SEMESTER TWO

MATHEMATICS
SPECIALIST
UNITS 3 & 4

2020

SOLUTIONS

Calculator-Free Solutions

1. $u=sec(\theta) \rightarrow \frac{du}{d\theta}=\frac{-\sin\theta}{-\cos^2\theta} \rightarrow du=sec\theta\tan\theta\,d\theta$ ✓

$u(\theta)=\frac{1}{\cos\theta} \rightarrow u(0)=1 \wedge u\left(\frac{\pi}{3}\right)=\frac{1}{0.5}=2$ ✓

$\therefore \int_0^{\frac{\pi}{3}} sec\theta\tan\theta\,d\theta=\int_1^2 du=[u]_1^2=2-1=1\,[4]$

2. $P(z)$ has real coefficients $\Rightarrow$ $\bar{z}=2-i$ is also a solution. ✓

$\therefore (z-2-i)(z-2+i)=z^2-4z+5$ is a factor of $P(z)$ ✓

	z^2	$-2z$	5		
z^2-4z+5	z^4	$-6z^3$	$18z^2$	$-30z$	25
	z^4	$-4z^3$	$5z^2$		
	0	$-2z^3$	$13xz^2$	$-30z$	25
		$-2z^3$	$8z^2$	$-10z$	
		0	$5z^2$	$-20z$	25
			$5z^2$	$-20z$	25
			0	0	0

✓✓

$\therefore z^2-2z+5=0$

$$z=\frac{2\pm\sqrt{4-20}}{2}=1\pm 2i$$

Solutions are $z=1\pm 2i, 2\pm i$ ✓ [5]

3. $\frac{2x^2+1}{x(x^2+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x} \rightarrow 2x^2+1=x(Ax+B)+C(x^2+1)$ ✓

$x=0 \rightarrow 1=0+C \rightarrow C=1$ ✓

$x=1 \rightarrow 3=A+B+2$

$x=-1 \rightarrow 3=A-B+2$

$\therefore A=1, B=0$ ✓

$$\therefore \int_1^2 \frac{2x^2+1}{x^3+x}\,dx=\int_1^2\left(\frac{x}{x^2+1}+\frac{1}{x}\right)dx=\left[\frac{1}{2}\ln(x^2+1)+\ln|x|\right]_1^2$$

$¿\frac{1}{2}\ln(5)+\ln(2)-\frac{1}{2}\ln(2)=\frac{1}{2}\ln 10=\ln\sqrt{10}$ as required ✓ [5]

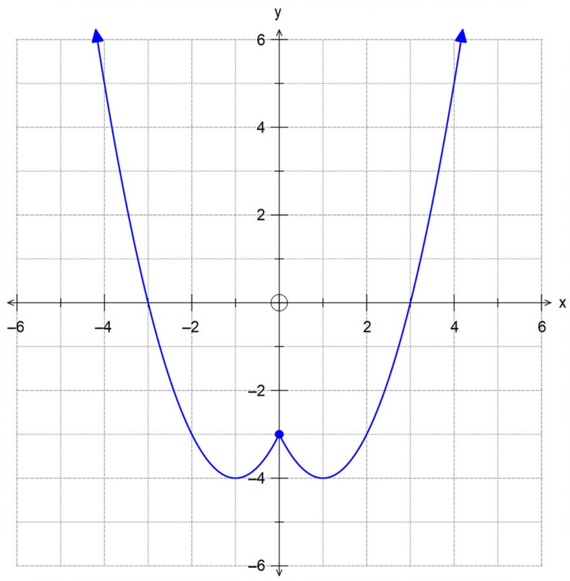
4. (a) $f(g(x))=f(-|x|)=|x|^2-2|x|-3=x^2-2|x|-3$ ✓

For $x\geq 0, f(g(x))=x^2-2x-3=(x-1)^2+4$

which has a turning point at $(1,-4)$

$\therefore D_x=R$ and $R_y=\{y\in R: y\geq -4\}$ ✓✓

(b)



