MAST90105 Methods of Mathematical Statistics
Assignment 3, Semester 1, 2023
Due: Wednesday May 10, end of day.

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1. Let X and Y be two independent standard normal random variables.

   (a) Define $V_1 = XY^2$ and $V_2 = X^2Y$. Compute the covariance $\mathrm{Cov}(V_1, V_2)$. Are V_1 and V_2 independent? Explain why or why not. *Hint:* For a standard normal random variable Z, $E(Z^3) = 0,\ E(Z^4) = 3,\ E(Z^5) = 0,\ E(Z^6) = 15$.

   (b) Show that if a random variable W has the moment generating function $M_W(t) = h(t^2)$ where h is a continuous function which is 3 times differentiable at $t = 0$, then the skewness of W is zero.

   (c) Find the moment generating function of $W = X^2 - Y^2$. Find the skewness of W.

2. Let $X_1, \dots, X_n$ be a random sample from a discrete distribution with the probability mass function $f(k) = \Pr(X = k)$:

$$f(0) = p^3, \quad f(1) = 3p^2q, \quad f(2) = 3pq^2, \quad f(3) = q^3,$$

   where $0 < p < 1$ is an unknown parameter and $q = 1 - p$. Assume that $n = 10$ and the observed data are $1, 1, 1, 0, 3, 2, 2, 1, 2, 0$.

   (a) Find the method of moments estimate of p.

   (b) Find the maximum likelihood estimate of p, $\widehat{p}_{ML}$. Show that the likelihood function attains its maximum at $p = \widehat{p}_{ML}$.

   (c) Assume the prior density of the parameter p is

$$f(p) = \begin{cases} 6p(1-p), & 0 < p < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Find the posterior density of p and the posterior mean for the observed data. Is this prior distribution a conjugate prior for these data?

3. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with the density function:

$$f_X(x;\lambda) = \begin{cases} C\lambda^2 x \exp\{-(\lambda x)^4 + (\lambda x)^2\}, & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\exp(z) = e^z$ is the exponential function, $\lambda > 0$ is an unknown parameter, and $C = 1.156$ is the normalizing constant. Assume that $n = 10$ and the observed data are 1.15, 0.95, 0.93, 0.81, 1.22, 0.67, 1.56, 1.33, 1.01, 0.45.

(a) Find the method of moments estimate of λ. *Hint:* $\int_0^\infty t^2 \exp\{-t^4 + t^2\} dt = 0.719$.

(b) Find the maximum likelihood estimate of λ, $\widehat{\lambda}_{ML}$.