Christ Church Grammar School

2016
Investigation 1

Year 12 MATHEMATICS METHODS
Section One: Calculator-free

Student name Solution

Teacher name ______________________

Time and marks available for this section
Reading time before commencing work: 2 minutes
Working time for this section: 15 minutes
Marks available: 15 marks

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: None

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. Write your answers in this Question/Answer Booklet.

2. Answer all questions.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that **you do not use pencil**, except in diagrams.

Question 1 **(7 marks)**

Determine $\frac{dy}{dx}$ in each of the following.

(a) $y = \sin\left(\frac{\pi}{2}\right)$ (1 mark)

$$\frac{dy}{dx} = 0$$

(b) $y = \sin(2x - 1)$ (2 marks)

$$\frac{dy}{dx} = 2\cos(2x - 1)$$

(c) $y = \cos^2 7x$ (2 marks)

$$\frac{dy}{dx} = 2\cos 7x \, . \, (-\sin 7x) \, . \, 7$$

$$= -14 \cos 7x \sin 7x$$

(d) $y = x^2 \cos 3x$ (2 marks)

$$\frac{dy}{dx} = 2x \cos 3x + x^2(-\sin 3x)(3)$$

$$= 2x\cos 3x - 3x^2 \sin 3x$$

Question 2 **(3 marks)**

Given that

$$\frac{d}{dx}(\sin x) = \cos x \text{ and } \frac{d}{dx}(\cos x) = -\sin x,$$

use the quotient rule to find

$$\frac{d}{dx}(\tan x).$$

Express your answer as a **single** trigonometry function.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

Question 8 **(7 marks)**

With the help of some of the following limits,

$$\lim_{h\to 0}\frac{\sin h}{h} = 1, \quad \lim_{h\to 0}\frac{\sin kh}{kh} = 1, \quad \lim_{h\to 0}\frac{1-\cos h}{h} = 0 \quad \text{and} \quad \lim_{h\to 0}\frac{1-\cos kh}{kh} = 0$$

where k is any real number,

determine **using first principles** the derivative of $f(x) = \sin 4x$ with respect to x.

The first step has been completed for you.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin 4(x+h) - \sin 4x}{h}$$

$$= \lim_{h\to 0}\frac{\sin 4x \cos 4h + \cos 4x \sin 4h - \sin 4x}{h}$$

$$= \lim_{h\to 0}\left(\cos 4x \frac{\sin 4h}{h} - \frac{\sin 4x(1-\cos 4h)}{h}\right)$$

$$= 4\cos 4x \lim_{h\to 0}\frac{\sin 4h}{4h} - 4\sin 4x \lim_{h\to 0}\frac{1-\cos 4h}{4h}$$

$$= 4\cos 4x(1) - 4\sin 4x(0)$$

$$= 4\cos 4x$$

Question 7 **(8 marks)**

(a) Using the fact that $\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} f(x) \lim_{x\to 0} g(x)$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$,

evaluate $\lim_{x\to 0} \frac{\sin^2 x}{2x^2}$. (3 marks)

$$\lim_{x\to 0} \frac{\sin^2 x}{2x^2} = \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x}$$
$$= \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{x}$$
$$= \frac{1}{2} \cdot 1 \cdot 1$$
$$= \frac{1}{2}$$

(b) Given that $\cos(x+y) = \cos x \cos y - \sin x \sin y$,

show that $\cos 2x = 1 - 2\sin^2 x$. (2 marks)

$$\cos 2x = \cos(x+x)$$
$$= \cos x \cos x - \sin x \sin x$$
$$= \cos^2 x - \sin^2 x$$
$$= (1 - \sin^2 x) - \sin^2 x$$
$$= 1 - 2\sin^2 x$$

(c) Using the results in (a) and (b) and showing full working, evaluate the limit

$\lim_{x\to 0} \frac{1-\cos 2x}{4x^2}$. (3 marks)

$$\lim_{x\to 0} \frac{1-\cos 2x}{4x^2} = \lim_{x\to 0} \frac{1-(1-2\sin^2 x)}{4x^2}$$
$$= \lim_{x\to 0} \frac{2\sin^2 x}{4x^2}$$
$$= \lim_{x\to 0} \frac{\sin^2 x}{2x^2}$$
$$= \frac{1}{2}, \text{ from (a)}$$

Question 3 **(5 marks)**

Using the concept of first principles, express each of the following limits as an appropriate derivative.

