Rossmoyne Senior High School

Semester One Examination, 2015

Question/Answer Booklet

MATHEMATICS SPECIALIST 3C
Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ____

Your name ____

Time allowed for this section
Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	6	6	50	50	33⅓
Section Two: Calculator-assumed	13	13	100	100	66⅔
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2015*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section Two: Calculator-assumed **(100 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 7 **(6 marks)**

(a) Determine $g(4)$ if $g(t)=\int_1^t \frac{1}{x}\,dx$, $t>0$. (2 marks)

$g(t)=\ln(t)$

$g(4)=\ln 4$

(b) Let $y=\log_{10} 2x$.

Show that $y=\frac{\ln 2x}{\ln 10}$ and hence determine the exact value of $\frac{dy}{dx}$ when $x=3$. (4 marks)

$y=\log_{10} 2x$

$10^y=2x$

$y\ln 10=\ln 2x$

$y=\frac{\ln 2x}{\ln 10}$

$\frac{dy}{dx}=\frac{2}{2x\ln 10}\Big|_{x=3}$

$=\frac{1}{3\ln 10}$

Question 8 **(8 marks)**

A curve is defined parametrically as $x = t^2 + t$ and $y = 2t - 1$, where $-1 \le t \le 2$.

(a) Determine an expression for $\frac{dy}{dx}$ in terms of t. (2 marks)

$$\frac{dx}{dt} = 2t + 1 \qquad \frac{dy}{dt} = 2$$

$$\frac{dy}{dx} = \frac{2}{2t+1}$$

(b) Determine the equation of the tangent to the curve when $x = 2$. (4 marks)

$$t^2 + t = 2$$
$$t^2 + t - 2 = 0$$
$$(t+2)(t-1) = 0$$
$$\cancel{t = -2},\ t = 1$$

$$y = 2 - 1 = 1$$

$$\frac{dy}{dx} = \frac{2}{2+1} = \frac{2}{3}$$

$$y - 1 = \frac{2}{3}(x - 2)$$
$$y = \frac{2}{3}x - \frac{1}{3}$$

(c) Determine the coordinates of the point on the curve where the tangent is parallel to the y-axis. (2 marks)

$$\frac{dy}{dx} = \frac{2}{2t+1}$$

$$2t + 1 = 0 \Rightarrow t = -0.5$$

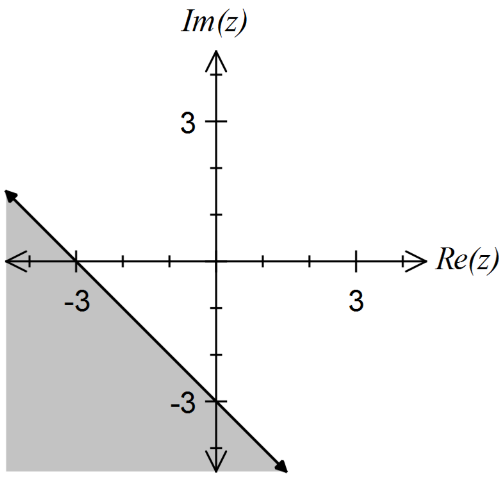
$$x = -0.25,\ y = -2$$

At $(-0.25, -2)$

Question 9 **(8 marks)**

Draw sketches of the following sets of points in the complex plane.

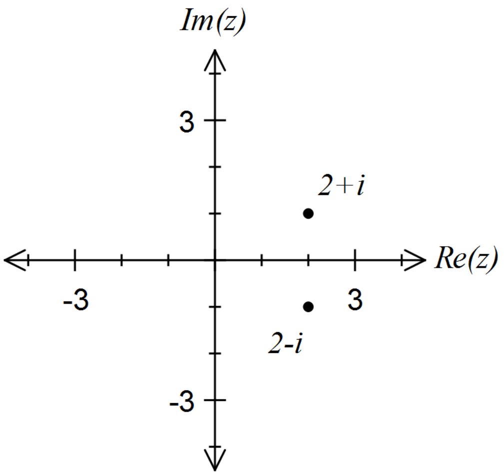
(a) $\{z: \ |z+3+3i| \le |z|\}$ (3 marks)



