MATHEMATICS METHODS

MAWA Semester 1 (Unit3) Examination 2017

Calculator-Assumed

Marking Key

The release date for this exam and marking scheme is

- **the end of week 8 of term 2, 2017**

Section Two: Calculator-assumed (100 Marks)

Question 8

Solution

$V=\frac{1}{3}\pi r^2 h=\frac{4}{3}\pi h^3$

When $V=60, h=(\frac{3\times 60}{4\times\pi})^{1/3}\approx 2.4286$

and $\frac{dV}{dh}=4\pi h^2\approx 4\pi\times 2.4286^2\approx 74.1$

$\delta V\approx\frac{dV}{dh}\delta h$

Since $\delta V=1$, $\delta h\approx 1/74.1\approx 0.0134$

So the height increases by about 13 millimetres

Marking key/mathematical behaviours	Marks
• expresses the volume as a function of height only	1
• evaluates h	1
• differentiates correctly and evaluates $\frac{dV}{dh}$	1+1
• uses increments formula correctly	1
• gives correct answer	1

Question 9(a)

Solution

$f'(x)=10e^{-x}\cos x-10e^{-x}\sin x$

$f''(x)=10e^{-x}$ ¿

Marking key/mathematical behaviours	Marks
• determines $f'(x)$ correctly	1
• determines $f''(x)$ correctly	1

$A(k)=\int_0^k x^2e^{-x}dx$

¿ $-e^{-x}(x^2+2x+2)\Big]_0^k$

¿ $-e^{-k}(k^2+2k+2)$

Marking key/mathematical behaviours	Marks
• correctly integrates and substitutes limits	1
• finds $A(k)$	1

• Recognises the situation with replacement as producing a Binomial distribution	1
• Determines the mean and SD of z	1+1
• Compares the means and draw the appropriate conclusion	1
• Compares the SD and draws the appropriate conclusion	1

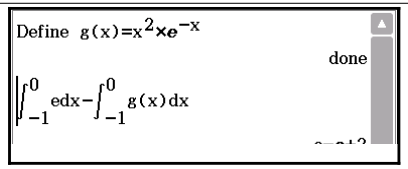
Question 18(a)

Solution

Determines the point of intersection of the curve and the line as $(-1, e)$

$$A = \int_{-1}^{0} e\,dx - \int_{-1}^{0} g(x)\,dx$$

= 2 sq units



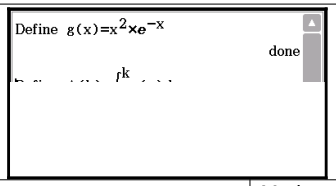
Marking key/mathematical behaviours	Marks
• determines both parts of the integral statement	2
• evaluates the area correctly	1

Question 18(b)

Solution

$$A(5) = \int_{0}^{5} g(x)\,dx$$

= 1.7507 sq units



Marking key/mathematical behaviours	Marks
• determines the integral statement	1
• evaluates the area correctly	1

Question 18(c)

Solution

$$f'(x) = e^{-x}(x^2+2x+2) + (2x+2)(-e^{-x})$$

¿ $x^2e^{-x} + 2xe^{-x} + 2e^{-x} - 2xe^{-x} - xe^{-x}$

¿ x^2e^{-x}

¿ $g(x)$

Marking key/mathematical behaviours	Marks
• uses the product rule and correctly differentiates	1+1
• simplifies and shows equality	1

Question 18(d)

Solution

Question 9(b)

Solution

$f'(x) = 10e^{-x}$¿ when $\cos x = \sin x$, i.e. $x = \pi/4$

$f''(x) = -20e^{-x}\cos x$ and so $f''(\pi/4) < 0$

so $f(\pi/4)$ is a local maximum.

Since there is only one local maximum and $f(0) = f(\pi) = 0$

$f(\pi/4) = \dfrac{10e^{-\pi/4}}{\sqrt{2}}$ is the maximum.

