SHENTON COLLEGE

2020 YEAR 12 MATHEMATICS: METHODS
Test 2 (Integration)

$\overline{\quad}$ 46

NAME: SOLUTIONS

TEACHER: AI FRIDAY WHITE

Calculator-Free | Formula sheet provided | Working time: 20 minutes | Marks: $\overline{\quad}$ 23

-1 overall +c

QUESTION 1

[13 marks – 2, 2, 3, 3, 1, 2]

Determine the following.

a) $\int 3x^2 - \frac{1}{\sqrt{x}} + x - 8 \ dx$

$= \int 3x^2 - x^{-\frac{1}{2}} + x - 8 \ dx$

$= x^3 - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^2}{2} - 8x + c$

$= x^3 - 2\sqrt{x} + \frac{x^2}{2} - 8x + c$ ✓✓

-1 per error

b) $\int -2\cos x \sin^4 x \ dx$

$= \int -2\cos x(\sin x)^4 \ dx$

$= -2\left(\frac{\sin^5 x}{5}\right) + c$ ✓✓

$= \frac{-2\sin^5 x}{5} + c$

c) $\int_{-\pi}^{\pi} \cos 3x \ dx$

$= \int_{-\pi}^{\pi} \frac{1}{3}(3)\cos 3x \ dx$

$= \frac{1}{3}\left[\sin 3x\right]_{-\pi}^{\pi}$ ✓ antidifferentiates correctly

$= \frac{1}{3}\left(\sin 3\pi - \sin(-3\pi)\right)$ ✓ substitutes bounds

$= \frac{1}{3}(0-0)$

$= 0$ ✓ correct value

d) $\int_0^1 (x^2 - x)^2 \ dx$

$= \int_0^1 x^4 - 2x^3 + x^2 \ dx$ ✓ expands binomial

$= \left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]_0^1$ ✓ antidifferentiates correctly

$= \frac{1}{5} - \frac{1}{2} + \frac{1}{3}$

$= \frac{6-15+10}{30}$

$= \frac{1}{30}$ ✓ correct value

e) $\frac{d}{dx}\left(\int_{\pi}^{x} \sin t \ dt\right)$

$= \sin x$ ✓

f) $\int_0^{\pi} \frac{d}{dt}\left(-\cos\frac{t}{2}\right) dt$

$= \left[-\cos\frac{t}{2}\right]_0^{\pi}$ ✓ $-\cos\frac{t}{2}$

$= -\cos\frac{\pi}{2} - (-\cos 0)$

$= 0 + 1$

$= 1$ ✓ substitutes bounds and evaluates

QUESTION 2

[6 marks – 1, 2, 3]

Given that $\int_{-1}^{2} f(x)\,dx = 6$ and $\int_{6}^{2} f(x)\,dx = -8$, evaluate the following definite integrals.

a) $\int_{2}^{-1} f(x)\,dx$	$= -\int_{-1}^{2} f(x)\,dx$ $= -6$ ✓
b) $\int_{-1}^{6} f(x)dx$	$= \int_{-1}^{2} f(x)\,dx + \int_{2}^{6} f(x)\,dx$ $= \int_{-1}^{2} f(x)\,dx - \int_{6}^{2} f(x)\,dx$ ✓ applies linearity properties correctly $= 6 - (-8)$ $= 14$ ✓ correct value
c) $\int_{6}^{2} 3f(x) - 4\,dx$	$= 3\int_{6}^{2} f(x)\,dx - \int_{6}^{2} 4\,dx$ ✓ applies linearity properties correctly $= 3(8) - [4x]_{6}^{2}$ ✓ antidifferentiates $\int 4\,dx$ $= -24 - (8 - 24)$ $= -8$ ✓ correct value

QUESTION 3

[4 marks]

Given that $f'(x) = \frac{6-x^4}{x^2}$ and $f(x)$ passes through the point $(3, -9)$, determine $f(x)$.

$f'(x) = \frac{6}{x^2} - x^2$

$= 6x^{-2} - x^2$ ✓ splits fraction of $f'(x)$ and simplifies.

$f(x) = \int 6x^{-2} - x^2\,dx$

$= \frac{6x^{-1}}{-1} - \frac{x^3}{3} + c$

$= \frac{-6}{x} - \frac{x^3}{3} + c$ ✓ antidifferentiates

$-9 = \frac{-6}{3} - \frac{3^3}{3} + c$ ✓ substitutes point and solves for c.