✓
Parabolic shape

✓
Mirror image over the y-axis

✓
Local min at $(\pm 1,-4)$, and roots at $x=\pm 3$

(c) $f(x)$ has its turning point at $(-1,-4)$, $\therefore k=-1$ ✓

$\therefore f^{-1}(x)=\sqrt{x+4}-1$ ✓

(d) (i) $g(f^{-1}(0))=g(1)=-|1|=-1$ ✓

(ii) $g(f^{-1}(f(x+1)))=g(x+1)=-|x+1|$ ✓ [10]

5. (a) R is a solid sphere centred at $(1,1,0)$ with radius 3 units. ✓✓

$|r|_{max}=\left|\begin{matrix}1\\1\\0\end{matrix}\right|+3=\sqrt{2}+3$ units ✓

(b) xz plane $\rightarrow y=0$, therefore the intersection is given by

$(x-1)^2+(0-1)^2+(z)^2=9$

$(x-1)^2+z^2=8$ ✓

$\therefore$ circle centred at $(1,0,0)$ and radius $2\sqrt{2}$ units. ✓

5. (c) The line is given by $r=\begin{pmatrix}5+\lambda\\2\\-\lambda\end{pmatrix}$, hence:

$$\left|\begin{pmatrix}5+\lambda\\2\\-\lambda\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix}\right|=3$$

$\therefore (4+\lambda)^2+1^2+(-\lambda)^2=3^2$ ✓

$2\lambda^2+8\lambda+8=0$

$2(\lambda+2)^2=0 \rightarrow \therefore \lambda=-2$ ✓

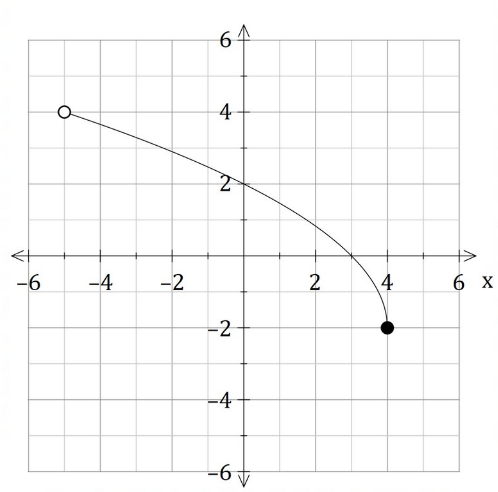
Unique solution implies the line is tangent to the sphere R ✓

$$r(-2)=\begin{pmatrix}5-2\\2\\-2\end{pmatrix}=\begin{pmatrix}3\\2\\2\end{pmatrix}$$

Point of tangency is $(3,2,2)$. ✓ [9]

6. (a)

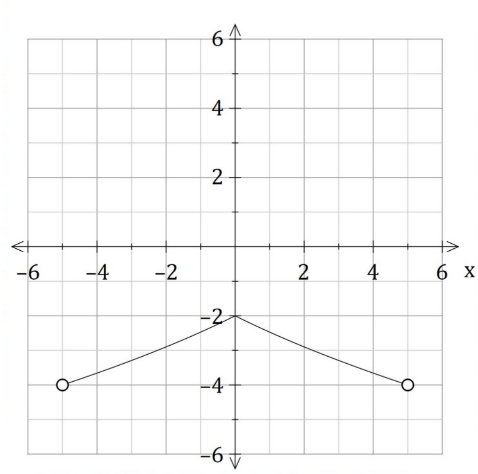
-f(x)



✓ Square root function with correct orientation, root at $x=3$ and y-intercept at $y=2$

