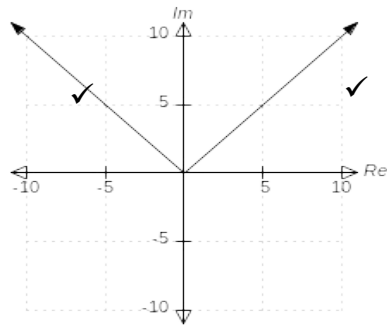


# **SPECIALIST 3 &4 PRACTICE**

## **SOLUTIONS**

1. (a)

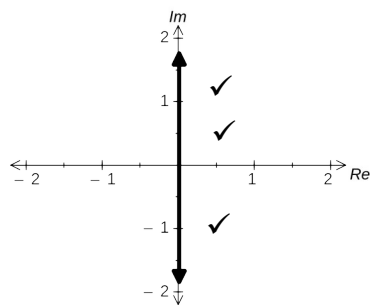


Let  $z = x + y i$

Then  $\text{Im}(z) = y$  and  $\text{Re}(z) = x$

Hence,  $\text{Im}(z) = |\text{Re}(z)|$  becomes  
 $y = |x|.$

(b)



Let  $z = x + y i .$

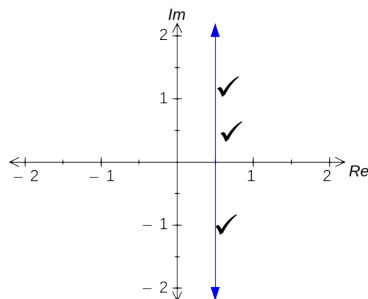
$\Rightarrow \text{Im}(z) = y$

$|z| = \sqrt{x^2 + y^2}$

Hence  $\text{Im}(z) = |z|$  becomes

$$\begin{aligned} \sqrt{x^2 + y^2} &= y \\ \Rightarrow x &= 0 \end{aligned}$$

(c)



Let  $z = x + y i.$

Then  $z + \frac{1}{z} = 1$  becomes

$$\begin{aligned} x + y i + (x - y i) &= 1 \\ \Rightarrow 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

[8]

2. (a)  $z = \frac{-2\pi \pm \sqrt{4\pi^2 - 20\pi^2}}{2}$   
 $= -\pi \pm 2\pi i$

✓

✓✓

(b)  $z^3 = e^3 \text{cis}(\pi + 2n\pi)$

✓

$z = e \text{cis}\left(\frac{\pi}{3} + \frac{2n\pi}{3}\right)$

✓

$z = e\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), -e, e\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

✓✓

[7]

3. (a)  $\{x : x > 0, x \neq \frac{1}{e}\}$  ✓✓

(b)  $y \neq 0$  ✓✓

(c) Yes, as  $f$  is a one-to-one and onto function. ✓✓  
(or the graph of  $y = f(x)$  passes the horizontal line test.)

Write  $x = \frac{1}{1 + \ln y}$  ✓

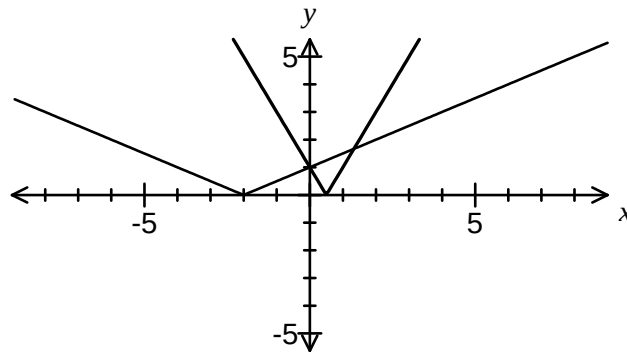
$$1 + \ln y = \frac{1}{x}$$

$$y = e^{1/x - 1} \quad \checkmark$$

[8]

4. **(10 marks)**

The graph of  $f(x) = |2x - 1|$  is shown below.



