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Mathematics Specialist
Unit 4
2017

TEST 5

SOLUTIONS

Student name: ________________________ **Teacher name:** ________________________

Time allowed for this task: **40 *minutes***, in class, under test conditions
Calculator-Assumed

Materials required:

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters, SCSA Formula Sheet.

Special items: Classpad Calculator and Scientific Calculator. Drawing instruments, templates

Marks available: ***39 marks***

Task weighting: **8%**

Question 1. **(7 marks)**

(a) A particle undergoing simple harmonic motion with a period of 5 seconds is observed to move in a straight line, oscillating 3.6 m either side of a central position. Determine the speed of the particle when it is 3 m from the central position. (3 marks)

Solution
$\omega=\frac{2\pi}{5} \quad v^2=\left(\frac{2\pi}{5}\right)^2\left(3.6^2-3^2\right)$ $\lvert v\rvert=2.50\text{ m/s}$
Specific behaviours
✓ determines ω ✓ substitutes into velocity equation ✓ evaluates speed

(b) Another particle moving in a straight line experiences an acceleration of $x+2.5$ ms^{-2}, where x is the position of the particle at time t seconds.

Given that when $x=1$, the particle had a velocity of 2 ms^{-1}, determine the velocity of the particle when $x=2$. (4 marks)

Solution
$\frac{1}{2}v^2=\int x+2.5\,dx \quad v^2=2\left(\frac{x^2}{2}+2.5x\right)+c$ $x=1, v=2\Rightarrow c=-2$ $x=2\Rightarrow v^2=2\left(\frac{2^2}{2}+2.5(2)\right)-2\Rightarrow v=\pm2\sqrt{3}$
Specific behaviours
✓ uses appropriate form of acceleration ✓ integrates ✓ evaluates constant ✓ states all possible values of v

Question 2. (8 marks)

Lengths of climbing rope produced by a manufacturer over a long production run have breaking strengths that are normally distributed with a mean of 180.2 kg and standard deviation of 9.5 kg.

(a) Determine the probability that the mean breaking strength of a randomly chosen sample of 10 lengths will be less than 175 kg. (3 marks)

$$sd = \frac{9.5}{\sqrt{10}} \approx 3.004$$

$$X \sim N(180.2, 3.004^2)$$

$$P(X < 175) = 0.0417$$

(b) At the start of a production run, a supervisor at the factory randomly samples 20 lengths and after testing, determines that the mean breaking strength of the sample is 176.9 kg. Construct a 90% interval estimate for the population mean based on this sample. (2 marks)

$$176.9 \pm 1.645 \frac{9.5}{\sqrt{20}} = (173.41, 180.39)$$

(c) If the supervisor repeated the same sampling process in (b) every day for 30 consecutive days, how many of the intervals constructed would be expected to include the known mean breaking strength of 180.2 kg? (1 mark)

$0.9 \times 30 = 27$ intervals.

(d) How large a sample should the supervisor take so that the width of a 95% confidence interval for the mean breaking strength has a width of no more than 5 kg? (2 marks)

$$n = \left(\frac{1.96 \times 9.5}{2.5}\right)^2$$
$$= 55.47$$

Hence sample 56 lengths.

