Rossmoyne Senior High School

Semester One Examination, 2014

Question/Answer Booklet

MATHEMATICS SPECIALIST 3C

Section One: Calculator-free

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section
Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	50	33⅓
Section Two: Calculator-assumed	13	13	100	100	66⅔
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2013*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

Section One: Calculator-free (50 Marks)

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 (5 marks)

Determine the exact area bounded by the graph of $f(x) = \cos^3(x)$ and the x-axis for $0 \le x \le 5\pi$.

Area will be 10 times area for $0 \le x \le \frac{\pi}{2}$

$$u = \sin x$$
$$du = \cos x dx$$

$$\cos^3 x dx = (1 - \sin^2 x)\cos x dx = (1 - u^2)du$$

$$\sin(0) = 0$$
$$\sin(\tfrac{\pi}{2}) = 1$$

$$\int_0^1 1 - u^2 du = \left[u - \frac{u^3}{3} \right]_0^1 = \frac{2}{3}$$

Hence area = $\frac{20}{3}$ sq units

Question 2 **(5 marks)**

(a) The vectors $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + a\mathbf{k}$ are perpendicular. Determine the value of a.

(1 mark)

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 2 \\ a \end{bmatrix} = 0 \Rightarrow 3 - 2 + 2a = 0 \Rightarrow a = -\tfrac{1}{2}$$

(b) Determine whether the two lines $\mathbf{r} = 8\mathbf{i} - \mathbf{j} - 8\mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{k})$ and $\mathbf{r} = \mathbf{j} - 3\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ intersect. If they do intersect, state the position vector of their point of intersection. If they do not intersect, justify your answer. (4 marks)

$\mathbf{i}$: $8 + 2\lambda = \mu$

$\mathbf{j}$: $-1 = 1 - \mu \Rightarrow \mu = 2,\ \lambda = -3$

$\mathbf{k}$: $-8 - 3(-3) = -3 + 2(2) \Rightarrow 1 = 1 \Rightarrow$ intersect

$$\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \text{intersect at } 2\mathbf{i} - \mathbf{j} + \mathbf{k}$$

Question 3 **(10 marks)**

Determine the following:

(a) $\int e^x \sin(e^x)dx$ (2 marks)

$$\boxed{-\cos(e^x)+c}$$

(b) $\int_{-1}^{0} x\sqrt{x+1}dx$ (4 marks)

$$u=x+1 \Rightarrow du=dx,\quad -1+1=0,\ 0+1=1$$

$$\int_0^1 (u-1)u^{0.5}du=\int_0^1 (u^{1.5}-u^{0.5})du$$

$$=\left[\frac{2u^{2.5}}{5}-\frac{2u^{1.5}}{3}\right]_0^1$$

$$=\frac{2}{5}-\frac{2}{3}$$

$$=-\frac{4}{15}$$

(c) $\int_{e}^{e^3} \frac{1}{x\log_e(x)}dx$ (Hint: Let $u=\log_e(x)$) (4 marks)

$$u=\ln x \rightarrow du=\frac{1}{x}dx,\ u(e)=1,\ u(e^3)=3$$

$$\int_1^3 \frac{1}{u}du=\left[\ln|u|\right]_1^3=\ln 3-\ln 1=\ln 3$$

Question 4 **(7 marks)**

(a) If $z = 3 - 4i$, determine the reciprocal, $\frac{1}{z}$. (2 marks)

$$\begin{aligned}\frac{1}{z} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{3+4i}{3^2+4^2} \\ &= \frac{3}{25} + \frac{4i}{25}\end{aligned}$$

(b) Let the non-zero complex number $z = a + bi$. Show that $\frac{1}{a+bi} = \frac{\bar{z}}{|z|^2}$. (3 marks)

$$\begin{aligned}LHS &= \frac{1}{a+bi} \\ &= \frac{1}{a+bi} \times \frac{a-bi}{a-bi} \\ &= \frac{a-bi}{a^2+b^2} \\ &= \frac{\bar{z}}{|z|^2} \\ &= RHS\end{aligned}$$

(c) Describe the geometrical relationship between any non-zero complex number and its reciprocal. (2 marks)

The reciprocal z^{-1} is the conjugate of z but multiplied by scale factor of $\frac{1}{|z|}$.

