PERTH MODERN SCHOOL
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INDEPENDENT PUBLIC SCHOOL

Semester One Examination, 2020

Question/Answer booklet

SPECIALIST MATHS
UNIT 3
Section One:
Calculator-free

Your Name

Your Teacher's Name

Time allowed for this section
Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Mark	Max	Question	Mark	Max
1			5		
2			6		
3			7		
4					

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	50	35
Section Two: Calculator-assumed	10	10	100	100	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section One: Calculator-free **(50 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 50 minutes.

Question 1 **(6 marks)**

Consider the polynomial $P(z)=z^4-z^3+3z^2-5z-10$.

(a) Determine $P(-1)$ (1 mark)

Solution
$P(-1)=z^4-z^3+3z^2-5z-10=1+1+3+5-10=0$
Specific behaviours
✓ states zero

(b) Show that $(z-\sqrt{5}i)$ is a factor of $P(z)$. (2 marks)

Solution
$P(\sqrt{5}i)=z^4-z^3+3z^2-5z-10=25+5\sqrt{5}i-15-5\sqrt{5}i-10=0$
Specific behaviours
✓ subs correct value for z ✓ shows that all 5 terms cancel to zero (simply stating is not enough)

(c) Determine all the roots to $P(z)=0$ (3 marks)

Solution

$P(z) = z^4 - z^3 + 3z^2 - 5z - 10 = (z+1)(z-a)(z-\sqrt{5}i)(z+\sqrt{5}i)$ $z = 0 \quad -10 = -5a \quad a = 2$ $z = -1, \pm\sqrt{5}i, 2$
Specific behaviours
✓ states real result as one root ✓ uses conjugate ✓ states all four correct roots

Question 2 **(10 marks)**

Consider the functions $g(x) = x^2$ & $f(x) = \dfrac{1}{\sqrt{x+5}}$.

(a) Determine the natural domain and range of $f(x)$. (2 marks)

Solution
$x > -5$ $y > 0$
Specific behaviours
✓ states domain ✓ states range

(b) Determine the rule for $g \circ f(x)$ and state its natural domain and range. (3 marks)

Solution
$g \circ f(x) = \dfrac{1}{x+5}$ $domain : x > -5$ $range : y > 0$
Specific behaviours
✓ states rule ✓ states natural domain of f ✓ states range

(c) Determine the rule and natural domain for $f \circ f(x)$. Explain why the composite exists.
(3 marks)

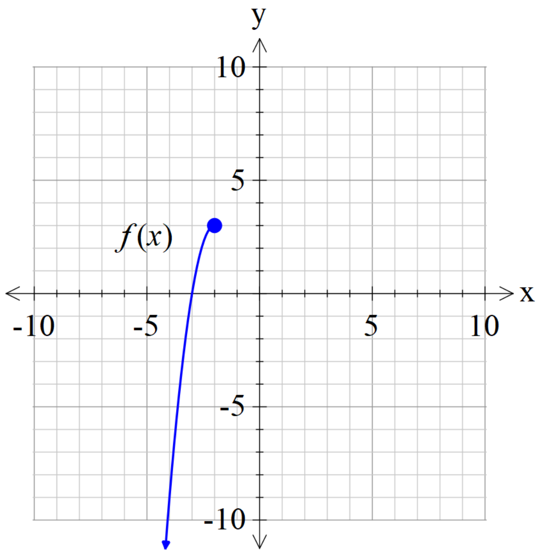
Solution
$f \circ f(x) = \dfrac{1}{\sqrt{\dfrac{1}{\sqrt{x+5}}+5}}$ $domain : x > -5$ $r_f : y > 0$ $d_f : x > -5$ $r_f \subseteq d_f \therefore exists$
Specific behaviours
✓ states un-simplified rules ✓ states domain ✓ shows relevant domain and range and rule for existence

(d) Does $g \circ f(x) = \dfrac{1}{x+5}$? Justify. (2 marks)

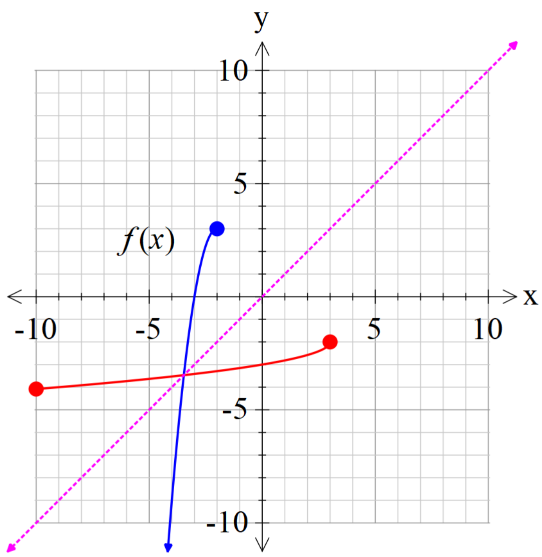
Solution
No as natural domain is $x \neq -5$ which is different to composite.
Specific behaviours
✓ states no with a reason ✓ correct reason

Question 3 **(9 marks)**

Consider the function $f(x)$ which is drawn below for $x \le -2$.



