MATHEMATICS SPECIALIST 3CD
INTEGRATION BY PARTS

Can you determine $\int x\cos x\,dx$?

One possible method would be trial and error. Obviously the appropriate result would have the form $x \sin x$

Test this by differentiating with respect to x which results in the expression

$$1.\sin x + x\cos x$$

which looks right but has the extra term $\sin x$.

Try $x\sin x + \cos x$

Testing this by differentiating with respect to x results in the expression

$$1.\sin x + x\cos x - \sin x \quad \text{or} \quad x\cos x \quad \text{as required.}$$

$$\therefore \int x\cos x\,dx = x\sin x + \cos x + c$$

1. [3 marks]

Determine $\int x\sin x\,dx$

Try $x\cos x$: Differentiating gives $1.\cos x + x.(-\sin x)$

Try $-x\cos x + \sin x$: Differentiating gives $(-1).\cos x - x.(-\sin x) + \cos x$

or $x\sin x$

$$\therefore \int x\sin x\,dx = -x\cos x + \sin x + c$$ ✓✓✓

An alternative method for finding integrals of this type utilises the product rule.

$$\frac{d}{dx}(uv) = v.\frac{du}{dx} + u.\frac{dv}{dx}$$

Integrating with respect to x gives

$$\int \frac{d}{dx}(uv)dx = \int v.\frac{du}{dx}dx + \int u.\frac{dv}{dx}dx$$

ie $\quad uv = \int v.\frac{du}{dx}dx + \int u.\frac{dv}{dx}dx \quad$ or rearranging

$$\int u.\frac{dv}{dx}dx = uv - \int v.\frac{du}{dx}dx\,.$$

To use integration by parts to determine $\int x\cos x\,dx$

Let $u = x$ and $\frac{dv}{dx} = \cos x$

then $\frac{du}{dx} = 1$ and $v = \sin x$ (neglecting the constant term)

Using $\int u.\frac{dv}{dx}dx = uv - \int v.\frac{du}{dx}dx$

$$\int x\cos x\,dx = x\sin x - \int \sin x.1\,dx$$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + c \quad \text{as before.}$$

2. [4 marks]

Show that the result would not change if we used $v = \sin x + k$ in place of

$v = \sin x$.

Let $u = x$ and $\frac{dv}{dx} = \cos x$

then $\frac{du}{dx} = 1$ and $v = \sin x + k$

$$\int x \cos x \, dx = x.(\sin x + k) - \int (\sin x + k).1 dx$$ ✓
$$= x \sin x + x.k - (-\cos x + k.x) + c$$ ✓
$$= x \sin x + kx + \cos x - kx + c$$ ✓
$$= x \sin x + \cos x + c$$ ✓

Generally u is the function which produces a simpler result when differentiated while $\frac{dv}{dx}$ is the more complex part which can still be integrated.

Integrate by parts, each of the following.

3. [4, 4, 4, 4, 4, 8 marks]

(a) $\int x e^x dx$

Let $u = x$ and $\frac{dv}{dx} = e^x$

then $\frac{du}{dx} = 1$ and $v = e^x$

$$\int x e^x dx = x.e^x - \int e^x .1 dx$$ ✓✓
$$= x.e^x - \int e^x dx$$ ✓
$$= x.e^x - e^x + c$$ ✓

(b) $\int 3x \sin x \, dx$

Let $u = 3x$ and $\frac{dv}{dx} = \sin x$

then $\frac{du}{dx} = 3$ and $v = -\cos x$

$$\int 3x \sin x \, dx = 3x.(-\cos x) - \int (-\cos x).3$$ ✓✓
$$= -3x.\cos x + 3\int \cos x \, dx$$ ✓
$$= -3x \cos x + 3 \sin x + c$$ ✓

(c) $\int x\sqrt{2x-1}\, dx$

Let $u = x$ and $\frac{dv}{dx} = \sqrt{2x-1}$

then $\frac{du}{dx} = 1$ and $v = \frac{2.(2x-1)^{\frac{3}{2}}}{3..2}$

$$\int x\sqrt{2x-1}\,dx = x.\frac{2.(2x-1)^{\frac{3}{2}}}{3.2} - \int \frac{2.(2x-1)^{\frac{3}{2}}}{3.2} \quad ✓✓$$

$$= \frac{x.(2x-1)^{\frac{3}{2}}}{3} - \int \frac{(2x-1)^{\frac{3}{2}}}{3}dx \quad ✓$$

$$= \frac{x.(2x-1)^{\frac{3}{2}}}{3} - \frac{(2x-1)^{\frac{5}{2}}}{15} + c \quad ✓$$

(d) $\int x^3 \ln x\, dx$

Let $u = \ln x$ and $\frac{dv}{dx} = x^3$

then $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^4}{4}$

$$\int x^3 \ln x dx = \ln x.\frac{x^4}{4} - \int \frac{x^4}{4}.\frac{1}{x}dx \quad ✓✓$$

$$= \frac{x^4}{4}.\ln x - \frac{1}{4}\int x^3 dx \quad ✓$$

$$= \frac{x^4}{4}.\ln x - \frac{x^4}{16} + c \quad ✓$$

(e) $\int 3x(2x+3)^5\, dx$

Let $u = 3x$ and $\frac{dv}{dx} = (2x+3)^5$

then $\frac{du}{dx} = 3$ and $v = \frac{(2x+3)^6}{6.2}$

$$\int 3x(2x+3)^5\, dx = 3x.\frac{(2x+3)^6}{12} - \int \frac{(2x+3)^6}{12}.3dx \quad ✓✓$$

$$= \frac{x.(2x+3)^6}{4} - \frac{1}{4}\int (2x+3)^6 dx \quad ✓$$

$$= \frac{x.(2x+3)^6}{4} - \frac{1}{4}\frac{(2x+3)^7}{7.2} + c$$

$$= \frac{x.(2x+3)^6}{4} - \frac{(2x+3)^7}{56} + c \quad ✓$$

(f) $\int e^x \cos x\, dx$ using $u = e^x$ and $\frac{dv}{dx} = \cos x$

Let $u = e^x$ and $\frac{dv}{dx} = \cos x$

then $\frac{du}{dx} = e^x$ and $v = \sin x$

$$\int e^x \cos x \, dx = e^x.\sin x - \int \sin x.e^x \, dx$$ ✓✓

$$= e^x.\sin x - \left(e^x.(-\cos x) - \int e^x.(-\cos x)\, dx\right)$$ ✓✓

$$= e^x.\sin x + e^x \cos x - \int e^x \cos x \, dx + c$$ ✓

$$ie \quad 2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + c$$ ✓✓

$$\therefore \quad \int e^x \cos x \, dx = \frac{1}{2}(e^x \sin x + e^x \cos x) + c$$ ✓

IMPORTANT RESULTS:

$\frac{d}{dx} e^x = e^x$	$\int e^x \, dx = e^x + c$
$\frac{d}{dx} \ln x = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln x + c$