Week 6 - Applications of Integration

Tags

Mass–Density Formula of a Circular Object

Theorem 2.1 Finding the Area between 2 Curves

If $f(x) \geq g(x)$ and both are continuous

The area between the 2 curves and $x = a$ and $x = b$ is

$\int_a^b f(x) - g(x) dx$

Theorem 2.2 Finding the Area of a Region between Curves that cross

If $f(x)$ and $g(x)$ are continuous

The area between the curves and $x = a$ and $x = b$

Is still $\int_a^b |f(x) - g(x)| dx = \int_a^c f(x) - g(x) dx + \int_c^b f(x) - g(x) dx$

Theorem 2.3 Finding the Area Between 2 Curves, Integrating along the y-axis

If $u(y) \geq v(y)$ are continuous functions

The area between the curves and $y = c$ and $y = d$ is

$\int_c^d |u(y) - v(y)| dy$

Disk Method

If $f(x)$ is continuous and non-negative

Volume of the solid of revolution formed by $f(x)$ and lines $x = a$ and $x = b$ is

$V = \int_a^b \pi(f(x))^2 dx$

Disk Method for Solids of Revolution around y-axis

$V = \int_c^d \pi(u(y))^2 dy$

Washer Method

Volume of revolution when there are 2 functions and $f(x) \geq g(x)$

$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]dx$

Proof:

Washer Method for Solids of Revolution around y-axis

$V = \int_c^d \pi[(u(y))^2 - (v(y))^2]dy$

Volume of Revolution with a different axis of revolution

Instead of $V = \int_a^b \pi(f(x))^2 dx$

If the axis is $x = e$

$V = \int_a^b \pi(f(x) - e)^2 - (-e)^2 dx$

Theorem 2.4 Arc Length

$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$

Proof:

We start by looking at the distance between 2 points. By Pythagoras' theorem, this is $\sqrt{(\Delta x)^2 + (\Delta y)^2}$

If we make the x distances constant, then this becomes:

$\sqrt{(\Delta x)^2 + (\Delta y_i)^2}$

$= \Delta x \sqrt{1 + (\frac{\Delta y_i}{\Delta x})^2}$

According to the Mean Value Theorem, there is a point such that $f'(x_i^*) = \frac{\Delta y_i}{\Delta x}$, so

$= \Delta x \sqrt{1 + (f'(x))^2}$

If we add up all the lengths of the line segments

$L \approx \Sigma_{i=1}^{n} \sqrt{1 + (f'(x))^2} \Delta x$

This is a Riemann sum

Taking the limit as $n \to \infty$

$L = \lim_{n \to \infty} \Sigma_{i=1}^{n} \sqrt{1 + (f'(x))^2} \Delta x$

$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$

Theorem 2.5 Arc Length for a y-function

$L = \int_c^d \sqrt{1 + (g'(y))^2} dy$

Proof:

Same proof as Theorem 2.4

Theorem 2.6 Surface Area of Revolution

Around x-axis

$A = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$

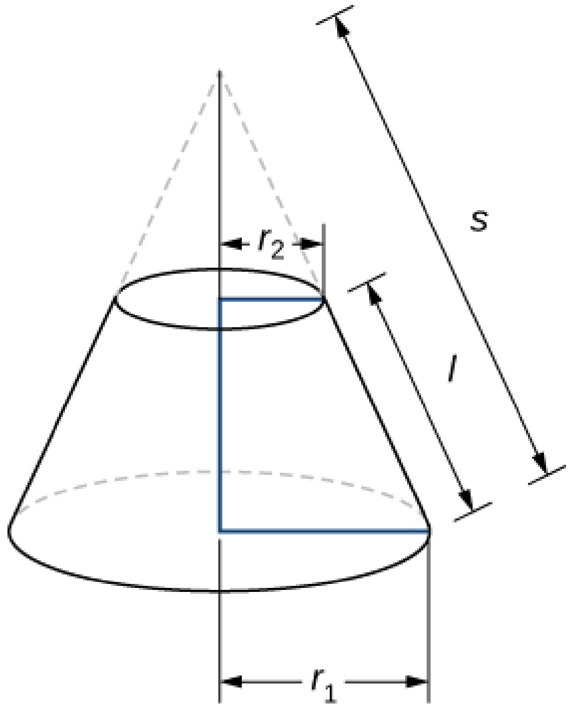
Around y-axis

$A = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} dy$

Proof:

