

**Papers written by
Australian Maths
Software**

SEMESTER TWO

MATHEMATICS SPECIALIST

REVISION 2

UNIT 3-4

2016

SOLUTIONS

Section One

1. (6 marks)

$$(a) \quad z = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{1+2i+i^2}{1-i^2} = \frac{2i}{2} = i \quad \checkmark$$

$$(i) \quad 1 + \frac{1}{z} = 1 + \frac{1}{i} \times \frac{i}{i} = 1 - i \quad \checkmark$$

$$(ii) \quad \frac{\bar{z}}{|z|^2} = \frac{-i}{|i|^2} = \frac{-i}{1} = -i \quad \checkmark \checkmark$$

$$(b) \quad \left\{ z : \frac{\pi}{4} \leq \arg(z) < \frac{3\pi}{4} \cap 1 < \operatorname{Im}(z) \leq 3 \right\} \quad \checkmark \checkmark$$

2. (6 marks)

$$(a) \quad \sin(x-y) + \cos(x+y) = 1$$

$$\left(1 - \frac{dy}{dx}\right) \cos(x-y) - \left(1 + \frac{dy}{dx}\right) \sin(x+y) = 0 \quad \checkmark \checkmark$$

$$\frac{dy}{dx} (-\cos(x-y) - \sin(x+y)) + \cos(x-y) - \sin(x+y) = 0 \quad \checkmark$$

$$\frac{dy}{dx} (-\cos(x-y) - \sin(x+y)) = \sin(x+y) - \cos(x-y)$$

$$\frac{dy}{dx} = \frac{\sin(x+y) - \cos(x-y)}{-\cos(x-y) - \sin(x+y)} \quad \checkmark$$

$$(b) \quad \text{At } \left(\frac{\pi}{2}, 0\right) \quad \frac{dy}{dx} = \frac{\sin\left(\frac{\pi}{2} + 0\right) - \cos\left(\frac{\pi}{2} - 0\right)}{-\cos\left(\frac{\pi}{2} - 0\right) - \sin\left(\frac{\pi}{2} + 0\right)} \quad \checkmark$$

$$\frac{dy}{dx} = \frac{1 - 0}{-1}$$

$$\frac{dy}{dx} = -1 \quad \checkmark$$

3. (10 marks)

(a) $\int \tan(x) dx$

$$= \int \frac{\sin(x)}{\cos(x)} dx$$

put $u = \cos(x)$

$$\frac{du}{dx} = -\sin(x)$$

$$-du = \sin(x) dx$$

$$\equiv \int \frac{du}{u}$$

$$= -\ln(u) + c$$

$$\equiv -\ln(\cos(x)) + c$$

(b) $\int_1^2 \frac{\ln(x) dx}{x}$

put $u = \ln(x)$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{dx}{x}$$

If $u = e^2$, $u = \ln(e^2) = 2$

If $u = e$, $u = \ln(e) = 1$

$$= \int_1^2 u du$$

$$= \frac{1}{2} [u^2]_1^2$$

$$= \frac{1}{2} (4 - 1)$$

$$= 1.5$$

$$(c) \int_0^{\frac{\pi}{3}} \sin(3x) \times e^{\cos(3x)} dx$$

$$\text{put } u = \cos(3x)$$

$$\frac{du}{dx} = -3 \sin(3x)$$

$$\frac{du}{-3} = \sin(3x) dx$$

$$\text{If } x = \frac{\pi}{3}, u = -1$$

$$\text{If } x = 0, u = 1$$

$$= \int_1^{-1} e^u \frac{du}{-3}$$

$$= -\frac{1}{3} [e^u]_1^{-1}$$

$$= -\frac{1}{3} \left(\frac{1}{e} - e \right)$$

$$= \frac{1}{3} \left(e - \frac{1}{e} \right)$$

4. (3 marks)

The other four roots are $-\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{2}$

$$z = cis\left(\frac{\pi}{6} + \frac{k2\pi}{6}\right)$$

$$z^6 = cis(\pi + 2k\pi)$$

$$z^6 = \cos(\pi) + i \sin(\pi + 2k\pi)$$

$$z^6 = -1$$

Check

$$k=0 \quad z = cis\left(\frac{\pi}{6}\right)$$

$$k=1 \quad z = cis\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = cis\left(\frac{\pi}{2}\right) \quad \text{add multiples of } \frac{\pi}{3}$$

$$k=-1 \quad z = cis\left(\frac{\pi}{6} - \frac{\pi}{3}\right)$$

$$k=2 \quad z = cis\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) = cis\left(\frac{5\pi}{6}\right)$$

$$k=-2 \quad z = cis\left(\frac{\pi}{6} - \frac{2\pi}{3}\right) = cis\left(-\frac{\pi}{2}\right)$$

$$k=-3 \quad z = cis\left(\frac{\pi}{6} - \frac{3\pi}{3}\right) = cis\left(-\frac{5\pi}{6}\right)$$

