Workshop 4

COMP90051 Statistical Machine Learning

Semester 2, 2024

Learning outcomes

At the end of this workshop you should:

- Be able to explain how the regularization helps to solve the issues in linear regression
- Be able to implement the Ridge Regression
- Be able to implement the Lasso Regression
- Be able to explain the bias-variance trade-off

Regularization

- Regularisation: introduce an additional condition into the system
- The original problem is to minimise $\|\mathbf{y} - \mathbf{X}\mathbf{w}\|_2^2$
- The regularised problem is to minimise

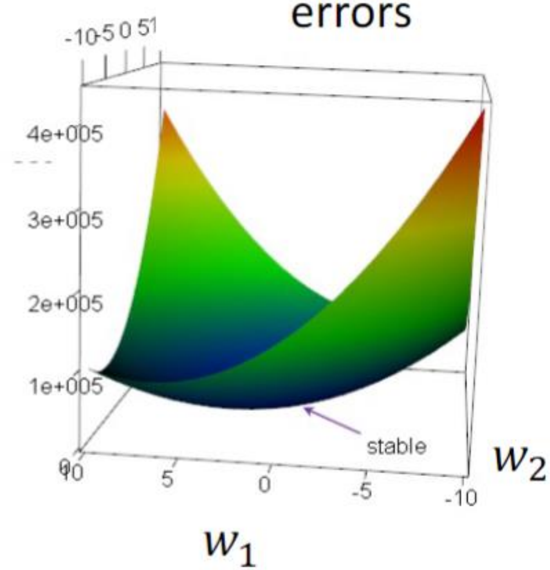
$$\|\mathbf{y} - \mathbf{X}\mathbf{w}\|_2^2 + \lambda\|\mathbf{w}\|_2^2 \text{ for } \lambda > 0$$

- The solution is now

$$\hat{\mathbf{w}} = (\mathbf{X}'\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}'\mathbf{y}$$

- This formulation is called ridge regression
  - Turns the ridge into a deep, singular valley
  - Adds λ to eigenvalues of $\mathbf{X}'\mathbf{X}$: makes invertible

sum of squared errors

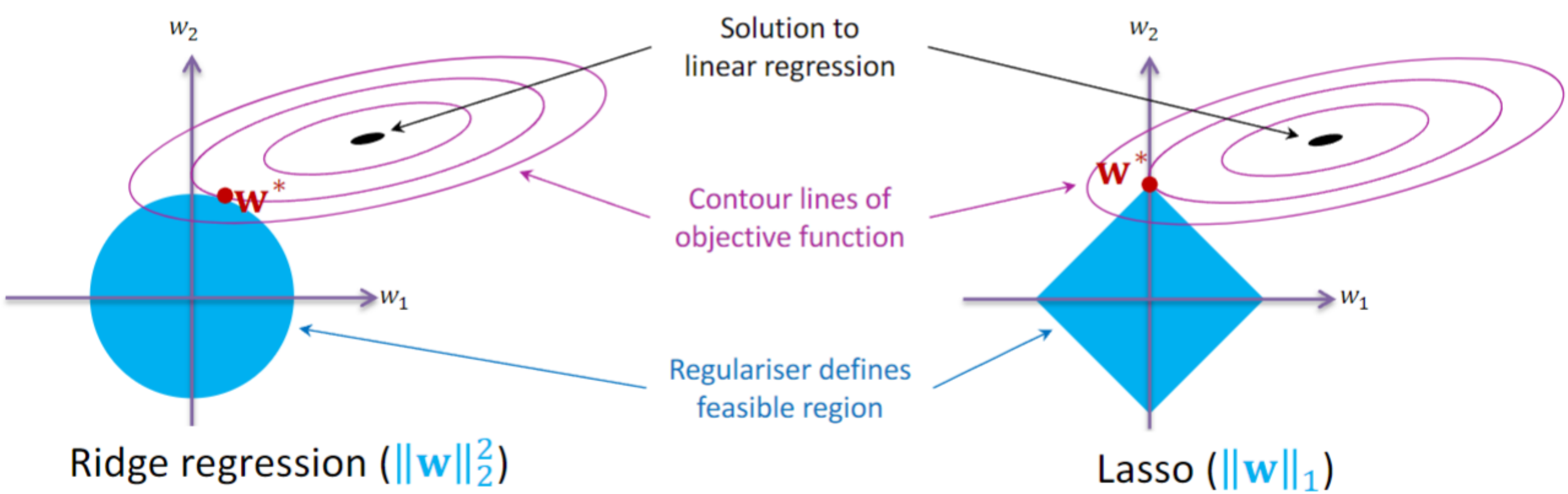


strictly convex

Ridge vs Lasso

- For illustrative purposes, consider a *modified problem*:

  minimise $\|\mathbf{y}-\mathbf{X}\mathbf{w}\|_2^2$ *subject to* $\|\mathbf{w}\|_2^2 \leq \lambda$ for $\lambda > 0$

