PERTH MODERN SCHOOL
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Semester One Examination, 2022

Question/Answer booklet

MATHEMATICS SPECIALIST
UNIT 3

Section Two:
Calculator-assumed

Your Name

Your Teacher's Name

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Marks	Max	Question	Marks	Max
9			16		
10			17		
11			18		
12			19		
13			20		
14			21		
15			22		

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	14	14	100	97	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **(97 Marks)**

This section has **14** questions. Answer **all** questions. Write your answers in the spaces provided.

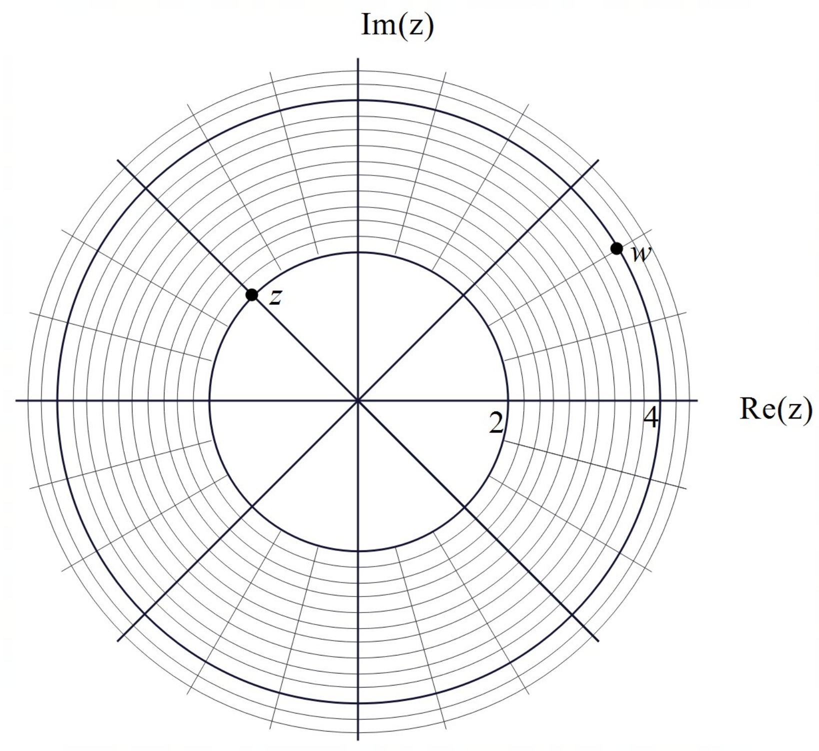
Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 100 minutes.

Question 9 **(7 marks)**

Consider the complex numbers $z \,\&\, w$ plotted on the Argand plane below.



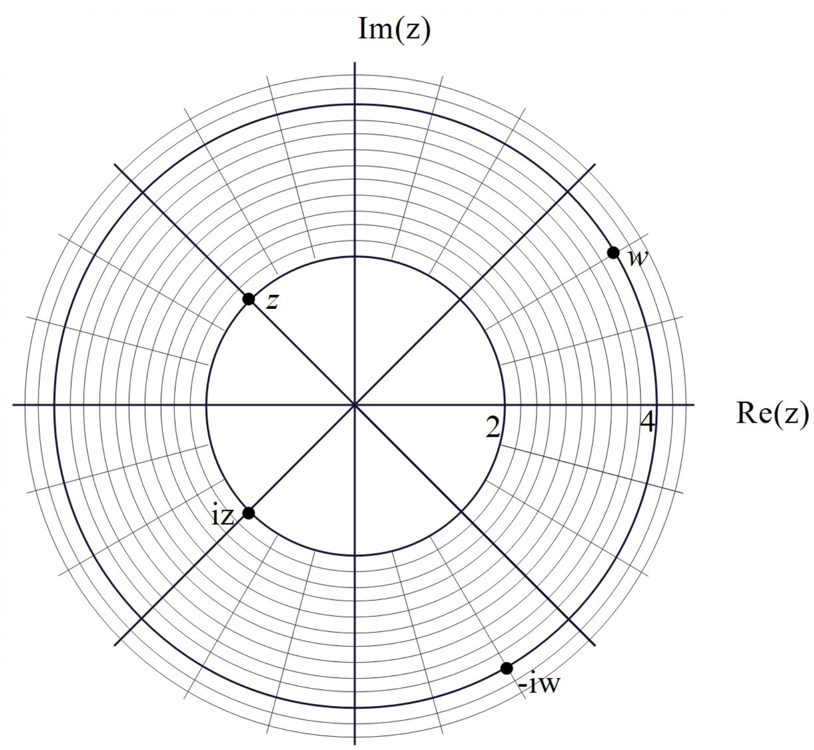
a) Express z in polar form with principal argument. (2 marks)

Solution
$z = 2cis(\frac{9\pi}{12}) \quad or 2cis(\frac{3\pi}{4})$
Specific behaviours
P states argument (unsimplified) P states modulus

b) Express w in cartesian form. (2 marks)

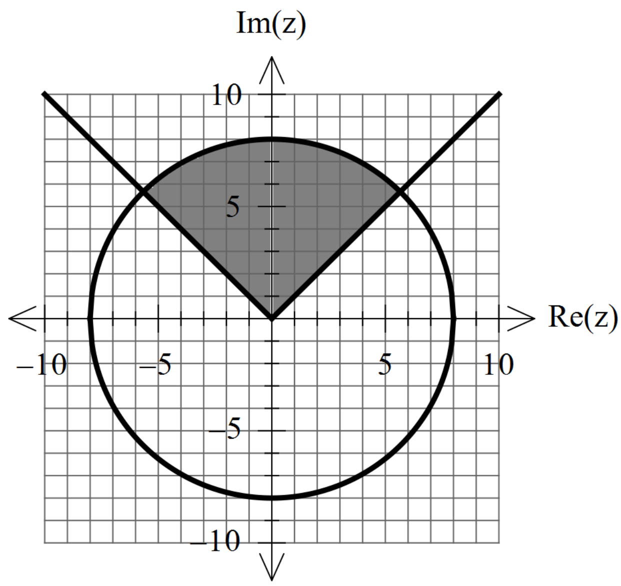
Solution
$w = 4cis\left(\frac{\pi}{6}\right) = 4\frac{\sqrt{3}}{2} + 4\frac{1}{2}i = 2\sqrt{3} + 2i$
Specific behaviours
P polar form P exact cartesian form

c) Plot iz & $-iw$ on the axes above. (3 marks)

Solution

Specific behaviours
P uses rotation around origin for both P correct argument of iz P correct argument of -iw

Question 10 **(10 marks)**

Consider the region shaded in the Argand plane below.



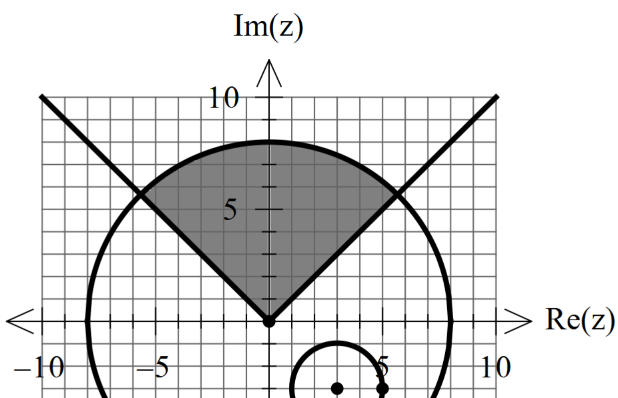
a) In terms of z, describe the region of complex numbers shaded above. (4 marks)

Solution
$\|z\|\leq 8 \cap \left(\frac{\pi}{4}\leq Arg(z)\leq\frac{3\pi}{4}\right)$

Specific behaviours
P uses the form $\|z-c\|\leq r$ P states $c=0, r=8$ P uses form $\theta_1\leq Arg(z)\leq\theta_2$ P uses $\frac{\pi}{4}$ & $\frac{3\pi}{4}$

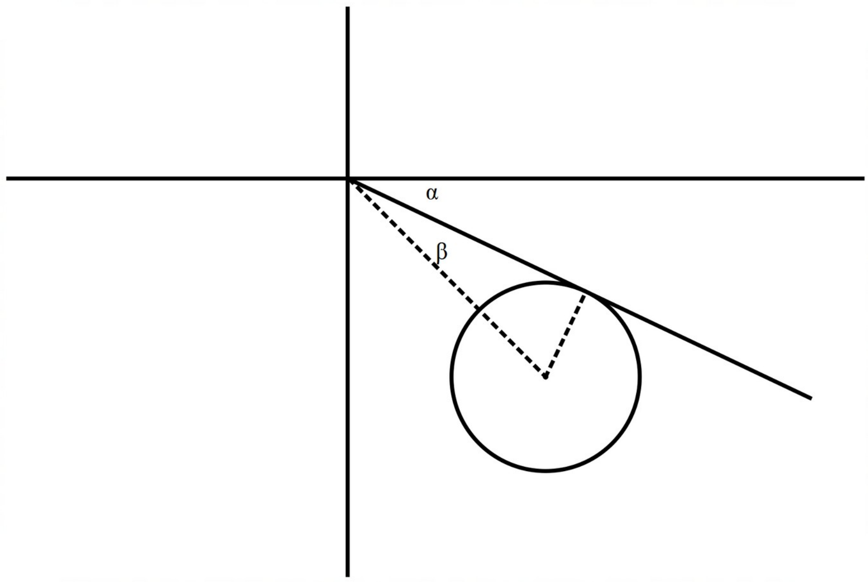
b) i) Sketch the locus w such that $|w-3+3i|=2$. (3 marks)

