MATHEMATICS SPECIALIST UNIT 3

Semester One

2017

SOLUTIONS

Calculator-free Solutions

1. (a) $P(2i)=2(2i)^3-4(2i)^2+8(2i)-16$ ✓

¿ $2(-8i)-4(-4)+16i-16$ ✓

¿ $-16i+16+16i-16=0$ as required

(b) $z_2=\overline{z_1}=-2i$ ✓

Using factorisation:

$P(z)=2(z+2i)(z-2i)(z-k)$ ✓

¿ $2(z^2+4)(z-k)=2(z^3-2z^2 8z-18)$

$\therefore k=2$ ✓

$\therefore P(z)=2(z+2i)(z-2i)(z-2)$ ✓

OR using division of polynomials:

$$
\begin{array}{r|rrrr}
 & 2z & -4 & & \\
\hline
z^2+4 & 2z^3 & -4z^2 & +8z & -16 \\
 & 2z^3 & & +8z & \\
\hline
 & 0 & -4z^2 & 0 & -16 \\
 & & -4z^2 & & -16 \\
\hline
 & & 0 & & 0
\end{array}
$$
✓✓

$\therefore P(z)=(z^2+4)(2z-4)=2(z+2i)(z-2i)(z-2)$ ✓ [6]

2. (a) $f(z)=z^5-32i=0$

$z^5=32i=32\,cis\left(\frac{\pi}{2}+2\pi k\right)$ with $k=0,\pm1,\pm2$ ✓

$\therefore z=2\cos\left(\frac{\pi+4\pi k}{10}\right)$ with $k=0,\pm1,\pm2$ ✓

hence,

$k=0\rightarrow z_0=2\cos\left(\frac{\pi}{10}\right)$

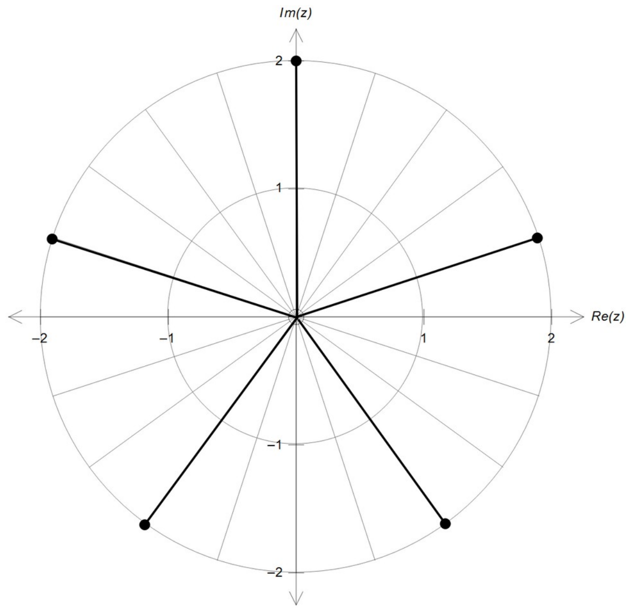
$k=1\rightarrow z_1=2\cos\left(\frac{\pi}{2}\right)$

$k=-1\rightarrow z_2=2\cos\left(\frac{-3\pi}{10}\right)$

$k=2\rightarrow z_3=2\cos\left(\frac{9\pi}{10}\right)$

$k=-2 \rightarrow z_0=2\cos\left(\frac{-7\pi}{10}\right)$ ✓✓

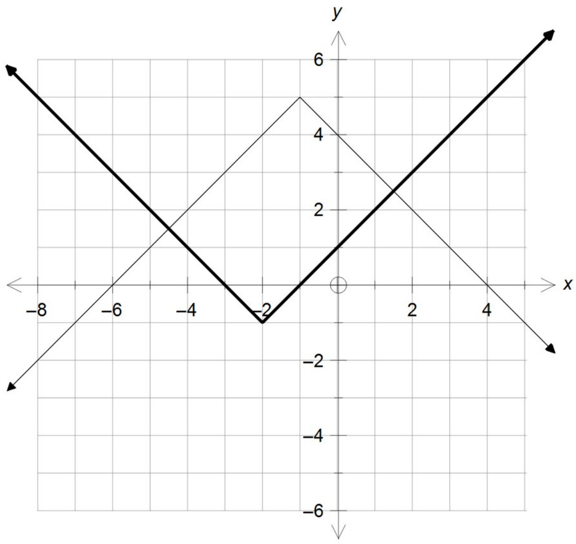
2. (b)



