

No other items may be taken into the examination room. It is your responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor before reading any further.

Important note to candidates

Special items: drawing instruments, emplates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

To be provided by the candidate
Formula Sheet (relating from Section One)
This Question/Answer Booklet
To be provided by the supervisor
Materials required/recommended for this section

Working time for Section One hundred minutes
Reading time before commencing work: ten minutes
Working time for Section Two: one hundred minutes

Time allowed for this section

Your name _____

In words _____

Student Number: _____
In figures _____

MATHEMATICS
METHODS
UNIT 3
Calculator-assumed
Section Two:

SOLUTIONS

Semester 1 Examination, 2016
Question/Answer Booklet



Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	48	35
Section Two: Calculator-assumed	13	13	100	101	65
Total				149	100

Question number: _____

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.
2. Write your answers in this Question/Answer Booklet.
3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.
4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
 - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
 - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.
5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.
6. It is recommended that you **do not use pencil**, except in diagrams.
7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

- (a) A recent news report said that it took 34 months for the population of Australia to increase from 23 to 24 million people.
- Assuming that the rate of growth of the population can be modelled by the equation
- $$\frac{dp}{dt} = kp$$
- where P is the population of Australia at time t months, determine the value of the constant k .
- Assume that the population of Australia is 23 million at time $t = 0$.
- Using the equation $P = P_0 e^{kt}$, calculate the time taken for the population to increase from 23 million to 24 million.
- (5 marks)

Solution	$P = P_0 e^{kt}$
	$24 = 23e^{kt}$
	$k = 0.001252$
specific behaviours	uses growth and decay equations substitutes correctly solves for k

- (b) Assuming the current rate of growth continues, how long will it take for the population to increase from 24 million to 25 million people?
- (2 marks)

Solution	$25 = 24e^{0.001252t}$
	$t = 32.6$ months
specific behaviours	solves for t sets up correct values in equation, using k to at least 3sf
solves for t	

Question 10

A small object is moving in a straight line with acceleration $a = 6t + k \text{ ms}^{-2}$, where t is the time in seconds and k is a constant. When $t = 1$ the object was stationary and had a displacement of 4 metres relative to a fixed point O on the line. When $t = 2$ the object had a velocity of 1 ms^{-1} .

- (a) Determine the value of k and hence an equation for the velocity of the object at time t . (4 marks)

Solution
$v = 3t^2 + kt + c$
$t = 1, 3 + k + c = 0$
$t = 2, 12 + 2k + c = 1$
$k = -8$
$c = 5$
$v = 3t^2 - 8t + 5$
Specific behaviours
✓ antidifferentiates acceleration, adding constant
✓ derives simultaneous equations from information
✓ solves equations
✓ writes velocity equation

- (b) Determine the displacement of the object when $t = 2$. (3 marks)

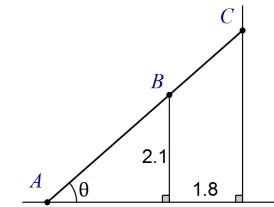
Solution
$s = t^3 - 4t^2 + 5t + c$
$t = 1, 4 = 1 - 4 + 5 + c$
$c = 2$
$s = t^3 - 4t^2 + 5t + 2$
$s(2) = 8 - 16 + 10 + 2$
$= 4 \text{ m}$
Specific behaviours
✓ antidifferentiates velocity
✓ determines constant
✓ evaluates displacement

See next page

(7 marks)

Question 21

A vertical wall, 2.1 metres tall, stands on level ground and 1.8 metres away from the wall of a house. A ladder, of negligible width, leans at an angle of θ to the ground and just touches the ground, wall and house, as shown in the diagram.



- (a) Show that the length of the ladder, L , is given by
$$L = \frac{2.1}{\sin \theta} + \frac{1.8}{\cos \theta}. \quad (3 \text{ marks})$$

Solution
$\sin \theta = \frac{2.1}{AB} \Rightarrow AB = \frac{2.1}{\sin \theta}, \cos \theta = \frac{1.8}{BC} \Rightarrow BC = \frac{1.8}{\cos \theta}$
$L = AB + BC$
$= \frac{2.1}{\sin \theta} + \frac{1.8}{\cos \theta}$
Specific behaviours
✓ shows on diagram and writes length of AB
✓ shows on diagram and writes length of BC
✓ sums lengths to obtain total, using labels added to diagram

- (b) Use a calculus method to determine the length of the shortest ladder that can touch the ground, wall and house at the same time. (4 marks)

