20. (a) Binomial (100, 0.02) ✓
$\mu = np = 2$ ✓
$\sigma = \sqrt{2(0.98)} = 1.4$ ✓

(b) $P(X \geq 5) = 1 - P(X \leq 4) = 0.0508$ ✓✓

(c) $n = 2000$, $p = 0.02$, $X = 40$
90% interval is 0.0149 to 0.0251 from CAS ✓✓

(d) $P(X = 2) = {}^3C_2\,(0.9)^2(0.1) = 0.243$ or from CAS ✓✓

(e) $0.0149 \times 2000 \approx 30$
$0.0251 \times 2000 \approx 50$ ✓✓

(f) Interval is from 30 to 50.
Sample 2 is outside. (57 > 50)
Sample 3 is outside. (28 < 30) ✓✓ [13]

MATHEMATICS
METHODS
UNITS 3 & 4

Semester Two

2017

SOLUTIONS

Calculator-free Solutions

1. (a) $\frac{d}{dx}(e^{\cos x} + 5)$

$= -\sin x \, e^{\cos x}$ ✓✓

(b) $\int (\sin x \, . \, e^{\cos x}) \; dx$

$= -\int (-\sin x \, . e^{\cos x}) \; dx$

$= -e^{\cos x} + c$ ✓✓ [4]

2. (a) $f'(x) = 2e^{2x} - \frac{1}{x}$ ✓

For max/min, $2e^{2x} - \frac{1}{x} = 0$ ✓

∴ $2xe^{2x} - 1 = 0$ ✓

$x = 0.5 \, e^{-2x}$

(b) $f''(x) = 4e^{2x} + \frac{1}{x^2}$

Since expression > 0 for all x values, ✓

then stationary point is a minimum. ✓ [5]

3. (a) $A = \int_0^k (2 - e^{-x}) - x \; dx$ ✓✓

(b) $A = \left[2x - \frac{e^{-x}}{-1} - \frac{x^2}{2}\right]_0^k$ ✓

$= (2k + e^{-k} - \frac{k^2}{2}) - 1$ ✓✓ [5]

4. (a) $\frac{5x^2}{2} - \frac{\sin 5x}{5} + c$ ✓✓

(b) $\left[\frac{e^{2x}}{2} - \frac{2x^{1.5}}{3}\right]_0^4$ ✓

$= \left[\frac{e^8}{2} - \frac{16}{3}\right] - \left[\frac{1}{2} - 0\right] = 0{\cdot}5e^8 - \frac{35}{6}$ ✓✓

(c) $2 \sin 2x$ ✓✓ [7]

5. (a) $x = \sin 2t + e^{-2t} + c$ ✓

$x(0) = 0+1 + c = 1$ ∴ $c = 0$ ✓

$x = \sin 2t + e^{-2t}$ ✓

(b) $a = -4\sin 2t + 4e^{-2t}$ ✓✓

(c) Assume $a = -k^2x$

Then $-4 \sin 2t + 4 e^{-2t} = -k^2(\sin 2t + e^{-2t})$

This leads to the result that $k^2 = 4$ and $k^2 = -4$.

Hence, relationship is false.

Or $a = -4(\sin 2t - 4e^{-2t}) \neq -4(\sin 2t + 4e^{-2t}) = -2^2x$ ✓✓ [7]

16. (a) $\frac{1}{3}$ ✓

(b) $\frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$ ✓✓

(c) $\mu = 6$ and $\text{Var}(X) = \int_3^9 \frac{1}{6}(x - 6)^2 \, dx = 3$

$s(x) = \sqrt{3}$ or $s(X) = \frac{6}{\sqrt{12}}$ by formula ✓✓

(d) $\mu(y) = 2(6) + 5 = 17$ ✓

$s(y) = 2\sqrt{3}$ ✓ [7]

17. (a) $P(390 < X < 410) = 0.15852$ ✓✓

(b) $P(X < 400 + k) = 0.96$ ✓

∴ $400 + k = 487.534$

∴ $k = 87.534$ ✓

(c) $m = 810$ $s = 100$ ✓✓ [6]

