PRESBYTERIAN LADIES' COLLEGE
A COLLEGE OF THE UNITING CHURCH IN AUSTRALIA

MATHEMATICS DEPARTMENT

Year 12 MATHEMATICS SPECIALIST

TEST 4: DIFFERENTIATION AND DIFFERENTIAL EQUATIONS

DATE: 28th June 2016 Name___________________________

Reading Time: 3 minutes

SECTION ONE: CALCULATOR FREE

TOTAL: 33 marks

EQUIPMENT: Pens, pencils, pencil sharpener, highlighter, eraser, ruler, SCSA formula sheet.

WORKING TIME: 30 minutes (maximum)

SECTION TWO: CALCULATOR ASSUMED

TOTAL: 25 marks

EQUIPMENT: Pens, pencils, pencil sharpener, highlighter, eraser, ruler, drawing instruments, templates, up to 3 Calculators,

1 A4 page of notes (one side only), SCSA formula sheet.

WORKING TIME: 20 minutes (minimum)

SECTION 1 Question	Marks available	Marks awarded	SECTION 2 Question	Marks available	Marks awarded
1	6		5	8	
2	6		6	8	
3	11		7	9	
4	10				
Total	33			25	

Section One: Calculator-free **[33 marks]**

This section has **four (4)** questions. Answer **all** questions. Write your answers in the spaces provided.

Question 1 **[6 marks]**

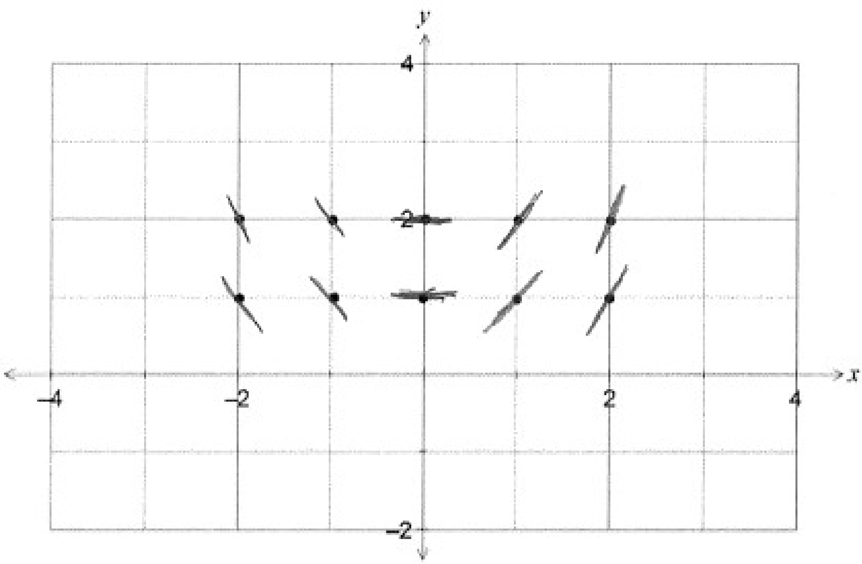
A first order differential equation is given by $\frac{dy}{dx} = xy$.

(a) Use the equation to complete the table below. [2]

x	-2	-1	0	1	2	3
y	2	2	2	2	2	3
$\frac{dy}{dx}$	-4	-2	0	2	4	9

(2) {-0.5 per error, down to zero}

(b) Create a slope field on the 10 points on the graph below. [2]



(2) {-0.5 per error, down to zero}

(c) Find the solution that passes through the point given by $x=1$ and $y=1$. [2]

$$\frac{dy}{dx} = xy \Rightarrow \frac{1}{y}\frac{dy}{dx} = x \Rightarrow \ln y = \frac{x^2}{2} + c$$ (1)

$$(1,1) \Rightarrow 0 = \frac{1}{2} + c \Rightarrow c = -\frac{1}{2} \Rightarrow y = e^{\frac{x^2-1}{2}}$$ (1)

Question 2 [6 marks]

A function is defined parametrically by the equations $x(t)=t^2+2t$ and $y(t)=t^3-9t$

(a) Find $\frac{dy}{dx}$ in terms of t [2]

$$\frac{dy}{dt}=3t^2-9, \quad \frac{dx}{dt}=2t+2 \quad \Rightarrow \quad \frac{dy}{dx}=\frac{3t^2-9}{2t+2}=\frac{3(t^2-3)}{2(t+1)}$$

(1) (1)

(b) By finding the second derivative, $\frac{d^2y}{dx^2}$ in terms of t, show that there are no points of inflection on this curve. [4]

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{3(t^2-3)}{2(t+1)}\right)\times\frac{dt}{dx}$$

(1)

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{3}{2}\left(\frac{2t(t+1)-(t^2-3)}{(t+1)^2}\right)\times\frac{1}{2(t+1)}=\frac{3}{4}\left(\frac{t^2+2t+3}{(t+1)^3}\right)$$

(1) (1)

As $t^2+2t+3\neq 0 \quad \Rightarrow \quad \frac{d^2y}{dx^2}\neq 0 \quad \Rightarrow$ No points of inflection.