NB

$z+3+3i = z-(-3-3i)$

(b) $\{z: z^2 = 4z-5\}$ (2 marks)

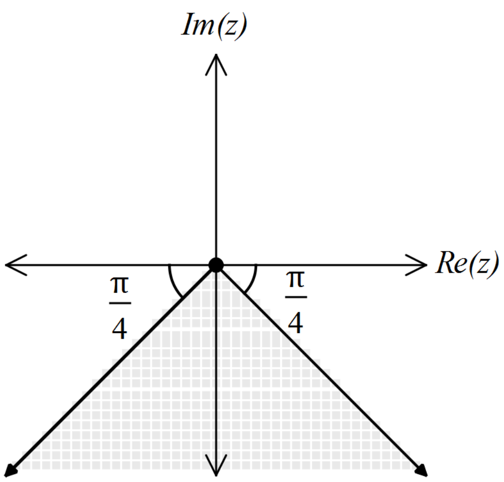


NB

$z^2 - 4z + 5 = 0$

$z = 2+i,\ z = 2-i$

(c) $\left\{z: -\frac{\pi}{4} \le \arg(iz) \le \frac{\pi}{4}\right\}$ (3 marks)



Question 10 **(6 marks)**

(a) If $y = \sin(x+y)$, show that $\dfrac{dy}{dx} = \dfrac{\cos(x+y)}{1-\cos(x+y)}$. (3 marks)

$$\frac{dy}{dx} = \left(1+\frac{dy}{dx}\right)\cos(x+y)$$

$$\frac{dy}{dx} = \cos(x+y) + \frac{dy}{dx}\cos(x+y)$$

$$\frac{dy}{dx}\big(1-\cos(x+y)\big) = \cos(x+y)$$

$$\frac{dy}{dx} = \frac{\cos(x+y)}{1-\cos(x+y)}$$

(b) Determine the gradient of the curve $y = xy + \dfrac{1}{x}$ at the point where $x = 2$. (3 marks)

When $x = 2,\ y = 2y + \dfrac{1}{2} \ \Rightarrow \ y = -\dfrac{1}{2}$

$$\frac{dy}{dx} = y + x\frac{dy}{dx} - \frac{1}{x^2}$$

$$\frac{dy}{dx} = -\frac{1}{2} + 2\frac{dy}{dx} - \frac{1}{2^2}$$

$$\frac{dy}{dx} = \frac{3}{4}$$

Question 11 **(7 marks)**

The points A and B have position vectors $3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ respectively.

(a) Determine a vector equation for the straight line passing through A and B. (2 marks)

$$\begin{aligned}\overrightarrow{AB} &= \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} - (3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \\ &= -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}\end{aligned}$$

$$\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \lambda(-2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})$$

(b) Write your answer to (a) in its parametric equivalent and hence, or otherwise, express the Cartesian equation of the line in the form $\dfrac{x-a}{p} = \dfrac{y-b}{q} = \dfrac{z-c}{r}$. (3 marks)

$$\begin{aligned}x &= 3 - 2\lambda \\ y &= -2 + 5\lambda \\ z &= 2 - 4\lambda\end{aligned}$$

$$\lambda = \frac{x-3}{-2} = \frac{y+2}{-5} = \frac{z-2}{4}$$

(c) Determine a unit vector parallel to the straight line in (a). (2 marks)

