Rossmoyne Senior High School

Year 12 Trial WACE Examination, 2015

Question/Answer Booklet

MATHEMATICS
SPECIALIST 3CD
Section One:
Calculator-free

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	50	33⅓
Section Two: Calculator-assumed	13	13	100	100	66⅔
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section One: Calculator-free **(50 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(5 marks)**

Determine the matrix A, given that $A\begin{bmatrix} 6 & 5 \\ -1 & 1 \end{bmatrix} - 3A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$.

$$AB - 3A = C$$
$$A(B - 3I) = C$$
$$A = C(B - 3I)^{-1}$$

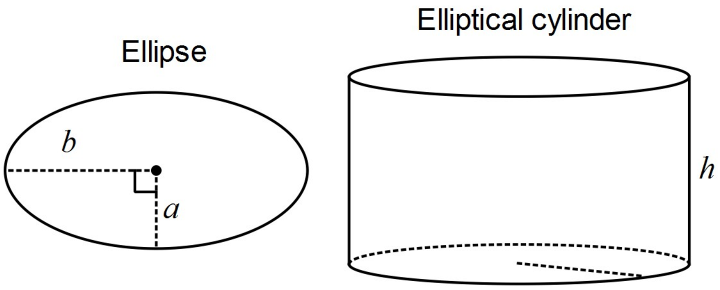
$$B - 3I = \begin{bmatrix} 6 & 5 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$$

$$(B - 3I)^{-1} = \frac{1}{-1}\begin{bmatrix} -2 & -5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 11 \end{bmatrix}$$

Question 2 **(6 marks)**

An elliptical cylinder is such that the sum of the height, h, and the length of the minor semi-axis, a, is 30 cm. The length of the major semi-axis, b, is twice the length of the minor semi-axis.



$$V = \pi abh$$

Use calculus to determine the dimensions of the elliptical cylinder that maximise its volume and state the maximum volume.

$$V = \pi a(2a)(30 - a)$$
$$= 60\pi a^2 - 2\pi a^3$$

$$\frac{dV}{da} = 120\pi a - 6\pi a^2$$

$$120\pi a - 6\pi a^2 = 0$$
$$6\pi a(20 - a) = 0 \Rightarrow a = 0,\ a = 20$$

$a = 20$ cm
$b = 40$ cm
$h = 10$ cm

$V = 8000\pi \text{ cm}^3$

Question 3 **(5 marks)**

Using the substitution $u = 1 + x$ or otherwise, determine $\int_{-1}^{0} x\sqrt{1+x}\ dx$.

$x = u - 1,\ du = dx$

$x = -1,\ u = 0$ and $x = 0,\ u = 1$

$$\int_{-1}^{0} x\sqrt{1+x}\ dx = \int_{0}^{1} (u-1)u^{0.5}du$$

$$= \int_{0}^{1} (u^{1.5} - u^{0.5})du$$

$$= \left[\frac{2u^{2.5}}{5} - \frac{2u^{1.5}}{3}\right]_{0}^{1}$$

$$= \frac{2}{5} - \frac{2}{3} = -\frac{4}{15}$$

Question 4 **(11 marks)**

(a) Express the following in the form $a+bi$:

(i) $\left(1-i\sqrt{3}\right)^7$. (4 marks)

$$\begin{aligned}\left(1-i\sqrt{3}\right)^7 &= \left[2cis\left(-\frac{\pi}{3}\right)\right]^7 \\ &= 2^7 cis\left(-\frac{7\pi}{3}\right) \\ &= 2^7 cis\left(-\frac{\pi}{3}\right) \\ &= 2^6 \times 2cis\left(-\frac{\pi}{3}\right) \\ &= 64-64\sqrt{3}i\end{aligned}$$

