Christ Church Grammar School

Mathematics Department
Year 11 Mathematics Methods

Note on marking:
-1 mark at most in Section Two for missing units -1 mark at most in Section Two for incorrect rounding. If a question does not specify how rounding should occur, students need to round to give an answer which is consistent with the rounded answer in the solution.

Semester Two Examination, 2018

Question/Answer booklet

MATHEMATICS METHODS UNITS 1 AND 2

Section Two: Calculator-assumed

SOLUTIONS

Student Name ________________________________

Time allowed for this section

Reading time before commencing work:	ten minutes
Working time:	one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items:	pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters
Special items:	drawing instruments, templates, notes on two unfolded sheets of A4 paper (both sides), and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	14	14	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet, preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that **you do not use pencil**, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Additional working space

Question number: __________

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **fourteen (14)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(6 marks)**

(a) A sequence is defined by $T_{n+1} = T_n - 3.7,\ T_1 = 685$. Determine

(i) the value of T_{90}. (1 mark)

Solution
$T_{90} = 355.7$
Specific behaviours
✓ correct value

(ii) the sum of the first 90 terms of this sequence. (2 marks)

Solution
$S_{90} = \frac{90}{2} \times (685 + 355.7)$ $= 46\,831.5$
Specific behaviours
✓ uses sum formula   ✓ correct sum   Note: allow follow through if answer from (a)(i) was incorrect

(b) Another sequence is defined by $T_n = 685(0.8)^{n-1}$. Determine

(i) the value of T_4. (1 mark)

Solution
$T_4 = 350.72$
Specific behaviours
✓ correct value

(ii) the value that the sum of the first n terms of the sequence approaches as $n \to \infty$. (2 marks)

Solution
$S_\infty = \frac{685}{1 - 0.8}$ $= 3\,425$
Specific behaviours
✓ uses sum to infinity formula   ✓ correct value

Question 10 **(5 marks)**

(a) Given that the graph of $y = x^3 - 6x^2 + kx - 4$ has exactly one point at which the gradient is zero, determine the value of k. (3 marks)

Solution
$\frac{dy}{dx} = 3x^2 - 12x + k$
So for turning point $3x^2 - 12x + k = 0$ For exactly one solution $b^2 - 4ac = 0$, so $(-12)^2 - 4 \times 3 \times k = 0$ Therefore $144 - 12k = 0$, so $k = 12$
Specific behaviours
✓ correctly calculates $\frac{dy}{dx}$ in terms of k ✓ substitutes correct values into $b^2 - 4ac = 0$ formula ✓ calculates correct value of k

(b) Calculate the greatest and least values of $4 - 3x^2 + x^3$, for $-2 \le x \le 3$. (2 marks)

Solution
Using fMax() and fMin() functionality or similar on CAS calculator greatest value = 4, least value = -16
Specific behaviours
✓ correct greatest value ✓ correct least value

Additional working space

Question number: _________

Additional working space

Question number: __________

Question 11 **(6 marks)**

A mobile phone retailer classified recent sales of 375 phones by the age of customer and if the phone was bought outright or on a plan. A summary of the data is shown in the table below.

	Aged under 30	Aged 30 or over	Total
Bought outright	p	94	r
Bought on a plan	115	q	224
Total			375

(a) Determine the values of p, q and r shown in the table. (3 marks)

Solution
$q = 224 - 115 = 109$ $r = 375 - 224 = 151$ $p = 151 - 94 = 57$
Specific behaviours
✓ each correct value

(b) A recent sale is selected at random from those recorded above. Event A occurs if the customer was aged under 30 and event B occurs if the phone was bought outright. Determine the following probabilities:

(i) $P(A \cup B)$. (1 mark)

Solution
$P(A \cup B) = \dfrac{375 - 109}{375} = \dfrac{266}{375} \quad (= 0.709\bar{3})$
Specific behaviours
✓ correct probability

(ii) $P(\bar{A}\,|\bar{B}\,)$. (2 marks)

Solution
$P(\bar{A}\,|\bar{B}\,) = \dfrac{109}{224} \quad (\approx 0.487)$
Specific behaviours
✓ numerator   ✓ denominator