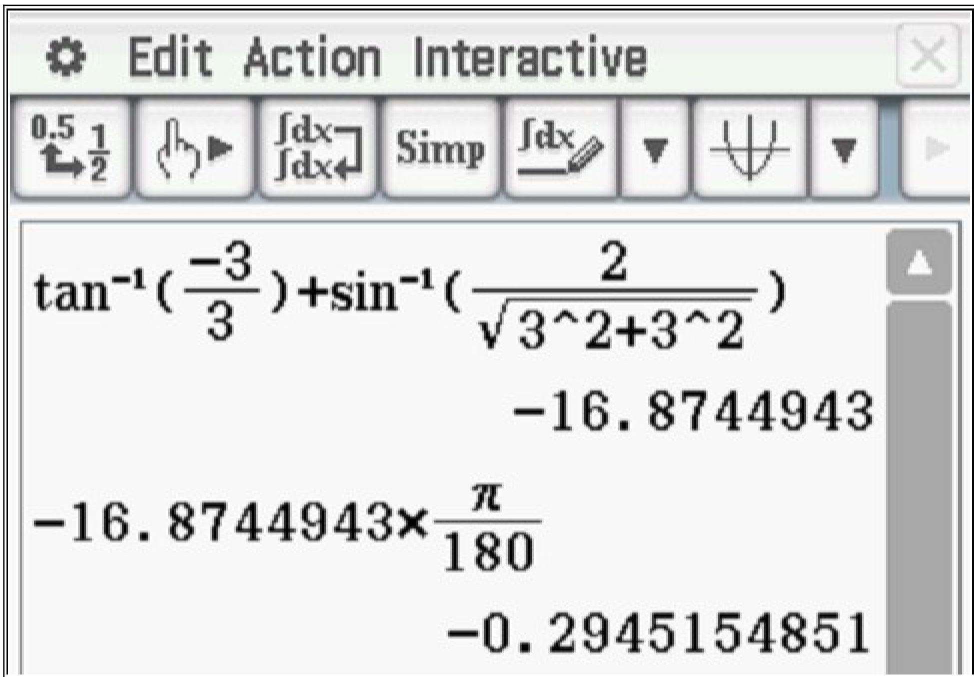
Solution



Specific behaviours
P plots a circle P centre P radius

ii) Determine the maximum value of Arg(w) (3 marks)

Solution




$\tan(\alpha+\beta)=\frac{-3}{3}$

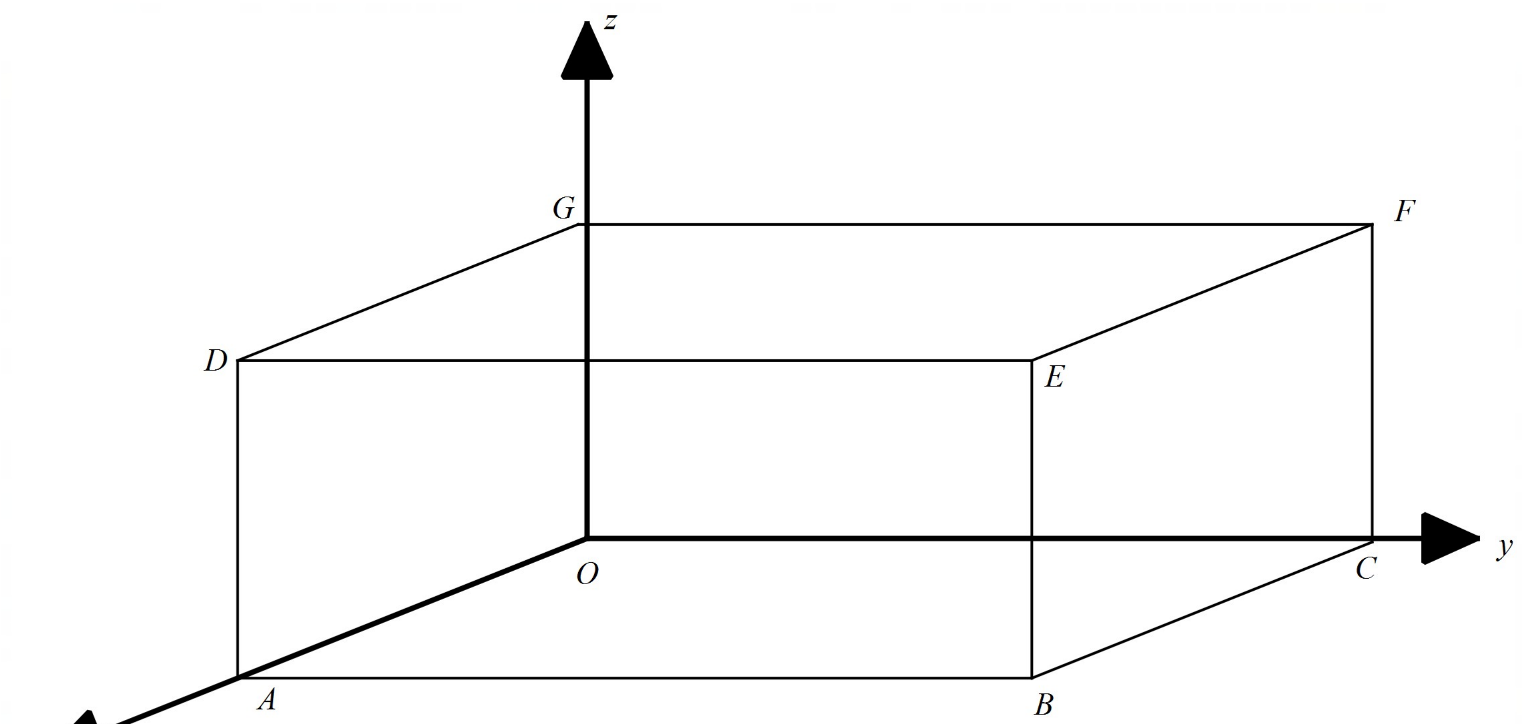
$\sin\beta=\frac{2}{\sqrt{3^2+3^2}}$

Specific behaviours

P finds argument of circle centre
P uses tangent to find angle beta
P determines max argument in radians or degrees (4^{th} quadrant)

Question 11 (8 marks)

Consider a rectangular prism $OABCDEFG$,as shown below, with $A(5,0,0), C(0,7,0) \& G(0,0,4)$



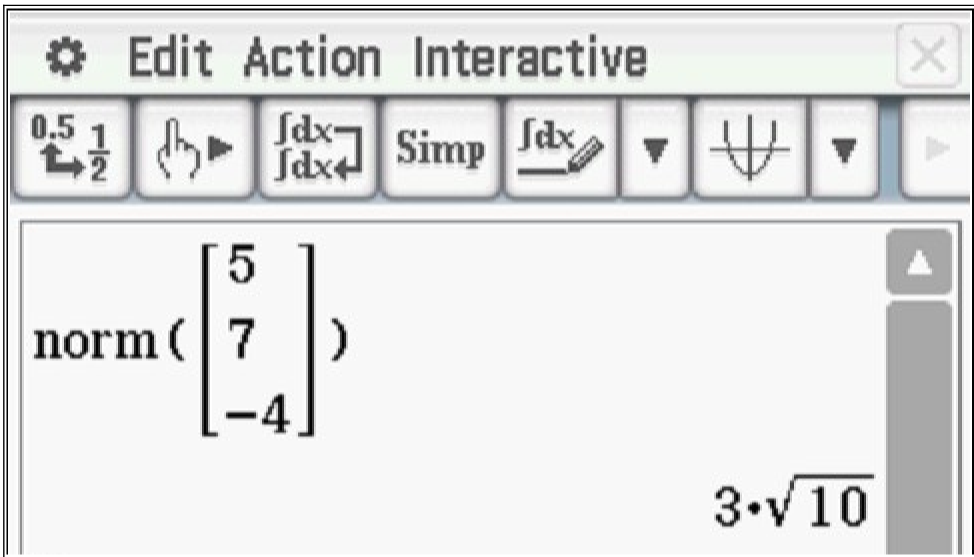
a) Prove that the diagonals $\overline{AF}\ \&\ \overline{GB}$ bisect each other using vector methods. (4 marks)

Solution
$\vec{AF} = \begin{pmatrix} 0 \\ 7 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \\ 4 \end{pmatrix}$ $\vec{GB} = \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ -4 \end{pmatrix}$ $Let\ m = midpoint\ AF$ $Let\ n = midpoint\ GB$ $\vec{OM} = \vec{OA} + \frac{1}{2}\vec{AF} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -5 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{7}{2} \\ 2 \end{pmatrix}$ $\vec{ON} = \vec{OG} + \frac{1}{2}\vec{GB} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 5 \\ 7 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{7}{2} \\ 2 \end{pmatrix}$ $ON = OM$ QED
Specific behaviours
P determines diagonals P defines two midpoints P determines position vector of one midpoint P determines position vector of both midpoints and shows identical

Q11 continued-

b) Determine the exact vector equation of a sphere that goes through all vertices of the rectangular prism $OABCDEFG$. (4 marks)

Solution
Midpoint of diagonals (5/2,7/2,2)



$$\left| r - \begin{pmatrix} \frac{5}{2} \\ \frac{7}{2} \\ 2 \end{pmatrix} \right| = \frac{1}{2} 3\sqrt{10}$$

Specific behaviours
P identifies centre P determines diameter or radius P uses vector equation of a sphere P gives exact equation

Question 12 **(9 marks)**

Consider rockets $A \& B$ that are ignited at the same times from the following positions and constant velocities. (At time $t = 0$)

$$r_A = \begin{pmatrix} 12 \\ -18 \\ 20 \end{pmatrix} km \quad , r_B = \begin{pmatrix} -35 \\ 22 \\ -8 \end{pmatrix} km$$

$$v_A = \begin{pmatrix} -7 \\ 14 \\ 2 \end{pmatrix} km/h \quad , v_B = \begin{pmatrix} 15 \\ -7 \\ 5 \end{pmatrix} km/h$$

