

PERTH MODERN SCHOOL

UNIT 3CD MAS – 2014

TEST 1

POLAR COORDINATES, COMPLEX NUMBERS & VECTORS

NAME: \_\_\_\_\_

DATE: Thurs. 13<sup>th</sup> Feb.

Total: 43 marks

Time: 45 min.

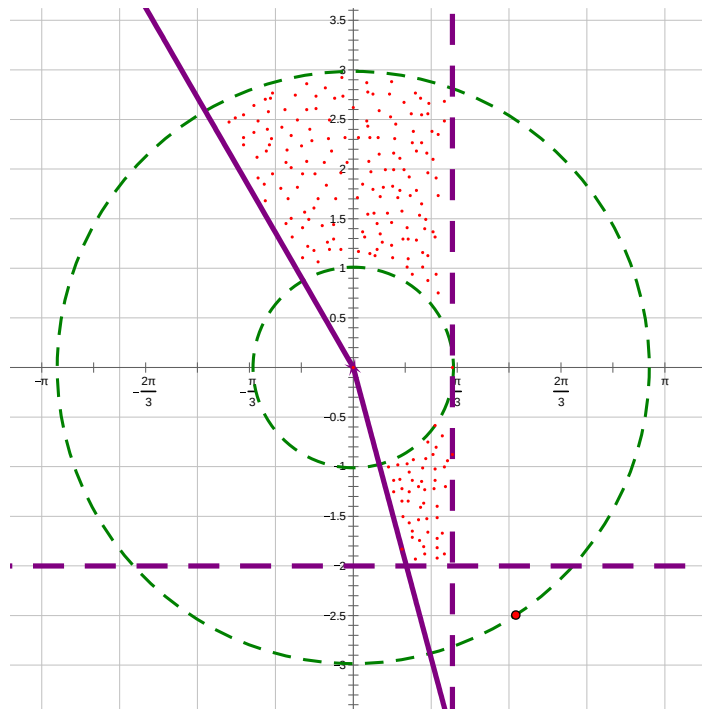
1.  $Z$  is a complex number. Sketch the region given by

[5]

$\operatorname{Re}(Z) < 1$  and  $\operatorname{Im}(Z) > -2$  ☒

and  $1 < |Z| < 3$  ☒

and  $-\frac{5\pi}{12} \leq \operatorname{Arg} Z \leq \frac{2\pi}{3}$  ☒



☒ ☒ for correct shading

2. Express  $Z = -1 - \sqrt{3}i$  in polar form.

[2]

$$|Z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} \quad \tan \theta = \sqrt{3} \text{ Quadrant 3}$$

$$= 2 \quad \checkmark \quad \therefore \theta = -\frac{2\pi}{3}$$

$$\therefore Z = 2\text{cis}\left(-\frac{2\pi}{3}\right) \quad \checkmark$$

3. If  $Z_1 = 5\text{cis}\frac{\pi}{6}$  and  $Z_2 = 2\text{cis}\frac{\pi}{12}$ , then prove  $Z_1 Z_2 = 5\sqrt{2}(1 + i)$

[4]

$$Z_1 Z_2 = 5\text{cis}\frac{\pi}{6} \times 2\text{cis}\frac{\pi}{12}$$

$$= 10\text{cis}\left(\frac{\pi}{6} + \frac{\pi}{12}\right)$$

$$= 10\text{cis}\frac{\pi}{4} \quad \checkmark \quad \checkmark$$

$$= 10\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= 10\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$\checkmark$

$$= \frac{10}{\sqrt{2}}(1 + i)$$

$$\therefore Z_1 Z_2 = 5\sqrt{2}(1 + i)$$

$\checkmark$

4. Find  $Z$  if  $Z\bar{Z} + 2Z = \frac{1}{4} + i$

[6]

Let  $Z = a + bi$  ☒

$$Z\bar{Z} + 2Z = (a + bi)$$

i.e.  $(a + bi)(a - bi) + 2(a + bi) = \frac{1}{4} + i$

i.e.  $a^2 + b^2 + 2a + 2bi = \frac{1}{4} + i$  ☒

Compare real, imaginary

$$a^2 + b^2 + 2a = \frac{1}{4} \quad \text{①} \quad \text{☒$$

$$2b = 1 \quad \text{②}$$

$\therefore b = \frac{1}{2}$  ☒

sub  $b = \frac{1}{2}$  in ①

$$a^2 + \left(\frac{1}{2}\right)^2 + 2a = \frac{1}{4}$$

$$a^2 + 2a = 0$$

$$a(a + 2) = 0$$

$\therefore a = 0 \quad a = -2$  ☒

Hence  $Z = -2 + \frac{i}{2}$  ☒

5.