See next page

Question 12 **(11 marks)**

(a) Calculate the area of the minor segment that subtends an arc of $108°$ in a circle of diameter 130 cm. (2 marks)

Solution
$$108° = \frac{3\pi}{5}, \qquad r = \frac{130}{2} = 65$$ $$A = \frac{1}{2}(65)^2\left(\frac{3\pi}{5} - \sin\frac{3\pi}{5}\right) \approx 1\,973 \text{ cm}^2$$
Specific behaviours
✓ converts angle, uses correct radius ✓ calculates area

(b) A chord of length 56 cm subtends an angle of $\frac{\pi}{7}$ at the centre of a circle. Calculate the radius of the circle. (2 marks)

Solution
Using the cosine rule: $$56^2 = r^2 + r^2 - 2r^2\cos\left(\frac{\pi}{7}\right)$$ $$r \approx 126 \text{ cm}$$
Specific behaviours
✓ uses appropriate formula ✓ calculates radius

Additional working space

Question number: _________

Question 22 **(5 marks)**

Consider the curve

$$y = \frac{8}{x^2}$$

(a) Determine the equation of the tangent line to this curve at the point where $x = a$, giving your answer in terms of a. (4 marks)

Solution
$\frac{dy}{dx} = \frac{-16}{x^3}$ at $x = a$, $\frac{dy}{dx} = \frac{-16}{a^3}$, $y = \frac{8}{a^2}$ so using $y = mx + c$, $\frac{8}{a^2} = \frac{-16}{a^3} \times a + c$, so $c = \frac{24}{a^2}$ so tangent is $y = \frac{-16}{a^3}x + \frac{24}{a^2}$ or $16x + a^3y = 24a$ (either representation is acceptable)
Specific behaviours
✓ correct expression for $\frac{dy}{dx}$ ✓ correct values of $\frac{dy}{dx}$ and y at $x = a$ ✓ correct value of constant c in $y = mx + c$ ✓ correct final equation for tangent line

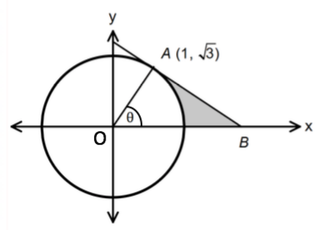
(b) Give the coordinates of the x intercept for the tangent line in part (a). (1 mark)

Solution
$0 = \frac{-16}{a^3}x + \frac{24}{a^2}$ $x = \frac{3a}{2}$ So coordinates are $\left(\frac{3a}{2}, 0\right)$
Specific behaviours
✓ gives correct coordinates

End of questions

Question 12 continued

(c) The point A with coordinates $(1, \sqrt{3})$ lies on a circle with centre at the origin, O. A tangent to the circle is drawn at A and this intersects with the x-axis at the point B. Angle $AOB = \theta$.



(i) Calculate θ in radians and the length of the radius of the circle. (3 marks)

Solution
$tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$, so $\theta = \frac{\pi}{3}$ radians (i.e. 1.047 radians) $r^2 = 1^2 + (\sqrt{3})^2$, so radius is 2 units
Specific behaviours
✓ correct trigonometric ratio involving θ ✓ correct value of θ in radians   ✓ correct value of radius

(ii) Hence, calculate the area, correct to 2 decimal places, of the shaded region. (4 marks)

Solution
Area of sector = $\frac{1}{2}r^2\theta = \frac{1}{2} \times 4 \times \frac{\pi}{3} = \frac{2\pi}{3}$ $\cos\left(\frac{\pi}{3}\right) = \frac{2}{OB}$, so $OB = 2 \div \cos\left(\frac{\pi}{3}\right) = 4$ Area of triangle $OAB = \frac{1}{2} \times 4 \times \sqrt{3} = 2\sqrt{3}$ Shaded area = Area of triangle OAB – Area of sector = $2\sqrt{3} - \frac{2\pi}{3} = 1.37$ square units (to 2 dp)
Specific behaviours
✓ correct area of sector   ✓ correct length OB ✓ correct area of triangle AOB ✓ correct shaded area

