Kennedy Baptist College

Semester One Examination, 2023

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words

Your name

Number of additional answer booklets used (if applicable):

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	50	35
Section Two: Calculator-assumed	12	12	100	100	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Supplementary page

Question number: _________

Question 7 **(7 marks)**

An 8 cm length of thin straight wire is bent once and laid on a level surface to form side BC and diagonal CE of rectangle $BCDE$. Let the length of $BC = x$.

(a) Show that the area of the rectangle is $x\sqrt{64-16x}$ cm^2. (3 marks)

Solution
$BE^2 = CE^2 - BC^2$ $= (8-x)^2 - x^2$ $= 64 - 16x + x^2 - x^2$ $BE = \sqrt{64-16x}$ $\text{Area} = BC \times BE$ $= x\sqrt{64-16x}$
Specific behaviours
✓ indicates correct length of diagonal CE ✓ derives expression for length of BE ✓ derives expression for area

(b) Determine the maximum possible area of the rectangle. (4 marks)

Solution
$A = x\sqrt{64-16x}$ $\dfrac{dA}{dx} = \sqrt{64-16x} + x \times \dfrac{1}{2}\dfrac{-16}{\sqrt{64-16x}}$ For maximum require $\frac{dA}{dx} = 0$: $\sqrt{64-16x} - \dfrac{8x}{\sqrt{64-16x}} = 0$ $\dfrac{8x}{\sqrt{64-16x}} = \sqrt{64-16x}$ $8x = 64 - 16x$ $24x = 64$ $x = \dfrac{8}{3}$ $A = \dfrac{8}{3}\sqrt{64 - 16 \times \dfrac{8}{3}}$ $= \dfrac{8}{3}\sqrt{\dfrac{3 \times 64 - 2 \times 64}{3}}$ $= \dfrac{64}{3\sqrt{3}} = \dfrac{64\sqrt{3}}{9}$ cm^2
Specific behaviours
✓ indicates use of product rule ✓ correct expression for derivative ✓ equates derivative to zero and solves for x ✓ substitutes and simplifies to obtain maximum area

End of questions

Section One: Calculator-free **35% (50 Marks)**

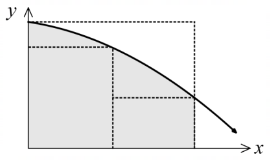
This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

The curve $y = 15 - 2x - x^2$ is shown, with a bounding rectangle and two inscribed rectangles of equal width.

The shaded region is bounded by the curve, the x-axis, the y-axis and the line $x = 2$.



(a) Use areas of rectangles to explain why the area of the shaded region must be between 19 and 30 square units. (3 marks)

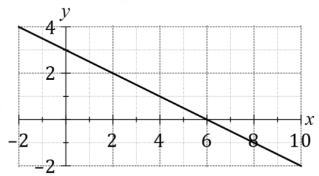
Solution
Points on curve: $(0,15)$, $(1,12)$, $(2,7)$. Area of bounding rectangle is $2 \times 15 = 30$, which is greater than shaded area. Area of LH rectangle is $1 \times 12 = 12$, RH rectangle is $1 \times 7 = 7$ and their sum is $12 + 7 = 19$, which is less than shaded area. Hence area of the shaded region is between 19 and 30 square units.
Specific behaviours
✓ derives area of bounding rectangle ✓ derives sum of inscribed rectangles ✓ explanation

(b) Determine the area of the shaded region. (3 marks)

Solution
$A = \int_0^2 (15 - 2x - x^2)\,dx$ $= \left[15x - x^2 - \frac{x^3}{3}\right]_0^2$ $= 30 - 4 - \frac{8}{3} = 23\frac{1}{3} = \frac{70}{3}\ \mathrm{u}^2$
Specific behaviours
✓ writes correct integral ✓ correct antiderivative ✓ substitutes bounds to obtain correct area

Question 6 **(6 marks)**

The graph of the linear function $y = f(x)$ is shown.



