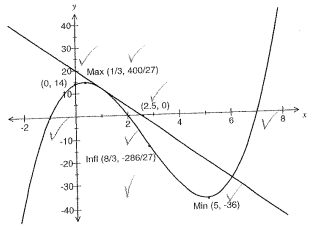
Question 19 **(8 marks)**

A function $f(x)$ has derivative given by $f'(x) = 3x^2 - 16x + 5$.

Another function $g(x) = 20 - 8x$ is a tangent to $f(x)$ in the first quadrant.

Sketch the curves $f(x)$ and $g(x)$, showing the **exact** coordinates of all axis-intercepts, turning points and points of inflection.



$f(x) = x^3 - 8x^2 + 5x + c$

$3x^2 - 16x + 5 = -8$ when $x = 1$ or $x = 13/3$

$g(1) = 12 \Rightarrow$ first quadrant, $g(13/3) = -44/3 \Rightarrow$ not first quadrant.

$f(1) = 12 \Rightarrow c = 14$
$f(x) = x^3 - 8x^2 + 5x + 14 \Rightarrow$ y-intercept at (0, 14)
$f(x) = (x+1)(x-2)(x-7) \Rightarrow$ roots at (-1, 0) (2, 0) and (7, 0)

$3x^2 - 16x + 5 = 0$ when $x = 1/3$ or $x = 5$
Max at (1/3, 400/27) and min at (5, -36).

$f''(x) = 6x - 16$
$= 0$ when $x = 8/3 \Rightarrow$ Pt of inflection at (8/3, -286/27)

$g(x)$ has axis-intercepts at (0, 20) and (2.5, 0)

$3x^2 - 16x + 5 = -8$

$x = (1)$ or $x = \left(\frac{13}{3}\right)$

$f(1) \quad 1^3 - 8 + 5 + c$

$12 = -2 + c$

$\therefore c = 14$

See next page

Question 20 **(7 marks)**

A teacher introduced the following probability experiment to her class. Five cards with the letters A, B, C, D and E are thoroughly shuffled and then the letter on the top card noted. This trial is repeated a total of 20 times to complete the experiment.

Let X be the random variable 'the number of times the card with the letter A is drawn in one experiment'.

(a) Explain why X is a discrete random variable, and state the parameters of the binomial distribution which X follows. (2 marks)

X is a drv because:
it can only take specific integer values
the associated probability distribution sums to 1

$X \sim Bin(20, \tfrac{1}{5})$

(b) Find $P(0 < X \le 4)$. (1 mark)

$\boxed{P(1 \le X \le 4) = 0.6181}$

(c) A large number of students each carry out the experiment above k times and then they share with their class the mean of their k experiments, $\bar{X}$. If approximately 90% of the means of the students' experiments are less than 4.354, use the central limit theorem to estimate k. (4 marks)

$np = 20 \times 0.2 = 4$
$np(1-p) = 20 \times 0.2 \times .8 = 3.2$
$\bar{X} \sim N(4, \frac{3.2}{k})$ by CLT

If $Z \sim N(0,1)$ then $P(Z < 1.282) = 0.9$
Given $P(\bar{X} < 4.354) = 0.9$
$\dfrac{4.354 - 4}{\sqrt{\frac{3.2}{k}}} = 1.282$
$k = 41.94$
$k = 42$

End of questions

MATHS 3CD

REVISION BOOKLET 3

Name : ______________________

Section One: Calculator-free

SOLUTIONS with COMMENT

(40 Marks)

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

POOR

Questions done poorly!

Question 1 (4 marks)

Determine the minimum and maximum values of $f(x)=2x^3-3x^2-12x+27$ over the interval $-3 \le x \le 3$.

$\Rightarrow f'(x) = 6x^2 - 6x - 12$ ✓(skill)

$= 6(x^2 - x - 2)$

$= 6(x+1)(x-2)$

Now: $f'(x) = 0$ when $x=-1$ or $x=2$ ✓ (key concept)

Thus: $f(-3) = 2(-3)^3 - 3(-3)^2 - 12(-3) + 27$

$= -54 - 27 + 36 + 27$

$= -18$

$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 27$

$= -2 - 3 + 12 + 27$

$= 34$ local max

$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 27$

$= 16 - 12 - 24 + 27$

$= 7$ local min

$f(3) = 2(3)^3 - 3(3)^2 - 12(3) + 27$

$= 54 - 27 - 36 + 27$

$= 18$ (routine) ✓ # crunching

$\therefore$ min. value is -18

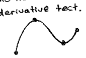
max. value is 34 (conclusion) ✓

COMMENT:

- Method

  Find local max/min using differential calculus and compare with the interval end points.

