PERTH MODERN SCHOOL
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Semester One Examination, 2019

Question/Answer booklet

Yr 12 SPECIALIST
UNIT 3

Section Two:
Calculator-assumed

Your Name

Your Teacher's Name

Time allowed for this section

Reading time before commencing work:	ten minutes
Working time:	one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items:	pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters
Special items:	drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	49	34.5
Section Two: Calculator-assumed	13	13	100	93	65.5
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **(93 Marks)**

This section has **13** questions. Answer **all** questions. Write your answers in the spaces provided.

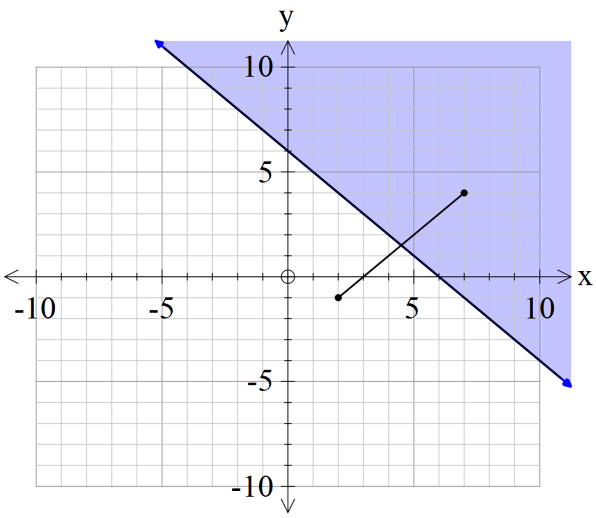
Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 100 minutes.

Question 8 **(6 marks)**

a) Sketch the following region in the complex plane, $|z-2+i| \geq |z-7-4i|$ (3 marks)

Solution

Specific behaviours
✓ plots points ✓ draws perpendicular bisector ✓ shades correct region and includes line

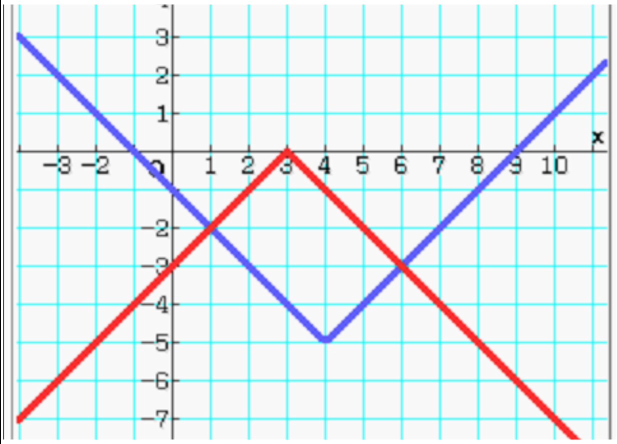
b) Determine the cartesian equation of $|z-2+i| = |z-7-4i|$ (3 marks)

Solution
Midpoint $\left(\frac{2+7}{2},\frac{-1+4}{2}\right)\Rightarrow\left(\frac{9}{2},\frac{3}{2}\right)$ $gradient=\frac{4--1}{7-2}=\frac{5}{5}=1$ $perpendicular \quad m=-1$ $y=-x+c$ $\frac{3}{2}=-\frac{9}{2}+c$ $c=6$ $y=-x+6$
Specific behaviours
✓ uses midpoint ✓ uses perpendicular gradient ✓ states cartesian gradient OR ✓uses subs z=x+iy ✓determines magnitude of both sides ✓squares both sides and simplifies to find cartesian rule

Question 9 **(6 marks)**

a) Given that $|x-4|-5 \le -|x+a|$, where a is a constant, is only true for $1 \le x \le 6$, determine the value of a.

(3 marks)

Solution
 At x=1 -2=-|1+a| a=-3 or 1 At x=6 -3=-|6+a| a=-3 or -9 Common value a=-3
Specific behaviours
✓ equates y values at x=1 ✓ equates y values at x=6 ✓ determines common value of a

b) Given that $|2x+6| = a|x+b|+c$, where $a, b \,\&\, c$ are constants, is only true for $-3 \le x \le 2$, determine the values of $a, b \,\&\, c$. (3 marks)

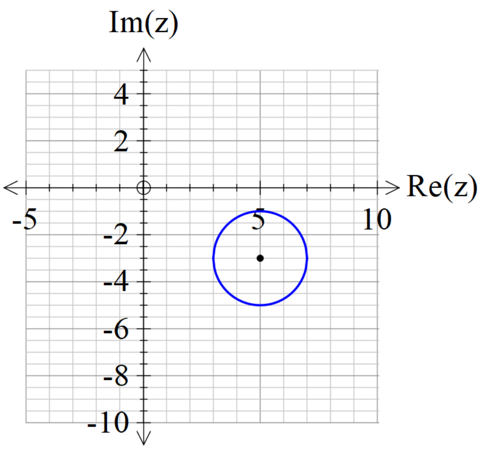
Solution

 $y = -2\|x - 2\| + 10$
Specific behaviours
✓ determines a ✓ determines b ✓ determines c

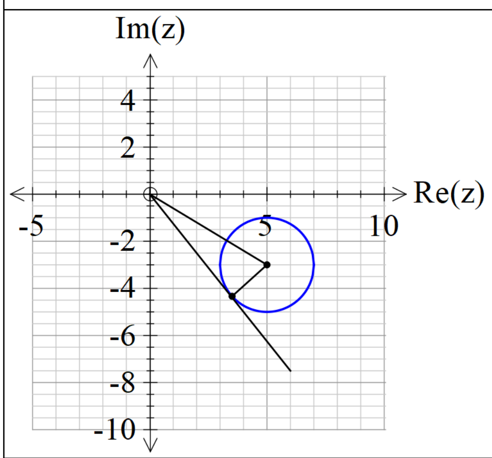
Question 10 **(8 marks)**

Consider the locus of points on $|z-5+3i|=2$ in the complex plane.

a) Sketch this locus below. (2 marks)

Solution

Specific behaviours
✓ circle shape ✓ correct centre and radius

b) Determine the minimum principal $Arg(z)$ on this locus. (3 marks)

Solution


$$-\tan^{-1}\left(\frac{3}{5}\right)-\sin^{-1}\left(\frac{2}{\sqrt{5\text{^}2+3\text{^}2}}\right)$$

$$-0.890525278$$

$$-\tan^{-1}\left(\frac{3}{5}\right)-\sin^{-1}\left(\frac{2}{\sqrt{5\text{^}2+3\text{^}2}}\right)$$

