1.

(a)

$Pr\{H \text{ on the } k\text{-th flip} | H \text{ is observed on the } (k-1)\text{-st flip}\} = 1 - \frac{1}{3} = \frac{2}{3}$

$Pr(X=1) = Pr(T) = \frac{1}{2}$

$Pr(X=2) = Pr(H) \cdot Pr(T \text{ on the 2nd flip} | H \text{ is observed on the 1st flip}) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

$Pr(X=3) = Pr(H)\, Pr(H \text{ on the 2nd flip} | H \text{ is observed on the 1st flip})\, Pr(T \text{ on the 3rd flip} | H \text{ is observed on the 2nd flip}) = \frac{1}{2} \times \frac{2}{3} \times \frac{1}{3} = \frac{1}{9}$

$Pr(X=4) = \frac{1}{2} \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{27}$

$Pr(X=5) = \frac{1}{2} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{81}$

.... the same way to calculate result whe $x \geq 5$

PMF : $Pr(X=k) \begin{cases} \frac{1}{2} & k=1 \\ \frac{1}{2}\left(\frac{2}{3}\right)^{k-2}\frac{1}{3} = \frac{1}{6}\left(\frac{2}{3}\right)^{k-2} & k = 2, 3, 4, \dots \end{cases}$

(b) $M(t) = E(e^{tX}) = \sum_{x=1}^{n} e^{tx} f(x)$

$$= \frac{1}{2} \times e^t + \sum_{x=2}^{n} e^{tx} \times \frac{1}{6}\left(\frac{2}{3}\right)^{x-2}$$

$$= \frac{1}{2}e^t + \frac{1}{6}\sum_{x=2}^{n} e^{tx}\left(\frac{2}{3}\right)^{x-2}$$

$$= \frac{1}{2}e^t + \frac{1}{6}\sum_{x=2}^{n} e^{tx} \times \left(\frac{2}{3}\right)^{x} \times \left(\frac{2}{3}\right)^{-2}$$

$$= \frac{1}{2}e^t + \frac{3}{8}\sum_{x=2}^{n}\left(\frac{2}{3}e^t\right)^{x}$$

$$= \frac{1}{2}e^t + \frac{3}{8} \times \frac{\left(\frac{2}{3}e^t\right)^2}{1 - \frac{2}{3}e^t}$$

$$= \frac{\frac{1}{2}e^t - \frac{1}{6}e^{2t}}{1 - \frac{2}{3}e^t}$$

X doesn't have negative binomial distribution. because if $X \sim NB(r, p)$ $M(t) = \left(\frac{pe^t}{1-qe^t}\right)^r$, but MGF of X doesn't satisfy this form.

(c)

$$M'(t) = \frac{\left(\frac{1}{2}e^t - \frac{1}{3}e^{2t}\right)\left(1 - \frac{2}{3}e^t\right) - \left(\frac{1}{2}e^t - \frac{1}{6}e^{2t}\right)\left(-\frac{2}{3}e^t\right)}{\left(1 - \frac{2}{3}e^t\right)^2}$$

$$E(X) = M'(0) = \frac{(\frac{1}{2}-\frac{1}{3})(1-\frac{2}{3}) - (\frac{1}{2}-\frac{1}{6})(-\frac{2}{3})}{(1-\frac{2}{3})^2} = \frac{5}{2} = 2.5$$

2. (a)

We can divide the time period 10:00 ~ 12:00 into 2 stages:

1st stage: from 10:00 – 11:00 ($0<t<1$) 2nd stage: from 11:00 ($t \geq 1$)

Let X_i be the number of phone calls during i-th stage.

during 1st stage: from 10:00 to 11:00, only c_1 is operating, so $X_1 \sim Pois(\lambda_1)$

accordingly, T_1 follows exponential distribution with rate of λ_1

CDF of T_1: $F_1(t) = P_r(T_1 \leq 1) = 1 - e^{-\lambda_1 t}$

during 2nd stage: from 11:00 ($t \geq 1$), c_1, c_2 are operating together, so

$X_2 \sim Pois(\lambda_1 + \lambda_2)$, accordingly, T_1 follows exponential distribution with the rate of $\lambda_1 + \lambda_2$. CDF of T_1 $F_1(t) = P_r(T_1 \leq t) = 1 - e^{-(\lambda_1+\lambda_2)t}$

$$F_1(t) \begin{cases} 1 - e^{-\lambda_1 t} & \text{1st stage} \quad 0<t<1 \\ 1 - e^{-(\lambda_1+\lambda_2)t} & \text{2nd stage} \quad t \geq 1 \end{cases}$$

(b)

E(number of calls during 10:00 ~ 12:00)

= E(number of calls in the 1st stage) + E(number of calls in the 2nd stage)

$= \lambda_1 + \lambda_1 + \lambda_2$

$= 2\lambda_1 + \lambda_2$

3.

