Government of Western Australia
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-free

Sample WACE Examination 2016

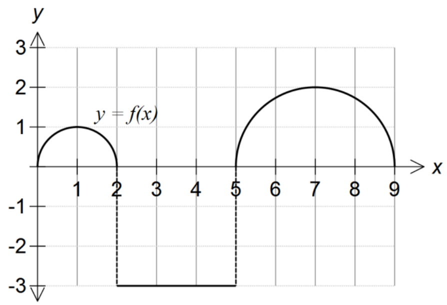
Marking Key

Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

2014/10410

Question 1 **(4 marks)**

Use the graph of $y = f(x)$ to calculate the following definite integrals.



(a) $\int_0^5 f(x)dx$ (2 marks)

Solution
$\int_0^5 f(x)dx =$ Area of semicircle $-$ area of square $= \frac{\pi}{2} - 9$
Specific behaviours
✓ expresses integral as area of semicircle – area of square ✓ calculates integral correctly

(b) $\int_0^9 f(x)dx$ (2 marks)

Solution
$\frac{\pi}{2} - 9 + \frac{4\pi}{2} = \frac{5\pi}{2} - 9$
Specific behaviours
✓ uses additivity of integrals to write sum of areas ✓ calculates integral between 5 and 9 correctly

Question 2 **(7 marks)**

(a) Solve, exactly, each of the following equations.

(i) $\log_x 4 = 2$ (2 marks)

Solution
$\log_x 2^2 = 2$ or $x^2 = 4 \quad x > 0$
$2\log_x 2 = 2 \qquad \Rightarrow x = 2$
$\therefore x = 2$
Specific behaviours
✓ uses log laws or definition of exponential
✓ correctly evaluates value of x

(ii) $e^{2x} = 5$ (2 marks)

Solution
$\ln e^{2x} = \ln 5$
$2x = \ln 5$
$x = \dfrac{\ln 5}{2}$
Specific behaviours
✓ applies logarithms to both sides of equation
✓ uses log laws correctly to determine exact value of x

(b) If $\log a + \log a^2 + \log a^3 + ... + \log a^{50} = k \log a$, determine k. (3 marks)

Solution
$\log a + \log a^2 + \log a^3 + ... + \log a^{50}$
$= 1\log a + 2\log a + 3\log a + ... + 50\log a$
$= (1+2+3+...+50)\log a$
$= (25\times 51)\log a$
$= 1275 \log a$
$\therefore k = 1275$
Specific behaviour
✓ applies log laws to simplify expression
✓ factorises expression
✓ evaluates k

Question 3 **(5 marks)**

A curve has a gradient function $\frac{dA}{dt} = 60 - 3at^2$, where a is a constant.

Given that the curve has a maximum turning point when $t = 2$ and passes through the point (1, 62), determine the equation of the curve.

Solution
$\frac{dA}{dt} = 60 - 3at^2$ At $t = 2$, $\frac{dA}{dt} = 0 = 60 - 12a$ $\therefore\ a = 5$ Hence $A = 60t - 5t^3 + c$ substituting $(1,\ 62)$ into the equation $62 = 60 - 5 + c$ $\therefore c = 7$ so $A = 60t - 5t^3 + 7$

Specific behaviours
✓ substitutes $t = 2$ into $\frac{dA}{dt} = 0$ ✓ evaluates a ✓ anti-differentiates $\frac{dA}{dt}$ correctly ✓ substitutes (1, 62) and evaluates c ✓ states the equation of the curve

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(b) This simulation in part (a) is repeated another 100 times and the proportion (p) of even numbers is recorded for each simulation. Comment on the key features of a typical graph, showing the results of the 100 simulations. (3 marks)

Solution
The graph in part (a) illustrates a typical result of the proportion of even numbers when the simulation is repeated 100 times. The distribution in part (b) should reflect a binomial distribution since we are counting how many even numbers (as opposed to odd numbers) occur per simulation. This distribution tends towards a normal distribution as the number of simulations increases. Hence the frequency distribution is roughly normal centred around $p = 0.5$.
Specific behaviours
✓ states that the given graph is based on a uniform or constant distribution and reflects the result of only one simulation ✓ states that the distribution for part (a) approximates a Binomial distribution as n increases ✓ states that the distribution is centred about 0.5

Question 4 **(5 marks)**

Harry fires an arrow at a target n times. The probability, p, of Harry hitting the target is constant and all shots are independent.

