- $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ is a 2×3 matrix

- $\text{rank}(A) = 2$ because $c_1(1,1,0) = c_2(0,1,1)$
$\Rightarrow c_1 = c_2 = 0$

- $AA^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$|AA^T - \lambda I| = 0 \iff \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\iff (2-\lambda)^2 - 1 = 0 \iff \lambda^2 - 4\lambda + 3 = 0$

$\Rightarrow \lambda = \dfrac{4 \pm \sqrt{4}}{2} = \begin{cases} 1 = \lambda_2 \\ 3 = \lambda_1 \end{cases}$

- $v_1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$ associated to $\lambda_1 = 3$:

$AA^T v_1 = 3 v_1 \Rightarrow \begin{cases} 2v_{11} + v_{12} = 3v_{11} \\ v_{11} + 2v_{12} = 3v_{22} \end{cases}$

$\Rightarrow v_{11} = v_{12} \Rightarrow v_1 = \begin{pmatrix} v_{11} \\ v_{11} \end{pmatrix} = v_{11}\boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

Find e.ve of norm 1 :

$\left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = \sqrt{1+1} = \sqrt{2} \Rightarrow \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ is e.ve

with e.val 3, of norm 1.

- $v_2 = \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$ associated to $\lambda_2 = 1$:

$AA^T v_2 = 1 \cdot v_2 \Rightarrow \begin{cases} 2v_{21} + v_{22} = v_{21} \\ v_{21} + 2v_{22} = v_{22} \end{cases} \Rightarrow v_{21} = -v_{22}$

$\Rightarrow v_2 = v_{21}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Find e.ve of norm 1 :

$\left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\| = \sqrt{2} \Rightarrow$ e.ve of norm 1 $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$$\Rightarrow \Gamma = (v_1 \; v_2) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} \lambda_1^{1/2} & 0 \\ 0 & \lambda_2^{1/2} \end{pmatrix} = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix}$$

- Now find Δ : e.ve of $A^T A$

$$A^T A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

* e-ve of $A^T A$ corresponding to $\lambda_1 = 3$:

$$v_1 = \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix} \quad \text{s.t.} \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix} = 3 \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix}$$

$$\Rightarrow \begin{cases} v_{11} + v_{12} = 3 v_{11} \\ v_{11} + 2 v_{12} + v_{13} = 3 v_{12} \\ v_{12} + v_{13} = 3 v_{13} \end{cases} \Rightarrow \begin{cases} v_{12} = 2 v_{11} \\ v_{12} = v_{11} + v_{13} \\ v_{12} = 2 v_{13} \end{cases}$$

$$\Rightarrow \begin{cases} v_{12} = 2 v_{11} \\ v_{13} = v_{11} \end{cases} \Rightarrow v_1 = \begin{pmatrix} v_{11} \\ 2 v_{11} \\ v_{11} \end{pmatrix} = v_{11} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Norm the vector to be of norm 1 :

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} / \underbrace{\sqrt{1 + 2^2 + 1}}_{= \sqrt{6}} = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

* e-ve of $A^T A$ corresponding to $\lambda_1 = 1$: $v_2 = \begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix}$

$$\text{s.t.} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix} = \begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix} \Rightarrow \begin{cases} v_{21} + v_{22} = v_{21} \\ v_{21} + 2 v_{22} + v_{23} = v_{22} \\ v_{22} + v_{23} = v_{23} \end{cases}$$

$$\Rightarrow \begin{cases} v_{22} = 0 \\ v_{23} = - v_{21} \end{cases} \Rightarrow v_2 = \begin{pmatrix} v_{21} \\ 0 \\ -v_{21} \end{pmatrix} = v_{21} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

normalise to be of norm 1 : $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} / \sqrt{1+0+1} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$

Therefore, $\Delta = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ 2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} \end{pmatrix}$

and $A = \Gamma \; \Lambda \; \Delta^T$

$$\underset{2 \times 3}{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}} = \underset{2 \times 2}{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}} \underset{2 \times 2}{\begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix}} \underset{2 \times 3}{\begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}}$$