$\dfrac{e^{-\pi/4}}{\sqrt{2}} \approx 3.224$ correct to 4 significant figures

Marking key/mathematical behaviours	Marks
• finds zeros of $f'(x)$	1
• obtains $f''(\pi/4) < 0$ and uses second derivative test correctly	1+1
• obtains exact value of $f(\pi/4)$	1
• evaluates $f(\pi/4)$ correct to 4 significant figures	1

Question 9(c)

Solution

$f''(x) = -20e^{-x}\cos x = 0$ when $x = \pi/2$

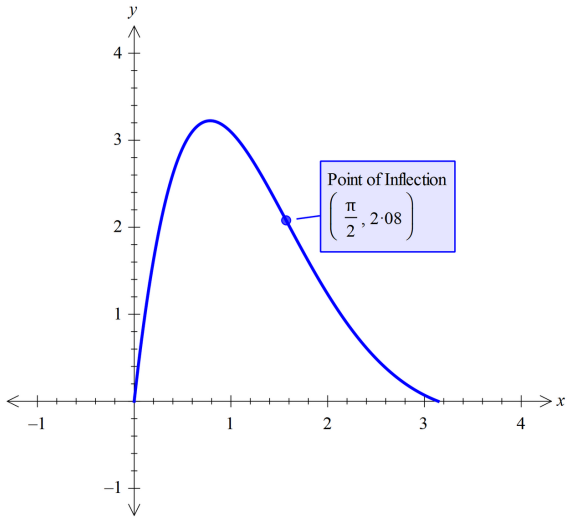
Since $f''(x)$ changes sign at $x = \pi/2$, there is a point of inflection at $x = \pi/2$

The point of inflection has coordinates ¿

Marking key/mathematical behaviours	Marks
• finds the zero of $f''(x)$ for $0 \le x \le \pi$	1
• finds correct answer	1

Question 9(d)

Solution



Marking key/mathematical behaviours	Marks
• graph has right 'shape', correctly plotted maximum turning point (at $x=\pi/4\approx0.78$), non-zero derivatives at end points	1+1+1
• inflection point indicated correctly	1

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MARKING KEY

Question 17(d)

Solution

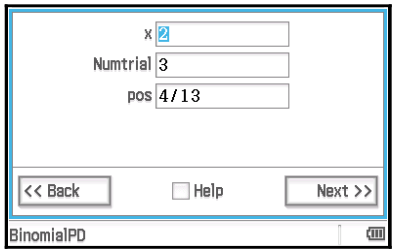
(i) Without replacement, the distribution of $P(Z)$ is Binomial with $n=3$ and $p=\frac{4}{13}$

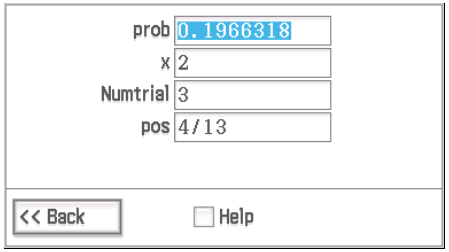
Hence the mean = $np=\frac{12}{13}=0.92308$, and the standard deviation is

$$\sqrt{np(1-p)}=\sqrt{3\times\frac{4}{13}\times\frac{9}{13}}=\sqrt{\frac{108}{169}}=0.79941$$

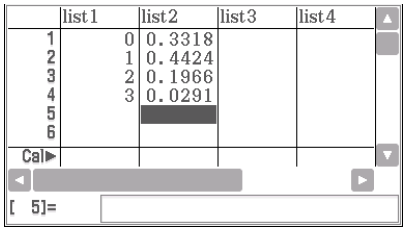
Alternatively, may determine the probability density function for $P(Z)$ either as per the table below or using Statistics App of a CAS calculator