(ii) $e^{\left(3+i\frac{\pi}{2}\right)}$. (3 marks)

$$\begin{aligned}e^{\left(3+i\frac{\pi}{2}\right)} &= e^3 e^{\left(i\frac{\pi}{2}\right)} \\ &= e^3 cis\left(\frac{\pi}{2}\right) \\ &= e^3 i\end{aligned}$$

(b) Determine all solutions to the equation $z^3 = -64$, giving your answers in the form $re^{i\theta}$. (4 marks)

$$z^3 = -64$$

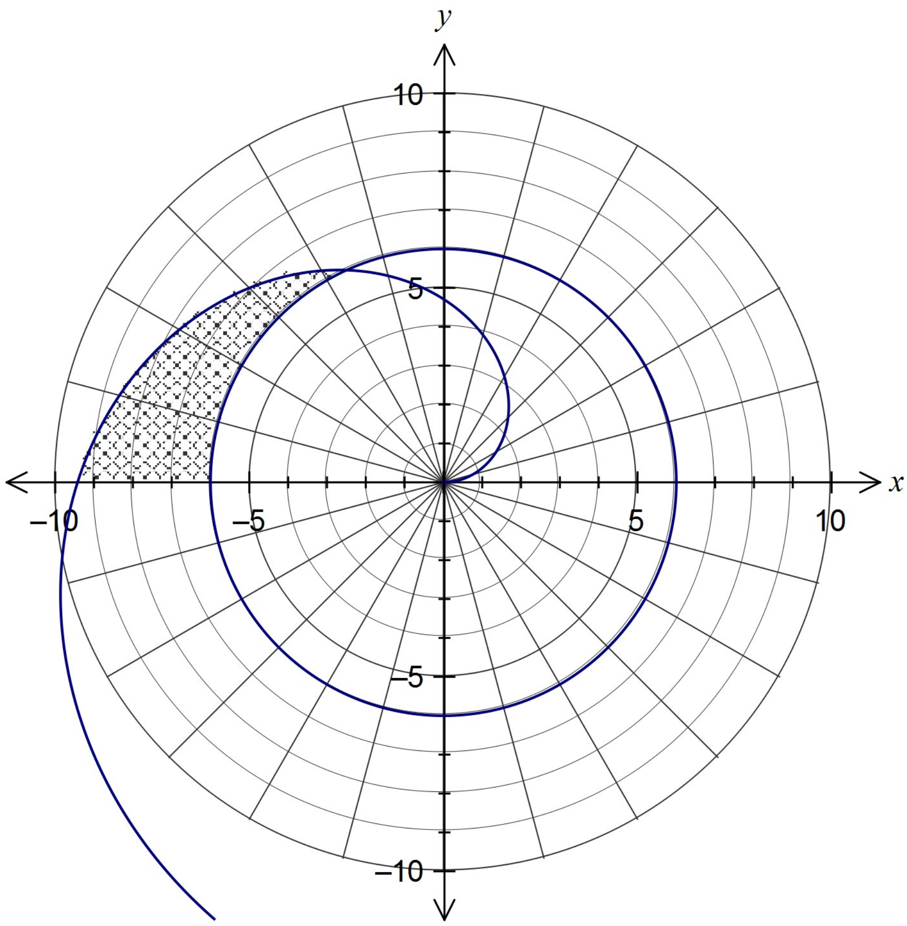
$$z_1 = -4$$
$$= 4cis(\pi) = 4e^{i\pi}$$

$$z_2 = 4cis\left(\pi - \frac{2\pi}{3}\right)$$
$$= 4cis\left(\frac{\pi}{3}\right) = 4e^{i\frac{\pi}{3}}$$

$$z_3 = 4cis\left(-\frac{\pi}{3}\right) = 4e^{-i\frac{\pi}{3}}$$

Question 5 **(8 marks)**

(a) On the axes below shade the region satisfied by $r \geq 6$, $r \leq 3\theta$ and $\frac{\pi}{2} \leq \theta \leq \pi$. (3 marks)



For any curve $r = f(\theta),\ \alpha \le \theta \le \beta$, the area bounded by the curve and the rays $\theta = \alpha$ and $\theta = \beta$ is given by $\int_{\alpha}^{\beta}\left(\frac{1}{2}r^2\right)\,d\theta$.

