PERTH MODERN SCHOOL
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INDEPENDENT PUBLIC SCHOOL

Semester One Examination, 2019

Question/Answer booklet

MATHEMATICS METHODS
UNIT 3
Section One:
Calculator-free

Your Name: ______________________________

Your Teacher's Name: ______________________________

Time allowed for this section
Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Marks	Max	Question	Mark	Max
1		6	5		6
2		6	6		10
3		8	7		10
4		6			

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	52	34
Section Two: Calculator-assumed	13	13	100	103	66
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2019*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section One: Calculator-free **(52 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 50 minutes.

Question 1 **(6 marks)**

Differentiate the following with respect to x:

(a) $e^x \sin\left(x^2+2\right)$ (2 marks)

Solution
$\frac{d}{dx}\left(e^x \sin\left(x^2+2\right)\right) = e^x\, 2x\cos\left(x^2+2\right) + e^x \sin(x^2+2)$ $= e^x\left[2x\cos\left(x^2+2\right) + \sin(x^2+2)\right]$
Specific behaviours
✓ uses product rule ✓ differentiates sine term correctly with chain rule(no need to factorise)

(b) $\frac{\cos x}{x^2+5}$ (2 marks)

Solution
$\frac{d}{dx}\left(\frac{\cos x}{x^2+5}\right) = \frac{\left(x^2+5\right)(-\sin x) - (2x)\cos x}{\left(x^2+5\right)^2}$
Specific behaviours
✓ uses quotient rule(or appropriate product rule)

✓ differentiates cosine term

(c) $\int^{x} t^{3}dt$ (2 marks)

Solution
$\frac{d}{dx}\int^{x} t^{3}dt = x^{3}$
Specific behaviours
✓ uses fundamental theorem ✓ obtains correct expression

Question 2 **(6 marks)**

“Blood flow” is defined as the volume V of blood flowing through an artery per unit of time. It can be modelled by the formula $V = k\,r^{3}$, where r is the radius of the artery and k is a constant.

(a) By what fraction is the blood flow in the artery reduced when its radius is halved? (2 marks)

Solution
$k\left(\frac{r_o}{2}\right)^{3} = \frac{kr_o}{8} = \frac{V_o}{8}$ Reduced by $\frac{7}{8}$
Specific behaviours
✓ cubes one half ✓ states $\frac{7}{8}$

(b) Use the incremental formula to estimate the percentage increase required in the radius of a partially clogged artery to produce a 6% increase in the blood flow. (3 marks)

Solution
$\Delta V \approx \frac{dV}{dr}\Delta r$ $=3kr^2\Delta r$ $\frac{\Delta V}{V}=\frac{3kr^2\Delta r}{kr^3}=3\frac{\Delta r}{r}$ $0.06=3\frac{\Delta r}{r}$ $\frac{\Delta r}{r}=0.02$ 2% *increase*
Specific behaviours
✓ uses incremental formula ✓obtains expression for change in Volume/Radius ✓ obtains percentage change in radius(accept 0.02)

(c) Explain why the incremental formula does not give a good estimate for the change in V in part (a). (1 mark)

Solution
$\frac{\Delta r}{r}=50\%$ which is too large Incremental formula only useful for small percentage changes
Specific behaviours
✓ states reasonable explanation

Question 3 **(8 marks)**

Determine the following:

(a) $\frac{d}{dx}\left(e^{2x}\sin 3x\right)$ (2 marks)

Solution
$\frac{d}{dx}\left(e^{2x}\sin 3x\right)=2e^{2x}\sin 3x+e^{2x}3\cos 3x$ $=e^{2x}\left(2\sin 3x+3\cos 3x\right)$
Specific behaviours
✓ uses product rule ✓ diff sine term with chain rule(no need to factorise)

(b) $\frac{d}{dx}\left(e^{2x}\cos 3x\right)$ (2 marks)

Solution
$\frac{d}{dx}\left(e^{2x}\cos 3x\right)=2e^{2x}\cos 3x-e^{2x}3\sin 3x$ $=e^{2x}\left(2\cos 3x-3\sin 3x\right)$
Specific behaviours
✓ uses product rule ✓ diff cosine term with chain rule(no need to factorise)

Hence, determine the following integral by considering **both expressions** above.

