



PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Year 12 Test Four 2022

Student name: _____ Teacher name: _____

Task type: Response

Time allowed for this task: ____40____ mins

Number of questions: ____6____

Materials required: Upto 3 Calculators with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: ____40____ marks

Task weighting: ____10____%

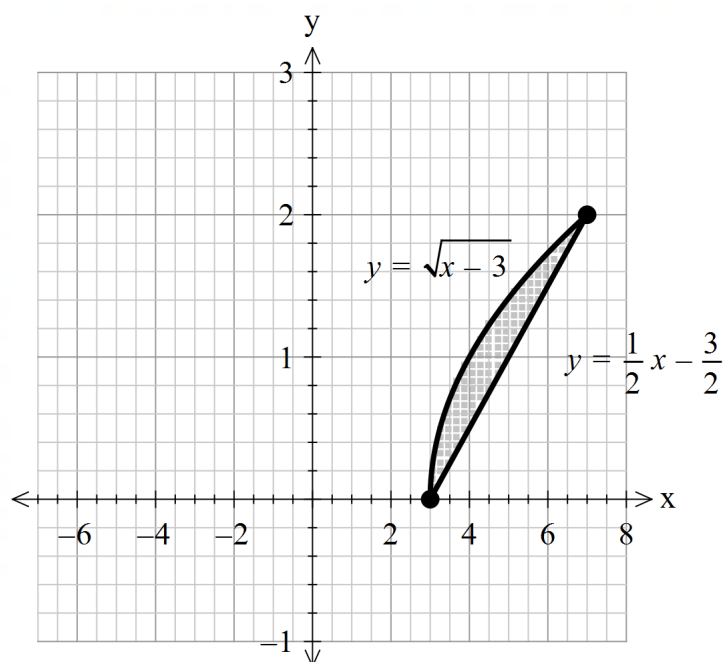
Formula sheet provided: Yes

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (5 marks)

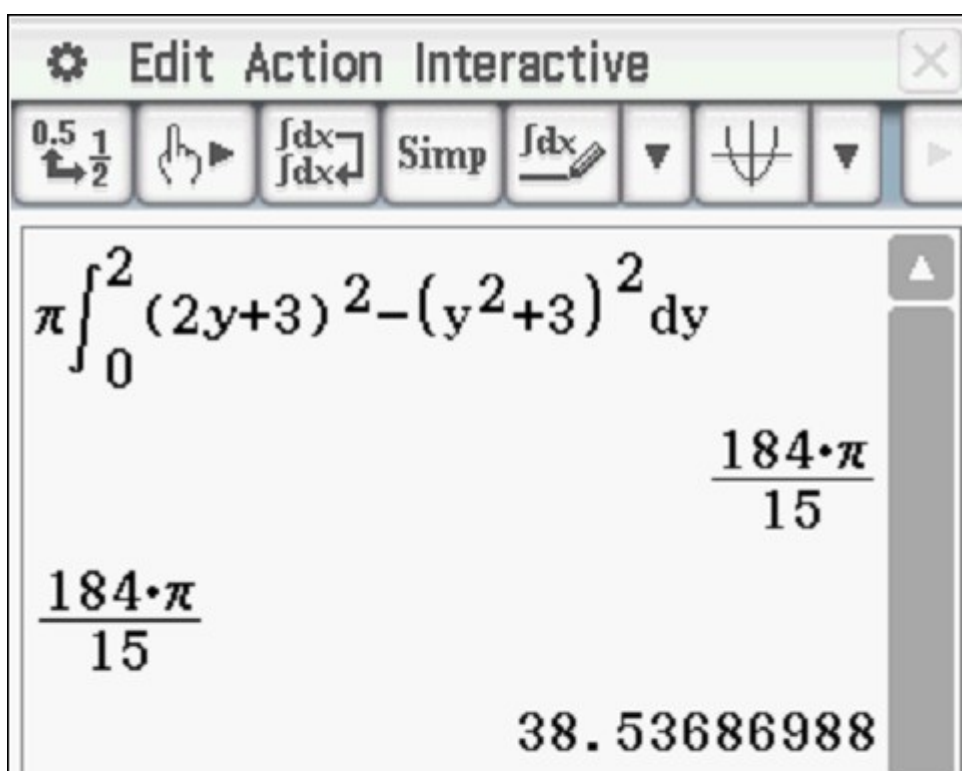
Determine the volume of the solid formed by rotating the area enclosed between

$y = \sqrt{x-3}$ & $y = \frac{1}{2}x - \frac{3}{2}$ about the y axis, as shown below.



