THE UNIVERSITY OF MELBOURNE

Student ID: 1407870

The University of Melbourne

Mid Semester 1 Assessment — 2019

School of Mathematics and Statistics

MAST90105 Methods of Mathematical Statistics

Exam duration: **3 hours**
Reading time: **15 minutes**
This paper has **4 pages** including this page

Authorised materials:
The calculator authorised at the University of Melbourne is the CASIO FX82 and this is permitted.
Two A4 double-sided handwritten sheets of notes.

Instructions to invigilators:
Sixteen-page script books shall be supplied to each student.
Students may not take this paper with them at the end of the exam.

Instructions to students:
There are 8 questions. All questions may be attempted.
The number of marks for each question is indicated after the question.
The total number of marks available is 100.
Your raw mark of this exam will be multiplied with 0.35 before being added to your final subject mark.

This exam paper is not to be held by Baillieu Library.

Pr(mutation in gene) = 0.02

1. A mutation in a certain gene can occur with probability 0.02. The probability of a rear disease in a person with mutation in this gene is 0.10, and this probability is 0.002 otherwise.

   (a) Find the probability that a random person has this disease.

   (b) Find the probability that a person with the disease has a mutation in this gene.

   (c) Consider a group of three patients with the disease. Find the probability that two of them have mutations in this gene.

[4 + 2 + 4 = 10 marks]

2. In Sydney area, the number of earthquakes during next t years, X_t, follows a Poisson process with rate 1 per year.

   (a) Find the probability that there will be no earthquakes in Sydney next year.

   (b) Let T_0 be the time (in years) until first year without an earthquake. Find $\Pr(2 \leq T_0 \leq 5)$.

   (c) Find $\Pr(X_3 \geq 3 | X_3 \geq 2)$.

   (d) Find $\Pr(2 \leq X_2 \leq 4 | X_5 = 7)$.

[2 + 3 + 5 + 5 = 15 marks]

3. There are three coins: one coin is fair, and the other two are biased. The probability that the first biased coin shows tail and head is 1/3 and 2/3, respectively, and the probability that the second biased coin shows tail and head is 2/3 and 1/3, respectively. We randomly select one coin and flip it two times. Let X be the number of tails in the two flips.

   (a) Find the range and probability mass function (pmf) of X.

   (b) Find $\mathrm{E}(X)$, $\mathrm{Var}(X)$ and the moment generating function of X.

[5+5 = 10 marks]

4. The moment generating function of a random variable X is

$$M(t) = C\left(\frac{e^t + e^{-t}}{3}\right)^2 + \left(\frac{e^{t/2} + e^{-t/2}}{3}\right)^2.$$

   (a) Find the constant C and the probability mass function of X.

   (b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

[5 + 5 = 10 marks]

5. A device has two components that work independently of each other. This device fails if at least one of these components fail. The lifetimes (times to failure, measured in years) of these two components, T_1, T_2, follow an exponential distribution with $\Pr(T_k \leq t) = 1 - e^{-tk}$, $k = 1, 2$.

   (a) Find the probability that exactly one of the two components fails in one year.

   (b) Let T_0 be the lifetime of the device. Find the cumulative distribution function of T_0, $\Pr(T_0 \leq t)$.

   (c) What is the expected lifetime of the device?

[5 + 5 + 2 = 12 marks]

6. Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} C, & \text{if } -1 \leq x \leq 0, \\ 2C, & \text{if } \quad 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the constant C.

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(c) Find the cumulative distribution function of X, $F(x)$.

(d) Find the median of X.

(e) Let $Y = X^2$. Find the cumulative distribution function of Y, $G(y)$.

(f) Find the probability density function of Y, $g(y)$.

[2 + 5 + 4 + 4 + 5 + 2 = 22 marks]

7. Let X_1, X_2 be two independent Bernoulli random variables with probability of success $p = 1/2$. Define two new random variables $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \max\{X_1, X_2\}$.

(a) Find the joint probability mass function of Y_1, Y_2.

(b) Find $\mathrm{E}(Y_1), \mathrm{E}(Y_2), \mathrm{Var}(Y_1)$ and $\mathrm{Var}(Y_2)$.

(c) Find $\mathrm{Cov}(Y_1, Y_2)$. Are the variables Y_1, Y_2 independent? Why or why not?

[4 + 4 + 3 = 11 marks]

8. Let Z_0, Z_1 and Z_2 be three independent standard normal random variables.

(a) Define two random variables $W_1 = Z_1 + \rho Z_0$ and $W_2 = Z_2 + \rho Z_0$. Find ρ such that the correlation $\mathrm{Corr}(W_1, W_2) = 0.2$.

(b) Define two random variables $Y_1 = Z_1 + 2Z_2$, $Y_2 = 2Z_1 - Z_2$. Find $\mathrm{E}(Y_1 | Y_2 = 1)$.

[5 + 5 = 10 marks]

`Total marks = 100`

End of the Questions.
Formulas are on the next page.

求CDF，推导技巧的hint   cov计算公式

Table XII: Discrete Distributions

Probability Distribution and Parameter Values	Probability Mass Function	Moment-Generating Function	Mean $E(X)$	Variance $\mathrm{Var}(X)$	Examples
Bernoulli $0 < p < 1$ $q = 1 - p$	$p^x q^{1-x}, \quad x = 0, 1$	$q + pe^t$	p	pq	Experiment with two possible outcomes, say success and failure, $p = P(\text{success})$
Binomial $n = 1, 2, 3, \ldots$ $0 < p < 1$	$\binom{n}{x} p^x q^{n-x}$, $x = 0, 1, \ldots, n$	$(q + pe^t)^n$	np	npq	Number of successes in a sequence of n Bernoulli trials, $p = P(\text{success})$
Geometric $0 < p < 1$ $q = 1 - p$	$q^{x-1} p$, $x = 1, 2, \ldots$	$\dfrac{pe^t}{1 - qe^t}$	$\dfrac{1}{p}$	$\dfrac{q}{p^2}$	The number of trials to obtain the first success in a sequence of Bernoulli trials
Hypergeometric $x \le n, x \le N_1$ $n - x \le N_2$ $N = N_1 + N_2$ $N_1 > 0, \quad N_2 > 0$	$\dfrac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}$		$n\left(\dfrac{N_1}{N}\right)$	$n\left(\dfrac{N_1}{N}\right)\left(\dfrac{N_2}{N}\right)\left(\dfrac{N-n}{N-1}\right)$	Selecting r objects at random without replacement from a set composed of two types of objects
Negative Binomial $r = 1, 2, 3, \ldots$ $0 < p < 1$	$\binom{x-1}{r-1} p^r q^{x-r}$, $x = r, r+1, \ldots$	$\dfrac{(pe^t)^r}{(1 - qe^t)^r}$	$\dfrac{r}{p}$	$\dfrac{rq}{p^2}$	The number of trials to obtain the rth success in a sequence of Bernoulli trials
Poisson $0 < \lambda$	$\dfrac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, \ldots$	$e^{\lambda(e^t - 1)}$	λ	λ	Number of events occurring in a unit interval, events are occurring randomly at a mean rate of λ per unit interval
Uniform $m > 0$	$\dfrac{1}{m}, \quad x = 1, 2, \ldots, m$		$\dfrac{m+1}{2}$	$\dfrac{m^2 - 1}{12}$	Select an integer randomly from $1, 2, \ldots, m$

