PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Test 2 Year 12

Student name: ______________ Teacher name: ______________

Task type:	**Response/Investigation**
Reading time for this test :	**5 mins**
Working time allowed for this task:	**40 mins**
Number of questions:	**7**
Materials required:	Upto 3 classpads/calculators
Standard items:	Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters
Special items:	Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations
Marks available:	**41 marks**
Task weighting:	**13%**
Formula sheet provided:	**no but formulae stated on page 2**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Useful formulae

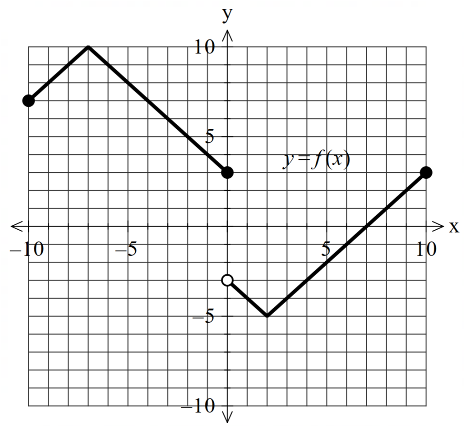
Complex numbers

Cartesian form	
$z = a + bi$	$\bar{z} = a - bi$
$\text{Mod}(z) = \|z\| = \sqrt{a^2 + b^2} = r$	$\text{Arg}(z) = \theta, \quad \tan\theta = \frac{b}{a}, \quad -\pi < \theta \le \pi$
$\|z_1 z_2\| = \|z_1\|\|z_2\|$	$\left\|\frac{z_1}{z_2}\right\| = \frac{\|z_1\|}{\|z_2\|}$
$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$	$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
$z\bar{z} = \|z\|^2$	$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{\|z\|^2}$
$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$	$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
Polar form	
$z = a + bi = r(\cos\theta + i\sin\theta) = r\,cis\,\theta$	$\bar{z} = r\,cis\,(-\theta)$
$z_1 z_2 = r_1 r_2\,cis\,(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,cis\,(\theta_1 - \theta_2)$
$cis(\theta_1 + \theta_2) = cis\,\theta_1\,cis\,\theta_2$	$cis(-\theta) = \frac{1}{cis\,\theta}$
De Moivre's theorem	
$z^n = \|z\|^n\,cis\,(n\theta)$	$(cis\,\theta)^n = \cos n\theta + i\sin n\theta$
$z^{\frac{1}{q}} = r^{\frac{1}{q}}\left(\cos\frac{\theta + 2\pi k}{q} + i\sin\frac{\theta + 2\pi k}{q}\right)$, for k an integer	

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$$

Q1 (2 & 3 = 5 marks)

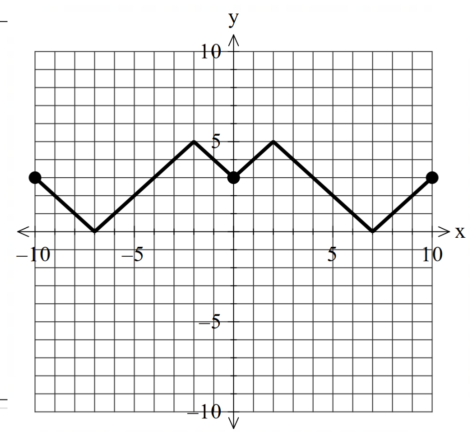
Consider the function $f(x)$ plotted below.



a) Solve for $|f(x)|=5$.

x=-2,2
Specific behaviours
P one value P exactly two values

b) Sketch $y=|f(|x|)|$ on the axes below.



Specific behaviours
P y intercept P Absolute value used to reflect negative y values in x axis P reflection of right to give new left side

Q2 (2, 3 & 3 = 8 marks)

Consider the functions $f(x)=\frac{1}{\sqrt{2x-9}}$ and $g(x)=\frac{1}{3x-1}$.

a) Determine the natural domain and range of $g(x)$.

$d_g : x \neq \frac{1}{3}$ $r_g : y \neq 0$
Specific behaviours
P domain P range

b) Does $f \circ g(x)$ exist over the natural domain of $g(x)$? Explain.