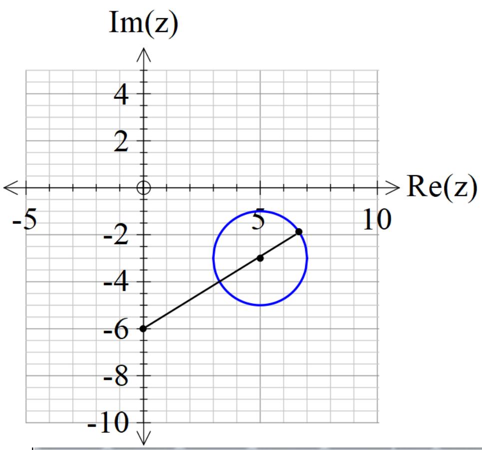
$$-51.02333998$$

Specific behaviours

✓ determines argument of centre
✓ adds argument with triangle of tangent
✓ states the principal argument(does not need to be rounded)

c) Determine the maximum value of $|z+6i|$ on this locus. (3 marks)

Solution



$$\sqrt{5\text{^}2+(6-3)\text{^}2}+2$$

$$\sqrt{34}+2$$

$$\sqrt{34}+2$$

$$7.830951895$$

Specific behaviours

✓ recognizes distance from -6i
✓ determines distance to centre
✓ determines maximum distance

Question 11 **(5 marks)**

Show that the line $x = 2 + 2t, y = -1 + 3t, z = -\frac{5}{2}t$ is parallel to the plane $10x - 5y + 2z = 0$ and determine its distance from the plane.

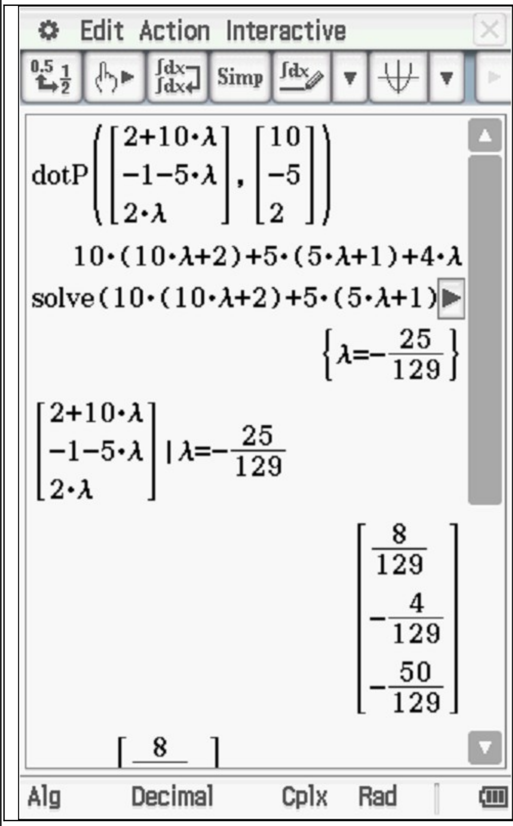
Solution

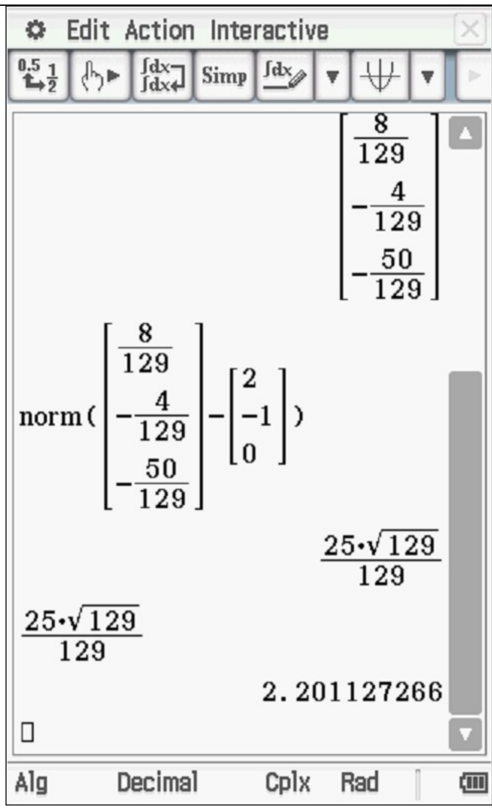
$$r = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ \frac{-5}{2} \end{pmatrix}$$

$$\begin{pmatrix} 10 \\ -5 \\ 2 \end{pmatrix} . \begin{pmatrix} 2 \\ 3 \\ \frac{-5}{2} \end{pmatrix} = 20 - 15 - 5 = 0 \quad \therefore parallel$$

$$r = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 10 \\ -5 \\ 2 \end{pmatrix} \quad subs \quad r . \begin{pmatrix} 10 \\ -5 \\ 2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 + \lambda 10 \\ -1 - 5\lambda \\ 2\lambda \end{pmatrix} . \begin{pmatrix} 10 \\ -5 \\ 2 \end{pmatrix} = 0$$





Specific behaviours

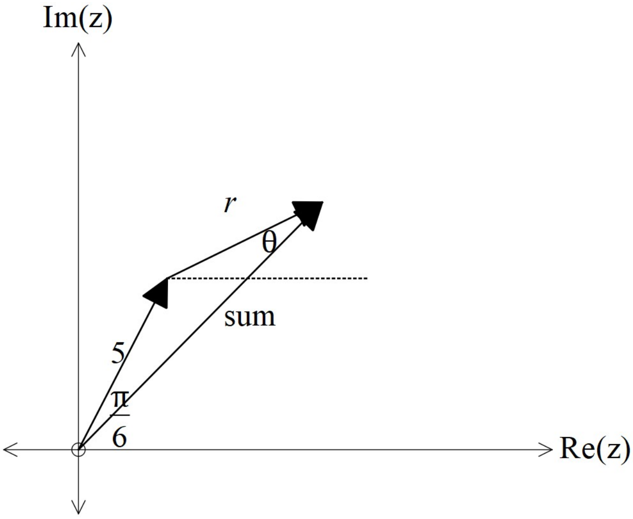
✓ identifies vector parallel to line

✓ shows that normal perpendicular to line
✓ uses a normal line through a point on line
✓solves for where line meets the plane
✓determines distance from plane.
OR
✓determines point on plane
✓determines vector between point on plane and point on line
✓determines unit normal vector
✓uses dot product to determine distance
✓determines distance

Question 12 **(5 marks)**

Let $z = r cis\theta$, where $0 < \theta < \frac{\pi}{2}$, consider the sum $z + 5cis\frac{\pi}{6}$.

