$(a+bi)^3 = a^3 + 3a^2bi + 3a(-b^2) - b^3i = a^3 - 3ab^2 + i(3a^2b - b^3)$ $= -13\sqrt{2} + \sqrt{5}i$ $\left|-13\sqrt{2} + \sqrt{5}i\right| = \left|a+bi\right|^3$ $\sqrt{169(2)+5} = \left(\sqrt{a^2+b^2}\right)^3$ $a^2 + b^2 = (343)^{\frac{1}{3}} = 7$
Specific behaviours
✓ expands cubic (no need to simplify) ✓ determines real and imaginary parts of z cubed ✓ rearranges to obtain expression of a squared plus b squared ✓ shows that 7 is cube root of 343 NOTE: any statement that is not supported receives zero marks)

PERTH MODERN SCHOOL
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Course Specialist Test 1 Year 12

Student name: ______________ Teacher name: ______________

Task type: **Response/Investigation**

Reading time for this test : 5 mins

Working time allowed for this task: 40 mins

Number of questions: **7**

Materials required: No cals allowed!!

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, NO notes allowed!

Marks available: **41 marks**

Task weighting: **13%**

Formula sheet provided: no, but formulae stated on page 2

Note: All part questions worth more than 2 marks require working to obtain full marks.

Useful formulae

Complex numbers

Cartesian form	
$z = a + bi$	$\bar{z} = a - bi$
$\text{Mod}(z) = \lvert z \rvert = \sqrt{a^2 + b^2} = r$	$\text{Arg}(z) = \theta, \quad \tan\theta = \dfrac{b}{a}, \quad -\pi < \theta \le \pi$
$\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \lvert z_2 \rvert$	$\left\lvert \dfrac{z_1}{z_2} \right\rvert = \dfrac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$
$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$	$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
$z\bar{z} = \lvert z \rvert^2$	$z^{-1} = \dfrac{1}{z} = \dfrac{\bar{z}}{\lvert z \rvert^2}$
$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$	$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
Polar form	
$z = a + bi = r(\cos\theta + i\sin\theta) = r\,cis\,\theta$	$\bar{z} = r\,cis\,(-\theta)$
$z_1 z_2 = r_1 r_2\,cis\,(\theta_1 + \theta_2)$	$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\,cis\,(\theta_1 - \theta_2)$
$cis(\theta_1 + \theta_2) = cis\,\theta_1\,cis\,\theta_2$	$cis(-\theta) = \dfrac{1}{cis\,\theta}$
De Moivre's theorem	
$z^n = \lvert z \rvert^n\,cis\,(n\theta)$	$(cis\,\theta)^n = \cos n\theta + i\sin n\theta$
$z^{\frac{1}{q}} = r^{\frac{1}{q}}\left(\cos\dfrac{\theta + 2\pi k}{q} + i\sin\dfrac{\theta + 2\pi k}{q}\right)$, for k an integer	

$$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$m\dfrac{\sqrt{3}}{1} = -1$ $m = -1/\sqrt{3} = \tan\theta$ $\theta = \dfrac{5\pi}{6}, \dfrac{-\pi}{6}$
Specific behaviours
✓ determines gradient of tangent ✓ determines min argument

d) State the maximum value of $Arg(z)$

Solution
Max = $\dfrac{5\pi}{6}$ See above
Specific behaviours
✓ determines gradient of tangent ✓ determines max argument

Q7 (4 marks)

In the following simultaneous equations, $a \,\&\, b$ are real numbers.

$$a^3 = 3ab^2 - 13\sqrt{2}$$

$$b^3 = 3a^2 b - \sqrt{5}$$

In order to determine the value of $a^2 + b^2$ from these equations, it is useful to consider the complex expansion for $(a+bi)^3$. Hence or otherwise, determine the exact value of $a^2 + b^2$.

(Note: answers without working will receive zero marks)

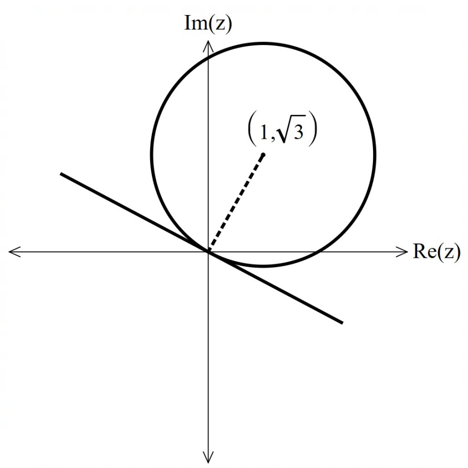
Solution

Specific behaviours
✓ circle with centre coordinates stated ✓ goes through origin

b) State the maximum value of $|z|$

Solution
$|z| = 4$
Specific behaviours
✓ states maximum

c) State the minimum value of $Arg(z)$

Solution


No cals allowed!!

