Semester Two Examination, 2019

Question/Answer booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section One: Calculator-free

If required by your examination administrator, please place your student identification label in this box

Student number: In figures [ ][ ] [ ][ ][ ] [ ][ ][ ]

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

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4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Markers use only		
Question	Maximum	Mark
1	4	
2	6	
3	7	
4	7	
5	7	
6	8	
7	6	
8	7	
S1 Total	52	
S1 Wt (×0.6731)	35%	
S2 Wt	65%	
Total	100%	

(c) Interpret the value of a and the value of b in the context of this model. (2 marks)

Solution
$a = 1\,000$ represents the initial population $b \approx 1.02$is the growth constant - the population is growing by approximately 2% per year.
Specific behaviours
✓ interpretation for a ✓ interpretation for b that includes annual growth rate

(d) Use the model to determine

(i) the population when $t = 45$. (1 mark)

Solution
$P = 1000(1.02)^{45} \approx 2\,443$
Specific behaviours
✓ correct value

(ii) the number of years for the population to reach $10\,000$. (1 mark

Solution
$10000 = 1000(1.02)^t$ $t \approx 116$ years
Specific behaviours
✓ number of years

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(4 marks)**

Determine the following:

(a) $\int 12(2x+1)^2\,dx$. (2 marks)

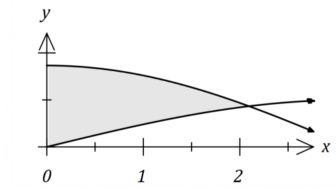
(b) $\dfrac{d}{dx}\cos(2x+1)$. (1 mark)

(c) $\dfrac{d}{dx}\int_3^x (2t+1)\,dt$. (1 mark)

Question 2 **(6 marks)**

Let $f(x)=\sqrt{3}\cos\left(\frac{x}{2}\right)$ and $g(x)=\sin\left(\frac{x}{2}\right)$.

The shaded region on the graph below is enclosed by $x=0$, $y=f(x)$ and $y=g(x)$.

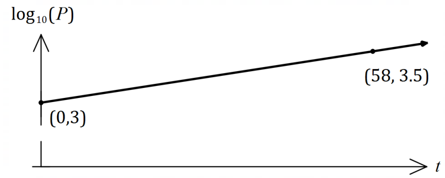


(a) Show that $f\left(\frac{2\pi}{3}\right)=g\left(\frac{2\pi}{3}\right)$. (2 marks)

(b) Determine the area of the shaded region. (4 marks)

Question 21 **(9 marks)**

The population of a species P can be modelled by the equation $P=ab^t$, where a and b are constants and t is the number of years since the population was first recorded. The graph below shows the linear relationship between t and $\log_{10}P$ for the population over the past 60 years and passes through the points $(0,3)$ and $(58,3.5)$.



(a) Write an equation relating $\log_{10}P$ and t. (2 marks)

Solution
$\frac{3.5-3}{58-0}=\frac{1}{116}$ $\log_{10}P=\frac{1}{116}t+3$
Specific behaviours
✓ gradient   ✓ equation

(b) Determine the value of a and the value of b. (3 marks)

Solution
$P=ab^t \Rightarrow \log_{10}P=\log_{10}a+t\log b$ $\log_{10}a=3 \Rightarrow a=10^3=1000$ $\log_{10}b=\frac{1}{116} \Rightarrow b=10^{\left(\frac{1}{116}\right)}\approx 1.020$
Specific behaviours
✓ forms log equation   ✓ value of a

✓ value of

Question 20 **(9 marks)**

Researchers in a large city wish to determine a 95% confidence interval for p, the proportion of citizens who had used the city library at least once during the previous year. The margin of error of the interval is to be no more than 6%.

(a) If the researchers had no reliable estimate for p, determine the sample size they should take, noting **all** assumptions made. (5 marks)

Solution
$z_{95\%}=1.96, E=0.06, p=0.5$ $n=\dfrac{1.96^2(0.5)(0.5)}{0.06^2}=267$ Assumed that: 1. $p=0.5$ for conservative estimate.   2. Sample will be a simple random sample of citizens.   3. Sample is large enough so that the sampling distribution is close approximation to a normal distribution.
Specific behaviours
✓ uses correct parameters   ✓ calculates sample size   ✓ notes assumed value for p ✓ notes need for a random sample   ✓ notes need for large enough sample

(b) The researchers were given access to data from a random sample of 165 citizens collected a few years earlier. Of these, 36 had used the city library at least once during the previous year.

(i) Determine the margin of error for a 95% confidence interval for p based on this sample. (2 marks)

Solution
$z_{95\%}=1.96, n=165, \hat{p}=36\div165=0.218$ $E=1.96\sqrt{\dfrac{0.218(1-0.218)}{165}}=0.063$
Specific behaviours
✓ uses correct parameters   ✓ calculates margin of error

(ii) The researchers used this data to decrease the sample size calculated in part (a). By how much did the sample size decrease? (2 marks)

Solution
$n=\dfrac{1.96^2(0.218)(1-0.218)}{0.06^2}=182$ Decrease is $267-182=85$
Specific behaviours
✓ new sample size   ✓ decrease

Question 3 **(7 marks)**

(a) Write $2\log_4 3-\log_4 6-1$ in the form $\log_4 k$. (3 marks)

(b) Solve for x the equation $e^{2x-3}=9$. (2 marks)

(c) Determine $\dfrac{d}{dx}\left(\log_e\left(\dfrac{1}{x^6+5}\right)\right)$. (2 marks)

Question 4 **(7 marks)**

The velocity of a small body moving in a straight line at time t seconds is given by

$$v=\frac{8}{t+3}\text{ m/s}, t\geq 0.$$

(a) Determine the velocity of the body when its acceleration is -0.5 m/s^2. (4 marks)

(b) Calculate the distance travelled by the body in the first 3 seconds. (3 marks)

Question 19 **(9 marks)**

A particle moves along the x-axis with initial position $x(0)=1.5$ m and velocity $v(0)=-2.4$ m/s.