(a) Sketch $y = f^{-1}(x)$ on the axes above. (2 marks)

Solution

Specific behaviours
✓ appears to be reflected in line y=x ✓ endpoints clearly plotted to scale

(b) Given that $f(x) = -3x^2 - 12x - 9, x \le -2$, determine the rule for $y = f^{-1}(x)$ and state the domain and range. (4 marks)

Solution
$f(x) = -3x^2 - 12x - 9, x \le -2$ $d_{f^{-1}} : x \le 3$ $r_{f^{-1}} : y \le -2$ $x = -3y^2 - 12y - 9$ $3y^2 + 12y + 9 + x = 0$ $y = \frac{-12 \pm \sqrt{144 - 4(3)(9+x)}}{6} = \frac{-12 \pm \sqrt{36 - 12x}}{6} = \frac{-12 \pm 2\sqrt{3(3-x)}}{6}$ $f^{-1}(x) = \frac{-6 - \sqrt{3(3-x)}}{3}$
Specific behaviours
✓ states domain and range of inverse ✓ swaps x and y ✓ solves for rule with both signs ✓ states correct rule (may be un-simplified)

(c) Determine the exact solution(s) to $f(x) = f^{-1}(x)$ if any. (3 marks)

Solution
$x = -3x^2 - 12x - 9, x \le -2$ $0 = -3x^2 - 13x - 9$ $3x^2 + 13x + 9 = 0$ $x = \frac{-13 \pm \sqrt{169 - 4(3)(9)}}{6}$ $= \frac{-13 - \sqrt{61}}{6}$
Specific behaviours
✓ states an equation that will solve for x ✓ solves for x using quadratic formula with working shown ✓ discards the invalid result and states correct exact value (may be un-simplified)

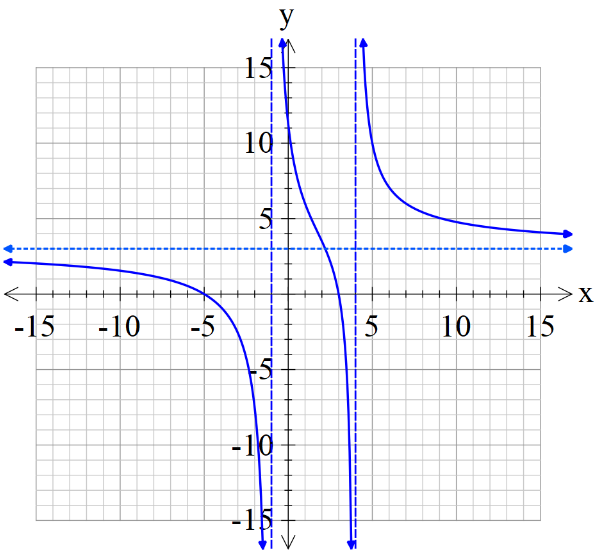
Question 4 **(3 marks)**

Consider the complex equation $z^n = 1 + i$ for any positive integer $n \geq 3$. The n roots are designated $z_0, z_1, z_2 \ldots z_{n-1}$.

Let $p = z_0 \times z_1 \times z_2 \ldots z_{n-1}$, determine $|p|$ for any positive integer $n \geq 3$. Explain.

Solution
$z^n = 1 + i = \sqrt{2}\,cis(\frac{\pi}{4} + 2m\pi) \quad m = 0, \pm 1, \pm 2 \ldots$ $z = 2^{\frac{1}{2n}} cis\left(\frac{\pi}{4n} + \frac{2m\pi}{n}\right)$ $p = 2^{\frac{1}{2n}} cis\left(\frac{\pi}{4n}\right) \times 2^{\frac{1}{2n}} cis\left(\frac{\pi}{4n} + \frac{2\pi}{n}\right) \ldots 2^{\frac{1}{2n}} cis\left(\frac{\pi}{4n} + \frac{2(n-1)\pi}{n}\right)$ $|p| = \left(2^{\frac{1}{2n}}\right)^n = \sqrt{2}$
Specific behaviours
✓ uses De Moivers   ✓ shows that there are n terms with each modulus being $2^{\frac{1}{2n}}$ ✓ states the required result

Question 5 **(5 marks)**



The function $f(x) = \dfrac{ax^2 + bx + c}{x^2 + px + q}$ is drawn to the left where $a, b, c, p \,\&\, q$ are all integers.