Table XIII: Continuous Distributions

Probability Distribution and Parameter Values	Probability Density Function	Moment-Generating Function	Mean $E(X)$	Variance $\mathrm{Var}(X)$	Examples
Beta $0 < \alpha$ $0 < \beta$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	$X = X_1/(X_1 + X_2)$, where X_1 and X_2 have independent gamma distributions with same θ
Chi-square $r = 1, 2, \ldots$	$\lambda e^{-\lambda x}$ $\dfrac{x^{r/2-1} e^{-x/2}}{\Gamma(r/2) 2^{r/2}}$, $0 < x < \infty$	$\dfrac{1}{(1-2t)^{r/2}}, \quad t < \dfrac{1}{2}$	r	$2r$	Gamma distribution, $\theta = 2$, $\alpha = r/2$; sum of squares of r independent $N(0, 1)$ random variables
Exponential $0 < \theta$	$\dfrac{1}{\theta} e^{-x/\theta}, \quad 0 \le x < \infty$	$\dfrac{1}{1 - \theta t}, \quad t < \dfrac{1}{\theta}$	θ	θ^2	Waiting time to first arrival when observing a Poisson process with a mean rate of arrivals equal to $\lambda = 1/\theta$
Gamma $0 < \alpha$ $0 < \theta$	$\dfrac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha}$, $0 < x < \infty$	$\dfrac{1}{(1-\theta t)^\alpha}, \quad t < \dfrac{1}{\theta}$	$\alpha\theta$	$\alpha\theta^2$	Waiting time to αth arrival when observing a Poisson process with a mean rate of arrivals equal to $\lambda = 1/\theta$
Normal $-\infty < \mu < \infty$ $0 < \sigma$	$\dfrac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$, $-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2	Errors in measurements; heights of children; breaking strengths
Uniform $-\infty < a < b < \infty$	$\dfrac{1}{b-a}, \quad a \le x \le b$	$\dfrac{e^{tb} - e^{ta}}{t(b-a)}, \quad t \ne 0$; $1, \quad t = 0$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$	Select a point at random from the interval $[a, b]$

λe^{-1}

$Pr($ $|$

1. A mutation in a certain gene can occur with probability 0.02. The probability of a rear disease in a person with mutation in this gene is 0.10 and this probability is 0.002 otherwise.

   (a) Find the probability that a random person has this disease.

   (b) Find the probability that a person with the disease has a mutation in this gene.

   (c) Consider a group of three patients with the disease. Find the probability that two of them have mutations in this gene.

$Pr(\text{mutation}) = 0.02 \quad Pr(\overline{\text{mutation}}) = 0.98$

$Pr(\text{disease} \mid \text{mutation}) = 0.1$

$Pr(\text{disease} \mid \overline{\text{mutation}}) = 0.002$

ⓐ $Pr(\text{disease}) = 0.02 \times 0.1 + 0.98 \times 0.002 = 3.96 \times 10^{-3}$

ⓑ $Pr(\text{mutation} \mid \text{disease}) = \dfrac{Pr(\text{mutation} \cap \text{disease})}{3.96 \times 10^{-3}} = \dfrac{Pr(\text{disease} \mid \text{mutation}) Pr(\text{mutation})}{3.96 \times 10^{-3}} = \dfrac{0.02 \times 0.1}{3.96 \times 10^{-3}}$

≈ 0.51

ⓒ $C_3^2 (0.51)^2 \times (1 - 0.51)$

2. In Sydney area, the number of earthquakes during next t years, X_t, follows a Poisson process with rate 1 per year.

   (a) Find the probability that there will be no earthquakes in Sydney next year.

   (b) Let T_0 be the time (in years) until first year without an earthquake. Find $\Pr(2 \leq T_0 \leq 5)$.

   (c) Find $\Pr(X_3 \geq 3 | X_3 \geq 2)$.

   (d) Find $\Pr(2 \leq X_2 \leq 4 | X_5 = 7)$.

ⓐ $P(X_1 = 0) = \dfrac{\lambda^k e^{-\lambda}}{k!} = \dfrac{1^0 e^{-1}}{0!} = \dfrac{1}{e}$

ⓑ $Pr(T_1 \leq 5) - Pr(T_1 \geq 2) = Pr(T_1 \leq 5) - (1 - Pr(T_1 \leq 2))$

$= Pr(T_1 \leq 5) - 1 + Pr(T_1 \leq 2)$

$\frac{1}{\theta} e^{-\frac{x}{\theta}} = \lambda e^{-\lambda x}$

$= e^{-x}$

$\int_0^x e^{-x} dx = -e^{-x} \Big|_0^x = -e^{-x} + 1$

$= 1 - e^{-5} - 1 + 1 - e^{-2}$

$= 1 - \frac{1}{e^5} + \frac{1}{e^2}$

$\dfrac{P(X_3 \geq 3)}{P(X_3 \geq 2)}$

$\dfrac{e^{-\lambda} \lambda^x}{k!}$

$\dfrac{e^{-3} 3^5}{} + \dfrac{e^{-3} 3^4}{} + \dfrac{e^{-3} 3^3}{}$

(c) $\lambda t = 3$

$= \frac{\frac{\quad}{5!} \cdot \frac{\quad}{4!} \cdot \frac{\quad}{3!}}{\frac{e^{-5} 5^7}{7!}}$

memorylessness

$1 - Pr(X_3 \le 3)$

$1 - [Pr(X_3=0) + Pr(X_3=1) + Pr(X_3=2) + Pr(X_3=3)]$

$\lambda t = 3 \quad \lambda t = 3$

$\frac{P(2 \le X_2 \le 4, X_5 = 7)}{P(X_5=7)} = \frac{P(X_2=2, X_5=7) + P(X_2=3, X_5=7) + P(X_2=4, X_5=7)}{P(X_5=7)}$

$P(X_2=2) \cancel{P(X_3=5) + P(X_3=4) + P(X_3=3)}$

$=$

(d) $Pr(X_2=2) + Pr(X_2=3) + Pr(X_2=4)$

3. There are three coins: one coin is fair, and the other two are biased. The probability that the first biased coin shows tail and head is 1/3 and 2/3, respectively, and the probability that the second biased coin shows tail and head is 2/3 and 1/3, respectively. We randomly select one coin and flip it two times. Let X be the number of tails in the two flips.

(a) Find the range and probability mass function (pmf) of X.