The acceleration of the particle after t seconds is given by $a(t)=m-0.4t$ m/s^2.

Between $t=2$ and $t=5$ the particle undergoes a change in displacement of 69 m.

(a) Determine the value of the constant m. (4 marks)

Solution
$v(t)=\int a(t)\,dt¿mt-0.2t^2-2.4$ $\Delta x=\int_2^5 v(t)\,dt¿10.5m-15$ $10.5m-15=69\Rightarrow m=8$
Specific behaviours
✓ expression for velocity   ✓ expression for Δx ✓ evaluates Δx ✓ correct value

(b) Determine

(i) the maximum velocity of the particle. (2 marks)

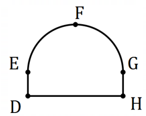
Solution
$a(t)=0\Rightarrow 8-0.4t=0\Rightarrow t=20$ $v(20)=7(20)-0.3(20)^2-2.4=77.6\text{ m/s}$
Specific behaviours
✓ solves $a=0$ for t ✓ correct velocity

(ii) the distance of the particle from the origin after 3 seconds. (3 marks)

Solution
$\Delta x=\int_0^3 v(t)\,dt=27$ $x(3)=27+1.5=28.5\text{ m}$
Specific behaviours
✓ indicates method   ✓ change in displacement   ✓ correct distance

Question 18 **(7 marks)**

When seen from above, an evaporation tank of area 480 m^2 has the shape of rectangle $EDGH$ and semicircle EFG of radius r.



(a) If length $DE=x$, express x in terms of r and hence show that the perimeter, P m, of the tank is given by (3 marks)

$$P=\frac{480}{r}+2r+\frac{\pi r}{2}$$

Solution
$A=480=2rx+\frac{\pi r^2}{2}\Rightarrow x=\frac{240}{r}-\frac{\pi r}{4}$ $P=\pi r+2r+2x¿\pi r+2r+2\left(\frac{240}{r}-\frac{\pi r}{4}\right)$ $¿\frac{480}{r}+2r+\frac{\pi r}{2}$
Specific behaviours
✓ expression for area   ✓ transposes for x ✓ substitutes into expression for perimeter

(b) Use a calculus method to determine the minimum perimeter of the tank. (4 marks)

Solution
$\frac{dP}{dx}=\frac{-480}{r^2}+2+\frac{\pi}{2}$ $\frac{dP}{dx}=0\Rightarrow r^2=\frac{960}{4+\pi}\Rightarrow r=\frac{8\sqrt{15}}{\sqrt{\pi+4}}\,(\approx 11.59)$ $P_{MIN}=8\sqrt{15\pi+60}\,\text{m}\ (\approx 82.8\,\text{m})$
Specific behaviours
✓ computes first derivative   ✓ solves equal to zero   ✓ chooses positive root   ✓ correct minimum perimeter

Question 5 **(7 marks)**

The random variable X has probability density function $f(x)$ shown below, where k is a positive constant.

$$f(x)=\begin{cases} kx+\frac{1}{30} & ¿0\le x\le 5 \\ 0 & ¿\text{elsewhere} \end{cases}$$

(a) Deduce that $k=\frac{1}{15}$. (3 marks)

(b) Determine the value of a if $P(1<X<a)=\frac{3}{5}$. (4 marks)

Question 6 **(8 marks)**

Let $f(x)=(1+x)e^{-3x}$.

(a) Determine the coordinates of the stationary point of the graph of $y=f(x)$ and use the second derivative test to determine its nature. (6 marks)

(b) Determine the coordinates of the point of inflection of the graph of $y=f(x)$. (2 marks)

Question 17 **(7 marks)**

A citrus farm grows Verna lemons. Their weights are normally distributed with a mean of 153 g and a standard deviation of 9.4 g.

(a) Determine the probability that

(i) a randomly chosen lemon has a weight less than 150 g. (1 mark)

Solution
$P(W<150)=0.3748$
Specific behaviours
✓ correct probability

(ii) in a random sample of 10 lemons, exactly 1 has a weight less than 150 g. (2 marks)

Solution
$X\quad B(10,0.3748)$ $P(X=1)=0.0547$
Specific behaviours
✓ indicates distribution with parameters ✓ correct probability

The farm classifies their lemons by size, so that the ratio of the number of small to medium to large lemons is $2:4:5$.

(b) Determine the upper and lower bounds for the weight of a medium sized lemon. (2 marks)

Solution
$P(W<l)=\frac{2}{11}\Rightarrow l=144.5\text{ g}$ $P(W<u)=\frac{6}{11}\Rightarrow u=154.1\text{ g}$ Hence $144.5\le w\le 154.1$ g
Specific behaviours
✓ indicates correct method ✓ correct bounds

(c) Determine the probability that when lemons are picked at random, the first medium lemon is chosen on the 6^{th} pick. (2 marks)

Solution
$P=\left(\frac{7}{11}\right)^5\left(\frac{4}{11}\right)\approx 0.0379$
Specific behaviours
✓ indicates correct method ✓ correct probability

(c) Determine the coordinates of the maximum of the graph of $y=A(x)$. (2 marks)

Solution
Maximum at $x=20$. $A(20)=\frac{1}{2}(15)(5)-12.5=37.5-12.5=25$. At $(20,25)$.
Specific behaviours
✓ x-coordinate

correct coordinates

(d) Determine the root of the graph of $y=A(x)$ for $x>10$. (2 marks)

Solution
$A(k)=0$ $25+\frac{1}{2}(6)(-3)+(k-26)(-3)=0\Rightarrow k=\frac{94}{3}$ Root when $x=\frac{94}{3}$.
Specific behaviours
✓ indicates area above¿area below axis

correct root

Question 7 **(6 marks)**

Let $f(x)=\frac{x}{x+1}$.