(a) Sketch a diagram of this sum in the complex plane. (2 marks)

Solution

Specific behaviours
✓ denotes angles and lengths of arrows ✓ adds numbers as vectors

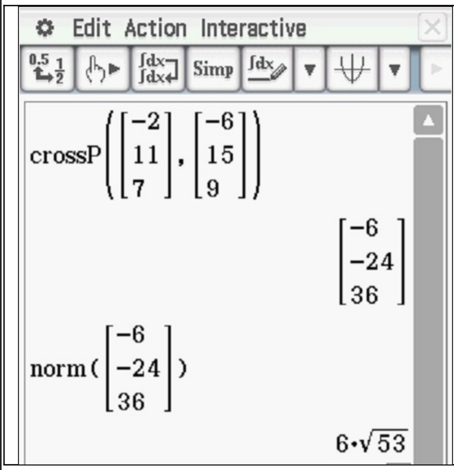
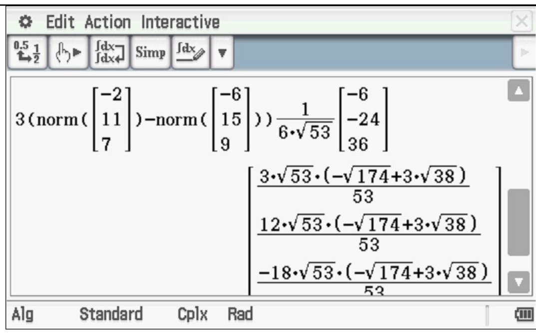
(b) Obtain an expression for the $\left|z + 5cis\frac{\pi}{6}\right|$ in terms of $r \,\&\, \theta$. (3 marks)

Solution
$\left|z + 5cis\frac{\pi}{6}\right| = \sqrt{5^2 + r^2 - 2(5)r\cos(\frac{\pi}{3} + \frac{\pi}{2} + \theta)}$ $= \sqrt{5^2 + r^2 - 10r\cos(\frac{5\pi}{6} + \theta)}$
Specific behaviours
✓ uses cosine rule ✓ determines opposite angle to modulus

✓ obtains correct expression

Question 13 (6 marks)

a) Determine a vector that is perpendicular to both $p=\begin{pmatrix}-2\\11\\7\end{pmatrix}$ and $q=\begin{pmatrix}-6\\15\\9\end{pmatrix}$ and has a magnitude of $3(|p|-|q|)$.(Do not simplify) (3 marks)

Solution
 

Specific behaviours
✓ uses cross product ✓ uses unit vector ✓ uses magnitudes of all vectors to determine correct vector

b) Let $a=\begin{pmatrix}2\\-2\\-2\end{pmatrix}$ and $b=\begin{pmatrix}3\\m\\3\end{pmatrix}$ where m is a real constant. In terms of m, determine an expression for the angle between $a\,\&\,b$. (3 marks)

Solution

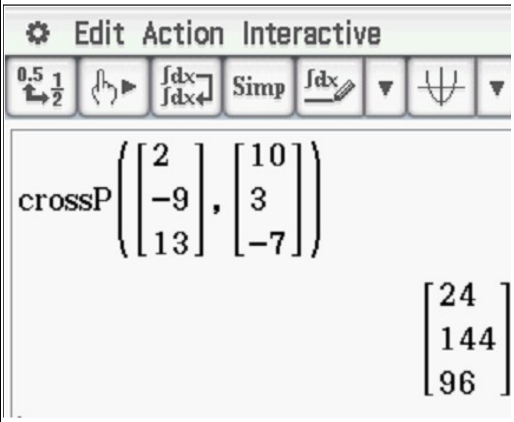
$-2m = \sqrt{12}\sqrt{9+m^2+9}\cos(Angle)$ $Angle = \text{Cos}^{-1}\left(\dfrac{-2m}{\sqrt{12(18+m^2)}}\right)$
Specific behaviours
✓ uses dot product ✓ determines magnitude of both vectors ✓ determine an inverse cosine expression

Question 14 **(10 marks)**

Consider a plane defined by $r = \begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -9 \\ 13 \end{pmatrix} + \lambda \begin{pmatrix} 10 \\ 3 \\ -7 \end{pmatrix}$.

a) Determine a normal vector to this plane. (1 mark)

Solution

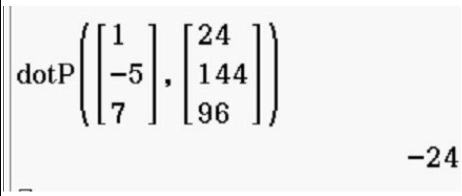


Specific behaviours

✓ uses cross product

b) Determine the cartesian equation of this plane. (2 marks)

Solution



$24x + 144y + 96z = -24$

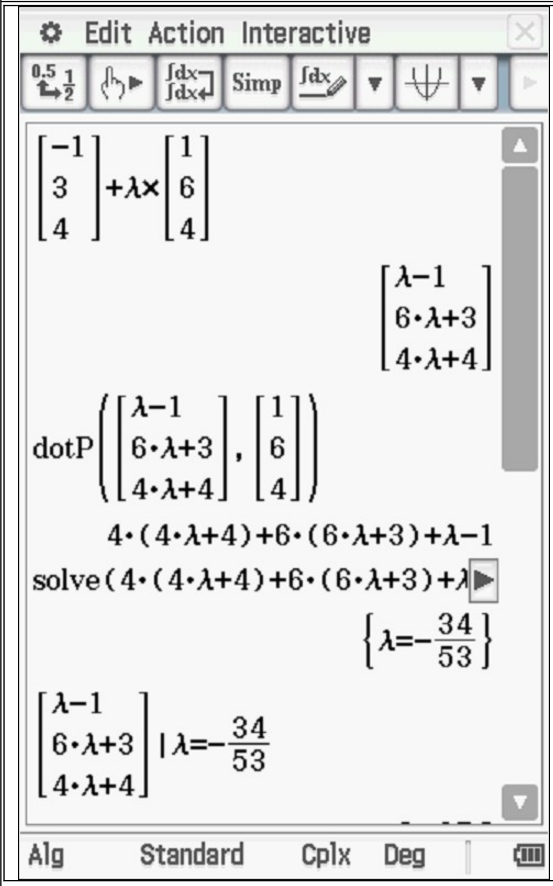
$x + 6y + 4z = -1$

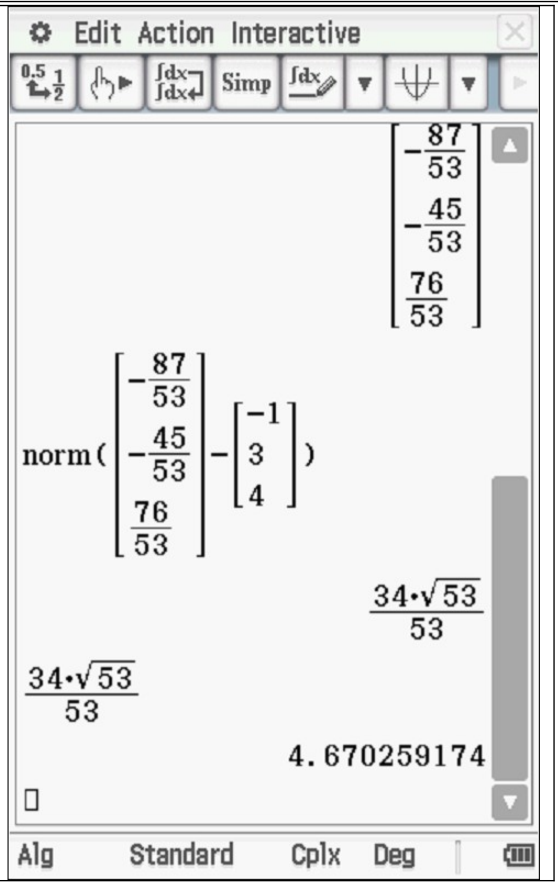
Specific behaviours

✓ uses dot products with normal
✓ determines a cartesian equation

c) Show how to determine the distance of point $P(-1,3,4)$ from the plane above **using** scalar dot product and the normal vector. (4 marks)