(b) Find $E(X)$, $Var(X)$ and the moment generating function of X.

[5+5 = 10 marks]

(a)

$X = \{0, 1, 2\}$

$Pr(X=0) = P(H|fair) \cdot P(fair) P(H|fair) + P(H|baised_1) \cdot P(baised_1) P(H|baised_1) + P(H|baised_2) \cdot P(baised_2) P(H|baised_2)$

$= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} =$

$\frac{1}{12} + \frac{4}{27} + \frac{1}{27} = \frac{5}{27} + \frac{1}{12} = \frac{5 \times 12 + 27}{27 \times 12} = \frac{87}{324} = \frac{29}{108}$

$P(X=2) = P(T|fair) \cdot P(fair) P(T|fair) + P(T|baised_1) P(baised_1) P(T|baised_1) + P(T|baised_2) \cdot P(baised_2) P(T|baised_2)$

$= \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{1}{12} + \frac{1}{27} + \frac{4}{27} = \frac{29}{108}$

$P(X=1) = 1 - \frac{58}{108} = \frac{50}{108}$

X	0	1	2

Pr	$\frac{29}{108}$	$\frac{50}{108}$	$\frac{29}{108}$	1

b $E(X) = \frac{50}{108}\times 1 + \frac{29}{108}\times 2 = \frac{50}{108} + \frac{58}{108} = 1.$

$E(X^2) = \frac{50}{108} + \frac{29}{108}\times 2^2 = \frac{166}{108}$

$Var = E(X^2) - E^2(X) = \frac{166}{108} - 1 = \frac{58}{108}$

MGF: $E(e^{tX}) = e^0 \times \frac{29}{108} + e^t\frac{50}{108} + e^{2t}\frac{29}{108}$

5+5 = $C = \frac{5}{4}$

4. The moment generating function of a random variable X is MGF $M(0)=1.$

$\sum e^{tx} f(x)$

$$M(t) = C\left(\frac{e^t+e^{-t}}{3}\right)^2 + \left(\frac{e^{t/2}+e^{-t/2}}{3}\right)^2.$$

$C\frac{4}{9}+\frac{4}{9}=1$ $\quad C\frac{4}{9}=\frac{5}{9}$

(a) Find the constant C and the probability mass function of X.

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

$c\left(\frac{e^{2t}+2+e^{-2t}}{9}\right) + \left(\frac{e^t+2+e^{-t}}{9}\right)$

$\frac{2}{9}(1+c)=0$

$c=-1.$

$E(e^{tX}) = \frac{ce^{2t} + e^t + e^{-t} + ce^{-2t} + 2(1+c)}{9}$

$E'(e^{tX}) = \frac{2c}{9}e^{2t} + \frac{1}{9}e^t - \frac{1}{9}e^{-t} - \frac{2c}{9}e^{-2t} + 0 = \frac{2c}{9} + \frac{1}{9} - \frac{1}{9} - \frac{2c}{9} = 0$

$E''(0) = \frac{4c}{9}e^{2t} + \frac{1}{9}e^t + \frac{1}{9}e^{-t} + \frac{4c}{9}e^{-2t}$

5. A device has two components that work independently of each other. This device fails if at least one of these components fail. The lifetimes (times to failure, measured in years) of these two components, T_1, T_2, follow an exponential distribution with $\Pr(T_k \le t) = 1 - e^{-tk}$, $k = 1, 2$.

(a) Find the probability that exactly one of the two components fails in one year.

(b) Let T_0 be the lifetime of the device. Find the cumulative distribution function of T_0, $\Pr(T_0 \le t)$.

(c) What is the expected lifetime of the device?

$Pr(T_1 \le 1)\cdot Pr(T_2 > 1) + Pr(T_2 \le 1)\, Pr(T_1 > 1)$

$\ldots + (1-e^{-1\times 2})\,[1 - Pr(T_1 \le 1)]$

$= (1-e^{-1})\cdot[1-F_{r}(\ldots)]$

$= (1-e^{-1})(1-1+e^{-2}) + (1-e^{-2})(1-1+e^{-1})$

$= e^{-2} - e^{-3} + e^{-1} - e^{-3} = e^{-1} + e^{-2} - 2e^{-3}$

PDF: $\lambda e^{-\lambda x} = (1+2)e^{-3x} = 3e^{-3x} =$

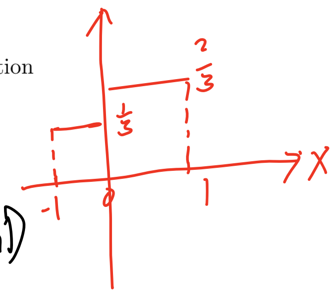
CDF $\quad 1-3e^{-3x}$

$y = (0,1] =$

(c) $\frac{1}{3}$

6. Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} C, & \text{if } -1 \le x \le 0, \\ 2C, & \text{if } \quad 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$



(a) Find the constant C.
(b) Find E(X) and Var(X).
(c) Find the cumulative distribution function of X, $F(x)$.
(d) Find the median of X.
(e) Let $Y = X^2$. Find the cumulative distribution function of Y, $G(y)$.
(f) Find the probability density function of Y, $g(y)$.

$X(-1,0)\,(0,1)$

$Y(0,1)$

(a) $\int_{-\infty}^{+\infty} f(x)\,dx = 1 \Rightarrow \int_{-1}^{0} c\,dx + \int_{0}^{1} 2c\,dx = 1 \qquad cx\Big|_{-1}^{0} + 2cx\Big|_{0}^{1} = 1$

$c + 2c = 1 \Rightarrow c = \frac{1}{3}$

(b) $E(x) = \int_{-1}^{0} \frac{1}{3}x\,dx + \int_{0}^{1} \frac{2}{3}x\,dx = \frac{1}{6}x^2\Big|_{-1}^{0} + \frac{1}{3}x^2\Big|_{0}^{1} = -\frac{1}{6} + \frac{1}{3} = \frac{1}{6}$

$E(x^2) = \int_{-1}^{0} \frac{1}{3}x^2\,dx + \int_{0}^{1} \frac{2}{3}x^2\,dx = \frac{1}{9}x^3\Big|_{-1}^{0} + \frac{2}{9}x^3\Big|_{0}^{1} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$

$\mathrm{Var}(x) = E(x^2) - E^2(x) = \frac{1}{3} - \frac{1}{36} = \frac{12-1}{36} = \frac{11}{36}$

$\frac{1}{3}x\Big|_{-1}^{x} = \frac{1}{3}x - \frac{1}{3}$

(c)

$$F(x) = \begin{cases} 0, & x < -1, \\ \int_{-1}^{x} \frac{1}{3}\,dx & -1 \le x < 0, = \frac{1}{3}(x+1) \\ \int^{x} \frac{2}{3}\,dx + \int_{-1}^{0} \frac{1}{3}\,dx, & 0 \le x < 1, = \frac{2}{3}x + \frac{1}{3} \end{cases}$$