(1)

Question 3 [11 marks]

The equation of a curve in the plane is $x^2+3y^2+2xy=12$.

(a) Show that for all points on the curve $(3y+x)\frac{dy}{dx}=-x-y$. [4]

Differentiation implicitly gives

$2x+6y\frac{dy}{dx}+2y+2x\frac{dy}{dx}=0$ (1) (1) $\Rightarrow$ $2(3y+x)\frac{dy}{dx}=-2(x+y)$ (1) $\Rightarrow$ $(3y+x)\frac{dy}{dx}=-x-y$ (1)

as required

(b) Find the equation of the tangent to the curve at the point $(0,2)$. [3]

$\frac{dy}{dx}=\frac{-x-y}{(3y+x)}$ (1) $\Rightarrow$ at $(0,2)$ the gradient is $m=\frac{-0-2}{3\times 2+0}=-\frac{1}{3}$ (1)

$\Rightarrow$ Equation is $y=-\frac{1}{3}x+2$ (1) or $x+3y=6$

(c) At what points on the curve is the tangent parallel to the *y*-axis? [4]

Tangent is parallel to $y-$ axis when $3y+x=0$. (0.5)

$\Rightarrow y=-\frac{x}{3}$ (0.5)

Subbing into original equation gives

$x^2+3\left(-\frac{x}{3}\right)^2+2x\left(-\frac{x}{3}\right)=12$ (1) $\Rightarrow$ $x^2+\left(\frac{x^2}{3}\right)-\left(\frac{2x^2}{3}\right)=12$ $\Rightarrow$ $2x^2=36$ $\Rightarrow$

$x=\pm 3\sqrt{2}$ (1)

Thus points are $(3\sqrt{2},-\sqrt{2})$ and $(-3\sqrt{2},\sqrt{2})$ (1)

Question 4 [10 marks]

The volume *V* of blood flowing through an artery in unit time can be modelled by the formula $V = \mathbf{kr^4}$, where *r* is the radius of the artery and *k* is a constant.

(a) What is the effect on the volume of blood flow if the radius of the artery is halved?

[2]

If $V_0 = kr_0^4$ and $r_1 = \frac{r_0}{2}$ $\Longrightarrow$ $V_1 = kr_1^4 = \frac{kr_0^4}{16} = \frac{1}{16}V_0$ i.e. the volume is 1/16th of the original volume. (1) (1)

(b) Use the incremental formula to estimate the percentage decrease in the radius of a partially clogged artery that will produce a 10% decrease in the flow of blood.

[5]

$$\frac{\Delta V}{V} = 0.1 \Rightarrow \Delta V = 0.1V$$ (1)

$$\Delta V \approx \frac{dV}{dr}\Delta r = 4kr^3 \Delta r \Longrightarrow \frac{\Delta V}{V} = \frac{4kr^3}{kr^4}\Delta r \Longrightarrow \frac{\Delta V}{V} = \frac{4\Delta r}{r} \Longrightarrow$$

(1) (1)

$$\frac{\Delta r}{r} = \frac{0.1}{4} = 0.025$$ (1)

Thus, the radius that will produce a 10% decrease in flow of blood is reduced by 2.5%. (1)

(c) Show that the incremental formula gives a physically absurd estimate for the change in *V* resulting from a halving of the radius of the artery. Explain why this estimate is so poor compared to the true answer found in (a).

[3]

Halving the radius mean that $\frac{\Delta r}{r} = 0.5 \Rightarrow \Delta r = 0.5r$ (1)

$\frac{\Delta V}{V} \approx \frac{4kr^3}{kr^4} \times \Delta r = \frac{4kr^3}{kr^4} \times 0.5r = 2$ which is a 200% reduction, which is not possible. (1)

The reason for this estimate being absurd is because Δr is not small compared to r. (1)

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NAME:____

Section Two: Calculator-assumed **[25 marks]**

This section has **three (3)** questions. Answer **all** questions. Write your answers in the spaces provided

Question 5 [8 marks]

The needle in a sewing machine moves vertically with simple harmonic motion, and the distance between the highest and lowest positions of the tip is 8 mm.

The height of the tip of the needle above its mid-point position *t* seconds after it starts to move is *x(t)* mm, where *x(t)* satisfies the differential equation

$$\frac{d^2x}{dt^2} = -16\pi^2 x.$$

(a) Determine $x(t)$, given that the needle starts at its highest point. [3]

Amplitude = 4 mm and $k = 4\pi \Rightarrow x(t) = 4\cos(4\pi t)$
(1) (1) (1)

(b) How long does it take for the needle to return to its highest point? [2]