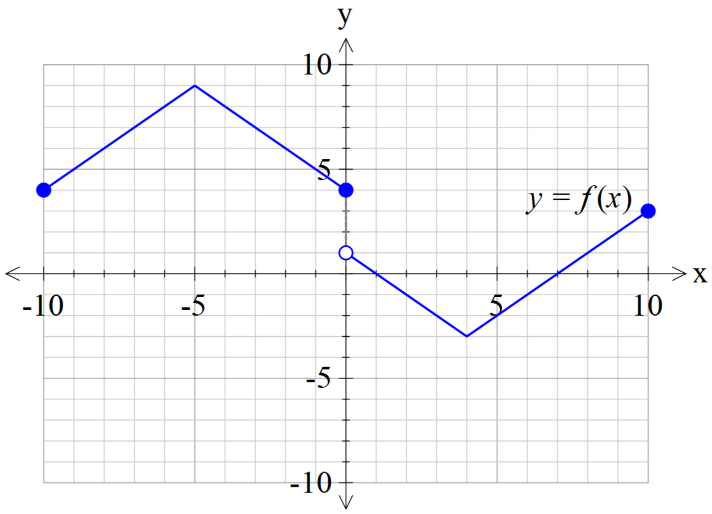
Solution

a	b	c	p	q
3	6	-45	-3	-4

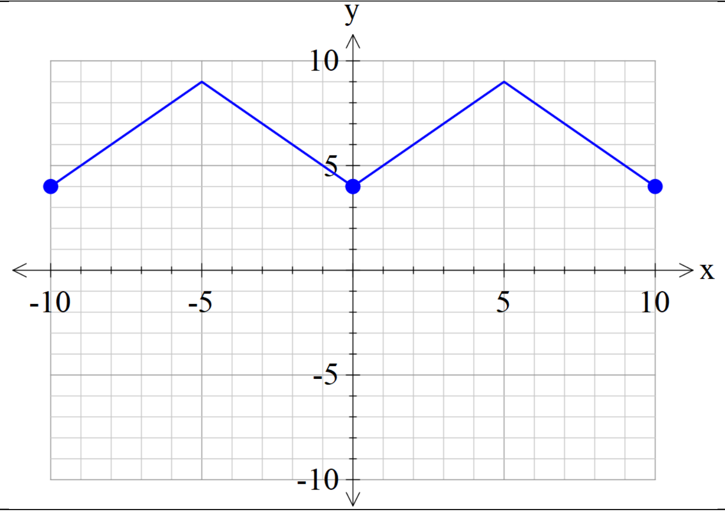
Specific behaviours
✓ one mark for each correct value

Question 6 (8 marks)

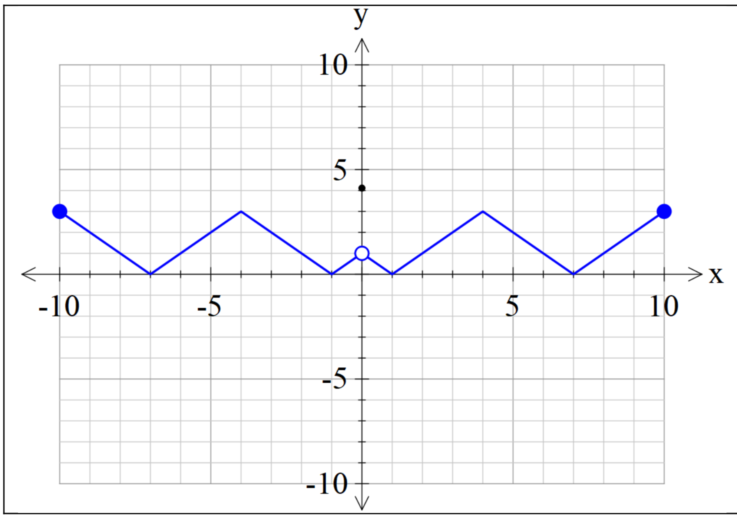
Consider the function $f(x)$ which is drawn below and is defined for $-10 \le x \le 10$.



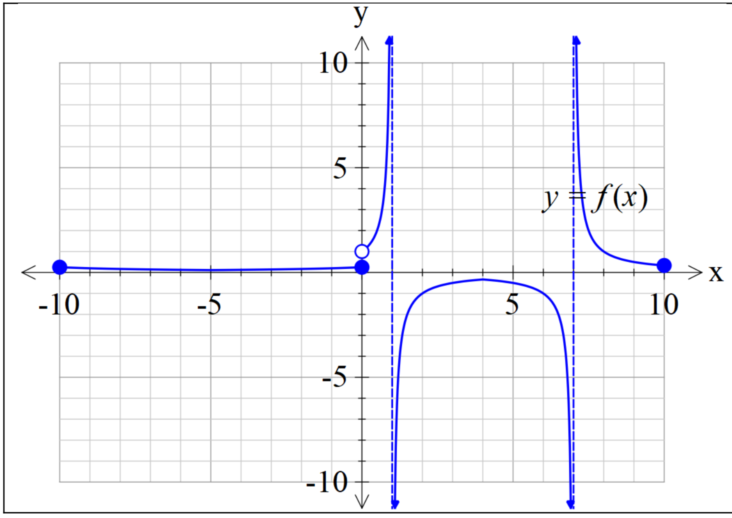
(a) Sketch $y = f(-|x|)$ on the axes below. (2 marks)