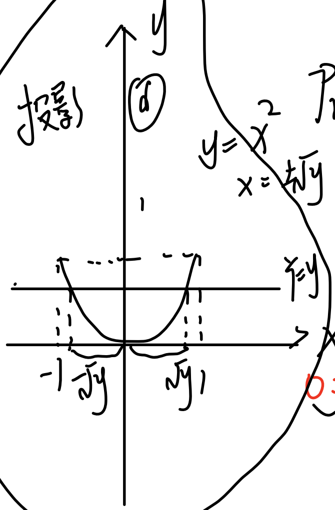
$\int_{0}^{x} \frac{2}{3}\,dx + \frac{1}{3}$

$$\begin{cases} \ldots \\ 1, & x \geq 1 \end{cases}$$

$$\frac{2}{3}x + \frac{1}{3} = \frac{1}{2}$$

$$\frac{2}{3}x = \frac{1}{6}$$

$$x_m = \frac{1}{4}$$



$x \in (-1,0) \cup (0,1)$

$Y = X^2 \Rightarrow (0,1)$

$Pr(Y \leq y) = Pr(X^2 \leq y)$

$y = x^2$

$x = \pm\sqrt{y}$

$y < 0$时，$F_Y(y) = 0$

$y > 1$时，$F_Y(y) = 1$

$0 \leq y < 1$时　左闭右开

$F_Y(y) = Pr\{-\sqrt{y} \leq x \leq \sqrt{y}\} = \int_{-\sqrt{y}}^{0} \frac{1}{3}dx + \int_{0}^{\sqrt{y}} \frac{2}{3}dx$

用f(x)积分 PDF

$= \frac{1}{3}\sqrt{y} + \frac{2}{3}\sqrt{y}$

$= \sqrt{y}$

$f(y) = (F_Y(y))' = \begin{cases} \frac{1}{2} \times \frac{1}{\sqrt{y}} & 0 \leq x < 1 \end{cases}$

$Y =$

设 $X \sim E(2)$ 证 $Y = 1 - e^{-2X}$ 服从 $U(0,1)$

(把分布函数里的x改为X，新随机变量服从的分布)

$-e^{-2x} \leq y - 1$

$e^{-2x} \geq 1 - y$

$-2x \geq \ln(1-y)$

$x \leq -\frac{1}{2}\ln(1-y)$

$$f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{else} \end{cases}$$

$F_Y(y) = Pr\{Y \leq y\} = Pr\{1 - e^{-2X} \leq y\} = Pr\{X \leq \frac{1}{2}\ln(\frac{1}{1-y})\}$

$$\int_0^{\frac{1}{2}\ln(\frac{1}{1-y})} 2e^{-2x} = -e^{-2x}\Big|_0^{\frac{1}{2}\ln(\frac{1}{1-y})}$$

$y \in (0,+\infty)$　$x \to +\infty \to e^{-2x} = 0$)　下界　$= \quad e^{-2 \times \frac{1}{2}\ln(\frac{1}{1-y})} \quad -(-1)$

$x \in [0,1) \Rightarrow$ $\quad 1 \Rightarrow y(0,1)$

$x \to 0 \to e^{-2x} = 1$ $\quad$ [illegible] 0

$$y(0,1) \to \begin{cases} y \le 0 & 0 \\ y \ge 1 & 1 \\ \underbrace{0 \le y \le 1}_{y} \end{cases}$$

$$\begin{cases} 0 & y \le 0 \\ y & 0 < y < 1 \\ 1 & y \ge 1 \end{cases}$$

$$= -e^{\ln(1-y)} + 1$$

$$= 1 - (1-y) = y$$

$$f(x) = \begin{cases} x & 0 \le x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & \text{其他} \end{cases}$$

① 求分布函数 $F_X(x)$

$\frac{1}{2}x^2$

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ \int_0^x x, & 0 \le x < 1 \\ \int_0^x 2-x + \int_0^1 x, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases} \Rightarrow \begin{cases} 0, & x \le 0 \\ \frac{1}{2}x^2, & 0 \le x < 1 \\ -\frac{1}{2}x^2 + 2x - 1 & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

PDF of X: $f(x) = \begin{cases} \frac{1}{9}x^2 & 0<x<3 \\ 0 & 其他 \end{cases}$ 令 $Y = \begin{cases} 2 & x \le 1 \\ X & 1<X<2 \\ 1 & X \ge 2 \end{cases}$

求 $F_Y(y)$

(标准答案 左闭右开) $1 \le Y \le 2$ => $y<1$ $F(y)=0$ $y \ge 2$ $F(y)=1$

$1 < Y < 2$ => $F_Y(Y \le y)$ => $P\{Y=1\} + P\{1<Y$

rv

7. Let X_1, X_2 be two independent Bernoulli random variables with probability of success $p = 1/2$. Define two new random variables $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \max\{X_1, X_2\}$.

0, 1

(a) Find the joint probability mass function of Y_1, Y_2.

(b) Find $E(Y_1), E(Y_2), Var(Y_1)$ and $Var(Y_2)$.

(c) Find $Cov(Y_1, Y_2)$. Are the variables Y_1, Y_2 independent? Why or why not?

(a)

Y_1	0	1
Pr	$\frac{3}{4}$	$\frac{1}{2}\times\frac{1}{2}$

Y_2	0	1
Pr	$\frac{1}{2}\times\frac{1}{2}$	$\frac{3}{4}$

Y_2 \ Y_1	0	1	$Pr(Y_2)$
0	$\frac{1}{4}$	0	$\frac{1}{4}$
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$
$Pr(Y_1)$	$\frac{3}{4}$	$\frac{1}{4}$	1

(b) $E(Y_1) = \frac{1}{4}$ $E(Y_1^2) = \frac{1}{4}$

$E(Y_2) = \frac{3}{4}$ $E(Y_2^2) = \frac{3}{4}$

$Var(Y_1) = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$

$Var(Y_2) = \frac{3}{4} - \frac{9}{16} = \frac{12}{16} - \frac{9}{16} = \frac{3}{16}$

$Y_1 = 0$

$Y_2 = 0 \Rightarrow \{min, max\} = 0$

$X_1 = 0$ $X_2 = 0$ $= \frac{1}{2} \times \frac{1}{2}$

③

$$cov(Y_1,Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = \frac{1}{4} - \frac{1}{4}\times\frac{3}{4} = \frac{1}{16}$$

$$E(Y_1Y_2) = \frac{1}{4}$$

不独立 $P(Y_1=0)\times P(Y_2=0) = \frac{3}{4}\times\frac{1}{4} = \frac{3}{16} \neq P(Y_1=0, Y_2=0) = \frac{1}{4}$

[4 + 4 + 3 = 11 marks]

8. Let Z_0, Z_1 and Z_2 be three independent standard normal random variables.

   (a) Define two random variables $W_1 = Z_1 + \rho Z_0$ and $W_2 = Z_2 + \rho Z_0$. Find ρ such that the correlation $\mathrm{Corr}(W_1, W_2) = 0.2$.

