PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Test 1 Year 12

Student name: ______________ Teacher name: ______________

Task type: **Response**

Reading time for this test : 5 mins

Working time allowed for this task: 40 mins

Number of questions: 7

Materials required: No cals allowed!!

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **42 marks**

Task weighting: **13%**

Formula sheet provided: no

Note: All part questions worth more than 2 marks require working to obtain full marks.

Useful formulae

Complex numbers

Cartesian form	
$z = a + bi$	$\bar{z} = a - bi$
$\text{Mod}(z) = \lvert z \rvert = \sqrt{a^2 + b^2} = r$	$\text{Arg}(z) = \theta, \quad \tan\theta = \dfrac{b}{a}, \quad -\pi < \theta \le \pi$
$\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \lvert z_2 \rvert$	$\left\lvert \dfrac{z_1}{z_2} \right\rvert = \dfrac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$
$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$	$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
$z\bar{z} = \lvert z \rvert^2$	$z^{-1} = \dfrac{1}{z} = \dfrac{\bar{z}}{\lvert z \rvert^2}$
$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$	$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
Polar form	
$z = a + bi = r(\cos\theta + i\sin\theta) = r\,cis\,\theta$	$\bar{z} = r\,cis\,(-\theta)$
$z_1 z_2 = r_1 r_2\,cis\,(\theta_1 + \theta_2)$	$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\,cis\,(\theta_1 - \theta_2)$
$cis(\theta_1 + \theta_2) = cis\,\theta_1\,cis\,\theta_2$	$cis(-\theta) = \dfrac{1}{cis\,\theta}$
De Moivre's theorem	
$z^n = \lvert z \rvert^n\,cis\,(n\theta)$	$(cis\,\theta)^n = \cos n\theta + i\sin n\theta$
$z^{\frac{1}{q}} = r^{\frac{1}{q}}\left(\cos\dfrac{\theta + 2\pi k}{q} + i\sin\dfrac{\theta + 2\pi k}{q}\right)$, for k an integer	

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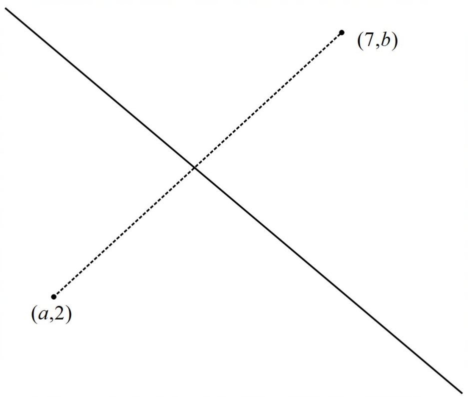
Specific behaviours
✓ identifies two major points in terms of a&b OR subs z=x+iy into both sides ✓ uses midpoint and subs into line equation OR squares both sides and eliminates squared terms. ✓ uses perpendicular gradient and major points OR subs eqn of line ✓ sets up two simultaneous eqns for a&b ✓solves for both a & b

Solution



Midpoint $\left(\frac{a+7}{2}, \frac{2+b}{2}\right)$

Gradient of dotted line $\frac{b-2}{7-a}$ perpendicular to locus line

$\frac{b-2}{7-a} = \frac{3}{4}$

$4b - 8 = 21 - 3a$

$b = \frac{29-3a}{4} = \frac{87-9a}{12}$

$6\left(\frac{b+2}{2}\right) = -8\left(\frac{7+a}{2}\right) + 61$

$3b + 6 = -28 - 4a + 61$

$b = \frac{27-4a}{3} = \frac{108-16a}{12}$

$87 - 9a = 108 - 16a$

$7a = 21$

$a = 3$

$b = 5$

No cals allowed!!

Q1 (2, 2, 2 & 2 = 8 marks)

If $z = 3 + 4i$ and $w = 1 - i$ determine the following exactly.

a) zw

Solution
$zw = 7 + i$
Specific behaviours
✓ real part ✓ imaginary part

b) z^2w

Solution
$z^2w = (3+4i)^2(1-i) = (9-16+24i)(1-i)$ $= (-7+24i)(1-i)$ $= 17 + 31i$
Specific behaviours
✓ real part ✓ imaginary part

c) $\frac{1}{z}$

Solution
$\frac{1}{3-4i}\frac{3+4i}{3+4i} = \frac{3+4i}{25}$
Specific behaviours
✓uses conjugate ✓ states result

d) $\frac{z}{w}$

Solution
$\frac{3+4i}{1-i}\frac{1+i}{1+i} = \frac{-1+7i}{2}$

Specific behaviours
✓uses conjugate ✓ states result

Q2 (4 marks)

Determine all possible real number pairs $a \& b$ such that $\frac{22-3i}{a+i}=5+bi$.

