SCHOOL

Semester One Examination, 2013

Question/Answer Booklet

MATHEMATICS SPECIALIST 3C

SOLUTIONS

Section One: Calculator-free

Student Number:	In figures	
	In words	______________________________
	Your name	______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	50	33
Section Two: Calculator-assumed	12	12	100	100	67
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2013*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

Section One: Calculator-free (50 Marks)

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 (4 marks)

Prove by deduction that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Question 2 **(8 marks)**

Two complex numbers are given by $w = 2cis\frac{\pi}{3}$ and $z = \sqrt{3} - i$.

(a) Express in polar form

(i) z (1 mark)

$$\boxed{2cis\left(-\frac{\pi}{6}\right)}$$

(ii) $w \cdot z$ (1 mark)

$$\boxed{4cis\frac{\pi}{6}}$$

(b) State the

(i) argument of $\frac{z}{w}$ (1 mark)

$$\boxed{-\frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{2}}$$

(ii) modulus of $\frac{1}{z}$ (1 mark)

$$\boxed{\frac{1}{2}}$$

(c) Show that the product $w \cdot \overline{w}$ is purely real. (2 marks)

$$\begin{aligned} 2cis\left(\frac{\pi}{3}\right) \times 2cis\left(-\frac{\pi}{3}\right) &= 4cis0 \\ &= 4\cos(0) + 4i\sin(0) \\ &= 4(1) + 4i(0) \\ &= 4 \end{aligned}$$

(d) For what positive values of n is v_n purely real, given $v_{n+1} = v_n \times w$, $v_1 = w$? (2 marks)

$v_2 = 4cis\left(\frac{2\pi}{3}\right)$

$v_3 = 8cis\left(\frac{3\pi}{3}\right) = -8$

$v_4 = ...$

When $n = 3, 6, 9, 12, ...$

Hence n must be a multiple of 3.

Question 3 **(9 marks)**

(a) Find $\frac{dy}{dx}$ for the following:

(i) $y = e^{x^2 \cos 3x}$ (2 marks)

$$\frac{dy}{dx} = e^{x^2 \cos 3x}\left(2x\cos 3x - 3x^2 \sin 3x\right)$$
$$= xe^{x^2 \cos 3x}\left(2\cos 3x - 3x\sin 3x\right)$$

(ii) $y = \log_4\left(1 + x^3\right)^2$ (2 marks)

$$y = \frac{\ln(1+x^3)^2}{\ln 4}$$
$$y = \frac{2\ln(1+x^3)}{2\ln 2}$$
$$\frac{dy}{dx} = \frac{3x^2}{\ln 2 \times (1+x^3)}$$

(b) Determine

(i) $\int \sin 2x \cdot e^{\cos 2x}\left(5 + e^{\cos 2x}\right)^3 \, dx$ (3 marks)

$$u = 5 + e^{\cos 2x} \Rightarrow du = -2\sin 2x \cdot e^{\cos 2x} dx$$
$$\int \sin 2x \cdot e^{\cos 2x}\left(5 + e^{\cos 2x}\right)^3 \, dx$$
$$= -\frac{1}{2}\int u^3 du$$
$$= -\frac{1}{2}\frac{u^4}{4} + c$$
$$= \frac{-\left(5 + e^{\cos 2x}\right)^4}{8} + c$$

(ii) $\int \frac{2}{x\ln x} \, dx$ (2 marks)

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$
$$\int \frac{2}{x\ln x}\, dx = 2\int \frac{1}{u} du$$
$$= 2\ln u + c$$
$$= 2\ln(\ln x) + c$$

Question 4 **(8 marks)**

(a) Find the values of λ and μ when $\lambda(\mathbf{a}+\mathbf{b}) - (1+2\mu)\mathbf{a} = (\mu+2)\mathbf{b}$, given that $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. (3 marks)

$$
\begin{aligned}
&(\lambda - 1 - 2\mu)\mathbf{a} = (\mu + 2 - \lambda)\mathbf{b} \\
&\lambda - 1 - 2\mu = 0 \\
&\mu + 2 - \lambda = 0 \\
&\Rightarrow -\mu + 1 = 0 \\
&\mu = 1 \\
&\lambda = 3
\end{aligned}
$$

(b) P, Q and R are three collinear points with position vectors **p**, **q** and **r** respectively, where Q lies between P and R. If $|\overrightarrow{QR}| = \frac{1}{3}|\overrightarrow{PQ}|$, find **r** in terms of **p** and **q**. (2 marks)

$$
\begin{aligned}
\overrightarrow{OR} &= \overrightarrow{OP} + \overrightarrow{PQ} + \overrightarrow{QR} \\
\mathbf{r} &= \mathbf{p} + 4(\mathbf{r} - \mathbf{q}) \\
3\mathbf{r} &= 4\mathbf{q} - \mathbf{p} \\
\mathbf{r} &= \frac{4}{3}\mathbf{q} - \frac{1}{3}\mathbf{p}
\end{aligned}
$$