   (b) Define two random variables $Y_1 = Z_1 + 2Z_2$, $Y_2 = 2Z_1 - Z_2$. Find $\mathrm{E}(Y_1|Y_2 = 1)$.

ⓐ $$corr(W_1,W_2) = \frac{E(W_1W_2) - E(W_1)E(W_2)}{\sqrt{Var(W_1)}\sqrt{Var(W_2)}} = \frac{\rho^2}{\sqrt{1+\rho^2}\cdot\sqrt{1+\rho^2}} = 0.2$$

$$Cor(Z_1,Z_2) = \frac{cov(Z_1,Z_2)}{\sqrt{Var(Z_1)}\sqrt{Var(Z_2)}} = \frac{E(Z_1Z_2) - E(Z_1)E(Z_2)}{\sqrt{Var(Z_1)}\sqrt{Var(Z_2)}} = 0$$

$$E(W_1) = E(Z_1 + \rho Z_0) = E(Z_1) + \rho E(Z_0) = 0 + \rho\times 0 = 0$$

$$E(W_2) = E(Z_2 + \rho Z_0) = E(Z_2) + \rho E(Z_0) = 0$$

$$Var(W_1) = Var(Z_1) + \rho^2 Var(Z_0) = 1+\rho^2$$

$$Var(W_2) = Var(Z_2) + \rho^2 Var(Z_0) \quad 1+\rho^2$$

$$W_1W_2 = Z_1Z_2 + \rho Z_1Z_0 + \rho Z_2Z_0 + \rho^2 Z_0^2$$

$$\frac{\rho^2}{1+\rho^2} = 0.2$$

$$\rho^2 = 0.2 + 0.2\rho^2$$

$$0.8\rho^2 = 0.2$$

$$\rho^2 = \frac{1}{4} \quad \rho = \pm\frac{1}{2}$$

$$E(Z_0) = E(Z_1) = E(Z_2) = 0$$

$$Var(Z_0) = Var(Z_1) = Var(Z_2) = 1.$$

$$E(W_1W_2) = E(Z_1Z_2) + \rho E(Z_1Z_0) + \rho E(Z_2Z_0) + \rho^2 E(Z_0^2)$$

$$= 0 + 0 + 0 + \rho^2$$

$$Y_1 \sim (0, 1+4) \qquad Y_2 \sim (0, 4+1)$$

$$Y_1 \sim (0,5) \qquad Y_2 \sim (0,5)$$

$$E(Y_1 \mid Y_2 = 1) = \int_{-\infty}^{+\infty} y_1 \times f_{Y_1|Y_2}(y_1 \mid 1)\, dy_1$$

$$f_{Y_1|Y_2}(y_1 \mid y_2) = \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_2}(y)} = \int_{-\infty}^{+\infty} y_1$$

$$\frac{Y_1}{\sqrt{5}} = \frac{1}{\sqrt{5}}(Z_1 + 2Z_2) \sim N(0,1)$$

$$\frac{Y_2}{\sqrt{5}} = \frac{1}{\sqrt{5}}(2Z_1 - Z_2) \sim N(0,1)$$

$$= 5E\left(\frac{2}{5}Z_1^2 + \frac{3}{5}Z_1Z_2 - \frac{2}{5}Z_2^2\right)$$

$$= E(Z_1^2) + 3E(Z_1Z_2) - 2Z_2^2.$$

$$0 \qquad 0$$

$$= 0$$

$$E(Y_1) = 0$$

$$W_1 = (Z_1 + Z_2 + Z_3)^2$$

$$W_2 = (Z_1 + 2Z_2 - 3Z_3)^2.$$

不管多少N相加成[illegible] Bivariant normal

$$E(W_1 \mid W_2 = 2) = 3$$

$$E((Z_1 + Z_2 + Z_3)^2) \qquad E(Z_1^2) + 2E(Z_2^2) + 3E(Z_2^2)$$

$$1 + 1 + 3 = 5$$

$$E((Z_1 + Z_2 + Z_3)^2)$$

$$= E(Z_1 + Z_2 + Z_3)^2$$

$$= Var(Z_1 + Z_2 + Z_3)$$

$$= 3$$

$$E(W_1 \mid W_2 = 2)$$

$\mathrm{cov}(W_1, W_2) = E($ $\qquad$ $E(W_1)$

Moment generating functions and moments of random variables

Moment generating functions (MGFs) can be used to find moments of random variables, including the mean and variance. The k-th order moment of rv X, $\mathrm{E}(X^k)$, can be found by taking the k-th derivative of the MGF of X and setting $t = 0$.

If $M_X(t)$ is the MGF of X, then $\mathrm{E}(X^k) = M_X^{(k)}(t)|_{t=0}$, where $M_X^{(k)}(t)$ is the k-th order derivative of $M_X(t)$.

Some differentiation rules you may find useful:

- If $f(t) = t^n$, then $f'(t) = nt^{n-1}$
- If $f(t) = e^{\alpha t}$, then $f'(t) = \alpha e^{\alpha t}$
- If $f(t) = g(h(t))$, then $f'(t) = g'(y)|_{y=h(t)} \times h'(t)$ (chain rule)

Example 1 (Binomial distribution): $M_X(t) = (q + pe^t)^n$ where $p + q = 1$. We can write $M_X(t) = g(h(t))$ with $g(y) = y^n$ and $h(t) = q + pe^t$. We find $g'(y) = ny^{n-1}$ and $h'(t) = pe^t$ and therefore $M_X'(t) = g'(y)_{y=h(t)=q+pe^t} \times h'(t) = n(q + pe^t)^{n-1} \times pe^t = pne^t(q + pe^t)^{n-1}$.

To find the moments of X with the given MGF, you can take derivatives of $M_X(t)$ and set $t = 0$ or you can identify one of the distributions we considered in class. If you identify the distribution and its parameters correctly (in this example, binomial with parameters n and p), then you can use the formula for the mean and variance for this distribution: $\mu = np$ and $\sigma^2 = npq = np(1-p)$. No derivation is required in this case but you have to explain that MGF uniquely identifies the distribution and hence its moments.

Example 2 (Gamma distribution): $M_X(t) = 1/(1 - \theta t)^n$. We can write $M_X(t) = g(h(t))$ with $g(y) = y^{-n}$ and $h(t) = 1 - \theta t$. We find $g'(y) = -ny^{-n-1}$ and $h'(t) = -\theta$ and therefore $M_X'(t) = g'(y)_{y=h(t)=1-\theta t} \times h'(t) = -n(1-\theta t)^{-n-1} \times (-\theta) = n\theta(1-\theta t)^{-n-1}$.

