Mathematical Reasoning

These challenging questions are taken from the NSW HSC Mathematics Extension Examinations and hence NOT ALL questions here may be suitable.

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, show that

$$\cot\theta + \frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2}.$$

(ii) Use mathematical induction to prove that, for integers $n \geq 1$,

$$\sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^{n-1}}\cot\frac{x}{2^n} - 2\cot x.$$

(iii) Show that

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{2}{x} - 2\cot x.$$

(iv) Hence find the exact value of

$$\tan\frac{\pi}{4} + \frac{1}{2}\tan\frac{\pi}{8} + \frac{1}{4}\tan\frac{\pi}{16} + \cdots.$$

Answers

Question 8.

(a) (i) $\cot\theta + \frac{1}{2}\tan\frac{\theta}{2} = \frac{1-t^2}{2t} + \frac{t}{2} = \frac{1}{2t} = \frac{1}{2}\cot\frac{\theta}{2}$

(ii) $n=1$

LHS $= \tan\frac{x}{2} = \cot\frac{x}{2} - 2\cot x =$ RHS

by (i) with $\theta = x$.

Suppose

$$\sum_{r=1}^{k} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^{k-1}}\cot\frac{x}{2^k} - 2\cot x$$

Add $\frac{1}{2^k}\tan\frac{x}{2^{k+1}}$

$$\sum_{r=1}^{k+1} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^{k-1}}\cot\frac{x}{2^k} + \frac{1}{2^k}\tan\frac{x}{2^{k+1}} - 2\cot x$$

$$= \frac{1}{2^k}\cot\frac{x}{2^{k+1}} - 2\cot x$$

by (i) with $\theta = \frac{x}{2^k}$

The result follows by mathematical induction.

(a) (iii)

$$\lim_{n\to\infty}\sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \lim_{n\to\infty} \frac{1}{2^{n-1}}\cot\frac{x}{2^n} - 2\cot x$$

$$= \lim_{n\to\infty} \frac{2}{x}\cdot\cos\frac{x}{2^n}\cdot\frac{\frac{x}{2^n}}{\sin\frac{x}{2^n}} - 2\cot x$$

$$= \frac{2}{x}\cdot 1\cdot 1 - 2\cot x$$

$$= \frac{2}{x} - 2\cot x$$

(iv) Put $x = \frac{\pi}{2}$

$$\lim_{n\to\infty}\sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{\pi}{2^{r+1}} = \frac{4}{\pi} - 2\times 0 = \frac{4}{\pi}$$

Note there are no answers available for the following questions

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) It is given that $2\cos A\sin B = \sin(A+B) - \sin(A-B)$. (Do NOT prove this.)

Prove by induction that, for integers $n \geq 1$,

$$\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}.$$

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Use the binomial theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + b^n$$

to show that, for $n \geq 2$,

$$2^n > \binom{n}{2}.$$

(ii) Hence show that, for $n \geq 2$,

$$\frac{n+2}{2^{n-1}} < \frac{4n+8}{n(n-1)}.$$

(iii) Prove by induction that, for integers $n \geq 1$,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

(iv) Hence determine the limiting sum of the series

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots.$$

Question 7 (continued)

(c) The sequence $\{x_n\}$ is given by

$$x_1 = 1 \text{ and } x_{n+1} = \frac{4 + x_n}{1 + x_n} \text{ for } n \geq 1.$$

(i) Prove by induction that for $n \geq 1$

$$x_n = 2\left(\frac{1 + \alpha^n}{1 - \alpha^n}\right),$$

where $\alpha = -\frac{1}{3}$.

(ii) Hence find the limiting value of x_n as $n \to \infty$.

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) For each integer $n \geq 0$, let $I_n(x) = \int_0^x t^n e^{-t}\, dt$.

(i) Prove by induction that

$$I_n(x) = n!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)\right].$$

(ii) Show that

$$0 \leq \int_0^1 t^n e^{-t}\, dt \leq \frac{1}{n+1}.$$

(iii) Hence show that

$$0 \leq 1 - e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) \leq \frac{1}{(n+1)!}.$$

(iv) Hence find the limiting value of $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ as $n \to \infty$.