(a) $F_X(x) = \int_1^x \left(\frac{c}{\sqrt{x}} + \frac{1}{3x\sqrt{x}}\right) dx$

$F_X(4) = \int_1^4 \left(\frac{c}{\sqrt{x}} + \frac{1}{3x\sqrt{x}}\right) dx = 1$

$c\int_1^4 x^{-\frac{1}{2}} dx + \frac{1}{3}\int_1^4 x^{-\frac{3}{2}} dx = 1$

$2c \times \left(x^{\frac{1}{2}}\Big|_1^4\right) - \frac{2}{3}\left(x^{-\frac{1}{2}}\Big|_1^4\right) = 1$

$c = \frac{1}{3}$

(b) Let median of X be X_m

$F_X(X_m) = \int_1^{x_m} \frac{1}{3}\left(\frac{1}{\sqrt{x}} + \frac{1}{x\sqrt{x}}\right) dx = \frac{1}{2}$

$\int_1^{x_m} \frac{1}{\sqrt{x}} dx + \int_1^{x_m} \frac{1}{x\sqrt{x}} dx = \frac{3}{2}$

$\left(2x^{\frac{1}{2}}\Big|_1^{x_m}\right) + \left(-2x^{-\frac{1}{2}}\Big|_1^{x_m}\right) = \frac{3}{2}$

$2X_m^{\frac{1}{2}} - 2 + \left(-2X_m^{-\frac{1}{2}} + 2\right) = \frac{3}{2}$

$2X_m^{\frac{1}{2}} - 2X_m^{-\frac{1}{2}} = \frac{3}{2}$

$X_m = \frac{41 + 3\sqrt{73}}{32} \approx 2.082$

(c)

$E(X) = \int_1^4 \frac{x}{3}\left(\frac{1}{\sqrt{x}} + \frac{1}{x\sqrt{x}}\right) dx$

$= \int_1^4 \frac{1}{3}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

$= \frac{1}{3}\left(\frac{2}{3}x^{\frac{3}{2}}\Big|_1^4 + 2x^{\frac{1}{2}}\Big|_1^4\right)$

$= \frac{20}{9}$

$E(X^2) = \int_1^4 \frac{x^2}{3}\left(\frac{1}{\sqrt{x}} + \frac{1}{x\sqrt{x}}\right) dx$

$= \int_1^4 \frac{1}{3}\left(x^{\frac{3}{2}} + x^{\frac{1}{2}}\right) dx$

$= \frac{1}{3}\left(\frac{2}{5}x^{\frac{5}{2}}\Big|_1^4 + \frac{2}{3}x^{\frac{3}{2}}\Big|_1^4\right)$

$= \frac{256}{45}$

$Var(X) = E(X^2) - E^2(X) = \frac{256}{45} - \frac{400}{81} \approx 0.75$

(d) when $1<x<4$, $1<\sqrt{x}<2 \Rightarrow 1<y<2$

$F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2) = F_X(y^2)$

$f_Y(y) = F_Y'(y) = F_X'(y^2)\cdot 2y = f_X(y^2)\cdot 2y = \frac{1}{3}(\frac{1}{y}+\frac{1}{y^3})\times 2y = \frac{2}{3}(1+\frac{1}{y^2})$

when $x \le 1$ or $x \ge 4$ $f_X(x)=0$, so $f_Y(y)=0$

$$f_Y(y) = \begin{cases} \frac{2}{3}(1+\frac{1}{y^2}) & 1<y<2 \\ 0 & \text{otherwise.} \end{cases}$$

4.

(a)

$Z_1 = X+Y = \{-2,-1,0,1,2\}$

$Z_2 = XY = \{-1,0,1\}$

PMF of Z_1

Z_1	-2	-1	0	1	2
Pr	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

PMF of Z_2

Z_2	-1	0	1
Pr	$\frac{2}{9}$	$\frac{5}{9}$	$\frac{2}{9}$

$E(Z_1) = 0$

$E(Z_2) = 0$

joint PMF of $Pr(Z_1=z_1, Z_2=z_2)$

$Z_2 \backslash Z_1$	-2	-1	0	1	2	$Pr(Z_2)$
-1	0	0	$\frac{2}{9}$	0	0	$\frac{2}{9}$
0	0	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	0	$\frac{5}{9}$
1	$\frac{1}{9}$	0	0	0	$\frac{1}{9}$	$\frac{2}{9}$
$Pr(Z_1)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	1

$\Pr(Z_1=-2, Z_2=-1) = \Pr(x+y=-2, xy=-1) = 0 \quad \Pr(Z_1=-2, Z_2=0) = \Pr(x+y=-2, xy=0) = 0$