Similarly, we can find $M_X''(t) = n(n+1)\theta^2(1-\theta t)^{-n-2}$ and $M_X'''(t) = n(n+1)(n+2)\theta^3(1-\theta t)^{-n-3}$. It implies that $\mathrm{E}(X) = M_X'(t)|_{t=0} = n\theta$, $\mathrm{E}(X^2) = M_X''(t)|_{t=0} = n(n+1)\theta^2$ and $\mathrm{E}(X^3) = M_X'''(t)|_{t=0} = n(n+1)(n+2)\theta^3$. The variance is $\mathrm{Var}(X) = \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2 = n(n+1)\theta^2 - \{n\theta\}^2 = n\theta^2$.

Sum of geometric progression:

If $|q| < 1$, then $\sum_{k=m}^{\infty} q^k = q^m + q^{m+1} + q^{m+2} + \cdots = q^m/(1-q)$. You can use this result without proof. This property is useful to find moments and/or MGFs for some discrete distributions (such as the Poisson distribution we considered in class).

Example 3: Let X be a random variable such that $\Pr(X = 1/2^k) = 1/2^k$, $k = 1, 2, 3, \ldots$. That is, X takes values $1/2, 1/4, 1/8, \ldots$ with probabilities $1/2, 1/4, 1/8, \ldots$. Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ using the definition of the mean and variance.

[5+5 =

4. The moment generating function of a random variable X is

$$M(t) = C\left(\frac{e^t + e^{-t}}{3}\right)^2 + \left(\frac{e^{t/2} + e^{-t/2}}{3}\right)^2.$$

(a) Find the constant C and the probability mass function of X.

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(a) $M(0) = C\left(\frac{1+1}{3}\right)^2 + \left(\frac{1+1}{3}\right)^2 = \frac{4}{9}C + \frac{4}{9} = 1 \quad \frac{4}{9}C = \frac{5}{9} \quad C = \frac{5}{4}$

$\sum e^{tx} f(x) = \frac{5}{4} \times \frac{e^{2t} + 2 + e^{-2t}}{9} + \frac{e^t + 2 + e^{-t}}{9}$

$= \frac{\frac{5}{4}e^{2t} + \frac{5}{2} + 2 + \frac{5}{4}e^{-2t} + e^t + e^{-t}}{9}$

X	-2	-1	0	1	2
Pr	$\frac{5}{4}$	1	$\frac{9}{2}$	1	$\frac{5}{4}$

(0, 1)

$W = (Z_1 + Z_2 + Z_3)^2$

$W_2 = (Z_1 + 2Z_2 - 3Z_3)^2$

$E(W_1 \mid W_2 = 2)$

$\rho = E(W_1 W_2) - E(W_1)E(W_2)$

$\mathrm{Cov}((Z_1 + Z_2 + Z_3), (Z_1 + 2Z_2 - 3Z_3))$

$= \mathrm{Var}(Z_1) + 2\mathrm{Var}(Z_1) - 3\mathrm{Var}(Z_3) = 0$

$E(W_1) =$

Solution: To find $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$, we use the above result about the sum of geometric progression with $q = 1/4$ and $q = 1/8$, respectively:

$$\mathrm{E}(X) = \sum_{k=1}^{\infty} \frac{1}{2^k} \Pr\left(X = \frac{1}{2^k}\right) = \sum_{k=1}^{\infty} \frac{1}{2^k} \times \frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{1}{4} \times \frac{1}{1-1/4} = \frac{1}{3},$$

$$\mathrm{E}(X^2) = \sum_{k=1}^{\infty} \left(\frac{1}{2^k}\right)^2 \Pr\left(X = \frac{1}{2^k}\right) = \sum_{k=1}^{\infty} \frac{1}{4^k} \times \frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{1}{8^k} = \frac{1}{8} \sum_{k=0}^{\infty} \frac{1}{8^k} = \frac{1}{8} \times \frac{1}{1-1/8} = \frac{1}{7},$$

$$\mathrm{Var(X)} = \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2 = \frac{1}{7} - \frac{1}{9} = \frac{2}{63}.$$

Example 4 (optional! Questions of this type will NOT be tested on the exam): Let X be a random variable such that $\Pr(X = k) = 1/2^k$, $k = 1, 2, 3, \ldots$. That is, X takes values $1, 2, 3, \ldots$ with probabilities $1/2, 1/2^2, 1/2^3, \ldots$. Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ using a) the definition of the mean and variance; b) the MGF of X.

Solution: a) We write:

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{k=1}^{\infty} k \Pr(X = k) = \sum_{k=1}^{\infty} \frac{k}{2^k} = \sum_{k=0}^{\infty} \frac{k+1}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{k}{2^{k+1}} + \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \frac{k}{2^k} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{k}{2^k} + \frac{1}{2} \cdot \frac{1}{1-1/2} = \frac{1}{2}\mathrm{E}(X) + 1,
\end{aligned}$$

solving this equation we get $\mathrm{E}(X) = 2$.

$$\begin{aligned}
\mathrm{E}(X^2) &= \sum_{k=1}^{\infty} k^2 \Pr(X = k) = \sum_{k=1}^{\infty} \frac{k^2}{2^k} = \sum_{k=0}^{\infty} \frac{(k+1)^2}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{k^2}{2^{k+1}} + \sum_{k=0}^{\infty} \frac{2k}{2^{k+1}} + \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \frac{k^2}{2^k} + \sum_{k=0}^{\infty} \frac{k}{2^k} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{k^2}{2^k} + \sum_{k=1}^{\infty} \frac{k}{2^k} + \frac{1}{2} \cdot \frac{1}{1-1/2} \\
&= \frac{1}{2}\mathrm{E}(X^2) + \mathrm{E}(X) + 1,
\end{aligned}$$

solving this equation we get $\mathrm{E}(X^2) = 2\mathrm{E}(X) + 2 = 6$. Finally, $\mathrm{Var}(X) = \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2 = 6 - 4 = 2$. b) We find the MGF of X, $M_X(t)$:

$$M_X(t) = \mathrm{E}(e^{Xt}) = \sum_{k=1}^{\infty} e^{kt} \Pr(X = k) = \sum_{k=1}^{\infty} e^{kt}/2^k = \sum_{k=1}^{\infty} (e^t/2)^k = \frac{e^t}{2} \sum_{k=0}^{\infty} (e^t/2)^k = \frac{e^t/2}{1 - e^t/2}.$$

We can recognize the MGF of the geometric distribution with $p = 1/2$.

