St Mary's
ANGLICAN GIRLS' SCHOOL

2010

Semester II Examination

Question/answer booklet

MATHEMATICS SPECIALIST 3CDMAS

Section One

CALCULATOR-FREE

Time allowed for this section

Reading time before commencing work: 5 minutes
Working time for this section: 50 minutes

Material required/recommended for this section

To be provided by the supervisor
Question/answer booklet for Section One, containing a removable formula sheet which may also be used for Section Two.

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler

Important note to candidates

No other items may be taken into the examination room.
It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room.
If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be attempted	Working time	Marks available
Section One Calculator—free	**8**	**8**	**50 minutes**	**40**
Section Two Calculator—assumed	12	12	100 minutes	80
			Total marks	120

Instructions to candidates

1. Answer the questions in the spaces provided.

2. Spare answer pages are provided at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued i.e. give the page number.

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Correct answers given without supporting reasoning may not be allocated full marks. Incorrect answers given without supporting reasoning cannot be allocated any marks. If you repeat an answer to any question, ensure that you cancel the answers you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

1. [1, 1, 5 = 7 marks]

Determine the following integrals.

(a) $\int 5\cos(4-3x)\,dx$

$= \frac{5}{-3}\sin(4-3x) + C$ ✓ R/W

−1 for no "c"

(b) $\int \frac{2x}{3x^2+5}\,dx$

$= \frac{1}{3}\ln|3x^2+5| + C$ ✓ R/W

(c) $\int \frac{\sin^3 x}{\cos^4 x}\,dx$ let $u = \cos x$

$\frac{du}{dx} = -\sin x$ ✓

$\int \frac{\sin^3 x}{u^4} \times \frac{1}{-\sin x}\,du$ ✓

$u^2 = \cos^2 x$

$u^2 = 1 - \sin^2 x$

$\sin^2 x = 1 - u^2$

$= -\int \frac{\sin^2 x}{u^4}\,du$

$= -\int \frac{1-u^2}{u^4}\,du$ ✓

$= \int \frac{1}{u^2} - \frac{1}{u^4}\,du$

$= -u^{-1} + \frac{u^{-3}}{3} + C$ ✓

$= \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C$ ✓

2. [1, 3, 1 = 5 marks]

(a) Given matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ for which $\boldsymbol{AB} = \boldsymbol{C}$ and det $\boldsymbol{A}$ 0, express $\boldsymbol{B}$ in terms of $\boldsymbol{A}$ and $\boldsymbol{C}$.

$$AB = C$$
$$B = A^{-1}C \quad \checkmark$$

(b) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 3 & -3 & 2 \end{pmatrix}$, $\boldsymbol{D} = \begin{pmatrix} -4 & 13 & -7 \\ -2 & 7 & -4 \\ 3 & -9 & 5 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix}$.

(i) Find the matrix $\boldsymbol{DA}$

$$\begin{bmatrix} -4 & 13 & -7 \\ -2 & 7 & -4 \\ 3 & -9 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 3 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \checkmark$$

(ii) Find $\boldsymbol{B}$ if $\boldsymbol{AB} = \boldsymbol{C}$.

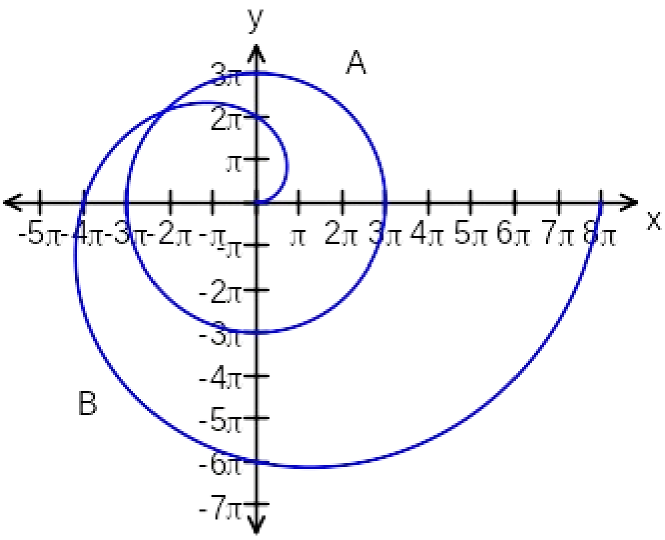
$$\begin{aligned} B &= A^{-1}C \\ &= DC \\ &= \begin{bmatrix} -4 & 13 & -7 \\ -2 & 7 & -4 \\ 3 & -9 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 10 \end{bmatrix} \quad \checkmark \\ &= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \checkmark \end{aligned}$$

(c) Find the coordinates of the point of intersection of the planes
$x + 2y + 3z = 5$, $2x - y + 2z = 7$ and $3x - 3y + 2z = 10$.

$$(1, -1, 2) \quad \checkmark$$

3. [2 marks]

Determine the equation of the graphs below in polar form.



