Compiled 3CDMAS questions

Wizard's solutions

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Notes (not the kind of which you take 4 pages into the exam, but notes on a few of the questions. These questions are identified by an *)

Version History

Calculus

Canning College 2010 S2 RF 8 c [4 marks]

Determine, in simplified form:

$$\int_0^4 \frac{x}{\sqrt{25-x^2}}\, dx$$

$$\int_0^4 \frac{x\,\mathrm{d}x}{\sqrt{25-x^2}} = \int_0^4 \frac{\mathrm{d}(x^2)}{2\sqrt{25-x^2}} = \int_4^0 \frac{\mathrm{d}(25-x^2)}{2\sqrt{25-x^2}} = \sqrt{25-x^2}\,\Big|_4^0 = 5-3=2$$

Canning College 2010 S2 RR 12 [4 marks]

Consider the function $P=2\pi\sqrt{\frac{t}{5}}$

Use a calculus method to determine the error in calculating P if t is measured to be 3 ± 0.1

$$\mathrm{d}P = \frac{2\pi}{\sqrt{5}} \cdot \frac{\mathrm{d}t}{2\sqrt{t}} \approx \frac{2\pi}{\sqrt{5}} \cdot \frac{0.1}{2\sqrt{3}} = \frac{\pi}{10\sqrt{15}}$$

Canning College 2010 S2 RR 13 [1, 3, 2, 2 marks]

An object is moving along the *x* axis such that its velocity after *t* seconds is given by

$$v=2\pi\cos 4\pi t+4\pi\cos 2\pi t$$

Given the object is initially at $x=4$, determine:
The maximum velocity of the object

$$\begin{aligned} v(t) &= 2\pi\cos(4\pi t) + 4\pi\cos(2\pi t) \\ &= 2\pi(2\cos^2(2\pi t) - 1) + 4\pi\cos(2\pi t) \\ &= 4\pi(\cos^2(2\pi t) + \cos(2\pi t)) - 2\pi \\ &= 4\pi((\cos(2\pi t) + 1/2)^2 - 1/4) - 2\pi \\ &\le 4\pi((1+1/2)^2 - 1/4) - 2\pi \qquad \text{since } \cos(2\pi t) \le 1 \\ &= 6\pi \end{aligned}$$

The time taken for the object to return to its starting position for the first time

$$x(t) = \int v\,\mathrm{d}t = \frac{1}{2}\sin(4\pi t) + 2\sin(2\pi t) + c$$

$$x(0) = 4 \implies c = 4, \text{ hence we solve for:}$$

$$\begin{aligned} 0 &= x(t) - 4 \\ &= \frac{1}{2}\sin(4\pi t) + 2\sin(2\pi t) \\ &= \frac{1}{2}(\sin(4\pi t) + 4\sin(2\pi t)) \\ &= \frac{1}{2}(2\sin(2\pi t)\cos(2\pi t) + 4\sin(2\pi t)) \\ &= \sin(2\pi t)(\cos(2\pi t) + 2) \end{aligned}$$

$$\implies t = \frac{\sin^{-1} 0}{2\pi} = \frac{k\pi}{2\pi} = \frac{k}{2}, \qquad k \in \mathbb{Z}^+ \cup \{0\}$$

$$\text{OR}$$

$$\cos(2\pi t) = -2, \text{ not possible for real } t$$

$$\text{Hence it takes } 1/2 \text{ sec to return to start position.}$$

The distance the object travels in the first 0.1 seconds (use your calculator but indicate the method used)

$$\begin{aligned} s &= \int_0^{0.1} |v|\,\mathrm{d}t \\ &= \int_0^{0.1} |2\pi\cos(4\pi t) + 4\pi\cos(2\pi t)|\,\mathrm{d}t \\ &= \int_0^{0.1} 2\pi\cos(4\pi t) + 4\pi\cos(2\pi t)\,\mathrm{d}t \qquad \text{since } v \geq 0 \text{ for } t \in [0, 0.1] \\ &= x(t)\Big|_0^{0.1} \\ &= \left[\frac{1}{2}\sin(4\pi t) + 2\sin(2\pi t)\right]_0^{0.1} \\ &= \frac{1}{2}\sin\frac{2\pi}{5} + 2\sin\frac{\pi}{5} \\ &= \frac{1}{2}\sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}} + 2\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}} \quad \text{metres, according to WolframAlpha} \end{aligned}$$