So the reciprocal z^{-1} will be the reflection of z in the real axis and of length $\frac{1}{|z|}$ times z.

Question 5 **(8 marks)**

(a) Use logarithmic differentiation to find $\frac{dy}{dx}$ when $y=(\cos x)^{\sin x}$. (4 marks)

$$\ln y=\sin x\cdot\ln\cos x$$
$$\frac{1}{y}\frac{dy}{dx}=\cos x\cdot\ln\cos x+\sin x\cdot\frac{-\sin x}{\cos x}$$
$$\frac{dy}{dx}=(\cos x)^{\sin x}\left(\cos x\cdot\ln\cos x-\frac{\sin^2 x}{\cos x}\right)$$
$$=(\cos x)^{\sin x+1}\cdot\ln\cos x-(\cos x)^{\sin x-1}\cdot\sin^2 x$$

(b) Determine the derivative of $f(x)=x^3-x$ from first principles. (4 marks)

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^3-(x+h)-(x^3-x)}{h}$$
$$=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x-h-x^3+x}{h}$$
$$=\lim_{h\to 0}\frac{(3x^2+3xh+h^2-1)h}{h}$$
$$=\lim_{h\to 0}3x^2+3xh+h^2-1$$
$$=3x^2-1$$

Question 6 **(7 marks)**

A curve is such that its coordinates x and y satisfy $\frac{dx}{dt} = -4\sin t$ and $\frac{dy}{dt} = 3\cos t$ for $0 \le t \le 2\pi$.

When $t = 0$, $x = 1$ and $y = -2$.

(a) Find the values of t for which the gradient of the curve is $\frac{\sqrt{3}}{4}$. (2 marks)

$$\boxed{\begin{aligned} -\frac{3\cos t}{4\sin t} &= \frac{\sqrt{3}}{4} \\ t &= \frac{2\pi}{3}, \frac{5\pi}{3} \end{aligned}}$$

(b) Determine the values of x and y when $t = \pi$. (3 marks)

$$\boxed{\begin{array}{ll} x = 4\cos t + c & y = 3\sin t + k \\ 1 = 4(1) + c & -2 = 3(0) + k \\ x = 4\cos t - 3 & y = 3\sin t - 2 \\ & \\ x(\pi) = -7 & y(\pi) = -2 \end{array}}$$

(c) Determine a Cartesian equation for the curve. (2 marks)

$$\boxed{\begin{array}{ll} \frac{x+3}{4} = \cos t & \frac{y+2}{3} = \sin t \\ \left(\frac{x+3}{4}\right)^2 + \left(\frac{y+2}{3}\right)^2 = 1 & \end{array}}$$

Question 7 **(8 marks)**

(a) Write down the sum of the series $1+3+5+7+9$. (1 mark)

$\boxed{25}$

(b) The sum, S_n, of the first n odd numbers from 1 to $2n-1$, is given by the recurrence relation $S_n = S_{n-1} + an - b,\ S_1 = 1$, where a and b are positive integers. Determine the values of a and b. (2 marks)

1
$1+3=4,\ 4=1+3$
$1+3+5=9,\ 9=4+5$
$1+3+5+7=16,\ 16=9+7$

$S_n = S_{n-1} + 2n - 1 \Rightarrow a=2,\ b=1$

(c) The sum, S_n, of the first n terms of the sequence $1+5+9+\ldots+(4n-3)$, is also given by a recurrence relation of the form $S_n = S_{n-1} + an - b,\ S_1 = 1$, where a and b are positive integers. Determine the values of a and b. (2 marks)

1
$1+5=6,\ 6=1+5$
$1+5+9=15,\ 15=6+9$
$1+5+9+13=28,\ 28=15+13$

$S_n = S_{n-1} + 4n - 3 \Rightarrow a=4,\ b=3$

(d) The n^{th} term of a sequence of figurate numbers is given by $T_n = 4n^2 - 3n$. Determine a recurrence relation for this sequence. (3 marks)

$T_1 = 1$
$T_2 = 10 = T_1 + 9$
$T_3 = 27 = T_2 + 17$
$T_4 = 52 = T_3 + 25$

$T_n = T_{n-1} + 8n - 7,\ T_1 = 1$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

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