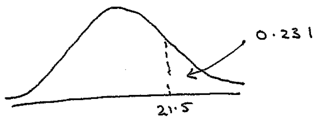
Question 20 **(5 marks)**

A random variable X is normally distributed such that the mean is twice the variance and the probability that X is greater than 21.5 is 0.231. Find the mean and standard deviation of X.

$\mu = 2\sigma^2$ ✓

$X \sim N(2\sigma^2, \sigma^2)$



$P(Z > z) = 0.231$

$z = 0.735558$ ✓

$z = \frac{x - \mu}{\sigma}$

$0.735558 = \frac{21.5 - 2\sigma^2}{\sigma}$ ✓

$2\sigma^2 + 0.735558\sigma - 21.5 = 0$ ✓

$\sigma = 3.09998$

$= 3.1 \text{ (1 d.p)}$ ✓

$\mu = 2 \times 3.1^2$ ✓

$= 19.22$ ✓

Section One: Calculator-free **(40 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the space provided.

Working time for this section is 50 minutes.

Question 1 **(4 marks)**

Determine the equation of the tangent to the curve $y = 1 - \frac{9}{2x-1}$ at the point (2, -2).

F/T

$y = 1 - 9(2x-1)^{-1}$ ✓

$\frac{dy}{dx} = 9(2x-1)^{-2} \cdot 2$

$= \frac{18}{(2x-1)^2}$ ✓

When $x = 2$ $\frac{dy}{dx} = 2$ ✓

Equation of tangent

$y = 2x + c$

$-2 = 2(2) + c$

$-6 = c$

Equation: $y = 2x - 6$ ✓

$9(2x-1)^{-2}$

$= \frac{9}{(2x-1)^2}$

$\frac{dy}{dx} = 1$

$y = x + c$

$-2 = \cancel{2} + c$

$c = -4$

$y = \cancel{2} - 4$

Question 2 **(4 marks)**

Differentiate the following, without simplifying:

(a) $y = \dfrac{3}{\sqrt{1+e^{5x}}}$ $= 3\left(1+e^{5x}\right)^{-1/2}$ (2 marks)

$$\frac{dy}{dx} = -\frac{3}{2}\left(1+e^{5x}\right)^{-3/2} . 5e^{5x}$$ ✓ ✓

(b) $y = \dfrac{x^3-4}{x-2}$ (2 marks)

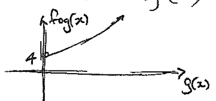
$$\frac{dy}{dx} = \frac{(x-2)3x^2 - (x^3-4).1}{(x-2)^2}$$ ✓✓

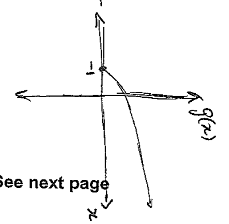
Question 3 **(4 marks)**

Determine the domain and range of $f \circ g(x)$, where $f(x) = 2^{x+2}$ and $g(x) = \sqrt{x+1}$.

Domain of $g(x)$ is $x \geq -1$ ✓
Range " $g(x)$ is $g(x) \geq 0$
Domain of $f \circ g(x)$ is $x \geq -1$ ✓
Range " " is $f(x) \geq 4$ ✓✓

$$f \circ g(x) = 2^{\sqrt{x+1}} + 2.$$





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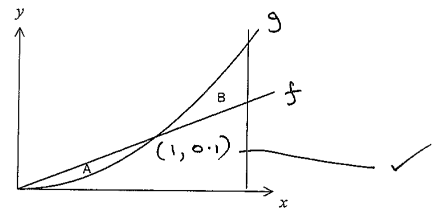
Question 19 **(7 marks)**

The graph below, not to scale, shows the functions $f(x) = \dfrac{x}{10}$, $g(x) = \dfrac{x^2}{10}$ and the line $x = 2$.



Region A is the area trapped by f and g.
Region B is the area trapped by f, g and the line $x = 2$.

