Mathematics Methods 3 and 4

Test 1 Calculator Free

Name: ....

SHENTON COLLEGE Teacher: Mrs Martin Dr Moore Mr Smith

Time Allowed : 30 minutes

Marks	/31

Materials allowed: Formulae Sheet provided.
Attempt all questions.
All necessary working and reasoning must be shown for full marks.
Where appropriate, answers should be given in exact values.
Marks may not be awarded for untidy or poorly arranged work.

Question 1 **[2, 2, 2 = 6 marks]**

Differentiate the following: Do not simplify your answers.

(a) $y = (x^2+1)(2x-3)$

$\frac{dy}{dx} = 2x(2x-3) + 2(x^2+1)$

✓ use of product rule
✓ correct

OR Expand first ✓
Diff ✓

(b) $y = \sqrt{e^x - 1}$ $= (e^x - 1)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(e^x - 1)^{-\frac{1}{2}}(e^x)$

✓ use of chain rule
✓ correct

(c) $y = \dfrac{\sin x}{1+\cos x}$

$$\frac{dy}{dx} = \frac{\cos x(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

✓ use of quotient
✓ correct

OR $y = \sin x\,(1+\cos x)^{-1}$

$$\frac{dy}{dx} = \cos x\,(1+\cos x)^{-1} + \sin(x)(-1)(1+\cos x)^{-2}(-\sin x)$$

✓ use of product
✓ correct

Question 2 **[2, 2 = 4 marks]**

If $f(x) = (1-x^2)^{\frac{3}{2}}$, find

(a) $f'(x)$ $= \frac{3}{2}(1-x^2)^{\frac{1}{2}}(-2x)$

$= -3x(1-x^2)^{\frac{1}{2}}$

✓ use of chain rule
✓ correct

(b) $f''(x)$ (Do not simplify)

$= -3(1-x^2)^{\frac{1}{2}} + (-3x)(\frac{1}{2})(1-x^2)^{-\frac{1}{2}}(-2x)$

✓ product
✓ chain

Question 3 **[3 marks]**

Show that for $y = \sin^4(x) + 2\cos^2(x) - \cos^4(x)$, $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = 4\sin^3(x)\cos x + 4\cos x(-\sin x) - 4\cos^3 x(-\sin x)$ ✓ Diff. correctly

$= 4\sin^3(x)\cos x - 4\sin x\cos x + 4\cos^3 x\sin x$

$= 4\sin x\cos x(\sin^2 x - 1 + \cos^2 x)$ ✓ Take out common factor

$= 4\sin x\cos x(1 - 1)$ ✓ using $\sin^2 x + \cos^2 x = 1$

$= 0$

Question 4 **[6 marks]**

For the function $f(x) = \frac{x^2}{e^x}$, find the co-ordinates of any stationary points and show use of calculus to determine their nature.

$f'(x) = \frac{2xe^x - x^2e^x}{e^{2x}} = 0$ ✓ Diff ✓ $=0$

$xe^x(2-x) = 0$

$x = 0, 2$ ✓ Solving correctly

$(0,0)$ $\left(2, \frac{4}{e^2}\right)$ ✓ Corresponding y values

Nature

at $(0,0)$

x	-1	0	1
$f'(x)$	$-$	0	$+$

$\therefore (0,0)$ Min T.P.

at $\left(2, \frac{4}{e^2}\right)$

x	1	2	3
$f'(x)$	$+$	0	$-$

$\therefore \left(2, \frac{4}{e^2}\right)$ Max. T.P.

✓ use of sign test

✓ Correct

OR

$f'(x) = \frac{2x - x^2}{e^x}$

$f''(x) = \frac{(2-2x)e^x - (2x - x^2)e^x}{e^{2x}}$

At $x = 0$ $f''(0) = \frac{2}{1} > 0$ $\therefore$ min TP

At $x = 2$ $f''(2) = \frac{-2e^2 - 0}{e^4} < 0$ $\therefore$ max TP

✓ use of $f''(x)$

✓ Correct

Question 4 **[2, 1, 2, 2, 1 = 8 marks]**

An approximation of the kangaroo population in a certain confined region is given by $f(t) = \frac{100000}{1 + 100e^{-0.3t}}$ where t is the time in years, $t \geq 0$.

(a) Find the approximate population at (i) $t = 0$ (ii) $t = 20$

(i) 990 ✓ (accept 1000)

(ii) (80136.22) Accept 80 136, 80 140, 80 100, 80 200, 80 000 ✓

(b) Find $f'(t)$

$= \frac{3\,000\,000\,e^{0.3x}}{(e^{0.3x} + 100)^2}$ ✓

(c) Find the rate of growth of the population when (i) $t = 0$ (ii) $t = 20$

(i) ~ 294 kangaroos/yr ✓

(ii) ~ 4775 roos/yr ✓

(Accept reasonable rounding)

(d) When was the population increasing at its fastest rate?

$f''(t) = 0$

$t = 15.4$ years ✓✓

(e) For what period of time is the rate of growth of the population increasing?

$0 < t < 15.4$ yrs ✓

Accept $t < 15.4$ yrs.

Question 2 **[4 marks]**

In the domain $0 < x < \pi$, show use of calculus to find the exact co-ordinates of the position on the curve $y = 3\sin(x) - \sin^3(x)$ where the tangent to the curve has a gradient of $\frac{3}{8}$.