(b) Determine the area of the shaded region in (a). (5 marks)

Circle and spiral intersect when $3\theta = 6 \Rightarrow \theta = 2$.

Hence $\alpha = 2,\ \beta = \pi$.

$$\text{Area} = \int_{2}^{\pi}\frac{1}{2}\left((3\theta)^2 - 6^2\right)d\theta$$

$$= \frac{1}{2}\left[3\theta^3 - 36\theta\right]_2^{\pi}$$

$$= \frac{3\pi^3 - 36\pi - (24 - 72)}{2}$$

$$= \frac{3\pi^3 - 36\pi + 48}{2} \text{ sq units}$$

Question 6 **(9 marks)**

The rate of change of the distance, x centimetres, between two particles t seconds after an experiment began, is given by the equation $t\dfrac{dx}{dt} = x\ln(t)$, where $t > 0$ and $x > 0$.

(a) Using the substitution $u = \ln(t)$, or otherwise, determine $\displaystyle\int \frac{\ln(t)}{t}\,dt$. (3 marks)

$$u = \ln(t) \;\Rightarrow\; du = \frac{1}{t}dt$$

$$\int u\,du = \frac{1}{2}u^2 + c$$

$$= \frac{1}{2}(\ln(t))^2 + c$$

(b) Determine an expression for x in terms of t, if the distance between the two particles after 1 second is 0.5 cm. (4 marks)

$$\int \frac{1}{x}\,dx = \int \frac{1}{t}\ln(t)\,dt$$

$$\ln(x) = \frac{1}{2}(\ln(t))^2 + c \quad (x > 0)$$

$$x = ke^{\frac{(\ln(t))^2}{2}}$$

$$x(1) = \tfrac{1}{2} \;\Rightarrow\; k = \tfrac{1}{2}$$

$$x = \frac{1}{2}e^{\frac{(\ln(t))^2}{2}}$$

(c) Show that when $t = 2$, $x = 2^{\left(\frac{\ln(2) - 2}{2}\right)}$. (2 marks)

$$x(2) = \frac{1}{2} e^{\frac{(\ln(2))^2}{2}}$$
$$= \frac{1}{2} e^{\ln(2)\frac{\ln(2)}{2}}$$
$$= 2^{-1}.2^{\frac{\ln(2)}{2}}$$
$$= 2^{\frac{\ln(2)}{2} - 1}$$
$$= 2^{\left(\frac{\ln(2) - 2}{2}\right)}$$

Question 7 **(6 marks)**

Prove by induction that $\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$ is always an integer when n is a positive integer.

When $n=1$, $\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}=\frac{1}{3}+\frac{1}{2}+\frac{1}{6}=1 \Rightarrow$ True when $n=1$.

Assume that when $n=k$, $\frac{k^3}{3}+\frac{k^2}{2}+\frac{k}{6}=N$, where N is an integer.

When $n=k+1$

$$\frac{(k+1)^3}{3}+\frac{(k+1)^2}{2}+\frac{k+1}{6}$$
$$=\frac{k^3+3k^2+3k+1}{3}+\frac{k^2+2k+1}{2}+\frac{k+1}{6}$$
$$=\frac{k^3}{3}+k^2+k+\frac{1}{3}+\frac{k^2}{2}+k+\frac{1}{2}+\frac{k}{6}+\frac{1}{6}$$
$$=\frac{k^3}{3}+\frac{k^2}{2}+\frac{k}{6}+k^2+2k+1$$
$=N+k^2+2k+1$, which is an integer.

Hence, by the principle of induction, the statement is true for all n.

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

Additional working space

Question number: _________

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