MATHEMATICS
SPECIALIST
UNIT 1

Semester One

2019

SOLUTIONS

Calculator-free Solutions

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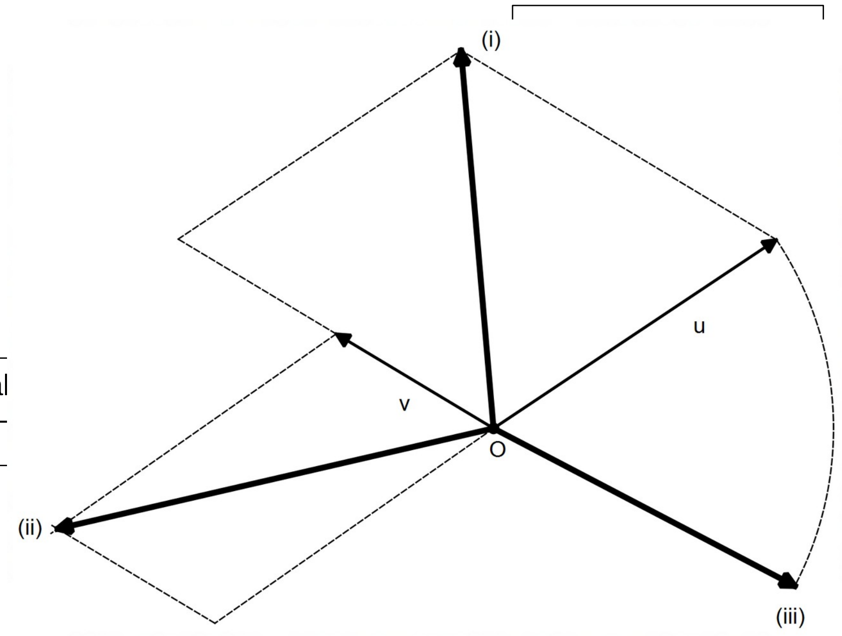
v and –



magnitude |u|

[6]

2. (a) (i) $\begin{pmatrix}3\\-5\end{pmatrix}=k\begin{pmatrix}4\\\alpha\end{pmatrix}$ ✓

$\therefore k=\frac{3}{4}\rightarrow\alpha=\frac{-5}{k}=\frac{-20}{3}$ ✓

(ii) $\begin{pmatrix}4\\\alpha\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}=0$ ✓

$\therefore 4+\alpha=0\rightarrow\alpha=-4$ ✓

(iii) $\overrightarrow{PQ}=\begin{pmatrix}1\\\alpha+5\end{pmatrix}$

since the *x*–coordinate is already 1 unit in length, then the y–coordinate must be zero. ✓

$\therefore\alpha=-5$ ✓

(iv) PQ as base $\Rightarrow$ |OP| = |OQ|

$\left|\begin{matrix}3\\-5\end{matrix}\right|=\left|\begin{matrix}4\\\alpha\end{matrix}\right|\rightarrow\sqrt{34}=\sqrt{16+\alpha^2}$ ✓

$\therefore\alpha^2=18\rightarrow\alpha=\pm3\sqrt{2}$ ✓

2. (b) (i) Solving simultaneously (any method, elimination shown below):

$$\begin{array}{l} u\text{¿}-6i-2j\times 3 \\ v\text{¿}\ 2i\ +3j\times 2 \end{array} \rightarrow \begin{array}{l} 3u\text{¿}-18i-6j \\ 2v\text{¿}\ 4i\ \ +6j \end{array}\downarrow\text{¿}$$

$$\therefore 3u+2v=-14i \rightarrow i=\frac{-3}{14}u-\frac{1}{7}v$$ ✓✓

similarly (or by substitution):

$$\begin{array}{l} u\text{¿}-6i-2j\times 1 \\ v\text{¿}\ 2i\ +3j\times 3 \end{array} \rightarrow \begin{array}{l} u\ \text{¿}-6i-2j \\ 3v\text{¿}\ 6i\ +9j \end{array}\downarrow\text{¿}$$

$$\therefore u+3v=7j \rightarrow j=\frac{1}{7}u+\frac{3}{7}v$$ ✓✓

(ii) $$r=14\left(\frac{-3}{14}u-\frac{1}{7}v\right)+7\left(\frac{1}{7}u+\frac{3}{7}v\right)$$

$$\therefore r=-2u+v$$ ✓✓ [14]