Let X be the number of times Harry hits the target in the n attempts.

The mean of X is 32 and the standard deviation is 4.

(a) State the distribution of X. (1 mark)

Solution
The distribution is binomial.
Specific behaviours
✓ identifies the correct distribution

(b) Determine n and p. (4 marks)

Solution
$32 = np \qquad 4 = \sqrt{np(1-p)}$ $4 = \sqrt{32(1-p)}$ $16 = 32(1-p)$ $p = \frac{1}{2}$ $32 = \frac{1}{2}n$ $\therefore n = 64$
Specific behaviours
✓ states the equation $32 = np$ ✓ states the equation $4 = \sqrt{np(1-p)}$ ✓ solves simultaneously for n and p ✓ elevates n and p correctly

Question 5 **(5 marks)**

The continuous random variable X is defined by the probability density function

$$f(x)=\begin{cases}\dfrac{q}{x} & 1\le x\le 3\\ 0 & \text{elsewhere.}\end{cases}$$

(a) Determine the exact value of q. (3 marks)

Solution
$\int_1^3 \frac{q}{x}\,dx=1$ $[q\ln x]_1^3=1$ $q\ln 3-q\ln 1=1$ $q=\frac{1}{\ln 3}$
Specific behaviours
✓ states correct integral   ✓ integrates $\frac{q}{x}$ correctly   ✓ calculates value of q exactly

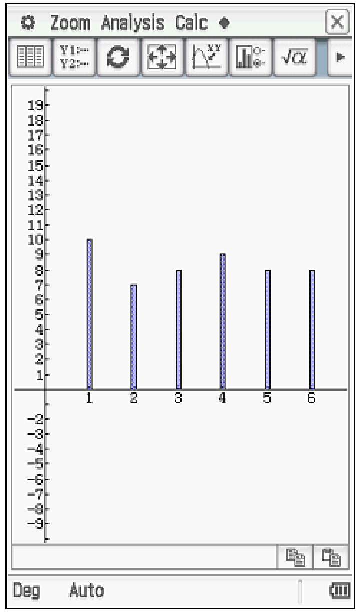
(b) Determine $P(2<X<3)$. (2 marks)

Solution
$\frac{1}{\ln 3}\int_2^3 \frac{1}{x}\,dx=\frac{1}{\ln 3}[\ln x]_2^3=1-\frac{\ln 2}{\ln 3}$
Specific behaviours
✓ states correct integral   ✓ calculates probability correctly

Question 9 **(5 marks)**

The graph on the calculator screen shot below shows the results of a simulation of the tossing of a standard six-sided die, 50 times.

Simulated results of 50 tosses of a standard six-sided die



(a) (i) Describe the type of probability distribution related to this simulation (1 mark)

(ii) Calculate the proportion of even numbers recorded in this simulation. (1 mark)

Solution
The probability distribution is uniform $p=24/50$
Specific behaviors
✓ recognises the distribution as uniform in nature   ✓ calculates p accurately

(b) Hence or otherwise determine the coordinates of the local maximum value of $f(x)$. (3 marks)

Solution

$$\frac{-3(x-1)(x-5)}{(x^2-x-2)^2}=0$$

$(x-1)(x-5)=0$

$x=1,5$

$\therefore$ maximum at $x=5$ and maximum value of $f(x)$ is $\frac{1}{3}$

Specific behaviours

✓ equates $f'(x)=0$

✓ solves for x

✓ calculates maximum value of $f(x)$ correctly

Question 6 **(6 marks)**

(a) Given $f'(x)=x^2\ln(2x+1)$, determine $f''(x)$. Do not simplify. (3 marks)

Solution

$$f''(x)=2x\ln(2x+1)+x^2\frac{2}{2x+1}$$

Specific behaviours

✓ uses product rule correctly

✓ differentiates x^2 correctly

✓ differentiates $\ln(2x+1)$ correctly

(b) Determine $f'(t)$, where $f(t)=t\sqrt{t}+\int_0^t \frac{dx}{1-x^2}$. (3 marks)