(a) Determine $f(x)$ and $f(x+\delta x)$ when $x=70$ and $\delta x=5$. (1 mark)

(b) Use $f(x)$ and the increments formula to estimate the difference between $\frac{89}{90}$ and $\frac{92}{93}$. (5 marks)

Question 8 **(7 marks)**

In a class of 21 students, 18 are right-handed.

(a) One student is selected at random from the class and the random variable X is the number of right-handed students in the selection. Determine the mean and standard deviation of X. (3 marks)

(b) Two students are selected at random from the class without replacement and the random variable Y is the number of right-handed students in the selection.

(i) Complete the probability distribution table below. (3 marks)

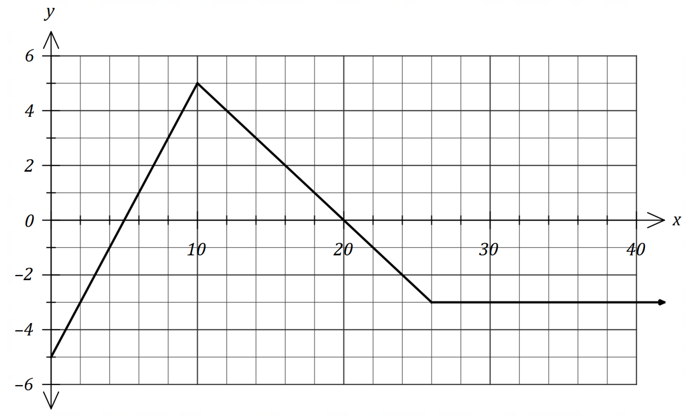
y	0	1	2
$P(Y=y)$			

(ii) Determine $E(Y)$. (1 mark)

End of questions

Question 16 **(8 marks)**

The graph of $y=f(x)$ is shown below.



(a) Determine $\int_3^9 f(x)\,dx$. (2 marks)

Solution
$\frac{1}{2}(4)(4)+\frac{1}{2}(2)(-2)=8-2=6$
Specific behaviours
✓ uses difference of areas ✓ correct value

Let $A(x)=\int_0^x f(t)\,dt$.

(b) Determine

(i) $A(5)$. (1 mark)

Solution
$\frac{1}{2}(5)(-5)=-12.5$
Specific behaviours
✓ correct value

(ii) $A'(5)$. (1 mark)

Solution
$A'(5)=f(5)=0$
Specific behaviours
✓ correct value

End of questions

Question 15 **(8 marks)**

It is known that 80% of a large population of animals carry microfilariae in their blood (are carriers). A student must simulate selecting animals that either are or are not carriers.

(a) Describe a method that the student could use. (2 marks)

Solution
Examples Use 5-sided spinner marked 1-5: 1-4 is carrier, 5 is not carrier. Use dice: 1-4 is carrier, 5 is not carrier, 6 ignore. Use random number generator, balls in hat, etc, etc.
Specific behaviours
✓ describes method ✓ indicates how long-term success of 80% is achieved

(b) The random variable X is the number of animals in a random sample of size 200 that are carriers. Describe the distribution of X and determine $E(X)$. (2 marks)

Solution
$X \sim B(200, 0.8)$ $E(X) = 200 \times 0.8 = 160$
Specific behaviours
✓ distribution with parameters ✓ expected value

225 students carry out the simulation so that they each have a sample of size 200. Then each student calculates $\hat{p}$, the proportion of animals in their sample that are carriers. The distribution of these 225 values of $\hat{p}$ will be approximately normal.

(c) Determine the parameters of the normal distribution the 225 values of $\hat{p}$ will approximate. (2 marks)

Solution
$\mu = 0.8$ $\sigma^2 = \frac{0.8(1-0.8)}{200} = \frac{1}{1250} = 0.0008 \; (\sigma \approx 0.0283)$
Specific behaviours
✓ mean ✓ variance (or sd)

(d) Briefly describe how the closeness of the normal approximation would change if

(i) the sample size was larger. (1 mark)

Solution
Becomes closer. ($np(1-p)$ increases)
Specific behaviours
✓ indicates closer

(ii) the percentage of animals that are carriers was higher. (1 mark)

Solution
Becomes less close. ($np(1-p)$ decreases as p moves away from 0.5)
Specific behaviours
✓ indicates less close

See next page

Semester Two Examination, 2019

Question/Answer booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section Two: Calculator-assumed

If required by your examination administrator, please place your student identification label in this box

Student number: In figures

In words ____________________

Your name ____________________

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To be provided by the candidate
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End of questions

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Markers use only		
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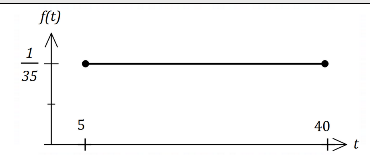
10	6	
11	6	
12	6	
13	8	
14	9	
15	8	
16	8	
17	7	
18	7	
19	9	
20	9	
21	9	
S2 Total	98	
S2 Wt (×0.6633)	65%	

Question 14 **(9 marks)**

The time taken to answer a customer call at a large business can be modelled by the continuous random variable T that is uniformly distributed between 5 and 40 seconds.

