Lecture 19. Bayesian classification

COMP90051 Statistical Machine Learning

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This lecture

- Bayesian ideas in discrete settings
  * Beta-Binomial conjugacy
  * Uniqueness up to proportionality
  * Sunrise example
  * Common conjugate pairs
- Bayesian logistic regression
  * Non-conjugacy
  * Pointer: Laplace approximation

How to apply Bayesian view to discrete data?

- First off consider models which *generate* the input
  - cf. *discriminative* models, which *condition* on the input
  - I.e., $p(y \mid \mathbf{x})$ vs $p(\mathbf{x}, y)$, Logistic Regression vs Naïve Bayes
- For simplicity, start with most basic setting
  - *n* coin tosses, of which *k* were heads
  - only have $\boldsymbol{x}$ (sequence of outcomes), but no 'classes' $\boldsymbol{y}$
- Methods apply to generative models over discrete data
  - e.g., topic models, generative classifiers
    (Naïve Bayes, mixture of multinomials)

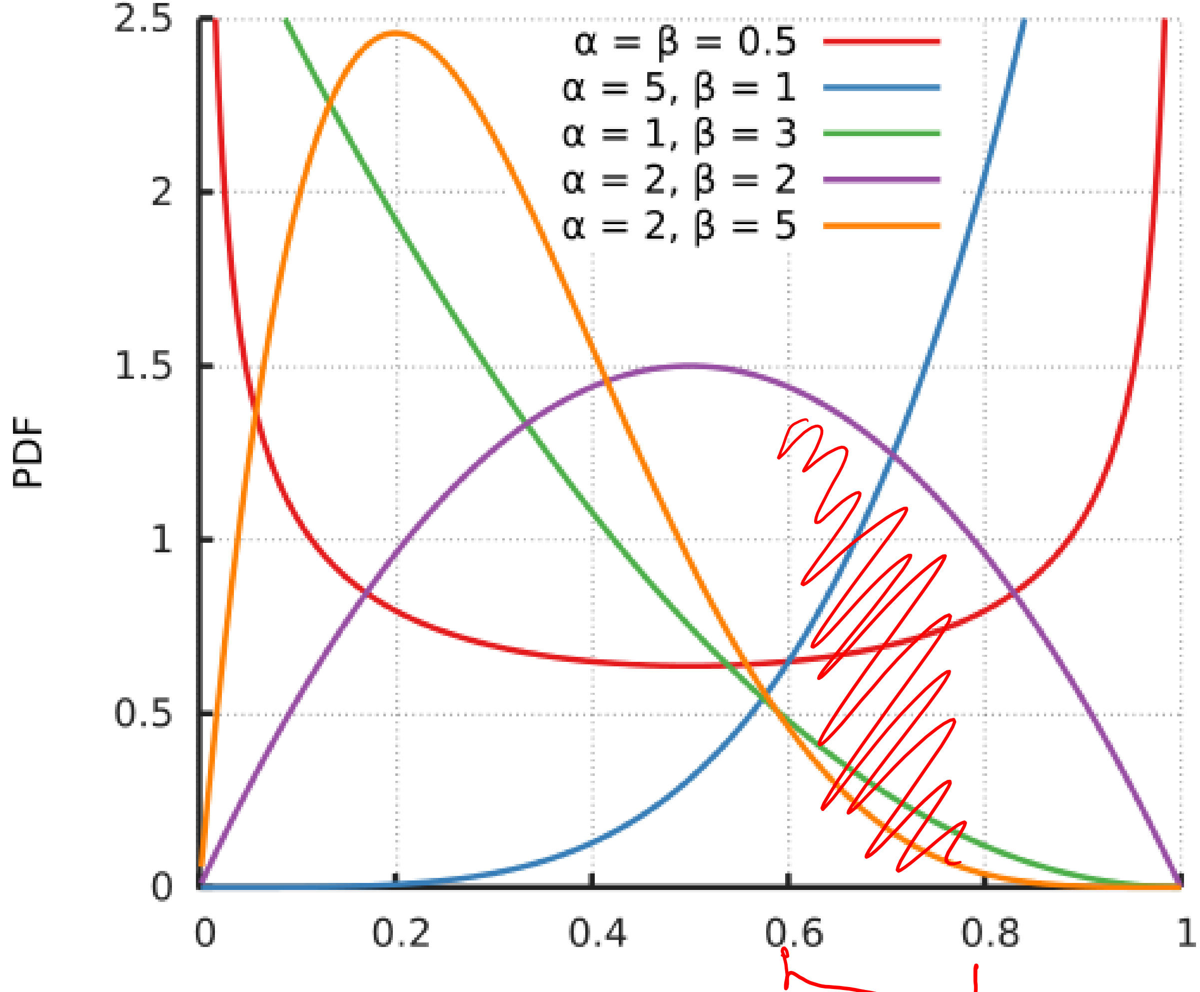
Discrete Conjugate prior: Beta-Binomial

- Conjugate priors also exist for discrete spaces
- Consider *n* coin tosses, of which *k* were heads
  - let p(head) = *q* from a single toss (*Bernoulli dist*)
  - Inference question is the coin biased, i.e., is *q ≈ 0.5*
- Several draws, use *Binomial dist*
  - and its conjugate prior, *Beta dist*

$$p(k|n,q) = \binom{n}{k} q^k (1-q)^{n-k}$$

$$\begin{aligned} p(q) &= \mathrm{Beta}(q; \alpha, \beta) \\ &= \frac{\gamma(\alpha+\beta)}{\gamma(\alpha)\gamma(\beta)} q^{\alpha-1}(1-q)^{\beta-1} \end{aligned}$$