Two more differentiation rules you may find useful:

- If $f(t) = g(t) \cdot h(t)$, then $f'(t) = g'(t) \cdot h(t) + g(t) \cdot h'(t)$

- If $f(t) = \frac{g(t)}{h(t)}$, then $f'(t) = \frac{g'(t) \cdot h(t) - g(t) \cdot h'(t)}{\{h(t)\}^2}$

We use the second rule first with $g(t) = 1$ and $h(t) = (1 - e^t/2)$ and then with $g(t) = e^t/2$ and $h(t) = (1 - e^t/2)^2$. To find $h'(t)$, chain rule can be used:

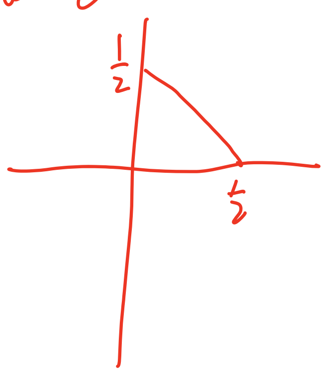
$$M_X(t) = -1 + \frac{1}{1 - e^t/2}, \quad M'_X(t) = \frac{e^t/2}{(1 - e^t/2)^2}, \quad M'_X(t)|_{t=0} = \frac{1/2}{(1 - 1/2)^2} = 2,$$

$$M''_X(t) = \frac{(e^t/2) \cdot (1 - e^t/2)^2 - (e^t/2) \cdot (-e^t/2) \cdot 2(1 - e^t/2)}{(1 - e^t/2)^4} = \frac{(e^t/2)(1 + e^t/2)}{(1 - e^t/2)^3},$$

$$\mathrm{E}(X^2) = M''_X(t)|_{t=0} = \frac{(1/2)(1 + 1/2)}{(1 - 1/2)^3} = 6, \quad \mathrm{Var}(X) = \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2 = 6 - 4 = 2.$$

If we recognize the geometric distribution from the MGF, we could avoid this derivation and use the results we proved in class: $\mathrm{E}(X) = 1/p = 1/(1/2) = 2$ and $\mathrm{Var}(X) = (1-p)/p^2 = (1 - 1/2)/(1/2)^2 = 2$. Again, in this case you have to explain that MGF ~~uniquely identifies~~ the distribution and hence its moments.

$f(x,y) = 8 \quad x,y > 0 \quad x + y < \frac{1}{2}$



$f_X(x) = \int_0^{\frac{1}{2} - x} 8 \, dy = 4 - 8x$

$0 < x < \frac{1}{2}$

$f_Y(y) = \int_0^{\frac{1}{2}}$

对称理指:
先保证这个 mean 存在

symmetric

$f(x)=f(-x)$

对称不一定导致 mean = median，首先保证 mean 存在

$\frac{1}{1+x^2}\cdot\frac{1}{\pi}$ (对称) around 0，但 mean 不存在，median = 0

$$E(X)=\int_{-\infty}^{+\infty}\frac{x\,dx}{1+x^2}\cdot\frac{1}{\pi} = \text{infinity}$$

$x^2=t$

$$=2\int_{0}^{+\infty}\frac{dy/2}{1+y}=\log(1+y)\Big|_{0}^{+\infty}$$

skewness = 0 不一定对称

shape is symmetric 是一个更强的假设

2. In Sydney area, the number of earthquakes during next t years, X_t, follows a Poisson process with rate 1 per year.

(a) Find the probability that there will be no earthquakes in Sydney next year.

(b) Let T_0 be the time (in years) until first year without an earthquake. Find $\Pr(2 \leq T_0 \leq 5)$.

(c) Find $\Pr(X_3 \geq 3 | X_3 \geq 2)$.

(d) Find $\Pr(2 \leq X_2 \leq 4 | X_5 = 7)$.

$$\Pr(2 \leq X_2 \leq 4 | X_5 = 7) = \frac{\Pr(X_2=2, X_5=7) + \Pr(X_2=3, X_5=7) + \Pr(X_2=4, X_5=7)}{\Pr(X_5=7)}$$

$$= \frac{\Pr(X_2=2)\cdot\Pr(X_3=5) + \Pr(X_2=3)\Pr(X_3=4) + \Pr(X_2=4)\Pr(X_3=3)}{\Pr(X_5=7)}$$

$$\frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(X_3=5) = \frac{e^{-3}3^5}{5!} \quad P(X_3=4) = \frac{e^{-3}3^4}{4!} \quad P(X_3=3) = \frac{e^{-3}3^3}{3!}$$

$$P(X_2=2) = \frac{e^{-2}2^2}{2!} \quad P(X_2=3) = \frac{e^{-2}2^3}{3!}$$

$$P(X_2=4) = \frac{e^{-2}2^4}{4!} \quad P(X_5=7) = \frac{e^{-5}5^7}{7!}$$

$$\frac{\frac{4\cancel{e}^{-2}}{2} \times \frac{\cancel{e}^{-3}3^5}{5!} + \frac{8\cancel{e}^{-2}}{3!} \times \frac{\cancel{e}^{-3}3^4}{4!} + \frac{\cancel{e}^{-2}2^4}{4!} \times \frac{\cancel{e}^{-3}3^3}{3!}}{\frac{\cancel{e}^{-5}5^7}{7!}}$$

$$\frac{\left(\frac{81 \times \cancel{3}}{5 \times 4 \times \cancel{3}} + \frac{\overset{9}{\cancel{81}} \times \cancel{8}}{\cancel{6} \times \cancel{4} \times \underset{2}{\cancel{6}}} + \frac{\overset{2}{\cancel{16}} \times \overset{3}{\cancel{27}}}{\cancel{4} \times \cancel{6} \times \cancel{6}_2}\right) \times 7!}{5^7}$$

$$\frac{81 \times 7 \times 6 \times 6 + 9 \times 7 \times 6 \times 5 \times 4 \times 3 + 2 \times 3 \times 7 \times 5 \times 6 \times 4 \times 3}{5^7}$$

$$\frac{58212}{78125} \approx 0.745$$

$E(X_i) =$

1 Skewness and kurtosis of the binomial distribution

Let $X \sim \text{Binom}(n, p)$, where n is the number of trials and p is the probability of success in each trial. We can write $X = X_1 + X_2 + \ldots + X_n$, where X_i are i.i.d. (independent and identically distributed) random variables with $Pr(X_i = 1) = p$ and $\Pr(X_i = 0) = 1 - p$. Let $Y_i = X_i - p$, then

$$\mathrm{E}(Y_i) = \mathrm{E}(X_i - p) = 0, \quad \mathrm{E}(Y_i^2) = \mathrm{E}[(X_i - p)^2] = \mathrm{Var}(X_i) = p(1-p),$$

$$\mathrm{E}(Y_i^3) = \mathrm{E}[(X_i - p)^3] = (1-p)^3 \cdot p + (0-p)^3 \cdot (1-p) = p(1-p)(1-2p),$$

$$\mathrm{E}(Y_i^4) = \mathrm{E}[(X_i - p)^4] = (1-p)^4 \cdot p + (0-p)^4 \cdot (1-p) = p(1-p)(1-3p+3p^2).$$

