PERTH MODERN SCHOOL
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Semester Two Examination, 2021

Question/Answer booklet

SPECIALIST MATHEMATICS
UNITs 3&4
Section Two:
Calculator-assumed

Your Name

Your Teacher's Name

Time allowed for this section
Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Marks	Max	Question	Marks	Max
9			16		
10			17		
11			18		
12			19		
13			20		
14			21		

15			

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	50	35
Section Two: Calculator-assumed	13	13	100	101	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **(101 Marks)**

This section has **13** questions. Answer **all** questions. Write your answers in the spaces provided.

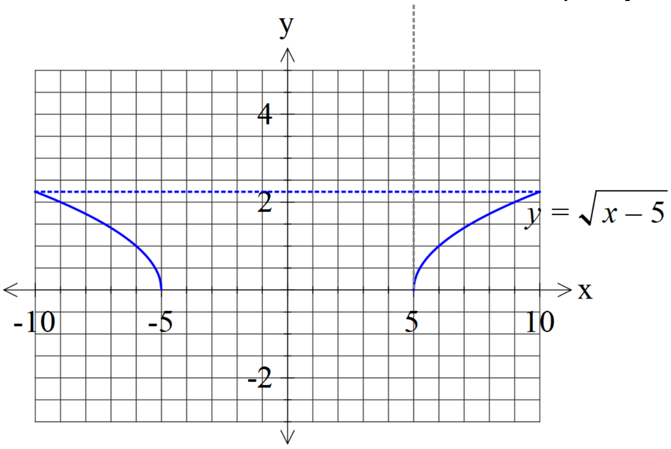
Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

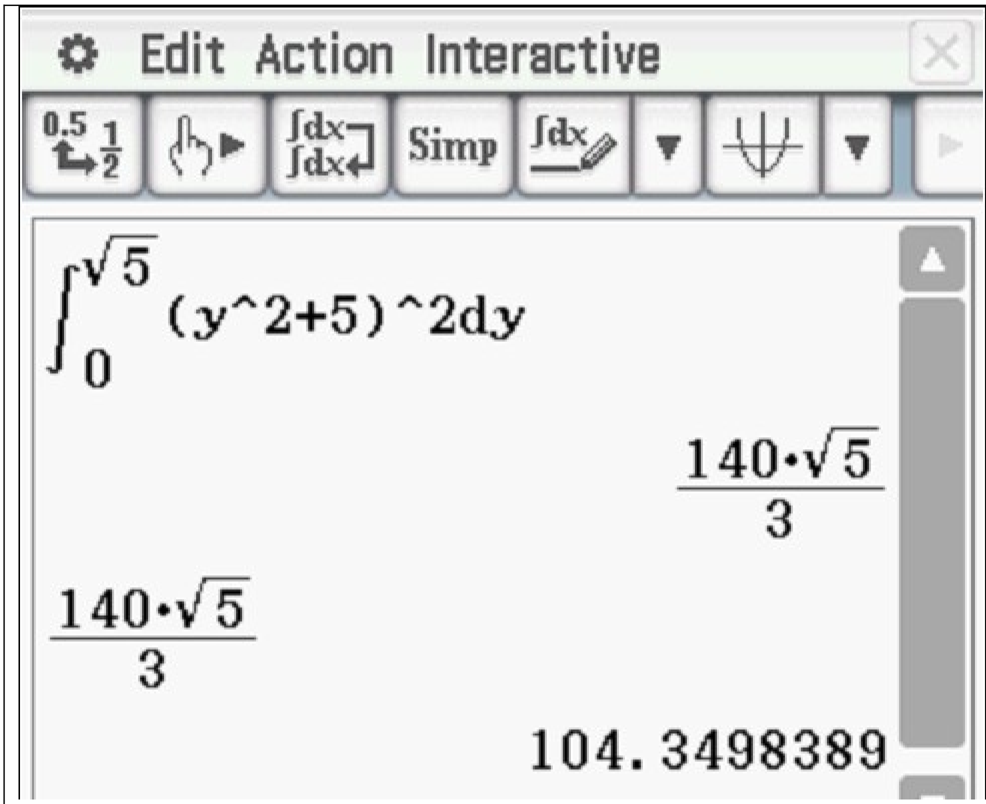
Working time: 100 minutes.

Question 9 **(4 marks)**

A glass bowl is formed by rotating the curve $y = \sqrt{x - 5}$ cm from $5 \le x \le 10$ cm about the y axis as seen below. Determine the maximum capacity in litres given that $1cm^3 \equiv 1ml$.



Solution
$\int_0^{\sqrt{5}} \pi x^2 dy$ $\int_0^{\sqrt{5}} \pi (y^2 + 5)^2 dy$

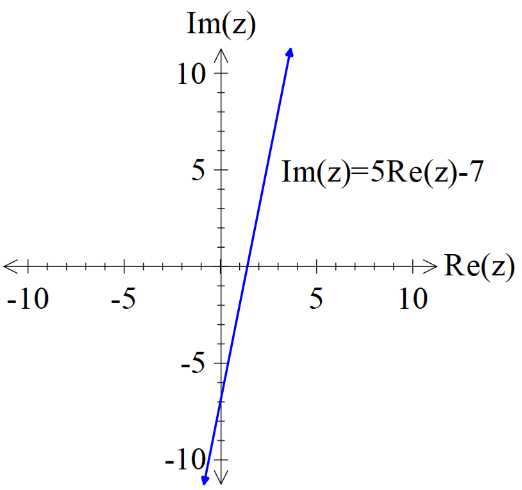


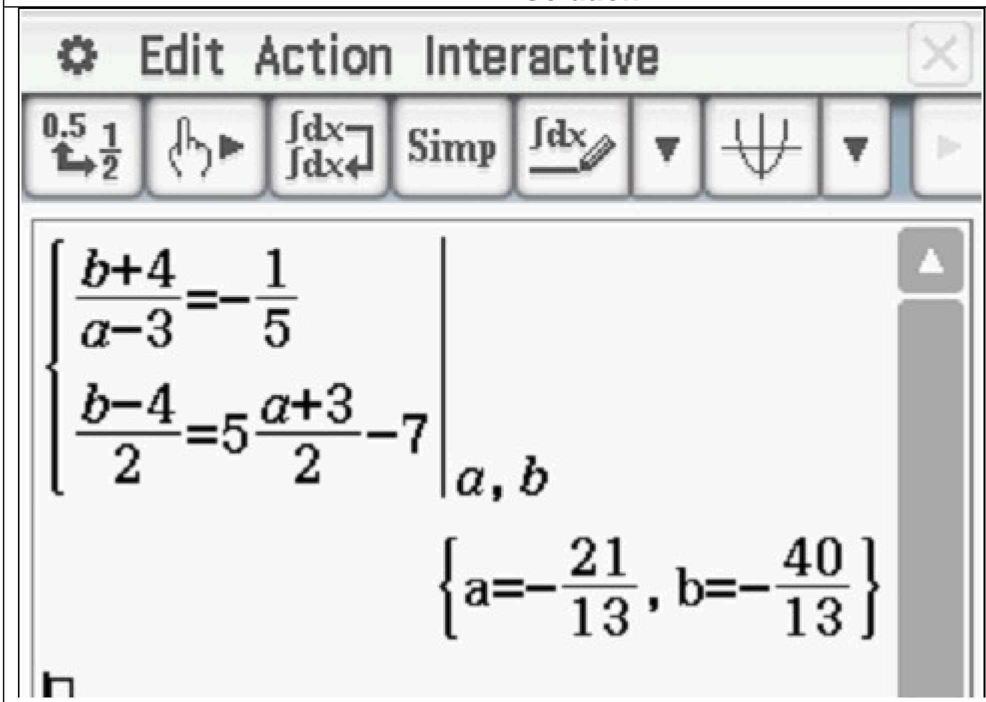
Volume- 327.8 cubic cm
Capacity =327.8 ml