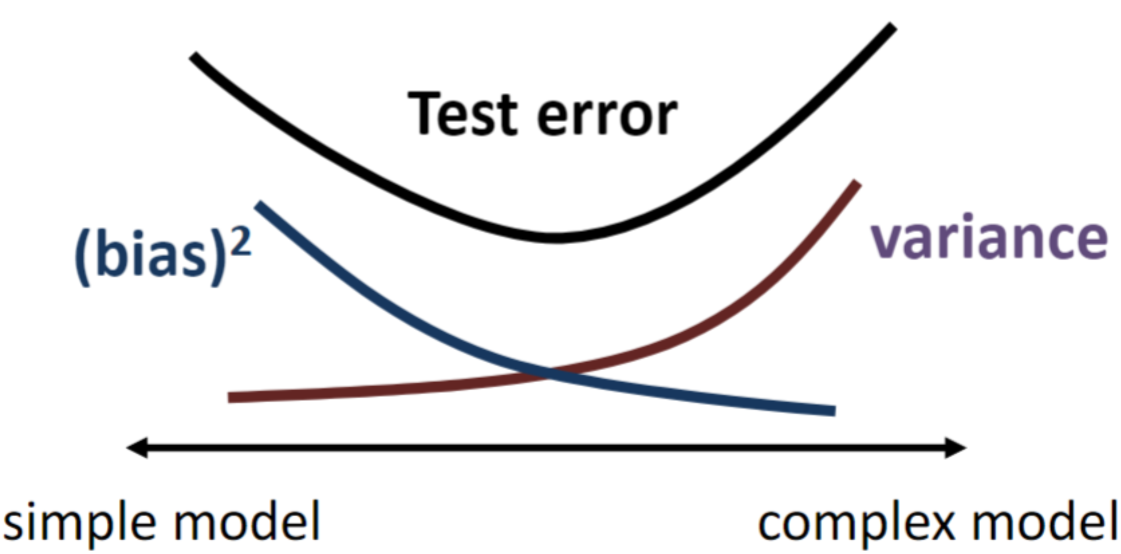


- Lasso (L_1 regularisation) encourages solutions to sit on the axes

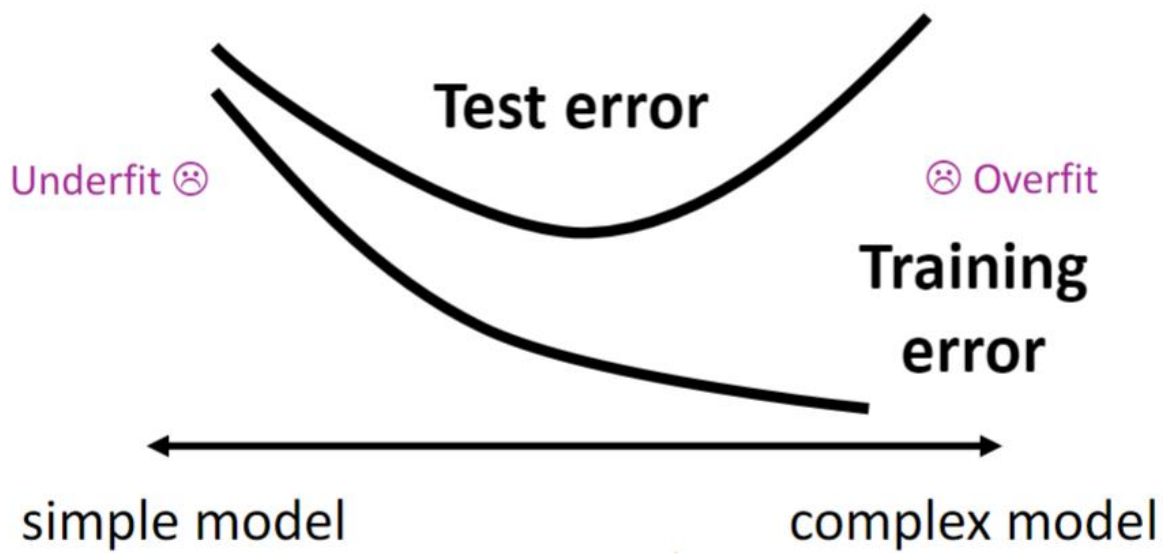
  → Some of the weights are set to zero → Solution is sparse

Bias-Variance Trade-off

- simple model ➔ high bias, low variance
- complex model ➔ low bias, high variance



Bias-Variance Trade-off



$$\|y - Xw\|_2^2 + \alpha\|w\|_2^2$$

$\nabla_w Aw = A$

$\nabla_w w^T x = \nabla_w x^T w = x^T$

$\nabla_w w^T A w = w^T (A^T + A)$

$$= (y - Xw)^T (y - Xw)^T + \alpha w^T w$$

$$= y^T y - y^T X w - w^T X^T y + w^T X^T X w + \alpha w^T w \rightarrow \alpha w^T I w$$

$$\nabla_w \mathcal{L}(w) = \underset{0}{} - 2y^T X + 2w^T X^T X + \alpha w^T (I^T + I)$$

Worksheet 4

$$= 0 - 2y^T X + 2w^T X^T X + \alpha w^T (I^T + I) = 0$$

$$=$$

$$2\alpha w^T + 2w^T X^T X = 2y^T X$$

$$w^T (2\alpha + 2X^T X) = 2y^T X$$

$$w^T = y^T X (\alpha + X^T X)^{-1}$$

☆☆☆可逆矩阵的逆的T 不变

$$w^* = (X^T X + \alpha I)^{-1} X^T y$$