Beta distribution



Sourced from https://en.wikipedia.org/wiki/Beta_distribution

Beta-Binomial conjugacy

$$p(k|n,q) = \binom{n}{k} q^k (1-q)^{n-k}$$

$$p(q) = \mathrm{Beta}(q; \alpha, \beta)$$

$$= \frac{\gamma(\alpha+\beta)}{\gamma(\alpha)\gamma(\beta)} q^{\alpha-1}(1-q)^{\beta-1}$$

Sweet! We know the normaliser for Beta

Bayesian posterior

$$p(q|k,n) \propto p(k|n,q)p(q)$$

$$\propto q^k (1-q)^{n-k} q^{\alpha-1}(1-q)^{\beta-1}$$

$$= q^{k+\alpha-1}(1-q)^{n-k+\beta-1}$$

$$\propto \mathrm{Beta}(q; k+\alpha, n-k+\beta)$$

trick: ignore constant factors (normaliser)

Uniqueness up to normalisation

- A trick we've used many times:

  When an unnormalized distribution is proportional to a recognised distribution, we say it must be that distribution

- If $f(\theta) \propto g(\theta)$ for g a distribution, $\dfrac{f(\theta)}{\int_\Theta f(\theta)d\theta} = g(\theta)$.

- <u>Proof</u>: $f(\theta) \propto g(\theta)$ means that $\exists C$

$$f(\theta) = C \cdot g(\theta)$$

$$\int_\Theta f(\theta)d\theta = C \int_\Theta g(\theta)d\theta = C$$

and the result follows from LHS1/LHS2 = RHS1/RHS2

Laplace’s Sunrise Problem

Every morning you observe the sun rising. Based solely on this fact, what’s the probability that the sun will rise tomorrow?

- Use Beta-Binomial, where *q* is the Pr(sun rises in morning)
  - posterior $$p(q|k,n) = \mathrm{Beta}(q; k+\alpha, n-k+\beta)$$
  - n = k = observer’s age in days
  - let $\alpha = \beta = 1$ (*uniform* prior)
- Under these assumptions