The acceleration of the object at t = 2 seconds

$$a(2) = v'(2) = -8\pi^2(\sin(4\pi t) + \sin(2\pi t))|_2 = 0$$

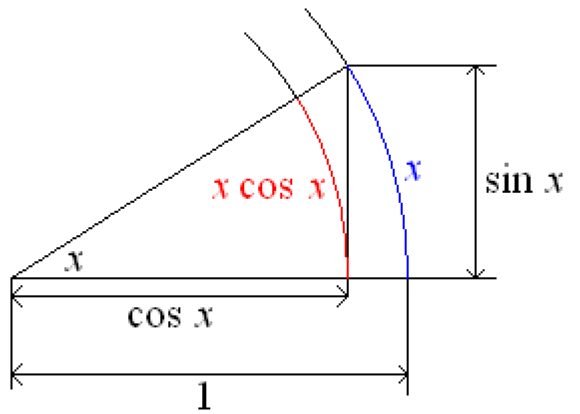
Edwest 2011 S2 RF 2 a i [2 marks]

Find: $\frac{d}{dx}\int_1^{2x}\frac{1-u^2}{\sin u}$ du

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^{2x}\frac{1-u^2}{\sin u}\,\mathrm{d}u=\left[\frac{1-u^2}{\sin u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\right]_{u=1}^{2x}=2\cdot\frac{1-4x^2}{\sin(2x)}$$

Edwest 2011 S2 RF 6 [4, 2 marks]

Establish the inequalities $x\cos x<\sin x<x$ for $0<x<\frac{\pi}{2}$ using ideas related to the unit circle



$$\therefore x\cos x<\sin x<x \qquad \text{for } x\in(0,\pi/2)$$

Use the above result to establish $\lim_{x\to0}\frac{\sin x}{x}$

$$x\cos x\le\sin x\le x \qquad \text{for } x\in[0,\pi/2]$$

$$\implies \cos x\le\frac{\sin x}{x}\le 1$$

$$\implies \lim_{x\to0}\cos x\le\lim_{x\to0}\frac{\sin x}{x}\le\lim_{x\to0}1$$

$$\implies 1\le\lim_{x\to0}\frac{\sin x}{x}\le 1$$

$$\implies \lim_{x\to0}\frac{\sin x}{x}=1$$

The original solution (with ‘less than’ instead of ‘less than or equal to’) is wrong, because

$1<\lim_{x\to0}\frac{\sin x}{x}<1$ implies $1<1$, a contradiction.

Edwest 2011 S2 RR 10 [7 marks]

Police Forensic Investigators are called late at night to investigate a murdered person in a suburban house. To get an idea of when the person died, the investigators use Newton's Law of Cooling which states that the rate of change of the temperature of a body is proportional to the difference between its own temperature and the ambient temperature (temperature of the surroundings). The investigators note the body's temperature when they arrived at 3:15am was 17.4°C and at 4:15am was 15.0°C. To estimate the time of death, the investigators assume the room temperature that night remained a constant 10°C and that the person's body had a temperature of 37.0°C at the time of death. Use Newton's Law of Cooling and the supplied information to estimate the time of death to the nearest 5 minutes.

Let $T(t)$ denote the temperature T of the body at time t after 12 am.

Then $\dfrac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_\infty)$ where T_∞ is the ambient temperature of the room.

$$\implies \frac{\mathrm{d}T}{T - T_\infty} = k\,\mathrm{d}t$$

$$\implies \log(T - T_\infty) = kt + \log c \quad \text{for some } c$$

Working in minutes and degrees Celsius, we are given

$$T_\infty = 10,\ T(3\cdot 60 + 15) = 17.4 \text{ and } T(4\cdot 60+15) = 15.0$$

Substituting gives:

$$\begin{cases} \log(17.4 - 10) = k(3\cdot 60 + 15) + \log c \\ \log(15.0 - 10) = k(4\cdot 60 + 15) + \log c \end{cases}$$

$$\implies 60k = \log 5 - \log 7.4 = -\log(7.4/5) = -\log 1.48$$

$$\implies k = \frac{-\log 1.48}{60}$$

$$\implies \log c = \log 5 - \frac{-\log 1.48}{60}\cdot(4\cdot 60 + 15) = \log 5 + \frac{17}{4}\log 1.48$$

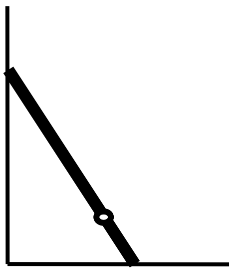
The time of death occurred at $T = 37.0$:

$$\begin{aligned} t &= \frac{1}{k}(\log(T - T_\infty) - \log c) \\ &= \frac{-60}{\log 1.48}\left(\log(37.0 - 10) - \log 5 - \frac{17}{4}\log 1.48\right) \\ &\approx -\left(3 + \frac{5}{60} + \frac{40.5}{60^2}\right) \end{aligned}$$

which is about 3 minutes before midnight.

