Papers written by
Australian Maths
Software

SEMESTER ONE

REVISION 2

MATHEMATICS METHODS

UNIT 3

2016

SOLUTIONS

SECTION ONE

1. (8 marks)

(a) $y = 2(10 - x)^3$

$\frac{dy}{dx} = -6(10-x)^2$ ✓✓ -1/error (2)

(b) $y = e^{-x}(\cos(x))$

$\frac{dy}{dx} = -e^{-x}(\cos(x)) + e^{-x}(-\sin(x))$ ✓ ✓

$\frac{dy}{dx} = -e^{-x}(\cos(x) + \sin(x))$ ✓

(3)

(c) $y = \frac{\tan(x)}{x}$

$\frac{dy}{dx} = \frac{x\sec^2(x) - 1\times\tan(x)}{x^2}$ ✓ ✓ OR $\frac{dy}{dx} = \frac{\frac{x}{\cos^2(x)} - 1\times\tan(x)}{x^2}$ (2)

$\frac{dy}{dx} = \frac{1}{x^2}\left(\frac{x - \sin(x)\cos(x)}{\cos^2(x)}\right)$

2. (6 marks)

(a) (i) $\int (5-2x)^5\,dx = \frac{(5-2x)^6}{-12} + c$ ✓ ✓ (2)

(ii) $\int (4e^{2x} - \cos(2x))dx = 2e^{2x} - \frac{\sin(2x)}{2} + c$ ✓ ✓ (2)

(b) $\frac{dy}{dx} = \sin(x) + e^x$

$y = \int (\sin(x) + e^x)\,dx$

$y = -\cos(x) + e^x + c$

$(0,0)$ belongs to the function

$0 = -1 + 1 + c \Rightarrow c = 0$

$y = -\cos(x) + e^x$

(2)

(c)

$$E(Y) = 3\times(1.875) - 2$$
$$E(Y) = 3.625$$
$$Sd(Y) = 3\times(1.053268722)$$
$$Sd(Y) \approx 3.16$$

(2)

20. (5 marks)

$p = 0.15$

$P(X = 2) = \binom{n}{2}(0.15)^2(0.85)^{n-2}$ ✓✓

Want

$\binom{n}{2}(0.15)^2(0.85)^{n-2} \geq 0.9$ *or* $\frac{n(n-1)}{2}(0.15)^2(0.85)^{n-2} \geq 0.9$

Solve the equation $\binom{n}{2}(0.15)^2(0.85)^{n-2} = 0.9.$

If n is a fraction, round up and Paul has his answer.

(5)

END OF SECTION TWO

3. (7 marks)

(a) $$\int_0^1 \frac{x^2 + x^3 - 3x}{x}\,dx = \int_0^1 x + x^2 - 3\,dx$$
$$= \left[\frac{x^2}{2} + \frac{x^3}{3} - 3x\right]_0^1$$
$$= \left(\frac{1}{2} + \frac{1}{3} - 3\right) - (0)$$
$$= -2\frac{1}{6}$$

(3)

(b) $$\int_2^3 (1-2x)^3\,dx = -\frac{1}{8}\times\left[(1-2x)^4\right]_2^3$$
$$= -\frac{1}{8}\times(625 - 81)$$
$$= -68$$

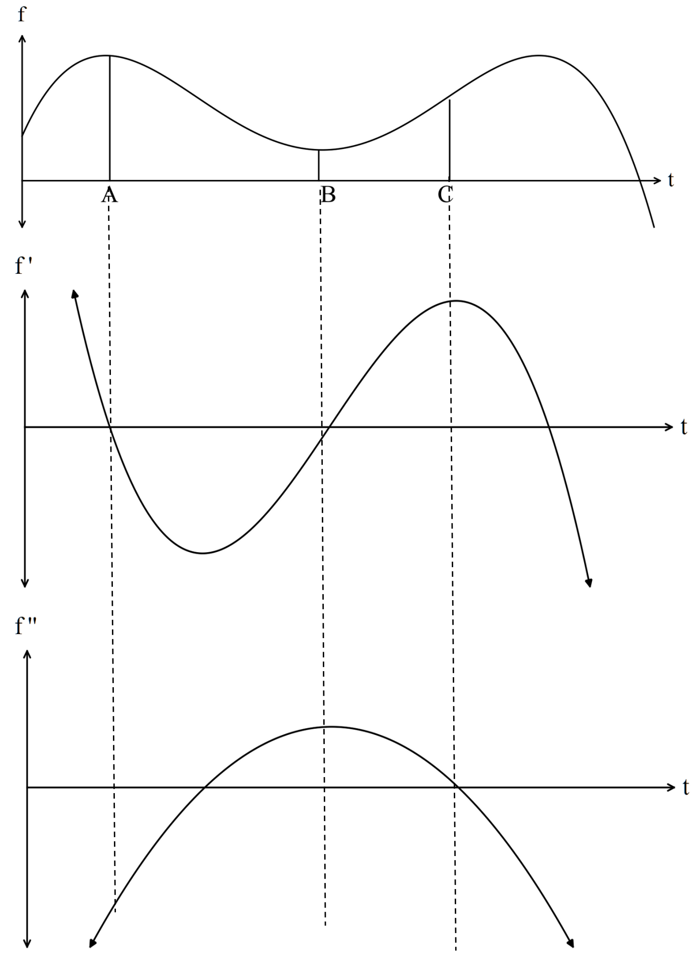
(2)

