PERTH MODERN SCHOOL
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Semester Two Examination, 2023

Question/Answer booklet

MATHEMATICS METHODS UNITS 3&4

Section Two: Calculator-assumed

SOLUTIONS

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	51	34.5
Section Two: Calculator-assumed	11	11	100	97	65.5
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65.5% (97 Marks)**

This section has **eleven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 8 **(8 marks)**

The manager of a local bank wants to know what proportion of their customers never use cash and has asked two clerks to collect sample data by standing in the bank foyer for an hour before lunch on a Friday morning and questioning as many people as they can.

(a) Identify and explain two possible sources of bias with this sampling procedure. (4 marks)

Solution
The sample is taken at a fixed time – only customers who visit the bank on that day and at that time will be sampled, leading to undercoverage. The sample is taken at a fixed location – only customers who actually visit the bank will be sampled, again leading to undercoverage of all other customers.
Specific behaviours
✓ identifies a source of bias ü explains how the source introduces bias ✓ identifies a second source of bias ü explains how the second source introduces bias

(b) Briefly describe a sampling procedure that the manager could use in order to minimise all sources of bias. (2 marks)

Solution
Randomly select individuals from a list of all bank customers and contact them by their preferred means to obtain their response to the question.
Specific behaviours
✓ indicates use of random sampling ü indicates random sample to be drawn from all customers

From the 150 responses obtained using a reliable sampling procedure, the manager was presented with the confidence interval $(0.1652, 0.2748)$ for the proportion of their customers who never use cash.

(c) Determine the number of customers in the sample who said they never use cash. (2 marks)

Solution
$p = (0.1652 + 0.2748) \div 2 = 0.22$ $n = 0.22 \times 150 = 33$ customers.
Specific behaviours
✓ indicates correct sample proportion ü correct number of customers

Question 9 **(6 marks)**

The loudness, measured in decibels, of a sound is given by:

$L = 10\ln\left(\frac{I}{I_0}\right)$ where *I* is the intensity measured in $watt/m^2$. I_0 is 10^{-12} which is barely audible.

(a) Ear damage occurs if the intensity of a sound is greater than $8.1 \times 10^{-9}\, watt/m^2$.
What is the maximum loudness of a sound to prevent damage? (2 marks)

Solution
$L = 10\ln\left(8.1 \times 10^{3}\right)$ ¿ 89.996
Specific behaviours
✓ States correct substitution. ✓ States correct answer (no penalty for SFs)

(b) If the loudness of a sound at a concert is 110 decibels, find the intensity of the sound.
Give your answer in scientific notation rounded to 2 decimal places. (2 marks)

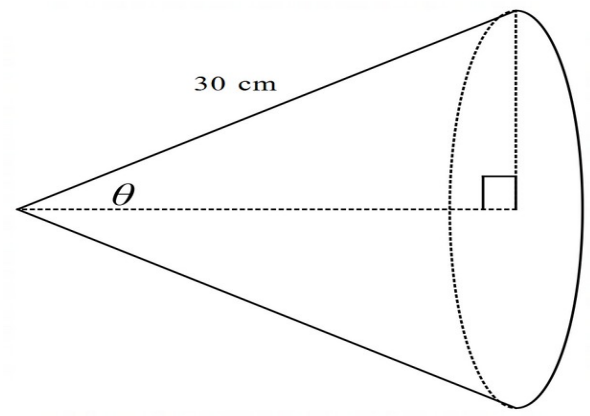
Solution
$110 = 10\ln\left(10^{12} I\right)$ $I = 5.99 \times 10^{-8}$
Specific behaviours
✓ States correct equation. ✓ States correct answer

(c) If the intensity of a sound is tripled, what effect does this have on the loudness.
Justify your answer. (2 marks)

Solution
$L = 10\ln\left(\frac{3I}{10^{-12}}\right)$ $L = 10\left(\ln(3) + \ln\left(\frac{I}{10^{-12}}\right)\right)$ Increases by $10\ln 3$ = $10.986\, db$
Specific behaviours
✓ Shows use of intensity tripling ✓ States correct answer

Question 10 **(7 marks)**

A line segment 30 cm in length forms an angle of θ with the x-axis. A cone is formed by rotating this segment around the x-axis.