(a) Find the areas of regions A and B. (3 marks)

$$\text{Area A} = \int_0^1 \frac{x}{10} - \frac{x^2}{10}\,dx = 0.01\dot{6} \quad ✓ \left(\tfrac{1}{60}\right)$$

$$\text{Area B} = \int_1^2 \frac{x^2}{10} - \frac{x}{10}\,dx = 0.08\dot{3} \quad ✓ \left(\tfrac{1}{12}\right)$$

(b) $f(x)$ is modified to become the line $f(x) = kx$, so that the area of region A is exactly the same as the area of region B. Determine the value of k. (4 marks)

$$kx = \frac{x^2}{10}$$

$$10kx - x^2 = 0$$

$$x(10k - x) = 0$$

$$x = 10k \quad ✓$$

$$✓ \int_0^{10k} kx - \frac{x^2}{10}\,dx = \int_{10k}^{2} \frac{x^2}{10} - kx\,dx \quad ✓$$

$$\frac{50k^3}{3} = \frac{4-30k}{15} + \frac{50k^3}{3}$$

$$4 - 30k = 0$$

$$k = \frac{4}{30} \text{ or } \frac{2}{15} \quad ✓$$

See next page

Question 18 **(7 marks)**

Climbing rope produced by a manufacturer is known to be such that over a long production run, ten-metre lengths have breaking strengths that are normally distributed with a mean of 180.2kg and standard deviation of 9.5kg.

(a) Find the probability that a randomly chosen ten-metre length will have a breaking strength of less than 160kg. (1 mark)

X = breaking strength $\quad X \sim N(180 \cdot 2, 9 \cdot 5^2)$

$P(X < 160) = 0 \cdot 016738$ ✓

(b) At the start of a production run, a quality control officer at the factory randomly samples 20 ten-metre lengths and after testing, determines that the mean breaking strength of the sample is 176.9kg. Construct a 90% confidence interval for the population mean based on this sample. (3 marks)

$\mu = 176 \cdot 9 \qquad n = 20$

$176 \cdot 9 - 1 \cdot 645 \times \frac{9 \cdot 5}{\sqrt{20}} < \mu < 176 \cdot 9 + 1 \cdot 645 \times \frac{9 \cdot 5}{\sqrt{20}}$ ✓✓

$173 \cdot 41 < \mu < 180 \cdot 39$ ✓

(c) If the quality control officer repeated the same sampling process in (b) every day for 30 consecutive days, how many of the intervals constructed would be expected to include the known mean breaking strength of 180.2kg? You may assume there were no problems with the manufacturing process. (1 mark)

$0 \cdot 9 \times 30 = 27$ ✓

(d) How large a sample should the quality control officer take, if the probability that the estimated mean breaking strength is in error by more than 2.5kg is to be at most 5%? (2 marks)

95% C I .

$1 \cdot 96 \times \frac{9 \cdot 5}{\sqrt{n}} < \underbrace{\bar{x} - \mu}_{2 \cdot 5} < 1 \cdot 96 \times \frac{9 \cdot 5}{\sqrt{n}}$

✓

$n = 55 \cdot 4$

need 56 minimum ✓

Question 4 **(6 marks)**

In a foreign country, a student had a number of \$5, \$2 and \$1 notes with a total value of \$40. The number of \$1 notes was one more than the total number of \$2 and \$5 notes, with a total of 19 notes altogether.

Let x, y and z be the number of \$5, \$2 and \$1 notes respectively.

(a) Write down three equations using the above information. (2 marks)

$5x + 2y + z = 40$

$x + y + z = 19$ ✓✓

$z = x + y + 1$ −1 each error.

$-x - y + z = 1$

(b) Solve the system of equations in part (a). (4 marks)

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 19 \\ 5 & 2 & 1 & 40 \end{pmatrix}$$

$$\begin{array}{c} \\ R_1 - R_2 \\ 5R_1 - R_3 \end{array} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & -2 & -20 \\ 0 & 3 & -6 & -45 \end{pmatrix}$$

$-2z = -20 \Rightarrow z = 10$

$3y - 6z = -45$ ✓✓✓✓

$3y = 15$ −1 each error.