(a) Sketch a diagram of the associated probability density function for T. (3 marks)

Solution



Specific behaviours

✓ indicates scale on y-axis
✓ indicates endpoints on x-axis
✓ horizontal line segment

(b) Determine $P(T<32|T>11$¿. (2 marks)

Solution

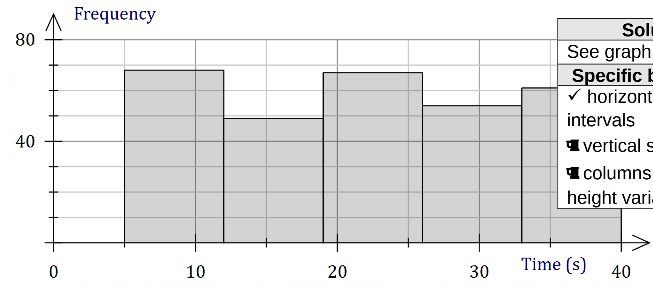
$$P=\frac{P(11<T<32)}{P(T>11)}=\frac{32-11}{40-11}=\frac{21}{29}$$

Specific behaviours

✓ indicates correct method
✓ correct probability

(c) A simulation involves taking a random sample from the uniform distribution, recording the time and repeating a total of 300 times. The times are then grouped into 5 equal width classes, from which a frequency histogram is constructed.

(i) Sketch a likely histogram on the axes below. (3 marks)



Solution

See graph.

Specific behaviours

✓ horizontal scale and intervals
✓ vertical scale
✓ columns with slight height variations

(ii) Briefly explain how your sketch would change if the simulation was repeated a second time. (1 mark)

Solution

Would expect about the same again, but with slightly different frequencies in each class.

Specific behaviours

✓ reasonable explanation

Question 13 **(8 marks)**

The table below shows the probability distribution for a random variable X.

x	0	1	2	3
$P(X=x)$	$2k^2+2k$	k^2	$2k^2+k$	k

(a) Determine the value of the constant k. (2 marks)

Solution
$2k^2+2k+k^2+2k^2+k+k=1$ $5k^2+4k-1=0$ $(5k-1)(k+1)=0 \Rightarrow k=\frac{1}{5}$
Specific behaviours
✓ sums probabilities to 1 ✓ solves equation for k

(b) Determine $E(X)$ and $Var(X)$. (3 marks)

Solution
$P(X=0)=\frac{12}{25}, P(X=1)=\frac{1}{25}, P(X=2)=\frac{7}{25}, P(X=3)=\frac{5}{25},$ $E(X)=\frac{6}{5}=1.2, Var(X)=\frac{38}{25}=1.52$
Specific behaviours
✓ evaluates probabilities ✓ correct mean ✓ correct variance

(c) Given that $E(aX+b)=5$ and $Var(aX+b)=38$, determine all possible values of the constants a and b. (3 marks)

Solution
Using variance: $\frac{38}{25}\times a^2=38 \Rightarrow a=\pm 5$ Using mean: $5\left(\frac{6}{5}\right)+b=5 \Rightarrow b=-1$ or $-5\left(\frac{6}{5}\right)+b=5 \Rightarrow b=11$ $(a=5, b=-1)$ or $(a=-5, b=11)$
Specific behaviours
✓ one value for a ✓ one value for b

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(6 marks)**

The graph of $y=f(x)$, where $f(x)=5\log_e(x-a)$, has a root at $x=5$.

(a) Determine the value of the constant a and hence state the equation of the asymptote of the graph. (2 marks)

(b) Determine the exact coordinates of the point on the graph where $f'(x)=\frac{1}{5}$. (3 marks)

(c) The graph of the equation $y=f(x)$ is the same shape as the graph with the equation $y=\log_e g(x)$. State a suitable function for $g(x)$. (1 mark)

Question 10 **(6 marks)**

A machine is set to fill bottles with more than the stated capacity. The random variable X mL is the amount it overfills bottles and has probability density function $f(x)$ shown below.

$$f(x)=\begin{cases}\frac{\sqrt{3x-6}}{6} & ¿2\leq x\leq 5¿\,\text{otherwise}¿ \\ ¿ & ¿\end{cases}$$

(a) Determine $E(X)$. (2 marks)

(b) Determine $Var(X)$. (2 marks)

(c) The amount another machine overfills bottles is given by $Y=1+0.25X$. Determine

(i) $E(Y)$. (1 mark)

(ii) $Var(Y)$. (1 mark)

Question 12 **(6 marks)**

An opinion poll found that 206 out of 358 people supported a policy to increase the minimum wage, from which a 99% approximate confidence interval for the population proportion was calculated to be

$$(0.508, 0.643)$$

(a) Show how this interval was calculated. (4 marks)

Solution
$\hat{p}=\frac{206}{358}\approx 0.5754$ $\sigma_{\hat{p}}=\sqrt{\frac{(0.5754)(1-0.5754)}{358}}\approx 0.0261$ $z_{0.99}\approx 2.576\, E=2.576\times 0.0261\approx 0.0673$ $0.5754\pm 0.0673=(0.5081, 0.6427)\approx(0.508, 0.643)$
Specific behaviours
✓ indicates proportion   ✓ indicates standard deviation   ✓ uses z-score for 95% to determine margin of error   ✓ uses margin of error to obtain interval

(b) If 18 similar opinion polls were taken and each time a 99% confidence interval calculated, determine the probability that all 18 intervals contain the true population proportion. (2 marks)

Solution
Let the rv X be the # of intervals containing the true proportion, then $X \quad B(18, 0.99)$ $P(X=18)=0.8345$
Specific behaviours
✓ indicates distribution with parameters   ✓ correct probability

Question 11 **(6 marks)**

A water tank sprung a leak. The amount of water W remaining in the tank t minutes after the leak began can be modelled by the equation $W = 30e^{-kt}$ kilolitres, where k is a constant.