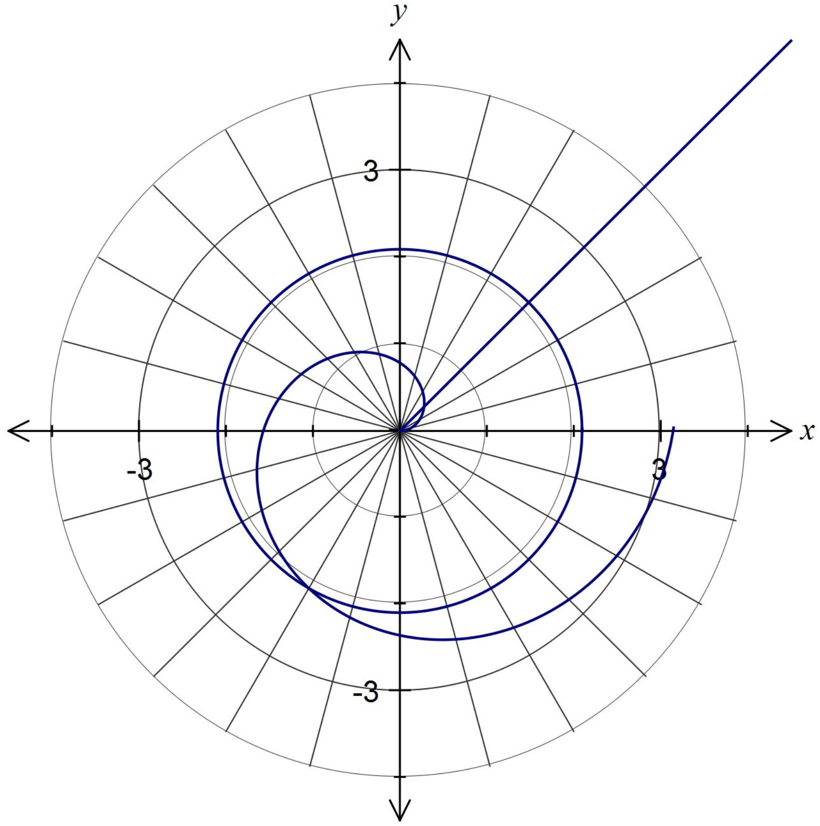
$$|\overrightarrow{AB}| = \sqrt{(-2)^2 + 5^2 + (-4)^2}$$

$$\begin{aligned}\hat{\mathbf{r}} &= \frac{1}{3\sqrt{5}}(-2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}) \\ &= \frac{\sqrt{5}}{15}(-2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})\end{aligned}$$

Question 12 **(8 marks)**

(a) Draw sketches of the polar graphs given by $\theta = \frac{\pi}{4}$, $r = \frac{2\pi}{3}$ and $r = \frac{\theta}{2}$ on the axes below.

(5 marks)



(b) If P is the point of intersection of the graphs of $\theta = \frac{\pi}{4}$ and $r = \frac{2\pi}{3}$, and Q is the point of intersection of the graphs of $r = \frac{2\pi}{3}$ and $r = \frac{\theta}{2}$, determine the length of PQ.

(3 marks)

$$P: \; r = \frac{2\pi}{3}, \;\; \theta = \frac{\pi}{4}$$

$$Q: \; r = \frac{2\pi}{3}, \;\; \theta = \frac{4\pi}{3}$$

$$PQ^2 = \left(\frac{2\pi}{3}\right)^2 + \left(\frac{2\pi}{3}\right)^2 - 2\left(\frac{2\pi}{3}\right)\left(\frac{2\pi}{3}\right)\cos(2$$

$$= 2\left(\frac{2\pi}{3}\right)^2\left(1 - \cos\left(\frac{11\pi}{12}\right)\right)$$

$$PQ = 4.153$$

Question 13 **(9 marks)**

A plane Π contains the two lines $\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(-\mathbf{i} + \mathbf{j} + 3\mathbf{k})$.