Solution
$L = 2.1(\sin \theta)^{-1} + 1.8(\cos \theta)^{-1}$
$\frac{dL}{d\theta} = -2.1\cos \theta (\sin \theta)^{-2} - 1.8(-\sin \theta)(\cos \theta)^{-2}$
$= \frac{1.8\sin^3 \theta - 2.1\cos^3 \theta}{\sin^2 \theta \cos^2 \theta}$
$\frac{dL}{d\theta} = 0 \Rightarrow 1.8\sin^3 \theta - 2.1\cos^3 \theta = 0$
$\tan^3 \theta = \frac{2.1}{1.8} \Rightarrow \theta = \tan^{-1} \sqrt{\frac{2.1}{1.8}} \approx 0.8111$
$L(0.8111) \approx 5.51 \text{ metres}$
Specific behaviours
✓ shows first derivative of L (may use CAS, but must show key results)
✓ solves derivative equal to 0
✓ obtains acute angle solution
✓ substitutes into equation to obtain minimum length

End of questions

(7 marks)

Question 11
It is known that 15% of Year 12 students in a large country study advanced mathematics.A random sample of n students is selected from all Year 12's in this country, and the random variable X is the number of those in the sample who study advanced mathematics.

(2 marks)

(a) Complete the distribution of X .Solution
 $X \sim B(n, 0.15)$ - binomial distribution with n trials and $p = 0.15$ \checkmark states binomial distribution
 \checkmark specific behavioursSee table
 \checkmark calculates at least 3 correct values \checkmark calculates all values correctlyFor what value(s) of x is the function $A(x)$ increasing?(b) If $n = 22$, determine the probability that

(i) three of the students in the sample study advanced mathematics.

- (ii) more than three of the students in the sample study advanced mathematics.
 $P(X > 4) = 0.4248$
 \checkmark evaluates probability
 \checkmark specific behaviours
 \checkmark Solution
 $P(X = 3) = 0.2370$
 \checkmark evaluates probability
 \checkmark specific behaviours
 \checkmark Solution
 $P(X = 0) = 0.0280$
 \checkmark evaluates probability
 \checkmark specific behaviours
 \checkmark Solution
 \checkmark none of the students in the sample study advanced mathematics.
 \checkmark one of these samples has no students who study advanced mathematics.

- (c) If ten random samples of 22 students are selected, determine the probability that at least one of these samples has no students who study advanced mathematics.
 $Y \sim B(10, 0.028)$
 $P(Y \geq 1) = 0.247$
 \checkmark evaluates probability
 \checkmark specific behaviours
 \checkmark Solution
 \checkmark one of these samples has no students who study advanced mathematics.

- (d) If ten random samples of 22 students are selected, determine the probability that at least one of these samples has no students who study advanced mathematics.
 $A(x) = \int_{-\infty}^x f(t) dt = \frac{1}{2} - \frac{x^2}{2}$
 \checkmark states function in terms of x
 \checkmark specific behaviours
 \checkmark Solution
 $x = 4$
 \checkmark states value
 \checkmark specific behaviours
 \checkmark Solution
 $A(0) = 0$
 \checkmark when $A(x) = 0$
 \checkmark determine

Question 20
It is known that 15% of Year 12 students in a large country study advanced mathematics.A random sample of n students is selected from all Year 12's in this country, and the random variable X is the number of those in the sample who study advanced mathematics.

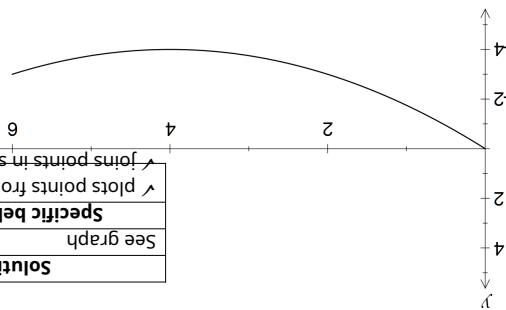
(7 marks)

(a) Complete the table below.

Solution
 X 0 1 2 3 4 5 6
 $A(x)$ 0 0.15 0.3 0.375 0.4248 0.4248 0.4248

- (b) For what value(s) of x is the function $A(x)$ increasing?
 \checkmark calculates all values correctly
 \checkmark calculates at least 3 correct values
 \checkmark specific behaviours
 \checkmark Solution
 \checkmark For what value(s) of x is the function $A(x)$ increasing?

