PERTH MODERN SCHOOL
Exceptional schooling. Exceptional students.
INDEPENDENT PUBLIC SCHOOL

Semester One Examination, 2023

Question/Answer booklet

MATHEMATICS METHODS
UNIT 3
Section Two:
Calculator-assumed

Your Name: ______________________________

Your Teacher's Name: ______________________________

Time allowed for this section
Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Marks	Max	Question	Marks	Max
7		8	13		12
8		6	14		11
9		12	15		8
10		10			
11		10			
12		13			

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	6	6	50	51	35
Section Two: Calculator-assumed	10	10	100	90	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2019*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **(90 Marks)**

This section has **ten** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 100 minutes.

Question 7 **(8 marks)**

58 mg of a radioisotope with a half-life of 63 hours was injected into a patient before a CT scan. The mass M of the radioisotope decays continuously so that t hours after administration, the mass remaining is given by $M=M_0e^{-kt}$, where M_0 and k are constants.

(a) Determine the value of the constants M_0 and k. (3 marks)

Solution
$t=0\Rightarrow M=M_0=58$ $\frac{M}{M_0}=0.5=e^{-63k}\Rightarrow k=0.011$
Specific behaviours
ü states M_0 ü equation for k ü value of k

(b) Determine the mass of the radioisotope that remains in the patient exactly 6 days after their injection. (1 mark)

Solution
$t=6\times 24=144\text{ h}, M=58e^{-0.011\times 144}=11.9\text{ mg}$
Specific behaviours
ü calculates mass M

(c) Eventually, the mass of the remaining radioisotope falls to 2 mg.

(i) Determine how long after their injection that this occurs. (2 marks)

Solution
$2=58e^{-0.011t}\Rightarrow t=306\text{ h}$
Specific behaviours
ü substitutes to form equation ü uses CAS to solve for t

(ii) Determine the rate at which the radioisotope is decaying at this time. (2 marks)

Solution
$\frac{dM}{dt}=-kM¿-0.011\times 2=-0.022\text{ mg/h}$ Decay rate is 0.022 mg/h
Specific behaviours
ü uses rate of change equation ü correct rate

Question 8 **(6 marks)**

A barrel is filled with 34 balls numbered with the integers $1, 2, 3, \ldots, 33, 34$, but otherwise identical.

Let the random variable X be the number on a ball drawn at random from the barrel.

(a) Explain why X has a uniform distribution. (1 mark)

Solution
Every outcome is equally likely.
Specific behaviours
ü reasonable explanation indicating equally likely outcomes

(b) Determine the expected value of X. (1 mark)

Solution
Using the symmetry of a uniform distribution, $E(X) = 17.5$
Specific behaviours
ü correct value

Let the random variable Y take the value 1 when $X < 10$ and the value 0 otherwise.

(c) State the particular name given to two-outcome random variables such as Y. (1 mark)

Solution
Bernoulli random variable.
Specific behaviours
ü correct name

(d) Determine $P(Y=1)$. (1 mark)

Solution
$P(Y=1) = \dfrac{9}{34}$
Specific behaviours
ü correct probability

(e) Three balls are drawn at random from the barrel. Determine the probability that exactly two of the balls are marked with single digit numbers. (2 marks)

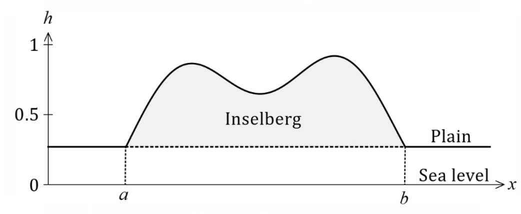
Solution
$W \quad B\left(3, \dfrac{9}{34}\right), P(W=2) = 0.1546$ Alternative: $p = \left(\dfrac{9}{34}\right)^2 \times \dfrac{25}{34} \times 3 = \dfrac{6075}{39304} = 0.1546$
Specific behaviours
ü indicates correct method   ü correct probability

Additional working space

Question number: ________

Question 9 **(12 marks)**

A vertical cross section through the highest point of an inselberg, a mountain range that rises above a surrounding level plain, is shown in the figure below.



