PRESBYTERIAN LADIES' COLLEGE

A COLLEGE OF THE UNITING CHURCH IN AUSTRALIA

MATHEMATICS DEPARTMENT

Year 12 MATHEMATICS SPECIALIST

TEST 2: VECTORS

DATE: 3rd March 2016 Name___________________________

Reading Time: 3 minutes

SECTION ONE: CALCULATOR FREE

TOTAL: 25 marks

EQUIPMENT: Pens, pencils, pencil sharpener, highlighter, eraser, ruler, SCSA formula sheet.

WORKING TIME: 25 minutes (maximum)

SECTION TWO: CALCULATOR ASSUMED

TOTAL: 28 marks

EQUIPMENT: Pens, pencils, pencil sharpener, highlighter, eraser, ruler, drawing instruments, templates, up to 3 Calculators,

1 A4 page of notes (one side only), SCSA formula sheet.

WORKING TIME: 25 minutes (minimum)

SECTION 1 Question	Marks available	Marks awarded	SECTION 2 Question	Marks available	Marks awarded
1	5		6	9	
2	6		7	7	
3	4		8	12	
4	6				
5	4				
Total	25			28	

Section One: Calculator-free **[25 marks]**

This section has **five (5)** questions. Answer **all** questions. Write your answers in the spaces provided

Question 1 **[5 marks]**

A straight line passes through the points $P\left(2,-3\right)$ and $Q\left(5,3\right)$.

(a) Find the vector equation of the line in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. [2]

$$\overrightarrow{AB} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$ **(1)**

$\Longrightarrow$ $r = \begin{pmatrix} 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ **OR** Suitable alternative **(1)**

(b) Find the equation of the line through P and Q in parametric form. [1]

$x = 5 + \lambda$

$y = 3 + 2\lambda$ **OR** Suitable alternative **(1)**

(c) Find the equation of the line through P and Q in Cartesian form. [2]

$\lambda = x - 5$

$\lambda = \dfrac{y-3}{2}$ **(1)**

$\Longrightarrow$ $x - 5 = \dfrac{y-3}{2} \Rightarrow y = 2x - 7$ **(1)**

Question 2 [6 marks]

The point A lies on the line with equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j})$ and the point B has position vector $4\mathbf{i} - 5\mathbf{j}$. Use a method involving a dot product to determine the position vector of A so that the distance from A to B is a minimum. [6]

$$a = \begin{pmatrix} 2+2\lambda \\ 1-\lambda \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$$

$\Rightarrow \quad {}_A r_B = \begin{pmatrix} 2\lambda-2 \\ 6-\lambda \end{pmatrix}$ (1)

At point of closest approach

$\begin{pmatrix} 2\lambda-2 \\ 6-\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 0$ (1)

$\Rightarrow \quad 4\lambda-4-6+\lambda=0$ (1)

$\Rightarrow \quad \lambda=2$ (1)

$\Rightarrow \quad a = \begin{pmatrix} 2+2\times2 \\ 1-2 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$

(1) (1)

Question 3 **[4 marks]**

Point A has position vector $\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$ and point B has position vector $\begin{pmatrix} 6 \\ 5 \\ -6 \end{pmatrix}$. Find the position vector of the point P that divides AB internally in the ratio $2:3$.

$$\overrightarrow{AB} = \begin{pmatrix} 6 \\ 5 \\ -6 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -10 \end{pmatrix}$$

(1)

$$\overrightarrow{AP} = \frac{2}{5} \begin{pmatrix} 5 \\ 0 \\ -10 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$$

(1)

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$$

(1) (1)

Question 4 [6 marks]

(a) Find a vector perpendicular to the two vectors:

$\overrightarrow{OP} = \boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$

$\overrightarrow{OQ} = -2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ [3]

$$\overrightarrow{OP}\times\overrightarrow{OQ} = i(1) - j(-3) + k(-5) = \begin{pmatrix}1\\-3\\-5\end{pmatrix}$$

(1) (1) (1)

(b) If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are position vectors for the points P and Q, use your answer to part (a), or otherwise, to find the area of the triangle OPQ. [3]

Area = $\frac{1}{2}|OP|\times|OQ|\times\sin(\theta)$ (1)

= $\frac{1}{2}|\overrightarrow{OP}\times\overrightarrow{OQ}| = \frac{1}{2}\left|\begin{pmatrix}1\\-3\\-5\end{pmatrix}\right| = \frac{\sqrt{35}}{2}$ units^2.

(1) (1)

Question 5 [4 marks]

Points P and Q have coordinates $(3, 1, -2)$ and $(4, 2, -1)$ respectively.

(a) Write a vector equation for the line passing through P and Q. [2]

$$\overrightarrow{PQ} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$ (1)

$$r = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$ (1)

(b) Show that the vector $2i - j - k$ is perpendicular to the line through P and Q. [1]

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 2 - 1 - 1 = 0$$ (1)

$$\Rightarrow \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \perp \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

(c) Write down a vector equation of the plane containing P and Q with $2i - j - k$ as its normal vector. [1]

$$r \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \quad r \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 7$$

(1)

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Section Two: Calculator-assumed **[25 marks]**

This section has **three (3)** questions. Answer **all** questions. Write your answers in the spaces provided

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Question 6 **[9 marks]**

Two rockets are fired from different positions at the same time. Rocket 1 leaves from position $-7i + 9j - 5k$ km at a velocity of $5i - 4j + 2k$ km/min and Rocket 2 leaves from position $-6i - 5j + 2k$ km at a velocity of $9i + 6j - 3k$ km/min. Each rocket leaves a trail of smoke and, although the rockets do not collide, their smoke trails do intersect.

