Christ Church Grammar School

2017
Investigation 2

MATHEMATICS METHODS Year 12

Part One:
Take home investigation

Student name ______________________________

Teacher name ______________________________

Time allowed for this section

Working time for this section: 2 weeks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, and up to three calculators approved for use in the WACE examinations

Important note to candidates

This investigation has 2 parts: a take home investigation that the student needs to research and complete at home, and an in-class validation that will be completed in test conditions during class.

The take home investigation will not be assessed but has to be completed prior to the validation. Solution for the take home investigation will be provided.

The in-class validation will be assessed and it constitutes **7%** of the course mark.

This paper is the take home investigation.
It is handed out on Mon 12 June 2017.

The in-class validation will be on Tuesday 27 June 2017.

Instructions to candidates

1. Write your answers in this Question/Answer Booklet.

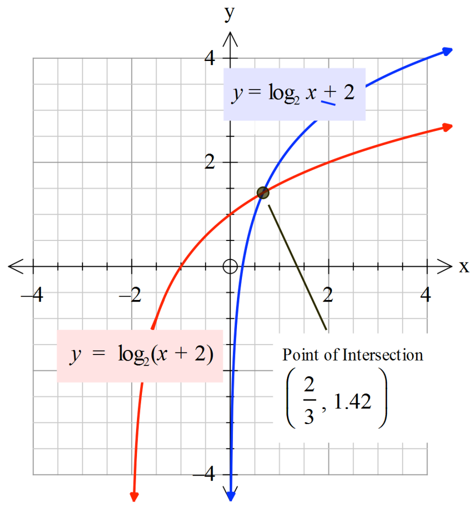
2. Answer all questions.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that **you do not use pencil**, except in diagrams.

Question 8(a)

(i) and (ii)



Question 8(b)

(i)

Function	Domain	Range	Asymptote
$y = \log_2 x + 2$	$x > 0$	$y \in R$	$x = 0$
$y = \log_2(x + 2)$	$x > -2$	$y \in R$	$x = -2$

(ii)

The graph of $y = \log_2 x + 2$ is the graph of $y = \log_2 x$ translated up 2 units.
The graph of $y = \log_2(x+2)$ is the graph of $y = \log_2 x$ translated to the left 2 units.

Question 8(c)

(i) $y = \log_2(x+2)$,

$x + 2 = 2^y$

(ii) $y = \log_2(x) + 2$,

$y - 2 = \log_2(x)$

$2^{y-2} = x$

(iii) From (ii) $x = 2^{y-2} = \frac{2^y}{2^2}$

ie. $x = \frac{2^y}{4}$

$4x = 2^y$

From (i) $x + 2 = 2^y$ so $4x = x + 2$

$x = \frac{2}{3}$

(iv) $y = \log_2(x) + 2$, so $y = \log_2\left(\frac{2}{3}\right) + 2 = 1.415$ (to 3 d.p)

Question 5

Semi-logarithm paper squashes the y axis into a manageable scale so it is possible to use all parts of the number range on the graph.

Question 6

$log_c(y) = log_c\left(A(b)^t\right)$

$= log_c(A) + log_c\left((b)^t\right)$

$log_c(y) = log_c(A) + t\,log_c(b)$

This is of the form

$y = mx + b$

where $log_c(A)$ is constant, and $log_c(b)$ the constant gradient.

Question 7

Solution		
(a)	False	$\log_3 9 = 2$ and $\log_9 3 = 0.5$
(b)	False	$\log_e x^0 = \log_e 1 = 0$
(c)	True	$\ln\left(\frac{1}{x^2}\right) = \ln(x^{-2}) = -2\ln(x)$
(d)	True	Let $e^{\ln(b)} = x$, then $\ln(b)=\log_e(x)$ so $x = b$
(e)	False	$\log_2(x+1)$ is defined for x > -1

LOGARITHMIC FUNCTIONS

Aim of Investigation

The aim of this investigation is to examine the inverse relationship between exponentials and logarithms, interpret and use logarithmic scales, use the algebraic properties of logarithms to solve equations and in proofs, and identify contexts suitable for modelling using logarithmic functions and use them to solve practical problems.