(a) Write down a vector equation of the plane Π. (1 mark)

$$\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) + \mu(-\mathbf{i} + \mathbf{j} + 3\mathbf{k})$$

(b) The point $8\mathbf{i} + 2\mathbf{j} + c\mathbf{k}$ lies in the plane Π. Determine the value of the constant c.
(3 marks)

Equating $\mathbf{i}$ and $\mathbf{j}$ coefficients:

$$1 + 2\lambda - \mu = 8$$
$$-1 + 3\lambda + \mu = 2$$

$$\lambda = 2,\ \mu = -$$

$$c = 2 + 2(-$$

(c) The vector $a\mathbf{i} + b\mathbf{j} + \mathbf{k}$ is perpendicular to the plane Π. Determine the values of the constants a and b. (3 marks)

$$\begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = 0 \Rightarrow 2a + 3b - 1 = 0$$

$$\begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = 0 \Rightarrow -a + b + 3 = 0$$

$$a = 2,\ b = -1$$

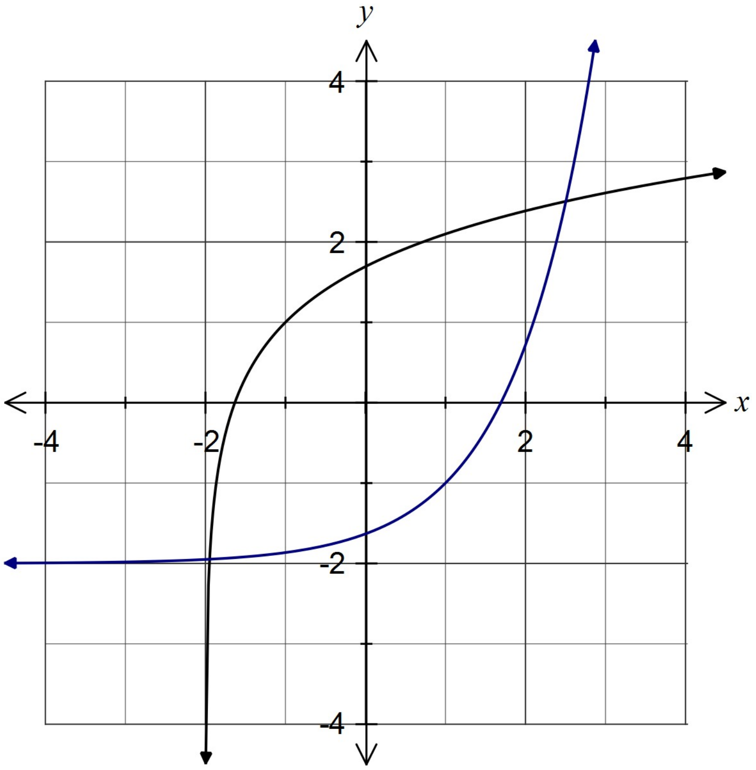
(d) State the equation of the plane Π in the form $\mathbf{r} \cdot \mathbf{n} = k$. (2 marks)

$$\mathbf{r} \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} - \mathbf{j}$$

$$\mathbf{r} \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 5$$

Question 14 **(7 marks)**

The graph of the function f, defined by $f(x) = \ln(x+2)+1$, is shown below.



(a) Determine the function f^{-1}, the inverse of f, and add a sketch of the graph of $y = f^{-1}(x)$ to the axes above. (4 marks)

$$x = \ln(y+2)+1$$
$$e^{x-1} = y+2$$

$$f^{-1}(x) = e^{x-1} - 2$$

(b) Determine the area of the region bounded by the graphs of f and f^{-1}. (3 marks)

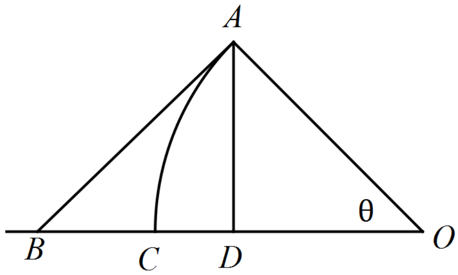
Functions f and f^{-1} intersect when $x = -1.94753$ and $x = 2.50524$.

$$\text{Area} = \int_{-1.94753}^{2.50524} \left((\ln(x+2)+1) - (e^{x-1} - 2)\right)dx$$
$$= 11.3889$$
$$\approx 11.4 \text{ sq units}$$

Question 15 **(6 marks)**

In the diagram, arc AC subtends θ radians ($\theta < \pi/2$) in a circle of radius 1 unit and centre O.

