Rossmoyne Senior High School

Year 12 Trial WACE Examination, 2015

Question/Answer Booklet

MATHEMATICS
SPECIALIST 3CD
Section Two:
Calculator-assumed

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	50	33⅓
Section Two: Calculator-assumed	13	13	100	100	66⅔
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section Two: Calculator-assumed **(100 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 8 **(6 marks)**

If $y = e^{10t}$ and $t = \log_{10}\left(x^2\right)$, determine the exact value of $\dfrac{dy}{dx}$ when $x = \sqrt{10}$.

$x = \sqrt{10} \;\Rightarrow\; t = \log_{10} 10 = 1$

$$t = \frac{2\ln x}{\ln 10}$$

$$\frac{dt}{dx} = \frac{2}{x\ln 10} = \frac{2}{\sqrt{10}\ln 10}$$

$$\frac{dy}{dt} = 10e^{10t} = 10e^{10}$$

$$\frac{dy}{dx} = 10e^{10} \times \frac{2}{\sqrt{10}\ln 10}$$
$$= \frac{2\sqrt{10}e^{10}}{\ln 10}$$

Question 9 **(8 marks)**

The popularity of four contestants at the end of each round of a talent show can be modelled by the matrix equation below, where A_n, B_n, C_n and D_n represent the percentage of all votes cast for each of the four contestants at the conclusion of round n.

$$\begin{bmatrix} A_{n+1} \\ B_{n+1} \\ C_{n+1} \\ D_{n+1} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.7 & 0.2 & 0.1 \\ 0 & 0.1 & 0.6 & 0 \\ 0.1 & 0.1 & 0 & 0.8 \end{bmatrix} \times \begin{bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{bmatrix}$$

(a) Explain the significance of the three zero elements in the transition matrix. (2 marks)

No one who votes for A or D in one round will vote for C in the next round and no one who votes for C in one round will vote for D in the next round.

At the end of round two, the percentage of votes cast for A, B, C and D were 19%, 30%, 5% and 36% respectively.

(b) If the competition finishes at the end of round six, determine which competitor will be the most popular at that time and state the percentage of votes cast for this competitor. (3 marks)

$$\begin{bmatrix} A_6 \\ B_6 \\ C_6 \\ D_6 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.7 & 0.2 & 0.1 \\ 0 & 0.1 & 0.6 & 0 \\ 0.1 & 0.1 & 0 & 0.8 \end{bmatrix}^4 \times \begin{bmatrix} 19 \\ 30 \\ 15 \\ 36 \end{bmatrix} = \begin{bmatrix} 32.6 \\ 27.9 \\ 8.2 \\ 31.3 \end{bmatrix}$$

A is most popular, with 32.6% of votes.

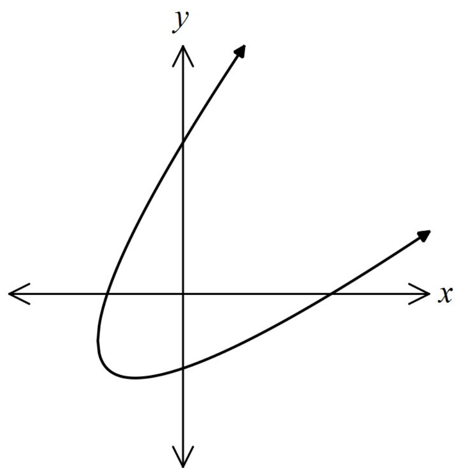
(c) Rank the competitors in order of popularity at the end of round one. Justify your answer. (3 marks)

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.7 & 0.2 & 0.1 \\ 0 & 0.1 & 0.6 & 0 \\ 0.1 & 0.1 & 0 & 0.8 \end{bmatrix}^{-1} \times \begin{bmatrix} 19 \\ 30 \\ 15 \\ 36 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \\ 20 \\ 40 \end{bmatrix}$$

Ranking is D, B, C and A.

Question 10 **(7 marks)**

The graph of $x+y=(x-y)^2-2$ is shown below.



