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Section 5.4 2, 10, 29

5.4 #2:

Suppose b_1' b_2, b_3… is a sequence defined as follows:

$B_1 = 4$, $b_2 = 12$

$B_k = b_{k-2} + b_{k-1}$ for all integers $k \geq 3$.

Prove that b_n is divisible by 4 for all integers $n \geq 1$.

Let b_1' b_2, b_3… be the sequence defined as follows: $B_1 = 4$, $b_2 = 12$ and $B_k = b_{k-2} + b_{k-1}$ for all integers $k \geq 3$. And let P(n) be the property b_n is divisible by 4 for all integers $n \geq 1$.

We can easily establish that P(1) and P(2) are true, because P(10 = 4, and 4 / 4 is 1. P(2) is = 12 and 12 / 4 = 3, hence both are divisible.

We must show for all integers $k \geq 3$, that if P(i) is true for all integers I from 1 through k, then P(k+1) is also true.

Let k be any specific but arbitrarily chosen integer with $k \geq 3$, and suppose that

$B_k = b_{k-2} + b_{k-1}$ is divisible by 4.

We must show that $B_{(k+1)}$is also divisible by 4.

$B_{(k+1)} = B_{k-1} + b_k$

We know what b_k and b_{k-1} are divisible by 4 through the strong inductive hypothesis.

Because $B_k$and B_{k-1} are both divisible by 4, $B_{(K+1)}$ will also be divisible by 4 as the sum of 2 numbers divisible by 4.

Thus, because $B_{(k+1)}$ is divisible by 4, B_n is divisible by 4 for all integers $n \geq 1$.

10. Use strong mathematical induction to prove that P(n) is true for all integers $n \geq 14$.

Let P(n) be “any collection of n coins that can be obtained using a combination of 3 cent and 5 cent coins”.

P(10) can be obtained with 2 5 cent coins.

P(11) can be obtained with 1 5 cent coin and 2 3 cent coins.

P(12) can be obtained with 4 3 cent coins.

P(13) can be obtained by 2 5 cent coins and 1 3 cent coins.

P(14) can be obtained with 1 5 cent coin and 3 3 cent coins.

P(15) can be obtained with 5 3 cent coins.

P(16) can be obtained with 2 5 cent coins and 2 3 cent coins.

P(17) can be obtained with 1 5 cent coin and 4 3 cent coins.

P(18) can be obtained with 6 3 cent coins.

We can thus infer that P(n) is true for $12 \leq n \leq 18$

Suppose that N is true for $12 \leq n \leq k$, where $K \geq 14$.

Assuming the inductive hypothesis, we can show that N is also true for P(K+1)

Using the inductive hypothesis, P(K-2) holds, because K is ≥ 14, so $K - 2 \geq 12$ and we have already proved P(n) true for all values from 10 to 18. We can add a 3 cent coin to P(K-2) to obtain (K+1). We know that P(K-1) is true because $K \geq 14$, so $K - 1 \geq 13$, and P(13) has already been proven to be true. To obtain (K+1),we can alternatively replace a 3 cent coin with a 5 cent coin from P(K-1).

We have proven methods to obtain P(K+1) as true from both P(K-1) and P(K-2). Because P(n) is true for P(15) and P(16), P(K-1) and P(K-2) is true for values where $K \geq 14$, and thus we can obtain P(K+1) as true for any value of $K \geq 14$.

Thus, P(n) is true whenever $K \geq 14$.

29. It is a fact that every integer $n \geq 1$ can be written in the form

$C_r * 3^r + c_{r-1} * 3^{r-1} + \ldots + c_2 * 3^2 + c_1 * 3 + c_0$,

Let P(n) be the property "n can be written in the form $C_r * 3^r + c_{r-1} * 3^{r-1} + \ldots + c_2 * 3^2 + c_1 * 3 + c_0$" for all integers $n \geq 1$. R is a non-negative integer. c_r is either 1 or 2 and c_i is either 0, 1, or 2 for all integers I = 0, 1, 2, … , r-1. C_0 is the coefficient of the right most term.

Showing that P(1) and P(2) are true.

P(1) can be true when C_r is 1, r is 0, and c_0 is 1. This gives us $1 = (1 * 3_0)$, which is true, because the right side simplifies to 1 *1, which is 1, and the left side is already 1, so both the left and right hands of the equations are equivalent.