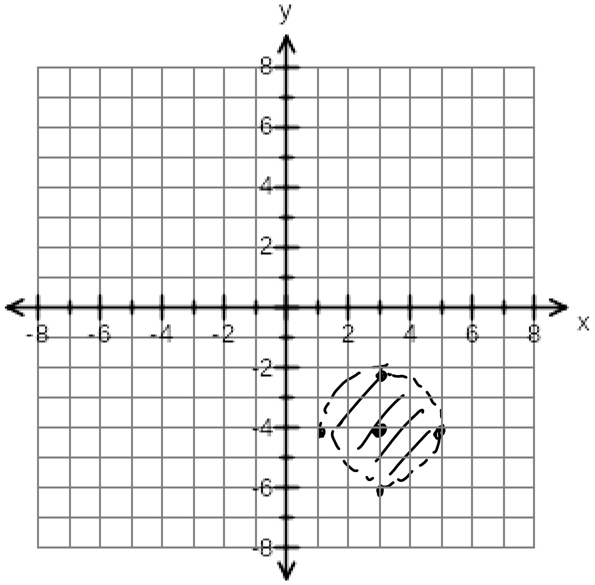
A $r = 3\pi$ ✓

B $r = 4\theta$ ✓

4. [2, 3 = 5 marks]

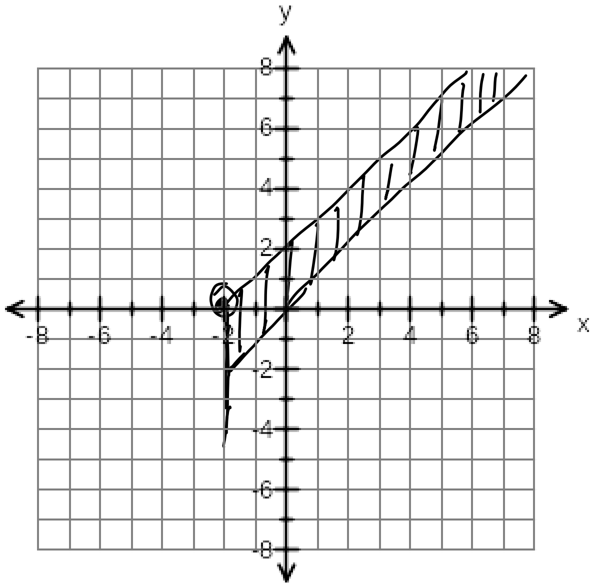
Sketch the following regions in the complex plane.

(a) $|z-3+4i|<2$



Dotted
Centre $(3, {}^-4)$ √√
Radius 2

(b) ¿ ¿ $\mathrm{Im}(z) \geq \mathrm{Re}(z)$ ¿¿



open circle on $(-2, 0)$ ($-\frac{1}{2}$ if not shown)
√ line
√ line
√ shading

5. [5 marks]

Find the coordinates of the points where the tangent to the curve $x^2+2xy+3y^2=18$ is horizontal.

$$2x+2y+2x\frac{dy}{dx}+6y\frac{dy}{dx}=0 \quad \checkmark$$

$$\frac{dy}{dx}=\frac{-2x-2y}{(2x+6y)}=0 \quad \checkmark$$

$$x=-y \quad \checkmark$$

$$x^2-2x^2+3x^2=18$$

$$2x^2=18$$

$$x=\pm 3 \quad \checkmark$$

$$(3,-3) \text{ and } (-3,3) \quad \checkmark$$

6. [3 marks]

The plane $6x-2y+z=11$ contains the line $x-1=\frac{y+1}{2}=\frac{z-3}{k}$. Find k.