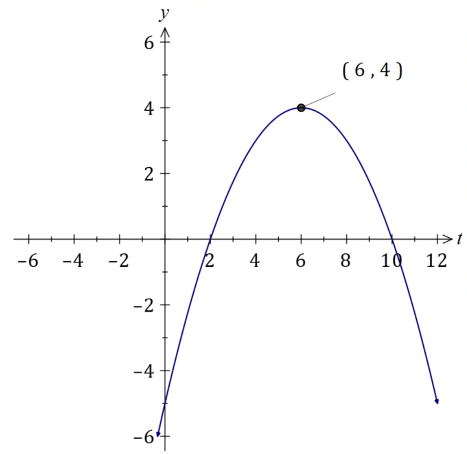
Another function is defined as

$$A(t) = \int_2^t f(x)\,dx$$

(a) Using the graph of $y = f(x)$, or otherwise, evaluate $A(2)$ and $A(6)$. (2 marks)

Solution
$A(2) = 0, \qquad A(6) = \frac{1}{2} \times 4 \times 2 = 4$
Specific behaviours
✓ one correct value ✓ second correct value

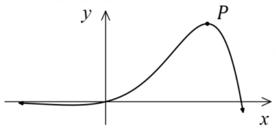
(b) Sketch the graph of $y = A(t)$ on the axes below. (4 marks)



Solution	
Sketch is easiest using the idea that $A(t)$ is the area beneath $f(x)$ from 2 to t, and is a parabolic function with maximum when $t = 6$, root at $t = 2$ and vertical intercept $A(0) = -5$.	$A(t) = \int_2^t 3 - \frac{x}{2}\,dx$ $= \left[3x - \frac{x^2}{4}\right]_2^t = 3t - \frac{t^2}{4} - 5$
Specific behaviours	
✓ maximum turning point ✓ roots ✓ vertical intercept ✓ smooth parabolic curve	

Question 5 **(8 marks)**

The graph of $y = e^{6x}\sin(6x)$ is shown.



(a) Determine the x-coordinate of point P, the first local maximum of the curve as x increases from 0. (4 marks)

Solution

$$\frac{dy}{dx} = 6e^{6x} \times \sin(6x) + e^{6x} \times 6\cos(6x)$$

At P slope is zero:

$$6e^{6x}(\sin(6x) + \cos(6x)) = 0$$
$$\sin(6x) + \cos(6x) = 0$$
$$\sin(6x) = -\cos(6x)$$
$$\tan(6x) = -1$$
$$6x = \frac{3\pi}{4}$$
$$x = \frac{\pi}{8}$$

Specific behaviours

✓ indicates correct use of product rule
✓ correct expression for y'
✓ sets $y' = 0$ and simplifies to $\tan(6x) = -1$
✓ correct x-coordinate

(b) Determine the value of $\frac{d^2y}{dx^2}$ when $x = \frac{3\pi}{2}$ and hence describe the concavity of the curve at this point. (4 marks)

Solution

$$\frac{dy}{dx} = 6e^{6x}(\sin(6x) + \cos(6x))$$
$$\frac{d^2y}{dx^2} = 36e^{6x}(\sin(6x) + \cos(6x)) + 6e^{6x}(6\cos(6x) - 6\sin(6x))$$
$$= 72e^{6x}\cos(6x)$$

When $x = \frac{3\pi}{2}$

$$\frac{d^2y}{dx^2} = 72e^{6\times\frac{3\pi}{2}}\cos\left(6 \times \frac{3\pi}{2}\right)$$
$$= 72e^{9\pi}\cos(9\pi)$$
$$= -72e^{9\pi}$$

Since $\frac{d^2y}{dx^2} < 0$, then the curve is concave down when $x = \frac{3\pi}{2}$.

Specific behaviours

✓ indicates correct use of product rule
✓ correct expression for y''
✓ evaluates y'' at required ordinate
✓ correctly describes concavity of curve

Question 2 **(8 marks)**

The probability function for the random variable X is $P(X = x) = \begin{cases} k^2 - k + x, & x = 0 \\ 5k^2x, & x = 1 \\ 0, & \text{otherwise.} \end{cases}$

(a) Determine the value of the constant . (4 marks)

Solution

$$P(X = 0) + P(X = 1) = 1$$
$$k^2 - k + 5k^2 = 1$$
$$6k^2 - k - 1 = 0$$
$$(3k + 1)(2k - 1) = 0$$
$$k = -\frac{1}{3}, k = \frac{1}{2}$$

$$k = -\frac{1}{3} \Rightarrow P(X = 0) = \frac{4}{9}, P(X = 1) = \frac{5}{9}$$
$$k = \frac{1}{2} \Rightarrow P(X = 0) = -\frac{1}{4}, P(X = 1) = \frac{5}{4}$$

Ignore $k = \frac{1}{2}$ as we require $0 \le p \le 1$ and hence $k = -\frac{1}{3}$.