(a) Add the function  $g(x) = \frac{|x+2|}{2}$  to the graph. (1 mark)

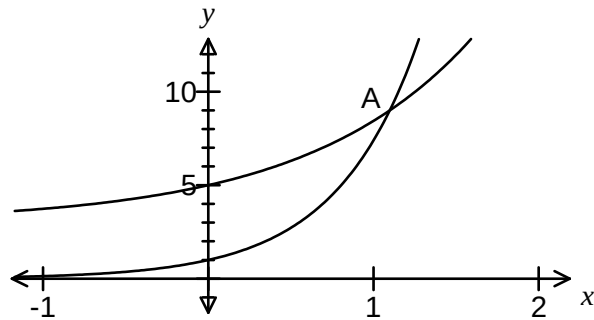
(b) Solve  $f(x) - g(x) \geq 0$ . (3 marks)

$$f(x) = g(x) \text{ when } x = 0 \text{ and when}$$

$$0.5x + 1 = 2x - 1 \Rightarrow x = \frac{4}{3}$$

$$\text{Hence } x \leq 0 \text{ or } x \geq \frac{4}{3}$$

The graphs of  $y = e^{2x}$  and  $y = 2e^x + 3$  intersect at the point A, shown on the graph below.



- (c) Show that the  $x$ -coordinate of A is  $\log_e 3$ . (3 marks)

$$\begin{aligned} e^{2x} &= 2e^x + 3 \\ e^{2x} - 2e^x - 3 &= 0 \\ (e^x + 1)(e^x - 3) &= 0 \\ \therefore e^x &= 3 \Rightarrow x = \ln 3 \end{aligned}$$

- (d) Determine the exact area, in simplest form, of the region bounded by the two curves and the  $y$ -axis. (3 marks)

$$\begin{aligned} &\int_0^{\ln 3} (2e^x + 3 - e^{2x}) dx \\ &= \left[ 2e^x + 3x - \frac{e^{2x}}{2} \right]_0^{\ln 3} \\ &= \left[ 6 + 3\ln 3 - \frac{9}{2} \right] - \left[ 2 + 0 - \frac{1}{2} \right] \\ &= 3\ln 3 \end{aligned}$$

5.

**Question 2****(7 marks)**

Two complex numbers are given by  $z = 2\text{cis}\frac{\pi}{3}$  and  $w = \sqrt{3} - i$ .

- (a) Determine  $\arg \frac{z}{w}$  (2 marks)

$$\begin{aligned}\arg \frac{z}{w} &= \arg \frac{2\text{cis}\frac{\pi}{3}}{2\text{cis}\left(-\frac{\pi}{6}\right)} \\ &= \frac{\pi}{3} + \frac{\pi}{6} \\ &= \frac{\pi}{2}\end{aligned}$$

- (b) Evaluate  $\left| w \times \overline{w \times z} \right|$ . (3 marks)

$$\begin{aligned}\left| w \times \overline{w \times z} \right| &= \left| w \times \overline{w} \times \overline{z} \right| \\ &= |w|^2 \times |\overline{z}| \\ &= 4 \times 2 \\ &= 8\end{aligned}$$

- (c) Find the complex number  $u$  given that  $\frac{z \times u}{2} = \text{cis}\left(\frac{3\pi}{4}\right)$ . (2 marks)

$$\begin{aligned}u &= \frac{2\text{cis}\frac{3\pi}{4}}{2\text{cis}\frac{\pi}{3}} \\ &= \text{cis}\frac{5\pi}{12}\end{aligned}$$

---

6. (a)  $I = \int x^2 - x^2(1+x^3)^{\frac{1}{2}} dx$       or       $\int x^2(1-y^{\frac{1}{2}}) \frac{dy}{3x^2}$       ✓

$= \frac{x^3}{3} - \frac{1}{3} \int 3x^2(1+x^3)^{\frac{1}{2}} dx$        $= \frac{1}{3} \int (1-y^{\frac{1}{2}}) dy$       ✓

$= \frac{x^3}{3} - \frac{1}{3} \left[ \frac{(1+x^3)^{\frac{3}{2}}}{\frac{3}{2}} \right] + C$        $= \frac{1}{3} (y - \frac{2}{3} y^{\frac{3}{2}}) + C$       ✓

