Perth Modern School

PERTH MODERN SCHOOL
Exceptional schooling. Exceptional students.

Semester One Examination, 2019

Question/Answer booklet

MATHEMATICS UNIT

Section One: Calculator-free

Your Name _____

Your Teacher's Name_____

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Mark	Max	Question	Mark	Max
1		6	5		9
2		5	6		7
3		6	7		7
4		10			

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	50	35
Section Two: Calculator-assumed	14	14	100	105	65
			Total	155	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section One: Calculator-free **(50 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(6 marks)**

(a) Differentiate $\frac{x}{e^x}$ and simplify using the quotient rule. (3 marks)

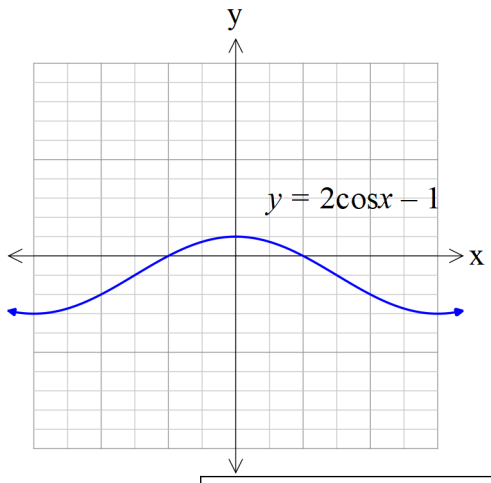
Solution
$\frac{d}{dx}\left(\frac{x}{e^x}\right) = \frac{e^x - xe^x}{e^{2x}} = \frac{1-x}{e^x}$
Specific behaviours
✓ correct numerator ✓correct denominator ✓ simplifies

(b) Hence, find the definite integral $\int_0^1 \frac{1-x}{2e^x}\,dx$. (3 marks)

Solution
$\frac{1}{2}\int_0^1 \frac{1-x}{e^x}dx = \frac{1}{2}\left[\frac{x}{e^x}\right]_0^1 = \frac{1}{2}\left[\frac{1}{e} - 0\right] = \frac{1}{2e}$
Specific behaviours
✓ takes out factor of 1/2 ✓ uses fundamental theorem ✓uses limits correctly

Question 2 **(5 marks)**

Determine the area between the x-axis and the curve $y=2\cos(x)-1$ for $0 \leq x \leq \pi$.



Solution
$2\cos x - 1 = o$ $\cos x = \frac{1}{2}$ $x = \frac{\pi}{3}$ $\int_0^{\frac{\pi}{3}} 2\cos x - 1 dx + \left\| \int_{\frac{\pi}{3}}^{\pi} 2\cos x - 1 dx \right\|$ $\left[2\sin x - x\right]_0^{\frac{\pi}{3}} + \left\| \left[2\sin x - x\right]_{\frac{\pi}{3}}^{\pi} \right\|$ $\left(\frac{2\sqrt{3}}{2} - \frac{\pi}{3}\right) - \left(-\pi - \frac{2\sqrt{3}}{2} + \frac{\pi}{3}\right)$ $2\sqrt{3} + \frac{\pi}{3}$
Specific behaviours
✓ works in radians ✓ determines exact x intercept ✓ breaks into two integrals, above and below areas ✓integrates correctly ✓ determines total area

Question 3 **(6 marks)**

Determine whether the following could be a probability density function on the given interval. Explain with reasons.

(a) $f(x)=\frac{2}{3}(x^2-x)$ *for the interval* $-1\le x\le 2.$ (2 marks)

Solution
Quadratic is less than zero between x equal to zero and one hence not a pdf
Specific behaviours
✓ states no with a reason ✓ correct reason

(b) $f(x)=\begin{cases}3x^2 \text{ for } -1\le x\le 0\\ 0 \text{ otherwise}\end{cases}$ (2 marks)

Solution
$\int_{-1}^{0}3x^2dx=\left[x^3\right]_{-1}^{0}=0--1=1$ Yes as all positive with area equaled to one
Specific behaviours
✓ states yes with a reason ✓ correct reason

(c) $f(x)=\frac{-1}{x^2+1}\quad ,-1<x<1$. (2 marks)

Solution
No as f(x) is always negative
Specific behaviours
✓ No with reason ✓ correct reason