✓ Open boundary at $x=-5$ closed boundary at $x=4$

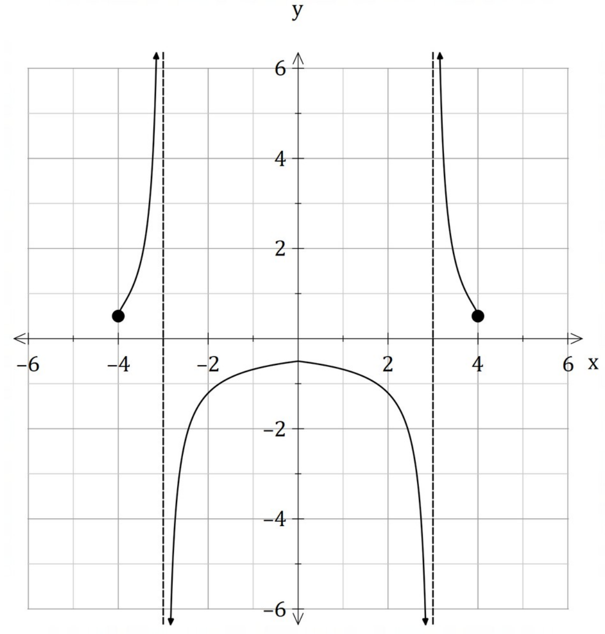
f(-|x|)



✓ Mirror image over the y-axis, completely below the x-axis

✓ y-intercept at $y=-2$, open boundaries at ± 5.

6. (b)

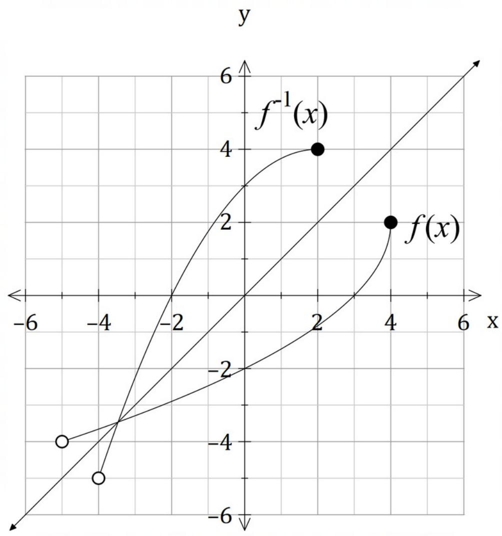


✓
Poles at $x=\pm 3$

✓
Closed boundaries at $x=\pm 4$
Y-intercept at $y=-0.5$

✓
Behaviour on either side of both poles

(c) $f(x)$ and $f^{-1}(x)$ are shown below, hence, for $f^{-1}(x)$:



✓
$D_x=\{x \in R:-4<x \le 2\}$

✓
$R_y=\{y \in R:-5<y \le 4\}$

[9]

7. (a) (i) The internal parabolic outline is given by $y=\frac{x^2}{4}-4$

$\therefore x^2=4(y+4)$

Hence, the volume is given by:

$$v_y=\pi\int_0^{21} x^2\,dy=\pi\int_0^{21} 4(y+4)\,dy=4\pi\int_0^{21}(y+4)\,dy$$

$\therefore f(y)=y+4$ as required

7. (a) (ii)

$$V_y = 4\pi \int_0^{21} (y+4)\,dy = 4\pi \left[\frac{y^2}{2} + 4y\right]_0^{21}$$

$$\therefore V_y = 4\pi \left[y\left(\frac{y}{2}+4\right)\right]_0^{21} = 4\pi \times 21 \times \left(\frac{21}{2}+4\right)$$

¿ $2\pi \times 21 \times 29 = 1218\pi$ cm^3 ✓

(b) $\mathrm{V} = \mathrm{V}_{\text{curved side}} + \mathrm{V}_{\text{base}}$

$= (\mathrm{V}_{\text{outter shell}} - \mathrm{V}_{\text{inner shell}}) + \mathrm{V}_{\text{base}}$

$$\therefore V = \pi \int_0^{21} 5(y+5)\,dy - \pi \int_0^{21} 4(y+4)\,dy + \pi \int_{-1}^{0} 5(y+5)\,dy$$