Specific behaviours
✓ uses correct formula ✓ writes a correct definite integral ✓ determines volume ✓ states capacity with units(ml or litres only)

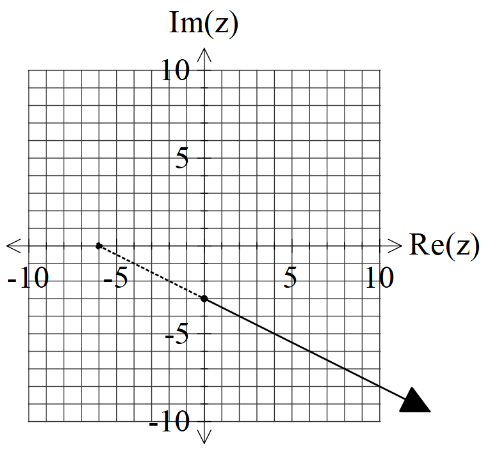
Question 10 **(7 marks)**

a) Consider the locus $|z-3+4i|=|z-a-bi|$ where $a\,\&\,b$ are real constants. See diagram below. Given that this locus is also given by $\text{Im}(z)=5\text{Re}(z)-7$, determine the exact values of $a\,\&\,b$ and plot this point on the axes below. (4 marks)



Solution

Specific behaviours
✓ sets up one equation ✓ sets up two equations ✓ solves for a ✓ solves for b

b) Sketch the locus $|z+6| = 3\sqrt{5} + |z+3i|$ on the axes below. (3 marks)

Solution

Specific behaviours
✓ plots pivotal pts on both axes ✓ dotted line or no line for Re(z)<0 ✓ correct line for Re(z)>0

Question 11 **(4 marks)**

Consider the following complex numbers.

$$|p| = 7 \quad Arg(p) = -\frac{\pi}{4}$$

$$q = \frac{1-\sqrt{3}i}{(1-i)w} \quad q = \frac{5}{p}$$

Determine w in cartesian form.

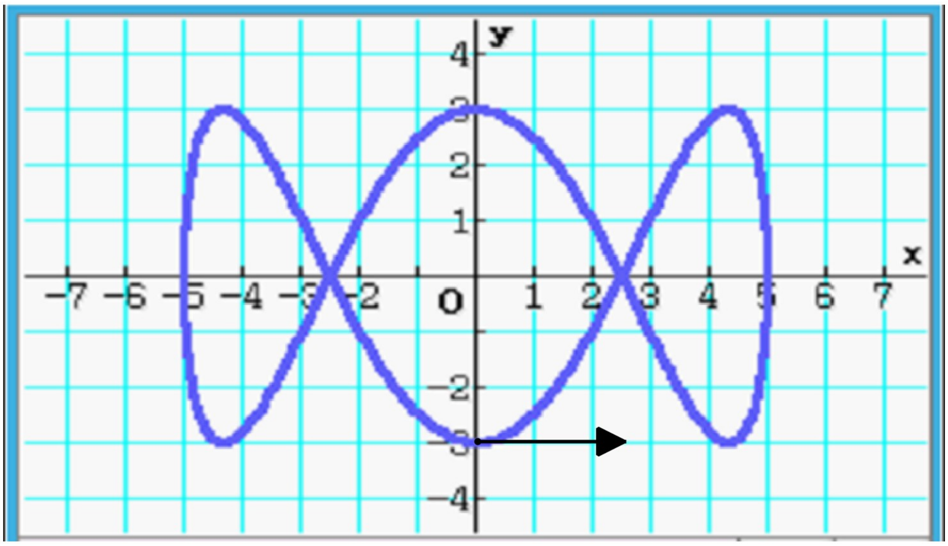
Solution
$p = 7cis\left(-\frac{\pi}{4}\right)$ $\lvert q\rvert = \frac{5}{7} \ldots Arg(q) = \frac{\pi}{4}$ $\lvert q\rvert = \frac{2}{\sqrt{2}\lvert w\rvert} \ldots \lvert w\rvert = \frac{14}{5\sqrt{2}} = \frac{7\sqrt{2}}{5}$ $\frac{\pi}{4} = \frac{-\pi}{3} + \frac{\pi}{4} - Arg(w)$ $Arg(w) = \frac{-\pi}{3}$ $w = \frac{7\sqrt{2}}{5} cis\left(\frac{-\pi}{3}\right) = \frac{7\sqrt{2}}{5}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
Specific behaviours
✓ determines arg of q ✓ determines arg of w ✓ gives w in polar form ✓ gives w in cartesian form

Question 12 **(11 marks)**

Consider a racing car that follows the following path on a surface.

The car's position vector is given by $r = \begin{pmatrix} 5\sin\left(\frac{t}{3}\right) \\ -3\cos(t) \end{pmatrix} km$ at time t hours.

a) Determine the initial velocity and position and mark the direction on the diagram above. (4 marks)

Solution
$r = \begin{pmatrix} 5\sin\left(\frac{t}{3}\right) \\ -3\cos(t) \end{pmatrix}$ $\dot{r} = \begin{pmatrix} \frac{5}{3}\cos\left(\frac{t}{3}\right) \\ 3\sin(t) \end{pmatrix}$ $\dot{r}(0) = \begin{pmatrix} \frac{5}{3} \\ 0 \end{pmatrix} \ldots r(0) = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$ 
Specific behaviours
✓ diff to find velocity   ✓ states initial velocity   ✓ states initial position   ✓ shows initial position and direction on diagram

b) Determine the time taken to complete one circuit. (hours) (2 marks)