- Recall shape of a cubic with positive leading coefficient. this avoids the need for 1st or 2nd derivative test.



- Common error

  students confuse $f(x)$ with $f'(x)$

- Be organised on the page. Remember the best way to improve your mathematics is to write is well!
- Be sure to state your result/conclusion.
- Many confused WHEN ($x=-1$) with WHAT (max is 34)

Question 2 (5 marks)

Determine $\frac{dy}{dx}$ in terms of x for each of the following.

(a) $y = x(1+2e^{3x})$ (2 marks)

$\Rightarrow \frac{dy}{dx} = 1(1+2e^{3x}) + x(2e^{3x}.3)$ ✓

$= 1 + 2e^{3x} + 6x e^{3x}$ ✓

COMMENT:

Product Rule and Chain Rule with Exp.

Even if you expand the bracket as in: $y = x + 2xe^{3x}$ you still have the product rule, etc

(b) $y = \int_1^x (t^2+t-1)dt$ (1 mark)

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\int_1^x (t^2+t-1)\,dt\right)$

$= x^2 + x - 1$ ✓

Fundamental Theorem of Calculus (FTC)

Many students antidiff then diff, with only some getting back to where they started! is going in circles.

In general FTC:

$\frac{d}{dx}\int_c^{g(x)} f(t)\,dt = f(g(x)).g'(x)$

$= (x^2+x-1).1$

(c) $y = z^3 - z$ and $z = x^2 - 9$ (2 marks)

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

$= (3z^2-1)(2x)$ ✓

$= (3(x^2-9)^2 - 1)(2x)$

$= 6x(x^2-9)^2 - 2x$ ✓

Possible to substitute at the outset:

$y = (x^2-9)^3 - (x^2-9)$

$\Rightarrow \frac{dy}{dx} = 3(x^2-9)^2.2x - 2x$

$= 6x(x^2-9)^2 - 2x$

as before.

Either method requires an understanding of Chain Rule just different notational appreciation.

(d) What is the probability that a pallet contains at least one bottle with less than the stated contents? (2 marks)

$1 - 0.8867^{48} = 1 - 0.0031$

$= 0.9969$ ✓ ✓

(e) The bottling company randomly choose a pallet from the stockyard. The mean content of all the bottles from this pallet is 389.9 mL.

(i) Construct a 90% confidence interval for the mean content of all bottles. (3 marks)

$n = 24 \times 48$

$= 1152$ bottles ✓

$389.9 \pm 1.645\frac{8.15}{\sqrt{1152}}$ ✓

$= 389.9 \pm 0.395$

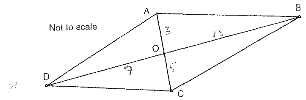
$= (389.5,\ 390.3)$ ✓

(ii) Should the interval be of concern to the bottling company? (1 mark)

✓ Yes. The interval does not come close to containing the accepted plant mean of 391 and so under filling may be commonplace.

Question 18 (5 marks)

The diagonals AC and BD of a quadrilateral ABCD intersect at O.



If OA = 3 cm, OB = 15 cm, AC = 8 cm and BD = 24 cm, prove that AD is parallel to BC.