✓ magnitude = 2

✓ $\frac{2\pi}{5}$ radians apart

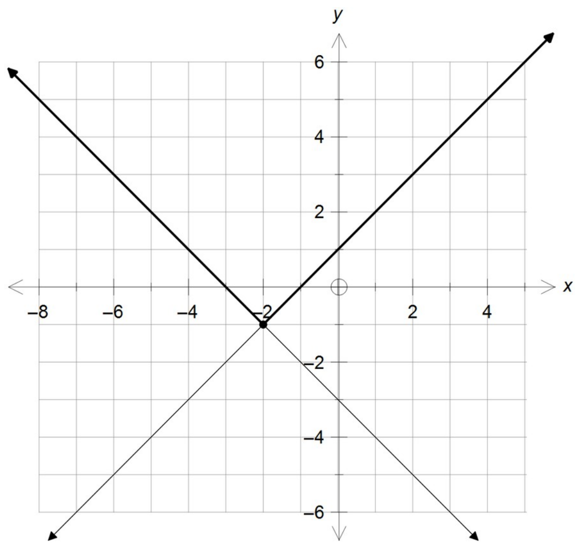
3. (a)



✓ correct x position

✓ correct y position

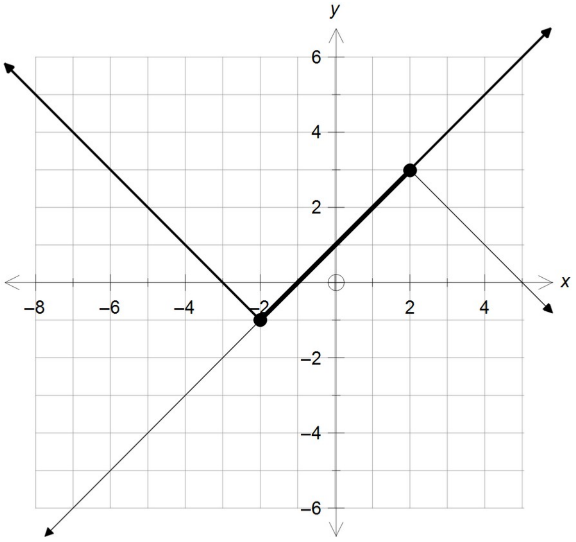
(b) (i)



$y = -1 - |x+2|$

$\therefore a = -1$ and $b = 2$ ✓✓

3. (b) (ii)



$y=3-|x-2|$

$\therefore a=3$ and $b=-2$ ✓✓ [2]

4. (a) $h(x)=f(f(x))=\dfrac{1}{1+f(x)}$ with $x\neq-1$

$\therefore 1+f(x)\neq 0$ ✓

$f(x)=\dfrac{1}{1+x}\neq-1$

$\therefore x\neq-2$ and $x\neq-1$ ✓✓

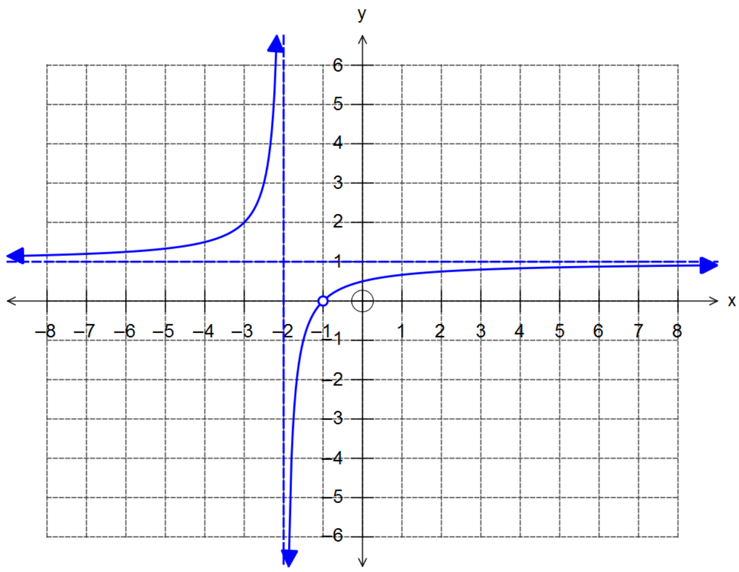
$f(x)\neq 0$ on its natural domain, hence ✓