5. (6 marks)

(a) Let $P(z) = z^3 - 2z^2 - 3z + 10$ and $Q(z) = z^3 - 7z^2 + 17z - 15$

$$P(-2) = -8 - 8 + 6 + 10 = 0$$

$$\therefore z = -2$$

$$\begin{array}{r} -2 \overline{) 1 \quad -2 \quad -3 \quad 10} \\ \underline{1 \quad -4 \quad 5 \quad 0} \end{array}$$

$$\therefore P(z) = (z + 2)(z^2 - 4z + 5)$$

$$z = -2 \text{ or } (z^2 - 4z + 5) = 0$$

$$z = \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$z = -2 \text{ or } z = \frac{4 \pm 2i}{2}$$

$$z = -2 \text{ or } z = 2 \pm i$$

$$P(3) = 27 - 63 + 51 - 15 = 78 - 78 = 0$$

$$\therefore z = 3$$

$$\begin{array}{r} 3 \overline{) 1 \quad -7 \quad 17 \quad -15} \\ \underline{1 \quad -4 \quad 5 \quad 0} \end{array}$$

$$\therefore P(z) = (z - 3)(z^2 - 4z + 5)$$

$$z = -3 \text{ or } (z^2 - 4z + 5) = 0$$

which is the same quadratic factor
so there are two common roots.

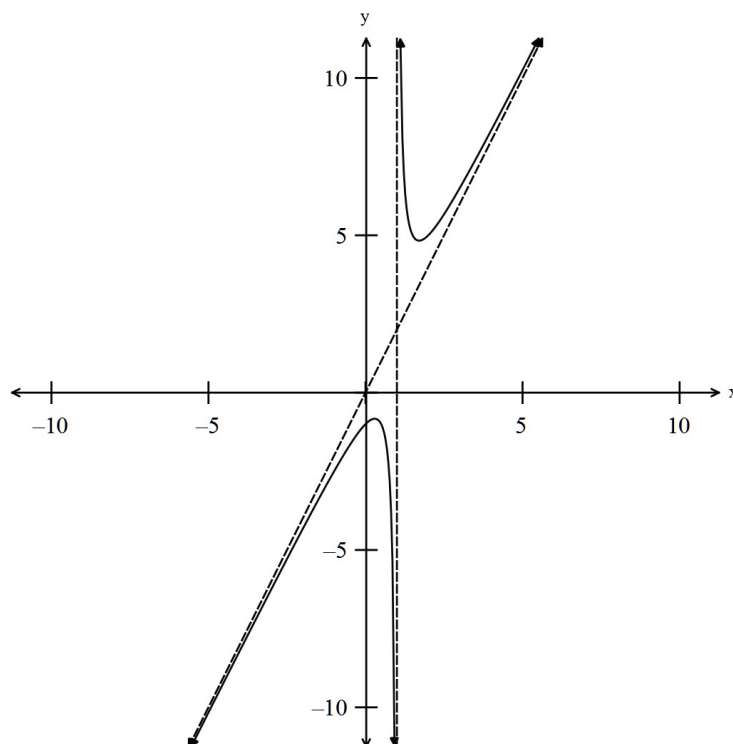
(b) $(z - i)(z - 2 + i)(z - 3 - i)$

$$= (z - i)(z^2 - 5z + 7 - i)$$

$$= z^3 + (-5 - i)z^2 + (7 + 4i)z - 1 - 7i$$

6. (6 marks)