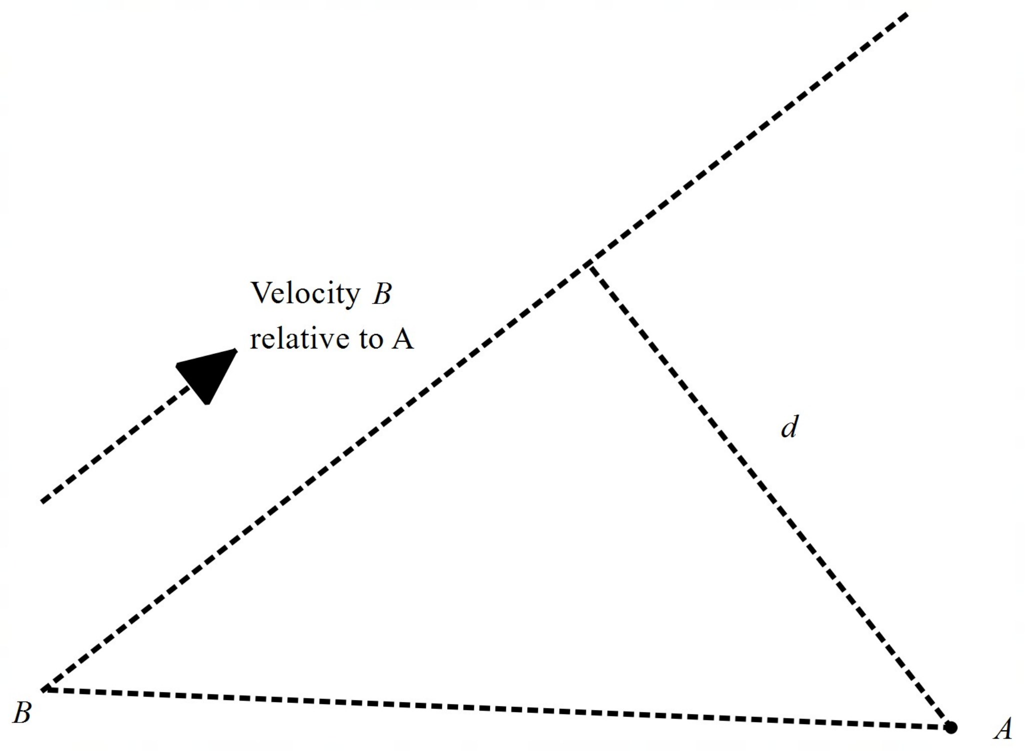
a) Prove using vector methods that the two rockets do not meet. (3 marks)

Solution

$r_A = \begin{pmatrix} 12 \\ -18 \\ 20 \end{pmatrix} km \quad , r_B = \begin{pmatrix} -35 \\ 22 \\ -8 \end{pmatrix} km$ $v_A = \begin{pmatrix} -7 \\ 14 \\ 2 \end{pmatrix} km/h \quad , v_B = \begin{pmatrix} 15 \\ -7 \\ 5 \end{pmatrix} km/h$ $\begin{pmatrix} 12 \\ -18 \\ 20 \end{pmatrix} + t\begin{pmatrix} -7 \\ 14 \\ 2 \end{pmatrix} = \begin{pmatrix} -35 \\ 22 \\ -8 \end{pmatrix} + t\begin{pmatrix} 15 \\ -7 \\ 5 \end{pmatrix}$ $12 - 7t = -35 + 15t$ $22t = 47 \quad t = \frac{47}{22}$ $-18 + 14t = 22 - 7t$ $21t = 40 \ t = \frac{40}{21}$
Specific behaviours
P determines vector equation in terms of t for one rocket P determines vector equation in terms of t for both rockets P shows that t values different for 2 components

b) Determine the closest approach between the two rockets **using vector methods**.

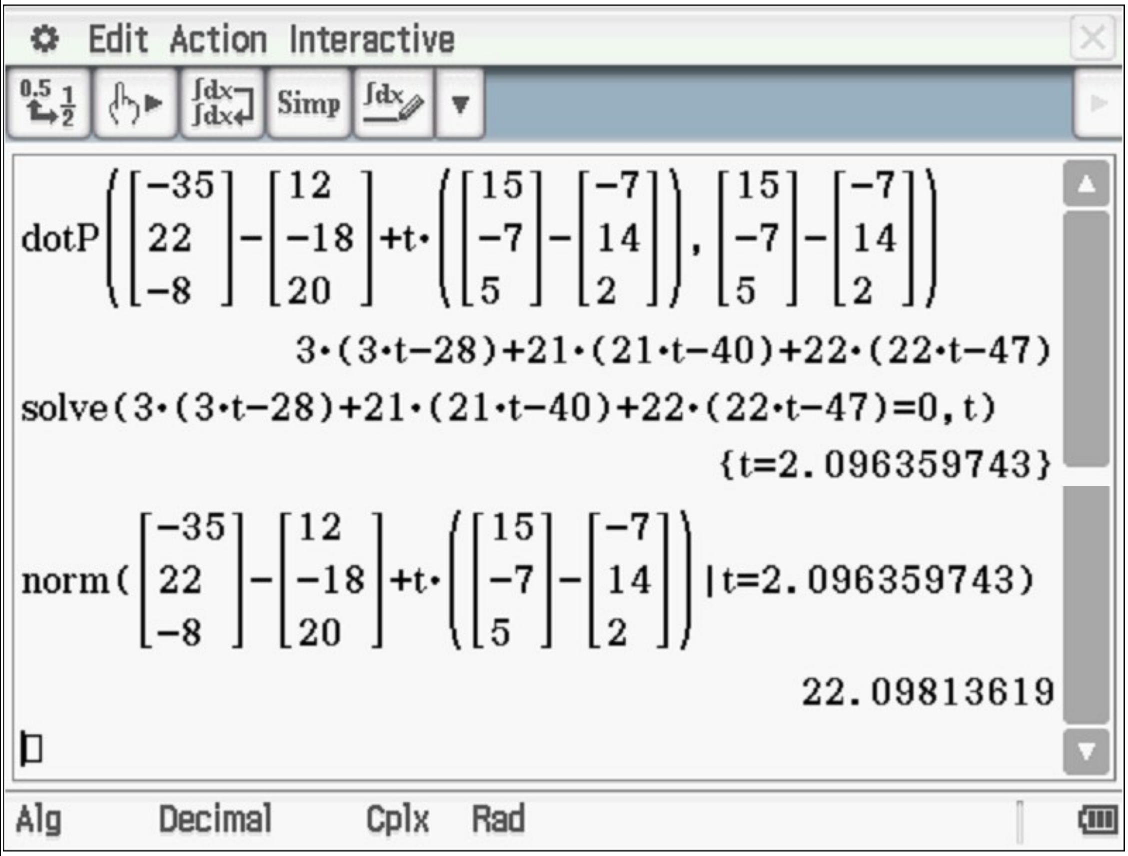
(4 marks)



Solution

$\vec{d} = AB + tV_{rel}$

$d \cdot V_{rel} = 0$

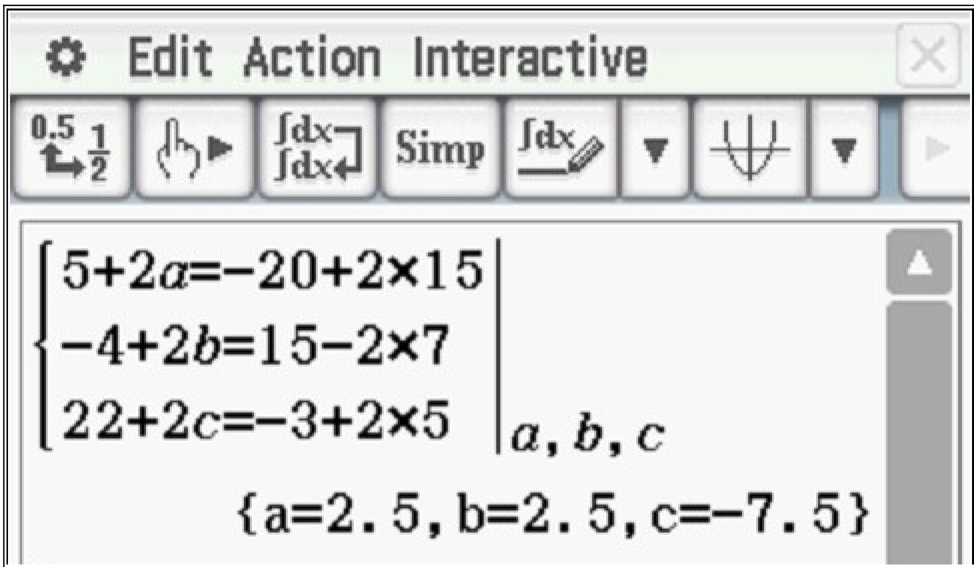


Specific behaviours
P uses relative velocity P determines separation vector P uses dot product to find time P determines approx. closest distance Note-max 2 marks if calculus used

c) At time $t = 1$ hour, rocket A will change its velocity so that it will collide with rocket B at time $t = 3$ hours. Determine this new constant velocity of rocket A to 2 decimal places.

