



PERTH MODERN SCHOOL
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Independent Public School

Course _____ **Specialist** _____ **Year** 12

Student name: _____ Teacher name: _____

Date: 24 Feb

Task type: _____ **Response**

Time allowed for this task: 45 mins

Number of questions: 8

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: 50 marks

Task weighting: 10%

Formula sheet provided: Yes/No

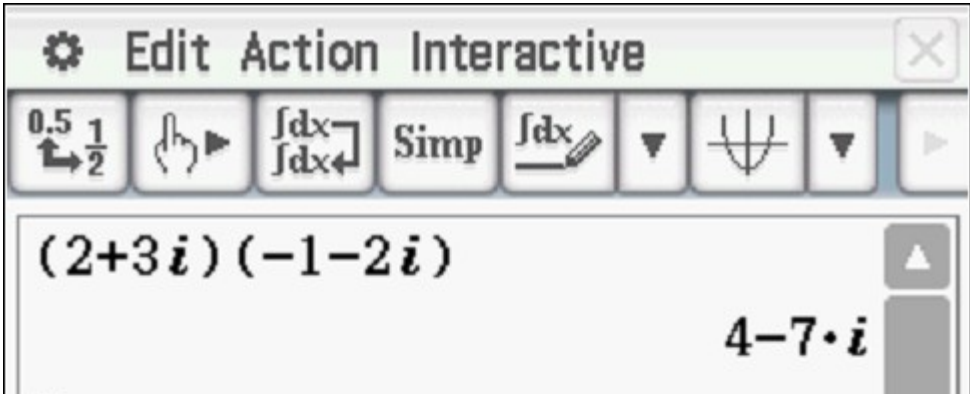
Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (3.1.1, 3.1.2, 3.1.3)

(2, 2, 3 & 3 = 10 marks)

If $z = 2 + 3i$ and $w = -1 + 2i$ determine exactly the following. (Simplify)

a) \overline{zw}

b) Solution

Specific behaviours
<ul style="list-style-type: none"> ✓ determines conjugate of w ✓ determines product

b) \overline{ww}

Solution
$1^2 + 2^2 = 5$
Specific behaviours
<ul style="list-style-type: none"> ✓ gives a real result ✓ determines product

c) $w \div \overline{w}$

Solution
$\frac{-1+2i}{-1-2i} \times \frac{-1+2i}{-1+2i} = \frac{1-4-4i}{1+4} = \frac{-3-4i}{5}$
Specific behaviours
<ul style="list-style-type: none"> ✓ multiplies by conjugate over conjugate ✓ evaluates numerator ✓ evaluates denominator

