# MATHEMATICS SPECIALIST 3C/3D
# MARKING KEY

# MATHEMATICS: SPECIALIST
# 3CMAS/3DMAS
# RESOURCE-FREE

## Question 1 [4 marks]

Given $m = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $p = \mathbf{i} - \mathbf{j}$ **and** $n = 7\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ :

**(a)  Find** $2m - 3p$

| 2 marks | Description |
|---|---|
| 1 | $2m - 3p = 2\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ |
| 1 | $= \begin{pmatrix} 1 \\ 9 \\ 2 \end{pmatrix}$ |

**(b)   Find** $|n|$ **.**

| 2 marks | Description |
|---|---|
| 1 + 1 | $|n| = \sqrt{(7)^2 + 3^2 + 2^2} = \sqrt{62}$ |

## Question 2 [6 marks]

**(a)** $\int (e^x + x^e)\, dx$

| 3 marks | Description |
|---|---|
| ( 1 + 1 + 1) | $I = e^x + \dfrac{x^{e+1}}{e+1} + c$ |

**(b)  Show that** $\displaystyle\int_0^1 \frac{x}{1+x^2}\, dx = \ln\sqrt{2}.$

| 3 marks | Description |
|---|---|
| 1 + 1 | $\displaystyle\int_0^1 \frac{x}{1+x^2}\, dx = \left[ \frac{1}{2} \ln(1+x^2) \right]_0^1$ |
| 1 | $= \frac{1}{2}\left[ \ln 2 - \ln 1 \right] = \frac{1}{2}\ln(2) = \ln\sqrt{2}$ |

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**Question 3 [7 marks]**

**(a) Find an expression for $\dfrac{dy}{dx}$ and hence find the equation of the tangent to the curve at the point where: $\theta = \pi$**

| 5 marks | Description |
|---|---|
| 1 | $\dfrac{dx}{d\theta} = 2\cos\theta$ |
| 1 | and $\quad \dfrac{dy}{d\theta} = 2(3\sin\theta)\cos\theta$ |
| 1  1 | $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx}$  $= 6\sin\theta\cos\theta \times \dfrac{1}{2\cos\theta}$  $= 3\sin\theta$ |
| 1 | At $\theta = \pi$, $\ x = 0, y = 0$ and $\dfrac{dy}{dx} = 3\sin\pi = 0$.  Hence the equation of the tangent is $y = 0$ |

**(b) Determine the Cartesian equation of the curve.**

| 2 marks | Description |
|---|---|
| 1 | From above $\ \sin\theta = \dfrac{x}{2}$ and $\ \sin^2\theta = \dfrac{y}{3}$ or $y = 3\left(\dfrac{x}{2}\right)^2$ |
| 1 | So $\left(\dfrac{x}{2}\right)^2 = \left(\dfrac{y}{3}\right)$ or, directly $\ y = \dfrac{3x^2}{4}$ |

**Question 4 [10 marks]**

**(a) Find the solution to $\dfrac{dy}{dx} = 4y - 2yx$ satisfying $y(0) = 3$.**

| 5 marks | Description |
|---|---|
| 1 | Factorising: $\dfrac{dy}{dx} = (4 - 2x)y$ |
| 1 | separating variables: $\int \dfrac{dy}{y} = \int (4 - 2x)dx$ |
| 1  1 | $ln|y| = 4x - x^2 + c$  $\therefore y(x) = Be^{4x - x^2}$ |
| 1 | From $y(0) = 3$ we find B=3, so that $y(x) = 3e^{4x - x^2}$ |

**(b)** Use algebra and explain your reasoning, solve *exactly* the inequality |sin *2x*| $\leq$ sin *x* over the interval $0 \leq x \leq \pi$.

| 5 marks | Description |
|---|---|
| 1 | If $|\sin 2x| \leq \sin x$ then $2\sin x |\cos x| \leq \sin x$ |
| 1 | as $\sin x \geq 0$ across the whole of the interval |
| 1 | Thus need $|\cos x| \leq 1/2$ or $x = 0$ or $x = \pi$ |
| 2 | implying that $\dfrac{\pi}{3} \leq x \leq \dfrac{2\pi}{3}$ or $x = 0$ or $x = \pi$ <br><br> (-1 for each incorrect solution or inequality written as 1.05<*x*<2.09). |