3.5 kL of water was lost from the tank in the first 10 minutes.

(a) Determine the value of k. (2 marks)

Solution
$30 - 3.5 = 30e^{-10k}$ $k = 0.0124$
Specific behaviours
✓ substitutes values into equation   ✓ solves for k

(b) How many kilolitres of water leaked from the tank during the first 2 hours? (2 marks)

Solution
$W(120) = 30e^{-0.0124(120)}$ ¿ 6.77 $30 - 6.77 = 23.23$ kL leaked.
Specific behaviours
✓ amount remaining   ✓ amount leaked

(c) At what time, to the nearest minute, was the instantaneous rate of water loss 186 litres per minute? (2 marks)

Solution
$\frac{dW}{dt} = -0.186 = -0.0124\,W$ $W = 15$ $15 = 30e^{-0.0124t}\ t = 55.9 \approx 56$ minutes
Specific behaviours
✓ calculates amount of water   ✓ solves for t

CALCULATOR-ASSUMED 15 METHODS UNITS 3 AND 4

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SN025-145-2 See next page

Question 12 **(6 marks)**

An opinion poll found that 206 out of 358 people supported a policy to increase the minimum wage, from which a 99% approximate confidence interval for the population proportion was calculated to be

$$(0.508, 0.643)$$

(a) Show how this interval was calculated. (4 marks)

(b) If 18 similar opinion polls were taken and each time a 99% confidence interval calculated, determine the probability that all 18 intervals contain the true population proportion. (2 marks)

Question 10 **(6 marks)**

A machine is set to fill bottles with more than the stated capacity. The random variable X mL is the amount it overfills bottles and has probability density function $f(x)$ shown below.

$$f(x)=\begin{cases}\dfrac{\sqrt{3x-6}}{6} & ¿2\le x\le 5¿\,\text{otherwise}¿ \\ ¿ & ¿\end{cases}$$

(a) Determine $E(X)$. (2 marks)

Solution
$\int_2^5 x\cdot f(x)\,dx=\frac{19}{5}=3.8\,\text{mL}$
Specific behaviours
✓ correct integral
✓ correct mean

(b) Determine $Var(X)$. (2 marks)

Solution
$\int_2^5 \left(x-\frac{19}{5}\right)^2\cdot f(x)\,dx=\frac{108}{175}\approx 0.6171\,\text{mL}^2$
Specific behaviours
✓ correct integral
✓ correct variance

(c) The amount another machine overfills bottles is given by $Y=1+0.25X$. Determine

(i) $E(Y)$. (1 mark)

Solution
$1+0.25\left(\frac{19}{5}\right)=\frac{39}{20}=1.95\,\text{mL}$
Specific behaviours
✓ correct mean

(ii) $Var(Y)$. (1 mark)

Solution
$(0.25)^2\times\frac{108}{175}=\frac{27}{700}\approx 0.0386\,\text{mL}^2$
Specific behaviours
✓ correct variance

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(6 marks)**

The graph of $y=f(x)$, where $f(x)=5\log_e(x-a)$, has a root at $x=5$.

(a) Determine the value of the constant a and hence state the equation of the asymptote of the graph. (2 marks)

Solution
$x-a=1 \Rightarrow a=5-1=4$ Asymptote is $x=4$.
Specific behaviours
✓ determines value of a ✓ equation of asymptote

(b) Determine the exact coordinates of the point on the graph where $f'(x)=\frac{1}{5}$. (3 marks)

Solution
$f'(x)=\frac{5}{x-4}$ $\frac{5}{x-4}=\frac{1}{5} \Rightarrow x=29$ $y=5\ln(29-4)=5\ln 25=5\ln 5^2=10\ln 5$ At $(29, 10\ln 5)$.
Specific behaviours
✓ indicates $f'(x)$ ✓ solves for x ✓ exact coordinates

(c) The graph of the equation $y=f(x)$ is the same shape as the graph with the equation $y=\log_e g(x)$. State a suitable function for $g(x)$. (1 mark)

Solution
$g(x)=(x-4)^5$
Specific behaviours
✓ correct function (accept translations)

Question 13 **(8 marks)**

The table below shows the probability distribution for a random variable X.

x	0	1	2	3
$P(X=x)$	$2k^2+2k$	k^2	$2k^2+k$	k

(a) Determine the value of the constant k. (2 marks)

(b) Determine $E(X)$ and $Var(X)$. (3 marks)

(c) Given that $E(aX+b)=5$ and $Var(aX+b)=38$, determine all possible values of the constants a and b. (3 marks)

Question 14 **(9 marks)**

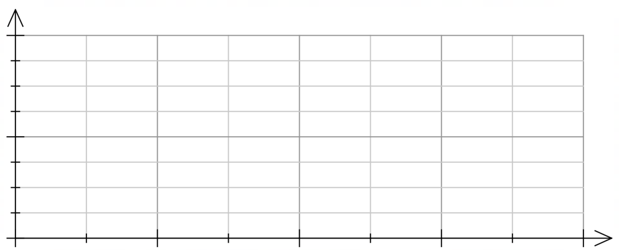
The time taken to answer a customer call at a large business can be modelled by the continuous random variable T that is uniformly distributed between 5 and 40 seconds.

(a) Sketch a diagram of the associated probability density function for T. (3 marks)

(b) Determine $P(T<32|T>11$¿. (2 marks)

(c) A simulation involves taking a random sample from the uniform distribution, recording the time and repeating a total of 300 times. The times are then grouped into 5 equal width classes, from which a frequency histogram is constructed.

