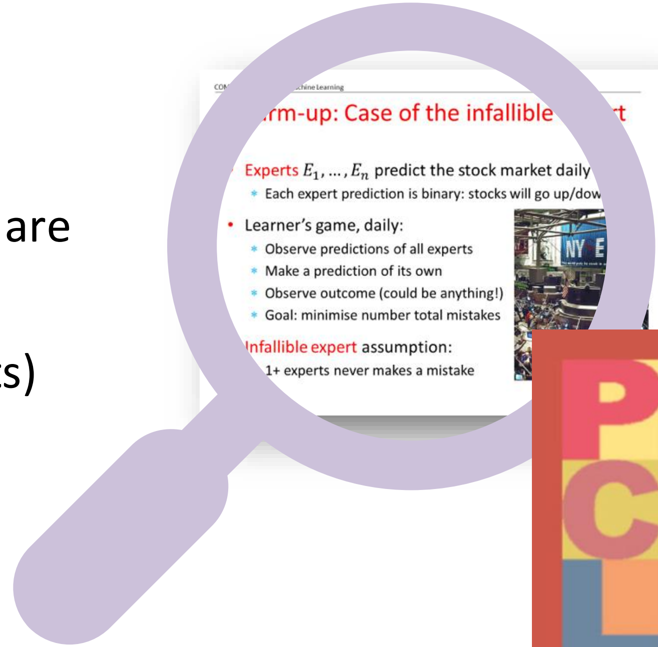
Intuition: Halving (e.g. tree data structures!)? Expect to see log

Proof

- Loop invariant: If algorithm makes a mistake, then at least $|E|/2$ experts must have been wrong
- I.e. for every incorrect prediction, E reduced by at least half. I.e. after M mistakes, $|E| \leq n/2^M$
- By infallibility, at all times $1 \leq |E|$
- Combine to $1 \leq |E| \leq n/2^M$, then solve for M.

$|E|$ is the potential function

How is this “online learning”?

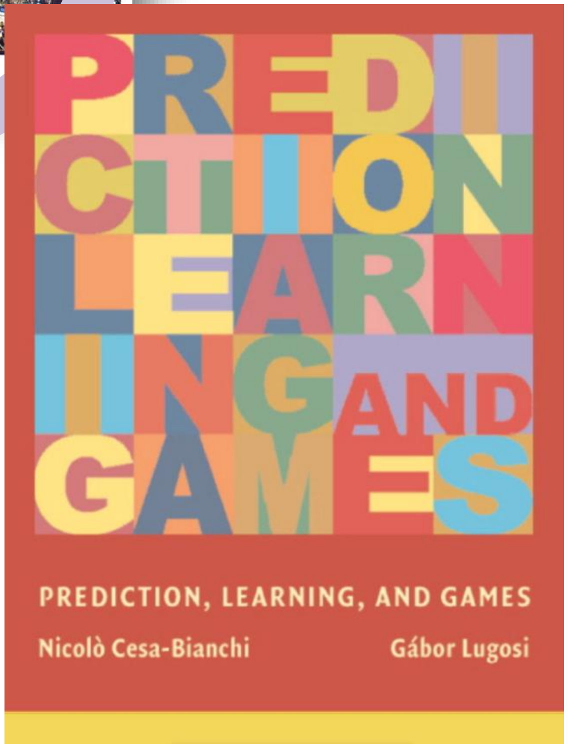


Learning

- Weights on which experts are worth listening to
- (Infallible case: 0/1 weights)
- Making predictions/ taking actions
- Incurring loss (so far 0/1)
- IID “Distribution” replaced by adversarial outcomes

Online

- A repeated game



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Mini Summary

- Learning with expert advice paradigm
  - Abstraction of online learning problem
  - Adversarial feedback
  - Later: Applications abound
- Bounds on mistakes (later losses) “easy”
  - Involve “potential function” technique
  - Later: interested in scaling with best expert performance

Next: Imperfect experts. Down weight don’t drop bad experts

Imperfect experts and the Halving Algorithm

Similar proof technique; similar algorithm; much more interesting setting

No one's perfect

- No more guarantee of an infallible expert
- What breaks?
  - We could end up with $E = \emptyset$, how to predict then?
  - No sense: "Zero tolerance" dropping experts on a mistake
- Very general setting / very few assumptions
  - Not assuming anything about expert error rates
  - Not assuming anything about correlation of expert errors
  - Not assuming anything about outcome observations. Not even stochastic (could be adversarial!)

Imperfect experts: Halving Algorithm

1. Initialise $w_i = 1$ weight of expert E_i
2. Repeat per round
   a) Observe predictions E_i for all $i \in \{1, \dots, n\}$
   b) Make **weighted** majority prediction $\arg\max_{y \in \{-1,1\}} \sum_{i \in E} w_i 1[E_i = y]$
   c) Observe correct outcome
   d) Downweigh each mistaken expert E_i $w_i \leftarrow w_i / 2$

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Mistake Bound for Halving

Proposition: If the best expert makes m mistakes, then weighted majority vote makes $M \leq 2.4(m + \log_2 n)$ mistakes.

