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Week 4 - Applications of Derivatives

Tags	

Linear Approximations

Linear Approximation/Tangent Line Approximation

$L(x) = f(a) + f'(a)(x-a)$

$\Delta y = f(a+dx) - f(a)$

$dy = f'(a)dx$

Error

Measurement Error

dx or Δx

Propagated Error

$\Delta y = f(a+dx) - f(a)$

$\Delta y \approx dy = f'(a)dx$

Relative Error

$\frac{\Delta q}{q}$*100= Percentage Error

Extrema

Absolute Maximum	$f(c) \geq f(x)$ for all x

Absolute Minimum	$f(c) \leq f(x)$ for all x

If $f(x)$ has an absolute minimum or maximum, $f(x)$ has an absolute extremum

Theorem 4.1 Extreme Value Theorem

If $f(x)$ is a continuous function over the closed, bounded interval $[a, b]$, then there is a point in $[a, b]$ at which $f(x)$ has an absolute minimum and absolute maximum

Requirements:

1. Continuous Function
2. Closed interval

Endpoint Extrema

If an absolute extremum occurs at an endpoint, this is an endpoint extremum

Critical Points

c is a critical point if either

1. $f'(c) = 0$
2. $f'(c)$ is undefined

Theorem 4.1 Fermat's Theorem

If $f(x)$ has a local extremum at c and is differentiable at c, then $f'(c) = 0$

Critical points are not necessarily local extrema

Proof:

Since $f(x)$ is differentiable at c:

$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

This exists, therefore both one-sided and two-sided limits exist

$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$ (*)

$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$ (**)

If we assume $f(x)$ to be a local maximum, $f(x) - f(c) \leq 0$, for x near c

However, $x > c$, therefore $\frac{f(x)-f(c)}{x-c} \leq 0$

From (*), we then know that $f'(c) \leq 0$

(**) would show that $x < c$, which would state that $\frac{f(x)-f(c)}{x-c} \geq 0$

Which could mean $f'(c) \geq 0$

$0 \leq f'(c) \leq 0$ therefore, $f'(c) = 0$

Theorem 4.3 Location of Absolute Extrema

The absolute extrema of $f(x)$ must occur at endpoints or critical points

Rolle's Theorem

If $f(x)$ is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b), such that $f(a) = f(b)$, there exists at least one c, such that $f'(c) = 0$

Proof

Let $k = f(a) = f(b)$

1. $f(x) = k$ for all x

In this scenario, $f'(x) = 0$ for all x in the interval

2. There exists x, such that $f(x) > k$
3. There exists x, such that $f(x) < k$

2. and 3. are the same:

By the Extreme Value Theorem, an absolute minimum/maximum exists.

The maximum value must be greater than k, in case 2, since there exists $f(x) > k$

The minimum value must be less than k, in case 3, since there exists $f(x) < k$

Therefore, the max/min values do not occur at the endpoints, but at an interior point

Because $f(x)$ has a max/min at an interior point c, and $f(x)$ is differentiable at c, by Fermat's Theorem, $f'(c) = 0$

Mean Value Theorem

If $f(x)$ is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b) then there exists a point c such that:

$f'(c) = \frac{f(b)-f(a)}{b-a}$

Proof

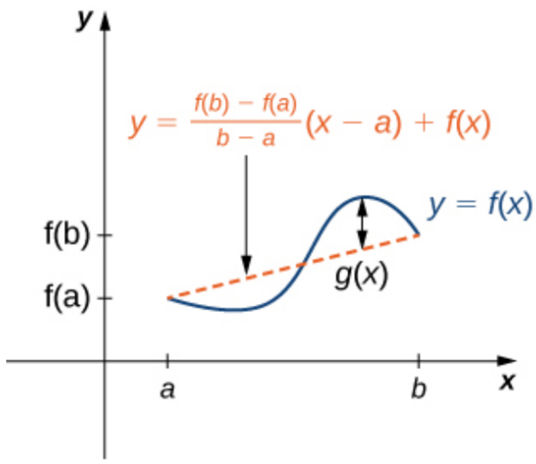
There is a straight line connecting $(a, f(a))$ and $(b, f(a))$