The skewness and kurtosis of X is $\mu_3 = \mathrm{E}\{[(X-\mu)/\sigma]\}^3$ and $\mu_4 = \mathrm{E}\{[(X-\mu)/\sigma]^4\}$, where $\mu = \mathrm{E}(X) = np$ and $\sigma^2 = \mathrm{Var}(X) = np(1-p)$. Let $q = 1-p$. Using the multinomial theorem, we find:

$$\begin{aligned}
\mu_3 &= \mathrm{E}\left[\frac{X-np}{\sqrt{np(1-p)}}\right]^3 = \mathrm{E}\left[\frac{Y_1+\ldots+Y_n}{\sqrt{np(1-p)}}\right]^3 = [np(1-p)]^{-3/2} \cdot \mathrm{E}[(Y_1+\ldots+Y_n)^3] \\
&= [np(1-p)]^{-3/2} \cdot \left[\sum_{i=1}^n \mathrm{E}(Y_i^3) + 3\sum_{i\neq j} \mathrm{E}(Y_i Y_j^2) + 6 \sum_{i<j<k} \mathrm{E}(Y_i Y_j Y_k)\right] \\
&= [np(1-p)]^{-3/2} \cdot \left[\sum_{i=1}^n \mathrm{E}(Y_i^3)\right] = [np(1-p)]^{-3/2} \cdot np(1-p)(1-2p) = \frac{q-p}{\sqrt{npq}},
\end{aligned}$$

and

$$\begin{aligned}
\mu_4 &= \mathrm{E}\left[\frac{X-np}{\sqrt{np(1-p)}}\right]^4 = \mathrm{E}\left[\frac{Y_1+\ldots+Y_n}{\sqrt{np(1-p)}}\right]^4 = [np(1-p)]^{-2} \cdot \mathrm{E}[(Y_1+\ldots+Y_n)^4] \\
&= [np(1-p)]^{-2} \cdot \left[\sum_{i=1}^n \mathrm{E}(Y_i^4) + 4\sum_{i\neq j} \mathrm{E}(Y_i Y_j^3) + 6\sum_{i<j} \mathrm{E}(Y_i^2 Y_j^2)\right. \\
&+ \left. 12 \sum_{i<j, i\neq k, j\neq k} \mathrm{E}(Y_i Y_j Y_k^2) + 24 \sum_{i<j<k<l} \mathrm{E}(Y_i Y_j Y_k Y_l)\right] \\
&= [np(1-p)]^{-2} \cdot \left[\sum_{i=1}^n \mathrm{E}(Y_i^4) + 6\sum_{i<j} \mathrm{E}(Y_i^2 Y_j^2)\right] \\
&= [np(1-p)]^{-2} \cdot [np(1-p)(1-3p+3p^2) + 3n(n-1)p^2(1-p)^2] = 3 + \frac{1-6pq}{npq},
\end{aligned}$$

because the expectations of all cross products involving odd degrees of Y_i are zero: due to independence of Y_i, Y_j for $i \neq j$, we have $\mathrm{E}(Y_i Y_j) = \mathrm{E}(Y_i)\mathrm{E}(Y_j) = 0$, $\mathrm{E}(Y_i Y_j^2) = \mathrm{E}(Y_i)\mathrm{E}(Y_j^2) = 0$, etc.

From these formulas we can see that:

- $\mu_3 \to 0$ and $\mu_4 \to 3$ if $n \to \infty$

- $\mu_3 = 0$ if $p = q$, that is, $p = 1/2$
- $\mu_3 \to \infty$ ($\mu_3 \to -\infty$) if $p \to 0$ ($p \to 1$)
- $\mu_4 \to \infty$ if $p \to 0$ or $p \to 1$
- μ_4 is minimized if $p = 1/2$

2 MGF of the standardized binomial distribution

Again, for $X \sim \text{Binom}(n, p)$, define the standardized variable

$$Y = \frac{X - \mu}{\sigma} = \frac{X - np}{\sqrt{npq}}.$$

Note that $Y = \frac{\sum_{i=1}^n (X_i - p)}{\sqrt{npq}}$ where X_i, $i = 1, \ldots, n$, are i.i.d. random variables with $\Pr(X_i = 0) = 1 - p$ and $\Pr(X_i = 1) = p$. Using independence of $X_1, \ldots, X_n$, we find:

$$\begin{aligned} M_Y(t) &= \mathrm{E}\left[\exp\left\{\frac{(\sum_{i=1}^n X_i) - np}{\sqrt{npq}} \cdot t\right\}\right] = \exp\left[-\frac{\sqrt{np} \cdot t}{\sqrt{q}}\right] \cdot \mathrm{E}\left[\exp\left\{\frac{\sum_{i=1}^n X_i}{\sqrt{npq}} \cdot t\right\}\right] \\ &= \exp\left[-\frac{\sqrt{np} \cdot t}{\sqrt{q}}\right] \cdot \prod_{i=1}^n \mathrm{E}\left[\exp\left\{\frac{X_i}{\sqrt{npq}} \cdot t\right\}\right] = \exp\left[-\frac{\sqrt{np} \cdot t}{\sqrt{q}}\right] \cdot \prod_{i=1}^n M_{X_i}\left(\frac{t}{\sqrt{npq}}\right), \end{aligned}$$

where the MGF of X_i is $M_{X_i}(t) = pe^t + q$. We find that

$$\begin{aligned} M_Y(t) &= \exp\left[-\frac{\sqrt{np} \cdot t}{\sqrt{q}}\right] \cdot \left(p \exp\left\{\frac{t}{\sqrt{npq}}\right\} + q\right)^n \\ &= \left(p \exp\left\{\frac{t}{\sqrt{npq}} - \frac{\sqrt{p} \cdot t}{\sqrt{nq}}\right\} + q \exp\left\{-\frac{\sqrt{p} \cdot t}{\sqrt{nq}}\right\}\right)^n \\ &= \left(p \exp\left\{\frac{\sqrt{q} \cdot t}{\sqrt{np}}\right\} + q \exp\left\{-\frac{\sqrt{p} \cdot t}{\sqrt{nq}}\right\}\right)^n. \end{aligned}$$

We use the Taylor expansion of exponent function for small x:

$$\exp(x) = 1 + x + \frac{x^2}{2} + O(x^3),$$

where $O(x^3)$ includes cubic and higher order terms and therefore

$$\begin{aligned} M_Y(t) &= \left(p\left[1 - \frac{\sqrt{q} \cdot t}{\sqrt{np}} + \frac{q \cdot t^2}{2np}\right] + q\left[1 - \frac{\sqrt{p} \cdot t}{\sqrt{nq}} + \frac{p \cdot t^2}{2nq}\right] + O\left(\frac{1}{n\sqrt{n}}\right)\right)^n \\ &= \left(1 + \frac{t^2}{2n} + O\left(\frac{1}{n\sqrt{n}}\right)\right)^n \to \exp\left(\frac{t^2}{2}\right) \quad \text{as} \quad n \to \infty. \end{aligned}$$

We can see that the MGF of the standardized binomial random variable, Y, converges to the MGF of some other random variable. This random variable has a normal distribution and we will learn more about this distribution in Module 3.