(4 marks)

Solution



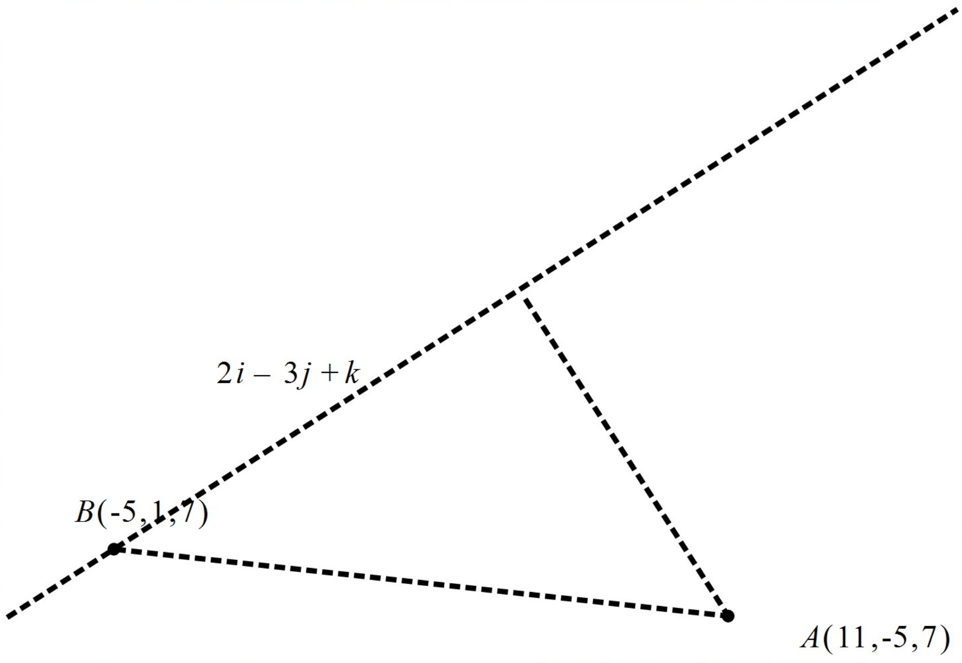
Velocity = $\begin{pmatrix} \frac{5}{2} \\ \frac{5}{2} \\ -\frac{15}{2} \end{pmatrix} km/h$

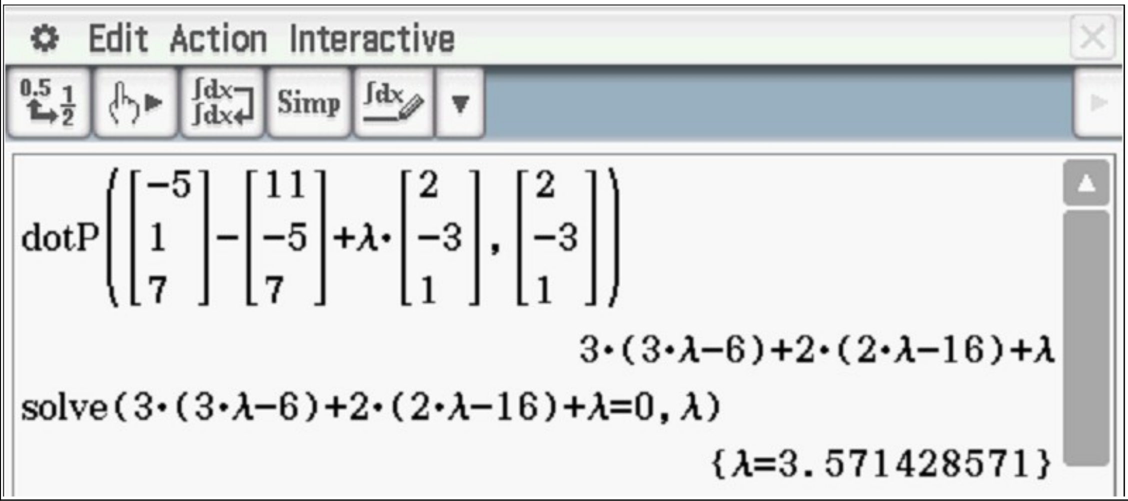
Specific behaviours
P determines positions at t=1 P sets up one component equation form collision 2 hours later P sets up 3 component equations P solves for velocity components rounded to 2 dp, no need for units

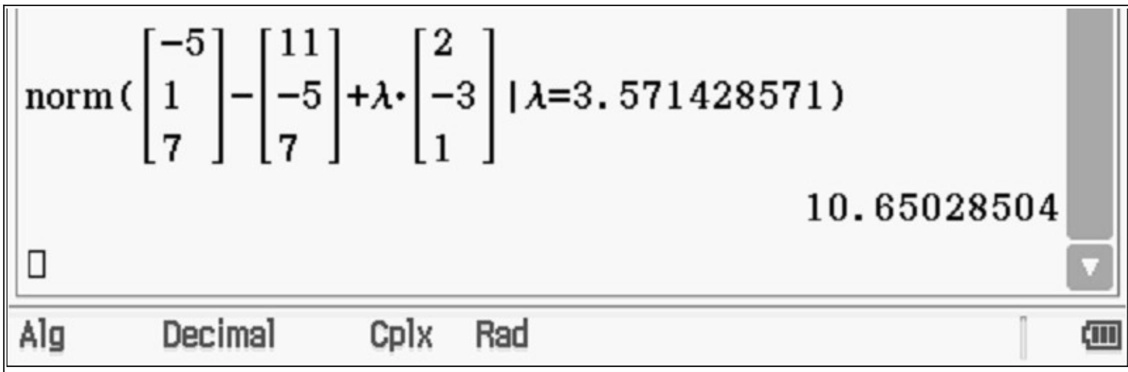
Question 13 **(6 marks)**

Consider the line $r = \begin{pmatrix} -5 \\ 1 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and the point $A(11, -5, 7)$.

a) Determine the distance of point A to the line using vector **dot** product. (3 marks)



Solution
$\vec{d} = AB + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$
$d.\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$




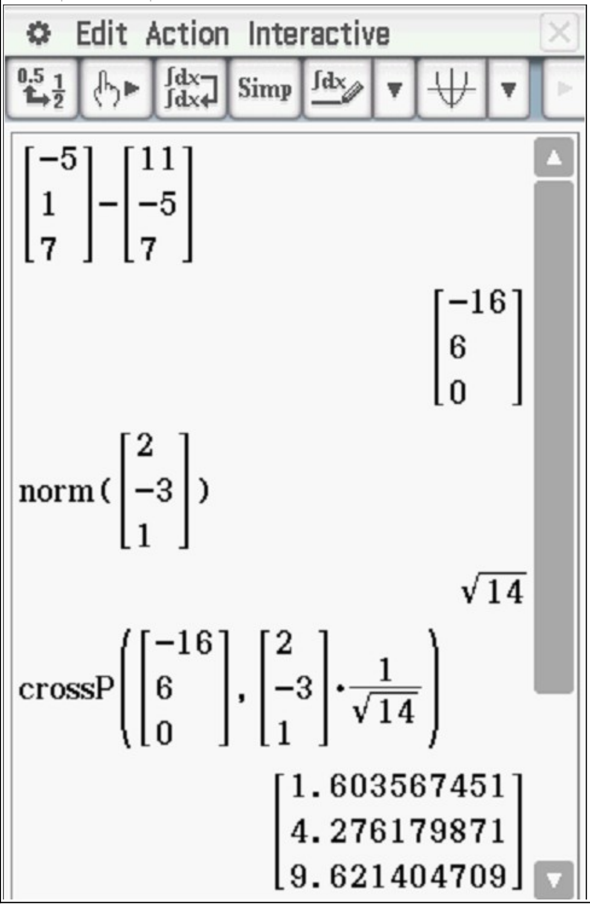
Specific behaviours

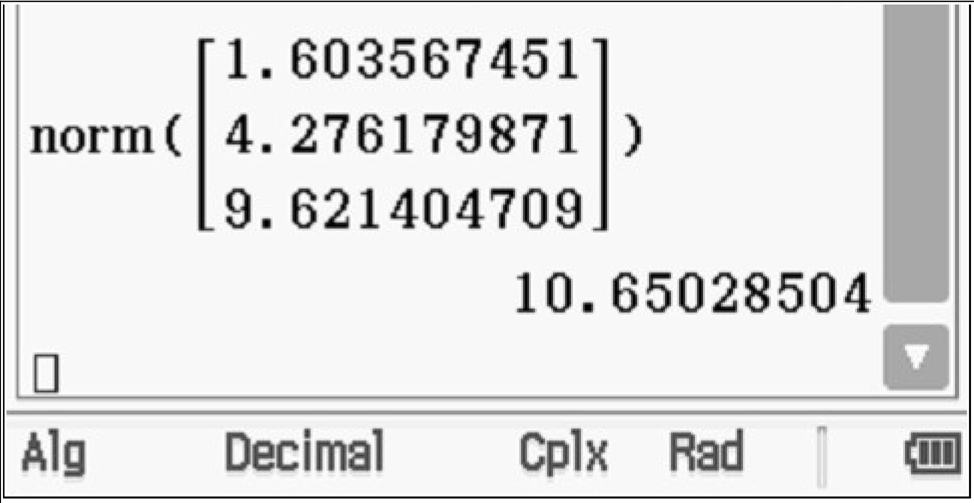
P sets up a separation vector
P uses dot product
P determines distance approx

b) Determine the distance of point A to the line using vector **cross** product. (3 marks)

