Week 8 - Differential Equations

Tags	

Differential Equation	Equation involving an unknown function$y = f(x)$ and one or more of its derivatives.
Solution to Differential Equation	A function $y = f(x)$ that satisfies the differential equation when $f(x)$ and its derivatives are substituted into the equation.
Order of Differential Equation	Highest order of any derivative
General Solution	Solution with a constant C
Particular Solution	Solution without constant C
Direction/Slope Fields	Mathematical object used to graphically represent solutions to first-order differential equations
Equilibrium Solution	A solution to the differential equation of the form $y = c$

Direction/Slope Fields

Solutions

Solution	Definition	Actual Meaning
Asymptotically Stable Solution	$y = k$ is an asymptotically stable solution to the differential equation if there exists $\epsilon > 0$ such that for any value $c \in (k{-}\epsilon, k + \epsilon)$the solution to the initial-value problem $y' = f(x, y)$, $y(x0) = c$ approaches k as $x \to \infty$	The differential equation stabilises to k as $x \to \infty$
Asymptotically Unstable Solution	$y = k$ is an asymptotically unstable solution to the differential equation if there exists $\epsilon > 0$ such that for any value $c \in (k{-}\epsilon, k + \epsilon)$ the solution to the initial-value problem $y' =$	The differential equation never stabilises to k as $x \to \infty$

	$f(x, y)$, $y(x0) = c$ never approaches k as $x \to \infty$	
Asymptotically Semi-Stable Solution	$y = k$ is an asymptotically semi-stable solution to the differential equation if it is neither asymptotically stable nor asymptotically unstable	

Theorem 4.1 Euler's Method

To approximate a solution to $y' = f(x, y)$, $y(x_0) = y_0$

$x_n = x_0 + nh$

$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

Where h is the step size and n is an integer representing the number of steps

Separation of Variables

Separable Differentiable Equation	Any equation that can be written in the form $y' = f(x)g(y)$

Strategy

1. Check for any values of y that make $g(y) = 0$. These are the constant solutions.
2. Rewrite the differential equation $\dfrac{dy}{g(y)} = f(x)dx$
3. Integrate both sides
4. Solve for y if possible
5. If an initial condition exists, substitute to find the particular solution

Applications of Separation of Variables

Solution Concentration

$$\frac{du}{dt} = Inflow\ Rate - Outflow\ Rate$$

Newton's Law of Cooling

$$\frac{dT}{dt} = k(T - T_s)$$

Logistic Equation

$$\frac{dP}{dt} = rP(1 - \frac{P}{K})$$

Carrying Capacity	Maximum population of an organism that the environment can sustain indefinitely	K
Growth Rate	Rate at which the population grows	r

Theorem 4.2 Solution of the Logistic Differential Equation

The solution to the initial value of a logistic differential equation is:

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$

First Order Differential Equations

Linear

$a(x)y' + b(x)y = c(x)$

Standard Form

$y' + a(x)y = c(x)$ - Make the coefficient of y' 1 and have all y terms on one side

Solving a First Order Differential Equation

1. Put the equation into standard form
2. Calculate the integrating factor $\mu(x) = e^{\int p(x)dx}$
3. Multiply both sides by the integrating factor $\mu(x)$
4. Integrate both sides
5. Divide by integrating factor $\mu(x)$
6. If there is an initial condition, determine the constant C