The height of the plain and the inselberg above sea level h, in kilometres, is given by

$$h(x)=\begin{cases} x-\frac{1}{5}\left(x^2+2+\sin\left(\frac{13x}{4}\right)\right) & a\le x\le b \\ 0.27 & \text{otherwise}\end{cases}$$

where x is the horizontal displacement in kilometres from an arbitrary origin.

(a) Determine the value of a and the value of b, the x displacements where the inselberg meets the surrounding plain. (2 marks)

Solution
$x-\frac{1}{5}\left(x^2+2+\sin\left(\frac{13x}{4}\right)\right)=0.27$ Using CAS to solve results in $a=0.88$ and $b=4.04$.
Specific behaviours
ü writes equation ü states both values (correct to two decimal places)

(b) Use calculus to determine the cross-sectional area of the inselberg shaded in the figure above. (3 marks)

Solution
$A=\int_{0.88}^{4.04}\left(x-\frac{1}{5}\left(x^2+2+\sin\left(\frac{13x}{4}\right)\right)-0.27\right)dx¿1.417\text{ km}^2$
Specific behaviours
ü correct integrand ü correct bounds of integration ü correct area, with units (must be km^2)

(c) Use calculus to

(i) Using calculus determine and justify the maximum height of the inselberg above the surrounding plain. (7 marks)

Solution
$$h'(x)=1-\frac{8x+13\cos\left(\frac{13x}{4}\right)}{20}$$ Using CAS to solve $h'(x)=0$ gives $x=1.625$, $x=2.397$, $x=3.238$. $$h''(x)=\frac{1}{80}\left(169\sin\left(\frac{13x}{4}\right)-32\right)$$ $$h''(1.625)=-2.18$$ $$h''(2.397)=1.71$$ $$h''(3.238)=-2.28$$ As the sign of the second derivative at this stationary point is negative then the curve is concave down and thus a maximum. $$h(1.625)=0.865,\, h(3.238)=0.919$$ Hence maximum height is 919 m above sea level, which is $919-270=649$ m above plain.
Specific behaviours
ü obtains first derivative of h ü shows all solutions to $h'(x)=0$ ü obtains and shows second derivative function   ü calculates second derivative of all stationary points   ü uses sign of second derivative for justification

Additional working space

Question number: ________

Additional working space

Question number: ________

Question 10 **(10 marks)**

(a) Use the quotient rule to show that $\frac{d}{dx}\left(\frac{4x+2}{e^{0.5x}}\right)=\frac{3}{e^{0.5x}}-\frac{2x}{e^{0.5x}}$. (3 marks)

Solution
$u=4x+2,\ u'=4,\ v=e^{0.5x},\ v'=0.5e^{0.5x}$ Using the quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{u'v-uv'}{v^2}¿\frac{4e^{0.5x}-(4x+2)0.5e^{0.5x}}{(e^{0.5x})^2}¿\frac{4-0.5(4x+2)}{e^{0.5x}}$ $¿\frac{4-2x-1}{e^{0.5x}}=\frac{3}{e^{0.5x}}-\frac{2x}{e^{0.5x}}$
Specific behaviours
ü correct derivatives for u, v ü clearly shows use of quotient rule ü clear simplification steps to obtain required result

(b) Use your result from part (a) to show that $\int \frac{2x}{e^{0.5x}}dx=\frac{-4x}{e^{0.5x}}-\frac{8}{e^{0.5x}}+c$, where c is a constant. (3 marks)

