PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Year 12 Test Three 2022

Student name: _______________ Teacher name: _______________

Task type: **Response**

Time allowed for this task: _____40______ mins

Number of questions: _____6______

Materials required: NO classpads nor calculators

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: __40____ marks

Task weighting: _10___%

Formula sheet provided: Yes

Note: All part questions worth more than 2 marks require working to obtain full marks.

NO classpads nor calculators!

Q1 (3 & 3= 6 marks) (3.3.9-3.3.10)

a) Solve the following set of linear equations.

$$3x - 2y + z = -8$$
$$x + 2y - 3z = -14$$
$$2x + y - z = -9$$

Solution
$\begin{bmatrix} 1 & 2 & -3 & -14 \\ 2 & 1 & -1 & -9 \\ 3 & -2 & 1 & -8 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -3 & -14 \\ 0 & 3 & -5 & -19 \\ 0 & 8 & -10 & -34 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -3 & -14 \\ 0 & 3 & -5 & -19 \\ 0 & -2 & 0 & -4 \end{bmatrix}$ $-2y = -4 \quad , y = 2$ $6 - 5z = -19 \quad , z = 5$ $x + 4 - 15 = -14 \quad , x = -3$
Specific behaviours
✓ eliminates one variable from two equations   ✓ eliminates two variables from one equation   ✓ solves for all 3 variables

b) Consider the system below,

$$3x - 2y + z = p$$
$$x + 2y - 3z = -14$$
$$2x + y + qz = -9$$

Determine the values of $p \,\&\, q$ such that there are:

i) Unique solution

ii) Infinite solutions

iii) No solutions.

Solution

$\begin{bmatrix} 1 & 2 & -3 & -14 \\ 2 & 1 & q & -9 \\ 3 & -2 & 1 & p \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -3 & -14 \\ 0 & 3 & -6-q & -19 \\ 0 & 8 & -10 & -42-p \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -3 & -14 \\ 0 & 3 & -6-q & -19 \\ 0 & 0 & -18-8q & -26+3p \end{bmatrix}$ $i)q \neq \frac{-18}{8}\left(-\frac{9}{4}\right)$ $ii)q = \frac{-18}{8} \;\&\; p = \frac{26}{3}$ $iii)q = \frac{-18}{8} \;\&\; p \neq \frac{26}{3}$
Specific behaviours
✓ eliminates two variables from one equation   ✓ determines values for uniqueness   ✓ determines all values for infinite and no solutions

Q2 (2, 2, 2 & 3 = 9 marks) (3.3.11, 3.3.13)

A particle moves such that at time t seconds the velocity is $v = \begin{pmatrix} t \\ -t^2 \\ -3 \end{pmatrix} m/s$. The particle is initially at the origin.

Determine:

a) The position vector at time $t = 1$ second.

Solution
$v = \begin{pmatrix} t \\ -t^2 \\ -3 \end{pmatrix} m/s$ $r = \begin{pmatrix} \frac{1}{2} \\ \frac{-t^3}{3} \\ -3t \end{pmatrix} + c$ $c = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $r = \begin{pmatrix} \frac{1}{2} \\ \frac{-1}{3} \\ -3 \end{pmatrix}$
Specific behaviours
✓ integrates and states a constant C   ✓ states r with t=1

b) The acceleration of the particle at $t = 1$ second.

Solution

$a = \begin{pmatrix} 1 \\ -2t \\ 0 \end{pmatrix} m/s$ $a = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$
Specific behaviours
✓ diff v ✓ states with t=1

c) The speed of the particle at $t = 2$ seconds.