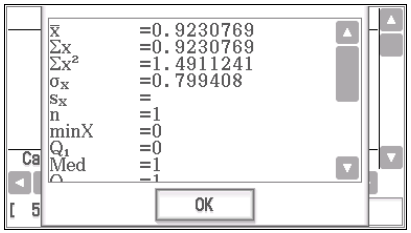
z	0	1	2	3
$P(Z=z)$	$\frac{9}{13}\times\frac{9}{13}\times\frac{9}{13}$ ≈0.33182	$3\times\frac{4}{13}\times\frac{9}{13}\times\frac{9}{13}$ ≈0.44242	$3\times\frac{4}{13}\times\frac{4}{13}\times\frac{9}{13}$ ≈0.19663	$\frac{4}{13}\times\frac{4}{13}\times\frac{4}{13}$ ≈0.00291





The mean and the standard deviation may then be easily determined on a CAS calculator as shown below





(ii) Now, comparing with the mean and standard deviation without replacement as given it may be concluded that the means with and without replacement are the same but the standard deviation without replacement < standard deviations with replacement.

Marking key/mathematical behaviours	Marks

Question 17(b)

Solution

$P(x=1)$ = probability of selecting one red and one black marble

$$= \frac{{}^4C_1 \times {}^9C_1}{{}^{13}C_2}$$

$$= \frac{6}{13}$$

Alternatively, on a CAS

nCr(4,1)×nCr(9,1)/nCr(13,2) $\quad \frac{6}{13}$

Marking key/mathematical behaviours	Marks
• States the correct method of selecting a red and a black marble	1
• States the correct result	1

Question 17(c)

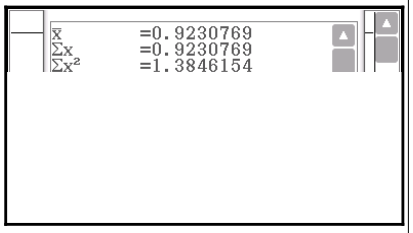
Solution

y	0	1	2	3
$P(Y=y)$	$\frac{{}^4C_0 \times {}^9C_3}{{}^{13}C_3} = \frac{42}{143}$	$\frac{{}^4C_1 \times {}^9C_2}{{}^{13}C_3} = \frac{72}{143}$	$\frac{{}^4C_2 \times {}^9C_1}{{}^{13}C_3} = \frac{27}{143}$	$\frac{{}^4C_3 \times {}^9C_0}{{}^{13}C_3} = \frac{2}{143}$

Using the statistics App on the calculator, mean

= 0.9230769 and

standard deviation = 0.7297564



	list1	list2	list3	list4
1	0	42/143		
2	1	72/143		
3	2	27/143		
4	3	2/143		

Marking key/mathematical behaviours	Marks
• Identifies the correct number of values for y	1
• Correctly calculates the probability for one of the y values	1
• Correctly states all four values	1
• States the mean of y	1
• States the standard deviation of y	1

Question 10(a)

Solution

Isotope α decays faster because it takes less time for it to lose half of its weight.

Marking key/mathematical behaviours	Marks
• justifies answer correctly.	1

Question 10(b)

Solution

$A_0 = A(0) = 8.5$

$e^{-5.8k} = \frac{1}{2}$

$\Longrightarrow k = \frac{\ln 2}{5.8} \approx 0.1195$ (calculator)

Marking key/mathematical behaviours	Marks
• evaluates A_0 correctly	1
• uses $e^{-5.8k} = \frac{1}{2}$	1
• evaluates k accurately	1

Question 10(c)

Solution

$A(t) = 8.5e^{-0.1195t} = 0.001$

$\Longrightarrow t \approx \frac{\ln 0.001 - \ln 8.5}{-0.1195} \approx 75.7$ (or directly from calculator)

So it takes 75.7 years

Marking key/mathematical behaviours	Marks
• uses correct equation	1
• solves accurately	1

Question 10(d)

Solution

$B(t) = 2e^{-\omega t}$

where $e^{-7.6\omega} = \frac{1}{2}$ i.e. $\omega \approx 0.0912$ (calculator)

$8.5e^{-0.1195t} = 2e^{-0.0912t}$

$\Longrightarrow \ln 8.5 - 0.1195t = ¿ \ln 2 - 0.0912t ¿$

$\Longrightarrow t=\frac{\ln 8.5-\ln 2}{0.1195-0.0912}\approx 50.065$ (or directly from calculator) So the amounts of the two isotope will be equal in weight 50.065 years from now.