	**1 mark reflects in y axis** **1 mark uses left hand side**

(b) Sketch $y = \left|f\left(|x|\right)\right|$ on the axes below. (3 marks)

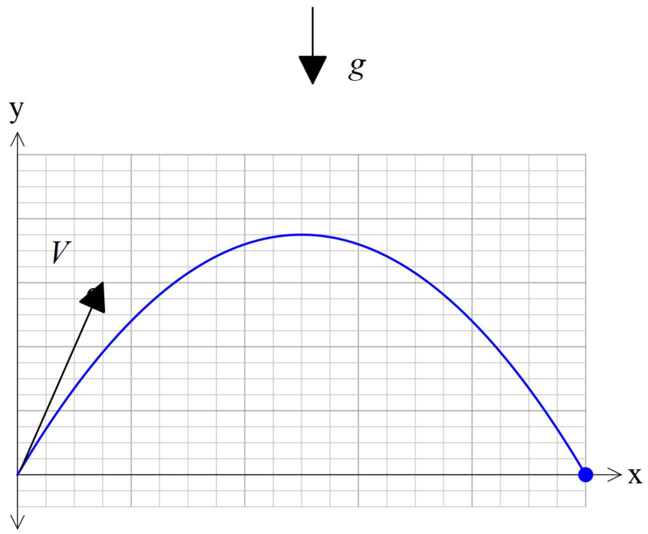
	**1 plots y intercept as 5 only** **1 mark uses right hand side** **1 mark reflects negative ys in x axis**

(c) Sketch $y = \dfrac{1}{f(x)}$ on the axes below. (3 marks)

	**1 mark for defined y intercept** **1 mark for both vertical asymptotes dotted** **1 mark for shape**

Question 7 **(9 marks)**

Consider a projectile that has an initial speed, $V\ m/s$, at an angle of α with the horizontal that moves with an acceleration of $\ddot{r} = \begin{pmatrix} 0 \\ -g \end{pmatrix} m/s^2$ where g is a constant.



a) If the projectile begins at the origin, show that a time, $t\ s$, and using vector calculus that the velocity vector is given by: (2 marks)

$$\dot{r} = \begin{pmatrix} V\cos\alpha \\ V\sin\alpha - gt \end{pmatrix} m/s$$

Solution
$\ddot{r} = \begin{pmatrix} 0 \\ -g \end{pmatrix} \quad \dot{r}(0) = \begin{pmatrix} V\cos\alpha \\ V\sin\alpha \end{pmatrix}$ $\dot{r} = \begin{pmatrix} 0 \\ -gt \end{pmatrix} + c$ $t = 0 \quad c = \begin{pmatrix} V\cos\alpha \\ V\sin\alpha \end{pmatrix}$ $\dot{r} = \begin{pmatrix} V\cos\alpha \\ V\sin\alpha - gt \end{pmatrix}$
Specific behaviours
✓ determine initial velocity   ✓ integrates and solves for vector constant

b) In terms of $V, g, \alpha \& t$ derive the cartesian equation of the projectile. (4 marks)

Solution
$\dot{r} = \begin{pmatrix} V\cos\alpha \\ V\sin\alpha - gt \end{pmatrix} \quad r(0) = \underset{\sim}{0}$ $r = \begin{pmatrix} Vt\cos\alpha \\ Vt\sin\alpha - \frac{g}{2}t^2 \end{pmatrix} + c \quad c = \underset{\sim}{0}$ $x = Vt\cos\alpha \quad y = Vt\sin\alpha - \frac{g}{2}t^2$ $t = \frac{x}{V\cos\alpha} \quad y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$
Specific behaviours
✓ integrates and shows vector constant ✓ obtains expression for t using x parametric equation ✓ subs t into y parametric equation ✓ derives above equation

c) Given that $V = \sqrt{5} \& g = 10$ show that α is a solution of the following equation.
$x^2\tan^2\alpha - x\tan\alpha + x^2 + y = 0$. (3 marks)

Solution
$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$ $y = x\tan\alpha - \frac{10x^2}{10}\sec^2\alpha$ $y = x\tan\alpha - \frac{10x^2}{10}(1+\tan^2\alpha)$ $x^2\tan^2\alpha - x\tan\alpha + x^2 + y = 0$
Specific behaviours
✓ uses secx ✓ uses identity for tan and sec ✓ subs values for V & g

End of questions