$$\left.\begin{array}{ll} x-1=t & x=1+t \\ y+1=2t & y=2t-1 \\ z-3=kt & z=kt+3 \end{array}\right\} \checkmark$$

$$6(1+t)-2(2t-1)+kt+3=11 \checkmark$$

$$\cancel{6}+\cancel{6}t-\cancel{4}t+\cancel{2}+\cancel{k}t+\cancel{3}=\cancel{11}$$

$$2t+kt=0$$

$$k=-2 \checkmark$$

7. [2, 2, 1, 2 = 7 marks]

Let $z = \frac{2-4i}{-3+i}$

(a) Express z in the form $x+iy$

$$\frac{(2-4i)}{(-3+i)}\frac{(-3-i)}{(-3-i)} = \frac{-6-4+12i-2i}{10} \checkmark$$

$$= -1+i \checkmark$$

(b) Show that $\sqrt{2}|z| = |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$

$|z| = \sqrt{2}$

$|\mathrm{Re}\, z| = 1$

$|\mathrm{Im}\, z| = 1$

$\therefore$ LHS $\sqrt{2} \times \sqrt{2} = 2$ ✓

RHS $1+1 = 2$ ✓

$\therefore$ RHS = LHS

(c) Express z in polar form.

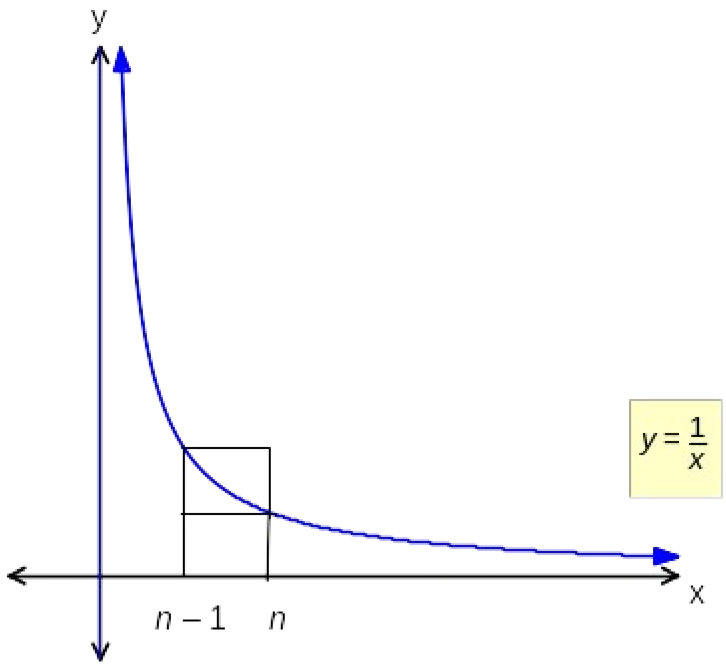
$\sqrt{2}\ \mathrm{Cis}\ \frac{3\pi}{4}$ ✓ R|W

(d) Find z^5

$(\sqrt{2})^5\ \mathrm{Cis}\ \frac{15\pi}{4}$ ✓

$= 4\sqrt{2}\ \mathrm{Cis}\ \frac{-\pi}{4}$ ✓

8. [6 marks]



The area of the region under the curve $y=\frac{1}{x}$ from $x=n-1$ to $x=n$ is in between the areas of the two rectangles shown in the diagram.

Show that $e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1}$

Area of small rectangle $= \frac{1}{n} \times (n-(n-1)) = \frac{1}{n}$ ½✓

Area of large rectangle $= \frac{1}{n-1} \times (1) = \frac{1}{n-1}$ ½✓

Area under curve $\int_{n-1}^{n} \frac{1}{x}\,dx = \ln x \Big|_{n-1}^{n}$

$= \ln(n) - \ln(n-1)$

$= \ln\left(\frac{n}{n-1}\right)$ ✓

$\therefore \frac{1}{n} < \ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$ $(\times n)$

This page is for extra working.

$$1 < n\ln\left(\frac{n}{n-1}\right) < \frac{n}{n-1} \quad \checkmark$$

$$1 < \ln\left(\frac{n}{n-1}\right)^n < \frac{n}{n-1} \quad \checkmark$$

$$e < \left(\frac{n}{n-1}\right)^n < e^{\frac{n}{n-1}} \quad \checkmark$$

$$e^{-1} > \left(\frac{n-1}{n}\right)^n > e^{-\frac{n}{n-1}}$$

$$e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1} \quad \checkmark$$

This page has been left blank intentionally.

