# SEMESTER TWO

# MATHEMATICS
# SPECIALIST
# UNITS 3 & 4

# 2017

# SOLUTIONS

***Calculator-free Solutions***

1. (a) $$\frac{2\text{cis}\,\frac{\pi}{4} \times 3\text{cis}\,\frac{\pi}{3}}{4\text{cis}\,\frac{\pi}{12}} = \frac{3}{2}\text{cis}\,\frac{\pi}{2} = \frac{3}{2}i$$ ✓

(b) $$\frac{1}{16}\text{cis}\left(-\frac{4\pi}{6}\right) = \frac{1}{16}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$ ✓

$$= -\frac{1}{32} - \frac{\sqrt{3}}{32}i$$ ✓

(c) $$\left(\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)\right)^{10}$$ ✓

$$= 32\left(\text{cis}\left(-\frac{5\pi}{2}\right) + i\sin\left(-\frac{5\pi}{2}\right)\right)$$ ✓

$$= -32i$$ ✓ [6]

2. $f(-3) = 0$ so $z + 3$ is a factor

$f(z) = 2(z+3)(z^2 + bz + c)$ ✓

$\therefore$ $c = 13$ and $b = -6$ by equating coefficients or division

$$\frac{z^3 - 3z^2 - 5z + 39}{z+3} = z^2 - 6z + 13$$ ✓

$$\therefore \quad z = \frac{6 \pm \sqrt{36 - 4(13)}}{2} = \frac{6 \pm \sqrt{-16}}{2}$$ ✓

$\therefore$ $z = -3$ or $z = 3 \pm 2i$ ✓ [4]

3. $$(z+1)^3 = 27\text{cis}\,\pi \rightarrow z + 1 = (27\text{cis}\,\pi)^{\frac{1}{3}}$$ ✓

$$\therefore \quad z + 1 = 3\text{cis}\left(-\frac{\pi}{3}\right) \text{ or } 3\text{cis}\,\frac{\pi}{3} \text{ or } 3\text{cis}\,\pi$$ ✓✓

$$\therefore \quad z = 3\text{cis}\left(-\frac{\pi}{3}\right) - 1 \text{ or } 3\text{cis}\,\frac{\pi}{3} - 1 \text{ or } 3\text{cis}\,\pi - 1$$ ✓ [4]

4. (a) $\dfrac{4x+3}{3x(3-2x)} = \dfrac{a}{3x} + \dfrac{b}{3-2x}$ ✓

$\therefore \quad 4x + 3 = a(3-2x) + b(3x)$

$\therefore \quad 4 = -2a + 3b$ and $3 = 3a \quad \rightarrow \quad a = 1$ and $b = 2$ ✓

$\therefore \quad \dfrac{4x+3}{3x(3-2x)} = \dfrac{1}{3x} + \dfrac{2}{3-2x}$ ✓

(b) $\displaystyle\int \frac{4x+3}{9x-6x^2}\,dx = \int \left(\frac{1}{3x} + \frac{2}{3-2x}\right)\,dx$ ✓

$= \dfrac{1}{3}\ln|x| - \ln|3-2x| + c$ ✓✓ [6]

5. (a) $\displaystyle\int \frac{\pi\sin x - \pi}{\sqrt{x+\cos x}}\,dx \qquad u = x + \cos x \rightarrow \frac{du}{dx} = 1 - \sin x$ ✓

$= \displaystyle -\pi\int \frac{1}{u^{\frac{1}{2}}}\,du \qquad$ as $dx = \dfrac{du}{1-\sin x}$ ✓

$= -\pi\left[2u^{\frac{1}{2}}\right] + c$ ✓

$= -2\pi\sqrt{x + \cos x} + c$ ✓

(b) $\displaystyle\int_0^{\frac{\pi}{4}} (\cos^2 x + 4\sin x\cos x + 4\sin^2 x)\,dx$ ✓