Solution

$$\int_0^2 \pi (2y+3)^2 dy - \int_0^2 \pi (y^2+3)^2 dy$$



Specific behaviours
<ul style="list-style-type: none"> ✓ uses correct integral type ✓ determines x as the subject for each graph ✓ sets up integrals for both functions with limits ✓ uses subtraction after squaring ✓ states approx. volume of solid

Q2 (5, 3 & 2= 10 marks)

- a) By using integration and partial fractions, show how to derive $N = \frac{a}{b + Ce^{-at}}$ from the differential equation $\frac{dN}{dt} = aN - bN^2$ ($a, b > 0$) and c is a constant

Solution
$\frac{dN}{dt} = aN - bN^2 = N(a - bN)$ $\frac{dN}{dt} \rightarrow 0, a - bN = 0 \therefore N < \frac{a}{b} \rightarrow a - bN > 0$ $\int \frac{dN}{N(a - bN)} = \int dt$ $\frac{1}{N(a - bN)} = \frac{c}{N} + \frac{d}{a - bN}$ $1 = c(a - bN) + dN$ $N = 0$ $1 = ca, c = \frac{1}{a}$ $N = \frac{a}{b}$ $1 = d \frac{a}{b}, d = \frac{b}{a}$ $\frac{1}{N} + \frac{b}{a - bN} dn = t + c$ $\frac{1}{a} \ln N - \frac{1}{a} \ln a - bN = t + c$ $\text{As } a - bN > 0$ $\ln N - \ln(a - bN) = at + c$ $\ln \frac{N}{a - bN} = at + c$ $\frac{N}{a - bN} = e^{at+c} = Ce^{at}, \frac{a - bN}{N} = Ce^{-at}$ $a - bN = NCe^{-at}$ $a = N(b + Ce^{-at}), N = \frac{a}{b + Ce^{-at}}$

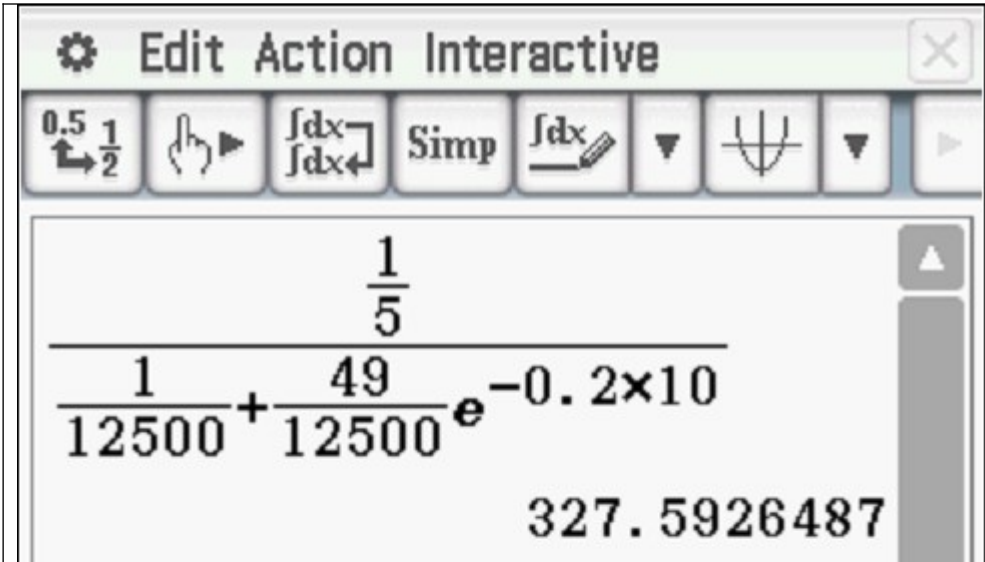
Specific behaviours
<ul style="list-style-type: none"> ✓ explains limit of N and sign of a-bN ✓ separates dN & dt and integrates ✓ uses partial fractions ✓ uses logs and obtains expression of N in terms of t ✓ shows derivation of final rule

