Tut 8

$F_1(x) = \frac{\sum_{i=1}^{n} \# x_i <= x}{n}$

$$\theta_0 = \theta(F_0) = u^3$$

Question 1 :

1):

$$\hat{\theta} = \theta(F_1) = \left[\int x \, dF_1(x)\right]^3 = \left[\int x \left(\frac{1}{n}\sum_{i=1}^{n} \delta(x_i - x)\, dx\right)\right]^3 = \left[\frac{1}{n}\sum_{i=1}^{n} x_i\right]^3$$

where F_1 is emprical distubution:

$$= \bar{x}^3$$

2):

$$b_1 = \mathbb{E}_{F_0}(\hat{\theta} - \theta_0) = \mathbb{E}_{F_0}(\hat{\theta}) - \theta_0 = \mathbb{E}_{F_0}(\bar{x}^3) - \theta_0$$

Let $\sigma^2 = \mathrm{var}(x_i) = \mathbb{E}[(x_i - u)^2]$ [I drop F_0 for convience]

$$r = \mathbb{E}[(x_i - u)^3]$$

$$\mathbb{E}[\bar{x}^3] = \mathbb{E}[(x_i - u + u)^3]$$

$$= \mathbb{E}[u^3 + 3u^2(x_i - u) + 3u(\bar{x} - u)^2 + (\bar{x} - u)^3]$$

$$= \underbrace{\mathbb{E}[u^3]}_{u^3} + 3u^2 \underbrace{\mathbb{E}[x_i - u]}_{=0} + 3u \underbrace{\mathbb{E}[(\bar{x} - u)^2]}_{A} + \underbrace{\mathbb{E}[(\bar{x} - u)^3]}_{B}$$

$$A = \mathrm{Var}(\bar{x}) = \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2} \times n \, \mathrm{var}(x_i) = \frac{\sigma^2}{n}$$

$$B = \frac{\mathbb{E}[(n\bar{x} - nu)^3]}{n^3} = \frac{1}{n^3} \mathbb{E}\left[\left(\sum_{i=1}^{n} x_i - nu\right)^3\right]$$

$$= \frac{1}{n^3} \mathbb{E}\left[\left(\sum_{i=1}^{n} (x_i - u)\right)^3\right]$$

$$= \frac{1}{n^3} \mathbb{E}\left(\sum_{i=1}^{n} (x_i - u)^3 + \sum_{i \neq j} \overbrace{(x_i - u)^2 (x_j - u)}^{\text{idpt}} \right)$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} \mathbb{E}\left[(x_i - u)^3\right] + \sum_{i = j} \mathbb{E}(x_i - u)^2 \underbrace{\mathbb{E}(x_j - u)}_{=0}$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} r$$

$$= \frac{r}{n^2}$$

$$\Rightarrow b_1 = u^3 + \frac{3u\sigma^2}{n} + \frac{r}{n^2} - u^3 = \frac{3u\sigma^2}{n} + \frac{r}{n^2}$$

3): Bootstrap estimate of b_1,

$$\hat{b}_1 = \mathbb{E}_{F_1}\left[\theta(\hat{F}_2) - \theta(F_1)\right]$$

$$= \mathbb{E}_{F_1}\left[\theta(F_2)\right] - \theta(F_1)$$

$$= \frac{3\bar{x}\hat{\sigma}^2}{n} + \frac{\hat{r}}{n^2}$$

(just replace $u \mapsto \bar{x}$, $\sigma^2 \mapsto \hat{\sigma}^2$, $r \mapsto \hat{r}$.)

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad \hat{\sigma}^2 = \int (x - \bar{x})^2 \, dF_1 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$\hat{r} = \int (x - \bar{x})^3 \, dF_1 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^3$$

4):

Bootstrap principle:

the population Equation: $\mathbb{E}[f_t(F_0, F_1) \mid F_0] = 0$

we solve via

the sample equation: $\mathbb{E}\{f_t(F_1, F_2) \mid F_1\} = 0$

For Bias correction:

$$f_t = (F_0, F_1) = \theta(F_1) - \theta(F_0) + t_0$$

$$f_t(F_1, F_2) = \theta(F_2) - \theta(F_1) + t_1$$

$$\mathbb{E}\{f_t(F_1, F_2) \mid F_1\} = 0$$

$$\Rightarrow \quad t_1 = \hat{t}_0 = \theta(F_1) - \mathbb{E}\{\theta(F_2) \mid F_1\}.$$

Then the bootstrap bias-reduced estimate is:

$$\theta(F_1) + \hat{t}_0 = 2\theta(F_1) - \mathbb{E}\{\theta(F_2) \mid F_1\}.$$

$$= \theta(F_1) - \mathbb{E}\{\theta(F_2) - \theta(F_1) \mid F_1\}$$

$$= \bar{x}^3 - \hat{b}_1$$

$$\boxed{= \bar{x}^3 - \frac{3\bar{x}\hat{\sigma}^2}{n} - \frac{\hat{\gamma}}{n^2}} \qquad \hat{\theta}_1$$

$\underline{\underline{5)}}$: $b_2 = \mathbb{E}_{F_0}(\hat{\theta}_1 - \theta_0)$

$$= \mathbb{E}(\hat{\theta}_1) - \theta_0$$

$$= \mathbb{E}\left(\bar{x}^3 - \frac{3\bar{x}\hat{6}^2}{n} - \frac{\hat{r}}{n^2}\right) - u^3$$