St Mary's
ANGLICAN GIRLS' SCHOOL

2010

Semester II Examination

Question/answer booklet

MATHEMATICS SPECIALIST 3CDMAS

Section Two

CALCULATOR-ASSUMED

Time allowed for this section

Reading time before commencing work: 10 minutes
Working time for this section: 80 minutes

Material required/recommended for this section

To be provided by the supervisor
Question/answer booklet for Section Two. Candidates may use the removable formula sheet from Section One.

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler
Special items: drawing instruments, templates, notes on up to two unfolded sheets of A4 paper, and up to three calculators, CAS, graphic or scientific, which satisfy the conditions set by the Curriculum Council for this course.

Important note to candidates

No other items may be taken into the examination room.
It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room.
If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this examination

	Number of questions available	Number of questions to be attempted	Working time	Marks available
Section One Calculator—free	8	8	50 minutes	40
Section Two Calculator—assumed	**12**	**12**	**100 minutes**	**80**
			Total marks	120

Instructions to candidates

1. Answer the questions in the spaces provided.

2. Spare answer pages are provided at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued i.e. give the page number.

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Correct answers given without supporting reasoning may not be allocated full marks. Incorrect answers given without supporting reasoning cannot be allocated any marks. If you repeat an answer to any question, ensure that you cancel the answers you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

9. [4, 3 = 7 marks]

(a) If z is a complex number and $|z+16| = 4|z+1|$, find the value of $|z|$.

$$|x+iy+16| = 4\,|x+iy+1|$$

$$(x+16)^2 + y^2 = 16\left[(x+1)^2 + y^2\right] \checkmark$$

$$x^2 + \cancel{32x} + \cancel{256} + y^2 = 16x^2 + \cancel{32x} + \cancel{16} + 16y^2$$

$$240 = 15x^2 + 15y^2 \checkmark$$

$$16 = x^2 + y^2 \checkmark$$

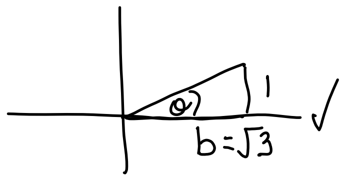
$$\therefore |z| = 4 \checkmark$$

(b) Given that $z = (b + i)^2$ where b is real and positive, find the **exact** value of b when $\arg z = \dfrac{\pi}{3}$.

$$\operatorname{Arg}(z) = \frac{\pi}{3}$$

$$\operatorname{Arg}(z^{1/2}) = \frac{\pi}{6} \checkmark \qquad \text{(Since } b>0\text{)}$$

$$\arg(b+i) = \frac{\pi}{6}$$



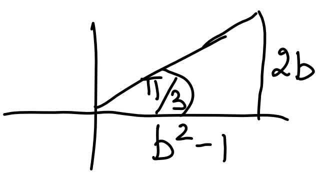
✓

$$b = \sqrt{3} \checkmark$$

Alternative ans

$$z = (b+i)^2 = b^2 + 2bi - 1$$

$$= (b^2-1) + 2bi$$



$$\tan\frac{\pi}{3} = \frac{\sqrt{3}}{1} = \frac{2b}{b^2-1}$$

Solve $b = \sqrt{3}$

10. [1, 1, 3, 4 = 9 marks]

Given that **p** = 4**i** + 2**j** + 3**k**, **q** = 6**i** + **j** – 2**k** and **r** = *x***i** + *y***j** + 4**k**, find:

(a) 2**p** – **q**

$$\underset{\sim}{8i} + 4\underset{\sim}{j} + 6\underset{\sim}{k} - (6\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$$

$$= 2\underset{\sim}{i} + 3\underset{\sim}{j} + 8\underset{\sim}{k} \quad \checkmark$$

(b) a unit vector in the direction of **p**

$$\frac{1}{\sqrt{29}}(4\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$$

(c) the value(s) of *x* and *y* if **r** has length 9 units and is perpendicular to **p**

$$\underset{\sim}{r} \cdot \underset{\sim}{p} = 0$$

$$4x + 2y + 12 = 0 \quad \checkmark$$

$$x^2 + y^2 + 16 = 81 \quad \checkmark$$

Solve

$$\left.\begin{array}{l} x = 1, \ y = -8 \\ x = -5{\cdot}8, \ y = 5{\cdot}6 \end{array}\right\} \checkmark$$

(d) Find the exact shortest distance between the point with position vector **p** and the plane $r \cdot (2i + 3j + k) = 5$.