[7]

8. (a) $a = \dfrac{dv}{dt} \times \dfrac{dx}{dx}$

¿ $\dfrac{dx}{dt} \times \dfrac{dv}{dx} = v\dfrac{dv}{dx}$ as required ✓

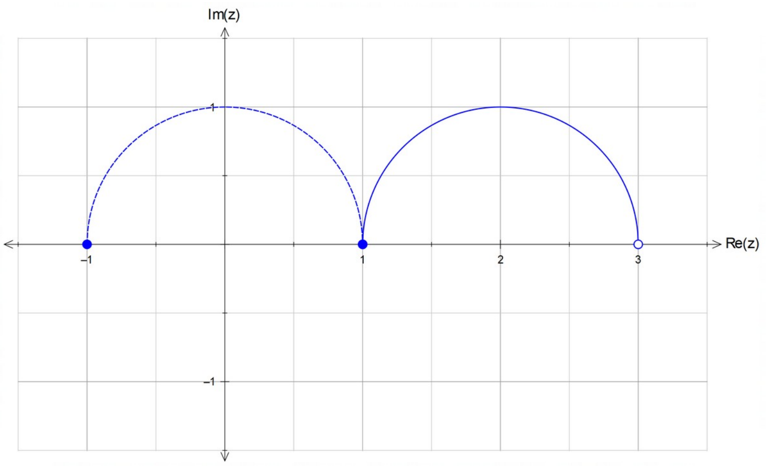
(b) $\dfrac{d}{dx} \rightarrow 4x^2 - v^2 = 1$

$8x - 2v\dfrac{dv}{dx} = 0$ ✓✓

$\therefore v\dfrac{dv}{dx} = a = 4x$ ✓ [4]

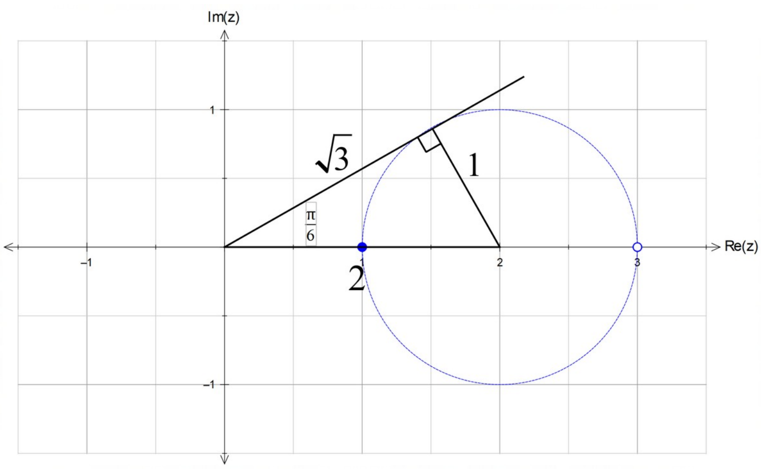
Calculator-assumed Solutions

9. (a)



✓ Semicircle ✓ Centred at (2,0) and radius 1 ✓ Discontinuous at $x=3$ (or $x=-1$ for FT marks if drawn on LHS)

(b)