$= \frac{x^3}{3} - \frac{2}{9} (1+x^3)^{\frac{3}{2}} + C$        $= \frac{1}{3} [ (1+x^3) - \frac{2}{3} (1+x^3)^{\frac{3}{2}} ] + C$       ✓

(b)  $I = \int \frac{e^{-x}}{1-e^{-x}} dx$       ✓

$= \ln |1-e^{-x}| + C$       ✓

(c)  $I = \int \cos x + i \sin x dx$

$= \sin x - i \cos x + C$       ✓

$= -i (\cos x + i \sin x) + C$

$= -i \operatorname{cis} x + C$       ✓

[8]

7. (a)

$$\begin{aligned}\frac{d}{dx} [\ln \cos^2 2x] &= \frac{d}{dx} [2 \ln \cos 2x] \\ &= 2 \times -\frac{2 \sin 2x}{\cos 2x}\end{aligned}$$

✓

$$= -4 \tan 2x$$

✓

(b) Since,

$$\frac{d}{dx} [\ln \cos^2 2x] = -4 \tan 2x$$

$$\int -4 \tan 2x \, dx = \ln \cos^2 2x + c_1$$

✓

$$\therefore \int \tan 2x \, dx = -\frac{1}{4} \ln \cos^2 2x + c_2$$

✓

[4]

8. (a)

(6 marks)

(a) Find the exact gradient of the curve  $y = 4^{3x-5}$  at the point (2, 4).

(3 marks)

$$\ln y = (3x - 5) \ln 4$$

$$\frac{1}{y} \frac{dy}{dx} = 3 \ln 4$$

$$\frac{dy}{dx} = 3y \ln 4 \Big|_{y=4}$$

$$= 12 \ln 4$$

---

(b) Evaluate  $\int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$ .

(3 marks)

$$\text{NB If } y = e^{\sqrt{x}} \text{ then } y' = \frac{1}{2\sqrt{x}} e^{\sqrt{x}}$$

$$\begin{aligned} 2 \int_0^4 \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx &= 2 \left[ e^{\sqrt{x}} \right]_0^4 \\ &= 2(e^2 - 1) \end{aligned}$$



8. (a)

$$\tan \theta = \frac{100}{x}$$

✓

$$\therefore \frac{1}{\cos^2 \theta} \frac{d\theta}{dt} = - \frac{100}{x^2} \frac{dx}{dt}$$

✓

$$\frac{1}{\cos^2 \theta} \frac{d\theta}{dt} = - \frac{100}{x^2} \frac{dx}{dt}$$

$$\text{When } t = 0, x = \sqrt{(120^2 - 100^2)} = \sqrt{4400} \quad \checkmark$$

When  $t = 5$ , horizontal distance covered by kite =  $10 \times 5 = 50$  m.

Hence, when  $t = 5$ ,  $x = \sqrt{4400} + 50 = 116.3325$  m.

$$\therefore \tan \theta = \frac{100}{116.3365} \rightarrow \theta = 40.68^\circ$$

✓

$\therefore$  when  $t = 5$ ,

$$\frac{1}{\cos^2 40.68^\circ} \frac{d\theta}{dt} = - \frac{100}{116.3325^2} \times 10$$

$$= -0.042 \text{ radians per second}$$

✓

(b)

$$s^2 = 100^2 + x^2$$

✓

$$\therefore 2s \frac{ds}{dt} = 2x \frac{dx}{dt}$$

✓

$$\frac{ds}{dt} = \frac{x}{s} \frac{dx}{dt}$$

$$\text{At } t = 5, x = 116.3325, s = \sqrt{(100^2 + 116.3325^2)} = 153.4055 :$$

✓

$$\therefore \frac{ds}{dt} = 7.58 \text{ m/sec}$$

✓

[9]

9. (a) (i)

$$\sqrt{2} \operatorname{cis} \frac{\pi}{4} \times 2 \operatorname{cis} \frac{\pi}{3} = 2\sqrt{2} \operatorname{cis} \frac{7\pi}{12}$$