Solution





Specific behaviours

✓ uses a normal vector through point
✓ solves for where line meets plane using dot product
✓ uses two points
✓ determines distance
OR
✓determines a point on plane
✓determines vector from P to this point
✓determines unit normal
✓dot product between these two vectors

d) Consider a general plane $Ax + By + Cz + D = 0$, where $A, B, C \& D$ are constants.

Show that the distance of point $Q(x_1, y_1, z_1)$ from this plane is given by the expression $\dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$

(3 marks)

Solution
Point on plane x=0=y $0 + 0 + Cz = D$ $z = \dfrac{-D}{C}$ $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \frac{-D}{C} \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 + \frac{D}{C} \end{pmatrix}$ $\begin{pmatrix} x_1 \\ y_1 \\ z_1 + \frac{D}{C} \end{pmatrix} \cdot \dfrac{1}{\sqrt{A^2 + B^2 + C^2}} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \dfrac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}$
Specific behaviours
✓ determines a point on plane   ✓ separation vector between this point and Q   ✓ uses dot product with unit normal

e)

Question 15 **(8 marks)**

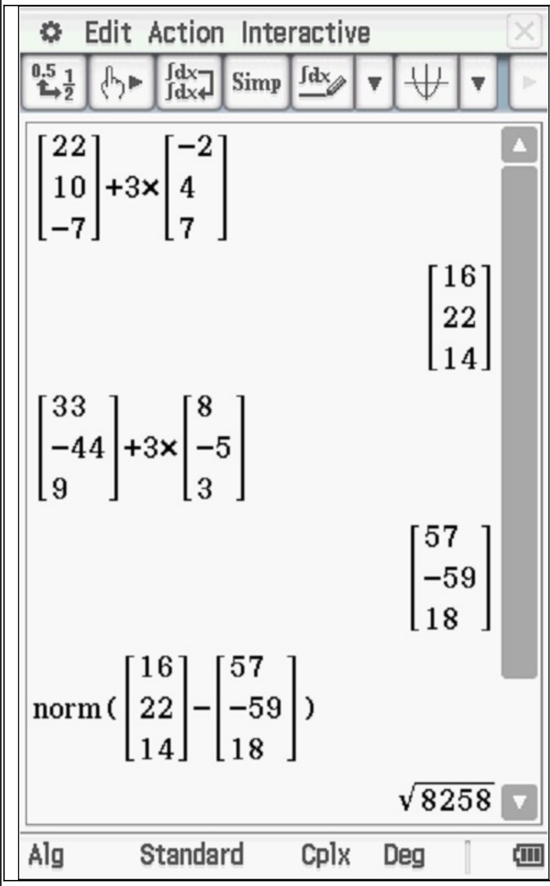
In deep space an astronaut is space walking outside a stationary space station. At time $t=0$ seconds the astronaut is positioned at $(22,10,-7)$ metres relative to the space station and is moving with a velocity of $\begin{pmatrix}-2\\4\\7\end{pmatrix}$ metres per second. A rogue satellite is observed to be at position $(33,-44,9)$ at time $t=0$ with a velocity of $\begin{pmatrix}8\\-5\\3\end{pmatrix}$ metres per second relative to the space station.

The satellite emits radiation and if the astronaut comes within 70 metres of the satellite the dosage will be harmful.

a) Determine the distance between the astronaut and satellite at $t=3$ seconds.

(3 marks)

Solution

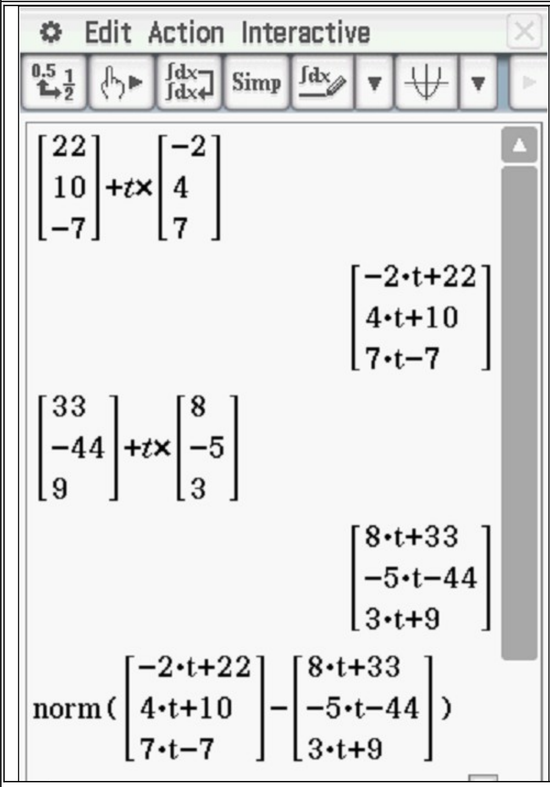


Specific behaviours

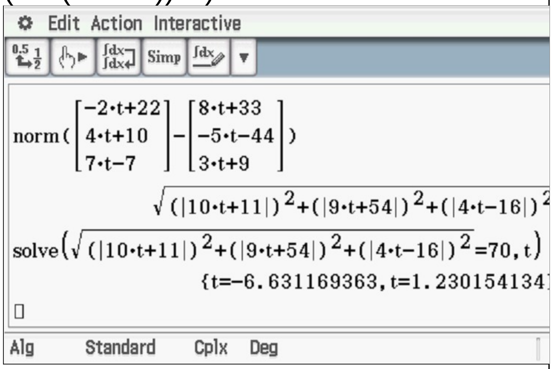
✓ determines position of astronaut at t=3
✓ determines position of satellite at t=3
✓ determines distance apart

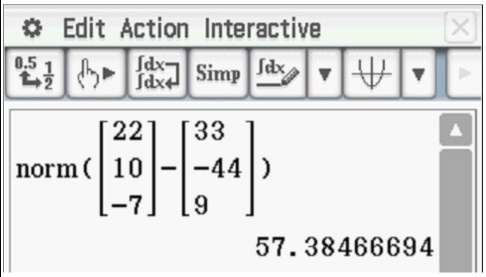
b) Determine if the astronaut is in danger and if so for how long in seconds, 2dp. (Justify your answer). (5 marks)

Solution



((abs(10*x+11))^2+(abs(9*x+54))^2+(abs(4*x-16))^2)^0.5





The first 1.23 seconds the astronaut is in danger.

