MATHEMATICS DEPARTMENT

Year 12 Methods - Test Number 3 - 2016

Integration and the Binomial Distribution

SOLUTIONS

1 The statement $\int_b^a f(x)dx = \int_a^b f(x)dx$ is not correct since $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

$\therefore$ **C** **[2 marks]**

2 The width of each rectangle is 0.2 and the centres are at 0.1, 0.3, 0.5, …, 0.9

The heights are $f(0.1)$, $f(0.3)$, $f(0.5)$, …, $f(1.9)$

Total area $= 0.2 \times 0.1^3 + 0.2 \times 0.3^3 + 0.2 \times 0.5^3 + 0.2 \times 0.7^3 + 0.2 \times 0.9^3 + 0.2 \times 1.1^3$
$+ 0.2 \times 1.3^3 + 0.2 \times 1.5^3 + 0.2 \times 1.7^3 + 0.2 \times 1.9^3$

$= 0.2 \times [0.1^3 + 0.3^3 + 0.5^3 + 0.7^3 + 0.9^3 + 1.1^3 + 1.3^3 + 1.5^3 + 1.7^3 + 1.9^3]$

$\therefore$ **D** **[2 marks]**

3 The algebraic area between $x = -4$ and $x = 1$ is negative, so $-\int_{-4}^{1} f(x)dx$ will give the physical area.

$\therefore$ **E** **[2 marks]**

4 Width of each rectangle is 0.5 units.

Heights are $f(0)$, $f(0.5)$, $f(1)$, $f(1.5)$

i.e. $4 - 0^2$, $4 - 0.5^2$, $4 - 1^2$, $4 - 1.5^2$

Total area $= 0.5 \times (4 - 0^2) + 0.5 \times (4 - 0.5^2) + 0.5 \times (4 - 1^2) + 0.5 \times (4 - 1.5^2)$

$= 0.5 \times [4 - 0^2 + 4 - 0.5^2 + 4 - 1^2 + 4 - 1.5^2$

$\therefore$ **A** **[2 marks]**

5 $\int_0^{\frac{\pi}{6}} \sin(x)dx = [-\cos(x)]_0^{\frac{\pi}{6}}$

$= -\cos\left(\frac{\pi}{6}\right) - [-\cos(0)]$

$= -\frac{\sqrt{3}}{2} + 1$

$= \frac{2}{2} - \frac{\sqrt{3}}{2}$

$= \frac{2-\sqrt{3}}{2}$

$\therefore$ **B** **[2 marks]**

6 $\int_0^2 [5f(x)+3]dx = 5\int_0^2 f(x)dx + \int_0^2 3dx$

$= 5\int_0^2 f(x)dx + [3x]_0^2 = 5\int_0^2 f(x)dx + 6$

$\therefore$ **D** **[2 marks]**

7 Area between x = 0 and x = 5 is $\int_0^5 f(x) - g(x)dx$

Area between x = 5 and x = 8 is $\int_5^8 g(x) - f(x)dx$ total area = $\int_0^5 f(x) - g(x)dx + \int_5^8 g(x) - f(x)dx$

$\therefore$ **C** **[2 marks]**

8 $\int_0^4 (6\sqrt{x} - x)dx = \int_0^4 (6x^{\frac{1}{2}} - x)dx$

$= \left[\frac{6 \times 2x^{\frac{3}{2}}}{3} - \frac{x^2}{2}\right]_0^4$

$= \left[4x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^4$

$= \left(4 \times 4^{\frac{3}{2}} - \frac{4^2}{2}\right) - (0-0)$

$= 4 \times 2^3 - 8$

$= 24$

$\therefore$ **D** **[2 marks]**

9 $\frac{d}{dx}e^{x^2-6x} = 2(x-3)\,e^{x^2-6x}$

So $\int 2(x-3)e^{x^2-6x}\,dx = e^{x^2-6x} + c$

So $\int (x-3)e^{x^2-6x}\,dx = \frac{1}{2}\int (x-3)e^{x^2-6x}\,dx$

$= \frac{1}{2}e^{x^2-6x} + c$

$\therefore$ **E** **[2 marks]**

10 This is a binomial experiment with $p = \frac{2}{3}$, $q = \frac{1}{3}$, n = 4 and x = 2.

$P(X=2) = \binom{4}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = \frac{8}{27}$

[3 marks]

Resource Rich Section

1 Let X be the number of prisoners who reoffend.

$n = 10$

$p = 0.68$

$P(X = x) = \binom{10}{x}(0.68)^x(0.32)^{10-x}$ **[1 mark]**

$P(X \geq 4) = 1 - P(X < 4)$

$= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)]$ **[1 mark]**

$= 1 - \left[\binom{10}{0}(0.32)^{10} + \binom{10}{1}(0.68)^1(0.32)^9 + \binom{10}{2}(0.68)^2(0.32)^8 + \binom{10}{1}(0.68)^3(0.32)^7\right]$

[1 mark]