### Question 5 [6 marks]
**(a) Verify that:**
$[z - \text{cis}(\theta)][z - \text{cis}(-\theta)] = z^2 - 2z\cos\theta + 1$.

| 3 marks | Description |
|---|---|
| 1 | $[z - \text{cis}(\theta)][z - \text{cis}(-\theta)] = z^2 - z[\text{cis}(\theta) + \text{cis}(-\theta)] + \text{cis}(\theta)\text{cis}(-\theta).$ |
| 1 <br><br> 1 | Now cis($\theta$)+cis($-\theta$) = $(\cos\theta + i\sin\theta) + [\cos(-\theta) + i\sin(-\theta)] = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$ and cis($\theta$)cis($-\theta$)= $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta = 1$ <br><br> so that $[z - \text{cis}(\theta)][z - \text{cis}(-\theta)] = z^2 - 2z\cos\theta + 1$      as required. |

**(b) Sketch the following subset of the complex plane:**
$\left\{ z : |z| \leq 1 \quad \text{and} \quad Re\, z > Im\, z \right\}$

| 3 marks | Description |
|---|---|
| 1 circle <br><br> 1 shading <br><br> 1 boundary |  |

**Question 7 [7 marks]**
**The sum of the products of two consecutive positive integers seems to have the form,**

$$1\times2 + 2\times3 + 3\times4 + \ldots = \frac{n^3 + 3n^2 + 2n}{3}$$

**Prove that this conjecture is true for all positive integers, *n*.**

| 7 marks | Description | |
|---|---|---|
| 1 | Firstly, show that the conjecture is true for $n = 1$<br>$LHS = 1(1+1) = 2$<br><br>$RHS = \frac{1\times2\times3}{3} = 2$<br><br>- as required<br>Now, assume the conjecture is true for $n$ terms, and consider the expression for $n+1$ terms | Firstly, show that the conjecture is true for $n = 1$<br>$LHS = 1(1+1) = 2$<br><br>$RHS = \frac{1^3 + 3\times1^2 + 2\times1}{3} = 2$<br><br>- as required<br>Now, assume the conjecture is true for $n$ terms, and consider the expression for $n+1$ terms |
| 1<br><br>1<br><br>1<br>1<br><br><br><br><br><br>1 | $1\times2 + 2\times3 + 3\times4 + \ldots + n(n+1) + (n+1)(n+2)$<br><br>$= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)$<br><br><br>$= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3}$<br><br><br>$= \frac{(n+1)(n+2)(n+3)}{3}$<br><br><br><br><br><br><br>which has the same form as for the $n^{th}$ term and so the statement is true for all $n$. | $1\times2 + 2\times3 + 3\times4 + \ldots + n(n+1) + (n+1)(n+2)$<br><br>$= \frac{n^3 + 3n^2 + 2n}{3} + (n+1)(n+2)$<br><br>$= \frac{n^3 + 3n^2 + 2n + 3(n^2 + 3n + 2)}{3}$<br><br>$= \frac{n^3 + 3n^2 + 2n + 3n^2 + 9n + 6)}{3}$<br><br>$= \frac{(n^3 + 3n^2 + 3n + 1) + (3n^2 + 8n + 5)}{3}$<br><br>$= \frac{(n+1)^3 + (3n^2 + 6n + 3) + 2n + 2}{3}$<br><br>$= \frac{(n+1)^3 + 3(n+1)^2 + 2(n+1)}{3}$<br><br>which has the same form as for the $n^{th}$ term and so the statement is true for all $n$. |

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# 3CMAS/3DMAS
# RESOURCE-RICH

**Question 1 [5 marks]**

**If** $y = \dfrac{1 + cos(2x)}{1 - cos(2x)}$ **, show that** $\dfrac{dy}{dx} = -\dfrac{2\cos x}{sin^3 x}$ **.**

| 5 marks | Description |
|---|---|
| 1<br>1 | $\dfrac{dy}{dx} = \dfrac{-2\sin(2x)(1 - \cos(2x)) - (1 + \cos(2x))(2\sin(2x))}{(1 - \cos(2x))^2}$ |
| 1<br>1 | $= \dfrac{-4\sin(2x)}{(2\sin^2 x)^2}$ |
| 1 | $= \dfrac{-8\sin x \cos x}{4\sin^4 x} = -\dfrac{2\cos x}{\sin^3 x}$ |

