PERTH MODERN SCHOOL Exceptional schooling. Exceptional students. Independent Public School	Year 12 Specialist TEST 4 Weds 28 Aug 2019 TIME: 50 minutes working Classpads allowed No notes allowed 45 marks 8 Questions

Name:________________________________

Teacher: _____________________________

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (3, 3 & 3 = 9 marks)
Determine the following integrals using the given substitutions.

a) $\int 3x(5x^2+1)^7\,dx \qquad u=5x^2+1$

Solution
$\int 3x(5x^2+1)^7\,dx \qquad u=5x^2+1$ $\int 3xu^7\frac{1}{10x}du=\frac{3}{10}\int u^7du=\frac{3u^8}{80}=\frac{3}{80}(5x^2+1)^8+C$
Specific behaviours
✓ subs du ✓ integrates wrt u ✓ expresses answer in terms of x only with a constant

b) $\int (5x-2)\sqrt{2x-1}\,dx \qquad u = 2x-1$

Solution

$$\int (5x-2)\sqrt{2x-1}\,dx \qquad u = 2x-1 \quad x = \frac{u+1}{2}$$

$$\int [5(\frac{u+1}{2}) - \frac{4}{2}]u^{\frac{1}{2}}\frac{1}{2}du = \int \frac{5u+1}{4}u^{\frac{1}{2}}du = \int \frac{5u^{\frac{3}{2}} + u^{\frac{1}{2}}}{4}du$$

$$\frac{3}{6}u^{\frac{5}{2}} + \frac{1}{6}u^{\frac{3}{2}} + c = \frac{1}{2}(2x-1)^{\frac{5}{2}} + \frac{1}{6}(2x-1)^{\frac{3}{2}} + c$$

Specific behaviours

✓ subs du
✓ integrates wrt u
✓ expresses answer in terms of x only (no need for constant)

c) $\int \sec^2 x \tan^8 x\,dx \qquad u = \tan x$

Solution

$$\int \sec^2 x \tan^8 x\,dx \qquad u = \tan x$$

$$\int \sec^2 x u^8 \frac{1}{\sec^2 x}du = \int u^8 du = \frac{u^9}{9} + C = \frac{\tan^9 x}{9} + C$$

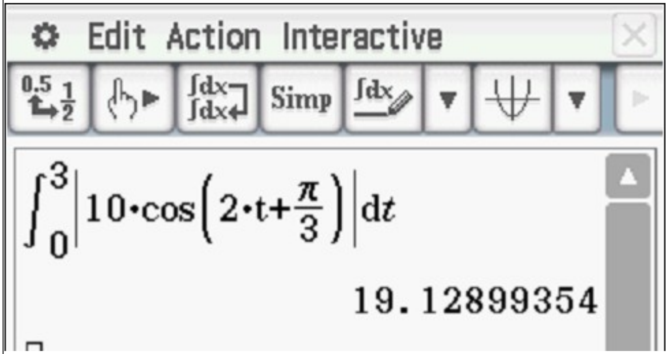
Specific behaviours

✓ subs du
✓ integrates wrt u
✓ expresses answer in terms of x only (no need for constant)

✓ uses separation of variables
✓ integrates correctly
✓ solves for constant in terms of A & n

c) Determine the distance travelled in the first 3 seconds.

Solution



Specific behaviours

✓ uses correct integral with absolute velocity
✓ states distance travelled

Q8 (4 marks)
A particle with displacement, x metres from the origin at time t seconds, has an acceleration given by $\ddot{x} = -n^2 x$. The amplitude of the motion is given by A metres.

Show by using integration that the speed, v metres per second, is given by $v^2 = n^2\left(A^2 - x^2\right)$.

Solution

$$v\frac{dv}{dx} = -n^2 x$$
$$\int v\,dv = \int -n^2 x\,dx$$
$$\frac{v^2}{2} = -n^2\frac{x^2}{2} + c \quad v^2 = -n^2x^2 + c$$
$$x = A,\ v = 0 \quad c = n^2A^2$$
$$v^2 = -n^2x^2 + n^2A^2 = n^2\left(A^2 - x^2\right)$$

Specific behaviours

✓ uses alternative expression for accleration

Q2 (3 marks)

Identical twins Sherry and Mary were both given the following integral to solve. $\int 2\sin x\cos x\,dx$
Sherry's solution was as follows.

$$\int 2\sin x\cos x\,dx \quad u = \sin x$$
$$\int 2u\cos x\frac{du}{\cos x}$$
$$\int 2u\,du = u^2 = \sin^2 x$$

While Mary's solution was to:

$$\int 2\sin x\cos x\,dx = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x$$

Explain why the solutions differ and state which is the correct answer. Show your reasoning.