The point B lies on OC extended so that AB is a tangent to the circle at A and the point D lies on OC so that AD is perpendicular to OC.



(a) By comparing the lengths of AB, AC and AD, establish the inequalities $\sin\theta < \theta < \tan\theta$. (3 marks)

$\tan\theta = \frac{AB}{AO} \Rightarrow AB = \tan\theta \ (AO = 1)$

$AC = r\theta = \theta \ (r = 1)$

$\sin\theta = \frac{AD}{AO} \Rightarrow AD = \sin\theta \ (AO = 1)$

Since it can be seen that $AD < AC < AB$

then $\sin\theta < \theta < \tan\theta$.

(b) Use the inequalities $\sin\theta < \theta < \tan\theta$ to show that as $\theta \to 0$, $\frac{\sin\theta}{\theta} \to 1$. (3 marks)

$\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$

$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

$1 > \frac{\sin\theta}{\theta} > \cos\theta$

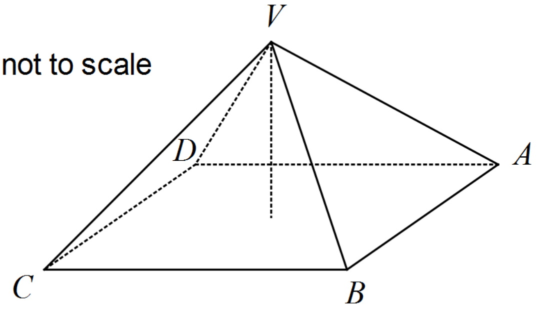
But as $\theta \to 0$ $\cos\theta \to 1$

and so by the sandwich principle, $\frac{\sin\theta}{\theta}$ –

Question 16 **(11 marks)**

A pyramid has its vertex V at $-4.5\mathbf{i}+6.5\mathbf{j}+7\mathbf{k}$ and the corners of its square base $ABCD$ at $\mathbf{i}+\mathbf{j}+2\mathbf{k}$, $-\mathbf{j}+4\mathbf{k}$, $2\mathbf{i}+6\mathbf{k}$ and $3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ respectively.

Diagram not to scale



(a) Determine the position vectors of

(i) M, the midpoint of side AB. (1 mark)

$$\frac{(\mathbf{i}+\mathbf{j}+2\mathbf{k})+(-\mathbf{j}+4\mathbf{k})}{2}=0.5\mathbf{i}+$$

(ii) N, the point of intersection of the diagonals of the square base. (1 mark)

Midpoint of AC:

$$\frac{(\mathbf{i}+\mathbf{j}+2\mathbf{k})+(2\mathbf{i}+6\mathbf{k})}{2}=1.5\mathbf{i}+0.5\mathbf{j}+4\mathbf{k}$$

(b) Show that VN is perpendicular to both AN and MN. (3 marks)

$$\overrightarrow{VN}=\begin{bmatrix}1.5\\0.5\\4\end{bmatrix}-\begin{bmatrix}-4.5\\6.5\\7\end{bmatrix}=\begin{bmatrix}6\\-6\\-3\end{bmatrix}$$