Q2 continued

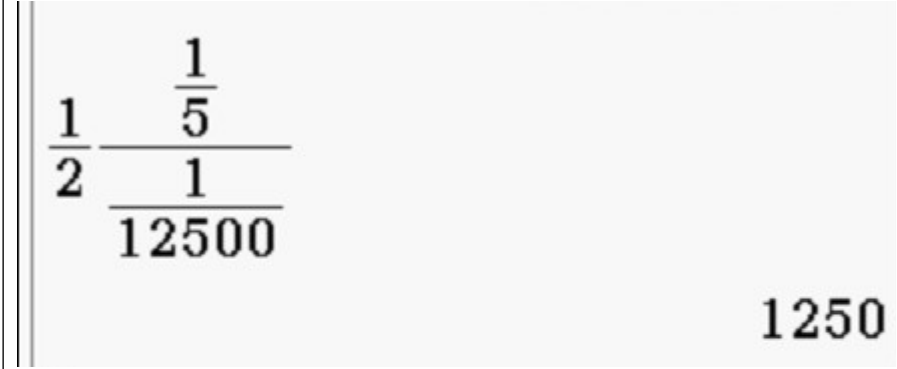
b) Let N equal the number of kangaroos living in a habitat after t years and

$\frac{dN}{dt} = \frac{1}{5}N - \frac{1}{12500}N^2$. If initially there are 50 kangaroos, determine the number in 10 years time.

Solution
$N = \frac{\frac{1}{5}}{\frac{1}{12500} + ce^{-0.2t}}$
<p>The screenshot shows a TI-Nspire calculator window titled "Edit Action Interactive". The main display shows the equation $50 = \frac{\frac{1}{5}}{\frac{1}{12500} + c}$ and the result $\left\{ c = \frac{49}{12500} \right\}$. The calculator interface includes a toolbar with various mathematical symbols and a scroll bar on the right.</p>

