Student Name: ____

Methodist Ladies' College Semester 2, 2010

3CD MATHEMATICS: SPECIALIST

Question/Answer Booklet – Section 1 – Calculators *NOT* allowed – Notes sheets *NOT* allowed

Teacher's Name: _________**SOLUTIONS**____________________

Time allowed for this paper

Section	Reading	Working
Calculator-free	5 minutes	50 minutes
Calculator-assumed	10 minutes	100 minutes

Materials required/recommended for this paper

Section One (Calculator-free): 40 marks

To be provided by the supervisor
Section One Question/Answer booklet Formula sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler

Section Two (Calculator-assumed): 80 marks

To be provided by the supervisor
Section Two Question/Answer booklet Formula sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this course.

Important Note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

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Instructions to candidates

1. **All** questions should be attempted.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare answer pages may be found at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued (i.e. give the page number).

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

Structure of this paper

Questions	Marks available	Your score
1	5	
2	7	
3	4	
4	7	
5	8	
6	6	
7	3	
Total:	**40**	
8	6	
9	8	
10	9	
11	10	
12	6	
13	7	
14	7	
15	6	
16	5	
17	10	
18	6	
Total:	**80**	
Total marks = 120		
		%

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Section One: Calculator-free **(40 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the space provided.

Suggested working time for this section is 50 minutes.

Question 1 **(5 marks)**

The transformation matrix **T** is defined by **T** = **AB**, where **A** and **B** are the transformations:

A: a rotation about the origin through 210° anticlockwise;

B: a reflection in the line through the origin that makes an angle of 120° with the x-axis.

Determine matrix **T** and describe **T** geometrically.

Solution
$T=\begin{bmatrix}\cos 210° & -\sin 210° \\ \sin 210° & \cos 210°\end{bmatrix}\begin{bmatrix}\cos 240° & \sin 240° \\ \sin 240° & -\cos 240°\end{bmatrix}$ $\text{¿}\begin{bmatrix}\frac{-\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}\frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$ $\text{¿}\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}$ **T** represents a reflection in the line $y=x$.
Specific behaviours
✓ defines rotation matrix in terms of trig functions   □ defines reflection matrix in terms of trig functions   □ correctly evaluates terms in rotation and reflection matrices   ✓ correctly calculates matrix **T**   ✓ correctly describes **T** geometrically

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Question 2 **(7 marks)**

Use an algebraic method to solve $3|x-2|>|2x+1|$.

Solution
Critical values: $x=\frac{-1}{2}$ and $x=2$ For $x<-\frac{1}{2}, 3(2-x)>-2x-1$ $\quad\quad$ $x<7$ i.e. $x<-\frac{1}{2}$ For $\frac{-1}{2}\le x\le 2, 3(2-x)>2x+1$ $\quad\quad$ $x<1$ i.e. $\frac{-1}{2}\le x<1$ For $x>2, 3(x-2)>2x+1$ $\quad\quad$ $x>7$ i.e. $x>7$ Hence, $x<1$ or $x>7$
Specific behaviours
✓ correct inequality for $x<-\frac{1}{2}$ ✓ correct solution for $x<-\frac{1}{2}$ ✓ correct inequality for $\frac{-1}{2}\le x\le 2$ ✓ correct solution for $\frac{-1}{2}\le x\le 2$ ✓ correct inequality for $x>2$ ✓ correct solution for $x>2$ ✓ correct solution for inequality (no mark if 'and' used)

Question 3 **(4 marks)**

A body in simple harmonic motion passes from rest to rest through a distance of 20 cm in 2.5 seconds. Find the maximum velocity this body attains.

Solution
Let $x=10\sin\left(\frac{2}{5}\pi t\right)$ $\quad v=4\pi\cos\left(\frac{2}{5}\pi t\right)$ Since $-4\pi\le v\le 4\pi$ $v_{max}=4\pi$ cm s^{-1}
Specific behaviours
✓ correct period ✓ correct amplitude ✓ differentiates correctly ✓ identifies max velocity Similarly using, $x=10\cos\left(\frac{2}{5}\pi t\right)$ or $x=10\sin\left(\frac{2}{5}\pi t+\alpha\right)$ or $x=10\cos\left(\frac{2}{5}\pi t+\beta\right)$

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Question 4 **(7 marks)**

Find the following antiderivatives.

