SEMESTER ONE

MATHEMATICS
SPECIALIST
UNIT 1

2020

SOLUTIONS

Calculator-free Solutions

1. (a) (i) **-b** ✓

(ii) $\frac{1}{2}\mathbf{a} + \mathbf{b}$ ✓

(b) $\overrightarrow{RX} = \overrightarrow{RP} + \frac{2}{7}\overrightarrow{PQ}$ ✓

$= -\mathbf{b} + \frac{2}{7}(\mathbf{b} - \mathbf{a})$ ✓

$= -\frac{2}{7}\boldsymbol{a} - \frac{5}{7}\boldsymbol{b}$ ✓ [5]

2. (a) (i) 5 ✓
(ii) 21 ✓

(b) (i) $x = 3$ ✓
(ii) $x = 4$ ✓
(iii) $x = 7$ ✓
(iv) $x = 2$ ✓

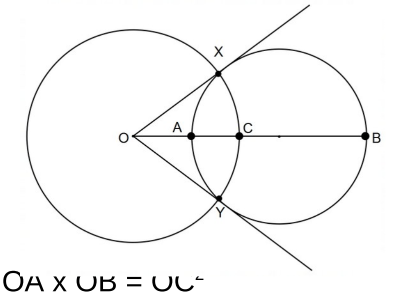
(c) (i) $\binom{7}{5} = 21$ ✓

(ii) $\binom{2}{1} \times \binom{5}{4} + \binom{2}{2} \times \binom{5}{3}$ ✓
$= 20$ ✓

(d) (i) ${}^5\mathbf{P}_5 = 120$ ✓

(ii) $4 \times 3 \times 2 \times 1 \times 2! = 48$ ✓✓ [12]

3. (a) (i)



✓

(ii) OA x OB = OC^2
But OC = OX
$\therefore$ OA x OB = OX^2 ✓
This is the converse of the secant/tangent theorem
$\therefore$ OX is a tangent ✓
OA x OB = OC^2
But OC = OY
$\therefore$ OA x OB = OY^2 ✓
This is the converse of the secant/tangent theorem
$\therefore$ OY is a tangent ✓

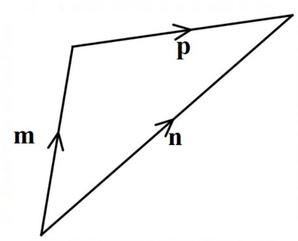
(b) $^{16}\mathbf{C}_{10}$ or $^{16}\mathbf{C}_{6}$ ✓✓

(c) (i) $^{16}\mathbf{C}_{10} \times {}^{16}\mathbf{C}_{4} \times {}^{2}\mathbf{C}_{2}$ ✓✓
(ii) 3! = 6 ways ✓ [10]

4. (a) (i) **-i** - 13**j** ✓
(ii) 5 ✓
(iii) $-9\boldsymbol{i} - 12\boldsymbol{j} + 2\boldsymbol{i} + 6\boldsymbol{j}$ ✓
$= -7\,\mathbf{i} - 6\,\mathbf{j}$ ✓
(iv) $\boldsymbol{b} - \boldsymbol{a} = -6\boldsymbol{i} + 2\boldsymbol{j}$ ✓
$|\boldsymbol{b} - \boldsymbol{a}| = \sqrt{40} = 2\sqrt{10}$ ✓
$$\therefore \hat{\boldsymbol{d}} = -\frac{3}{\sqrt{10}}\boldsymbol{i} + \frac{1}{\sqrt{10}}\boldsymbol{j} = \frac{\sqrt{10}}{10}(-3\boldsymbol{i} + \boldsymbol{j})$$ ✓

(b) $|\boldsymbol{a}| = |3\boldsymbol{i} - 6\boldsymbol{j}| = \sqrt{45} = 3\sqrt{5}$
$|\boldsymbol{c}| = |\boldsymbol{i} + 3\boldsymbol{j}| = \sqrt{10}$
$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -6 \end{pmatrix} = (3\sqrt{5})(\sqrt{10})\cos\theta$$ ✓
$$\therefore \cos\theta = \frac{3 - 18}{15\sqrt{2}} = -\frac{15}{15\sqrt{2}} = -\frac{1}{\sqrt{2}}$$ ✓
$$\therefore |\boldsymbol{c}|\cos\theta\,\hat{\boldsymbol{a}} = \sqrt{10} \times -\frac{1}{\sqrt{2}} \times \frac{1}{3\sqrt{5}}\begin{pmatrix} 3 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$ ✓✓

