Supplementary page

Question number: ____________________

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Semester Two Examination, 2021

Question/Answer Booklet

MATHEMATICS METHODS
ATAR Year 12
Section One: Calculator-free

Student Name: **SOLUTIONS**

Please circle your teacher's name

Teacher: Miss Hosking Miss Rowden

Time allowed for this paper
Reading time before commencing work: 5 minutes
Working time for paper: 50 minutes

Materials required/recommended for this paper
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

Number of additional answer booklets used (if applicable):	

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Suggested working time (minutes)	Marks available	Percentage of examination
Section One: Calculator free	8	8	50	51	35
Section Two: Calculator-assumed	13	13	100	97	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the ATAR course examinations are detailed in the *Year 12 Information Handbook 2021*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Supplementary pages for the use planning/continuing your answer to a question have been provided at the end of the Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Supplementary page

Question number: ____________________

Question 8 **(6 marks)**

In triangle ABC, the length of the side opposite angle A is given by $a=\sqrt{13-6\cos A}$ cm.

Use the increments formula to calculate the approximate change in length of a as the size of angle A decreases from $\frac{20\pi}{30}$ to $\frac{19\pi}{30}$.

Solution
$a=(13-6\cos A)^{\frac{1}{2}} \frac{da}{dA}=\frac{1}{2}(13-6\cos A)^{\frac{-1}{2}}(6\sin A)$ $¿\frac{3\sin A}{\sqrt{13-6\cos A}}$ When $\angle A=\frac{2\pi}{3}$: $\frac{da}{dA}=\frac{3\sin\frac{2\pi}{3}}{\sqrt{13-6\cos\frac{2\pi}{3}}}=3\frac{\sqrt{3}}{2}\div\sqrt{16}=\frac{3\sqrt{3}}{8}$ $\delta A=\frac{-\pi}{30}$ $\delta a\approx\frac{da}{dA}\times\delta A\approx\frac{3\sqrt{3}}{8}\times-\frac{\pi}{30}\approx-\frac{\sqrt{3}\pi}{80}\text{cm}$ Hence length decreases by approximately $\frac{\sqrt{3}\pi}{80}$ cm.
Specific behaviours
✓ indicates use of chain rule   ü correct derivative   ü evaluates derivative at initial angle   ü indicates incremental change   ü uses increments formula   ü states decrease in length with units

End of questions

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Section One: Calculator-free **35% (51 Marks)**

This section has eight (8) questions. Answer **all** questions. Write your answers in the spaces provided.

Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

Working time: 50 minutes.

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Question 1 **(5 marks)**

a) Determine $\int\frac{4x+1}{2x^2+x-5}dx, x>2$. (2 marks)

Solution
$\int\frac{4x+1}{2x^2+x-5}dx=\ln(2x^2+x-5)+c$
Specific behaviours
ü antiderivative   ü includes constant of integration

(b) The line $y=12-2x$ intersects the curve $y=\frac{10}{x}$ at $(1,10)$ and $(5,2)$. Determine the area trapped between line and the curve. (3 marks)

Solution
$A=\int_1^5 12-2x-\frac{10}{x}dx¿[12x-x^2-10\ln x]_1^5$ $¿[60-25-10\ln 5]-[12-1-0]¿24-10\ln 5\text{ sq units}$
Specific behaviours
✓ writes correct integral   ü antidifferentiates correctly   ü substitutes and simplifies

Question 2 **(5 marks)**

A summary of the lengths of a large sample of nails from a production line are shown below.

Length, L mm	Relative frequency
$147<L\le 148$	0.17
$148<L\le 149$	0.13
$149<L\le 150$	0.21
$150<L\le 151$	0.19
$151<L\le 152$	0.16
$152<L\le 153$	0.14

(a) What proportion of nails are longer than 149 mm? (1 mark)

Solution

$p=1-0.13-0.17=0.7$

Specific behaviours

✓ correct proportion

(b) Determine the probability that a randomly selected nail from the production line is longer than 150 mm given that it is no longer than 152 mm. (2 marks)

Solution

$$P(L>150 \vee L\le 152)=\frac{0.19+0.16}{1-0.14}=\frac{35}{86}$$

Specific behaviours

✓ indicates use of correct relative frequencies

ü simplifies to proper fraction

(c) State, with reasons, whether the data suggests that the nail lengths are normally distributed. (2 marks)

Solution

Not normally distributed. The relative frequencies do not reflect the bell shaped outline of a normal distribution and appear closer to a uniform distribution.