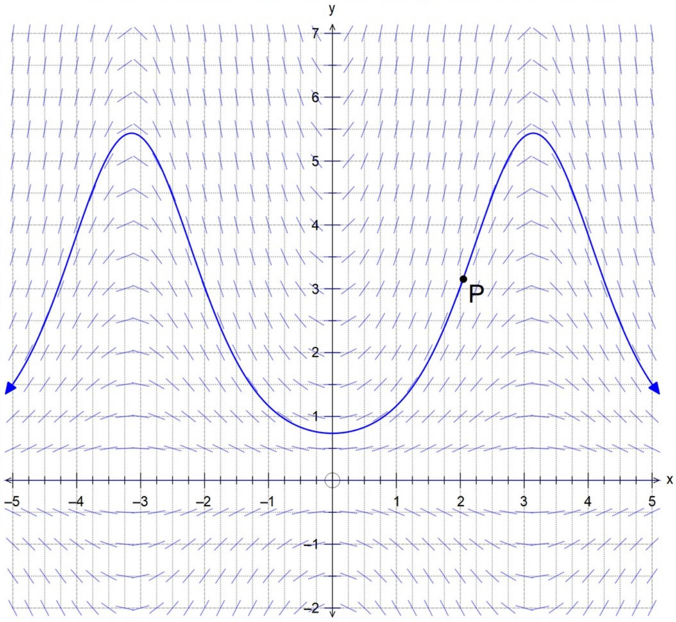
From diagram, $arg(z-2)_{max}=\frac{\pi}{6}$ and $|z-2|=\sqrt{3}$ ✓✓

OR

$arg(z-2)_{max}=\pi$ and $|z-2|=\sqrt{3}$ for follow through marks if the circle is drawn on the LHS of the y-axis. [5]

10. (a) $\left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{2},2\right)}=2\times\sin\frac{\pi}{2}=2$ ✓

(b)



✓
Follows the isoclines to show a cyclic curve
(must be wider along the y-axis, it cannot look like a sinusoid)

✓
Passes through the point P

(c) $\frac{dy}{dx}=y\sin x\rightarrow\frac{dy}{y}=\sin x\,dx$

$$\therefore\int\frac{1}{y}\,dy=\int\sin x\,dx$$

$\therefore\ln|y|=-\cos x+C_1$ ✓

$y=e^{-\cos x+C_1}=e^{C_1}\times e^{-\cos x}=C_2e^{-\cos x}$

$\left(\frac{\pi}{2},2\right)\rightarrow2=C_2\times e^0\rightarrow C_2=2$

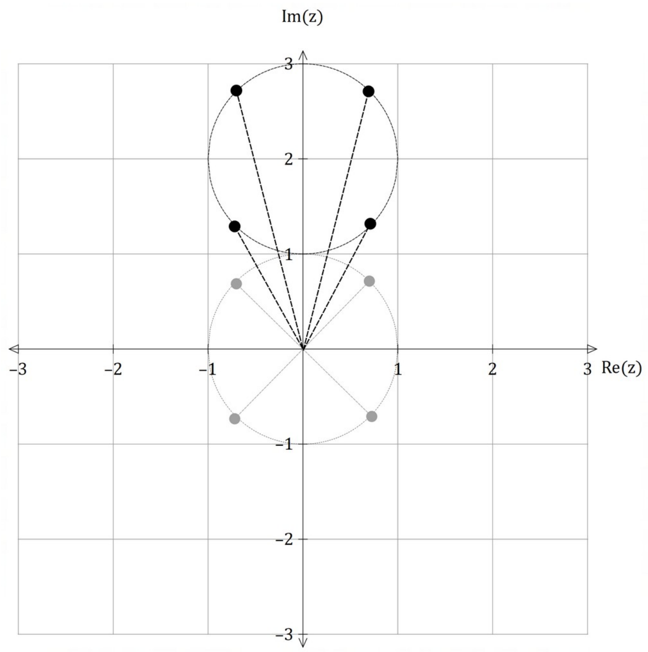
$\therefore y=2e^{-\cos x}$ ✓

11. (a) (i) $z^4=-1=cis(\pi+2k\pi),k=0,\pm1,-2$ ✓

$\therefore z=cis\left(\frac{\pi+2k\pi}{4}\right),k=0,\pm1,-2$ ✓

$\therefore z=cis\left(\frac{\pi}{4}\right),cis\left(\frac{3\pi}{4}\right),cis\left(\frac{-\pi}{4}\right),cis\left(\frac{-3\pi}{4}\right)$ ✓✓

11. (a) (ii) Solutions from (a)(i) move two units up:



✓
4 solutions equidistant from 2i

✓✓
equally spaced at $\frac{\pi}{2}$ radians from each other within the circle centred at $2i$

(b) (i) $z=\frac{\sqrt{3}+i}{1-i}\times\frac{1+i}{1+i}=\left(\frac{\sqrt{3}-1}{2}\right)+\left(\frac{\sqrt{3}+1}{2}\right)i$

$\therefore \Re(z)=\frac{\sqrt{3}-1}{2}\wedge\Im(z)=\frac{\sqrt{3}+1}{2}$ ✓✓

(or CAS)

(ii) $z=\frac{\sqrt{3}+i}{1-i}=\frac{2\,cis\left(\frac{\pi}{6}\right)}{\sqrt{2}\,cis\left(\frac{-\pi}{4}\right)}=\frac{2}{\sqrt{2}}cis\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\sqrt{2}\,cis\left(\frac{5\pi}{12}\right)$ ✓✓

$\therefore \sqrt{2}\cos\left(\frac{5\pi}{12}\right)=\Re(z)=\frac{\sqrt{3}-1}{2}$ ✓

$\cos\left(\frac{5\pi}{12}\right)=\frac{\sqrt{3}-1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}(\sqrt{3}-1)$ as required ✓

12. (a) $r(t)=\int v(t)dt=\left(\int 2t\,dt\right)i+\left(\int \sin t\,dt\right)j$

¿ $(t^2+C_1)i+(-\cos t+C_2)j$ ✓

$r(0)=\begin{pmatrix}C_1\\-1+C_2\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}\rightarrow \begin{matrix}C_1=2\\C_2=2\end{matrix}$ ✓

$\therefore r(t)=(t^2+2)i+(2-\cos t)j$ ✓

(b) $a(t)=\frac{d}{dt}v(t)=2i+\cos t\,j$ ✓

$\therefore a(0)=2i+\cos 0\,j=2i+j$ ✓

$\therefore |a(0)|=\begin{vmatrix}2\\1\end{vmatrix}=\sqrt{5}\approx 2.24\,m/s^2$ ✓

12. (c) $y_{max}=2-(-1)=3$ for $\cos t=-1 \rightarrow t=\pi$ ✓

$\therefore a=\pi^2+2$ and $b=3$ ✓✓

(d) $\int_0^{10}\sqrt{4t^2+\sin^2 t}\,dt\approx 100.43\,m$ ✓✓✓

13. (a) $\overline{T}$ is approximately a normal distribution by the Central Limit Theorem since $n=100>30$ ✓

$\therefore \overline{T}\ \ N\left(37,\frac{0.25^2}{100}\right)=N\left(37,6.25\times 10^{-4}\right)$ ✓

with $\sigma(\overline{T})=\frac{1}{40}=0.025$ ✓

(b) $P(\overline{T}>37.005)=P\left(z>\frac{37.005-37}{0.0025}\right)=P(z>0.2)$ ✓

$\therefore P(z>0.2)\approx 0.42074$ ✓

(c) Since $n>100 \Rightarrow \sigma(\overline{T})<0.025$

$\therefore$ the answer in (b) would be lower/smaller ✓

because the lower standard deviation of the sample mean ✓

(d) Require $P(37<\overline{T}<37.005)>0.45$

$\therefore P(0<z<k)>0.45$ ✓

$CAS \rightarrow k>1.6449$

$\frac{37.005-37}{\left(\frac{0.25}{\sqrt{n}}\right)}>1.6449$ ✓

$\rightarrow n>6764.2\approx n>6764$ patients ✓

[Accept 6764 and 6765 depending on accuracy used]

14. (a) $\overrightarrow{OA} = \begin{pmatrix} 10 \\ -10 \\ 0 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 16 \\ -8 \\ 6 \end{pmatrix}$ ✓

$\overrightarrow{OB} = \begin{pmatrix} -10 \\ 30 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 26 \\ 8 \end{pmatrix}$ ✓

