Christ Church
Grammar School

2016
UNIT TEST 5

MATHEMATICS METHODS Year 12

Section One:
Calculator-free

Student name Solution

Teacher name ____________________

Time and marks available for this section

Reading time before commencing work:	3 minutes
Working time for this section:	30 minutes
Marks available:	30 marks

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. Write your answers in this Question/Answer Booklet.

2. Answer all questions.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that **you do not use pencil**, except in diagrams.

Question 13 **(8 marks)**

An infertility study two years ago found that 308 out of 346 couples conceived naturally within the first year of trying.

(a) Use these figures to calculate the proportion of couples who conceived naturally within the first year. (1 mark)

$\hat{p} = \frac{308}{346} = 0.89$

A follow-up study is to be conducted, from which a 90% confidence interval for the population proportion will be calculated to confirm the results of the initial study.

(b) Use the sample proportion from (a) to calculate the smallest sample size necessary to ensure a margin of error of no more than 5%. (2 marks)

$1.645\sqrt{\frac{0.89(1-0.89)}{n}} = 0.05$

$n = 106$

In preparation for the follow-up study, a simulation was designed in which 140 random samples are selected from a population in which it was assumed that 85% of couples conceived naturally within the first year.

(c) Calculate a 90% confidence interval based on a sample size of 140 and the assumed 85% proportion. (2 marks)

$90\%\ CI = 0.85 \pm 1.645\sqrt{\frac{0.85 \times 0.15}{140}}$

$= (0.800, 0.900)$

(d) The results from running the simulation ten times are shown below.

Simulation	1	2	3	4	5	6	7	8	9	10
Number of couples conceiving	120	116	118	121	115	126	112	111	126	123
$\hat{p}$	0.86	0.83	0.84	0.86	0.82	0.9	0.8	0.79	0.9	0.88

Comment on these results. (2 marks)

90% of the simulations (9 out of 10 here) have a sample proportion falling within the 90% CI of (0.800, 0.900).

End of questions

Question 1 **(6 marks)**

(a) Determine $f'(\frac{\pi}{8})$ if $f(x) = \log_e(\sin(2x))$. (2 marks)

$f'(x) = \frac{2\cos 2x}{\sin 2x}$

$f'(\frac{\pi}{8}) = \frac{2\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}$

$= 2$

(b) A function is defined by $f(x) = (x-2)e^x$. Determine the coordinates and nature of all stationary points. (4 marks)

$f'(x) = (1)e^x + (x-2)e^x$

$f''(x) = e^x + (1)e^x + (x-2)e^x$

$f'(x) = 0 \Rightarrow e^x + (x-2)e^x = 0$

$e^x(1 + x - 2) = 0$

$x = 1$

$y = -e$

$f''(1) > 0 \Rightarrow (1, -e)$ is min pt.

See next page

Question 2 **(7 marks)**

The random variable X takes the values -1, 0, 1 and 2 only, with probabilities $2p$, p, $3p$ and $4p$ respectively.

(a) Determine $P(X \geq 0 \mid X \leq 1)$. (2 marks)

$$= \frac{p + 3p}{2p + p + 3p}$$

$$= \frac{4}{6}$$

(b) Determine the value of the constant p. (2 marks)

$$2p + p + 3p + 4p = 1$$

$$p = \frac{1}{10}$$

(c) Calculate $E(X)$. (2 marks)

$$= -2p + 0 + 3p + 8p$$

$$= 9p$$

$$= \frac{9}{10}$$

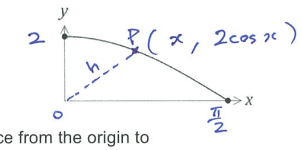
(d) Given that $\mathrm{Var}(X) = 1.29$, determine $Var(10X + 5)$ (1 mark)

$$= 10^2(1.29)$$

$$= 129$$

Question 12 **(6 marks)**

A function is defined as $f(x) = 2\cos(x)$ on the interval $0 \leq x \leq \frac{\pi}{2}$, and is shown in the graph below.