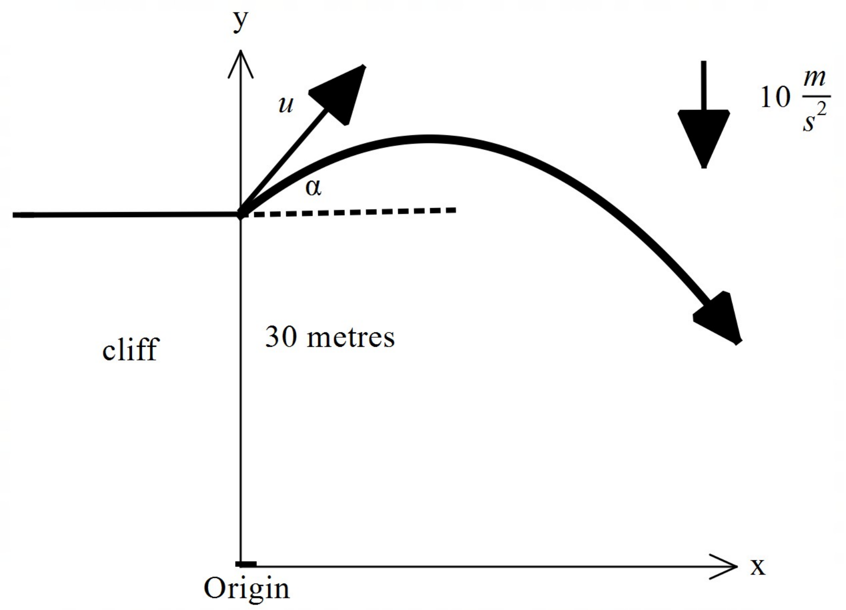
Solution
$v = \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix} m/s$ $\lvert v \rvert = \sqrt{4+16+9} = \sqrt{29}$
Specific behaviours
✓ determines velocity at t=2 ✓ determines speed, no need to simplfy

d) The times when the velocity is perpendicular to the acceleration.

Solution
$\begin{pmatrix} t \\ -t^2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2t \\ 0 \end{pmatrix} = t + 2t^3 = 0, t = 0$
Specific behaviours
✓ uses dot product ✓ equates to zero

✓ states one non negative result

Q3 (4, 3 & 2 = 9 marks) (3.3.12, 3.3.13, 3.3.15)
Consider a particle that is projected from the top of a cliff of height 30 metres with a speed of u metres per second at an angle of α to the horizontal. Assume that the acceleration is constant at $10\,m/s^2$ towards the centre of the Earth. Let the origin of cartesian axes be at the base of the cliff as shown below with the appropriate unit vectors $i\ \&\ j$.



Let $\ddot{r} = \begin{pmatrix} 0 \\ -10 \end{pmatrix} m/s^2$.

a) Using vector integration, show how to derive the position vector r at time t seconds in terms of $u\ \&\ \alpha$. Show all steps.

Solution
$\ddot{r} = \begin{pmatrix} 0 \\ -10 \end{pmatrix} m/s^2$ $\dot{r} = \begin{pmatrix} 0 \\ -10t \end{pmatrix} + \begin{pmatrix} u\cos\alpha \\ u\sin\alpha \end{pmatrix} = \begin{pmatrix} u\cos\alpha \\ u\sin\alpha - 10t \end{pmatrix}$ $r = \begin{pmatrix} ut\cos\alpha \\ ut\sin\alpha - 5t^2 \end{pmatrix} + c$ $\begin{pmatrix} 0 \\ 30 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + c, \quad c = \begin{pmatrix} 0 \\ 30 \end{pmatrix}$ $r = \begin{pmatrix} ut\cos\alpha \\ ut\sin\alpha - 5t^2 + 30 \end{pmatrix}$

Specific behaviours
✓ integrates acceleration with plus constant ✓ solves for constant in terms of two variables ✓ integrates velocity with plus constant ✓ solves for constant and states r in terms of t

b) Show how to derive the cartesian equation for the path of the particle in terms of $u \,\&\, \alpha$.

