

# **Test 4**

## **(Matrices, Exponentials and Logarithms, Functions)**



This assessment contributes 5% towards the final year mark.  
45 minutes are allocated for this task.

# **MARKING KEY and SOLUTIONS**

**Part A**     **The use of a CAS calculator is assumed.**  
(12 minutes permitted)

Do NOT turn over this page until you are instructed to do so.

1. A native reptile of MathMagic Isle called the hypottentot has a generation change every 2 years. The table below shows the survival rates, breeding rates and the initial population profile for 5 of the age groups :

Age (years)	0-2	2-4	4-6	6-8	8-10
Survival Rate	0.6	0.8	0.7	0.4	0
Breeding Rate	0.1	0.9	1.4	0.5	0.4
Initial Population	10	12	15	20	10

- a. State the Leslie matrix  $L$  for this colony of hypottentots.

$$L = \begin{bmatrix} 0.1 & 0.9 & 1.4 & 0.5 & 0.4 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix}$$

✓ 5 x 5 matrix with breeding rates in Row 1  
✓ Other elements correct

[2]

- b. Use matrix  $L$  to determine the population profile after 10 years.

**10 years is equivalent to 5 generations.**  
 **$P(5) = L^5 \times P(0)$**

✓ 5 generations or transitions  
✓ expression for  $P(5)$

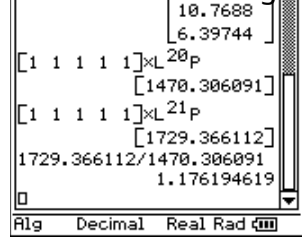
$$P(5) = \begin{bmatrix} 0.1 & 0.9 & 1.4 & 0.5 & 0.4 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix}^5 \begin{bmatrix} 10 \\ 12 \\ 15 \\ 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 57.16... \\ 33.29... \\ 19.31... \\ 10.74... \\ 6.30... \end{bmatrix}$$

✓ sensible conclusion using whole values

**So after 10 years there will be approximately 57 aged 0-2 yrs, 33 aged 2-4 yrs, 19 aged 4-6 yrs, 11 aged 6-8 yrs and 6 aged 8-10 yrs.**

[3]

- c. Determine the long term inter-generational growth rate, as a percentage.



**From CAS calculator, inter-generational growth rate is approx 17.6%.**

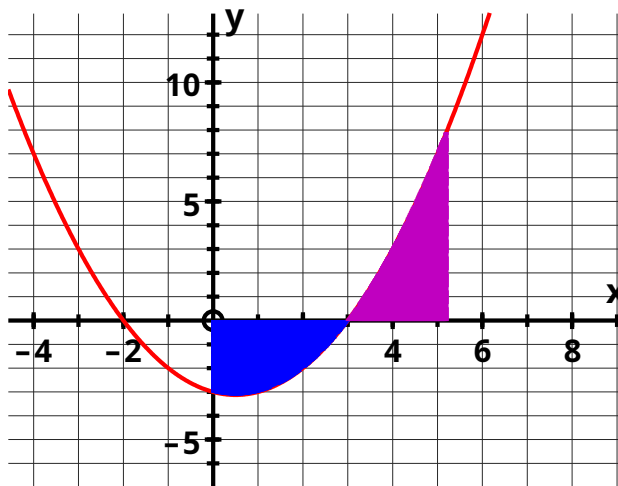
✓ percentage growth conclusion

- ✓ explores consecutive generations for a LARGE number of transitions.
- ✓ determines totals for each generation.

[3]

2. The graph of function  $f(x) = 0.5(x + 2)(x - 3)$  is shown below.

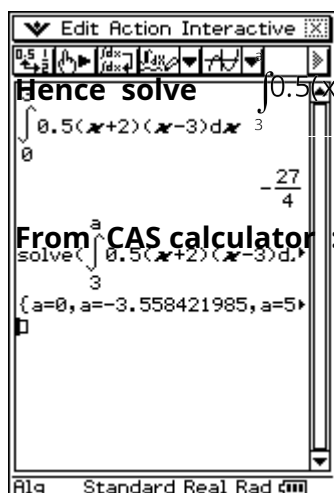
Region A is bounded by the curve, the x axis, and the lines  $x = 0$  and  $x = 3$ .  
Region B is bounded by the curve, the x axis, and the lines  $x = 3$  and  $x = a$ .



If the areas of regions A and B have the same area, determine the value of the constant  $a$  correct to 0.001.

$$\text{Area A} = - \int_0^3 0.5(x^2 - x - 6) dx = \frac{27}{4}$$

$$\text{Area B} = \int_3^a 0.5(x^2 - x - 6) dx$$



From CAS calculator :  $a = 5.058$  (3 d.p.)