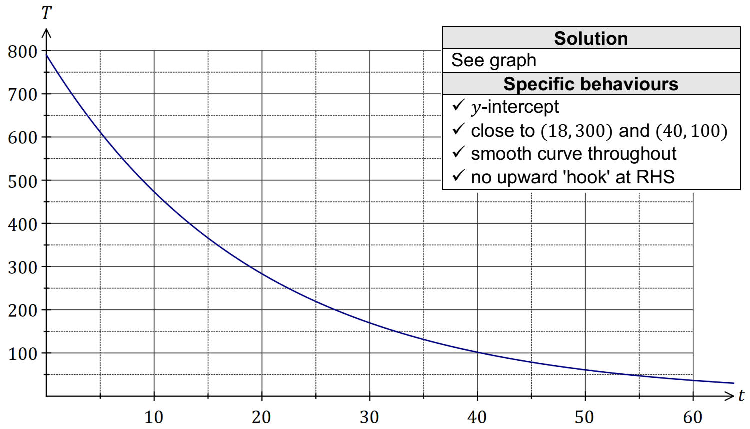
Question 13 **(9 marks)**

The temperature T of a cast taken out of an oven cools according to the model $T = 790(0.95)^t$, where t is the time in minutes since the cast was removed from the oven. T is measured in $°C$.

(a) Determine the fall in temperature of the cast during the first 6 minutes. (2 marks)

Solution
$T = 790(0.95)^6 \approx 581°C$ $\Delta T = 790 - 581 = 209°C$
Specific behaviours
✓ value of T when $t = 6$ ✓ correct drop

(b) Graph the temperature of the cast against time on the axes below. (4 marks)



Solution
See graph
Specific behaviours
✓ y-intercept ✓ close to $(18, 300)$ and $(40, 100)$ ✓ smooth curve throughout ✓ no upward 'hook' at RHS

(c) State the name of this type of function. (1 mark)

Solution
Exponential.
Specific behaviours
✓ correct name

(d) The temperature of the cast falls to room temperature of $20°C$.

(i) Determine the time taken for the cast to reach room temperature. (1 mark)

Solution
$790(0.95)^t = 20 \Rightarrow t = 71.7$ m
Specific behaviours
✓ correct time

(ii) Comment on the usefulness of the model for large values of t. (1 mark)

Solution
For large values of t the model shows that $T \to 0$ but the temperature of the cast only falls to $20°C$ and so model not valid for large t.
Specific behaviours
✓ states not valid, with reason

Question 21 continued

(c) Find the probability that a randomly chosen seventeen year old who has taken a driver's license test

(i) passed on the first attempt. (1 mark)

Solution
$0.2597 + 0.2726 = 0.5323$
Specific behaviours
✓ gives correct answer

(ii) failed on the first attempt or is male. (1 mark)

Solution
$0.2703 + 0.2597 + 0.1974 = 0.7274$
Specific behaviours
✓ gives correct answer

(iii) failed on the first attempt, given that he is a male. (2 marks)

Solution
$\dfrac{0.2703}{0.53} = 0.51$
Specific behaviours
✓ gives correct numerator in fraction ✓ gives denominator numerator in fraction Note: must give final answer (0.51). Student gets 1 mark deducted if numerator and denominator correct but final answer incorrect.

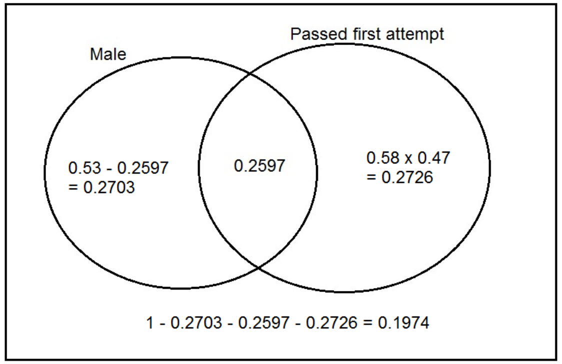
Question 21 **(8 marks)**

53% of seventeen year olds who take their driver's license test are male. 49% of seventeen year old males pass on the first attempt, while 58% of seventeen year old females pass on their first attempt.