$f \circ g(x)$ $to\, exist\; r_g \subseteq d_f$ $r_g : y \neq 0$ $d_f : x > \frac{9}{2}$ not $\geq$ $\therefore not$ $r_g \not\subset d_f$ "memorize" this
Specific behaviours
P states relevant domain and range P states reason to exist P states does not exist with a reason

c) Determine the largest possible domain for $f \circ g(x)$.

Working out space

End of test

Working out space

$$f \circ g(x) = \frac{1}{\sqrt{2\frac{1}{3x-1} - 9}} = \sqrt{\frac{3x-1}{11-27x}}$$

$$3x - 1 > 0 \Rightarrow x > \frac{1}{3}$$

$$11 - 27x > 0 \Rightarrow x < \frac{11}{27}$$

$$d: \frac{1}{3} < x < \frac{11}{27}$$

Specific behaviours

P states rule, no need to simplify OR gives reasoning
P determines lower limit of domain (non inclusive)
P determines upper limit of domain (non inclusive)
Do not award if inequality incorrect

Q3 (3, 3, & 2 = 8 marks)

Consider the function $f(x) = 3x^2 - 12x + 19, \ x \leq 2$.

a) Determine $f^{-1}(x)$ and state its domain.

$$f(x) = 3x^2 - 12x + 19, \ x \leq 2$$

$$x = 3y^2 - 12y + 19, y \leq 2$$

$$0 = 3y^2 - 12y + 19 - x$$

$$y = \frac{12 \pm \sqrt{144 - 12(19 - x)}}{6} = \frac{12 \pm \sqrt{12x - 84}}{6}$$

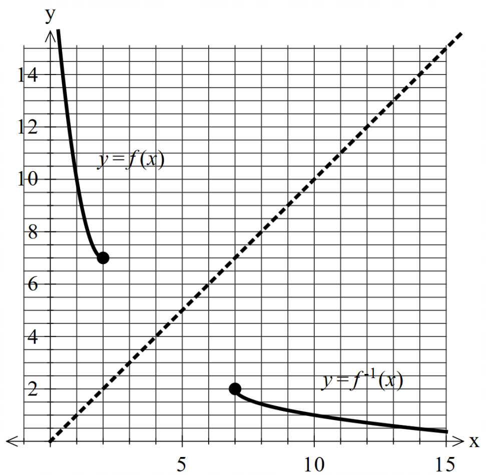
$$f^{-1}(x) = \frac{12 - 2\sqrt{3x - 21}}{6} = \frac{6 - \sqrt{3x - 21}}{3}, x \geq 7$$

Specific behaviours

P swaps x and y
P states inverse rule with initially two possibilities
P discards positive and states domain

Q3 continued

b) Sketch $f(x)$ & $f^{-1}(x)$ on the same set of axes below.

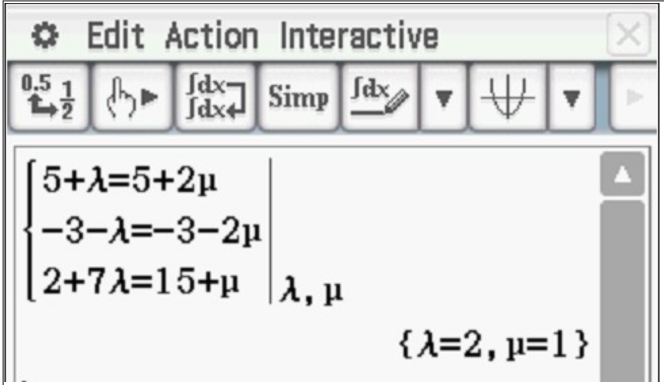


Specific behaviours
P sketches f with point (2,7) clearly plotted P sketches f(-1) with point (7,2) clearly plotted P both functions appear to be reflected in line y=x

c) Determine value(s) of x, if any, such that $f\circ f(x)=x$. Explain.

