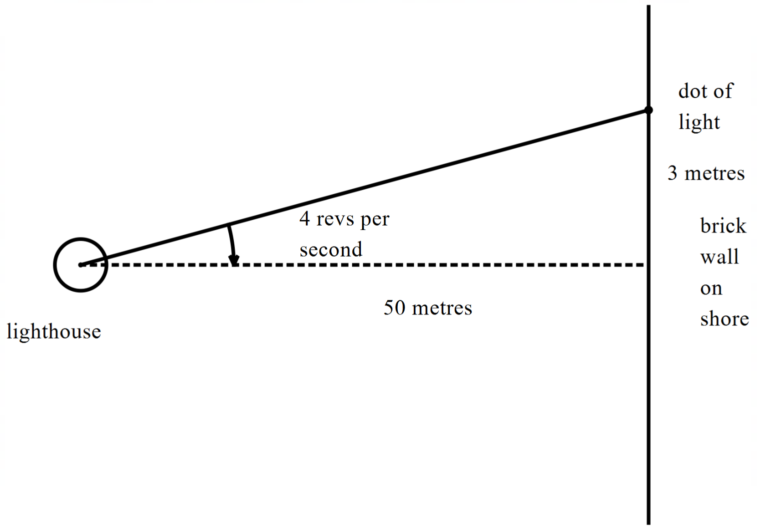
Q6 (5 marks) (4.1.1, 4.2.2)
Consider a lighthouse that is 50 metres away from the shore. On the shore is a long brick wall. The light on the lighthouse is rotating at 4 revolutions per second. Determine the exact speed of the dot of light on the wall at a point 3 metres from the point directly opposite the lighthouse as shown below.



Solution
$\tan\theta = \frac{x}{50}$ $\sec^2\theta\dot{\theta} = \frac{\dot{x}}{50}$ $50(1+\tan^2\theta)8\pi = \dot{x}$ $50\left(1+\left(\frac{3}{50}\right)^2\right)8\pi = \dot{x}$
Specific behaviours
✓ sets up equation between angle and distance along wall ✓ determines rate of angle in radians ✓ uses implicit diff or related rates to link all rates ✓ determines exact value of tan of angle ✓ states an exact expression of speed

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Course Specialist Year 12 Test Three 2022

Student name: ______________ Teacher name: ______________

Task type: **Response**

Time allowed for this task: _____40______ mins

Number of questions: _____6______

Materials required: NO classpads nor calculators

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: __40____ marks

Task weighting: _10___%

Formula sheet provided: Yes

Note: All part questions worth more than 2 marks require working to obtain full marks.

NO classpads nor calculators!

Q1 (3 & 3= 6 marks) (3.3.9-3.3.10)

a) Solve the following set of linear equations.

$3x - 2y + z = -8$

$x + 2y - 3z = -14$

$2x + y - z = -9$

Solution
$\begin{bmatrix} 1 & 2 & -3 & -14 \\ 2 & 1 & -1 & -9 \\ 3 & -2 & 1 & -8 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 & -3 & -14 \\ 0 & 3 & -5 & -19 \\ 0 & 8 & -10 & -34 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 & -3 & -14 \\ 0 & 3 & -5 & -19 \\ 0 & -2 & 0 & -4 \end{bmatrix}$
$-2y = -4 \quad , y = 2$
$6 - 5z = -19 \quad , z = 5$
$x + 4 - 15 = -14 \quad , x = -3$
Specific behaviours
✓ eliminates one variable from two equations
✓ eliminates two variables from one equation
✓ solves for all 3 variables

b) Consider the system below,

$3x - 2y + z = p$

$x + 2y - 3z = -14$

$2x + y + qz = -9$

Determine the values of $p \,\&\, q$ such that there are:

i) Unique solution

ii) Infinite solutions

iii) No solutions.