✓✓

(ii)

---


$$2(1+i) \left( \frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = (1-\sqrt{3}) + (1+\sqrt{3})i$$

✓✓

(iii)

$$2\sqrt{2} \operatorname{cis} \frac{7\pi}{12} = (1-\sqrt{3}) + (1+\sqrt{3})i$$

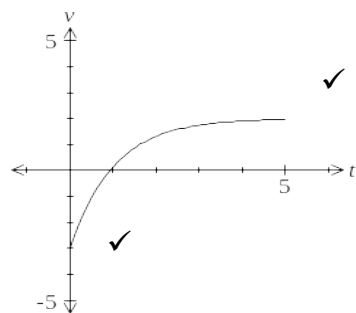
✓

$$2\sqrt{2} \left( \frac{\cos 7\pi}{12} + i \frac{\sin 7\pi}{12} \right) = (1-\sqrt{3}) + (1+\sqrt{3})i$$

$$\text{Hence, } \frac{\cos 7\pi}{12} = \frac{1}{2\sqrt{2}}(1-\sqrt{3}) \text{ or } \frac{\sqrt{2}}{4}(1-\sqrt{3})$$

✓

- (b) (i)  $2 + i$  and  $-2i$ . ✓✓
- (ii)  $f(z) = k(z - 2i)(z + 2i)(z - 2 + i)(z - 2 - i)$  ✓  
 $= k(z^2 - 4z + 9z^2 - 16z + 20)$  ✓  
 $f(1) = 20 \Rightarrow 20 = k(10)$   
Hence,  $k = 2$  ✓  
 $f(z) = 2z^4 - 8z^3 + 18z^2 - 32z + 40$   
 $\therefore a = 2, b = -8, c = 18, d = -32, e = 40$  ✓✓ [13]
10. (a) At the y-axis  $y = -\pi$  ✓  
 $\frac{dy}{dx} = \cos 2xy \left[ 2x \frac{dy}{dx} + 2y \right] + \pi(2\cos x \sin x)$  ✓✓  
 $\therefore \frac{dy}{dx} = (1) \times [-2\pi] - 0 = -2\pi$  ✓
- (b) If perpendicular  $m = \frac{1}{2\pi}$  ✓  
 $\therefore y = \frac{1}{2\pi}x - \pi$  ✓ [6]
11.  
 $z = \frac{2(a - i)(1 + i) + (3 - i)(1 - i)}{(1 - i)(1 + i)}$  ✓✓  
 $= (a + 2) + (a - 3)i$  ✓  
 $|z| = 5 \Rightarrow (a + 2)^2 + (a - 3)^2 = 25$  ✓  
 $a^2 - a - 6 = 0$  ✓  
 $a = -2, 3$  ✓✓ [7]
12. (a)



(b) Moves with constant velocity of 2 metres per second. ✓

(c)  $t = 0.91629 = 0.92$  seconds (Use GC, root) ✓

(d)  $\int_0^{0.91629} v \, dt = \int_{0.91629}^k v \, dt$  ✓

$$\therefore \left[ 5e^{-t} + 2t \right]_{0.91629}^k = 1.16742$$

-k

$$1.16742 = 5e^{-k} + 2k - 3.83258$$

$$k = 2.23 \quad (\text{use GC Solver}) \quad \checkmark$$

(e) Distance travelled  $= \int_0^5 |v| \, dt$  ✓

$$= 7.3685 = 7.4 \text{ metres} \quad \checkmark \checkmark \quad [11]$$

13. (a)  $x^3 - 5x^2 + 6x = 0$  ✓

$$x(x - 2)(x - 3) = 0 \quad \checkmark$$

$$\Rightarrow x = 0, 2, 3 \quad \checkmark$$

(b)

$$V = \pi \int_0^2 x^3 - 5x^2 + 6x \, dx + \pi \int_3^4 x^3 - 5x^2 + 6x \, dx$$

✓✓

$$= \pi \left[ \frac{x^4}{4} - \frac{5x^3}{3} + 3x^2 \right]_0^2 + \pi \left[ \frac{x^4}{4} - \frac{5x^3}{3} + 3x^2 \right]_3^4$$