$\frac{dy}{dx} = -3\cos x \sin^2 x + 3\cos x$ ✓ Derivative

$-3\cos x \sin^2 x + 3\cos x = \frac{3}{8}$ ✓ Showing $\frac{dy}{dx} = \frac{3}{8}$

$x = \frac{\pi}{3}$ ✓ Solving $x =$

$\left(\frac{\pi}{3}, \frac{9\sqrt{3}}{8}\right)$ ✓ Finding y value

Question 3 **[1, 1, 1, 1 = 4 marks]**

The ferris wheel "London Eye" contains 32 capsules. A person enters a capsule when it is at its lowest point, but still a certain distance above ground level. The height, h metres above the ground after t minutes is given by $h = 75 - 60\cos\left(\frac{2\pi t}{25}\right)$ (The height is defined by the distance from the centre of the capsule to the ground)

Determine:

(a) the maximum height of the capsule above the ground

135 m ✓

(b) the minimum height of the capsule above the ground

15 m ✓

(c) the time it takes for the capsule to complete one revolution of the wheel

25 minutes ✓

(d) the rate of change of the height of the capsule when $t = 10.5$ minutes.

7.26 ✓ m/min

Question 5 **[4 marks]**

A particle moves along a straight line such that its displacement, x metres at time t seconds is given by $x = 3\sin(2t) + 4$. Determine:

(a) an expression for the velocity of the particle at time t.

$\frac{dx}{dt} = v = \dot{x} = 3\cos(2t)(2)$

$= 6\cos(2t)$ ✓

(b) the maximum velocity of the particle

6 m/s ✓

(c) an expression for the acceleration of the particle at time t.

$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \ddot{x} = -6\sin(2t)(2)$

$= -12\sin(2t)$ ✓

(d) the acceleration when $t = \frac{3\pi}{4}$

$\ddot{x} = -12\sin\left(\frac{3\pi}{2}\right)$

$= 12 \text{ m/s}^2$ ✓

Question 6 **[3 marks]**

Given $m = 5v$, $v = 3h^2 - 2$ and $h = 2x^3$, find $\frac{dm}{dx}$ using the chain rule.

$\frac{dm}{dx} = \frac{dm}{dv} \times \frac{dv}{dh} \times \frac{dh}{dx}$ ✓ Chain rule

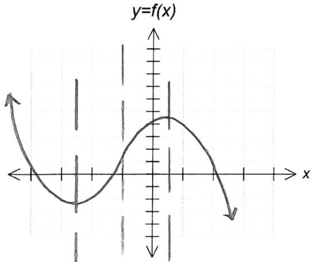
$= 5 \times 6h \times 6x^2$ ✓

$= 5 \times 6(2x^3) \times 6x^2$ ✓ Replacing $h = 2x^3$

$= 360x^5$

Question 7 **[5 marks]**

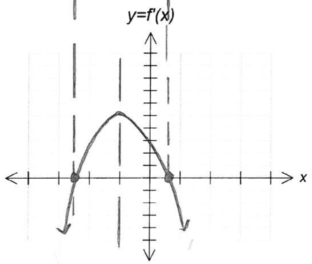
Given the graph of $y = f''(x)$, provide possible graphs of $y = f'(x)$ and $y = f(x)$



✓ Stationary points line up with their roots

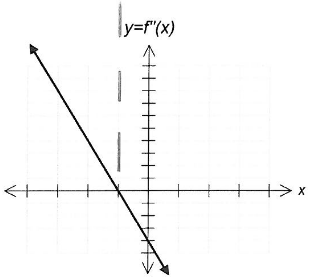
✓ gradients correct − + −

✓ Pt of Inflection lining up with $f''(x) = 0$



✓ stationary point lines up

✓ gradients correct either side



Mathematics Methods 3 and 4

Test 1 Calculator Assumed

SHENTON COLLEGE

Name: ....

Teacher: Mrs Martin Dr Moore Mr Smith

Time Allowed : 20 minutes

Marks	/24

Materials allowed: Classpad, calculator, one page of notes (one side)
Formulae Sheet provided. Attempt all questions.
All necessary working and reasoning must be shown for full marks.
Marks may not be awarded for untidy or poorly arranged work.

Question 1 **[2, 2, 1, 1, 1, 1 = 8 marks]**

The mass, m kg, of radioactive lead remaining in a sample t hours after observations began is given by $m = 2e^{-0.2t}$.

(a) Find the mass of lead, to the nearest gram, remaining after 12 hours.

0.181 kg ✓ ✓ rounded correctly

(b) Find how long it takes for the mass of lead to decay to half its value at $t = 0$.

$t = 3.47$ hours ✓✓
(any sensible rounding)

(c) Express the rate of decay as a function of t

$$\frac{dm}{dt} = -0.2 \times 2e^{-0.2t}$$
$$= -0.4e^{-0.2t} \checkmark$$

(d) Find the rate of decay at $t = 6$

-0.12 kg/hr ✓

Accept 0.12g/hr as the question says "Decay"

(e) Express the rate of decay as a function of m

$-0.2m$ ✓

(f) Find the rate of decay when there is 20 grams of lead remaining.

-0.2×20

-4 g/hr ✓

or (-0.004 kg/hr)