Proof

- Invariant: If algorithm makes a mistake, then weight of wrong experts is at least half the total weight $\mathrm{W} = \sum_{i=1}^{n} w_i$
- Weight of wrong experts reduced by $1/2$, therefore total weight reduced by at least $3/4$. I.e. after M mistakes, $W \leq n(3/4)^M$
- Best expert E_i has $w_i = (1/2)^m$
- Combine to $(1/2)^m = w_i \leq W \leq n(3/4)^M$
- Taking logs $-m \leq \log_2 n + M \log_2(3/4)$, solving $M \leq \frac{m+\log_2 n}{\log_2(4/3)}$

Compare, compare: What's going on?

- Price of imperfection (vs. infallibility) is $\mathcal{O}(m)$
  - Infallible case: $M \in \mathcal{O}(\log n)$
  - Imperfect case: $M \in \mathcal{O}(m + \log n)$
- Scaling to many experts is no problem
- Online learning vs. PAC frameworks

	Modelling of losses	Ultimate goal
PAC	i.i.d. losses *(due to e.g. Hoeffding)*	(For ERM; L6c) Small estimation error $R[f_m] - R[f^*]$. Bounded in terms of family's VC dimension
Online learning	Adversarial/arbitrary losses	Small $M - m$. Bounded in terms of number of experts.

Mini Summary

- Imperfect expert setting
  - Don’t drop bad experts, just halve their weight
  - Predict by weighted majority, not simply majority
  - Mistake bound follows similar “potential function” pattern!
- Learning with expert advice paradigm
  - Key difference to PAC is adversarial feedback
  - Similarity: Also concerned with performance relative to “best in class”

Next: Imperfect experts continued. Generalising halving.

From Halving to Multiplying weights by $1 - \varepsilon$

Generalising weighted majority.

Useful (but otherwise boring) inequalities

- Lemma 1: For any $\varepsilon \in [0,0.5]$, we have
$$-\varepsilon - \varepsilon^2 \leq \log_e(1-\varepsilon) < -\varepsilon$$

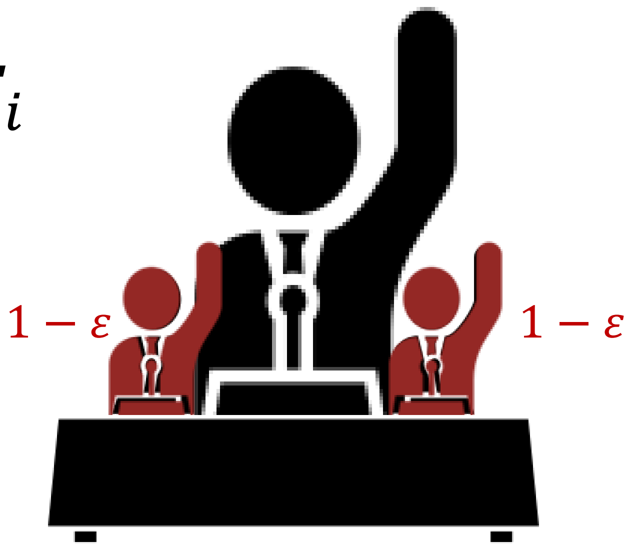
- Proof:
  * Upper bound by Taylor expansion, dropping all by first term (as they're negative)
  * Lower bound by convexity of $\exp(-\varepsilon - \varepsilon^2)$

- Lemma 2: For all $\varepsilon \in [0,1]$ we have,
$$1 - \varepsilon x > \begin{cases} (1-\varepsilon)^x, & \text{if } x \in [0,1] \\ (1+\varepsilon)^{-x}, & \text{if } x \in [-1,0] \end{cases}$$

- Proof: by convexity of the RHS functions

Weighted Majority Vote Algorithm

1. Initialise $w_i = 1$ weight of expert E_i
2. Repeat per round
   a) Observe predictions E_i for all $i \in \{1, \ldots, n\}$
   b) Make weighted majority prediction $\arg\max_{y\in\{-1,1\}} \sum_{i\in E} w_i 1[E_i = y]$
   c) Observe correct outcome
   d) Downweigh each mistaken expert E_i $w_i \longleftarrow (1-\varepsilon)w_i$



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Mistake Bound

Proposition: If the best expert makes m mistakes, then $(1-\varepsilon)$-weighted majority makes $M \leq 2(1+\varepsilon)m + (2\log_e n)/\varepsilon$ mistakes.

Bound improves dependence on *m* compared to halving. Why?

