SEMESTER TWO

MATHEMATICS
SPECIALIST
UNITS 3 & 4

2018

SOLUTIONS

Calculator-free Solutions

1. (a) $a = 3$ *and* $b \neq 2$. (The two planes are parallel.) ✓ (1)

(b)

$$\begin{array}{c c} \begin{bmatrix} 1 & 2 & 3 & -2 \\ -1 & -2 & 3 & -4 \\ 1 & 2 & -1 & -2 \end{bmatrix} & \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

$$\begin{array}{c c} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 6 & -6 \\ 0 & 0 & 4 & 0 \end{bmatrix} & \begin{array}{l} \\ R_1 + R_2 \quad ✓ \\ R_1 - R_3 \quad ✓ \end{array} \end{array}$$

$z = 0$

$y + 6z = -6 \quad y = -6$

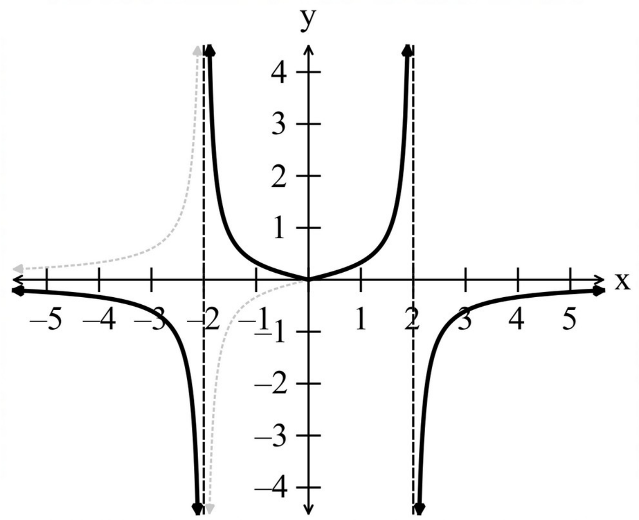
$x + 2y + 3z = -2$

$x - 12 + 0 = -2 \quad x = 10$ ✓

The point of intersection is (10,-6,0) ✓

(4)

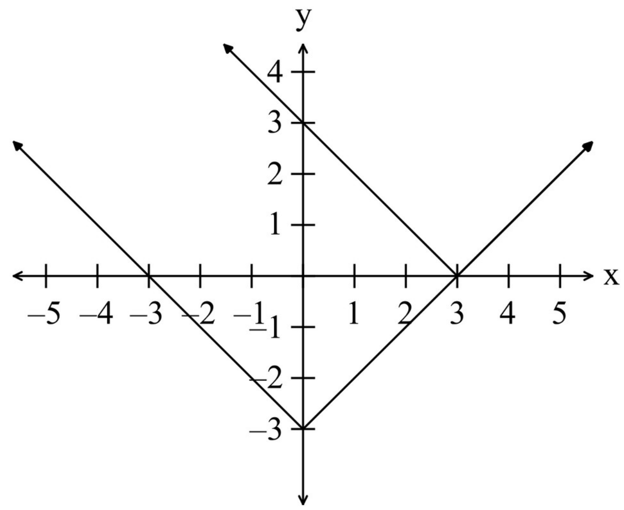
2. (a)



✓✓ -1/error

(2)

(b)



✓✓

i.e. $x \geq 3$ ✓ (3)

OR algebraically

2. (b)

$|x-3| = |x| - 3$

$|x-3| = \begin{cases} x-3 & for \ \ x \geq 3 \\ -x+3 & for \ \ x < 3 \end{cases}$

$|x| - 3 = \begin{cases} x-3 & for \ \ x \geq 0 \\ -x-3 & for \ \ x < 0 \end{cases}$

For $x < 0$	$0 < x \leq 3$	$x > 3$
$-x+3 = -x-3$	$-x+3 = x-3$	$x-3 = x-3$
$3 = -3$	$2x = 6$	$x = x$
	$x = 3$	Valid for all $x > 3$
no solution	$x = 3$	

