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Week 10 - Power Series

☰ Tags	

Power series centred at $x = 0$	$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$
Power series centred at $x = a$	$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$

Theorem 6.1: Convergence of a Power Series

The power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ satisfies **one** of the following:

1. The series converges at $x = a$ and diverges for all $x \neq a$
2. The series converges for all real numbers x
3. There exists a real number $R > 0$ such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$. At the x values where $|x - a| = R$, the series may converge or diverge

Proof:

Suppose the power series is centred at $a = 0$ (for a series centred at another value of a, it can be solved by using $y = x - a$ and consider the series $\sum_{n=1}^{\infty}$)

We must prove the following: if $d \neq 0$ exists, such that $\sum_{n=0}^{\infty} c_n d^n$ converges, then $\sum_{n=1}^{\infty} c_n x^n$ converges absolutely for all x where $|x| < |d|$

Since $\sum_{n=0}^{\infty} c_n d^n$ converges, the nth term $c_n d^n \to 0$ as $n \to \infty$

Therefore N exists such that $|c_n d^n| \leq 1$ for all $n \geq N$

$$|c_n x^n| = |c_n d^n| |\frac{x}{d}|^n$$

So, $|c_n x^n| \leq |\frac{x}{d}|^n$

$\sum_{n=1}^{\infty} |\dfrac{x}{d}|^n$ is a geometric series if $|\dfrac{x}{d}|^n < 1$

By the comparison test, we conclude that $\sum_{n=N}^{\infty} c_n x^n$ also converges for $|x| < |d|$

Since we can add a finite number of terms, $\sum_{n=0}^{\infty} c_n x^n$ to a convergent series, $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < |d|$

Interval of Convergence

The set of real numbers where $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges is the interval of convergence

If the series converges for $|x-a| < R$ and diverges for all $|x-a| > R$, R is the radius of convergence

If the series only converges at $x = a$, $R = 0$

If the series converges for all real numbers, $R = \infty$

Theorem 6.2: Combining Power Series

If $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to f and g respectively, with a shared interval of convergence I

1. $\sum_{n=0}^{\infty} (c_n x^n \pm d_n x^n)$ converges to $f \pm g$ on I
2. For any $m \geq 0$ and real number b, the power series $\sum_{n=0}^{\infty} b x^m c_n x^n$ converges to $b x^m f(x)$
3. For any $m \geq 0$ and real number b, $\sum_{n=0}^{\infty} c_n (b x^m)^n$ converges to $f(b x^m)$ where $b x^m$ is in I

Theorem 6.3: Multiplying Power Series

If $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to f and g respectively, with a shared interval of convergence I

Let $e_n = c_0 d_n + c1 d_{n-1} + c_2 d_{n-2} + ... + c_{n-1} d_1 + c_n d_0$

$e_n = \sum_{n=0}^{\infty} c_k d_{n-k}$

Then $(\sum_{n=0}^{\infty} c_n x^n)(\sum_{n=0}^{\infty} d_n x^n) = e_n x^n$

e_n converges to $f(x) \cdot g(x)$ on I

$\sum_{n=0}^{\infty} e_n x^n$ is known as the Cauchy product

Theorem 6.4: Term-by-Term Differentiation and Integration for Power Series

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges on the interval $(a-R, a+R)$

Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ...$

Then $f(x)$ is differentiable on the interval $(a-R, a+R)$

$f'(x) = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + ...$

And $f(x)$ is integrable

$\int f(x)dx = C + \sum_{n=0}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1}$

$\int f(x)dx = C + c_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + ...$

Theorem 6.5: Uniqueness of Power Series

If $\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} d_n(x-a)^n$

Then $c_n = d_n$ for all $n \geq 0$

Taylor Series

If $f(x)$ has derivatives of all orders at $x = a$, then the Taylor Series is:

$\sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + ... + \dfrac{f^{(n)}(a)}{n!}(x-a)^n + ...$

The n^{th} Taylor polynomial for f at a is

$p_n(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \dfrac{f'''(a)}{3!}(x-a)^3 + ... + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$

Maclaurin Series	The n^{th} Taylor polynomial for f at 0

Theorem 6.6: Uniqueness of Taylor Series

If $f(x)$ has a power series at a that converges to $f(x)$ on an open interval containing a, then that power series is the Taylor Series $f(x)$ at a

Theorem 6.7: Taylor's Theorem with Remainder

Let f be a function that can be differentiated $n+1$ times on an interval I which contains a

Let p_n be the n^{th} Taylor polynomial of f at a

Let $R_n(x) = f(x) - p_n(x)$ be the n^{th} remainder

Then c exists between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

If M exists where $|f^{(n+1)}(x)| \leq M$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Proof:

Let $g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t) - ... - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}$

$g(t)$ satisfies Rolle's Theorem ($g'(t) = 0$ at some point), since

$$g(a) = f(x) - f(a) - f'(a)(a-a) - ... + \frac{f^{(n)}(a)}{n!} - R_n(x)$$

$$g(a) = f(x) - p_n - R_n(x)$$

$$g(a) = 0$$

and

$$g(x) = f(x) - f(x) - 0 - ... - 0$$

$$g(x) = 0$$

Therefore, c exists such that $g'(c) = 0$

Using product rule:

$$\frac{d}{dt}\left[\frac{f^{(n)}(t)}{n!}(x-t)^n\right] = \frac{-f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

Therefore,

$$g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2\right] + \ldots + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^n\right] + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}$$

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}$$

By Rolle's Theorem, we know that c exists where $g'(c) = 0$, so

$$0 = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}$$

Rearranging:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Thus, if $M \geq |f^{(n+1)}(x)|$ exists:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Theorem 6.8: Convergence of Taylor Series

The Taylor Series converges to $f(x)$ for all x if and only if $\lim_{n\to\infty} R_n(x) = 0$

The Taylor series converges to $f(x) \iff \lim_{n\to\infty} R_n(x) = 0$

Binomial Series

The Maclaurin Series for $f(x) = (1+x)^r$ is the binomial series

It converges to f for $|x| < 1$

$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$

$$(1+x)^r = 1 + rx + \frac{r(r-r)}{2!}x^2 + ... + \frac{r(r-1)...(r-n+1)}{n!}x^n$$