(i) $OC = 8 - 3 = 5$ cm and $OD = 24 - 15 = 9$ cm ✓

(ii) $\triangle OAD$ is similar to $\triangle OCB$

because of two pairs of sides in same ratio and included angle equal. ✓

$OA = \frac{3}{5}OC$

$OD = \frac{3}{5}OB$

$\angle AOD = \angle COD$ ✓

(iii) $\angle OAD = \angle OCB$ (corresponding angles in similar triangles) ✓

(iv) $\angle CAD = \angle ACD$

and so AD is parallel to BC as alternate angles are equal. ✓

Question 16 **(7 marks)**

The velocity $v(t)$ ms^{-1} of a body moving along a straight track after t seconds, is given by

$$v(t)=\frac{t^2+2t+3}{(t+1)^2},\ t\ge 0.$$

(a) Find the acceleration of the body after 4 seconds. (1 mark)

$$\boxed{\begin{aligned} v'(4)&=-\frac{4}{125}\\ &=-0.032\text{ ms}^{-2}\end{aligned}}$$ ✓

(b) Explain why the body is never stationary over the given domain. (1 mark)

> The numerator of $v(t)$ has no real roots and so the velocity of the body can never be 0. ✓

(c) If $x(t)$ m is the displacement of the body from a fixed point on the track and $x(1)=5$ determine $x(4)$. (2 marks)

$$\boxed{\begin{aligned} x(4)&=x(1)+\int_1^4 v(t)dt\\ &=5+3.6\\ &=8.6\end{aligned}}$$ ✓ ✓

(d) The average speed of the body over the first T seconds is 1.2 ms^{-1}. Determine the value of T. (3 marks)

$$\boxed{\begin{aligned} \frac{\int_0^T v(t)dt}{T}&=1.2\\ \frac{T-\frac{2}{T+1}+2}{T}&=1.2\\ T&=9\end{aligned}}$$ ✓ ✓ ✓

Question 17 **(11 marks)**

A bottling machine fills bottles of water. The content, X mL, of the bottles is a normally distributed random variable with a mean of 391 mL and a standard deviation of 8.15 mL.

It is known that 1 out of every 200 bottles that the machine fills has less than the stated contents on the bottle label.

24 bottles are packed in a carton and 48 cartons are loaded onto a shipping pallet.

(a) What is the probability that a bottle contains more than 375 mL of water? (1 mark)

$$\boxed{\begin{aligned} &X\sim N(391,8.15^2)\\ &P(X>375)=0.9752\end{aligned}}$$ ✓

(b) What are the stated contents on the bottle label? (2 marks)

$$\boxed{\begin{aligned} &P(X<k)=0.005\\ &k=370.0\text{ mL}\end{aligned}}$$ ✓ ✓

(c) What is the probability that a carton does not contain any bottles with less than the stated contents? (2 marks)

$$\boxed{\begin{aligned} &C\sim B(24,0.005)\\ &P(C=0)=0.8867\end{aligned}}$$ ✓ ✓

Question 3 **(5 marks)**

Two independent events A and B are such that P(A) = 0.9 and P(B) = 0.4.

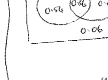
(a) Determine $P(\overline{A\cup B})$. (2 marks)

$P(A\cap B)=P(A)P(B)$ since independent
$=0.9\times0.4$
$=0.36$ ✓

$\Rightarrow P(A\cup B)=P(A)+P(B)-P(A\cap B)$
$=0.9+0.4-0.36$
$=0.94$

$\therefore P(\overline{A\cup B})=1-P(A\cup B)$
$=1-0.94$
$=0.06$ ✓

COMMENT:
If you don't like manipulating Prob Laws there is much ongoing merit in a Venn Diagram.



(b) Determine $P(\overline{B}\mid\overline{A}\cup B)=\dfrac{P(\overline{B}\cap(\overline{A}\cup B))}{P(\overline{A}\cup B)}$ (1 mark)

$=\dfrac{P(\overline{A\cup B})}{1-P(A\cap\overline{B})}$

$=\dfrac{0.06}{1-0.54}$

$=\dfrac{0.06}{0.46}=\dfrac{6}{46}=\dfrac{3}{23}$ ✓

This is where the picture of a Venn Diagram comes in handy!

Most difficult mark on the paper!

(c) Show that $\overline{A}$ and $\overline{B}$ are also independent. (2 marks)

$P(\overline{A}\cap\overline{B})=P(\overline{A\cup B})$ ✓
$=0.06$ from part (a)
$=0.1\times0.6$
$=P(\overline{A})\,P(\overline{B})$ ✓ QED.

The key step here is the opening line in order to establish the final line.