18. (a) $\dot{x} = 16 - 4t^3 + t^2$

$\ddot{x} = -12t^2 + 2t = 0$ ✓

∴ $t = 0$ or $\frac{1}{6}$ ✓

(b) $16 - 4t^3 + t^2 = 0$ ✓

$t = 1.68$ ✓

(c) $-12t^2 + 2t$ is a maximum when $-24t + 2 = 0$

ie when $t = \frac{1}{12}$ ✓

$x = 5.33$ ✓

(d) $\int_0^2 |\, 16 - 4t^3 + t^2 \,| \, dt = 22.323$ m ✓✓ [8]

19. (a) Normal curve ✓

(b) mean = 21 ✓

standard deviation = $\sqrt{(0.07)(300)(0.93)} = 4.4193$ ✓

(c) $\hat{p} = 0.07$ ✓

standard deviation = $\sqrt{\frac{(0.07)(0.93)}{300}} = 0.0147$ ✓✓

(d) This is a proportion of 0.15

$Z = \frac{0.15 - 0.07}{0.0147} = 5.44$

5.44 standard deviations above the mean is very unlikely.

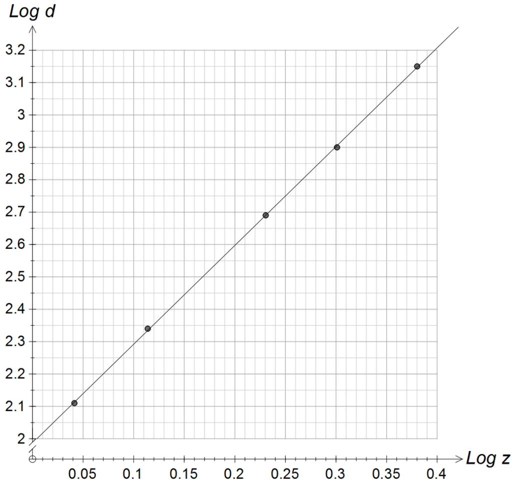
The testing method may need reviewing. ✓✓ [8]

14. (a)

z	1.1	1.3	1.7	2.0	2.4
d	130	220	490	800	1400
Log z	0.0414	0.1139	0.2305	0.3010	0.3802
Log d	2.11	2.34	2.69	2.90	3.15

✓✓

(b)



✓✓✓

(c) $\log d = 2 + 3\log z$ ✓✓

(d) $d = 100.z^3$ ✓✓ [9]

15. (a) $A = \frac{1}{2}\,3\,.5 \sin\theta = 7.5 \sin\theta$ ✓

$\frac{dA}{d\theta} = 7.5 \cos\theta$ ✓

(b) $\frac{dA}{dt} = \frac{dA}{d\theta} \cdot \frac{d\theta}{dt}$

$\therefore \frac{dA}{dt} = \left(7.5 \cos\frac{\pi}{2}\right)\pi = 0$ ✓✓

(c) Area has reached a maximum value. ✓ [5]

6. (a) $\log\left(\frac{a}{b}\right) + \log\left(\frac{b}{c}\right) + \log\left(\frac{c}{a}\right)$

$= \log\left(\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}\right)$ ✓

$= \log 1 = 0$ ✓

(b) $y = 1 - x$

$2^x = 3^{1-x}$

$x \log 2 = (1 - x)\log 3$ ✓

$x \log 2 + x \log 3 = \log 3$ ✓

$x = \frac{\log 3}{\log 2 + \log 3}$ ✓

$x = \frac{\log 3}{\log 6}$ as required ✓

$y = 1 - \frac{\log 3}{\log 6}$ ✓ [7]

7. (a) (i) 14 ✓

(ii) $14 + 6 - 6 = 14$ ✓✓

(iii) $2\int_0^8 f(x)\ dx = 40$ ✓✓

(b) $\int_8^{10} f(x)\ dx = -6$ ✓

$\int_4^8 f(x)\ dx = 6$ ✓

$\therefore$ $a = 4$, b = 8, $c = 10$ ✓ [8]