Learning Objectives

At the end of this investigation, you should be able to:

- define logarithms as indices: $a^x = b$ is equivalent to $x = \log_a b$ i.e.$a^{\log_a b} = b$
- establish and use the algebraic properties of logarithms
- examine the inverse relationship between logarithms and exponentials: $y = a^x$ is equivalent to $x = \log_a y$
- solve equations involving indices using logarithms
- identify the qualitative features of the graph of $y = \log_a x$ $(a > 1)$, including asymptotes, and of its translations $y = \log_a x + b$ and $y = \log_a(x - c)$
- solve simple equations involving logarithmic functions algebraically and graphically
- define the natural logarithm $\ln x = \log_e x$
- examine and use the inverse relationship of the functions $y = e^x$ and $y = \ln x$

Required Material

1. All the material contained in this booklet.

2. All the material found in:
   A.J. Sadler, Mathematics Methods Unit 4, Chapter 1

Instructions to Candidates

1. To achieve the objectives of this investigation and to prepare for the validation, students will have to work through all the notes and questions specified as Required Material.

2. Additional resources can be found in:
   O.T. Lee, WACE Revision Series, Mathematics Methods Year 12, Pages 6 – 17.
   Creelman Exam Questions, Mathematics Methods Units 3 & 4, Pages 94 – 102.

Data for Questions 1 to 5

The following chart shows the loudness levels, some of which can damage the human ear.

Chart of sound intensity levels (loudness)for environmental noise		
	Weakest sound heard	0 dB
	Rustling leaves	10 dB
	A whisper in library at 2 m	30 dB
	Converation at home	50 dB
	Conversation in restaurant	60 dB
	Passenger car at 80 kph at 6 m	70 dB
	Vacuum cleaner at 1m	70 dB
	Freeway at 20 m	73 dB
	Telephone dial tone	80 dB
At 90 - 95 dB sustained exposure may sustain hearing loss	Car wash at 6 m	90 dB
	Train whistle at 150 m	90 dB
	Hand drill	98 dB
	Lawn mower at 1M	105 dB
	Motorbike	100 dB
	Jet take off at 300 m	100 dB
	Sand blasting	115 dB
Threshold of discomfort	Thunderclap	120 dB
	Chain saw	120 dB
	Oxygen torch	120 dB
	Loud rock concert	115 dB
Pain threshold 130 dB	Pneumatic riveter	125 dB
	Aircraft carrier deck	140 dB
Eardrum rupture	Jet take off at 25 m	150 dB

The reference level of the intensity of sound, I_0 that all others are compared to is 10^{-12} watts/m ^{2}. It was chosen because it is the weakest intensity of sound that can be detected by the human ear.

Intensity, $I = \frac{Power}{Area}$ so I is measured in watts/m ^{2}.

$I_0 = 10^{-12}$ watts/m ^{2}.

The most intense sound that is not painful to humans is roughly 10 watts/m^2.
Since the human pain threshold at 10 watts/m ^{2} is 10,000,000,000,000 times greater than the reference level, it makes sense to use a logarithmic scale to discuss the intensity of sound, I. The sound intensity level, L, is a logarithmic measure given as

$L = 10 \log\left(\frac{I}{I_0}\right)$ and is measured in decibels (dB).

See next page

Question 4(a)

(i) a chain saw at 120 db

$$120 = 10\log\left(\frac{I}{I_o}\right)$$

$$12 = \log\left(\frac{I}{10^{-12}}\right)$$

$$10^{12} = \frac{I}{10^{-12}}$$

$$I = 1 watts / m^2$$

(ii) a vacuum cleaner at 1m at 70 dB

$$70 = 10\log\left(\frac{I}{I_o}\right)$$

$$7 = \log\left(\frac{I}{10^{-12}}\right)$$

$$I = 10^{-5}\ watts / m^2$$

(iii) rustling leaves at 10 dB

$$10 = 10\log\left(\frac{I}{I_0}\right)$$

$$1 = \log\left(\frac{I}{I_0}\right)$$

$$I = 10^{-11}\ watts/m^2$$

(iv) a telephone dial tone at 80 dB

$$80 = 10\log\left(\frac{I}{I_o}\right)$$

$$10^8 \times 10^{-12} = I$$

$$I = 10^{-4}\ watts / m^2$$

(v) a hand drill at 98 dB

$$98 = 10\log\left(\frac{I}{I_o}\right)$$

$$10^{9.8} \times 10^{-12} = I$$

$$I = 10^{-2.2}\ watts / m^2$$

(vi) an aircraft carrier deck at 140 dB

$$140 = 10\log\left(\frac{I}{I_o}\right)$$

$$10^{14} \times 10^{-12} = I$$

$$I = 10^{2}\ watts / m^2$$

(vii) the ticking of a watch with a decibel reading of 20 dB

$$20 = 10\log\left(\frac{I}{I_o}\right)$$

$$10^{2} \times 10^{-12} = I$$

$$I = 10^{-10}\ watts / m^2$$

Question 4(b)

Decibels are within a much more manageable range, for example for human hearing 0 to about 130. Humans can distinguish and associate sounds within such a range.