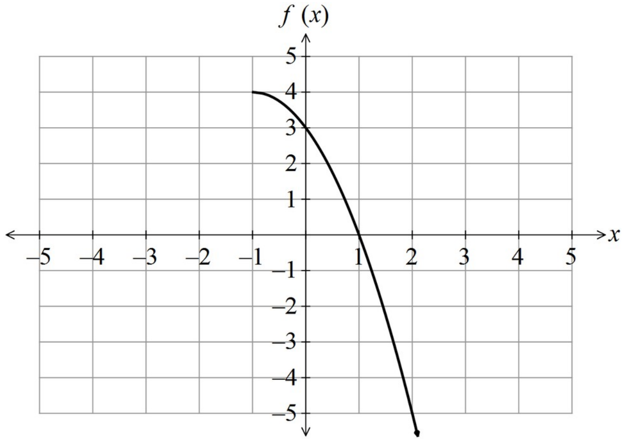
$= \displaystyle\int_0^{\frac{\pi}{4}} \left(1 + 2\sin 2x + 3\left(\frac{1-\cos 2x}{2}\right)\right)dx$ ✓

$= \left[\dfrac{5}{2}x - \cos 2x + \dfrac{3}{4}\sin 2x\right]_0^{\frac{\pi}{4}}$ ✓

$= \dfrac{5\pi}{8} + 1 - \dfrac{3}{4}$

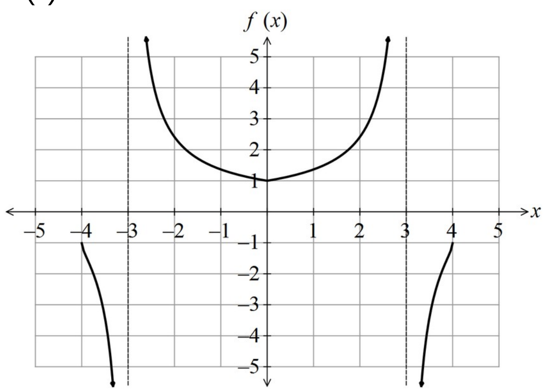
$= \dfrac{5\pi}{8} + \dfrac{1}{4}$ or $\dfrac{5\pi+2}{8}$ ✓ [8]

6. (a) (i)




✓

(ii)




✓✓✓✓

(b) $g(x) = (x-3)^2 - 7$ by completing the square ✓

So domain of $g(x)$ will be $x \leq 3$ or $x \geq 3$ (max allowable for $g$ and to have an inverse)

$\therefore \quad x = (y-3)^2 - 7$ ✓

$\therefore \quad y = g^{-1}(x) = 3 - \sqrt{x+7}$ or $3 + \sqrt{x+7}$ ✓ [8]

7. (a) (i) $C = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$ ✓

(ii) A and C ✓

(iii) $(x-2)^2 + (y+4)^2 + (z-6)^2 = 49$ ✓

(iv) $(3-2)^2 + (-4+4)^2 + (4-6)^2 = 5$

Since < 49, then inside ✓

(b) $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{bmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ 6 & -2 & 4 \\ -2 & -6 & 8 \end{bmatrix}$

$= \quad 8\underset{\sim}{\mathbf{i}} - 56\underset{\sim}{\mathbf{j}} - 40\underset{\sim}{\mathbf{k}}$ and $-8\underset{\sim}{\mathbf{i}} + 56\underset{\sim}{\mathbf{j}} + 40\underset{\sim}{\mathbf{k}}$ ✓✓

Or any other vector parallel to $\begin{pmatrix} 1 \\ -7 \\ -5 \end{pmatrix}$ [6]

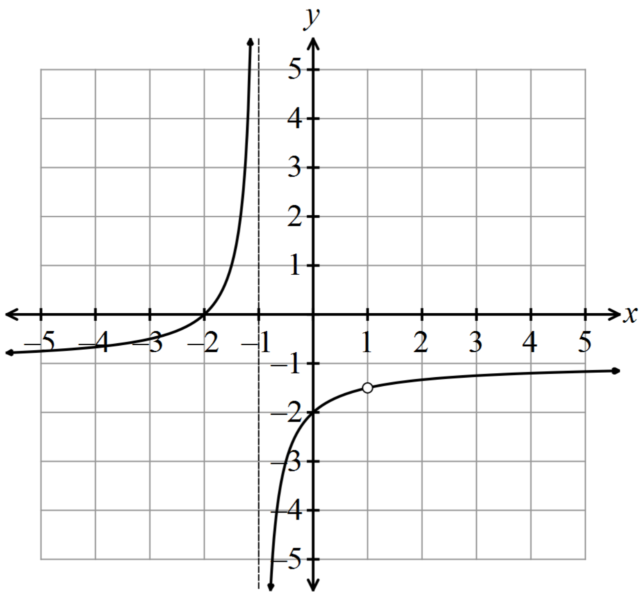
8. (a) $\dfrac{x^2 + x - 2}{1 - x^2} = -\dfrac{(x+2)(x-1)}{(x-1)(x+1)} = -\dfrac{x+2}{x+1}, \quad x \neq 1$ ✓