Solution
$22-3i=(5+bi)(a+i)=5a-b+i(ab+5)$ $22=5a-b,\quad b=5a-22$ $-3=ab+5=a(5a-22)+5=5a^2-22a+5$ $0=5a^2-22a+8=(5a-2)(a-4)$ $a=4, b=-2$ $a=\frac{2}{5}, b=-20$
Specific behaviours
✓ equates reals and imaginaries ✓ sets up an equation with only one variable ✓ solves for two a values ✓ solves for two b values

Q3 (2, 3 & 3 = 8 marks)

Consider the function $f(z)=z^3+2z^2+9z+18$.

a) Determine $f(3i)$.

Solution
$f(3i)=-27i+-18+27i+18=0$
Specific behaviours
✓ shows all 4 terms ✓ final answer of zero (Zero marks if all 4 terms not shown)

b) Hence solve $z^3+2z^2+9z+18=0$

Solution

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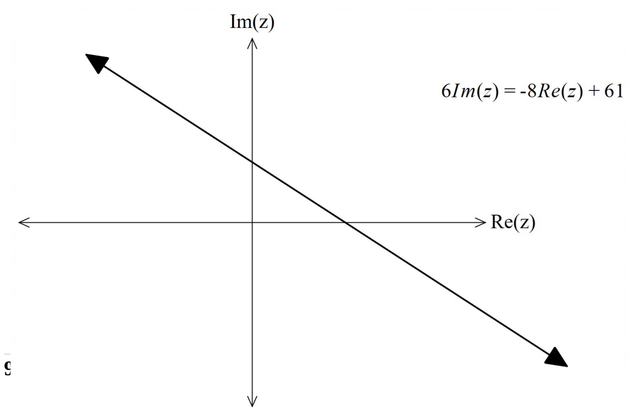
Solution
Specific behaviours
✓ scale shown with 6 roots equally spaced ✓ correct positions for all roots

c) Determine the area of the polygon formed by joining the points in (b) above.

Solution
$6\times\frac{1}{2}\left(4^{\frac{1}{6}}\right)^2 \sin\frac{\pi}{3}=3\frac{\sqrt{3}}{2}\left(4^{\frac{1}{3}}\right)$
Specific behaviours
✓ uses 6 equilateral triangles ✓ states exact value in surd form

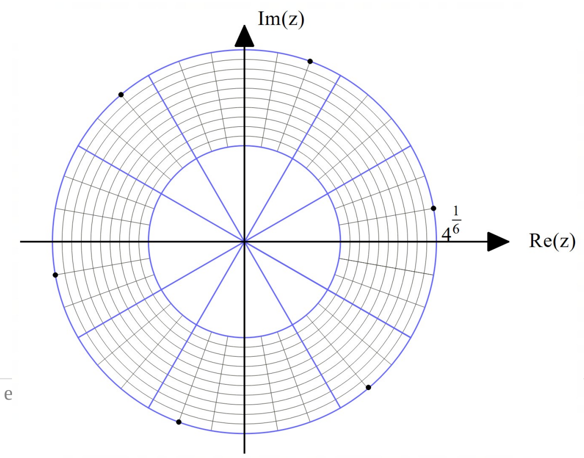
Q7 (5 marks)

The locus of $|z-a-2i|=|z-7-bi|$ where $a \& b$ are real constants is plotted below and can also be defined as $6\operatorname{Im}(z)=-8\operatorname{Re}(z)+61$. Determine the values of $a \& b$ showing full reasoning.