3. (a) (i) $$20!-18!=20\times 19\times 18!-18!$$ ✓

$$\text{¿}(20\times 19-1)\times 18!$$

$$\text{¿}(380-1)k=379k$$ ✓

(ii) $$\frac{{}^{20}P_3}{{}^{21}C_3}=\frac{20!}{3!}\div\frac{21!}{3!\times 18!}$$ ✓

$$\text{¿}\frac{20!}{3!}\times\frac{3!\times 18!}{21\times 20!}=\frac{k}{21}$$ ✓

(b) RHS $$\text{¿}\begin{pmatrix} n \\ n-r \end{pmatrix}=\frac{n!}{(n-r)!\times[n-(n-r)]!}$$ ✓

$$\text{¿}\frac{n!}{(n-r)![r]!}$$ ✓

$$\text{¿}\begin{pmatrix} n \\ r \end{pmatrix}=\text{¿}$$ LHS [6]

4. (a) If $m<1$, then $m>m^2$. ✓

It is NOT always true because it does not work for negatives. ✓

e.g. $m=-2<1 \rightarrow m^2=4>m \therefore$ false ✓

The converse is always true for $0<m<1$ ✓

(b) If the parallelogram is not a rectangle, then it does not have congruent diagonals. ✓

Yes it is always true as only squares and rectangles have congruent diagonals. ✓

(c) For all rational numbers ✓, there exists two integer numbers a and b ✓ such that p is the quotient of a and b. [8]

5. (a) (i) $2^6=1+6+15+20+15+6+1=64$ ✓

(ii) $11^5=(10+1)^5$

¿ $10^5+5\times10^4+10\times10^3+10\times10^2+5\times10+1^5$ ✓

¿ $100\,000+50\,000+10\,000+1\,000+50+1$

¿ 161051 ✓

(b) (i) $x=3$ since ${}^6C_3=20$ ✓

(ii) $x=7$ since ${}^7C_5=21$ ✓

(iii) $x=8$ since ${}^8C_2={}^8C_6$ ✓

(c) $(2x-y)^5$

¿ $(2x)^5+5(2x)^4(-y)^1+10(2x)^3(-y)^2+10(2x)^2(-y)^3+5(2x)^1(-y)^4+(-y)^5$ ✓

¿ $32x^5-80x^4y+80x^3y^2-40x^2y^3+10xy^4-y^5$ ✓✓

(d) (i) ${}^8C_5=56$ ✓

(ii) ${}^2C_2\times{}^6C_3=1\times20=20$ ✓

(iii) ${}^3C_2\times{}^5C_3+{}^3C_3\times{}^5C_2$ ✓

¿ $3\times10+1\times10=30+10=40$ ✓ [13]

6. (a) $\overrightarrow{AD}=\frac{2}{5}\overrightarrow{AB}=\frac{2}{5}(b-a)$ ✓

$\overrightarrow{CD}=\overrightarrow{CO}+\overrightarrow{OA}+\overrightarrow{AD}=\frac{-1}{2}b+a+\frac{2}{5}(b-a)$ ✓

$\therefore \overrightarrow{CD}=\frac{3}{5}a-\frac{1}{10}b$ ✓

(b) $\overrightarrow{OC}+\overrightarrow{CE}=\overrightarrow{OE}\rightarrow\overrightarrow{OC}+\beta\overrightarrow{CD}=\alpha\overrightarrow{OA}$ given

$\therefore \frac{1}{2}b+\beta\left(\frac{3}{5}a-\frac{1}{10}b\right)=\alpha a$ ✓

$\times10\rightarrow 5b+6\beta a-\beta b=10\alpha a$

$\rightarrow(6\beta-10\alpha)a=(\beta-5)b$

since a and b are non–parallel, then:

$\beta-5=0\rightarrow\beta=5$ ✓

$6\beta-10\alpha=0\rightarrow\alpha=\frac{3}{5}\beta=3$ ✓ [6]

<u>Calculator-assumed Solutions</u>

7. (a) ABC collinear $\Rightarrow$ AB // BC

$$\overrightarrow{AB}=\begin{pmatrix}-2\\4\end{pmatrix}-\begin{pmatrix}4\\-5\end{pmatrix}=\begin{pmatrix}-6\\9\end{pmatrix}$$

$$\overrightarrow{BC}=\begin{pmatrix}-6\\10\end{pmatrix}-\begin{pmatrix}-2\\4\end{pmatrix}=\begin{pmatrix}-4\\6\end{pmatrix}$$

$$\therefore \begin{pmatrix}-6\\9\end{pmatrix}=k\begin{pmatrix}-4\\6\end{pmatrix}\rightarrow \begin{matrix}k=\frac{-6}{-4}=\frac{3}{2}\\ k=\frac{9}{6}=\frac{3}{2}\end{matrix}$$ ✓✓

Since k is unique, then AB // BC and hence ABC collinear. ✓

(b) $|AB|=\left|\begin{matrix}-6\\9\end{matrix}\right|=3\left|\begin{matrix}-2\\3\end{matrix}\right|$ and $|BC|=\left|\begin{matrix}-4\\6\end{matrix}\right|=2\left|\begin{matrix}-2\\3\end{matrix}\right|$ ✓

$\therefore AB:BC=3:2$ ✓ [5]

8. (a) ∠PFO = 35° ✓

Because ΔOFP is isosceles since |OP| = |OF| = radii ✓

(b) ∠FEP = 55° ✓

Since ∠FOP = 110° from ΔOFP, and the angle at the centre is double the size of the angle at the edge. ✓

(c) ∠PQF = ∠FEP = 55° ✓

Angles at the circumference within the same segment are congruent. ✓

(d) ∠CFP = ∠FEP = 55° ✓

The alternate segment theorem ✓

(e) |GC| = |CF| = 11 – |FB| = 11 – 8 = 3 cm ✓

Tangents to a circle from the same external point are congruent. ✓

(f) $|AM| \times (|AM| + 2 \times \text{radius}) = |AH|^2$

$|AM| \times (|AM| + 8) = 5^2$ ✓

$|AM|^2 + 8|AM| - 25 = 0$

CAS $\Rightarrow$ $|AM| = -4 \pm \sqrt{41}$ ✓

$\therefore$ $|AM| = \sqrt{41} - 4 \approx 2.40$ cm only solution ✓ [13]

9. (a) (i) Divisible by 3 and 5 = divisible by 15

100 ÷ 15 = 6.6 $\Rightarrow$ only 6 elements are divisible by 15 ✓

Therefore, assuming every other element is chosen

instead of those 6, we need 100 – 6 +1 = 95 elements ✓

(ii) Divisible by 3 = 100 ÷ 3 = 33.3 $\Rightarrow$ 33 elements

Divisible by 5 = 100 ÷ 5 = 20 elements ✓

Divisible by 3 or 5 = 33 + 20 – 6 = 47 elements ✓

Assuming the other 53 elements are chosen first,

then 53 + 1 = 54 elements must be chosen ✓

(b) Assuming the highest numbers are chosen first:

100 + 99 + 98 + … + 91 + 90 = 955 ✓

If 89 is chosen next then the sum exceeds 1000. ✓

Therefore, a maximum of 11 elements must be chosen. ✓ [8]

10. (a) $n(M \cup C) = n(M) + n(C) - n(M \cap C)$ ✓

14 334 ✓ = 7 531 + 9 885 – $n(M \cap C)$

$\therefore n(M \cap C)$ = 3 082 households ✓

(b) $n(M \cup C \cup B) = n(M) + n(C) + n(B)$

$- n(M \cap C) - n(M \cap B) - n(C \cap B)$

$+ n(M \cap C \cap B)$ ✓

$\therefore n(M \cup C \cup B) =$ ¿ 7 531 + 9 885 + 4 977 – 3 082 – 2 252 – 4 310 + 1 724