Solution

$d = \left|AB \times \hat{b}\right|$

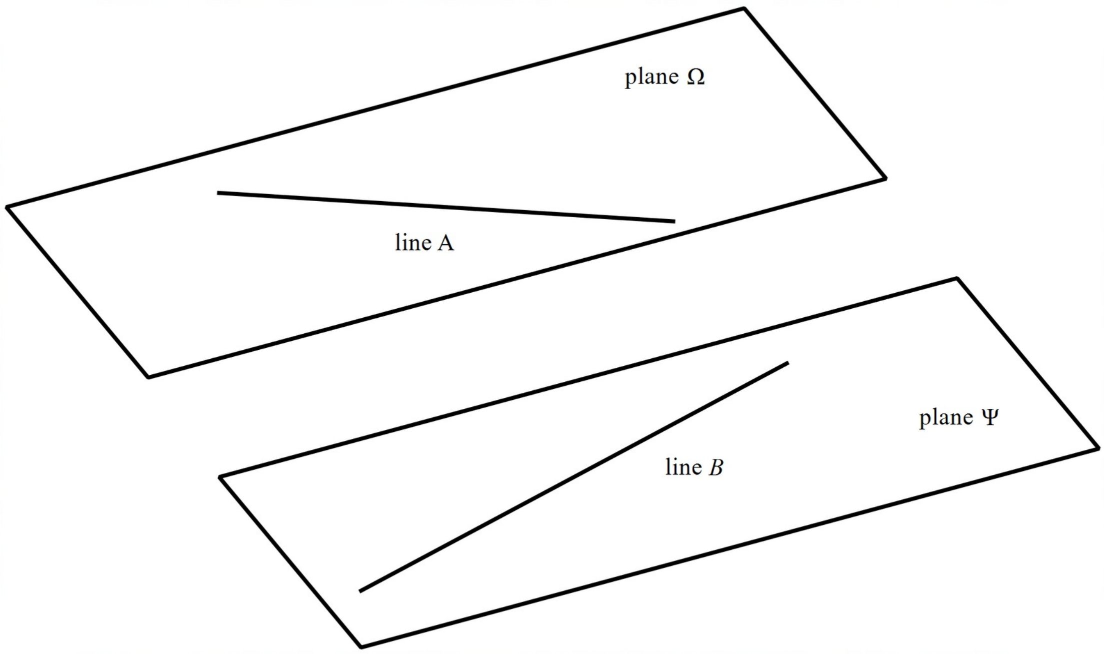




Specific behaviours
P sets up an expression with cross product P determines separation vector using cross product P determines approx. distance

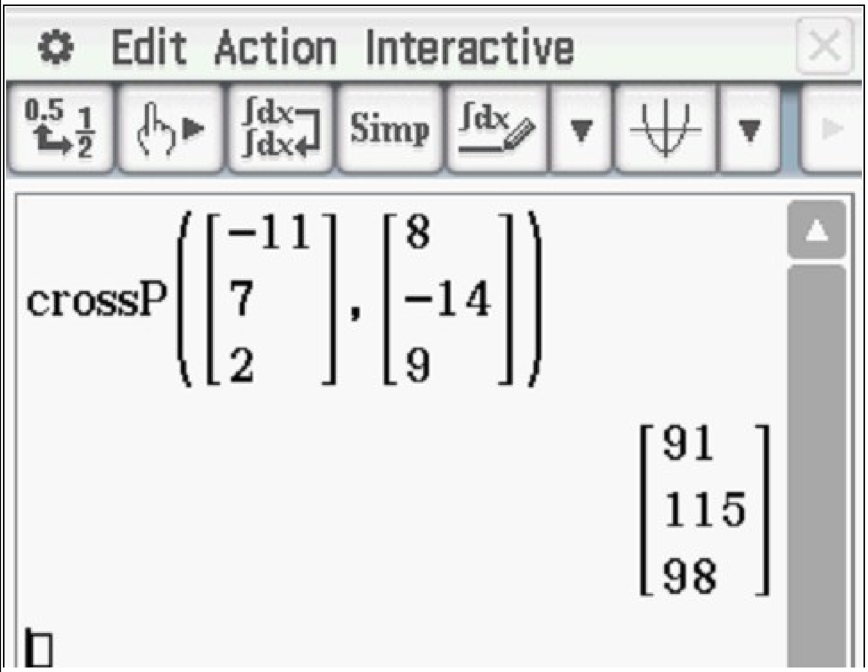
Question 14 **(7 marks)**

Consider the plane Ω which contains the line A, $r_A = \begin{pmatrix} -1 \\ 5 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} -11 \\ 7 \\ 2 \end{pmatrix}$ and the **parallel plane** Ψ which contains the line B, $r_B = \begin{pmatrix} 15 \\ -10 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -14 \\ 9 \end{pmatrix}$ as shown in the diagram below, (not drawn to scale).



a) Determine the cartesian equation of plane Ω (3 marks)

Solution



$$r.\begin{pmatrix}91\\115\\98\end{pmatrix}=\begin{pmatrix}-1\\5\\7\end{pmatrix}.\begin{pmatrix}91\\115\\98\end{pmatrix}=1170$$

$$91x+115y+98z=1170$$

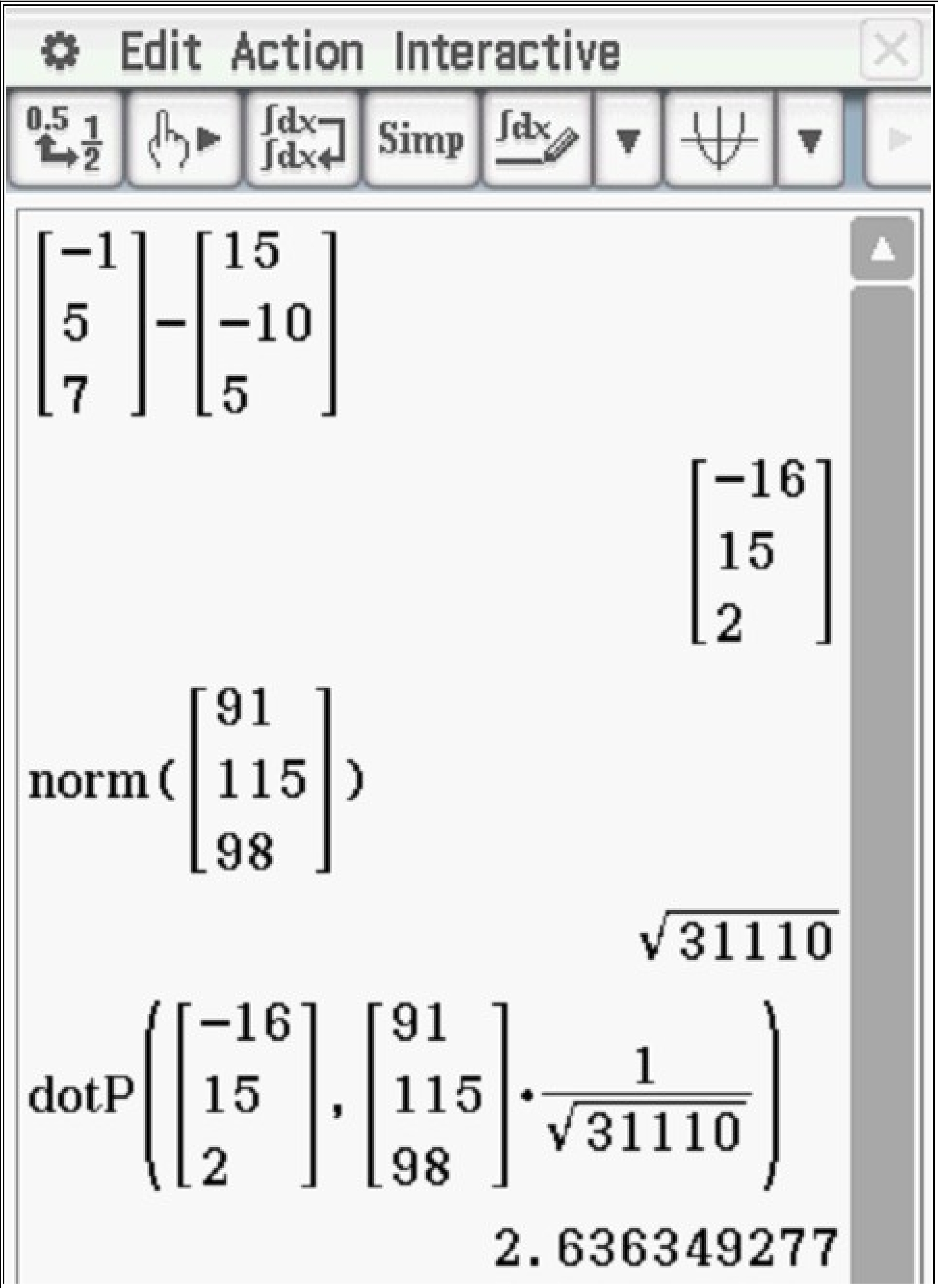
Specific behaviours

P determines normal vector
P determines vector equation
P determines cartesian

b) Determine the distance between the two planes. (4 marks)

Solution

$$d=\left|AB.\hat{n}\right|$$



Specific behaviours
P uses separation vector from a point on each plane P uses unit normal vector P uses dot product P determines approx. distance (must be positive)

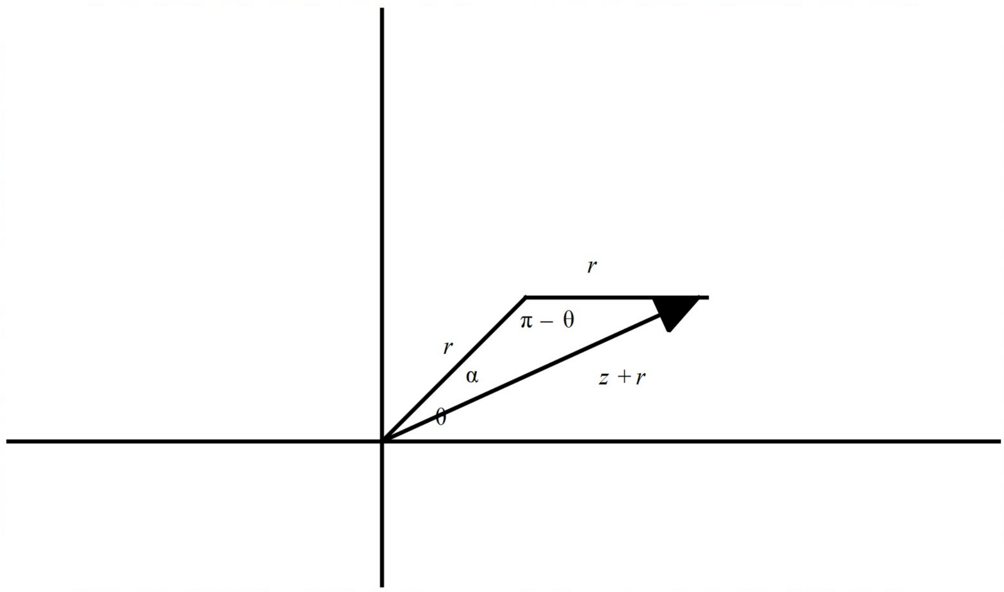
Question 15 **(6 marks)**

Let $z = r cis\theta$ where $0 < \theta < \frac{\pi}{2}$.