Some students confused independent events with mutually exclusive events where $P(A\cap B)=0$

Question 4 **(7 marks)**

Two functions are defined as $f(x)=\sqrt{x-1}$ and $g(x)=\dfrac{1}{x-1}$.

COMMENT:
Composite functions:

(a) Evaluate $g\circ f\left(\frac{13}{9}\right)=g\left(f\left(\frac{13}{9}\right)\right)$ (2 marks)

$=\dfrac{1}{\sqrt{\frac{13}{9}-1}-1}$ ✓

$=\dfrac{1}{\sqrt{\frac{4}{9}}-1}$

$=\dfrac{1}{\frac{2}{3}-1}=-3$ ✓

Could find $f\left(\frac{13}{9}\right)=\frac{2}{3}$ first;
then $g\left(\frac{2}{3}\right)=-3$

$\frac{13}{9}\xrightarrow{f}\frac{2}{3}\xrightarrow{g}-3$

(b) Determine in simplified form $g\circ g(x)$. (2 marks)

$=g(g(x))$

$=\dfrac{1}{\frac{1}{x-1}-1}$ ✓

$=\dfrac{1}{\frac{1-(x-1)}{x-1}}$

$=\dfrac{1}{\frac{2-x}{x-1}}=\dfrac{x-1}{2-x}$ ✓

Be aware of the alternative notation for composing functions

Yeah old algebraic fractions from Year 9 days!

(c) Determine the domain of $f(g(x))$. (3 marks)

$=\sqrt{\dfrac{1}{x-1}-1}$ ✓ $x\neq1$

Thus $\dfrac{1}{x-1}-1\geq0$

$\Rightarrow \dfrac{2-x}{x-1}\geq0$ using part (b)

$\therefore$ Domain is $1<x\leq2$ ✓ ✓

The two big issues are:
- can't divide by zero
- can't take the square root of a negative.

This beautiful question captured both these issues

Beware: $f(g(x))\neq g(f(x))$

The enthusiasts amongst you also determined the range; well done but alas no bonus marks!

Question 5 **(4 marks)**

Solve the system of equations
$c+2a=3+4b$ ··· Eq①
$a+2b+2c=4$ ··· Eq②
$5a+3c=5+2b$ ··· Eq③

$2a-4b+c=3$ ··· Eq ①
$a+2b+2c=4$ ··· Eq ②
$5a-2b+3c=5$ ··· Eq ③

Eq① + 2 Eq②
$4a+5c=11$ ✓

Eq② + Eq③
$6a+5c=9$ ✓

$-2a = 2$ ✓
$\therefore a = \underline{\underline{-1}}$
$\Rightarrow c = \underline{\underline{3}}$
$\Rightarrow b = \underline{\underline{-\frac{1}{2}}}$

$\therefore \underline{\underline{a=-1,\ b=-\frac{1}{2},\ c=3}}$ ✓

Comment:

Advice.
- Look before you leap!
- 'Elimination' tidier than substitution
- Be super neat / organised.
- Be alphabetic!
- Look to eliminate one variable twice so you now have two equations in two unknowns

Question 6 **(5 marks)**

(a) Determine $\int \frac{2e^{-0.2y}}{5}\,dy$.

$$= \frac{2e^{-0.2y}}{5(-0.2)} + c$$
$$= \underline{\underline{-2e^{-0.2y} + c}} \quad ✓$$

(b) Determine $\int (t-1)(1-2t+t^2)^3\,dt$.

$$= \frac{1}{2}\int 2(t-1)(1-2t+t^2)^3\,dt \quad ✓$$
$$= \underline{\underline{\frac{(1-2t+t^2)^4}{8} + c}} \quad ✓$$

(c) Evaluate $\int_1^6 \frac{3}{x^2}\,dx$.

$$= \int_1^6 3x^{-2}\,dx$$
$$= \left[\frac{3x^{-1}}{-1}\right]_1^6 \quad ✓$$
$$= \left[-\frac{3}{x}\right]_1^6$$
$$= -\frac{1}{2} - (-3)$$
$$= \underline{\underline{\frac{5}{2}}} \quad ✓$$

Comment. (1 mark)
Here we are antidifferentiating with Exp.
Yes $\frac{2}{5(-0.2)} = -2$
and without a calculator!