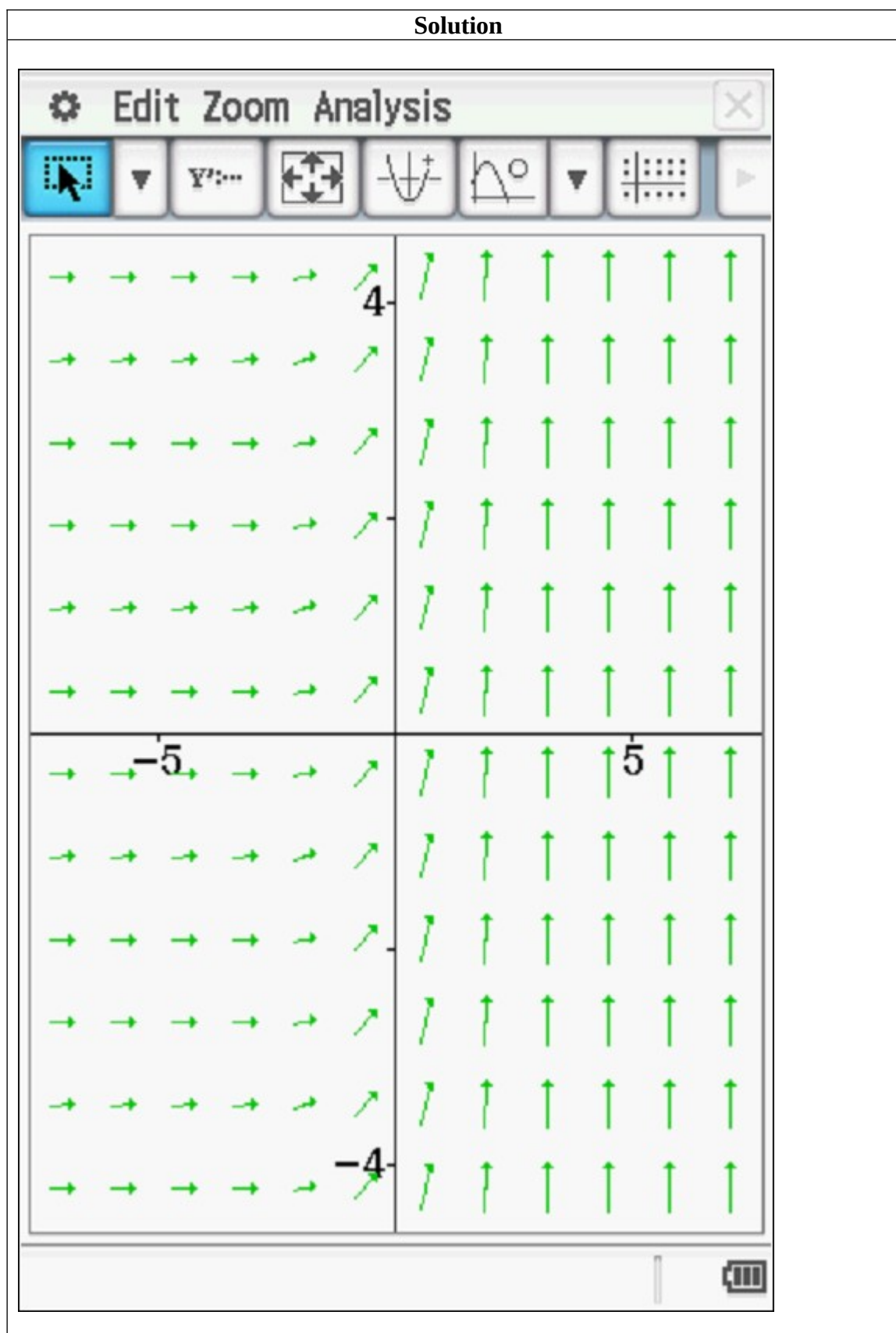
	
Specific behaviours	
<ul style="list-style-type: none"> ✓ solves for constant ✓ subs t=10 into correct expression ✓ states population (accept decimal) 	

c) Determine the size of the population at the maximum growth rate.

Solution	
$N = \frac{1}{2} \frac{a}{b}$ 	
Specific behaviours	
<ul style="list-style-type: none"> ✓ uses half of maximum ✓ States population 	

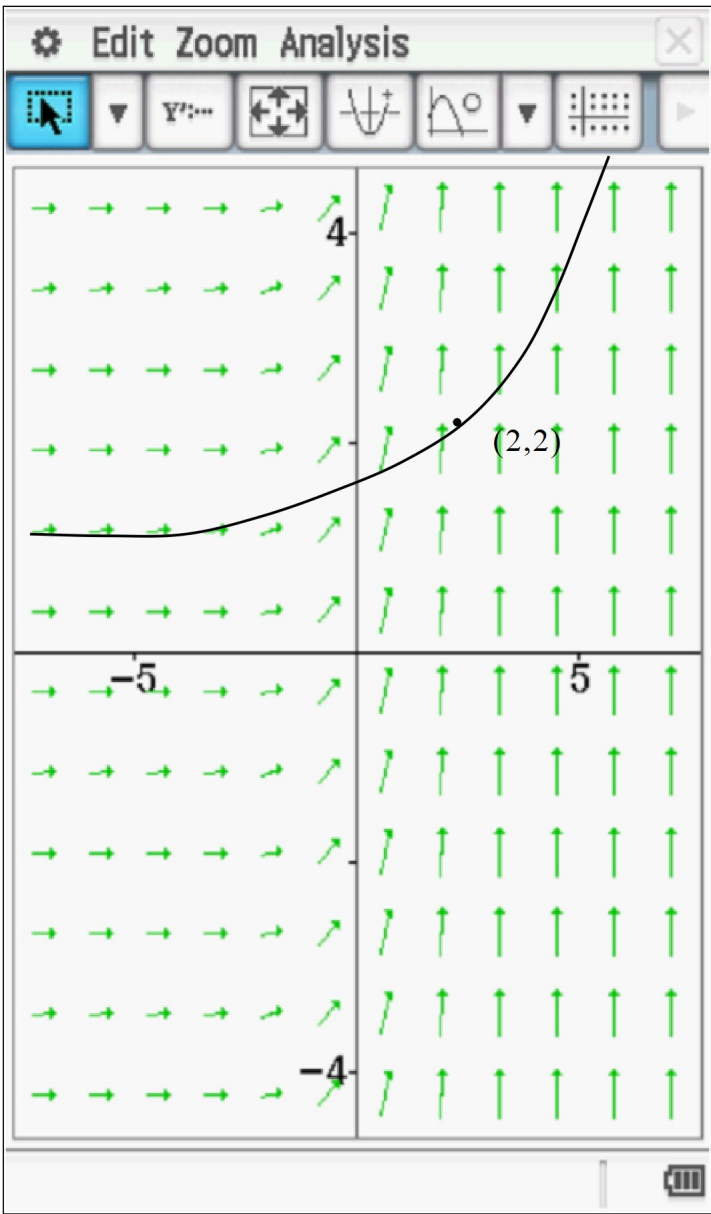
Q3 (3, 2 & 3 = 8 marks)