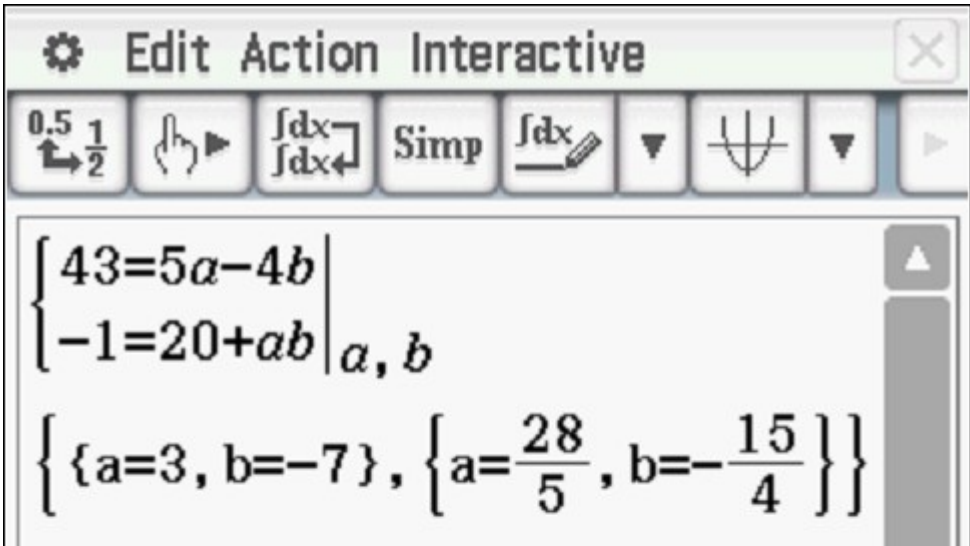
d) $\frac{1}{z} + \frac{1}{w}$

Solution
$\frac{1}{2+3i} \frac{2-3i}{2-3i} + \frac{1}{-1+2i} \frac{-1-2i}{-1-2i}$ $\frac{2-3i}{13} + \frac{-1-2i}{5}$ $\frac{10-15i}{65} + \frac{-13-26i}{65} = \frac{-3-41i}{65}$
Specific behaviours
<ul style="list-style-type: none"> ✓ uses idea of conjugates ✓ expresses both reciprocals with real denominator ✓ determines correct simplified sum

Q2 (3.1.3)

(3 marks)

Determine all possible real values of a & b such that $\frac{43-i}{a+4i} = 5+bi$

Solution
$(43-i) = (5+bi)(a+4i) = 5a - 4b + i(20+ab)$ $43 = 5a - 4b$ $-1 = 20 + ab$ 
Specific behaviours
<ul style="list-style-type: none"> ✓ sets up one equation involving a or b ✓ sets up two equations for a & b

✓solves for two pairs of real values for a & b
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Q3 (3.1.14, 3.1.15)

(3 & 3 = 6 marks)

Consider the quadratic equation $x^2 + bx + c = 0$ where b & c are real.

a) If one root of the above equation is $x = 4 - 2i$, determine b & c .

Solution
$(4 - 2i)^2 + b(4 - 2i) + c = 0$ $16 - 4 - 16i + 4b - 2bi + c = 0$ $12 + 4b + c = 0$ $-16 - 2b = 0$ $b = -8$ $c = -12 + 32 = 20$
Specific behaviours
<ul style="list-style-type: none"> ✓ sets up one equation involving c or b ✓ sets up two equations for c & b ✓ solves for one pair of real values for c & b

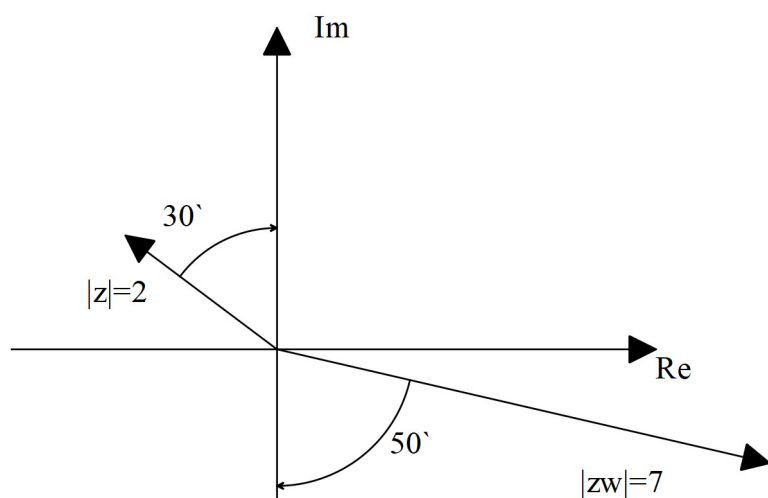
Consider the equation $x^3 + px^2 + qx + w = 0$ where p, q & w are real.

b) If the cubic equation above has roots $x = 2$ & $x = \sqrt{3}i$, determine p, q & w .

Solution
$(x - 2)(x - \sqrt{3}i)(x + \sqrt{3}i)$ $(x - 2)(x^2 + 3)$ $x^3 + 3x - 2x^2 - 6$ $x^3 - 2x^2 + 3x - 6$ $p = -2, q = 3$ & $w = -6$
Specific behaviours
<ul style="list-style-type: none"> ✓ recognises that conjugate also a root ✓ expands at least two linear factors ✓ solves for p, q & w

Q4 (3.1.3, 3.1.3, 3.1.3)

(2 marks)

Determine z & w in the form $rcis\theta$ with $-\pi < \theta \leq \pi$. (Note: diagram not drawn to scale)