(i) Sketch a likely histogram on the axes below. (3 marks)



(ii) Briefly explain how your sketch would change if the simulation was repeated a second time. (1 mark)

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

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Semester Two Examination, 2019

Question/Answer booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section Two: Calculator-assumed

SOLUTIONS

Student number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question 15 **(8 marks)**

It is known that 80% of a large population of animals carry microfilariae in their blood (are carriers). A student must simulate selecting animals that either are or are not carriers.

(a) Describe a method that the student could use. (2 marks)

(b) The random variable X is the number of animals in a random sample of size 200 that are carriers. Describe the distribution of X and determine $E(X)$. (2 marks)

225 students carry out the simulation so that they each have a sample of size 200. Then each student calculates $\hat{p}$, the proportion of animals in their sample that are carriers. The distribution of these 225 values of $\hat{p}$ will be approximately normal.

(c) Determine the parameters of the normal distribution the 225 values of $\hat{p}$ will approximate. (2 marks)

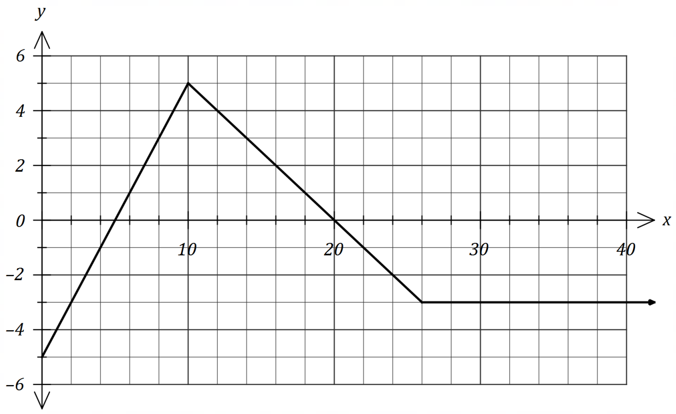
(d) Briefly describe how the closeness of the normal approximation would change if

(i) the sample size was larger. (1 mark)

(ii) the percentage of animals that are carriers was higher. (1 mark)

Question 16 **(8 marks)**

The graph of $y=f(x)$ is shown below.



(a) Determine $\int_3^9 f(x)\,dx$. (2 marks)

Let $A(x)=\int_0^x f(t)\,dt$.

(b) Determine

(i) $A(5)$. (1 mark)

(ii) $A'(5)$. (1 mark)

Question 8 **(7 marks)**

In a class of 21 students, 18 are right-handed.

(a) One student is selected at random from the class and the random variable X is the number of right-handed students in the selection. Determine the mean and standard deviation of X. (3 marks)

Solution

$$E(X)=p=\frac{18}{21}=\frac{6}{7}$$

$$Var(X)=p(1-p)=\frac{6}{7}\times\frac{1}{7}=\frac{6}{49}$$

Standard deviation ¿ $\sqrt{\frac{6}{49}}=\frac{\sqrt{6}}{7}$

Specific behaviours

✓ mean
✓ variance
✓ standard deviation

(b) Two students are selected at random from the class without replacement and the random variable Y is the number of right-handed students in the selection.

(i) Complete the probability distribution table below. (3 marks)

y	0	1	2
$P(Y=y)$	$1/70$	$18/70$	$51/70$

Solution

$$P(Y=2)=\frac{18}{21}\times\frac{17}{20}=\frac{6}{7}\times\frac{17}{20}=\frac{3}{7}\times\frac{17}{10}=\frac{51}{70}$$

$$P(Y=0)=\frac{3}{21}\times\frac{2}{20}=\frac{1}{7}\times\frac{1}{10}=\frac{1}{70}$$

$$P(Y=1)=1-\frac{51}{70}-\frac{1}{70}=\frac{18}{70}$$

Specific behaviours

✓ each correct probability

(ii) Determine $E(Y)$. (1 mark)

Solution

$$E(Y)=0+\frac{18}{70}+\frac{2(51)}{70}=\frac{120}{70}=\frac{12}{7}$$

Specific behaviours

✓ correct value

Question 7 **(6 marks)**

Let $f(x)=\frac{x}{x+1}$.

(a) Determine $f(x)$ and $f(x+\delta x)$ when $x=70$ and $\delta x=5$. (1 mark)

Solution
$f(70)=\frac{70}{71}, f(75)=\frac{75}{76}$
Specific behaviours
✓ both correct fractions

(b) Use $f(x)$ and the increments formula to estimate the difference between $\frac{89}{90}$ and $\frac{92}{93}$. (5 marks)

Solution
$f'(x)=\frac{1(x+1)-x(1)}{(x+1)^2}¿\frac{1}{(x+1)^2}$ Find δy when $x=89$ and $\delta x=3$. $\delta y \approx f'(x).\delta x \approx \frac{1}{(x+1)^2}\times\delta x \approx \frac{3}{90^2}\approx\frac{1}{2700}$ Difference is approximately $\frac{1}{2700}$.
Specific behaviours
✓ use of quotient rule for $f'(x)$ ✓ correct $f'(x)$ ✓ indicates values of x and δx ✓ uses increments formula   ✓ substitutes, simplifies and states difference

(c) Determine the coordinates of the maximum of the graph of $y=A(x)$. (2 marks)

(d) Determine the root of the graph of $y=A(x)$ for $x>10$. (2 marks)

Question 17 **(7 marks)**

A citrus farm grows Verna lemons. Their weights are normally distributed with a mean of 153 g and a standard deviation of 9.4 g.

(a) Determine the probability that

(i) a randomly chosen lemon has a weight less than 150 g. (1 mark)

(ii) in a random sample of 10 lemons, exactly 1 has a weight less than 150 g. (2 marks)

The farm classifies their lemons by size, so that the ratio of the number of small to medium to large lemons is $2:4:5$.