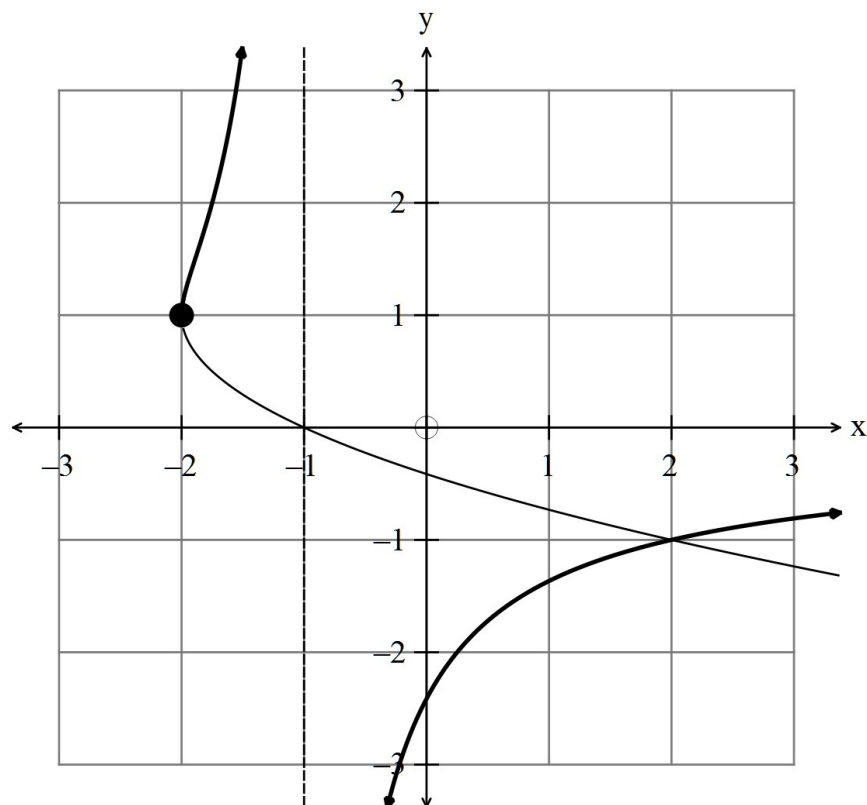
(a)



✓✓✓✓

y intercept, asymptotes, no x intercept

(b)



7. (11 marks)

(a) $\mathbf{r}(t) = (10\cos(t))\mathbf{i} + (6\sin(t))\mathbf{j}$

$\mathbf{v}(t) = (-10\sin(t))\mathbf{i} + (6\cos(t))\mathbf{j}$

$|\mathbf{v}(t)| = \sqrt{100\sin^2(t) + 36\cos^2(t)}$

$|\mathbf{v}(t)| = \sqrt{64\sin^2(t) + 36}$

(b) Max speed occurs when $\sin(t) = 1$ and is 10 m/sec

Min speed occurs when $\sin(t) = 0$ and is 6 m/sec

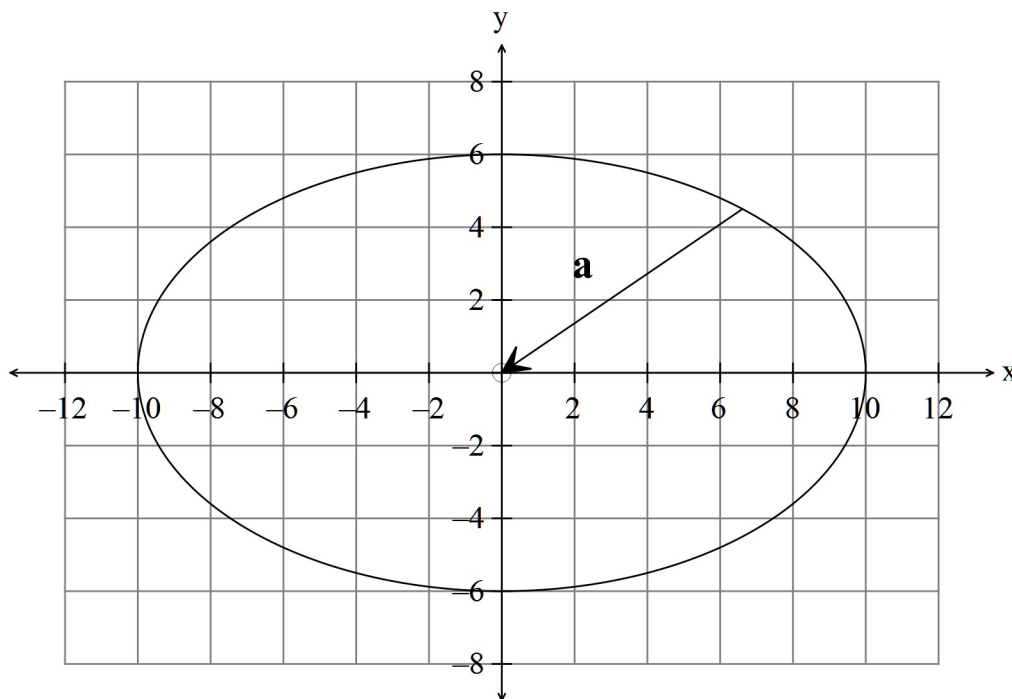
(c) Max : $\sin(t) = \pm 1$ when $t = \frac{\pi}{2}, \frac{3\pi}{2}$ at $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$. ✓