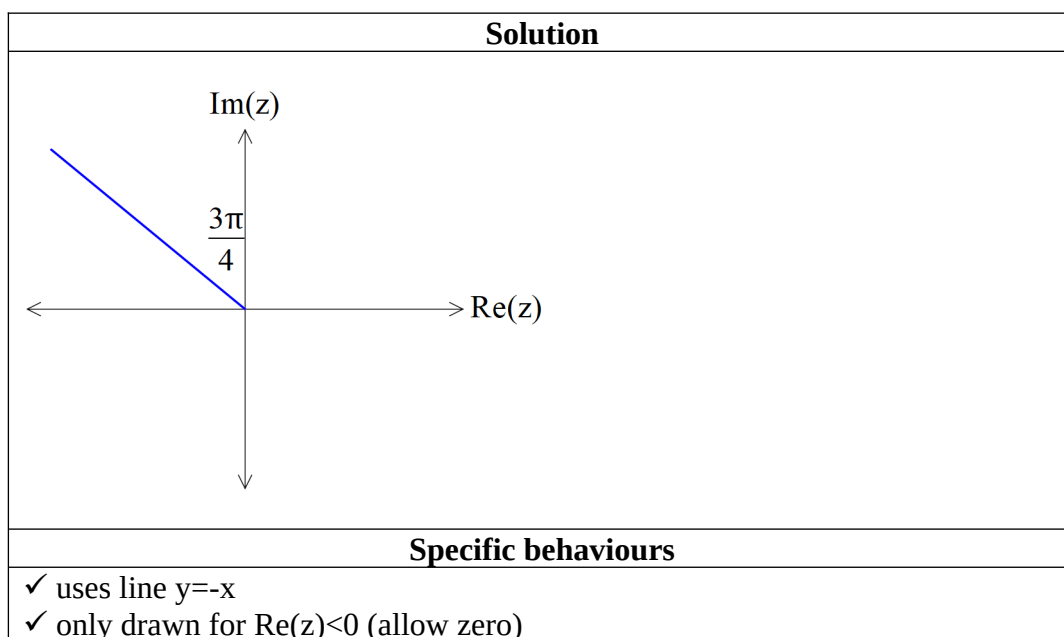
Solution
$z = 2cis120^\circ$ or $2cis \frac{2\pi}{3}$ $w = \frac{7}{2}cis(-160^\circ)$ or $\frac{7}{2}cis\left(-\frac{8\pi}{9}\right)$
Accept radians or degrees
Specific behaviours
<ul style="list-style-type: none"> ✓ determines z with principal argument ✓ determines w with principal argument

Q5 (3.1.10)

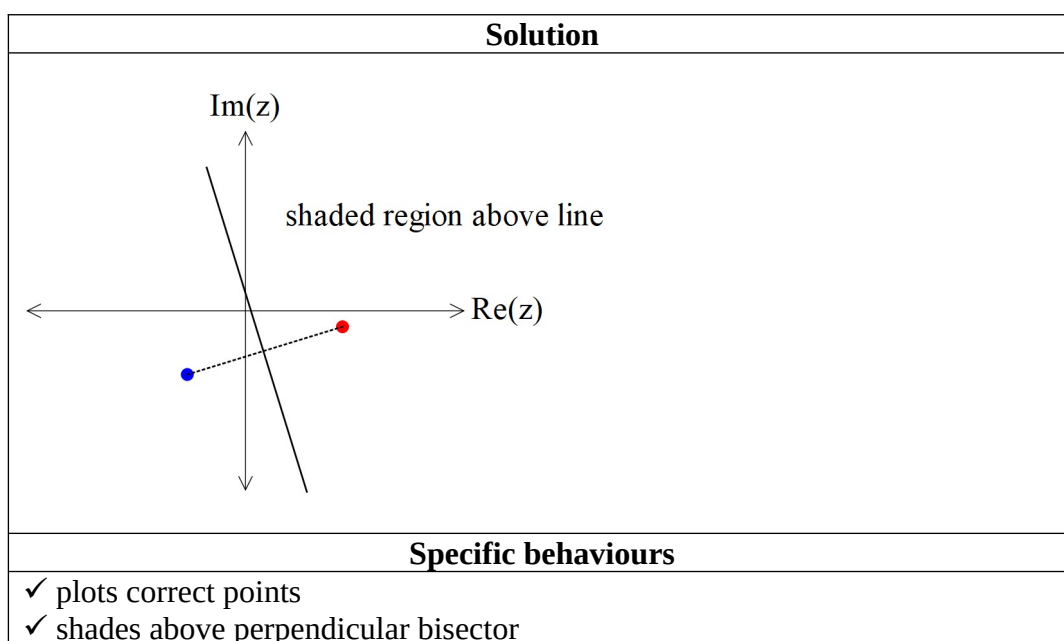
(2, 2 & 3 = 7 marks)