(a) From the population of seventeen year olds who have taken their driving test a random person is selected. Show that the probability that this person is a male who passed the test first time is 0.2597. (1 mark)

Solution
$0.53 \times 0.49 = 0.2597$
Specific behaviours
✓ shows correct calculation

(b) Complete the Venn diagram below to represent the relevant probabilities for seventeen year olds who have taken their driving test. (3 marks)



Solution
See Venn diagram
Specific behaviours
✓ correct 0.2703 value ✓ correct 0.2726 value ✓ correct 0.1914 value

See next page

Question 14 **(4 marks)**

Parallelogram $PQRS$ has side $QR = 24$ cm, side $RS = 39$ cm and an area of 460 cm^2. Determine the lengths of the diagonals of $PQRS$.

Solution
$\frac{1}{2}(24)(39)\sin x = \frac{460}{2}$ $x = 29.44°, 150.56°$ $L_1 = \sqrt{24^2 + 39^2 - 2(24)(39)\cos 29.44}$ ≈ 21.6 cm $L_2 = \sqrt{24^2 + 39^2 - 2(24)(39)\cos 150.56}$ ≈ 61.1 cm
Specific behaviours
✓ equation for half area ✓ both angles of parallelogram ✓ correct length of one diagonal ✓ second correct length

See next page

Question 15 **(8 marks)**

A council took a random sample of 125 and 172 properties from suburbs P and Q respectively. A total of 36 of the properties in the sample were in arrears with their rates, of which 21 of these properties were in suburb Q. 'In arrears' means that payment of rates is overdue.

(a) Council officers wanted to randomly choose 4 of the properties that were in arrears. How many different selections of properties are possible? (1 mark)

Solution
$\binom{36}{4} = 58\,905$
Specific behaviours
✓ correct number

(b) Determine the probability that one randomly chosen property from the sample

(i) is not in arrears and is in suburb Q. (2 marks)

Solution
$P = \dfrac{172-21}{125+172} = \dfrac{151}{297} \quad (\approx 0.508)$
Specific behaviours
✓ numerator ✓ denominator

(ii) is in suburb P given that it is in arrears. (2 marks)

Solution
$P = \dfrac{36-21}{36} = \dfrac{15}{36} = \dfrac{5}{12} \quad (= 0.41\bar{6})$
Specific behaviours
✓ numerator ✓ denominator Note: either simplified or non simplified fraction is acceptable

(c) Justifying your answer with conditional probabilities and rounding the conditional probabilities to 2 decimals places in your analysis, comment on whether being in arrears with rates is independent of the suburb the property is in. (3 marks)

Solution
$P(\text{Arrears}\mid P) = \dfrac{15}{125} = 0.12$ $P(\text{Arrears}\mid \text{Q}) = \frac{21}{172} = 0.12$ (to 2 dp) Hence being in arrears is independent of suburb, as the conditional probabilities are the same to 2 dp.
Specific behaviours
✓ calculates $P(\text{Arrears}\mid P)$ ✓ calculates $P(\text{Arrears}\mid Q)$ ✓ correct conclusion

Question 20 **(8 marks)**

A pyramid with a rectangular base of length L and width w has perpendicular height h. The length of the base is three times its width and the sum of the width, length and height is 180 cm.

(a) Calculate the length, height and volume of the pyramid when $w = 15$ cm. (2 marks)

Solution
$L = 3 \times 15 = 45, \quad h = 180 - 15 - 45 = 120$ $V = \frac{1}{3}(15 \times 45) \times 120 = 27\,000 \text{ cm}^3$
Specific behaviours
✓ correct length and height ✓ correct volume

(b) Show that the volume of the pyramid is given by $V = 180w^2 - 4w^3$. (2 marks)

Solution
$L = 3w, \quad h = 180 - w - 3w = 180 - 4w$ $V = \frac{1}{3}(w \times 3w)(180 - 4w)$ $= 180w^2 - 4w^3$
Specific behaviours
✓ expressions for length and height ✓ substitutes width, length and height correctly

(c) Use calculus to determine the maximum volume of the pyramid. (4 marks)

