# Complex Numbers

Questions are taken from VCE Secondary Papers

## 2009

### Question 1

Find all solutions to the equation $z^4 - z^2 - 6 = 0,\ z \in C$.

### Question 4

Given that $\cos(2\theta) = \frac{3}{4}$ where $\theta \in \left(\frac{3\pi}{4}, \pi\right)$, find $\text{cis}(\theta)$ in cartesian form.

## Answers

1. $z = \pm\sqrt{3}, \pm\sqrt{2}\,i$

4. $\text{cis}(\theta) = -\frac{\sqrt{7}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}i$

### Question 2

In the complex plane, $L$ is the line with equation $|z-1| = \left|z - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right|$.

a. Verify that the point (0, 0) lies on $L$.

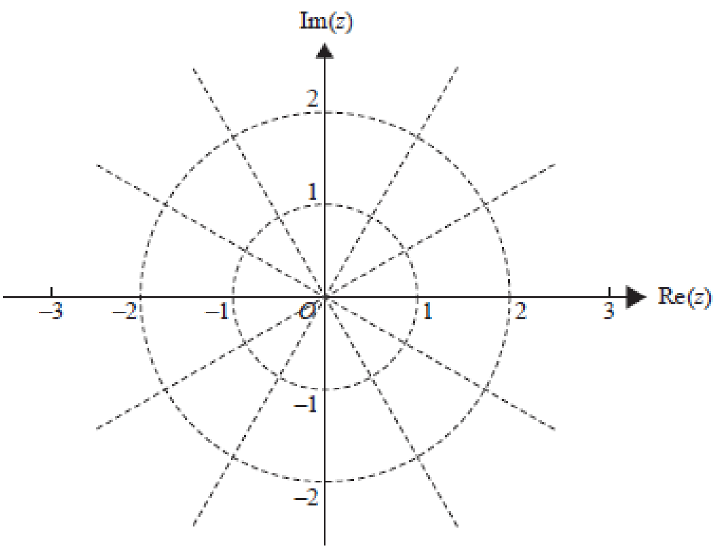
b. Show that the cartesian equation of $L$ is given by $y = \frac{1}{\sqrt{3}}x$.

The equation of the part of $L$ in the third quadrant of the complex plane can be written in the form $\text{Arg}(z) = \alpha$.

c. Write down the value of $\alpha$.

d. Find, in cartesian form, the point(s) of intersection of $L$ and the graph of $|z| = 2$.

e. Sketch $L$ and the graph of $|z| = 2$ on the argand diagram below.



**f.** Find the area in the first quadrant that is enclosed by $L$ and the graphs of $|z| = 2$, $|z| = 1$ and $\text{Arg}(z) = \frac{\pi}{3}$.

## Answers

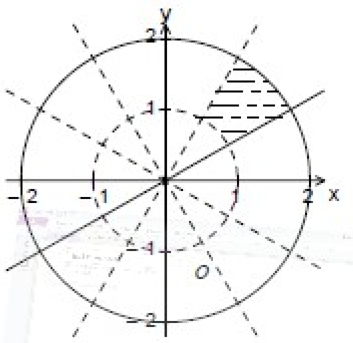
2. a. When (0, 0) is substituted, left side of the equation = $|-1| = 1$, right side of the equation = $\left|-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

b. $(x-1)^2 + y^2 = \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{\sqrt{3}}{2}\right)^2$, $x^2 - 2x + 1 + y^2 = x^2 - x + \frac{1}{4} + y^2 - \sqrt{3}y + \frac{3}{4}$, $-x = -\sqrt{3}y$ which gives the required result, $y = \frac{1}{\sqrt{3}}x$.

c. $\alpha = -\frac{5\pi}{6}$.

d. $x^2 + y^2 = 2^2$, $y = \frac{1}{\sqrt{3}}x$, solves to give $(\sqrt{3}, 1), (-\sqrt{3}, -1)$ as the coordinates of the points of intersection.

e.




f. $\text{Area} = \frac{1}{12} \times \left(\pi \times 2^2 - \pi \times 1^2\right) = \frac{\pi}{4}$

## 2008

**Question 10**

Let $w = 1 + ai$ where $a$ is a real constant.

