Week 9 - Sequence and Series

Tags	

Infinite Sequence	Ordered list of the form $\{a_n\} = a_1, a_2, a_3...$
n	Index variable of the sequence
a_n	Term of the sequence
Explicit Formulae	$a_n = f(n)$
Recurrence Relation	Subsequent terms are defined in earlier terms of the sequence
Geometric Sequence	Ratio of every pair of consecutive terms is the same $a_n = cr^n$
Arithmetic Sequence	Difference between every pair of consecutive terms is the same $a_n = bn + c$
Increasing Sequence	Sequence is increasing if for all $n \geq n_0$ $a_n \leq a_{n+1}$ for all
Decreasing Sequence	Sequence is decreasing if for all $n \geq n_0$ $a_n \geq a_{n+1}$ for all
Monotone Sequence	Sequence is monotone if it is either increasing or decreasing for all $n \geq n_0$

Limits

If the terms a_n become arbitrarily close to a finite number L, as n becomes sufficiently large, $\{a_n\}$ is a convergent sequence, and L is the limit

$\lim_{n \to \infty} a_n = L$

If it is not convergent, a_n is a divergent sequence

A sequence converges to a real number L if for all $\epsilon > 0$ there exists an integer N such that $|a_n - L| < \epsilon$ if $n \geq N$.

$\lim_{n \to \infty} a_n = L \; or \; a_n \to L$

Theorem 5.1: Limit of a Sequence Defined by a Function

Consider a sequence such that $a_n = f(n)$ for all $n \geq 1$, if a limit L exists

$\lim_{x\to\infty} f(x) = L$

then a_n converges and

$\lim_{n\to\infty} a_n = L$

Theorem 5.2: Algebraic Laws

If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$

1. $\lim_{n\to\infty} c = c$
2. $\lim_{n\to\infty} c \cdot a_n = c \lim_{n\to\infty} a_n = cA$
3. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$
4. $\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = A \cdot B$
5. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \dfrac{A}{B}$ provided $B \neq 0$ and each $b_n \neq 0$

Proof:

Proving 3.

Let $\epsilon > 0$

Since $\lim_{n\to\infty} a_n = A$, there exists a positive N_1 such that for all $n \geq N_1$

Since $\lim_{n\to\infty} b_n = B$, there exists a positive N_2 such that $|b_n - B| < \dfrac{\epsilon}{2}$ for all $n \geq N_2$

Let N be the largest of N_1 and N_2. Therefore $n \geq N$

$|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$

Theorem 5.3: Continuous Functions Defined on Convergent Sequences

Suppose a_n converges to L

Suppose f is a continuous function at L.

Then there exists an integer N such that f is defined at all values a_n for $n \geq N$ and the sequence $f(a_n)$ converges to $f(L)$

Proof:

Let $\epsilon > 0$

Since f is continuous at L, there exists $\delta > 0$ such that $|f(x) - f(L)| < \epsilon$ if $|x - L| < \delta$

Since the sequence $\{a_n\}$ converges to L, there exists N such that $|a_n - L| < \delta$ for $n \geq N$

Therefore, for all $n \geq N$, $|a_n - L| < \delta$, which implies $|f(a_n) - f(L)| < \epsilon$

We conclude that $\{f(a_n)\}$ converges to $f(L)$

Theorem 5.4: Squeeze Theorem

Suppose there exists an integer N such that

$a_n \leq b_n \leq c_n$ for all $n \geq N$

If there exists a real number L such that

$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$

Then b_n converges and $\lim_{n\to\infty} b_n = L$

Proof:

Let $\epsilon > 0$

Since a_n converges to L, there exists an integer N_1 such that $|a_n - L| < \epsilon$ for all $n \geq N_1$

Since c_n converges to L, there exists an integer N_2 such that $|c_n - L| < \epsilon$ for all $n \geq N_2$

By assumption, there exists an integer N such that $a_n \leq b_n \leq c_n$ for all $n \geq N$

Let M be the largest of N_1, N_2 and N

We must show that $|b_n - L| < \epsilon$ for all $n \geq M$:

$$-\epsilon < -|a_n - L| \leq a_n - L \leq b_n - L \leq c_n - L \leq |c_n - L| < \epsilon$$

Therefore, $-\epsilon < b_n - L < \epsilon$

Thus $|b_n - L| < \epsilon$ for all $n \geq M$

So, b_n converges to L

Bounded Sequences

Bounded above	a_n is bounded above if M exists such that $a_n \leq M$ for all positive integers n
Bounded Below	a_n is bounded below if M exists such that $a_n \geq M$ for all positive integers n
Bounded Sequence	a_n is bounded if it is bounded above **and** below
Unbounded Sequence	If it is not bounded, it is unbounded