(a) Determine the distance from the origin to

(i) the y-intercept. (1 mark)

2

(ii) the x-intercept. (1 mark)

$\frac{\pi}{2}$

(b) Show that the distance h from the origin to a point on the graph of f is given by (1 mark)

$$h = \sqrt{4\cos^2 x + x^2}$$

$$h^2 = x^2 + (2\cos x)^2$$

$$h = \sqrt{x^2 + 4\cos^2 x}$$

(c) Use calculus to determine the minimum distance from the origin to a point on the graph of f, giving your answer correct to three decimal places. (3 marks)

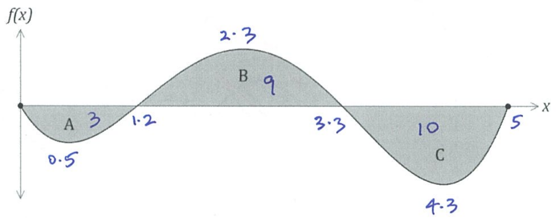
$$\frac{dh}{dx} = \frac{2x - 8\cos x \sin x}{2(x^2 + 4\cos^2 x)^{\frac{1}{2}}}$$

$$\frac{dh}{dx} = 0 \Rightarrow x = 1.2373$$

$$h = 1.400$$

Question 11 **(7 marks)**

The graph of the function f is shown below. The shaded regions A, B and C have areas of 3, 9 and 10 square units respectively. The function has roots when $x = 0, 1.2, 3.3$ and 5, and stationary points when $x = 0.5, 2.3$ and 4.3.



(a) Determine

(i) $\int_{1.2}^{5} f(x)\,dx$. $= 9 - 10$ (1 mark)

$= -1$

(ii) $\int_{0}^{3.3} 2f(x)\,dx$. $= 2(-3+9)$ (1 mark)

$= 12$

(iii) $\int_{0}^{5} 2 - f(x)\,dx$. $= 2[x]_0^5 - \int_0^5 f(x)\,dx$ (2 marks)

$= 10 - (-3+9-10)$

$= 14$

(b) If $F(x) = \int_0^x f(t)\,dt$, $0 \le x \le 5$, determine

(i) the maximum value of $F(x)$ and the value of x when this occurs. (2 marks)

$F(x)$ = net area under curve from 0 to x

max $F(x) = -3 + 9 = 6$

when $x = 3.3$

(ii) the number of solutions to $F(x) = 0$. (1 mark)

3 solutions ($x = 0, \sim 2, \sim 4$)

Question 3 **(8 marks)**

(a) A curve for which $\frac{dy}{dx} = -\frac{k}{x^3}$, where k is a constant, passes through the points (1, 18) and (4, 3). Determine the equation of the curve. (4 marks)

$y = \frac{k}{2x^2} + c$

$18 = \frac{k}{2} + c$ — ①

$3 = \frac{k}{32} + c$ — ②

① − ② $15 = \frac{15}{32}k$

$k = 32$

$c = 2$

$\therefore y = \frac{16}{x^2} + 2$

(b) Determine the exact value of the constant k for which $\int_1^k \frac{2}{3x-2}\,dx = 2$. (4 marks)

$\frac{2}{3}\left[\ln(3x-2)\right]_1^k = 2$

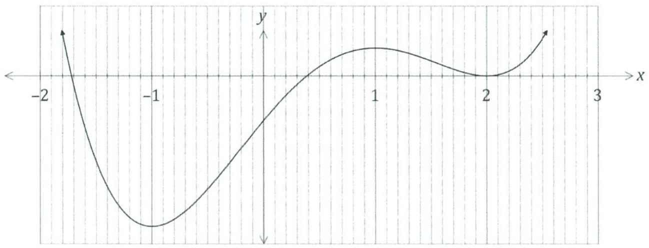
$\ln(3k-2) - 0 = 3$

$3k - 2 = e^3$

$k = \frac{e^3+2}{3}$

Question 4 **(5 marks)**

The graph of the derivative $f'(x)$ is shown below.



(a) State the number of local minima that f has. (1 mark)

1 ✓ ($x = 0.4$)

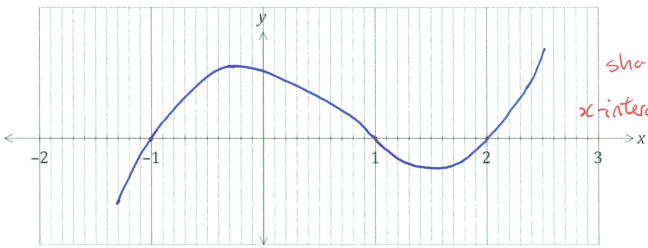
(b) For what values of x does f have a point of inflection? (1 mark)

