# SEMESTER TWO

# MATHEMATICS
# METHODS
# UNITS 3 & 4

# 2021

# SOLUTIONS

**Calculator-free Solutions**

1. (a) $(2\pi,\ 0)$

$$\frac{dy}{dx} = \frac{2\tan(x)}{x} + \frac{2\ln x}{\cos^2(x)}$$ ✓

When $x = 2\pi$, $m = 0 + 2\ln 2\pi$ ✓

$0 = 2\ln 2\pi(2\pi) + c \ \therefore c = -4\pi \ln 2\pi$

$y = (2\ln 2\pi)x - 4\pi \ln 2\pi$ ✓

(b) $2t\ln(t^4)\tan(t^2)$ ✓✓ [5]

2. (a) $\dfrac{1+2+3+4+5}{k} = 1$

$k = 15$ ✓

(b) $\dfrac{12}{15} = \dfrac{4}{5}$ ✓

(c) $\dfrac{1+4+9+16+25}{15} = \dfrac{55}{15} = \dfrac{11}{3}$ ✓

(d) (i) Expected value $\left(\dfrac{1}{10}\right)\left(\dfrac{11}{3}\right) + 2 = \dfrac{71}{30}$ ✓

Variance of $Y$: $\dfrac{1+8+27+64+125}{15} - \dfrac{121}{9} = \dfrac{14}{9}$

Standard deviation of $Y = \dfrac{\sqrt{14}}{3}$ ✓

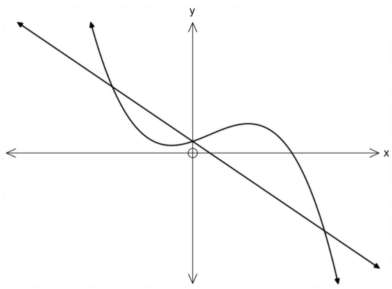
Standard deviation of $X = \dfrac{\sqrt{14}}{30}$ ✓

(ii) $P\left(X \le \dfrac{11}{5}\right) = P(Y \le 2) = \dfrac{3}{15}$ ✓✓ [8]

3. (a) $g''(x) = -2x + 1$ when $x > \dfrac{1}{2}$ $g(x)$ is concave down ✓✓

(b) Maximum ✓

(c)



Correct point of inflection ✓
Shape showing local minimum and maximum ✓ [5]

4. (a) $-1 = \log_c(2)$

$c^{-1} = 2 \quad \therefore c = \frac{1}{2}$ ✓

$b = -1$ ✓

$0 = \log_c x \quad \therefore x = 1 \rightarrow C(1, 0)$

$p(x) = 2a^x - 1$ given $(1,0)$

$2a - 1 = 0 \quad \therefore a = \frac{1}{2}$ ✓

(b) $= 3\log_5 5 + \log_5 2 - \frac{1}{3}(3)\log_5 2$ ✓

$= 3$ ✓

(c) $\log_4\left(x^2 - 6x\right) = 2$ ✓

$x^2 - 6x - 16 = 0$

$(x - 8)(x + 2) = 0$

$\therefore x = 8$ ✓ [7]

5. (a) 0.16 ✓
(b) 0.34 ✓
(c) $\frac{0.16}{0.5} = 0.32$ ✓✓ [4]

6. $\log_2 y = \frac{1}{4}\log_2 x + 3$

$\log_2 y = \frac{1}{4}\log_2 x + 3\log_2 2$ ✓

$\log_2 y = \log_2 8x^{0.25}$

$y = 8x^{\frac{1}{4}}$ ✓

$k = 8 \quad n = \frac{1}{4}$ ✓ [3]

7. $$m'(x) = \frac{(x^2-1)(2xe^{x^2-1}) - (e^{x^2-1})(2x)}{(x^2-1)^2}$$ ✓

$$= \frac{2xe^{x^2-1}(x^2-2)}{(x^2-1)^2}$$

Stationary points occur when $2xe^{x^2-1}(x^2-2)=0$ ✓

$\therefore$ $x = 0, \pm\sqrt{2}$ ✓

$\therefore$ $\left(0, -\frac{1}{e}\right)$ $(\sqrt{2}, e)$ $(-\sqrt{2}, e)$ ✓ [4]