Min : $\sin(t) = 0$ when $t = 0, \pi$ at $\begin{pmatrix} 10 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -10 \\ 0 \end{pmatrix}$.

$$(d) \quad \mathbf{a}(t) = (-10\cos(t))\mathbf{i} - (6\sin(t))\mathbf{j}$$

$$\mathbf{a}\left(\frac{\pi}{4}\right) = \left(-10\cos\left(\frac{\pi}{4}\right)\right)\mathbf{i} - \left(6\sin\left(\frac{\pi}{4}\right)\right)\mathbf{j} = -5\sqrt{2}\mathbf{i} - 3\sqrt{2}\mathbf{j}$$

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \left(10\cos\left(\frac{\pi}{4}\right)\right)\mathbf{i} + \left(6\sin\left(\frac{\pi}{4}\right)\right)\mathbf{j} = 5\sqrt{2}\mathbf{i} + 3\sqrt{2}\mathbf{j}$$



$$(e) \quad x = 10\cos(t), \quad y = 6\sin(t)$$

$$\sin^2(t) + \cos^2(t) = 1$$

$$\left(\frac{x}{10}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

8 (4 marks)

$$\int \frac{-3dx}{(x-2)(x+1)} = -3 \int \frac{dx}{(x-2)(x+1)}$$

$$\frac{1}{(x-2)(x+1)} = \frac{a}{(x-2)} + \frac{b}{(x+1)}$$

$$\frac{0x+1}{(x-2)(x+1)} = \frac{a(x+1)+b(x-2)}{(x-2)(x+1)}$$

$$= \frac{x(a+b)+(a-2b)}{(x-2)(x+1)}$$

Equating coefficients

$$a+b=0 \quad a-2b=1$$

$$a=-b \quad -3b=1$$

$$a=\frac{1}{3} \quad b=-\frac{1}{3}$$

$$\int \frac{-3dx}{(x-2)(x+1)} = -3 \left(\int \left(\frac{1/3}{(x-2)} - \frac{1/3}{(x+1)} \right) dx \right)$$

$$= \int \left(\frac{1}{(x+1)} - \frac{1}{(x-2)} \right) dx$$

$$= \ln(x+1) - \ln(x-2) + c$$

$$\int \frac{-3dx}{(x-2)(x+1)} = \ln \frac{(x+1)}{(x-2)} + c$$

END OF SECTION ONE

Section Two

9. (4 marks)

(a) $(x-1)^2 + (y-2)^2 + z^2 = 4$

At $y=3$, $(x-1)^2 + z^2 = 3$

which is a circle with centre $(1,3,0)$ with radius $\sqrt{3}$ and lying in the plane $y=3$.

(b) $x=1-t$ $x=-s$
 $y=1+2t$ $y=1+4s$
 $z=2+3t$ $z=3+5s$

If $x=x$ If $y=y$
 $1-t=-s$ $1+2t=1+4s$
 $t=1+s$ $2t=4s$

$t=2s$
 $\therefore 1+s=2s$
 $s=1, t=2$

If $s=1, z=2+6=8$

If $t=2, z=2+3(2)=8$ Same z value

Therefore the two lines do meet, and they meet at the point $P(-1,5,8)$

10. (10 marks)

Given $f(x)=x^2$, $g(x)=\frac{1}{x}$, $h(x)=\ln(x)$, $j(x)=2x-1$ and $k(x)=\sqrt{x}$

(a) A $h(f(x))=h(x^2)=\ln(x^2)$ ✓✓

B $j(g(x))=j\left(\frac{1}{x}\right)=\frac{2}{x}-1$ ✓✓

C $f(g(x))=f\left(\frac{1}{x}\right)=\frac{1}{x^2}$ OR $g(f(x))=g(x^2)=\frac{1}{x^2}$ ✓✓