(c) $$\int_{\pi/4}^{\pi} cos(2y)\,dy = \frac{1}{2}\times\left[sin(2y)\right]_{-\pi/4}^{\pi}$$
$$= \frac{1}{2}\times(0 - (-1))$$
$$= \frac{1}{2}$$

(2)

4. (9 marks)

(a)



✓✓

✓✓

(4)

18. (8 marks)

(a)

x	2	3	4	5	6	7	8	9	10
$P(X = x)$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{3}{24}$	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{3}{24}$	$\frac{2}{24}$	$\frac{1}{24}$

✓✓✓ (3)

(b) $P(\text{even score}) = \frac{1+3+4+3+1}{24} = \frac{12}{24}$ ✓ ✓ (2)

(c) $P(\text{score} < 6) = \frac{1+2+3+4}{24} = \frac{10}{24}$ ✓ ✓ (2)

(d) $a = 8$ ✓ (1)

19. (12 marks)

(a) The data does represent a discrete random variable as the probabilities add up to 1. ✓✓ (2)

(b) (i)

x	1	2	3	4
$P(X = x)$	0.5	0.25	0.125	**0.125**

✓✓

(2)

(ii) $P(x = 1 \text{ or } x = 2) = 0.75$ ✓ (1)

(iii) $P(x \geq 2) = 1 - 0.5 = 0.5$ ✓ (1)

(iv) $E(X) = 1\times0.5 + 2\times0.25 + 3\times0.125 + 4\times0.125 = 1.875$ ✓

$Var(X) = E(X^2) - (E(X))^2$

$E(X^2) = 1^2\times0.5 + 2^2\times0.25 + 3^2\times0.125 + 4^2\times0.125 = 4.625$

$Var(X) = 4.625 - 1.875^2$

$= 1.109375$

$Sd(X) = 1.053268722$

$Sd(X) \approx 1.05$

(4)

16. (11 marks)

(a) (i) $\int_a^b f(x)dx = \int_0^b f(x)dx - \int_0^a f(x)dx = 3.9 - 5.4 = -1.5$ (2)

✓ ✓

(ii) $Area = \int_0^a f(x)dx + \left|\int_a^b f(x)dx\right| = 5.4 + |-1.5| = 6.9$ units^2 (2)

✓ ✓

(b) (i) From below

$1\times1 + 1\times1.41 + 1\times1.73 = 4.14$ ✓✓

From above

$1\times1.41 + 1\times1.73 + 1\times2 = 5.14$ ✓✓

The average of the two estimates is 4.64 units^2 ✓ (5)

(ii) Area $= 4.6$ units^2 ✓✓ (2)

17. (5 marks)

(a)

x	1	2	3	4	5
$P(X = x)$	0.1	**0.3**	**0.3**	**0.2**	**0.1**

✓✓

(2)

(b) $P(x \geq 4) = 0.2 + 0.1 = 0.3$ (1)

(c) $P(x > 2) = 0.3 + 0.2 + 0.1 = 0.6$ (2)

(b) (i) At t = A, there is a maximum turning point where the gradient is zero and the second derivative is negative.

$f'(t) = 0 \quad and \quad f''(t) < 0.$ ✓✓ (2)

The particle changes direction so velocity changes from positive to negative. As the velocity is decreasing, there is negative acceleration that can be seen in f”.

(ii) At t = B, there is a minimum turning point where the gradient is zero and the second derivative is positive.

$f'(t) = 0 \quad and \quad f''(t) > 0.$ ✓✓ (2)

The particle changes direction so velocity changes from negative to positive. As the velocity is increasing, there is positive acceleration that can be seen in f”.

(iii) At t = C, there is a point of inflection where the second derivative is zero.