(c) $\int_0^{\frac{\pi}{2}} 13e^{2x}\cos 3x\,dx$ (4 marks)

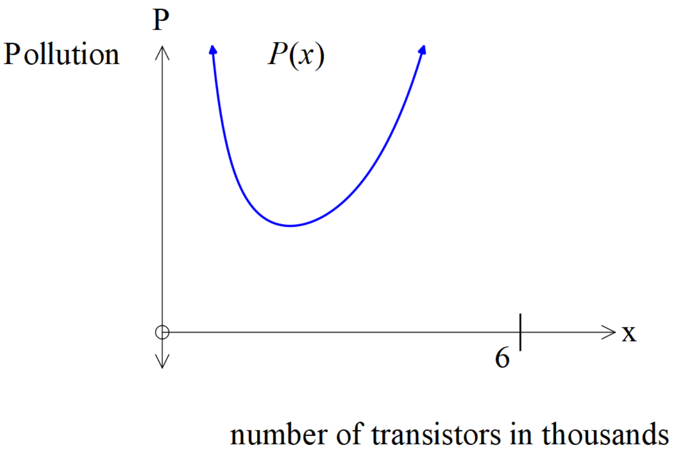
Solution
$3\frac{d}{dx}\left(e^{2x}\sin 3x\right)=e^{2x}\left(6\sin 3x+9\cos 3x\right)$ $2\frac{d}{dx}\left(e^{2x}\cos 3x\right)=e^{2x}\left(4\cos 3x-6\sin 3x\right)$ $3\frac{d}{dx}\left(e^{2x}\sin 3x\right)+2\frac{d}{dx}\left(e^{2x}\cos 3x\right)=13e^{2x}\cos 3x$ $\int 13e^{2x}\cos 3x dx=3e^{2x}\sin 3x+2e^{2x}\cos 3x=e^{2x}\left(3\sin 3x+2\cos 3x\right)$ $\int_0^{\frac{\pi}{2}} 13e^{2x}\cos 3x dx=e^{\pi}\left(-3\right)-\left(2\right)=-3e^{\pi}-2$
Specific behaviours
✓ modifies derivatives in a & b by multiplying by factors 2 & 3   ✓ obtains an equivalent expression of integral in terms of two derivatives   ✓ uses fundamental theorem   ✓obtains exact value for integral

Question 4 **(6 marks)**

The amount of pollution, P, in tonnes, to build x thousands number of transistors by an electronic manufacturer, is given by the following formula:

$$P(x)=\frac{e^x}{x^3}$$

where $0<x\leq 5.5$ thousands.



(a) Describe how the pollution changes as the number of transistors made, varies from $0 < x \leq 5.5$ thousands. (2 marks)

Solution
Pollution is initially very high but then decreases to a minimum value before then increasing again.
Specific behaviours
✓ initially high ✓ reaches a minimum then increases

(b) Using calculus, determine the number of transistors that will minimize the pollution produced. (4 marks)

Solution
$\frac{d}{dx}\left(\frac{e^x}{x^3}\right) = \frac{x^3e^x - e^x 3x^2}{x^6} = \frac{e^x x^2(x-3)}{x^6}$ $x=3 \quad \frac{d}{dx}\left(\frac{e^x}{x^3}\right) = 0$ $x=2 \quad \frac{d}{dx}\left(\frac{e^x}{x^3}\right) = \frac{e^2}{2^4}(-1) \quad < 0$ $x=4 \quad \frac{d}{dx}\left(\frac{e^x}{x^3}\right) = \frac{e^4}{4^4}(1) \quad > 0$ Therefore x=3 is a local minimum 3000 transistors will minimize pollution
Specific behaviours
✓ uses quotient rule(or appropriate product rule) ✓ equates derivative to zero and solves for x

✓ uses first derivative sign test (or second) ✓ states in thousands the number to minimize pollution

Question 5 **(6 marks)**

Twenty teachers have been marking the same set of exam papers and after double checking it was found that the teachers made the following number of errors,
1, 1, 3, 1, 4, 4, 1, 3, 5, 1, 2, 2, 4, 4, 5, 0, 6, 5, 5, 5.

Let $X = ¿$ the number of errors of a teacher.

(a) Construct a table that defines the **probability distribution** of X. (3 marks)

Solution
see table below
Specific behaviours
✓ correct values of x ✓ uses frequencies ✓ states probabilities(no need to simplify)

x	0	1	2	3	4	5	6
P(X=x)	$\frac{1}{20}$	$\frac{5}{20}$	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{5}{20}$	$\frac{1}{20}$

(b) Use the probability distribution above to show how to evaluate the expected value of X. State this value. (3 marks)

Solution

$E(x) = 0\times\frac{1}{20} + 1\times\frac{5}{20} + 2\times\frac{2}{20} + 3\times\frac{2}{20} + 4\times\frac{4}{20} + 5\times\frac{5}{20} + 6\times\frac{1}{20}$ $= \frac{62}{20} = \frac{31}{10} = 3.1$
Specific behaviours
✓ multiplies x by probability ✓ sums the above ✓states the expected value, no need to simplify.