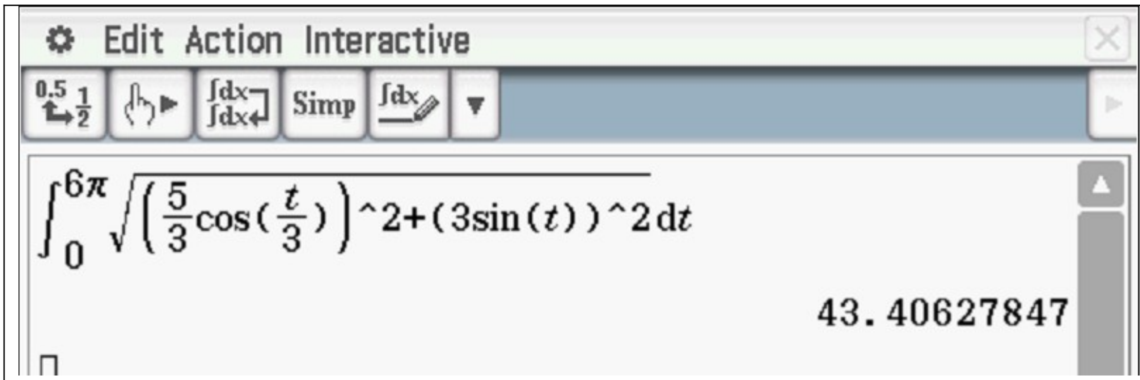
Solution
$LCM: 6\pi \; \& \; 2\pi = 6\pi$
Specific behaviours
✓ states periods in each direction ✓ states LCM (no need for units)

c) Determine the initial acceleration. (2 marks)

Solution
$r = \begin{pmatrix} 5\sin\left(\frac{t}{3}\right) \\ -3\cos(t) \end{pmatrix}$ $\dot{r} = \begin{pmatrix} \frac{5}{3}\cos\left(\frac{t}{3}\right) \\ 3\sin(t) \end{pmatrix}$ $\ddot{r} = \begin{pmatrix} \frac{-5}{9}\sin\left(\frac{t}{3}\right) \\ 3\cos(t) \end{pmatrix}$ $\ddot{r}(0) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$
Specific behaviours
✓ diff velocity ✓ states acceleration

d) Determine to the nearest metre the distance travelled in one circuit. (3 marks)

Solution



Distance = 43 406 metres

Specific behaviours
✓ uses the magnitude of velocity ✓ integrates with correct limits ✓ rounds to nearest metre with units

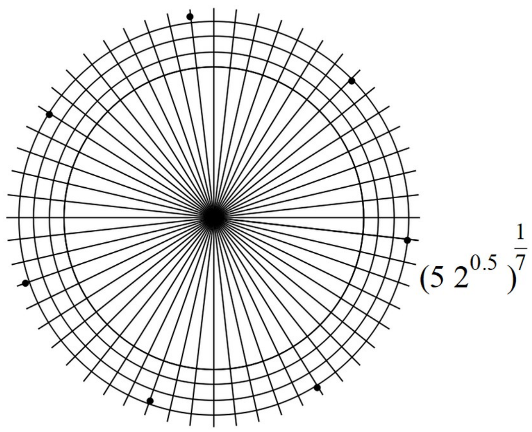
Question 13 (7 marks)

a) Determine the solutions to $z^7 = 5 - 5i$ in the form $z = r\,cis\theta$ with $-\pi < \theta \le \pi$. (4 marks)

Solution
$z^7 = 5 - 5i = 5\sqrt{2}cis\left(\frac{-\pi}{4} + 2n\pi\right) \quad n = 0, \pm1, \pm2...$ $z = \left(5\sqrt{2}\right)^{\frac{1}{7}} cis\left(\frac{-\pi}{28} + \frac{8}{28}n\pi\right)$ $z_1 = \left(5\sqrt{2}\right)^{\frac{1}{7}} cis\left(\frac{-\pi}{28}\right)$ $z_2 = \left(5\sqrt{2}\right)^{\frac{1}{7}} cis\left(\frac{7\pi}{28}\right)$ $z_3 = \left(5\sqrt{2}\right)^{\frac{1}{7}} cis\left(\frac{-9\pi}{28}\right)$ $z_4 = \left(5\sqrt{2}\right)^{\frac{1}{7}} cis\left(\frac{15\pi}{28}\right)$ $z_5 = \left(5\sqrt{2}\right)^{\frac{1}{7}} cis\left(\frac{-17\pi}{28}\right)$ $z_6 = \left(5\sqrt{2}\right)^{\frac{1}{7}} cis\left(\frac{23\pi}{28}\right)$ $z_7 = \left(5\sqrt{2}\right)^{\frac{1}{7}} cis\left(\frac{-25\pi}{28}\right)$
Specific behaviours

✓ uses De Moivre's ✓ uses correct modulus ✓ uses correct arguments ✓ uses principal argument with seven roots only

b) Plot these solutions on the axes below. (3 marks)

Solution
 v^2=
Specific behaviours
✓ states scale ✓ equally spaced ✓ correct positions

Question 14 (7 marks)

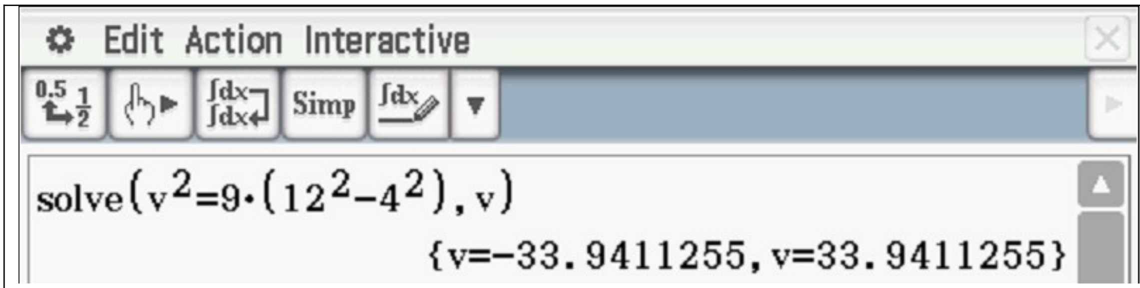
A particle moves in a straight line with the displacement from the origin, x metres satisfies the following differential equation at time t seconds.

$$\ddot{x} = -9x$$

The particle is a rest at $x = 12$ metres.