8. (a) A = Length x Width = $3x\cos 2x$ ✓

(b) $A = 3x \cos 2x$

$A' = 3x(-\sin 2x)(2) + 3\cos 2x = 0$ for max / min ✓

$6x\sin 2x = 3\cos 2x$ ✓

$2x \tan 2x = 1$ as required. ✓

(c) $A'' = -12 \sin 2x - 12x\cos 2x$ ✓

$= -12(\sin 2x + x\cos 2x)$ ✓

< 0 for maximum ✓

$\therefore$ $\sin 2x + x \cos 2x > 0$ as required [7]

Calculator-assumed Solutions

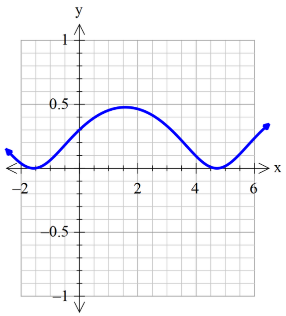
9. (a) Solve $0.9 = e^{-2k}$
$\therefore$ $k = 0.05268 = 0.0527$ (3 s.f.) ✓✓

(b) Solve $0.5 = e^{-0.05268\,t}$
$t = 13.153$ ✓✓
Half life is 13.153 years.

(c) $\dfrac{dM}{dt} = M_0\, e^{-kt}\,.\,(-k)$ ✓
At $t = 2$, $\dfrac{dM}{dt} = 20\,.\,e^{-0.0527\,(2)}\,.\,(-0.0527)$
$= -0.9486$ units of mass per year. ✓✓ [7]

10. (a) Only 2 results for each trial–single or married. ✓
(b) 0.6 ✓
(c) $\sqrt{(0.6)\dfrac{0.4}{40}} = \sqrt{0.006} = 0.07746$ ✓✓
(d) We can be 95% confident that the true proportion is p
where $0.6 - (1.96)(0.07746) < p < 0.6 + (1.96)(0.07746)$
ie $0.4482 < p < 0.7518$ ✓✓
(e) $n = \dfrac{(1.96)^2 \times 0.6 \times 0.4}{(0.05)^2}$ ✓
$n = 368.8$ ✓
Sample size needs to be 369. ✓ [9]

11. (a) From calculator, 0.142 ✓
(b) $\sin x$ has a minimum of –1. So, $2 + \sin x$ has a minimum of 1
So $\log(2 + \sin x)$ has a minimum of 0. ✓✓
(c)



✓✓✓

(d) By inspecting the graph, all of this curve is above the x axis. ✓
$\therefore$ since $\log\sqrt{2 + \sin x} = \dfrac{1}{2}\log(2 + \sin x)$ ✓
Area = 0.5(0.142) = 0.071 ✓ [9]

12. (a) Minimum when $f'(x) = 0$
$\therefore$ $2e^{2x} - 2ke^{-2x} = 0$ ✓
$\therefore$ $e^{4x} = k$
$\therefore$ $x = \dfrac{1}{4}\ln k$ ✓
$\therefore$ Minimum value is $e^{\frac{1}{2}\ln k} + k\,e^{-\frac{1}{2}\ln k}$
$= \sqrt{k} + \dfrac{k}{\sqrt{k}} = 2\sqrt{k}$ ✓
$\therefore$ Range is $y \geq 2\sqrt{k}$ ✓

(b) $\delta y \approx \dfrac{dy}{dx}.\delta x = (2e^{2x} - 6e^{-2x})(0.01)$ ✓✓
= 1.09 (2 decimal places) ✓
(c) $f(2) = 54.653$
$f(2.01) = 55.755$ ✓
Change is 1.10 (2 decimal places) ✓ [9]

13. (a) Not equally likely outcomes, so biased. ✓
(b) $\dfrac{7}{8}$ ✓
(c) E(X) = 1.875 Var(X) = $1.0533^2 = 1.109$ ✓✓
(d) $P(Y = 4) = {}^5C_4\,(0.25)^4(0.75) = \dfrac{15}{4^5} = 0.0146$ ✓✓
(e) P(five 4s) = ${}^5\mathbf{C}_5\left(\dfrac{1}{8}\right)^5\left(\dfrac{7}{8}\right)^0 = 0.00003$ ✓ ✓
[8]