**Question 2 [6 marks]**

**Given the following expression** $\dfrac{2x^3 - 3x^2 - 29x + 60}{40 + 6x - x^2}$

**(a) Simplify the expression.**

| 2 marks | Description |
|---|---|
| 1 | factorise as $\dfrac{(x + 4)(x - 3)(2x - 5)}{(10 - x)(4 + x)}$ or use CAS to give |
| 1 | $\dfrac{-(x - 3)(2x - 5)}{x - 10}$ or equivalent |

**(b) For which value(s) of $x$, if any, is the original expression not defined.**

| 2 marks | Description |
|---|---|
| 1 | $x = 10$ |
| 1 | $x = -4$ |

**(c) Compare the original expression and its simplified version. Are there any values of $x$, where these two expressions are unequal? Explain your answer.**
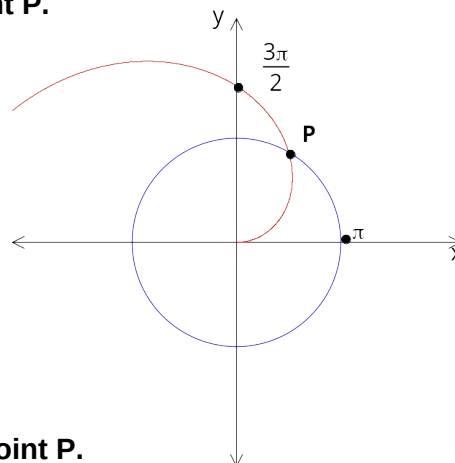
| 2 marks | Description |
|---|---|
| 1 | yes |
| 1 | the simplified expression has a value at $x = -4$, whereas the original expression is undefined at that value |

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**Question 3 [5 marks]**
**The diagram below shows the polar graphs of $r = k$ and $r = n\theta$ ($n$ an integer).**
**Also, the point of intersection of these two graphs is point P.**

**(a) Write down the values of $k$ and $n$.**

| 2 marks | Description |
|---|---|
| 1 | $k = \pi$ |
| 1 | and since $r = \dfrac{3\pi}{2}$ when $\theta = \dfrac{\pi}{2}$ , $n = 3$ |

**(b) Determine, EXACTLY, the Cartesian coordinates of point P.**

| 3 marks | Description |
|---|---|
| 1 | $r = \pi$ and $r = 3\theta \Rightarrow \pi = 3\theta$<br>i.e. $\theta = \dfrac{\pi}{3}$ so $P(\pi, \dfrac{\pi}{3})$. |
| 1 | Now convert to Cartesian –<br>$x = \pi\cos(\dfrac{\pi}{3}) = \dfrac{\pi}{2}$ and $y = \pi\sin(\dfrac{\pi}{3}) = \dfrac{\sqrt{3}}{2}\pi$ |
| 1 | $\Rightarrow P(\dfrac{\pi}{2}, \dfrac{\sqrt{3}}{2}\pi)$. |

**Question 4 [8 marks]**
**(a) By comparing:**
**(i) the areas of sector OPQ and triangle OPB**
**and**
**(ii) the lengths of PB and arc AB,**

**or otherwise, establish the inequalities $x\cos x < \sin x < x$.**

| 4 marks | Description |
|---|---|
| 1 | $OP = \cos x$ so the area of sector OPQ $= \dfrac{1}{2}x.\cos^2 x$ |
| 1 | $BP = \sin x$ so the area of $\Delta OPB = \dfrac{1}{2}\sin x.\cos x$ |
| 1 | $\therefore \ \dfrac{1}{2}x.\cos^2 x < \dfrac{1}{2}\sin x.\cos x \ \Rightarrow \ x.\cos x < \sin x$ |
| 1 | Now, the length of arc AB $= x$ so PB $<$ AB<br>$\Rightarrow \ \sin x < x$<br>$\Rightarrow \ x.\cos x < \sin x < x$ |

**(b)** Explain why the above result shows that $\dfrac{\sin x}{x} \to 1$ **as** $x \to 0$ **and hence,**

**determine:** $\displaystyle\lim_{x \to 0} \dfrac{2x}{\tan x}$ .

| 4 marks | Description |
|---|---|
| 1 | $x . \cos x < \sin x < x \quad \Rightarrow \quad \cos x < \dfrac{\sin x}{x} < 1$ |
| 1 | as $\quad x \to 0, \quad \cos x \to 1$ <br><br> so, by the sandwich principle $\quad \dfrac{\sin x}{x} \to 1$ |
| 1 | $\displaystyle\lim_{x \to 0} \dfrac{2x}{\tan x} = \lim_{x \to 0} \; 2 . \left( \dfrac{x}{\sin x} \right) . \cos x$ |
| 1 | $= 2.$ |