In general: (2 marks)
$$\int f'(x)\left(f(x)\right)^n dx = \frac{\left(f(x)\right)^{n+1}}{n+1} + c$$
May have to use Mult. Prop. of One to establish $f'(x)$

The marker needs to see you are aware of $+c$.

(2 marks)
Evaluate means to find the value

Often people differentiate when they should be antidiff. ax^n to obtain $\frac{ax^{n+1}}{n+1} + c$

Question 14 **(5 marks)**

A cubical six-sided dice is known to be biased. It is thrown 3 times and the number of sixes is noted. This experiment is then repeated 200 times in all and the results are shown in the table.

Number of sixes	0	1	2	3
Frequency	67	93	33	7

(a) What is the mean number of sixes? (1 mark)

$\boxed{\bar{x}=0.9}$

(b) What is the probability of obtaining a six when this dice is thrown? (1 mark)

If X is the random variable 'number of sixes in 3 throws of the dice', then assume that $X \sim \text{Bin}(3, p)$. $\bar{X}=np$ and so $p=\frac{0.9}{3}=0.3$

(c) Use a suitable binomial distribution to calculate the theoretical frequency distribution for the number of sixes in 200 such experiments and comment on how well your distribution models the experimental results above. (3 marks)

If $X \sim \text{Bin}(3, 0.3)$ then

$200\times P(X=0)=200\times 0.343=68.6$
$200\times P(X=1)=200\times 0.441=88.2$
$200\times P(X=2)=200\times 0.189=37.8$
$200\times P(X=3)=200\times 0.027=5.4$

The experimental and theoretical frequencies are very close to each other, suggesting that the use of the binomial model $X \sim \text{Bin}(3, 0.3)$ is appropriate.

Question 15 **(8 marks)**

(a) A team of 3 students is chosen at random from a group of 4 girls and 5 boys for a TV game show. What is the probability that the team chosen consists of more boys than girls? (2 marks)

$$P = \frac{{}^5C_3 \times {}^4C_0 + {}^5C_2 \times {}^4C_1}{{}^9C_3}$$
$$= \frac{25}{42}$$

0.5952

(b) In one of the games, the team choose one of four closed doors. The doors then open to reveal a prize placed at random behind just one of them. The team keep the prize if they are correct. How many rounds of this game must the team play so that the probability of them obtaining at least one prize is greater than 0.95? (3 marks)

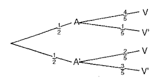
P(At least 1 prize) = 1 - P(No prizes)
$$1-\left(\frac{3}{4}\right)^n \geq 0.95$$
$$n \geq 10.4$$
Must play at least 11 rounds.

(c) At the close of the show, the team can select one of two boxes to keep as another prize. Inside each of the boxes are five sealed envelopes, each containing a voucher. In one of the boxes, four of the vouchers are worth \$10 000 and the fifth \$100, whilst in the other box two of the vouchers are worth \$10 000 and the other three, \$100 each.

The team is allowed to choose an envelope from one of the boxes and open it. They must then decide whether to keep that box or choose the other one. The team plan to keep the box that the envelope they opened came from if it contains a \$10 000 voucher. Otherwise they will take the other box.

What is the probability that the team wins more than \$30 000? (3 marks)

Let event A be choose box with four \$10 000 vouchers and event V be open envelope with a \$10 000 voucher inside. We need $P(A \cap V) + P(\bar{A} \cap \bar{V})$.



$$P(A \cap V) + P(\bar{A} \cap \bar{V}) = \frac{1}{2}\times\frac{4}{5} + \frac{1}{2}\times\frac{3}{5} = \frac{7}{10}$$

(c) Consider the situation where the price of apples fell to $1.70 per kg but the price of pears fell considerably more. Given that the vertex in part (b) still yielded the minimum cost, what would be the minimum price of pears on this day? (2 marks)

> Cost will be equal at both D and E.
>
> $35\times1.7+90k=50\times1.7+60k$
>
> $30k=25.5$
>
> $k=0.85$
>
> Minimum price will be $0.85

See next page

Question 13 **(5 marks)**

Two functions are defined by $f(x)=e^x$ and $g(x)=e^{1-2x}$.