$f(0)=1$ will not be generated by the second $f(x)$

$\therefore \{y: y\neq 1\wedge y\neq 0\}$ ✓

(b) $h(x)=\dfrac{1}{1+\left(\dfrac{1}{1+x}\right)}=\dfrac{1}{\dfrac{1+x+1}{1+x}}=\dfrac{x+1}{x+2}$

$\therefore h(x)=\dfrac{x+2-1}{x+2}=\dfrac{x+2}{x+2}-\dfrac{1}{x+2}=\dfrac{-1}{x+2}+1$ ✓



✓ asymptotes
x = -2 and y = 1

✓ discontinuity at
x = -1

✓ correct shape

4. (c) $x=1-\dfrac{1}{y+2}$

$\therefore\ y=\dfrac{-1}{x+1}-2=\dfrac{1-2x}{x-1}$ ✓

Domain $\{x : x\neq 1, x\neq 0\}$ ✓

Range $\{y : y\neq -2, y\neq -1\}$ ✓ [12]

5. (a) Cartesian equations of the planes:

$2x+y+z=1$
$4y+2z=2$
$3x+y-z=8$

<u>Using matrices:</u>

$$\begin{array}{cc}\begin{bmatrix}2 & 1 & 1 & 1\\ 0 & 4 & 2 & 2\\ 3 & 1 & -1 & 8\end{bmatrix} & \begin{array}{l}R_1\\ R_2\\ R_3\end{array}\end{array}$$

$$\begin{array}{cc}\begin{bmatrix}2 & 1 & 1 & 1\\ 0 & 2 & 1 & 1\\ 0 & -1 & -5 & 13\end{bmatrix} & \begin{array}{c}R_1\\ \frac{1}{2}R_2\\ 2R_3-3R_1\end{array}\end{array}$$ ✓

$$\begin{array}{cc}\begin{bmatrix}2 & 1 & 1 & 1\\ 0 & 2 & 1 & 1\\ 0 & 0 & -9 & 27\end{bmatrix} & \begin{array}{c}R_1\\ R_2\\ R_2+2R_3\end{array}\end{array}$$ ✓

$\therefore -9z=27, \quad z=-3$

$2y-3=1, \quad y=2$

$2x+2-3=1, \quad x=1$

Unique solution: $(1, 2, -3)$ ✓

(b) Fully simplified matrix in terms of a:

$$\begin{bmatrix}2 & 1 & 1 & 1\\ 0 & 2 & 1 & 1\\ 0 & 0 & (4a-5) & 27\end{bmatrix}$$ ✓

$\therefore 4a-5=0$ ✓

$a\neq\dfrac{5}{4}$ ✓ [6]

(Algebraic manipulation and substitution is also acceptable)

6. (a) Midpoint between $\begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix}$ is $\begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$ ✓✓

OR $\overrightarrow{AG} = \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}$, centre = $\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AG} = \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$ ✓✓

radius = $\frac{1}{2}\left|\overrightarrow{AG}\right| = \frac{1}{2}\begin{vmatrix} 4 \\ 4 \\ -4 \end{vmatrix} = 2\begin{vmatrix} 1 \\ 1 \\ -1 \end{vmatrix} = 2\sqrt{3}$ units ✓

$$\left| r - \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} \right| = 2\sqrt{3}$$ ✓

(b) $\overrightarrow{AF} = \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix}, \overrightarrow{AB} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$

$$\overrightarrow{AF} \times \overrightarrow{AB} = \begin{pmatrix} 0+16 \\ -16-0 \\ 16-0 \end{pmatrix} = \begin{pmatrix} 16 \\ -16 \\ 16 \end{pmatrix} = 16\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$ ✓✓

$$k = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 4$$ ✓

$$\therefore r \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 4$$ ✓ [8]