$$p(q|k) = \mathrm{Beta}(q; k+1, 1)$$

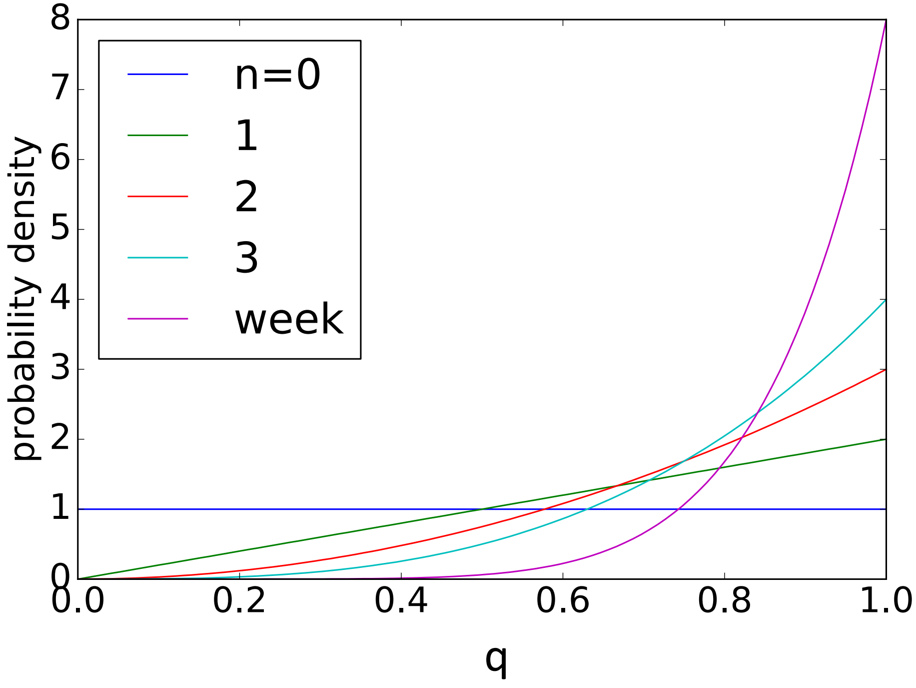
$$E_{p(q|k)}\left[q\right] = \frac{k+1}{k+2}$$

’smoothed’ count of days where sun rose / did not

Sunrise Problem (cont.)

Consider human-meaningful period

Day (n, k)	k+α	n-k+β	E[q]
0	1	1	0.5
1	2	1	0.667
2	3	1	0.75
…			
365	366	1	0.997
2920 (8 years)	2921	1	0.99997



Effect of prior diminishing with data, *but never disappears completely*.

Suite of useful conjugate priors

	likelihood	conjugate prior
regression	Normal	Normal (for mean)
	Normal	Inverse Gamma (for variance) or Inverse Wishart (covariance)
classification	Binomial	Beta
	Multinomial	Dirichlet
counts	Poisson	Gamma

Mini Summary

- Bayesian ideas in discrete settings
  - Beta-Binomial conjugacy
  - Uniqueness in proportionality
  - Sunrise example
  - Conjugate pairs