(a) Describe, in order, the transformations which must be applied to the graph of $f(x)$ to obtain the graph of $g(x)$. (2 marks)

> 1. Translate 1 unit to the left
>
> 2. Reflect in the y-axis and dilate horizontally by a scale factor of 1/2.

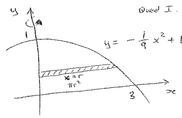
(b) Determine the domain and range of $g(f(x))$. (3 marks)

> Domain: $x\in\mathbb{R}$
>
> Range: $0<y<e$
>
> $x\to\infty \quad f(x)\to\infty \quad g(x)\to e^{-\infty}=0$
>
> $x\to-\infty \quad f(x)\to0 \quad g(x)\to e^1=e$

See next page

Question 7 **(4 marks)**

The region in the first quadrant bounded by $x=0$, $y=0$ and $y=1-\frac{x^2}{9}$ is rotated 360° about the y-axis. If x and y are distances measured in centimetres, determine the volume of the solid formed.



$V_y = \pi\int x^2\,dy$ ✓

$= \pi\int_0^1 (9-9y)\,dy$ ✓

$y=1-\frac{x^2}{9}$

$\Rightarrow 9y=9-x^2$

$\therefore x^2=9-9y$

$= 9\pi\int_0^1 (1-y)\,dy$

$= 9\pi\left[y-\frac{y^2}{2}\right]_0^1$ ✓

$= 9\pi\left(1-\frac{1}{2}\right)$

$= \frac{9\pi}{2}\ \text{cm}^3$ ✓

Comment

A sketch always helps!

the Formula is on the Formulae sheet!

Common Error: $\int_{y=0}^{y=\frac{1}{9}} f(y)\,dy$

Antidiff.!

Many students couldn't do: 'something' − ½ of 'something' = ½ of 'something' !!!

Pay attention to units

Some students produced negative volumes!

See next page

Question 8 **(6 marks)**

The variables k and m are both integers such that $m^2+3=2k$.

(a) Use counter-examples to disprove **two** of the three conjectures listed below. (2 marks)

- m can be any even integer.

Consider $m=2$ ✓ an even integer

$m^2+3 = 2^2+3 = 4+3 = 7 = 2(3.5) = 2k \Rightarrow k=3.5$ is not an integer.

∴ statement FALSE

COMMENT: Only need one counter example to do the job!

- m can be any odd integer.

N.B. this statement is true so no counter example exists! If you found one then take a long hard look at yourself!

- m must be a positive odd integer.

Consider $k=2$ an integer

$2k = 2(2) = 4 = 3+1^2$ or $3+(-1)^2$ ✓

$\Rightarrow m=1$ $\Rightarrow m=-1$

m can be negative ∴ FALSE

Note the linguistic difference between 'Can be' and 'must be'

(b) Using the fact that any odd integer can be written in the form $2n+1$ or otherwise and given m odd, prove that k is always the sum of three square numbers. (4 marks)

$2k = m^2+3$

$= (2n+1)^2+3$ ✓

$= 4n^2+4n+1+3$

$\therefore 2k = 4n^2+4n+4$

$\Rightarrow k = 2n^2+2n+2$ ✓

$= n^2+n^2+2n+1+1$ ✓ *

$= n^2+(n+1)^2+1^2$ ✓

i.e. the sum of three square numbers namely $n, n+1, 1$ QED

Many students read into the question that these three square numbers had to be consecutive which was not the case.

Advice

* Look for something subtle rather than masses of algebraic manipulation.

End of questions

Section Two: Calculator-assumed **(80 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 **(5 marks)**

In a production facility, the lengths of metal rods are recorded to the nearest 5 mm. The rounding error, E mm, is the difference of the actual rod length minus the rounded length and is uniformly distributed between -2.5 mm and 2.5 mm.