$\therefore \overrightarrow{AB} = \begin{pmatrix} -4 \\ 26 \\ 8 \end{pmatrix} - \begin{pmatrix} 16 \\ -8 \\ 6 \end{pmatrix} = \begin{pmatrix} -20 \\ 34 \\ 2 \end{pmatrix} \rightarrow \left|\begin{matrix} -20 \\ 34 \\ 2 \end{matrix}\right| \approx 39.50\,m$ ✓✓

14. (b) $\overrightarrow{OA}=\begin{pmatrix}20\\-10\\0\end{pmatrix}+t\begin{pmatrix}-2\\1\\3\end{pmatrix}=\begin{pmatrix}20-2t\\t-10\\3t\end{pmatrix}$

$\left|\overrightarrow{OA}\right|=\begin{vmatrix}20-2t\\t-10\\3t\end{vmatrix}=\sqrt{(20-2t)^2+(t-10)^2+(20t)^2}=30$ ✓

$\therefore CAS\to t=10s$ later

$\therefore \overrightarrow{OA}=\begin{pmatrix}0\\0\\30\end{pmatrix}$ ✓

$\therefore \overrightarrow{OB}(10)=\begin{pmatrix}-10\\30\\0\end{pmatrix}+10\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\30\end{pmatrix}$ ✓

$\therefore v_B=i-3j+3k$ m/s ✓

15. All coordinates in the first octant are positive. ✓

xy-plane intersection $(z=0)$:

$3+\lambda=0\to\lambda=-3$

$\therefore r=\begin{pmatrix}4+6\\2-3\\3-3\end{pmatrix}=\begin{pmatrix}10\\-1\\0\end{pmatrix}$ not in the first octant ✓

xz-plane intersection $(y=0)$:

$2+\lambda=0\to\lambda=-2$

$\therefore r=\begin{pmatrix}4+4\\2-2\\3-2\end{pmatrix}=\begin{pmatrix}8\\0\\1\end{pmatrix}$ in the first octant ✓

yz-plane intersection $(x=0)$:

$4-2\lambda=0\to\lambda=2$

$\therefore r=\begin{pmatrix}4-4\\2+2\\3+2\end{pmatrix}=\begin{pmatrix}0\\4\\5\end{pmatrix}$ in the first octant ✓

$\therefore \begin{pmatrix}8\\0\\1\end{pmatrix}-\begin{pmatrix}0\\4\\5\end{pmatrix}=\begin{pmatrix}8\\-4\\-4\end{pmatrix}\to\begin{vmatrix}8\\-4\\-4\end{vmatrix}=4\begin{vmatrix}2\\-1\\-1\end{vmatrix}=4\sqrt{6}\,units$ ✓✓

16. (a) $P(0)=\dfrac{80\,000}{5+3\times1}=10\,000\,cats$ ✓

(b) $P(t)_{max} = \dfrac{80000}{5+3\times 0} = 16\,000\, cats$ ✓

which is the carrying capacity of the system

16. (c) $P=\dfrac{80000}{5+3e^{-0.25t}} \to 5+3e^{-0.25t}=\dfrac{80000}{P}$

$\dfrac{dP}{dt}=-80000\left(5+3e^{-0.25t}\right)^{-2}\times -0.75e^{-0.25t}$ ✓

¿ $\dfrac{80000}{\left(5+3e^{-0.25t}\right)}\times\dfrac{1}{4}\times\dfrac{1}{\left(5+3e^{-0.25t}\right)}\times 3e^{-0.25t}$ ✓

¿ $\dfrac{P}{4}\times\dfrac{P}{80000}\times\left(\dfrac{80000}{P}-5\right)$ ✓✓

¿ $\dfrac{P}{4}\left(1-\dfrac{P}{16000}\right)$ as required

(d) $\left.\dfrac{dP}{dt}\right|_{t=0}=\dfrac{10000}{4}\left(1-\dfrac{10000}{16000}\right)=2500\times\dfrac{3}{8}=937.5\,cats/\,year$ ✓✓