- a) Sketch the slope field on the axes below for $\frac{dy}{dx} = 3^x$



Specific behaviours
<ul style="list-style-type: none"> ✓ left side near zero gradients ✓ 45 degrees on y axis i.e 1 ✓ right side approaches vertical lines, i.e infinite

b) Show the solution curve on the axes above that passes through point (2,2).

Solution

Specific behaviours
<ul style="list-style-type: none"> ✓ follows contours ✓ passes through (2,2)

c) Determine in cartesian form the solution curve for b above **without using a classpad**.

Hint – use logarithmic differentiation. Show all working.

Solution
$y = 3^x$ $\ln y = \ln 3^x = x \ln 3$ $\frac{1}{y} y' = \ln 3$ $y' = y \ln 3 = 3^x \ln 3$ $\int 3^x dx = \frac{1}{\ln 3} 3^x + c$ $\frac{dy}{dx} = 3^x$ $y = \frac{1}{\ln 3} 3^x + c$ $2 = \frac{9}{\ln 3} + c$ $c = 2 - \frac{9}{\ln 3}$ $y = \frac{1}{\ln 3} 3^x + 2 - \frac{9}{\ln 3}$
Specific behaviours
<ul style="list-style-type: none"> ✓ uses log diff to diff exponential ✓ uses integration ✓ solves for exact constant <p>Note max 1 out of 3 if log diff not shown</p>

Q4 (5 marks)

Determine expressions in terms of x & y only for $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in terms of x, y & y' using the following equation $x^3y^2 = 5 - xy$

Solution
$x^3y^2 = 5 - xy$ $x^3(2yy') + y^23x^2 = -xy' - y$ $y'(2x^3y + x) = -y(1 + 3x^2y)$ $y' = \frac{-y(1 + 3x^2y)}{(2x^3y + x)}$ $y'(2x^3y' + 6x^2y + 1) + (2x^3y + x)y'' = -y(3x^2y' + 6xy) - y'(1 + 3x^2y)$ $y'' = \frac{-y(3x^2y' + 6xy) - y'(1 + 3x^2y) - y'(2x^3y' + 6x^2y + 1)}{(2x^3y + x)}$
Specific behaviours
<ul style="list-style-type: none"> ✓ uses product rule on both sides for first derivative ✓ uses implicit diff in terms of x ✓ uses product/quotient rule for second derivative ✓ obtains an expression with second derivative ✓ makes second derivative subject in terms of x,y and first derivative

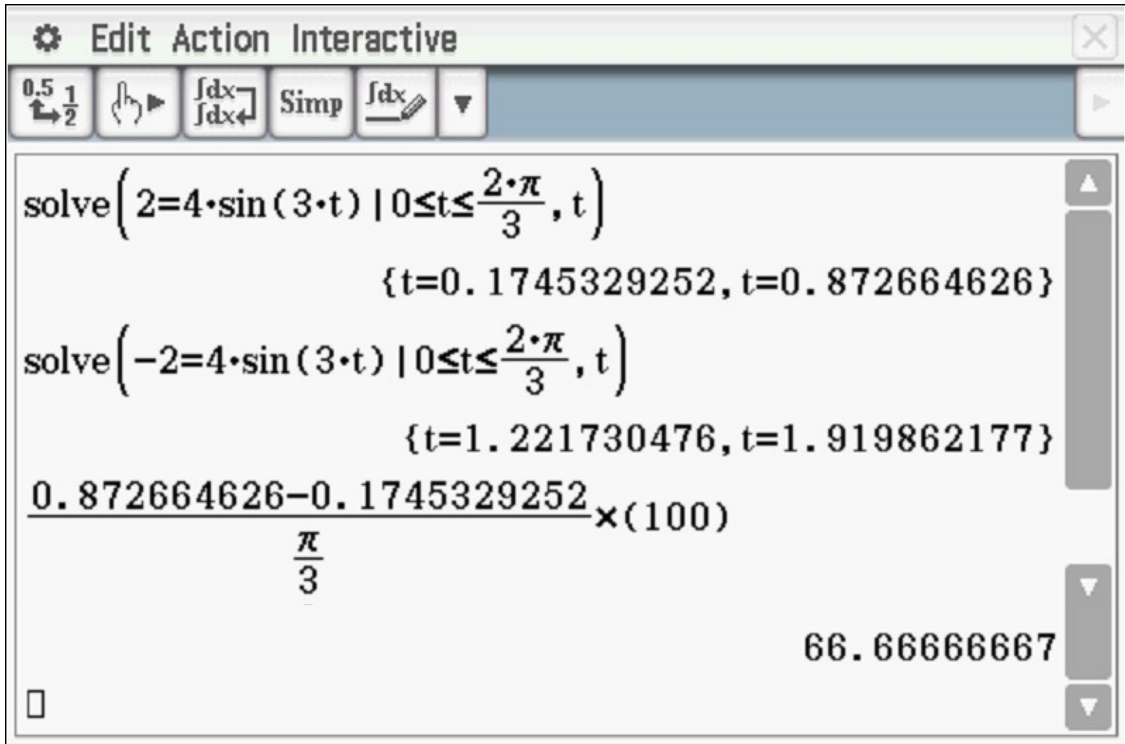
Q5 (3 & 3 = 6 marks)

