PERTH MODERN SCHOOL
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INDEPENDENT PUBLIC SCHOOL

Semester Two Examination, 2019

Question/Answer booklet

MATHEMATICS SPECIALIST
UNIT 2

Section One:
Calculator-free

Your Name________________________________

Your Teacher's Name________________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Mark	Max	Question	Mark	Max
1		3	6		8
2		6	7		6
3		7	8		4
4		8	9		3
5		7			

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	9	9	50	52	36
Section Two: Calculator-assumed	13	13	100	94	64
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2019*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section One: Calculator-free **(52 Marks)**

This section has **nine (9)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 50 minutes.

Question 1 **(3 marks)** (2.8)

Prove the following statement:

If a and b are each 1 less than a multiple of 3, then ab is 1 more than a multiple of 3.

Solution
Assume that a and b are each 1 less than a multiple of 3. Then $a=3k-1$ and $b=3l-1$ for some $k, l \in Z$. Hence $$ab=(3k-1)(3l-1)=9kl-3k-3l+1=3(3kl-k-l)+1$$ which is 1 more than a multiple of 3 since $3kl-k-l$ is an integer. QED
Specific behaviours
✓ writes a and b as $3k-1$ and $3l-1$ ✓ multiplies and simplifies to $3(3kl-k-l)+1$ ✓ concludes that ab is 1 more than a multiple of 3

[redacted] **(6 marks)** (4.2)

Consider the system of simultaneous linear equations:

$$3x - ay = 6 - 6x + 4y = b$$

a) Write down the matrix A such that the equation

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ b \end{bmatrix}$$

is equivalent to the system of equations above. (1 mark)

Solution
$\begin{bmatrix} 3 & -a \\ -6 & 4 \end{bmatrix}$
Specific behaviours
✓ writes correct matrix

b) Suppose that A is singular (non-invertible).

i. Determine the value of a (show working). (2 marks)

Solution
$A\,singular \Leftrightarrow det\,A = 0 \Leftrightarrow 3 \times 4 - (-a)(-6) = 0 \Leftrightarrow a = 2$
Specific behaviours
✓ attempts to solve $det\,A = 0$ for a ✓ states correct value

ii. State the possible number(s) of solutions that the system of equations could have with the value of a you just found. (2 marks)

Solution
Either no solutions or infinitely many solutions.
Specific behaviours
✓ states at least one of ‘no solutions’ and ‘infinitely many solutions’ ✓ states both possibilities No marks if implies that there can be exactly one solution

iii. State the number of solutions the system has if a has the value found above and $b = 11$. (1 mark)

Solution
No solutions.
Specific behaviours
✓ States no solutions.

[redacted] **(7 marks)** (4.5)

a) Write $3\cos 5x+3\sqrt{3}\sin 5x$ in the form $a\sin(bx+\alpha)$. (3 marks)

Solution
$a=\sqrt{3^2+(3\sqrt{3})^2}$¿$\sqrt{36}$¿$6 b=5 \alpha=\sin^{-1}\left(\frac{3}{6}\right)$¿$\frac{\pi}{6}$ So $3\cos 5x+3\sqrt{3}\sin 5x=6\sin\left(5x+\frac{\pi}{6}\right)$
Specific behaviours
✓ writes correct value for a ✓ writes correct value for b ✓ writes correct value for α

b) Hence, solve the equation $3\cos 5x+3\sqrt{3}\sin 5x=3\sqrt{3}$ for $\frac{-\pi}{2}\le x\le\frac{\pi}{2}$. (4 marks)

Solution
$3\cos 5x+3\sqrt{3}\sin 5x=3\sqrt{3}6\sin\left(5x+\frac{\pi}{6}\right)=3\sqrt{3}\sin\left(5x+\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$ $5x+\frac{\pi}{6}=\frac{\pi}{3}+k2\pi\vee 5x+\frac{\pi}{6}=\frac{2\pi}{3}+k2\pi, k\in Z x=\frac{\pi}{30},\frac{13\pi}{30},-\frac{11\pi}{30}\vee x=\frac{\pi}{10},\frac{5\pi}{10},-\frac{3\pi}{10}$ $x=\frac{-11\pi}{30},-\frac{3\pi}{10},\frac{\pi}{30},\frac{\pi}{10},\frac{13\pi}{30},\frac{\pi}{2}$
Specific behaviours
✓ equates expression from part (a) to $3\sqrt{3}$ ✓ obtains at least one general solution (using $\frac{\pi}{3}$ or $\frac{2\pi}{3}$) for $5x+\frac{\pi}{6}$ ✓ obtains at least 3 correct solutions for x (in correct domain) ✓ obtains 6 correct solutions for x (in correct domain)

Question 4 **(8 marks)** (5.1)

A 2×2 real matrix A can 'transform' a complex number if we view the complex number as a column vector. That is, for any complex number $z=a+bi$, the matrix A transforms z to $c+di$ where $\begin{bmatrix} c \\ d \end{bmatrix}=A\begin{bmatrix} a \\ b \end{bmatrix}$.

