Year 12 Mathematics Specialist 2017

Test Number 3: Vectors in 3 Dimensions

Resource Free

Name: _______SOLUTIONS_______________________ **Teacher: DDA**

Marks: **29**

Time Allowed: **30 minutes**

Instructions: You **ARE NOT** permitted any notes or calculator. Show your working where appropriate remembering you must show working for questions worth more than 2 marks.

Question 1 Circle the correct answer. **[1 marks]**

M (4, −1, −7) and which of the following points are on a line parallel with the *x*-axis?

A (−4, −1, −7)

B (3, −1, 7)

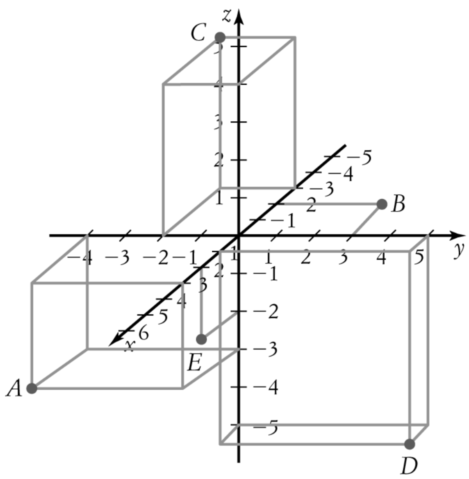
C (−4, 1, −7)

D (4, −1, 5)

E (4, 2, −7)

A line parallel with the x-axis will contain points with identical y and z coordinates.

Question 2 **[3 marks]**



With reference to the above diagram, answer the following questions.

a) Which point lies on the x-y plane? B

b) What are the coordinates of the point A? (4,-4,-3)

c) Find the exact distance from point D from the origin. $\sqrt{1^2+5^2+(-5)^2}=\sqrt{51}$

Question 3 **[2 marks]**

Find a vector equation of the straight line which passes through the points (3, 7,−2) and (4, 5, −3).

$$r=\begin{bmatrix}3\\7\\-2\end{bmatrix}+\lambda\left(\begin{bmatrix}4\\5\\-3\end{bmatrix}-\begin{bmatrix}3\\7\\-2\end{bmatrix}\right)$$

$$r=\begin{bmatrix}3\\7\\-2\end{bmatrix}+\lambda\left(\begin{bmatrix}1\\-2\\-1\end{bmatrix}\right)$$

Question 4 **[5 marks]**

A system of equations can have one unique, no or infinite solutions. The number of solutions for a system of equations having three unknown variables can be determined by understanding the following pattern occurring in a row of the augmented matrix in row echelon form.

0 0 # | #

0 0 0 | #

0 0 0 | 0

a) Identify which row indicates no solutions and show why.

0 0 0 | #

The # represents a number which is not 0. The row is equivalent to the equation: $0\,z=¿$

Dividing both sides by *z* $\Longrightarrow 0=¿$ which is not possible indicating there are no solutions.

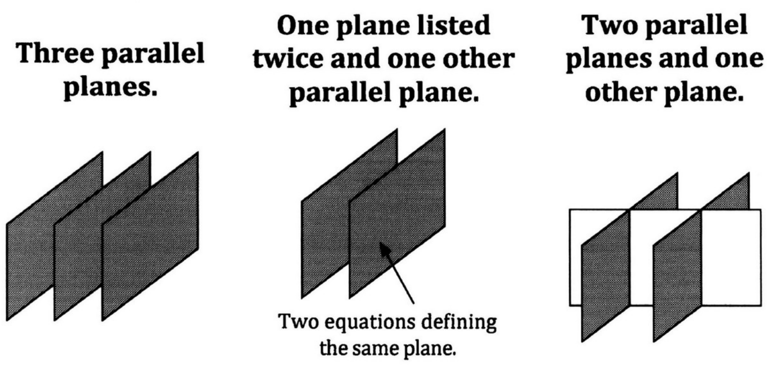
b) Identify which row indicates infinite solutions and show why.