Specific behaviours

✓ determines position of astronaut at t seconds
✓ determines position of satellite at t seconds
✓ determines distance apart in terms of t
✓ solves for time when distance apart equals 70 metres
✓determines time that astronaut is in danger

c)

Question 16 **(4 marks)**

Consider the two spheres $\left| r - \begin{pmatrix} 9 \\ -1 \\ 3 \end{pmatrix} \right| = \sqrt{21}$ and $x^2 + y^2 + z^2 + 8x - 2y + 2z = 66$.
Determine whether there are any common points on both spheres. Justify your answer.

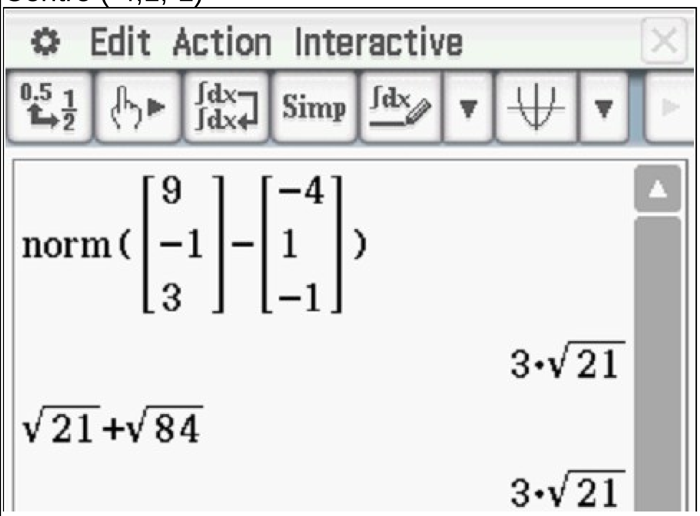
Solution

$x^2 + y^2 + z^2 + 8x - 2y + 2z = 66$

$x^2 + 8x + 16 - 16 + y^2 - 2y + 1 - 1 + z^2 + 2z + 1 - 1 = 66$

$(x+4)^2 + (y-1)^2 + (z+1)^2 = 66 + 16 + 1 + 1 = 84$

Centre (-4,1,-1)



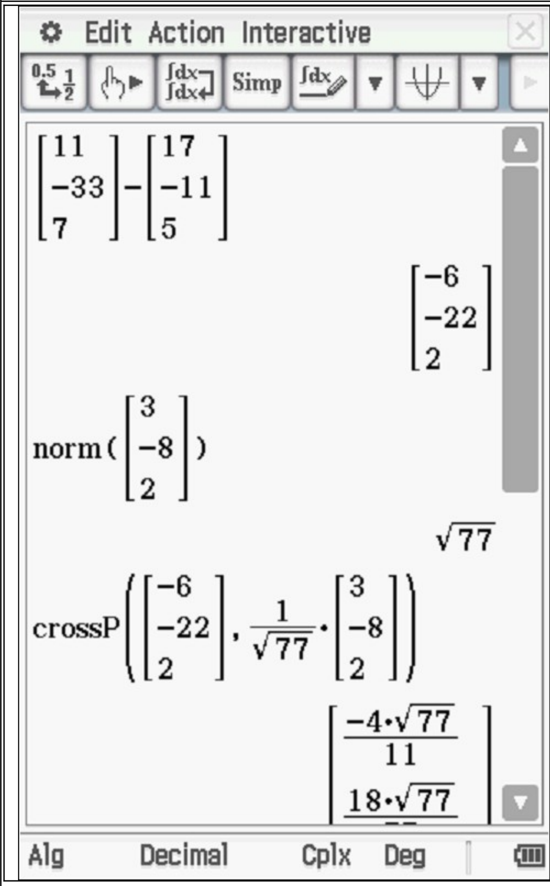
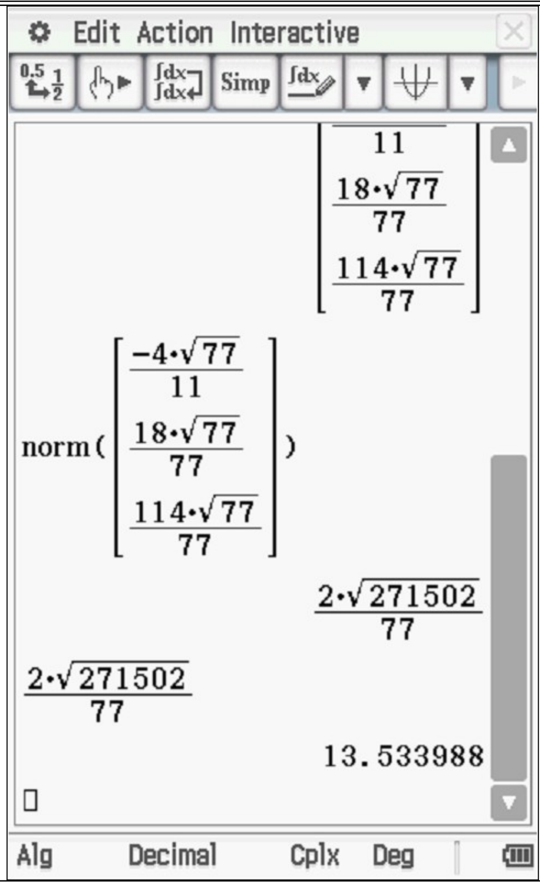
Only one common point as distance between centres equals sum of radii

Specific behaviours

✓ determines centre of second sphere
✓ determines radius of second sphere
✓ determines distance between centres
✓ shows that spheres touch at only one point

Question 17 **(4 marks)**

Show **using vector cross product**, how to determine the distance of point $A(11,-33,7)$ from the line $r = \begin{pmatrix} 17 \\ -11 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -8 \\ 2 \end{pmatrix}$.

