PERTH MODERN SCHOOL
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Independent Public School

Course ________Specialist_______ Year __12______

Student name: ______________ Teacher name: ______________

Date: Fri week 5

Task type: **Response**

Time allowed for this task: **_____45______ mins**

Number of questions: **_____6______**

Materials required: No cals

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, , and up to three calculators approved for use in the WACE examinations

NO NOTES ALLOWED

Marks available: **__39____ marks**

Task weighting: **_12___%**

Formula sheet provided: **Yes**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (4.1.2) (3, 3 & 3 = 9 marks)

Determine the following integrals showing full working.

a) $\int \frac{5x}{\sqrt{7x^2-3}}dx \quad u=7x^2-3$

Solution
$\int \frac{5x}{\sqrt{7x^2-3}}dx \quad u=7x^2-3$ $\int \frac{5x}{\sqrt{7x^2-3}}\frac{dx}{du}du=\int \frac{5x}{\sqrt{u}}\frac{1}{14x}du=\frac{5}{14}\int u^{-\frac{1}{2}}du$ $=\frac{5}{14}\left[2u^{\frac{1}{2}}\right]=\frac{5}{7}(7x^2-3)^{\frac{1}{2}}+c$
Specific behaviours
✓ uses change of variable and its derivative ✓ derives new integral ✓ obtains result with a constant

b) $\int (3x+2)(5x-1)^7 dx \quad u=5x-1$

Solution
$\int (3x+2)(5x-1)^7 dx \quad u=5x-1$ $\int \left(3\left(\frac{u+1}{5}\right)+2\right)u^7 \frac{1}{5}du$ $=\int \left(\frac{3u+13}{25}\right)u^7 du$ $=\frac{1}{25}\int 3u^8+13u^7 du$ $=\frac{1}{25}\left[\frac{u^9}{3}+\frac{13}{8}u^8\right]+c$ $\frac{1}{75}(5x-1)^9+\frac{13}{200}(5x-1)^8+c$
Specific behaviours
✓ uses change of variable and its derivative ✓ derives new integral

✓ obtains result in terms of x

c) $\int \frac{\sqrt{x}}{\sqrt{x}+7}\,dx$

Solution
$let\ u = \sqrt{x}+7$ $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ $\int \frac{\sqrt{x}}{\sqrt{x}+7}\frac{dx}{du}\,du = \int \frac{2x}{u}du = \int \frac{2(u-7)^2}{u}du$ $= 2\int \frac{u^2-14u+49}{u}du = 2\int u - 14 + \frac{49}{u}\,du = 2(\frac{u^2}{2} - 14u + 49\ln u) + c$ $= \left(\sqrt{x}+7\right)^2 - 28\left(\sqrt{x}+7\right)^2 + 98\ln\left(\sqrt{x}+7\right) + c$ Alternative $\int \frac{\sqrt{x}}{\sqrt{x}+7}dx \quad u = \sqrt{x}$ $\int \frac{u}{u+7}2udu = \int \frac{2u^2}{u+7}du = \int 2u - 14 + \frac{98}{u+7}du = u^2 - 14u + 98\ln\left\|u+7\right\| + c$ $= x - 14\sqrt{x} + 98\ln\left\|\sqrt{x}+7\right\| + c$ $\begin{array}{r} 2u - 14 \\ u+7 \overline{) \quad 2u^2} \\ 2u^2 + 14u \\ -14u - 98 \\ 98 \end{array}$
Specific behaviours
✓ uses change of variable and its derivative ✓ derives new integral ✓ obtains result in terms of x

Q2 (4.1.1 -4.1.3) (3, 3 & 3 = 9 marks)

Determine the following definite integrals showing full working.