7. (a) $x = 2\cos t \rightarrow \cos t = \frac{x}{2}$

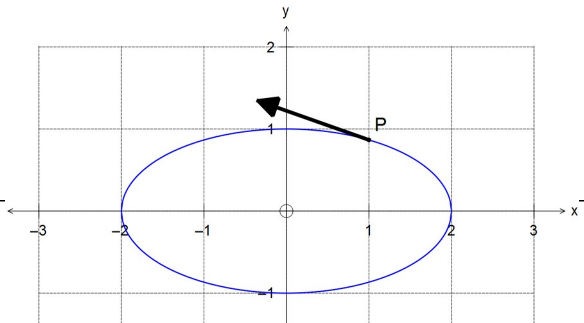
$$\sin^2 t + \cos^2 t = y^2 + \frac{x^2}{4} = 1$$ ✓✓

(b) $x = 1 \rightarrow 2\cos t = 1, \therefore t = \frac{\pi}{3}$ ✓

$\dot{r}(t) = -2\sin t\, i + \cos t\, j$ ✓

¿ $-2\sin\left(\frac{\pi}{3}\right)i + \cos\left(\frac{\pi}{3}\right)j$

¿ $-\sqrt{3}i + \frac{1}{2}j$ ✓



✓ [6]

Calculator-Assumed Solutions

8. (a) $x=2$ and $y=\frac{-x}{2}+1$ ✓✓

(b) $f(x)=\frac{k}{x-2}+\left(\frac{-x}{2}+1\right)$

$f(3)=\frac{k}{1}+1-\frac{3}{2}=\frac{3}{2}\rightarrow k=2$ ✓

$\therefore f(x)=\frac{2}{x-2}-\frac{x}{2}+1=\frac{-x^2+4x}{2x-4}$ ✓

$\therefore a=-1, b=4, c=0, m=2, n=-4$ ✓✓ [6]

9. (a) $v(t)=\int\left[(6t)i+2j\right]dt=(3t^2+A)i+(2t+B)j$ ✓

$v(0)=\begin{pmatrix}0+A\\0+B\end{pmatrix}=\begin{pmatrix}-2\\-3\end{pmatrix}$

$\therefore v(t)=(3t^2-2)i+(2t-3)j$ ✓

$r(t)=\int\left[(3t^2-2)i+(2t-3)j\right]dt$

¿ $(t^3-2t+C)i+(t^2-3t+D)j$ ✓

$r(0)=\begin{pmatrix}0+C\\0+D\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$\therefore r(t)=(t^3-2t)i+(t^2-3t)j$ ✓

(b) Condition to stop:

$|v(t)|=\begin{vmatrix}3t^2-2\\2t-3\end{vmatrix}=0$ ✓

$\therefore t=\pm\sqrt{\frac{2}{3}}$ for *x* axis and $t=\frac{3}{2}$ for *y* axis ✓

Since there is no unique solution, the particle does not come to a stop. ✓

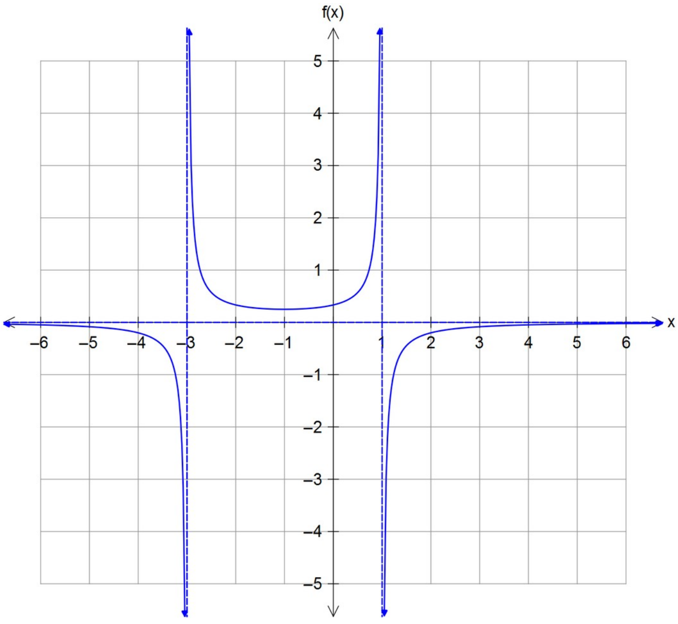
(c) Motion parallel to *x* axis = no *y* axis velocity