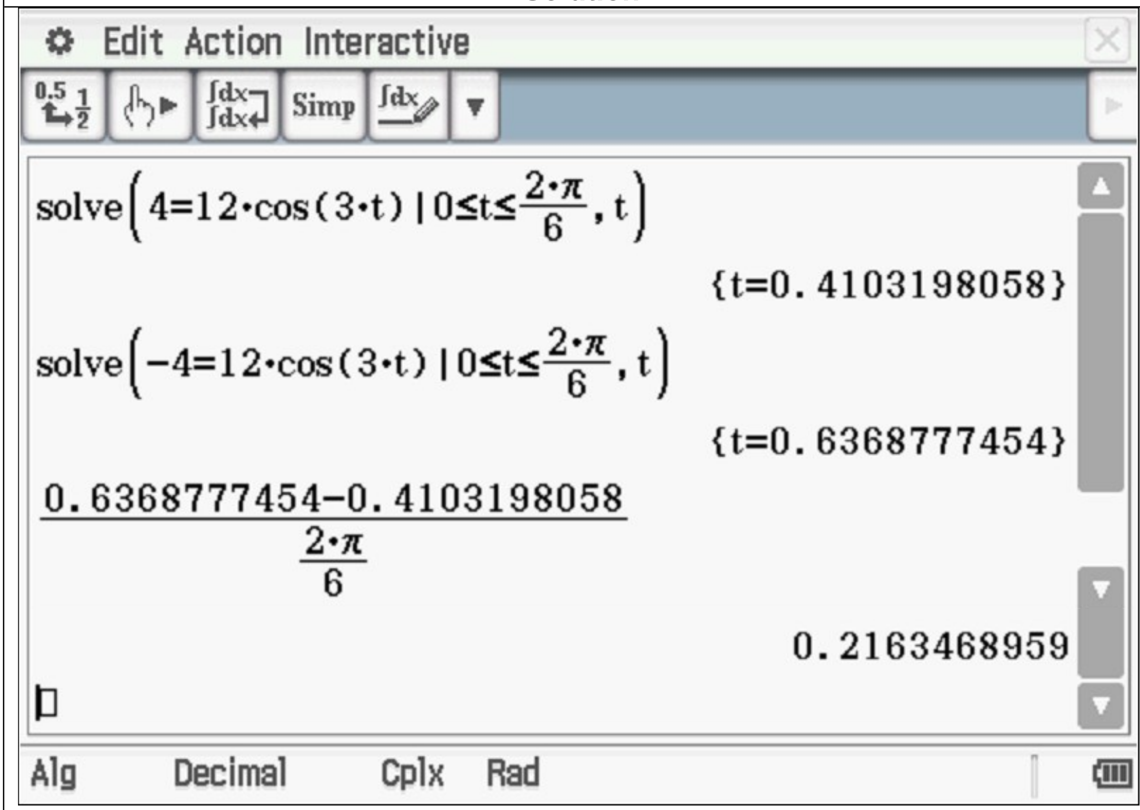
a) Determine the speed when $x = 4$ metres. (3 marks)

Solution
$v^2 = n^2\left(A^2 - x^2\right)$



Specific behaviours
✓ uses correct formula ✓ uses correct values for n & A ✓ states one positive value

b) Determine the percentage of the time that the object is less than 4 metres from the origin. (4 marks)

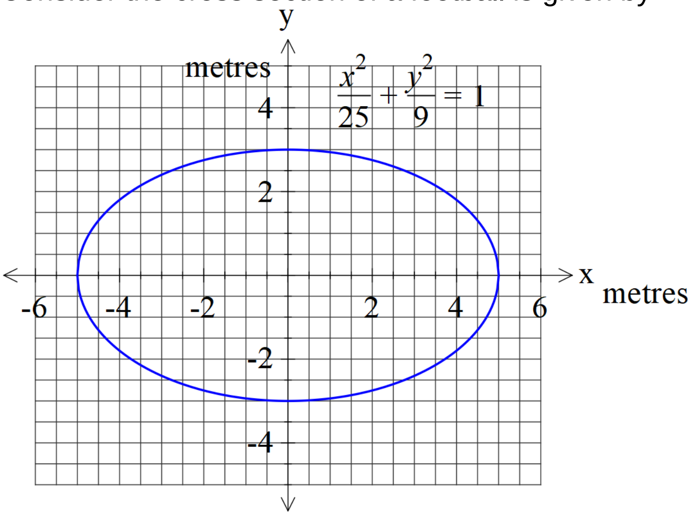
Solution


Specific behaviours
✓ solves for one time when x=4 ✓ solves for one time when x=-4 OR second time for x=4

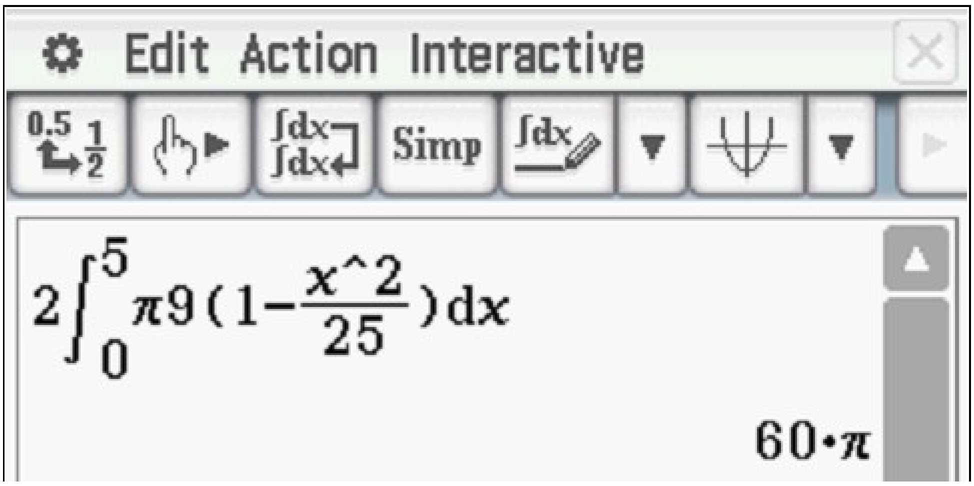
✓ determines interval
✓ divides by cycle length or part thereof for percentage

Question 15 **(4 marks)**

Consider the cross section of a football is given by $\frac{x^2}{25}+\frac{y^2}{9}=1$. See diagram below.



If the curve above is revolved around the x axis forming a 3-D football, determine the exact volume in cubic metres.

Solution
$2\int_0^5 \pi y^2 dx$ 
Specific behaviours
✓ uses volume of revolution
✓ uses correct limits
✓ writes an exact integral

✓ states exact volume (no need for units)

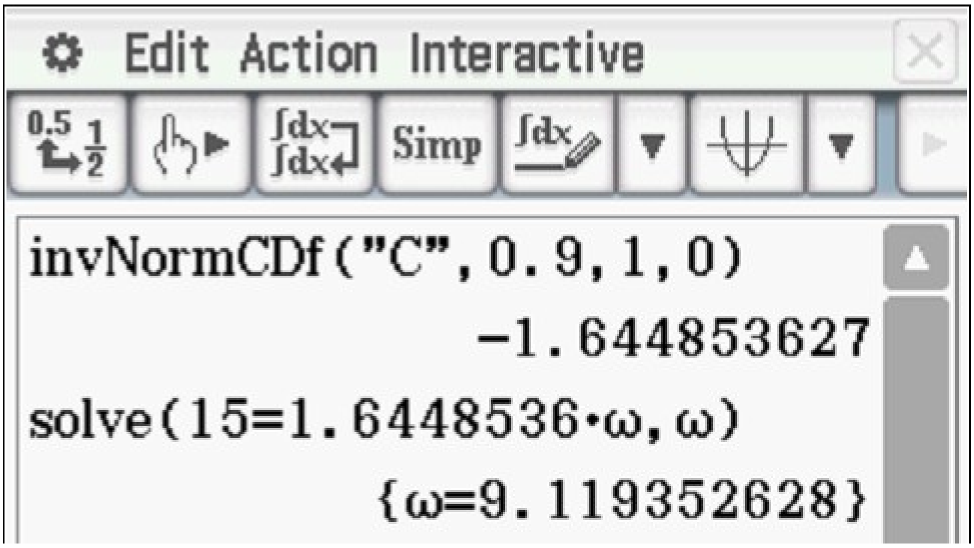
Question 16 **(7 marks)**

Car manufacturer Subaru makes engines for their BRZ sports car with μ equaling the population mean engine power in kilowatts for the engine and σ being the population standard deviation.
A sample of engines was examined and a 90% confidence interval for μ was given as $260 < \mu < 290$ kilowatts.