$\therefore \ \ x \geq 3$

3. (a) *(i)* $f(x) = ln(x)$ and $g(x) = x^2 - 1$

$y = f(g(x))$

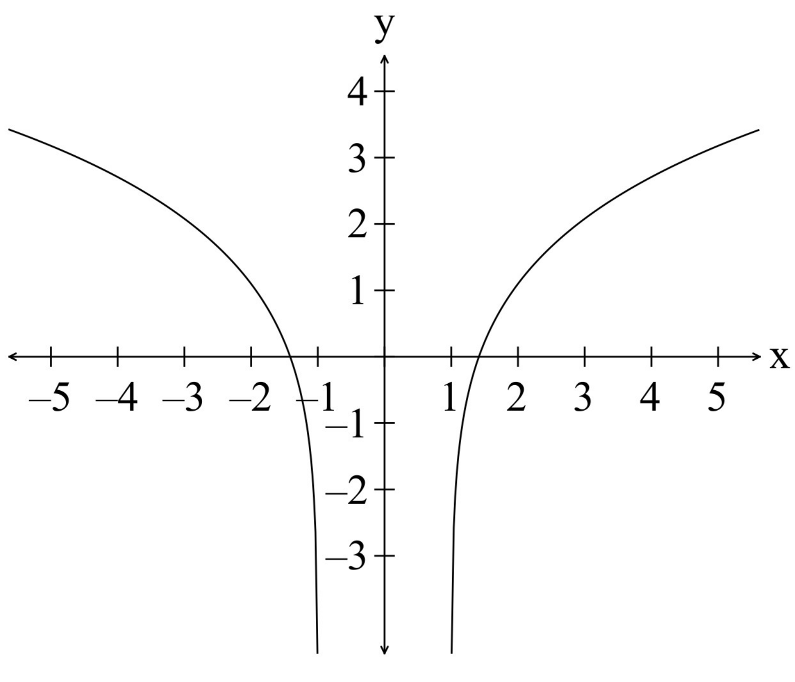
$y = f(x^2 - 1)$

$y = ln(x^2 - 1)$ which is defined for $x > 1$ or $x < -1$.

✓ ✓ ✓

(3)

(ii)



✓

Not one to one as $f(g(a)) = f(g(-a)) \ \ for \ \ |a| > 1$, i,e, $f(g(a))$ is not unique. ✓

A one to one function has a unique *y* value for very *x* value and vice versa.

(2)

(b) Let $y = f(g(x)) = ln(x^2 - 1)$

To obtain the inverse:

$x = ln(y^2 - 1)$ ✓

$y^2 - 1 = e^x$

$y^2 = e^x + 1$

$y = \pm\sqrt{e^x + 1}$

For $x > 1$, $y > \sqrt{1 + e}$ and for $x < -1$, $y > \sqrt{1 + \frac{1}{e}}$

$\therefore (f(g(x)))^{-1} = \sqrt{e^x + 1}$ for $x > 1$ OR $(f(g(x)))^{-1} = -\sqrt{e^x + 1}$ for $x < -1$ ✓✓

Accept either solution. (3)

4. (a) $2z^2 + bz + 2 = 0$

$\Delta = b^2 - 4ac$

$\Delta = b^2 - 16$

For complex roots, $\Delta < 0$

$b^2 - 16 < 0$

$-4 < b < 4$ ✓

(1)

(b) $z^4 + 2z^3 + z^2 + 8z - 12 = 0.$

Let $P(z) = z^4 + 2z^3 + z^2 + 8z - 12.$

$P(1) = 1 + 2 + 1 + 8 - 12 = 0$ ✓

$P(-3) = 81 - 54 + 9 - 24 - 12 = 0$ ✓

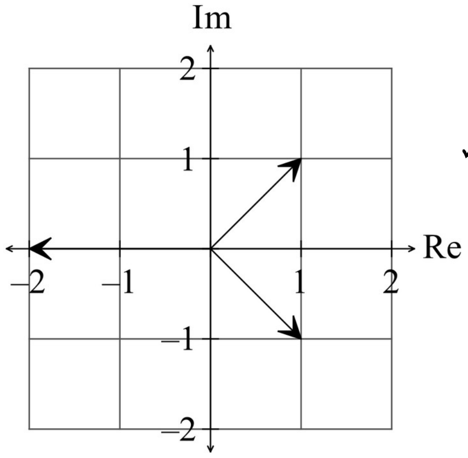
$$\begin{array}{r|rrrrr}
1 & 1 & 2 & 1 & 8 & -12 \\
 & \downarrow & 1 & 3 & 4 & 12 \\
\hline
-3 & 1 & 3 & 4 & 12 & 0 \\
 & \downarrow & -3 & 0 & -12 & \\
\hline
 & 1 & 0 & 4 & 0 &
\end{array}$$ ✓

$\therefore\ z = 1, z = -3$ or $z^2 + 4 = 0$

$\therefore\ z = 1, z = -3$ or $z = \pm 2i$ ✓

(4)

(c) (i)



✓

(1)

(ii) $z=1-i,\ z=1+i,\ z=-2$

$\therefore (z-1+i)(z-1-i)(z+2)=0$ ✓

The cubic equation is $z^3-2z+4=0$ ✓

(2)

(d)

$$\frac{\left(3cis\left(\frac{3\pi}{2}\right)\right)^2\times\sqrt{2cis\left(\frac{2\pi}{3}\right)}}{\sqrt{6}\,cis\left(\frac{7\pi}{6}\right)}$$

$$=\frac{9cis(3\pi)\times\sqrt{2}cis\left(\frac{\pi}{3}\right)}{\sqrt{6}\,cis\left(\frac{7\pi}{6}\right)}$$

$$=\frac{9(-1)cis\left(\frac{\pi}{3}-\frac{7\pi}{6}\right)}{\sqrt{3}}$$

$$=-3\sqrt{3}cis\left(-\frac{5\pi}{6}\right)$$

✓ ✓

$$=-3\sqrt{3}\left(cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right)$$

$$=-3\sqrt{3}\left(-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)$$

$$=\frac{9}{2}+i\frac{3\sqrt{3}}{2}$$

✓ ✓

$$\frac{\left(3cis\left(\frac{3\pi}{2}\right)\right)^2 \times \sqrt{2cis\left(\frac{2\pi}{3}\right)}}{\sqrt{6}\,cis\left(\frac{7\pi}{6}\right)}$$