$y = 5$

$x + y + z = 19$

$x + 5 + 10 = 19$

$x = 4$

$x = 4, \; y = 5, \; z = 10.$

Question 5 **(5 marks)**

Determine the following integrals:

(a) $\int (6x+9)(3x+x^2)^2\,dx$ (2 marks)

$$= 3\int (2x+3)(3x+x^2)^2\,dx \quad \checkmark$$

$$= \frac{3\,(3x+x^2)^3}{3} + c$$

$$= (3x+x^2)^3 + c \quad \checkmark$$

(b) $\int_1^4 3\sqrt{x}\,dx$ (3 marks)

$$\int_1^4 3x^{1/2}\,dx$$

$$= \left[2 \times \frac{3x^{3/2}}{3}\right]_1^4 \quad \checkmark$$

$$= \left[2(\sqrt{x})^3\right]_1^4 \quad \checkmark$$

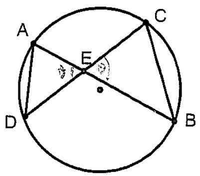
$$= 2(8) - 2(1)$$

$$= 14. \quad \checkmark$$

Question 17 **(7 marks)**

In the diagram below the chords AB and CD intersect at the point E.

The area of $\triangle EAD$ is 15cm^2.



(a) Explain why $s\angle EAD = s\angle ECB$ (1 mark)

Angles in same segment or Angles subtended by the same arc. ✓

(b) Prove that $\triangle EAD$ is similar to $\triangle ECB$. (3 marks)

$\angle EAD = \angle ECB$ ✓ (angles in same segment)

$\angle AED = \angle CEB$ ✓ (vert opp $\angle$s)

$\triangle AED \cong \triangle CEB$ ✓

or $\triangle EAD \cong \triangle ECB$ (AAA condition)

(c) Use your result from (b) to show that $AE \times BE = DE \times CE$. (1 mark)

$$\frac{AE}{CE} = \frac{DE}{BE}$$

$\therefore AE \times BE = DE \times CE$ ✓

(d) Find the area of $\triangle ECB$ if $CE = 2 \times AE$. (2 marks)

$BE = 2 \times DE$ (Similar $\triangle$'s)

$$\text{Area } \triangle ECB = \frac{1}{2} CE \cdot BE \cdot \sin\theta$$

$$= \frac{1}{2} \cdot 2 \cdot AE \cdot 2 \cdot DE \sin\theta \quad \checkmark$$

$$= 4 \times \frac{1}{2} AE \cdot DE \sin\theta$$

$$= 4 \times 15 = 60\text{ cm}^2 \quad \checkmark$$

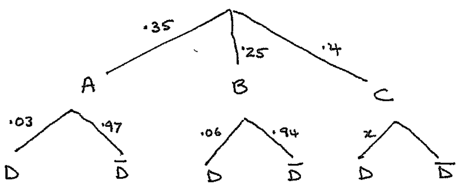
Question 16 **(7 marks)**

A factory uses three machines to produce one type of plastic bottle. Of the total production, machine A produces 35%, machine B produces 25% and machine C the rest. Due to the age of the machines, they all produce some defective bottles. Of their production, machines A and B produce 3% and 6% defective bottles respectively.

(a) Find the probability that a randomly selected bottle is produced by machine A and is defective. (1 mark)

$0.35 \times 0.03 = 0.0105$

(b) If the probability of a randomly selected bottle being defective is 0.0455, what percentage of the production of machine C is defective? (4 marks)



$P(\text{defective})$

$= \cdot 35 \times \cdot 03 + \cdot 25 \times \cdot 06 + \cdot 4 \times x$

$= 0.0455$

$\therefore P(C \text{ defective}) = 0.05$

$\therefore$ 5% defective

(c) Given that a randomly selected bottle is not defective, find the probability that it was produced by either machine A or machine B. (2 marks)

$P(\text{not defective}) = 1 - 0.0455$

$P(A \text{ or } B \mid \bar{D}) = \dfrac{P((A \text{ or } B) \cap \bar{D})}{P(\bar{D})}$

$= \dfrac{0.35 \times 0.97 + 0.25 \times 0.94}{1 - 0.0455}$

$= 0.6019.$

Question 6 **(4 marks)**

The volume, V in cm^3, of an object is changing with time, t in seconds, so that the volume at any time is given by $V = 5t + \frac{12}{t}$. Use the incremental formula to find the approximate change in volume of the object between $t = 2$ and $t = 2.01$ seconds.