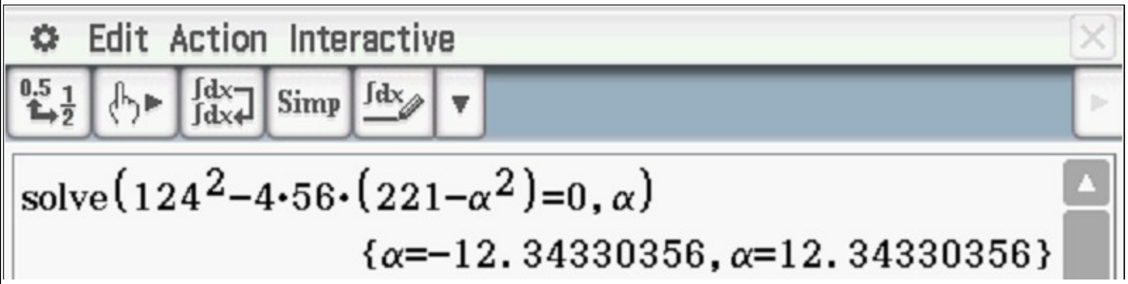
Solution
 

Specific behaviours
✓ de3termines vector from point on line to pt A ✓ uses cross product with vector parallel to line ✓ uses unit vector ✓ determines approx distance from pt A to line using cross product

Question 18 **(9 marks)**

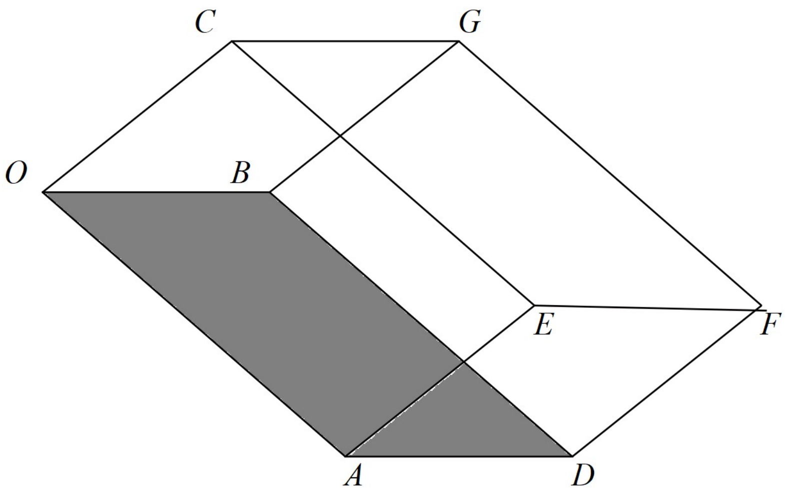
Consider the line $r = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}$ and the sphere $\left| r - \begin{pmatrix} 2 \\ -9 \\ 7 \end{pmatrix} \right| = \alpha$ where α is a constant.
Determine the values of α, to two decimal places, for each of the following scenarios:
(Justify your answers)

i) the line does not meet the sphere at all.
ii) the line meets the sphere at two points.
iii) the line is a tangent to the sphere.

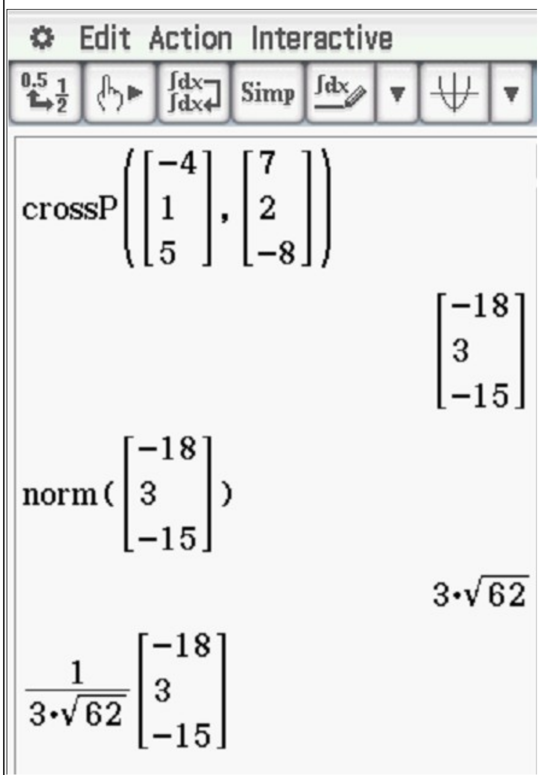
Solution
$\left\| \begin{pmatrix} -1+6\lambda \\ 5-2\lambda \\ 3+4\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ -9 \\ 7 \end{pmatrix} \right\| = \alpha$ $\left\| \begin{pmatrix} -3+6\lambda \\ 14-2\lambda \\ -4+4\lambda \end{pmatrix} \right\| = \alpha$ $\sqrt{(-3+6\lambda)^2 + (14-2\lambda)^2 + (-4+4\lambda)^2} = \alpha$ $9 - 36\lambda + 36\lambda^2 + 196 - 56\lambda + 4\lambda^2 + 16 - 32\lambda + 16\lambda^2 = \alpha^2$ $56\lambda^2 - 124\lambda + 221 - \alpha^2 = 0$ $\det = 124^2 - 4(56)(221 - \alpha^2)$    i) alpha less than 12.34 but greater than zero   ii) alpha greater than 12.34   iii) alpha equal to 12.34
Specific behaviours
✓ subs line into vector eqn of sphere   ✓determines magnitude of left hand side   ✓ derives a quadratic equation for λ ✓ determines an expression for det in terms of α ✓ equates det to zero and solves   ✓only accepts positive values   ✓determines values for not meeting   ✓ determines values for meeting at two points   ✓ determines value for scenario of tangent

Question 19 **(13 marks)**

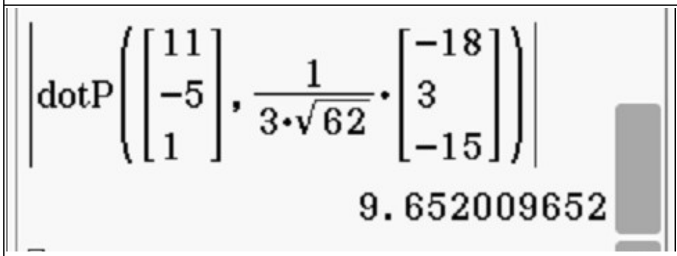
Consider a prism where opposite sides are congruent parallelograms(parallelepiped) with coordinates $O(0,0,0)$ $A(-4,1,5)$ $B(7,2,-8)$ $C(11,-5,1)$.