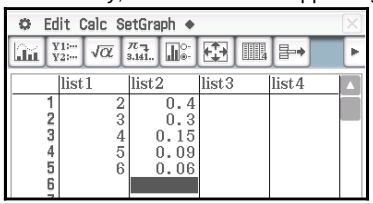
Marking key/mathematical behaviours	Marks
• determines ω accurately	1
• uses $8.5e^{-0.1195t}=2e^{-0.0912t}$	1
• solves accurately	1
• interprets solution correctly	1

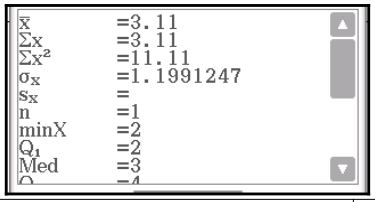
Question 11(a)

Solution

$$E(X)=2\times0.4+3\times0.3+4\times0.15+5\times0.09+6\times0.06$$
$$=3.11$$

Alternatively, use the statistics app of a graphic calculator – setting up list 2 as the frequency.





Marking key/mathematical behaviours	Marks
• Applies the expected value formula	1
• States the correct result	1

Question 11(b)

Solution

$$Var(X)=E(X^2)-(E(X))^2$$
$$=4\times0.4+9\times0.3+16\times0.15+25\times0.09+36\times0.06-3.11^2$$
$$=1.4379$$

Alternatively, use the statistics app of a graphic calculator – and square σ_x produced in part (a), namely $(1.1991247)^2=1.4379$

Marking key/mathematical behaviours	Marks
• Applies the expected value formula	1
• States the correct result	1

Question 11(c)

Solution

$$P(1.911\le X\le 4.309)$$
$$=P(2\le X\le 4)$$
$$=0.85$$

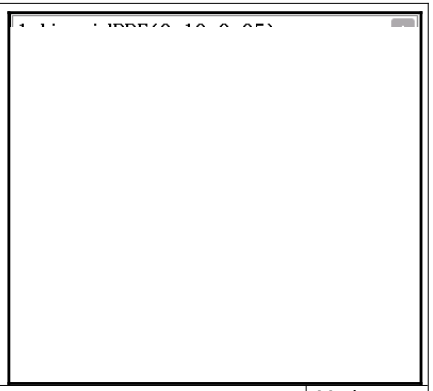
• States the correct result	1

Question 16(d)

Solution

Using P(at least 1)=1 – P(0) and testing
n = 13, 14, 15,16,17 etc.
Largest sample is 15.

OR using solve



Marking key/mathematical behaviours	Marks
• correctly uses complementary event and tests suitable numbers for n	1+1
• determines correct sample size	1

Question 17(a)

Solution

$$P(x=2)=\frac{4}{13}\times\frac{3}{12}=\frac{1}{13}\approx 0.0769$$

Alternatively,

$$P(x=2)=\frac{{}^4C_2}{{}^{13}C_2}=\frac{6}{78}\approx 0.0769$$

Marking key/mathematical behaviours	Marks
• States the number of ways of selecting two red	1
• States number of selection in sample space	1

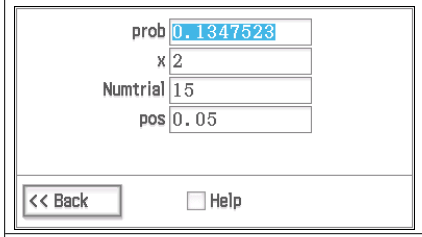
Question 16(a)

Solution

Probability that a computer is defective is 0.05

Probability that two are defective (out of 15)