Question 3(a) (cont'd)

Alternative method:

$$\frac{I_{rock\,concert}}{I_{converstaion\ at\ home}} = \frac{\frac{I_{rock\,concert}}{I_0}}{\frac{I_{converstaion\ at\ home\,t}}{I_0}}$$

$$115 = 10\log\left(\frac{I}{I_o}\right) \qquad 50 = 10\log\left(\frac{I}{I_o}\right)$$

$$11.5 = \log\left(\frac{I}{I_o}\right) \qquad 5 = \log\left(\frac{I}{I_o}\right)$$

$$10^{11.5} = \frac{I}{I_o} \qquad 10^{5} = \frac{I}{I_o}$$

$$\frac{I_{rock\,concert}}{I_{converstaion\ at\ home}} = \frac{10^{11.5}}{10^{5}} = 10^{6.5}$$

etc

Question 3(b)

$$\frac{I_{converstaion\ at\ restaurant}}{I_{converstaion\ at\ home}} = \frac{\frac{I_{rock\,concert}}{I_0}}{\frac{I_{converstaion\ at\ home\,t}}{I_0}}$$

$$115 = 10\log\left(\frac{I}{I_o}\right) \qquad 60 = 10\log\left(\frac{I}{I_o}\right)$$

$$11.5 = \log\left(\frac{I}{I_o}\right) \qquad 6 = \log\left(\frac{I}{I_o}\right)$$

$$10^{11.5} = \frac{I}{I_o} \qquad 10^{6} = \frac{I}{I_o}$$

$$\frac{I_{rock\,concert}}{I_{converstaion\ at\ home}} = \frac{10^{6}}{10^{5}} = 10^{1}$$

So the conversation in a restaurant is 10 times more intense than a conversation at home.

The lowest sound heard by man is

$$L = 10\log\left(\frac{I_0}{I_0}\right)$$
$$= 10\log(1)$$
$$L = 0\ dB$$

The loudest sound heard without pain (ie. the pain threshold) is 10 watts/m^2.

$$L = 10\log\left(\frac{I}{I_0}\right)$$
$$= 10\log\left(\frac{10}{10^{-12}}\right)$$
$$= 10\log\left(10^{13}\right)$$
$$= 130\log(10)$$
$$L = 130\ dB$$

Question 1

Prove that the pain threshold is 10^{13} times more intense than the lowest sound heard by man.

See next page

Question 2

(a) What is the sound intensity level that corresponds to a sound that has an intensity of 10^{-2} watts/m^2?

(b) What sound could this be from the table?

Solutions

Question 1

0 dB has $I = I_0 = 10^{-12}$ *watts/m^2*.
130 dB has $I = 10$ *watts/m^2*

$$\frac{I}{I_0} = \frac{10}{10^{-12}} = 10^{13}$$

i.e. the pain threshold is 10 000 000 000 000 times more intense than the lowest sound heard by man.

Question 2

(a) $\frac{I}{I_0} = \frac{10^{-2}}{10^{-12}} = 10^{10}$

$$L = 10\log\left(\frac{I}{I_0}\right)$$

$$L = 10\log\left(10^{10}\right)$$

$$L = 100\ dB$$

(b) The sound could be a motorbike or a jet taking off 300 m away.

Question 3(a)

$I_0 = 10^{-12}$ *watts/m^2*.