✓

$$= \frac{23\pi}{4}$$

✓

[7]

14.

$$\tan \theta = \frac{x}{2} \rightarrow x = 2 \tan \theta \quad \text{and} \quad \delta \theta = 0.01 \times \frac{\pi}{180}$$

✓

$$\therefore \frac{dx}{d\theta} = \frac{2}{\cos^2 \theta}$$

✓

$$\therefore \delta x = \frac{2}{\cos^2 \theta} \delta \theta$$

✓

$$= \frac{2}{\cos^2 2^\circ} \times 0.01 \times \frac{\pi}{180}$$

✓

-4

$$= 3.4949 \times 10^{-4} \text{ km}$$

$$= 0.349 \text{ metres}$$

✓

[5]

15. A line

(a)

Intersect when

$$\begin{bmatrix} -1 \\ 3 - 2\lambda \\ 2 + 4\lambda \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 33$$

$$\therefore \lambda = 3$$

$$\text{Hence point of intersection at } \begin{bmatrix} -1 \\ -3 \\ 14 \end{bmatrix}$$

---

(b) Find the acute angle between the line and plane.

(3 marks)

Find angle between line and normal to plane:

$$\cos \theta = \frac{\begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} \right\| \times \left\| \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\|}$$

$$\theta = 26.565$$

Hence angle between line and plane  $= 90 - 26.6 = 63.4^\circ$

16.

(10 marks)

A body, A, has an initial position of  $\begin{bmatrix} -7 \\ 21 \\ 6 \end{bmatrix}$  metres and is moving with a constant velocity of  $\begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$  metres per second.

- (a) A second body, B, is moving with constant velocity of  $\begin{bmatrix} 8 \\ 1 \\ -1 \end{bmatrix}$  metres per second and collides with body A after six seconds.

Determine the initial distance apart of body A and body B.

(4 marks)

A and B collide at

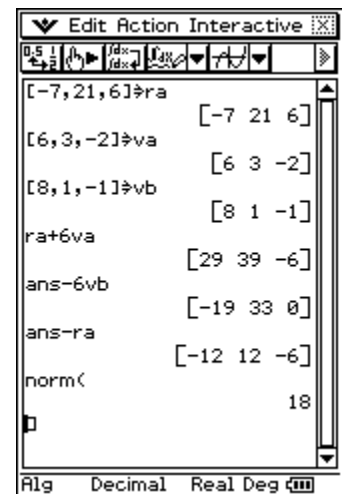
$$\begin{bmatrix} 7 \\ -21 \\ 6 \end{bmatrix} + 6 \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 29 \\ 39 \\ -6 \end{bmatrix}$$

Hence initial position of B is at

$$r_B + 6 \begin{bmatrix} 8 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 29 \\ 39 \\ -6 \end{bmatrix} \Rightarrow r_B = \begin{bmatrix} -19 \\ 33 \\ 0 \end{bmatrix}$$

Distance apart of A and B is

$$\left\| \begin{bmatrix} -19 \\ 33 \\ 0 \end{bmatrix} - \begin{bmatrix} -7 \\ 21 \\ 6 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -12 \\ 12 \\ -6 \end{bmatrix} \right\| = 18$$



- (b) A third body, C, is initially located at  $\begin{bmatrix} 5 \\ -10 \\ 1 \end{bmatrix}$  metres and is also moving with a constant velocity  $\begin{bmatrix} 2 \\ y \\ -3 \end{bmatrix}$ . After five seconds, the distance between bodies A and C is a minimum.

Find the value of  $y$  for which the speed of C is also a minimum.

(6 marks)

Let  $\mathbf{v}$  and  $\mathbf{r}$  be velocity and displacement of A relative to C

Then minimum distance apart when  $\mathbf{v}$  and  $\mathbf{r} + 5\mathbf{v}$  are perpendicular, ie the dot product of  $\mathbf{v}$  and  $\mathbf{r} + 5\mathbf{v}$  is zero.

