Rossmoyne Senior High School

Year 12 Trial WACE Examination, 2014

Question/Answer Booklet

MATHEMATICS 3C/3D

Section One:
Calculator-free

SOLUTIONS

Student Number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	50	33⅓
Section Two: Calculator-assumed	13	13	100	100	66⅔
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2013*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

Section One: Calculator-free **(50 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(3 marks)**

If $f'(x) = (x-1)e^{x^2-2x+1}$ and $f(1)=1$, determine $f(x)$.

$y = e^{x^2-2x+1} \Rightarrow y' = (2x-2)e^{x^2-2x+1}$

$f(x) = \frac{1}{2}e^{x^2-2x+1} + c$

When $x = 1$

$1 = \frac{1}{2}e^{0} + c \Rightarrow c = \frac{1}{2}$

$f(x) = \frac{1}{2}e^{x^2-2x+1} + \frac{1}{2}$

Question 2 **(5 marks)**

The function $f(x) = ax^2 + bx + c$, where a, b and c are constants, passes through the three points (3, 10), (2, -1) and (1, -6).

(a) Explain why a, b and c the satisfy the equation $9a + 3b + c = 10$. (1 mark)

When $x = 3$, $f(x) = 10$ and by substitution
$a(3)^2 + b(3) + c = 10 \Rightarrow 9a + 3b + c = 10$

(b) Write down another two equations satisfied by a, b and c. (1 mark)

$4a + 2b + c = -1$

$a + b + c = -6$

(c) Solve the above equations to determine the values of a, b and c. (3 marks)

$9a + 3b + c = 10 \quad (1)$
$4a + 2b + c = -1 \quad (2)$
$a + b + c = -6 \quad (3)$

$(1) - (2): 5a + b = 11 \quad (4)$
$(2) - (3): 3a + b = 5 \quad (5)$

$(5) - (4): 2a = 6 \Rightarrow a = 3$

$3(3) + b = 5 \Rightarrow b = -4$
$3 - 4 + c = -6 \Rightarrow c = -5$

Question 3 **(8 marks)**

(a) If $f(x) = (1-3x)^3 \cdot \sqrt{3x-2}$, evaluate $f'(1)$. (4 marks)

$$u = (1-3x)^3\Big|_{x=1} = -8$$
$$u' = 3(-3)(1-3x)^2\Big|_{x=1} = -36$$

$$v = (3x-2)^{0.5}\Big|_{x=1} = 1$$
$$v' = (0.5)(3)(3x-2)^{-0.5}\Big|_{x=1} = \tfrac{3}{2}$$

$$\begin{aligned} f'(1) &= (-36)(1) + (-8)(\tfrac{3}{2}) \\ &= -36 - 12 \\ &= -48 \end{aligned}$$

(b) Determine the area enclosed between the curve $y = 3(x-1)^2$ and the line $y = 3$. (4 marks)

$$3(x-1)^2 = 3$$
$$3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0 \Rightarrow x = 0, x = 2$$
$$\int_0^2 6x - 3x^2 \, dx = \left[3x^2 - x^3\right]_0^2$$
$$= (12-8) - (0-0) = 4 \text{ sq units}$$

Question 4 **(7 marks)**

Let $f(x)=\sqrt{2-x}$ and $g(x)=\dfrac{1}{x-2}$.

(a) Determine

(i) an expression for $f\circ g(x)$. (1 mark)

$$\boxed{f\circ g(x)=\sqrt{2-\frac{1}{x-2}}}$$

(ii) the domain of $f\circ g(x)$. (2 marks)

$$2-\frac{1}{x-2}=\frac{2x-5}{x-2}$$

$$\frac{2x-5}{x-2}\geq 0 \;\Rightarrow\; x<2 \text{ or } x\geq 2.5$$

(b) Determine

(i) the domain of $g\circ f(x)$. (2 marks)

$$g\circ f(x)=\frac{1}{\sqrt{2-x}-2} \;\Rightarrow\; \begin{cases} 2-x\neq 4 \;\Rightarrow\; x\neq -2 \\ 2-x\geq 0 \;\Rightarrow\; x\leq 2 \end{cases}$$

Hence $x\leq 2$ and $x\neq -2$

(ii) the range of $g\circ f(x)$. (2 marks)

For $x<-2$

$y>0$

For $-2<x\leq 2$

$g\circ f(2)=-\frac{1}{2} \;\Rightarrow\; y\leq -\frac{1}{2}$

Hence $y>0$ or $y\leq -\frac{1}{2}$

Question 5 **(7 marks)**

The radius, in centimetres, of a circular ink spot t seconds after it first appears is given by $r = \dfrac{3+4t}{2+t}$.