17. (a) 90% confidence interval for μ:

$P(-k<z<k)=0.9 \to k\approx 1.645$ ✓

$\therefore\ 20-1.645(2)<\mu<20+1.645(2)$ ✓

$16.71<\mu<23.29\,min$ ✓

(b) From July: $\dfrac{\sigma}{\sqrt{n}}=2 \to \sigma=2\sqrt{n}$

For August: $\sigma(\overline{X})=\dfrac{\sigma}{\sqrt{2n}}=\dfrac{1}{\sqrt{2}}\left(\dfrac{\sigma}{\sqrt{n}}\right)=\dfrac{1}{\sqrt{2}}(2)\approx 1.41\,min$ ✓✓

(c) C is the most accurate. ✓

Because it is based on the smallest standart error ✓

(d) We do not know which interval, if any, ✓

because we cannot determine the true

value of the population mean. ✓

18. (a) $-\omega^2=-9 \to \omega=3\,rad/s$ ✓

$\therefore T=\dfrac{2\pi}{\omega}=\dfrac{2\pi}{3}\approx 2.09\,sec$ ✓

(b) Let $x(t) = A\sin(3t)$ ✓

$\therefore x'(t) = 3A\cos(3t)$

$3A = \dfrac{3}{2} \rightarrow A = \dfrac{1}{2}m = 50\,cm$ ✓

18. (c) Use $v^2=\omega^2\left(A^2-x^2\right)$:

$6^2=\omega^2\left(A^2-\sqrt{3}^2\right)$ and $\left(6\sqrt{2}\right)^2=\omega^2\left(A^2-\sqrt{2}^2\right)$ ✓

$CAS \rightarrow \omega=3 \rightarrow T=\frac{\pi}{3}s \wedge A=2m$ ✓✓

19. (a) Outline of the circle in x/y plane given by:

$x^2+y^2=r^2 \rightarrow y=\pm\sqrt{r^2-x^2}$ ✓

$\therefore 2x+2y\frac{dy}{dx}=0 \rightarrow \frac{dy}{dx}=\frac{-x}{y}$ ✓

$\therefore A=\int_a^b 2\pi y\sqrt{1+\frac{x^2}{y^2}}\,dx$ ✓

(b) Use $y^2=r^2-x^2$:

$\therefore A=\int_a^b 2\pi y\sqrt{\frac{y^2+x^2}{y^2}}\,dx=\int_a^b 2\pi\sqrt{y^2+x^2}\,dx$ ✓

$¿\int_a^b 2\pi\sqrt{r^2-x^2+x^2}\,dx=\int_a^b 2\pi r\,dx$ ✓

$¿2\pi r\left[x\right]_a^b=2\pi r\left(b-a\right)$ as required.

(c) $a=-r \wedge b=r$:

$\therefore A=2\pi r\left(r-\left(-r\right)\right)=2\pi r\times 2r=4\pi r^2$ as required ✓

20. $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}=k\,for\,k\in R^{+¿¿}$ ✓

$\therefore\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=k^2$

$\frac{d}{dt}\rightarrow 2\left(\frac{dx}{dt}\right)\frac{d^2x}{dt^2}+2\left(\frac{dy}{dt}\right)\frac{d^2y}{dt^2}+2\left(\frac{dz}{dt}\right)\frac{d^2z}{dt^2}=0$ ✓✓

$2\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\\ \frac{dz}{dt}\end{pmatrix}\cdot\begin{pmatrix}\frac{d^2x}{dt^2}\\ \frac{d^2y}{dt^2}\\ \frac{d^2z}{dt^2}\end{pmatrix}=0$ ✓

$\therefore v\left(t\right)\cdot a\left(t\right)=0\rightarrow v\left(t\right)\perp a\left(t\right)$ as required ✓ [5]