(a) Show the volume of this cone can be written as: $V = 9000\pi \sin^2\theta(cos\theta)$.

(1 marks)

Solution
$V = \frac{1}{3}\pi r^2 h$ $r = 30\,sin\theta$ $h = 30\,cos\theta$ $V = \frac{1}{3}\pi(30\,sin\theta)^2(30\,cos\theta)$ $V = 9000\pi \sin^2\theta.(cos\theta)$
Specific behaviours
✓ States correct substitutions for $r \wedge h$

(b) Using calculus methods find the maximum possible volume of this cone.

(4 marks)

Solution
Maximum occurs when. $\frac{dV}{d\theta} = 0 = 9000\pi[-\sin^3\theta + 2\cos^2\theta.\,sin\theta]$ $\theta = 0.95532$ $\frac{d^2V}{d\theta^2} = 9000\,\pi$ ¿ $\frac{d^2V}{d\theta^2}(0.95332) = -2.306 < 0 \therefore Maximum\ value$ $V(0.95532) = 10882.7\,cm^3$
Specific behaviours
✓ States correct first derivative. ✓ Solves to find θ. ✓ Justifies maximum value with a value using second or first derivatives. ✓ States maximum value.

(c) If the size of the angle decreases by 0.02use the incremental formula to find the corresponding decrease in the volume of the cone when $\theta = \frac{\pi}{6}$.

(2 marks)

Solution
Maximum occurs when. $\frac{dV}{d\theta}(\frac{\pi}{6}) = 17671.46$ $\delta V = \frac{dV}{d\theta}\delta\theta$ $\delta V = 17671.46(0.02) = 353.43 cm^3$
Specific behaviours
✓ Determines first derivative at $\frac{\pi}{6}$ ✓ Substitutes correctly and finds correct change in V. (no need for units)

Question 11 **(7 marks)**

The air pressure, P kPa, inside the tyre of a motor vehicle t seconds after it was punctured can be modelled by the equation $P = a + 122e^{kt}$, where a and k are constants.

The initial pressure in the tyre was 220 kPa and after 8.5 seconds it had dropped to 142 kPa.

(a) Determine the value of a and the value of k. (3 marks)

Solution
$P(0) = a + 122 = 220 \Rightarrow a = 98$ $P(2) = 98 + 12e^{8.5k} = 142 \Rightarrow k = -0.12$
Specific behaviours
✓ correct value of a ü forms equation for k ü correct value of k

(b) Determine

(i) the pressure in the tyre after 5 seconds. (1 mark)

Solution
$P = 98 + 122e^{-0.12 \times 5} = 165.0\,\text{kPa}$
Specific behaviours
✓ correct pressure

(ii) the time taken for the pressure in the tyre to fall to 99 kPa. (1 mark)

Solution
$99 = 98 + 122e^{-0.12t} \Rightarrow t = 40.0\,\text{s}$
Specific behaviours
✓ correct time

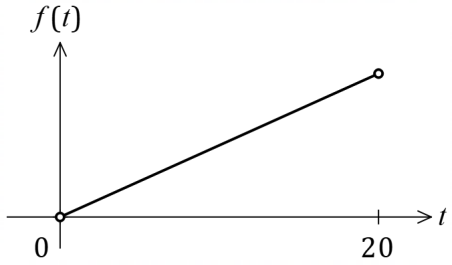
(c) Given that the pressure was falling at a rate of 8 kPa per second after 5 seconds, use the increments formula to estimate the pressure in the tyre after 5.1 seconds. (2 marks)

Solution
$\delta P \approx \frac{dP}{dt}\delta t \approx -8 \times 0.1 = -0.8$ $P = 165 - 0.8 = 164.2\,\text{kPa}$
Specific behaviours
✓ uses increments formula to estimate change ü adds change to previous pressure to obtain estimate(no need for units)

Question 12 **(8 marks)**

Alan works from home every Tuesday and always starts work after his digital clock first shows $8:10$ am and before it shows $8:30$ am.