Proof

- Whenever learner mistakes, at least half of total weight reduced by factor of $1-\varepsilon$. So after M mistakes, $W \leq n(1-\varepsilon/2)^M$
- Best expert E_i has $w_i = (1-\varepsilon)^m$
- Combine to $(1-\varepsilon)^m = w_i \leq W \leq n(1-\varepsilon/2)^M$
- Taking logs: $m\log_e(1-\varepsilon) \leq \log_e n + M\log_e(1-\varepsilon/2)$
- Lemma 1 replaces both $\log_e(1-\varepsilon)$: $-\mathrm{m}(\varepsilon+\varepsilon^2) \leq \log_e n - M\varepsilon/2$
- Solving for M proves the bound.

Dependence in m **provably near optimal**!

- New to lower bounds? example shows an analysis or even an algorithm can't do better than some limit
- Weighted majority almost achieves $2m$ dependence, with $2(1+\varepsilon)m$ (considering no. experts fixed)
- Example with $M = 2m$
  * Consider $n = 2$ with E_1 (E_2) correct on odd (even) days
  * Then best expert makes mistakes half the time
  * But after 1st round, for any ε, majority vote is wrong all the time, as incorrect expert gets more than half weight
- Consequence? Can't improve the constant 2 factor in m

Mini Summary

- Imperfect expert setting continued…
- From halving to multiplicative weights!
  - Mistake bound proved as usual via “potential function” trick
  - Bound’s dependence on best expert improved to $2 + \varepsilon$ factor
- Lower bound / impossibility result
  - Factor of 2 is optimal for (deterministic) multiplicative weights!

Next: Imperfect experts continued. Randomise*!!*

The probabilistic experts algorithm

wherein randomisation helps us do better!

Probabilistic experts algorithm

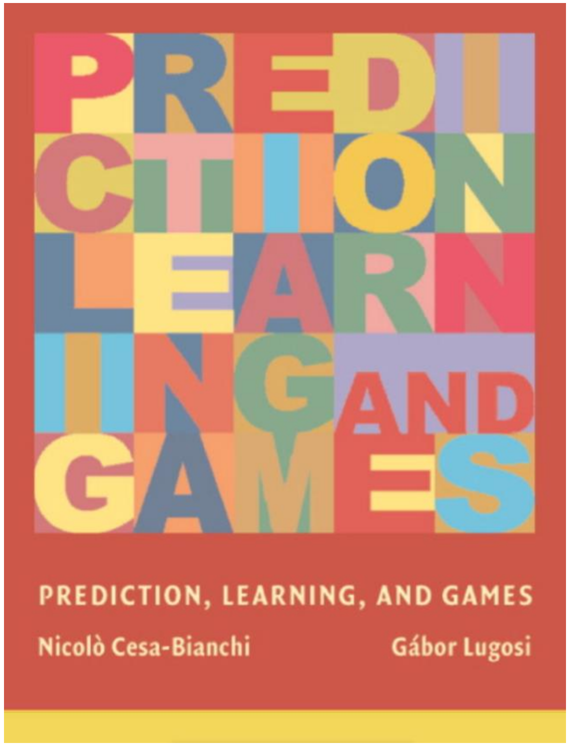
- Change 1 from mistakes: Loss $\ell_i^{(t)} \in [0,1]$ of E_i, round t
- Change 2: Randomised algorithm means, bounding expected losses (sound familiar? It should.... a risk!)

1. Initialise $w_i = 1$ weight of expert E_i
2. Repeat per round
   a) Observe predictions E_i for all $i \in \{1, \ldots, n\}$
   b) Predict E_i of expert i with probability $\frac{w_i}{W}$ where $W = \sum_{i=1}^{n} w_i$
   c) Observe losses
   d) Update each weight $w_i \longleftarrow (1-\varepsilon)^{\ell_i^{(t)}} w_i$

Probabilistic experts: Expected loss bound

- Proposition: Expected loss of the probabilistic experts algorithm is $L \leq \frac{\log_e n}{\varepsilon} + (1+\varepsilon)L^*$ where L^* is the minimum loss over experts.

- Proof: next, follows similar "potential" pattern
- Beats deterministic! Shaves off optimal constant 2
- Generalises in many directions. Active area of research in ML, control, economics, in top labs.