Lateral surface area of a cone (excludes base)

$= \pi r s$ where r is radius and s is slant height



The small cone and large cone are similar triangles so

$$\frac{r_2}{r_1} = \frac{s-l}{s}$$

Which leads to

$$s = \frac{r_1 l}{r_1 - r_2}$$

The lateral surface area of frustum

$=$ Lateral surface area of large cone $-$ Lateral surface area of small cone

$$= \pi r_1 s - \pi r_2 (s-l)$$

$$= \pi[r_1(\frac{r_1 l}{r_1 - r_2}) - r_2(\frac{r_1 l}{r_1 - r_2} - l)]$$

$$= \pi[\frac{r_1^2 l}{r_1 - r_2} - \frac{r_1 r_2 l}{r_1 - r_2} + r_2 l]$$

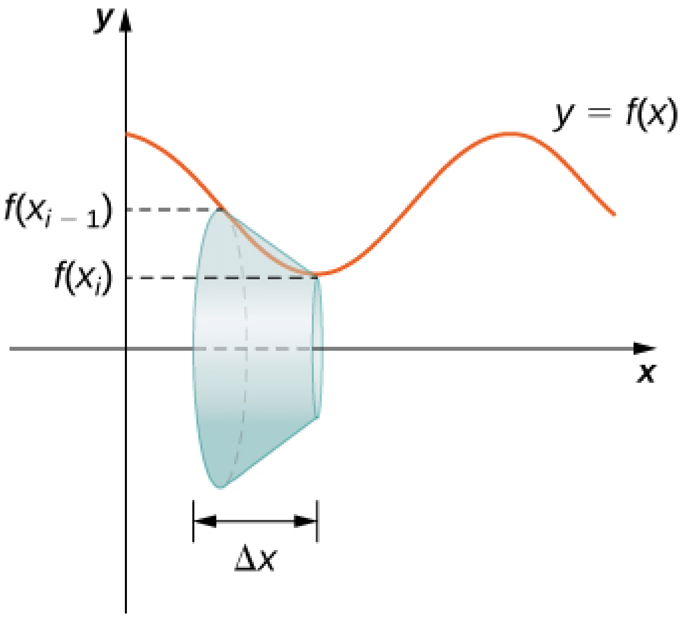
$$= \pi[\frac{r_1^2 l}{r_1 - r_2} - \frac{r_1 r_2 l}{r_1 - r_2} + \frac{r_2 l(r_1 - r_2)}{r_1 - r_2}]$$

$$= \pi[\frac{r_1^2 l}{r_1 - r_2} - \frac{r_1 r_2 l}{r_1 - r_2} + \frac{r_1 r_2 l}{r_1 - r_2} - \frac{r_2^2 l}{r_1 - r_2}]$$

$$= \pi[\frac{(r_1^2 - r_2^2) l}{r_1 - r_2}]$$

$$= \pi[\frac{(r_1 - r_2)(r_1 + r_2) l}{r_1 - r_2}]$$

$$= \pi(r_1 + r_2) l$$



Both radii are actually just the y values $f(x_i)$ and $f(x_{i-1})$

Therefore

$$s = \pi(r_1 + r_2) l$$

$$= \pi(f(x_i) + f(x_i) \sqrt{\Delta x^2 + (\Delta y_i)^2})$$

$$= \pi(f(x_i) + f(x_i) \Delta x \sqrt{1 + \frac{\Delta yi}{\Delta x}})$$

Again, using the Mean Value Theorem

$$= \pi(f(x_{i-1}) + f(x_i)) \Delta x \sqrt{1 + f'(x_i^*)^2}$$

By the Intermediate Value Theorem, there is a point x^{**} such that $f(x^{**}) = \frac{1}{2}[f(x_{i-1}) + f(x_i)]$

$$= 2\pi f(x^{**}) \Delta x \sqrt{1 + f'(x_i^*)^2}$$

Then the area over the whole revolution will be

$A \approx \Sigma_{i=1}^{n} 2\pi f(x^{**}) \Delta x \sqrt{1 + f'(x_i^*)^2}$

We can do a Riemann Sum because as $n \to \infty$, both x^* and x^{**} will approach x, since they are both in the range $[x_{i-1}, x_i]$