The probability density function for T, the time in minutes after $8:10$ that he starts work, is $f(t)$ and is displayed at right.



(a) Write the defining rule for the probability density function $f(t)$. (2 marks)

Solution

Let $f(t)=kt$ then:

$$\int_0^{20} kt\,dt=1\Rightarrow 200k=1\Rightarrow k=\frac{1}{200}=0.005$$

$$f(t)=\begin{cases}\dfrac{t}{200} & 0<t<20\\ 0 & \text{Otherwise}\end{cases}$$

Specific behaviours

✓ indicates area beneath $f(t)$ must be 1

ü correct function with correct inequalities (piecewise form /

(b) Determine the probability that on a randomly chosen Tuesday, Alan starts work after his clock first shows $8:23$ am. (2 marks)

Solution

$$\int_{13}^{20}\frac{t}{200}\,dt=\frac{231}{400}=0.5775$$

Specific behaviours

✓ writes correct integral with bounds

ü correct probability

(c) Determine the mean and standard deviation of T. (4 marks)

Solution

$$\overline{T}=\int_0^{20}\frac{t^2}{200}dt=\frac{40}{3}=13.\overline{3}$$

$$Var(T)=\int_0^{20}\left(\left(t-\frac{40}{3}\right)^2\times\frac{t}{200}\right)dt=\frac{200}{9}=22.\overline{2},\sqrt{\frac{200}{9}}=\frac{10\sqrt{2}}{3}\approx 4.714$$

Mean of T is $13.\overline{3}$ minutes and standard deviation is 4.714 minutes.

Specific behaviours

✓ correct integral for mean

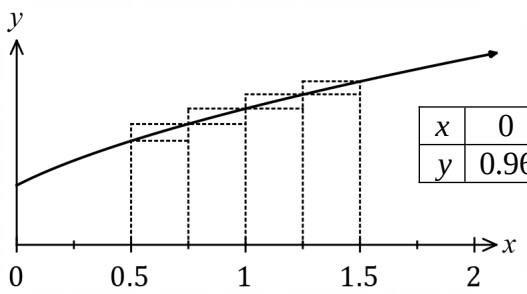
ü correctly evaluates mean

ü correct integral for variance

ü correctly evaluates standard deviation

Question 13 **(8 marks)**

The graph of $y=f(x)$ and a table of values for the function f are shown below.



x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
y	0.96	1.38	1.68	1.94	2.16	2.42	2.64	2.84	3.03

(a) By considering the areas of the rectangles shown, demonstrate and explain why 2.17 is a reasonable estimate for $\int_{0.5}^{1.5} f(x)\,dx$. (3 marks)

Solution

Area of inscribed rectangles: $A_I=0.25(1.68+1.94+2.16+2.42)=2.05$

Area of circumscribed rectangles: $A_C=0.25(1.94+2.16+2.42+2.64)=2.29$

Average of underestimate and overestimate: $(2.05+2.29)\div 2=2.17$

Specific behaviours

✓ shows calculation for inscribed rectangles
ü shows calculation for circumscribed rectangles
ü indicates one is underestimate, other is overestimate and averages

(b) Determine, with justification, estimates for

(i) $\int_{0.5}^{1.5} 4f(x)\,dx$. (1 mark)

Solution

$$I=4\int_{0.5}^{1.5} f(x)\,dx=4\times 2.17=8.68$$

Specific behaviours

✓ correct estimate, shows product

(ii) $\int_{0.5}^{1.5} f(x)+4\,dx$. (2 marks)

Solution

$$I=\int_{0.5}^{1.5} f(x)\,dx+\int_{0.5}^{1.5} 4\,dx=2.17+4=6.17$$

Specific behaviours

✓ indicates use of linearity to split integral
ü correct estimate

(iii) $\int_{0}^{1} f\left(\frac{x}{2}\right)dx$. (2 marks)

Solution

$$A_I=0.5(0.96+1.38)=1.17$$
$$A_C=0.5(1.38+1.68)=1.53$$
$$I=(1.17+1.53)\div 2=1.35$$

Specific behaviours

✓ indicates use of appropriate intervals
ü correct estimate

Question 14 **(9 marks)**

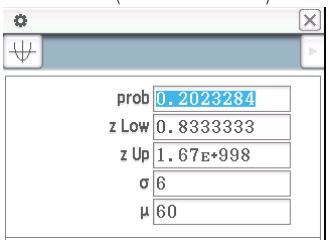
The quantity of juice, in ml, that can be extracted by a machine from different sizes of oranges follows a normal distribution as shown in the table below.