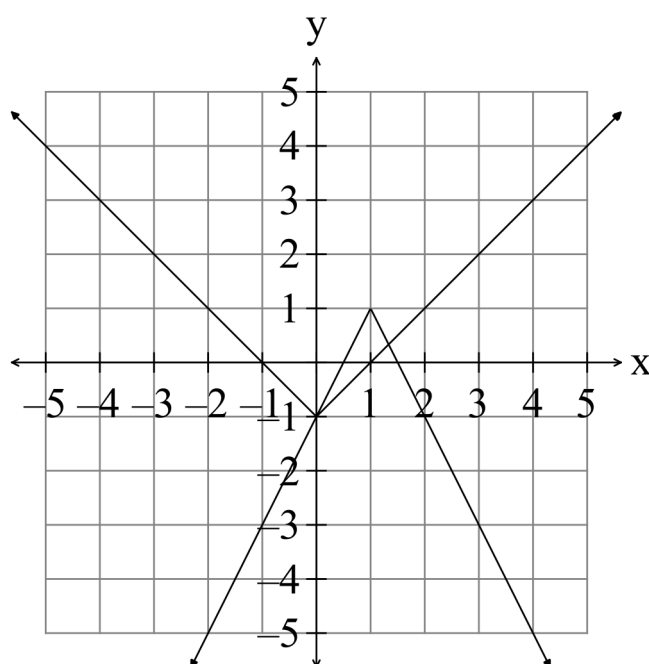
(b) D $g^{-1}(x)=\frac{1}{x}$

E $h^{-1}(x)=e^x$

F $k^{-1}(x)=x^2$ for $x \geq 0$

G $j^{-1}(x)=\frac{1+x}{2}$

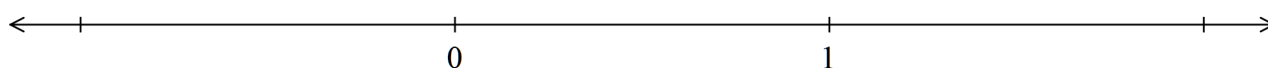
11. (5 marks)



Not a good method to find fractions

$$|x| - 1 = \begin{cases} x - 1 & \text{for } x \geq 0 \\ 1 - x & \text{for } x < 0 \end{cases}$$

$$1 - 2|x - 1| = \begin{cases} 3 - 2x & \text{for } x \geq 1 \\ -1 + 2x & \text{for } x < 1 \end{cases}$$



$$1 - x = -1 + 2x$$

$$2 = 3x$$

$$x = \frac{2}{3}$$

Not in given domain

$$x - 1 = -1 + 2x$$

$$x = 0$$

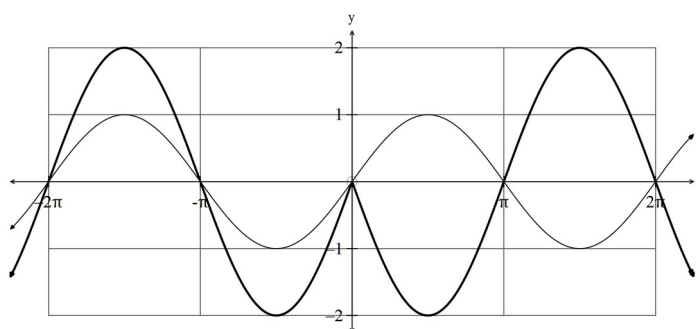
$$x - 1 = 3 - 2x$$

$$3x = 4$$

$$x = \frac{4}{3}$$

$$x = 0 \text{ OR } x = \frac{4}{3}$$

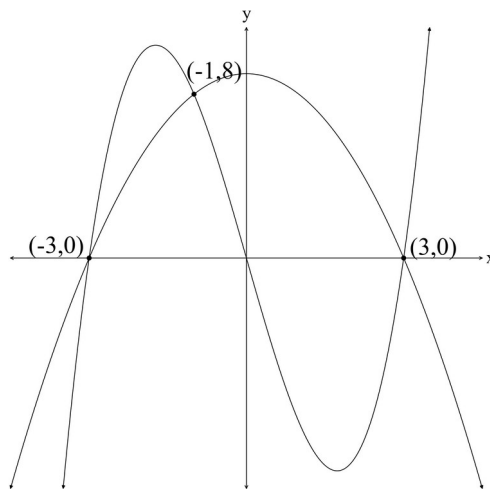
(b)



12. (8 marks)

(a) $z^5 - 1 = 0$
 $z^5 = 1 = (\text{cis}(0 + n(2\pi)))$
 $z = (\text{cis}(2n\pi))^{\frac{1}{5}}$
 $z = \left(\text{cis} \left(\frac{2n\pi}{5} \right) \right)$
 $n = 0 \quad z = (\text{cis}(0))$
 $n = 1 \quad z = \text{cis} \left(\frac{2\pi}{5} \right)$
 $n = 2 \quad z = \text{cis} \left(\frac{4\pi}{5} \right)$
 $n = -1 \quad z = \text{cis} \left(-\frac{2\pi}{5} \right)$
 $n = -2 \quad z = \text{cis} \left(-\frac{4\pi}{5} \right) \quad \checkmark\checkmark\checkmark \text{ -1/ error}$

(b) $f(x) = x(x-3)(x+3)$ and $g(x) = 9 - x^2$.