$x = -1, 1, 2$ (gradient of $f'(x) = 0$) ✓

(c) Explain whether or not f has a horizontal point of inflection. (1 mark)

Yes, at $x = 2$ where $f'(2) = 0$, $f''(2) = 0$ ✓

(d) Sketch the graph of $y = f''(x)$ on the axis below. (2 marks)



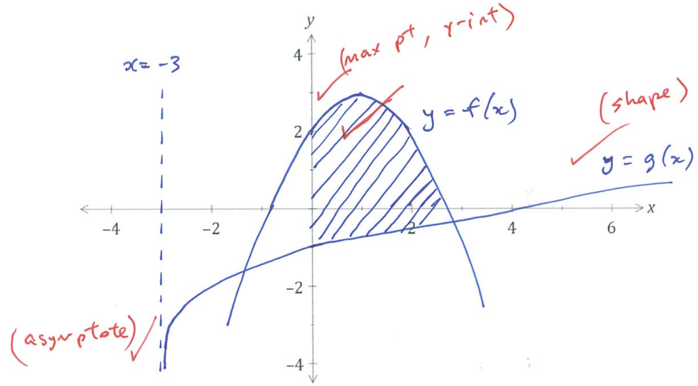
shape ✓

x-intercepts ✓

Question 10 **(8 marks)**

Consider the functions given by $f(x) = 3 - (1 - x)^2$ and $g(x) = \ln(x + 3) - 2$.

(a) Sketch both functions on the axes below, showing all asymptotes. (3 marks)



(b) Determine, to three decimal places, the values of x for which $f(x) = g(x)$. (1 mark)

$x = -1.086, 2.801$ ✓

(c) Let A be the region where $f(x) \geq g(x)$ and $x \geq 0$.

(i) Shade the region A on the graph above. (1 mark)

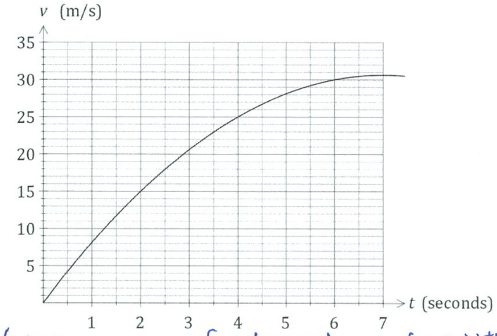
(ii) Write down an integral that represents the area of A, and evaluate this integral. (3 marks)

$$\text{area} = \int_0^{2.801} \left(3 - (1-x)^2 - \ln(x+3) + 2\right)\,dx$$ ✓ ✓

$= 7.623$ ✓

Question 9 **(10 marks)**

(a) The graph shows the velocity, in metres per second, of a car accelerating from rest. Use the graph to estimate the distance the car travelled in the first 6 seconds. (3 marks)



appropriate method

(Using area of trapezium of width 1 unit)

distance $= \frac{1}{2}(0+30) + 8 + 15 + 20.5 + 25 + 28$

$= 111.5$ (accept 110 to 113)

(b) A particle, initially at the origin, has velocity $3t^2 - 10t + 4$ metres per second at time t seconds, where $t \geq 0$.

(i) Determine the velocity of the body when it has no acceleration. (3 marks)

$\frac{dv}{dt} = 6t - 10 = 0$

$t = \frac{10}{6}$

$v = -\frac{13}{3}$ m/s

(ii) Determine the change in displacement of the body between $t = 1$ and $t = 5$. (2 marks)

change in displacement $= \int_1^5 3t^2 - 10t + 4 \, dt$

$= 20$ m

(iii) Determine the time(s) at which the body is at the origin for $t > 0$. (2 marks)

$0 + \int_0^t 3t^2 - 10t + 4 \, dt = 0$

$t = 1, 4$ s $(t > 0)$

Question 5 **(4 marks)**

Given that $F(x) = \int_0^x f(t)\,dt$, $\frac{d^2F}{dx^2} = 6x - 2$ and $F(3) = 6$, determine $f(x)$.

