(a)

$$L(\theta)=\prod_{i=1}^{n}\left(\frac{2x_i}{\theta^2}\right) \quad (0\le x\le\theta)$$

$$=\left(\frac{2}{\theta^2}\right)^n\prod_{i=1}^{n}x_i \quad (0\le x\le\theta)$$

we can see when θ increasing, $L(\theta)$ would become smaller

so θ should be equal to its minimum value to make sure $L(\theta)$ get its maximum value. then $\hat{\theta}_{mL}=\max(X_1,\cdots,X_n)=X_{(n)}$ (nth order statistics.)

(b)

$$E(X)=\int_0^\theta\frac{2x}{\theta^2}\cdot x\,dx=\int_0^\theta\frac{2x^2}{\theta^2}dx=\frac{2}{3}\frac{x^3}{\theta^2}\Big|_0^\theta=\frac{2}{3}\theta$$

$$\frac{2}{3}\theta=\frac{1}{n}\sum_{i=1}^{n}x_i=\bar{X} \qquad \hat{\theta}_{MM}=\frac{3}{2}\bar{X}$$

$$E(X^2)=\int_0^\theta\frac{2x}{\theta^2}\cdot x^2\,dx=\int_0^\theta\frac{2x^3}{\theta^2}dx=\frac{2}{4}\frac{x^4}{\theta^2}\Big|_0^\theta=\frac{1}{2}\theta^2$$

$$Var(X)=E(X^2)-E^2(X)=\frac{1}{2}\theta^2-\frac{4}{9}\theta^2=\frac{1}{18}\theta^2$$

$\therefore$ based on CLT: $\bar{X}\sim N\left(\frac{2}{3}\theta,\frac{\theta^2}{18n}\right)$

$$E(\hat{\theta}_{mm})=\frac{3}{2}E(\bar{X})=\theta \qquad Var(\hat{\theta}_{mm})=Var\left(\frac{3}{2}\bar{X}\right)=\frac{9}{4}Var(\bar{X})=\frac{9}{4}\times\frac{\theta^2}{18n}=\frac{\theta^2}{8n}$$

$\therefore \hat{\theta}_{mm}\sim N\left(\theta,\frac{\theta^2}{8n}\right)$

$\dfrac{\bar{X}-M}{\frac{6}{\sqrt{n}}}\sim N(0,1)$ when n is large enough

$$\therefore \frac{\hat{\theta}_{mm}-\theta}{\frac{\theta}{\sqrt{8n}}}\sim N(0,1)\Rightarrow P\left(Z_{0.025}<\frac{\hat{\theta}_{mm}-\theta}{\frac{\theta}{\sqrt{8n}}}<Z_{0.975}\right)=0.95$$

$$\Rightarrow P\left(Z_{0.025}<\frac{\hat{\theta}_{mm}-\theta}{\frac{\hat{\theta}_{mm}}{\sqrt{8n}}}<Z_{0.975}\right)=0.95$$

$$\Rightarrow P\left(\hat{\theta}_{mm}-Z_{0.975}\frac{\hat{\theta}_{mm}}{\sqrt{8n}}<\theta<\hat{\theta}_{mm}-Z_{0.025}\frac{\hat{\theta}_{mm}}{\sqrt{8n}}\right)$$

$\therefore$ 95% CI for θ is $\left(\hat{\theta}_{mm}\left(1-\frac{1.96}{\sqrt{8n}}\right),\ \hat{\theta}_{mm}\left(1+\frac{1.96}{\sqrt{8n}}\right)\right)$

$$\left(\frac{3}{2}\bar{X}\left(1-\frac{1.96}{\sqrt{8n}}\right),\ \frac{3}{2}\bar{X}\left(1+\frac{1.96}{\sqrt{8n}}\right)\right)$$

(c) $T = \hat{\theta}_{ML} = \max(X_1, X_2, \cdots, X_n) = X_{(n)}$

$$P(T \le t) = P(\max(X_1, X_2, \cdots, X_n) \le t) = \prod_{i=1}^{n} P(X_i \le t)$$

$$P(X_i \le t) = \int_0^t \frac{2x}{\theta^2} dx \quad (0 \le x \le \theta) = \frac{x^2}{\theta^2}\Big|_0^t = \frac{t^2}{\theta^2}$$