a) $\int_0^{\frac{\pi}{6}} \cos^2 4x\,dx$

Solution
$\int_0^{\frac{\pi}{6}} \cos^2 4x\,dx = \int_0^{\frac{\pi}{6}} \frac{\cos 8x + 1}{2}\,dx = \left[\frac{1}{16}\sin 8x + \frac{1}{2}x\right]_0^{\frac{\pi}{6}} = \left(\frac{-\sqrt{3}}{32} + \frac{\pi}{12}\right)$
Specific behaviours
✓ uses double angle formula for cosine ✓ obtains antiderivative ✓ obtains exact value

b) $\int_0^{\pi} \sin^3 2x\,dx$

Solution
$\int_0^{\pi} \sin^2 2x \sin 2x\,dx = \int_0^{\pi} (1 - \cos^2 2x)\sin 2x\,dx$ $= \int_0^{\pi} \sin 2x - \cos^2 2x \sin 2x\,dx$ $= \left[-\frac{1}{2}\cos 2x + \frac{1}{6}\cos^3 2x\right]_0^{\pi} = \left(\frac{-1}{2} + \frac{1}{6}\right) - \left(\frac{-1}{2} + \frac{1}{6}\right) = 0$
Specific behaviours
✓ uses trig identity ✓ obtains antiderivative ✓ subs limits to show value

c) $\int_0^{\frac{\pi}{4}} -12\tan^2 5x\,dx$

Solution
$\int_0^{\frac{\pi}{4}} -12\tan^2 5x\,dx$ $= \int_0^{\frac{\pi}{4}} -12(\sec^2 5x - 1)\,dx$ $= \left[\frac{-12}{5}\tan 5x + 12x\right]_0^{\frac{\pi}{4}} = \left(\frac{-12}{5} + 3\pi\right)$
Specific behaviours
✓ uses trig identity ✓ obtains antiderivative ✓ subs limits to show exact value

Q3 **(4 marks)**

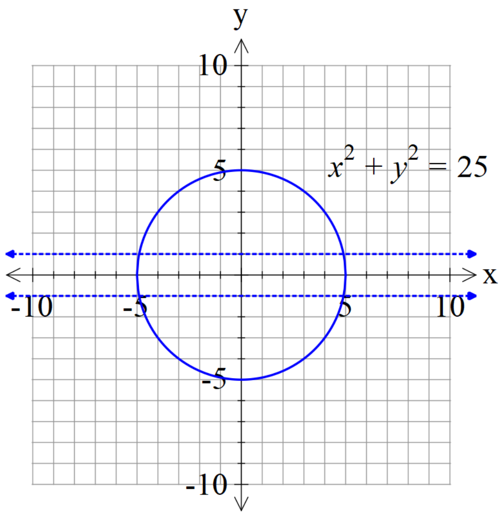
Determine the following integral showing full working.

$$\int \frac{x+7}{(x+1)(x-3)^2}\,dx$$

Solution
$\int \frac{x+7}{(x+1)(x-3)^2}\,dx$ $\frac{x+7}{(x+1)(x-3)^2} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$ $x+7 = A(x-3)^2 + B(x+1)(x-3) + C(x+1)$ $x=3$ $10 = 4C \quad C = \frac{5}{2}$ $x=-1$ $6 = 16A \quad A = \frac{3}{8}$ $x=0$ $7 = \frac{27}{8} - 3B + \frac{5}{2}$ $3B = \frac{27}{8} + \frac{20}{8} - \frac{56}{8} = -\frac{9}{8} \quad B = -\frac{3}{8}$ $\frac{3}{8}\ln\lvert x+1\rvert - \frac{3}{8}\ln\lvert x-3\rvert - \frac{5}{2}(x-3)^{-1} + C$ (Note: if only two terms used- max of 2 out of 4 marks)
Specific behaviours
✓ uses partial fractions ✓ sets up equations to solve for constants ✓ solves all 3 constants ✓ anti differentiates all terms (no need to add constant)

Q4 (4.1.5-4.1.6) (5 marks)

Consider a cylindrical drill of width 2 cm that carves a cavity inside a solid sphere of radius 5 cm as shown below. Determine the volume of the sphere remaining.(Simplify)



