TRINITY COLLEGE

Mathematics Methods 3 & 4
Test 1 2016

Section 1 Calculator Free

Differentiation, Anti-differentiation and their applications.

STUDENT'S NAME MARKING KEY

DATE: Friday 4th March **TIME:** 25 minutes **MARKS:** 27

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser, Formula sheet.

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (6 marks)

Differentiate the following. Do not simplify your answer.

(a) $y = (4x+7)^3(9x-5)$ let $u = (4x+7)^3$ $v = (9x-5)$ [3]

$u' = 3(4x+7)^2(4)$ $v' = 9$

$= 12(4x+7)^2$

$\frac{dy}{dx} = u'v + v'u$ ✓✓✓

$= 12(4x+7)^2(9x-5) + 9(4x+7)^3$

(b) $y = \frac{\sqrt{2x^5}}{\sqrt{x+7}}$ let $u = (2x^5)^{\frac{1}{2}}$ $u' = 5x^4(2x^5)^{-\frac{1}{2}}$ [3]

$v = (x+7)^{\frac{1}{2}}$ $v' = \frac{1}{2}(x+7)^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{u'v - v'u}{v^2}$

$= \frac{5x^4(2x^5)^{-\frac{1}{2}}(x+7)^{\frac{1}{2}} - \frac{1}{2}(x+7)^{-\frac{1}{2}}(2x^5)^{\frac{1}{2}}}{x+7}$ ✓✓ ✓

2. (3 marks)

Determine $\int 2x(7-3x^2)^4\,dx$

$$= \frac{2x(7-3x^2)^5}{5(6x)} + c$$

$$= \frac{(7-3x^2)^5}{-15} + c$$

3. (3 marks)

Given that $\int_1^a (2x-3)\,dx = 6$, determine a.

$$= \left[x^2 - 3x\right]_1^a$$

$$= (a^2 - 3a) - ((1)^2 - 3(1))$$

$\therefore\ a^2 - 3a + 2 = 6$

$a^2 - 3a - 4 = 0$

$\therefore\ a = -1$

$a = 4$

10. (6 marks)

A particle moves in rectilinear motion with a velocity of 7 m/s as it passes through a fixed point O.

t is the number of seconds since passing through O. Acceleration a is defined as $a = mt - n$, where m and n are constants.

When $t = 1$, the velocity is 12 m/s, and when $t = 7$ the particle is instantaneously at rest.

(a) Calculate the values of m and n. [3]

$V = \frac{mt^2}{2} - tn + c$

$V(0) = 7 \therefore c = 7$

$\therefore V(t) = \frac{mt^2}{2} - tn + 7$ ✓

$t=1 \quad V=12$

$12 = \frac{m}{2} - n + 7$ ✓

$t=7 \quad V=0$

$0 = \frac{49m}{2} - 7n + 7$

$m = -2$

$n = -6$ ✓

(b) Hence, determine the expression for the velocity as a function of time. [1]

$V(t) = -t^2 + 6t + 7$ ✓

(c) Determine when and where the maximum velocity is attained. [2]

max V occurs when $\frac{dv}{dt} = 0$

$\frac{dv}{dt} = -2t + 6$

$0 = -2t + 6$

$t = 3$ ✓

$x(t) = -\frac{1}{3}t^3 + 3t^2 + 7t$

$x(3) = 39$ ✓

4. (6 marks)

The air in a hot air balloon is being inflated such that the rate of change of its volume at any time t, minutes, is given as:

$$\frac{dV}{dt} = 3t^2 - 2t \qquad \text{for} \quad t \geq 0$$

If initially the balloon has 3 m^3 of air in it, determine:

(a) The rate of change in volume when $t = 1$. Explain the meaning of this. [2]

$\frac{dV}{dt} = 3(1) - 2(1)$

$= 1 m^3/min$ ✓

The balloon is instantaneously increasing in volume by $1 m^3/min$ at $t = 1$. ✓

(b) For what values of t the volume is increasing. [2]

$\frac{dV}{dt} > 0.$

$3t^2 - 2t > 0$ ✓

$t(3t-2) > 0$

$t > 0, \quad t > \frac{2}{3}$

$t < 0$

$\therefore t > \frac{2}{3}$ ✓

(c) The volume of the balloon after five minutes. [2]

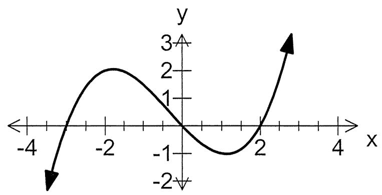
$V(t) = t^3 - t^2 + 3$ ✓

$V(5) = (5)^3 - (5)^2 + 3$

$= 103 m^3$ ✓

5. (4 marks)

The graph of $y = f(x)$ is shown below.



Given $\int_{-3}^{0} f(x)dx = 4$ and $\int_{0}^{2} f(x)dx = -1$, determine the following:

(a) $\int_{-3}^{2} f(x)dx$ $= 3$ ✓ [1]

(b) $\int_{0}^{2} 5f(x)dx$ $= 5\int f(x)\,dx$ [1]

$= 5(-1)$

$= -5$ ✓

(c) $\int_{-3}^{2} |f(x)|dx$ $= 5$ ✓ [1]

(d) The area enclosed by $f(x)$ and the x axis. [1]

5 units^2 ✓

(b) Using calculus techniques, determine the minimum time taken by the man to reach point B and the distance he would travel by foot to achieve this minimum time. [5]

$t(x) = \frac{1}{6}(x^2+9)^{\frac{1}{2}} + \frac{1}{8}(8-x)$

$t'(x) = \frac{1}{12}(x^2+9)^{-\frac{1}{2}}.(2x) - \frac{1}{8}$

$t'(x) = \dfrac{x}{6(x^2+9)^{\frac{1}{2}}} - \frac{1}{8}$ ✓ $\quad 0 = \dfrac{4x - 3\sqrt{x^2+9}}{24\sqrt{x^2+9}}$

Solve $t'(x) = 0$ $\quad x = \frac{9\sqrt{7}}{7}$

$x = \frac{9}{\sqrt{7}}$ via CP. ✓ $\quad (3.40)$

Verify $\frac{9}{\sqrt{7}}$ is a min.