$\therefore 2t-3=0\rightarrow t=\frac{3}{2}$ seconds ✓✓

$\therefore \left|v\left(\frac{3}{2}\right)\right|=\begin{vmatrix}3\left(\frac{3}{2}\right)^2-2\\0\end{vmatrix}=\frac{19}{4}=4.75$ ms^{-1} ✓

$$r\left(\frac{3}{2}\right)=\begin{pmatrix}\left(\frac{3}{2}\right)^3-2\left(\frac{3}{2}\right)\\ \left(\frac{3}{2}\right)^2-3\left(\frac{3}{2}\right)\end{pmatrix}=\begin{pmatrix}\frac{3}{8}\\ \frac{-9}{4}\end{pmatrix}=\frac{3}{8}i-\frac{9}{4}j$$ ✓ [11]

10. (a)



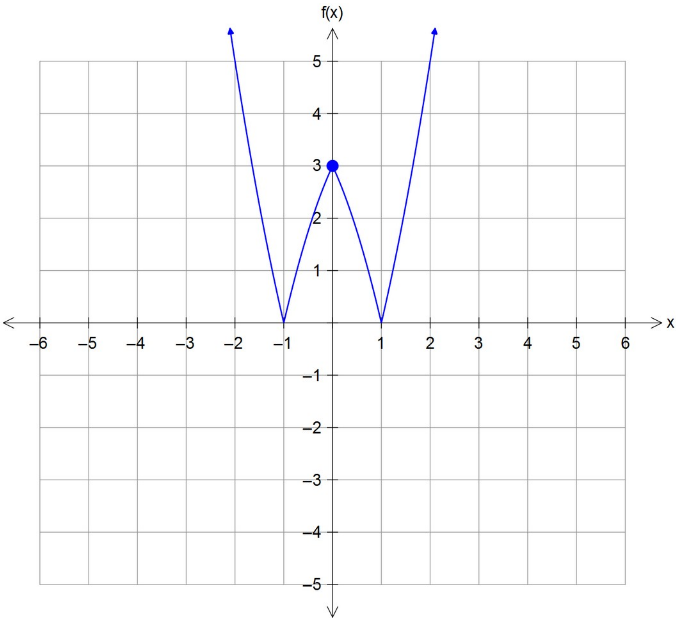
✓ Vertical asymptotes $x = -3$, $x = 1$

✓ Indicates $y \to 0^-$ as $|x| \to \infty$

✓ Shows local min at $x = -1$

✓ Correct curvature at $x = -3$ and $x = 1$

(b)



✓ Correct mirror image over x axis

✓ Correct mirror image over y axis

✓ Indicates points (0,3),(-1,0) and (1,0)

(c) Function can be inverted about its turning point.

i.e., for $x=-1 \to \therefore k=-1$ ✓

Domain $\{x : x \leq 4\}$ ✓

Range $\{y : y \geq -1\}$ ✓ [10]

11. (a) At the xz plane y = 0. ✓

$\therefore x^2+z^2-2x=7$

$(x-1)^2+z^2=8$ ✓

circle centred at $(1,0)$ and radius $2\sqrt{2}$ units ✓✓

(b) Completing the squares of the original sphere:

$(x-1)^2+(y+1)^2+z^2=9$

$\therefore \left| r-\begin{pmatrix}1\\-1\\0\end{pmatrix}\right|=3$ ✓

centre of sphere is $C(1,-1,0)$

$\therefore n=\overrightarrow{CP}=\begin{pmatrix}3\\1\\1\end{pmatrix}-\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}2\\2\\1\end{pmatrix}$ ✓

$k=n\cdot\overrightarrow{OP}=\begin{pmatrix}2\\2\\1\end{pmatrix}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix}=6+2+1=9$ ✓

$\therefore r\cdot\begin{pmatrix}2\\2\\1\end{pmatrix}=9$ and hence $2x+2y+z=9$ ✓ [8]