(a) Determine

(i) the time taken for the radius to double its initial value. (2 marks)

$$r(0) = \frac{3+4(0)}{2+(0)} = 1.5 \;\Rightarrow\; 2 \times r(0) = 3$$

$$\frac{3+4t}{2+t} = 3$$

$$3+4t = 6+3t$$

$$t = 3$$

(ii) the rate at which the radius is increasing when $t = 3$. (3 marks)

$$\frac{dr}{dt} = \frac{4(2+t) - 1(3+4t)}{(2+t)^2}$$

$$= \frac{5}{(2+t)^2}$$

When $t = 3$, $\dfrac{dr}{dt} = \dfrac{5}{5^2} = \dfrac{1}{5}$ cms^{-1}

(b) Use the formula $\delta r \approx \dfrac{dr}{dt} \cdot \delta t$ to approximate the increase in the radius of the ink spot between $t = 3$ and $t = 3.05$ seconds. (2 marks)

$$\delta r \approx \frac{1}{5} \times (3.05 - 3)$$

$$\approx \frac{0.05}{5}$$

$$\approx 0.01 \text{ cm}$$

Question 6 **(8 marks)**

Let $f(x)=\dfrac{x(12-x^2)}{3}$ and $g(x)=(ax+2)(x-7)$, where a is a constant.

(a) Show that $f'(x)=(2+x)(2-x)$. (2 marks)

$$f(x)=\frac{12x-x^3}{3}$$
$$f'(x)=4-x^2$$
$$=(2+x)(2-x)$$

(b) The tangents to the graphs of $y=f(x)$ and $y=g(x)$ are parallel when $x=3$. Determine the value of a. (3 marks)

$$f'(3)=(5)(-1)=-5$$

$$g'(x)=a(x-7)+1(ax+2)$$
$$g'(3)=-4a+3a+2$$
$$=2-a$$

$$2-a=-5 \Rightarrow a=7$$

(c) What are the minimum and maximum values of $f(x)$ over the domain $-1\le x\le 4$? (3 marks)

$f'(x)=0$ when $\cancel{x=-2}, x=2$

$$f(-1)=\frac{-11}{3}$$
$$f(2)=\frac{16}{3}$$
$$f(4)=\frac{-16}{3}$$

Maximum is $\dfrac{16}{3}$ and minimum is $\dfrac{-16}{3}$

Question 7 **(5 marks)**

In a pack of six identical rechargeable batteries, it is known that two are flat and the other four are fully charged.

Four batteries are removed at random from the pack.

(a) Determine the probability that

(i) three of them are fully charged. (2 marks)

$$\frac{4\times3\times2\times2}{6\times5\times4\times3}\times\binom{4}{1}=\frac{2}{15}\times4=\frac{8}{15}$$

$$\left(\text{or } \frac{{}^4C_3\,{}^2C_1}{{}^6C_4}\right)$$

(ii) no more than two of them are fully charged. (2 marks)

$$\frac{4\times3\times2\times1}{6\times5\times4\times3}\times\binom{4}{2}=\frac{1}{15}\times6=\frac{6}{15}$$

$$\left(\text{or } \frac{{}^4C_2\,{}^2C_2}{{}^6C_4}\right)$$

(b) The first battery removed is flat, the second is fully charged, but the condition of the last two removed is not known. Determine the probability that three out of the four batteries removed are fully charged. (1 mark)

$$\frac{3\times2}{4\times3}=\frac{1}{2}$$

$$\left(\text{or } \frac{{}^3C_2\,{}^1C_0}{{}^4C_2}\right)$$

Question 8 **(7 marks)**

Let $B = n^2 + an + a$, where a and n are both positive integers.

(a) For each conjecture below state whether it is true or false. If a conjecture is true, give an example that shows it is true. If a conjecture is false, give an example that shows it is false. (3 marks)

(i) B is always odd when n is even.

FALSE
$n = 2,\ a = 2 \Rightarrow B = 2^2 + 2 \times 2 + 2 = 10 \Rightarrow$ Even

(ii) B is always odd when a is odd.

TRUE
$n = 1,\ a = 1 \Rightarrow B = 1^2 + 1 \times 1 + 1 = 3 \Rightarrow$ Odd

(iii) B is always even when a is even.

FALSE
$n = 1,\ a = 2 \Rightarrow B = 1^2 + 2 \times 1 + 2 = 5 \Rightarrow$ Odd

(b) Prove the conjecture from part (a) that is true. (4 marks)

Prove if a is odd, then B is always odd:

Let $a = 2k + 1$, where k is any integer, $k \geq 0$.

Then

$$\begin{aligned} B &= n^2 + (2k+1)n + 2k + 1 \\ &= n^2 + 2kn + n + 2k + 1 \\ &= n(n+1) + 2(kn + k) + 1 \end{aligned}$$

$n(n+1)$ is always even (product of two consecutive integers)
$2(kn + k) + 1$ is always odd
Hence B is always odd (sum of odd and even always odd)

Additional working space

Question number: _________

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