- (c) On the axes below, sketch the graph of $y = A(x)$ for $0 \leq x \leq 6$.
 \checkmark plots points from table
 \checkmark specific behaviours
 \checkmark Solution
 \checkmark see graph
 \checkmark joins points in smooth curve



- (d) Determine the graph of $y = f(x)$ for $0 \leq x \leq 6$.
 \checkmark specific behaviours
 \checkmark Solution
 \checkmark evaluate inequality
 \checkmark specific behaviours
 \checkmark Solution
 \checkmark see graph
 \checkmark plots points from table
 \checkmark specific behaviours
 \checkmark Solution
 \checkmark evaluate probability
 \checkmark specific behaviours
 \checkmark Solution
 \checkmark one of the students in the sample study advanced mathematics.
 \checkmark one of these samples has no students who study advanced mathematics.

Question 12

(8 marks)

The height of grain in a silo, initially 0.4 m, is increasing at a rate given by $h'(t) = 0.55t - 0.05t^2$ for $0 \leq t \leq 11$, where h is the height of grain in metres and t is in hours.

- (a) At what time is the height of grain rising the fastest? (2 marks)

Solution
$h''(t) = 0.55 - 0.1t$
$0.55 - 0.1t = 0 \Rightarrow t = 5.5$ hours
Specific behaviours
✓ differentiates rate of change ✓ solves derivative equal to zero to obtain time.

- (b) Determine the height of grain in the silo after 11 hours. (3 marks)

Solution
$\Delta h = \int_0^{11} 0.55t - 0.05t^2 dt$
≈ 11.09
$h = 11.09 + 0.4$
$= 11.49$ m
Specific behaviours
✓ shows integral of rate of change ✓ evaluates integral to obtain change in height ✓ adds initial height

- (c) Calculate the time taken for the grain to reach a height of 4.45 m. (3 marks)

Solution
$\Delta h = 4.45 - 0.4 = 4.05$
$\Delta h = \int_0^k 0.55t - 0.05t^2 dt$
$= \frac{11k}{40} - \frac{k^3}{60}$
$\frac{11k}{40} - \frac{k^3}{60} = 4.05 \Rightarrow k = 4.5$ hours
Specific behaviours
✓ determines change in height ✓ evaluates integral using constant ✓ solves equation; ignoring solutions outside domain

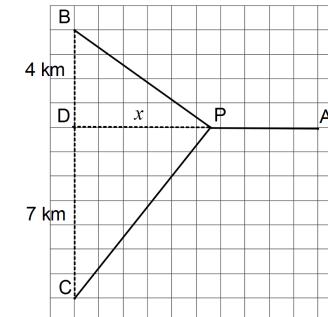
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Question 19

(7 marks)

Three telecommunication towers, A, B and C, each need underground power cable connections directly to a new power station, P, that is to be built x km from depot D on a 10 km road running east-west between D and A.

Tower B lies 4 km due north of depot D and tower C lies 7 km south of the depot, as shown in the diagram.



- (a) Determine an expression for the total length of underground cable required to connect A, B and C directly to P. (2 marks)

Solution
$L = \sqrt{16 + x^2} + \sqrt{49 + x^2} + 10 - x$
Specific behaviours
✓ uses Pythagoras' theorem for BP and CP ✓ determines correct expression

- (b) Show that the minimum length of cable occurs when $\frac{x}{\sqrt{16+x^2}} + \frac{x}{\sqrt{49+x^2}} = 1$. (3 marks)

Solution
$\frac{dL}{dx} = \frac{1}{2}(2x)(16+x^2)^{-0.5} + \frac{1}{2}(2x)(49+x^2)^{-0.5} - 1$
$\frac{dL}{dx} = 0 \Rightarrow \frac{x}{\sqrt{16+x^2}} + \frac{x}{\sqrt{49+x^2}} = 1$
Specific behaviours
✓ uses chain rule to determine derivative ✓ shows that derivative must equal zero ✓ simplifies equation to required result

- (c) Determine the minimum length of cable required, rounding your answer to the nearest hundred metres. (2 marks)

Solution
$x \approx 3.025536$
$L = \sqrt{16 + x^2} + \sqrt{49 + x^2} + 10 - x \Big _{x=3.025536} \approx 19.6$ km
Specific behaviours
✓ solves equation from (b) ✓ substitutes to find length

See next page

(a)