a) Determine an expression for $\frac{\sqrt{3}+i}{\bar{z}(1+i)}$ in terms of $r \,\&\, \theta$ only. (simplify) (3 marks)

Solution
$\frac{\sqrt{3}+i}{\bar{z}(1+i)} = \frac{2cis\frac{\pi}{6}}{rcis(-\theta)\sqrt{2}cis\frac{\pi}{4}} = \frac{\sqrt{2}}{r}cis\left(\frac{\pi}{6} - \frac{\pi}{4} + \theta\right) = \frac{\sqrt{2}}{r}cis\left(\theta - \frac{\pi}{12}\right)$
Specific behaviours
P converts all terms to polar (radians) P determines simplified modulus P determines simplified argument

b) Determine an expression for $Arg(z+r)$ in terms of θ only. (3 marks)

Solution

$Arg(z+r) = \theta - \alpha$ $\alpha = \frac{\theta}{2}$ (Isoceles) $Arg(z+r) = \frac{\theta}{2}$
Specific behaviours
P shows isosceles triangle

P determines all angles in triangle P determines argument in required form

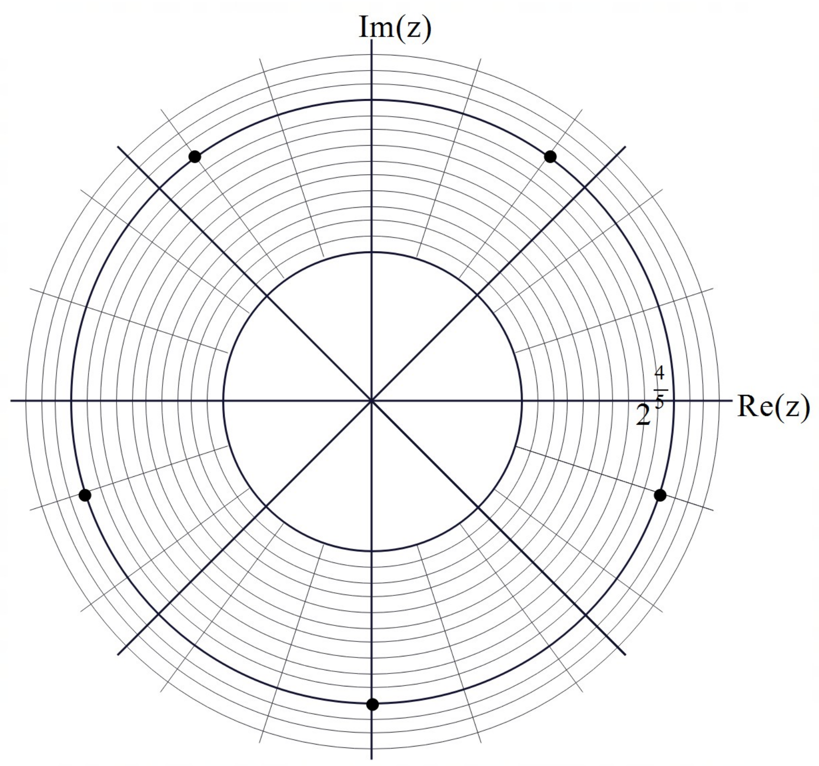
Question 16 **(9 marks)**

a) Determine the roots to $z^5 = -16i$ in the form $z = r cis\theta$ with $-\pi < \theta \leq \pi$. (4 marks)

Solution
$z^5 = -16i = 16cis\left(-\frac{\pi}{2} + 2n\pi\right) \cdots n = 0, \pm1, \pm2...$ $z = 2^{\frac{4}{5}} cis\left(-\frac{\pi}{10} + 2n\pi\frac{1}{5}\right) \cdots n = 0, \pm1, \pm2...$ $z = 2^{\frac{4}{5}} cis\left(-\frac{\pi}{10} + 4n\pi\frac{1}{10}\right) \cdots n = 0, \pm1, \pm2...$ $z_1 = 2^{\frac{4}{5}} cis\left(-\frac{\pi}{10}\right)$ $z_2 = 2^{\frac{4}{5}} cis\left(\frac{3\pi}{10}\right)$ $z_3 = 2^{\frac{4}{5}} cis\left(\frac{-5\pi}{10}\right) \quad or 2^{\frac{4}{5}} cis\left(\frac{-\pi}{2}\right)$ $z_4 = 2^{\frac{4}{5}} cis\left(\frac{7\pi}{10}\right)$ $z_5 = 2^{\frac{4}{5}} cis\left(-\frac{9\pi}{10}\right)$
Specific behaviours
P converts RHS to polar with positive modulus P shows use of De Moivres P determines modulus of each root P determines all principal arguments

Q16 continued-

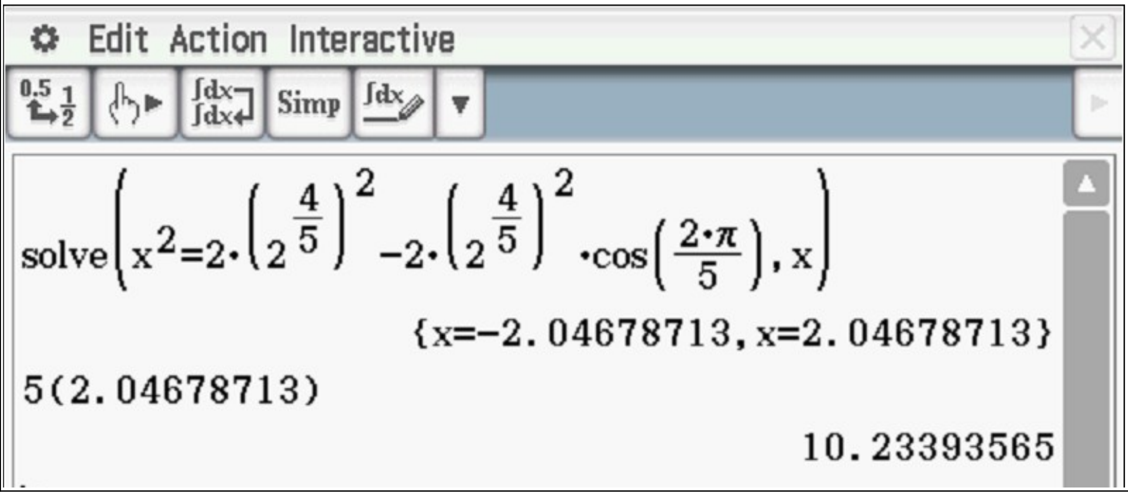
b) Plot the roots from part a on the axes below. (2 marks)



Solution
Specific behaviours
P scale shown and roots equally spaced P correct positions on graph axes

c) The roots above form a polygon, determine the perimeter of this polygon.

(3 marks)

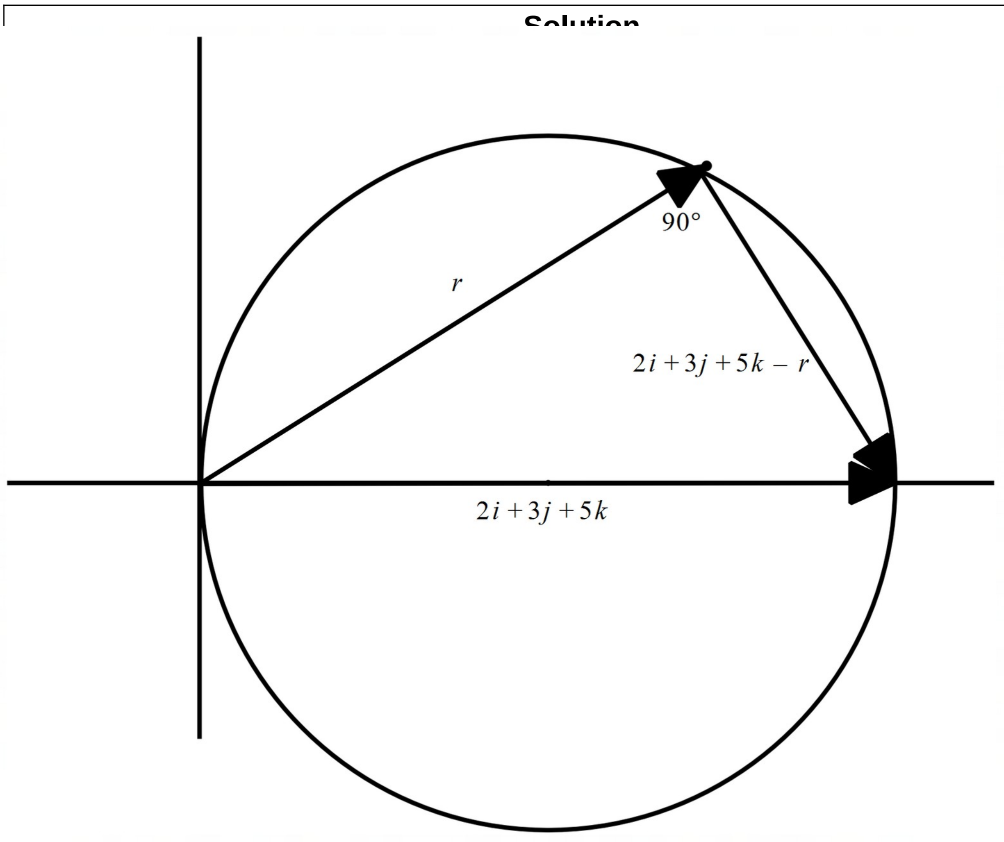
Solution


Specific behaviours
P uses cosine rule P determines the correct central argument P states approx. perimeter (no need for units)