✓ value for Region A

- ✓ expression for Region B
- ✓ equation for equal areas
- ✓ solves for a correct to 3 dp

[4]

Year 12 3CD Mathematics Specialist

Term 2, 2010

## Test 4

### (Matrices, Exponentials and Logarithms, Functions)

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**Part B**      **No calculator to be used.**  
(33 minutes permitted)

# MARKING KEY and SOLUTIONS

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3. Consider the following transformation matrices in the co-ordinate plane :

- R rotates  $180^\circ$  about the origin  
D dilates vertically about  $y = 0$  with factor 1.5  
S downward shear parallel to the vertical axis with factor 1

a. Give matrices R, D and S.

$$\mathbf{R} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

✓ each matrix correct

[3]

The 3 diagrams below show a parallelogram.

Draw the image of this parallelogram under the action of transformation :

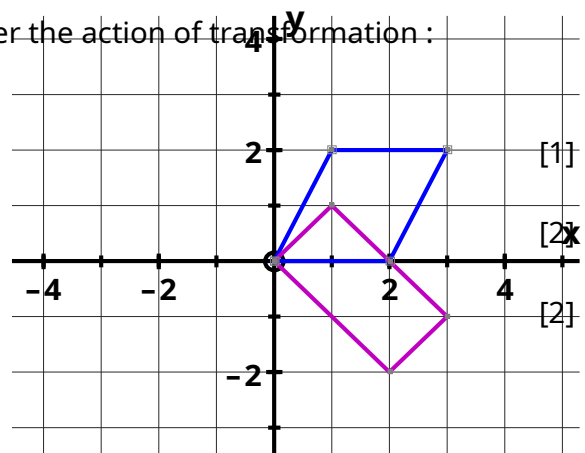
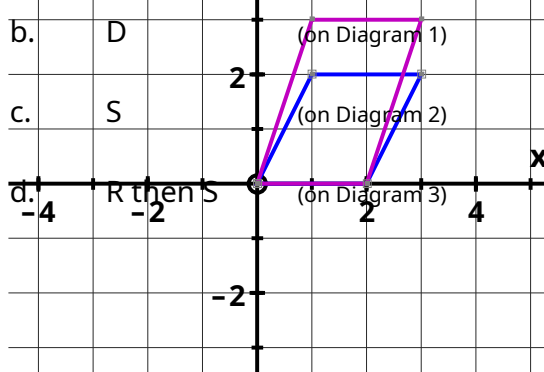


Diagram 1  
✓ shows the dilation correctly

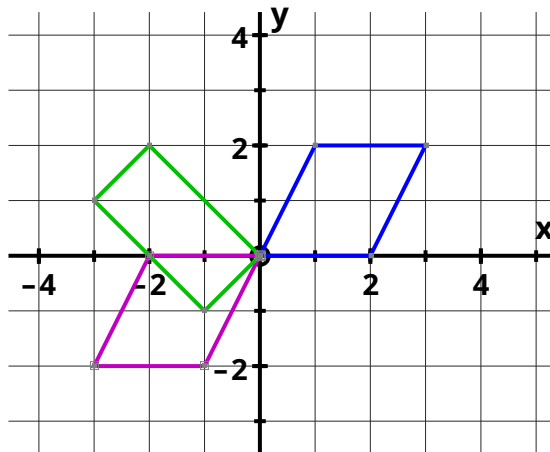


Diagram 2  
✓ shows image of (3,2) as (3,-1)  
✓ shows the TOTAL image correctly as a rectangle

Diagram 3

✓ shows the rotation correctly  
✓ shows the shear correctly

3. e. Does transformation matrix  $S$  change the area of any object it transforms? Explain.

$$S = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \det(S) = 1(1) - (-1)(0) = 1$$

Hence since  $|\det(S)| = 1$ , then the AREA of the image will NOT be any different from the area of the object.

✓ calculates the determinant  
✓ makes the correct conclusion

[2]

- f. If the parallelogram is transformed by matrix  $D$  then  $S$ , what matrix will return the resultant image back to the original parallelogram?