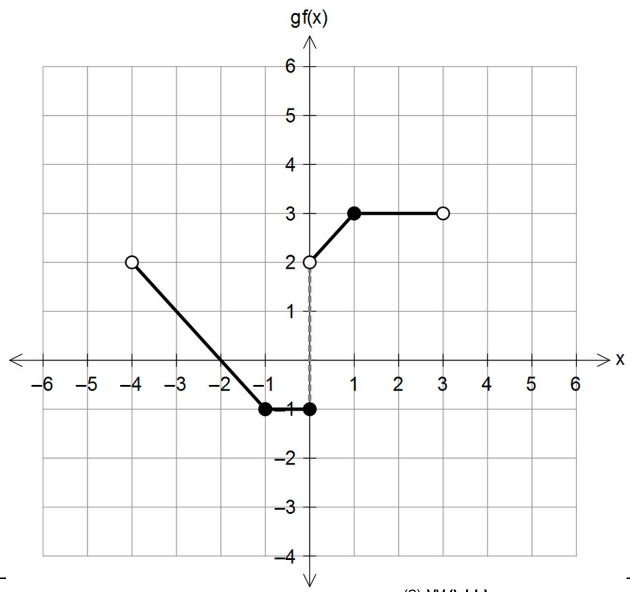
12. (a) $F(x)=[\cos(1-x)]^2=f[\cos(1-x)]=f[g(1-x)]=f[g(h(x))]$

$\therefore f(x)=x^2, g(x)=\cos x, h(x)=1-x$ ✓✓✓

(b) (i) $f(g(1))=f(2)=3$ ✓

$f(f(-3))=f(0)=-3$ ✓

(ii)



✓ correct sections

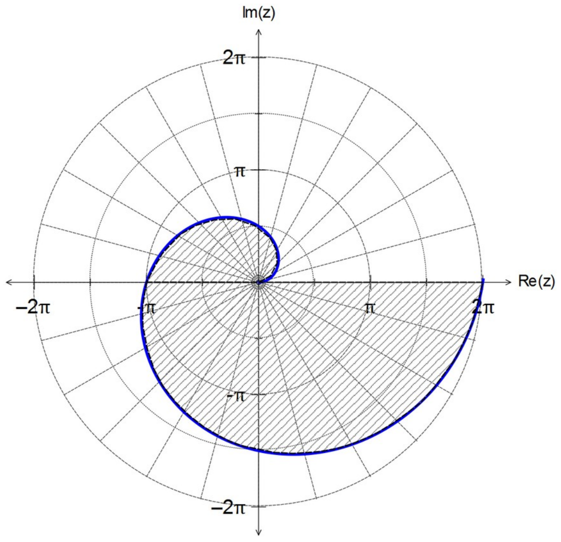
✓ correct y-intersect

✓ correct discontinuities

[8]

13. (a) $|z+2| \le |z-2+2i|$ ✓✓✓

(b) $r \le \theta \therefore$ region under spiral



✓ spiral r = θ

✓ region under spiral and x axis

(b) $\frac{z+1}{z+i} = \frac{(x+1)+yi}{x+i(y+1)}$ such that $z \ne -i$

CAS: $\text{¿}\left(\frac{x^2+y^2+x+y}{x^2+(y+1)^2}\right) + i\left(\frac{-x-y-1}{x^2(y+1)^2}\right)$ ✓

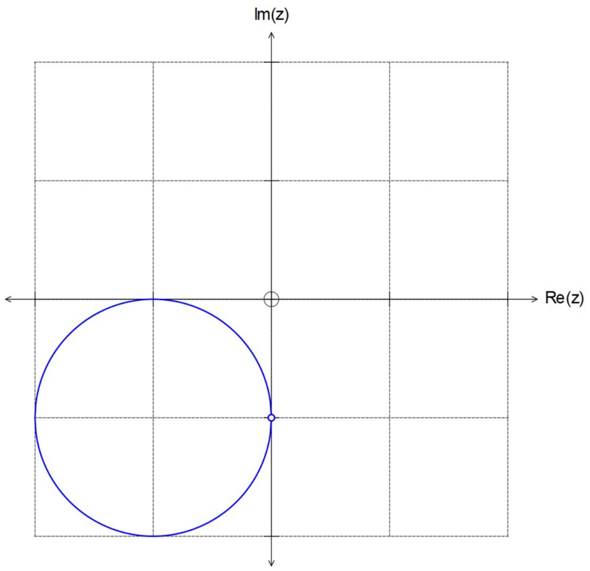
$Arg\left(\frac{z+1}{z+i}\right) = \frac{\pi}{4} \rightarrow \Re\left(\frac{z+1}{z+i}\right) = \Im\left(\frac{z+1}{z+i}\right)$