	Mean	Variance
Small	60	36
Large	80	σ^2

(a) What is the probability that more than 65 ml of juice can be extracted from one small orange? (2 marks)

Solution

$$P(X>65)=P\left(Z>\frac{65-60}{6}\right)=0.2023$$



Specific behaviours

✓ States correct inequality.
✓ States correct answer

It is known that 5% of large oranges produce more than 95 ml of juice.

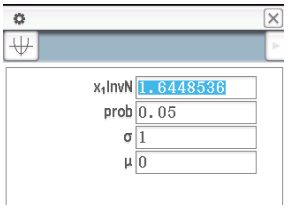
(b) Calculate the value of σ. (3 marks)

Solution

$$P(X>95)=P\left(Z>\frac{95-80}{\sigma}\right)=0.05$$

$$\frac{15}{\sigma}=1.64485$$

$$\sigma=9.12$$



Specific behaviours

✓ Obtains correct z value.
✓ States suitable equation.
✓ States correct answer

If I buy 4 small oranges

(c) (i) Determine the probability that at least three of the oranges produce more than 65 ml of juice. (2 marks)

Solution
$P(X \geq 3) = \binom{4}{3}(0.2023)^3(0.7977) + \binom{4}{4}(0.2023)^4$
¿ 0.0281
Specific behaviours
✓ Recognises binomial with correct coefficients.
✓ States correct answer

(ii) Find the probability that at least three oranges produce more than 65ml given that at least one of the four produces more than 65 ml. (2 marks)

Solution
$P(X \geq 3 \vee X \geq 1) = \frac{0.0281}{0.59509}$
¿ 0.0472
Specific behaviours
✓ Calculates P (X≥ 1¿
✓ States correct answer.

Question 15 **(8 marks)**

Repair tasks undertaken by technical staff who work at an IT company are assigned minor, major or critical status. Over the long term, 4% of the tasks have been critical, 24% major and the remainder minor.

(a) Assuming that the long-term proportions are correct, determine the smallest sample size required so that the width of a 90% confidence interval for the proportion of minor tasks is less than 0.093. (3 marks)

Solution
$p=1-0.04-0.24=0.72, E=0.093\div 2=0.0465$ $n>\dfrac{1.645^2(0.72)(0.28)}{0.0465^2}=252.3$ A sample size of 253 is required. Alternative Use p=0.5, n>312.9 A sample size of 313 is required.
Specific behaviours
✓ indicates correct margin of error ü uses appropriate formula ü correct smallest sample size by rounding up

At the end of one month, a manager suspects that the proportion of major tasks has changed and so she takes a random sample of 250 tasks from the last month, of which 45 were major.

(b) Use this sample to construct a 95% confidence interval for the proportion of major tasks. (3 marks)

Solution
$\hat{p}=45\div 250=0.18$ $0.18\pm 1.96\sqrt{\dfrac{0.18(0.82)}{250}}=0.18\pm 0.0476$ The 95% confidence interval is $(0.1324, 0.2276)$.
Specific behaviours
✓ indicates correct sample proportion ü indicates correct margin of error ü correct interval

(c) Does your confidence interval in part (b) support the managers suspicions? Justify your answer. (2 marks)

Solution
Yes – the interval does not contain the long-term proportion of 24% and so the sample supports the hypothesis that the proportion of major tasks has changed.
Specific behaviours
✓ indicates interval supports claim ü states interval does not contain long-term value

Question 16 **(8 marks)**

The acceleration a m/s^2 of a train moving in a straight line at time t seconds is given by

$$a=k+\frac{3}{8}\cos\left(\frac{\pi t}{12}\right), 0\le t\le 60.$$

Initially, the train was at an origin O and moving with a velocity of 2 m/s.

