



Semester Two Examination, 2020

Question/Answer booklet

**MATHEMATICS  
METHODS  
UNITS 3&4  
Section One:  
Calculator-free**

**SOLUTIONS**

WA student number:

In figures

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In words

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Your name

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**Time allowed for this section**

Reading time before commencing work:

five minutes

Working time:

fifty minutes

Number of additional  
answer booklets used  
(if applicable):

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**Materials required/recommended for this section**

***To be provided by the supervisor***

This Question/Answer booklet

Formula sheet

***To be provided by the candidate***

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener,  
correction fluid/tape, eraser, ruler, highlighters

Special items: nil

**Important note to candidates**

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

## Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
Total					100

## Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.
2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.
3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.
4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.
5. It is recommended that you do not use pencil, except in diagrams.
6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.
7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free

35% (52 Marks)

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1

(7 marks)

The function  $f$  is defined by  $f(x) = \frac{x^2 - 5}{3 - x}$ ,  $x \neq 3$ .

The second derivative of  $f$  is  $f''(x) = 8(3 - x)^{-3}$ .

Determine the coordinates and nature of all stationary points of the graph of  $y = f(x)$ .

Solution
$f'(x) = \frac{2x(3-x) - (-1)(x^2 - 5)}{(3-x)^2}$
$f'(x) = 0 \Rightarrow 6x - 2x^2 + x^2 - 5 = 0 \Rightarrow x^2 - 6x + 5 = 0 \Rightarrow (x-1)(x-5) = 0$ $x = 1, 5$
$f''(1) = \frac{8}{8} > 0 \Rightarrow \text{Min}, f''(5) = \frac{8}{-8} < 0 \Rightarrow \text{Max}$
$f(1) = \frac{-4}{2} = -2, f(5) = \frac{20}{-2} = -10$
$f(x)$ has a minimum at $(1, -2)$ and a maximum at $(5, -10)$ .
Specific behaviours
✓ indicates correct use of quotient rule ü correct $f'(x)$ ü equates numerator to zero ü determines x-coordinates of stationary points ü indicates correct use of second derivative for nature ü correct minimum ü correct maximum

## Question 2

(5 marks)

The rate of change of pressure in an air tank is given by  $P'(t) = -3e^{-0.05t}$ , where  $t$  is the time in minutes since it began emptying from an initial pressure of 70 psi.

- (a) Determine an expression for the pressure  $P$  in the tank at any time  $t, t \geq 0$ . (2 marks)

Solution
$P(t) = \frac{-3}{-0.05} e^{-0.05t} + c \quad \checkmark 60e^{-0.05t} + c$
$(0, 70) \Rightarrow 70 = 60e^0 + c \quad c = 10$
$P(t) = 10 + 60e^{-0.05t}$
Specific behaviours
$\checkmark$ correctly integrates $P'(t)$ $\checkmark$ correct expression for $P(t)$

- (b) Determine

- (i) the time taken for the pressure in the tank to fall to 40 psi. (2 marks)

Solution
$10 + 60e^{-0.05t} = 40 \quad e^{-0.05t} = 0.5$
$-0.05t = \ln 0.5 \quad t = -20 \ln 0.5 \quad (\checkmark 20 \ln 2)$
Specific behaviours
$\checkmark$ simplifies equation to $e^{-0.05t} = k$ $\checkmark$ correct time

- (ii) the minimum pressure in the tank for  $t \geq 0$ . (1 mark)

Solution
$t \rightarrow \infty, P \rightarrow 10 \text{ psi}$
Specific behaviours
$\checkmark$ correct pressure

**Question 3**

**(6 marks)**

The continuous random variable  $X$  takes values in the interval 1 to 5 and has cumulative distribution function  $F(x)$  where

$$F(x) = P(X \leq x) = \begin{cases} 0 & x < 1 \\ \frac{x-1}{4} & 1 \leq x \leq 5 \\ 1 & x > 5. \end{cases}$$

(a) Determine

(i)  $P(X \leq 3.5)$ .

(1 mark)

Solution
$P(X \leq 3.5) = \frac{3.5-1}{4} = \frac{2.5}{4} = \frac{5}{8} = 0.625$
Specific behaviours
✓ correct probability as fraction or decimal

(ii) the value of  $k$ , if  $P(X > k) = 0.85$ .

(2 marks)

Solution
$P(X \leq k) = 1 - P(X > k) = 1 - 0.85 = 0.15$
$\frac{k-1}{4} = 0.15 \Rightarrow k = 1.6$
Specific behaviours
✓ indicates $P(X \leq k)$
ü correct value

(b) Determine  $f(x)$ , the probability density function of  $X$ , and sketch the graph of  $y = f(x)$ .