**a.** Show that $|w^3| = (1 + a^2)^{\frac{3}{2}}$.

**b.** Find the values of $a$ for which $|w^3| = 8$.

**c.** Let $p(z) = z^3 + bz^2 + cz + d$ where $b$, $c$ and $d$ are non-zero real constants. If $p(z) = 0$ for $z = w$ and all roots of $p(z) = 0$ satisfy $|z^3| = 8$, find the values of $b$, $c$ and $d$ and show that these are the only possible values.

## Answers

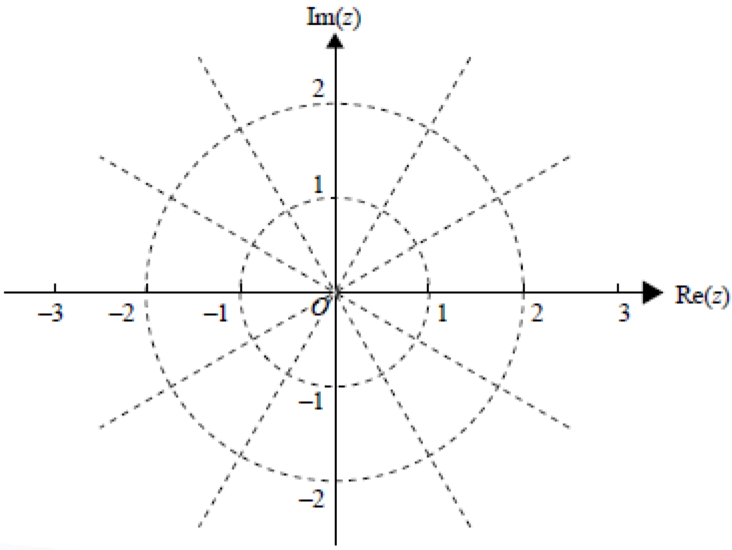
10. b. $\pm\sqrt{3}$

c. $b = -4, c = 8, d = -8$ or $p(z) = z^3 - 4z^2 + 8z - 8$

**Question 5**

**a.** Verify that $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ is one root of the equation $z^3 = i$.

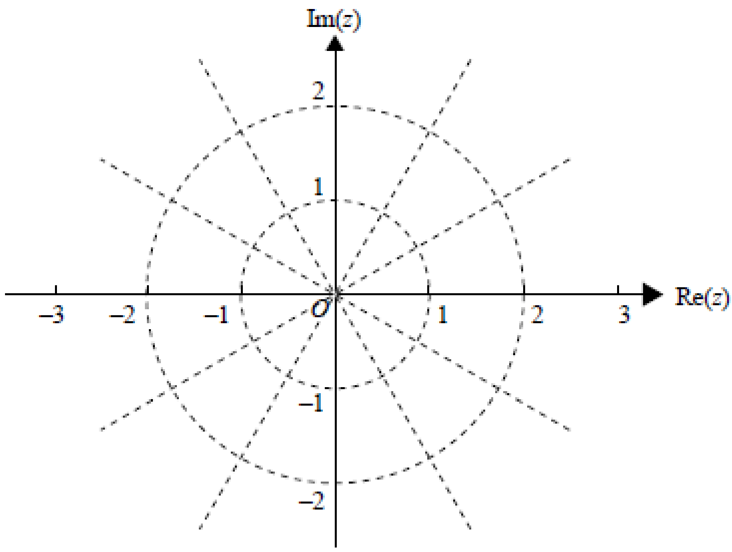
**b.** Plot the three roots of $i$ on the argand diagram below.




