MAST90105 Methods of Mathematical Statistics
Assignment 1, Semester 1 2023
Due date: Wednesday March 22, end of day.

- Your assignment should show all working and reasoning. Marks will be given for method as well as for correct answers. You may use R for calculations but must include the code that you input to get the results.
- Four assignments count for 20% of your assessment (5% each).
- **Please submit a scanned or other electronic copy of your work via the Canvas Learning Management System**
- **Late assignments will only be accepted under exceptional circumstances. A late penalty may be imposed.**

1. There are two coins in the bag: one coin is fair (shows T and H with $\Pr(T) = \Pr(H) = 0.5$), and the other one is biased (shows T and B with $\Pr(T) = 0.3$ and $\Pr(B) = 0.7$). A randomly selected coin is tossed and shows T. Find the probability that **the other** coin is biased.

2. Assume we have two bags with 1 red ball and 3 green balls in the first bag, and with 3 red balls and 1 green ball in the second bag. We randomly select a bag, and then randomly select two balls (a) with replacement; (b) without replacement from this bag. One of the two balls is green. Find the probability that the balls were drawn from the first bag.

3. A fair four-sided die marked with 1, 2, 2, 3 is rolled once and shows Y. This die is then rolled again Y times, and the minimum score (S_1) and the maximum score (S_2) in the last Y rolls are recorded. Find $\Pr(S_1 = 2)$ and $\Pr(S_2 = 2)$.

4. A fair cubic die marked with 1, 2, 3, 4, 5, 6 and fair four-sided die marked with 1, 2, 3, 4 are rolled and show S_6 and S_4, respectively. Assuming the outcomes in each roll are independent, find the probability mass function (pmf) of a random variable $X = S_6 - S_4$. Find the skewness of X.

5. Let X and Y be two independent random variables with the moment generating functions $M_X(t) = 1 - t + \alpha t^2 + \epsilon_1(t)$ ($\alpha > 0.5$) and $M_Y(t) = 1 + t + t^2 + \epsilon_2(t)$, where $\lim_{t\to 0} \epsilon_i(t) = 0$, $\lim_{t\to 0} \epsilon_i^{(k)}(t) = 0$ for $i = 1, 2$ and $k = 1, 2, 3$ (i.e., these functions $\epsilon_1(t), \epsilon_2(t)$ and their first three derivatives converge to zero as $t \to 0$). Define $Z = X + Y$. Find the maximum value of skewness of Z. Find the variance $Var(Z)$ of Z with maximum skewness.

4. A fair cubic die marked with 1, 2, 3, 4, 5, 6 and fair four-sided die marked with 1, 2, 3, 4 are rolled and show S_6 and S_4, respectively. Assuming the outcomes in each roll are independent, find the probability mass function (pmf) of a random variable $X = S_6 - S_4$. Find the skewness of X.

$X=0 \Rightarrow S_6 - S_4 = 0 \Rightarrow \frac{1}{6}\times\frac{1}{4}\times 4 = \frac{1}{6}$

$X=1 \Rightarrow S_6 - S_4 = 1 \Rightarrow S_6 = S_4 + 1 \Rightarrow \begin{cases}1+1\\2+1\\3+1\\4+1\end{cases} \Rightarrow \frac{1}{6}\times\frac{1}{4}\times 4 = \frac{1}{6}$

$X=2 \Rightarrow S_6 = S_4 + 2 \Rightarrow \begin{cases}1+2=3\\2+2=4\\3+2=5\\4+2=6\end{cases} \Rightarrow \frac{1}{6}\times\frac{1}{4}\times 4 = \frac{1}{6}$

$X=3 \Rightarrow S_6 = S_4 + 3 \Rightarrow \begin{cases}1+3=4\\2+3=5\\3+3=6\end{cases} \Rightarrow \frac{1}{6}\times\frac{1}{4}\times 3 = \frac{1}{8}$

$X=4 \Rightarrow S_6 = S_4 + 4 \Rightarrow \begin{cases}1+4=5\\2+4=6\end{cases} \Rightarrow \frac{1}{6}\times\frac{1}{4}\times 2 = \frac{1}{12}$

$X=5 \Rightarrow S_6 = S_4 + 5 \begin{cases}1+5\end{cases} \Rightarrow \frac{1}{6}\times\frac{1}{4} = \frac{1}{24}$

$X=-1 \Rightarrow S_6 - S_4 = -1 \Rightarrow S_6 = -1 + S_4 \Rightarrow \begin{cases}-1+2\\-1+3\\-1+4\end{cases} \Rightarrow \frac{1}{6}\times\frac{1}{4}\times 3 = \frac{1}{8}$

$X=-2 \Rightarrow \begin{cases}-2+4\\-2+3\end{cases} \Rightarrow \frac{1}{6}\times\frac{1}{4}\times 2 = \frac{1}{12}$

$X=-3 \Rightarrow \frac{1}{6}\times\frac{1}{4} \Rightarrow \frac{1}{24}$

X	-3	-2	-1	0	1	2	3	4	5
P	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{24}$

$\sigma^2 = E(X^2) - E^2(X)$

$E\left[\left[\frac{X-\mu}{\sigma}\right]^3\right]$

$E(X) = -\frac{1}{8} - \cancel{\frac{1}{6}} - \frac{1}{8} + \cancel{\frac{1}{6}} + \frac{1}{3} + \frac{3}{8} + \frac{1}{3} + \frac{5}{24} = \frac{1}{8} + \frac{2}{3} + \frac{5}{24} = \frac{3+16+5}{24} = 1$