$|KA| = \underset{\sim}{\hat{n}} \cdot (\vec{OA} - \vec{OB})$ where $\vec{OA}$ = point p
$\vec{OB}$ = point on plane

$\vec{OB} = \langle 1, 1, 0 \rangle$ ✓ (or similar)

$OA - OB = \langle 3, 1, 3 \rangle$ ✓

$\frac{1}{\sqrt{14}} \langle 2, 3, 1 \rangle \cdot \langle 3, 1, 3 \rangle$ ✓

$= \frac{12}{\sqrt{14}} \times \frac{\sqrt{14}}{\sqrt{14}}$

$= \frac{6}{7}\sqrt{14}$ ✓

11. [4, 3 = 7 marks]

(a) Find the roots of $z^4 = i$ in polar form.

$z^4 = \text{cis}\,\frac{\pi}{2}$ ✓

$z = \text{cis}\,\frac{\pi}{8}$ ✓

$z = \text{cis}\,\frac{\pi}{8}$, $\text{cis}\,\frac{5\pi}{8}$, $\text{cis}\,\frac{-3\pi}{8}$, $\text{cis}\,\frac{-7\pi}{8}$ ✓✓

(must adjust angle)

(b) For any complex number z, prove that $e^{z+2\pi i} = e^z$

LHS $e^z e^{2\pi i}$ ✓

$e^z(\cos 2\pi + i\sin 2\pi)$ ✓

$= e^z(1 + 0i)$

$= e^z$ ✓

$=$ RHS

12. [1, 2, 2 = 5 marks]

In Hollywood, there are three different make up brands that dominate the market: Revlon, Maybelline and Max Factor. People switch from one brand to another all the time.

If they use Revlon this week, there is 0.7 probability they will continue to use it next week, 0.2 probability they will switch to Maybelline, and 0.1 probability they will switch to Max Factor.

If they are now using Maybelline, there is 0.4 probability that they will switch to Revlon, 0.3 probability they will stay with Maybelline, and 0.3 probability they will switch to Max Factor.

If they are now using Max Factor, there is 0.2 probability they will switch to Revlon, 0.3 probability they will switch to Maybelline, and 0.5 probability they will stay with Max Factor.

(a) Express this information as a transition matrix.

$$T = \begin{array}{c} \\ R' \\ M' \\ MF' \end{array} \begin{array}{c} \begin{array}{ccc} R & M & MF \end{array} \\ \begin{bmatrix} 0.7 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{bmatrix} \end{array} \checkmark$$

(b) If an actress is using Maybelline, what is the probability she will be using Revlon 3 weeks later?

$$T^3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.482 \\ 0.254 \\ 0.264 \end{bmatrix} \quad \therefore\ 0.482 \checkmark$$

$\checkmark$

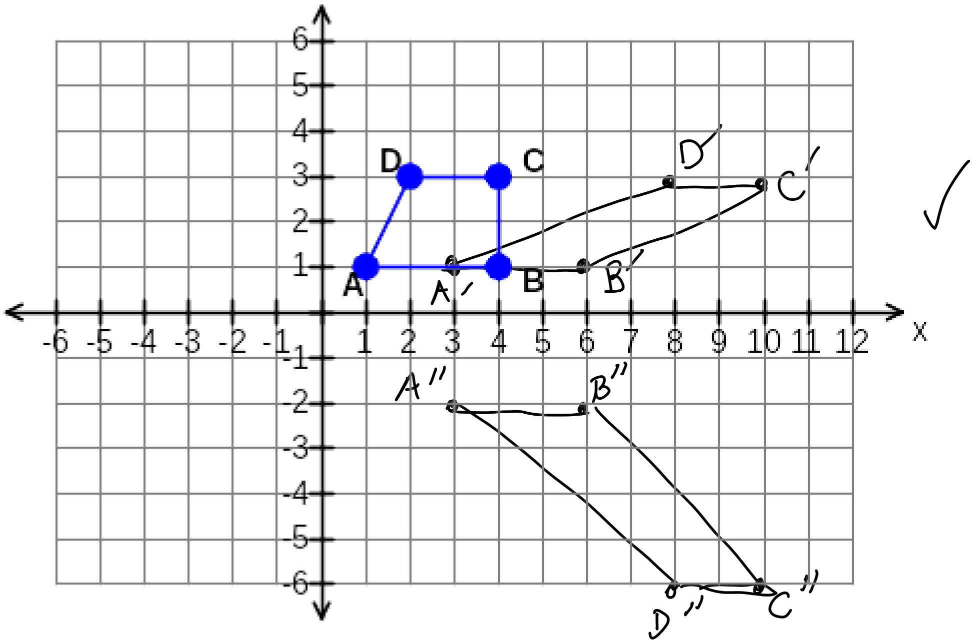
(c) If an actress starts with Maybelline, what is the probability she will be using Max Factor 4 weeks later?