(a) Find the coordinates of the point at which the smoke trails intersect. [4]

Rocket 1: $r = \begin{pmatrix} -7 \\ 9 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}$

Rocket 2: $r = \begin{pmatrix} -6 \\ -5 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 9 \\ 6 \\ -3 \end{pmatrix}$

At point of intersection: $-7 + 5\lambda = -6 + 9\mu$ and $9 - 4\lambda = -5 + 6\mu$ **(1)**

$\Rightarrow \quad \lambda = 2, \ \mu = 1$ **(1)**

This result for λ and μ gives the same z-component of -1. **(1)**

Thus, point of intersection is $(3, 1, -1)$ **(1)**

(b) Find the position of Rocket 1 three minutes after firing. [1]

For Rocket 1: $r(3) = \begin{pmatrix} -7 \\ 9 \\ -5 \end{pmatrix} + 3 \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix}$ **(1)**

(c) Find the shortest distance of Rocket 1 from the smoke trail of Rocket 2, three minutes after firing. Give your answer to the nearest metre. [4]

For Rocket 1 at $(8, -3, 1)$,

$$\overrightarrow{R_2 R_1} = \begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 9\mu-6 \\ 6\mu-5 \\ 3-3\mu \end{pmatrix} = \begin{pmatrix} 14-9\mu \\ 2-6\mu \\ 3\mu-1 \end{pmatrix}$$ **(1)**

Using CAS, $\left|\overrightarrow{R_2 R_1}\right|_{MIN} = 6.574$ km at $t=1.119$ $t=1.119$ minutes.
(1) **(1)**

Thus, shortest distance is 6 574 m. **(1)**

(Can use dot product also, for same result)

Question 7 [7 marks]

(a) The equation of a sphere is given by $x^2+y^2+z^2-6x+4y+8z=153$. Determine the vector equation of the sphere. [3]

$(x-3)^2+(y+2)^2+(z+4)^2=153+9+16+4=182$
(1) (1)

$\Rightarrow$ Equation of sphere is $\left|r-\begin{pmatrix}3\\-2\\-4\end{pmatrix}\right|=\sqrt{182}$. (1)

(b) Determine the position vector(s) of the points of intersection between the sphere and the line $r=-3i+5j+k+\lambda(-2i+j-2k)$. [4]

At point of intersection:

$$\left|\begin{pmatrix}-3-2\lambda\\5+\lambda\\1-2\lambda\end{pmatrix}-\begin{pmatrix}3\\-2\\-4\end{pmatrix}\right|=\sqrt{182}$$ (1)

$\Rightarrow$ $(-6-2\lambda)^2+(7+\lambda)^2+(5-2\lambda)^2=182$ (1)

$\Rightarrow$ $\lambda=-4,\ 2$ (1)

$\Rightarrow$ Position vectors of points of intersection are: $\begin{pmatrix}5\\1\\9\end{pmatrix}$ and $\begin{pmatrix}-7\\7\\-3\end{pmatrix}$ (1)

Question 8 [12 marks]

Let $r = \begin{pmatrix} 2t+5 \\ -2t-1 \\ t \end{pmatrix}, t \in R$, be an equation of line L.

The plane P has a normal vector $\begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$ and passes through the point $A(-1, 0, 4)$.

(a) Show that the point $B(9, -5, 2)$ lies on the line L. [2]

$2t+5=9 \quad \Rightarrow \quad t=2$ (1)

$$r(2) = \begin{pmatrix} 2\times2+5 \\ -2\times2-1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ -5 \\ 2 \end{pmatrix}$$

$\Rightarrow$ $B(9, -5, 2)$ lies on the line L (1)

(b) Give the normal vector equation of the plane P. [2]

$$r \cdot \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = -7$$

(1)

$\Rightarrow$ Normal vector equation of P is $r \cdot \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = -7$ (1)

(c) Find the shortest distance that plane P is from the origin. [2]

$|n| = \sqrt{26}$ (1) $\quad \Rightarrow \quad d = \dfrac{7}{\sqrt{26}}$ (1)

(d) Show that the line L meets the plane P at the point $C(1, 3, -2)$. [3]

At point of intersection, $\begin{pmatrix} 2t+5 \\ -2t-1 \\ t \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = -7$ (1)

$\Rightarrow \quad 6t + 15 + 8t + 4 - t = -7$

$\Rightarrow \quad t = -2$ (1)

$r(-2) = \begin{pmatrix} 2\times -2+5 \\ -2\times -2-1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ (1)

(e) Find the angle between the line L and the plane P. (Give your answer correct to 1 decimal place.) [3]

Direction of L is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Direction of normal is $n = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$ **(1)**

Angle between L and n is 31.8^0 **(1)**

Angle between L and P is $90^0 - 31.8^0 = 58.2^0$ **(1)**

END OF QUESTIONS