$\dfrac{dV}{dt} = 5 - \dfrac{12}{t^2}$

$-\frac{1}{2}$ not δ notation

$\delta V \simeq \dfrac{dV}{dt} \times \delta t$

$-\frac{1}{2}$ not $\cong$ or $\simeq$

$= \left(5 - \dfrac{12}{t^2}\right) \times 0.01$

When $t = 2$ $\quad \delta V = \left(5 - \dfrac{12}{2^2}\right) \times 0.01$

$= 0.02 \text{ cm}^3$

$-\frac{1}{2}$ not correct units

small % change = 2 marks only (0.125)

Question 7 **(6 marks)**

Solve for x the inequality $\frac{1}{2x-1} \geq \frac{2}{x+2}$.

$$\frac{1}{2x-1} - \frac{2}{x+2} \geqslant 0 \quad ✓$$

$$\frac{x+2-2(2x-1)}{(2x-1)(x+2)} \geqslant 0$$

$$\frac{4-3x}{(2x-1)(x+2)} \geqslant 0 \quad ✓✓$$

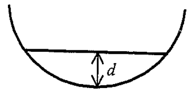
✓

$x < -2$	$-2 < x < \frac{1}{2}$	$\frac{1}{2} < x \leqslant \frac{4}{3}$	$x \geqslant \frac{4}{3}$
+ ve	− ve	+ ve	− ve

$x < -2$ ✓ $\qquad \frac{1}{2} < x \leqslant \frac{4}{3}$ ✓

Question 15 **(6 marks)**

When fluid rests in the bottom of a hemisphere of radius r, the volume of fluid V, can be calculated using the formula $V = \frac{\pi d^2(3r-d)}{3}$, where d is the depth of the fluid in cm.



If water is poured into a hemisphere of radius 45cm at a constant rate of 2 litres per minute, how fast is the depth of water increasing at the instant that the hemisphere contains 70L of water? (Note: 1cm^3 = 1mL)

$$\frac{dV}{dt} = 2 \text{ litres/min} = 2000 \text{ cm}^3/\text{min}. \quad ✓$$

$$\frac{dd}{dt} = \frac{dd}{dV} \times \frac{dV}{dt}.$$

$$V = \frac{\pi d^2(135-d)}{3} = 45\pi d^2 - \frac{\pi}{3}d^3 \quad ✓ \text{ subst } r = 45$$

$$\frac{dV}{dd} = 90\pi d - \pi d^2 \quad ✓$$

$$\frac{dd}{dt} = \frac{1}{90\pi d - \pi d^2} \times 2000 \quad ✓$$

When $V = 70$ litres

$$\frac{\pi d^2(135-d)}{3} = 70\,000 \quad ✓$$

$$d = 24.607$$

$$\frac{dd}{dt} = 0.396 \text{ cm/min} \quad ✓$$

Question 14 **(11 marks)**

A manufacturer of chocolate produces 3 times as many soft centred chocolates as hard centred ones. The chocolates are randomly packed in boxes of 20.

(a) Find the probability that in a box there are

(i) an equal number of soft centred and hard centred chocolates (3 marks)

$P(\text{soft}) = \frac{3}{4}$ $P(\text{hard}) = \frac{1}{4}$ ✓

Let H = no of hards in a box of 20

$H \sim Bin(20, \frac{1}{4})$ ✓ ✓

$P(H = 10) = 0.0099223.$

or $P(H=10) = {}^{20}C_{10}(\frac{3}{4})^{10}(\frac{1}{4})^{10} = 0.0099223.$

(ii) fewer than 5 hard centred chocolates. (1 mark)

$P(H \leq 4) = 0.41484$ ✓

(b) Determine the mean and standard deviation of the number of hard centred chocolates in a box of 20. (2 marks)

$\mu = 20 \times \frac{1}{4} = 5$ ✓

$\sigma = \sqrt{20 \times \frac{1}{4} \times \frac{3}{4}} = 1.9365$ ✓

(c) A random sample of 5 boxes is taken from the production line. Find the probability that exactly 3 of them contain fewer than 5 hard centred chocolates. (2 marks)

$p = 0.41484$ (use follow through from a ii)

$n = 5$

X = number with fewer than 5 in a sample of 5.