$f\circ f(x)=x$ results in $f(x)=f^{-1}(x)$ graphs overlapping at these points From graph above it is apparent that $f(x)\neq f^{-1}(x)$ therefore no solutions
Specific behaviours
P explains that $f\circ f(x)=x$ results in $f(x)=f^{-1}(x)$ P states no solution to equation with a reason

Working out space



Smoke trails meet at (7,-5,16)km

Specific behaviours
P uses vector equation of lines P uses two different parameter variables P states exact point in space, no need for units (Max 1 mark if only one parameter used)

Q4 (3 marks)

If $z = 27 cis \frac{7\pi}{8}$ is a solution to the equation $z^n = ir$ where r is a positive real number and n is a positive integer, determine the smallest possible value for r in the form 3^p. **Justify** your answer.

$z^n = 3^{3n} cis \frac{7\pi n}{8} = ir = r cis \frac{k\pi}{2}, k = 1,5,9,13,17,21...$

$\frac{7\pi n}{8} = \frac{k\pi}{2}$

$n = \frac{4k}{7}$

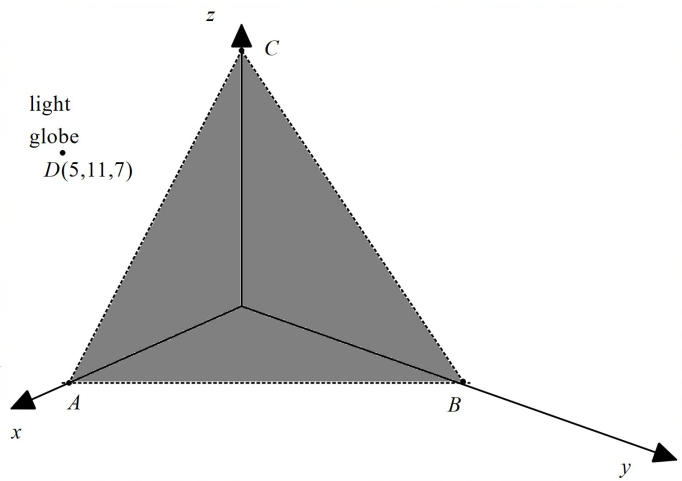
$smallest\ k = 21, n = 12$

$r = 3^{36}$

Specific behaviours
P establishes a relationship between n and k algebraically P determines smallest value for n P expresses r as a power of 3.

Q5 (3 & 3 = 6 marks)

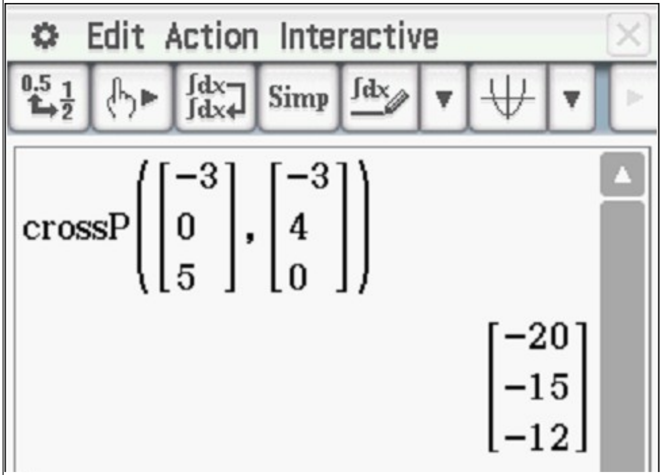
Consider a triangular plane with vertices $A(3,0,0), B(0,4,0) \& C(0,0,5)$ shaded as shown below. There is a light globe situated at point $D(5,11,7)$.



a) Determine the cartesian equation of the shaded plane ABC above.

$$AC = \begin{pmatrix}0\\0\\5\end{pmatrix} - \begin{pmatrix}3\\0\\0\end{pmatrix} = \begin{pmatrix}-3\\0\\5\end{pmatrix}$$

$$AB = \begin{pmatrix}0\\4\\0\end{pmatrix} - \begin{pmatrix}3\\0\\0\end{pmatrix} = \begin{pmatrix}-3\\4\\0\end{pmatrix}$$