$\approx 1 - (0.000\,011 + 0.000\,239 + 0.002\,288 + 0.012\,965)$

$= 0.9845$ **[1 mark]**

2 **a** For any binomial experiment $P(X = x) = \binom{n}{x} p\, q^{n-x}$

For this binomial experiment $P(X = x) = \binom{6}{x}(0.45)^x(0.55)^{6-x}$

$n = 6$ **[1 mark]**

b $p = 0.45$ **[2 marks]**

c $P(X = 0) = \binom{6}{0}(0.45)^0(0.55)^6 \approx 0.0277$

$P(X = 1) = \binom{6}{1}(0.45)^1(0.55)^5 \approx 0.1359$

$P(X = 2) = \binom{6}{2}(0.45)^2(0.55)^4 \approx 0.2780$

And so on.

x	0	1	2	3	4	5	6
p(x)	0.0277	0.1359	0.2780	0.3032	0.1861	0.0609	0.0083

[3 marks]

3 **a** This is an example of a binomial experiment.

$n = 7$

$p = 0.25$, $q = 0.75$

X = number of bullseyes

$P(\text{at least 2 bullseyes}) = P(X \geq 2)$

$= 1 - P(X < 2)$ **[1 mark]**

$= 1 - [P(X = 1) + P(X = 0)]$

$= 1 - \binom{7}{1}(0.25)^1(0.75)^6 - (0.75)^7$

$= 1 - 0.3134... - 0.1314...$

≈ 0.5551 **[1 mark]**

b $n = ?$

$p = 0.25$, $q = 0.75$

X = number of bullseyes

$P(X \geq 1) > 0.9$

$P(X = 1) + P(X = 2) + P(X = 3) + ... > 0.9$

$1 - P(X = 0) > 0.9$ **[1 mark]**

$1 - (0.75)^n > 0.9$

$-(0.75)^n > 0.9 - 1$

$(0.75)^n < -0.9 + 1$

$(0.75)^n < 0.1$

$n > 8.0039...$ (using a graphics calculator or trial and error) **[1 mark]**

The archer must take at least 9 shots to ensure the probability of scoring at least one bullseye is at least 0.9. **[1 mark]**

4 **a** $\int_1^3 (2x - 9)dx = \left[x^2 - 9x\right]_1^3$ **[1 mark]**

$= (3^2 - 9 \times 3) - (1^2 - 9 \times 1)$ **[1 mark]**

$= 9 - 27 - 1 + 9$

$= -10$ **[1 mark]**

b $\int_2^6 e^x\,dx = \left[e^x\right]_2^6$ **[1 mark]**

$= e^6 - e^2$ **[1 mark]**

$= e^2(e^4 - 1)$ **[1 mark]**

c $\int_0^\pi \cos(x)dx = \left[\sin(x)\right]_0^\pi$ **[1 mark]**

$= \sin(\pi) - \sin(0)$ **[1 mark]**

$= 0 - 0$

$= 0$ **[1 mark]**

d $\int_{-2}^1 (x^2 - 3x + 5)dx = \left[\frac{x^3}{3} - \frac{3x^2}{2} + 5x\right]_{-2}^1$ **[1 mark]**

$= \left(\frac{1^3}{3} - \frac{3 \times 1^2}{2} + 5 \times 1\right) - \left(\frac{(-2)^3}{3} - \frac{3(-2)^2}{2} + 5 \times -2\right)$ **[1 mark]**

$= \left(\frac{1}{3} - \frac{3}{2} + 5\right) - \left(\frac{-8}{3} - 6 - 10\right)$

$= \frac{1}{3} - \frac{3}{2} + 5 + \frac{8}{3} + 6 + 10$

$= 22\frac{1}{2}$ **[1 mark]**

5 **a** $\int_{-3}^{3} 2x^3\,dx = \left[\frac{2x^4}{4}\right]_{-3}^{3}$

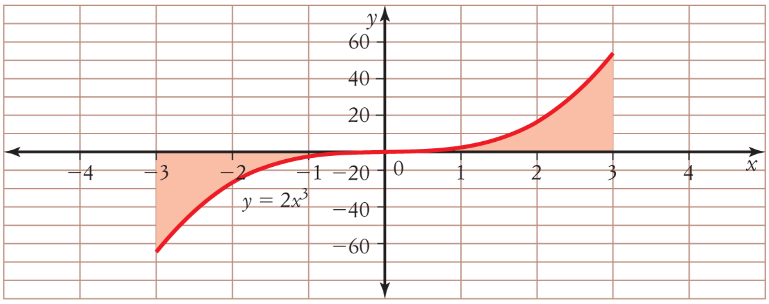
$= \left[\frac{x^4}{2}\right]_{-3}^{3}$ **[1 mark]**