Question 4 **(10 marks)**

(a) Given that $\log_3 x = 2, \log_7 343 = y \;\&\; \log_z 125 = 3$ determine $x + y - z$. (4 marks)

Solution
$x = 3^2$ $7^y = 343$ $z^3 = 125$ $x = 9$ $y = 3$ $z = 5$ $x + y - z = 7$
Specific behaviours
✓ shows x & z in power expressions ✓ shows y in power expressions ✓ solves for two variables ✓determines x+y-z

(b) Solve for x if $3\log_3(x) + \log_3(2x-1) - 2\log_3(x) = 1$ (4 marks)

Solution
$3\log_3 x + \log_3(2x - 1) - 2\log_3 x = 1$ $\log_3 \frac{x^3(2x - 1)}{x^2} = 1$ $x(2x - 1) = 3$ $2x^2 - x - 3 = 0$ $(2x - 3)(x + 1) = 0$ $x = \frac{3}{2}, -1$ $reject\ -1$
Specific behaviours
✓ changes all three terms to 1log(term)

✓ writes LHS as one log statement ✓ changes to a power statement and obtains a quadratic equation for x ✓states one positive value of x

Question 5 **(9 marks)**

(a) Determine $f(t)$ if $f''(t)=2e^t+3\sin(t)$ and $f(0)=0, f'(0)=0.$ (4 marks)

Solution
$f'(t)=2e^t - 3\cos(t)+c$ $0=2-3+c$ $c=1$ $f(t)=2e^t - 3\sin(t)+t+k$ $0=2+k$ $k=-2$ $f(t)=2e^t - 3\sin(t)+t-2$
Specific behaviours
✓ integrates second derivative ✓ solves for first constant ✓ integrates first derivative ✓solves for second constant

(b) Evaluate $\int\frac{2x}{5x^2-1}dx$ (3 marks)

Solution
$\int\frac{2x}{5x^2-1}dx=A\ln(5x^2-1)+c$ $diff \rightarrow \frac{A(10x)}{5x^2-1} \leftrightarrow equate\ 2=10A$ $A=\frac{1}{5}$ OR $\frac{1}{5}\int\frac{10x}{5x^2-1}dx=\frac{1}{5}\ln(5x^2-1)+c$
Specific behaviours
✓ uses ln ✓ shows derivation of 1/5 ✓adds a constant

(c) Determine $\frac{d}{dx}(x^3 \cos 2x)$ and simplify. (3 marks)

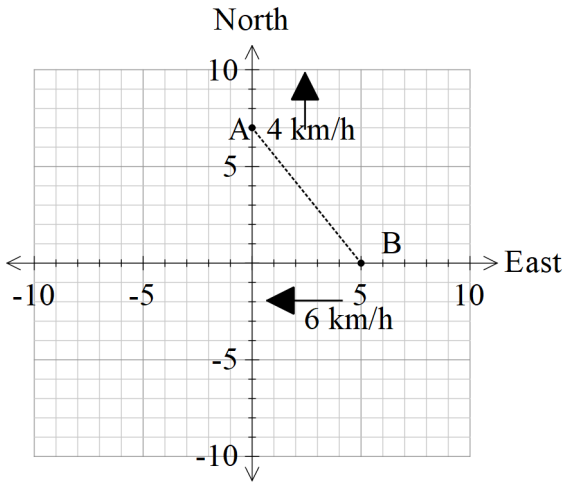
Solution
$\frac{d}{dx}(x^3 \cos 2x) = x^3(-2\sin 2x) + 3x^2 \cos 2x$ $= x^2(3\cos 2x - 2\sin 2x)$
Specific behaviours
✓ uses product rules ✓ differentiates all terms correctly ✓ factorises

Question 6 **(7 marks)**

Two very slow cars are positioned at time $t = 0$ hours at the positions shown in the diagram below. Car A is moving North at 4 km/h while car B is moving West at 6 km/h.

(a) Show that the distance between the two cars at time t hours is given by $D = \sqrt{(5 - 6t)^2 + (7 + 4t)^2}$ km.

(2 marks)



Solution
$x = 5 - 6t$ $y = 7 + 4t$ $D = \sqrt{x^2 + y^2}$
Specific behaviours
✓ determines expressions for x and y at time t ✓ uses Pythagoras

(b) Determine using calculus, the time in hours that the separation is a minimum.