$X \approx Bin(5, 0.4148)$ ✓ $P(X = 3) = 0.24445$ ✓

(d) A random sample of 30 boxes is taken from the production line. Find the probability that the mean number of hard centred chocolates per box in the sample exceeds 5.5. (3 marks)

$n = 30$

$\mu = 5$

$\sigma = 1.9365$

X = Sample means $\sim N\left(5, \frac{1.9365^2}{30}\right)$ ✓ ✓

$P(X > 5.5) = 0.07865$ ✓

See next page

Question 8 **(7 marks)**

Determine the coordinates of all roots, stationary points and points of inflection of the function $y = x^3(4+x)$. Justify the nature of the stationary points found using a standard test.

Roots $y = 0$ $x^3(4+x) = 0$

$x = 0$ or -4

Coordinates $(0, 0)$ $(-4, 0)$ ✓✓

$\frac{dy}{dx} = 12x^2 + 4x^3$

Stationary pts when $\frac{dy}{dx} = 0$

$4x^2(3+x) = 0$

$x = 0$ or -3

Sign test.

x	-4	-3	-1	0	1
$\frac{dy}{dx}$	$-ve$ (-64)	0	$+ve$ 8	0	$+ve$ 16

or $\frac{d^2y}{dx^2} = 24x + 12x^2$

When $x = -3$ $\frac{d^2y}{dx^2} = +ve$ $\therefore$ min ✓ ✓

$x = 0$ $\frac{d^2y}{dx^2} = 0$ — pt of inflection

Stat points at $(0,0)$ and $(-3,-27)$ ✓✓

Pt inflection $\frac{d^2y}{dx^2} = 24x + 12x^2 = 0$ $12x(2+x) = 0$

$x = 0$ or -2

Pts of inflection at $(0,0)$ $(-2,-16)$ ✓

End of questions

Section Two: Calculator-assumed **(80 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the space provided.

Working time for this section is 100 minutes.

Question 9 **(4 marks)**

The percentage of trees, P, in a plantation affected by a disease was changing with time, t in months, according to the relationship $\frac{dP}{dt} = -0.017P$.

(a) Was the health of the plantation getting better or worse? Briefly justify your answer by referring to the above relationship. (1 mark)

% affected is decreasing (-0.017)

$\therefore$ Health getting better ✓

(b) If 7.2% of the trees in the plantation were affected today, what percentage is expected to be affected by the disease in one and a half years time? (3 marks)

$P = P_0 e^{-0.017t}$

When $t = 0$ $P_0 = 7.2$

$P = 7.2e^{-0.017t}$ ✓

When $t = 18$ ✓ $P = 5.3\%$. ✓

Question 13 **(5 marks)**

A sub-committee of four, consisting of a chairperson, a secretary and two ordinary members is to be chosen from a committee of 20 people.

(a) Find the number of possible choices for

(i) the posts of chairperson and secretary, (1 mark)

$20 \times 19 = 380$ ✓

$({}^{20}C_1 \times {}^{19}C_1)$

(ii) the two ordinary members, (1 mark)

${}^{20}C_2 = 190$ ✓

(iii) the chairperson, secretary and two ordinary members. (1 mark)

$20 \times 19 \times {}^{18}C_2 = 58140$ ✓

(b) If all possible sub-committees are equally likely to be chosen, what is the probability that the chairman of the main committee is not selected in the sub-committee? (2 marks)

$$\frac{{}^{19}C_1 \times {}^{18}C_1 \times {}^{17}C_2}{{}^{20}C_1 \times {}^{19}C_1 \times {}^{18}C_2}$$ ✓

$= 0.8$ ✓

OR/ $\frac{19}{20} \times \frac{18}{19} \times \frac{17}{18} \times \frac{16}{17} = \frac{4}{5}$.

(c) How many of each pack should the apprentice buy to minimise the purchase cost and what is the minimum cost? (3 marks)

$C = 3.5x + 6.5y$ ✓

	$C = 3.5x + 6.5y$
(5, 6)	\$ 56.50
(0, 16)	\$ 104
(9, 4)	\$ 57.50
(21, 0)	\$ 73.50

✓

Should buy ✓
5 Best Buy and
6 Chef's Choice.

