Counting Techniques

Permutation:

The number of ways of arranging a set of objects

**Order does matter

1234≠4321

e.g. A B C D

Total =4x3x2x1

=24

=4!

P=Permutation

r=places

The number of permutations of r objects taken from n different objects is $\frac{n!}{(n-r)!}$.
We write this as ${}^{n}P_{r}$.

e.g. Any 4 letter permutation

26 25 24 23

$$\frac{26!}{(26-4)!} \qquad = \frac{26!}{22!}$$

→

26x25x24x23x22x21x20x19…

22x21x20x19x18x17x16x15…c

e.g. Bike Lock (allows to repeat number)

$${}^{10}P_4 = \frac{10!}{6!}$$

X Does not work

P= Does not allow repetition

10 10 10 10

Permutations of Objects, not all Different:

If n objects contain p of one kind, q of another, r of another etc., then there are

$$\frac{n!}{p!q!r!\ldots} \text{ arrangements of the } n \text{ objects.}$$

e.g.

EXAMPLE 5

How many ways can the nine letters of the word ISOSCELES be arranged in a row?
How many of these start with the L?

Solution

The word ISOSCELES involves 9 letters including 3 Ss and 2 Es.

$$\text{Number of arrangements} = \frac{9!}{3!2!} = 30\,240$$

There are 30 240 arrangements of the nine letters of the word ISOSCELES.

If we start with the L then we are arranging the other 8 letters, which include 3 Ss and 2 Es, and then placing an L at the front of each arrangement.

$$\text{Number of arrangements starting with the L} = \frac{8!}{3!2!} = 3360$$

There are 3360 arrangements that start with the L.

Addition Principle:

The addition principle

If there are a ways that event A can occur and b ways that event B can occur, then, provided A and B are mutually exclusive (i.e. A and B cannot occur together), $a + b$ is the number of ways either A or B can occur.

e.g.

A three-character code can either consist of three digits from 1, 2, 3, 4 and 5 or it can consist of three letters from A, B, C, D and E. How many codes are possible if digits can be used more than once in a code but letters cannot?

Solution

Number of possible three digit codes $= 5 \times 5 \times 5 = 125$

No. of ways for each digit		
5	5	5

Number of possible three letter codes $= 5 \times 4 \times 3 = 60$

No. of ways for each letter		
5	4	3

These events are mutually exclusive because a three character code cannot be both three digits and three letters. Thus 185 (= 125 + 60) codes can be formed.

U= union

∩= Intersection

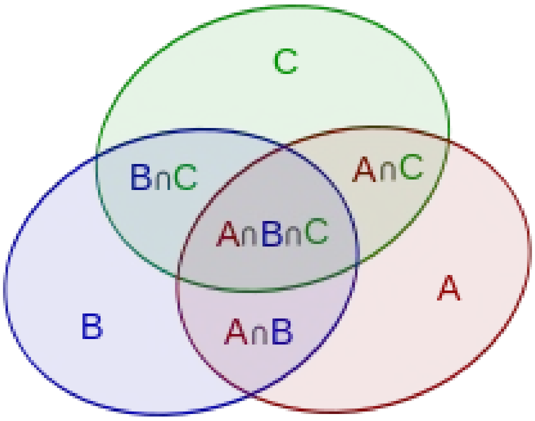
Inclusion- Exclusion Principle:

Is when we include the numbers of the first set, exclude the numbers of the two set intersection, then include the numbers of the 3 set intersection

https://www.youtube.com/watch?v=szUTQRJU76Q&index=27&list=FLmBm2NbG4Ovc8ep4wASKaFg&t=0s

-use this for when there are sets that are non-mutually and interlay

- we can either use the easier way, of drawing a ven diagram or we could also use the formula



|A|, | | means the elements of A

Don’t forget to subtract the 2 intersects from A∩B∩C

And there also might be some that are neither A, B or C

But there are also formulas

$$n(A \cup B) = n(A) + n(B)$$

$$n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

e.g.

a How many multiples of 2 are there between 1 and 89?

b How many multiples of 3 are there between 1 and 89?

c How many numbers between 1 and 89 are multiples of 2 or 3?