$$T^4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4918 \\ 0.2518 \\ 0.2564 \end{bmatrix} \quad \therefore\ 0.2564 \checkmark$$

$\checkmark$

13. [3, 3, 2 = 8 marks]

The following diagram shows a trapezium with vertices A (1, 1), B (4,1), C (4,3), D (2,3). It undergoes a transformation to the new shape A'(3, 1), B'(6, 1), C'(10, 3), D'(8, 3).



(a) Draw the new shape on the axes above, describe in words the transformation that has taken place and give the appropriate transformation matrix.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \checkmark$$

horizontal shear parallel to the x-axis
scale factor 2 ✓

(b) Shape $A'B'C'D'$ is then transformed by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ to shape $A''B''C''D''$.

(i) Draw shape $A''B''C''D''$ on the axes above.

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & 6 & 10 & 8 \\ 1 & 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 10 & 8 \\ -2 & -2 & -6 & -6 \end{bmatrix}$$

✓ Diagram on grid

(ii) Give a single transformation matrix required to return shape $A''B''C''D''$ to trapezium $ABCD$.

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \checkmark$$

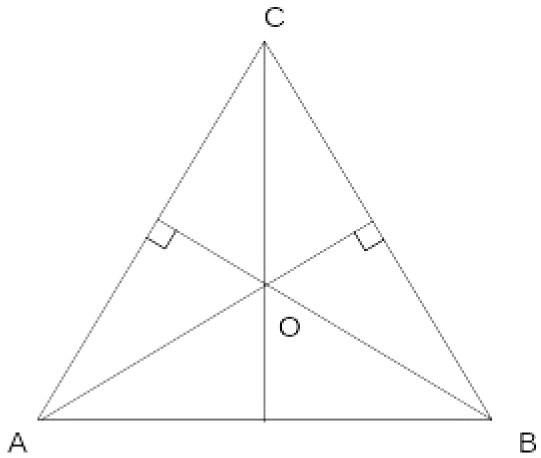
$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}^{-1} = -\frac{1}{2}\begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1/2 \end{bmatrix} \checkmark$$

(c) State a matrix transformation that will triple the area of the shape $A''B''C''D''$.

eg $\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$ $\checkmark\checkmark$

14. [5, 3 = 8 marks]

The diagram below shows an acute angled triangle ABC. O is the origin with $\overrightarrow{OA}=a,\ \overrightarrow{OB}=b\ \ and\ \ \overrightarrow{OC}=c$



(a) OA is perpendicular to BC and BO is perpendicular to AC. Use this to prove that $b\cdot c=a\cdot c$.

$\overrightarrow{OA}\circ\overrightarrow{BC}=0 \qquad \overrightarrow{BO}\circ\overrightarrow{AC}=0$

$\underset{\sim}{a}\circ(\underset{\sim}{c}-\underset{\sim}{b})=0$ ✓ $\quad -\underset{\sim}{b}\circ(\underset{\sim}{c}-\underset{\sim}{a})=0$ ✓

$\underset{\sim}{a}\cdot\underset{\sim}{c}-\underset{\sim}{a}\cdot\underset{\sim}{b}=0 \qquad -\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{a}\cdot\underset{\sim}{b}=0$

$\underset{\sim}{a}\circ\underset{\sim}{b}=\underset{\sim}{a}\cdot\underset{\sim}{c}$ ✓ $\qquad \underset{\sim}{a}\circ\underset{\sim}{b}=\underset{\sim}{b}\cdot\underset{\sim}{c}$ ✓

$\therefore\ \underset{\sim}{a}\cdot\underset{\sim}{c}=\underset{\sim}{b}\circ\underset{\sim}{c}$ ✓

as required to prove

(b) Prove that CO is perpendicular to AB.