Hale/St Mary's 2012 S2 RR 9 [5, 3, 4 marks]

A ladder, 2 metres long, has its base on level ground and its top resting against a vertical wall. A ring is fixed 0.5m from the base of the ladder as shown below. The ladder starts to slip down at a constant rate of 0.1m/s when it is $\sqrt{3}$ metres up the wall.



How fast (exact value) is the foot of the ladder moving away from the wall initally?

$$x^2 + y^2 = 2$$
$$\implies 2x\dot{x} + 2y\dot{y} = 0$$
$$\implies \dot{x}(0) = \frac{-y\dot{y}}{x} = \frac{-\sqrt{3} \cdot -0.1}{1} = \frac{\sqrt{3}}{10} \text{ m s}^{-1}$$

How fast is the ring moving down (vertically)?

$$y_{\text{ring}} = y/4 \text{ for all } t$$
$$\therefore \dot{y}_{\text{ring}} = \dot{y}/4 = -1/10/4 = -1/40 \text{ m s}^{-1}$$

How far is the ladder up the wall when the ring is moving with a speed of $\frac{1}{20}$ m/s ?

$$(x_{\text{ring}}, y_{\text{ring}}) = \left(\frac{3x}{4}, \frac{y}{4}\right) \text{ for all } t$$
$$\implies (\dot{x}_{\text{ring}}, \dot{y}_{\text{ring}}) = \left(\frac{3\dot{x}}{4}, \frac{\dot{y}}{4}\right) \text{ for all } t$$

Hence, when the ring has speed 1/20, we have:

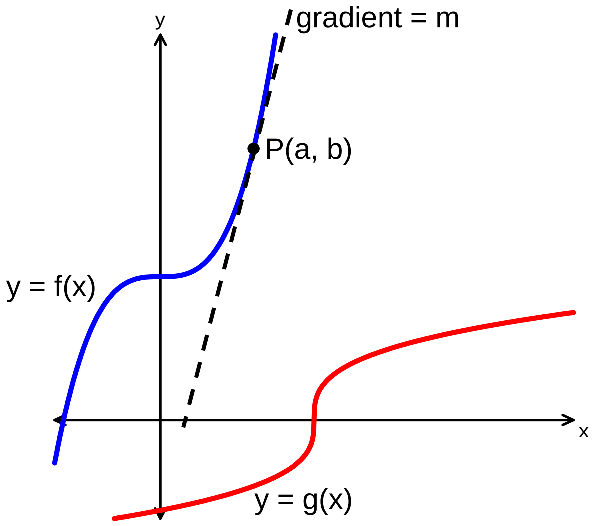
$$\left(\frac{3\dot{x}}{4}\right)^2 + \left(\frac{\dot{y}}{4}\right)^2 = \left(\frac{1}{20}\right)^2$$
$$\implies \dot{x}^2 = \frac{1}{3^2}\left(\frac{1}{5^2} - \dot{y}^2\right)$$
$$= \frac{1}{3^2}\left(\frac{1}{5^2} - \frac{1}{10^2}\right)$$
$$= \frac{1}{3 \cdot 10^2}$$
$$\implies \dot{x} = \frac{1}{10\sqrt{3}} \quad \left(\text{reject } \frac{-1}{10\sqrt{3}} < 0\right)$$
$$x^2 + y^2 = 2$$
$$\implies 2x\dot{x} + 2y\dot{y} = 0$$

$$\Longrightarrow \quad y\dot{y} = x\dot{x}$$
$$\Longrightarrow \quad (y\dot{y})^2 = (x\dot{x})^2$$
$$= x^2\dot{x}^2$$
$$= (2^2 - y^2)\dot{x}^2$$
$$\Longrightarrow \quad y^2(\dot{y}^2 + \dot{x}^2) = (2\dot{x})^2$$
$$\Longrightarrow \quad y = \frac{\pm 2\dot{x}}{\sqrt{\dot{y}^2 + \dot{x}^2}} = \frac{\pm 2/10/\sqrt{3}}{\sqrt{1/10^2 + 1/3/10^2}} = \pm 1$$

Reject $-1 < 0$, hence $y = 1$ m

Hale/St Mary's 2012 S2 RR 16 [1, 4, 4 marks]

The diagram below shows the graph of $y=f(x)$ and the graph of its inverse function $y=g(x)=f^{-1}(x)$



A point P (a,b) is on the graph of $y=f(x)$.
The tangent at P has a gradient m.