Specific behaviours

✓ states no

ü justifies response

Question 7 **(8 marks)**

The random variable X is defined by $P(X=x)=\begin{cases} k\log_3(x+2) & x=1,25,79 \\ ¿ & \text{elsewhere}¿ \end{cases}$

(a) Determine the value of the constant k. (2 marks)

Solution

$k(\log_3 3+\log_3 27+\log_3 81)=1$

$k(1+3+4)=1k=\frac{1}{8}$

Specific behaviours

✓ equation for k

ü correct value

(b) Calculate the expected value of X. (2 marks)

Solution

$E(X)=1\times\frac{1}{8}+25\times\frac{3}{8}+79\times\frac{1}{2}$

$¿\frac{76}{8}+\frac{79}{2}=9.5+39.5=49$

Specific behaviours

✓ indicates $\sum xp$

ü correct $E(X)$

The Bernoulli random variable Y is solely dependent on X, so that $Y=1$ when $X=1$, and $Y=0$ for all other values of X.

(c) Determine

(i) $P(Y=0)$. (1 mark)

Solution

$P(Y=0)=1-P(X=1)=\frac{7}{8}$

Specific behaviours

✓ correct probability

(ii) $E(Y)$. (1 mark)

Solution

$E(Y)=0\times\frac{7}{8}+1\times\frac{1}{8}=\frac{1}{8}$

Specific behaviours

✓ correct value

(iii) $Var(3Y+1)$. (2 marks)

Solution

$Var(Y)=\frac{7}{8}\times\frac{1}{8}=\frac{7}{64}$

$Var(3Y+1)=3^2\times\frac{7}{64}=\frac{63}{64}$

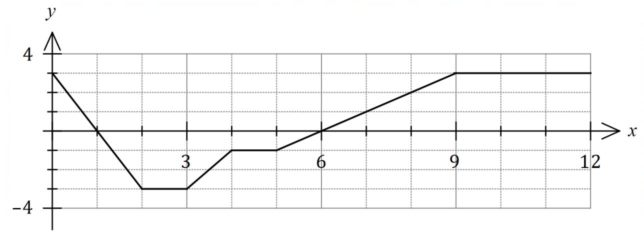
Specific behaviours

✓ $Var(Y)$

ü $Var(3Y+1)$

Question 6 **(6 marks)**

The graph of $y=f(x)$ consists of line segments, as shown below.



Evaluate each of the following:

(a) $\int_{7}^{10} f(x)dx$. (1 mark)

Solution

$\int ¿ 7$

Specific behaviours

✓ correct value

(b) $\int_{3}^{8} f(x)dx$. (2 marks)

Solution

$\int_{3}^{6} + \int_{6}^{8} ¿ -3.5+2=-1.5$

Specific behaviours

✓ indicates use of signed area

ü correct value

(c) $\int_{0}^{9} (f(x)+2)dx$. (3 marks)

Solution

$\int_{0}^{2} f + \int_{2}^{6} f + \int_{6}^{9} f + \int_{0}^{9} 2\,dx = 0-6.5+4.5+18=16$

Specific behaviours

✓ indicates use of additivity

✓ determines integral of 2 between 0 and 9

ü correct value

Question 3 **(7 marks)**

The curve $y=8x-\frac{4}{x^2}$ has one stationary point.

(a) Obtain expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. (2 marks)

Solution

$$\frac{dy}{dx}=8+\frac{8}{x^3}$$

$$\frac{d^2y}{dx^2}=\frac{-24}{x^4}$$

Specific behaviours

✓ first derivative

ü second derivative

(b) Determine the coordinates of the stationary point and determine its nature. (4 marks)

Solution

$$\frac{dy}{dx}=0 \Rightarrow 8+\frac{8}{x^3} \Rightarrow x=-1$$

$$y=-8-4=-12$$

Stationary point at $(-1,-12)$.

$$x=-1 \Rightarrow \frac{d^2y}{dx^2}=\frac{-24}{x^4}=-24$$

Hence stationary point is a maximum.