OR

$$=\frac{9cis(3\pi)\times\sqrt{2}\,cis\left(\frac{\pi}{3}\right)}{\sqrt{6}\,cis\left(\frac{7\pi}{6}\right)}$$

$$=3\sqrt{3}\,cis\left(\frac{13\pi}{6}\right)$$

$$=3\sqrt{3}\,cis\left(\frac{\pi}{6}\right)$$ ✓ ✓

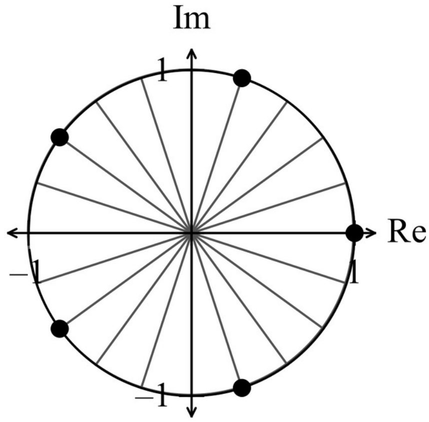
$$=3\sqrt{3}\cos\left(\frac{\pi}{6}\right)+i3\sqrt{3}\sin\left(\frac{\pi}{6}\right)$$

$$=3\sqrt{3}\times\frac{\sqrt{3}}{2}+i3\sqrt{3}\times\frac{1}{2}$$

$$=\frac{9}{2}+i\frac{3\sqrt{3}}{2}$$

✓ ✓ (4)

5. (a)



✓

(1)

(b) $z^5 = 1$

$z^5 = cis\,(0 + 2\pi)$

$z = (cis\,(0 + 2n\pi))^{\frac{1}{5}}$

$z = cis\left(\frac{2n\pi}{5}\right)$ ✓

$n = 0,\ z = cis(0)$

$n = 1,\ z = cis\left(\frac{2\pi}{5}\right)$

$n = 2,\ z = cis\left(\frac{4\pi}{5}\right)$

$n = 3,\ z = cis\left(\frac{6\pi}{5}\right) = cis\left(\frac{-4\pi}{5}\right)$

$n = 4,\ z = cis\left(\frac{8\pi}{5}\right) = cis\left(\frac{-2\pi}{5}\right)$ ✓✓ -1/error

(3)

6. (a) $x\sin(y) = 1$

$1 \times \sin(y) + x \times \cos(y) \times \frac{dy}{dx} = 0$ ✓✓ -1/error

$\frac{dy}{dx} = -\frac{\sin(y)}{x\cos(y)}$

$\frac{dy}{dx} = -\frac{\tan(y)}{x}$ ✓

(3)

(b) $\frac{dy}{dx}$ is not defined where $\tan(y)$ is not defined, i.e. at $y = \frac{\pi}{2}$

If $y = \frac{\pi}{2},\ x\sin\left(\frac{\pi}{2}\right) = 1 \Rightarrow x = 1$

The point is $\left(1, \frac{\pi}{2}\right)$. ✓

(1)

7. (a) (i) $\int_1^e \frac{(\ln(x))^2\,dx}{x}$

put $u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$ ✓

$du = \frac{dx}{x}$

If $x = e,\ u = 1$

If $x = 1,\ u = 0$ ✓

$= \int_0^1 u^2\,du$

$= \left[\frac{u^3}{3}\right]_0^1$

$= \frac{1}{3}$ ✓

(3)

(ii) $\int_0^{\frac{\pi}{12}} \sin^3(3x)\cos(3x)\,dx$

put $u = \sin(3x)$

$\frac{du}{dx} = 3\cos(3x)$

$\frac{du}{3} = \cos(3x)\,dx$ ✓

If $x = 0,\ u = 0$

If $x = \frac{\pi}{12},\ u = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ ✓

$= \int_0^{\frac{1}{\sqrt{2}}} \frac{u^3}{3}\,du$ ✓

$= \left[\frac{u^4}{12}\right]_0^{\frac{1}{\sqrt{2}}}$

$= \frac{1}{12}\left(\frac{1}{4} - 0\right)$

$= \frac{1}{48}$ ✓

(4)

(b)

$$\int \frac{1+\tan(x)}{1-\tan(x)}dx = \int \frac{\tan\left(\frac{\pi}{4}\right)+\tan(x)}{1-\tan\left(\frac{\pi}{4}\right)\tan(x)}dx$$

$$= \int \tan\left(\frac{\pi}{4}+x\right)dx \quad ✓$$

$$= \int \frac{\sin\left(\frac{\pi}{4}+x\right)}{\cos\left(\frac{\pi}{4}+x\right)}dx$$

$$\text{put } u = \cos\left(\frac{\pi}{4}+x\right) \quad ✓$$

$$\frac{du}{dx} = -\sin\left(\frac{\pi}{4}+x\right)$$

$$-du = \sin\left(\frac{\pi}{4}+x\right)dx \quad ✓$$

$$= \int -\frac{du}{u} \quad ✓$$

$$= -\ln(u)+c$$

$$= -\ln\left|\cos\left(\frac{\pi}{4}+x\right)\right|+c \quad ✓$$

(5)

End of Section One solutions.