$-9 = -2 - 9 + c$

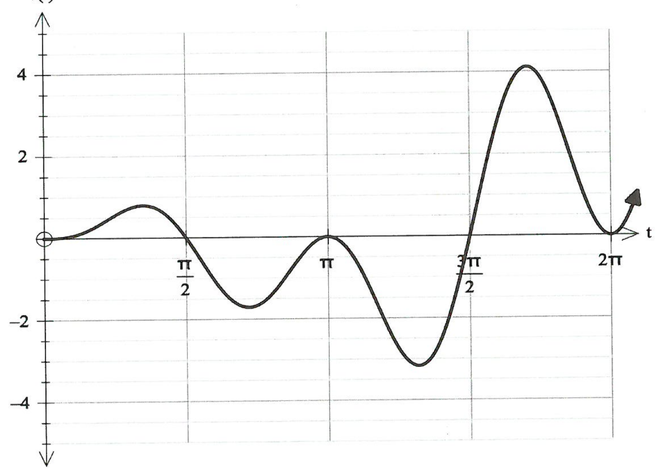
$c = 2$

$\therefore f(x) = \frac{-6}{x} + \frac{x^3}{3} + 2$ ✓ correct $f(x)$.

End of Calculator Free Section

QUESTION 7 **[4 marks – 1, 1, 2]**

The graph of $v(t) = 2x\sin^2 x\cos x$ as shown below displays the velocity of a body moving in rectilinear motion, in metres per second, for $0 \le t \le 2\pi$ seconds.



a) Explain the significance of the value of $\int_0^{\frac{3\pi}{2}} v(t)\ dt$ in relation to the body's movement.

The body's overall change in position from 0 secs to $\frac{3\pi}{2}$ secs. ✓

b) Explain the significance of the value of $\int_0^{\frac{\pi}{2}} v(t)\ dt - \int_{\frac{\pi}{2}}^{\pi} v(t)\ dt$ in relation to the body's movement.

The body's total distance travelled from 0secs to π secs. ✓

c) Calculate the total distance travelled between π seconds and 2π seconds.

$$-\int_{\pi}^{\frac{3\pi}{2}} 2x\sin^2 x\cos x\ dx + \int_{\frac{3\pi}{2}}^{2\pi} 2x\sin^2 x\cos x\ dx$$ ✓ correct integrals

$= 6.28$ m (2dp) ✓ correct distance

End of Calculator Assumed Section

2020 YEAR 12 MATHEMATICS: METHODS
Test 2 (Integration)

SHENTON COLLEGE

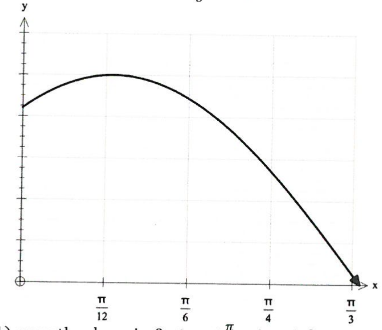
NAME: SOLUTIONS

TEACHER: AI FRIDAY WHITE

Calculator-Assumed Formula sheet provided Working time: 30 minutes Marks: ___ / 23

QUESTION 4 **[8 marks – 3, 2, 1, 2]**

a) Estimate the area under the curve of $y = \sin(2x+1)$ over the domain $0 \le x \le \frac{\pi}{3}$ using left rectangular strips of width $\frac{\pi}{12}$.



$A = \frac{\pi}{12}\left(\sin(1) + \sin\left(\frac{\pi}{6}+1\right) + \sin\left(\frac{\pi}{3}+1\right) + \sin\left(\frac{\pi}{2}+1\right)\right)$ ✓✓ heights of all rectangles correct

$= 0.8559$ units^2 (4dp)

✓ correct area

b) Estimate the area under the curve of $y = y = \sin(2x+1)$ over the domain $0 \le x \le \frac{\pi}{3}$ using right rectangular strips of width $\frac{\pi}{12}$.

$A = \frac{\pi}{12}\left(\sin\left(\frac{\pi}{6}+1\right) + \sin\left(\frac{\pi}{3}+1\right) + \sin\left(\frac{\pi}{2}+1\right) + \sin\left(\frac{2\pi}{3}+1\right)\right)$ ✓ heights of rectangles correct

$= 0.6480$ units^2 (4dp) ✓ correct area

c) Use your answers from part a) to b) to calculate an average estimated area.