a) Determine the sample mean for this confidence interval. (1 mark)

Solution
275 kilowatts
Specific behaviours
✓ states midpoint

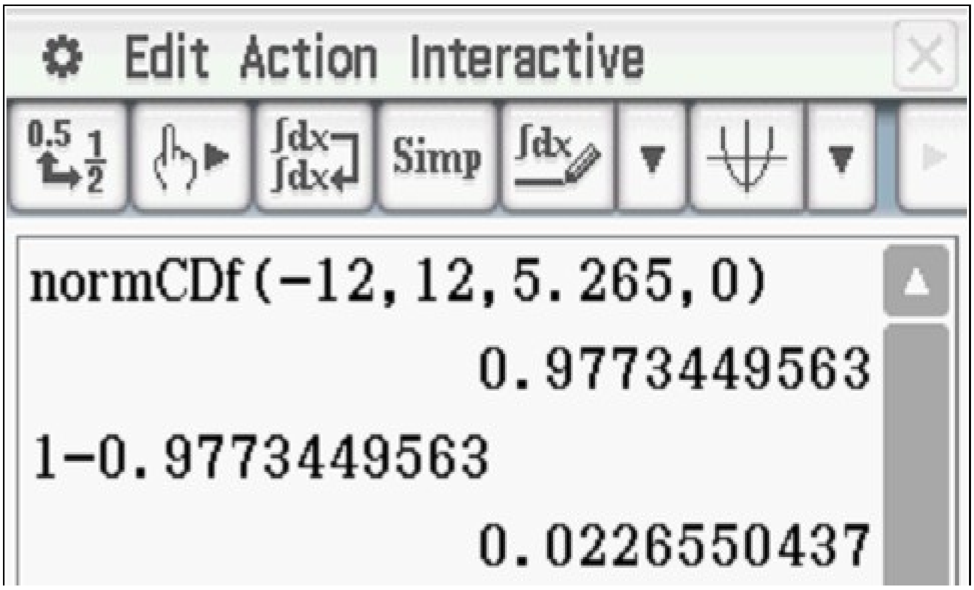
b) Determine the sample mean standard deviation for this confidence interval. (2 marks)

Solution

Specific behaviours
✓ determines z score ✓ states sample mean st dev

Another sample of engines was taken but this time the sample size is tripled.

c) Determine the probability that the sample mean of this larger sample will differ from μ by more than 12 kilowatts. (4 marks)

Solution
Change of origin let $\mu = 0$ Sample mean st dev = $\dfrac{9.119}{\sqrt{3}} \approx 5.265$



Specific behaviours
✓ determines new sample mean stdev
✓ uses change of origin (u=0)
✓ uses correct parameters for prob
✓ states final prob to at least 3 dp

Question 17 (11 marks)

A new species of tomato Type X has a weight that is normally distributed with mean u = 37.2 grams and standard deviation σ =11.9 grams.

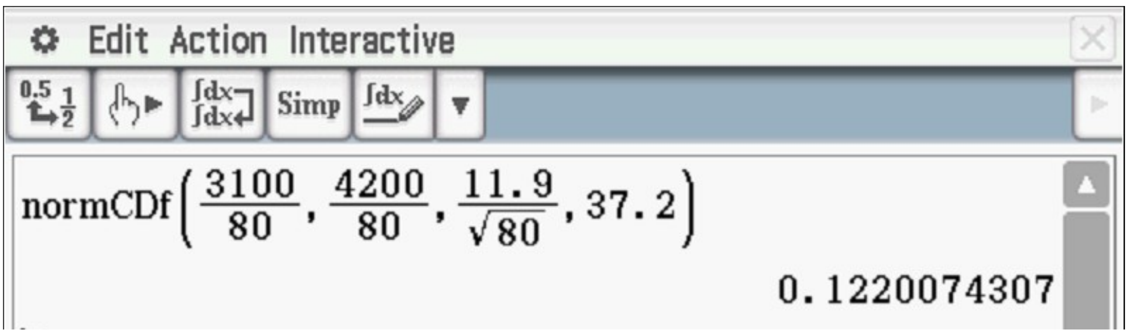
a) Determine the probability that a bunch of 80 Type X tomatoes will weigh between 3.1 kg and 4.2 kg. (4 marks)

Solution

Sample size = 80

$$\bar{X} \sim N\left(37.2.\left[\frac{11.9}{\sqrt{80}}\right]^2\right)$$

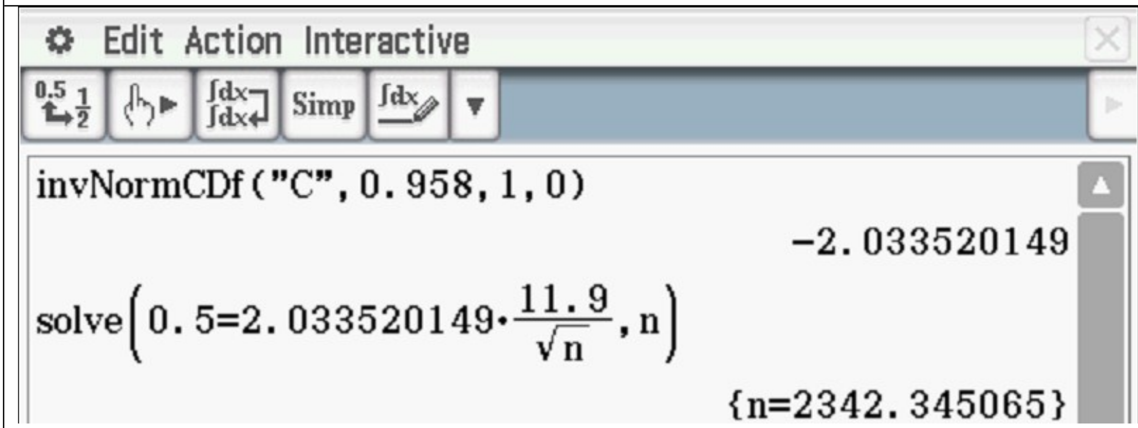
$$P\left(\frac{3100}{80} < \bar{X} < \frac{4200}{80}\right)$$



Specific behaviours

✓ defines sample mean distribution as Normal with mean
✓ determines limits for sample mean variable
✓ determines sample mean st dev
✓ determines prob to at least 3 dp