Consider the 3D position vectors

[8]

$$\overrightarrow{OA} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 5\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$

Determine:

a) the length of  $\overrightarrow{AB}$ .

$$\overrightarrow{AB} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k} \quad \checkmark$$

$$|\overrightarrow{AB}| = \sqrt{(3)^2 + (4)^2 + (-7)^2}$$

$$= \sqrt{74} \quad \checkmark$$

b)  $\angle AOB$  to the nearest degree.

$$\cos \angle AOB = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OA}| |\overrightarrow{OB}|}$$

$$= \frac{-5}{\sqrt{29}\sqrt{35}} \quad \checkmark$$

$$\therefore \angle AOB = 99^\circ \quad \checkmark$$

c) the vector equation of the line, in parametric form, through the points A and B.

$$\mathbf{r} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$$

$$\text{i.e. } \mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} \quad \checkmark$$

$$\text{i.e. } \mathbf{r} = (2 + 3\lambda)\mathbf{i} + (-3 + 4\lambda)\mathbf{j} + (4 - 7\lambda)\mathbf{k} \quad \checkmark$$

$$\text{or } x = 2 + 3\lambda \quad \checkmark$$

$$y = -3 + 4\lambda \quad \checkmark$$

$$z = 4 - 7\lambda$$

6.

[8]

A has the rectangular coordinates  $(-1, \sqrt{3})$  and B has polar coordinates  $(4, \frac{5\pi}{4})$ .

a) What are the exact polar coordinates of A? (1)

$$(2, 120^\circ)$$



$$\text{or } \left(2, \frac{2\pi}{3}\right)$$

b) What are the exact rectangular coordinates of B? (2)

$$x = 4\cos\left(\frac{5\pi}{4}\right) \Rightarrow -2\sqrt{2} \quad \checkmark$$

$$y = 4\sin\left(\frac{5\pi}{4}\right) \Rightarrow -2\sqrt{2} \quad \text{or} \quad (-2\sqrt{2}, -2\sqrt{2}) \quad \checkmark$$

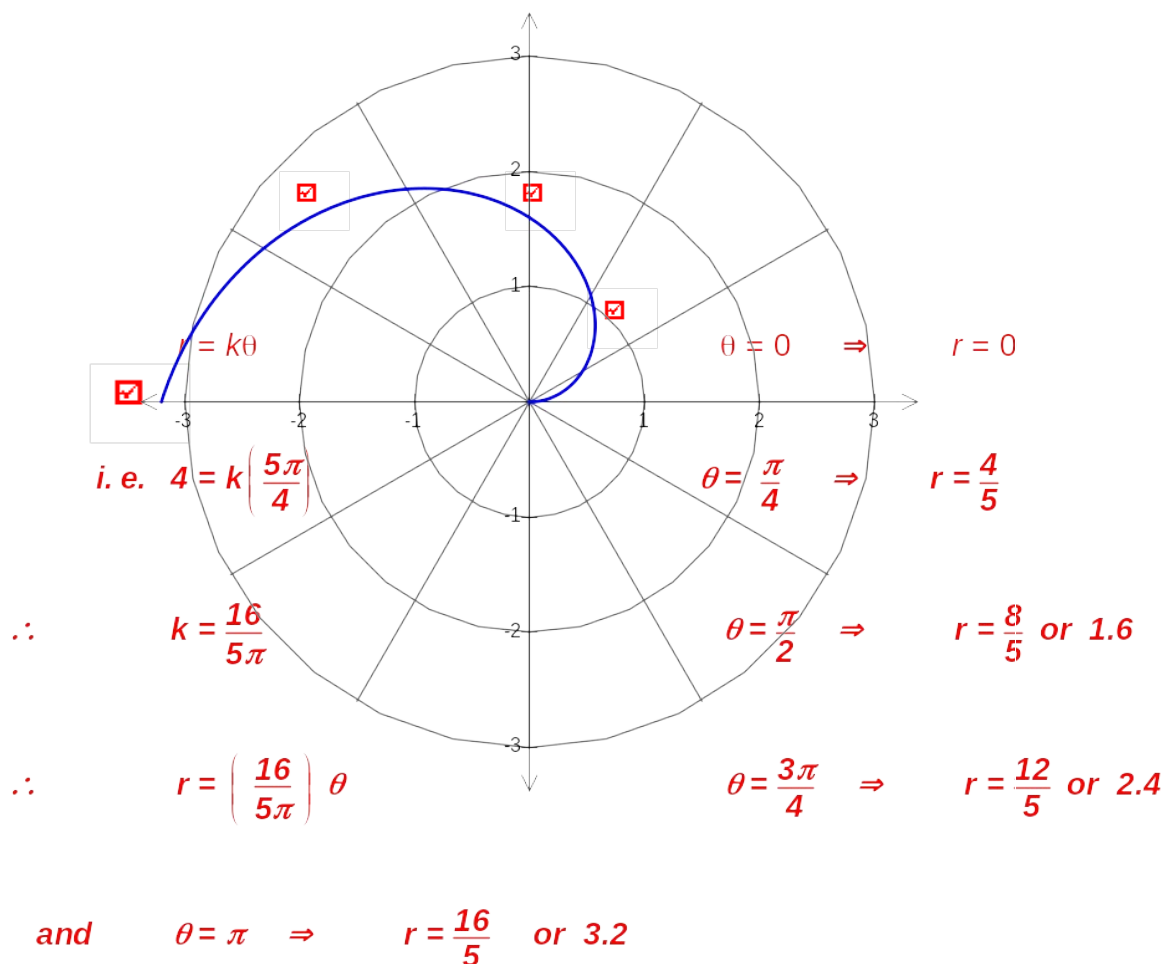
c) The graph of the polar equation  $r = k\theta$  passes through the point B.