Calculator-assumed Solutions

8. (a) $\frac{dy}{dx} = ax^2$ ✓ ✓ (2)

(b) $\frac{dy}{dx} = ax^2$

$\int dy = \int (ax^2)\,dx$ ✓

$y = \frac{ax^3}{3} + c$

Using $P(0,-1)$,

$-1 = 0 + c$

Using $P(1,0)$,

$0 = \frac{a}{3} - 1$

$a = 3$ ✓

$\therefore y = x^3 - 1$ ✓

(3)

9. (a) $|z - 2i| \le 4 \cap \frac{\pi}{3} < Arg(z) < \frac{2\pi}{3} \cap \text{Im}(z) \ge 2$ (4)

✓ ✓ ✓ ✓ correct inequalities

(b) $\frac{3+4i}{2+ai} = \frac{3+4i}{2+ai} \times \frac{2-ai}{2-ai}$

$= \frac{6+8i-3ai-4ai^2}{4+a^2}$

$= \frac{6+4a}{4+a^2} + i\left(\frac{8-3a}{4+a^2}\right)$ ✓

To be wholly real, $8 - 3a = 0$, $a = \frac{8}{3}$ ✓ (2)

10. (a) $\mathbf{r}(t) = 5cos(t)\mathbf{i} + 3sin(t)\mathbf{j}$

$x = 5cos(t)$ and $y = 3sin(t)$ ✓

$cos^2(t) + sin^2(t) = 1$

$\therefore \left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ ✓

(2)

(b) $\mathbf{r}(t) = 5cos(t)\mathbf{i} + 3sin(t)\mathbf{j}$

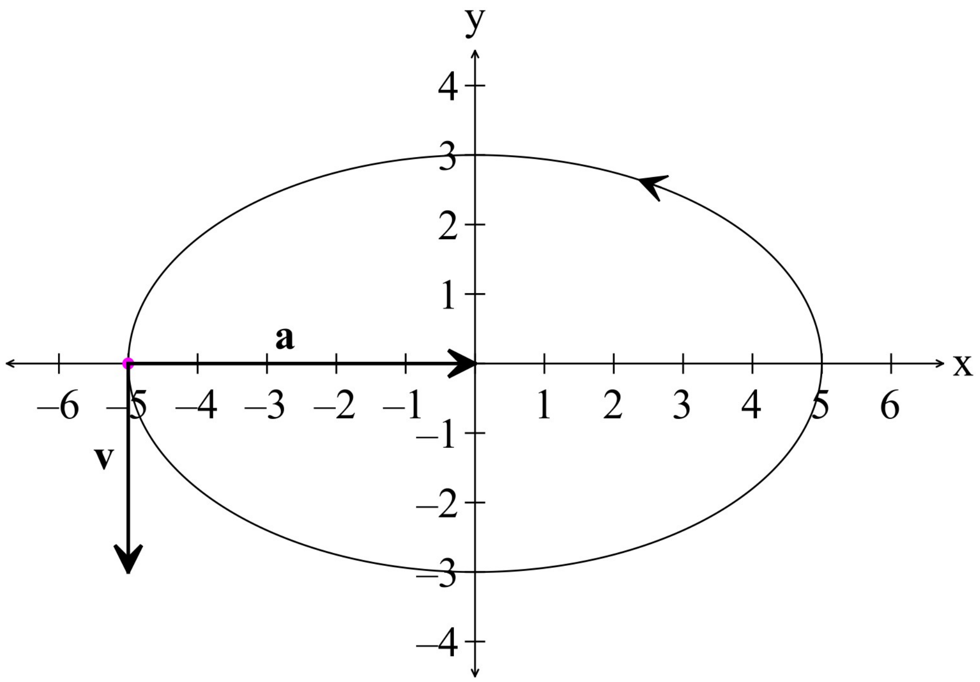
$\mathbf{v}(t) = -5sin(t)\mathbf{i} + 3cos(t)\mathbf{j}$ ✓

$\mathbf{a}(t) = -5cos(t)\mathbf{i} - 3sin(t)\mathbf{j}$ ✓

(2)

(c) $\mathbf{r}(\pi)=\begin{pmatrix}-5\\0\end{pmatrix}$, $\mathbf{v}(\pi)=\begin{pmatrix}0\\-3\end{pmatrix}$, $\mathbf{a}(t)=\begin{pmatrix}5\\0\end{pmatrix}$

✓



✓✓ two vectors

(3)

(d) $\mathbf{r}(t)\bullet\mathbf{v}(t)=0$

i.e. $\begin{pmatrix}5cos(t)\\3sin(t)\end{pmatrix}\bullet\begin{pmatrix}-5sin(t)\\3cos(t)\end{pmatrix}=0$ ✓