(a) Determine the coordinates of all axes intercepts. (2 marks)

$x=0$

$y=y^2-2$

$(y+1)(y-2)=0 \Rightarrow y=-1,2$

$y=0 \Rightarrow x=-1,2$ (symmetry)

At $(0,-1)$, $(0,2)$, $(2,0)$, $(-1,0)$

(b) Show that the gradient at any point on the curve is given by $\frac{dy}{dx}=\frac{2y-2x+1}{2y-2x-1}$. (3 marks)

$x+y=(x-y)^2-2$

$x+y=x^2-2xy+y^2-2$

$1+y'=2x-2y-2xy'+2yy'$

$2y-2x+1=(2y-2x-1)y'$

$$\frac{dy}{dx}=\frac{2y-2x+1}{2y-2x-1}$$

(c) Determine the equation of the tangent to the curve at the point where the curve intersects the positive y-axis. (2 marks)

At (0, 2):

$$\frac{dy}{dx} = \frac{2(2)+1}{2(2)-1} = \frac{5}{3}$$

$$y = \frac{5}{3}x + 2$$

Question 11 **(8 marks)**

The vector equation of plane P is $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$.

(a) Show that the vector $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ is perpendicular to plane P. (3 marks)

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = -2 - 1 + 3 = 0 \Rightarrow \text{ perpendicular}$$

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = 1 + 2 - 3 = 0 \Rightarrow \text{ perpendicular}$$

Since $(-2\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$ are not parallel and $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ is perpendicular to both, then it is also perpendicular to plane P.

(b) Determine a vector equation of the plane in the form $\mathbf{r} \cdot \mathbf{n} = c$. (2 marks)

$$\mathbf{r} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\mathbf{r} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = 10$$

(c) Determine where the line $\mathbf{r} = -\mathbf{j} + \mathbf{k} + \alpha(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ intersects the plane P. (3 marks)

$$\begin{bmatrix} 2\alpha \\ 2\alpha - 1 \\ \alpha + 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = 10$$

$2\alpha + 1 - 2\alpha + 3\alpha + 3 = 10$

$3\alpha = 6 \Rightarrow \alpha = 2$

Intersect at

$-\mathbf{j} + \mathbf{k} + 2(2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$

Question 12 **(7 marks)**

Let a, b and c be integers such that $a^2 + b^2 = c^2$.

(a) State which **one** of the following statements is true, and supply counterexamples to disprove the two false statements:

(3 marks)

(i) c is always even.

$3^2 + 4^2 = 5^2 \Rightarrow c = 5$ and is odd.

(ii) At least one of the integers a or b must be odd.

$6^2 + 8^2 = 10^2 \Rightarrow a = 6,\ b = 8$ and neither is odd.

(iii) At least one of the integers a or b must be even.

True

(b) Prove, by contradiction, the true statement from (a). (4 marks)

Assume that a is not even **and** b is not even.

Then $a = 2n + 1$ and $b = 2m + 1$, where n and m are integers.

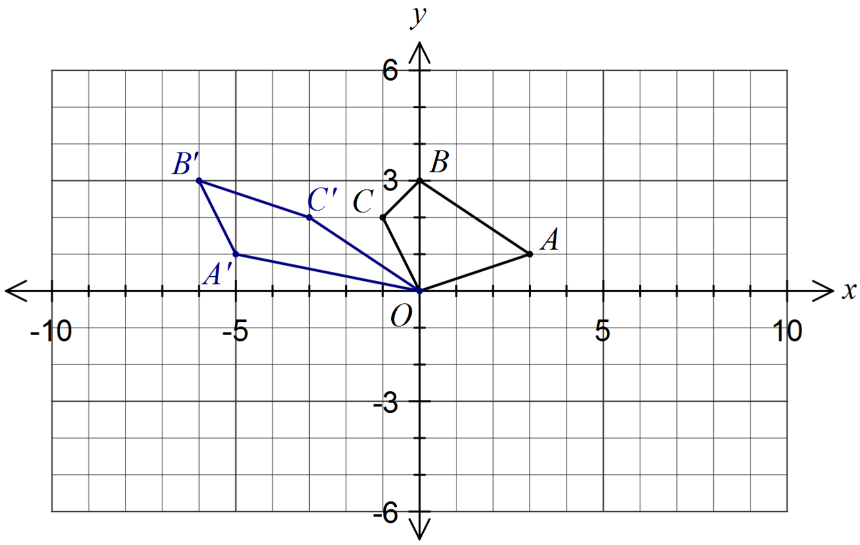
$$a^2 + b^2 = c^2$$
$$(2n + 1)^2 + (2m + 1)^2 = c^2$$
$$4n^2 + 4n + 1 + 4m^2 + 4m + 1 = c^2$$
$$2(2n^2 + 2n + 2m^2 + 2m + 1) = c^2$$