(d) By how much can the price of a 'Best Buy' pack rise without changing the optimum number of packs found in your answer to (c)? (2 marks)

Let new price of Best Buy be \$a

$C = ax + 6.5y$

$5a + 39 = 16 \times 6.5$
$a = 13$ ✓

The price difference is $13 - 3.50$
$= \$9.50$ ✓

∴ The price can rise by up to \$9.50, but not including \$9.50, before the optimum number of packs changes.

($-\frac{1}{2}$ if not less than)

See next page

Question 10 **(5 marks)**

(a) A and B are two independent events such that $P(A) = 0.2$ and $P(B) = 0.15$.

Evaluate

(i) $P(A|B) = P(A) = 0.2$ ✓ (1 mark)

(ii) $P(A \cap B) = P(A) \times P(B)$ (1 mark)
$= 0.2 \times 0.15 = 0.03$ ✓

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (1 mark)
$= 0.2 + 0.15 - 0.03$
$= 0.32$ ✓

(b) The probability that a door to door salesman convinces a customer to buy is 0.4.

Assuming that sales are independent, find the probability that the salesman makes at least one sale before reaching the fourth house. (2 marks)

$P(\text{make no sale}) = 0.6$
$P(\text{makes 0 sales in first 3}) = 0.6^3$
$\therefore P(\text{makes at least 1}) = 1 - 0.6^3$ ✓
$= 0.784.$ ✓

OR.
$P(1 \text{ sale}) = {}^3C_1 (0.4)(0.6)^2$
$P(2 \text{ ''}) = {}^3C_2 (0.4)^2(0.6)$
$P(3 \text{ ''}) = {}^3C_3 (0.4)^3(0.6)^0$
$P(\text{at least 1 sale}) = 0.784$

OR
$P(\text{customer buys}) = 0.4$
X = number of sales in 3 trials
$X \sim Bin(3, 0.4)$ ✓ $P(X \geq 1) = 0.784$ ✓

If $X \sim Bin(4$
$P(X \geq 1) = .87$

See next page

Question 11 **(6 marks)**

A body moves in a straight line so that its displacement, $x(t)$ metres, from a fixed point after t seconds is given by $x(t) = t^3 - 9t^2 + 24t$, for $0 \le t \le 5$.

(a) When is the body stationary? (2 marks)

$v(t) = 3t^2 - 18t + 24$ ✓ $0 \le t \le 5$

When $v(t) = 0$ $3t^2 - 18t + 24 = 0$

$t^2 - 6t + 8 = 0$

$(t-2)(t-4) = 0$

$t = 2$ or 4 s. ✓

(b) When is the body moving fastest? (2 marks)

$V_{max} = 6t - 18 = 0$

$t = 3$ secs. ✓

Consider $t = 0$ $v = 24$

$t = 3$ $v = -3$

$t = 5$ $v = 9$

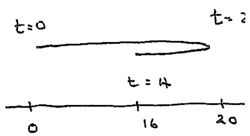
Fastest at $t = 0$ sec. ✓

(c) Calculate the distance travelled by the body in the first four seconds. (2 marks)

$t = 0$ $x(t) = 0$ ✓

$t = 2$ $x(t) = 20$

$t = 4$ $x(t) = 16$



Distance travelled

$= 20 + 4$

$= 24m$ ✓

Question 12 **(10 marks)**

Every weekday the chef at a restaurant sends out an apprentice to the local market to spend as little as possible and at the same time come back with at least 16kg of onions, at least 17kg of carrots and at least 21kg of potatoes.

One stall at the market sells 'Best Buy' packs consisting of 2kg of onions, 1kg of carrots and 1kg of potatoes for \$3.50 each. Another stall sells 'Chefs Choice' packs consisting of 1kg of onions, 2kg of carrots and 3kg of potatoes for \$6.50 each.

The apprentice buys x 'Best Buy' packs and y 'Chefs Choice packs.

(a) Write down three inequalities to represent the above constraints, apart from $x \ge 0$ and $y \ge 0$. (2 marks)

	Best Buy (x)	Chef's Choice (y)
Onions	2	1
Carrots	1	2
Potatoes	1	3

$2x + y \ge 16$ ✓✓

$x + 2y \ge 17$

$x + 3y \ge 21$

(b) Complete the constraints on the graph below and indicate the feasible region. (3 marks)