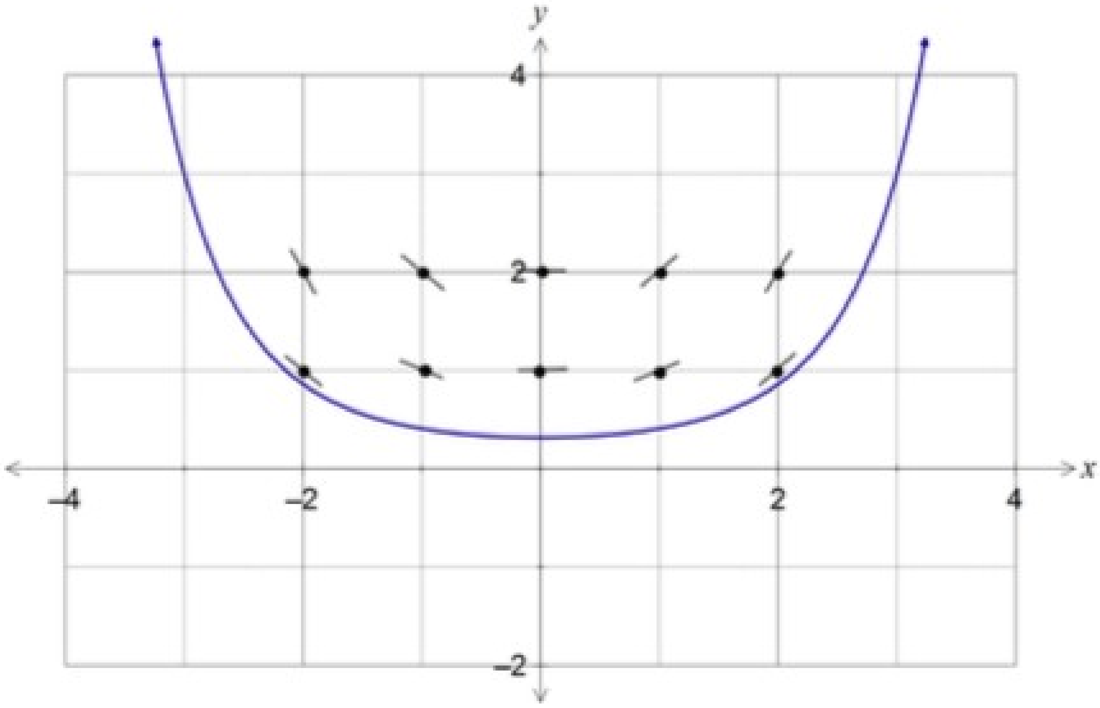
Question 3. **(12 marks)**

(a) A first-order differential equation is given by $\frac{dy}{dx} = \frac{xy}{2}$.

(i) Use the equation to complete the table below. (2 marks)

x	-2	-1	0	1	2	3
y	2	2	2	2	2	3
$\frac{dy}{dx}$	**-2**	**-1**	**0**	**1**	**2**	**4.5**

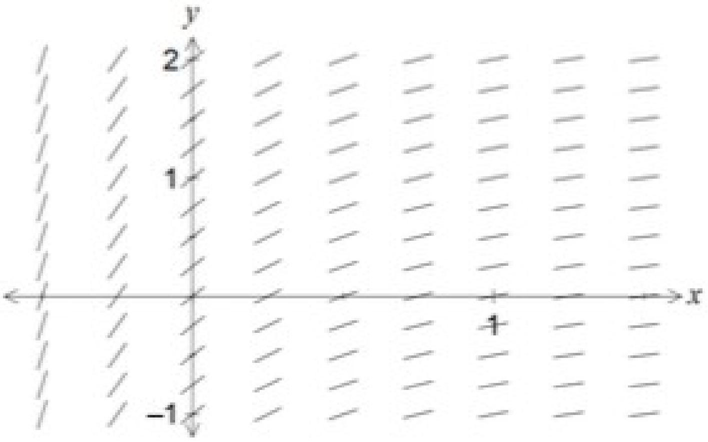
(ii) Create a slope field on the 10 points on the graph below. (2 marks)



(iii) On the graph, sketch the solution curve to the differential equation that passes through the point (3, 3). (2 marks)

Through (3, 3), u-shaped, symmetrical and close to that shown.

(b) The differential equation for a curve passing through the point (0, 1) is given by $\frac{dy}{dx} = \frac{2}{x^2 + 2x + 1}$. The slope field for the differential equation is shown below.



(i) Use the incremental formula $\delta y = \frac{dy}{dx} \times \delta x$, with $\delta x = 0.1$, to calculate an estimate for the y-coordinate of the curve when $x = 0.2$. (4 marks)

Using Euler's method with $\delta x = 0.1$:

x	y	$\frac{dy}{dx}$	$\delta y = \frac{dy}{dx} \times \delta x$
0	1	2	0.2
0.1	1.2	1.653	0.1653
0.2	1.3653		

Estimate is $y = 1.3653$, to four decimal places.

