PERTH MODERN SCHOOL
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Independent Public School

Mathematics Specialist Year 11

Student name: ______________________ Teacher name: _____________

Date: Friday 23rd July 2021

Task type:	**Response**
Time allowed:	**45 minutes**
Number of questions:	**6**
Materials required:	Calculator with CAS capability (to be provided by the student)
Standard items:	Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters
Special items:	Drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations
Marks available:	**40 marks**
Task weighting:	**10%**

Formula sheet provided: Yes

Note: All part questions worth more than 2 marks require working to obtain full marks.

1. [7 marks]
Use mathematical induction to prove that

$$3\times5+6\times6+9\times7+\ldots+3n(n+4)=\frac{n(n+1)(2n+13)}{2}$$

for all positive integers n.

Solution

Let $P(n)$ denote the proposition '$3\times5+6\times6+9\times7+\ldots+3n(n+4)=\frac{n(n+1)(2n+13)}{2}$' for all positive integers n.

With $n=1$,

$$LHS\ of\ P(1)=3\times5¿15$$

$$RHS\ of\ P(1)=\frac{1(1+1)(2\times1+13)}{2}¿15$$

Hence LHS=RHS, and so $P(1)$ is true.

Now assume that $P(k)$ is true for some positive integer k. Then

$$3\times5+6\times6+9\times7+\ldots+3k(k+4)=\frac{k(k+1)(2k+13)}{2}.$$

Now

$$LHS\ of\ P(k+1)=3\times5+6\times6+9\times7+\ldots+3k(k+4)+3(k+1)((k+1)+4)$$
$$¿\frac{k(k+1)(2k+13)}{2}+3(k+1)((k+1)+4)¿\frac{k(k+1)(2k+13)}{2}+3(k+1)(k+5)$$
$$¿\frac{k(k+1)(2k+13)}{2}+\frac{6(k+1)(k+5)}{2}¿\frac{k(k+1)(2k+13)+6(k+1)(k+5)}{2}$$
$$¿\frac{(k+1)[k(2k+13)+6(k+5)]}{2}¿\frac{(k+1)(2k^2+19k+30)}{2}¿\frac{(k+1)(k+2)(2k+15)}{2}$$
$$¿\frac{(k+1)((k+1)+1)(2(k+1)+13)}{2}¿RHS\ of\ P(k+1)$$

Hence $P(k+1)$ is true.

We have shown that $P(1)$ is true, and that if $P(k)$ is true for some positive integer n then $P(k+1)$ is also true. Hence, by the principle of mathematical induction, $P(n)$ is true for all positive integers n.

Specific behaviours

✓ proves $P(1)$ by **evaluating LHS and RHS separately**
✓ assumes $P(k)$ is true
✓ writes LHS of $P(k+1)$ using RHS of $P(k)$
✓ simplifies expression algebraically to one fraction
✓ writes numerator with a factor of $(k+1)$
✓ obtains expression for RHS of $P(k+1)$ **written in terms of** $k+1$

✓ writes conclusion for whole proof (accept just the second sentence without the first)

2. [2 marks]
A question in a Specialist exam paper asked students to prove the following statement:

‘$3n$ is odd if and only if n is odd (where n is an integer)’.

One student wrote the answer below. Explain clearly why they should **not** receive full marks for this answer.

Proof:
We prove the contrapositive. Assume that n is an even integer. Then $n=2k$ for some integer k. Now

$$3n=3(2k)¿2(3k)$$

which is even since $3k$ is an integer. Hence if n is even then $3n$ is even, which implies that $3n$ is odd if and only if n is odd.

Solution
The student has proved only the statement ‘if $3n$ is odd then n is odd’. However, since the original statement involves the phrase ‘if and only if’, it is also necessary to prove the statement ‘if n is odd then $3n$ is odd’.
Specific behaviours
✓ Notes that statement involves ‘if and only if’, or describes as an equivalence statement ✓ Explains that the student should also have proved that ‘if n is odd then $3n$ is odd’, or refers to the ‘backward direction’

3. [9 = 3+3+3 marks]
Write whether each of the following statements is true or false, and prove or disprove it accordingly.

a) For all positive real numbers x

$$x^3-x\geq x^2-x$$

Solution
The statement is **false**, and is disproved with the following counterexample: Let $x=\frac{1}{2}$. Then LHS = $\frac{1}{8}-\frac{1}{2}=\frac{-3}{8}$ and RHS = $\frac{1}{4}-\frac{1}{2}=\frac{-1}{4}$, meaning that $x^3-x<x^2-x$ in this case.

Hence the statement is false.
Specific behaviours
✓ states false ✓ states counterexample with a particular value of x ✓ shows that for that value of x, $x^3 - x < x^2 - x$. [Alternatively give 2nd and 3rd marks if successfully argues false for any value of x with $0 < x < 1$.]

b) There exist distinct prime numbers p and q such that $p - q = 2$.

Solution
The statement is **true**, and is proved with the following example: Let $p = 5$ and $q = 3$. Then $p - q = 2$.
Specific behaviours
✓ states true ✓✓ states example with values of p and q such that $p - q = 2$

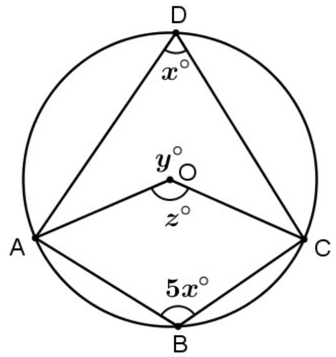
c) There exist distinct prime numbers p and q such that $p^2 - q^2 = 2$.