Therefore

$A = \lim n \to \infty \Sigma_{i=1}^{n} 2\pi f(x^{**}) \Delta x \sqrt{1 + f'(x_i^*)^2}$

$A = \int_a^b (2\pi f(x) \Delta x \sqrt{1 + f'(x_i)^2}) dx$

Theorem 2.7 Mass-Density Formula of a One-Dimensional Object

Let $\rho(x)$ denote a linear density function, giving the density of the object at point x along the x axis

Mass $m = \int_a^b \rho(x) dx$

Proof:

We treat a rod as if it had no thickness

$m_i \approx \rho(x_i^*)(x_i - x_{i-1})$

$m_i \approx \rho(x_i^*) \Delta x$

$m = \Sigma_{i=1}^{n} m_i \approx \rho(x_i^*) \Delta x$

This is a Riemann sum, taking the limit as $n \to \infty$

$m = \lim_{n \to \infty} \Sigma_{i=1}^{n} \rho(x_i^*) \Delta x$

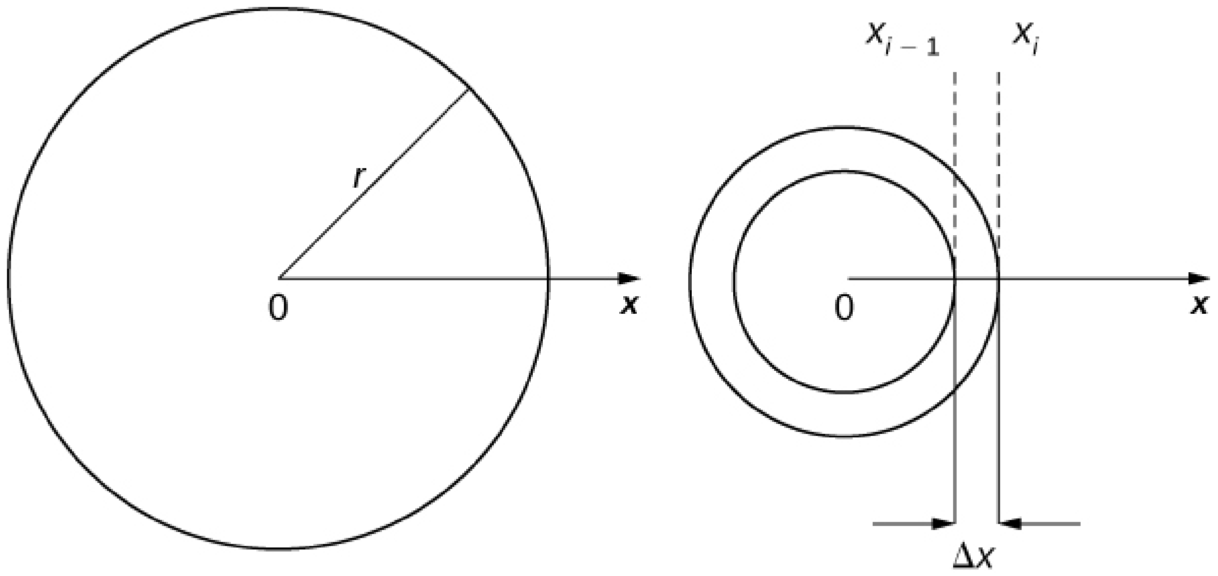
$m = \int_a^b \rho(x) dx$

Theorem 2.8 Mass–Density Formula of a Circular Object

Let $\rho(x)$ be an integrable function representing radial density of a disk r

Mass $m = \int_0^r 2\pi x \rho(x) dx$

Proof:



(a) (b)

Area $A = \pi(x_i^2) - \pi(x_{i-1})^2$

$= \pi[x_i^2 - x_{i-1}^2]$

$= \pi(x_i + x_{i-1})(x_i - x_{i-1})$

$= \pi(x_i + x_{i+1})\Delta x$

$x_i^* \approx \dfrac{(x_i + x_{i-1})}{2}$, so

$\approx 2\pi x_i^* \Delta x$

Using $\rho(x^*)$ to approximate the density of the washer

$m \approx \Sigma_{i=1}^{n} m_i \approx 2\pi x_i^* \rho(x_i^*)\Delta x$

This is a Riemann sum, hence as $n \to \infty$

$m = \lim_{n\to\infty} \Sigma_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x$