(a) State the probability density function for E. (2 marks)

$$\boxed{f(x)=\begin{cases}\frac{1}{5} & -2.5\le x\le 2.5\\ 0 & \text{Elsewhere}\end{cases}}$$

(b) Determine

(i) $P(E=1)$ (1 mark)

$$\boxed{0}$$

(ii) $P(E>-1.5\,|\,E\le 2)$ (1 mark)

$$\boxed{\frac{2--1.5}{2--2.5}=\frac{3.5}{4.5}=\frac{7}{9}}$$

(c) What is the probability that a randomly chosen rod with a recorded length of 135 mm has a real length of a least 136 mm? (1 mark)

$$\boxed{P(E>1)=\frac{2.5-1}{5}=\frac{1.5}{5}=\frac{3}{10}}$$

Question 10 **(6 marks)**

From an analysis of the median house price (M) in a city on July 1 each year from 1980 until 2010, it was observed that $\frac{dM}{dt}=0.0772M$, where t is the time in years since July 1 1980.

(a) According to this model, how long did it take for house prices to double? (2 marks)

$$\boxed{\begin{aligned}&M=M_0e^{-0.0772t}\\&2=e^{-0.0772t}\\&t=8.98\text{ years}\end{aligned}}$$

It was also observed that the median house price was $440 000 in 2008.

(b) What was the instantaneous rate of change of the median house price at this time? (1 mark)

$$\boxed{440000\times 0.0772=\$33968\text{ per year}}$$

(c) What was the median house price in 1988, to the nearest thousand dollars? (2 marks)

$$\boxed{\begin{aligned}M&=440000e^{-0.0772t}\\&=440000e^{-0.0772(-20)}\\&=\$93951\\&=\$94000\end{aligned}}$$

$M_0 = 50662$

(d) What was the average rate of change of the median house price between 1988 and 2008? (1 mark)

$$\boxed{\frac{440000-94000}{20}=\$17300\text{ per year}}$$

Question 11 **(6 marks)**

Oil is poured onto the surface of a large tank of water at a rate of 0.7 cm^3 per second. It spreads out on the surface to form a circular slick of uniform thickness 1.5 mm which can be modelled by a thin cylindrical shape.

(a) At what rate is the radius of the slick increasing one minute after pouring began? (4 marks)

$$\boxed{\begin{aligned}V_{cyl}&=\pi r^2h\\&=0.15\pi r^2 \qquad 60\times 0.7=0.15\pi r^2\ \Rightarrow\ r=9.441\\ \frac{dV}{dr}&=0.3\pi r\\&=0.3\pi(9.441)\\&=8.898\\ \frac{dr}{dt}&=\frac{dr}{dV}\times\frac{dV}{dt}\\&=\frac{1}{8.898}\times 0.7\\&=0.0787\text{ cm per second}\end{aligned}}$$

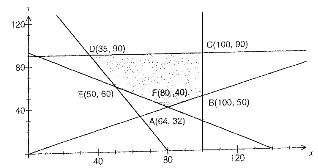
(b) Use the incremental formula $\partial y\approx\frac{dy}{dx}\times\partial x$ to estimate the time the slick will take to increase in radius from 55 cm to 55.5 cm. (2 marks)

$$\boxed{\begin{aligned}\partial V&\approx\frac{dV}{dr}\times\partial r\\&=0.3\pi(55)\times 0.5\\&=25.9\text{ cm}^3\\ \partial t&=25.9\div 0.7\\&\approx 37\text{ seconds}\end{aligned}}$$

Question 12 **(7 marks)**

A drink company make a fresh fruit drink every day using a combination of apples and pears. The recipe requires that the weight of apples must be no more than twice that of pears and at the same time the weight of the pears together with twice the weight of apples must be at least 160kg. Daily supplies are limited to 100kg of apples and 90kg of pears.

With x representing the weight of apples used and y the weight of pears, the feasible region for this information is shown on the graph below.



From a practical point of view, the company have another constraint such that twice the weight of the apples added to three times the weight of pears must be at least 280kg.

(a) Add this fifth constraint to the graph above and clearly label the vertices of the new feasible region. (3 marks)

Add $2x+3y\ge 280$.
Intersects with $y=0.5x$ at (80, 40)
Intersects with $2x+y=160$ at (50, 60)

(b) If the price of apples is $1.80 per kg and pears $2.20 per kg, find the minimum daily cost of fruit whilst satisfying all the above constraints. (2 marks)

D(35, 90) cost is $261 E(50, 60) cost is $222 F(80, 40) cost is $232
Minimum cost is $222.