RTP $\overrightarrow{CO}\circ\overrightarrow{AB}=0$

LHS $-\underset{\sim}{c}\circ(\underset{\sim}{b}-\underset{\sim}{a})$ ✓

$\left.\begin{aligned} &=-\underset{\sim}{c}\circ\underset{\sim}{b}+\underset{\sim}{c}\cdot\underset{\sim}{a} \\ &=-\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{a}\circ\underset{\sim}{c}\end{aligned}\right\}$ ✓

$=-\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{c}$ since $\underset{\sim}{a}\cdot\underset{\sim}{c}=\underset{\sim}{b}\cdot\underset{\sim}{c}$ ✓ from (a)

$=0$

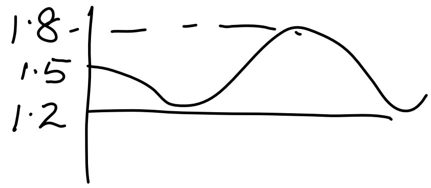
$\therefore$ perpendicular as RTP.

15. [3, 3, 2 = 8 marks]

A life jacket has fallen off the side of a boat in the middle of a lake. The life jacket moves in simple harmonic motion because of the waves. The distance between the lake floor and the life jacket varies between 1.2m and 1.8m and the period of motion is 3 seconds.

At t = 0 seconds the life jacket is 1.5m from the bottom of the lake and is moving in a downward direction.

(a) Find an expression for the position of the life jacket, x, with respect to the bottom of the lake at any time t seconds.



$\frac{2\pi}{n} = 3 \quad n = \frac{2\pi}{3}$

$x = -0\cdot3 \sin\left(\frac{2\pi t}{3}\right) + 1\cdot5$ ✓ ✓ ✓

(b) Find the maximum velocity of the life jacket and the first time at which this occurs.

$\dot{x} = -0\cdot3 \times \frac{2\pi}{3} \cos\left(\frac{2\pi}{3} t\right)$ ✓

$\dot{x}_{max} = 0\cdot3 \times \frac{2\pi}{3} = 0\cdot2\pi$ ✓ when $\frac{2\pi}{3} t = \pi$

$t = 3/2$ secs ✓

(c) Find the total distance the life jacket has travelled in the first 10 seconds.

$\int_0^{10} \left| -0\cdot3 \times \frac{2\pi}{3} \cos\left(\frac{2\pi}{3} t\right) \right| dt$ ✓

$= 3\cdot94$ m ✓

16. [7 marks]

At 9am a spaceship is travelling through the universe with a constant velocity (10**i** - 5**j** + 2**k)** km/h and is at the point with position vector (8**i** + 6**j** - 3**k**) km. At 10am an enemy spaceship flying with constant velocity of (15**i** - 8**j** + 2**k)** km/h is sighted at the point (7**i** - 5**j** + **k**) km. Use the scalar product method to determine the minimum distance between the two spaceships and at what time this occurs.

At 10am $r_A = 10\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k} + 8\underset{\sim}{i} + 6\underset{\sim}{j} - 3k$

$= 18\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ ✓

$r_B = 7\underset{\sim}{i} - 5\underset{\sim}{j} + \underset{\sim}{k}$

${}_A r_B(t) = \langle 18, 1, -1\rangle + t\langle 10, -5, 2\rangle -$

$(\langle 7, -5, 1\rangle + t\langle 15, -8, 2\rangle$

$= \langle 11 - 5t, 6 + 3t, -2\rangle$ ✓✓

${}_A v_B = \langle 10, -5, 2\rangle - \langle 15, -8, 2\rangle$

$= \langle -5, 3\ 0\ \rangle$ ✓

Closest point $\quad {}_A\underset{\sim}{r}_B \cdot {}_A\underset{\sim}{v}_B = 0$

$\langle 11 - 5t, 6 + 3t, -2\rangle \cdot \langle -5, 3, 0\rangle = 0$

$37 = 34t$

$t = 1.088$ ✓ after 10 am

$\therefore$ 11·05am ✓

$|{}_A r_B| = 10.99$ km ✓

17. [4, 1 = 5 marks]

The concentration of a drug in the bloodstream is given by the differential equation:

$\frac{dC}{dt} = 0.3(1 - C)$ where t is time in seconds.