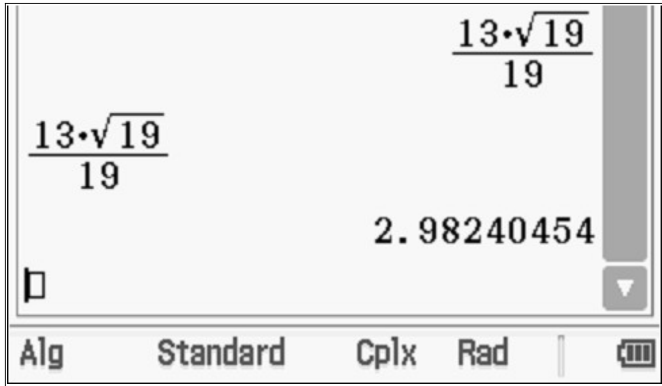


$$r.\begin{pmatrix}-20\\-15\\-12\end{pmatrix} = \begin{pmatrix}0\\0\\5\end{pmatrix}.\begin{pmatrix}-20\\-15\\-12\end{pmatrix} = -60$$

$$-20x - 15y - 12z = -60$$

Specific behaviours

P uses cross product using any two sides of triangle
P sets up vector equation of plane
P derives cartesian equation (or any multiple)



Specific behaviours

P determines expression for separation vector
P uses scalar dot product
P states approx. distance, no need for units

b) Determine the exact point in space, if any, where the smoke trails overlap at some time in the first 6 hours. (3 marks)

c

$$r_A = \begin{pmatrix}5\\-3\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\7\end{pmatrix}$$

$$r_B = \begin{pmatrix}5\\-3\\15\end{pmatrix} + \mu\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$r_A = r_B$$

$$\begin{bmatrix}5\\-3\\15\end{bmatrix}-\begin{bmatrix}5\\-3\\2\end{bmatrix}+t\times\left(\begin{bmatrix}2\\-2\\1\end{bmatrix}-\begin{bmatrix}1\\-1\\7\end{bmatrix}\right)$$

$$\begin{bmatrix}t\\-t\\-6\cdot t+13\end{bmatrix}$$

$$\text{dotP}\left(\begin{bmatrix}t\\-t\\-6\cdot t+13\end{bmatrix},\begin{bmatrix}2\\-2\\1\end{bmatrix}-\begin{bmatrix}1\\-1\\7\end{bmatrix}\right)$$

$$6\cdot(6\cdot t-13)+2\cdot t$$

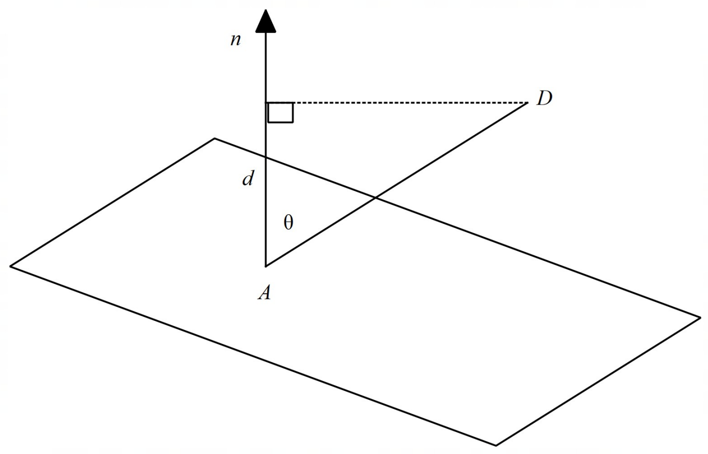
$$\text{solve}(6\cdot(6\cdot t-13)+2\cdot t=0,t)$$

$$\left\{t=\frac{39}{19}\right\}$$

$$\text{norm}\left(\begin{bmatrix}t\\-t\\-6\cdot t+13\end{bmatrix}\Big|\,t=\frac{39}{19}\right)$$

Q5 continued

b) Determine the distance of the globe to the shaded plane ABC.