Note: In mathematics we take the word ‘or’ to mean ‘one or the other or both’.
In other words we take ‘or’ to mean ‘at least one of’.
Thus for two events A and B we take ‘A or B’ to mean $A \cup B$.

Solution

a The multiples of 2 between 1 and 89 are: 2, 4, 6, 8, … 88.
44 numbers in total.

b The multiples of 3 between 1 and 89 are: 3, 6, 9, 12, … 87.
29 numbers altogether.

c The numbers between 1 and 89 that are multiples of both 2 and 3 are:

6, 12, 18, 24, … 84

14 numbers altogether.

Thus the required number $= 44 + 29 - 14$
$= 59$

There are 59 numbers between 1 and 89 that are multiples of 2 or 3.

Arrangements of Objects with Some Restrictions Imposed:

This is for when there are certain restrictions where certain things (e.g. numbers or letter) can be.

- The best way of thinking about is that there are n-1 options and n-1 places if a object has to go in a specific place

- And if two objects have to be together then u count them as just one, but also remember that there are n numbers of places that they could be.

e.g.

How many different *five* letter 'words' can be formed using the letters of the word FASHION if the middle letter in each arrangement must be the S and no word may feature a letter used more than once?

Solution

There is only one choice for the middle letter, an S.

There then remain six choices for the first, F, A, H, I, O, N, five for the second, four for the fourth and three for the last.

Total number of arrangements $= 6 \times 5 \times 1 \times 4 \times 3$
$= 360$

No. of ways for each letter				
?	?	1	?	?
6	5	1	4	3

e.g.

a How many five digit *even* numbers can be made using the digits 3, 4, 5, 6 and 7 if no digit may feature more than once in a number?

b How many of the numbers from **a** are greater than 70 000?

Solution

a Start by choosing the last digit, which must be either the 4 or the 6, to ensure an even number:

No. of ways for each digit				
?	?	?	?	2

Left hand digit is then chosen from remaining 4, next digit from remaining 3 and so on:

4	3	2	1	2

The number of possible five digit even numbers
$= 4 \times 3 \times 2 \times 1 \times 2$
$= 48$

b The last digit must be either the 4 or the 6 and the first digit must be the 7.

No. of ways for each digit				
1	?	?	?	2

The next digit is then chosen from the remaining 3 and so on:

1	3	2	1	2

Number that are even and greater than 70 000
$= 1 \times 3 \times 2 \times 1 \times 2$
$= 12.$

e.g.

Seven files, A, B, C, D, E, F and G are to be arranged on a shelf.

a In how many ways can this be done?

b In how many of these arrangements is file A next to file B?

A	B	C	D	E	F	G

Solution

a The first file can be chosen in 7 ways, the next in 6, the next in 5 and so on.

Total number of arrangements $= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ i.e. 7!, or 5040.

b If we imagine files A and B tied together we then have six things to arrange. However, A and B could be tied together in 2 ways (AB or BA).

Total number of arrangements $= (6 \times 5 \times 4 \times 3 \times 2 \times 1) \times 2$ i.e. $6! \times 2$, or 1440.

Additive and Multiplicative Reasoning:

You add how many times A can happen but not B and B can happen and not A and How many both can happen

e.g.

Start with an A and end with an F: $1 \times 4 \times 3 \times 1 = 12$
Start with an A and not end with an F: $1 \times 4 \times 3 \times 4 = 48$
Do not start with an A but end with an F: $4 \times 4 \times 3 \times 1 = 48$

Thus number that start with an A or end with an F: $= 12 + 48 + 48$
$= 108$, as before.

a How many six digit even numbers can be made using the digits 1, 2, 3, 4, 5 and 6 if no digit may be used more than once in a number?

b How many of the numbers from **a** are bigger than 400 000?

Solution

a Start by choosing the last digit, which must be 2, 4 or 6, to ensure the number is even:

?	?	?	?	?	3

Left hand digit is then chosen from remaining 5, next digit from remaining 4 and so on:

5	4	3	2	1	3

The number of possible 6 digit even numbers $= 5 \times 4 \times 3 \times 2 \times 1 \times 3$
$= 360$

There are 360 six digit even numbers that can be made using the digits 1, 2, 3, 4, 5 and 6.

b The problem now is that if we choose the last digit from the 2, 4 or 6 the number of choices for the first digit is 3, if the last digit was the 2, and 2 if the last digit was the 4 or the 6.