The assumption implies that c^2, and hence c, must be even, but the counterexample from (a)(i) shows that this is not true, and so we have a contradiction.

Thus, either a is even or b is even.

Question 13 **(9 marks)**

The diagram below shows the quadrilateral $OABC$ of area 6 square units, where O is at the origin and A, B and C have coordinates (3, 1), (0, 3) and (-1, 2).



(a) $OABC$ is transformed by a linear transformation P to $OA'B'C'$, where the coordinates of A' and B' are (-5, 1) and (-6, 3) respectively.

(i) Determine the transformation matrix P. (2 marks)

$$P\begin{bmatrix}3 & 0\\1 & 3\end{bmatrix}=\begin{bmatrix}-5 & -6\\1 & 3\end{bmatrix}$$

$$P=\begin{bmatrix}-5 & -6\\1 & 3\end{bmatrix}\times\begin{bmatrix}3 & 0\\1 & 3\end{bmatrix}^{-1}$$

$$=\begin{bmatrix}-1 & -2\\0 & 1\end{bmatrix}$$

(ii) State the coordinates of C' and draw the quadrilateral $OA'B'C'$ on the diagram above. (2 marks)

$$\begin{bmatrix}-1 & -2\\0 & 1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}-3\\2\end{bmatrix}\Rightarrow C'(-3,\ 2)$$

(b) Transformation P is the equivalent of transformation Q followed by transformation R, where R is a reflection in the y-axis.

(i) Determine the appropriate transformation matrix for Q. (2 marks)

$$R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = RQ$$
$$Q = PR^{-1}$$
$$= \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

(ii) Describe transformation Q. (1 mark)

Shear, factor 2, parallel to the x-axis.

(c) $OA'B'C'$ is transformed by the matrix $S = \begin{bmatrix} -3 & 15 \\ -6 & 3 \end{bmatrix}$ to $OA''B''C''$.

Determine the area of $OA''B''C''$. (2 marks)

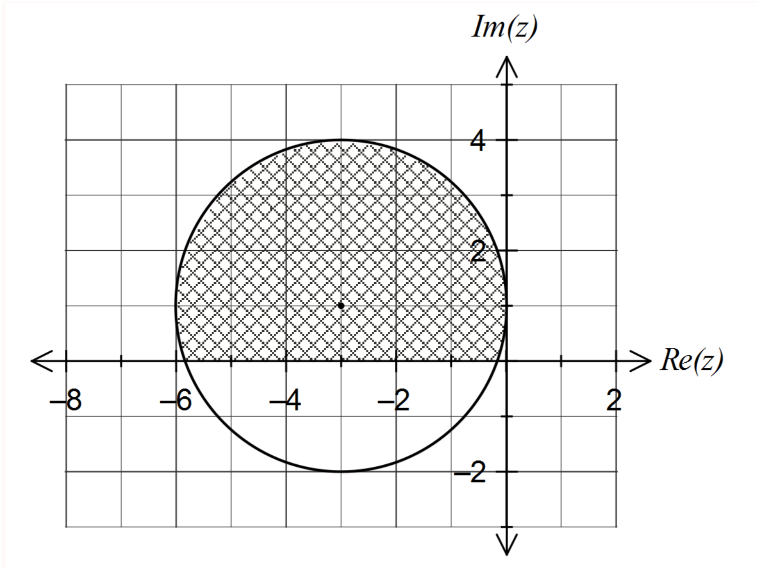
Original area of OABC is 6 sq units.