$f''(t) = 0.$ ✓ (1)

There is also a turning point on the f' graph. On the f graph it can be seen that while the particle still has a positive gradient; before the point of inflection the gradient was increasing but decreasing after the point of inflection.

Hence the turning point on the f' graph.

As $f'(x) = 0$ it follows that $f''(x) = 0.$.

5. (5 marks)

Given $\int_0^2 f(x)dx = 3$ and $\int_2^3 f(x)dx = 5$

(a) $\int_0^3 2f(x)dx = 2\left(\int_0^2 f(x)dx + \int_2^3 f(x)dx\right) = 2\times8 = 16$ ✓ (1)

(b) $\int_2^3 1 - 4f(x)dx = \int_2^3 1dx - 4\int_2^3 f(x)dx$

$= [x]_2^3 - 4\times5$

$= 1 - 20$

$= -19$

(2)

(c) $\int_1^2 \left(\frac{f(x)}{2}+x\right)dx = \frac{1}{2}\times\int_1^2 (f(x))dx + \int_1^2 x\,dx$

$= \frac{3}{2}+\frac{1}{2}\times\left[x^2\right]_1^2$

$= \frac{3}{2}+\frac{3}{2}$

$=3$

(2)

6. (8 marks)

(a) (i) $\int_{\pi/2}^{x^2} cos(t)dt = \left[sin(t)\right]_{\pi/2}^{x^2}$

$= sin(x^2) - 1$ (2)

(ii) $\frac{d}{dx}\left(\int_{\pi/2}^{x^2} cos(t)dt\right) = 2x\,cos(x^2)$ (2)

✓ ✓

(b) (i) $F'\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right) = tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ (2)

✓ ✓

(ii) $\int_1^4 f(x)dx = \left[\sqrt{x}\right]_1^4 = 2-1 = 1$ (2)

✓ ✓

7. (7 marks)

(a) Given $f(x) = sin(x)$ and $g(x) = \sqrt{x}$

(i) $y = g(f(x)) = g(sin(x)) = \sqrt{sin(x)}$. ✓ ✓ (2)

(ii) $\frac{dy}{dx} = \frac{cos(x)}{2\sqrt{sin(x)}}$ ✓

$At\ x = \frac{\pi}{2},\ \frac{dy}{dx} = 0$ ✓ (2)

(b) $y = sin(\sqrt{x}) \Rightarrow \frac{dy}{dx} = cos(\sqrt{x})\times\frac{1}{2\sqrt{x}} = \frac{cos(\sqrt{x})}{2\sqrt{x}}$ (2)

✓ ✓

END OF SECTION ONE

14. (5 marks)

(a) $y = e^{sin(x)}$

(i) $\frac{dy}{dx} = cos(x)\times e^{sin(x)}$ ✓

(1)

(ii) $\int_0^{\pi/2}\left(cos(x)\times e^{sin(x)}\right)dx = \left[e^{sin(x)}\right]_0^{\pi/2}$

$= e - 1$

(2)

(b) $\int_2^3 \frac{1-x^2}{\sqrt{1+x^2}}dx = -1.95\ (2dp)$ ✓ ✓ (2)

15. (8 marks)

(a)

$1965\quad t=0\quad P=715.2$ (million)

$1979\quad t=14\quad P=969$

$969 = 715.2(r)^{14}$

$r = 1.02193$

$2016\quad t=51$

$P_{2016} = 715.2(1.02193)^{51}$

$P_{2016} = 2162.25$

The expected population in 2016 was 2162.25 million people. (3)

(b)

$1979\quad t=0\quad P=969$

$2015\quad t=36\quad P=1401.6$

$1401.6 = 969(r)^{36}$

$r = 1.010305661$

The annual rate of growth from 1979 was 1.03% (rather than 2.193%). (3)

(c) From 1979,

$2016\quad t=37$

$P = 969(1.010305661)^{37}$

$P = 1416$

The difference in population is 746.2 million people. (2)

12. (7 marks)

(a) $f'(x) = \lim_{h \to 0} \frac{2^{(x+h)} - 2^x}{h}$

$= \lim_{h \to 0} \frac{2^x \times 2^h - 2^x \times 1}{h}$

$= \lim_{h \to 0} \frac{2^x (2^h - 1)}{h}$ *** ✓✓ (2)

$= 2^x \times 0.693$ *** ✓✓ (2)

Therefore if $f(x) = 2^x$

then $f'(x) = 2^x \times 0.693$ *** (1)