Sketch the following regions in the complex plane showing major features.

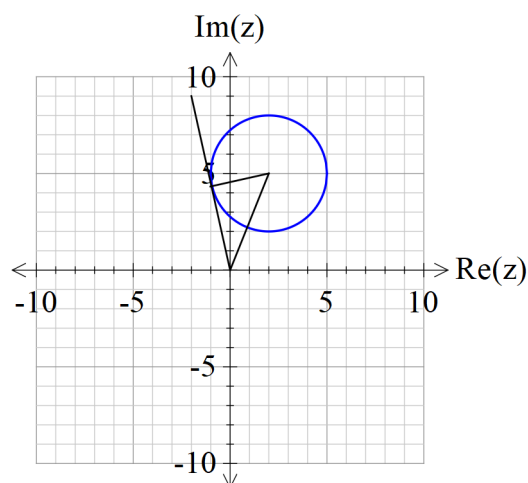
a) $\text{Arg}(z) = \frac{3\pi}{4}$



b) $|z + 3 + 4i| \geq |z - 5 + i|$



- c) Consider all the complex numbers z that satisfy $|z - (2 + 5i)| = 3$, determine the maximum possible value of $\text{Arg}(z)$, giving your answer in radians correct to two decimal places.

Solution

Edit Action Interactive

$\tan^{-1}\left(\frac{5}{2}\right) + \sin^{-1}\left(\frac{3}{\sqrt{2^2 + 5^2}}\right)$
 1.7811627

Specific behaviours

- ✓ determines argument of centre of circle
- ✓ uses left tangent idea for max argument
- ✓ solves for max argument in radians (no need to round to 2 dp)

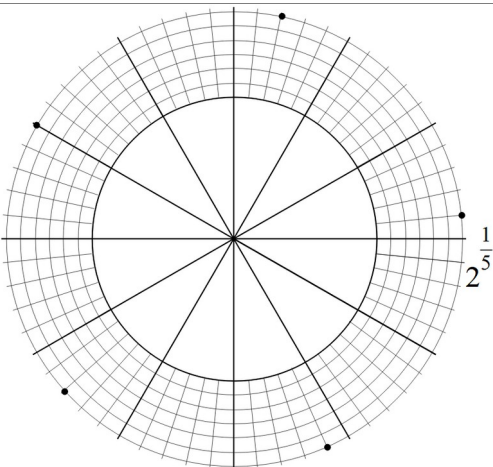
Q6 (3.1.7, 3.1.12)

(4 & 3=7 marks)

a) Determine all the roots of $z^5 = \sqrt{3} + i$ expressing in the form $rcis\theta$ with $-\pi < \theta \leq \pi$.

Solution
$z^5 = \sqrt{3} + i = 2cis\left(\frac{\pi}{6} + 2n\pi\right), n = 0, \pm 1, \pm 2, \dots$
$z = 2^{\frac{1}{5}} cis\left(\frac{\pi}{30} + \frac{2n\pi}{5}\right) = 2^{\frac{1}{5}} cis\left(\frac{\pi}{30} + \frac{12n\pi}{30}\right)$
$z_1 = 2^{\frac{1}{5}} cis\left(\frac{\pi}{30}\right)$
$z_2 = 2^{\frac{1}{5}} cis\left(\frac{13\pi}{30}\right)$
$z_3 = 2^{\frac{1}{5}} cis\left(-\frac{11\pi}{30}\right)$
$z_4 = 2^{\frac{1}{5}} cis\left(-\frac{23\pi}{30}\right)$
$z_5 = 2^{\frac{1}{5}} cis\left(\frac{25\pi}{30}\right) \text{ or } 2^{\frac{1}{5}} cis\left(\frac{5\pi}{6}\right)$
Specific behaviours
<ul style="list-style-type: none"> ✓ expresses right hand side into polar form ✓ uses De Moivre's theorem ✓ obtains five distinct roots in polar form ✓ uses principal arguments for all roots

b) Plot all of these roots on the diagram below.