$z^6 = 2 + 2\sqrt{3}i = 4cis\left(\frac{\pi}{3} + 2n\pi\right) \quad n = 0, \pm1, \pm2, \ldots$
$z = 4^{\frac{1}{6}} cis\left(\frac{\pi}{18} + \frac{2n\pi}{6}\right) = 4^{\frac{1}{6}} cis\left(\frac{\pi}{18} + \frac{6n\pi}{18}\right)$
$z_1 = 4^{\frac{1}{6}} cis\left(\frac{\pi}{18}\right)$
$z_2 = 4^{\frac{1}{6}} cis\left(\frac{7\pi}{18}\right)$
$z_3 = 4^{\frac{1}{6}} cis\left(\frac{-5\pi}{18}\right)$
$z_4 = 4^{\frac{1}{6}} cis\left(\frac{13\pi}{18}\right)$
$z_5 = 4^{\frac{1}{6}} cis\left(\frac{-11\pi}{18}\right)$
$z_6 = 4^{\frac{1}{6}} cis\left(\frac{-17\pi}{18}\right)$
Specific behaviours
✓converts RHS to polar form ✓ shows use of De Moivres ✓ states 6 roots with same modulus ✓ all roots equally spaced ✓all arguments in Principal form

b) Plot these points on the axes below.



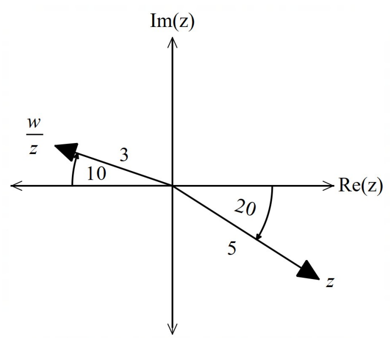
$z^3 + 2z^2 + 9z + 18 = (z - a)(z - 3i)(z + 3i) = (z - a)(z^2 + 9)$ $2 = -a, a = -2$ $z = -2, \pm 3i$
Specific behaviours
✓ uses conjugate ✓ shows full factorisation of f ✓ states all 3 roots

c) Consider $g(z) = (z^2 + bz + c)(z^2 + dz + e)$ where $b, c, d \,\&\, e$ are real constants and $g(3+i) = g(2-3i)$. Determine the values of $b, c, d \,\&\, e$.

Solution
$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$ $\alpha = 3+i, \beta = 3-i$ $z^2 - 6z + 10$ $\alpha = 2-3i, \beta = 2+3i$ $z^2 - 4z + 13$
Specific behaviours
✓ uses conjugates ✓ shows factorisation of each quadratic ✓ states all 4 constant values

Q4 (3 marks)
Use the diagram below to determine the complex number w in polar form with a principal argument.

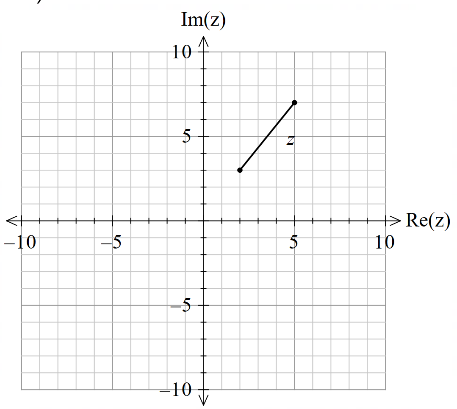
(diagram not drawn to scale)



Solution
$z = 5cis20$ $w = rcis\theta$ $\frac{w}{z} = 3cis170 = \frac{r}{5}cis(\theta - 20)$ $r = 15, \theta = 150$ $w = 15cis150$
Specific behaviours
✓ determines r of w ✓ determines argument of w ✓ states w in polar form

Q5 (2 & 3 = 5 marks)
Sketch the following regions on the axes below.

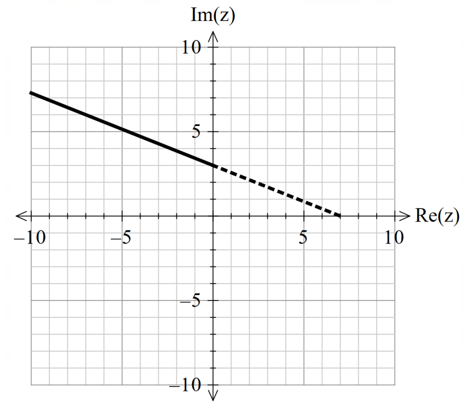
a) $|z - 2 - 3i| + |z - 5 - 7i| = 5$



Solution

Specific behaviours
✓ shows a line segment of length 5 units ✓ plots correct endpoints of closed line segment (includes endpoints)

b) $|z - 7| = |z - 3i| + \sqrt{58}$



Solution
Specific behaviours
✓ shows a line segment open (i.e arrow) ✓ plots y coordinate ✓ plots a dotted line segment to 7 on real axis

Q6 (5, 2 & 2 = 9 marks)

a) Solve $z^6 = 2 + 2\sqrt{3}i$ in polar form with principal arguments.

Solution