P has determinant of -1, so P does not change area.

S has determinant 81, so area of $OA''B''C''$ is $6 \times 81 = 486$ sq units.

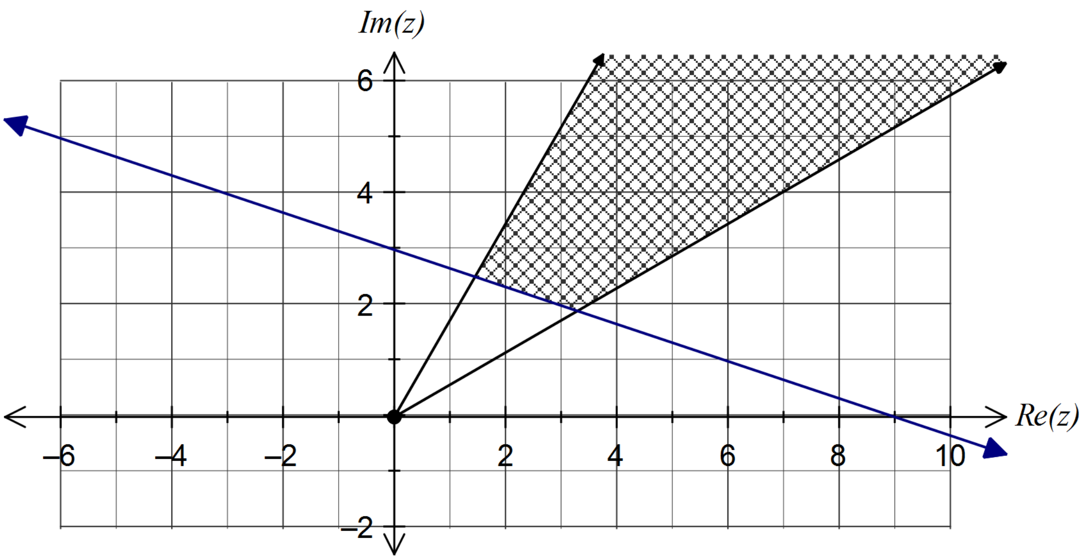
Question 14 **(9 marks)**

(a) Use inequalities to describe the shaded region in the complex plane below. (3 marks)



$\{ \mathrm{Im}(z) \geq 0 \} \cap \{ |z + 3 - i| \leq 3 \}$

(b) The set of complex numbers that satisfy $\frac{\pi}{6} \leq \arg(z) \leq \frac{\pi}{3}$ is shown below.



(i) Add the set of complex numbers $|z + 1| \geq |z - 1 - 6i|$ to the diagram and clearly shade the region that satisfies both inequalities. (3 marks)

(ii) Determine, in exact form, the minimum value of $\text{Im}(z)$ in the region that satisfies both inequalities. (3 marks)

$(i)\ y = \tan\left(\frac{\pi}{6}\right)x \Rightarrow x = \sqrt{3}y$

$(ii)\ y = 3 - \frac{1}{3}x \Rightarrow x = 9 - 3y$

Min Im(z) is y-coord at intersection of (i) and (ii)

$$\sqrt{3}y = 9 - 3y \Rightarrow y = \frac{9}{3+\sqrt{3}} = \frac{3\left(3-\sqrt{3}\right)}{2}$$

Question 15 **(7 marks)**

At 0830 on a windless day, a light aircraft takes off from an airport and flies due east at a constant 300 km/h. At the same instant, a passenger aircraft is 50 km due south of the airport and is flying directly towards the airport at a constant speed of 400 km/h.

