Question 2:

$y = \mathbb{1}_n\beta + \varepsilon$, $\beta \in \mathbb{R}^1$

$y \in \mathbb{R}^n$.

$\varepsilon \sim N(0, \sigma^2 I_n)$

$\boxed{\mathbb{1}_n = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}_{n\times1}}$ X

$y_1, \dots, y_n$.

a): $\hat\beta = (X^TX)^{-1}X^Ty$

$= (\underbrace{\mathbb{1}_n^T\mathbb{1}_n}_{n})^{-1}\underbrace{X^Ty}_{\sum_{i=1}^n y_i} = \frac{1}{n}\sum_{i=1}^n y_i = \bar y$

b): $\hat\beta$ is mean of $y_1, \dots, y_n$.

$\hat y = X\hat\beta = \mathbb{1}_n\bar y = \begin{pmatrix}\bar y\\ \vdots\\ \bar y\end{pmatrix}_{n\times1}$.

c): $H = X\underbrace{(X^TX)^{-1}}_{n}X^T$

$= \frac{1}{n}XX^T$

$= \frac{1}{n}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}(1, \dots, 1)$

$= \frac{1}{n}\underbrace{\begin{pmatrix}1 & \cdots & 1\\ \vdots & & \vdots\\ 1 & \cdots & 1\end{pmatrix}_{n\times n}}_{J_n} = \frac{1}{n}J_n$.

d): $\sum (y_i - \bar y)^2$

$= \underbrace{[y_1-\bar y, \dots, y_n-\bar y]}_{1\times n}\cdot\begin{bmatrix}y_1-\bar y\\ \vdots\\ y_n-\bar y\end{bmatrix}_{n\times1}$

$= (y-\bar y)^T(y-\bar y)$ $\quad \bar y = \hat y$

$= (y-\hat y)^T(y-\hat y)$ $\quad y = \begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}$ $\quad \bar y\mathbb{1}_n = \begin{pmatrix}\bar y\\ \vdots\\ \bar y\end{pmatrix}_{n\times1}$

$= (y-Hy)^T(y-Hy)$

$= y^T(I-H)y$

e): $\bar y = Hy$;

$\sum(y_i-\bar y)^2 = y^T(I-H)y$

since $cov(Hy, (I-H)y) = 0$?

$\Rightarrow$ Hy is idpt with $(I-H)y$

$\Rightarrow$ Hy is idpt with $y^T\underbrace{(I-H)^T(I-H)}_{(I-H)}y$.

$\Rightarrow$ $\bar y$ is idpt with $\sum(y_i-\bar y)^2$

f): $\bar y \sim N(\beta, \frac{\sigma^2}{n})$.

$\frac{\sum(y_i-\bar y)^2}{\sigma^2} \overset{?}{\sim} \chi^2(n-1)$

$\sum_{i=1}^n (y_i-\bar y)^2 = \sum_{i=1}^n (y_i - \beta + \beta - \bar y)^2$

$\chi^2(n-1) \sim \boxed{\frac{\sum_{i=1}^n (y_i-\beta)^2}{\sigma^2}} + \boxed{\sum_{i=1}^n (\bar y-\beta)^2}$

$y_i \sim N(\beta, \sigma^2)$ $\quad\sim\ \chi^2(n)$

$\bar y \sim N(\beta, \frac{\sigma^2}{n})$

$\frac{n(\bar y-\beta)^2}{\sigma^2} = \frac{(\bar y-\beta)^2}{\sigma^2/n} \sim \chi^2(1)$

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