Solution
$\frac{d}{dx}\left(\frac{4x+2}{e^{0.5x}}\right)=\frac{3}{e^{0.5x}}-\frac{2x}{e^{0.5x}}$ Hence $\int \frac{d}{dx}\left(\frac{4x+2}{e^{0.5x}}\right)dx=\int \frac{3}{e^{0.5x}}dx-\int \frac{2x}{e^{0.5x}}dx$ $\frac{4x+2}{e^{0.5x}}=\frac{-6}{e^{0.5x}}-\int \frac{2x}{e^{0.5x}}dx+c$ $\int \frac{2x}{e^{0.5x}}dx=\frac{-6}{e^{0.5x}}-\frac{4x+2}{e^{0.5x}}+c¿\frac{-4x}{e^{0.5x}}-\frac{8}{e^{0.5x}}+c$
Specific behaviours
ü uses result from (a), wrapping integrals around terms ü simplifies two integrals, including constant ü rearranges for required integral and simplifies

(c) The height h of a plant, initially 9 cm, is changing at a rate given by $\frac{dh}{dt}=\frac{2t}{e^{0.5t}}$ cm/day, for $t \geq 0$.

(i) Determine an equation to model the height of the plant as a function of time and hence determine its height after 7 days. (3 marks)

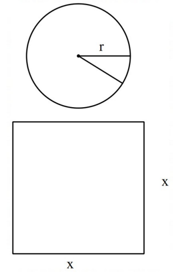
Solution
$h=\frac{-4t-8}{e^{0.5t}}+c \quad c=9-\frac{-8}{e^{0}}=17$ $h(t)=\frac{-4t-8}{e^{0.5t}}+17$ $h(7)=15.9\,\text{cm}$
Specific behaviours
ü uses result from (b)   ü evaluates constant c ü correct height

(ii) According to the model, what height will the plant never exceed? (1 mark)

Solution
As $t\to\infty$, $h\to 17$ cm.   Height will not exceed 17 cm.
Specific behaviours
ü correct height

Question 15 **(8 marks)**

The diagram below shows a logo which includes a circle of radius r cm and a square with side $x$cm.



Before construction, a stencil was made, and the total perimeter of both figures was found to be 60cm.

(a) By expressing r in terms of x, show that the total area of both shapes is given by:

$$A=x^2+\frac{900-120x+4x^2}{\pi}$$

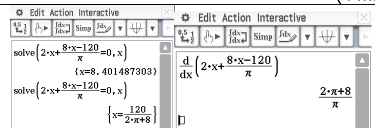
(4 marks)

$60=4x+2\pi r$ $r=\frac{60-4x}{2\pi}=\frac{30-2x}{\pi}$	$Area=x^2+\pi r^2$ $Area=x^2+\pi\left(\frac{30-2x}{\pi}\right)^2$ $¿\, x^2+\frac{900-120x+4x^2}{\pi}$

Specific behaviours	Mark allocation
Creates equation for Perimeter	1
Rearranges r in terms of x	1
Substitutes into Area	1
Simplifies for correct answer	1

(b) Using calculus determine the value of x which minimises the area.

(4 marks)

$\frac{dA}{dx}=2x+\frac{1}{\pi}(-120+8x)$ $2x+\frac{1}{\pi}(-120+8x)=0$ Solve using CAS: $x=8.4$ $\frac{d^2A}{dx^2}=\frac{2\pi+8}{\pi}\ ¿0$ $\therefore$ Minimum value	

Specific behaviours	Mark allocation
Correctly calculates derivative	1
Uses CAS to solve for x correctly	1
Calculates second derivative	1
Justifies answer	1

(c) Determine the expected value of X and interpret your answer in the context of this question.

(2 marks)

$E(X)=0(0.4)+1(0.18)+2(0.25)+3(0.13)+4(0.04)=1.23$

Usually, he damages approximately 1 tyre per day.

Specific behaviours	Mark allocation
Calculates correct answer	1
Makes a suitable statement interpreting the answer.	1

(d) Calculate the Standard deviation of X.

(2 marks)

$E(X)=1.23$

Variance $=0.4¿$

Std deviation$¿\sqrt{1.4771}$ $¿1.22$

Specific behaviours	Mark allocation
Uses correct equation for variance	1
Calculates correct answer	1

(e)

The factory owner decides to introduce a points scheme using the rule $Y=20-5X$.
Find the points Jake will expect to receive and state the variance of his points score.