Rock concert L = 115 dB   Conversation at home L = 50 dB

$$115 = 10\log\left(\frac{I}{I_o}\right) \qquad 50 = 10\log\left(\frac{I}{I_o}\right)$$

$$11.5 = \log\left(\frac{I}{10^{-12}}\right) \qquad 5 = \log\left(\frac{I}{10^{-12}}\right)$$

$$10^{11.5} = \frac{I}{10^{-12}} \qquad 10^{5} = \frac{I}{10^{-12}}$$

$$I_{Rock\,concert} = 10^{-0.5} \qquad I_{Rock\,concert} = 10^{-7}$$

$$\frac{I_{rock\,concert}}{I_{converstaion\ at\ home}} = \frac{10^{-0.5}}{10^{-7}}$$

$$= 10^{6.5}$$

$$= 3\,162\,278$$

So the rock concert is 3 million times more intense than a conversation at home.

Question 8 (continued)

(c)

(i) Complete the following to make a true statement.

When $y = \log_2(x+2)$,

$x + 2 =$

(ii) Complete the following to make a true statement.

When $y = \log_2(x) + 2$,

$x =$

(iii) Begin the process of determining the simultaneous solution(s) of the equations, $y = \log_2(x+2)$ and $y = \log_2(x) + 2$ algebraically, by forming and solving an equation in one variable, *x.*

(iv) Use the solution(s) of the equation formed in part (iii) to determine the corresponding solution(s) for y.

End of questions

Question 3

(a) How many times more intense is the sound of a loud rock concert than the sound of a conversation at home?

Hint : Find $\frac{I_{rock\,concert}}{I_{converstaion\ at\ home}}$

(b) How many times more intense is the sound of a conversation in a restaurant than the sound of a conversation at home?

Question 4

Determine the intensity in watts/m ^{2} of the following sounds

(a) (i) a chain saw

(ii) a vacuum cleaner at 1m

(iii) rustling leaves

(iv) a telephone dial tone

(v) a hand drill

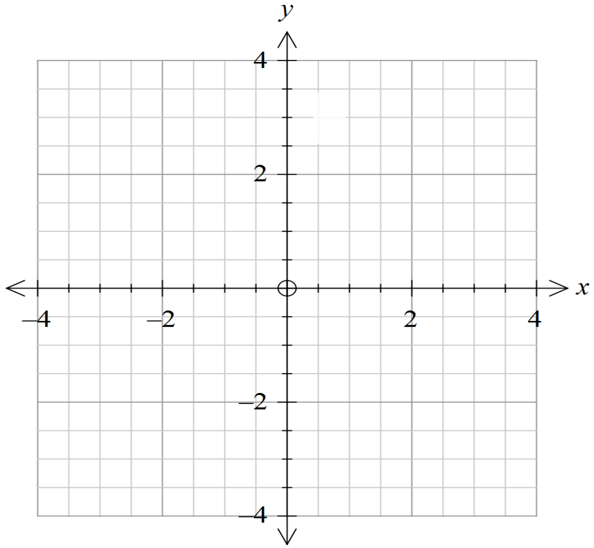
(vi) an aircraft carrier deck

(vii) the ticking of a watch with a sound intensity level of 20 dB

(b) Explain the reason for using sound intensity levels in decibels rather than intensity in watts/m ^{2}.

Question 8

(a) (i) Sketch the graph of each of the functions $y = \log_2(x+2)$ and $y = \log_2(x) + 2$ on the axes below.



(ii) State the co-ordinates of all points of intersections of the graphs.

(b) (i) State the domain, range and the equations of any asymptotes for each function.

(ii) Explain how the graph of each function is related to the graph of $y = \log_2(x)$

Question 7

True or false? Justify your decision.