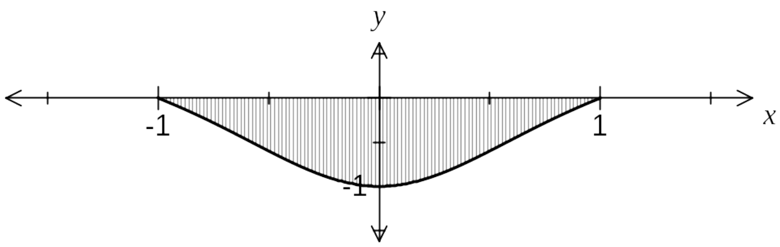
(c) Point C has position vector $-11\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ and point D has position vector $-3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Find the position vector of the point P that divides $\overrightarrow{CD}$ internally in the ratio 3:1. (3 marks)

$$
\overrightarrow{OD} - \overrightarrow{OC} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -11 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 4 \end{bmatrix}
$$

$$
\overrightarrow{OP} = \begin{bmatrix} -11 \\ 5 \\ -2 \end{bmatrix} + \left(\frac{3}{1+3}\right)\begin{bmatrix} 8 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}
$$

Question 5 **(6 marks)**

The graph of $y = 1 - \dfrac{2}{1+x^2}$ is shown for $-1 \le x \le 1$.



(a) Using the substitution $x = \tan\theta$ or otherwise, show that $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$. (4 marks)

$$x = \tan\theta$$
$$1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$$
$$dx = \sec^2\theta\, d\theta$$
$$\begin{aligned}\int \frac{1}{1+x^2}\,dx &= \int \frac{1}{\sec^2\theta}\sec^2\theta\, d\theta \\ &= \int d\theta \\ &= \theta + c \\ &= \tan^{-1} x + c\end{aligned}$$

(b) Find the exact area of the shaded region above. (2 marks)

$$\begin{aligned}\int_{1}^{1} 1 - \frac{2}{1+x^2}\,dx &= \left[x - 2\tan^{-1} x\right]_{-1}^{1} \\ &= \left(1 - 2\times\frac{\pi}{4}\right) - \left(-1 + 2\times\frac{\pi}{4}\right) \\ &= 2 - \pi\end{aligned}$$

Area $= \pi - 2$ sq units

Question 6 **(7 marks)**

A function is defined as $f(x)=\int_0^x(\cos 2t)(\sin^2 2t)\,dt$ for $0 \le x \le \frac{\pi}{2}$.

(a) Determine the x-coordinates of the stationary values of $f(x)$. (2 marks)

Require $f'(x)=0$

$f'(x)=\cos 2x\sin^2 2x$

$=0$ when $x=0, \frac{\pi}{4}, \frac{\pi}{2}$.

(b) Express $f(x)$ in terms of x only. (3 marks)

If $y=\sin^3 2t$ then $\frac{dy}{dt}=6\cos 2t\sin^2 2t$

Hence $f(x)=\frac{1}{6}\int_0^x \cos 2t\sin^2 2t\,dt$

$=\frac{1}{6}\left[\sin^3 2t\right]_0^x$

$=\frac{\sin^3 2x}{6}$

(c) Find the maximum value of $f(x)$. (2 marks)

$f(0)=0$

$f\left(\frac{\pi}{4}\right)=\frac{1}{6}$

$f\left(\frac{\pi}{2}\right)=0$

Hence maximum value is $\frac{1}{6}$

Question 7 **(8 marks)**

Two particles, A and B, are tracked in a laboratory experiment.

Particle A was observed to have initial position $2\mathbf{i} - \mathbf{j} + 8\mathbf{k}$ cm and constant velocity vector $-2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ cm/s.

Particle B was observed to have initial position $-2\mathbf{i} + 3\mathbf{j} - 9\mathbf{k}$ cm and constant velocity vector $3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ cm/s.

If the particles continue with these velocities, find the minimum distance between them in the subsequent motion and the time when this occurs.

Let $\mathbf{v}$ and $\mathbf{r}$ be velocity and displacement of A relative to B

$$\mathbf{v} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ -6 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 17 \end{bmatrix}$$

$$t\mathbf{v} + \mathbf{r} = \begin{bmatrix} -5t + 4 \\ 2t - 4 \\ -6t + 17 \end{bmatrix}$$

$$\begin{bmatrix} -5 \\ 2 \\ -6 \end{bmatrix} \bullet \begin{bmatrix} -5t + 4 \\ 2t - 4 \\ -6t + 17 \end{bmatrix} = 25t - 20 + 4t - 8 + 36t - 102$$

$65t - 130 = 0$ when $t = 2$ seconds

$$2\mathbf{v} + \mathbf{r} = \begin{bmatrix} -10 + 4 \\ 4 - 4 \\ -12 + 17 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 5 \end{bmatrix}$$

$$|\,2\mathbf{v} + \mathbf{r}\,| = \sqrt{36 + 25} = \sqrt{61} \text{ cm}$$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

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