(3 marks)

Solution
$f(x) = F'(x) = \begin{cases} \frac{1}{4} & 1 \leq x \leq 5 \\ 0 & \text{elsewhere} \end{cases}$
Specific behaviours
✓ $f(x)$ (no penalty for just $f(x) = \frac{1}{4}$ if sketch correct)
ü draws $y = 0.25$ between endpoints

## Question 4

(7 marks)

(a) Determine an expression for  $f'(x)$  when

(i)  $f(x) = \ln(1 - \cos 3x)$ .

(2 marks)

Solution
$f'(x) = \frac{3 \sin 3x}{1 - \cos 3x}$
Specific behaviours
✓ numerator ü denominator

(ii)  $f(x) = e^{5x}(5 - 2x)^3$ .

(3 marks)

Solution
$f'(x) = 5e^{5x}(5 - 2x)^3 + e^{5x} \cdot 3(-2)(5 - 2x)^2$ $= 5e^{5x}(5 - 2x)^3 - 6e^{5x}(5 - 2x)^2$ <p><i>N.B. Simplifies to <math>(19 - 10x)(5 - 2x)^2 e^{5x}</math></i></p>
Specific behaviours
ü derivative of $e^{5x}$ ü derivative of $(5 - 2x)^3$ ü correct expression using product rule

(b) For the positive number  $x$ , let  $A(x) = \int_0^x (8 - 2t^2) dt$ .Determine the value(s) of  $x$  for which  $\frac{dA}{dx} = 0$ .

(2 marks)

Solution
$\frac{dA}{dx} = \frac{d}{dx} \int_0^x (8 - 2t^2) dt = 8 - 2x^2$ $\therefore 2x^2 = 8 = 2^3 \Rightarrow x^2 = 3 \Rightarrow x = \sqrt{3}$
Specific behaviours
✓ expression for $A'(x)$ ü correct value of $x$

Question 5

(7 marks)

(a) Simplify  $\log 8 + 2 \log 5 - \log 2$ .

(2 marks)

Solution
$\log 8 + \log 5^2 - \log 2 = \log (8 \times 25 \div 2) \checkmark \log 10^2$ $\checkmark 2$
Specific behaviours
$\checkmark$ expresses as single log $\ddot{u}$ simplifies to number

(b) Given that  $\log_a x = 1.4$ , determine the value of  $\log_a x \sqrt{x}$ .

(2 marks)

Solution
$\log_a x \sqrt{x} = \log_a x + \log_a \sqrt{x} \checkmark \log_a x + 0.5 \log_a x$ $\checkmark 1.5 \log_a x \checkmark 1.5 \times 1.4 \checkmark 2.1$
Specific behaviours
$\checkmark$ obtains multiple of $\log_a x$ $\ddot{u}$ correct value

(c) Determine the solution to the equation  $5^{2x} = 2^{3-x}$  in the form  $x = \frac{\log a}{\log b}$ .

(3 marks)

Solution
$2x \log 5 = (3-x) \log 2$ $2x \log 5 = 3 \log 2 - x \log 2$ $2x \log 5 + x \log 2 = 3 \log 2$ $x(2 \log 5 + \log 2) = 3 \log 2 \quad x = \frac{\log 2^3}{\log (5^2 \times 2)}$ $x = \frac{\log 8}{\log 50}$
Specific behaviours
$\checkmark$ writes as log equation $\ddot{u}$ factors out $x$ $\ddot{u}$ solves and simplifies into required form

## Question 6

(7 marks)

The discrete random variable  $X$  is defined by

$$P(X=x) = \begin{cases} \frac{x+k}{3x+2} & x=0,1 \\ \text{elsewhere} \end{cases}$$

- (a) Determine the value of the constant
- $k$
- .

(2 marks)

Solution
$\frac{k}{2} + \frac{1+k}{5} = 1 \Rightarrow 5k + 2 + 2k = 10 \Rightarrow 7k = 8 \Rightarrow k = \frac{8}{7}$
Specific behaviours
✓ forms equation using $x=0$ and $x=1$ ü correct value

- (b) Determine

- (i)
- $P(X=0)$
- .

(1 mark)

Solution
$P(X=0) = \frac{8}{7} \div 2 = \frac{4}{7}$
Specific behaviours
✓ correct probability

- (ii)
- $E(3X-1)$
- .

(2 marks)

Solution
$E(X) = p = P(X=1) = 1 - \frac{4}{7} = \frac{3}{7}$ $E(3X-1) = 3 \times \frac{3}{7} - 1 = \frac{2}{7}$
Specific behaviours
✓ indicates $E(X)$ ü correct value

- (iii)
- $\text{Var}(3X-1)$
- .