Transformation  $T$ :  $D$  then  $S$  i.e.  $T = SD$

We require  $T^{-1}$ .

$$T = SD = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1.5 \end{bmatrix}$$

$$T^{-1} = \frac{1}{1.5 - 0} \begin{bmatrix} 1.5 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

- ✓ interprets matrix  $SD$
- ✓ calculates matrix  $SD$
- ✓ calculates the INVERSE to return image to the original

[3]

See next page

4. Find the following indefinite integrals, using an appropriate Calculus technique :  
[ONE mark will be given for a correct answer only]

$$\begin{aligned} \text{a. } \int \frac{4x + 8}{(x^2 + x)^5} dx &= \int (4x + 8)(x^2 + 4x)^{-5} dx \\ &= (4x + 8) \cdot \frac{(x^2 + 4x)^{-4}}{(-4)(2x + 4)} + c \\ &= \frac{(x^2 + 4x)^{-4}}{(-2)} + c \\ &= -\frac{1}{2(x^2 + 4x)^4} + c \end{aligned}$$

- ✓ integrates power (-5)
- ✓ derivative factor cancels
- ✓ simplifies expression

[3]

b. 
$$\int \frac{e^{2x}}{e^{2x} + 1} dx = \int e^{2x} (e^{2x} + 1)^{-1} dx$$

✓ recognises reciprocal function  
✓ In anti-derivative

$$= \frac{e^{2x} \ln|e^{2x} + 1|}{e^{2x} \cdot 2} + c$$

$$= \frac{1}{2} \ln(e^{2x} + 1) + c$$

✓ divide by derivative factor to simplify correctly

[3]

c. 
$$\int 30x\sqrt{x+2} dx$$
 Put  $u = x+2$   $\frac{du}{dx} = 1$

$$= \int 30(u-2)\sqrt{u} du$$

✓ express x in terms of u  
✓ multiply integrand

$$= \int 30(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}) du$$

$$= \frac{30u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{60u^{\frac{3}{2}}}{\frac{3}{2}} + c$$

✓ anti-derivative

$$= 12(x+2)^{\frac{5}{2}} - 40(x+2)^{\frac{3}{2}} + c$$

✓ express in terms of x

$$= 12\sqrt{(x+2)^5} - 40\sqrt{(x+2)^3} + c$$

✓ ONE mark penalty in Question 4 for lack of an integration constant

[4]

5. Evaluate the following definite integrals, using the given substitution and an anti-derivative technique : [ONE mark will be given for a correct answer only]

a. 
$$\int_{\frac{1}{6}}^{\frac{1}{3}} \frac{dx}{\sqrt{1-9x^2}}$$

Put  $3x = \cos \theta$

$$3 \cdot \frac{dx}{d\theta} = -\sin \theta$$

$$dx = -\frac{\sin \theta}{3} d\theta$$

x	$\frac{1}{6}$	$\frac{1}{3}$
$\theta$	$\frac{\pi}{3}$	0



$$= \int_{\frac{\pi}{3}}^0 - \frac{\sin \theta}{3\sqrt{1 - \cos^2 \theta}} d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin \theta}{3 \sin \theta} d\theta \quad \text{since for } 0 < \theta < \frac{\pi}{2}, \quad \sin \theta > 0$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{3} d\theta$$

$$= \left[ \frac{\theta}{3} \right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{9}$$

- ✓ express dx in terms of dθ
- ✓ change limits of integration
- ✓ simplifies integrand using the identity for  $1 - \cos^2 \theta$
- ✓ evaluates correctly

[4]

5. b. Given that  $\frac{d}{du} [\tan^{-1}(u)] = \frac{1}{1+u^2}$  evaluate  $\int_0^{\frac{\pi}{4}} \tan^2 \theta d\theta$   
by using  $u = \tan \theta$ .

$$\frac{du}{d\theta} = \sec^2 \theta \quad \therefore \quad d\theta = \frac{du}{\sec^2 \theta}$$

$$\int_0^{\frac{\pi}{4}} \tan^2 \theta \, d\theta = \int_0^1 \frac{u^2 \, du}{\sec^2 \theta}$$

$\theta$	0	$\frac{\pi}{4}$
$u$	0	1

$$= \int_0^1 \frac{u^2 \, du}{u^2 + 1}$$

$$= \int_0^1 \frac{u^2 + 1}{u^2 + 1} \, du - \int_0^1 \frac{1}{u^2 + 1} \, du$$

$$= \int_0^1 1 \, du - \int_0^1 \frac{1}{u^2 + 1} \, du$$

$$= \left[ u \right]_0^1 - \left[ \tan^{-1}(u) \right]_0^1$$

$$= 1 - \tan^{-1}(1) + \tan^{-1}(0)$$

$$= 1 - \frac{\pi}{4}$$

- ✓ expresses  $d\theta$  in terms of  $du$
- ✓ change limits of integration
- ✓ uses identity for  $\sec^2 \theta = 1 + \tan^2 \theta$
- ✓ splits integrand into 2 parts
- ✓ anti-derivatives correct (using the given result for inverse tangent)
- ✓ evaluates correctly

[6]

End of Assessment Task