Find the matrix A which (according to this rule) will transform any complex number z to:

a) $3z$ (2 marks)

Solution
Multiplication by 3 corresponds to dilation by factor 3. $$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$
Specific behaviours
✓ identifies corresponding transformation ✓ writes correct matrix

b) $\bar{z}$ (2 marks)

Solution
Conjugation corresponds to reflection in x-axis. $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
Specific behaviours
✓ identifies corresponding transformation ✓ writes correct matrix

c) iz (2 marks)

Solution
Multiplying by i corresponds to rotating anti-clockwise by $90°$. $$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
Specific behaviours
✓ identifies corresponding transformation ✓ writes correct matrix

d) $i\bar{z}$ (2 marks)

Solution
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\times\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
Specific behaviours
✓ writes product of matrices OR identifies corresponding transformation sequence

✓ writes correct matrix

[redacted] **(7 marks)** (6.2)

Evaluate the following for complex numbers $z=2+5i$ and $w=1-4i$

a) $z-w$ (2 marks)

Solution
$z-w=2+5i-(1-4i)¿1+9i$
Specific behaviours
✓ writes correct calculation ✓ writes correct answer

b) $z(w+\overline{w})$ (2 marks)

Solution
$z(w+\overline{w})=(2+5i)(1-4i+1+4i)¿4+10i$
Specific behaviours
✓ writes correct expression for conjugate of w ✓ writes correct answer

c) $\frac{w}{z}$ (3 marks)

Solution
$\frac{w}{z}=\frac{1-4i}{2+5i}¿\frac{(1-4i)(2-5i)}{(2+5i)(2-5i)}¿\frac{2-8i-5i+20i^2}{2^2+5^2}¿-\frac{18}{29}-\frac{13}{29}i$
Specific behaviours
✓ multiplies numerator and denominator by $\overline{w}$ ✓ expands numerator and denominator correctly ✓ writes correct answer

Question 6 **(8 marks)** (5.2)

In this question, a *proper* factor is a factor greater than 1.

Assume that a and b are both integers, and consider the following statement:

If ab has no proper square factors, then neither a nor b has a proper square factor.

a) Prove the statement using the method of proof by contradiction. (3 marks)

Solution
Assume that ab has no proper square factors, but that either a or b has a proper square factor. Then either $a=nk^2$ or $b=ml^2$ for some $k, l, m, n \in Z$. Hence either $ab=nk^2b$ or $ab=aml^2$, and in each case ab has a proper square factor. This is a contradiction, and so neither a nor b has a proper square factor.
Specific behaviours
✓ assumes negation of statement ✓ writes a and/or b as a product of an integer and a square ✓ shows that ab must therefore have a square factor and notes contradiction Accept argument for a alone having a square factor together with recognition that a similar argument will apply if b has a square factor.

b) Write the converse of the statement. (2 marks)

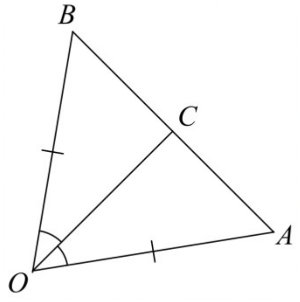
Solution
If neither a nor b has a proper square factor, then ab has no square factors.
Specific behaviours
✓ ✓ writes correct converse statement

c) Write whether the converse is true or false and prove or disprove it accordingly. (3 marks)

Solution
False. E.g. if $a=6$ and $b=10$, then neither a nor b has a proper square factor, but $ab=60=2^2\times 15$.
Specific behaviours
✓ states false ✓✓ gives correct counterexample

(6 marks) (4.2)

Let O be the origin, let A and B be points such that $OA = OB$, and let C be a point on $\overline{AB}$ such that $\overline{OC}$ bisects $\angle AOB$.