Question 6 **(10 marks)**

In a shop an average of 1 out of 5 pay with cash, 3 out of every 5 customers use a credit card, and the rest use a debit card to pay. A single customer is selected from the store.The random variable X is defined as the number of customers who pay with cash.

(a) Complete the probability distribution for X shown below. (2 marks)

Solution

x	0	1
$P(X=x)$	$\frac{4}{5}$	$\frac{1}{5}$

Specific behaviours

✓ determines for x=0
✓ determines for both variables

(b) State the distribution of X. (1 mark)

Solution
A Bernoulli distribution
Specific behaviours
✓ states Bernoulli

(c) Determine the mean and standard deviation of the distribution. (2 marks)

Solution
Mean = $\frac{1}{5}$

Standard deviation = $\sqrt{\frac{1}{5}(\frac{4}{5})} = \sqrt{\frac{4}{25}} = \frac{2}{5}$
Specific behaviours
✓ states the mean ✓ states the simplified standard deviation

Four customers are waiting in a queue to pay. The random variable Y is defined as number of customers from this queue who pay with **credit card**.

(d) State the distribution of Y, including its parameters. (2 marks)

Solution
$Y \sim Binomial(4, \frac{3}{5})$
Specific behaviours
✓ states Binomial ✓ states both parameters

(e) Evaluate the probability of at most one customer paying with credit card.
(No need to simplify)

(3 marks)

Solution
$P(Y \leq 1) = P(Y = 0) + P(Y = 1)$ $\left(\frac{2}{5}\right)^4 + 4\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^3 = \frac{16+96}{5^4} = \frac{112}{625}$
Specific behaviours
✓ uses Y=0, Y=1 only ✓ uses binomial formula ✓ derives an expression for the sum of these two probabilities

Question 7 **(10 marks)**

Consider a smooth and continuous function $f(x)$ where $f(-5)=0=f(0)=f(7)=f(10)$, $f(-3)=22$ and $f(4)=-13$. It is known that $f(x)\geq 0$ for $-5\leq x\leq 0$ and $7\leq x\leq 10$ with $f(x)<0$ for all other values. It is also given that $\int_{-5}^{10} f(x)dx=19$ and $\int_{0}^{7} f(x)dx=-6$ and $\int_{7}^{10} f(x)dx=15$.

Determine the following:

(a) $\int_{-3}^{4} f'(x)dx$ (2 marks)

Solution
$\int_{-3}^{4} f'(x)dx=\left[f(x)\right]_{-3}^{4}=f(4)-f(-3)=-13-22=-35$
Specific behaviours
✓ uses fundamental theorem ✓ determines integral

(b) $\int_{-5}^{7} f(x)dx$ (2 marks)

Solution

$\int_{-5}^{7} f(x)dx = \int_{-5}^{10} f(x)dx - \int_{7}^{10} f(x)dx$ $= 19 - 15 = 4$ OR $\int_{-5}^{0} f(x)dx + \int_{0}^{7} f(x)dx + \int_{7}^{10} f(x)dx = 19$ $\int_{-5}^{0} f(x)dx - 6 + 15 = 19$ $\int_{-5}^{0} f(x)dx = 10$ $\int_{-5}^{7} f(x)dx = 10 - 6 = 4$
Specific behaviours
✓ uses linearity for integrals(addition or subtraction) ✓ determines final integral

(c) the stationary points of $g(x)$ given that $g(x) = \int_{-5}^{x} f(t)\,dt$ with $-5 \le x \le 10$ (3 marks)

Solution
$g'(x) = f(x)$ $f(x) = 0$ $x = -5, 0, 7 \,\&\, 10$
Specific behaviours
✓ uses fundamental theorem ✓ equates f(x) to zero ✓solves for all 4 stationary points

It is known that $f'(x) \le 0$ for $-3 \le x \le 4$ and $f'(x) \ge 0$ for $-5 \le x \le -3$.

(d) Determine the area under the curve $y = f'(x)$ from $-5 \le x \le 4$. (3 marks)

Solution
$area = \int_{-5}^{-3} f'(x)dx + \left\| \int_{-3}^{4} f'(x)dx \right\|$ $= f(-3) - f(-5) + \left\| f(4) - f(-3) \right\|$ $= 22 - 0 + \left\| -13 - 22 \right\| = 57\,sq\,units$

Specific behaviours
✓ divides into two intervals with absolute value for one interval ✓ uses fundamental theorem ✓calculates area

End of Questions

Additional working space

Question number: ________