To cope with this we consider mutually exclusive events.

Ending with 2 and bigger than 400 000:

3	4	3	2	1	1

Ending with 4 and bigger than 400 000:

2	4	3	2	1	1

Ending with 6 and bigger than 400 000:

2	4	3	2	1	1

The number of possible 6 digit even numbers $> 400\,000 = 72 + 48 + 48$
$= 168$

There are 168 six digit even numbers bigger than 400 000 that can be made using the digits 1, 2, 3, 4, 5 and 6.

e.g.

Five members of a basketball team all have to stand in a line for a photograph.

The players are: Alex, Keith, Mark, Rani and Steve.

How many arrangements are there in which

a Rani is in the middle?

b Alex is at the left end?

c Mark is at the right end?

d At least one of **b** and **c** occur?

e Keith and Steve are next to each other?

f Keith and Steve are not next to each other?

Solution

a Middle place filled in one way:

		1		

Then fill left most space from remaining 4 players, next from remaining 3 etc.

4	3	1	2	1

Number of possible arrangements is $4 \times 3 \times 1 \times 2 \times 1 = 24$.

b Alex at left end then fill spaces from left:

1	4	3	2	1

Number of possible arrangements is $1 \times 4 \times 3 \times 2 \times 1 = 24$.

c Mark at right end then fill spaces from left:

4	3	2	1	1

Number of possible arrangements is $4 \times 3 \times 2 \times 1 \times 1 = 24$.

d Alex at left end and Mark not at right end:

1	3	2	1	3

Mark at right end and Alex not at left end:

3	3	2	1	1

Alex at left end and Mark at right end:

1	3	2	1	1

Number of possible arrangements is $18 + 18 + 6 = 42$.

Or: Add answers from **b** and **c** then subtract the 'and situation' that would otherwise be counted twice: $24 + 24 - 6 = 42$, as before.

e 'Tie' Keith and Steve together (2 ways). We now have 4 'things' to arrange.
Number of possible arrangements is $2 \times 4 \times 3 \times 2 \times 1 = 48$.

f There are $5 \times 4 \times 3 \times 2 \times 1$ (= 120) arrangements altogether and 48 have Keith and Steve together. Thus 72 (= 120 – 48) must have Keith and Steve not together.
There are 72 arrangements in which Keith and Steve are not together.

Note the use of the complementary event in part **f**.

How many three digit codes can be made using the digits 0, 1, 2, 3, 4 and 5 if there are no restrictions on using a digit more than once?

How many of these codes **a** start with a 3? **b** start with a 4? **c** do not start with a 4? **d** start with a 3 or a 4? **e** start with a 3 and end with a 4? **f** start with a 3 or end with a 4?

Solution

Number of ways each digit of code can be chosen: 6 6 6

The number of possible codes is $6 \times 6 \times 6 = 216$

a If the code must start with a 3: 1 6 6

The number of possible codes is $1 \times 6 \times 6 = 36$

b If the code must start with a 4: 1 6 6

The number of possible codes is $1 \times 6 \times 6 = 36$

c If the code must not start with a 4: 5 6 6

The number of possible codes is $5 \times 6 \times 6 = 180$

Alternatively: 36 of the 216 codes start with a 4 therefore 180 (= 216 – 36) do not start with a 4. (This alternative method uses the idea of the *complementary* event.)

d If the code must start with a 3 or a 4: 2 6 6

The number of possible codes is $2 \times 6 \times 6 = 72$

Alternatively, using additive reasoning because start with 3 and start with a 4 are mutually exclusive:

Number of codes that start with a 3 or a 4 is $36 + 36 = 72$

e If the code must start with a 3 and end with a 4: 1 6 1

The number of possible codes is $1 \times 6 \times 1 = 6$

f If the code must start with a 3: 1 6 6

The number of possible codes is $1 \times 6 \times 6 = 36$

If the code must end with a 4: 6 6 1

The number of possible codes is $6 \times 6 \times 1 = 36$

'Start with 3' and 'end with a 4' are not mutually exclusive. They can occur together, as in the code 314. The answer will *not* be 36 + 36.