(c)



✓ [12]

5. (a) $\angle ABC = 90°$ ✓

Thale's Theorem ✓

(b) $\angle CAD = 68°$ ✓

Revolution equal to 360 degrees ✓

(c) $\angle OCB = 45°$ ✓

Equal angles at base of isosceles triangle ✓ [6]

6. Assume $\exists\ x, y \in \mathbb{Z} : 8x - 24y = 5$ ✓

$\therefore 8(x - 3y) = 5$

$\therefore x - 3y = \frac{5}{8}$ ✓

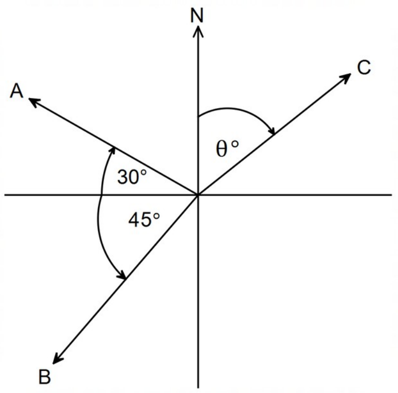
Since $x, y \in \mathbb{Z}$, then $(x - 3y) \in \mathbb{Z}$, but $\frac{5}{8} \notin \mathbb{Z}$. ✓

This is a contradiction as $(x - 3y)$ cannot both belong and not belong to the integer set. ✓

:. False, there are no integers for which this statement is true. ✓ [5]

Calculator-Assumed Solutions

7.



$A = \begin{pmatrix} -30\sin(60) \\ 30\cos(60) \end{pmatrix}$ $B = \begin{pmatrix} -22\sin(45) \\ -22\cos(45) \end{pmatrix}$ $C = \begin{pmatrix} 42\sin(\theta) \\ 42\cos(\theta) \end{pmatrix}$ ✓✓

$30\cos(60) - 22\cos(45) + 42\cos(\theta) = 0$ ✓

$15 - 11\sqrt{2} + 42\cos(\theta) = 0$

$\cos(\theta) = \frac{11\sqrt{2} - 15}{42}$

$\theta = 89.24° \therefore$ bearing $= 089.24°T$ ✓

$-30\sin(60) - 22\sin(45) + 42\sin(89.24°) = v$ ✓

$v = 0.46$ km/h ✓ [6]

8. (a) (i) **LM** parallel to x–axis so $b = 5$ ✓

$(a - 1)^2 + (5 - 2)^2 = 5^2$ ✓

$(a - 1)^2 = 16$

$a - 1 = \pm 4$

$a = 5$ or $a = -3$, but $a > 3 \therefore a = 5$ ✓

M = (5, 5)

(ii) $\boldsymbol{KM} \cdot \boldsymbol{NM} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 5 - c \\ 5 \end{pmatrix} = 0$ ✓

$20 - 4c + 15 = 0$

$35 = 4c$

$c = \dfrac{35}{4}$ ✓

(b) (i) $x = 5$ ✓
$y = 2$ ✓

(ii) If $y = 2$ & $x \geq 6$, then ***BC // AD*** ✓
If ***AB // DC*** ✓

$k\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 - x \\ 4 - y \end{pmatrix}$ ✓

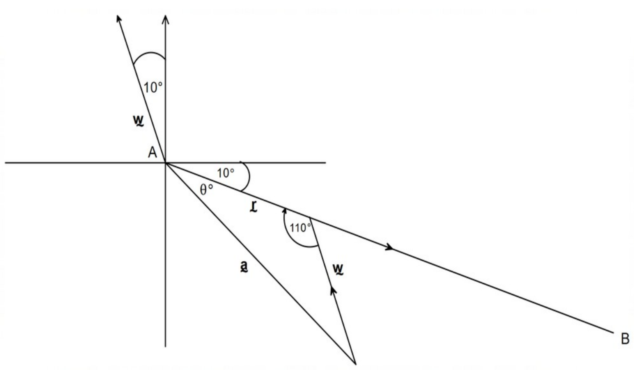
$2(6 - x) = (4 - y)$

$2x - y = 8$ ✓

(iii) $\mathbf{OA} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \mathbf{OB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2\mathbf{OA}$ ✓