Theorem 5.5: Convergent Sequences are bounded

If a_n converges, it is bounded

a_n converges → It is bounded

Theorem 5.6: Monotone Convergence Theorem

If a_n is bounded, and there exists a positive integer n_0 such that a_n is monotone for all $n \geq n_0$, then a_n converges

a_n bounded ∧ a_n is monotone → a_n converges

Infinite Series

$\Sigma_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$

The partial sum $S_k = \Sigma_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_k$

If the **sequence** of the partial sum converges, the infinite sum will as well

If a series converges to S, S is called the sum of the series

$\Sigma_{n=1}^{\infty} a_n = S$

If the **sequence** of a partial sum diverges, we have the divergence of a series

Theorem 5.7: Algebraic Properties of Convergent Series

Let $\Sigma_{n=1}^{\infty} a_n$ and $\Sigma_{n=1}^{\infty} b_n$ be convergent series

1. $\Sigma_{n=1}^{\infty}(a_n + b_n)$ converges and $\Sigma_{n=1}^{\infty}(a_n + b_n) = \Sigma_{n=1}^{\infty} a_n + \Sigma_{n=1}^{\infty} b_n$ **Sum Rule**
2. $\Sigma_{n=1}^{\infty}(a_n - b_n)$ converges and $\Sigma_{n=1}^{\infty}(a_n - b_n) = \Sigma_{n=1}^{\infty} a_n - \Sigma_{n=1}^{\infty} b_n$ **Difference Rule**
3. $\Sigma_{n=1}^{\infty} c \cdot a_n$ converges and $\Sigma_{n=1}^{\infty} c \cdot a_n = c\Sigma_{n=1}^{\infty} a_n$ **Constant Multiple Rule**

Geometric Series

$\Sigma_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ...$

If $|r| < 1$, the series converges, and $\Sigma_{n=1}^{\infty} ar^{n-1} = \dfrac{a}{1-r}$ for $|r| < 1$

If $|r| \geq 1$, the series diverges

Telescoping Series

Most terms cancel, leaving some of the first terms and some of the last terms

Theorem 5.8: Divergence Test

If $\lim_{n\to\infty} a_n = c \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then the series $\Sigma_{n=1}^{\infty} a_n$ diverges

$\lim_{n\to\infty} a_n = c \neq 0 \vee \lim_{n\to\infty} a_n$ does not exist → $\Sigma_{n=1}^{\infty} a_n$ diverges

Theorem 5.9: Integral Test

Suppose $\Sigma_{n=1}^{\infty} a_n$ is a series with positive terms

Assume that

1. f is a continuous function

2. f is decreasing
3. $f(n) = a_n$ for all integers $n \geq N$

Then $\Sigma_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(x)dx$ **both** converge or diverge

p Series

For any real number, $\Sigma_{n=1}^{\infty} \frac{1}{n^p}$ is called a p series

$p > 1$	Converges
$p \leq 1$	Diverges

Theorem 5.10: Remainder Estimate from the Integral Test

If $\Sigma_{n=1}^{\infty} a_n$ is a convergent series with positive terms

Assume that

1. f is continuous
2. f is decreasing
3. $f(n) = a_n$ for all integers $n \geq 1$

Let S_N be the N^{th} partial sum of $\Sigma_{n=1}^{\infty} a_n$ for all positive integers N

$S_N + \int_{N+1}^{\infty} f(x)dx < \Sigma_{n=1}^{\infty} a_n < S_N + \int_N^{\infty} f(x)dx$

If we let the remainder $R_N = \Sigma_{n=1}^{\infty} a_n - S_N = \Sigma_{n=N+1}^{\infty} a_n$ then

$\int_{N+1}^{\infty} f(x)dx < R_N < \int_N^{\infty} f(x)dx$

Theorem 5.11: Comparison Test

1. Suppose N exists such that $0 \leq a_n \leq b_n$ for all $n \geq N$. If $\Sigma_{n=1}^{\infty} b_n$ converges then $\Sigma_{n=1}^{\infty} b_n$ converges
2. Suppose N exists such that $a_n \geq b_n \geq 0$ for all $n \geq N$. If $\Sigma_{n=1}^{\infty} b_n$ diverges, then $\Sigma_{n=1}^{\infty} a_n$ diverges

Proof:

Let S_k be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$

Let $L = \sum_{n=1}^{\infty} b_n$

Since $a_n \geq 0$, the partial sums of a_n are increasing

Since $a_n \leq b_n$ for all $n \geq N$:

$\sum_{n=N}^{\infty} a_n \leq \sum_{n=N}^{\infty} b_n \leq \sum_{n=1}^{\infty} b_n = L$

Hence for all $k \geq 1$

$S_k = (a_1 + a_2 + ... + a_{N-1}) + \sum_{n=N}^{\infty} a_n \leq (a_1 + a_2 + ...a_{N-1}) + L$

Since$a_1 + a_2 + ... + a_{N-1}$ is a finite number, S_k is bounded above

Therefore, S_k is an increasing sequence that is bounded above

By the Monotone Convergence Theorem, S_k convergences, and thus $\sum_{n=1}^{\infty} a_n$ converges

The proof of 2. is the contrapositive of this.