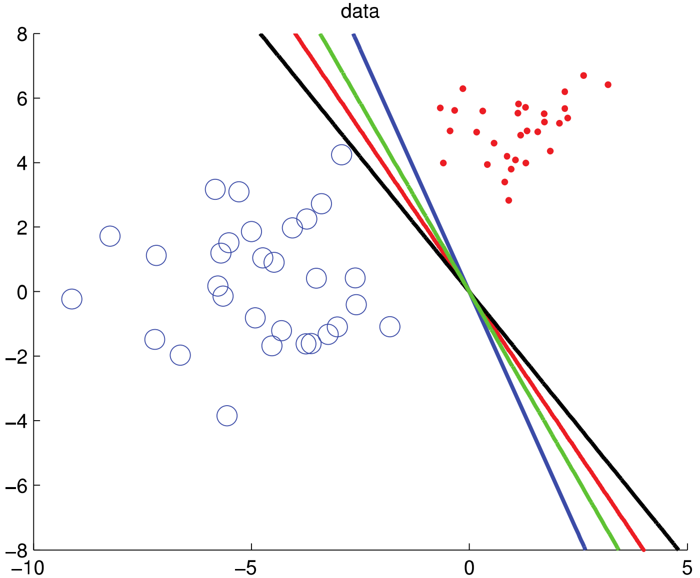
Next time: Bayesian logistic regression

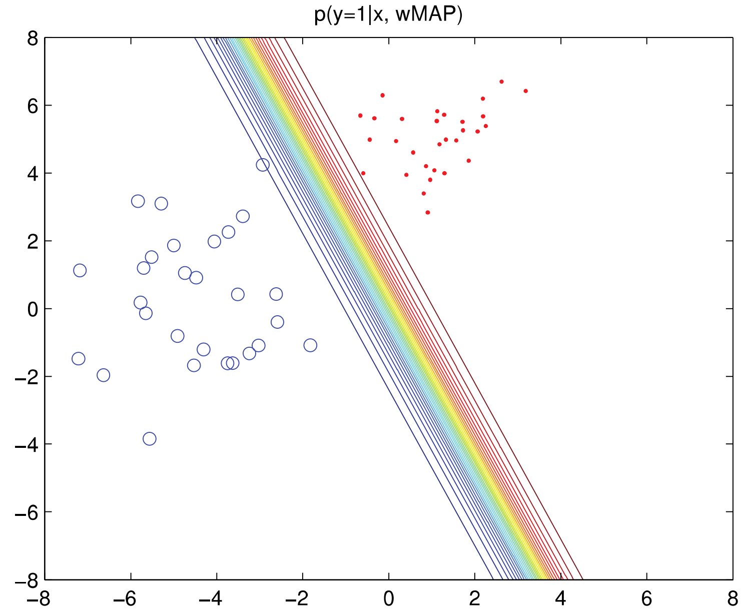
Bayesian Logistic Regression

Discriminative classifier, which conditions *on inputs. How can we do Bayesian inference in this setting?*

Now for Logistic Regression…

- Similar problems with parameter uncertainty compared to regression
  - although predictive uncertainty in-built to model outputs





Murphy Fig 8.5 & 8.6 p257-8

No conjugacy

- Can we use conjugate prior? E.g.,
  - Beta-Binomial for *generative* binary models
  - Dirichlet-Multinomial for multiclass (similar formulation)
- Model is *discriminative*, with parameters defined using logistic sigmoid*

$$p(y|q, \mathbf{x}) = q^y (1-q)^{1-y}$$
$$q = \sigma(\mathbf{x}'\mathbf{w})$$

  - need prior over ***w***, not *q*
  - no known conjugate prior (!), thus use a Gaussian prior
- Approach to inference: Monte Carlo sampling

* *Or softmax for multiclass; same problems arise and similar solution*

Approximation

- No known solution for the normalising constant

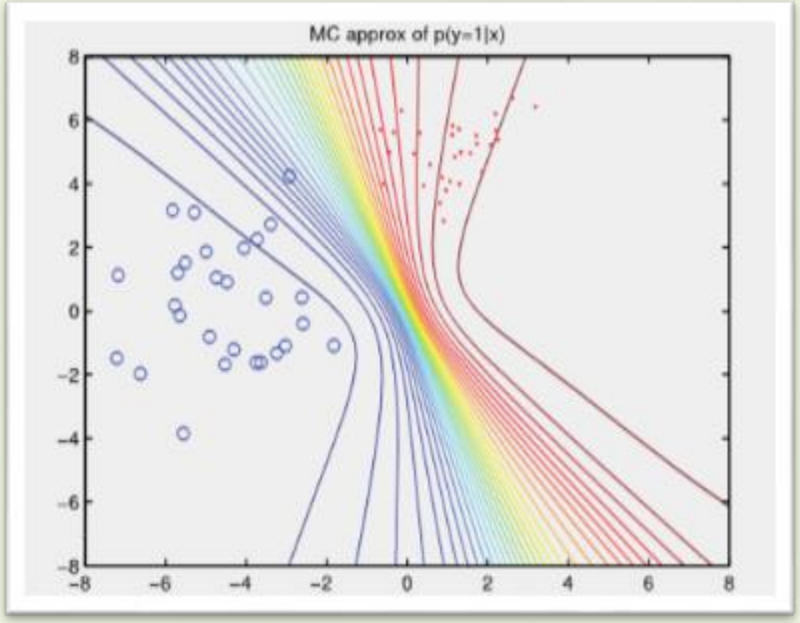
$$p(\mathbf{w}|\mathbf{X},\mathbf{y}) \propto p(\mathbf{w})p(\mathbf{y}|\mathbf{X},\mathbf{w})$$

$$= \mathrm{Normal}(\mathbf{0},\sigma^2\mathbf{I}) \prod_{i=1}^{n} \sigma(\mathbf{x}_i'\mathbf{w})^{y_i}(1-\sigma(\mathbf{x}_i'\mathbf{w}))^{1-y_i}$$

- Resolve by *approximation*

***Laplace approx.*:**

- assume posterior ≃ Normal about mode
- can compute normalisation constant, draw samples etc.
- Tractable MAP provides parameters for this (Normal) approximate posterior



Murphy Fig 8.6 p258

Mini Summary

- Bayesian ideas in discrete settings
  - Beta-Binomial conjugacy
  - Conjugate pairs; Uniqueness in proportionality
- Bayesian classification (logistic regression)
  - Non-conjugacy necessitates approximation

Next time: probabilistic graphical models