$\therefore$ ***OA*** || ***OB*** $\therefore$ A and B are collinear ✓ [13]

9. (a)



✓✓

(b) $\dfrac{\sin(\theta)}{190} = \dfrac{\sin 110}{40}$

$\therefore \quad \theta = 10.95°$ ✓

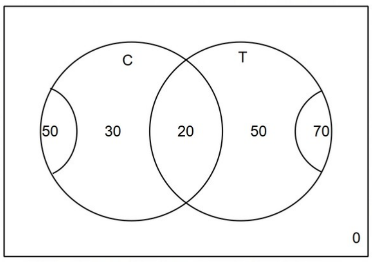
$\therefore$ Bearing = 110°T ✓

(c) $940^2 = 190^2 + |\underset{\sim}{r}|^2 - 2 \times \underset{\sim}{r} \times 190\cos 110°$ ✓

$\therefore \quad |\underset{\sim}{r}| = 857.9$ km/h ✓

$\therefore$ Time taken = $\dfrac{4350}{857.9} = 5$ h 4 mins ✓ [7]

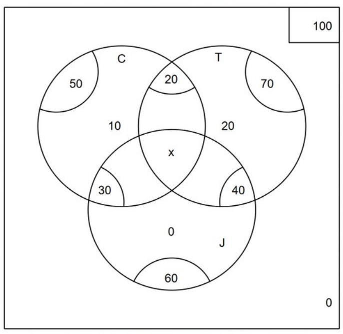
10. (a)



$\therefore$ 50 like tea only ✓

(b) None like neither ✓

(c)



n(J∪T∪C) = n(T)+n(C)+n(J)-n(T∩C)-n(T∩J)-n(J∩C)+n(T∩C∩J)

$\therefore \quad 100 = 70 + 50 + 60 - 20 - 40 - 30 + x$ ✓

$\therefore \quad = 90 + x$ ✓

$\therefore \quad x = 10$ ✓

(d) (i) $\dfrac{70}{100}$ ✓

(ii) $\dfrac{20}{100}$ ✓

(e) 41 ✓ [8]

11. (a) $\overrightarrow{OE} = \frac{1}{2}\overrightarrow{OB}$ ✓

$\overrightarrow{OE} = \frac{1}{2}\left(\boldsymbol{c} + \frac{1}{3}\boldsymbol{a}\right)$

$\overrightarrow{OE} = \frac{1}{2}\boldsymbol{c} + \frac{1}{6}\boldsymbol{a}$ ✓

$\overrightarrow{OF} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CA}$ ✓

$\overrightarrow{OF} = \boldsymbol{c} + \frac{1}{2}(\boldsymbol{a} - \boldsymbol{c})$

$\overrightarrow{OF} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c}$ ✓

(b) $\overrightarrow{CE} = \overrightarrow{OE} - \overrightarrow{OC} = \frac{1}{2}\boldsymbol{c} + \frac{1}{6}\boldsymbol{a} - \boldsymbol{c} = \frac{1}{6}\boldsymbol{a} - \frac{1}{2}\boldsymbol{c}$ ✓

$\overrightarrow{BF} = \overrightarrow{OF} - \overrightarrow{OB} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c} - \boldsymbol{c} - \frac{1}{3}\boldsymbol{a} = \frac{1}{6}\boldsymbol{a} - \frac{1}{2}\boldsymbol{c}$ ✓

$\therefore \overrightarrow{CE} = \overrightarrow{BF}$

$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c} - \frac{1}{2}\boldsymbol{c} - \frac{1}{6}\boldsymbol{a} = \frac{1}{3}\boldsymbol{a}$ ✓

$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = \boldsymbol{c} + \frac{1}{3}\boldsymbol{a} - \boldsymbol{c} = \frac{1}{3}\boldsymbol{a}$ ✓ [8]