Solution
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ut\cos\alpha \\ ut\sin\alpha - 5t^2 + 30 \end{pmatrix}$ $t = \dfrac{x}{u\cos\alpha}$ $y = u\sin\alpha \dfrac{x}{u\cos\alpha} - 5\dfrac{x^2}{u^2\cos^2\alpha} + 30$ $y = x\tan\alpha - 5\dfrac{x^2}{u^2\cos^2\alpha} + 30$
Specific behaviours
✓ expresses t in terms of x ✓ subs into y parametric equation ✓ states cartesian equation without any reference to t

c) Set up an equation in terms of $u \,\&\, \tan\alpha$ ONLY, but do not solve, that would allow the range (x) to be determined where the particle hits the floor from the base of the cliff.

Solution
$0 = x\tan\alpha - 5\dfrac{x^2}{u^2\cos^2\alpha} + 30$ $0 = x\tan\alpha - \dfrac{5x^2}{u^2}\sec^2\alpha + 30$ $0 = x\tan\alpha - \dfrac{5x^2}{u^2}(1 + \tan^2\beta) + 30$
Specific behaviours
✓ uses y=0 ✓ uses tan only with reference to angle in two terms of equation

Q4 (4 marks) (4.2.1)

If $y^2 - \sin x = 1 - 5y$, determine $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in terms of x & y only.

Solution
$y^2 - \sin x = 1 - 5y$ $2yy' - \cos x = 5y'$ $y'(2y - 5) = \cos x$ $y' = \frac{\cos x}{(2y-5)}$ $2yy'' + y'2y' + \sin x = 5y''$ $y''(2y-5) = -\sin x - 2(y')^2$ $y'' = \frac{-\sin x - 2(y')^2}{(2y-5)} = \frac{-\sin x - 2\left(\frac{\cos x}{(2y-5)}\right)^2}{(2y-5)} = \frac{-\sin x(2y-5)^2 - 2\cos^2 x}{(2y-5)^3}$
Specific behaviours
✓ implicit diff of original equation ✓ obtains expression for first derivative ✓ implicit diff involving first derivative (or first implicit equation) shown ✓ expression of second derivative in terms of x & y only, no need to simplify

Q5 (3 & 4 = 7 marks) (4.2.1)

Determine the following integrals:

a) $\int_0^1 \frac{5x}{\sqrt{x+1}}dx \quad u = x+1$

Solution
$\int_0^1 \frac{5x}{\sqrt{x+1}}dx \quad u = x+1, \frac{du}{dx} = 1, x = u - 1$ $\int_1^2 \frac{5x}{\sqrt{u}}\frac{dx}{du}du = \int_1^2 \frac{5(u-1)}{\sqrt{u}}du = 5\int_1^2 u^{\frac{1}{2}} - u^{\frac{-1}{2}}du = 5\left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right] = 5\left[\frac{2}{3}2^{\frac{3}{2}} - 2(2^{\frac{1}{2}}) - \frac{2}{3} + 2\right]$ $5\left[\frac{1}{3}2^{\frac{5}{2}} - (2^{\frac{3}{2}}) + \frac{4}{3}\right]$
Specific behaviours
✓ changes to variable u and du ✓ changes limits to u