CDF of $\hat{\theta}_{ML}$:

$$P(T \le t) = \begin{cases} 0 & t < 0 \\ \left(\frac{t^2}{\theta^2}\right)^n & , 0 < t < \theta \\ 1 & t \ge \theta \end{cases}$$

$$F_n(y) = P(W_n \le y) = P(n(\theta - \hat{\theta}_{ML}) \le y) = P((\theta - \hat{\theta}_{ML}) \le \frac{y}{n}) = P(\hat{\theta}_{ML} \ge \theta - \frac{y}{n})$$

$$F_n(y) = 1 - P(\hat{\theta}_{ML} \le \theta - \frac{y}{n}) = 1 - \left[\frac{(\theta - \frac{y}{n})}{\theta}\right]^{2n} = 1 - (1 - \frac{\frac{y}{\theta}}{n})^{2n} = 1 - (1 - \frac{\frac{2y}{\theta}}{2n})^{2n}$$

$$\lim_{n \to \infty} F_n(y) = 1 - e^{-\frac{2y}{\theta}}$$

The assumption of asymptotic normality is violated here.

① the range of rv is depend on parameter θ

② $\hat{\theta}_{ML}$ is not differentiable

(d)

$$P(\frac{W_n}{\theta} \le y) = P(W_n \le y\theta) = 1 - e^{-2y}$$

so $\frac{W_n}{\theta} \sim Exp(\lambda = 2)$ a: 2.5% quantile of $Exp(2)$. b: 97.5% quantile of $Exp(2)$

$$P\{a < \frac{W_n}{\theta} < b\} = 0.95$$

$$P\{a \le \frac{n(\theta - \hat{\theta}_{ML})}{\theta} \le b\} = 0.95 \Rightarrow P\{a \le \frac{n(\theta - \hat{\theta}_{ML})}{\hat{\theta}_{ML}} \le b\} = 0.95$$

$$\Rightarrow P\left\{\frac{a\hat{\theta}_{ML}}{n} \le \theta - \hat{\theta}_{ML} \le \frac{b\hat{\theta}_{ML}}{n}\right\} = 0.95 \Rightarrow P\left\{\hat{\theta}_{ML}(1 + \frac{a}{n}) \le \theta \le \hat{\theta}_{ML}(1 + \frac{b}{n})\right\} = 0.95$$

in R command: a = qexp(0.025, rate=2) ≈ 0.0127 . b = qexp(0.975, rate=2) ≈ 1.844

so 95% CI of θ is $(X_{(n)}(1 + \frac{0.0127}{n}), X_{(n)}(1 + \frac{1.844}{n}))$

(e)

$n=2$ CDF of $\hat{\theta}_{mL}$: $P(T \le t) = \frac{t^4}{\theta^4}$ $(0<t<\theta)$

$\alpha = P\{\hat{\theta}_{mL} > c \mid \theta=1\} = 1 - P(\hat{\theta}_{mL} \le c \mid \theta=1) = 1 - c^4$

$\beta = P(\hat{\theta}_{mL} \le c \mid \theta=2) = \frac{c^4}{2^4}$

$\alpha^2+\beta^2 = (1-c^4)^2 + (\frac{c^4}{16})^2$ Let $c^4 = u$

$\alpha^2+\beta^2 = (1-u)^2 + \frac{u^2}{16^2} = 1-2u+u^2+\frac{u^2}{16^2}$

When $u = -\frac{-2}{2(1+\frac{1}{16^2})}$, $\alpha^2+\beta^2$ gets its minimum value

$c^4 = \frac{16^2}{16^2+1}$ $c \approx 0.999$

assuming H_0 is true , and observed value $\hat{\theta}_{mL}$ is 0.99

$\therefore$ P-value: $1-P(\frac{(0.99)^4}{\theta^4} \le t \mid H_0) = 1 - 0.99^4 \approx 0.039 < 5\%$

So we have strong confidence to reject H_0.