Using $n(A \cup B) = n(A) + n(B) - n(A \cap B)$:

Number of codes starting with 3 or end with 4 is $36 + 36 - 6 = 66$

Or, considering 'start with a 3 or end with a 4' as three mutually exclusive events:

- Start with a 3 and end with a 4: 1 6 1
- Start with a 3 and not end with a 4: 1 6 5
- End with a 4 but not start with a 3: 5 6 1

Number of codes starting with 3 or end with 4 is $6 + 30 + 30 = 66$, as before.

A **combination** is a **selection** – the order does not matter.

A **permutation** is an **arrangement** – the order does matter.

Combinations:

There are ${}^nC_r$ combinations of r objects chosen from n different objects

where ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$.

In the example of choosing a group of 3 letters from 4 letters our initial listing of 24 arrangements reduced to just 4 selections because, when order became unimportant, each column of 6 (= 3!) arrangements reduced to just one selection.

$$\begin{aligned} {}^4C_3 &= \frac{{}^4P_3}{3!} \\ &= \frac{4!}{(4-3)!3!} \end{aligned}$$

To apply this thinking to combinations of r objects chosen from n different objects

$$\begin{aligned} {}^nC_r &= \frac{{}^nP_r}{r!} \\ &= \frac{n!}{(n-r)!\,r!} \end{aligned}$$

e.g.

How many different ways can a group of 6 people, 3 male and 3 female, be selected from 8 males and 9 females?

Solution

	Male		Female
From	8	and from	9
choose	3	choose	3

$$\text{Number of ways} = {}^8C_3 \times {}^9C_3 = 4704$$

A group of three males and three females can be selected from eight males and nine females in 4704 ways.

A normal pack of playing cards consists of 52 cards arranged in four suits.

Hearts	A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
Diamonds	A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
Spades	A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠
Clubs	A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣

These cards are well shuffled and 7 cards are randomly dealt to form a 'hand'.

a How many different 7 card hands are there?

How many of these hands contain:

b the ace of hearts (A♥)?

c the ace and two of hearts (A♥ and 2♥)?

d exactly 3 of the 4 kings?

e at least one ace?

Solution

It is the cards that make up the hand that is important, not the order in which they are dealt. Thus this situation involves combinations.

a

From	52
Choose	7

$$\text{Number of hands} = {}^{52}C_7 = 133\,784\,560$$

b

	A♥	Others
From	1	51
Choose	1	6

$$\text{Number of hands} = {}^1C_1 \times {}^{51}C_6 = 18\,009\,460$$

c

	A♥	2♥	Others
From	1	1	50
Choose	1	1	5

$$\text{Number of hands} = {}^1C_1 \times {}^1C_1 \times {}^{50}C_5 = 2\,118\,760$$

d

	K	Others
From	4	48
Choose	3	4

$$\text{Number of hands} = {}^4C_3 \times {}^{48}C_4 = 778\,320$$

e Number of hands with *no aces*:

	A	Others
From	4	48
Choose	0	7

$$\text{Number of hands} = {}^4C_0 \times {}^{48}C_7 = 73\,629\,072$$

$$\text{Number with at least one ace} = 133\,784\,560 - 73\,629\,072 = 60\,155\,488$$

Note carefully the use of the complementary event in part **e**.

Alternatively we could consider hands with 1 ace + hands with 2 aces + etc. i.e.
${}^4C_1 \times {}^{48}C_6 + {}^4C_2 \times {}^{48}C_5 + {}^4C_3 \times {}^{48}C_4 + {}^4C_4 \times {}^{48}C_3 = 60\,155\,488$, as before.

A subcommittee of five people is to be chosen from the following twelve people:

Alex	Ben	Chris	Dave	Eric	Frank
Gemma	Hetti	Icolyn	Jenny	Kym	Louise

How many different subcommittees are possible in each of the following cases?

a There are no restrictions as to the make up of the subcommittee.

b Jenny and Eric must both be on the subcommittee.

c Ben and Gemma must either both be on the subcommittee or neither be on the subcommittee.

d Dave and Icolyn must not both be on the subcommittee. Dave can be, or Icolyn can be, but not both.