(Give your answers to the nearest unit)

(2 marks)

$E(Y)=20-5(1.23)=13.85=14$

Variance$¿5^2(1.22^2)=37$

Specific behaviours	Mark allocation
Calculates new Expected Value correctly	1
Calculates correct answer for variance	1

Question 11 **(10 marks)**

Initially a water tank contains 300 *Litres*.
Between $t=0$ and $t=12$ hours , water is pumped into the tank at the rate of

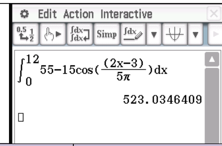
$$W(t)=55-15\cos\left(\frac{2t-3}{5\pi}\right) litres/hour$$

At the same time 50 $litres/hour$ is pumped from the tank.

(a) What is the total number of litres of water pumped into the tank during the first twelve hours.

(2 marks)

$$\int_0^{12} 55-15\cos\frac{2t-3}{5\pi}=523$$



Specific behaviours	Mark allocation
States correct integral with limits	1
States correct answer	1

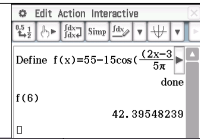
(b) Is the level of water rising or falling when $t=6$?

(2 marks)

$f(6)=42.4$, water is being removed 50L per hour.

$42.4<50$

$\therefore$ level is decreasing

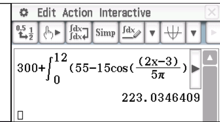


Specific behaviours	Mark allocation
Calculates correct rate	1
States level is falling	1

(c) How many litres are in the tank at $t=12$? (3 marks)

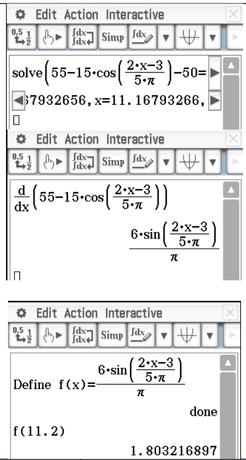
At t¿ 12, the tank contains

$$300+\int_0^{12}\left(55-15\cos\left(\frac{2t-3}{5\pi}¿\right)-50\right)dt=223¿$$



Specific behaviours	Mark allocation
States correct integral with limits	2
States correct answer	1

(d) When is the amount of the water in the tank at a minimum? (3 marks)

Amount of water at any time t is given by A(t) $=300+\int_0^t \left(55-15\cos\left(\frac{2t-3}{5\pi}\right)-50\right)dt$ Minimum occurs when $A'(t)=0$ Solve $55-15\cos\left(\frac{2t-3}{5\pi}\right)-50=0$ $t=11.2$ $A''(t)=\frac{6\sin\left(\frac{2t-3}{5\pi}\right)}{\pi}$ $A''(11.2)>0$ $\therefore$ a minimum	

Specific behaviours	Mark allocation
States correct equation to solve	1
Finds correct value for t	1
Justifies answer (must show the evaluation of the second derivative or first derivative sign test)	1

Question 14 **(11 marks)**

Jake works in a tyre factory. The number X of damaged tires that he makes on a random day has the **cumulative** probability distribution

x	0	1	2	3	4
$P(X\le x)$	0.4	0.58	0.83	0.96	1

(a) Determine the probability that Jake produces:

(i) more than two damaged tyres. (1 mark)

$$P(X\ge 3)=0.17$$

Specific behaviours	Mark allocation
States correct probability.	1

(ii) three damaged tyres. (1 mark)

$$P(X=3)=0.13$$

Specific behaviours	Mark allocation
States correct probability	1

(b) Calculate $P(X\ge 1 \vee X<4)$.

(3 marks)

$$P(X\ge 1 | X<4)=\frac{P(1\le X\le 3)}{P(X\le 3)}$$
$$= \frac{0.56}{0.96}$$
$$= \frac{7}{12}$$

Specific behaviours	Mark allocation
States correct conditional probability formula	1
Correct numerator	1
States correct answer	1

A Mathematics teacher asks each of her 20 students to write down the fraction they obtained from rolling the two dice.