(ii) Explain whether the estimate in (a) is an over- or under-estimate for the y-coordinate. (2 marks)

Over-estimate, as the slope field shows the curve has a positive gradient but is concave down between $x = 0$ and $x = 0.2$.

Question 4. **(12 marks)**

(a) $\frac{dN}{dt}=kN\left(\frac{K-N}{K}\right)$ for $r=0.1$ and $K=100$

$$\frac{dN}{dt}=0.1N\left(1-\frac{N}{100}\right)=\frac{0.1N}{100}(100-N)$$

$$so\ \frac{dt}{dN}=\frac{100}{0.1}\left(\frac{1}{N(100-N)}\right)$$

$$0.001t=\int\left(\frac{1}{N(100-N)}\right)dN \quad ✓$$

Using partial fractions,

$$\frac{1}{N(100-N)}=\frac{a}{N}+\frac{b}{100-N}$$

$$=\frac{a(100-N)+bN}{N(100-N)}$$

$$\frac{0\times N+1}{N(100-N)}=\frac{N(b-a)+100a}{N(100-N)}$$

$$0=b-a \text{ and } 1=100a$$

$$a=b \text{ and } a=0.01=b \quad ✓$$

$$\frac{1}{N(100-N)}=\frac{1}{100N}+\frac{1}{100(100-N)}$$

so $0.001t=\int\left(\frac{1}{N(100-N)}\right)dN$ becomes

$$\frac{0.1t}{100}=\int\frac{1}{100N}+\frac{1}{100(100-N)}dN$$

$$\frac{0.1}{\cancel{100}}t=\frac{1}{\cancel{100}}\left[\int\frac{1}{N}dN+\int\frac{1}{(100-N)}dN\right]$$

$$0.1t=\ln N+(-1)\ln(100-N)+C$$

$$0.1t-C=\ln\left(\frac{N}{100-N}\right) \quad ✓$$

$$\frac{N}{100-N} = e^{0.1t-C}$$

$$\frac{100-N}{N} = Ae^{-0.1t} \quad where\ A = e^{C}$$

$$At\ t = 0, N = N_0$$

$$\frac{100-N_0}{N_0} = Ae^{0} = A \quad \text{so} \quad A = \frac{100-N_0}{N_0}$$ ✓

$$\frac{100-N}{N} = Ae^{-0.1t}$$

Rearrange the formula to get N

$$100 - N = NAe^{-0.1t}$$

$$100 = N\left(1 + Ae^{-0.1t}\right)$$

$$\therefore\ N = \frac{100}{1 + Ae^{-0.1t}} \text{ with } A = \frac{100-N_0}{N_0}$$

✓ (5)

(b) (i) $K=610 \Rightarrow N=\dfrac{610}{1+Ae^{-kt}}$ with $A=\dfrac{610-N_0}{N_0}$

At $t=0,\ N_0=300$ (Jan 2015)

$A=\dfrac{610-300}{300}=1.03$ ✓

$N=\dfrac{610}{1+1.03e^{-kt}}\quad k=?$

At $t=12,\ N_{12}=400$ (Jan 2016) ✓

$400=\dfrac{610}{1+1.03e^{-k\times 12}}$

$1+1.03e^{-k\times 12}=\dfrac{610}{400}\Rightarrow k=0.05642890327$ ✓

$\therefore N=\dfrac{610}{1+1.03e^{-0.05642890324t}}$

At $t=24,\ N_{24}=?$ (Jan 2017) ✓

$N=\dfrac{610}{1+1.03e^{-0.05642890324\times 24}}$

$N=481.55$

The number of trout in the dam in January 2017 is expected to be close to 482. ✓

(5)

(ii) $N=600,\ t=?$

$600=\dfrac{610}{1+1.03e^{-0.05642890324\times t}}$ ✓

$1+1.03e^{-0.05642890324\times t}=\dfrac{610}{600}$

$t=73.14$

i.e. expect 600 trout in the dam in February 2021. ✓ (2)