$\therefore$ $x$ = -1 is vertical asymptote and $x$ = 1, (1, -1.5) is a hole. ✓✓

since $\lim_{x \to \pm\infty} G(x) = -1$ , horizontyal asymptote at $y = -1$ ✓

$y$-intercept = -2 and $x$-intercept = -2 ✓

(b)




✓✓✓ [8]

***Calculator-assumed Solutions***

9. (a) $g \circ f(x) = \dfrac{1}{[f(x)]^2} = \dfrac{1}{\left(\sqrt{x^2-1}\right)^2} = \dfrac{1}{x^2-1}$ ✓

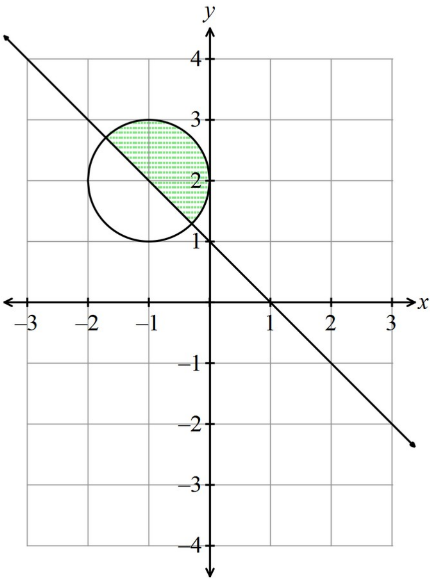
$g \circ g(x) = \dfrac{1}{\left(\dfrac{1}{x^2}\right)^2} = x^4$ ✓

(b) (i) $\{x : x < -1,\ x > 1\}$ ✓✓

(ii) y > 0 ✓

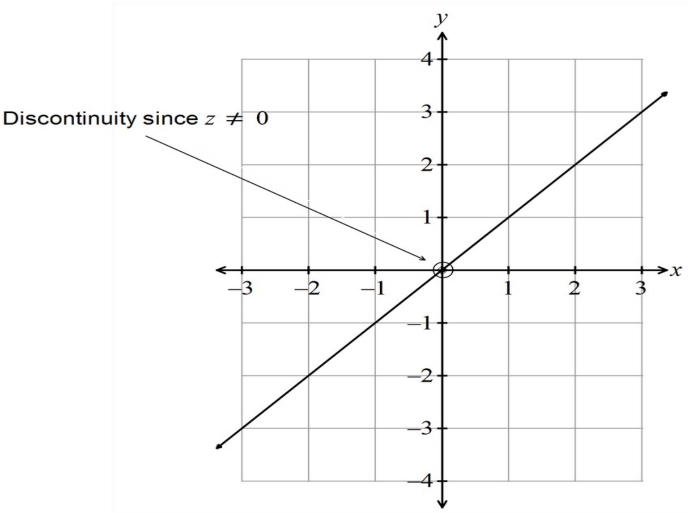
(c) $h(x) = \sqrt{x-4}$ ✓ [6]

10. (a) | $z$ - (-1+2i) | ≤ 1 is the region inside the circle centre (-1, 2) and radius 1
| $z$ + i | ≥ | $z$ - (2 + i) | is the region on one side of the line $y = -x + 1$,
containing (1, 1)



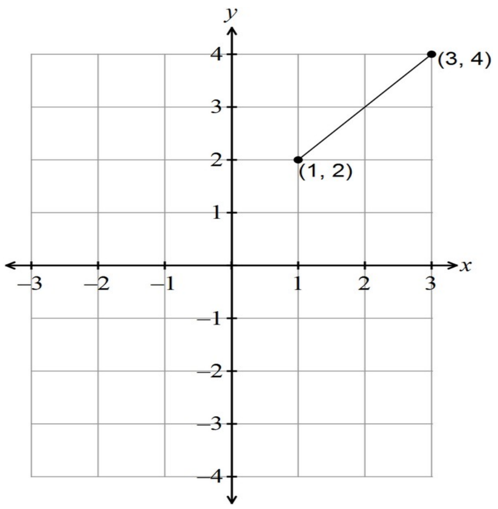

✓✓✓✓

(b) $\dfrac{z}{\bar{z}} = i$ is the line $y = x$, since $x + y$i = i($x - y$i) = $y + x$i




