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Week 5 - Integration

Tags	

Sigma Notation

$\Sigma_{i=1}^{n} c = nc$

$\Sigma_{i=1}^{n} ca_i = c\Sigma_{i=1}^{n} a_i$

$\Sigma_{i=1}^{n} (a_i \pm b_i) = \Sigma_{i=1}^{n} a_i \pm \Sigma_{i=1}^{n} b_i$

$\Sigma_{i=1}^{n} a_i = \Sigma_{i=1}^{m} a_i + \Sigma_{i=m+1}^{n} a_i$

$$\Sigma_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\Sigma_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Sigma_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Partition	A set of points that divide an interval into subintervals
Regular Partition	Sub intervals all have the same width

Left-Endpoint Approximation

For each subinterval, create a rectangle with width Δx and height $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval

The area of the subinterval is $f(x_{i-1})\Delta x$

Hence the total area under the curve is approximated by $\Sigma_{i=1}^{n} f(x_{i-1})\Delta x$

Right-Endpoint Approximation

For each subinterval create a rectangle with width Δx and height $f(x_i)$, which is the function value at the right endpoint of the subinterval

The area of the subinterval is $f(x_i)\Delta x$

Hence the total area under the curve is approximated by $\Sigma_{i=1}^{n} f(x_i)\Delta x$

Riemann Sums

Area ~ $\Sigma_{i=1}^{n} f(x_i^*)\Delta x$

Where x_i^* is any value of x in the subinterval

Exact area is

$\lim_{n\to\infty} \Sigma_{i=1}^{n} f(x_i^*)\Delta x$

Can choose overestimates or underestimates by using the minimum values in the subdivisions

Definite Integral

Definite Integral	Number
Indefinite Integral	Family of functions

$\int_a^b f(x)dx = \lim_{n\to\infty} \Sigma_{i=1}^{n} f(x_i^*)\Delta x$

$f(x)$ is the integrand

x is the variable of integration

Theorem 1.1 Continuous Functions are Integrable

If $f(x)$ is continuous, then $f(x)$ is integrable

Area

If part of a function goes below the x axis, then this area becomes negative

$\int_a^b f(x)dx = A_1 - A_2$ provides the net signed area

$\int_a^b |f(x)|dx = A_1 + A_2$ provides the total area

Properties of the Definite Integral

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
4. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Theorem 1.2 Comparison Theory

1. If $f(x) \geq 0$ for $a \leq x \leq b$,

   $\int_a^b f(x)dx \geq 0$

2. If $f(x) \geq g(x)$ for $a \leq x \leq b$

   $\int_a^b f(x)dx \geq \int_a^b f(g)dx$

3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$

   $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Average Value of a Function

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Theorem 1.3 Mean Value Theorem for Integrals

If $f(x)$ is continuous over an interval $[a, b]$, some point c exists such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

$$\int_a^b f(x)dx = f(c)(b-a)$$

Proof:

Since $f(x)$ is continuous, by the extreme value theorem, there are minimum and maximum values m and M

Thus, for all x: $m \leq f(x) \leq M$

By the comparison theorem,

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

$$m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M$$

By the intermediate value theorem, there is c such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Theorem 1.4 Fundamental Theorem of Calculus Part 1

If $f(x)$ is continuous over an interval $[a, b]$, and

$$F(x) = \int_a^x f(t)dt$$

Then $F'(x) = f(x)$

Proof:

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt]$$

$$= \lim_{h\to 0} \frac{1}{h}[\int_a^{x+h} f(t)dt + \int_x^a f(t)dt]$$

$$= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt$$

Which is just the average value of the function $f(x)$ over the interval $[x, x+h]$

According to the Mean Value Theorem for Integrals, there is some c, such that

$f(c) = \frac{1}{h}\int_x^{x+h} f(t)dt$

Since c is between x and h, $c \to x$ when $h \to 0$

$\lim_{h\to 0} f(c) = \lim_{c\to x} = f(x)$

$F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(x)dx$

$= \lim_{h\to 0} f(c)$

$= f(x)$

Theorem 1.5 Fundamental Theorem of Calculus, Part 2

$\int_a^b f(x)dx = F(b) - F(a)$

Proof:

Split the interval $[a, b]$ into regular partitions, then

$F(b) - F(a) = F(x_n) - F(x_0)$

$= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + ... + [F(x_1) - F(x_0)]$

$= \Sigma_{i=1}^n [F(x_i) - F(x_{i-1})]$

$F(x)$ is an antiderivative of $f(x)$, so by the Mean Value Theorem, there is some c_i between $[x_{i-1}, x_i]$

$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x$

$F(b) - F(a) = \Sigma_{i=1}^n f(c_i)\Delta x$

$F(b) - F(a) = \lim_{n\to\infty} \Sigma_{i=1}^n f(c_i)\Delta x$

$= \int_a^b f(x)dx$

Theorem 1.6 Net Change Theorem

The new value of a changing quantity is the initial value + integral of the rate of change

$F(b) = F(a) + \int_a^b F'(x)dx$

$\int_a^b F'(x)dx = F(b) - F(a)$

Integrating Odd and Even Functions

Even Functions

$f(-x) = f(x)$

$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$

Odd Functions

$f(-x) = -f(x)$

$\int_{-a}^a f(x)dx = 0$

Theorem 1.7 Substitution with Indefinite Integrals

Let $u = g(x)$ where $g'(x)$ is continuous

$\int f(g(x))g'(x)dx = \int f(u)du$

$= F(u) + c$

$= f(g(x)) + c$

Proof:

$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$

$\int \frac{d}{dx}F(g(x))dx = \int f(g(x))g'(x)dx$

$F(g(x)) + c = \int f(g(x))g'(x)$

Substitute $u = g(x)$ and $du = g'(x)dx$

$\int f(g(x))g'(x)dx$

$= \int f(u)du$

$= F(u) + c$

$= F(g(x)) + c$

Theorem 1.8 Substitution with Definite Integrals

$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$