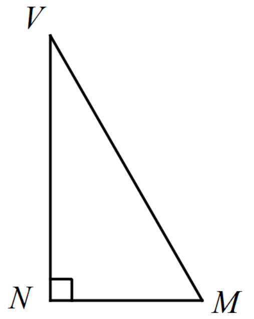
$$\overrightarrow{AN}=\begin{bmatrix}1.5\\0.5\\4\end{bmatrix}-\begin{bmatrix}1\\1\\2\end{bmatrix}=\begin{bmatrix}0.5\\-0.5\\2\end{bmatrix}$$

$$\overrightarrow{VN}\cdot\overrightarrow{AN}=\begin{bmatrix}6\\-6\\-3\end{bmatrix}\cdot\begin{bmatrix}0.5\\-0.5\\2\end{bmatrix}=3+3-6=0 \Rightarrow \text{perpend}$$

$$\overrightarrow{MN}=\begin{bmatrix}1.5\\0.5\\4\end{bmatrix}-\begin{bmatrix}0.5\\0\\3\end{bmatrix}=\begin{bmatrix}1\\0.5\\1\end{bmatrix}$$

$$\overrightarrow{VN}\cdot\overrightarrow{MN}=\begin{bmatrix}6\\-6\\-3\end{bmatrix}\cdot\begin{bmatrix}1\\0.5\\1\end{bmatrix}=6-3-3=0 \Rightarrow \text{perpendi}$$

(c) Calculate the angle between the square base $ABCD$ and the sloping face ABV. (3 marks)



Require angle VMN:

Use $\overrightarrow{MN}$ from (b)

$$\overrightarrow{MV} = \begin{bmatrix} -4.5 \\ 6.5 \\ 7 \end{bmatrix} - \begin{bmatrix} 0.5 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 6.5 \\ 4 \end{bmatrix}$$

Using CAS, angle between $\overrightarrow{MN}$ and $\overrightarrow{MV}$ is 80.54°

(d) Determine the volume of the pyramid. (3 marks)

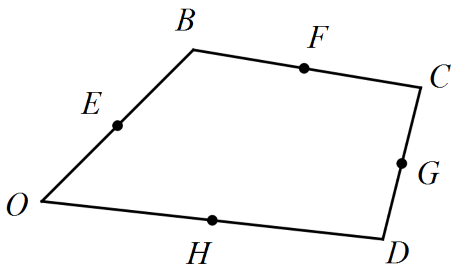
$$\begin{aligned} V &= \tfrac{1}{3}Ah \\ &= \frac{1}{3} \times |AB|^2 \times |VN| \end{aligned}$$

Using CAS, $|AB| = 3$, $|VN| = 9$

$$\begin{aligned} V &= \frac{1}{3} \times 3^2 \times 9 \\ &= 27 \text{ cubic units} \end{aligned}$$

Question 17 **(7 marks)**

In the diagram below, E, F, G and H are midpoints of the sides of the quadrilateral $OBCD$.



Let $\overrightarrow{OB} = 2\mathbf{b}$, $\overrightarrow{OC} = 2\mathbf{c}$ and $\overrightarrow{OD} = 2\mathbf{d}$.

(a) Show that $\overrightarrow{OF} = \mathbf{b} + \mathbf{c}$. (2 marks)

$$\begin{aligned}\overrightarrow{OF} &= \overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{BC}\\ &= \overrightarrow{OB} + \tfrac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OB}\right)\\ &= 2\mathbf{b} + \tfrac{1}{2}(2\mathbf{c} - 2\mathbf{b})\\ &= \mathbf{b} + \mathbf{c}\end{aligned}$$

(b) Determine $\overrightarrow{OG}$ in terms of $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. (2 marks)

$$\begin{aligned}\overrightarrow{OG} &= \overrightarrow{OD} + \tfrac{1}{2}\overrightarrow{DC}\\ &= \overrightarrow{OD} + \tfrac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OD}\right)\\ &= 2\mathbf{d} + \tfrac{1}{2}(2\mathbf{c} - 2\mathbf{d})\\ &= \mathbf{d} + \mathbf{c}\end{aligned}$$

(c) Prove that $EFGH$ is a parallelogram. (3 marks)

$$\begin{aligned}\overrightarrow{EF} &= \overrightarrow{OF} - \overrightarrow{OE}\\ &= \mathbf{b} + \mathbf{c} - \mathbf{b}\\ &= \mathbf{c}\end{aligned}$$

$$\begin{aligned}\overrightarrow{HG} &= \overrightarrow{OG} - \overrightarrow{OH}\\ &= \mathbf{d} + \mathbf{c} - \mathbf{d}\\ &= \mathbf{c}\end{aligned}$$

Hence EF is parallel to HG and of equal length and so $EFGH$ must be a parallelogram.