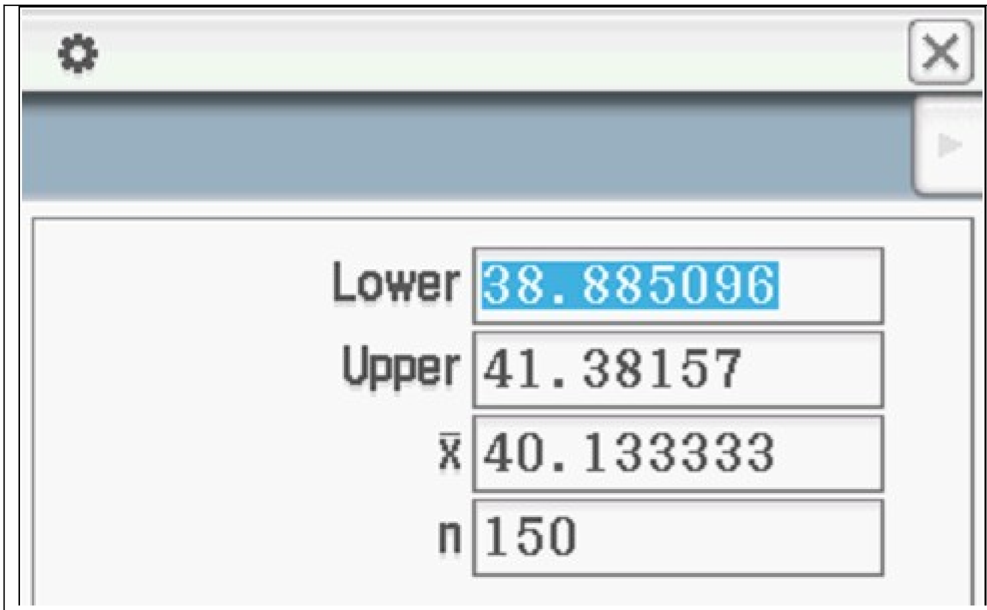
b) If the probability that a new sample of Type X tomatoes has a mean weight that differs from u by more than 0.5 grams is 4.2%, determine the sample size n. (3 marks)

Solution

Sample size = 2343
Specific behaviours
✓ determines z score
✓ sets up equation for n
✓ solves for n and rounds up

A rival species of tomato Type Y has a standard deviation of 7.8 grams (one tomato).
A bunch of 150 Type Y tomatoes has a weight of 6.02 Kg. The people who produce Type Y tomatoes claim that their tomatoes are heavier than Type X tomatoes.

c) Show calculations that would allow better comment on which tomato is heavier. (4 marks)

Solution
95 % confidence interval for type Y
$\overline{Y} = \frac{6020}{150} \approx 40.13$
$\overline{Y} \sim N\left(40.13, \left[\frac{7.8}{\sqrt{150}}\right]^2\right)$

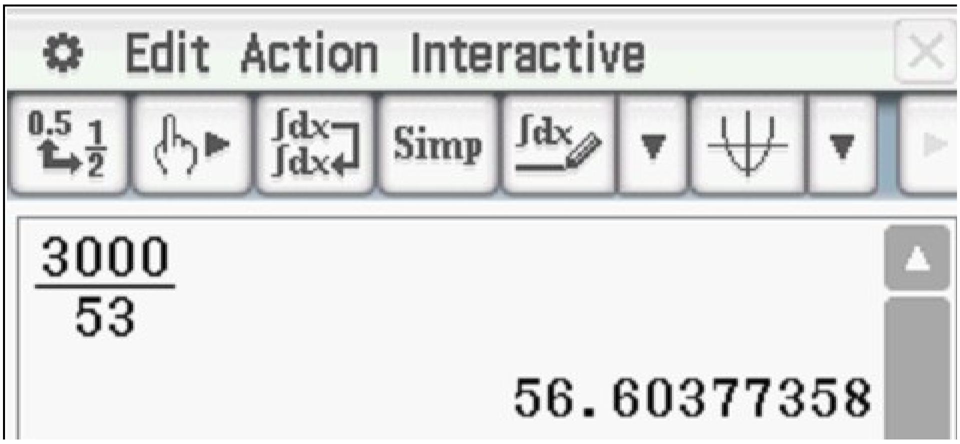


Specific behaviours
✓ states normal distribution with new sample mean ✓ shows calculation for new standard deviation ✓ determines an appropriate confidence interval ✓ **must show that old pop mean does not fit in interval** and hence Y is heavier OR ✓ Must state that not every interval contains the true value of pop mean and therefore no inference can be made.

Question 18 **(14 marks)**

The number of algae, N thousands, in a habitat at time t days is given by $N = \dfrac{3000}{1+52e^{-0.28t}}$.

a) Determine the initial number of algae. (2 marks)

Solution
 Initial = 56.6 thousand or 56604
Specific behaviours
✓ subs t=0 ✓ states with units

b) Determine the limiting number of algae after many decades. (2 marks)

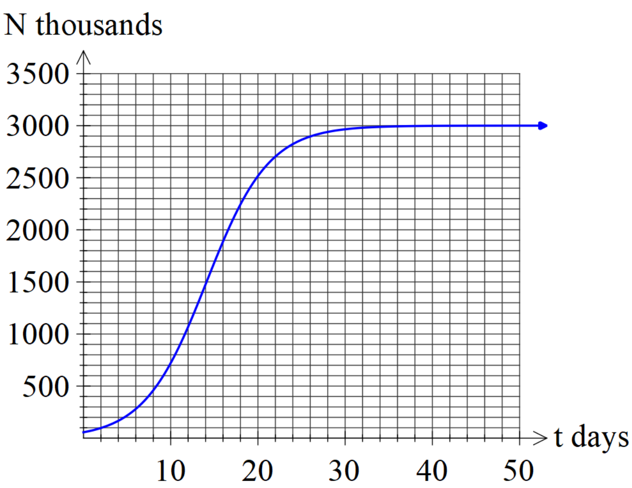
Solution
3000 thousand OR 3,000,000
Specific behaviours
✓ exponential term ignored ✓ states with units

c) Express the rate of growth in the form $\dfrac{dN}{dt} = rN(k - N)$ stating the values of the constants r & k. (2 marks)

Solution

$k = 3000$ $rk = 0.28$ $r = \frac{7}{75000}$ $\frac{dN}{dt} = \frac{7}{75000} N(3000 - N)$
Specific behaviours
✓ determines k ✓ determines r and writes differential equation with known values

d) Sketch the graph of $N \& t$ on the axes below and explain what is happening. (4 marks)

Solution

Specific behaviours
✓ concave up until around t around 14 days (Or rate of growth accelerating) ✓ inflection pt around 15 days OR peak rate of growth ✓ P concave down after 14 days OR rate of growth decelerating ✓ horizontal, asymptote at N=30000

e) If the rate of growth was given by $\frac{dN}{dt} = aN - bN^2$ where $a \& b$ are positive constants, show using integration and partial fractions how to derive $N = \frac{a}{b + Ce^{-at}}$ with constant C. (4 marks)