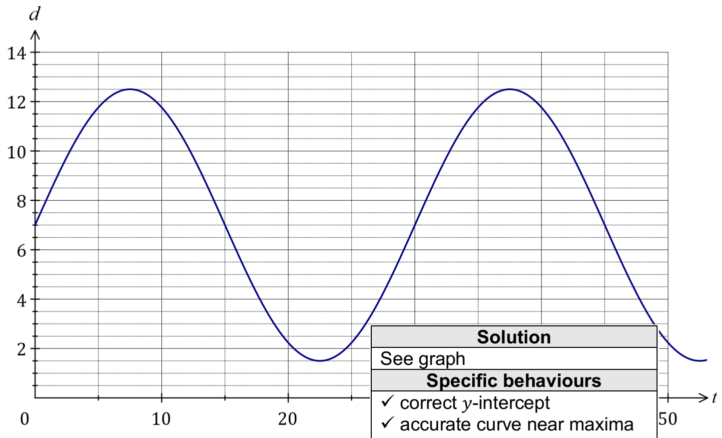
Solution
$\dfrac{dV}{dw} = 360w - 12w^2$ $360w - 12w^2 = 0 \Rightarrow w = 0, 30$ V: 29, 30, 31 $\dfrac{dV}{dW}$: $+$, 0, $-$ Sign test indicates $V = 30$ is maximum ($w = 0$ gives zero volume) $V_{max} = 180(30)^2 - 4(30)^3 = 54\,000 \text{ cm}^3$
Specific behaviours
✓ correct derivative using given variables ✓ solves derivative equal to zero ✓ uses sign test to show maximum at $w = 30$ ✓ correct maximum volume

Question 19 **(8 marks)**

The height, h metres, above level ground of a seat on a steadily rotating giant viewing wheel at the Perth Show, t seconds after observations began was given by

$$h = 5.5\sin\left(\frac{\pi t}{15}\right) + 7, \qquad t \geq 0.$$

(a) Draw the graph of the height of the seat against time on the axes below. (4 marks)



Solution
See graph
Specific behaviours
✓ correct y-intercept ✓ accurate curve near maxima ✓ accurate curve near minimum ✓ smooth curve throughout

(b) How long did the Ferris wheel take to complete one revolution? (1 mark)

Solution
30 seconds
Specific behaviours
✓ correct time

(c) At what time, when the seat was rising, did it first reach a height of 5 metres? (1 mark)

Solution
$t = 28.2$ s
Specific behaviours
✓ time that rounds to 28 s

(d) Determine the change in height of the seat between $t = 95$ and $t = 96$, giving your answer rounded to the nearest cm. (2 marks)

Solution
$h(96) = 12.23, \qquad h(95) = 11.76$ $\delta h = 12.23 - 11.76 = 0.47$ m
Specific behaviours
✓ determines both heights ✓ states difference to nearest cm

See next page

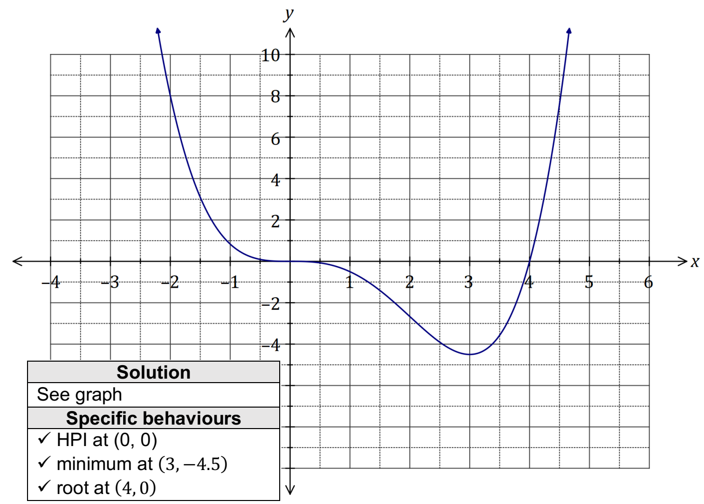
Question 16 **(6 marks)**

A function is defined by $f(x) = \frac{x^4}{6} - \frac{2x^3}{3}$.