Solution

$$f(t)=t^{\frac{3}{2}}+\int_0^t \frac{dx}{1-x^2}$$

$$f'(t)=\frac{3t^{\frac{1}{2}}}{2}+\frac{d}{dt}\int_0^t \frac{dx}{1-x^2}$$

$$f'(t)=\frac{3t^{\frac{1}{2}}}{2}+\frac{1}{1-t^2}$$

Specific behaviours

✓ differentiates $t\sqrt{t}$ correctly

✓ use the theorem $F'(x)=\frac{d}{dx}\left(\int_a^x f(t)dt\right)=f(x)$ correctly

✓ states $f'(t)$ correctly

Question 7 **(9 marks)**

A particle moves in a straight line according to the function $x(t) = e^{\sin t},\ t \geq 0$, where t is in seconds and x is in metres.

(a) Determine the velocity function for this particle. (3 marks)

Solution
Velocity $= x'(t)$ $= \cos t \times e^{\sin t}$
Specific behaviours
✓ relates velocity to the first derivative of $x(t)$ ✓ determines the derivative of $\sin t$ ✓ applies the chain rule and states the correct derivative

(b) Determine the rate of change of the velocity at any time, $t \geq 0$ seconds. (3 marks)

Solution
Rate of change of velocity $= x''(t)$ $= -\sin t \times (e)^{\sin t} + (\cos t)^2 \times (e)^{\sin t}$
Specific behaviours
✓ states that the rate of change of the velocity = $f''(x)$ ✓ determines the derivatives of $\sin x$ and $\cos x$ correctly   ✓ applies the chain rule and states the correct derivative

(c) Evaluate exactly $\int_0^{\frac{\pi}{2}} x'(t)dt.$ (2 marks)

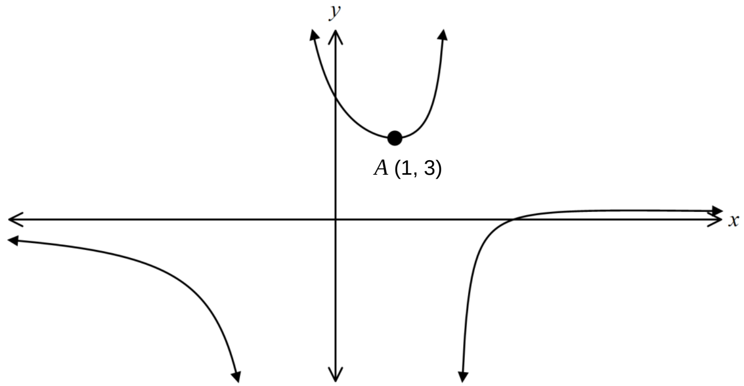
Solution
$\int_0^{\frac{\pi}{2}} x'(t)dt = \left[x(t)\right]_0^{\frac{\pi}{2}}$ $= (e)^{\sin\left(\frac{\pi}{2}\right)} - (e)^{\sin 0}$ $= e - 1$
Specific behaviours
✓ uses $x(t)$ and the correct limits   ✓ evaluates the integral correctly

(d) Interpret the answer to part (c) in terms of the context of the particle moving according to the function $x(t) = e^{\sin t},\ t \geq 0$ seconds. (1 mark)

Solution
$\int_0^{\frac{\pi}{2}} x'(t)dt$ = the change in displacement of the particle between 0 and $\frac{\pi}{2}$ seconds.
Specific behaviours
✓ interprets the result correctly, referring to displacement

Question 8 **(6 marks)**

Consider the graph of $f(x) = \dfrac{3x-9}{x^2-x-2}$ shown below with a local minimum at A (1, 3).



(a) Show that $f'(x) = \dfrac{-3(x-1)(x-5)}{(x^2-x-2)^2}$. (3 marks)

Solution
$f'(x) = \dfrac{3(x^2-x-2)-(3x-9)(2x-1)}{(x^2-x-2)^2}$ $= \dfrac{3x^2-3x-6-(6x^2-21x+9)}{(x^2-x-2)^2}$ $= \dfrac{-3x^2+18x-15}{(x^2-x-2)^2}$ $= \dfrac{-3(x-1)(x-5)}{(x^2-x-2)^2}$
Specific behaviours
✓ uses quotient rule correctly   ✓ differentiates each of the terms correctly   ✓ simplifies correctly