Q1 (2, 2, 2 & 2 = 8 marks)

If $z = 5 - 4i$ and $w = 2 + 3i$ determine the following:

a) zw

Solution
$(5 - 4i)(2 + 3i) = 10 + 12 - 8i + 15i$ $= 22 + 7i$
Specific behaviours
✓ real part ✓ Imaginary part

b) $\dfrac{1}{w}$

Solution
$\dfrac{1}{2+3i}\dfrac{2-3i}{2-3i} = \dfrac{2-3i}{13}$
Specific behaviours
✓ uses conjugate ✓ express answer

c) $\dfrac{\overline{z}}{w}$

Solution
$\dfrac{5+4i}{2+3i}\dfrac{2-3i}{2-3i} = \dfrac{22-7i}{13}$
Specific behaviours
✓ numerator ✓ denominator

d) $z^2\overline{w}$

Solution

$(5-4i)^2(2-3i)=(25-16-40i)(2-3i)$

$(9-40i)(2-3i)$

$=18-120-80i-27i$

$=-102-107i$

Specific behaviours

✓ evaluates square term
✓ determines answer

Q2 (2 & 3 = 5 marks)

a) Determine the complex roots of $3z^2+z+2=0$.

Solution

$3z^2+z+2=0$

$z=\dfrac{-1\pm\sqrt{1-24}}{6}$

$z=\dfrac{-1\pm\sqrt{23}i}{6}$

Specific behaviours

✓ uses quadratic formula
✓ has two complex roots

b) Use the quadratic equation to prove that if a quadratic equation with real coefficients has any non-real roots then it must have two complex roots and they must be conjugates of each other.

Solution

$ax^2+bx+c=0$

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$b^2-4ac=-n^2=i^2n^2$

$x=\dfrac{-b\pm\sqrt{i^2n^2}}{2a}=\dfrac{-b\pm in}{2a}$

$(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta$

$-(\alpha+\beta)=-2\,\mathrm{Re}\,al,\ \alpha\beta=|z|^2$

$f(z)=a(z^2-2z+2)(z^2-4z+13)$

$z=0, f(z)=52\ \therefore a=2$

$f(z)=2(z^4-6z^3+23z^2-34z+26)$

$a=2$

$b=-12$

$c=46$

$d=-68$

$e=52$

Specific behaviours

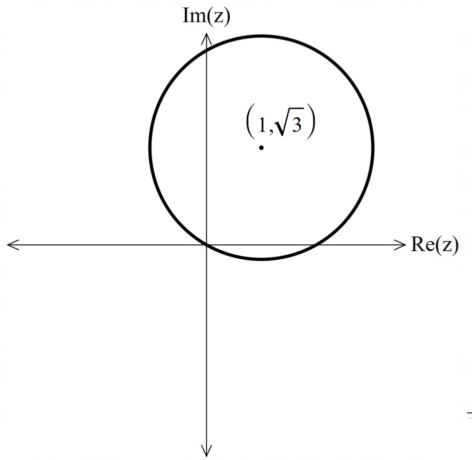
✓ shows reasoning for determining value of a
✓ uses one quadratic factor
✓ uses two quadratic factors
✓ shows reasoning in determining quadratic factors (i.e roots in brackets0
✓ shows reasoning on how to determine quartic polynomial.
Note: Any statement of values without reasoning will NOT receive any marks!

Q6 (2, 1, 2 & 2 = 7 marks)

Consider the locus of complex numbers z that satisfy $|z-1-\sqrt{3}i|=2$.

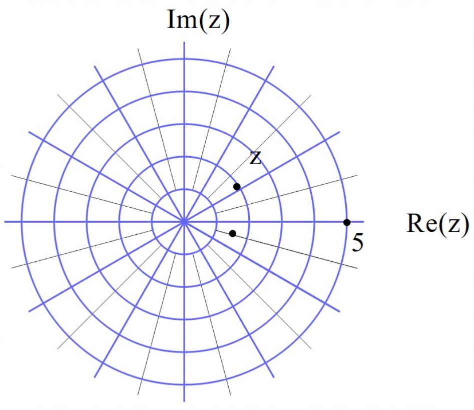
a) Sketch the locus on the axes below.