Specific behaviours

✓ equates first derivative to zero and solves

ü calculates coordinates

ü evaluates second derivative at point

ü states maximum

(c) Explain why the curve has no point of inflection. (1 mark)

Solution

There is no value of x for which $\frac{d^2y}{dx^2}=0$.

Specific behaviours

✓ explains using second derivative

Question 4 **(7 marks)**

(a) Let $F(x)=\int_0^x \sin 2\theta \, d\theta$.

Express $F(x)$ as a function of x and hence evaluate $F\left(\frac{\pi}{6}\right)$.

(3 marks)

Solution
$F(x)=\left[\frac{-1}{2}\cos 2\theta\right]_0^x$
$\text{¿}-\frac{1}{2}\cos 2x-\left(\frac{-1}{2}\right)\text{¿}\frac{1}{2}(1-\cos 2x)$
$F\left(\frac{\pi}{6}\right)=\frac{1}{2}\left(1-\cos\frac{\pi}{3}\right)=\frac{1}{4}$
Specific behaviours
✓ correct antiderivative
ü correct function
ü evaluates

(b) Let $g(x)=\frac{e^{2x-1}}{2x+1}$.

(i) Show that $g'(x)=\frac{4xe^{2x-1}}{(2x+1)^2}$. (2 marks)

Solution
$g'(x)=\frac{2e^{2x-1}(2x+1)-e^{2x-1}(2)}{(2x+1)^2}\text{¿}\frac{4xe^{2x-1}}{(2x+1)^2}$
Specific behaviours
✓ shows correct u' and v'
ü shows correct structure of quotient rule

(ii) Hence, or otherwise, evaluate $\int_0^1 \frac{xe^{2x-1}}{(2x+1)^2}dx$. (2 marks)

Solution
$\frac{1}{4}\int_0^1 \frac{4xe^{2x-1}}{(2x+1)^2}dx=\frac{1}{4}\left[\frac{e^{2x-1}}{2x+1}\right]_0^1\text{¿}\frac{e}{12}-\frac{1}{4e}$
Specific behaviours
✓ indicates correct antiderivative
ü evaluates

Question 5 **(7 marks)**

(a) By first using log laws, or otherwise, determine $\frac{d}{dx}\left(\ln\left(e^{3x}\sqrt{x^2+3}\right)\right)$ in simplest form.

(3 marks)

Solution
$\ln\left(e^{3x}\sqrt{x^2+3}\right)=\ln e^{3x}+\ln\left(\sqrt{x^2+3}\right)\text{¿}3x+\frac{1}{2}\ln(x^2+3)$
$\frac{d}{dx}\left(3x+\frac{1}{2}\ln(x^2+3)\right)=3+\frac{x}{x^2+3}\text{¿}\frac{3x^2+x+9}{x^2+3}$
Specific behaviours
✓ uses one log law appropriately
ü uses second log law appropriately
ü correctly differentiates (and simplifies to either of two forms shown)

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(b) The function $f(x)=x^2\ln(2x)$ for $x>0$ has one stationary point, a global minimum.

Determine the minimum value of the function. (4 marks)

Solution
$f'(x)=2x\ln(2x)+x^2\left(\frac{1}{x}\right)\text{¿}2x\ln(2x)+x\text{¿}x\text{¿}$
Stationary when:
$f'(x)=0\Rightarrow\ln(2x)=\frac{-1}{2}2x=e^{\frac{-1}{2}}x=\frac{e^{\frac{-1}{2}}}{2}$
Minimum value:
$f\left(\frac{e^{\frac{-1}{2}}}{2}\right)=\frac{e^{-1}}{4}\ln\left(e^{\frac{-1}{2}}\right)\text{¿}\frac{1}{4e}\times-\frac{1}{2}=\frac{-1}{8e}$
Specific behaviours
✓ uses product rule correctly
ü obtains derivative
ü obtains root of derivative
ü calculates minimum value