✓✓

(c)




✓✓ [8]

11. (a) $x = \sin\frac{\pi y}{4}$

$\therefore \quad 1 = \frac{\pi}{4}\cos\left(\frac{\pi y}{4}\right)\frac{dy}{dx}$ ✓

$\therefore \quad \frac{dy}{dx} = \frac{4}{\pi\cos\left(\frac{\pi y}{4}\right)}$ ✓

(b) $\int_0^4 \sin\left(\frac{\pi y}{4}\right)dy$ ✓

$= \left[-\frac{4}{\pi}\cos\left(\frac{\pi y}{4}\right)\right]_0^4$ ✓

$= -\frac{4}{\pi}(-2) = \frac{8}{\pi}$ ✓ [5]

12. (a) $\frac{dy}{dx} = ax + b$ where $a > 0$ ✓

Since $\frac{dy}{dx} = 0$ when $x = -1$ then $-a + b = 0 \rightarrow a = b$ ✓

$\therefore \quad \frac{dy}{dx} = a(x+1)$ or $ax + a$ ✓

(b) $\frac{dy}{dx} = a = \frac{1}{4}$ when $x = 0$ ✓

and $y = \frac{x^2}{8} + \frac{1}{4}x + c$

$\therefore \quad (0,1) \rightarrow c = 1$ ✓

$\therefore \quad y = \frac{x^2}{8} + \frac{x}{4} + 1$ ✓ [6]

13. (a) $t = 0 \to 10 = \dfrac{a}{1+b}$

$\therefore \quad t \to \infty \therefore 2000 = \dfrac{a}{1}$

$\therefore \quad a = 2000$ and $b = 199$ ✓✓

(b) Increasing at greatest rate at oblique point of inflection.

$\therefore \quad \dfrac{d^2P}{dt^2} = 0 \to t = 0{\cdot}5293$ years and $P = 1000$ ✓✓✓ [5]

14. (a) $\dfrac{d}{dx}\left\{\dfrac{1}{2}(25 - x^2)\right\} = -x$ ✓

$v\dfrac{dv}{dx} = (25 - x^2)^{\frac{1}{2}} \times \dfrac{1}{2}(25 - x^2)^{-\frac{1}{2}} \times (-2x) = -x$ ✓

(b) $v(-4) \to v = 3$ cm/sec , a = 4 cm/sec$^2$ ✓✓

(c) $v = 0 \to 25 - x^2 = 0$

$x = \pm 5$ cm and a = $\mp 5$ cm/sec$^2$ ✓✓

(d) $a = 2 \to x = -2$ and $v = \sqrt{21}$ cm/sec ✓✓ [8]

15.

$L_1 = L_1 \to x + 2y - 3z = a$

$L_2' = -2L_1 + L_2 \to 2y - 5z = -2a + b$

$L_3' = -L_1 + L_3 \to -4y + 10z = -a + c$ ✓

and then

$L_1 = L_1 \to x + 2y - 3z = a$

$L_2' = L_2' \to 2y - 5z = -2a + b$

$L_3'' = 2L_2' + L_3' \to 0 = -5a + 2b + c$ ✓

No solution $\to a \neq \dfrac{2b + c}{5}$ ✓✓ [4]

