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Week 7 - Techniques of Integration

Tags

Theorem 3.1 Integration by Parts

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives

$\int udv = uv - \int vdu$

Proof:

If $h(x) = f(x)g(x)$

Then $h'(x) = f'(x)g(x) + g'(x)f(x)$

$\int h(x)dx = \int f'(x)g(x) + g'(x)f(x)dx$

Rearranging:

$\int f(x)g'(x)dx = f(x)g(x) - \int g'(x)f(x)dx$

By then substituting $u = f(x)$ and $v = g(x)$:

$\int udv = uv - \int vdu$

Theorem 3.2 Integration by Parts for Definite Integrals

Let $u = f(x)$ and $v = g(x)$

$\int_a^b udv = [uv]_a^b - \int_a^b vdu$

Integrating Products and Powers of $\sin x$ and $\cos x$

If the powers of $\cos^p x$ or $\sin^p x$ is odd, convert to $\cos^{p-1} x \cos x$ or $\cos^{p-1} x \cos x$

If the powers of $\cos^p x$ or $\sin^p x$ is even, convert to $(\frac{\cos(2x)-1}{2})^{\frac{p}{2}}$ or $(\frac{1-\sin(2x)}{2})^{\frac{p}{2}}$

Integrating Products of $\sin(ax)$ and $\cos(bx)$

$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a–b)x)–\frac{1}{2}\cos((a+b)x)$
$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a–b)x)+\frac{1}{2}\sin((a+b)x)$
$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a–b)x)+\frac{1}{2}\cos((a+b)x)$

Integrating Products and Power of $\tan x$ and $\sec x$

If power of $\sec^j x$ is even	$\sec^j x \rightarrow (\tan^2 x + 1)^{(j-2)/2} \sec^2 x$
If power of $\tan^k x$ is odd	$\tan^k x \sec^j x \rightarrow (\sec^2 x - 1)^{(k-1)/2} \sec^{j-1} x \sec x \tan x$

Rule: Reduction Formulas for $\int \sec^n xdx$ and $\int \tan^n xdx$

$\int \sec^n xdx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2} xdx$

$\int \tan^n xdx = \frac{1}{n-1} tan^{n-1} x - \int \tan^{n-2} xdx$

Trigonometric Substitutions for Integration

Integral	Substitution

$\int \sqrt{a^2 - x^2} dx$	$x = a \sin u$
$\int \sqrt{a^2 + x^2} dx$	$x = a \tan u$
$\int \sqrt{x^2 - a^2} dx$	$x = a \sec u$

Solving Partial Fractions

1. Make sure the degree of the numerator > degree of denominator
2. Factorise denominator
3. If denominator has repeated factor $(ax+b)^n$ then the decomposition must contain

   $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_n}{(ax+b)^n}$$

4. For irreducible quadratic factors (determinant<0), there must be a term:

   $$\frac{Ax+B}{ax^2+bx+c}$$

5. For irreducible $(ax^2+bx+c)^n$, the decomposition must include:

   $$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + ... + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Improper Integrals

Integrating over an Infinite Interval

Let $f(x)$ be continuous, then

$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$

$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$

If the limit exists→the improper limit converges

$\int_{-\infty}^\infty f(x)dx = \lim_{t\to\infty} \int_0^t f(x)dx + \lim_{t\to-\infty} \int_t^0 f(x)dx$

Integrating a Discontinuous Integrand

1. Let $f(x)$ be continuous over $[a, b)$ (not including b)

   $\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$

2. Let $f(x)$ be continuous over $(a, b]$ (not including a)

   $\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$

If the limit exists, in these 2 cases the improper integral converges

3. If $f(x)$ is continuous over $[a, b]$ except at a point c in $[a, b]$, then

   $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Provided both integrals converge, the original integral converges

Theorem 3.7 Comparison Theorem

Let $f(x)$ and $g(x)$ be continuous over $[a, \infty)$, assuming $0 \leq f(x) \leq g(x)$

1. If $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx = \infty$ → $\int_a^\infty g(x)dx = \lim_{t\to\infty} \int_a^t g(x) = \infty$
2. If $\int_a^\infty g(x)dx = \lim_{t\to\infty} \int_a^t g(x) = L$ → $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx = M$ where $M \leq L$