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Proof: Upper bounding potential function

- Learner's round t expected loss: $L_t = \frac{\sum_{i=1}^{n} w_i^{(t)} \ell_i^{(t)}}{W(t)}$
- By Lemma 2, since losses are [0,1]:
  updated $w_i^{(t+1)} \leftarrow (1-\varepsilon)^{\ell_i^{(t)}} w_i^{(t)} \leq \left(1-\varepsilon \ell_i^{(t)}\right) w_i^{(t)}$
- Rearrange to obtain recurrence relation:

$$W(t+1) \leq \sum_{i=1}^{n} \left(1-\varepsilon \ell_i^{(t)}\right) w_i^{(t)} = \sum_{i=1}^{n} w_i^{(t)} \left(1-\varepsilon \frac{\sum_{i=1}^{n} w_i^{(t)} \ell_i^{(t)}}{\sum_{i=1}^{n} w_i^{(t)}}\right)$$

$$= W(t)\,(1-\varepsilon L_t)$$

- Initialisation gave $W(0) = n$, so telescoping we get:

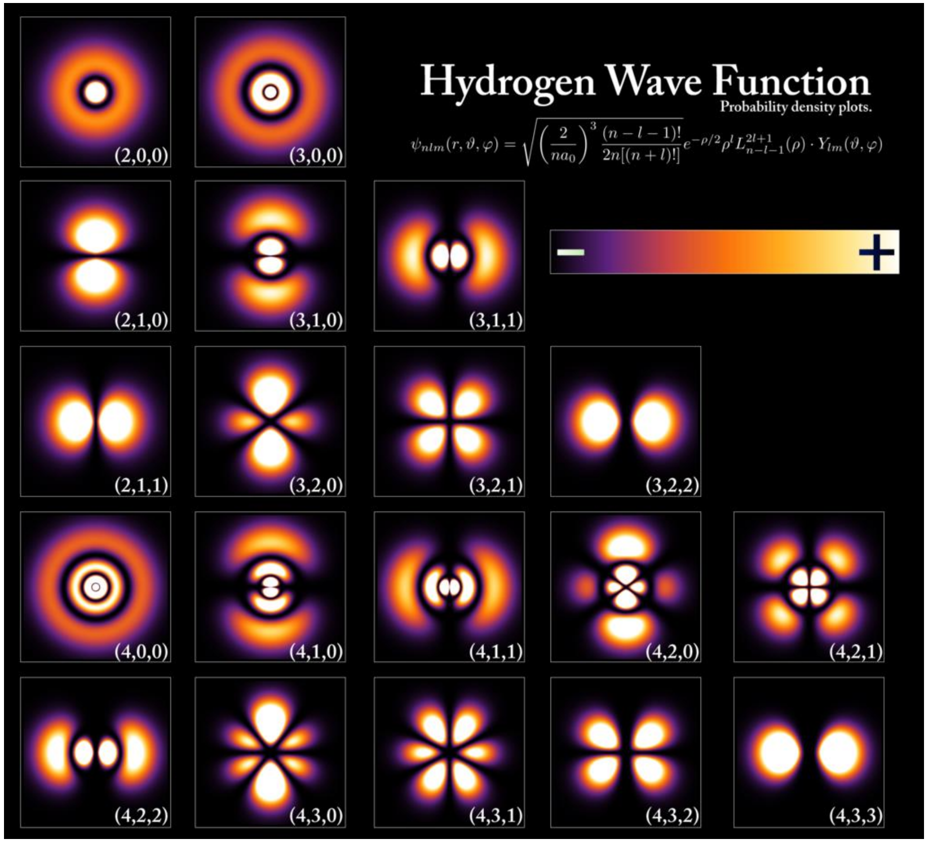
$$W(T) \leq n \prod_{t=1}^{T} (1-\varepsilon L_t)$$

Proof: Lower bounding potential, Wrap up

- Proved upper bound: $W(T) \leq n \prod_{t=1}^{T}(1 - \varepsilon L_t)$
- Lower bound from best expert total loss L^*:
$$W(T) \geq (1-\varepsilon)^{L^*}$$
- Combining bounds and taking log's:
$$L^* \log_e(1-\varepsilon) \leq \log_e n + \sum_{t=1}^{T} \log_e(1-\varepsilon L_t)$$
- By Lemma 1: $-L^*(\varepsilon + \varepsilon^2) \leq \log_e n - \varepsilon \sum_{t=1}^{T} L_t$
- Linearity of expectation $L = \sum_{t=1}^{T} L_t$, rearranging:
$$L \leq \frac{\log_e n}{\varepsilon} + (1+\varepsilon)L^*$$

Applications of multiplicative weights [Kale thesis 2007]

- Learning quantum states from noisy measurements
- Derandomising algorithms
- Solving certain zero-sum games
- Fast graph partitioning
- Fast solving of semidefinite programming problems
- Portfolio optimisation
- A basis for boosting
- Sparse vector technique in differential privacy



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Mini Summary

- Introducing randomisation to learning with experts
  - Algorithm choosing a random expert to follow
  - Weights become probabilities
  - Mistakes generalise to losses
- Loss bound
  - Have to bound expected loss (hey, risk!!)
  - Shaves off that 2 factor. Proves that randomisation really helps!

Next: Only observe reward of chosen expert → bandits!