See next page

evaluates integral and have See next page

factors fraction out of integral

 $\int_1^2 f(x) dx = \frac{1}{2}(-12 + 7 - 5 + 32) = 11$

Solution

specific behaviours

evaluates sums each part

 $\int_3^4 f(x) dx + \int_3^4 3 dx = (-5 + 32) + (9) = 36$

Solution

specific behaviours

evaluates integral into two parts

 $\int_2^2 \frac{f(x)}{2} dx$

Solution

specific behaviours

evaluates equations over interval (may choose to use CAS)

equations equations

evaluates limits

shows antiderivative

writes integral

solves equations over interval

evaluates integral

chooses regions C, D and E

adds unsigned values

 $5 + 32 + 21 = 58$ sq units

Determine the values of

from $x = 0$ to $x = 4$ area of region enclosed between the graph of $y = f(x)$ and the x -axis

adds signed values

assigns sign to all areas

specific behaviours

 $-12 + 7 - 5 + 32 - 21 = 1$ Determine the value of $\int_0^2 f(x) dx$

21 square units respectively.

The area of regions enclosed by the x -axis and the curve, A, B, C, D and E, are 12, 7, 5, 32 and

21 square units respectively.

The graph of the function $y = f(x)$ is shown below for $-2 \leq x \leq 4$.

Question 18

(6 marks)

CALCULATOR-ASSUMED

METHODS UNIT 3

Question 14

(14 marks)

- (a) Determine the mean of a Bernoulli distribution with variance of 0.24. (3 marks)

Solution
$p(1 - p) = 0.24$
$p = 0.4, 0.6 \Rightarrow$ mean is either 0.4 or 0.6
Specific behaviours
✓ writes variance equation
✓ solves equation
✓ states both values of p are possible means

- (b) A Bernoulli trial, with probability of success p , is repeated n times. The resulting distribution of the number of successes has an expected value of 5.76 and a standard deviation of 1.92. Determine n and p . (4 marks)

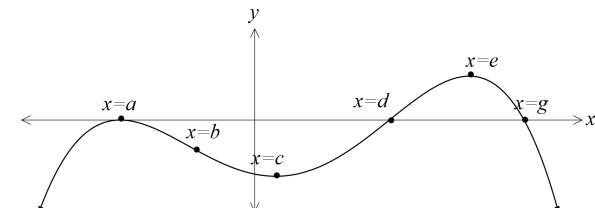
Solution
$X \sim B(n, p)$
$np = 5.76$
$np(1 - p) = 1.92^2$
$1 - p = 1.92^2 \div 5.76 = 0.64$
$p = 0.36$
$n = 16$
Specific behaviours
✓ identifies distribution of successes as binomial
✓ states equation for mean
✓ states equation for variance (or standard deviation)
✓ solves equations for n and p

See next page

Question 17

(8 marks)

The graph of $y = f'(x)$, the derivative of a polynomial function f , is shown below. The graph of $y = f'(x)$ has stationary points when $x = a$, $x = c$ and $x = e$, points of inflection when $x = b$ and $x = d$, and roots when $x = a$, $x = d$ and $x = g$, where $a < b < c < d < e < g$.



- (a) For what value(s) of x does the graph of $y = f(x)$ have a point of inflection? (1 mark)

Solution
$x = a, c, e$
Specific behaviours
✓ states all three values

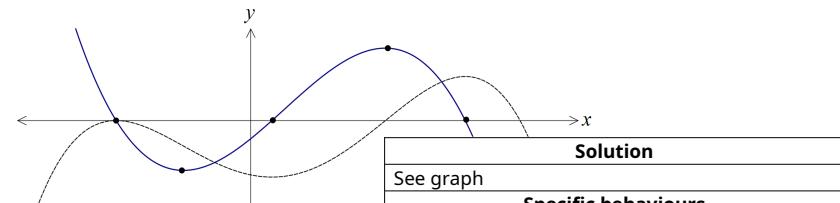
- (b) Does the graph of $y = f(x)$ have a local maximum? Justify your answer. (2 marks)

Solution
Yes, as x increases through $x = g$, the gradient of f changes from positive to zero to negative, indicating a local maximum.
Specific behaviours
✓ responds yes, indicating when $x = g$
✓ explains reason

- (c) Does the graph of $y = f(x)$ have a horizontal point of inflection? Justify your answer. (2 marks)

Solution
Yes, as x increases through $x = a$, the gradient of f changes from negative to zero to negative, indicating a horizontal pt of inflection.
Specific behaviours
✓ responds yes, indicating when $x = a$

- (d) On the axis below, sketch a possible graph of $y = f''(x)$. The graph of $y = f'(x)$ is shown with a broken line for your reference. (3 marks)



Solution
See graph
Specific behaviours
✓ correctly aligns three roots with turning pts
✓ correctly aligns min and max with pts of

See next page

(i)

(2 marks)

- The probability that a student misses their bus to school is 0.2, and the probability that they miss the bus on any day is independent of whether they missed it on the previous day.
- Over five consecutive weekdays, what is the probability that the student only misses the bus on Tuesday?