$\frac{0.8559 + 0.6480}{2} = 0.7519$ units^2 (4dp) ✓

d) Evaluate the actual area under the curve. Suggest one way that you could modify the process you completed from parts a) to c) so that your estimation is closer to this result.

$\int_0^{\frac{\pi}{3}} \sin(2x+1)\ dx = 0.7696$ units^2 (4dp) ✓ correct actual area

- Use more rectangles of thinner width
- use a midpoint approximation

✓ any one sensible suggestion

QUESTION 5

[5 marks – 2, 3]

Consider the cubic function $y = -x^3 + 2x^2 + 8x$.

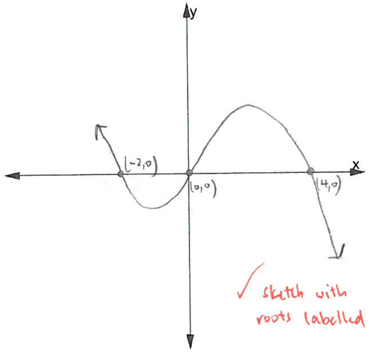
a) Determine the roots of the function and hence draw a sketch of the cubic on the axes provided, with its roots clearly labelled.
Note: you do not need to determine any other key features of the graph.

$-x^3 + 2x^2 + 8x = 0$

$x = -2, 0, 4$

$\therefore$ roots at $(-2,0)$, $(0,0)$ and $(4,0)$

✓ solves for roots



✓ sketch with roots labelled

b) Show the use calculus to determine the exact area bound by the curve and the x-axis.

$-\int_{-2}^{0} -x^3+2x^2+8x \, dx + \int_0^4 -x^3+2x^2+8x \, dx$ ✓ integrals correct

$= -\left[\frac{-x^4}{4} + \frac{2x^3}{3} + 4x^2\right]_{-2}^{0} + \left[\frac{-x^4}{4} + \frac{2x^3}{3} + 4x^2\right]_0^4$ ✓ antidifferentiates correctly

$= -\left(0 - \left(-4 - \frac{16}{3} + 16\right)\right) - 64 + \frac{128}{3} + 64 - 0$

$= 49\frac{1}{3}$ units^2

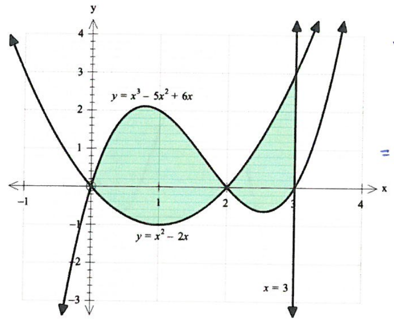
OR $\frac{148}{3}$ units^2 ✓ correct exact area

QUESTION 6

[6 marks – 2, 2, 2]

Show how you would use integrals to calculate the following shaded areas.
Note: You do not need to evaluate the areas.

a)

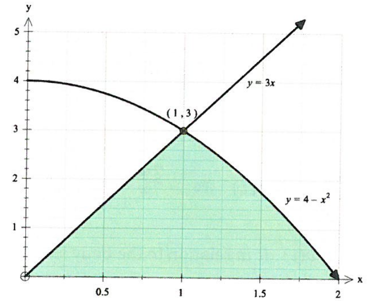


$\int_0^2 x^3 - 5x^2 + 6x - (x^2 - 2x) \, dx$

$+ \int_2^3 x^2 - 2x - (x^3 - 5x^2 + 6x) \, dx$

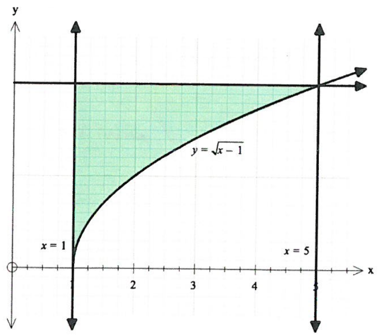
$= \int_0^2 x^3 - 6x^2 + 8x \, dx + \int_2^3 -x^3 - 4x^2 - 8x \, dx$ ✓ ✓

b)



$\int_0^1 3x \, dx + \int_1^2 4 - x^2 \, dx$ ✓ ✓

c)



$y = \sqrt{5-1}$
$y = 2$ ✓ determines equation of horizontal line

$\int_1^5 2 \, dx - \int_1^5 \sqrt{x-1} \, dx$

$= \int_1^5 2 - \sqrt{x-1} \, dx$ ✓