= 14 473 that have all three ✓

Therefore, 16 366 – 14 473 = 1 893 households have neither ✓ [6]

11. (a) (i) ${}^{36}P_4 = 1\,413\,720 \vee \left({}^{36}C_4 \times 4!\right)$ ✓

(ii) ${}^{10}C_2 \times {}^{26}C_2 \times 4! = 351\,000$ ✓✓

(iii) ${}^{5}C_1 \times {}^{34}P_2 \times {}^{4}C_1 = 22\,440$ ✓✓

(b) II and III ✓✓

(c) ${}^{x+1}P_3 = {}^{4}C_3 \times {}^{x}P_2$

$$\frac{(x+1)!}{(x+1-3)!} = 4 \times \frac{x!}{(x-2)!}$$ ✓

$$\frac{(x+1)\times x!}{(x-2)!}=4\frac{x!}{(x-2)!}$$ ✓

$$(x+1)=4\rightarrow x=3$$ ✓

11. (d) LHS $\text{¿}\dfrac{n!}{(n-2)!}+2n\times\dfrac{(n-1)!}{(n-1)!}$ ✓

$$\text{¿}\frac{n!}{(n-2)!}\times\frac{(n-1)}{(n-1)}+\frac{2n!}{(n-1)!}$$ ✓

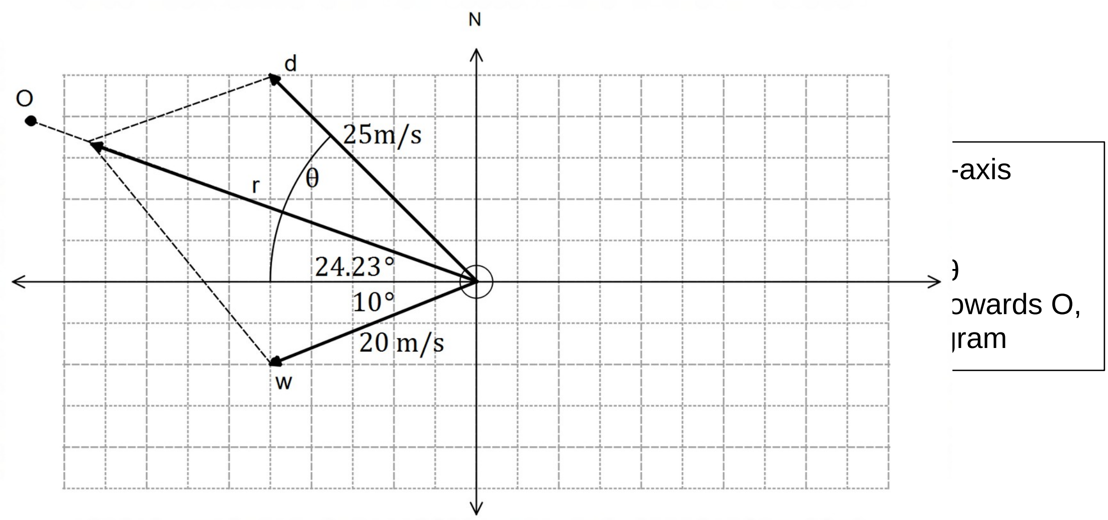
$$\text{¿}\frac{n!\times(n-1)+2n!}{(n-1)!}=\frac{n!(n-1+2)}{(n-1)!}$$ ✓

$$\text{¿}\frac{n!\times(n+1)}{(n+1-2)!}=\frac{(n+1)!}{(n+1-2)}={}^{n+1}P_2=\text{¿}\ \text{RHS}$$ ✓ [14]

12. (a) $w=\begin{pmatrix}-20\cos 10°\\-20\sin 10°\end{pmatrix}$

Hovering speed $\text{¿}-w=\begin{pmatrix}20\cos 10°\\20\sin 10°\end{pmatrix}$ ✓✓

(b) (i)