Solution

$$\int \frac{8x^2 - 6x + 5}{(x-2)(x^2+1)}\,dx$$

Solution
$\int \frac{8x^2 - 6x + 5}{(x-2)(x^2+1)}\,dx$
$\frac{8x^2 - 6x + 5}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx + C}{(x^2+1)}$
$8x^2 - 6x + 5 = A(x^2+1) + (Bx + C)(x-2)$
$x = 2$
$25 = 5A \quad , A = 5$
$x = 0$
$5 = 5 - 2C \quad , C = 0$
$x = 1$
$7 = 10 + B(-1) , B = 3$
$\int \frac{8x^2 - 6x + 5}{(x-2)(x^2+1)}\,dx = \int \frac{5}{x-2} + \frac{3x}{x^2+1}\,dx = 5\ln\lvert x-2\rvert + \frac{3}{2}\ln\lvert x^2+1\rvert + c$
Specific behaviours
✓ expresses as two partial fractions with THREE constants
✓ solves two constants
✓ solves all three constants showing derivation for all
✓ obtains expression for integral

✓ obtains numerical value(unsimplfied)

b)

$$\begin{bmatrix} 1 & 2 & -3 & -14 \\ 2 & 1 & q & -9 \\ 3 & -2 & 1 & p \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & -14 \\ 0 & 3 & -6-q & -19 \\ 0 & 8 & -10 & -42-p \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & -14 \\ 0 & 3 & -6-q & -19 \\ 0 & 0 & -18-8q & -26+3p \end{bmatrix}$$

$$i)\, q \neq \frac{-18}{8}\left(-\frac{9}{4}\right)$$

$$ii)\, q = \frac{-18}{8} \;\&\; p = \frac{26}{3}$$

$$iii)\, q = \frac{-18}{8} \;\&\; p \neq \frac{26}{3}$$

Specific behaviours
✓ eliminates two variables from one equation ✓ determines values for uniqueness ✓ determines all values for infinite and no solutions

Q2 (2, 2, 2 & 3 = 9 marks) (3.3.11, 3.3.13)

A particle moves such that at time t seconds the velocity is $v=\begin{pmatrix} t \\ -t^2 \\ -3 \end{pmatrix} m/s$. The particle is initially at the origin.
Determine:

a) The position vector at time $t=1$ second.

Solution
$v=\begin{pmatrix} t \\ -t^2 \\ -3 \end{pmatrix} m/s$
$r=\begin{pmatrix} \frac{1}{2} \\ \frac{-t^3}{3} \\ -3t \end{pmatrix}+c$
$c=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
$r=\begin{pmatrix} \frac{1}{2} \\ \frac{-1}{3} \\ -3 \end{pmatrix}$
Specific behaviours
✓ integrates and states a constant C ✓ states r with t=1

b) The acceleration of the particle at $t=1$ second.

Solution

Q4 (4 marks) (4.2.1)

If $y^2-\sin x=1-5y$, determine $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in terms of x & y only.

Solution
$y^2-\sin x=1-5y$
$2yy'-\cos x=5y'$
$y'(2y-5)=\cos x$
$y'=\frac{\cos x}{(2y-5)}$
$2yy''+y'2y'+\sin x=5y''$
$y''(2y-5)=-\sin x-2(y')^2$
$y''=\frac{-\sin x-2(y')^2}{(2y-5)}=\frac{-\sin x-2\left(\frac{\cos x}{(2y-5)}\right)^2}{(2y-5)}=\frac{-\sin x(2y-5)^2-2\cos^2 x}{(2y-5)^3}$
Specific behaviours
✓ implicit diff of original equation ✓ obtains expression for first derivative ✓ implicit diff involving first derivative (or first implicit equation) shown ✓ expression of second derivative in terms of x & y only, no need to simplify

Q5 (3 & 4 = 7 marks) (4.2.1)
Determine the following integrals:

a) $\int^{1}\frac{5x}{\sqrt{x+1}}dx \quad u=x+1$

Solution
$\int^{1}\frac{5x}{\sqrt{x+1}}dx \quad u=x+1,\frac{du}{dx}=1,x=u-1$
$\int^{2}\frac{5x}{\sqrt{u}}\frac{dx}{du}du=\int^{2}\frac{5(u-1)}{\sqrt{u}}du=5\int^{2}u^{\frac{1}{2}}-u^{\frac{-1}{2}}du=5\left[\frac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right]=5\left[\frac{2}{3}2^{\frac{3}{2}}-2(2^{\frac{1}{2}})-\frac{2}{3}+2\right]$
$5\left[\frac{1}{3}2^{\frac{5}{2}}-(2^{\frac{3}{2}})+\frac{4}{3}\right]$
Specific behaviours
✓ changes to variable u and du ✓ changes limits to u

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$a = \begin{pmatrix} 1 \\ -2t \\ 0 \end{pmatrix} m/s$ $a = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$
Specific behaviours
✓ diff v ✓ states with t=1

c) The speed of the particle at $t = 2$ seconds.