$-25cos(t)5sin(t)+9cos(t)5sin(t)=0$

$-16cos(t)5sin(t)=0$

$-8sin(2t)=0$ ✓

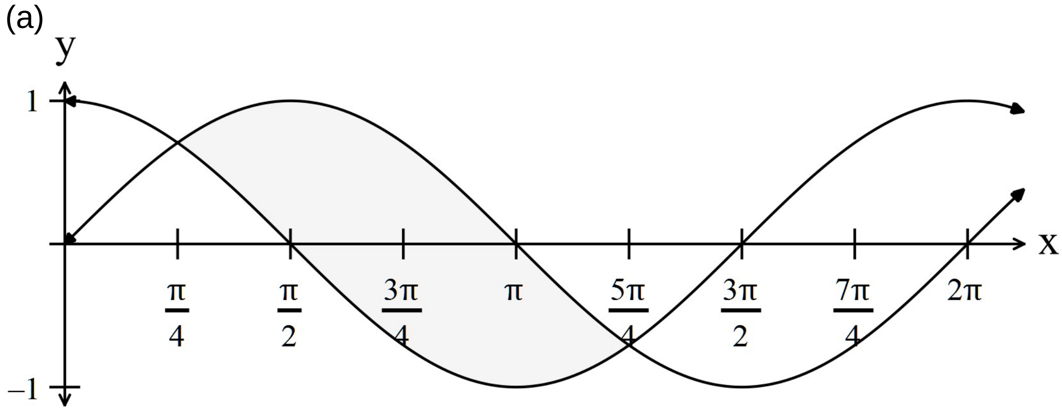
$2t=0,\pi,2\pi,3\pi.....$

$t=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}.....$

✓

The next time the direction of travel is perpendicular to the velocity is at $t=\frac{3\pi}{2}$. (3)

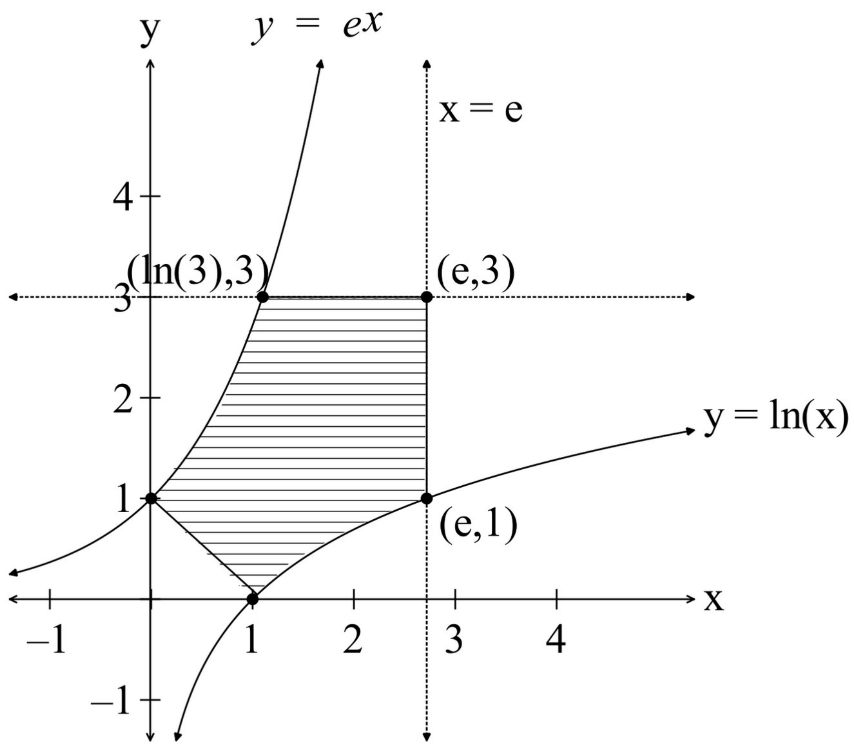
11. (a)



✓ ✓ ✓

Area = $\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin(x) - \cos(x))dx = 2.83$ units^2 (3)

(b)



$$\int_0^{\ln(3)} (e^x)dx + \int_{\ln(3)}^{e} (3)dx - \int_0^1 (1 - x)dx - \int_1^{e} \ln(x)dx$$

✓ ✓ ✓ ✓ (4)

12. (a)

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{\delta V}{\delta r} \approx \frac{dV}{dr}$ ✓

$\delta V \approx \frac{dV}{dr} \times \delta r = 4\pi r^2 \times \delta r$ ✓

$\delta r = 0.1\text{cm}$ and $r = 50\text{cm}$

$\delta V \approx 4\pi \times 50^2 \times 0.1$

$\delta V \approx 1000\pi\,\text{cm}^3 = 3141.59\,\text{cm}^3$ ✓

(3)

(b) (i) Show that $V = \pi\left(50h^2 - \frac{h^3}{3}\right)$.

$$x^2+(y-50)^2=50^2 \quad \checkmark$$

$$V=\int_0^h \pi \times x^2 dy$$

$$V=\pi\int_0^h\left(2500-(y-50)^2\right)dy \quad \checkmark$$

$$V=\pi\left(-\frac{h^3}{3}+50h^2\right) \quad \checkmark$$

$$V=\pi\left(50h^2-\frac{h^3}{3}\right)$$

(3)

(ii) $V=\pi\left(50h^2-\frac{h^3}{3}\right)$, $\frac{dV}{dt}=25\,cm^3s^{-1}$, $\frac{dh}{dt}=?$

$$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$$

$$\frac{dV}{dt}=\pi\left(100h-h^2\right)\frac{dh}{dt}$$

$$\frac{dh}{dt}=\frac{\frac{dV}{dt}}{\pi\left(100h-h^2\right)}$$

Need h at $t=100$s, i.e. when $V=25\times100=2500$

$$V=\pi\left(50h^2-\frac{h^3}{3}\right)$$

$$2500=\pi\left(50h^2-\frac{h^3}{3}\right)$$

$h=-3.938$ or $h=4.044317015$ or $h=149.894$

But $0<h<50$, so $h=4.044317015$ ✓

$$\therefore \frac{dh}{dt}=\frac{25}{\pi\left(100\times4.044317015-4.044317015^2\right)} \quad \checkmark$$

$$\frac{dh}{dt}=0.02\text{ cm s}^{-1} \quad \checkmark$$

(4)