Consider a particle that is moving with SHM such that $\ddot{x} = -9x$ with a maximum speed of 12 m/s.

a) Determine the exact speed when the particle is half of an amplitude from the origin.

Solution
$\ddot{x} = -9x$ $n = 3$ $12 = nA = 3A$ $A = 4$ $v^2 = n^2 (A^2 - x^2) = 9(16 - 4)$ $v = \sqrt{108}$
Specific behaviours
<ul style="list-style-type: none"> ✓ determines n & A ✓ uses correct formula ✓ states exact speed

b) Determine the percentage of the time that the particle is more than half an amplitude from the centre.

Solution
<p>Let $x = A \sin nt = 4 \sin 3t$</p>  <p> $\text{solve}\left(2=4 \cdot \sin(3 \cdot t) \mid 0 \leq t \leq \frac{2 \cdot \pi}{3}, t\right)$ $\{t=0.1745329252, t=0.872664626\}$ $\text{solve}\left(-2=4 \cdot \sin(3 \cdot t) \mid 0 \leq t \leq \frac{2 \cdot \pi}{3}, t\right)$ $\{t=1.221730476, t=1.919862177\}$ $\frac{0.872664626 - 0.1745329252}{\frac{\pi}{3}} \times (100)$ 66.66666667 </p>
Specific behaviours

- ✓ determines a model for displacement and period
- ✓ solves for times at half an amplitude in one cycle/(half cycle)
- ✓ determines percentage of time

Q6 (4 & 2 = 6 marks)

The motion of a bullet through a wall is modelled by the equation $a = -25(v + 75)^2$, $v > 0$ where $a \text{ m/s}^2$ is its acceleration and $v \text{ m/s}$ its velocity t seconds after impact. Initially the speed is 300 and is at the origin ($x = 0$ metres)

a) Determine x in terms of v .

Solution
$v \frac{dv}{dx} = -25(v + 75)^2, \quad v > 0$ $\int \frac{v}{(v + 75)^2} dv = \int -25 dx$ <p>let $y = v + 75$</p> $\int \frac{y - 75}{y^2} dy = \int y^{-1} - 75y^{-2} dy = \ln y + 75y^{-1} = \ln v + 75 + \frac{75}{v + 75} = -25x + c$ <p>$x = 0, v = 300$</p> $\ln 375 + \frac{75}{375} = c$ $x = \frac{1}{-25} \left(\ln v + 75 + \frac{75}{v + 75} - \ln 375 - \frac{75}{375} \right)$
Specific behaviours
<ul style="list-style-type: none"> ✓ uses dv & dx and separates variables v & x ✓ Integrates both sides ✓ changes variable to integrate dv ✓ solves for exact constant

Q6 continued-

b) Determine how far the bullet penetrates the wall before coming to rest to the nearest mm.

Solution
$x = \frac{1}{-25} \left(\ln 75 + 1 - \ln 375 - \frac{75}{375} \right)$ $\approx 0.0324 \text{ m}$ $\approx 32 \text{ mm}$

Specific behaviours
✓ subs $v=0$ ✓ rounds to nearest mm