Question 18 **(7 marks)**

As water slowly leaks out of a hole in the bottom of a cylindrical tank, the rate of change of the depth of water in the tank, h cm, can be modelled by the differential equation $\frac{dh}{dt} = -k\sqrt{h}$, $0 \le t \le 900$ where t is the time in seconds and $k = 0.003$.

Initially, the depth of water in the can was 36 cm.

(a) Show that $h = \left(6 - \frac{kt}{2}\right)^2$. (4 marks)

$$\int \frac{dh}{\sqrt{h}} = \int -k\,dt$$

$$2\sqrt{h} = -kt + c$$

$$2\sqrt{36} = c \Rightarrow c = 12$$

$$\sqrt{h} = \frac{12 - kt}{2}$$

$$h = \left(6 - \frac{kt}{2}\right)^2$$

(b) Determine the rate at which the depth of water in the tank is decreasing after ten minutes. (3 marks)

$$h(600) = \left(6 - \frac{0.003 \times 600}{2}\right)^2$$

$$= 26.01$$

$$\frac{dh}{dt} = -0.003 \times \sqrt{26.01}$$

$$= -0.0153$$

Decreasing at 0.0153 cr

Question 19 **(10 marks)**

(a) Consider the mathematical statements $a \cdot b = b \cdot a$ and $|a \cdot b| = |a| \cdot |b|$, for all real numbers a and b. Briefly explain why one of these statements is an axiom and the other is a theorem. (2 marks)

The first statement $a \cdot b = b \cdot a$ is an axiom as it is considered to be a basic truth, whereas the second statement is a theorem as it can be proved using axioms and other theorems.

(b) One of the following three mathematical statements is true and the other two are false.

Determine which the true statement is and justify your answer. (3 marks)

A: If n is a positive integer, then $n^2 + 4n + 4$ is never a multiple of 5.

B: If n is a positive integer, then $n^2 + 3n + 13$ is a prime number.

C: If n is a positive integer, then $n^2 + n - 1$ is never a multiple of 3.

C is the true statement as counterexamples exist for statements A and B:

A: When $n = 9, 10, 13, 18...$ statement is false

B: When $n = 3, 8, 13, ...$ statement is false

(c) Prove by exhaustion that $\left|\frac{p}{q}\right| = \frac{|p|}{|q|}$ for all real numbers p and q, $q \neq 0$. (5 marks)

Four cases to consider:

1. $p \geq 0,\ q > 0$ then $\left|\frac{p}{q}\right| = \frac{p}{q} = \frac{|p|}{|q|}$ since $|p| = p,\ |q| = q$.
2. $p \geq 0,\ q < 0$ then $\left|\frac{p}{q}\right| = -\frac{p}{q} = \frac{p}{-q} = \frac{|p|}{|q|}$ since $|p| = p,\ |q| = -q$.
3. $p < 0,\ q > 0$ then $\left|\frac{p}{q}\right| = -\frac{p}{q} = \frac{-p}{q} = \frac{|p|}{|q|}$ since $|p| = -p,\ |q| = q$.
4. $p < 0,\ q < 0$ then $\left|\frac{p}{q}\right| = \frac{p}{q} = \frac{-p}{-q} = \frac{|p|}{|q|}$ since $|p| = -p,\ |q| = -q$.

Hence $\left|\frac{p}{q}\right| = \frac{|p|}{|q|}$ for all real numbers p and q. $q \neq 0$.

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

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