This line has slope $\frac{f(b)-f(a)}{b-a}$

Which means it has equation $y = \frac{f(b)-f(a)}{b-a}(x - a) + f(a)$

$g(x)$ will represent the vertical difference between $(x, f(x))$ and (x, y)

$g(x) = f(x) - [\frac{f(b)-f(a)}{b-a}(x - a) + f(a)]$



$g(a) = g(b) = 0$

Since $f(x)$ is differentiable, so is $g(x)$

$f(x)$ and $g(x)$ are then also both continuous

Therefore, $g(x)$ satisfies Rolle's Theorem

Consequently, there must be some c, such that $g'(c) = 0$

Since $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$

$g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$

Since $g'(c) = 0$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

Theorem 4.6 Functions with Derivative = 0

If $f'(x) = 0$ for all x, then $f(x) = c$ for all x

Proof:

Since $f(x)$ is differentiable, $f(x)$ is also continuous

Suppose there is some $f(a) \neq f(b)$

$\frac{f(b)-f(a)}{b-a} \neq 0$

Since $f(x)$ is differentiable, by the Mean Value Theorem, there exists

$f'(d) = \frac{f(b)-f(a)}{b-a}$

$f'(d) \neq 0$ which contradicts the assumption

Theorem 4.7 Constant Difference Theorem

If $f(x)$ and $g(x)$ are differentiable and $f'(x) = g'(x)$ for all x, then $f(x) = g(x) + c$

Proof:

Let $h(x) = f(x) - g(x)$

Then $h'(x) = f'(x) - g'(x)$

By Theorem 4.6, $h(x) = d$

Therefore $f(x) = g(x) + d$

Theorem 4.8 Increasing and Decreasing Functions

Let $f(x)$ must continuous over a closed interval and differentiable in the open interval

i. If $f'(x) > 0$ in the interval, then $f(x)$ is increasing in the interval
ii. If $f'(x) < 0$ in the interval, then $f(x)$ is decreasing in the interval

Proof (for case i., case ii. is the same):

Suppose $f(x)$ is not an increasing function, with $a < b$ and $f(a) \geq f(b)$

Since $f(x)$ is differentiable, by the Mean Value Theorem

$f'(c) = \frac{f(b)-f(a)}{b-a} \leq 0$

However, $f'(x) > 0$ for all x

Therefore, there is a contradiction, so $f(x)$ must be an increasing function

Theorem 4.9 First Derivative Test

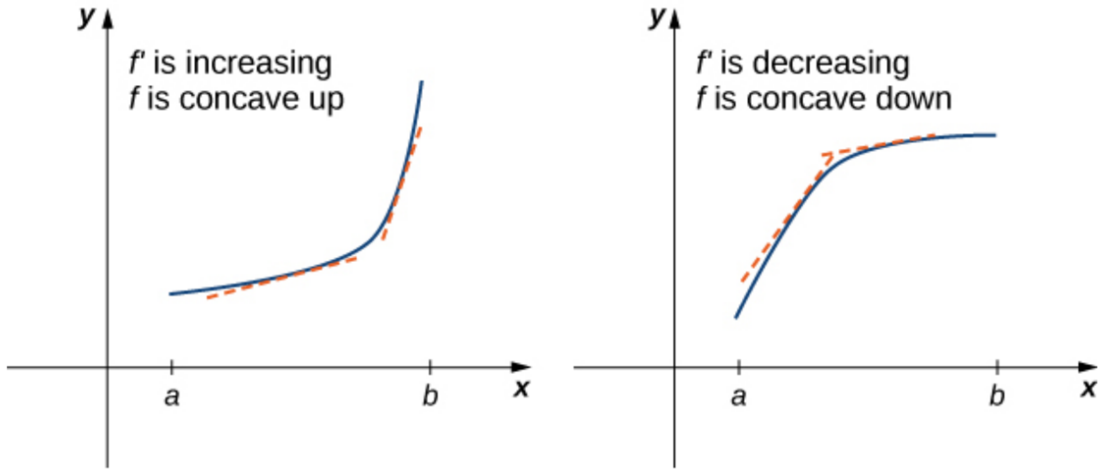
Suppose that $f(x)$ is a continuous function over an interval, containing a critical point c. If $f(x)$ is differentiable, then $f(x)$ satisfies one of the following:

1. If $f'(x)$ changes sign from positive when $x < c$ to negative when $x > c$ then c is a local maximum
2. If $f'(x)$ changes sign from negative when $x < c$ to positive when $x > c$ then c is a local minimum
3. If $f'(x)$ has the same sign when $x < c$ and when $x > c$ then c is neither a local maximum nor minimum

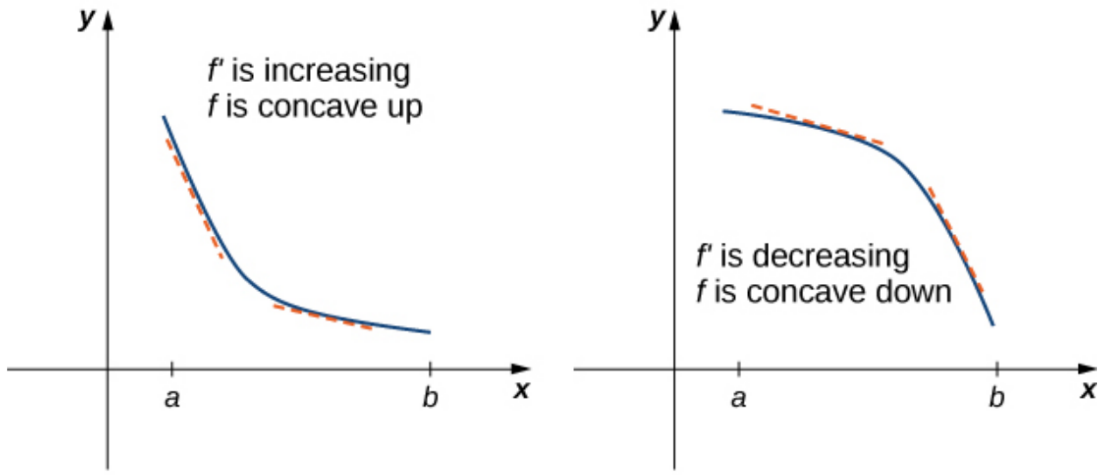
Concavity

If $f'(x)$ is increasing, $f(x)$ is concave up

If $f'(x)$ is decreasing, $f(x)$ is concave down



(a) (b)



Theorem 4.10 Test for Concavity

i. If $f''(x) > 0$ in the interval, then $f(x)$ is concave up

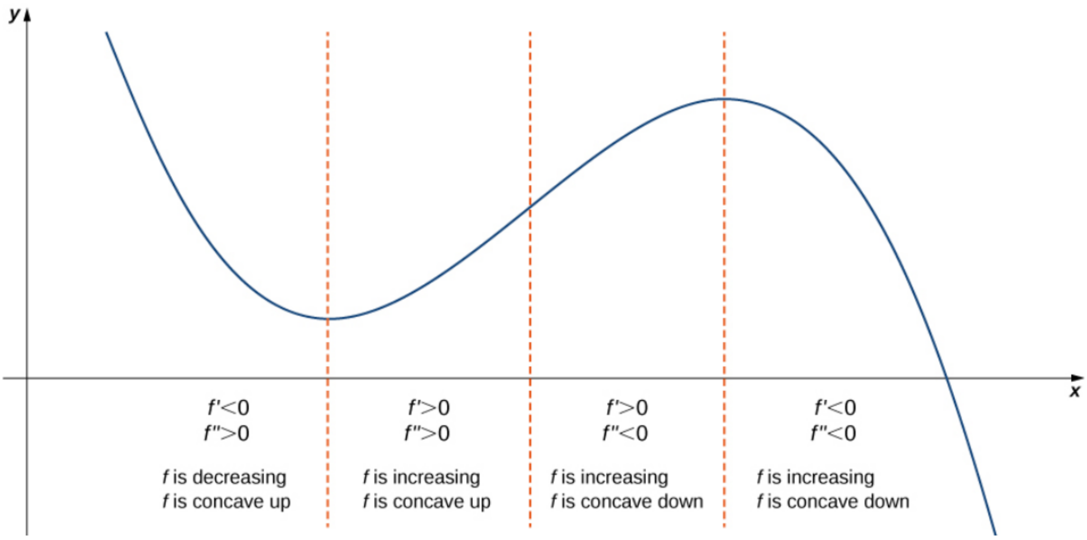
ii. If $f''(x) < 0$ in the interval, then $f(x)$ is concave down