Solution

Both missing constants
Constant differ

Both answers correct as $\frac{-1}{2}\cos 2x = \frac{-1}{2}\left(1 - 2\sin^2 x\right) = \sin^2 x + C$

Specific behaviours

✓ mentions that constants missing
✓ states that constants are different
✓ shows that both expressions differ by an added constant

Q3 (3 & 4 = 7 marks)
Determine the following integrals showing all working.

a) $\int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x - \sin x}\,dx$

Solution

$$\int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x - \sin x}\,dx = \left[-\ln\left|\cos x - \sin x\right|\right]_0^{\frac{\pi}{2}} = (\ln 1) - (-\ln 1) = 0$$

Specific behaviours

✓ integrates using ln
✓ uses absolute value
✓ determines result

Q7 (2, 3 & 2 = 7 marks)

A particle with displacement, x metres from the origin at time t seconds, moves such that $x = 5\sin\left(2t + \frac{\pi}{3}\right)$.

a) Show that the motion is simple harmonic.

Solution

$$x = 5\sin\left(2t + \frac{\pi}{3}\right)$$

$$\dot{x} = 10\cos\left(2t + \frac{\pi}{3}\right)$$

$$\ddot{x} = -20\sin\left(2t + \frac{\pi}{3}\right) = -4x \quad \therefore SHM$$

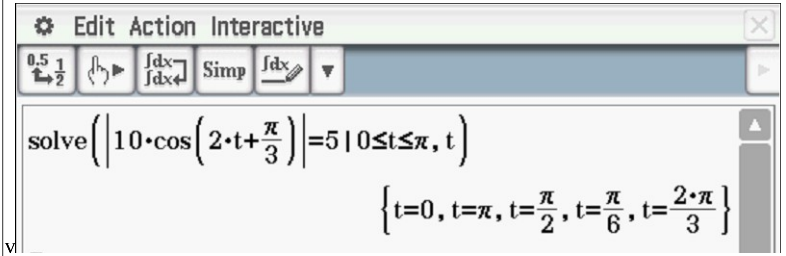
Specific behaviours

✓ obtains acceleration function
✓ shows correct differential equation for SHM

b) Determine the first two times that the speed is exactly half of the maximum speed.

Solution

t=0 v=5m/s



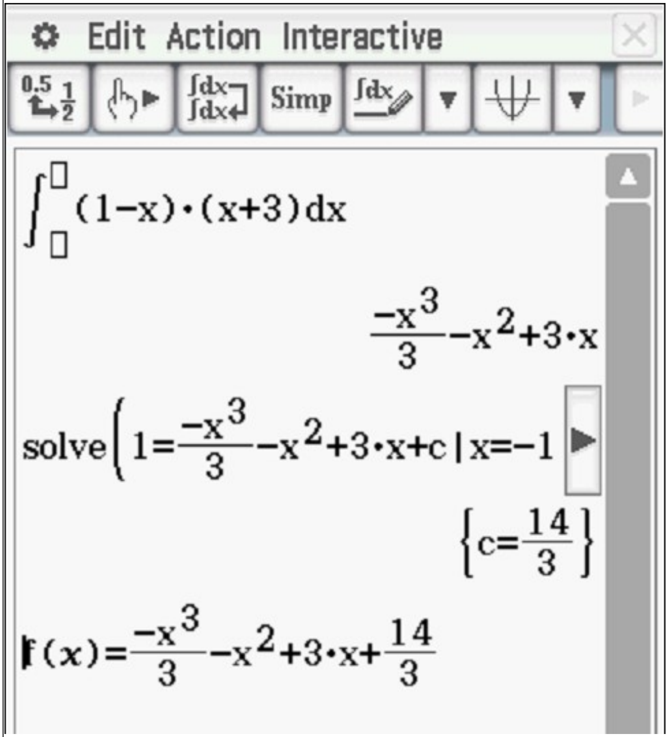
First two times are $0 \ \& \ \frac{\pi}{6}$

Specific behaviours

✓ states initial time
✓ uses negative velocity for second time
✓ solves for second time, approx

b) Given that point A (-1,1) is a known point on our solution, show this curve on the slope field above and give the equation.

