1. (a) $cov(V_1,V_2) = cov(XY^2, X^2Y) = E(XY^2 X^2Y) - E(XY^2)E(X^2Y)$

$\because X, Y$ are independent from each other $\therefore$ any function of X, Y are independent.

$\therefore cov(V_1,V_2) = E(X^3)E(Y^3) - E(X)E(Y^2)E(X^2Y)$ $\quad \because E(X^3)=0 \quad E(X)=0 \quad \therefore cov(V_1,V_2)=0$

$V_1^2 = X^2Y^4 \quad V_2^2 = X^4Y^2 \quad 6=15 \quad 4=3$

$cov(V_1^2, V_2^2) = E(X^6Y^6) - E(X^2)E(Y^4)E(X^4)E(Y^2)$

$= E(X^6)E(Y^6) - E(X^2)E(Y^4)E(X^4)E(Y^2)$

$= 15^2 - 3^2 = 216 \neq 0$

so V_1^2 and V_2^2 are dependent, so any function of V_1, V_2 are also dependent as well. We can say V_1, V_2 are dependent.

(b) $M = E(W) = M_W'(t)\big|_{t=0} = 2th'(t^2)\big|_{t=0} = 0$

$E(W^2) = M_W''(t)\big|_{t=0} = [2h'(t^2) + 2t \times h''(t^2) \times 2t]\big|_{t=0} = 2h'(0)$

$E(W^3) = M_W'''(t)\big|_{t=0} = [2h''(t^2)\times 2t + 8th''(t^2) + 4t^2 \times h'''(t^2)\times 2t]\big|_{t=0} = 0$

$6^2 = Var(W) = E(W^2) - E^2(W) = 2h'(0)$

skewness of W: $E[(\frac{W-M}{6})^3] = \frac{1}{6^3}E[(W-M)^3] = \frac{1}{6^3}E(W^3 - 3MW^2 + 3WM^2 - M^3)$

$= \frac{1}{6^3}[E(W^3) - E(3MW^2) + E(3WM^2) - E(M^3)]$

$= \frac{1}{6^2}[0 - 3ME(W^2) + E(W)\cdot 3M^2 - E(M^3)]$

$= 0$

hence we can conclude if W is a rv whose $M_W(t) = h(t^2)$ is a continous function which is 3 times differentiable at $t=0$, then the skewness of W is zero.

(c) $M_W(t) = E[e^{t(X^2-Y^2)}] = E(e^{tX^2}/e^{tY^2}) = E(e^{tX^2}) \times E(e^{-tY^2})$

$X^2 \sim \chi^2(1) \quad Y^2 \sim \chi^2(1) \quad \therefore M_W(t) = \frac{1}{(1-2t)^{\frac{1}{2}}} \times \frac{1}{(1+2t)^{\frac{1}{2}}} = (1-4t^2)^{-\frac{1}{2}}$

$M_W'(t) = -\frac{1}{2}(1-4t^2)^{-\frac{3}{2}} \times (-8t) \qquad E(W) = M_W'(t)\big|_{t=0} = 0$

$M_W''(t) = 4(1-4t^2)^{-\frac{3}{2}} + 4t \times (-\frac{3}{2}) \times (1-4t^2)^{-\frac{5}{2}} \times (-8t) \qquad E(W^2) = M_W''(t)\big|_{t=0} = 4$

$$M'''_w(t) = 4\times(-\tfrac{3}{2})\times(1-4t^2)^{-\frac{5}{2}}\times(-8t) + 96t(1-4t^2)^{-\frac{5}{2}} + 48t^2\times(-\tfrac{5}{2})(1-4t^2)^{-\frac{7}{2}}\times(-8t)$$

$$E(w^3) = M'''_w(t)\Big|_{t=0} = 0$$

skewness of w: $E[(\frac{w-\mu}{\sigma})^3] = \frac{1}{\sigma^3}E(w^3) = 0$

2.

(a) $\bar{X} = E(X) \Rightarrow \frac{\sum_{i=1}^{10} x_i}{10} = \sum_{i=1}^{n} x_i f(x_i)$

$\bar{X} = \frac{\sum_{i=1}^{10} x_i}{10} = \frac{4\times1+3\times2+3}{10} = 1.3$

According to PMF of X, we can conclude that $X \sim \text{Binomial}(3, 1-p)$

PMF of X:

X	0	1	2	3
q_i	$\binom{3}{0}p^{3-0}q^0$	$\binom{3}{1}p^{3-1}q^1$	$\binom{3}{2}p^{3-2}q^2$	$\binom{3}{3}p^{3-3}q^3$

So $E(X) = np = 3(1-p)$ $\quad 3(1-p) = 1.3 \Rightarrow \hat{p} = \frac{1.7}{3} \approx 0.567$

(b) $L(p) = \prod_{i=1}^{n} f(x_i; p) = (p^3)^2 \times (3p^2q)^4 \times (3pq^2)^3 \times q^3$

$= 3^7 p^{17} q^{13}$

$= 3^7 p^{17} (1-p)^{13}$

$\ln(L(p)) = 7\ln 3 + 17\ln p + 13\ln(1-p)$

$\ln'(L(p)) = \frac{17}{p} - \frac{13}{1-p}$ $\quad$ Let $\ln'(L(p)) = 0$: $\frac{17}{p} - \frac{13}{1-p} = 0$ $\quad \hat{p} \approx 0.567$