Question 17 **(3 marks)**

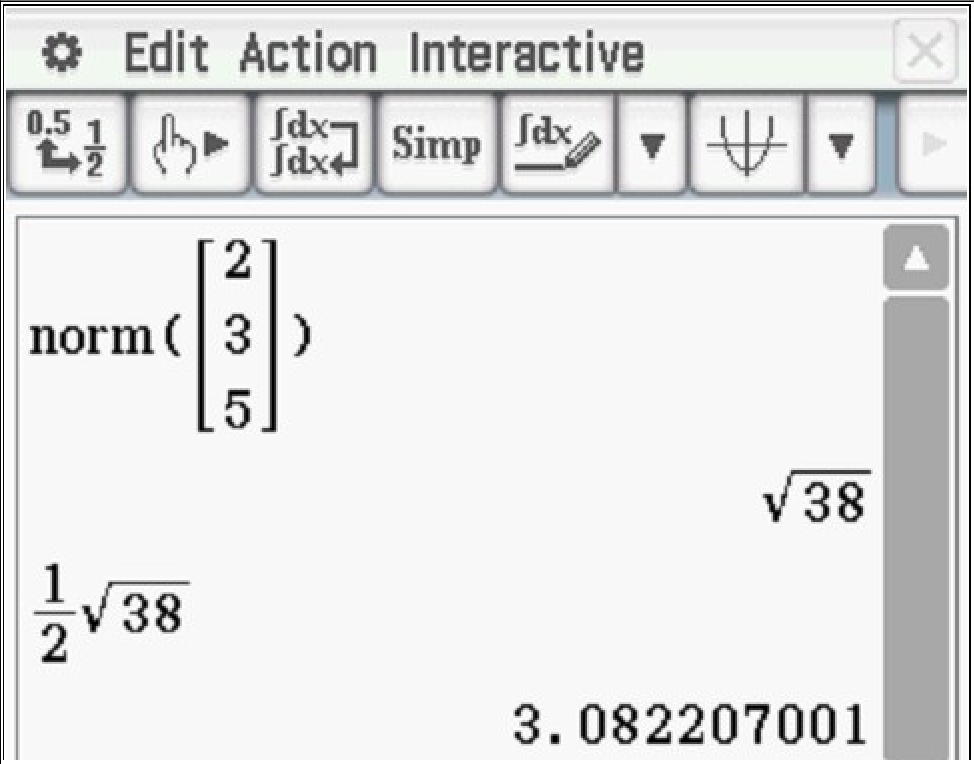
Consider the locus of points that satisfy $r.(2i+3j+5k-r)=0$. Describe this locus identifying all major features.

Solution



Sphere

Centre $\left(1, \frac{3}{2}, \frac{5}{2}\right)$



Specific behaviours
P states a sphere P states coordinates of centre P states approx. radius (no units)

Question 18 **(7 marks)**

Consider the line $r = \begin{pmatrix} -1 \\ \alpha \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix}$, α is a constant and the sphere $\left| r - \begin{pmatrix} -2 \\ 3 \\ -11 \end{pmatrix} \right| = 20$.

Determine all possible values of α using **vector methods** such that:

i) The line is a tangent to the sphere.
ii) The line passes through the sphere.
iii) The line misses the sphere completely.

Solution

$$\left|\begin{pmatrix}-1+7\lambda\\ \alpha\\ 5-2\lambda\end{pmatrix}-\begin{pmatrix}-2\\ 3\\ -11\end{pmatrix}\right|=20$$

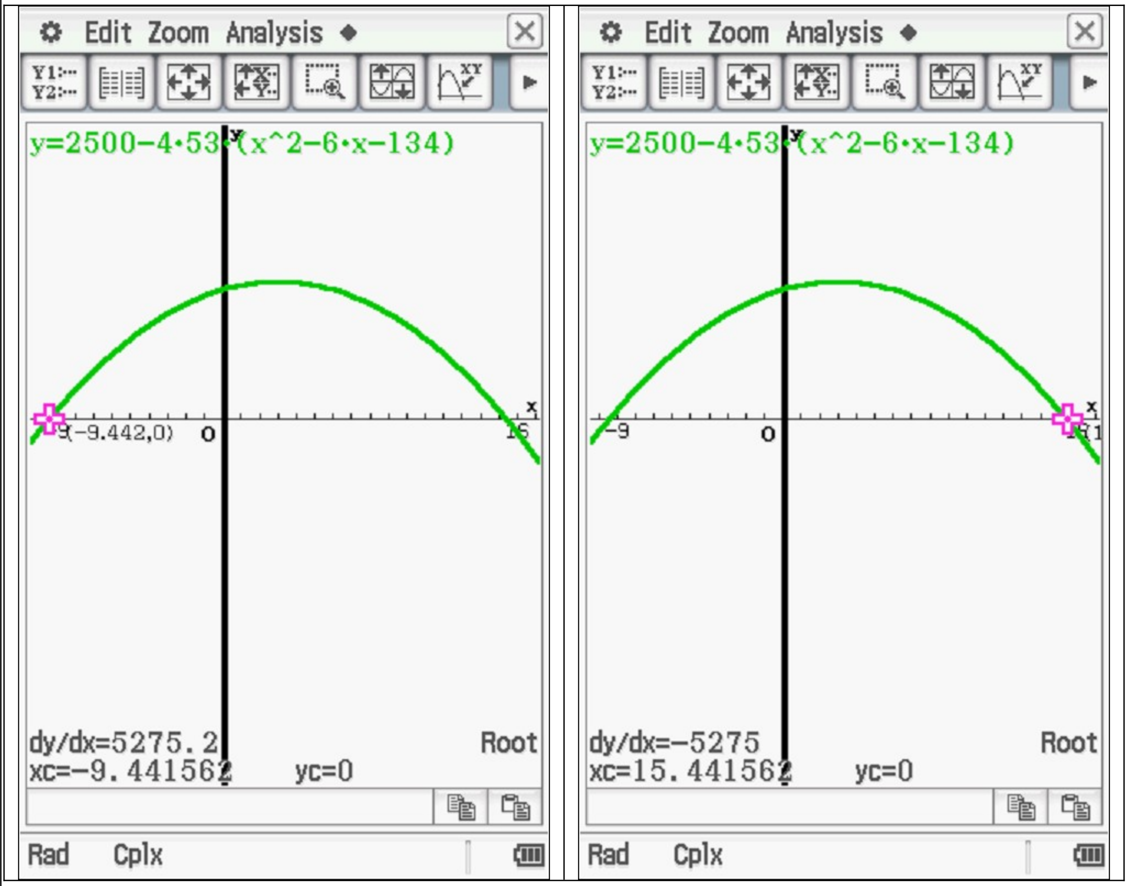
$$\left|\begin{pmatrix}1+7\lambda\\ \alpha-3\\ 16-2\lambda\end{pmatrix}\right|=20$$

$$(1+7\lambda)^2+(\alpha-3)^2+(16-2\lambda)^2=400$$

$$1+14\lambda+49\lambda^2+\alpha^2-6\alpha+9+256-64\lambda+4\lambda^2-400=0$$

$$53\lambda^2-50\lambda+\alpha^2-6\alpha-134=0$$

$$\Delta=2500-4(53)(\alpha^2-6\alpha-134)$$



Tangent α =-9.44,15.44

Passes through $-9.44<\alpha<15.44$

Not meeting $\alpha<-9.44,\alpha>15.44$

Specific behaviours

P states an equation with α & λ using vectors````````

P states a discriminant with α only

P graphs this discriminant

P uses three scenarios for this discriminant

P states all values for tangent

P states all values for passing through

Pstates all values for missing sphere

Question 19 **(4 marks)**

Simplify $\left(\dfrac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}\right)^n$ showing all reasoning.

Solution
$\left(\dfrac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}\right)^n$ $\left(\dfrac{1+2\cos^2\theta - 1 + i2\sin\theta\cos\theta}{1+2\cos^2\theta - 1 - i2\sin\theta\cos\theta}\right)^n$ $\left(\dfrac{2\cos^2\theta + i2\sin\theta\cos\theta}{2\cos^2\theta - i2\sin\theta\cos\theta}\right)^n$ $\left(\dfrac{2\cos\theta\left[\cos\theta + i\sin\theta\right]}{2\cos\theta\left[\cos-\theta + i\sin-\theta\right]}\right)^n$ $\left(\dfrac{cis\theta}{cis-\theta}\right)^n$ $cis(2\theta n)$
Specific behaviours
P uses double angle formula for cosine P takes out common factors P uses de Moivres or conjugates to simplify **inside** brackets P states simplified result

Question 20 **(4 marks)**

Consider $z = 2 - 2\sqrt{3}i$.

a) Show that $\left(z^n\right) - \left(\bar{z}\right)^n = -2^{2n+1} i \sin\left(\frac{n\pi}{3}\right)$ where n is a positive integer. (3 marks)

Solution
$z = 2 - 2\sqrt{3}i = 4cis\left(-\frac{\pi}{3}\right)$ $z^n - \left(\bar{z}\right)^n = 4^n\left[cis\left(-\frac{\pi n}{3}\right) - cis\left(\frac{\pi n}{3}\right)\right]$ $2^{2n} i\left[\sin\left(-\frac{\pi n}{3}\right) - \sin\left(\frac{\pi n}{3}\right)\right]$ $2^{2n} i\left[-\sin\left(\frac{\pi n}{3}\right) - \sin\left(\frac{\pi n}{3}\right)\right]$ $2^{2n+1} i\left[-\sin\left(\frac{\pi n}{3}\right)\right]$ $-2^{2n+1} i\left[\sin\left(\frac{\pi n}{3}\right)\right]$
Specific behaviours
P converts to polar and uses de Moivres P shows that sin(-x)=-sinx (must show this step) P simplifies to required form

b) Determine the positive integer values of n such that $\left(z^n\right) - \left(\bar{z}\right)^n = 0$ (1 mark)

Solution
$\left(z^n\right) - \left(\bar{z}\right)^n = -2^{2n+1} \sin\left(\frac{n\pi}{3}\right) = 0$ $\frac{n\pi}{3} = m\pi \quad m = 0, \pm 1, \pm 2 \ldots$ $n = 3m = +3, +6 + 9 \ldots$
Specific behaviours
P states that n are multiples of 3 (do not penalize for negatives)

Question 21 **(9 marks)**

Consider the following system of linear equations.

$$x + y - 2z = -1$$
$$x + 3y + z = 0$$
$$-2x - 4y + z = 1$$

a) Show **without** the use of a classpad, that there are infinite solutions. (3 marks)

Solution
$\begin{bmatrix} 1 & 1 & -2 & -1 \\ 1 & 3 & 1 & 0 \\ -2 & -4 & 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & -2 & -3 & -1 \\ 0 & -2 & -3 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & -2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
Specific behaviours
P eliminates one variable from one equation   P eliminates one variable from two equations   P shows that two equations are identical or row of zeros

b) Give a geometric interpretation to the solution above.