(ii) $w=\begin{pmatrix}-20\cos 10°\\-20\sin 10°\end{pmatrix} d=\begin{pmatrix}-25\cos\theta\\25\sin\theta\end{pmatrix} r=\begin{pmatrix}-r\cos 24.23°\\r\sin 24.23\end{pmatrix}$ ✓✓✓

(iii) $\begin{matrix}-r\cos 24.23° & \text{¿}-20\cos 10° & -25\cos\theta\\ r\sin 24.23° & \text{¿}-20\sin 10° & 25\sin\theta\end{matrix}$

$$\therefore \begin{matrix}25^2\cos^2\theta & \text{¿} & (r\cos 24.23°+20\cos 10°)^2\\ 25^2\sin^2\theta & \text{¿} & (r\sin 24.23°+20\sin 10°)^2\end{matrix}$$

$$\rightarrow 25^2=(r\cos 24.23°+20\cos 10°)^2+(r\sin 24.23°+20\sin 10°)^2$$ ✓✓

CAS $\rightarrow r=38.8621$ m/s OR $r=-5.7897$ m/s ✓

$\rightarrow \theta=46.64°$ OR $\theta=2.94°$ ✓

$\therefore time=\frac{d}{v}=\frac{\sqrt{1200^2+540^2}}{38.8621}=33.86$ seconds ✓

bearing ¿ $270°+\theta=308.86°T$ ✓ [14]

13. (a) $\sum F_y = 300\sin 62° + 252\sin 56°$ ✓

¿$473.8\,N$ ✓

Since $473.8\,N < 500\,N$ the machinery is not moving upwards ✓

(b) No horizontal component needed $\Rightarrow$ $\sum F_x = 0$

$\therefore\ 400\cos 62° = x\cos 56°$ ✓

$\rightarrow x = \dfrac{400\cos 62°}{\cos 56°} = 335.82\,N$ ✓

(c) $\sum F_y = 400\sin 62° + 335.82\sin 56°$ ✓

¿$631.59\,N$ ✓ [7]

14. (a) (i) Let $n \in N$ with $n = 2k+1 =$¿ odd ✓

Then $n^2 + 1 = (2k+1)^2 + 1$

¿$4k^2 + 4k + 2$ ✓

¿$2(2k^2 + 2k + 1)$ ✓

Since 2 is a factor, then n^2+1 is divisible by 2, and

hence the conjecture is true $\forall\, n \in N$ ✓

(ii) Contrapositive statement:

"if n^2+1 is odd, then *n* is even." ✓

Let $n^2 + 1 = odd = 2k + 1$

$\therefore\ n^2 = 2k$ ✓

$\Rightarrow n^2 = even \Rightarrow n = even$ ✓

Since the contrapositive statement is true $\forall\, n \in N$,
then the original conjecture is true $\forall\, n \in N$ ✓

(b) A $\Rightarrow$ B:
If the quadrilateral has two diagonals that intersect at right angles, then the quadrilateral is a rhombus, which implies it does have two pairs of parallel sides.

$\therefore$ A $\Rightarrow$ B is true ✓

B $\Rightarrow$ A:
If the quadrilateral has two pairs of parallel sides then it is a parallelogram, which does not necessarily imply it is a rhombus, and therefore it does not necessarily have diagonals that intersect at right angles.

$\therefore$ B $\Rightarrow$ A is false ✓

Therefore, A $\Leftrightarrow$ B is a false statement. ✓

(c) Assume that n is odd and n^2 is even. ✓

Then $\exists k \in N : n = 2k+1$

$\rightarrow n^2 = (2k+1)^2 = 4k^2 + 4k + 1$ ✓

¿ $2(2k^2+2k)+1 = 2m+1 = odd$ ✓

Since n^2 is both even and odd simultaneously, this is a contradiction ✓

and therefore the original conjecture must be true $\forall n \in N, n$ even. [15]

15. (a) (i) $|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$ ✓

(ii) $\overrightarrow{AB} \cdot \overrightarrow{AB} = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$