$= {}^{15}C_2 \times (0.05)^2 \times (0.95)^{13} = 0.1347523$



Marking key/mathematical behaviours	Marks
• Recognises the Binomial distribution	1
• Uses the appropriate binomial parameters	1
• States the correct value for the required probability	1

Question 16(b)

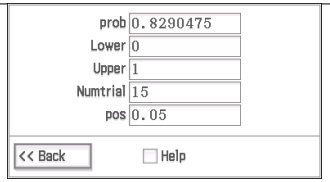
Solution

Probability that less than two computers are defective

means that one is defective or none are defective

$= {}^{15}C_0 \times (0.05)^0 \times (0.95)^{15} + {}^{15}C_1 \times (0.05)^1 \times (0.95)^{14}$

$= 0.8290475$



Marking key/mathematical behaviours	Marks
• Recognises equivalence to one or none, defective	1
• Determines the correct probability	2

Question 16(c)

Solution

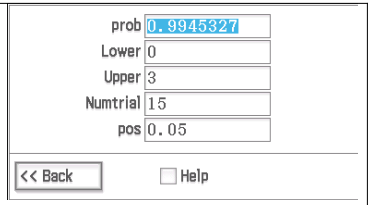
Probability that four or more are defective

$= 1 - P(\text{less than 4 are defective})$

$= 1 - P(x \leq 3)$

$= 1 - 0.9945327$

$= 0.0054673$



Marking key/mathematical behaviours	Marks
• Recognises equivalent to less than 4 defective (3 or less are defective)	1

Marking key/mathematical behaviours	Marks
• correctly calculates end points	1
• correctly converts to discrete values	1
• correctly calculates probability	1

Question 12(a)

Solution

The random variable X represents the probability that a respondent regularly uses social media. Let $x = 0$ to represent that a respondent does NOT regularly use social media, and $x = 1$ to represent that a respondent does regularly use social media.
Then, the probability distribution will be defined as in the table below.

x	0	1
$P(X = x)$	0.35	0.65

Marking key/mathematical behaviours	Marks
• Defines $P(X = x)$	1
• Provides the correct values for the Probability of the two possible values of x.	1

Question 12(b)

Solution

The random variable X , produces a Bernoulli distribution

Marking key/mathematical behaviours	Marks
• Indicates a Bernoulli distribution	1

Question 12(c)

Solution

(i) $\frac{250}{1000} = 0.25$

(ii) $\frac{250}{650} = \frac{5}{13}$

Marking key/mathematical behaviours	Marks
• States the correct result for (i)	1
• States the correct result for (ii)	1

Question 15(a)

Solution

$$\int_0^9 k x^{\frac{1}{2}} dx = 27$$

$$\int_0^9 k x^{\frac{1}{2}} dx = \frac{2}{3} k \sqrt{x^3} \Big]_0^9$$

$$= 18k$$

$$18k = 27$$

$$k = \frac{3}{2}$$

Marking key/mathematical behaviours	Marks
• correctly integrates	1
• correctly substitutes limits	1
• correctly solves	1

Question 15(b)

Solution

(i) $a = -2.658$, $b = 0$, $c = 0.978$

(ii) $\int_{-2.658}^{0} e^x - 1 - 2\sin x \, dx + \int_{0}^{0.978} 2\sin x - e^x + 1 \, dx$

(iii) Area = 2.244 square units

Marking key/mathematical behaviours	Marks
• states correct values of a, b and c for part (i)	3
• states correct integral for part (ii)	2
• correctly solves for the area in part (iii)	1

Question 14(a)(i)

Solution

$$\frac{dy}{dx} = e^x + x e^x - e^x$$

$$= x e^x$$

Marking key/mathematical behaviours	Marks
• correctly differentiates using the product rule	1

Question 14(a)(ii)