If  $k > 0$ , determine the value of  $k$ .

Then, on the axes below, sketch the graph of  $r = k\theta$  for  $0 \leq \theta \leq \pi$ ,

showing important features.  $\frac{1}{2}$  mark for each important feature

(5)





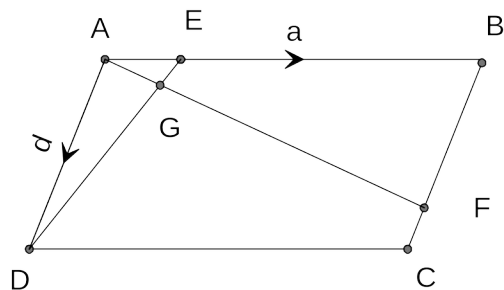
7. [10]

$ABCD$  is a parallelogram with points  $E$  and  $F$  such that  $\overrightarrow{AE} : \overrightarrow{EB} = 1 : 4$  and  $\overrightarrow{BF} : \overrightarrow{FC} = 3 : 1$ .

$\overrightarrow{ED}$  and  $\overrightarrow{AF}$  intersect each other at  $G$ .

Let  $\overrightarrow{AB} = a$  and  $\overrightarrow{AD} = d$ .

a) Complete the diagram below with the information given above. (2)



- b) Determine the ratios in which  $\overrightarrow{AF}$  and  $\overrightarrow{ED}$  intersect each other, if the intersection point is at G. (8)

$$\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF}$$

$$\overrightarrow{ED} = \overrightarrow{EA} + \overrightarrow{AD}$$

$$= a + \frac{3}{4} \overrightarrow{BC}$$

$$= -\frac{1}{5}a + d$$

$$= a + \frac{3}{4}d \quad \boxed{\checkmark}$$

$$\overrightarrow{EG} = k\overrightarrow{ED}$$

$$\overrightarrow{AG} = h\overrightarrow{AF}$$

$$= k \left( -\frac{1}{5}a + d \right) \quad \boxed{\checkmark}$$

$$= h \left( a + \frac{3}{4}d \right)$$

$$\overrightarrow{AG} = \overrightarrow{AE} + \overrightarrow{EG}$$

$$\therefore ha + \frac{3}{4}d = \frac{1}{5}a - \frac{1}{5}ka + kd \quad \boxed{\checkmark}$$

$$\boxed{\checkmark}$$

$$\boxed{\checkmark}$$

$$a : h = \frac{1}{5} - \frac{1}{5}k \quad \textcircled{1} \quad \text{and} \quad d : k = \frac{3}{4}h \quad \textcircled{2}$$

$$\text{solving } h = \frac{4}{17}, \quad k = \frac{3}{17} \quad \boxed{\checkmark}$$

$$\therefore G \text{ divides } \overrightarrow{AF} \text{ in the ratio } 4 : 13 \quad \boxed{\checkmark}$$

$$\text{and } G \text{ divides } \overrightarrow{ED} \text{ in the ratio } 3 : 14 \quad \boxed{\checkmark}$$