State the value of $g(f(a))$

$$g(f(a)) = f^{-1}(f(a)) = a$$

Show that $g'(b)=\frac{1}{m}$

$$
\begin{aligned}
g'(b) &= \left.\frac{\mathrm{d}(g(x))}{\mathrm{d}x}\right|_{x=b} \\
&= \left.\frac{\mathrm{d}(g(x))}{\mathrm{d}(f(g(x)))}\right|_{x=b} \qquad \text{since } f(g(x)) = x \\
&= \left.\left(\frac{\mathrm{d}(f(g(x)))}{\mathrm{d}(g(x))}\right)^{-1}\right|_{x=b} \\
&= \left.(f'(g(x)))^{-1}\right|_{x=b} \\
&= (f'(g(b)))^{-1} \\
&= (f'(a))^{-1} \\
&= m^{-1}
\end{aligned}
$$

Find the coordinates of the point of intersection of the tangent at P and the tangent at $x=b$ on the graph of $y=g(x)$ in terms of a, b and m (assume $m \neq -1$)

$$\text{Tangent 1: } y-b = f(a)+f'(a)\cdot(x-a) = (x-a)m$$
$$\text{Tangent 2: } y-a = g(b)+g'(b)\cdot(x-b) = (x-b)/m$$
$$\implies \begin{cases}(y-b) = m(x-a)\\ (x-b) = m(y-a)\end{cases}$$
$$\implies \begin{cases} mx-y = ma-b\\ -x+my = ma-b\end{cases}$$
$$\begin{aligned}\implies \begin{pmatrix}x\\y\end{pmatrix} &= \begin{pmatrix}m & -1\\ -1 & m\end{pmatrix}^{-1}\begin{pmatrix}ma-b\\ ma-b\end{pmatrix}\\ &= \frac{1}{m^2-1}\begin{pmatrix}m & 1\\ 1 & m\end{pmatrix}\cdot(ma-b)\begin{pmatrix}1\\1\end{pmatrix}\\ &= \frac{ma-b}{(m-1)(m+1)}\begin{pmatrix}m+1\\ m+1\end{pmatrix}\\ &= \frac{ma-b}{m-1}\begin{pmatrix}1\\1\end{pmatrix}\end{aligned}$$

Thus intersection occurs at

$$(x,y) = \left(\frac{ma-b}{m-1}, \frac{ma-b}{m-1}\right)$$

Penrhos/MLC 2010 S2 RF 5 b [5 marks]

Evaluate, using the substitution $x=\sin\theta$

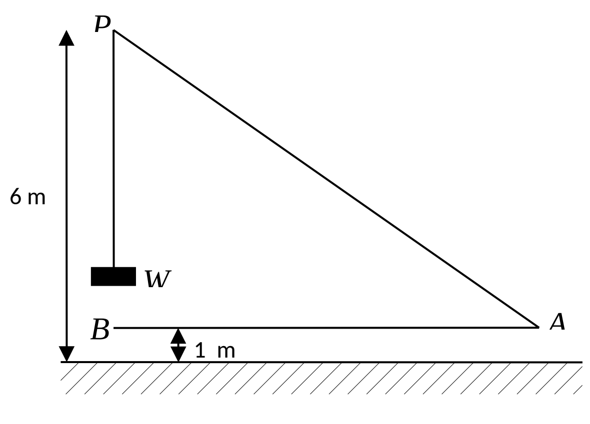
$$\int_0^{0.5} \frac{x}{\sqrt{1-x^2}}\,\mathrm{dx}$$

$$\int_0^{0.5}\frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} = \int_0^{\pi/6}\frac{\sin\theta\,\mathrm{d}(\sin\theta)}{\sqrt{1-\sin^2\theta}} = \int_0^{\pi/6}\frac{\sin\theta\cos\theta\,\mathrm{d}\theta}{\cos\theta} = -\cos\theta\Big|_0^{\pi/6} = 1-\frac{\sqrt{3}}{2}$$

Penrhos/MLC 2010 S2 RR 18 [2, 4 marks]

A weight W is attached to a rope 16 m long that passes over a pulley at point P, 6 m above the ground. The other end of the rope is attached to a truck at a point A, 1 m above the ground, as shown in the diagram.

Show that $y=\sqrt{25+x^2}-11$ represents the distance in metres the weight is above point B, given x metres represents the horizontal distance from point B to the truck.



$$16 = \overline{WP} + \overline{PA} = (6-1-y) + \sqrt{5^2+x^2}$$
$$\therefore y = \sqrt{25+x^2} - 11$$

If the truck moves away at the rate of 3 m/s,
how fast is the weight rising when it is 2 m above the ground?