(a) Determine the velocity of the train after 12 seconds when the constant $k=0.8$. (3 marks)

Solution

$$v(t)=\int a(t)\,dt=0.8t+\frac{9}{2\pi}\sin\left(\frac{\pi t}{12}\right)+c$$

$$v(0)=2\Rightarrow c=2$$

$$v(12)=0.8(12)+\frac{9}{2\pi}\sin(\pi)+2=11.6\text{ m/s}$$

Specific behaviours

✓ determines antiderivative of a

ü evaluates constant of integration

ü correct velocity

After 24 seconds the displacement of the train relative to the origin O was 120 m.

(b) Determine the value of the constant k and hence calculate, to the nearest metre, the displacement of the train after 40 seconds. (5 marks)

Solution

$$v=\int a(t)\,dt=kt+\frac{9}{2\pi}\sin\left(\frac{\pi t}{12}\right)+c, v(0)=2\Rightarrow c=2$$

$$v(t)=kt+\frac{9}{2\pi}\sin\left(\frac{\pi t}{12}\right)+2$$

$$s(t)=\frac{kt^2}{2}-\frac{54}{\pi^2}\cos\left(\frac{\pi t}{12}\right)+2t+C, s(0)=0\Rightarrow C=\frac{54}{\pi^2}$$

$$s(24)=120\Rightarrow\frac{k(24)^2}{2}-\frac{54}{\pi^2}\cos(2\pi)+2(24)+\frac{54}{\pi^2}=120\Rightarrow k=\frac{1}{4}=0.25$$

$$s(40)=\frac{1}{4}\times\frac{(40)^2}{2}-\frac{54}{\pi^2}\cos\left(\frac{40\pi}{12}\right)+2(40)+\frac{54}{\pi^2}=\frac{81}{\pi^2}+280\approx 288\text{ m}$$

Specific behaviours

✓ uses result from (a) to obtain $v(t)$

✓ determines antiderivative of $v(t)$

ü evaluates constant of integration and forms equation for k

ü evaluates

Question 17 **(9 marks)**

Members of a toy library may take home up to 5 toys per visit. The following frequency table shows the number of toys borrowed by a random sample of 80 members.

Toys borrowed	0	1	2	3	4	5
Frequency	0	10	26	28	12	4

You may assume that relative frequencies obtained from the above data are reliable point estimates of probabilities and that the number of toys borrowed by any two members are independent.

(a) Determine the probability that a member borrows fewer than 5 toys, given that they borrowed at least 3 toys. (2 marks)

Solution

$$p=\frac{28+12}{28+12+4}=\frac{40}{44}=\frac{10}{11}\approx 0.9091$$

Specific behaviours

✓ correct denominator

ü correct probability

(b) Determine the probability that at least 3 of the next 5 borrowers take home an even number of toys. (3 marks)

Solution

$$Y \quad B(5, p), p=\frac{26+12}{80}=\frac{38}{80}=0.475$$

$$P(Y\geq 3)=0.4532$$

Specific behaviours

✓ indicates use $B(5, p)$

ü indicates correct value of p

ü correct probability

(c) Show that the mean of the random variable X, the number of toys borrowed by a member, is 2.675. (2 marks)

Solution

$$E(X)=1\times 0.125+2\times 0.325+3\times 0.35+4\times 0.15+5\times 0.05$$

$$¿ 0.125+0.65+1.05+0.6+0.25 ¿ 2.675$$

Specific behaviours

✓ shows correct expression for mean

ü shows result of each product and their sum

Observations indicate that members spend 7 minutes at the library plus 3 minutes per toy chosen.