Solution
$x^2 + 1 = 25$ $x = \pm\sqrt{24} = \pm 2\sqrt{6}$ $V = 2\pi \int_0^{2\sqrt{6}} y^2 - 1\,dx = 2\pi \int_0^{2\sqrt{6}} 24 - x^2\,dx = 2\pi \left[24x - \frac{x^3}{3} \right]_0^{2\sqrt{6}} = 2\pi \left(48\sqrt{6} - \frac{48\sqrt{6}}{3} \right)$ $= \frac{4\pi 48}{3}\sqrt{6} \quad accept \quad or \; \frac{192\pi\sqrt{6}}{3}$ (Note- max 2 out of 5 if they have removed all of sphere between y=-1 and y=1)
Specific behaviours
✓ solves for when y=1 ✓ uses solid of revolutions integral ✓ sets up correct integral for volume remaining ✓ anti-differentiates ✓simplifies to one term/surd

Q5 (4.2.4) (4 marks)

Determine the solution to the following differential equation $yx^2\frac{dy}{dx}=\frac{x+x^3}{(5y^2+1)^4}$ given that (1,1) is a known point.(No need to simplify)

Solution
$yx^2\frac{dy}{dx}=\frac{x+x^3}{(5y^2+1)^4}$ $\int y(5y^2+1)^4\,dy=\int\frac{1}{x}+x\,dx$ $\frac{(5y^2+1)^5}{50}=\ln x+\frac{x^2}{2}+c$ $\frac{6^5}{50}-\frac{1}{2}=c$ $\frac{(5y^2+1)^5}{50}=\ln x+\frac{x^2}{2}+\frac{6^5}{50}-\frac{1}{2}$
Specific behaviours
✓ separates variables under integration ✓ integrates y terms ✓ integrates x terms ✓ solves for constant (unsimplified)

Q6 (4.2.6) (1, 5 & 2 = 8 marks)

Consider the differential equation $\frac{dN}{dt}=aN-bN^2$ with $a\,\&\,b$ positive constants.

a) Determine the limiting value for N as $t\to\propto$

Solution
$\frac{dN}{dt}=aN-bN^2=(a-bN)N=0$ $N=\frac{a}{b}$ $N<\frac{a}{b}$
Specific behaviours

✓ states value for N

b) Show how to derive using integration and partial fractions that the general solution is

$$N = \frac{a}{b + Ce^{-at}}$$

Solution
$\frac{dN}{dt} = aN - bN^2 = (a - bN)N$ $\int \frac{dN}{(a - bN)N} = \int dt$ $\frac{1}{(a - bN)N} = \frac{C}{a - bN} + \frac{D}{N}$ $1 = CN + (a - bN)D$ $N = 0$ $1 = aD \quad D = \frac{1}{a}$ $N = \frac{a}{b}$ $1 = C\frac{a}{b} \quad C = \frac{b}{a}$ $\int \frac{\frac{b}{a}}{a - bN} + \frac{\frac{1}{a}}{N} dN = t + c$ $-\frac{1}{a}\ln\|a - bN\| + \frac{1}{a}\ln N = t + c$ $\ln\frac{N}{a - bN} = at + c \quad Note: a - bN > 0 \quad as \quad N < \frac{a}{b}$ $Ce^{at} = \frac{N}{a - bN}$ $Ce^{-at} = \frac{a - bN}{N}$ $NCe^{-at} = a - bN$ $NCe^{-at} + bN = a$ $N = \frac{a}{b + Ce^{-at}}$
Specific behaviours

✓ separates variables ✓ sets up partial fractions for N ✓ shows how to find constants for partial fractions ✓ states that a-bN>0 ✓derives logistical formula

c) Consider $\frac{dN}{dt} = 5N - 3N^2$ with an initial value of $N = 1$. Determine N when t=50. (No need to simplify)

Solution
$N = \frac{5}{3 + Ce^{-5t}}$ $1 = \frac{5}{3 + C} \quad C = 2$ $N = \frac{5}{3 + 2e^{-250}}$
Specific behaviours
✓ solves for constant ✓ expresses exact value at t=50