(d) Determine the probability that at least half of the class obtained a fraction where the numerator is greater than or equal to the denominator. (3 marks)

Solution	Specific behaviours
$Let\ X\ be\ the\ number\ of\ students\ who\ obtained\ a\ fraction \geq 1$ $X \quad Bin\left(20, \frac{7}{12}\right)$ $P(X \geq 10) = 0.8372$	✓ Defines the appropriate random variable and states correct binomial distribution. ✓ States the correct probability statement. ✓ Determines the probability.

(e) The teacher asks each student one-by-one what fraction they obtained. The teacher dismisses the class when *exactly* 5 students have a fraction of less than 1.

Determine the probability that the teacher will need to ask *exactly* ten students before dismissing the class. (3 marks)

Solution	Specific behaviours
¿ $the\ first\ 9\ students, 4\ must\ have\ a\ fraction < 1, with\ the\ 10th\ student\ having\ a\ fraction < 1.$ $Let\ Y\ be\ the\ number\ of\ students\ who\ obtained\ a\ fraction < 1$ $Y \quad Bin\left(9, \frac{5}{12}\right)$ $P(exactly\ 5) = P(Y=4) \times \frac{5}{12}$ ¿ $0.2565 \times \frac{5}{12} = 0.1069$	✓ Defines the appropriate random variable and states correct binomial distribution. ✓ Determines probability of 4 out of 9 having a fraction < 1. ✓ Determines the final probability.

Question 12 **(13 marks)**

The Lupu Bridge in Shanghai was the longest steel arch bridge when it opened in 2003.

The main arch can be represented by the graph of the function

$$f(x) = 500 - 200\left(e^{\frac{x}{400}} + e^{\frac{-x}{400}}\right)$$

where x is the distance, in metres, from the middle, and $f(x)$ is the distance, in metres, above the Hangpu River.

The bridge is symmetrical about the vertical axis.

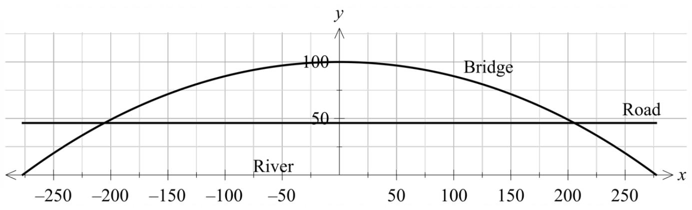
(a) Determine $f'(x)$ writing your answer in the form $f'(x) = a\left(e^{\frac{x}{400}} - e^{\frac{-x}{400}}\right)$, where a is a rational number. (2 marks)

Solution	Specific behaviours
$f'(x) = -200\left(\frac{1}{400}e^{\frac{x}{400}} - \frac{1}{400}e^{\frac{-x}{400}}\right)$ $f'(x) = \frac{-1}{2}\left(e^{\frac{x}{400}} - e^{\frac{-x}{400}}\right)$	✓ Correctly differentiates $e^{\frac{x}{400}}$ and $e^{\frac{-x}{400}}$. ✓ Determines $a = \frac{-1}{2}$ and writes answer in required form.

(b) Using calculus, verify that the **maximum** height of the bridge is 100 m. (4 marks)

Solution	Specific behaviours
$0 = \frac{-1}{2}\left(e^{\frac{x}{400}} - e^{\frac{-x}{400}}\right)$ $x = 0$ $f''(x) = \frac{-1}{800}\left(e^{\frac{x}{400}} + e^{\frac{-x}{400}}\right)$ $f''(0) = \frac{-1}{400} < 0\ i.e.\ concave\ down$ $f(0) = 100$ $\therefore Maximum\ height\ is\ 100\ m$	✓ Uses first derivative to determine x coordinate of stationary point. ✓ Determines second derivative. ✓ Determines $f''(0)$ to verify the stationary point is a maximum. ✓ Concludes maximum height is 100 m.