**Question 5 [7 marks]**
**Calculate the minimum distance they are apart and the time, after the aircraft leave their initial positions, when this distance is reached after the aircraft leave their initial positions.**

| 7 marks | Description |
|---|---|
| 1 | The displacement between the two toys at any time $t$ is given by: <br> $\mathbf{d} \;=\; (\mathbf{r_A} + \mathbf{v_A}\, t) - (\mathbf{r_B} + \mathbf{v_B}\, t)$ <br> $= \begin{pmatrix} 5 - t \\ -1 - 2t \\ 2 + 4t \end{pmatrix} - \begin{pmatrix} -3 \\ -3t \\ 4 + 6t \end{pmatrix}$ |
| 1 | $= \begin{pmatrix} 8 - t \\ -1 + t \\ -2 - 2t \end{pmatrix}$ where $t$ represents time in seconds |
| 1 | Hence $\qquad\qquad |\mathbf{d}|^2 = (8 - t)^2 + (-1 + t)^2 + (-2 - 2t)^2$ <br> $= 6t^2 - 10t + 69$ |
| 2 <br><br> 2 | Now by completing the square, using calculus or a graphics calculator (most likely) we find that the minimum value of $|d|^2$ is when $t = \dfrac{5}{6}$ seconds at which time $|d|^2 = 64\dfrac{5}{6}$ . <br> Hence the minimum distance they are apart is 8.05 m (to 2 dec pls) and the time taken is $\dfrac{5}{6}$ or $0.8\dot{3}$ seconds. |

**Question 6 [9 marks]**

**(a)  Show that** $M(t) = Ce^{-kt}$ **satisfies the differential equation for any constant *C*.**

| 2 marks | Description |
|---|---|
| 2 | If $M(t) = C\exp(-kt)$<br><br>then $\dfrac{dM}{dt} = -kC\exp(-kt) = -kM$ and so satisfies the DE. |

**(b)  Find *k* and *C*.**

| 3 marks | Description |
|---|---|
| 1 | If $M = 150$ when $t = 2$ then $150 = C\exp(-2k)$ and if $M = 100$ when $t = 3.5$ then $100 = C\exp(-3.5k)$ or |
| 1 | Dividing the equations gives $3/2 = \exp(1.5k)$ so $k = (2/3)\ln(3/2) \approx 0.27031$.<br><br>an alternative solution method for *k* $\displaystyle\int_{100}^{150}\frac{dM}{M} = \int_{3.5}^{2} -kdt$<br><br>solving for *k* gives 0.27031 |
| 1 | $C \approx 257.56$. |

**(c)  How long will it take for the mass of the substance to reduce to 40 grams?**

| 2 marks | Description |
|---|---|
| 1 | When $M = 40$ have $40 = C\exp(-kt) \Rightarrow -kt = \ln(40/C)$ so that $t \approx 6.890$. |
| 1 | Hence substance reduces to 40 grams after approximately 6.89 years. |

**(d)  Find the radioactive half-life of the substance.**

| 2 marks | Description |
|---|---|
| 1 | Initial mass of substance is *C*; half-life is defined to be time taken for mass to drop to $C/2$. |
| 1 | $C/2 = C\exp(-kt) \Rightarrow -kt = \ln(1/2) \Rightarrow t = k^{-1}\ln 2 \approx 2.564$ years. |

**Question 7 [7 marks]**

Consider the curve $x^2 y - 4y = b$ where *b* is a real value.