$\therefore \overrightarrow{EF} = \overrightarrow{CB}$

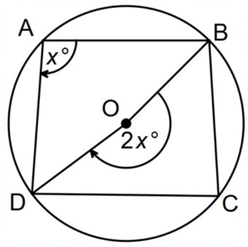
12. (a) $3 > 2$ but $-3 < -2$ is false ✓

(b) “If the triangle is not isosceles, then the triangle does not have two equal sides.” ✓
Yes is it always true since the original implication is always true by definition of isosceles triangles. ✓

(c) “If n is divisible by 4, then n is divisible by 8”. ✓
The converse is not always true, as shown by the counter-example where n = 12 ✓

(d) FOR ALL natural numbers p, EXISTS a real number q, such that q is one less than p. ✓✓ [7]

13. (a)



Let $\angle BAD = x^\circ$
$\therefore$ reflex $\angle BOD = 2x^\circ$ Angle at the centre theorem ✓
Similarly let $\angle BCD = y^\circ$
$\therefore$ $\angle BOD = 2y^\circ$ Angle at the centre teorem ✓
Since $2x + 2y = 360$
then $x + y = 180^\circ$ ✓
Hence $\angle BAD + \angle BCD = 180^\circ$ ✓
QED

(b) $\angle EAG + \angle BCD = 180^\circ$ Cyclic quadrilateral
$\therefore$ $\angle EAG \equiv \angle ECF$ ✓
In ΔEAG and ΔECF
$\angle AEG \equiv \angle CEF$ Bisector ✓
$\angle EAG \equiv \angle ECF$ Proven ✓
$\therefore$ $\angle CFE \equiv \angle BFG$ Vertically Opposite
$\therefore$ $\angle AGF \equiv \angle BFG$ QED ✓

(c) (i) Converse ✓
(ii) Yes ✓ [10]

14. (a) (i) $\boldsymbol{b} \cdot \boldsymbol{c} = 0$

$\therefore$ $-8x - 8 = 0$ ✓
$\therefore$ $x = -1$ ✓

(ii) $\boldsymbol{a} \cdot \boldsymbol{c} = |\boldsymbol{a}| \times |\boldsymbol{c}| \times \cos(60)$ ✓

$\therefore$ $\begin{pmatrix} 5 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} x \\ -8 \end{pmatrix} = \frac{1}{2} \times \sqrt{34} \times \sqrt{x^2 + 64}$ ✓

$\therefore$ $5x + 24 = \frac{1}{2} \times \sqrt{34} \times \sqrt{x^2 + 64}$

$\therefore$ $x = -0.13$ ✓

(iii) $\mathbf{a} - \mathbf{b} = 13\mathbf{i} - 4\mathbf{j}$ ✓

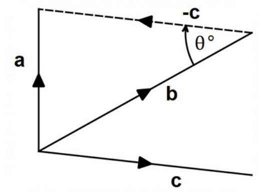
$\therefore$ $|\mathbf{a} - \mathbf{b}| = \sqrt{185}$

$\therefore$ $|\mathbf{c}| = \sqrt{x^2 + 64}$

$\therefore$ $\sqrt{x^2 + 64} = 2\sqrt{185}$ ✓

$\therefore$ $x = \pm 26$ ✓

(b) (i)



✓✓

(ii) cos(0) = 1 ✓

So, the dot product of a vector with itself is its magnitude squared multiplied by 1.

(iii) $\boldsymbol{a} \cdot \boldsymbol{a} = (\boldsymbol{b} - \boldsymbol{c}) \cdot (\boldsymbol{b} - \boldsymbol{c})$

$\therefore |\boldsymbol{a}|^2 = (\boldsymbol{b} - \boldsymbol{c}) \cdot (\boldsymbol{b} - \boldsymbol{c})$

$\therefore |\boldsymbol{a}|^2 = \boldsymbol{b} \cdot \boldsymbol{b} - 2(\boldsymbol{b} \cdot \boldsymbol{c}) + \boldsymbol{c} \cdot \boldsymbol{c}$ ✓

$\therefore |\boldsymbol{a}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{c}|^2 - 2 \times |\boldsymbol{b}| \times |\boldsymbol{c}| \times \cos\theta$ ✓