8. (a) $$p(x) = \begin{cases} \frac{1}{20} & 0 \le x \le 20 \\ 0 & x > 20 \end{cases}$$ ✓

(b) $$\frac{\frac{2}{20}}{\frac{17}{20}} = \frac{2}{17}$$ ✓✓

(c) 10 minutes ✓ [4]

9. $$\int_{-1}^{1} k(1-x^2) - 2k(x^2-1)\ dx = 3k\int_{-1}^{1}(1-x^2)\ dx$$ ✓

$$3k\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 8$$ ✓

$4k = 8$

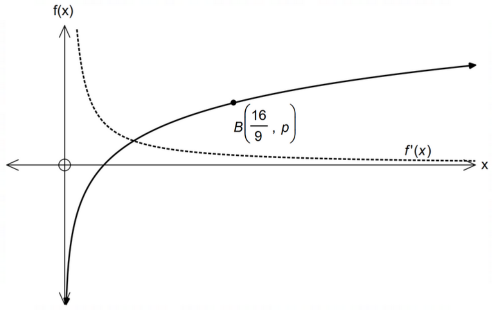
$k = 2$ ✓ [3]

10. (a) $0 < x \le 1$ ✓✓

(b) $$\left(\frac{4}{3}\right)^p = \frac{16}{9}$$ ✓

$p = 2$ ✓

(c) (i)



✓

10. (c) (ii) $\log_{\frac{4}{3}} x = \dfrac{\ln x}{\ln \frac{4}{3}}$

$f'(x) = \dfrac{1}{x\ln \frac{4}{3}} = \dfrac{1}{x(\ln 4 - \ln 3)}$ ✓✓ [7]

**Calculator-Assumed Solutions**

11. (a) $M_P(t) = M_R(t) - M_c(t) = -0.4t + 8 - 0.3t - 2$ ✓

Maximum profit occurs when $M_P(t) = -0.7t + 6 = 0$

therefore $t$ = 8.57 years ✓

(b) $\int_0^{8.57} -0.7t + 6 \; dt = 25.714285$ ✓

Maximum profit = $25714285-$12000000 = $13 714 285 ✓ [4]

12. (a) Discrete data; Independent events ✓

Only two outcomes: under the limit or over the limit. ✓

(b) (i) $X \sim Bin(10,\ 0.05) \quad Y \sim Bin(10,\ 0.013)$ ✓

$P(X=3) + P(X=2)\,P(Y=1) + P(X=1)\,P(Y=2) + P(Y=3)$ ✓

(0.01048)+(0.07463)(0.11556)+(0.31512)(0.00685)+(0.00024)
= 0.02150 ✓

(ii) $P(1 \le Y \le 10) = 0.12265$ ✓

(iii) X~Bin(275, 0.05) E (X) = 13.75

∴ 13 drivers. ✓

Variance = 275 x 0.05 x 0.95 = 13.0625

Standard deviation = 3.6142 ✓

(iv) $Y \sim Bin(n,\ 0.013) \quad P(Y \ge 1) \le 0.6$

$1 - P(X = 0) \le 0.6$ ✓

$P(Y = 0) \ge 0.4$

$0.987^n \ge 0.4 \quad \therefore n \le 70.025$ ✓

Therefore the maximum value of $n$ = 70 ✓ [11]

13. Point of intersection:

$x^3 - 4x^2 + 3x + 1 = x^2 - 3x + 1$

$x = 0$ or 2 or 3

Area of shaded part:

$\int_0^3 |(x^3 - 4x^2 + 3x + 1) - (x^2 - 3x + 1)| \; dx = \frac{37}{12}$ units$^2$ ✓✓ Alternate

Shaded area under line:

$\int_0^2 1 - x - \left(x^2 - 3x + 1\right) \; dx = \frac{4}{3}$ units$^2$ ✓

Fraction: $\frac{16}{37}$ ✓ [4]

14. (a) AC = $t - 30$ $\therefore p^2 = 30^2 - (t - 30)^2$ ✓

$p^2 = -t^2 + 60t$ ✓

Volume of cone = $\frac{1}{3}\pi r^2 h$

$V(t) = \frac{1}{3}\pi\left(-t^2 + 60t\right)(t) = -\frac{\pi t^3}{3} + 20\pi t^2$ ✓

(b) $V'(t) = 40\pi t - \pi t^2 = 0$ ✓

$t = 0$ or 40 ✓

$V''(t) = 40\pi - 2\pi t$

$V''(40) = -40\pi \therefore$ maximum at $t = 40$ cm ✓

(c) Volume of cone at $t = 40$ cm is $\frac{32000\pi}{3} = 33510.322$ cm$^3$ ✓

Volume of sphere = 113 097.336 cm$^3$

Percentage = 29.63% ✓

(d) $\frac{\delta r}{r} = -0.015$ $\quad \frac{dV}{dr} = 4\pi r^2$ $\quad \frac{\delta V}{\delta r} \approx \frac{dV}{dr}$

$\frac{\delta V}{V} \approx \frac{4\pi r^2}{\frac{4}{3}\pi r^3} \times \delta r$ ✓

$\frac{\delta V}{V} \approx \frac{3\delta r}{r}$

$\frac{\delta V}{V} \approx 3(-0.015)$ ✓

Approximately 4.5% decrease in volume. ✓ [11]

15. (a)
$$50 = 100\left(1 - e^{3k}\right)$$
$$k = -0.23105$$ ✓

(b) Less than 5 minutes: $P = 100(1 - e^{5 \times -0.23105}) = 68.5\%$ ✓
Percentage 5 minutes or longer = 31.5% ✓

(c) (i) $t$ is a continuous random variable where $0 \le t \le 30$.
The function in the domain $0 \le t \le 30$ is positive.
The probability at $t$ = 0 is 0 and $t$ = 30 is 1,
therefore is cumulative. ✓✓

(ii) $P(t) = 1 - e^{-0.23105(15)} = 0.96896$ ✓

(iii) $\dfrac{0.68502}{0.90079} = 0.7605$ ✓✓ [8]

16. $T(t) = 21e^{-kt} + 4$

$9.8 = 21e^{-kt} + 4$ ✓

$6.5 = 21e^{-k(t+15)} + 4$ ✓

$k = 0.056104 \qquad t = 22.9334$ ✓

The liquid was placed in the fridge at 11:37 am ✓ [4]

17. (a) 6 year old tree is growing at a rate of 17.69 cm/year
50 year old tree is growing at a rate of 2.277 cm/year ✓
7.8 times faster ✓

(b) (i) Convenience sample: this sample may not be representative of all the six year old trees in the plantation.
The sample is not large enough. ✓✓

(ii) Stratified sample where a number of trees from each area according to the size of the area are chosen at random.
Or
Systematic and Array sample: Every $n$th tree starting at a randomly assigned tree as one walks down each row. ✓✓

(c) (i) $P(Z > z) = 0.0094 \quad \therefore z = 2.3495$ ✓

$$\frac{326 - \mu}{12} = 2.3495$$ ✓

$\mu = 297.8$ cm ✓

(ii) According to the model, the height of a six year old tree is 297.8 cm and the mean of the sample is the same, therefore the model is suitable. ✓

(iii) $X \sim (5,\ 0.0094) \quad P(X = 0) = 0.95388$ ✓

(d)
$$\frac{132}{400} \pm 2.58\sqrt{\frac{\left(\frac{132}{400}\right)\left(\frac{268}{400}\right)}{400}} = (0.269,\ 0.391)$$ ✓✓

With a 99% confidence level, between 26.9% and 39.1% of the mature trees can be used for luxury furniture. ✓ [14]