Rewrite this as $(y+11)^2 = x^2 + 25 \qquad ...[1]$

and observe that $x = \sqrt{(y+11)^2 - 25} = \sqrt{(2-1+11)^2 - 25} = \sqrt{119}$

$$[1] \implies 2(y+11)\dot{y} = 2x\dot{x}$$
$$\implies \dot{y} = \frac{x\dot{x}}{y+11} = \frac{3\sqrt{119}}{2-1+11} = \frac{\sqrt{119}}{4} \text{ m s}^{-1}$$

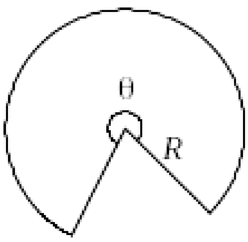
Mt Lawley 2011 S2 RF 2 b [3 marks]

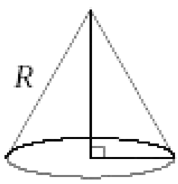
Evaluate
$$\int_1^{e^2} \frac{(\ln x)^2}{x} dx$$

$$\int_1^{e^2} \frac{(\ln x)^2 \, \mathrm{d}x}{x} = \int_1^{e^2} (\ln x)^2 \, \mathrm{d}(\ln x) = \left.\frac{(\ln x)^3}{3}\right|_{e^0}^{e^2} = \frac{2^3 - 0^3}{3} = \frac{8}{3}$$

Mt Lawley 2011 S2 RF 5 [3, 4 marks]

A minor sector of angle $2\pi - \theta$ is removed from a circular piece of paper of radius R. The two straight edges of the remaining major sector are pulled together to form a right circular cone, with a slant height of R.





Show that the volume of the cone is given by $V = \dfrac{R^3\theta^2\sqrt{4\pi^2 - \theta^2}}{24\pi^2}$

Cone has radius r given by $2\pi r = R\theta$

$$\implies r = \frac{R\theta}{2\pi}$$

Cone has height

$$\begin{aligned}
h &= \sqrt{R^2 - r^2} \\
&= \sqrt{R^2 - \left(\frac{R\theta}{2\pi}\right)^2} \\
&= \sqrt{\left(\frac{R}{2\pi}\right)^2 ((2\pi)^2 - \theta^2)} \\
&= \frac{R}{2\pi}\sqrt{4\pi^2 - \theta^2} \\
\therefore V &= \frac{1}{3}\pi r^2 h \\
&= \frac{1}{3}\pi \left(\frac{R\theta}{2\pi}\right)^2 \cdot \frac{R}{2\pi}\sqrt{4\pi^2 - \theta^2} \\
&= \frac{R^3\theta^2}{24\pi^2}\sqrt{4\pi^2 - \theta^2}
\end{aligned}$$

Assuming the radius, R, of the circular piece of paper to be fixed, show the exact value of θ which maximises the volume of the cone is $\frac{2\sqrt{2}\pi}{\sqrt{3}}$

Maximising V is equivalent to maximising $\dfrac{24\pi^2 V}{R^3} = \theta^2\sqrt{4\pi^2 - \theta^2}$

$$\begin{aligned}
\text{As usual, set } 0 &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\theta^2\sqrt{4\pi^2-\theta^2}\right) \\
&= 2\theta\sqrt{4\pi^2-\theta^2} + \frac{\theta^2 \cdot -2\theta}{2\sqrt{4\pi^2-\theta^2}} \\
&= \frac{\theta}{\sqrt{4\pi^2-\theta^2}}\left(2(4\pi^2-\theta^2)-\theta^2\right) \\
&= \frac{1}{\sqrt{4\pi^2-\theta^2}} \cdot \theta \cdot \left(8\pi^2 - 3\theta^2\right)
\end{aligned}$$

Reject $\dfrac{1}{\sqrt{4\pi^2-\theta^2}} = 0$ and $\theta = 0$, thus

$$8\pi^2 - 3\theta^2 = 0$$

$$\implies \theta = \sqrt{\frac{8\pi^2}{3}} = \frac{2\pi\sqrt{2}}{\sqrt{3}} \quad \left(\text{reject } \frac{-2\pi\sqrt{2}}{\sqrt{3}} < 0\right)$$

Hence $\theta = \dfrac{2\pi\sqrt{2}}{\sqrt{3}}$ maximises volume,

and nobody can be bothered doing the second derivative test.

But the proof is incomplete until that is done.