$\therefore \dfrac{|\boldsymbol{a}|^2 - |\boldsymbol{b}|^2 - |\boldsymbol{c}|^2}{2 \times |\boldsymbol{b}| \times |\boldsymbol{c}|} = \cos\theta$ ✓ [14]

15. (a)

$\boldsymbol{F}_1 = -\dfrac{3\sqrt{2}}{2}\mathbf{i} + \dfrac{3\sqrt{2}}{2}\mathbf{j}$ ✓

$\boldsymbol{F}_2 = \dfrac{3\sqrt{3}}{2}\mathbf{i} + \dfrac{3}{2}\mathbf{j}$ ✓

$\boldsymbol{F}_3 = -2\mathbf{i} - 2\sqrt{3}\mathbf{j}$ ✓

$\therefore \quad \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \begin{pmatrix} -1.523 \\ 0.157 \end{pmatrix}$ ✓

$\therefore \quad \tan^{-1}\left(\dfrac{0.157}{1.523}\right) = 5.89°$ ✓

$\therefore$ 276°T ✓

(b) **a** + **b** = -5**i** + **j** ✓

c = 5**i** - **j** ✓

| c | = $\sqrt{26}$ ✓

Direction = 101°T ✓ [10]

16. let $\angle BEQ = \alpha \therefore \angle QBE = \alpha$ and let $\angle QEC = \beta \therefore \angle QCE = \beta$ ✓

$\therefore \angle BEC = \alpha + \beta$, $\angle BQE = 180 - 2\alpha$ and $\angle EQC = 180 - 2\beta$ ✓

$\therefore 180 - 2\alpha + 180 - 2\beta = 180$ ✓

$\therefore 180 = 2\alpha + 2\beta$

$\therefore 90 = \alpha + \beta = \angle BEC \therefore \perp$ ✓ [4]

17. $$\frac{n!}{(n-3)!} \div \frac{n!}{(n-5)!5!} = \frac{1}{5}$$ ✓

$\therefore$ $$\frac{(n-5)!5!}{(n-3)!} = \frac{1}{5}$$

$\therefore$ $$\frac{120}{(n-3)(n-4)} = \frac{1}{5}$$ ✓

$\therefore$ $600 = n^2 - 7n + 12$ ✓

$\therefore$ $n^2 - 7n - 588 = 0$ ✓

$\therefore$ $(n-28)(n+21) = 0 \rightarrow n = 28$ ✓ [5]

18. (a) Independent term is the middle term. ✓

$\therefore$ $$T_6 = \binom{10}{5}(x)^5\left(\frac{2}{x}\right)^5$$ ✓

$\therefore$ 8064 ✓

(b)

$$(a+\sqrt{3})^n = a^n + \binom{n}{1}a^{n-1}(\sqrt{3}) + \binom{n}{2}a^{n-2}(\sqrt{3})^2 + \binom{n}{3}a^{n-2}(\sqrt{3})^3 + \binom{n}{4}a^{n-3}(\sqrt{3})^4 + \ldots$$

$$(a+\sqrt{3})^n = a^n + n\sqrt{3}a^{n-1} + \binom{n}{2}3a^{n-2} + \binom{n}{3}3\sqrt{3}a^{n-2} + \binom{n}{4}9a^{n-3} + \ldots$$ ✓

$$(a-\sqrt{3})^n = a^n + \binom{n}{1}a^{n-1}(-\sqrt{3}) + \binom{n}{2}a^{n-2}(-\sqrt{3})^2 + \binom{n}{3}a^{n-2}(-\sqrt{3})^3 + \binom{n}{4}a^{n-3}(-\sqrt{3})^4 + \ldots$$

$$(a-\sqrt{3})^n = a^n - n\sqrt{3}a^{n-1} + \binom{n}{2}9a^{n-2} - \binom{n}{3}3\sqrt{3}a^{n-2} + \binom{n}{4}9a^{n-3} + \ldots$$ ✓

Hence,

$$(a+\sqrt{3})^n + (a-\sqrt{3})^n = 2a^n + 6\binom{n}{2}a^{n-2} + 18\binom{n}{4}a^{n-4} + \ldots$$ ✓✓

$\therefore$ $\forall_n$ n is an integer ✓ [8]