$\therefore x^2+y^2+x+y = -x-y-1$ ✓

$(x+1)^2+(y+1)^2 = 1$ ✓

circle centred at $(1,1)$ and radius $r=1$ ✓

with discontinuity at $z=-i$ ✓



[10]

14. (a) height = vertical velocity × time = 90 × 4 = 360 m ✓

(b) Detonation at $\overrightarrow{OA} = \begin{pmatrix} 80 \\ -120 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 30 \\ -10 \\ 90 \end{pmatrix} = \begin{pmatrix} 200 \\ -160 \\ 360 \end{pmatrix}$ ✓

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{pmatrix} -200 \\ 350 \\ 20 \end{pmatrix} - \begin{pmatrix} 200 \\ -160 \\ 360 \end{pmatrix} = \begin{pmatrix} -400 \\ 510 \\ -340 \end{pmatrix}$ ✓

direct distance = $\left|\overrightarrow{AP}\right| = \left|\begin{matrix} -400 \\ 510 \\ -340 \end{matrix}\right| = 731.92$ m ✓

time difference ¿ $\frac{731.92}{340} = 2.15$ seconds ✓

(c) speed ¿ $\left|\begin{matrix} 30 \\ -10 \\ 90 \end{matrix}\right| = 10\sqrt{91} = 95.39$ ms^{-1} ✓

projection vector onto xy plane ¿ $\begin{pmatrix} 30 \\ -10 \\ 0 \end{pmatrix}$ ✓

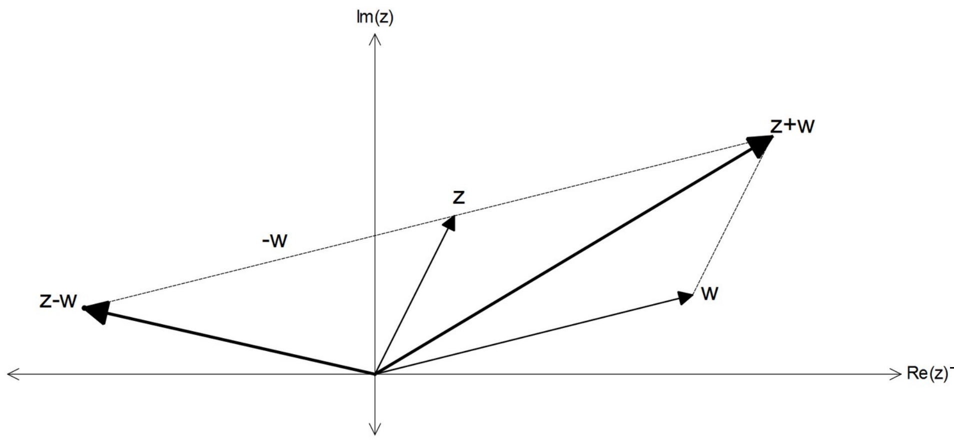
$\therefore \begin{pmatrix} 30 \\ -10 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 30 \\ -10 \\ 90 \end{pmatrix} = \left|\begin{matrix} 30 \\ -10 \\ 0 \end{matrix}\right| \times \left|\begin{matrix} 30 \\ -10 \\ 90 \end{matrix}\right| \times \cos\theta$ ✓

$\therefore \cos\theta = \frac{10}{\sqrt{910}} \rightarrow \theta = 70.64°$ ✓

(d) distance vector ¿ $\begin{pmatrix} 200 \\ -160 \\ 360 \end{pmatrix} - \begin{pmatrix} 120 \\ -40 \\ 0 \end{pmatrix} = \begin{pmatrix} 80 \\ -120 \\ 360 \end{pmatrix}$ ✓

velocity ¿ $\frac{1}{4}\begin{pmatrix} 80 \\ -120 \\ 360 \end{pmatrix} = \begin{pmatrix} 20 \\ -30 \\ 90 \end{pmatrix}$ ms^{-1} ✓ [11]

15. (a) (i)