Determine the rate, to the nearest km/h, that the distance between the two aircraft is changing at the instant the passenger plane is 30 km from the airport.

Let a and b be distance of light and passenger aircraft from airport respectively.

$$a = 300t$$

$$b = 50 - 400t$$

$$30 = 50 - 400t \Rightarrow t = \frac{1}{20} \text{ h}$$

$$\begin{aligned} x^2 &= a^2 + b^2 \\ &= (300t)^2 + (50 - 400t)^2 \end{aligned}$$

$$x = 50\sqrt{100t^2 - 16t + 1}$$

$$\frac{dx}{dt} = \frac{25(200t - 16)}{\sqrt{100t^2 - 16t + 1}}$$

$$t = \frac{1}{20}$$

$$\begin{aligned} \frac{dx}{dt} &= -100\sqrt{5} \text{ km/h} \\ &\approx -223.6 \text{ km/h} \end{aligned}$$

Distance is decreasing at 224 km/h.

Question 16 **(9 marks)**

The path of a particle is given by the parametric equations $x = 2\cos(2t)$ and $y = \cos^2(2t) - 1$, where x and y are measured in metres and t is the time in seconds since motion began.

(a) State the minimum and maximum values of x and y. (2 marks)

$$x_{MIN} = -2,\ x_{MAX} = 2,\ y_{MIN} = -1,\ y_{MA}$$

(b) Determine the value of $\frac{dy}{dx}$ at A, the position of the particle when $t = \frac{\pi}{6}$. (2 marks)

$$\frac{dx}{dt} = -4\sin(2t)$$

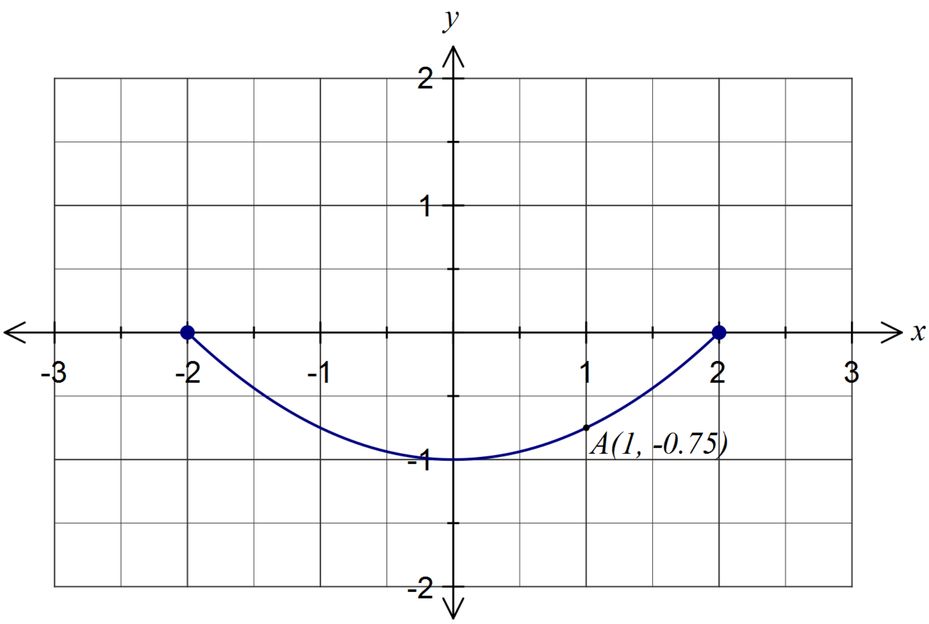
$$\frac{dy}{dt} = -4\cos(2t)\sin(2t)$$

$$\frac{dy}{dx} = \cos(2t)\Big|_{t=\frac{\pi}{6}} = \frac{1}{2}$$

(c) Determine the Cartesian equation of the path of the particle. (2 marks)

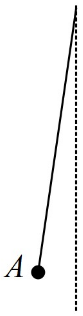
$$\cos(2t) = \frac{x}{2} \Rightarrow y = \left(\frac{x}{2}\right)^2 - 1$$

(d) Sketch the path of the particle, indicating the position of A. (3 marks)



Question 17 **(8 marks)**

A simple pendulum has a point mass on the end of a long string.