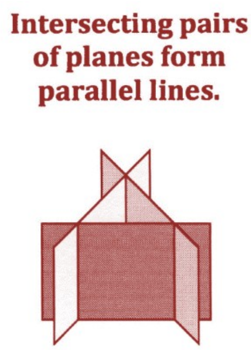
0 0 0 | 0

. The row is equivalent to the equation: $0\,z=0$

$\Longrightarrow z$ could take any real value. This indicates there are infinite possible solutions.

c) Given a system of equations with three unknown variables that has no solutions, a possible geometric explanation is that two or three of the planes represented by the equations are parallel as depicted in the images following.





What other geometric possibility could explain no solutions.

Question 5 **[6 marks]**

(a) Solve the system of equations. (3 marks)

$$x + y + z = 4$$
$$3x - y + z = 8$$
$$2x - y + z = 0$$

$x + y + z = 4 \quad \ldots (1)$

$3x - y + z = 8 \quad \ldots (2)$

$2x - y + z = 0 \quad \ldots (3)$

Solution

Consider $(2)-(3) \quad \therefore \quad x = 8$

$(1): \quad y + z = -4$

$(2): \quad -y + z = -16$

$(1)+(2): \quad 2z = -20$

$\therefore \quad z = -10$

$\therefore \quad y = 6$

Hence the solution is $x = 8$
$y = 6$
$z = -10$

$$\begin{array}{cl} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 3 & -1 & 1 & 8 \\ 2 & -1 & 1 & 0 \end{array}\right] & \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

$$\begin{array}{cl} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -4 & -2 & -4 \\ 0 & -3 & -1 & -8 \end{array}\right] & \begin{array}{l} R_1 \\ R_4 = R_2 - 3R_1 \\ R_5 = R_3 - 2R_1 \end{array} \end{array}$$

$$\begin{array}{cl} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -1 & -2 \\ 0 & -3 & -1 & -8 \end{array}\right] & \begin{array}{l} R_1 \\ R_6 = 0.5R_4 \\ R_5 \end{array} \end{array}$$

$$\begin{array}{cl} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 0.5 & -5 \end{array}\right] & \begin{array}{l} R_1 \\ R_6 \\ R_7 = R_5 - 1.5R_1 \end{array} \end{array}$$

$$0.5z = -5$$
$$z = -10$$

$$-2y - (-10) = -2$$
$$-2y = -12$$
$$y = 6$$

$$x + 6 - 10 = 4$$
$$x = 8$$

Suppose that the third equation in part (a) is changed to $2x - y + kz = 0$. The first two equations remain unchanged.

(b) Determine the value of the constant k so that the changed system of equations has no solution. (3 marks)

System is now : $x + y + z = 4 \quad \ldots (1)$

$3x - y + z = 8 \quad \ldots (2)$

$2x - y + kz = 0 \quad \ldots (3)$

Consider $(1)+(2)$: $4x + 2z = 12 \quad \ldots(4)$

$(2)-(3)$: $x + (1-k)z = 8 \quad \ldots(5)$

From (5): $x = 8 - (1-k)z$ substituting into (4): $4(8-(1-k)z) + 2z = 12$

i.e. $4kz - 2z = -20$

i.e. $z(4k-2) = -20$

Hence for there to be no solution we require $4k - 2 = 0$

i.e. $k = \frac{1}{2}$

$$\begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 3 & -1 & 1 & | & 8 \\ 2 & -1 & k & | & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 0 & -4 & -2 & | & -4 \\ 0 & -3 & k-2 & | & -8 \end{bmatrix} \begin{matrix} R_1 \\ R_4 = R_2 - 3R_1 \\ R_5 = R_3 - 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 0 & -2 & -1 & | & -2 \\ 0 & -3 & k-2 & | & -8 \end{bmatrix} \begin{matrix} R_1 \\ R_6 = 0.5R_4 \\ R_5 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 0 & -2 & -1 & | & -2 \\ 0 & 0 & k-0.5 & | & -5 \end{bmatrix} \begin{matrix} R_1 \\ R_6 \\ R_7 = R_5 - 1.5R_6 \end{matrix}$$

No solutions if $k - 0.5 = 0 \quad i.e. \ k = 0.5$

Question 6 **[7 marks]**

Points A, B have respective position vectors $\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}$.