✓ z + w

✓ z – w in the correct location

15. (a) (ii) $|z+w|^2=|z-w|^2$

$(z+w)^2=(z-w)^2$ ✓

$z^2+2zw+w^2=z^2-2zw+w^2$ ✓

$4zw=0$ and hence $wz=0$ ✓

if z, w are treated as vectors, then $z \perp w$ ✓

and hence $|arg(z)-arg\ (w)|=\frac{\pi}{2}$

(b) Let $u=x+yi$

$|i+x+yi|^2+|i-x-yi|^2$

¿ $\left(\sqrt{x^2+(y+1)^2}\right)^2+\left(\sqrt{(-x)^2+(1-y)^2}\right)^2$ ✓

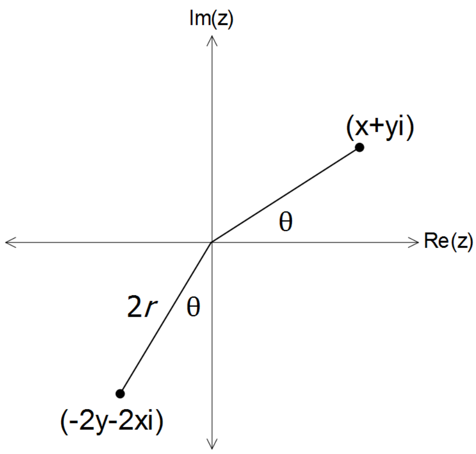
¿ $x^2+y^2+2y+1+x^2+1-2y+y^2$ ✓

¿ $2+2(x^2+y^2)=2+2(1)=4$ ✓✓

(c) (i) $iz^2=cis\left(\frac{\pi}{2}\right)r^2cis(2\theta)=r^2cis\left(2\theta+\frac{\pi}{2}\right)$

$\therefore |iz^2|=r^2$ and $arg(iz^2)=2\theta+\frac{\pi}{2}$ ✓✓

(ii)



$\therefore |-2y-2xi|=|-2(y+xi)|=2r$ ✓

$arg(-2y-2xi)=-\left(\theta+\frac{\pi}{2}\right)$ ✓ [14]

16. (a) CAS $\rightarrow Q(z)=z^2+z-1$ ✓✓

(OR, using division of polynomials)

(b) $(z^3+1)(z^2+z-1)=0$

$z^3=-1$

$\therefore z_1=-1, z_2=\frac{1}{2}+\frac{i\sqrt{3}}{2}, z_3=\frac{1}{2}-\frac{i\sqrt{3}}{2}$ ✓✓✓

$z=\frac{-1\pm\sqrt{1+4}}{2}$

$\therefore z_4=\frac{-1+\sqrt{5}}{2}, z_5=\frac{-1-\sqrt{5}}{2}$ ✓✓ [7]

17. (a) By De Moivre's theorem

$(\cos\theta+i\sin\theta)^4=\cos(4\theta)+i\sin(4\theta)$ ✓

¿ $\cos^4\theta+4i\sin\theta\cos^3\theta-6\sin^2\theta\cos^2\theta-4i\cos\theta\sin^3\theta+\sin^4\theta$ ✓

$\therefore \cos(4\theta)=\Re\left[(\cos\theta+i\sin\theta)^4\right]$

¿ $\cos^4\theta-6\sin^2\theta\cos^2\theta+\sin^4\theta$ ✓

¿ $\cos^4\theta-6\cos^2\theta(1-\cos^2\theta)+(1-\cos^2\theta)^2$ ✓

$\therefore \cos(4\theta)=8\cos^4\theta-8\cos^2\theta+1$ ✓

(b) Let $x=\cos\theta$, then ✓

$8x^4-8x^2+1=0$ when $\cos 4\theta=0$ ✓

$\therefore 4\theta=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}$

and $\therefore \theta=\frac{\pi}{8},\frac{3\pi}{8},\frac{5\pi}{8},\frac{7\pi}{8}$

$\therefore x=\cos\frac{\pi}{8},\cos\frac{3\pi}{8},\cos\frac{5\pi}{8},\cos\frac{7\pi}{8}$ ✓ [8]

18. (a) If the dot product is zero, then a is perpendicular to the plane normal to b and c. ✓✓

If the cross product is zero, then b and c are parallel ✓✓

(b) LHS ¿ $(a+b)\times(a-b)$

¿ $a\times a-a\times b+b\times a-b\times b$ ✓

¿ $0-(-(b\times a))+b\times a-0$ ✓

¿ $b\times a+b\times a$ ✓

¿ $2(b \times a)=$ ¿ RHS [7]