Solution



Specific behaviours

✓ shows curve on slope field going through pt A
✓ integrates slope field
✓ solves for constant

Q3 cont-

b) $\displaystyle\int \frac{6x^3+11x^2+15x+20}{(x+1)^2(x^2+4)}\,dx$ (4 marks)

Solution

$$\frac{6x^3+11x^2+15x+20}{(x+1)^2(x^2+4)}=\frac{a}{x+1}+\frac{b}{(x+1)^2}+\frac{cx+d}{x^2+4}$$

$6x^3+11x^2+15x+20=a(x+1)(x^2+4)+b(x^2+4)+(cx+d)(x+1)^2$

$x=-1$

$10=5b \quad b=2$

$x=0$

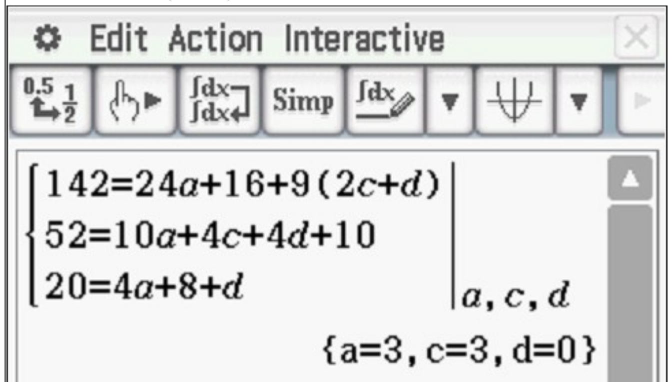
$20=4a+8+d$

$x=1$

$52=10a+4c+4d+10$

$x=2$

$142=24a+16+9(2c+d)$



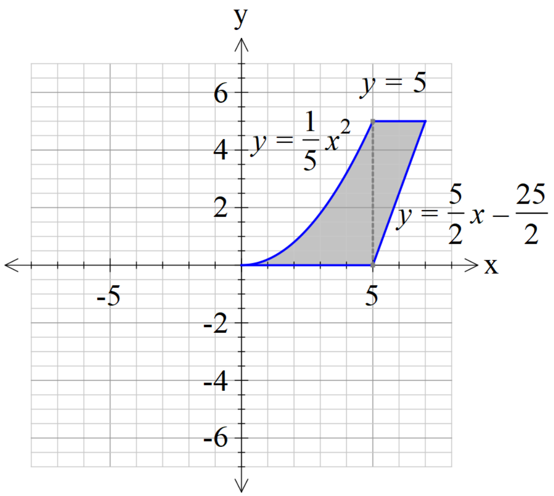
$\displaystyle\int \frac{3}{x+1}+\frac{2}{(x+1)^2}+\frac{3x}{x^2+4}\,dx$

$=3\ln|x+1|-2(x+1)^{-1}+\frac{3}{2}\ln|x^2+4|+c$

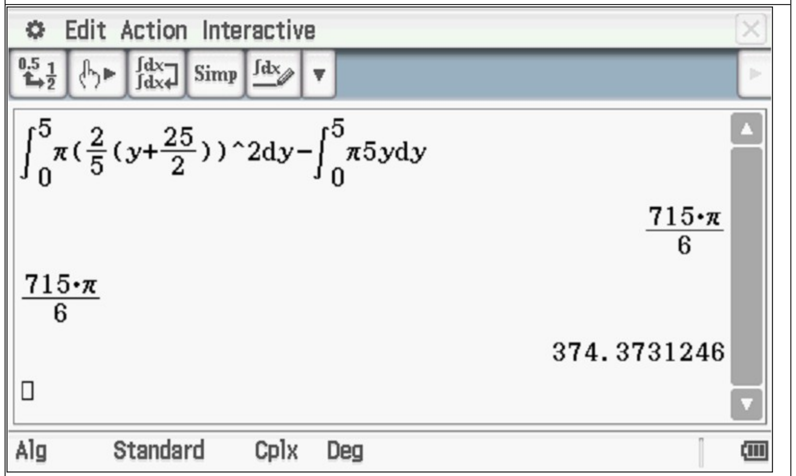
Specific behaviours

✓ uses correct partial fractions with 4 constants
✓ solves for at least one constant
✓ sets up simultaneous equations for other constants
✓ integrates correctly (no need to add c)

Q4 (5 marks)
The shaded region is rotated about the y axis. Determine the volume of the resulting solid.