$d = |\vec{AD}|\cos\theta$

$= |\vec{AD}\,\hat{n}|$

$(\hat{n} = \frac{n}{|n|})$

$$\text{dotP}\left(\begin{bmatrix}5\\11\\7\end{bmatrix}-\begin{bmatrix}3\\0\\0\end{bmatrix},\begin{bmatrix}20\\15\\12\end{bmatrix}\cdot\frac{1}{\sqrt{20^2+15^2+12^2}}\right)$$

$$10.42160654$$

Specific behaviours
P determines AD vector or line through pt D P uses scalar dot product with normal or line meeting plane P determines approx. distance

Q6 (5 marks)

Consider the line A $r = \begin{pmatrix} -3 \\ 0 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$ and the sphere B $\left| r - \begin{pmatrix} -3 \\ \alpha \\ 1 \end{pmatrix} \right| = 10$ where α is a real constant.
Determine all possible values of α, to one decimal place such that:

i) the line misses the sphere.
ii) the line just touches the sphere.
iii) the line pierces the sphere at two points.

$$r = \begin{pmatrix} -3 \\ 0 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3+5\lambda \\ -\lambda \\ 11+2\lambda \end{pmatrix}$$

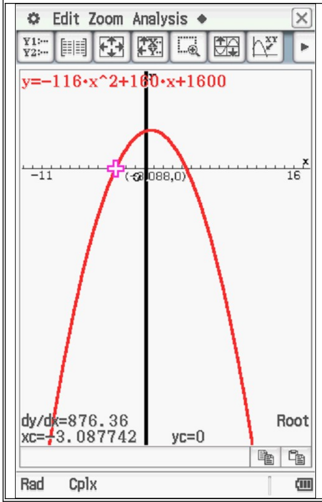
$$\left| \begin{pmatrix} -3+5\lambda \\ -\lambda \\ 11+2\lambda \end{pmatrix} - \begin{pmatrix} -3 \\ \alpha \\ 1 \end{pmatrix} \right| = 10$$

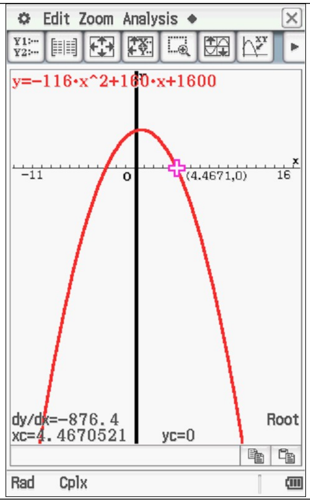
$$\left| \begin{pmatrix} 5\lambda \\ -\lambda-\alpha \\ 10+2\lambda \end{pmatrix} \right| = 10$$

$$\sqrt{25\lambda^2 + \lambda^2 + 2\lambda\alpha + \alpha^2 + 100 + 40\lambda + 4\lambda^2} = 10$$

$$30\lambda^2 + (2\alpha + 40)\lambda + \alpha^2 = 0$$

$$\Delta = (2\alpha + 40)^2 - 4(30)(\alpha^2) = -116\alpha^2 + 160\alpha + 1600$$





$\Delta < 0$ *misses* $\alpha < -3.1, \alpha > 4.5$

$\Delta = 0$ *touches* $\alpha = -3.1, \alpha = 4.5$

$\Delta > 0$ *pierces* $-3.1 < \alpha < 4.5$

Specific behaviours
P sets up an equation for $\lambda \,\&\, \alpha$
P states a quadratic equation or uses shortest distance approach
P uses discriminant expression or compares distances to radius
P states values of α for all three scenarios
P states a condition for **each** of the three scenarios to determine values

Q7 (3 & 3 = 6 marks)

Consider two rockets $A \,\&\, B$ that are ignited at the same time from different positions and move with constant velocities as shown below.

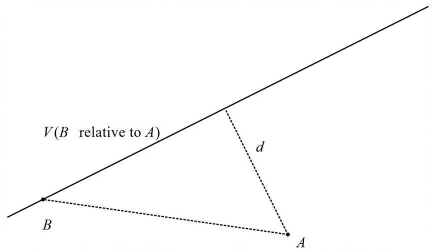
$$r_A = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} km \quad , v_A = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} km/h$$

$$r_B = \begin{pmatrix} 5 \\ -3 \\ 15 \end{pmatrix} km \quad , v_B = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} km/h$$

Both rockets leave a smoke trail that stays in the air for at least 6 hours.

a) Determine the distance of the closest approach between the rockets using scalar dot product

(3 marks)



$d = AB + t\,{}_BV_A$

$d \cdot {}_BV_A = 0$