MAST90105 Methods of Mathematical Statistics

Assignment 4, Semester 1, 2023

Due: Sunday May 28, end of day.

Please submit a scanned or other electronic copy of your work via the Canvas Learning Management System

1. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with the probability density function

$$f_X(x;\theta) = \begin{cases} \frac{2x}{\theta^2}, & 0 \le x \le \theta, \\ 0, & \text{otherwise} \end{cases}$$

Here, $\theta > 0$ is an unknown parameter.

(a) Find the maximum likelihood estimator of θ, $\widehat{\theta}_{ML}$.

(b) Find a method of moments estimator of θ, $\widehat{\theta}_{MM}$. Using approximate normality of $\widehat{\theta}_{MM}$, construct an approximate two-sided 95% confidence interval for θ.

(c) Find the cumulative distribution function (CDF) of $\widehat{\theta}_{ML}$ and use it to find the CDF of $W_n = n(\theta - \widehat{\theta}_{ML})$, $F_n(y)$. Find the limiting distribution $F_\infty(y) = \lim_{n\to\infty} F_n(y)$. Explain why $\widehat{\theta}_{ML}$ does not have asymptotic normal distribution. **Hint:** $\lim_{n\to\infty}(1 - k/n)^n = e^{-k}$.

(d) Using the limiting CDF $F_\infty(y)$, show that W_n/θ is a pivot and find its limiting distribution. Use it to construct an approximate two-sided 95% confidence interval for θ.

(e) Now assume that $n = 2$ and we want to test the null hypothesis $H_0 : \theta = 1$ against the alternative $H_1 : \theta = 2$. We reject H_0 in favor of H_1 if $\widehat{\theta}_{ML} > c$, where $c < 1$ is some constant. Find c that minimizes $\alpha^2 + \beta^2$ where α and β are the type 1 and type 2 errors of this test, respectively. Find the p-value of this test if the observed value $\widehat{\theta}_{ML} = 0.99$. Do we reject H_0 at the 5% significance level? **Hint:** here, you need to use the exact distribution $\widehat{\theta}_{ML}$, not the limiting one.