(a) Find the solution to the differential equation for $C(t)$ if initially $C = 0$.

$$\int \frac{dc}{1-c} = \int 0.3\,dt$$

$$-\ln|1-c| = 0.3t + C \checkmark$$

$$\ln(1-c) = -0.3t + F$$

$$1 - c = Ae^{-0.3t} \checkmark$$

at $t=0$ $c=0$ $1 = A \checkmark$

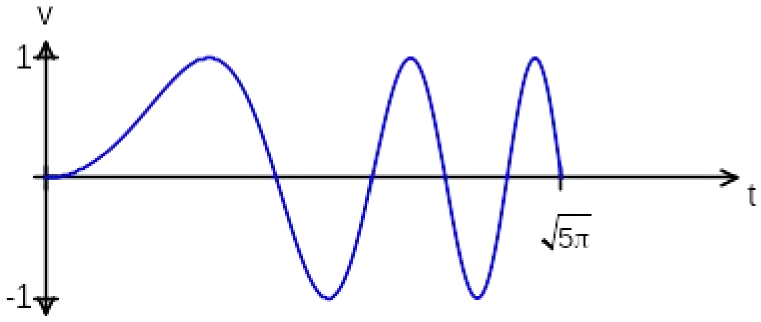
$$\therefore C = 1 - e^{-0.3t} \checkmark$$

(b) After how long will $C = 0.5$?

$$0.5 = 1 - e^{-0.3t}$$

Solve $t = 2.31$ secs $\checkmark$

18. [2, 2, 2 = 6 marks]



A particle moves along the *x*-axis so that its velocity *v* m/s at time *t* (seconds) is given by $v = \sin(t^2)$. The graph of *v* is shown above for $0 \le t \le \sqrt{5\pi}$. The position of the particle at time *t* = 0 is 5 m to the right of the origin.

(a) Find the acceleration of the particle at time *t* = 3.

$$a = \frac{dv}{dt} = 2t\cos(t^2)\Big|_{t=3} \checkmark$$

$$= -5.47 \text{ m/s}^2 \checkmark$$

(b) Find the total distance travelled by the particle from time *t* = 0 to *t* = 3.

$$\int_0^3 |\sin(t^2)|\, dt \checkmark$$

$$= 1.702\text{m} \checkmark$$

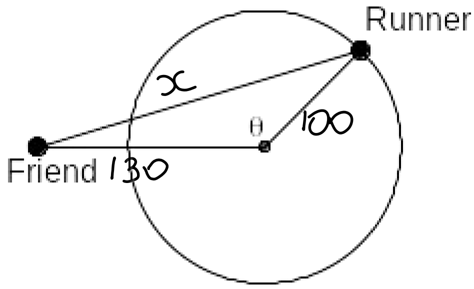
(c) For $0 \le t \le \sqrt{5\pi}$, find the time *t* at which the particle is furthest to the right.

$$t = \sqrt{\pi}$$

$$= 1.772 \text{ seconds } \checkmark\checkmark$$

19. [4 marks]

A runner sprints around a circular track of radius 100 metres at a constant speed of 7 m/s, giving an angular velocity $\frac{d\theta}{dt}=0.07$ rad/s. The runner's friend is standing at a distance 130 m from the center of the track. How fast is the distance between the two friends changing when the distance between them is 130 m?



$x^2 = 130^2 + 100^2 - 2\times130\times100\cos\theta$ ✓

Diff wrt time

$2x\frac{dx}{dt} = 2\times130\times100\sin\theta\frac{d\theta}{dt}$ ✓

When $x = 130$ $\theta = 1.176$ ✓

$\frac{dx}{dt} = \frac{\cancel{2}\times\cancel{130}\times100\sin1.176\times0.07}{\cancel{2}\times\cancel{130}}$

$= 6.46\text{ m/s}$ ✓

20. [6 marks]

Using mathematical induction, prove that the number $2^{2n}-3n-1$ is divisible by 9 for n = 1, 2, ...

Let $n=1$ $\quad 2^2-3-1=0$ $\therefore$ divisible by 9. ✓

Assume true for $n=k$ $\quad 2^{2k}-3k-1$ is divisible by 9 ✓

Let $n=k+1$

$2^{2(k+1)}-3(k+1)-1$ ✓

$= 2^{2k+2}-3k-3-1$

$= 4\times 2^{2k}-3k-4$ ✓

$= (4\times 2^{2k}-12k-4)+9k$ ✓

$= 4(2^{2k}-3k-1)+9k$

Since $2^{2k}-3k-1$ is divisible by 9 and $9k$ is divisible by 9, true for $n=k+1$ ✓

$\therefore$ True for all n

END OF EXAM