13. (a) Let $x=A\sin(nt)$

$$f=\frac{n}{2\pi}$$

$$2=\frac{n}{2\pi}$$

$$n=4\pi$$

$\therefore\quad x=10\sin(4\pi t)$ ✓

$v=4\pi\times10\cos(4\pi t)$ ✓

$v=40\pi\cos(4\pi t)$

The maximum speed is $40\pi\,\text{cms}^{-1}$. ✓

This occurs when $\cos(4\pi t)=1$, so $\sin(4\pi t)=0$,
i.e. the displacement from the origin is 0 cm. ✓

(4)

(b) (i) If $v = 0, t = ?$

$0 = 15t(4 - t)$

$t = 0$ or $t = 4$

It takes 4 hours to travel from P to Q. ✓

$x = \int_0^4 (60t - 15t^2)dt = 160\,\text{km}$ ✓

(2)

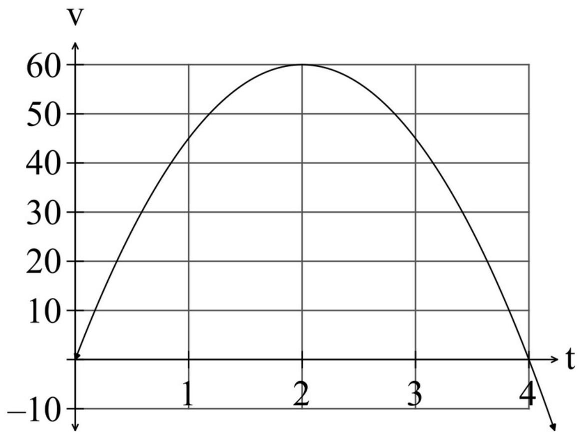
(ii) $v = 60t - 15t^2$

$a = 60 - 30t$

At $t = 0, a = 60\ km\ hour^{-2}$. ✓

(1)

(iii) $v = 60t - 15t^2 = 15t(4 - t)$



The maximum velocity occurs after two hours.
The maximum speed is 60 km hour^{-1}. ✓ (1)

(c) $a = 3x^2\ m\,s^{-2}$

$$\frac{v^2}{2} = \int (3x^2)\,dx \quad ✓$$

$$\frac{v^2}{2} = x^3 + c$$

$At\ t = 0,\ v = -\sqrt{2}\text{ m s}^{-1}$ and $x = 1$

$$\frac{(-\sqrt{2})^2}{2} = 1^3 + c$$

$$c = 0 \quad ✓$$

$$\frac{v^2}{2} = x^3$$

$$v = \pm\sqrt{2x^3}$$

If $x = 1$ *then* $v = -\sqrt{2}\text{ m s}^{-1}$ *so* $v = -\sqrt{2x^3}$ ✓

$$\frac{dx}{dt} = -\sqrt{2}x^{\frac{3}{2}}$$

$$\frac{dt}{dx} = -\frac{1}{\sqrt{2}}x^{-\frac{3}{2}}$$

$$t = -\frac{1}{\sqrt{2}}\int x^{-\frac{3}{2}}dx \quad ✓$$

$$-\sqrt{2}t = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}$$

$$\frac{\sqrt{2}}{2}t = x^{-\frac{1}{2}}$$

$$x = \left(\frac{t}{\sqrt{2}}\right)^{-2}$$

$$x = \frac{2}{t^2}\text{ m} \quad ✓$$

(5)

14. (a) (i) $-2x + y + 3z = d$ ✓

$A(1,2,3) \Rightarrow -\cancel{2+2} + 9 = d$

$\therefore\ -2x + y + 3z = 9$ ✓

(2)

(ii) $B(-2,-1,2)\quad -2x + y + 3z = 4 - 1 + 6 = 9$ ✓

$\therefore B \in \text{plane}$

(1)

(iii) $$\mathbf{r}(t) = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}-2\\1\\3\end{pmatrix} \quad ✓$$

NB Any point can be used i.e. $\begin{pmatrix}1\\2\\3\end{pmatrix}$ can be ANY point. (1)

(b) $\left|\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}\right| = 3$ ✓ is a sphere of centre (0,1,-2) with a radius of 3

$z = 1$ is a plane

When the plane intersects with the sphere $x^2 + (y-1)^2 + (z+2)^2 = 9$, we get a circle equation $x^2 + (y-1)^2 + (1+2)^2 = 9 \Rightarrow x^2 + (y-1)^2 = 0$ which is a circle with centre $(0,1,1)$ with zero radius, i.e. the intersection is the point $(0,1,1)$. ✓ ✓ (3)

(c) (i) $\mathbf{r}(t) = \begin{pmatrix} 10 \\ 5 \\ 3 \end{pmatrix} + t\begin{pmatrix} 8 \\ -4 \\ 8 \end{pmatrix}$ ✓