(a) $\int e^{\sin x}\cos x\,dx$ [1]

Solution
$\int e^{\sin x}\cos x\,dx = e^{\sin x} + c$
Specific behaviours
✓ correct antiderivative including the constant

(b) $\int \frac{x^2}{x^3+2}\,dx$ [2]

Solution
$\int \frac{x^2}{x^3+2}\,dx = \frac{1}{3}\int \frac{3x^2}{x^3+2}\,dx$ ¿ $\frac{1}{3}\ln\lvert x^3+2\rvert + c$
Specific behaviours
✓ recognises form $\frac{f'(x)}{f(x)}$ □ gives correct solution

(c) $\int \sin^3 x\,dx$ by using the substitution $u=\cos x$ [4]

Solution
$\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx$ $u=\cos x$ ¿ $-\int (1-u^2)\,du$ $du = -\sin x\,dx$ ¿ $-\left[u-\frac{u^3}{3}\right]+c$ $\sin^2 x = 1-\cos^2 x = 1-u^2$ ¿ $-\left(\cos x - \frac{\cos^3 x}{3}\right)+c$
Specific behaviours
✓ correctly substitutes for $\sin^2 x$ in terms of u □ correctly substitutes for $\sin x\,dx$ in terms of u □ correct antiderivative   ✓ answer given correctly in terms of x

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Question 5 **(8 marks)**

(a) Given that $xy+\cos y-x^2=1$, find $\frac{dy}{dx}$. [3]

Solution
$y+x\frac{dy}{dx}-\sin y\frac{dy}{dx}-2x=0$ $\frac{dy}{dx}=\frac{2x-y}{x-\sin y}$
Specific behaviours
✓ correctly applies product rule ✓ correctly differentiates implicitly ✓ correctly rearranges equation for $\frac{dy}{dx}$

(b) Evaluate $\int_0^{0.5}\frac{x}{\sqrt{1-x^2}}dx$ using the substitution $x=\sin\theta$. [5]

Solution
$\int_0^{0.5}\frac{x}{\sqrt{1-x^2}}dx=\int_0^{\frac{\pi}{6}}\frac{\sin\theta}{\cos\theta}\cos\theta\, d\theta$ $\quad x=\sin\theta$ $=\int_0^{\frac{\pi}{6}}\sin\theta\, d\theta$ $\quad dx=\cos\theta\, d\theta$ $=[-\cos\theta]_0^{\frac{\pi}{6}}$ $\quad x=0,\theta=0$ $=1-\frac{\sqrt{3}}{2}$ $\quad x=0.5,\theta=\frac{\pi}{6}$
Specific behaviours
✓ correctly substitutes for x ✓ correctly substitutes for dx ✓ correct upper and lower limits ✓ correct antiderivative ✓ correct answer

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Question 6 **(6 marks)**

(a) Given that $y=\ln x$, where $x>0$, show that the small increment in y corresponding to a small positive increment of h in x is approximately $\frac{h}{x}$. [2]

Solution
$\delta y \approx \frac{dy}{dx}\delta x$ $\delta y \approx \frac{1}{x}h$ $\delta y \approx \frac{h}{x}$
Specific behaviours
✓ chooses increment formula ✓ correct substitutions for derivative and for δx

The difference between the approximate and true increments, for a fixed value of h, is denoted by $f(x)$, so that

$$f(x)=\frac{h}{x}-\{\ln(x+h)-\ln x\}.$$

(b) Show that $f'(x)=\frac{-h^2}{x^2(x+h)}$. [3]

Solution
$f'(x)=\frac{-h}{x^2}-\left\{\frac{1}{x+h}-\frac{1}{x}\right\}$ ¿ $\frac{-h(x+h)-x^2+x(x+h)}{x^2(x+h)}$ ¿ $-\frac{h^2}{x^2(x+h)}$
Specific behaviours
✓ differentiates power function correctly ✓ differentiates log function correctly ✓ simplifies correctly

(c) Explain how this result shows that $f(x)$ decreases as x increases. [1]

Solution
$f'(x)=\frac{-h^2}{x^2(x+h)}$ As x increases, $f'(x)<0$, hence f is decreasing.

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Specific behaviours
interprets negative derivative correctly

Question 7 **(3 marks)**

Show that $\tan\theta + \frac{\cos\theta}{\sin\theta} = \frac{2}{\sin 2\theta}$.

Solution
$\tan\theta + \frac{\cos\theta}{\sin\theta} = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$ $¿\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$ $¿\frac{1}{\sin\theta\cos\theta}$ $¿\frac{2}{\sin 2\theta}$
Specific behaviours
✓ common denominator   ✓ use identity $\sin^2\theta + \cos^2\theta = 1$ ✓ use indentity $2\sin\theta\cos\theta = \sin 2\theta$

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End of questions

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Additional working space

Question number(s):_______________

Additional working space

Question number(s):______________

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