Solution

Specific behaviours
<ul style="list-style-type: none"> ✓ shows scale ✓ plots one root correctly

✓all five roots equally spaced

Q7 (3.2.1, 3.2.2)

(1, 2, 2 & 2 = 7 marks)

Consider the functions $f(x) = \sqrt{x-8}$ & $g(x) = x^3$.a) Give the defining rule for $f \circ g(x)$.

Solution
$f \circ g(x) = \sqrt{x^3 - 8}$
Specific behaviours
✓ states rule

a) Does $f \circ g(x)$ exist over the natural domain of $g(x)$? Explain

Solution
$r_g : \mathbb{R}$ $d_f : x \geq 8$ $r_g \not\subset d_f \therefore \text{does not exist}$
Specific behaviours
✓ determine appropriate domain and range ✓ shows that condition not meet for natural domain of g

b) State the natural domain and range for $f \circ g(x)$.

Solution
$f \circ g(x) = \sqrt{x^3 - 8}$ $x^3 \geq 8$ $x \geq 2$ $y \geq 0$
Specific behaviours
✓ states natural domain ✓ states range

Consider the function $h(x) = x - 8$.

- c) Does the function $[f(x)]^2 = h(x)$? Justify your answer.

Solution
$[f(x)]^2$ $d : x \geq 8$ $h(x)$ $d : \mathbb{R}$ <i>not equal as different domains</i>
Specific behaviours
<ul style="list-style-type: none"> ✓ states not equal with a reason ✓ shows that domains are different

Q8 (3.2.3, 3.2.4)

(2 & 3, 3 = 8 marks)

Consider the function $f(x)$ drawn below.

- a) Sketch $y = f^{-1}(x)$ on the axes above.

b) Solution
Specific behaviours
<ul style="list-style-type: none"> ✓ reflects in line $y=x$ ✓ inverse contains pt (5,3)

- b) Given that $f(x) = -2x^2 + 12x - 13$, $x \leq 3$, determine the defining rule for $y = f^{-1}(x)$.

Solution
$y \leq 3$ $x = -2y^2 + 12y - 13$ $-2y^2 + 12y - 13 - x = 0$ $y = \frac{-12 \pm \sqrt{144 - 4(-2)(-13-x)}}{-4} = \frac{-12 \pm \sqrt{144 - 104 - 8x}}{-4}$ $= \frac{-12 \pm 2\sqrt{10 - 2x}}{-4} = \frac{-6 \pm \sqrt{2(5-x)}}{-2}$ $f^{-1}(x) = 3 - \sqrt{\frac{5-x}{2}}$
Specific behaviours
<ul style="list-style-type: none"> ✓ interchanges x and y to solve for inverse ✓ states possible rules for inverse ✓ states correct rule with negative only

- c) Consider the function $h(x) = ax^3$ where a is a positive constant. Solve in terms of a , the solution(s) to $h(x) = h^{-1}(x)$.

c) Solution
$h(x) = h^{-1}(x) = x \quad \text{or} \quad ax^3 = \sqrt[3]{\frac{x}{a}} \Rightarrow a^3 x^9 = \frac{x}{a} \Rightarrow a^4 x^9 - x = 0 \Rightarrow x(a^4 x^8 - 1) = 0$ $ax^3 = x$ $x(ax^2 - 1) = 0$ $x = 0, \frac{1}{\sqrt{a}}, -\frac{1}{\sqrt{a}}$
Specific behaviours
<ul style="list-style-type: none"> ✓ sets up equation in terms of x & a ✓ factorises equation ✓ states all three x values in terms of a.

Working out space