Solution



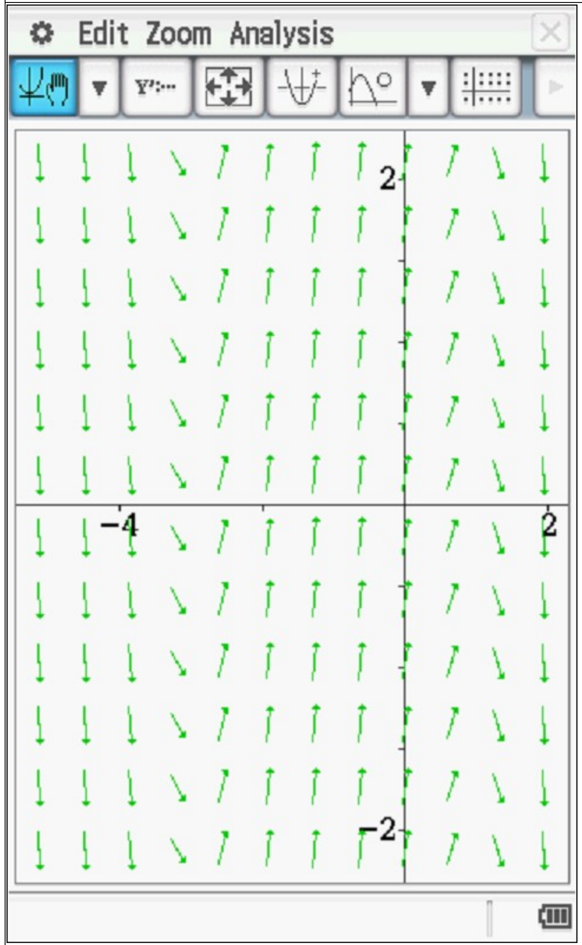
Specific behaviours

✓ uses correct integral around y axis
✓ uses correct limits
✓ sets up a difference calculation with volumes
✓ states correct approx. volume

Q6 (3 & 3 = 6 marks)

a) Sketch the slope field for $\frac{dy}{dx} = (1 - x)(x + 3)$ on the axes below.

Solution



Specific behaviours

✓ shows one zero
✓ shows both zeros
✓ pattern agrees with above

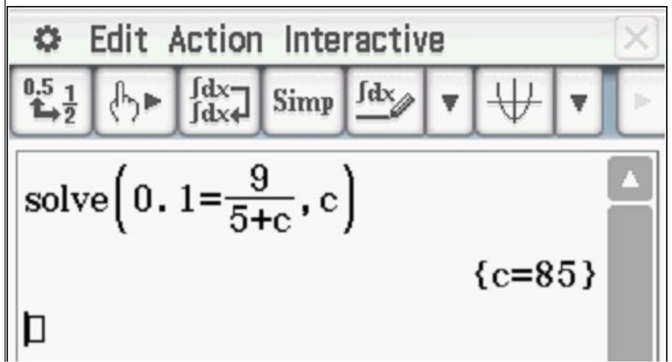
Q5 (1 & 4 = 5 marks)

The mass, N grams, of a gas produced in a factory at time t seconds can be modelled by the logistical formula $\frac{dN}{dt} = 9N - 5N^2$ with an initial mass of 0.1 grams.

a) Determine the limiting mass as $t \to \infty$.

Solution
$0 = 9N - 5N^2 = N(9-5N)$ $N = \frac{9}{5}$
Specific behaviours
✓ states limiting value

b) Show that $N = \frac{9}{5 + ce^{-9t}}$ and determine the constant.

Solution
$\frac{dN}{dt} = 9N - 5N^2 = N(9-5N)$ $\int \frac{dN}{N(9-5N)} = \int dt$ $\frac{1}{N(9-5N)} = \frac{a}{N} + \frac{b}{9-5N}$ $1 = a(9-5N) + bN$ $N = 0$ $1 = 9a \quad a = \frac{1}{9}$ $N = \frac{9}{5}$ $1 = b\frac{9}{5} \quad b = \frac{5}{9}$ $\int \frac{\frac{1}{9}}{N} + \frac{\frac{5}{9}}{9-5N}\, dN = \frac{1}{9}\ln\lvert N\rvert - \frac{1}{9}\ln\lvert 9-5N\rvert = t + c$ $-\ln\left\lvert\frac{N}{9-5N}\right\rvert = -9t + c \quad as\ N < \frac{9}{5} \therefore 9 - 5N > 0$ $\frac{9-5N}{N} = Ce^{-9t}$ $9 - 5N = NCe^{-9t}$ $9 = (5 + Ce^{-9t})N$ $N = \frac{9}{5 + Ce^{-9t}}$ 
Specific behaviours
✓ separates variables ✓ sets up partial fractions ✓ integrates and shows why absolute value not needed ✓ solves for constant