(a) Determine the vector equation for the sphere that has $\overline{AB}$ as its diameter. (3 marks)

Centre point $C = \frac{1}{2}\left(\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

Radius $r = |\overrightarrow{AC}| = \left|\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\right| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

Equation for circle with diameter $\overline{AB}$: $\left| \underset{\sim}{r} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \right| = \sqrt{6}$

✓ determines the position vector for the centre correctly
✓ determines the radius correctly
✓ forms the vector equation for the sphere correctly

If point O is the origin, consider the plane that contains the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

(b) Determine the vector equation for this plane in the form $\underset{\sim}{r} \cdot \underset{\sim}{n} = c$. (4 marks)

Use $\underset{\sim}{n} = \overrightarrow{OA} \times \overrightarrow{OB} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0(5)-(-2)(3) \\ 0(3)-4(5) \\ 4(-2)-0(0) \end{pmatrix} = \begin{pmatrix} 6 \\ -20 \\ -8 \end{pmatrix}$ or $\begin{pmatrix} 3k \\ -10k \\ -4k \end{pmatrix}$

Since $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in$ plane, then $c = 0$ i.e. equation of plane is $\underset{\sim}{r} \cdot \begin{pmatrix} 6 \\ -20 \\ -8 \end{pmatrix} = 0$

✓ uses the idea of the cross product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ to determine the normal
✓ determines the cross product correctly
✓ states that the constant $c = 0$
✓ forms the vector equation for the plane correctly

Question 7 **[3 marks]**

Given $a = ¿ 1, -1, 1 > ¿$ and $a \times b = ¿ 1,4,3 > ¿$, find a possible vector b.

Let $b = ¿ a, b, c > ¿$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -c-b \\ a-c \\ b+a \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

(1) ..... $-c = 1 + b$

(2) ..... $-c = 4 - a$

(3) ..... $a + b = 3$

$$1 + b = 4 - a$$
$$b = 3 - a$$

$$a + 3 - a = 3$$
$$3 = 3$$

i.e. a has infinite solutions

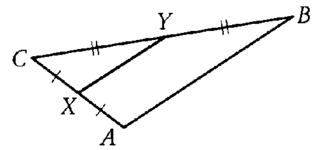
Let $a = 1, b = 2, -c = 3, c = -3$ $\quad \therefore b = ¿ 1,2, -3 > ¿$

Check: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 1+3 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

Question 8 **[2 marks]**

In ΔABC shown, X is the midpoint of CA and Y is the midpoint of CB.

Prove that $\mathbf{XY} = \frac{1}{2}\mathbf{AB}$



$$\overrightarrow{CA} = 2\overrightarrow{CX}, \overrightarrow{CB} = 2\overrightarrow{CY}$$

$$\overrightarrow{XY} = -\overrightarrow{CX} + \overrightarrow{CY}$$

$$\overrightarrow{AB} = -\overrightarrow{CA} + \overrightarrow{CB} = -2\overrightarrow{CX} + 2\overrightarrow{CY} = 2\left(-\overrightarrow{CX} + \overrightarrow{CY}\right) = 2\overrightarrow{XY}$$

$$\therefore \overrightarrow{XY} = \frac{1}{2}\overrightarrow{AB}$$

Year 12 Mathematics Specialist 2017

Test Number 3: Vectors in 3 Dimensions

Resource Rich

Name: ____________SOLUTIONS______________________________ **Teacher: DDA**

Marks: **17**

Time Allowed: **15 minutes**

Instructions: You are permitted 1 A4 pages of notes and your calculators. Show your working where appropriate remembering you must show working for questions worth more than 2 marks.

Question 9 **[3 mark]**

a) Find the magnitude of the vector product of vectors of magnitudes 15 and 9 if the angle between them is 75°.

≈ 130.4

$15\times9\times\sin(75)$

130.3999865

b) Describe the direction of the vector product of the vectors in a).

It's direction is perpendicular to both the original vectors (or perpendicular to the plane containing the original vectors)

c) Find the cross product of **u** = (3, −1, 5) and **v** = (−4, 2, −3).

$$\text{crossP}\left(\begin{bmatrix}3\\-1\\5\end{bmatrix},\begin{bmatrix}-4\\2\\-3\end{bmatrix}\right)$$

$$\begin{bmatrix}-7\\-11\\2\end{bmatrix}$$

Question 10 **[8 mark]**

An object moves such that its position vector r m, at time t s, is such that the velocity vector, $\dot{r}$ m/s, is given by $\dot{r} = \begin{bmatrix} 4\cos 2t \\ 3 \end{bmatrix} (t > 0)$.