Inflection Point

If $f(x)$ is continuous at a, and $f(x)$ changes concavity at a, the point $(a, f(a))$ is an inflection point

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

Table 4.1 What Derivatives Tell Us about Graphs



Theorem 4.11 Second Derivative Test

Suppose $f'(c) = 0$, $f''(x)$ is continuous over an interval containing c

i. If $f''(c) > 0$, then $f(x)$ has a local minimum at c

ii. If $f''(c) < 0$, then $f(x)$ has a local maxim at c

iii. If $f''(c) = 0$, then the test is inconclusive

Limits at Infinity

If the values of $f(x)$ become arbitrarily close to L as x becomes sufficiently large, we say the function $f(x)$ has a limit at infinity $\lim_{x\to\infty} f(x) = L$

If the values of $f(x)$ become arbitrarily close to L for $x < 0$ as $|x|$ becomes sufficiently large, we say the function $f(x)$ has a limit at negative infinity

$\lim_{x\to-\infty} f(x) = L$

$y = L$ is the horizontal limit of $f(x)$

Limit at infinity

$f(x)$ has a limit at infinity if L exists such that for all $\epsilon > 0$, there exists $N > 0$ such that

$|f(x) - L| < \epsilon$

Example

Prove $\lim_{x\to\infty} 2 + \frac{1}{x} = 2$

Let $\epsilon > 0$. Let $N > 1/\epsilon$. Therefore, for all $x > N$

$|2 + \frac{1}{x} - 2| = |\frac{1}{x}| = \frac{1}{x} < \frac{1}{N} = \epsilon$

Infinite Limit at Infinity

$f(x)$ has an infinite limit at infinity if for all $M > 0$ there exists an $N > 0$ such that $f(x) > M$ for all $x > N$

Example

Prove $\lim_{x\to\infty} x^3 = \infty$

Let $M > 0$. Let $N = \sqrt[3]{M}$, then for all $x > n$

$x^3 > N^3 = (\sqrt[3]{M})^3 = M$

Therefore $\lim_{x\to\infty} x^3 = \infty$

End Behaviour

1. The function approaches a horizontal asymptote $y = L$
2. The function approaches $\pm\infty$
3. The function does not approach a finite limit, nor does it approach ∞ or $-\infty$, in this case the function may exhibit oscillatory behaviour

Theorem 4.12 L'Hôpital's Rule ($\frac{0}{0}$ case)

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$

Proof:

$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ so $f(a) = g(a) = 0$

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x)-f(a)}{g(x)-g(a)}$ Since $f(a) = g(a) = 0$

$= \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$

$= \frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}}$

$= \frac{f'(a)}{g'(a)}$

$= \frac{\lim_{x \to a} f'(x)}{\lim_{x \to a} g'(x)}$

$= \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Theorem 4.13 L’Hôpital’s Rule ($\frac{\infty}{\infty}$ case)

If $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Antiderivative

$F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$

Theorem 4.14 General Form of an Antiderivative

Let $F(x)$ be the antiderivative of $f(x)$, then

1. For each constant c, the function $F(x) + c$ is also an antiderivative of $f(x)$
2. If $G(x)$ is an antiderivative of $f(x)$, there is a constant d, for which $G(x) = F(x) + d$

Indefinite Integrals

$\int f(x) dx = F(x) + c$

An integral is the most general antiderivative

Integrand	Variable of Integration
$f(x)$	x

Theorem 4.15 Power Rules for Integrals

$\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$

Theorem 4.16 Properties of Indefinite Integrals

Sums and Differences

$\int (f(x) \pm g(x)) dx = F(x) \pm G(x) + c$

Constant Multiples

$\int kf(x) dx = kF(x) + c$