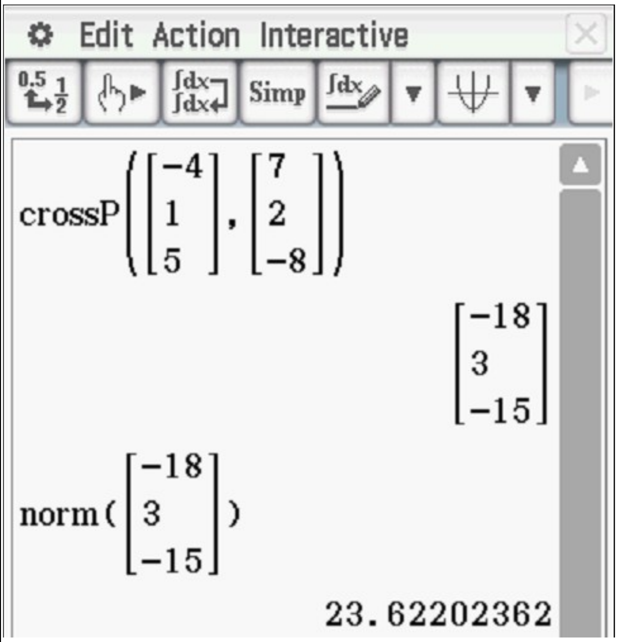
i) Determine a unit normal vector to the base $OADB$. (3 marks)

Solution

Specific behaviours
✓ determines two vectors in base plane ✓ uses cross product ✓ determines unit vector

ii) Using this unit normal, determine the distance between the sides $OADB \,\&\, CEFG$.
(Hint-use vector OC) (3 marks)

Solution

Specific behaviours
✓ determines OC vector ✓ uses dot product ✓ determines distance

iii) Show using cross product how to determine the area of the base $OADB$.
(2 marks)

Solution

Specific behaviours
✓ cross product of OA & OB vectors ✓ determines magnitude

iv) Hence or otherwise, determine the volume of the prism. (3 marks)

Solution
$\text{norm}\left(\begin{bmatrix}-18\\3\\-15\end{bmatrix}\right)$
23.62202362
$\text{dotP}\left(\begin{bmatrix}11\\-5\\1\end{bmatrix}, \frac{1}{3\cdot\sqrt{62}}\cdot\begin{bmatrix}-18\\3\\-15\end{bmatrix}\right)$
-9.652009652
$\left\lvert\text{dotP}\left(\begin{bmatrix}11\\-5\\1\end{bmatrix}, \frac{1}{3\cdot\sqrt{62}}\cdot\begin{bmatrix}-18\\3\\-15\end{bmatrix}\right)\right\rvert$
9.652009652
$23.62202362\times 9.652009652$
228
Specific behaviours
✓ uses V=A*H ✓ states area and height ✓ determines volume

v) In terms of the vectors $OA, OB \,\&\, OC$ write an expression using cross and dot products to represent the volume of the prism. (2 marks)

Solution
$\lvert OA \times OB . OC \rvert$
Specific behaviours
✓ uses cross and dot product ✓ uses absolute value

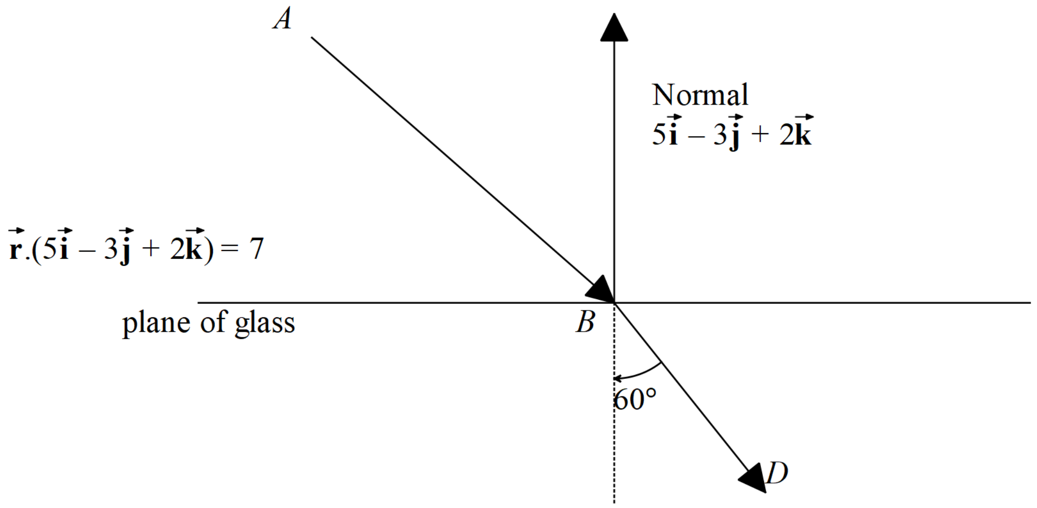
vi)

Question 20 **(9 marks)**

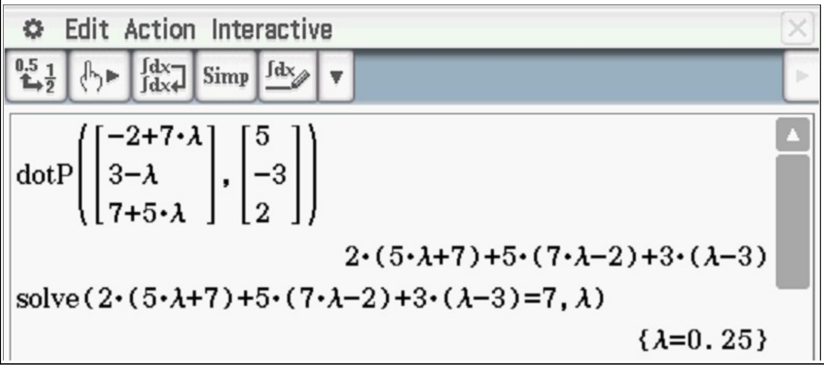
Consider a single photon of light that is released from a box positioned at point A $(-2, 3, 7)$ and moves in a direction of $\begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix}$ hitting a planar sheet of glass at point B. The planar sheet of glass is given by $r.\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} = 7$. The photon is refracted, that is changes direction, through the glass such that the angle with the normal is 60° and passes through point D.

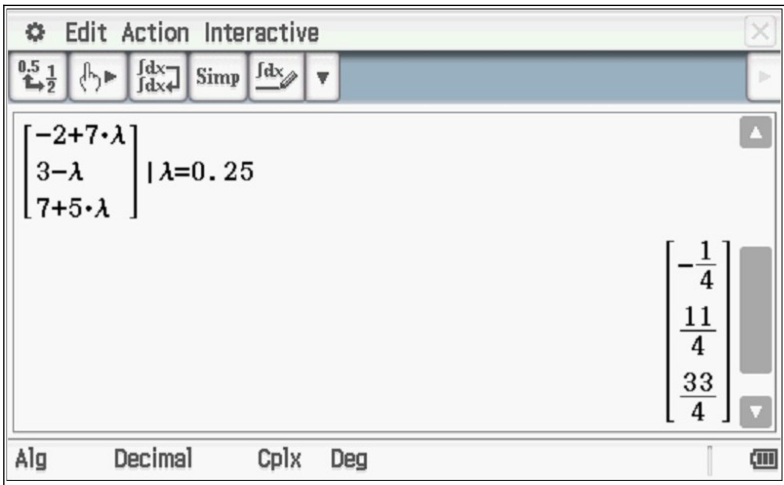
It is given that the vectors AB, BD and the normal are all in the same plane.