Solution
The statement is **false**. Let p and q be distinct prime numbers. Then $p^2 - q^2 = (p+q)(p-q)$ Since p and q are distinct primes, $p + q \geq 5$ and $p - q \geq 1$, and so $p^2 - q^2 \geq 5$. Hence there do not exist distinct prime numbers p and q with $p^2 - q^2 = 2$.
Specific behaviours
✓ states false ✓ factorises $p^2 - q^2$ using difference of squares ✓ argues that $p^2 - q^2$ cannot equal 2.

4. [6 marks]

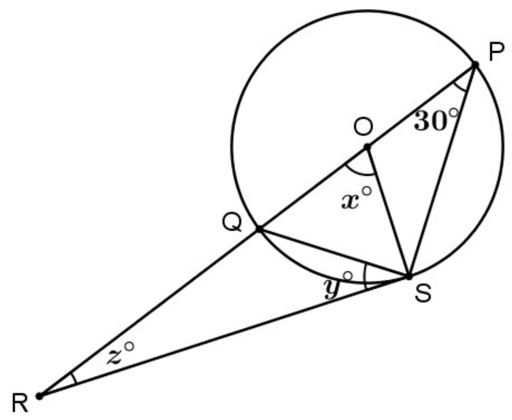
Find the values of x, y and z in each of the following:

a) A, B, C and D all lie on the circle with centre O:



Solution
$5x + x = 180 x = 30$ $z = 2 \times 30 \text{¿} 60$ $y = 360 - 60 \text{¿} 300$
Specific behaviours
✓✓✓ 1 mark per correct value

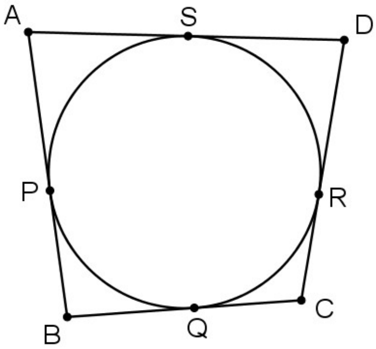
b) $\overline{RS}$ is tangent to the circle with centre O.



Solution
$x = 2 \times 30 \text{¿} 60$ $y = 30$ $z = 180 - 90 - 60 \text{¿} 30$
Specific behaviours

✓✓✓ 1 mark per correct value

5. [5 marks]
$ABCD$ is a quadrilateral such that each of the four sides is tangent to the same circle, at the points P, Q, R and S, as illustrated below. If $AB = 15$, $BC = 10$ and $CD = 12$, find the length AD.



Solution
Since the sides are tangent to the circle, we may write: $w = AS = AP$, $x = BQ = BP$, $y = CQ = CR$ and $z = DS = DR$. Thus $w + x = 15$ (1) $x + y = 10$ (2) $y + z = 12$ (3) Adding equations (1) and (3) gives $w + x + y + z = 27$ and subtracting equation (2) gives $w + z = 17$ Hence $AD = 17$.
Specific behaviours
✓ uses theorem for tangent segments from the same point ✓ identifies segments of equal lengths ✓ sets up equations for side lengths using sums of segment lengths ✓ solves set of equations for $w + z$ ✓ states correct value [Accept alternative methods.]

6. [11 = 3+4+4 marks]
Solve each of the following trigonometric equations for x in the stated domain.
Show all working to support your answers.
a) $2\cos(x) = \sqrt{3}$ for $0 \le x \le 2\pi$

Solution

$$2\cos(x)=\sqrt{3}\cos(x)=\frac{\sqrt{3}}{2}x=\frac{\pi}{6}\vee\frac{11\pi}{6}$$

Specific behaviours

✓ isolates cos (x)
✓ states at least one correct solution
✓ states two correct solutions
[No marks for answers only]

b) $\sin\left(x+\frac{\pi}{4}\right)=\frac{-1}{\sqrt{2}}$ for $-\pi\le x\le\pi$

Solution

$$\sin\left(x+\frac{\pi}{4}\right)=\frac{-1}{\sqrt{2}}x+\frac{\pi}{4}=\frac{5\pi}{4}+k2\pi\text{¿}x+\frac{\pi}{4}=\frac{7\pi}{4}+k2\pi$$

Hence

$$x=\pi+k2\pi$$
$$\text{¿}x=\frac{3\pi}{2}+k2\pi$$

With $k=0$, $x=\pi$ or $x=\frac{3\pi}{2}$

With $k=-1$, $x=-\pi$ or $x=\frac{-\pi}{2}$

Hence $x=-\pi,-\frac{\pi}{2}$ or π

Specific behaviours

✓ isolates $\sin\left(x+\frac{\pi}{4}\right)$
✓ states at least one correct solution for $x+\frac{\pi}{4}$
✓ states at least one correct solution or for x
✓ states all three correct solutions for x
[No marks for answers only]

c) $\frac{1}{\sqrt{3}}\tan(5x)=1$ for $0\le x\le\pi$

Solution

$$\tan(5x)=\sqrt{3}5x=\frac{\pi}{3}+k\pi x=\frac{\pi}{15}+k\frac{\pi}{5}$$

Letting $k=0,1,2,3$ and 4 we obtain: $x=\frac{\pi}{15}, \frac{4\pi}{15}, \frac{7\pi}{15}, \frac{10\pi}{15}$ and $\frac{13\pi}{15}$
Specific behaviours
✓ isolates $\tan(5x)$ ✓ obtains $\frac{\pi}{3}$ as a solution for $5x$ ✓ obtains $\frac{\pi}{15}$ as a solution for x ✓ states all five correct solutions for x [No marks for answers only]