**(a)  Determine the equation of the tangent line to this curve at the point ($x_1$, $y_1$).**

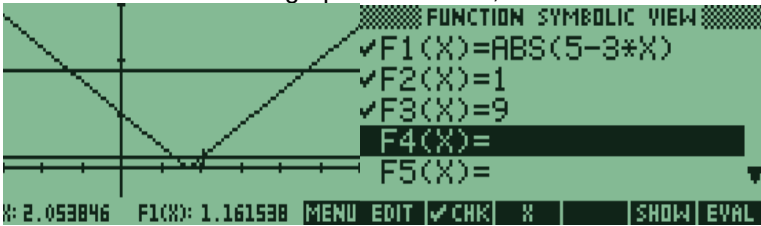| 4 marks | Description |
|---|---|
| | Differentiating implicitly and re-arranging to find the derivative |
| | $x^2 y - 4y = b$ |
| 1 | $2xy + x^2 \dfrac{dy}{dx} - 4\dfrac{dy}{dx} = 0$ |
| 1 | $\dfrac{dy}{dx}(x^2 - 4) = -2xy$ |
| 1 | $\dfrac{dy}{dx} = \dfrac{-2xy}{x^2 - 4} = \dfrac{2xy}{4 - x^2}$ |
| **OR** | **OR** rearranging first : |
| 1 | $y = \dfrac{b}{x^2 - 4}$ |
| 1 | $y' = (-1)b(x^2 - 4)^{-2}.2x = \dfrac{-2bx}{(x^2 - 4)^2}$ |
| 1 | thus, at *($x_1$,$y_1$)*, gradient is $\dfrac{2x_1 y_1}{4 - x_1^{\ 2}}$ |
| 1 | then the equation of the tangent to the curve at *($x_1$,$y_1$)* is $y - y_1 = \dfrac{2x_1 y_1}{4 - x_1^{\ 2}}(x - x_1)$ |

**(b)  What is the restriction on *b* so that the curve has a vertical tangent line at the point *($x_1$,$y_1$)*?**

| 3 marks | Description |
|---|---|
| 1 | There will be a vertical tangent when $x^2 - 4 = 0$ so |
| 2 | $4y - 4y = b$ and $b = 0$ |

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**Question 8**
**Solve the inequality** $1 < |5 - 3x| < 9$ **algebraically.**

Solving $1 < |5 - 3x| < 9$

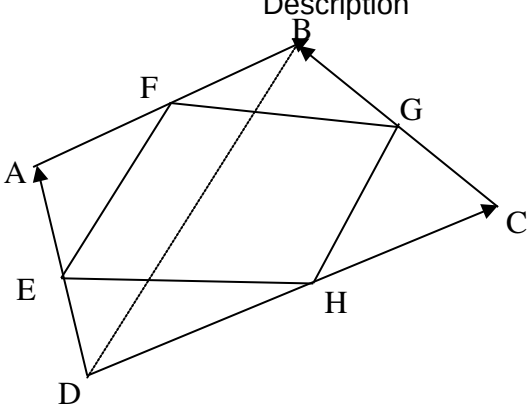| 5 Marks | Description |
|---|---|
| 1 | If $1 < |5 - 3x| < 9$ then: |
|  | (i) either 5-3x > 0, *x* < 5/3 and $1 < 5 - 3x$ and $5 - 3x < 9$ |
| 1 | $\therefore x < 4/3$ and $x > -4/3$ |
|  | the intersection is $-4/3 < x < 4/3$ |
| 1 | or $5 - 3x < 0$, x > 5/3 and $1 < -(5 - 3x)$ and $-(5 - 3x) < 9$ |
|  | $\therefore 2 < x$ and $x < 14/3$ |
| 1 | the intersection is $2 < x < 14/3$ |
| *or* | |
| 1 set up<br><br>1 for the<br>modulus graph<br><br>1 for lines | If answer is based on a graphical solution, such as the one shown below:<br> |
| 1 | so $-4/3 < x < 4/3$ or $2 < x < 14/3$ |
| 1 | Correct intervals $-\dfrac{4}{3} < x < \dfrac{4}{3}$ **or** $2 < x < \dfrac{14}{3}$, stated exactly. |

## Question 9 [6 marks]
**Use a vector method to prove that the mid-points of the sides of a quadrilateral form a parallelogram.**