$t''\left(\frac{9}{\sqrt{7}}\right) = \frac{7\sqrt{7}}{1152} > 0 \quad \therefore \frac{9}{\sqrt{7}}$ is a min. ✓

(0.0161)

time taken $= \frac{1}{6}\left[\left(\frac{9}{\sqrt{7}}\right)^2+9\right]^{\frac{1}{2}} + \frac{1}{8}\left(8-\frac{9}{\sqrt{7}}\right)$

$= \frac{2}{\sqrt{7}} + 0.5748$

$= 1.3307$

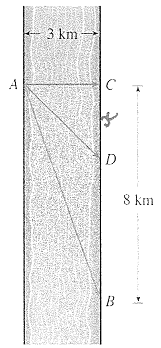
Running dist $= 8 - \frac{9}{\sqrt{7}}$

$= 4.598$ km

It would take the man 1.3307 ✓ hours to reach B. He would run 4.598 km. ✓

9. (7 marks)

A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible.



He could proceed in any of three ways:

1. Row his boat directly across the river to point C and then run to B
2. Row directly to B
3. Row to some point D between C and B and then run to B

(a) Given that $time = \frac{distance}{speed}$ and x is the distance from C to D, show that the time (t) taken for the man to travel from A to B can be represented by the equation. [2]

$$t = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$$

$$t = \frac{\overline{AD}}{6} + \frac{\overline{DB}}{8}$$

Given $x = \overline{CD}$

$\overline{AD} = \sqrt{3^2 + x^2}$

$\overline{AD} = \sqrt{x^2+9}$

$\therefore \quad t = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$

$\overline{DB} = 8 - \overline{CD}$

$\overline{DB} = 8 - x$ ✓✓

6. ($\cancel{5}$ 6 marks)

Given the function $y = (x+2)(x^2-4x+4)$.

(a) Determine the gradient of the tangent to the curve at $x = 3$. [3]

$$\frac{dy}{dx} = 1(x^2-4x+4) + (2x-4)(x+2) \checkmark$$
$$= x^2-4x+4 + (2x^2-8)$$
$$= 3x^2-4x-4 \checkmark$$
$$\left.\frac{dy}{dx}\right|_{x=3} = 3(9) - 4(3) - 4$$

gradient $= 11$ ✓

(b) Using calculus techniques, determine the nature of the stationary point at $x = 2$. [3]

$$\frac{d^2y}{dx^2} = 6x - 4 \checkmark$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 6(2) - 4$$
$$= 8 \checkmark$$

$\frac{d^2y}{dx}$ is positive $\therefore$ minimum T.P. ✓

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Section 2 Calculator Assumed

Differentiation, Anti-differentiation and their applications.

STUDENT'S NAME MARKING KEY

DATE: Friday 4th March **TIME:** 25 mins **MARKS:** 23

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser, Formula sheet.

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

7. (6 marks)

The volume Vcm^3 of water in a vessel is given by $V = \frac{1}{6}\pi x^3$, where x cm is the depth of the water in the cylinder in cm.

(a) Determine an approximation for the change in depth when the volume of water changes from 200 to 210 cm^3. [3]

$V = 200$ ✓

$x = \sqrt[3]{\frac{1200}{\pi}}$ or 7.26

$\frac{dV}{dx} = \frac{\pi x^2}{2}$

$\therefore \frac{dx}{dV} = \frac{2}{\pi x^2}$

$\Delta x = \frac{2}{\pi(7.26)^2} \times 10$ ✓

$\Delta x = 0.121$ cm ✓

(b) Determine the percentage change in the volume of the vessel if the depth has increased by 6%. [3]

$\Delta V = \frac{\pi x^2}{2} \times 0.06x$

$= 0.03\pi x^3$ ✓

$\frac{\Delta V}{V} = \frac{0.03\cancel{\pi x^3}}{\frac{1}{6}\cancel{\pi x^3}}$ ✓

$\frac{\Delta V}{V} = 0.18$

$= 18\%$ ✓

8. (4 marks)

A company manufacturing a new bike determines that the marginal cost (in dollars) for the production of the n^{th} unit is given by the expression:

$$\frac{dC}{dn} = \frac{200000}{(n+20)^2}$$

(a) The initial set up cost is \$ 10 000 (i.e. the cost of producing no bikes is \$ 10 000). Show that the expression for the total cost of producing n bikes is:

$$C = \frac{-200000}{n+20} + 20000$$ [2]

$C = \int \frac{dC}{dn}$

$= \frac{-200\,000}{n+20} + c$ ✓

$C(0) = 10\,000$

$10\,000 = \frac{-200\,000}{20} + c$

$c = 20000$ ✓

$\therefore C = \frac{-200000}{n+20} + 20000$

(b) If the company sells each bike for \$200, how many bikes must be sold before it first makes a profit? [2]

Solve $200n = \frac{-200000}{n+20} + 20\,000$ ✓

$n = 90.99$

The company must sell 91 bikes before making a profit. ✓