(b) Determine the upper and lower bounds for the weight of a medium sized lemon. (2 marks)

(c) Determine the probability that when lemons are picked at random, the first medium lemon is chosen on the 6^{th} pick. (2 marks)

Question 6 **(8 marks)**

Let $f(x)=(1+x)e^{-3x}$.

(a) Determine the coordinates of the stationary point of the graph of $y=f(x)$ and use the second derivative test to determine its nature. (6 marks)

Solution
$f'(x)=e^{-3x}-3(1+x)e^{-3x}$
$f'(x)=0\Rightarrow(-3x-2)e^{-3x}=0\Rightarrow x=\frac{-2}{3},\ y=\frac{e^2}{3}$
$f''(x)=-3e^{-3x}+3(-3x-2)e^{-3x}\text{¿}(9x+3)e^{-3x}$
$f''\left(\frac{-2}{3}\right)=-3e^2\Rightarrow\max$
Stationary point is at $\left(\frac{-2}{3},\frac{e^2}{3}\right)$ and is a maximum.
Specific behaviours
✓ correct $f'(x)$
■ equates $f'(x)$ to zero and obtains x-coordinate
■ obtains y-coordinate
■ obtains $f''(x)$
■ indicates sign of $f''(x)$ at point

(b) Determine the coordinates of the point of inflection of the graph of $y=f(x)$. (2 marks)

Solution
$(9x+3)e^{-3x}=0\Rightarrow x=\frac{-1}{3}$
$f\left(\frac{-1}{3}\right)=\frac{2e}{3}$
Point of inflection at $\left(\frac{-1}{3},\frac{2e}{3}\right)$
Specific behaviours
✓ solves $f''(x)=0$

Question 5 **(7 marks)**

The random variable X has probability density function $f(x)$ shown below, where k is a positive constant.

$$f(x)=\begin{cases} kx+\frac{1}{30} & \text{¿} 0\le x\le 5 \\ 0 & \text{¿elsewhere} \end{cases}$$

(a) Deduce that $k=\frac{1}{15}$. (3 marks)

Solution

$$\int_0^5 kx+\frac{1}{30}\,dx=\left[\frac{kx^2}{2}+\frac{x}{30}\right]_0^5 \text{¿} \frac{25k}{2}+\frac{1}{6}$$

$$\frac{25k}{2}+\frac{1}{6}=125k=\frac{5}{3}\Rightarrow k=\frac{1}{15}$$

Specific behaviours

✓ integrates $f(x)$
evaluates definite integral
equates to 1 and shows steps to solve for k

(b) Determine the value of a if $P(1<X<a)=\frac{3}{5}$. (4 marks)

Solution

$$\int_1^a \frac{x}{15}+\frac{1}{30}\,dx=\left[\frac{x^2}{30}+\frac{x}{30}\right]_1^a \text{¿} \frac{1}{30}(a^2+a)-\frac{1}{15}$$

$$\frac{1}{30}(a^2+a)-\frac{1}{15}=\frac{3}{5}a^2+a-20=0(a+5)(a-4)=0$$

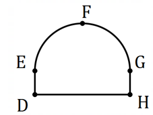
$$a=4$$

Specific behaviours

✓ integrates $f(x)$
evaluates definite integral
equates to probability and simplifies quadratic
factorises and states the only valid value of a

Question 18 **(7 marks)**

When seen from above, an evaporation tank of area 480 m^2 has the shape of rectangle $EDGH$ and semicircle EFG of radius r.



(a) If length $DE=x$, express x in terms of r and hence show that the perimeter, P m, of the tank is given by (3 marks)

$$P=\frac{480}{r}+2r+\frac{\pi r}{2}$$

(b) Use a calculus method to determine the minimum perimeter of the tank. (4 marks)

Question 19 **(9 marks)**

A particle moves along the x-axis with initial position $x(0)=1.5$ m and velocity $v(0)=-2.4$ m/s.

The acceleration of the particle after t seconds is given by $a(t)=m-0.4t$ m/s^2.

Between $t=2$ and $t=5$ the particle undergoes a change in displacement of 69 m.

(a) Determine the value of the constant m. (4 marks)

(b) Determine

(i) the maximum velocity of the particle. (2 marks)

(ii) the distance of the particle from the origin after 3 seconds. (3 marks)

Question 4 **(7 marks)**

The velocity of a small body moving in a straight line at time t seconds is given by

$$v=\frac{8}{t+3}\text{ m/s}, t\geq 0.$$

(a) Determine the velocity of the body when its acceleration is -0.5 m/s^2. (4 marks)

Solution
$\frac{dv}{dt}=\frac{d}{dt}\left((t+3)^{-1}\right)¿-8(t+3)^{-2}$ $\frac{-1}{2}=\frac{--8}{(t+3)^2}(t+3)^2=16t=-3\pm4t=1$ $v(1)=8\div4=2\text{ m/s}$
Specific behaviours
✓ correctly differentiates   ✓ equates to required value and simplifies   ✓ indicates time   ✓ correct velocity

(b) Calculate the distance travelled by the body in the first 3 seconds. (3 marks)

Solution
$\int_0^3 \frac{8}{t+3}dt=[8\ln(t+3)]_0^3¿8(\ln6-\ln3)$ $¿8\ln2\text{ m}$
Specific behaviours
✓ writes definite integral   ✓ correct antiderivative   ✓ substitutes bounds and simplifies

Question 3 **(7 marks)**

(a) Write $2\log_4 3 - \log_4 6 - 1$ in the form $\log_4 k$. (3 marks)