Solution

a

From	12
Choose	5

$$\text{No. of subcommittees} = {}^{12}C_5 = 792$$

b

	J & E	Others
From	2	10
Choose	2	3

$$\text{No. of subcommittees} = {}^2C_2 \times {}^{10}C_3 = 120$$

c B and G both on OR B and G both not on

B and G both on

	B & G	Others
From	2	10
Choose	2	3

B and G both not on

	B & G	Others
From	2	10
Choose	0	5

$$\text{No. of subcommittees} = {}^2C_2 \times {}^{10}C_3 + {}^2C_0 \times {}^{10}C_5 = 372$$

d

	D not I			I not D			Neither		
	D	I	Others	D	I	Others	D	I	Others
From	1	1	10	1	1	10	1	1	10
Choose	1	0	4	0	1	4	0	0	5

$$\text{No. of subcommittees} = {}^1C_1 \times {}^1C_0 \times {}^{10}C_4 + {}^1C_0 \times {}^1C_1 \times {}^{10}C_4 + {}^1C_0 \times {}^1C_0 \times {}^{10}C_5 = 672$$

Alternatively, for part **d**, we could find the number of committees containing both Dave and Icolyn and take this from the total number of subcommittees:

$$\text{No. of subcommittees} = {}^{12}C_5 - {}^2C_2 \times {}^{10}C_3 = 792 - 120 = 672 \quad \text{as before.}$$

How many different 5 letter arrangements can be made each consisting of 5 different letters of the alphabet, with exactly one of the 5 being a vowel?

Solution

Method 1: Suppose the vowel is first.

Number of arrangements with vowel first $= 5 \times 21 \times 20 \times 19 \times 18 = 718\,200$

But the vowel could be in any of the five positions

Number of arrangements with one vowel $= 5 \times 718\,200 = 3\,591\,000$

Method 2: Choose one vowel. Number of ways $= {}^5C_1$
Choose four consonants. Number of ways $= {}^{21}C_4$
Arrange the five items. Number of ways $= 5!$
Number of arrangements with one vowel $= {}^5C_1 \times {}^{21}C_4 \times 5! = 3\,591\,000$ as before.

- Note especially method 2 above in which the number of arrangements have been determined using a 'choose and then arrange' approach.

How many three letter permutations are there of the letters of the word PARALLEL?

Solution

Parallel involves 8 letters including 2 As and 3 Ls.

P	A	R	L	E
	A		L	
			L	

Consider mutually exclusive situations:

Arrangements with

3 letters, all different: $5 \times 4 \times 3 = 60$
2 As and one other, AA?, A?A, ?AA $3 \times 4 = 12$
2 Ls and one other, LL?, L?L, ?LL $3 \times 4 = 12$
3 Ls $1 = 1$
$60 + 12 + 12 + 1 = 85$

There are 85 permutations altogether.

Or, using a 'choose and then arrange' approach:

Arrangements with

3 letters, all different: ${}^5C_3 \times 3! = 60$

2 As and one other, AA?, A?A, ?AA ${}^4C_1 \times \frac{3!}{2!} = 12$

2 Ls and one other, LL?, L?L, ?LL ${}^4C_1 \times \frac{3!}{2!} = 12$

3 Ls $= 1$

$60 + 12 + 12 + 1 = 85$

There are 85 permutations altogether, as before.

* * One With Subsets

Including A itself and the empty set how many subsets can be made using the elements of set A where A = {a, e, i, o, u}?

Solution

There exists
1 subset of A with no elements. ${}^5C_1$ subsets with 1 element.
${}^5C_2$ subsets with 2 elements. ${}^5C_3$ subsets with 3 elements.
${}^5C_4$ subsets with 4 elements. ${}^5C_5$ subsets with 5 elements.

$$\begin{aligned}\text{Total number of subsets} &= 1 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 \\ &= 1 + 5 + 10 + 10 + 5 + 1 \\ &= 32\end{aligned}$$

Alternatively, when forming a subset of {a, e, i, o, u} we can consider each element on an 'include it or not include it' basis. We can include the letter a or not, we can then include the letter e or not, we can then include the letter i or not, etc. Thus there are 2 ways of dealing with each letter.

$$\begin{aligned}\text{Thus number of subsets} &= 2 \times 2 \times 2 \times 2 \times 2 \\ &= 2^5 \\ &= 32\end{aligned}$$

If we include the empty set and the set itself then a set with n different elements has 2^n possible subsets.