The mass, initially stationary, is released from position A. The mass swings a total distance of 48 cm away from A and then swings back again, returning to its initial position after a total of six seconds. The pendulum repeats this motion, with no measurable loss in amplitude from swing to swing.

The horizontal distance, x cm, of the point mass from A, t seconds after release, can be modelled by the equation $x = a\cos(nt - b) + c$, where a, n, b and c are positive constants.

(a) Express the acceleration $\frac{d^2x}{dt^2}$ in terms of x, n and c. (2 marks)

$$\frac{dx}{dt} = -na\sin(nt - b)$$

$$\frac{d^2x}{dt^2} = -n^2a\cos(nt - b)$$
$$= -n^2(x - c)$$

(b) Determine the values of the positive constants a, n, b and c. (4 marks)

Amplitude of motion: $a = 48 \div 2 = 24$.

Period: $n = \frac{2\pi}{6} = \frac{\pi}{3}$.

Horizontal shift: $b = \pi$.

Vertical shift: $c = 24$.

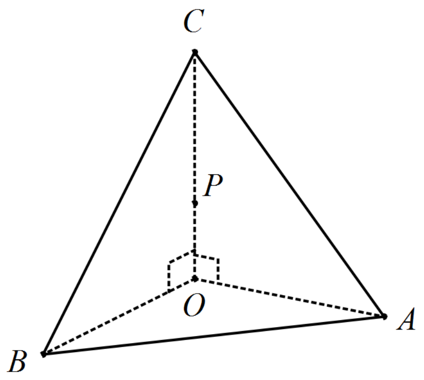
(c) At what time(s) during the first ten seconds is the mass moving with maximum speed? (2 marks)

From (a), $x - c = 0$ when $x = 24$.

When $t = 1.5,\ 4.5 \text{ and } 7.5 \text{ seconds}$.

Question 18 **(5 marks)**

Pyramid $OABC$ is such that edge OC is perpendicular to the triangular base OAB, angle AOB is obtuse and the length of OC is greater than that of OA and OB. Let $\overrightarrow{OA}=$¿$\mathbf{a}$, $\overrightarrow{OB}=$¿$\mathbf{b}$ and $\overrightarrow{OC}=$¿$\mathbf{c}$.



Suppose that $\angle AOB=\theta$ and P is a point on OC such that $\overrightarrow{OP}=\mu\overrightarrow{OC}$ and $\angle APB=\frac{\pi}{2}$.

(a) Express $\overrightarrow{AP}$ and $\overrightarrow{BP}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. (1 mark)

$\overrightarrow{AP}=-\mathbf{a}+\mu\mathbf{c}$. $\overrightarrow{BP}=-\mathbf{b}+\mu\mathbf{c}$

(b) Prove that when $\angle APB=\frac{\pi}{2}$, $\cos\theta=\frac{-\mu^2|\mathbf{c}|^2}{|\mathbf{a}|\times|\mathbf{b}|}$. (4 marks)

$\mathbf{AP}\cdot\mathbf{BP}=0$

$(-\mathbf{a}+\mu\mathbf{c})\cdot(-\mathbf{b}+\mu\mathbf{c})=0$

$\mathbf{a}\cdot\mathbf{b}-\mu\mathbf{a}\cdot\mathbf{c}-\mu\mathbf{b}\cdot\mathbf{c}+\mu^2\mathbf{c}\cdot\mathbf{c}=0$

$|a|\times|b|\cos\theta+\mu^2|c|^2=0$ since $\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{c}=0$

$\cos\theta=\frac{-\mu^2|\mathbf{c}|^2}{|\mathbf{a}|\times|\mathbf{b}|}$

Question 19 **(8 marks)**

(a) Using your calculator, or otherwise, determine the real part of

(i) $(x+iy)^3$. (1 mark)