Solution

$$\int_0^1 \left(x e^x + x^3\right) dx = \left[x e^x - e^x + \frac{x^4}{4}\right]_0^1$$

$$= e - e + \frac{1}{4} - (0 - 1 + 0)$$

$$= \frac{5}{4}$$

Marking key/mathematical behaviours	Marks
• correctly integrates using the substitution from (i)	1
• evaluates correctly	1

Question 14(b)

Solution

$$\int_1^4 \left(2f(x) + 3x\right) dx = 2\int_1^4 f(x) + x\, dx + \int_1^4 x\, dx$$

$$= 2 \times (2) + \left[\frac{x^2}{2}\right]_1^4$$

$$= 4 + 8 - \frac{1}{2}$$

$$= 11\frac{1}{2}$$

Marking key/mathematical behaviours	Marks
• expands the integral, clearly displaying integration rules	1
• integrates x correctly	1
• correctly evaluates	1

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Question 12(d)

Solution

$$\frac{250}{550} \times \frac{250}{550} = \left(\frac{5}{11}\right)^2$$

Marking key/mathematical behaviours	Marks
• Recognises the correct probability	1
• Applies the multiplication principle	1

Question 13(a)

Solution

Probability of Susan not stopping

$\Rightarrow$ Lights not Red at all three traffic signals

$\Rightarrow P(X = 0) = 0.85 \times 0.85 \times 0.85$

$= 0.614125$

Marking key/mathematical behaviours	Marks
• Indicates that the probability of light not being red is 0.85	1
• Applies the multiplication principle and states the correct result	1

Question 13(b)

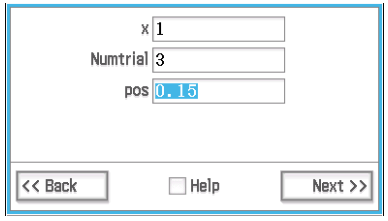
Solution

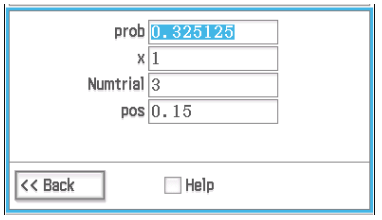
x	0	1	2	3
$P(X = x)$	0.614125	$3 \times (0.85)^2 \times 0.15$ $= 0.325125$	$3 \times (0.15)^2 \times 0.85$ $= 0.057357$	$(0.15)^3$ $= 0.003375$

Alternatively,

Use the binomial distribution app on a CAS calculator, e.g. for

$x = 2$





Marking key/mathematical behaviours	Marks
• Indicates use of the Binomial probability distribution	1
• States the correct probabilities for $x = 1, 2$ and 3	1+1+1

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Question 13(c)

Solution

As the distribution is Binomial with $n = 3$ and $p = 0.15$

$E(x) = np = 3 \times 0.15 = 0.45$

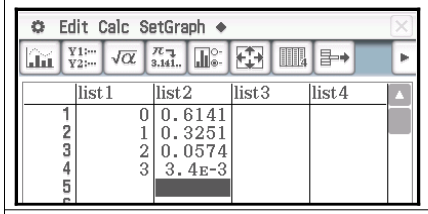
$SD(x) = \sqrt{np(1-p)} = \sqrt{3 \times 0.15 \times 0.85} \approx 0.61847$

Or alternative 2,

$$E(X) = 0 \times 0.614125 + 1 \times 0.325125 + 2 \times 0.057375 + 3 \times 0.003375$$
$$= 0.45$$

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= 1 \times 0.325125 + 4 \times 0.057375 + 9 \times 0.003375 - 0.45^2$$
$$= 0.585 - 0.45^2$$
$$= 0.3825$$
$$\Rightarrow SD \approx 0.61847$$

And alternative 3, use a CAS calculator and enter the data from part (b) into list1 and list2 of the Statistics App as indicated below and with list 2 set as the frequency use the one-variable stat calculations to write down the mean and SD



Marking key/mathematical behaviours	Marks
• States the correct mean	1
• States the correct standard deviation	1

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