$\ln''(L(p)) = -\frac{17}{p^2} - \frac{13}{(1-p)^2}$ $\quad$ Let $p = 0.567$ so $\ln''(L(p)) < 0$

MLE of p is $\hat{p} = 0.567$

(c) $f(p) = 6p(1-p)$ $\quad (0<p<1)$

$f(x, p) = \prod_{i=1}^{n} f(x_i) \times f(p) = [f(p)]^{10}[f(0)]^2[f(1)]^4[f(2)]^3[f(3)]$

$= 2^{10}3^{10}p^{10}(1-p)^{10}\, p^6\, 3^4 p^8 q^4\, 3^3 p^3 q^6\, q^3$

$= 2^{10}3^{17}p^{27}(1-p)^{23}$

$$m(x) = \int_0 f(x,p)\,dp = \int_0^1 2^{10}3^{17} p^{27}(1-p)^{23}\,dp = 2^{10}3^{17}\int_0^1 p^{27}(1-p)^{23}\,dp = 2^{10}3^{17}\frac{\Gamma(28)\Gamma(24)}{\Gamma(52)}$$

$$f(p|x) = \frac{f(x,p)}{m(x)} = \frac{2^{10}3^{17}p^{27}(1-p)^{23}}{\left(\frac{2^{10}3^{17}\Gamma(28)\Gamma(24)}{\Gamma(52)}\right)} = \frac{\Gamma(52)\,p^{27}(1-p)^{23}}{\Gamma(28)\,\Gamma(24)} = \frac{51!\,p^{27}(1-p)^{23}}{27!\,24!}$$

so the posterior density of p follows Beta distribution. $f(p|x) \sim Be(28,24)$

posterior mean : $\frac{\alpha}{\alpha+\beta} = \frac{28}{52} \approx 0.538$

$f(p)$ can be rewritten as $\frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} p^{2-1}(1-p)^{2-1}$, so $f(p)$ follows Beta distribution as well.

$f(p) \sim Be(2,2)$

So, we can conclude that the prior distribution $f(p)$ is a conjugate prior for these data, because $f(p)$ and $f(p|x)$ have same type of distribution, namely Beta distribution.

3.

(a) $\bar{x} = \left(\sum_{i=1}^{10} x_i\right)/10 = 1.008$

$$E(x) = \int_0^{+\infty} x\, f_X(x;\pi)\,dx = \int_0^{+\infty} c\pi^2 x^2 \exp\{-(\pi x)^4 + (\pi x)^2\}\,dx$$

$\pi x = t$

$(\pi x)'dx = dt$

$\pi dx = dt$

$dx = \frac{1}{\pi}dt$

$$= \int_0^{+\infty} \frac{c}{\pi} t^2 \exp\{-t^4 + t^2\}\,dt$$

$$= \frac{c}{\pi}\int_0^{+\infty} t^2 \exp\{-t^4 + t^2\}\,dt$$

$$= \frac{c}{\pi} \times 0.719$$

$$= \frac{0.831164}{\pi}$$

$$\frac{0.831164}{\pi} = 1.008 \qquad \hat{\pi} \approx 0.825$$

(b)

$$l(\pi) = \prod_{i=1}^{10} c\pi^2 x_i \exp\{-(\pi x_i)^4 + (\pi x_i)^2\}$$

$$= c^{10}\pi^{20}\left(\prod_{i=1}^{10} x_i\right) \exp\left\{-\sum_{i=1}^{10}(\pi x_i)^4 + \sum_{i=1}^{10}(\pi x_i)^2\right\}$$

$$\ln(L(\pi)) = 10\ln C + 20\ln\pi + \sum_{i=1}^{10}\ln x_i - \sum_{i=1}^{10}(\pi x_i)^4 + \sum_{i=1}^{10}(\pi x_i)^2$$

$$= 10\ln C + 20\ln\pi + \sum_{i=1}^{10}\ln x_i - \pi^4\sum_{i=1}^{10}x_i^4 + \pi^2\sum_{i=1}^{10}x_i^2$$

$$\ln'(L(\pi)) = \frac{20}{\pi} - 4\pi^3\sum_{i=1}^{10}x_i^4 + 2\pi\sum_{i=1}^{10}x_i^2$$

$$\ln'(L(\pi)) = 0 \Rightarrow \quad 20 - 4\pi^4\sum_{i=1}^{10}x_i^4 + 2\pi^2\sum_{i=1}^{10}x_i^2 = 0 \quad (\pi > 0)$$

$$\sum_{i=1}^{10}x_i^2 = 11.1084 \qquad \sum_{i=1}^{10}x_i^4 = 16.2916$$

$$\hat{\pi} \approx \sqrt{0.75} \approx 0.866$$

$\ln''(L(\pi)) = -\frac{20}{\pi^2} - 12\pi^2\sum_{i=1}^{10}x_i^4 + 2\sum_{i=1}^{10}x_i^2$ Let π be 0.866 $\ln''(L(\pi)) < 0$

so the MLE of π is $\hat{\pi} = 0.866$