① $\mathbb{E}(\bar{x}^3) = u^3 + \frac{3u6^2}{n} + \frac{r}{n^2}$

② $\mathbb{E}(\bar{x}\hat{6}^2) = \mathbb{E}\left[\bar{x} \cdot \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]$

③ $\mathbb{E}(\hat{r})$

For $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - 2x_i\bar{x} + \bar{x}^2\right]$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - 2n\bar{x}^2 + n\bar{x}^2\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{1}{n}n\bar{x}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

$\Rightarrow$ ② $= \mathbb{E}[\bar{x}\hat{6}^2] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} x_i^2\bar{x} - \bar{x}^3\right]$

$$= \frac{1}{n}\left\{\mathbb{E}\left[\bar{x} \cdot \sum_{i=1}^{n} x_i^2\right]\right\} - \mathbb{E}[\bar{x}^3]$$

For $\mathbb{E}[\bar{X} \cdot \sum_{i=1}^{n} X_i^2] = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\{X_j \sum_{i=1}^{n} X_i^2\}$

for $\mathbb{E}(X_j \sum_{i=1}^{n} X_i^2)$

$$= \mathbb{E}\Big(X_j^3 + \sum_{i \neq j}^{n} \overbrace{X_j X_i^2}^{\text{idpt}}\Big)$$

$$= \mathbb{E}(X_j^3) + \sum_{i \neq j}^{n} \underbrace{\mathbb{E}(X_j)}_{= u} \underbrace{\mathbb{E}(X_i^2)}_{= (6^2 + u^2)}$$

using n=1 in the expression of E { (\bar_X)^3 } gives E {X^3_j}

$$= \underline{u^3 + 3u6^2 + r + (n-1)u(u^2 + 6^2)}$$

Not dependent on j

$$\Rightarrow \mathbb{E}(\bar{X} \cdot \sum_{i=1}^{n} X_i^2) = u^3 + 3u6^2 + r + nu^3 - u^3 + nu6^2 - u6^2$$

$$= 3n6^2 + r + nu^3 + nu6^2 - u6^2$$

(2) $$= \frac{3n6^2}{n} + \frac{r}{n} + u^3 + u6^2 - \frac{u6^2}{n} - u^3 - \frac{3u6^2}{n} - \frac{r}{n^2}$$

$$= u6^2 + \frac{1}{n}(r - u6^2) - \frac{r}{n^2}$$

③ $\mathbb{E}(\hat{r}) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^3\right)$

$$= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(x_i-\bar{x})^3$$

$$= \mathbb{E}\left[(x_i-\bar{x})^3\right]$$

$$= \mathbb{E}\left[(x_i-u+u-\bar{x})^3\right]$$

$$= \mathbb{E}\left\{(x_i-u)^3 - 3(x_i-u)^2(u-\bar{x}) + 3(x_i-u)(\bar{x}-u)^2 - (\bar{x}-u)^3\right\}$$

$$= \mathbb{E}\{(x_i-u)^3\} - 3\underbrace{\mathbb{E}\{(x_i-u)^2(u-\bar{x})\}}_{C} + 3\underbrace{\mathbb{E}\{(x_i-u)(\bar{x}-u)^2\}}_{D}$$

$$- \mathbb{E}\{(\bar{x}-u)^3\}$$

$$= r - 3C + 3D - \frac{r}{n^2}$$

$$C = \mathbb{E}\left\{(x_j-u)^2(\bar{x}-u)\right\}$$

$$= \frac{1}{n}\mathbb{E}\left[(x_j-u)^2\left(\sum_{i=1}^{n}x_i - u\right)\right]$$

idpt

$$= \frac{1}{n}\mathbb{E}\left[(x_j-u)^3 + \sum_{i\neq j}^{n}(x_j-u)^2(x_i-u)\right]$$

$$= \frac{1}{n}r + \sum_{i\neq j}^{n}\mathbb{E}(x_j-u)^2\underbrace{\mathbb{E}(x_i-u)}_{=0}$$

$$= \frac{r}{n}$$

$$D = \mathbb{E}\{(x_j - u)(\bar{x} - u)^2\}$$

$$= \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\,[(x_j - u)(\bar{x} - u)^2]$$

$$= \frac{1}{n} \cdot \mathbb{E}\left(\sum_{j=1}^{n} (x_j - u)(\bar{x} - u)^2\right)$$

$$= \frac{1}{n} \cdot \mathbb{E}\left[\left(\sum_{j=1}^{n} x_j - nu\right)(\bar{x} - u)^2\right]$$

$$= \frac{1}{n}\,\mathbb{E}\left[(n\bar{x} - nu)(\bar{x} - u)^2\right]$$

$$= \mathbb{E}\,(\bar{x} - u)^3$$

$$= \frac{r}{n^2}$$

$$b_2 = u^3 + \frac{3u6^2}{n} + \frac{r}{n^2} - \frac{3}{n}\left(u6^2 + \frac{r}{n} - \frac{u6^2}{n} - \frac{r}{n^2}\right)$$

$$- \frac{1}{n^2} r\left(1 - \frac{3}{n} + \frac{2}{n^2}\right) - u^3$$

$$= \frac{3}{n^2}(u6^2 - r) + \frac{6r}{n^3} - \frac{2r}{n^4}$$

6): Compare b_1 and b_2

- the Bias of the estimator $\tilde{\theta}_1$ is $O\left(\frac{1}{n}\right)$
- The Bias of the estimator $\hat{\theta}$ is $O\left(\frac{1}{n^2}\right)$

$\Rightarrow$ Bootstrap Bias correction reduces the order magnitude of the Bias by the factor $\frac{1}{n}$