The x values of the intersection points are -3, -1 and 3 \checkmark

Area =

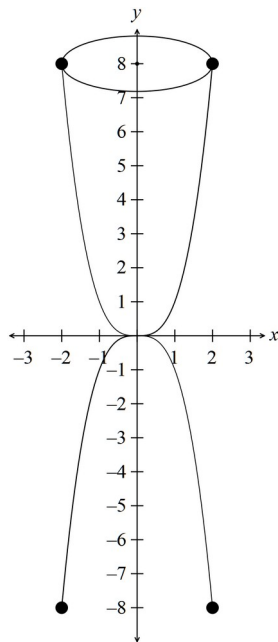
$$\int_{-3}^{-1} (x^3 - 9x^2) - (9 - x^2) dx + \int_{-1}^3 (9 - x^2) - (x^3 - 9x^2) dx = 6\frac{2}{3} + 42\frac{2}{3} = 49\frac{1}{3} \text{ units}^2$$

\checkmark \checkmark

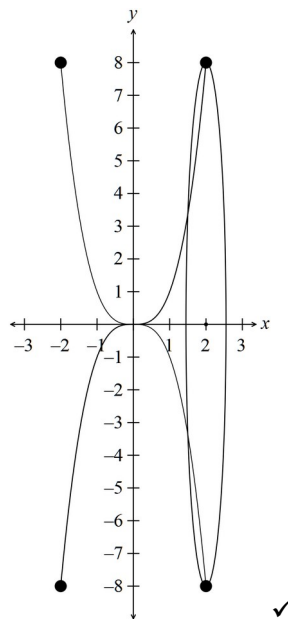
OR $\text{Area} = \int_{-3}^3 |(x^3 - 9x^2) - (9 - x^2)| dx = 49\frac{1}{3} \text{ units}^2$

13. (9 marks)

(a) About the x axis:



About the y axis



$$\checkmark\checkmark V_{y \text{ axis}} = 2 \int_0^8 \pi y^{\frac{2}{3}} dy = 120.64 \text{ units}^3$$

$$V_{x \text{ axis}} = 2 \int_0^2 \pi x^6 dx = 114.89 \text{ units}^3 \quad \checkmark\checkmark$$

There is a different shape generated so the volumes will be different.

(b) The graphs shade the same area above and below the x axis.

$$V = 2 \int_0^8 \pi y^{\frac{2}{3}} dy - 2 \int_0^8 \pi \left(\frac{y}{4} \right)^2 dy$$

\checkmark
 $\checkmark\checkmark$

14. (12 marks)

(a) (i) $v = \sqrt{4 - 2x}$

$$a = \frac{d}{dx} \left(\frac{v^2}{2} \right)$$

$$a = \frac{d}{dx} \left(\frac{4 - 2x}{2} \right) = \frac{d}{dx} (2 - x)$$

$$a = -1 \text{ cm s}^{-2}$$

$$\text{At } t = 2, a = -1 \text{ cm s}^{-2}$$

(ii) $v = \sqrt{4 - 2x}$

$$\text{At } v = 3, \quad 9 = 4 - 2x$$

$$2x = -5$$

$$x = -2.5$$

$$(b) \quad \frac{dy}{dx} = 2xy - y$$

$$\int \frac{dy}{y} = \int (2x - 1) dx$$

$$\ln(y) = x^2 - x + c$$

$$(1, e) \quad \ln e = 1 - 1 + c \rightarrow c = 1$$

$$\ln(y) = x^2 - x + 1$$

$$y = e^{x^2 - x + 1}$$

$$(c) \quad V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 2\text{cm}^3 / \text{s} \quad \frac{dr}{dt} = ? \text{ at } r = 10\text{cm}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \checkmark \checkmark$$

$$2 = 4\pi 10^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{200\pi} \text{cm} / \text{s}$$

15. (7 marks)

$$(a) \quad \mathbf{r}(t) = \mathbf{AB} + t\mathbf{AB} + s\mathbf{AC}$$

$$\mathbf{r}(t) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + s \begin{pmatrix} 6 \\ 3 \\ 10 \end{pmatrix} \quad \checkmark \checkmark$$

(b) (i) No solution where we have $2p = 0$ and $1 - q \neq 0$ i.e. $p = 0, q \neq 1$ $\checkmark \checkmark$

(ii) An infinite number of solutions if $2p = 0$ and $1 - q = 0$ i.e. $p = 0, q = 1$

$\checkmark \checkmark$

(iii) Exactly one solution if $2p \neq 0$ i.e. $p \neq 0$ \checkmark

16. (8 marks)

$$(a) \quad E(\bar{X}) = \mu = 15, \quad \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{3}{\sqrt{16}} = \frac{3}{4}$$