$= \frac{3^4}{2} - \frac{(-3)^4}{2}$

$= \frac{81}{2} - \frac{81}{2}$

= 0 **[1 mark]**

b



$A = 2\int_{0}^{3} 2x^3\,dx$ **[1 mark]**

$= \int_{0}^{3} 4x^3\,dx$

$= \left[\frac{4x^4}{4}\right]_{0}^{3}$

$= \left[x^4\right]_{0}^{3}$ **[1 mark]**

$= 3^4 - 0^4$

= 81

The area is 81 units^2. **[2 marks]**

6 $\int_{0}^{1}(5x^3 - 2x^2 + x - 2)\,dx - \int_{0}^{1}(x^3 - 5x^2 + 4)\,dx$

$= \int_{0}^{1}(4x^3 + 3x^2 + x - 6)\,dx$ **[1 mark]**

$= \left[x^4 + x^3 + \frac{x^2}{2} - 6x\right]_{0}^{1}$ **[1 mark]**

$= (1^4 + 1^3 + \frac{1^2}{2} - 6 \times 1) - (0^4 + 0^3 + \frac{0^2}{2} - 6 \times 0)$

$= 1 + 1 + \frac{1}{2} - 6 - 0$

$= -3\frac{1}{2}$ **[1 mark]**

7 **a** $\int_{-1}^{3}(6x^2 + 4x - 1)dx = \left[\frac{6x^3}{3} + \frac{4x^2}{2} - x\right]_{-1}^{3}$

$= \left[2x^3 + 2x^2 - x\right]_{-1}^{3}$ **[1 mark]**

$= \left[2 \times 3^3 + 2 \times 3^2 - 3\right] - \left[2 \times (-1)^3 + 2 \times (-1)^2 + 1\right]$

= 69 - 1

= 68 **[1 mark]**

b $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 6\cos(3x)dx = \left[\frac{6\sin(3x)}{3}\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$

$= \left[2\sin(3x)\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$ **[1 mark]**

= 2 sin(π) - 2 sin(-π)

= 0 - 0

= 0 **[1 mark]**

c $\int_{2}^{5}\frac{dx}{(x+3)^2} = \int_{2}^{5}(x+3)^{-2}\,dx$

$= \left[\frac{(x+3)^{-1}}{1 \times -1}\right]_{2}^{5}$

$= \left[\frac{-1}{x+3}\right]_{2}^{5}$ **[1 mark]**

$= \frac{-1}{8} - \frac{-1}{5}$

$= \frac{3}{40}$ **[1 mark]**

8 $\frac{dy}{dx} = 8x - 7$

$y = 4x^2 - 7x + c$ **[1 mark]**

$y = 13$ when $x = -1$, so $13 = 4 \times (-1)^2 - 7 \times -1 + c$

$13 = 11 + c$

$c = 2$ **[1 mark]**

$y = 4x^2 - 7x + 2$ **[1 mark]**

9 Draw a sketch of $y = x^2 - 4x - 12$.

Identify the key features.

The graph is a parabola.

Let $y = 0$, $x^2 - 4x - 12 = 0$

$(x + 2)(x - 6) = 0$

$x = -2$ or 6 **[1 mark]**

Zeros are located at (-2, 0) and (6, 0).

The function has a minimum.

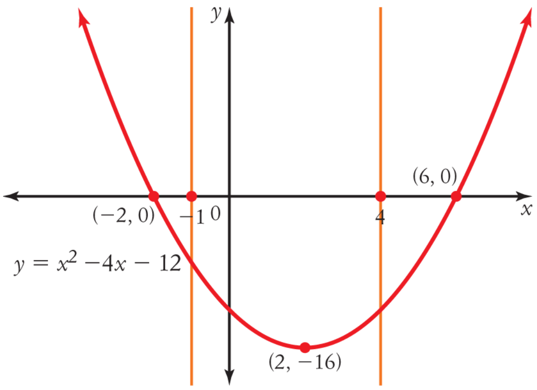
Find the derivative: $y' = 2x - 4$

Let $y' = 0$: $0 = 2x - 4$

$x = 2$

When $x = 2$: $y = -16$ **[1 mark]**

Minimum at (2, -16)



Required area = $\int_{-1}^{4} (x^2 - 4x - 12)dx$

$= \left[\frac{x^3}{3} - \frac{4x^2}{2} - 12x\right]_{-1}^{4}$

$= \left[\frac{x^3}{3} - 2x^2 - 12x\right]_{-1}^{4}$

$= \left(\frac{4^3}{3} - 2 \times 4^2 - 12 \times 4\right) - \left(\frac{(-1)^3}{3} - 2 \times (-1)^2 - 12 \times -1\right)$ **[1 mark]**

$= -\frac{176}{3} - \frac{29}{3}$

$= -\frac{205}{3}$

$= -68\frac{1}{3}$

The negative sign means the area is below the *x*-axis.

Area = $68\frac{1}{3}$ units^2 **[1 mark]**

10 Total change = $\int_{0}^{3} F'(t)dt$ **[1 mark]**

$= \int_{0}^{3} 100e^{0.2t}\, dt$ **[1 mark]**

= 381....

In the first 3 hours, about 381 L flowed into the tank. **[2 marks]**