16. (a) $x = A\cos(nt)$ since start at end point

$\therefore \quad x = 15\cos\left(\frac{\pi t}{5}\right)$ ✓✓

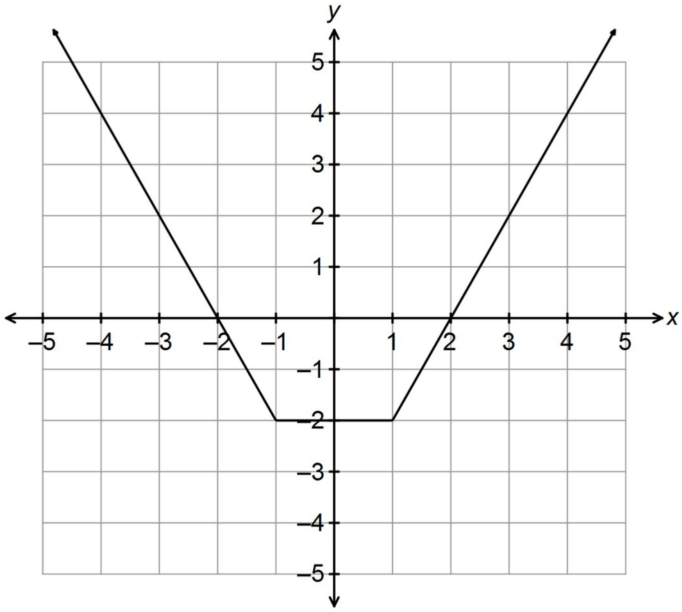
and $\frac{dx}{dt} = -3\pi\sin\left(\frac{\pi t}{5}\right)$

$\therefore$ Max speed = $3\pi$ m/sec ✓

(b) $\frac{d^2x}{dt^2} = -\frac{3\pi^2}{5}\cos\left(\frac{\pi t}{5}\right)$

$\therefore$ Min acceleration = $-\frac{3\pi^2}{5}$ m/sec$^2$ is at $x = 15$ m ✓✓ [5]

17. (a)




✓

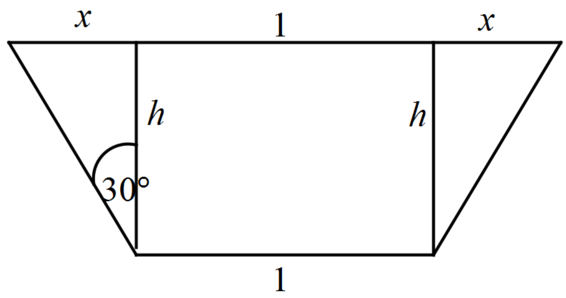
(b) $a = 1$ ✓

$b = 2$ (or -2) ✓

$h(0) = 4 - c = -2 \;\therefore\; c = 6$ ✓

(c) $d > -6$ ✓✓ [6]

18. (a)




$V = \frac{1}{2}(1 + 2x)h \times 2$ and $\tan 30° = \frac{x}{h} \rightarrow x = \frac{h}{\sqrt{3}}$ ✓

$\therefore \quad V = \left(1 + \frac{h}{\sqrt{3}}\right)h \times 2$ ✓

$\therefore \quad V = 2h\left(\frac{\sqrt{3} + h}{\sqrt{3}}\right)$ ✓

(b) $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{\mathrm{dt}}$ and $\frac{dV}{dh} = \frac{4h}{\sqrt{3}} + 2$ ✓

$\frac{dV}{dh}\Big|_{h = 0 \cdot 4} = 2 \cdot 924$ ✓

$\therefore \quad -0 \cdot 05 = 2 \cdot 924 \frac{dh}{dt} \rightarrow \frac{dh}{dt} = -0 \cdot 017$ m/hr ✓

(c) $-0 \cdot 05 = \left(\frac{4h}{\sqrt{3}} + \frac{2\sqrt{3}}{\sqrt{3}}\right)\frac{dh}{\mathrm{dt}}$ ✓

$\therefore \quad \frac{dh}{dt} = \frac{-0 \cdot 05\sqrt{3}}{4h + 2\sqrt{3}} = -\frac{\sqrt{3}}{80h + 40\sqrt{3}}$ ✓

(d) $\int (-\sqrt{3})\ \mathrm{dt} = \int (80h + 40\sqrt{3})\ \mathrm{dh}$

$\therefore \quad -\sqrt{3}\, t = 40h^2 + 40\sqrt{3}\, h + c$ ✓

Since $t = 0, h = 1 \rightarrow c = 40 + \frac{40\sqrt{3}}{3}$ ✓

$\therefore \quad t = 40 + \frac{40\sqrt{3}}{3} - \frac{40\sqrt{3}\, h^2}{3} - 40h$ ✓ [11]

19. (a) (i) $\frac{\sigma}{\sqrt{65}} = 0{\cdot}031$ ✓

$\therefore$ $P(\bar{x} \le 1{\cdot}15) = 0{\cdot}053$ using $N(1{\cdot}2, 0{\cdot}031^2)$ ✓✓