On arrival at the nest"

$$x = x \quad 10 + 8t = 170$$
$$8t = 160$$
$$t = 20$$
$$y = y \quad 5 - 4t = -75$$
$$4t = 80$$
$$t = 20$$
$$z = z \quad 3 + 8t = 163$$
$$8t = 160$$
$$t = 20$$

✓ method

The eagle takes 20 seconds to reach the nest. ✓ (2)

(ii) $$\mathbf{r}(15)=\begin{pmatrix}10\\5\\3\end{pmatrix}+15\begin{pmatrix}8\\-4\\8\end{pmatrix}=\begin{pmatrix}130\\-55\\123\end{pmatrix}$$

$$\mathbf{r}(20)=\begin{pmatrix}10\\5\\3\end{pmatrix}+20\begin{pmatrix}8\\-4\\8\end{pmatrix}=\begin{pmatrix}170\\-75\\163\end{pmatrix} \checkmark$$

OR

$$\left|V_{eagle}\right|=\sqrt{8^2+4^2+8^2} \checkmark$$
$$=12\text{m/s} \checkmark$$
$$12\times 5=60\text{ seconds} \checkmark$$

$$\mathbf{r}(20)-\mathbf{r}(15)=\begin{pmatrix}170\\-75\\163\end{pmatrix}-\begin{pmatrix}130\\-55\\123\end{pmatrix}=\begin{pmatrix}40\\-20\\40\end{pmatrix} \checkmark$$

$$\left|\mathbf{r}(20)-\mathbf{r}(15)\right|=\sqrt{40^2+(-20)^2+40^2}$$
$$=60\text{m}$$

The eagle is 60 m from the nest. ✓ (3)

(iii) Assume $t=0$ at this time.

$$\mathbf{r}_e(t)=\begin{pmatrix}130\\-55\\123\end{pmatrix}+t\begin{pmatrix}10\\-5\\2\end{pmatrix} \qquad \mathbf{r}_c(t)=\begin{pmatrix}170\\-75\\163\end{pmatrix}+t\begin{pmatrix}4\\-2\\-4\end{pmatrix}$$

$$\mathbf{r}_e(t)=\begin{pmatrix}130+10t\\-55-5t\\123+2t\end{pmatrix} \qquad \mathbf{r}_c(t)=\begin{pmatrix}170+4t\\-75-2t\\163-4t\end{pmatrix}$$

At the point of intersection,

$x=x$	$y=y$	$z=z$	
$130+10t=170+4t$	$-55-5t=-75-2t$	$123+2t=163-4t$	✓✓
$6t=40$	$20=3t$	$40=6t$	
$t=\frac{20}{3}$	$t=\frac{20}{3}$	$t=\frac{20}{3}$	✓

Therefore it takes $6\frac{2}{3}$ seconds for the eagle to catch the crow.
To determine the position vector:

$$\mathbf{r}_e\left(\frac{20}{3}\right)=\begin{pmatrix}130\\-55\\123\end{pmatrix}+\frac{20}{3}\begin{pmatrix}10\\-5\\2\end{pmatrix} \checkmark$$

$$\mathbf{r}_e\left(\frac{20}{3}\right)=\begin{pmatrix}196\frac{2}{3}\\-88\frac{1}{3}\\136\frac{1}{3}\end{pmatrix} \checkmark$$

(5)

15. (a) (i)

$$z+\frac{1}{z}=\cos(x)+\cancel{i\sin(x)}+\cos(x)-\cancel{i\sin(x)}$$
$$z+\frac{1}{z}=2\cos(x)$$
✓ (1)

(ii)

$$z-\frac{1}{z}=\cancel{\cos(x)}+i\sin(x)-\left(\cancel{\cos(x)}-i\sin(x)\right)$$
$$z-\frac{1}{z}=2i\sin(x)$$
✓ (1)

(iii)

$$z^n=\left(cis(x)\right)^n=cis(nx)=\cos(nx)+i\sin(nx)$$
$$\frac{1}{z^n}=\frac{1}{\left(cis(nx)\right)}=cis(-nx)=\cos(nx)-i\sin(nx)$$
$$z^n+\frac{1}{z^n}=\cos(nx)+\cancel{i\sin(nx)}+\cos(nx)-\cancel{i\sin(nx)}$$
$$z^n+\frac{1}{z^n}=2\cos(nx)$$
✓ (1)

(iv)

$$z^n-\frac{1}{z^n}=\cancel{\cos(nx)}+i\sin(nx)-\left(\cancel{\cos(nx)}-i\sin(nx)\right)$$
$$z^n-\frac{1}{z^n}=2i\sin(nx)$$
✓ (1)

(b) Show that $8\sin^4(\theta)=\cos(4\theta)-4\cos(2\theta)+3$

$$8\sin^4(\theta)=8\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^4$$
✓

$$=\frac{8}{16i^4}\left(z^4+4z^3\left(-\frac{1}{z}\right)+6z^2\left(-\frac{1}{z}\right)^2+4z\left(-\frac{1}{z}\right)^3+\left(-\frac{1}{z}\right)^4\right)$$
✓

$$=\frac{1}{2}\left(z^4-4z^2+6-\frac{4}{z^2}+\frac{1}{z^4}\right)$$
$$=\frac{1}{2}\left(\left(z^4+\frac{1}{z^4}\right)-4\left(z^2+\frac{1}{z^2}\right)+6\right)$$
✓

$$=\frac{1}{2}\left(2\cos(4\theta)-4\left(2\cos(2\theta)\right)+6\right)$$
✓

$$\therefore\ 8\sin^4(\theta)=\cos(4\theta)-4\cos(2\theta)+3$$

(4)