(ii) only misses the bus on Tuesday?

(2 marks)

Determine $P(Y \geq 0 Y \leq 1)$	
Solution	
$P(Y = y)$	0.4
y	-2
$P(Y = y)$	0.2
y	-1
$P(Y = y)$	0
y	1
$P(Y = y)$	0.1
y	2
$P(Y = y)$	0.2

Question 16

The discrete random variable Y has the probability distribution shown in the table below.

Determine $P(Y \geq 0 Y \leq 1)$	
Solution	
$P(Y = y)$	0.2
y	1

12

CALCULATOR-ASSUMED

METHODS UNIT 3

(i)

(2 marks)

(ii) only misses the bus on Tuesday?

(2 marks)

(i) $E(Y)$	
Solution	
$E(Y) = -2(0.4) + (-1)(0.2) + (0)(0.1) + (1)(0.1) + (2)(0.2)$	= -0.8 - 0.2 + 0.1 + 0.4
	= -0.5
	✓ forms products to obtain expected value
	✓ applies both linear changes to obtain expected value

(c) Calculate $E(1 - 2Y)$	
Solution	
$E(1 - 2Y) = 1 - 2(-0.5) = 2$	
	✓ applies both linear changes to obtain expected value

(d) Calculate $Var(Y)$	
Solution	
$Var(Y) = (0.4)(-1.5)^2 + (0.2)(-0.5)^2 + (0.1)(1.5)^2 + (0.1)(2.5)^2$	= 49
	= 20
	= 2.45
	✓ uses correct formula

(e) Calculate $Var(1 - 2Y)$	
Solution	
$Var(1 - 2Y) = (-2)^2 \times Var(Y) = \frac{49}{5} = 9.8$	
	✓ applies square of multiplier to original variance

See next page

See next page

Question 15

(9 marks)

$$x(t) = \frac{t^2 + 3}{t + 1}$$

A particle moves in a straight line according to the function $x(t) = \frac{t^2 + 3}{t + 1}$, $t \geq 0$, where t is in seconds and x is the displacement of the particle from a fixed point O , in metres.

- (a) Determine the velocity function, $v(t)$, for the particle. (2 marks)

Solution

$$\begin{aligned} v(t) &= \frac{d}{dt} x(t) \\ &= \frac{t^2 + 2t - 3}{(t + 1)^2} \end{aligned}$$

Specific behaviours

- ✓ relates velocity to first derivative of displacement wrt t
- ✓ determines the first derivative

- (b) Determine the displacement of the particle at the instant it is stationary. (2 marks)

Solution

$$v(t) = 0 \Rightarrow t^2 + 2t - 3 = 0 \Rightarrow t = 1$$

$$x(1) = 2 \text{ m}$$

Specific behaviours

- ✓ solves $v=0$ over domain
- ✓ determines displacement

- (c) Show that the acceleration of the particle is always positive. (2 marks)

Solution

$$\begin{aligned} a(t) &= \frac{d}{dt} v(t) \\ &= \frac{8}{(t + 1)^3} \\ t \geq 0 &\Rightarrow a(t) \geq 0 \end{aligned}$$

Specific behaviours

- ✓ determines acceleration function
- ✓ shows that acceleration always positive for $t > 0$

- (d) After five seconds, the particle has moved a distance of k metres.

$$(i) \quad k \neq \int_0^5 v(t) dt$$

(1 mark)

Solution

Integral will calculate change in displacement, but since particle turned around after one second, this will not be the same as distance travelled.

Specific behaviours

✓ explains change in displacement not distance travelled in this instance

- (ii) Calculate k . (2 marks)

Solution

$$\begin{aligned} k &= \left| \int_0^1 v(t) dt \right| + \int_1^5 v(t) dt \\ &= 1 + \frac{8}{3} \\ k &= \frac{11}{3} \end{aligned}$$

Specific behaviours

- ✓ separates into two integrals
- ✓ determines k