(ii) Mean of Joe's catch ~$N(1{\cdot}2, 0{\cdot}031^2)$ ✓

$\therefore$ $P\left(\frac{75}{65} \le \text{Mean} \le \frac{80}{65}\right) = 0{\cdot}771$ ✓✓

(b) $\frac{\sigma z}{\sqrt{n}} < 0{\cdot}05$ with $\sigma = 0{\cdot}25$ and $z = 2{\cdot}326$

$\therefore$ $n > 135{\cdot}3$ ✓

$\therefore$ Need to catch 136 fish. ✓

(c) $\bar{x} = \frac{270}{220} = 1{\cdot}227$ kg and $\sigma = 0{\cdot}25$ and $n = 220$

$z = 1{\cdot}96$

$\bar{x} \pm z\frac{\sigma}{\sqrt{n}} = 1{\cdot}1976 \rightarrow 1{\cdot}2564$ ✓✓

$\therefore$ $\mu = 1{\cdot}2$ is within 95% CI so James' claim is not accepted. ✓

$\therefore$ Cannot conclude that crayfish in James' area are significantly bigger ✓ [12]

20. (a) $x = 2\sin t\cos t$ and $y = \sin t$

$\therefore$ $x^2 = 4\sin^2 t\cos^2 t$ ✓

$\therefore$ $x^2 = 4y^2(1 - y^2)$ ✓

(b) $\underset{\sim}{\mathbf{v}}(t) = \begin{pmatrix} 2\cos 2t \\ \cos t \end{pmatrix}$

$\therefore$ $-0{\cdot}5 = \sin 2t \rightarrow t = \frac{7\pi}{12}, \frac{11\pi}{12}, \ldots$ ✓✓

$t = \frac{11\pi}{12} \rightarrow \underset{\sim}{\mathbf{v}} = \begin{pmatrix} 2\cos\frac{11\pi}{6} \\ \cos\frac{11\pi}{12} \end{pmatrix}$ ✓

$\therefore$ Speed = 1.98 m/sec ✓

(c) Distance = $\int_0^{2\pi} \sqrt{4\cos^2 2t + \cos^2 t}\ dt$ ✓✓

= 9.43 m ✓ [9]

21. (a) $\overrightarrow{BD} \times \overrightarrow{BE} = 12i - 6j + 12k$ and $\overrightarrow{AB} = 2i - j + 2k$

$\therefore \quad k = 6$ ✓✓

(b) $\underset{\sim}{r} \cdot (2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}) = c$ with $c = (3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}) \cdot (2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}) = 5$ ✓

$\underset{\sim}{r} \cdot (2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}) = 5$ ✓

(c) $\underset{\sim}{r} = (\underset{\sim}{i} - 3\underset{\sim}{k}) + \lambda(2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})$ ✓

(d) Area ($\Delta\, DBE$) = 9 and |AC| = 9 and |AB| = 3

The distance from A to each triangle has a scale of 1:3 ✓

$\therefore$ Area ($\Delta CFG$) $= 3^2 \times 9 = 81$ units$^2$ ✓

(e) If scale factor = $16 = 4^2 \rightarrow \overrightarrow{AH} = 4\,\overrightarrow{AB}$ ✓

$$\overrightarrow{OA} + \overrightarrow{AH} = \overrightarrow{OH} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + 4\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$ ✓

Now $\underset{\sim}{r} \,.\, \underset{\sim}{n} = \lambda$

$$\therefore \quad \underset{\sim}{r} \cdot \underset{\sim}{n} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 32$$

$\therefore \quad 2x - y + 2z = 32$ ✓ [10]

22. (a) $v = \int_a^b [\pi(r^2 - x^2)]\ dx$ ✓

(b) $V = \int_9^{11} \{\pi(11^2 - x^2)\}\ dx$ or $\int_{-11}^{-9} \{\pi(11^2 - x^2)\}\, dx$ ✓

$= \dfrac{124\pi}{3}$ or $129{\cdot}856$ cm$^3$ ✓

(c) $\dfrac{dy}{dx} = \dfrac{1}{2}(r^2 - x^2)^{-\frac{1}{2}}(-2x)$ ✓

$\therefore \quad A = \int_9^{11} \left\{ 2\pi\sqrt{11^2 - x^2} \times \sqrt{\dfrac{11^2}{11^2 - x^2}} \right\} dx$ ✓ [5]