✓ ✓ ✓

$$(b) \quad \mu = E(\bar{X}) = 20$$

$$\sigma_{\bar{x}}^2 = 10, \quad \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

$$\sqrt{10} \times \sqrt{40} = \sigma$$

$$\sigma = 20$$

$$(c) \quad \mu = 182, \quad \sigma = 10$$

$$E(\bar{X}) = 182, \quad \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{9}} = \frac{10}{3}$$

$$P(X > 182) = 0.184$$

$$\text{normCDF}\left(185, \infty, \frac{10}{3}, 182\right) = 0.18406$$

17. (7 marks)

$$(a) \quad 99\% \text{ confidence limits} \Rightarrow z = 2.5758$$

$$10 \pm 2.5758 \times \frac{2}{\sqrt{100}}$$

$$9.4848 \leq \mu \leq 10.5152$$

$$(b) \quad \mu = \frac{8.5 + 14.5}{2} = 11.5 \quad \checkmark \checkmark$$

$$95\% \text{ confidence limits} \Rightarrow z = 1.96$$

$$11.5 + 1.96 \times \frac{\sigma}{\sqrt{n}} = 14.5 \quad (\text{Assume } n=100 \text{ from (a)})$$

$$\sigma = 15.306$$

18. (12 marks)

$$(a) \quad N(2) = \frac{8}{1 + 128.866e^{-3.529 \times 2}} = 7.20144 \approx 7 \quad \checkmark \checkmark$$

$$(b) \quad (i) \quad N = \frac{8}{1 + 128.866e^{-3.529t}}$$

$$\frac{dN}{dx} = -8(1 + 128.866e^{-3.529t})^{-2} \times (0 + 128.866e^{-3.529t} \times (-3.529))$$

$$\frac{dN}{dx} = \frac{3638.144912e^{-3.529t}}{(1 + 128.866e^{-3.529t})^2}$$

At $t = 5$

$$\frac{dN}{dx} = 0.000079 \approx 0$$

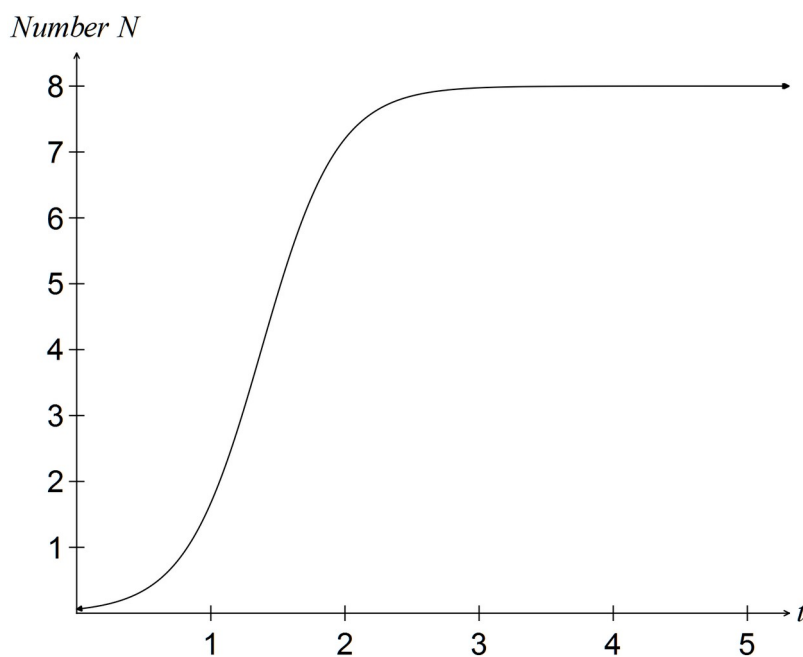
$$(ii) \quad \frac{dN}{dx} = \frac{3638.144912e^{-3.529t}}{(1 + 128.866e^{-3.529t})^2}$$