Solution
$2\log_4 3 - \log_4 6 - 1 = 2\log_4 3 - \log_4 6 - \log_4 4$ $¿\log_4 3^2 - (\log_4 6 + \log_4 4) ¿ \log_4 \left(\frac{9}{6\times 4}\right) ¿ \log_4 \left(\frac{3}{8}\right)$
Specific behaviours
✓ uses $\log_a a = 1$ ✓ uses $x\log_a y = \log_a y^x$ ✓ uses $\log_a x \pm \log_a y$

(b) Solve for x the equation $e^{2x-3} = 9$. (2 marks)

Solution
$2x - 3 = \ln 9 2x = 2\ln(3) + 3$ $x = \ln(3) + 1.5$
Specific behaviours
✓ expresses using natural log   ✓ simplifies

(c) Determine $\frac{d}{dx}\left(\log_e\left(\frac{1}{x^6+5}\right)\right)$. (2 marks)

Solution
$\log_e\left(\frac{1}{x^6+5}\right) = -\ln(x^6+5)$ $\frac{d}{dx}(-\ln(x^6+5)) = \frac{-6x^5}{x^6+5}$
Specific behaviours
✓ uses log law   ✓ correct derivative

Question 20 **(9 marks)**

Researchers in a large city wish to determine a 95% confidence interval for p, the proportion of citizens who had used the city library at least once during the previous year. The margin of error of the interval is to be no more than 6%.

(a) If the researchers had no reliable estimate for p, determine the sample size they should take, noting **all** assumptions made. (5 marks)

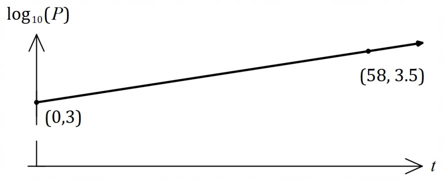
(b) The researchers were given access to data from a random sample of 165 citizens collected a few years earlier. Of these, 36 had used the city library at least once during the previous year.

(i) Determine the margin of error for a 95% confidence interval for p based on this sample. (2 marks)

(ii) The researchers used this data to decrease the sample size calculated in part (a). By how much did the sample size decrease? (2 marks)

Question 21 **(9 marks)**

The population of a species P can be modelled by the equation $P = a\,b^t$, where a and b are constants and t is the number of years since the population was first recorded. The graph below shows the linear relationship between t and $\log_{10} P$ for the population over the past 60 years and passes through the points $(0, 3)$ and $(58, 3.5)$.



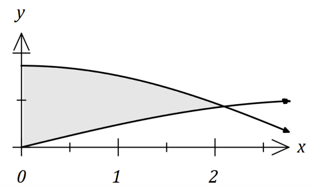
(a) Write an equation relating $\log_{10} P$ and t. (2 marks)

(b) Determine the value of a and the value of b. (3 marks)

Question 2 **(6 marks)**

Let $f(x) = \sqrt{3}\cos\left(\frac{x}{2}\right)$ and $g(x) = \sin\left(\frac{x}{2}\right)$.

The shaded region on the graph below is enclosed by $x = 0$, $y = f(x)$ and $y = g(x)$.



(a) Show that $f\left(\frac{2\pi}{3}\right) = g\left(\frac{2\pi}{3}\right)$. (2 marks)

Solution
$f\left(\frac{2\pi}{3}\right) = \sqrt{3}\cos\left(\frac{\pi}{3}\right) = \sqrt{3} \times \frac{1}{2} = \frac{\sqrt{3}}{2}$ $g\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ Hence $f\left(\frac{2\pi}{3}\right) = g\left(\frac{2\pi}{3}\right)$.
Specific behaviours
✓ evaluates $f(x)$

✓ evaluates , stating same as

(b) Determine the area of the shaded region. (4 marks)

Solution
$\int_0^{\frac{2\pi}{3}} \sqrt{3}\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) dx$ $¿\left[2\sqrt{3}\sin\left(\frac{x}{2}\right) + 2\cos\left(\frac{x}{2}\right)\right]_0^{\frac{2\pi}{3}}$ $¿\left[2\sqrt{3}\sin\left(\frac{\pi}{3}\right) + 2\cos\left(\frac{\pi}{3}\right)\right] - \left[2\sqrt{3}\sin(0) + 2\cos(0)\right]$ $¿\left(2\sqrt{3} \times \frac{\sqrt{3}}{2} + 2 \times \frac{1}{2}\right) - 2¿3 + 1 - 2 = 2$ sq units
Specific behaviours
✓ writes correct integral ✓ integrates correctly ✓ substitutes correctly

✓ correct area

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(4 marks)**

Determine the following:

(a) $\int 12(2x+1)^2\,dx$. (2 marks)

Solution
$\frac{12}{2\times 3}(2x+1)^3 = 2(2x+1)^3 + c$
Specific behaviours
✓ integrates correctly ✓ includes constant

(b) $\frac{d}{dx}\cos(2x+1)$. (1 mark)

Solution
$-2\sin(2x+1)$
Specific behaviours
✓ correct derivative

(c) $\frac{d}{dx}\int_3^x (2t+1)\,dt$. (1 mark)

Solution
$2x+1$
Specific behaviours
✓ correct use of fundamental theorem

(c) Interpret the value of a and the value of b in the context of this model. (2 marks)

(d) Use the model to determine

(i) the population when $t = 45$. (1 mark)

(ii) the number of years for the population to reach $10\,000$. (1 mark)

End of questions

Semester Two Examination, 2019

Question/Answer booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section One: Calculator-free

SOLUTIONS

Student number: In figures

In words ______________________________________

Your name ______________________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

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This Question/Answer booklet
Formula sheet

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Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

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