$\frac{dF}{dx} = \frac{d}{dx}\int_0^x f(t)\,dt$

$= f(x)$, using FTC

$\frac{d^2F}{dx^2} = f'(x) = 6x - 2$

$f(x) = 3x^2 - 2x + c$

$F(3) = 6$

$\int_0^3 3x^2 - 2x + c \, dx = 6$

$\left[x^3 - x^2 + cx\right]_0^3 = 6$

$27 - 9 + 3c = 6$

$c = -4$

$\therefore f(x) = 3x^2 - 2x - 4$

Additional working space

Question number: _________

Question 8 **(6 marks)**

A continuous random variable X has a probability density function defined by

$$f(x) = \begin{cases} kx & 1 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

where a and k are constants, and $a > 1$.

(a) Show that $k = \frac{2}{a^2-1}$. (2 marks)

$$\int_1^a kx \, dx = 1$$

$$\left[k\frac{x^2}{2} \right]_1^a = 1$$

$$\frac{k}{2}(a^2 - 1) = 1$$

$$k = \frac{2}{a^2-1}$$

(b) Given that $E(X) = 2.8$

(i) determine the value of a. (2 marks)

$$\frac{2}{a^2-1}\int_1^a x^2 \, dx = 2.8$$

$$a = 4$$

(ii) calculate $\text{Var}(X)$. (2 marks)

$$\text{Var}(X) = \frac{2}{4^2-1} \int_1^4 (x-2.8)^2 \, x \, dx$$

$$= \frac{33}{50} \quad (0.66)$$

Question 7 **(7 marks)**

A random survey was carried out to estimate the proportion of subscribers to a pay TV channel who had watched the last episode of a particular program. It was found that 98 out of 685 people surveyed had watched the episode.

(a) Determine $\hat{p}$, the sample proportion of those who had watched the episode. (1 mark)

$\hat{p} = \frac{98}{685}$

$= 0.143$

(b) Calculate the approximate margin of error for a 98% confidence interval estimate for p, the true proportion of subscribers. (3 marks)

$ME = 2.32635 \times \sqrt{\frac{0.143(1-0.143)}{685}}$

$= 0.0311$

(c) Determine a 98% confidence interval for p. (1 mark)

$98\%\ CI = 0.143 \pm 0.0311$

$= (0.112, 0.174)$

(d) If 20 similar surveys were carried out and a 98% confidence interval for p was calculated from each survey, determine the probability that fewer than 18 of the intervals will contain the true value of p. (2 marks)

let Y = no. of CI containing p

$Y \sim Bin(20, 0.98)$

$P(Y < 18) = 0.007$

To be continued on Thursday Assessment Period …

Christ Church Grammar School

2016

UNIT TEST 5

MATHEMATICS METHODS Year 12

Section Two:
Calculator-assumed

Student name Solution

Teacher name ______________

Time and marks available for this section

Reading time before commencing work:	2 minutes	+ 5 min
Working time for this section:	15 minutes	+ 45 min
Marks available:	15 marks	+ 45 marks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, and up to three calculators approved for use in the WACE examinations

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. Write your answers in this Question/Answer Booklet.

2. Answer all questions.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that **you do not use pencil**, except in diagrams.

Question 6 **(8 marks)**

Plastic lawn edging is supplied in nominal 2.1 m length rolls. The actual length, X metres, of a roll may be modelled by a normal distribution with mean 2.15 and standard deviation 0.03.

(a) Determine the probability that a randomly selected roll has length

(i) greater than 2.1 m. (1 mark)

$P(X > 2.1) = 0.9522$

(ii) less than 2.15 m given that it is greater than 2.1 m. (2 marks)

$P(X < 2.15 \mid X > 2.1)$

$= \dfrac{P(2.1 < X < 2.15)}{P(X > 2.1)} = \dfrac{0.4522}{0.9522}$

$= 0.4749$

(b) Determine the value of k given that 95% of rolls have lengths that exceed k m. (1 mark)

$P(X > k) = 0.95$

$k = 2.1007$

(c) A customer buys 10 rolls of lawn edging. Determine the probability that at least nine of the rolls have lengths of at least 2.1 m. (2 marks)

let Y = no. of rolls with length > 2.1 m

$Y \sim Bin(10, 0.9522)$

$P(Y \geq 9) = 0.9203$

(d) If the manufacturer wanted the lengths of at least 99% of rolls to exceed the nominal length, determine the required mean of the normal distribution, if the standard deviation remained at 0.03 m. (2 marks)

Suppose $X \sim N(\mu, 0.03^2)$

$P(X > 2.1) = 0.99$

$\dfrac{2.1 - \mu}{0.03} = -2.3263$

$\mu = 2.170$