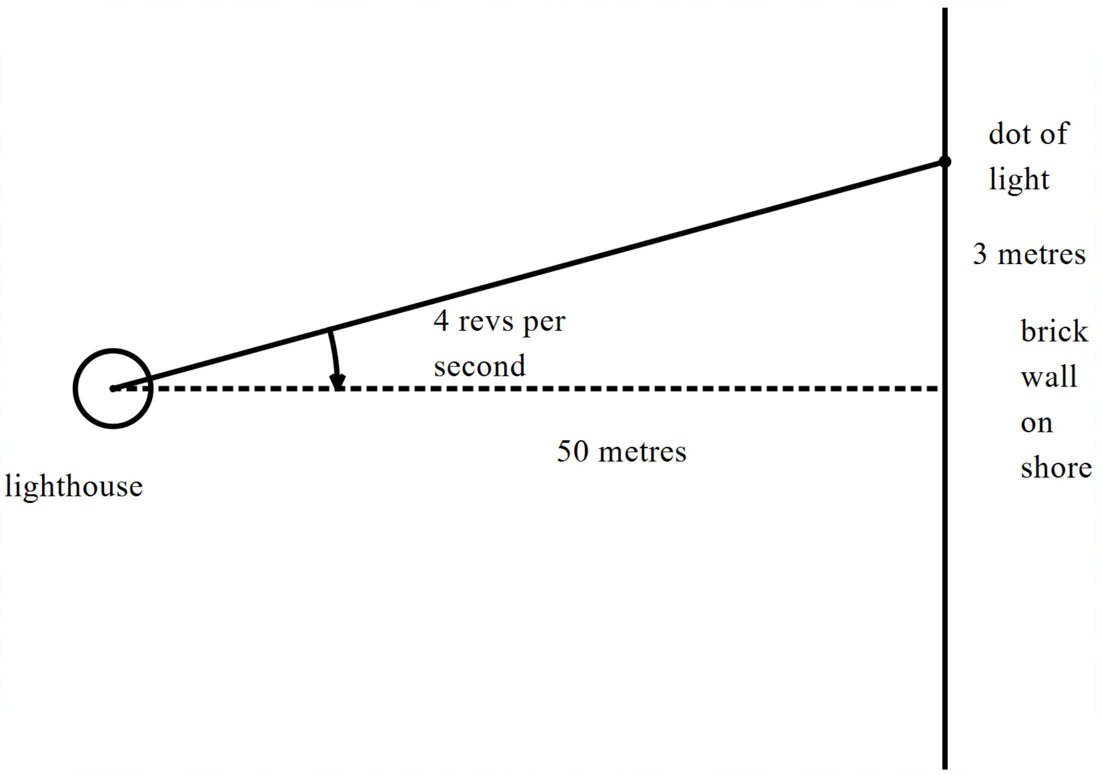
✓ obtains numerical value(unsimplfied)

b)

$$\int \frac{8x^2 - 6x + 5}{(x-2)(x^2+1)} dx$$

Solution
$\int \frac{8x^2 - 6x + 5}{(x-2)(x^2+1)} dx$ $\frac{8x^2 - 6x + 5}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{(x^2+1)}$ $8x^2 - 6x + 5 = A(x^2+1) + (Bx+C)(x-2)$ $x = 2$ $25 = 5A \quad , A = 5$ $x = 0$ $5 = 5 - 2C \quad , C = 0$ $x = 1$ $7 = 10 + B(-1), B = 3$ $\int \frac{8x^2 - 6x + 5}{(x-2)(x^2+1)} dx = \int \frac{5}{x-2} + \frac{3x}{x^2+1} dx = 5\ln\lvert x-2\rvert + \frac{3}{2}\ln\lvert x^2+1\rvert + c$
Specific behaviours
✓ expresses as two partial fractions with THREE constants ✓ solves two constants ✓ solves all three constants showing derivation for all ✓ obtains expression for integral

Q6 (5 marks) (4.1.1, 4.2.2)
Consider a lighthouse that is 50 metres away from the shore. On the shore is a long brick wall. The light on the lighthouse is rotating at 4 revolutions per second. Determine the exact speed of the dot of light on the wall at a point 3 metres from the point directly opposite the lighthouse as shown below.



Solution
$\tan\theta = \frac{x}{50}$ $\sec^2\theta\dot{\theta} = \frac{\dot{x}}{50}$ $50(1+\tan^2\theta)8\pi = \dot{x}$ $50\left(1+\left(\frac{3}{50}\right)^2\right)8\pi = \dot{x}$
Specific behaviours
✓ sets up equation between angle and distance along wall ✓determines rate of angle in radians ✓ uses implicit diff or related rates to link all rates ✓ determines exact value of tan of angle ✓ states an exact expression of speed