The graph below shows the Lupu Bridge and the road, which is 46 m above the river.



(c) Determine, correct to the nearest 100 m², the cross-sectional area between the road, the bridge and water. (4 marks)

Solution	Specific behaviours
$500-200\left(e^{\frac{x}{400}}+e^{\frac{-x}{400}}\right)=0$ $x=-277.26, 277.26$	✓ Determines x intercepts of the bridge.
$500-200\left(e^{\frac{x}{400}}+e^{\frac{-x}{400}}\right)=46$ $x=-205.58, 205.58$	✓ Determines x coordinates of the points of intersection of the bridge and road.
$A=46(2\times 205.58)+¿$ $2\int_{205.58}^{277.26} 500-200\left(e^{\frac{x}{400}}+e^{\frac{-x}{400}}\right)dx$ $A=22400\,m^2$	✓ Writes an integral to determine the area. ✓ Determines the area, correct to the nearest 100 m².

An observation deck is positioned at the top of the bridge. To access the deck, visitors need to climb the arch. The distance, D, travelled along the arch is given by

$$D(t)=\int_0^t \sqrt{\frac{1}{2}+\frac{e^{\frac{x}{200}}}{4}+\frac{e^{\frac{-x}{200}}}{4}}\,dx$$

where D is measured in metres, and t is measured in seconds.

(d) Determine the speed, $s=\frac{dD}{dt}$ of the visitors, when they are 2 minutes into their ascent. (3 marks)

Solution	Specific behaviours
$\frac{d}{dt}\int_0^t \sqrt{\frac{1}{2}+\frac{e^{\frac{x}{200}}}{4}+\frac{e^{\frac{-x}{200}}}{4}}\,dx=\sqrt{\frac{1}{2}+\frac{e^{\frac{t}{200}}}{4}+\frac{e^{\frac{-t}{200}}}{4}}$ $When\, t=120\, s=1.05\, m\, s^{-1}$	✓ Uses Fundamental Theorem to determine expression in terms of t for $\frac{dD}{dt}$. ✓ Substitutes $t=120$. ✓ Determines speed and states correct units.

Question 13 **(12 marks)**

A Mathematics teacher gives a student two fair six-sided dice. One die is coloured red and the other coloured blue.

The student rolls the two dice, and then writes down the following fraction:

$$Fraction(F)=\frac{Number\ on\ the\ red\ die}{Number\ on\ the\ blue\ die}$$

(a) Show that the probability of getting a fraction less than 1 is $\frac{5}{12}$. (2 marks)

Solution	Specific behaviours
$Number\ on\ red<Number\ on\ blue$ $There\ are\ 5+4+3+2+1=15\ ways$ $this\ can\ occur\left(i.e.\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\ldots,\frac{5}{6}\right)$	✓ Recognises that number on red < number on blue.
$Hence, P(F<1)=\frac{15}{36}=\frac{5}{12}$	✓ Shows how to get probability of $\frac{15}{36}$.

The teacher draws up the following table on the board.

Fraction F	$F<1$	$1\le F<2$	$2\le F<3$	$3\le F<6$	$F=6$
$P(F)$	$\frac{5}{12}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{4}{36}=\frac{1}{9}$	$\frac{1}{36}$

(b) Complete the table above, providing the missing probabilities. (2 marks)

Solution	Specific behaviours
$See\ \textbf{entries}\in table$	✓ Determines $P(F=6)=\frac{1}{36}$. ✓ Determines $P(3\le F<6)$.

(c) Determine the probability that a student obtains a fraction that is at least 2, given that the fraction is less than 3. (2 marks)

Solution	Specific behaviours
$P(F\ge 2\mid F<3)=\frac{P(2\le F<3)}{P(F<3)}$	✓ Determines correct numerator.
$¿\frac{\frac{1}{9}}{\frac{5}{12}+\frac{1}{3}+\frac{1}{9}}=\frac{4}{31}=0.1290$	✓ Determines correct denominator and obtains final answer.