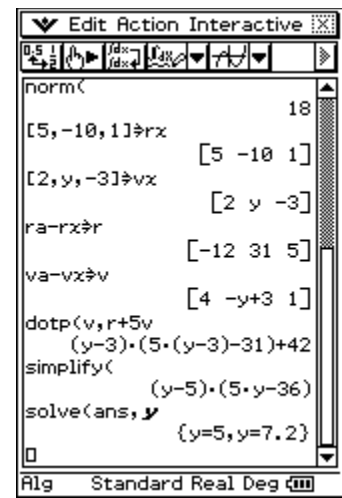
$$\mathbf{v} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ y \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3-y \\ 1 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} -7 \\ 21 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ -10 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 31 \\ 5 \end{bmatrix}$$

$$\mathbf{r} + 5\mathbf{v} = \begin{bmatrix} 8 \\ 46-5y \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 3-y \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 46-5y \\ 10 \end{bmatrix} = 5y^2 - 61y + 180$$

$$(y-5)(5y-36)=0 \text{ when } y=5 \text{ or } y=7.2$$





17. (a)

$$v = -\pi \sin \frac{\pi t}{2} + \pi \sin \left( 1 + \frac{\pi t}{2} \right)$$

✓

$$\begin{aligned} a &= -\frac{\pi^2}{2} \cos \frac{\pi t}{2} + \frac{\pi^2}{2} \cos \left( 1 + \frac{\pi t}{2} \right) \\ &= -\frac{\pi^2}{4} x \quad \therefore \text{SHM} \end{aligned}$$

✓

(b)

$$\begin{aligned} v &= 0 \text{ when } -\pi \sin \frac{\pi t}{2} + \pi \sin \left( 1 + \frac{\pi t}{2} \right) = 0 \\ \therefore \sin \frac{\pi t}{2} &= \sin \left( 1 + \frac{\pi t}{2} \right) \end{aligned}$$

✓

$$\therefore t = 0.68$$

✓

(c) Maximum velocity = 3.01

✓✓

$$(d) \quad a = -\frac{\pi^2}{4}(2) = -\frac{\pi^2}{2}$$

✓✓

[8]

18. (a)  $2(-0.5) + A = 0 \rightarrow A = 1$ 

✓

$$1 - B = 0 \rightarrow B = 1$$

✓

$$f(0) = -\frac{C}{-B} = 2 \quad \therefore C = 2$$

✓

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(b) Using GC

D(-0.90, -0.97), E(2.15, 0), F(-0.18, 1.80)

✓✓✓

[6]

19.

**Question 14**

**(8 marks)**

The temperature in a restaurant cool room is set to 4°C. One day, the refrigerator unit was turned back on after the temperature in the cool room had risen to 27°C due to cleaning and maintenance work. After 15 minutes, the temperature in the cool room had dropped to 11°C, with the temperature,  $T$ , falling according to the model

$$\frac{dT}{dt} = k(T - 4)$$

where  $t$  is the time in minutes since the refrigerator unit was turned back on.

(a) Find the value of  $k$  and express  $T$  as a function of  $t$ .

**(5 marks)**

$$\begin{aligned}\int \frac{1}{T - 4} dT &= \int k dt \\ \ln(T - 4) &= kt + c \\ T &= ae^{kt} + 4 \\ T(0) &= 27 \Rightarrow a = 23 \\ T(15) &= 11 \Rightarrow 11 = 23e^{15k} + 4 \Rightarrow k = -0.0793 \\ T &= 23e^{-0.0793t} + 4\end{aligned}$$

- (b) If the temperature continues to fall in this way, how long before the temperature in the cool room registers  $4^{\circ}\text{C}$ , to the nearest degree? (3 marks)

Temperature must drop to  $4.5^{\circ}\text{C}$

$$4.5 = 23e^{-0.0793t} + 4$$

$$t = 48.3$$

After 48.3 minutes.

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