(a) Use the derivative $f'(x)$ to determine the coordinates of all stationary points of the function. (3 marks)

Solution
$f'(x) = \frac{2}{3}x^3 - 2x^2$ $\frac{2}{3}x^3 - 2x^2 = 0 \Rightarrow x = 0, x = 3$ $f(0) = 0, \qquad f(3) = -4.5$ Stationary points at $(0,0)$ and $(3,-4.5)$
Specific behaviours
✓ correct derivative ✓ correct zeros of derivative ✓ correct coordinates

(b) Sketch the graph of $y = f(x)$ on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ HPI at (0, 0) ✓ minimum at $(3,-4.5)$ ✓ root at $(4,0)$

See next page

Question 17 **(8 marks)**

John has two, initially empty, water containers, which are being filled with water. The amount of water added to container A each minute follows an arithmetic sequence, with 3 mL poured in during the first minute and 6 mL poured in during the second minute. The amount of water added to container B each minute follows a geometric sequence, with 2 mL poured in during the first minute and 2.2 mL poured in during the second minute.

(a) The amount of water poured into container B during the n^{th} minute is given by $a(r)^{n-1}$. State the value of the constants a and r. (2 marks)

Solution
$a = 2$ $r = \frac{2.2}{2} = 1.1$
Specific behaviours
✓ value of a ✓ value of r

(b) How long does it take to fill container A with 360 mL of water? (2 marks)

Solution
$\frac{n}{2}\big(2(3) + (n-1)(3)\big) = 360$ $n = 15$ minutes
Specific behaviours
✓ uses sum formula ✓ correct time

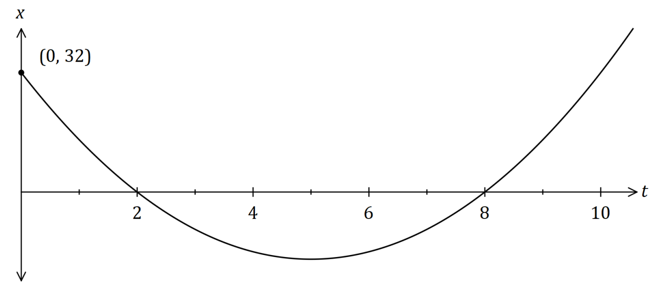
(c) John measures the amount of water in each of the two containers at the end of each minute. He finds that container B first holds more than container A at the end of minute m. Calculate the value of m. (4 marks)

Solution
Solve on CAS: $\frac{2(1.1^n - 1)}{1.1 - 1} = \frac{n}{2}(2 \times 3 + (n-1) \times 3)$ $n = 0, 0.29, 58.4$ Discount $n = 0$ and $n = 0.29$ as at end of first second A contains $3\ ml$ and B contains $2\ ml$. Therefore $m = 59$.
Specific behaviours
✓ uses correct initial equation ✓ obtains $n = 0, 0.29, 58.4$ solutions to initial equation ✓ discounts $n = 0$ and $n = 0.29$ with correct reasons ✓ gives final answer $m = 59$

Question 18 **(6 marks)**

A small body moves in a straight line so that its displacement x from a fixed point O after t seconds is given by $x = at^2 + bt + c$ metres.

The position-time graph of the body is shown below.



(a) Determine the values of the constants a, b and c. (3 marks)

Solution
$x = a(t-2)(t-8)$ $32 = a(-2)(-8) \Rightarrow a = 2$ $x = 2(t^2 - 10t + 16)$ $= 2t^2 - 20t + 32$ $a = 2, \quad b = -20, \quad c = 32$
Specific behaviours
✓ writes equation using roots ✓ uses y-intercept to find a ✓ expands and states three values

(b) Determine the displacement of the body when its velocity is 24 ms^{-1}. (3 marks)

Solution
$v = 4t - 20$ $4t - 20 = 24 \Rightarrow t = 11$ $x(11) = 2(11-2)(11-8) = 54$ m
Specific behaviours
✓ equation for velocity ✓ solves for time ✓ substitutes for displacement