3️⃣

# Week 3 - Differentiation

Tags

## Difference Quotient

$Q = \frac{f(x)-f(a)}{x-a}$

## Tangent

$f'(a) = m = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

$f'(a) = m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

## Theorem 3.1 Differentiability Implies Continuity

If $f(x)$ is differentiable at $a$ then $f(x)$ is continuous at $a$

Proof:

To prove that $f(x)$ is continuous at $a$, we need to show that $\lim_{x \to a} f(x) \to f(a)$

$\lim_{x \to a} f(x) = \lim_{x \to a} f(x) - f(a) + f(a)$

$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a}.(x-a) + f(a)$

$= (\lim_{x \to a} \frac{f(x)-f(a)}{x-a}).(\lim_{x \to a} x - a) + \lim_{x \to a} f(a)$

$= f'(a).0 + f(a)$

$= f(a)$

Therefore, since $f(a)$ is defined and $\lim_{x \to a} f(x) = f(a)$, we conclude that $f(x)$ is continuous at $a$ $\square$

## Theorem 3.2 The Constant Rule

$\frac{d}{dx}c = 0$

# Theorem 3.3/3.7/3.12 Power Rule

$\frac{d}{dx}x^n = nx^{n-1}$

# Theorem 3.4 Sum, Difference and Constant Multiple Rules

$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

$\frac{d}{dx}kf(x) = k\frac{d}{dx}f(x)$

# Theorem 3.5 Product Rule

$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$

# Theorem 3.6 Quotient Rule

$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

# Theorem 3.8/3.9 Trigonometric Derivatives

$\frac{d}{dx}\sin(x) = \cos(x)$

$\frac{d}{dx}\cos(x) = -\sin(x)$

To prove this:

We need to know that $\lim_{h\to 0}\frac{\sin(h)}{h} = 1$

$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$

$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$

$= \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)}{h} + \frac{\cos(x)\sin(h)}{h}$

$= \lim_{h\to 0}\sin(x)(\frac{\cos(h)-1}{h}) + \cos(x)\frac{sin(h)}{h}$

$= \sin(x) \cdot 0 + \cos(x) \cdot 1$

$= \cos(x)$

$\frac{d}{dx} \tan(x) = \sec^2(x)$

$\frac{d}{dx} \cot(x) = -\csc^2(x)$

$\frac{d}{dx} \sec(x) = \sec(x)\tan(x)$

$\frac{d}{dx} \csc = -\csc(x)\cot(x)$

# Theorem 3.10 Chain Rule

$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

$\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du}$

Proof:

$h(x) = f(g(x))$

$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x-a}$

$= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x)-g(a)} \cdot \frac{g(x)-g(a)}{x-a}$

$= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x)-g(a)} \cdot g'(a)$

let $y = g(x)$ and $b = g(a)$:

$= \lim_{y \to b} \frac{f(y) - f(g(b)}{y-b} = f'(b) = f'(g(a))$

Therefore

$= f'(g(a)) \cdot g'(a)$

# Theorem 3.11 Inverse Function Theorem

$\frac{d}{dx} f^{-1}(x) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

$g'(x) = \frac{1}{(g^{-1})'(g(x))}$

# Theorem 3.14 Derivative of the Natural Exponential Function

$\frac{d}{dx} e^{g(x)} = e^{g(x)} g'(x)$

Proof, assuming $e$ is the only value of $b$ for which this holds:

$B(x) = b^x$

$B'(x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$

$= \lim_{h \to 0} \frac{b^x b^h - b^x}{h}$

$= \lim_{h \to 0} \frac{b^x (b^h - 1)}{h}$

$= b^x \lim_{h \to 0} \frac{b^h - 1}{h}$

$b^x B'(0)$

# Theorem 3.15 Derivative of the Natural Logarithm Function

$\frac{d}{dx} \ln(g(x)) = \frac{1}{g(x)} g'(x)$

Proof:

$y = \ln(x)$

$e^y = x$

$e^y \frac{d}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y}$

Then, substituting $y = ln(x)$

$\frac{d}{dx} = \frac{1}{x}$

# Theorem 3.16 Derivatives of General Exponential and Logarithmic Functions

If $y = \log_b x$

$\frac{dy}{dx} = \frac{1}{x \ln(b)}$

More generally:

If $h(x) = \log_b(g(x))$

$h'(x) = \frac{g'(x)}{g(x) \ln(b)}$

Proof:

$y = \log_b x$

$b^y = x$

$\ln(b^y) = \ln(x)$

$y\ln(b) = \ln(x)$

$y = \frac{\ln(x)}{\ln(b)}$

$\frac{dy}{dy} = \frac{1}{x\ln(b)}$

If $y = b^x$

$\frac{dy}{dx} = b^x\ln(b)$

More generally:

If $h(x) = b^{g(x)}$

$h'(x) = b^{g(x)}g'(x)\ln(b)$

Proof:

$y = b^x$

$\ln(y) = x\ln(b)$

$\frac{1}{y}\frac{dy}{dx} = \ln(b)$

$\frac{dy}{dx} = y\ln(b) = b^x\ln(b)$