Solution



Specific behaviours
✓ argument ✓ modulus

d) $\frac{z}{(1+i)}$

Solution


Specific behaviours
✓ argument ✓ modulus

Q5 (5 marks)

Consider the polynomial $f(z) = az^4 + bz^3 + cz^2 + dz + e$ where $a, b, c, d \& e$ are real numbers.

Given that $f(1+i) = 0 = f(2-3i)$

and $f(0) = 52$

Determine the values of $a, b, c, d \& e$.

(Note: answers without working will receive zero marks)

Solution

Specific behaviours
✓ sets up equation with a negative discriminant ✓ uses $i^2 = -1$ with discriminant ✓ derives two complex roots which are conjugates of each other

Q3 (4 marks)

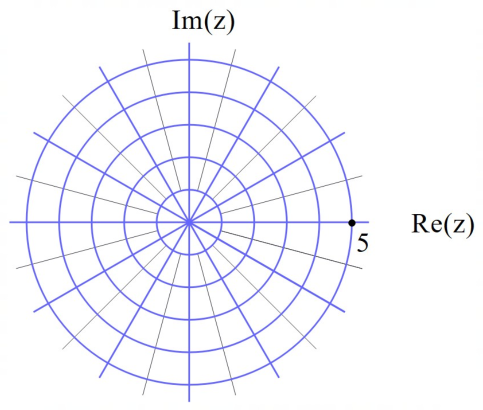
Determine all possible real number pairs $a \& b$ such that $\frac{31-29i}{a+2i} = 3+bi$.

Solution
$\frac{31-29i}{a+2i} = 3+bi$ $31-29i = (3+bi)(a+2i) = 3a-2b+i(ab+6)$ $31 = 3a-2b$ $-29 = ab+6$ $ab = -35, \quad a = \frac{-35}{b}$ $31 = \frac{-105}{b} - 2b$ $31b = -105 - 2b^2$ $2b^2 + 31b + 105 = 0$ $(2b+21)(b+5) = 0$ $b = -5, \frac{-21}{2}$ $a = 7, \frac{70}{21}$ $(7,-5) \& \left(\frac{70}{21}, \frac{-21}{2}\right)$

Specific behaviours
✓ sets up equation and equates real and imaginary ✓ obtains two simultaneous equations ✓ solves for one pair of values ✓ solves for two pairs of values

Q4 (2, 2, 2 & 2 = 8 marks)

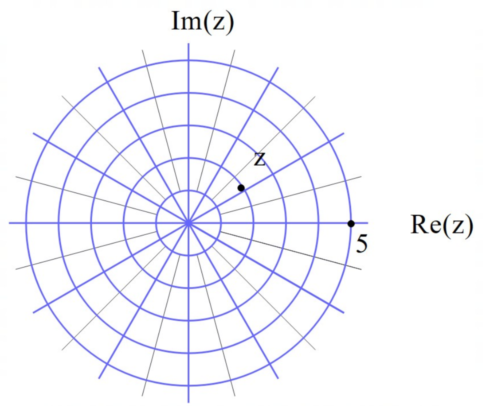
Consider the complex number $z = \sqrt{3} + i$.



Plot the following on the axes above.

a) z

Solution



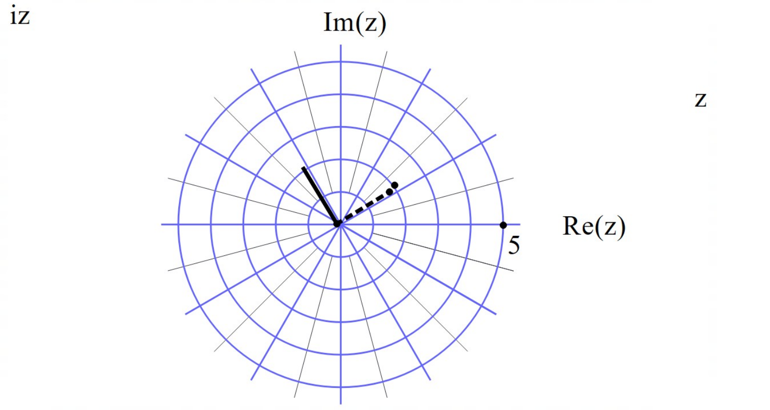
Specific behaviours

✓ argument

✓ length of 2 units

b) iz

Solution



Specific behaviours

✓ uses right angle
✓ rotates anticlockwise with unchanged length

c) $(1+i)z$

Solution