Theorem 5.12: Limit Comparison Test

Let $a_n, b_n \geq 0$ for all $n \geq 1$

1. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = L \neq 0$ then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or diverge
2. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
3. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

Alternating Series

Series whose terms alternate sign

$\Sigma_{n=1}^{\infty}(-1)^{n+1} b_n = b_1 + b_2 + b_3 + ...$

Alternating Series Test

Alternating series converges if:

1. $0 \leq b_{n+1} \leq b_n$ for all $n \geq 1$

2. $\lim_{n\to\infty} b_n = 0$

Proof:

Consider the odd terms S_{2k+1} for $\sum_{n=1}^{\infty}(-1)^n b_n$ when $k \geq 0$

$$S_{2k+1} = S_{2k-1} - \frac{1}{2k} + \frac{1}{2k+1} < S_{2k-1}$$

Therefore, S_{2k+1} is a decreasing sequence

Also, $S_{2k+1} = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + ... + (\frac{1}{2k-1} - \frac{1}{2k}) + \frac{1}{2k+1} > 0$

Therefore, S_{2k+1} is bounded below

Therefore, according to the Monotone Convergence Theorem, S_{2k+1} converges

Similarly, the even terms are bounded above because

$$S_{2k} = S_{2k-2} + \frac{1}{2k-1} - \frac{1}{2k} > S_{2k-2}$$

$S_{2k} = 1 + (-\frac{1}{2} + \frac{1}{3}) + ... + (-\frac{1}{2k-2} + \frac{1}{2k-1}) - \frac{1}{2k} < 1$

Hence, by the Monotone Convergence Theorem, S_{2k} also converges

Since

$$S_{2k+1} = S_{2k} + \frac{1}{2k+1}$$

$$\lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty} S_{2k} + \lim_{k\to\infty} \frac{1}{2k+1}$$

Let $S = \lim_{k\to\infty} S_{2k+1}$

$\lim_{k\to\infty} S_{2k} = S$, since $\lim_{k\to\infty} \dfrac{1}{2k+1} = 0$

The odd and even terms both converge to S, so the sequence of partial sums converges to S, thus the alternating series converges to S

Theorem 5.14: Remainders in Alternating Series

Let S denote the sum of the series, and S_N the N^{th}partial sum

The remainder $R_N = S - S_N$ satisfies

$|R_N| \leq b_{N+1}$

Absolute Convergence

Absolute Convergence	$\sum_{n=1}^{\infty} \lvert a_n \rvert$ converges
Conditional Convergence	$\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} \lvert a_n \rvert$ diverges

Theorem 5.15: Absolute Convergence Implies Convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges $\sum_{n=1}^{\infty} a_n$ converges

$\sum_{n=1}^{\infty} |a_n|$ converges → $\sum_{n=1}^{\infty} a_n$ converges

Proof:

Assume $\sum_{n=1}^{\infty} |a_n|$ converges

$a_n = |a_n|$ or $a_n = -|a_n|$

Therefore, $|a_n| + a_n = 2|a_n|$ or $|a_n| + a_n = 0$

$2\sum_{n=1}^{\infty} |a_n|$ converges

By using the comparison test, $\sum_{n=1}^{\infty}(|a_n| + a_n)$ also converges

By using algebraic properties:

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(|a_n| + a_n) - \sum_{n=1}^{\infty} |a_n|$

will also thus converge

Theorem 5.16: Ratio Test

Let $\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$

1. If $0 \leq \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely
2. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges
3. If $\rho = 1$, the test does not provide any information

Theorem 5.17: Root Test

Let $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

1. If $0 \leq \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely

2. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges
3. If $\rho = 1$, the test does not provide any information

Choosing a Convergence Test

p series	Check the power p
Geometric	Check the ratio r, to see if it converges
Alternating	Alternating series test
Similar to p series or geometric	Try comparison or limit comparison test
$a_n = b_n^n$	Root Test
Other powers	Ratio Test
Factorials	Ratio Test
Other scenarios	Divergence Test or Integral Test