Show that this is indeed a minimum. (5 marks)

Solution
$D = \sqrt{(5 - 6t)^2 + (7 + 4t)^2}$ $\frac{dD}{dt} = \frac{2(5 - 6t)(-6) + 2(7 + 4t)4}{2\sqrt{(5 - 6t)^2 + (7 + 4t)^2}} = \frac{-60 + 72t + 56 + 32t}{2\sqrt{(5 - 6t)^2 + (7 + 4t)^2}} = \frac{104t - 4}{2\sqrt{(5 - 6t)^2 + (7 + 4t)^2}}$ $\frac{dD}{dt} = 0 \quad , 104t - 4 = 0 \Rightarrow t = \frac{4}{104} = \frac{1}{26} h$ $t = 0 \quad 104t - 4 = -4 \Rightarrow -ve$ $t = 1 \quad 104t - 4 = 100 \Rightarrow +ve$ $\therefore local \min$

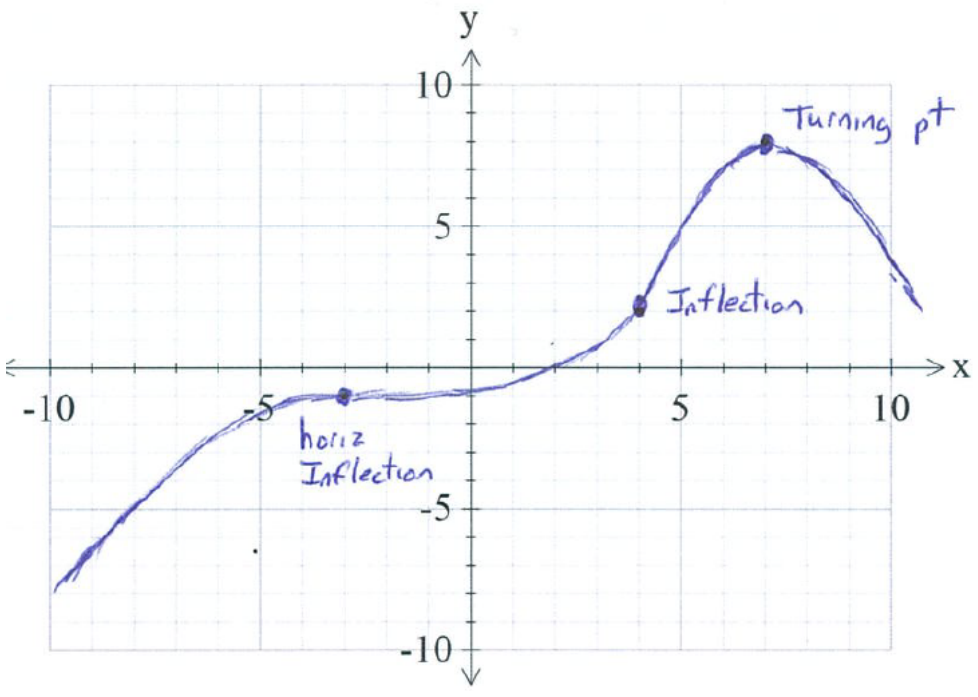
Specific behaviours
✓attempts to differentiate and equates to zero ✓ obtains derivative with correct denominator ✓ obtains derivative with correct numerator ✓solves for t ✓uses first derivative test showing actual values

Question 7 **(7 marks)**

Sketch the graph of a continuous function $y=f(x)$ which satisfies all the following conditions:

- Domain of $f(x)$ is $-10 \le x \le 10$
- $f(-3)=-1$, $f(4)=2$, $f(7)=8$
- $f'(x)\ge 0$ for $-10 \le x \le 7$, $f'(x)<0$ for $7 < x \le 10$
- $f'(-3)=0=f'(7)$
- $f''(-3)=0=f''(4)$
- $f'(4)\neq 0$

Solution



Specific behaviours
✓ non negative gradients between x values -10 & 7 ✓ two defined points ✓ three defined points ✓horizontal inflection at x=-3 ✓ non stationary inflection at x=4 ✓turning point at x=7 ✓negative gradients for x>7

Additional working space

Question number: _____