Solution

$$\frac{dN}{dt} = aN - bN^2 = N(a - bN)$$

$$\frac{dN}{dt} = 0 \cdots a - bN = 0 \cdots N = \frac{a}{b} \cdots N < \frac{a}{b}$$

$$\int \frac{dN}{N(a - bN)} = \int dt$$

$$\frac{1}{N(a - bN)} = \frac{d}{N} + \frac{e}{a - bN}$$

$$1 = d(a - bN) + eN$$

$$N = 0$$

$$1 = da \cdots d = \frac{1}{a}$$

$$N = \frac{a}{b}$$

$$1 = e\frac{a}{b} \cdots e = \frac{b}{a}$$

$$\int \frac{1/a}{N} + \frac{b/a}{a - bN}\,dN = t + c$$

$$\ln|N| - \ln|a - bN| = at + c \cdots N < \frac{a}{b} \cdots a - bN > 0$$

$$\ln\frac{N}{a - bN} = at + c$$

$$\frac{N}{a - bN} = Ce^{at}$$

$$\frac{a - bN}{N} = Ce^{-at}$$

$$a - bN = Ce^{-at}$$

$$N = \frac{a}{b + Ce^{-at}}$$

Specific behaviours

✓ separates variables and uses partial fractions

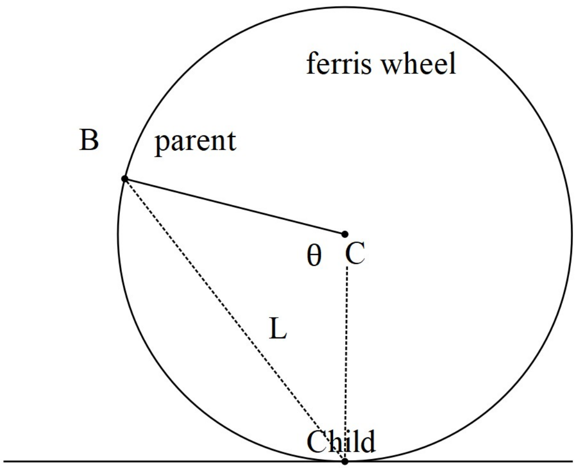
✓ shows how to find constants for partial fractions

✓ integrates and explains why absolute values not needed using limiting values

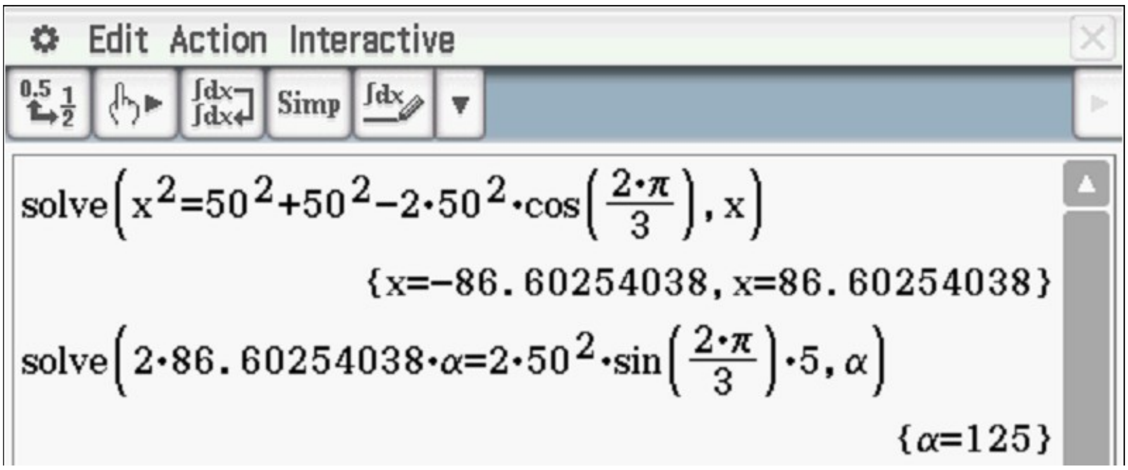
✓ rearranges index form into required rule

Question 19 (8 marks)

Consider a parent riding on a Ferris wheel looking down at her child who is left at the entrance to the Ferris wheel. Assume that the Ferris wheel moves with constant angular speed, $\frac{d\theta}{dt}=5$ rads/sec, and a radius of 50 metres. Let the distance of direct eye contact from parent to child be represented as L metres.



a) Determine $\frac{dL}{dt}$ when $\theta=\frac{2\pi}{3}$. (4 marks)

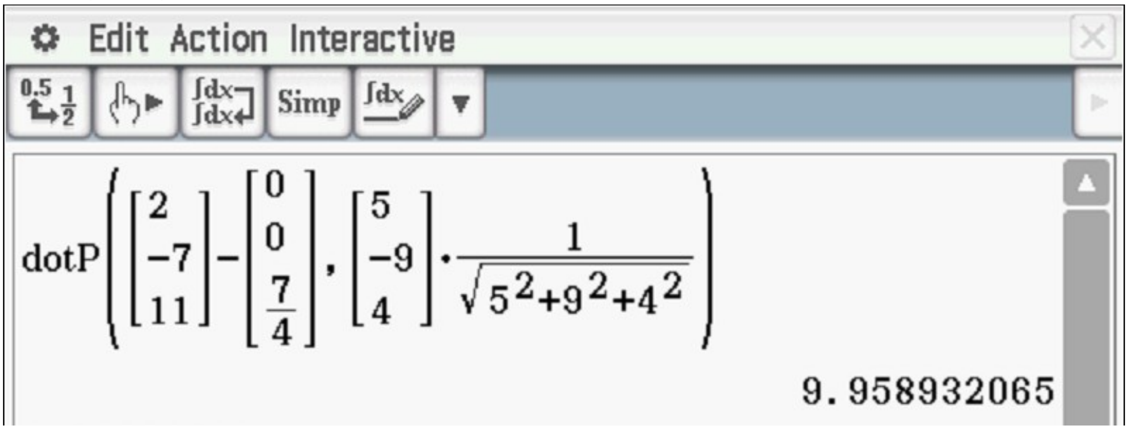
Solution
$L^2=50^2+50^2-2(50^2)\cos\theta$ $2L\dot{L}=2(50^2)\sin\theta(\dot{\theta})$ 
Specific behaviours
✓ determines L ✓ uses implicit diff ✓ sets up equation for derivative ✓ solves for derivative

b) Determine the $\frac{d^2L}{dt^2}$ when $\theta = \frac{2\pi}{3}$. (4 marks)

Solution
$2L\dot{L} = 2(50^2)\sin\theta(\dot{\theta})$ $2\dot{L}\dot{L} + 2L\ddot{L} = 2(50^2)\sin\theta(\ddot{\theta}) + 2(50^2)\cos\theta(\dot{\theta})(\dot{\theta})$ $\text{solve}\left(2\cdot 125^2 + 2\cdot 86.6025\cdot\beta = 2\cdot 50^2\cdot\cos\left(\frac{2\cdot\pi}{3}\right)\cdot 25, \beta\right)$ $\{\beta = -541.2661297\}$
Specific behaviours
✓ uses implicit to expression in (a) ✓ Left side correct ✓ Right side correct ✓ solves for second derivative