$x^3 - 3xy^2$

(ii) $(\cos\theta + i\cdot\sin\theta)^3$. (1 mark)

$\cos^3\theta - 3\cos\theta\cdot\sin^2\theta$

(b) Using the identity $(\cos\theta + i\cdot\sin\theta)^n = \cos n\theta + i\cdot\sin n\theta$ and the result from (a)(ii), show that $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$. (3 marks)

$\cos 3\theta + i\cdot\sin 3\theta = (\cos\theta + i\cdot\sin\theta)^3$

Equate real parts:

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\cdot\sin^2\theta\\ &= \cos^3\theta - 3\cos\theta\cdot(1-\cos^2\theta)\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

$$4\cos^3\theta = \cos 3\theta + 3\cos\theta$$

$$\cos^3\theta = \tfrac{1}{4}\cos 3\theta + \tfrac{3}{4}\cos\theta$$

(c) Hence show that $\int_0^{\pi/2} \cos^3\theta\, d\theta = \frac{2}{3}$. (3 marks)

$$\int_0^{\pi/2} \cos^3\theta\, d\theta = \frac{1}{4}\int_0^{\pi/2} \cos 3\theta + 3\cos\theta\, d\theta$$

$$= \frac{1}{4}\left[\frac{\sin 3\theta}{3} + 3\sin\theta\right]_0^{\pi/2}$$

$$= \tfrac{1}{4}\left((3 - \tfrac{1}{3}) - (0)\right)$$

$$= \tfrac{2}{3}$$

Question 20 **(9 marks)**

A small body A leaves position $10\mathbf{i} + 20\mathbf{j} - 10\mathbf{k}$ metres, and moves with a constant velocity of $2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ metres per second.

(a) A small body B leaves $10\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}$ meters five seconds later and collides with A after another ten seconds. Determine the constant velocity of B. (3 marks)

$$R_A = \begin{bmatrix} 10 \\ 20 \\ -10 \end{bmatrix} + 15\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 40 \\ 95 \\ 35 \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ -5 \\ -5 \end{bmatrix} + 10V_B = \begin{bmatrix} 40 \\ 95 \\ 35 \end{bmatrix}$$

$$V_B = \frac{1}{10}\left(\begin{bmatrix} 40 \\ 95 \\ 35 \end{bmatrix} - \begin{bmatrix} 10 \\ -5 \\ -5 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 10 \\ 4 \end{bmatrix}$$

(b) Another small body C leaves $-15\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$ metres at the same time as A, moving with a constant velocity of $x\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ metres per second. In the subsequent motion, the minimum distance between A and C occurs exactly four seconds later. Determine all possible values of x. (6 marks)

$${}_AV_B = \begin{bmatrix} 2 - x \\ 5 - 6 \\ 3 - -1 \end{bmatrix} = \begin{bmatrix} 2 - x \\ -1 \\ 4 \end{bmatrix}$$

$${}_AR_B = \begin{bmatrix} 10 - -15 \\ 20 - 6 \\ -10 - -5 \end{bmatrix} = \begin{bmatrix} 25 \\ 14 \\ -5 \end{bmatrix}$$

$${}_AV_B \cdot ({}_AR_B + t\,{}_AV_B) = 0 \text{ when } t = 4$$

$$\begin{bmatrix} 2 - x \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 25 + 4(2 - x) \\ 14 - 4 \\ -5 + 16 \end{bmatrix} = \begin{bmatrix} 2 - x \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 33 - 4x \\ 10 \\ 11 \end{bmatrix} = 0$$

$$(2 - x)(33 - 4x) - 10 + 44 = 0$$
$$4x^2 - 41x + 100 = 0$$
$$(x - 4)(4x - 25) = 0$$

$$x = 4,\ x = 6.25$$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

Additional working space

Question number: _________

Additional working space

Question number: _________

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