Let $a = \overrightarrow{OA}$, $b = \overrightarrow{OB}$ and $c = \overrightarrow{OC}$.

a) Show that $a \cdot c = b \cdot c$. (3 marks)

Solution
Let $\theta = \angle COB$. Then $\angle COA = \theta$. Now $a \cdot c = \vert a \vert \vert c \vert \cos\theta$ and $b \cdot c = \vert b \vert \vert c \vert \cos\theta$. Since $\vert a \vert = ¿ b \vee ¿$ it follows that $a \cdot c = b \cdot c$.
Specific behaviours
✓ writes expressions for $a \cdot c$ and $b \cdot c$ ✓ uses the fact that $\angle COB = \angle COA$ ✓ uses the fact that $\vert a \vert = ¿ b \vee ¿$

b) **Hence**, prove that $\overline{OC}$ is perpendicular to $\overline{AB}$. (3 marks)

Solution
$\overrightarrow{AB} = b - a$ $(b - a) \cdot c = b \cdot c - a \cdot c$ $¿ 0$ (by part a) Hence $\overline{AB}$ is perpendicular to $\overline{OC}$.
Specific behaviours
✓ writes $\overrightarrow{AB}$ as $b - a$ ✓ attempts to determine dot product $(b - a) \cdot c$ ✓ shows that $(b - a) \cdot c = 0$

[redacted] **(4 marks)** (2.0)

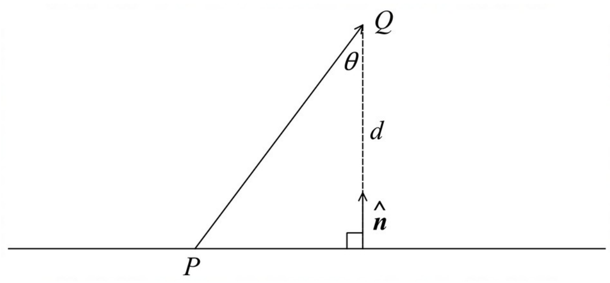
Prove the following identity.

$$\frac{\sin 7\theta - \sin 2\theta}{\cos 2\theta + \cos 7\theta} = \tan \frac{5\theta}{2}$$

Solution
$$LHS = \frac{\sin 7\theta - \sin 2\theta}{\cos 2\theta + \cos 7\theta} ¿ \frac{2\sin\left(\frac{5\theta}{2}\right)\cos\left(\frac{9\theta}{2}\right)}{2\cos\left(\frac{9\theta}{2}\right)\cos\left(\frac{5\theta}{2}\right)} ¿ \frac{\sin\left(\frac{5\theta}{2}\right)}{\cos\left(\frac{5\theta}{2}\right)} ¿ \tan\left(\frac{5\theta}{2}\right) ¿ RHS$$ Hence LHS = RHS QED
Specific behaviours
✓ rearranges RHS (or LHS) to obtain LHS (or RHS)   ✓ correctly uses sum-to-product identities   ✓ cancels factor of $2\cos\left(\frac{9\theta}{2}\right)$ ✓ simplifies to RHS (or LHS)

(3 marks) (1.5)

Let l be a line containing a point P, and let Q be a point not on l. Suppose that $\hat{n}$ is a unit vector perpendicular to the line l. Prove that the perpendicular distance from Q to l is $\left|\overrightarrow{PQ}\cdot\hat{n}\right|$.

Solution
   Distance from Q to l is $d=\left\|\overrightarrow{PQ}\right\|\cos\theta$ $\overrightarrow{PQ}\cdot\hat{n}=\left\|\overrightarrow{PQ}\right\|\left\|\hat{n}\right\|\cos\theta¿\left\|\overrightarrow{PQ}\right\|\cos\theta$ Hence $d=\overrightarrow{PQ}\cdot\hat{n}$ (or $-\overrightarrow{PQ}\cdot\hat{n}$ if $\hat{n}$ has opposite direction).
Specific behaviours
✓ writes $\overrightarrow{PQ}\cdot\hat{n}$ as $\left\|\overrightarrow{PQ}\right\|\left\|\hat{n}\right\|\cos\theta$ ✓ uses the fact that $\left\|\hat{n}\right\|=1$ ✓ uses the fact that the shortest distance from Q to l is $\left\|\overrightarrow{PQ}\right\|\cos\theta$ (Note that only $d=\overrightarrow{PQ}\cdot\hat{n}$ is required for marks.)

END OF SECTION ONE

Additional working space

Question number: ________

Additional working space

Question number: ________