$m = \int_0^r 2\pi x \rho(x) dx$

Work

Work done by a force $F(x)$ from point a to b is

$\int_a^b F(x)dx$

Proof:

$W_i \approx F(x_i^*)(x_i - x_{i-1})$

$W_i \approx F(x * i)\Delta x$

$W = \Sigma_{i=1}^{n} W_i \approx \Sigma_{i=1}^{n} F(x_i^*)\Delta x$

Which is a Riemann sum, hence as $n \to \infty$

$W = \lim_{n\to\infty} \Sigma_{i=1}^{n} F(x_i^*)\Delta x$

$W = \int_a^b F(x)dx$

Pumping Problems

Work for pumping water from the initial distance of h_0 to the bottom h is:

$W = \int_{h_0}^{h} \pi\rho r^2 x dx$

Proof:

Density equation

$$\rho = \frac{m}{V} \to m = \rho V \text{ 1.}$$

Force equation

$F = mg$

Plugging in 1.:

$F = \rho V g$ 2.

Work equation

$W = Fd$

Plugging in 2.:

$W = \rho V g d$

$W_i = \rho\pi r^2 x_i^* \Delta x$

$\Sigma_{i=1}^{n} W_i = \Sigma_{i=1}^{n} \rho\pi r^2 x_i^* \Delta x$

This is a Riemann sum, hence as $n \to \infty$

$W = \lim_{n\to\infty} \Sigma_{i=1}^{n} \rho\pi r^2 x_i^* \Delta x$

$W = \int_{h_0}^{h} \pi\rho r^2 x dx$

Hydrostatic Force

Force = Pressure*Area*Distance below water

$F_i = \rho As$

Assuming the thickness is thin enough, we can assume a constant force on the slice:

$= \rho[w(x_i^*)\Delta x]s(x_i^*)$

$F \approx \Sigma_{i=1}^{n} F_i = \Sigma_{i=1}^{n} \rho[w(x_i^*)\Delta x]s(x_i^*)$

This is a Riemann sum, so as $n \to \infty$

$\lim_{n\to\infty} \Sigma_{i=1}^{n} \rho[w(x_i^*)\Delta x]s(x_i^*) = \int_a^b \rho w(x)s(x)dx$

Theorem 2.9 Centre of Mass of Objects on a Line

Moment $M = \Sigma_{i=1}^{n} m_i x_i$

Centre of mass $\bar{x} = \frac{M}{m}$

Theorem 2.10 Centre of Mass of Objects in a Plane

Moments

$M_x = \Sigma_{i=1}^{n} m_i x_i$

$M_y = \Sigma_{i=1}^{n} m_i y_i$

Coordinates of Centre of Mass

$\bar{x} = \frac{M_x}{m}$

$\bar{y} = \frac{M_y}{m}$

Theorem 2.11 Symmetry Principle

If a region R is symmetric about a line l, then the centroid of R lies on l

Theorem 2.12 Centre of Mass of a Thin Plate in the xy Plane

ρ is the density of the lamina

Mass of the Lamina

$m = \rho \int_a^b f(x) dx$

Moments of the Lamina

$M_x = \rho \int_a^b \frac{f(x)^2}{2} dx$

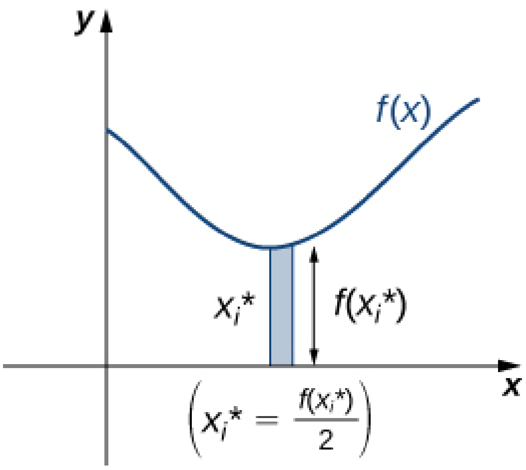
$M_y = \rho \int_a^b f(x) x dx$

Centres of Mass of the Lamina

$\bar{x} = \frac{M_x}{m}$

$\bar{y} = \frac{M_y}{m}$

Proof:



Partition the lamina and let $x_i^* = \dfrac{x_{i+1} + x_i}{2}$ which is the midpoint