**c.** Find the points of intersection of the curves given by

$$|z-i|=1 \text{ and } \operatorname{Re}(z) = -\frac{1}{\sqrt{3}}\operatorname{Im}(z).$$

**d.** Sketch the curves given by the relations $|z-i|=1$ and $\operatorname{Re}(z) = -\frac{1}{\sqrt{3}}\operatorname{Im}(z)$ on the argand diagram below.




**e.** On the argand diagram above shade the region given by

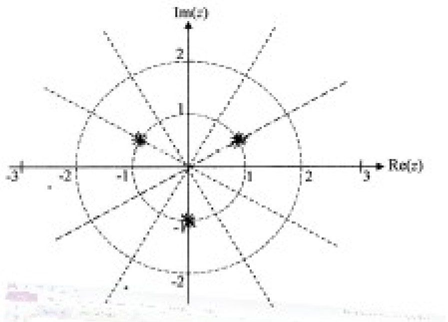
$$\{z:|z-i|\le 1\}\cap\left\{z:0\le \operatorname{Arg}(z)\le\frac{2\pi}{3}\right\}.$$

**f.** Find the area of the shaded region in part **e.**, correct to two decimal places.

## Answers

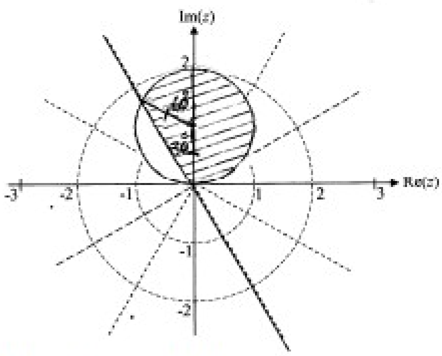
5. a. $\frac{\sqrt{3}}{2}+\frac{1}{2}i=\text{cis}\left(\frac{\pi}{6}\right)$, $\left(\text{cis}\left(\frac{\pi}{6}\right)\right)^3=\text{cis}\left(\frac{3\pi}{6}\right)=i$

b.




c. $(0,0)$, $\left(-\frac{\sqrt{3}}{2},\frac{3}{2}\right)$ or equivalent complex numbers.

d.




Circle and straight line

e. Shaded region on the diagram above

f. 2.53

## 2007

**Question 1**

Express $\frac{2\sqrt{3}+2i}{1-\sqrt{3}i}$ in polar form.

**Question 2**

**a.** Show that $\sqrt{5}-i$ is a solution of the equation $z^3-(\sqrt{5}-i)z^2+4z-4\sqrt{5}+4i=0$.

**b.** Find all other solutions of the equation $z^3-(\sqrt{5}-i)z^2+4z-4\sqrt{5}+4i=0$.

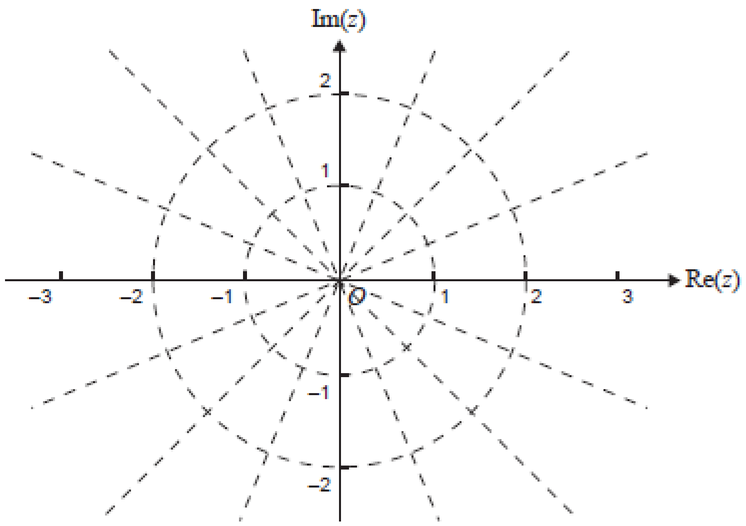
## Answers

1. $2\text{cis}\left(\frac{\pi}{2}\right)$ 2. b. $z=\pm 2i$

## 2006

**Question 5**

**a.** **i.** Let $z_1=\text{cis}\left(\frac{\pi}{4}\right)$. Plot and label carefully the points $-z_1$, $\overline{z_1}$ and $-\overline{z_1}$ on the Argand diagram below.