$E(X^2) = \frac{9}{24} + \frac{4}{12} + \frac{1}{8} + \frac{1}{6} + \frac{4}{6} + \frac{9}{8} + \frac{16}{12} + \frac{25}{24} = \frac{17}{24} + \frac{3}{24} + \frac{20}{24} + \frac{27}{24} + \frac{32}{24} + \frac{25}{24}$

$= \frac{40}{24} + \frac{59}{24} + \frac{25}{24} = \frac{65+59}{24} = \frac{124}{24}$

$\sigma = \frac{10}{\sqrt{24}} = \frac{10}{2\sqrt{6}} = \frac{5}{\sqrt{6}}$

$Var(X)\ E(X^2) - E^2(X) = \frac{124}{24} - \frac{24}{24} = \frac{100}{24}$

$\left[\frac{(-3-1)\sqrt{6}}{5}\right]^3 \times \frac{1}{24} + \left[\frac{-3\sqrt{6}}{5}\right]^3 \times \frac{1}{12} + \left[\frac{-2\sqrt{6}}{5}\right]^3 \times \frac{1}{8} + \left[\frac{-\sqrt{6}}{5}\right]^3 \times \frac{1}{6} + \left(\frac{\sqrt{6}}{5}\right)^3 \times \frac{1}{6} + \left(\frac{2\sqrt{6}}{5}\right)^3 \times \frac{1}{8} + \left(\frac{3\sqrt{6}}{5}\right)^3 \times \frac{1}{12} + \left(\frac{4\sqrt{6}}{5}\right)^3 \times \frac{1}{24}$

$= 0$

1. There are two coins in the bag: one coin is fair (shows T and H with $\Pr(T) = \Pr(H) = 0.5$), and the other one is biased (shows T and B with $\Pr(T) = 0.3$ and $\Pr(B) = 0.7$). A randomly selected coin is tossed and shows T. Find the probability that **the other** coin is biased.

$0.5 \times 0.3 = 0.15$

2. Assume we have two bags with 1 red ball and 3 green balls in the first bag, and with 3 red balls and 1 green ball in the second bag. We randomly select a bag, and then randomly select two balls (a) with replacement; (b) without replacement from this bag. One of the two balls is green. Find the probability that the balls were drawn from the first bag.

① with replacement

$$\frac{C_1^1 \times C_3^1 + C_3^1 \times C_3^1}{C_2^1 C_4^1 C_4^1} = \frac{1\times 3 + 9}{2\times 4\times 4} = \frac{12}{32} = \frac{3}{8}$$

② without replacement

$$\frac{C_1^1 \times C_3^1 + C_3^1 C_2^1}{C_2^1 C_4^1 C_3^1} = \boxed{\frac{3+6}{2\times 4\times 3}} = \frac{1+2}{2\times 4} = \frac{3}{8}$$

first bag.

3. A fair four-sided die marked with 1, 2, 2, 3 is rolled once and shows Y. This die is then rolled again Y times, and the minimum score (S_1) and the maximum score (S_2) in the last Y rolls are recorded. Find $\Pr(S_1 = 2)$ and $\Pr(S_2 = 2)$.

$P(Y=2) = \frac{1}{2}$ $P(Y=3) = \frac{1}{4}$ 4

$P(Y=1) = \frac{1}{4}$

all of events are independent.

$-1)\ P(S_1=2 \mid Y=1)$

$Y=1: \Pr(S_1=2) = \Pr(S=2) \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

$Y=2: \Pr(S_1=2) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{4} = \frac{1}{2}\left(\frac{1}{4}+\frac{1}{8}\right) = \frac{1}{2}\times\frac{3}{8} = \frac{3}{16}$

$Y=3: \Pr(S_1=2) = \frac{1}{4}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{4}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{4} + \frac{1}{4}\times\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4} = \frac{1}{4}\left(\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\right)$

$= \frac{1}{4}\left(\frac{3}{16}+\frac{1}{32}\right)$

$= \frac{1}{4}\times\frac{7}{32} = \frac{7}{128}$

1 1 1 3

$Y=2 \quad P_r(S_2=2) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = 16$

$Y=3 : P_r(S_2=2) = \frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}\times\frac{1}{2}+\frac{1}{4}\times\frac{1}{4}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{4}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{7}{128}$

5. Let X and Y be two independent random variables with the moment generating functions $M_X(t) = 1 - t + \alpha t^2 + \epsilon_1(t)$ $(\alpha > 0.5)$ and $M_Y(t) = 1 + t + t^2 + \epsilon_2(t)$, where $\lim_{t\to 0} \epsilon_i(t) = 0$, $\lim_{t\to 0} \epsilon_i^{(k)}(t) = 0$ for $i = 1, 2$ and $k = 1, 2, 3$ (i.e., these functions $\epsilon_1(t), \epsilon_2(t)$ and their first three derivatives converge to zero as $t \to 0$). Define $Z = X + Y$. Find the maximum value of skewness of Z. Find the variance $Var(Z)$ of Z with maximum skewness.

$E(Z) = E(X+Y) = E(X)+E(Y)$

$E(X) = M_X'(t) = 0 - 1 + 2\alpha t +$