(b) $f'(3) = 2^3 \times 0.693 = 5.544$ or 5.545 if all dp used on calculator. (2)

13. (11 marks)

(a) (i) $x = 3\sin\left(2 \times \frac{\pi}{4}\right) = 3$ ✓ (1)

(ii) $x = 3\sin(2t)$

$v = \frac{dx}{dt} = 6\cos(2t)$

$a = \frac{dv}{dt} = -12\sin(2t)$

(2)

(iii) $a = -12\sin(2t) = -4(3\sin(2t)) = -4x$ where $k = 4$ ✓ (1)

(b) $f(x) = \sin(2x) \Rightarrow f'(x) = 2\cos(2x)$

$g(x) = \cos(x) \Rightarrow g'(x) = -\sin(x)$

Given $2\cos(2x) = -\sin(x)$

$x = 1.002967$

(3)

(c) (i) and (vi) have the same derivative because $\cos^2(x) = 1 - \sin^2(x)$ ✓✓

(ii) and (v) have the same derivative as they differ only by a constant. ✓✓

(iii) and (iv) do NOT have the same derivative. They are horizontal translations of each other. ✓✓ (4)

SECTION TWO

8. (8 marks)

(a)

Price per entry	Number of people attending	Revenue
\$26	97	\$2 522
\$25	100	\$2 500
\$24	103	\$2 472

(2)

(b)

$\$(25+x)$	$100 - 3x$	$R(x) = (25+x)(100-3x)$

(1)

(c) Maximum area occurs when $R'(r) = 0$ and $R''(r) < 0$

$R(x) = (25+x)(100-3x)$

$R(x) = -3x^2 + 25x + 2500$

$R'(x) = -6x + 25$

$R''(x) = -6$

If $R'(x) = 0$ then $0 = -6x + 25$

$x = 4\frac{1}{6}$

$R''\left(4\frac{1}{6}\right) = -6 < 0$ so maximum

At $x = 4$, $R = \$2552$

At $x = 5$ $R = \$2550$

For maximum revenue, the committee should charge \$29 (= 25+4). (5)

9. (10 marks)

(a) (i) $p = 0.7$

$P(\text{all four tyres will still be OK})$

$= P(x = 4) = (0.7)^4 = 0.2401$ ✓✓ (2)

(ii) $p = 0.3$

$P(\text{at least one tyre will need replacing})$

$= 1 - P(x = 0)$

$= 1 - 0.2401$ (2)

$= 0.7599$

(iii) $p = 0.3$

$P(\text{exactly one tyre will need replacing})$

$= P(x = 1)$ (2)

$= 0.4116$

(b) (i) $E(X) = np = 4 \times 0.3 = 1.2$ ✓ ✓ (2)

(ii) $Var(X) = npq = 4 \times 0.3 \times 0.7 = 0.84$ ✓ ✓ (2)

10. (4 marks)

$V = \dfrac{4\pi r^3}{3}$

$\dfrac{dV}{dr} = 4\pi r^2$

$\delta V \approx \dfrac{dV}{dr} \times \delta r$

$\delta V \approx 4\pi r^2 \times \delta r$

$At\ \delta x = -0.5\,mm,\ r = 6\,mm$

$\delta V \approx 4\pi 6^2 \times (-0.5) = -72\pi$

$\delta V \approx 226.19\,mm^3$

The decrease in volume when the diameter is 11 mm rather than 12 mm is $226.19\,mm^3$.

(4)

11. (6 marks)

(a) $a = 2 - t$

$v = \int (2 - t)\,dt$

$v = 2t - \dfrac{t^2}{2} + c_1$

$At\ t = 1, v = 1.5\,ms^{-1}$

$1.5 = 2 - \dfrac{1}{2} + c_1 \Rightarrow c_1 = 0$

$v = 2t - \dfrac{t^2}{2}$

$x = \int \left(2t - \dfrac{t^2}{2}\right) dt$ (3)

$x = t^2 - \dfrac{t^3}{6} + c_2$

$At\ t = 1,\ x = \dfrac{5}{6} m$

$\dfrac{5}{6} = 1 - \dfrac{1}{6} + c_2 \Rightarrow c_2 = 0$

$x = t^2 - \dfrac{t^3}{6}$

(b) Changes direction at $v = 0$ for $t > 0$.

$0 = 2t - \dfrac{t^2}{2}$

$0 = t\left(2 - \dfrac{t}{2}\right)$

$t = 0\ or\ t = 4\ but\ t > 0$

$t = 4$

$x(4) = 4^2 - \dfrac{4^3}{6}$

$x = 5.\dot{3}\ m$

(3)