(1 mark)

Solution
All 3 planes meet along a line
Specific behaviours
P states common line to all 3 planes

c) Determine a vector equation for all solutions. (3 marks)

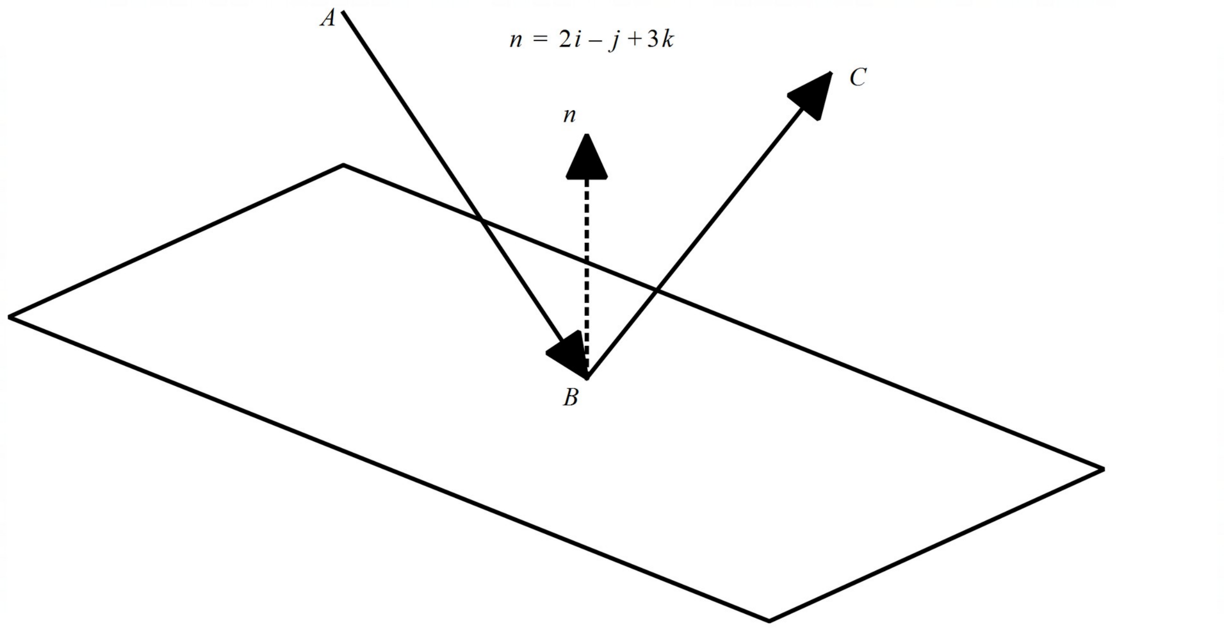
Solution
$\begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & -2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $let\ z = t$ $-2y - 3t = -1$ $y = \frac{1}{2} - \frac{3t}{2}$ $x + \frac{1}{2} - \frac{3t}{2} - 2t = -1$ $x = \frac{-3}{2} + \frac{7t}{2}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{-3}{2} + \frac{7t}{2} \\ \frac{1}{2} - \frac{3t}{2} \\ t \end{pmatrix} = \begin{pmatrix} \frac{-3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{7}{2} \\ -\frac{3}{2} \\ 1 \end{pmatrix}$
Specific behaviours
P uses one variable as parameter P determines parametric equations for other two variables P sets up a vector equation

d) If there is the restriction $-3 \le z \le 5$, determine the range of values for $x \,\&\, y$. (2 marks)

Solution
$-12 \le x \le 16$ $-7 \le y \le 5$
Specific behaviours
P states x range P states y range

Question 22 **(8 marks)**

Consider a projectile fired from a toy gun which moves at a constant velocity $\begin{pmatrix} -1 \\ 5 \\ -7 \end{pmatrix} m/s$ and rebounds off a plastic flat board with its speed unchanged. See diagram below.



Not drawn to scale

The projectile moves in the direction AB and rebounds in the direction BC with the same speed. The flat board has the equation $r.\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 5$. The angle of the incoming path AB and the normal $\underset{\sim}{n}$ is equal to the angle of the outcoming path BC and the normal. Both paths AB & BC and the normal exist in the same plane.

Determine the velocity of the reflected projectile and the angle with the normal above to 2 decimal places.

Solution

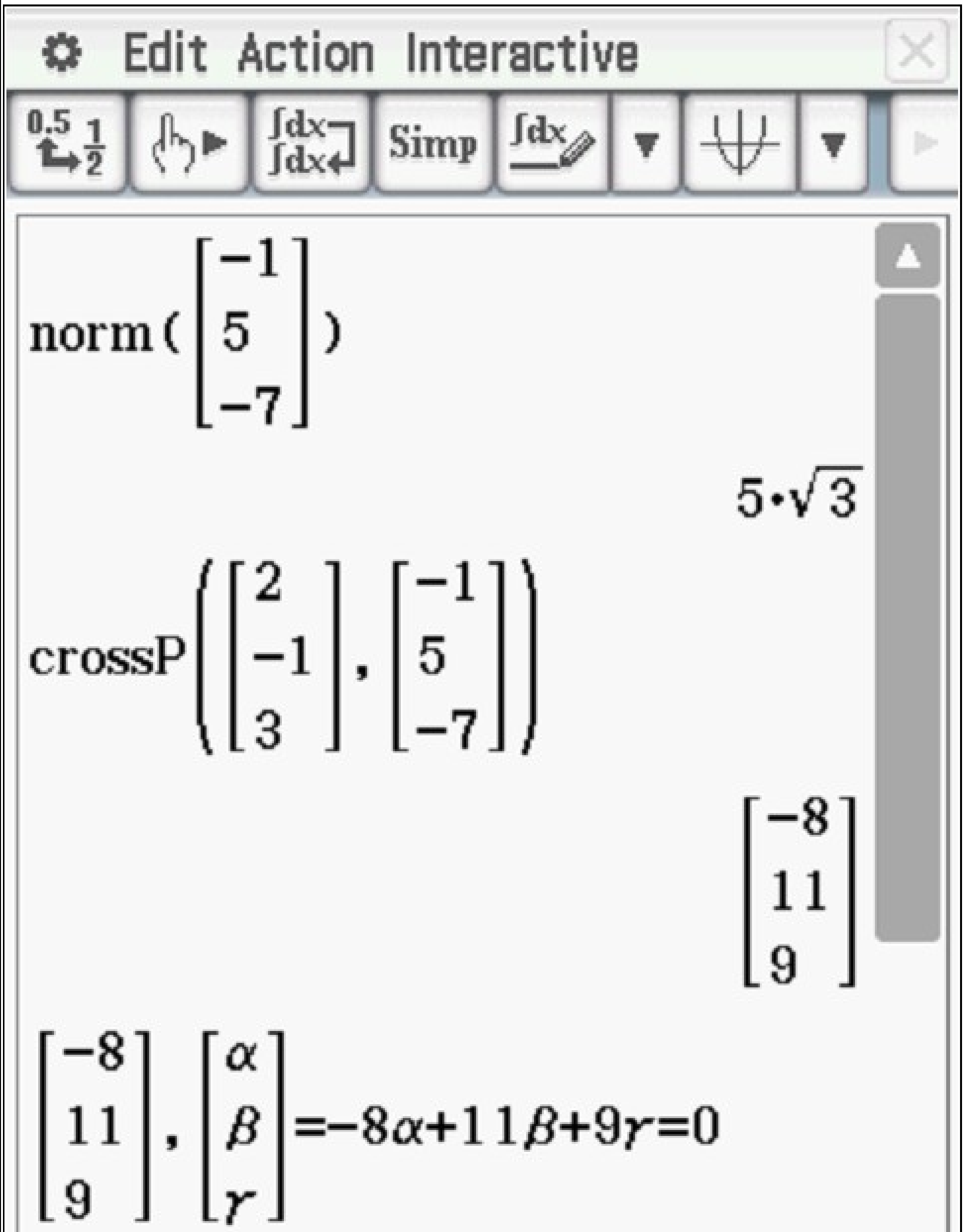
$$Vout = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} m/s$$

$$\left\| \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \right\| = \sqrt{1^2 + 5^2 + 7^2} = 5\sqrt{3}$$

$$\left| \begin{pmatrix} -1 \\ 5 \\ -7 \end{pmatrix} . n \right| = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} . n$$

$$\left| \begin{pmatrix} -1 \\ 5 \\ -7 \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right| = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad \textit{obtuse vs acute}$$

$$-(-2-5-21) = 2\alpha - \beta + 3\gamma = 28$$



Q21 continued-

Working out space

Working out space

Working out space