P(2) can be true when C_r is 2, r is 0, and c_0 is 2. This gives us $2 = (2 * 3_0)$, which is true because the right side simplifies to 2 * 1, which is 2, and the left side is already 2, so both the left and right hand sides of the equations are equivalent.

Now we must show that for all integers where $k \geq 2$, if P(i) is true for all integers I from 1 through k, then P(k+1) is also true.

Let K be a specific but arbitrarily chosen integer such that $k \geq 2$. P(k+1) will be true if it can be written as the sum of powers of 3, so we must demonstrate this is the case no matter what value K has in the case of (K+1).

Let us look at 3 cases.

In the first case, suppose that K+1 is divisible by 3. That is, imagine an integer X such that (K+1) = 3X.

X = (K+ 1) / 3. By the inductive hypothesis, $K \geq 2$, so $K + 1 \geq 3$, and therefore, both X and $(K + 1) / 3 \geq 1$. (K+1) / 3 is an integer by definition, because X, its equivalent, is also an integer.

Therefore, (K+1) / 3 can be written in the base form of $(K+1) /3 = C_r * 3^r + c_{r-1} * 3^{r-1} + \ldots + c_2 * 3^2 + c_1 * 3 + c_0$, with the previous numerical constraints on C_r, C_i, C_0 and r.

We can then derive (K+1) by multiplying both sides of the equation by 3, which gives us

$C_r * 3^{r+1} + c_{r-1} * 3^r + \ldots c_2 * 3^2 + c_1 * 3 + c_0 * 3$ Essentially, the exponents of the left most terms involving R increase by 1 until the equation enters constant territory, such as $c_2 * 3^2$. Because this is in the form involving powers of 3's required by P(n), we can say that P(K+1) is true when K+1 is divisible by 3.

In the second case, suppose that K + 1 is divisible by 3 with a remainder of 1. That is, imagine an integer X such that (K+1) = 3X + 1. We can subtract 1 from both sides to give us K = 3X. And thus, X = K / 3. By the inductive hypothesis, K / 3 must be ≥ 1 because K is ≥ 2, and k divides into 3 with a remainder of 1, which would indicate that K is ≥ 4. K /3 is an integer because it is equivalent to X, which is defined as an integer. Because K / 3 is an integer ≥ 1, it can be written in the base form $K / 3 = C_r * 3^r + c_{r-1} * 3^{r-1} + \ldots + c_2 * 3^2 + c_1 * 3 + c_0$.

To get K + 1 from K / 3, we can multiply both sides of the equation by 3, and then add 1, which gives us

$C_r * 3^{r+1} + c_{r-1} * 3^r + \ldots c_2 * 3^2 + c_1 * 3 + c_0 * 3 + 1$. This is still a sum of powers of 3, because $c_0 * 3$ could be rewritten as $3^1 * C_0$, and the +1 at the end can be rewritten as 3^0.

In the final case, suppose that K + 1 is divisible by 3 with a remainder of 2. That is, imagine an integer X such that (K+1) = 3X + 2. Solving for X, we get X = (K – 1) / 3. (K-1) / 3 is ≥ 1

because by the inductive hypothesis, X ≥ 2, and divides into 3 with a remainder of 2, indicating K ≥ 5. (K – 1) / 3 is also an integer because it is equivalent to X, which is defined as an integer.

Because (K – 1) / 3 is ≥ 1, and an integer, it can be written in the base form (K – 1) / 3 = $C_r * 3^r + c_{r-1} * 3^{r-1} + \ldots + c_2 * 3^2 + c_1 * 3 + c_0$

We can derive K+1 by multiplying both sides by 3 and then adding 2, giving us a final result of $C_r * 3^{r+1} + c_{r-1} * 3^r + \ldots c_2 * 3^2 + c_1 * 3 + c_0 * 3 + 2$. This is a sum of powers of 3, because 3 can be represented as $2 * 3^0$, which is equivalent to $3^0 + 3^0$.

It must be the case that either K, K +1, or K-1 are divisible by 3, and we have demonstrated that no matter what case is applicable, K+1 can be represented as a sum of powers of 3 where K ≥ 2, as was to be shown.

Therefore, P(K+1) is true.

For this proof, I used the hint on the back of the book on Appendix B A-40. It reads as follows: "Hint: In the inductive step, divide into cases depending on whether k can be written as k = 3x or k = 3x + 1 or k = 3x+2 for some integer x."