Hence determine the limit by evaluating the derivative.

(a) (2 marks)

$$\lim_{h\to 0} \frac{\cos 3(x+h) - \cos 3x}{h}$$

$$= \frac{d}{dx} \cos 3x$$
$$= -3\sin 3x$$

(b) (3 marks)

$$\lim_{h\to 0} \frac{(x+h)^3 \sin 2(x+h) - x^3 \sin 2x}{h}$$

$$= \frac{d}{dx}\left(x^3 \sin 2x\right)$$
$$= 3x^2 \sin 2x + x^3 . \cos 2x . 2$$
$$= 3x^2 \sin 2x + 2x^3 \cos 2x$$

Question 6 **(4 marks)**

Find the co-ordinates of the point on the curve $y = \sin^2 2x$ where the gradient is 2 and $0 \le x < \frac{\pi}{2}$. Give your answer in **exact values** (decimal answer will be penalised.)
You may use your ClassPad but essential steps and working must be shown in the space below.

$$\frac{dy}{dx} = 4\sin 2x \cos 2x \checkmark$$

$$4\sin 2x \cos 2x = 2 \checkmark$$

$$x = \frac{\pi}{8}$$

$$y = \frac{1}{2}$$

$\therefore$ co-ordinates $= \left(\frac{\pi}{8}, \frac{1}{2}\right)$ ✓ ✓

Question 5 **(5 marks)**

Find the equation of the tangent to the curve $y = x^2 \sin\frac{x}{2}$ at the point $x = 2\pi$.

You may use your ClassPad to determine derivatives and required values, but the essential steps and working must be shown in the space below.

$$\frac{dy}{dx} = 2x\sin\frac{x}{2} + \frac{1}{2}x^2\cos\frac{x}{2}$$

$$\left.\frac{dy}{dx}\right|_{x=2\pi} = -2\pi^2$$

$$x = 2\pi, \quad y = 0$$

Equation of tangent $y = -2\pi^2 x + c$

$$0 = -2\pi^2(2\pi) + c$$

$$c = 4\pi^3$$

$$\therefore \quad y = -2\pi^2 x + 4\pi^3$$

Christ Church Grammar School

2016

Investigation 1

Year 12 MATHEMATICS METHODS

Section Two:
Calculator-assumed

Student name Solution

Teacher name ____________________

Time and marks available for this section

Reading time before commencing work:	3 minutes
Working time for this section:	30 minutes
Marks available:	30 marks

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, and up to three calculators approved for use in the WACE examinations

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. Write your answers in this Question/Answer Booklet.

2. Answer all questions.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that **you do not use pencil**, except in diagrams.

Question 4 **(6 marks)**

(a) The gradient function of a curve is given by

$$\lim_{h\to 0}\frac{\sin 2(x+h) - \sin 2x}{h}$$

Write down the equation of the curve. (2 marks)

$y = \sin 2x$ ✓ ✓

(b) It is given that

$$y = \sin^2\theta \quad \text{and} \quad \theta = (\pi - x)\,.$$

Use the chain rule to determine the derivative of y with respect to x. (4 marks)

$\frac{dy}{d\theta} = 2\sin\theta\cos\theta$ ✓

$\frac{d\theta}{dx} = -1$ ✓

$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$

$= 2\sin\theta\cos\theta\,.\,(-1)$ ✓

$= -2\sin(\pi - x)\cos(\pi - x)$ ✓

or $= -\sin 2(\pi - x)$