18. Over - estimate = 0.2(0.127+0.172+0.184+0.184+0.181) = 0.1696 ✓
Under - estimate = 0.2(0+0.127+0.172+0.181+0.173) = 0.1306 ✓
Average = 0.1501 units$^2$ ✓ [3]

19. (a) (i) $n > 30$
$np = 300 \times 0.12 = 36 \quad npq = 300 \times 0.12 \times 0.88 = 31.68$
$np > 10 \quad npq > 10$ therefore
a normal distribution can apply. ✓✓

(ii) $\hat{p} \sim N\left(0.12,\ 0.01876^2\right)$ ✓✓

(iii) $P\left(\hat{p} > 0.125\right) = 0.3949$ ✓

(iv) $P\left(\hat{p} < k\right) = 0.25 \quad k = 0.1073$ ✓
10.7% ✓

(b) (i) $\int_5^{18} \frac{1}{15}\, dx = \frac{13}{15}$ ✓

(ii) Expected waiting time= 12.5 days ✓

Normal distribution $\hat{p} \sim N\left(0.5,\ 0.07906^2\right)$ ✓✓

(iii) $P\left(\hat{p} < 0.7\right) = 0.9943$ ✓ [12]

20. (a) $\ln\left(\sqrt{\frac{x+1}{x^3}}\right) = \frac{1}{2}\ln(x+1) - \frac{3}{2}\ln x$ ✓

$\frac{d}{dy}\left(\frac{1}{2}\ln(x+1) - \frac{3}{2}\ln x\right) = \frac{1}{2}\left(\frac{1}{x+1}\right) - \frac{3}{2x}$ ✓✓

$= \frac{1}{2x+2} - \frac{3}{2x}$

(b) (i) $pH = -\log_{10}\left[H_3O^+\right]$
Hydronium ions range from
$\left[H_3O^+\right] = 10^{-4.5} = 0.00003162$ moles per litre ✓
to
$\left[H_3O^+\right] = 10^{-4.7} = 0.000019953$ moles per litre ✓

(ii) $pH = -\log_{10}\left[H_3O^+\right] = -\log_{10} 4.8 \times 10^{-8}$ ✓
$pH = 7.32$ therefore it is not acidic. ✓ [7]

21. (a) $p = \dfrac{28}{80}$ ✓

(b) (i) (0.2455, 0.4545) ✓✓

(ii) ME= $1.960\sqrt{\dfrac{\frac{28}{80}\left(\frac{52}{80}\right)}{80}} = 0.1045$ ✓

(iii) 95% the width is 0.209
99% the width is 0.2747 ✓
$0.209x = 0.2747$
$x = 1.3144$ ✓
Therefore an increase of 31.45%

(c) $1.960\sqrt{\dfrac{\frac{28}{80}\left(\frac{52}{80}\right)}{n}} = 0.02$ ✓
$n = 2184.9$ ✓
Therefore sample size of 2185 ✓

(d) 0.95 x 40 = 38
∴ Approximately 38 would contain the true proportion ✓ [10]

22. (a) $-192\cos(4t) - 80\sin(4t) = 0$
$t = 1.27679\ (max)$ ✓
$v(t) = 20\cos(4t) - 48\sin(4t) + c$
$v(t) = 20$ when $t = 0$ therefore c = 0 ✓
$v(1.27679) = 52$ units/sec ✓

(b) $v(2) = -50.4$ units/sec ✓
$a(2) = -51.2$ units/sec$^2$ ✓
The velocity and the acceleration are both negative, therefore the particle is moving faster to the left.
∴ Increasing speed ✓

(c) $x(t) = 5\sin 4t + 12\cos 4t + c$
$x(0) = 12 \quad \therefore c = 0$ ✓
$x(2) = 3.2008$ units ✓

(d) $\int_4^5 20\cos(4t) - 48\sin(4t)\ dt = 22.393$ units ✓✓

(e) $\int_0^5 |\ 20\cos(4t) - 48\sin(4t)\ |\ dt = 160.538$ units ✓✓ [12]

End of Questions