(d) Determine the mean and standard deviation of the random variable T, the time in minutes spent by members at the toy library. (2 marks)

Solution
$T = 3X + 7$ $E(T) = 3(2.675) + 7 = 15.025 \text{ minutes}$ $\text{sd}_T = 3 \times \sqrt{1.069375} \approx 3.102 \text{ minutes}$
Specific behaviours
✓ correct mean ü correct standard deviation

Question 18 **(19 marks)**

A researcher wishes to estimate, with 95% confidence, the proportion of people who do not have a land line phone. She decides to survey people outside her local mobile phone store every morning for a week around lunchtime and asks each person whether they have a landline phone.

(a) Identify and explain two sources of bias in the proposed sampling method. (4 marks)

Solution
Single location: only restricting to local area, excludes people living elsewhere. Restriction of time: not everyone will be shopping at lunchtime, not everyone is included in the sample. Restriction of participants: People who own landlines only, unlikely to visit near a mobile phone store.
Specific behaviours
✓ Identifies a source of bias. ✓ Explains how the source is biased. ✓ Identifies a second source of bias. ✓ Explains how the source is biased.

She interviewed 560 people, of these 420 did not have a land line phone.

(b) Calculate a 95% confidence interval of the true proportion of people who do not have a land line. (3 marks)

Solution
$\hat{P}=\frac{420}{560}=0.75$ $npq=560(0.75)(0.25)=105$ $CI=0.75\pm1.96\left(\frac{\sqrt{105}}{560}\right)=(0.7141,0.7859)$
Specific behaviours
✓ Substitutes correctly for $\hat{p}$ ✓ Calculates S_e ✓ Calculates correct confidence interval

(c) Which of the statements below is a valid interpretation of this confidence interval? (1 mark)

(i) There is a 95% chance that the true value of p lies within the interval.

(ii) If many different samples of size 560 were selected and based on each sample, a confidence interval was constructed, 95% of the time the true value of p would lie within the interval.

(iii) If many different random samples of size 560 were selected and based on each sample, a confidence interval was constructed, in the long term 95% of the confidence intervals would contain the true value of p.

Solution
Option (iii).
Specific behaviours
✓ States correct statement.

(d) A previous study shows 60% of those interviewed did not have a land line. The researcher wishes to be accurate within 2% of the true proportion.

Find the minimum sample size required. (2 marks)

Solution
$n = p(1-p)\left(\frac{Z}{E}\right)^2 = (0.6)(0.4)¿$ $\therefore 2305$
Specific behaviours
✓ Substitutes correctly into formula.   ✓ States correct answer by rounding up.   Note: accept any z cutoff from 90,95 or99

(e) The results from the previous studies were lost. The researcher needs to produce some evidence to justify a minimum sample size for a 95% confidence level.

What assumption does the researcher need to make and what effect if any would this have on the minimum sample size required to be accurate within 2% of the true population. (2 marks)

Solution
Assume $\hat{p} = 0.5$ Changing $\hat{p}$, n needs to increase as accuracy would decrease without a corresponding increase
Specific behaviours
✓ States correct assumption.   ✓ Makes a statement regarding n having to increase.

(f) The researcher was not pleased with her results and decided to complete another study. 85 of those surveyed, stated they had a landline.
A confidence interval for those having a landline was created: $0.1267 < p < 0.2132$

(i) Find the value for n. (2 marks)

Solution
$\hat{p} = \frac{0.1267 + 0.2132}{2} = 0.170.17 = \frac{85}{n}$ $n = 500$
Specific behaviours
✓ Determines the value for $\hat{p}$.   ✓ Calculates new value for n

(ii) The interval has a% confidence level. Find the value of a (3 marks)

Solution
$0.0433 = z\sqrt{\frac{(0.17)(0.83)}{500}}$ $z = 2.57756$ $a = 99\%$
Specific behaviours
✓ States S_e ✓ Substitutes correctly   ✓ Calculates value for a

Note: use follow through with different rounding for precentage

(iii) The researcher decides because of time constraints to either reduce the confidence level or decrease the sample size. What effects would these have on the margin of error?

(2 marks)

Solution
Decrease confidence level: margin of error decreases. Decrease sample size: margin of error increases.
Specific behaviours
✓ States correct reason for confidence level. ✓ States correct reason for sample size.

Supplementary page

Question number: _____