Construct a rectangle and let $f(x_i^*)$ be the height of the rectangle

Therefore, the centre of mass will be $(x_i^*, \dfrac{f(x_i^*)}{2})$

Mass of the rectangle will be $\rho f(x_i^*) \Delta x$ where ρ is the density

Mass $m \approx \Sigma_{i=1}^{n} \rho f(x_i^*) \Delta x$

This is a Riemann sum, hence as $n \to \infty$

$$m = \lim_{n \to \infty} \Sigma_{i=1}^{n} \rho f(x_i^*) \Delta x = \rho \int_a^b f(x) dx$$

Finding Moment

Moment = Mass*Distance to centre of mass

$$M = \rho f(x_i^*) \Delta x \frac{f(x_i^*)}{2}$$

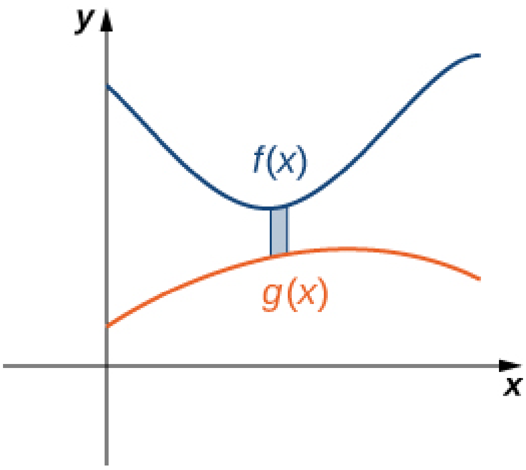
Taking a Riemann sum:

$$M_y = \lim_{n \to \infty} \Sigma_{i=1}^{n} \rho f(x_i^*) \Delta x \frac{f(x_i^*)^2}{2} = \rho \int_a^b f(x) \frac{f(x)^2}{2} dx$$

Similarly

$$M_x = \rho \int_a^b x f(x) dx$$

Theorem 2.13: Centre of Mass of a Lamina Bounded by Two Functions



Mass

Mass $m = \rho \int_a^b f(x) - g(x) dx$

Proof

The height of the of the rectangle is $f(x_i^*) - g(x_i^*)$

Therefore the area is $[f(x_i^*) - g(x_i^*)] \Delta x$

Hence the mass is $m = \rho \int_a^b f(x) - g(x) dx$

Moment

$M_x = \rho \int_a^b x(f(x) - g(x))dx$

$M_y = \rho \int_a^b \frac{1}{2}[f(x)^2 - g(x)^2]dx$

Proof

The moment is found by multiplying the area $[f(x_i^*) - g(x_i^*)]\Delta x$ by the distance $\dfrac{f(x_i^*) + g(x_i^*)}{2}$

Which gives $\frac{1}{2}[f(x)^2 - g(x)^2]\Delta x$ for M_y

and $x[f(x)^2 - g(x)^2]\Delta x$ for M_x

Theorem 2.14 Theorem of Pappus for Volume

Let R be a region in the plane and let l be a line in the plane that does not intersect R. Then the volume of the solid of revolution formed by revolving R around l is equal to the area of R multiplied by the distance d travelled by the centroid of R.

Proof:

The area of the region between $f(x)$ and $g(x)$ is $\int_a^b f(x) - g(x)dx$

If the axis of rotation is the y axis, the distance travelled by the centroid of the region depends only on $\bar{x}$, which is

$\bar{x} = \frac{M_y}{m}$

where

$m = \rho \int_a^b f(x) - g(x)dx$

$M_y = \rho \int_a^b x[f(x) - g(x)]dx$

Then

$$d = 2\pi \frac{\rho \int_a^b x[f(x) - g(x)]}{\rho \int_a^b f(x) - g(x)dx}$$

Since

$d = 2\pi\bar{x}$, because the distance travelled is a circumference of a circle around the y-axis

Thus

$dA = \rho \int_a^b x[f(x) - g(x)]dx$

Using the method of cylindrical shells, we get

$V = \rho \int_a^b x[f(x) - g(x)]dx$

$V = d \cdot a$

Exponential Growth Model

Systems that exhibit exponential growth:

$y = y_0 e^{kt}$

Exponential Decay Model

Systems that exhibit exponential decay:

$y = y_0 e^{-kt}$

Half-Life

- Time taken for the quantity to halve

$\lambda = \frac{\ln 2}{k}$