$\Pr(Z_1=-2, Z_2=1) = \Pr(x+y=-2, xy=1) = \Pr(x=-1, y=-1) = \frac{1}{9}$

$\Pr(Z_1=-1, Z_2=-1) = \Pr(x+y=-1, xy=-1) = 0 \quad \Pr(Z_1=-1, Z_2=1) = \Pr(x+y=-1, xy=1) = 0$

$\Pr(Z_1=-1, Z_2=0) = \Pr(x+y=-1, xy=0) = \Pr(\{x=-1, y=0\}, \{x=0, y=-1\}) = \frac{2}{9}$

$\Pr(Z_1=0, Z_2=-1) = \Pr(x+y=0, xy=-1) = \Pr(\{x=-1, y=1\}, \{x=1, y=-1\}) = \frac{2}{9}$

$\Pr(Z_1=0, Z_2=0) = \Pr(x+y=0, xy=0) = \Pr(x=0, y=0) = \frac{1}{9}$

$\Pr(Z_1=0, Z_2=1) = \Pr(x+y=0, xy=1) = 0 \quad \Pr(Z_1=1, Z_2=-1) = \Pr(x+y=1, xy=-1) = 0$

$\Pr(Z_1=1, Z_2=0) = \Pr(x+y=1, xy=0) = \Pr(\{x=0, y=1\}, \{x=1, y=0\}) = \frac{2}{9}$

$\Pr(Z_1=1, Z_2=1) = \Pr(x+y=1, xy=1) = 0 \quad \Pr(Z_1=2, Z_2=-1) = \Pr(x+y=2, xy=-1) = 0$

$\Pr(Z_1=2, Z_2=0) = \Pr(x+y=2, xy=0) = 0 \quad \Pr(Z_1=2, Z_2=-1) = \Pr(x+y=2, xy=1) = \frac{1}{9}$

$E(Z_1Z_2) = \sum\sum z_1 z_2 \Pr(z_1, z_2) = 0 \qquad E(Z_1) = E(Z_2) = 0$

$$\mathrm{Cor}(Z_1, Z_2) = \frac{\mathrm{Cov}(Z_1, Z_2)}{\sqrt{\mathrm{Var}(Z_1)}\sqrt{\mathrm{Var}(Z_2)}} = \frac{E(Z_1Z_2) - E(Z_1)E(Z_2)}{\sqrt{\mathrm{Var}(Z_1)}\sqrt{\mathrm{Var}(Z_2)}} = 0$$

Z_1, Z_2 are not independent, we can use equation $P(A)P(B) = P(AB)$ to check independence, take $P(Z_1=-2, Z_1=-1)$ as an example:

$$P(Z_1=-2, Z_1=-1) = 0 \neq P(Z_1=-2)P(Z_1=-1) = \frac{2}{81}$$

$\therefore$ It doesn't satisfy the requirement of independence.

(b) $W = Z_1 + Z_2 = \{-1, 0, 1, 3\}$

PMF of W:

W	-1	0	1	3
$\Pr$	$\frac{5}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

$\Pr(W=-1) = \Pr(\{Z_1=0, Z_2=-1\}, \{Z_1=-1, Z_2=0\}, \{Z_1=-2, Z_1=1\}) = \frac{5}{9}$

$\Pr(W=0) = \Pr(\{Z_1=0, Z_2=0\}, \{Z_1=-1, Z_2=1\}, \{Z_1=+1, Z_2=-1\}) = \frac{1}{9}$

$\Pr(W=1) = \Pr(\{Z_1=1, Z_2=0\}, \{Z_1=0, Z_2=1\}, \{Z_1=2, Z_2=-1\}) = \frac{2}{9}$

$Pr(w=3) = Pr(\{z_1=2, z_2=1\}) = \frac{1}{9}$

$E(w) = (-1)\times\frac{5}{9} + 1\times\frac{2}{9} + 3\times\frac{1}{9} = 0$ $\quad E(w^2) = \frac{5}{9} + \frac{2}{9} + 3^2\times\frac{1}{9} = \frac{16}{9}$

$Var(w) = E(w^2) - E^2(w) = \frac{16}{9}$ $\quad \sigma = \sqrt{var(w)} = \frac{4}{3}$

Skewness of w: $E\left[\left[\frac{w-\mu}{\sigma}\right]^3\right] = E\left[\frac{w^3}{\sigma^3}\right] = \frac{1}{\sigma^3}E(w^3) = \left(\frac{3}{4}\right)^3\times\frac{24}{9} = \frac{9}{8}$

$E(w^3) = -1\times\frac{5}{9} + 1\times\frac{2}{9} + 27\times\frac{1}{9} = \frac{27-3}{9} = \frac{24}{9} = \frac{8}{3}$