| 6 marks | Description |
|---|---|
| |  |
| 1 | *Appropriate diagram* |
| 1 | *(symbolic set-up)* ABCD is a quadrilateral.<br>E, F, G and H are mid-points of the sides of the quadrilateral.<br>let $\vec{DA} = \underset{\sim}{u}$, let $\vec{AB} = \underset{\sim}{v}$, let $\vec{DC} = \underset{\sim}{w}$ and $\vec{CB} = \underset{\sim}{x}$ |
| 4 | *(clear, logical proof)*<br>$\vec{HG} = \frac{1}{2}\underset{\sim}{w} + \frac{1}{2}\underset{\sim}{x} = \frac{1}{2}\left(\underset{\sim}{w} + \underset{\sim}{x}\right)$ and $\vec{EF} = \frac{1}{2}\underset{\sim}{u} + \frac{1}{2}\underset{\sim}{v} = \frac{1}{2}\left(\underset{\sim}{u} + \underset{\sim}{v}\right)$<br><br>$\vec{DB} = \underset{\sim}{w} + \underset{\sim}{x} = \underset{\sim}{u} + \underset{\sim}{v}$<br><br>$\vec{HG} = \frac{1}{2}\vec{DB}$ *and* $\vec{EF} = \frac{1}{2}\vec{DB}$<br><br>So $\vec{HG} = \vec{EF}$ and EFGH is a parallelogram . |

## Question 10 [12 marks]
**(a)  Suppose that *z* is a complex number with modulus *r* and argument $\theta$.**
**Express in terms of *r* and $\theta$ the modulus and argument of each of the complex numbers $z_1, z_2, z_3$ and $z_4$, where:**

**(i)  $z_1 = \bar{z}$**

| 2 marks | Description |
|---|---|
| 1 | $|z_1| = r$ |
| 1 | $\arg z_1 = -\theta$ |

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**(ii)** $z_2 = -z^{-1}$

| 3 marks | Description |
|---|---|
| 1 | $\left| z_2 \right| = r^{-1}$ |
| 1+1 | $arg\ z_2 = \pi - \theta$ |

**(iii)** $z_3$ **and** $z_4$ **are the square roots of z.**

| 3 marks | Description |
|---|---|
| 1 | $\left| z_3 \right| = \left| z_4 \right| = \sqrt{r}$ |
| 1 | $arg\ z_3 = \dfrac{\theta}{2}$ |
| 1 | $arg\ z_4 = \dfrac{\theta}{2} + \pi$ |

**(b)  Indicate as precisely as you can on the diagram above the locations of the complex numbers** $z_1, z_2, z_3$ **and** $z_4$**, as defined in part (a).**

| 5 marks | Description |
|---|---|
| (*1 mark - for each point correctly located and labelled*) |  |
| 4 | |

**Question 11 [10 marks]**

**(a)  If initially there are 150 kids, 200 yearlings, 200 adults and 80 old goats what is the distribution after 20 years?**

| 3 marks | Description |
|---|---|
| 1 for 20th power<br><br>1 layout + multiply<br><br>1 interpret solution | $L^{20}\begin{bmatrix}150\\200\\200\\80\end{bmatrix} = \begin{bmatrix} 4.4357\times10^{-2} & 5.2449\times10^{-2} & 8.4967\times10^{-2} & 2.7992\times10^{-2} \\ 3.6389\times10^{-2} & 4.4357\times10^{-2} & 4.8701\times10^{-2} & 1.4451\times10^{-2} \\ 2.0232\times10^{-2} & 3.9189\times10^{-2} & 4.4356\times10^{-2} & 1.2266\times10^{-2} \\ 1.9626\times10^{-2} & 2.4901\times10^{-2} & 4.4787\times10^{-2} & 1.4919\times10^{-2} \end{bmatrix}\begin{bmatrix}150\\200\\200\\80\end{bmatrix} = \begin{bmatrix}36.376\\25.226\\20.725\\18.075\end{bmatrix}$<br><br>*2 for knowing what to do, 1 for the result which can be done by calculator* |

**(b)  Determine the net reproduction rate of the population of female goats.**

| 3 marks | Description |
|---|---|
| 1<br>1<br><br>1 | $R = a_1 + a_2 b_1 + a_3 b_1 b_2 + a_4 b_1 b_2 b_3$<br>$= 0 + 0 + 1.2 \times .65 \times .7 + .5 \times .65 \times .7 \times .8$<br>$= 0.826$ |

**(c)  Assuming there are enough male goats to fulfil their part, will the population of female goats become stable over time (that is, survive in the long run)?**

| 4 marks | Description |
|---|---|
| 1 | using previous result in (a), |
| 1<br><br>1 | and finding a higher power of L in the recursive relationship, indicates that this population appears to be increasing without limit, which is an unrealistic expectation. |
| 1 | However, based on the assumption of ample food and no external threats, this population would survive in the 'long term'. |

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