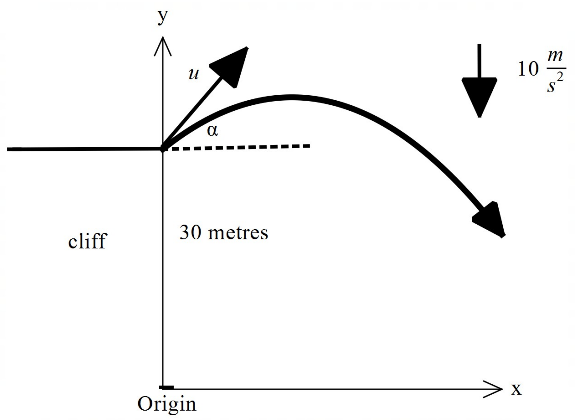
Solution
$v = \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix} m/s$ $\lvert v \rvert = \sqrt{4+16+9} = \sqrt{29}$
Specific behaviours
✓ determines velocity at t=2 ✓ determines speed, no need to simplfy

d) The times when the velocity is perpendicular to the acceleration.

Solution
$\begin{pmatrix} t \\ -t^2 \\ -3 \end{pmatrix} . \begin{pmatrix} 1 \\ -2t \\ 0 \end{pmatrix} = t + 2t^3 = 0, t = 0$
Specific behaviours
✓ uses dot product ✓ equates to zero

✓ states one non negative result

Q3 (4, 3 & 2 = 9 marks) (3.3.12, 3.3.13, 3.3.15)
Consider a particle that is projected from the top of a cliff of height 30 metres with a speed of u metres per second at an angle of α to the horizontal. Assume that the acceleration is constant at $10\,m/s^2$ towards the centre of the Earth. Let the origin of cartesian axes be at the base of the cliff as shown below with the appropriate unit vectors $i\ \&\ j$.



Let $\ddot{r} = \begin{pmatrix} 0 \\ -10 \end{pmatrix} m/s^2$.

a) Using vector integration, show how to derive the position vector r at time t seconds in terms of $u\ \&\ \alpha$. Show all steps.

Solution

$$\ddot{r} = \begin{pmatrix} 0 \\ -10 \end{pmatrix} m/s^2$$

$$\dot{r} = \begin{pmatrix} 0 \\ -10t \end{pmatrix} + \begin{pmatrix} u\cos\alpha \\ u\sin\alpha \end{pmatrix} = \begin{pmatrix} u\cos\alpha \\ u\sin\alpha - 10t \end{pmatrix}$$

$$r = \begin{pmatrix} ut\cos\alpha \\ ut\sin\alpha - 5t^2 \end{pmatrix} + c$$

$$\begin{pmatrix} 0 \\ 30 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + c, \quad c = \begin{pmatrix} 0 \\ 30 \end{pmatrix}$$

$$r = \begin{pmatrix} ut\cos\alpha \\ ut\sin\alpha - 5t^2 + 30 \end{pmatrix}$$

Specific behaviours

✓ integrates acceleration with plus constant
✓ solves for constant in terms of two variables
✓ integrates velocity with plus constant
✓ solves for constant and states r in terms of t

b) Show how to derive the cartesian equation for the path of the particle in terms of $u\ \&\ \alpha$.

Solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ut\cos\alpha \\ ut\sin\alpha - 5t^2 + 30 \end{pmatrix}$$

$$t = \frac{x}{u\cos\alpha}$$

$$y = u\sin\alpha\frac{x}{u\cos\alpha} - 5\frac{x^2}{u^2\cos^2\alpha} + 30$$

$$y = x\tan\alpha - 5\frac{x^2}{u^2\cos^2\alpha} + 30$$

Specific behaviours

✓ expresses t in terms of x
✓ subs into y parametric equation
✓ states cartesian equation without any reference to t

c) Set up an equation in terms of $u\ \&\ \tan\alpha$ ONLY, but do not solve, that would allow the range (x) to be determined where the particle hits the floor from the base of the cliff.

Solution

$$0 = x\tan\alpha - 5\frac{x^2}{u^2\cos^2\alpha} + 30$$

$$0 = x\tan\alpha - \frac{5x^2}{u^2}\sec^2\alpha + 30$$

$$0 = x\tan\alpha - \frac{5x^2}{u^2}(1 + \tan^2\beta) + 30$$

Specific behaviours

✓ uses y=0
✓ uses tan only with reference to angle in two terms of equation