(2 marks)

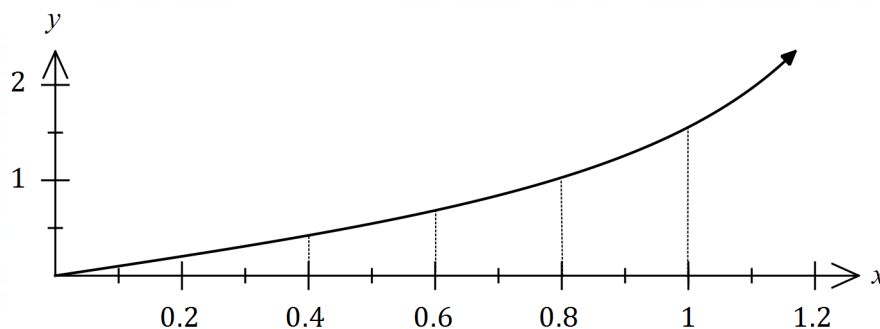
Solution
$\text{Var}(X) = p(1-p) = \frac{3}{7} \times \frac{4}{7} = \frac{12}{49}$ $\text{Var}(3X-1) = 3^2 \times \frac{12}{49} = \frac{108}{49}$
Specific behaviours
✓ indicates $\text{Var}(X)$ ü correct value



Question 7

(7 marks)

The graph and a table of values for  $y=f(x)$  is shown below, where  $f(x)=\tan x$ .



$x$	$y$
0.2	0.2
0.4	0.42
0.6	0.68
0.8	1.03
1	1.56
1.2	2.57

Let  $I = \int_{0.4}^1 \tan x \, dx$ .

- (a) By using the information shown and considering sums of the form  $\sum_i f(x_i) \delta x_i$ , explain why  $I > 0.426$ . (3 marks)

Solution
With $\delta x = 0.2$ , $x_1 = 0.4$ , $x_2 = 0.6$ and $x_3 = 0.8$ then $\sum_i f(x_i) \delta x_i = 0.2(0.42 + 0.68 + 1.03) \checkmark 0.2(2.13) \checkmark 0.426$ Hence $I$ must exceed this value as it is the area of inscribed rectangles that underestimate the area under the curve.
Specific behaviours
✓ indicates $x$ -ordinates for inscribed rectangles $\ddot{u}$ shows sum of $f(x_i) \delta x_i$ $\ddot{u}$ explains inequality

- (b) In a similar manner to (a), determine the best estimate for the value of the constant  $U$ , where  $I < U$ . (2 marks)

Solution
With $\delta x = 0.2$ , $x_1 = 0.6$ , $x_2 = 0.8$ and $x_3 = 1$ then $U = \sum_i f(x_i) \delta x_i = 0.2(0.68 + 1.03 + 1.56) \checkmark 0.2(3.27)$ $\checkmark 0.654$
Specific behaviours
✓ indicates $x$ -ordinates for circumscribed rectangles $\ddot{u}$ value of $U$

- (c) Use your previous answers to determine a numerical estimate for  $I$  and explain whether your estimate is smaller or larger than the exact value of  $I$ . (2 marks)

Solution
$I = \frac{0.426 + 0.654}{2} = 0.54$ This value slightly overestimates the exact value of $I$ as the curve is concave upwards.
Specific behaviours
✓ correctly averages values $\ddot{u}$ states overestimate with reason

## Question 8

(6 marks)

The acceleration at time  $t$  seconds of a small body travelling in a straight line is given by

$$a(t) = \frac{-3}{\sqrt{2t+3}} \text{ cm/s}^2, t \geq 0.$$

When  $t=3$  the body was at the origin and 8 seconds later its displacement was 30 cm.

Determine the velocity of the body when  $t=6.5$ .

Solution
$v(t) = \int -3(2t+3)^{-\frac{1}{2}} dt \quad \checkmark \quad \frac{-3}{\frac{1}{2} \times 2} (2t+3)^{\frac{1}{2}} + c \quad \checkmark \quad -3(2t+3)^{\frac{1}{2}} + c$
$\Delta x = \int_3^{3+8} -3(2t+3)^{\frac{1}{2}} + c dt \quad \checkmark \quad \left[ \frac{-3}{\frac{3}{2} \times 2} (2t+3)^{\frac{3}{2}} + ct \right]_3^{11}$
$\checkmark \quad \left[ -(2t+3)^{\frac{3}{2}} + ct \right]_3^{11} \quad \checkmark \quad [-125 + 11c] - [-27 + 3c] \quad \checkmark \quad 8c - 98$
<p>But <math>\Delta x = 30</math></p> $8c - 98 = 30 \quad 8c = 128 \quad c = 16$
$v(6.5) = -3(2(6.5)+3)^{\frac{1}{2}} + 16 \quad \checkmark \quad -12 + 16 \quad \checkmark \quad 4 \text{ cm/s}$
Specific behaviours
<ul style="list-style-type: none"> <li>✓ antiderivative of <math>a(t)</math></li> <li>ü integral for change in displacement <math>\Delta x</math></li> <li>ü antiderivative of <math>v(t)</math></li> <li>ü simplifies equation for <math>c</math></li> <li>ü uses given <math>\Delta x</math> to determine value of <math>c</math></li> <li>ü correct velocity</li> </ul>

Supplementary page

Question number: \_\_\_\_\_