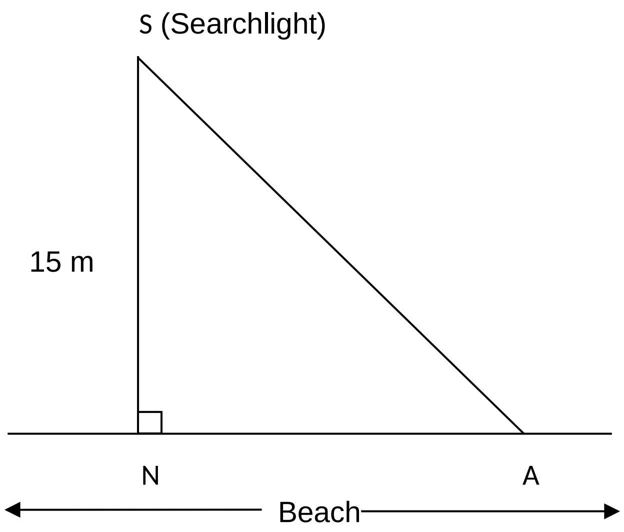
$T = \frac{2\pi}{4\pi} = 0.5$ seconds. i.e it takes half a second to return to the highest point.
(1) (1)

(c) How far does the tip travel in the first 0.3 seconds? [3]

$v(t) = \frac{dx}{dt} = -16\pi\sin(4\pi t) \Rightarrow d = \int_0^{0.3} |-16\pi\sin(4\pi t)|\, dt = 8.764$ mm.
(1) (1) (1)

Question 6 [8 marks]

A searchlight S is just above sea level and is revolving in the horizontal plane. The searchlight is located 15 metres out to sea from the nearest point N on a straight beach. S and N are in the same horizontal plane and the searchlight rotates at 2 revolutions per minute.



Determine the rate at which the beam of light is moving along the beach when:

(a) the beam illuminates the beach at a point A such that the angle SAN is 30° [6]

Let $\angle ASN = \theta$ and $AN = x$. (1)

Also, 2 revolutions per minute $\Rightarrow \frac{d\theta}{dt} = 4\pi$ radians per minute. (1)

$\tan\theta = \frac{x}{15}$ (1) $\Rightarrow \frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{15}\frac{dx}{dt}$ (1) $\Rightarrow \frac{dx}{dt} = \frac{60\pi}{\cos^2\theta}$ m/minute. (1)

When $\angle SAN = 30^0$, $\theta = \frac{\pi}{3}$ and $\cos^2\theta = \frac{1}{4}$ $\Rightarrow \frac{dx}{dt} = 240\pi$ m/minute or $\frac{dx}{dt} = 4\pi$ m/sec. (1)

(b) the beam illuminates at a point B on the beach 39 metres from S. [2]

When AS = 39 m, $\cos\theta = \frac{15}{39} = \frac{5}{13}$ (1) $\Rightarrow \frac{dx}{dt} = \frac{60\pi}{\left(\frac{5}{13}\right)^2} = 21.237$ m/second. (1)

(or 1274.230 m/minute)

Question 7 [9 marks]

The expected uptake of a new model of smart phone in a country, currently with one million models in use, can be modelled by the logistic equation $\frac{dx}{dt}=\frac{x(20-x)}{250}$, where x is the total number of models in millions and t is the time in weeks.

(a) Express x as a function of t in the form $x=\frac{a}{1+be^{-ct}}$ where a, b and c are positive constants. [5]

This is logistic model of the form $\frac{dx}{dt}=px-qx^2$ which has solution $x=\frac{p}{q+\left(\frac{p}{x(0)}-q\right)e^{-pt}}$ **(1)**

In this case,

$p=\frac{20}{250}$, $q=\frac{1}{250}$, $x(0)=1$ $\Rightarrow$ $x=\frac{\frac{20}{250}}{\frac{1}{250}+\frac{19}{250}e^{-\frac{20}{250}t}}=\frac{20}{1+19e^{-0.08t}}$

(1) **(1)** **(1)**

i.e. $a=20,\ b=19,\ c=0.08$

(1)

OR

$\int\frac{dx}{x(20-x)}=\int\frac{dt}{250}$ **(1)** $\Rightarrow$ $\int\frac{1}{x}+\frac{1}{20-x}\,dx=\int\frac{2}{25}\,dt$ **(1)** $\Rightarrow$ $\ln x-\ln(20-x)=.08t+c$ **(1)**

$t=0,\ x=1$ $\Rightarrow$ $c=-\ln 19$ $\Rightarrow$ $x=\frac{20}{1+19e^{-0.08t}}$ **(1)**

i.e. $a=20,\ b=19,\ c=0.08$

(1)

(b) Calculate

(i) the expected number of models in use after 30 weeks. [1]

t = 30 $\Rightarrow$ x = 7.343 million models. **(1)**

(ii) the week during which the number of models in use is increasing at the greatest rate. [3]

For $\frac{dx}{dt}$ to be maximised, $\frac{d^2x}{dt^2} = 0 \Rightarrow x = 10$ **(1)**

$\frac{20}{1 + 19e^{-0.08t}} = 10$ **(1)** $\Rightarrow$ $t = 36.8$ i.e. the 37th week. **(1)**

END OF QUESTIONS