Specific behaviours

✓ sums probabilities to 1 and forms quadratic equation
✓ solves for both values of k
✓ indicates check of both values of k
✓ correct value of k

(b) Determine the mean and variance of X. (2 marks)

Solution

$$E(X) = p = \frac{5}{9}, \quad \mathrm{Var}(X) = p(1 - p) = \frac{5}{9} \times \frac{4}{9} = \frac{20}{81}$$

Specific behaviours

✓ mean
✓ variance

(c) The random variable $Y = 3X + 1$. Determine the mean and variance of Y. (2 marks)

Solution

$$E(Y) = 3E(X) + 1 = \frac{8}{3}, \quad \mathrm{Var}(Y) = 3^2 \times \mathrm{Var}(X) = \frac{20}{9}$$

Specific behaviours

✓ mean
✓ variance

Question 3 **(6 marks)**

(a) Determine $f'(x)$ when $f(x) = \dfrac{5+\cos(x)}{5+\sin(2x)}$. There is no need to simplify the derivative.

(2 marks)

Solution
$f'(x) = \dfrac{-\sin(x) \times (5+\sin(2x)) - (5+\cos(x)) \times 2\cos(2x)}{(5+\sin(2x))^2}$
Specific behaviours
✓ correct use of quotient rule ✓ correct $f'(x)$

(b) Let $y = \cos(x)$, so that when $x = 30°,\ y \approx 0.8660$. Given that $1° \approx 0.017$ radians, use the increments formula to determine an approximate value for $\cos(29°)$. (4 marks)

Solution
When $x = 30°$ and decreases to $29°$ then $\delta x = -1° \approx -0.017$ radians. $\delta y \approx \dfrac{dy}{dx}\delta x$ $\approx -\sin(x)\,\delta x$ $\approx -\sin(30°) \times -0.017$ $\approx 0.5 \times 0.017$ ≈ 0.0085 Hence $\cos(29°) \approx 0.8660 + 0.0085 \approx 0.8745$.
Specific behaviours
✓ correct value of δx ✓ uses increments formula to obtain expression for δy ✓ obtains value of δy ✓ obtains approximation

Question 4 **(9 marks)**

The function $f(x)$ is defined for $x > -2.5$, has derivative $f'(x) = \dfrac{6}{(2x+5)^2}$, and passes through the point $(-2, 3)$.

(a) Determine the rate of change of $f'(x)$ when $x = -1$. (3 marks)

Solution
$f'(x) = 6(2x+5)^{-2}$ $f''(x) = 6(-2)(2)(2x+5)^{-3}$ $= -24(2x+5)^{-3}$ $f''(-1) = -24(3)^{-3} = -\dfrac{24}{27} = -\dfrac{8}{9}$
Specific behaviours
✓ indicates correct use of chain rule ✓ obtains correct derivative ✓ substitutes and obtains correct value

(b) Determine $f(x)$. (4 marks)

Solution
$f(x) = \int 6(2x+5)^{-2}dx$ $= \dfrac{6}{(-1)(2)}(2x+5)^{-1} + c$ $= -3(2x+5)^{-1} + c$ $f(-2) = 3 \Rightarrow -3(2(-2)+5)^{-1} + c = 3 \Rightarrow c = 3 + 3 = 6$ $f(x) = -\dfrac{3}{2x+5} + 6$
Specific behaviours
✓ attempts to obtain antiderivative, with constant ✓ correct antiderivative ✓ indicates use of point to evaluate constant ✓ correct function

(c) Determine $\dfrac{d}{dt}\displaystyle\int_t^{-1} (3x - f'(x))\,dx$. (2 marks)

Solution
$\dfrac{d}{dt}\displaystyle\int_t^{-1}\big(3x - f'(x)\big)\,dx = -\dfrac{d}{dt}\displaystyle\int_{-1}^{t}\big(3x - f'(x)\big)\,dx$ $= f'(t) - 3t$ $= \dfrac{6}{(2t+5)^2} - 3t$
Specific behaviours
✓ adjusts integral so that variable is upper bound ✓ applies fundamental theorem to obtain correct result