**ii.** Write down the complex equation of the straight line which passes through the points $z_1$ and $-z_1$, in terms of $\overline{z_1}$.

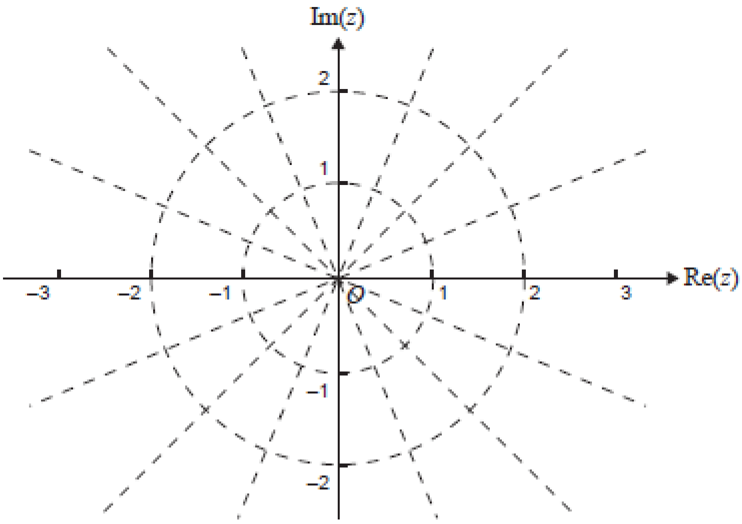
**b.** Use a double angle formula to show that the exact value of $\cos\left(\frac{\pi}{8}\right)=\frac{\sqrt{2+\sqrt{2}}}{2}$.
Explain why any values are rejected.

**c.** **Hence** show that the exact value of $\sin\left(\frac{\pi}{8}\right)=\frac{\sqrt{2-\sqrt{2}}}{2}$.

**d.** Evaluate $\left(\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}i\right)^7$, giving your answer in polar form.

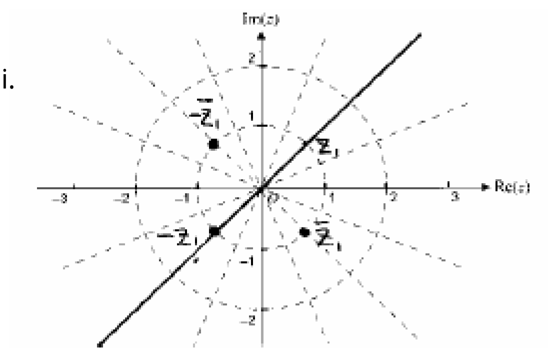
**e.** For what values of $n$ is $\left(\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}i\right)^n$ a real number?

**f.** Plot the roots of $z^8 = 1$ on the Argand diagram below.




## Answers

5. a. i.




ii. $|z - \overline{z}_1| = |z + \overline{z}_1|$

b. Solving the equation $\cos\left(\frac{\pi}{4}\right) = 2\cos^2\left(\frac{\pi}{8}\right) - 1$ for $\cos\left(\frac{\pi}{8}\right)$ gave the stated result.

c. $\sin^2\left(\frac{\pi}{8}\right) = 1 - \cos^2\left(\frac{\pi}{8}\right)$ yields the stated result.

d. $\text{cis}\left(\frac{7\pi}{8}\right)$

e. $n = 8k,\ k \in Z$ (the set of integers)

f.