Question 20 **(8 marks)**

a) Determine the distance of Point A(2, -7, 11) to the plane $5x - 9y + 4z = 7$ showing full reasoning and working. (4 marks)

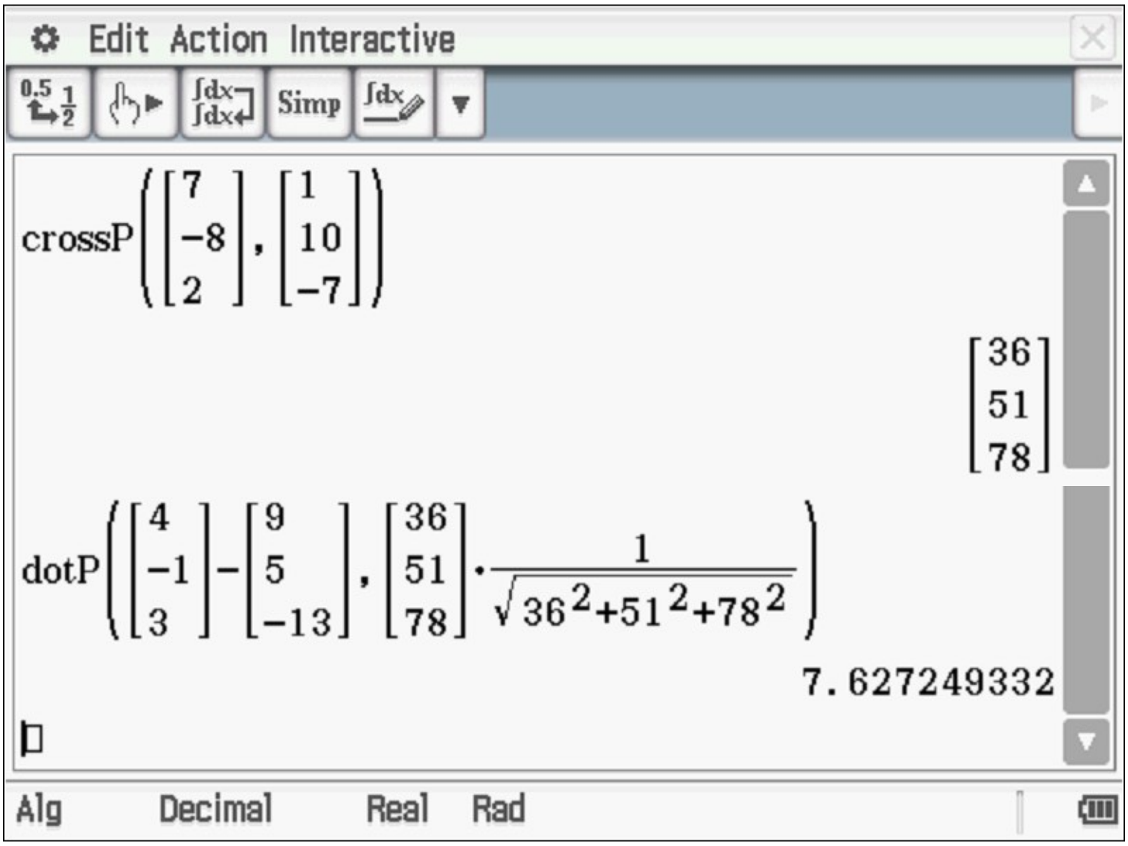
Solution
Point on plane B $\begin{pmatrix} 0 \\ 0 \\ \frac{7}{4} \end{pmatrix}$ $dist = \left\| AB.\hat{n} \right\|$ 

Specific behaviours
✓ determines any point on plane OR uses line ✓ uses dot product ✓ derives an expression for distance (Do not accept formula that is not derived ✓ determines approx. distance

b) Consider the lines below and determine minimal distance between them. (4 marks)

$$r_a = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -8 \\ 2 \end{pmatrix}$$

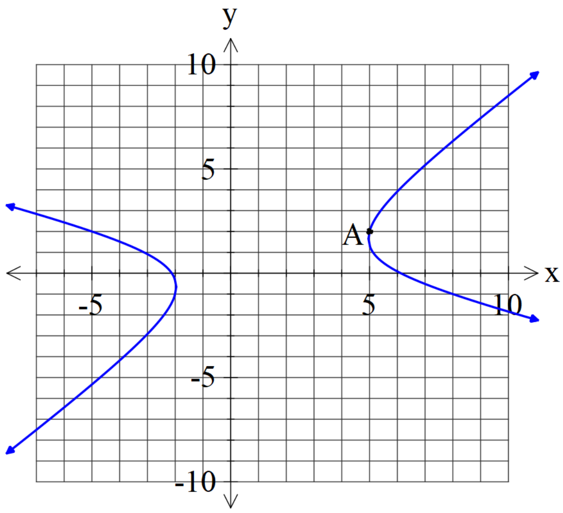
$$r_b = \begin{pmatrix} 9 \\ 5 \\ -13 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 10 \\ -7 \end{pmatrix}$$

Solution

Specific behaviours
✓ uses cross product ✓ determines vector between points on each plane ✓ uses dot product with unit normal (Do not accept formula not derived) ✓ determines approx. distance

Question 21 **(9 marks)**

Consider the locus defined by $3y^2 + 4x = x^2 + 2xy - 13$ which contains point A(5,2). See diagram below.

End of questions



a) Determine the equation of the tangent at point A. Show full reasoning and working without the use of a classpad. (4 marks)

Solution
$3y^2 + 4x = x^2 + 2xy - 13$ $6y\dot{y} + 4 = 2x + 2x\dot{y} + 2y$ $(6y - 2x)\dot{y} = 2(x + y) - 4$ $\dot{y} = \frac{2(x+y)-4}{(6y-2x)} = \frac{10}{2} = 5$ $y = 5x + c$ $2 = 25 + c$ $c = -23$ $y = 5x - 23$
Specific behaviours
✓ uses implicit diff ✓ left side correct ✓ right side correct and solves for derivative ✓ states equation of tangent

b) Determine $\frac{d^2y}{dx^2}$ at point A. Show full reasoning and working. (3 marks)

Solution

$(6y - 2x)\dot{y} = 2(x+y) - 4$ $(6y - 2x)\ddot{y} + (6\dot{y} - 2)\dot{y} = 2(1+\dot{y})$ $(2)\ddot{y} + (28)(5) = 12$ $\ddot{y} = -64$
Specific behaviours
✓ uses implicit diff of result above ✓ sets up equation for second derivative ✓ solves for second derivative

c) Determine the relationship between $x \,\&\, y$ at the points where the tangent is vertical. (2 marks).

Solution
$\dot{y} = \frac{2(x+y) - 4}{(6y - 2x)}$ $6y - 2x = 0$ $y = \frac{1}{3}x$
Specific behaviours
✓ uses denominator only ✓ gives un simplified relationship between x & y

(2

End of questions

Additional working space

Question number:

Additional working space

Question number:

Additional working space

Question number:

Acknowledgements

Published by the School Curriculum and Standards Authority of Western Australia
27 Walters Drive
OSBORNE PARK WA 6017