16. (a)

$$\frac{dN}{dt} = \frac{0.01N}{1000}(1000 - N)$$

so $\dfrac{dt}{dN} = \dfrac{1000}{0.01}\left(\dfrac{1}{N(1000-N)}\right)$

$$0.00001t = \int\left(\frac{1}{N(1000-N)}\right)dN \quad ✓$$

$$\frac{1}{N(1000-N)} = \frac{a}{N} + \frac{b}{1000-N}$$

$$= \frac{a(1000-N)+bN}{N(1000-N)}$$

$$\frac{0 \times N + 1}{N(1000-N)} = \frac{N(b-a)+1000a}{N(1000-N)}$$

Equating coefficients

$0 = b - a$ and $1 = 1000a$

$a = b$ and $a = 0.001 = b$ ✓

$$0.00001t = \int\left(\frac{1}{N(1000-N)}\right)dN = \int\left(\frac{a}{N} + \frac{b}{1000-N}\right)dN$$

$$0.00001t = \int\left(\frac{0.001}{N}\right)dN + \int\left(\frac{0.001}{1000-N}\right)dN$$

$$\frac{0.00001}{0.001}t = \ln(N) - \ln(1000-N) + c$$

$$0.01t = \ln(N) - \ln(1000-N) + c \quad ✓$$

$$0.01t - c = \ln\left(\frac{N}{1000-N}\right)$$

$$\frac{N}{1000-N} = e^{0.01t-c}$$

$$\frac{1000-N}{N} = Ae^{-0.01t} \text{ where } A = e^{c} \quad ✓$$

At $t = 0$, $N = 50$ and $\dfrac{1000-50}{50} = A \times e^{0} = A$

$\therefore A = 19$

$$\frac{1000-N}{N} = 19e^{-0.01t}$$

$$\Rightarrow 1000 - N = N \times 19e^{-0.01t}$$

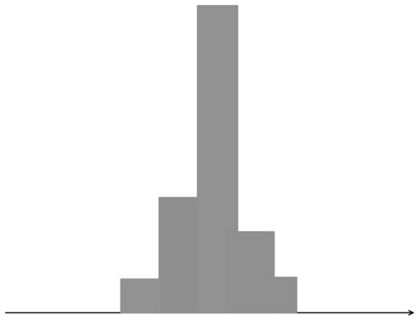
$$N(1+19e^{-0.01t}) = 1000 \quad ✓$$

$$N = \frac{1000}{(1+19e^{-0.10t})}$$

(5)

(b) If $t = 20$ then $N(20) = \dfrac{1000}{(1+19e^{-0.10\times 20})} = 60.4014851377 \approx 60$ ✓ ✓ (2)

17. (a)

(i)  ✓ (1)

(ii) The samples will each have a mean not too far from the mean of the population.
They will cluster about the population mean hence a tightly clustered (but not necessarily completely symmetrical) histogram is required. ✓ (1)

(b) (i) $\bar{x} = 41$ grams, $sd = \frac{\sigma}{\sqrt{n}} = \frac{1}{10}$ ✓ ✓ (2)

(ii) $P\left(\bar{x} < 40\right) = 7.62 \times 10^{-24}$ ✓ ✓ (2)

(iii) $\bar{x} = 41.8$ grams, $sd = \frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$ ✓

$\bar{x} \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$

$41.8 \pm 1.96 \times \frac{1}{3}$

$41.147 \le \mu \le 42.453$ ✓

$41.147 > 41$

Yes, there is a significant difference at the 95% level so the machine needs adjusting. ✓

(3)

(c) $\bar{x} = 4.8$ years, $\sigma = 1.5$ years, $n = 100$

$\bar{x} - k \times \frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + k \times \frac{\sigma}{\sqrt{n}}$

$\frac{\sigma}{\sqrt{n}} = \frac{1.5}{\sqrt{100}} = \frac{1.5}{10} = 0.15; \quad k = 1.96$ ✓

$4.8 - 1.96 \times 0.15 \le \mu \le 4.8 + 1.96 \times 0.15$

✓ ✓

$4.506 \le \mu \le 5.094$

4 years 6 months $\le \mu \le$ 5 years 1 month

(3)

(d) $\bar{x} - k \times \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + k \times \frac{\sigma}{\sqrt{n}}$ ✓

$k = 2.576$ for 99% confidence limits ✓

$k \times \frac{\sigma}{\sqrt{n}} = 8$

$2.576 \times \frac{10}{\sqrt{n}} = 8$ ✓

$\sqrt{n} = \frac{2.576 \times 10}{8}$

$n = 10.3684$ ✓

✓

You need a sample size of at least 11 to be 99% confident that the mean of the sample s within 8 grams of the population mean. (5)

End of Section Two