a) When $t = 0$ the object has position vector vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix} m$, with respect to the origin, O. Find the position vector of the object when $t = \pi$. [3 marks]

$\underset{\sim}{r}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\, dt$

$\underset{\sim}{r}(t) = \begin{pmatrix} 2\sin 2t \\ 3t \end{pmatrix} + \underset{\sim}{c}$

$\therefore \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \underset{\sim}{c}$

$\therefore \underset{\sim}{r}(t) = \begin{pmatrix} 2\sin 2t + 2 \\ 3t - 1 \end{pmatrix} m$

$\therefore \underset{\sim}{r}(\pi) = \begin{pmatrix} 2(0) + 2 \\ 3\pi - 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3\pi - 1 \end{pmatrix} m$

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∫([4cos(2x) 3]
                 [2·sin(2·x) 3·x]
[2·sin(2·x)+2 3·x−1]|x=π
                        [2 3·π−1]
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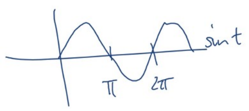
b) Find the speed and position vector of the object the first time, $t > 0$, for which the velocity of the object is perpendicular to the acceleration of the object. [5 marks]

$\underset{\sim}{a}(t) = \dot{\underset{\sim}{v}}(t) = \begin{pmatrix} -8\sin 2t \\ 0 \end{pmatrix} m/s^2$

$\underset{\sim}{v}(t) \perp$ to $\underset{\sim}{a}(t) \Rightarrow \begin{pmatrix} 4\cos 2t \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -8\sin 2t \\ 0 \end{pmatrix} = 0$

$-32\cos 2t \sin 2t = 0$

$\sin 4t = 0$



$\therefore 4t = \pi$

$t = \frac{\pi}{4}$ s

$\underset{\sim}{r}(\frac{\pi}{4}) = \begin{pmatrix} 2\sin \frac{\pi}{2} + 2 \\ 3\frac{\pi}{4} - 1 \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{3\pi}{4} - 1 \end{pmatrix} m$

Speed $= \left|\underset{\sim}{v}(\frac{\pi}{4})\right| = \left|\begin{pmatrix} 4\cos \frac{\pi}{2} \\ 3 \end{pmatrix}\right| = \left|\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right| = 3 m/s$

```
diff([4cos(2x) 3]
                  [−8·sin(2·x) 0]
dotP([−8·sin(2·x) 0],[4cos(▸
                 −32·cos(2·x)·sin(2·x)
solve(
                  {x=π·constn(1)/4}
[2·sin(2·x)+2 3·x−1]|x=π/4
                        [4 3·π/4−1]
[4cos(2x) 3]|x=π/4
                                [0 3]
norm(
                                    3
```

Question 11 **[4 marks]**

A disc with a radius of 12 cm turns with an angular velocity of 3 radians per second.

The position vector of a particle on the edge of this disc at time t seconds is $r(t)$ m where

$$r(t)=0.12\cos(3t)i+0.12\sin(3t)$$

Show that $a(t)=-kr(t)$ for k a positive scalar constant and determine its value.

What does the result $a(t)=-kr(t)$ mean in terms of the direction of a?

$$v(t)=\dot{r}(t)=\begin{bmatrix}-0.36\sin(3t)\\0.36\cos(3t)\end{bmatrix}$$

$$a(t)=\dot{v}(t)=\begin{bmatrix}-1.08\cos(3t)\\-1.08\sin(3t)\end{bmatrix}=-9\begin{bmatrix}0.12\cos(3t)\\0.12\sin(3t)\end{bmatrix}$$

$$k=9$$

$a(t)$ is in the opposite direction of $r(t)$.

Therefore $a(t)$ is always directed towards (0,0), the centre of the circle.

Question 12 **[2 marks]**

An object travels a distance of 35 m in 2.7 s as it moves around a circle of radius 2 m. Find, correct to 2 decimal places, the angular velocity, ω, and speed, $|v(t)|$, over that time period.

$$|v(t)| = \frac{35}{2.7} = \frac{350}{27} \approx 12.96\ m/s$$

$$|v(t)| = r\omega$$

$$\therefore \frac{350}{27} = 2\omega$$

$$\therefore \omega \approx 6.48\ rad/sec$$

Question 11 **[7 marks]**