(a) $\log_3 9 = \log_9 3$

(b) $\log_e x^0 = 1$ for $x \neq 0$

(c) $\ln\left(\frac{1}{x^2}\right) = -2\ln(x)$

(d) $e^{\ln(b)} = b$

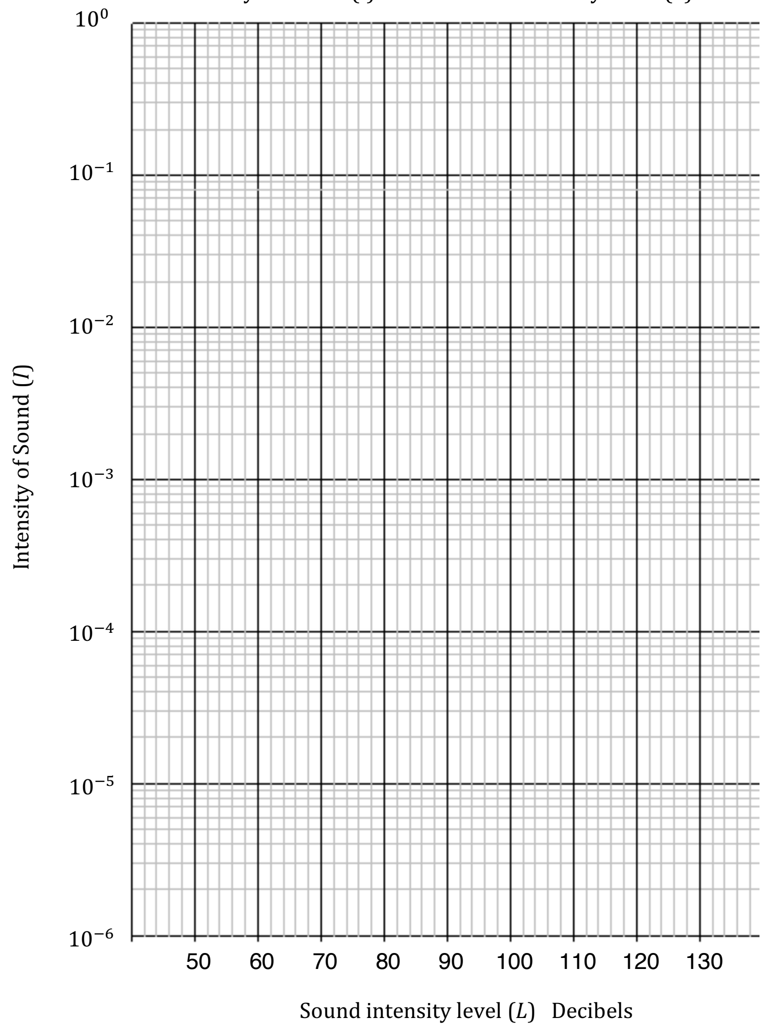
(e) $\log_2(x+1)$ is defined for $x \geq -1$

Question 5

Use the semi – logarithm graph paper (one axis has a logarithmic scale) to plot some of the intensities of the sounds in **Question 4** against their sound intensity levels and comment on the shape of the graph.

What is the advantage of using this graph paper for the data from Question 4?

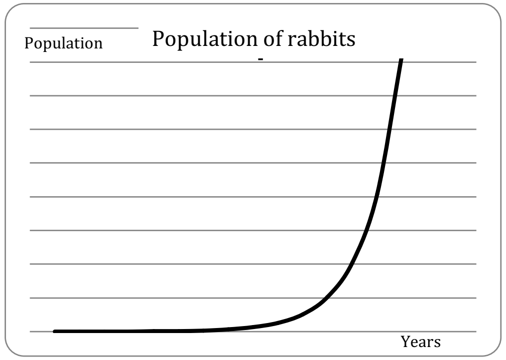
Intensity of Sound (I) versus Sound Intensity Level (L)



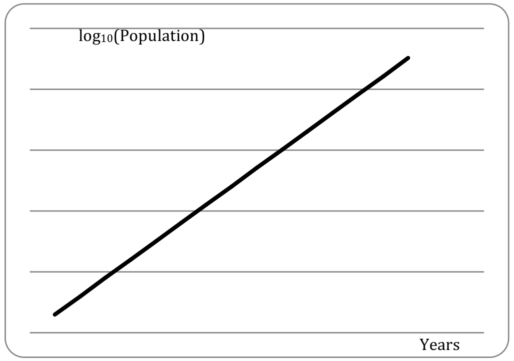
Semi-logarithm graphs are useful when graphing data that increases exponentially.

For example, a population of rabbits that is doubling every year.

Year	Population
1	1
2	2
3	4
4	8
5	16
6	32
7	64
8	128
9	256
10	512
11	1024
12	2048
13	4096
14	8192
15	16384



Year	$\log_{10}$(Population)
1	0.30103
2	0.60206
3	0.90309
4	1.20412
5	1.50515
6	1.80618
7	2.10721
8	2.40824
9	2.70927
10	3.0103
11	3.31133
12	3.61236
13	3.91339
14	4.21442
15	4.51545



If the function is exponential, then using semi-logarithm paper makes the graph linear.

Question 6

Prove that if the function is exponential, say $y = A(b)^t$ then the function $\log_c(y)$ graphed against t is linear.