At $t = 3$

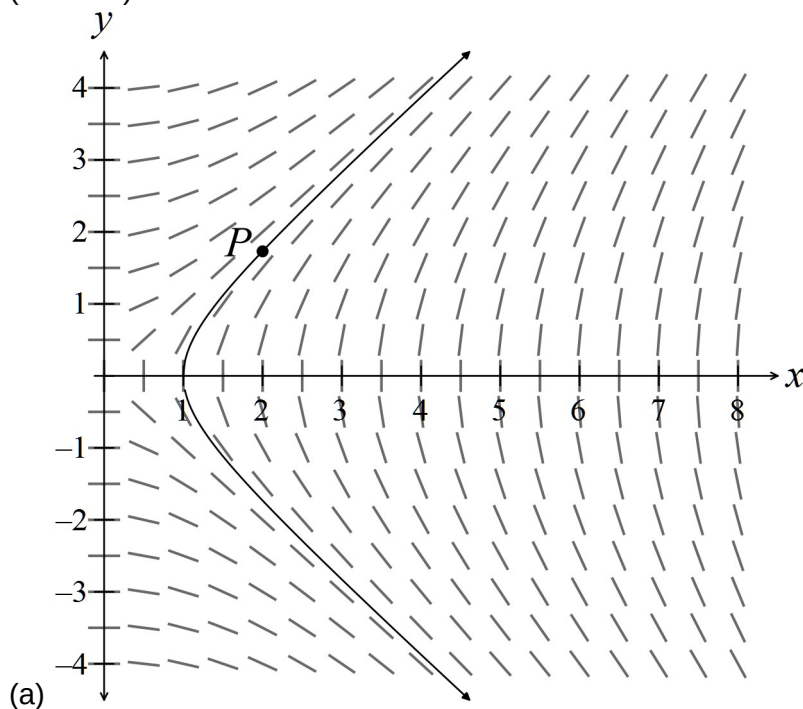
$$\frac{dN}{dx} = 0.09123$$

Positive so increasing.

(c)



19. (4 marks)



- (b) The shape of the graph changes as the starting point changes.
 At (4,0), the gradient is not defined. At (4,1), the gradient is steep, but becoming less steep as the y value increases; at (4,3) the gradient is close to 1
 As the y value decreases from y = 0, the slope changes from undefined to close to -1. ✓✓

20. (11 marks)

$$(a) \quad \frac{\left(\operatorname{cis}\left(\frac{3\pi}{4}\right) \right)^{-4} \times \left(\frac{1+i}{1-i} \right)^2}{\sqrt{\operatorname{cis}(2\pi)}}$$

$$= \frac{(\operatorname{cis}(-3\pi)) \times \left(\frac{2i}{-2i} \right)}{(1)^{\frac{1}{2}}}$$

$$= -1 \times \operatorname{cis}(-3\pi)$$

$$= -\operatorname{cis}(\pi)$$

$$= 1$$

(b) Prove that $\cos^4(\theta)\sin^3(\theta) = -\frac{1}{64}\sin(7\theta) - \frac{1}{64}\sin(5\theta) + \frac{3}{64}\sin(3\theta) + \frac{3}{64}\sin(\theta)$

Let $z = \cos(\theta) + i\sin(\theta)$

$$\frac{1}{z} = \cos(\theta) - i\sin(\theta)$$

Add $z + \frac{1}{z} = 2\cos(\theta)$ Subtract $z - \frac{1}{z} = 2i\sin(\theta)$

$$z^n = (\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

$$\frac{1}{z^n} = (\cos(\theta) + i\sin(\theta))^{-n} = \cos(-n\theta) + i\sin(-n\theta)$$

$$\therefore \frac{1}{z^n} = \cos(n\theta) - i\sin(n\theta)$$

$$z^n + \frac{1}{z^n} = 2\cos(n\theta) \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin(n\theta)$$

$$\begin{aligned} \cos^4(\theta)\sin^3(\theta) &= \left(\frac{1}{2} \left(z + \frac{1}{z} \right) \right)^4 \left(\frac{1}{2i} \left(z - \frac{1}{z} \right) \right)^3 \\ &= \frac{1}{16} \times \left(-\frac{1}{8i} \right) \left(z^2 - \frac{1}{z^2} \right)^3 \left(z + \frac{1}{z} \right) \\ &= -\frac{1}{128i} \left(z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6} \right) \left(z + \frac{1}{z} \right) \\ &= -\frac{1}{128i} \left(z^7 - 3z^3 + \frac{3}{z} - \frac{1}{z^5} + z^5 - 3z + \frac{3}{z^3} - \frac{1}{z^7} \right) \\ &= -\frac{1}{128i} \left(\left(z^7 - \frac{1}{z^7} \right) + \left(z^5 - \frac{1}{z^5} \right) - 3 \left(z^3 - \frac{1}{z^3} \right) - 3 \left(z - \frac{1}{z} \right) \right) \\ &= -\frac{1}{128i} (2i\sin(7\theta) + 2i\sin(5\theta) - 3 \times 2i\sin(3\theta) - 3 \times 2i\sin(\theta)) \\ \cos^4(\theta)\sin^3(\theta) &= -\frac{1}{64} \times \sin(7\theta) - \frac{1}{64} \times \sin(5\theta) + \frac{3}{64} \times \sin(3\theta) + \frac{3}{64} \times \sin(\theta) \end{aligned}$$

END OF SECTION TWO