cross-section



a) Determine the point B. (3 marks)

Solution


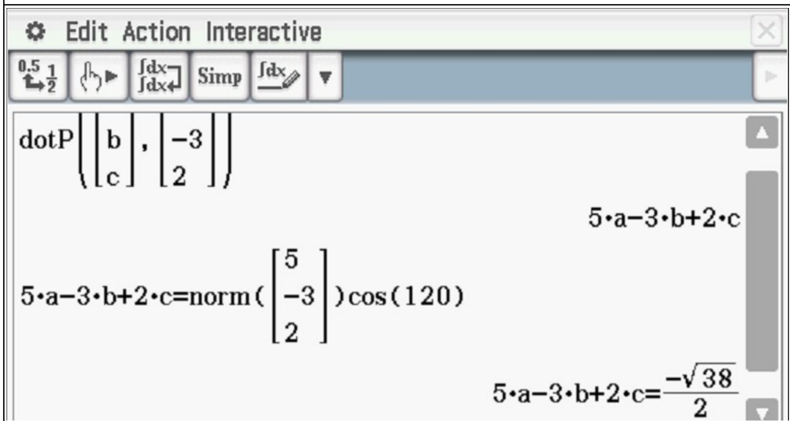


Specific behaviours
✓ determines vector eqn of line of photon ✓ sets up linear equation with vector eqn of plane ✓ solves for point B

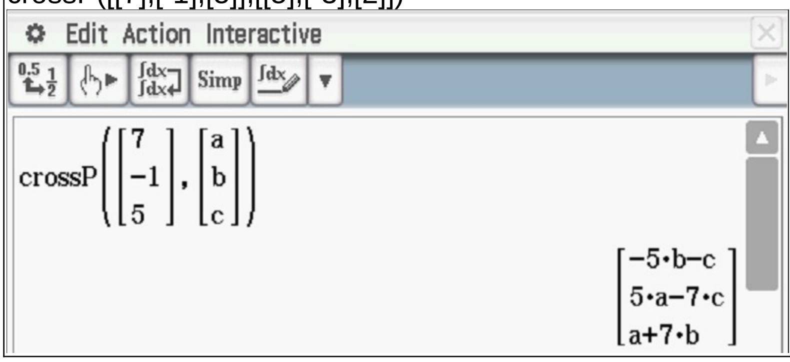
Let BD represent a unit vector and be represented as $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with $a^2+b^2+c^2=1^2$

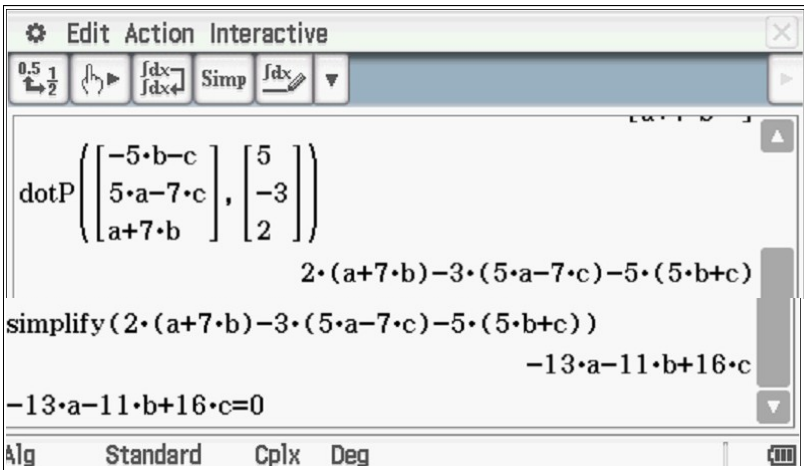
b) Determine two other independent equations for $a, b \,\&\, c$. (4 marks)

Solution

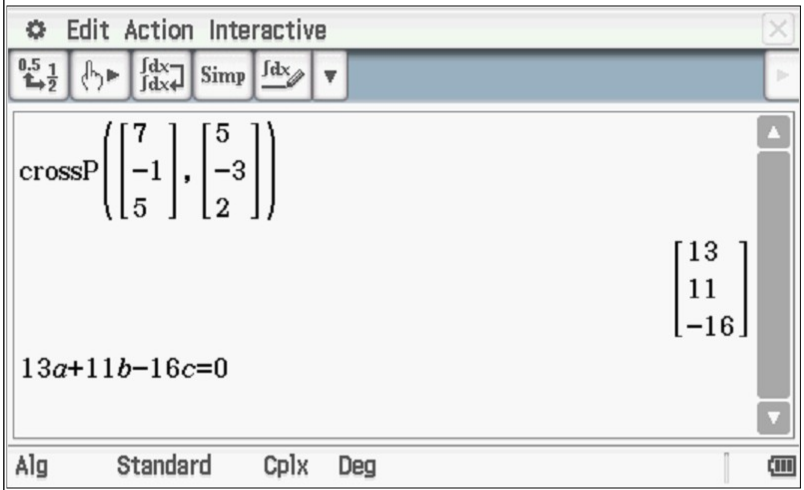


crossP([[7],[-1],[5]],[[5],[-3],[2]])





OR

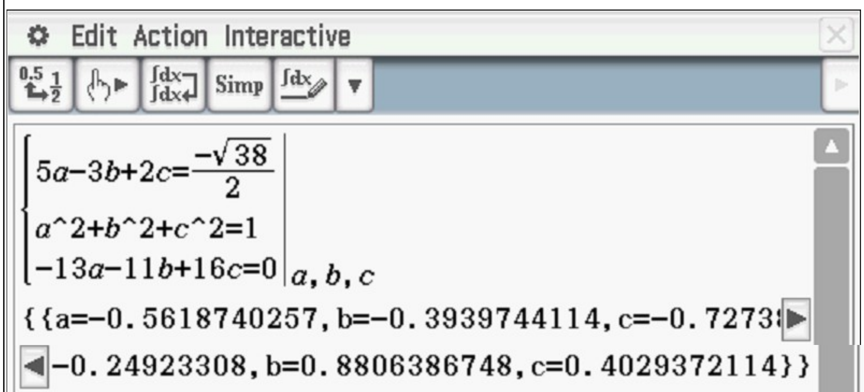


Specific behaviours
✓ uses dot product with BD and normal ✓ obtains an eqn in terms of a,b&c ✓ uses cross product with BD ✓ uses dot product of this cross product with vector in plane to derive second eqn

Question continued

c) Hence solve for vector BD using your CAS calculator (simultaneous) to 2 decimal places.

Solution



Specific behaviours
✓ uses simultaneous mode on CAS with three independent eqns for a.b&c

✓ solves for at least one set of values NOTE: Follow through will only occur if correct ideas were used in setting up equations in b above.

Additional working space

Question number: __________

Additional working space

Question number: __________