LHS ¿ $(\overrightarrow{OB} - \overrightarrow{OA}) \cdot (\overrightarrow{OB} - \overrightarrow{OA})$ ✓

¿ $\overrightarrow{OB} \cdot \overrightarrow{OB} - \overrightarrow{OB} \cdot \overrightarrow{OA} - \overrightarrow{OA} \cdot \overrightarrow{OB} + \overrightarrow{OA} \cdot \overrightarrow{OA}$ ✓

¿ $|OB|^2 + |OA|^2 - 2\overrightarrow{OA} \cdot \overrightarrow{OB}$ ✓

$\therefore |OB|^2 + |OA|^2 - 2\overrightarrow{OA} \cdot \overrightarrow{OB} = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$

$\rightarrow -2\overrightarrow{OA} \cdot \overrightarrow{OB} = -2|OA||OB|\cos\theta$ ✓

$\rightarrow \overrightarrow{OA} \cdot \overrightarrow{OB} = |OA||OB|\cos\theta$ as required

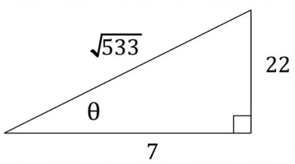
(b) (i) $\begin{pmatrix} 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{vmatrix} 4 \\ 5 \end{vmatrix} \times \begin{vmatrix} 2 \\ -3 \end{vmatrix} \cos\theta$

$8 - 15 = \sqrt{41} \times \sqrt{13} \cos\theta$ ✓

$\therefore \cos\theta = \frac{-7}{\sqrt{533}}$ ✓

Since $\cos\theta < 0 \Rightarrow \theta$ is obtuse ✓

(ii)



$\Rightarrow \sin\theta = \frac{22}{\sqrt{533}}$ ✓

$\therefore$ area ΔOAB ¿ $\frac{1}{2}|OA||OB|\sin\theta$

¿ $\frac{1}{2}\sqrt{41} \times \sqrt{13} \times \frac{22}{\sqrt{533}} = 11$ units^2 ✓ [10]

16. P, Q, R and S are the midpoints of their respective sides:

$\overrightarrow{OP}=\begin{pmatrix}-0.5\\4\end{pmatrix}\overrightarrow{OQ}=\begin{pmatrix}6\\1.5\end{pmatrix}\vec{¿}=\begin{pmatrix}1.5\\-4\end{pmatrix}\overrightarrow{OS}=\begin{pmatrix}-5\\-1.5\end{pmatrix}$ ✓

Therefore:

$\overrightarrow{PQ}=\begin{pmatrix}6\\1.5\end{pmatrix}-\begin{pmatrix}-0.5\\4\end{pmatrix}=\begin{pmatrix}6.5\\-2.5\end{pmatrix}$

$\overrightarrow{SR}=\begin{pmatrix}1.5\\-4\end{pmatrix}-\begin{pmatrix}-5\\-1.5\end{pmatrix}=\begin{pmatrix}6.5\\-2.5\end{pmatrix}$ ✓

$\overrightarrow{PQ}=\overrightarrow{SR}\Rightarrow\therefore\overrightarrow{PQ}\,//\,\overrightarrow{SR}$ ✓

$\overrightarrow{PS}=\begin{pmatrix}-5\\-1.5\end{pmatrix}-\begin{pmatrix}-0.5\\4\end{pmatrix}=\begin{pmatrix}-4.5\\-5.5\end{pmatrix}$

$\overrightarrow{QR}=\begin{pmatrix}1.5\\-4\end{pmatrix}-\begin{pmatrix}6\\1.5\end{pmatrix}=\begin{pmatrix}-4.5\\-5.5\end{pmatrix}$ ✓

$\overrightarrow{PS}=\overrightarrow{QR}\Rightarrow\therefore\overrightarrow{PS}\,//\,\overrightarrow{QR}$ ✓

Since $\overrightarrow{PQ}\,//\,\overrightarrow{SR}$ and $\overrightarrow{PS}\,//\,\overrightarrow{QR}$ $\Rightarrow$ $PQRS$ is a parallelogram [5]