Trinity College

Semester Two Examination, 2017

Question/Answer booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section One: Calculator-free

SOLUTIONS

Student Number: In figures ☐☐ ☐☐☐ ☐☐☐

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	97	65
				Total	100

Instructions to candidates

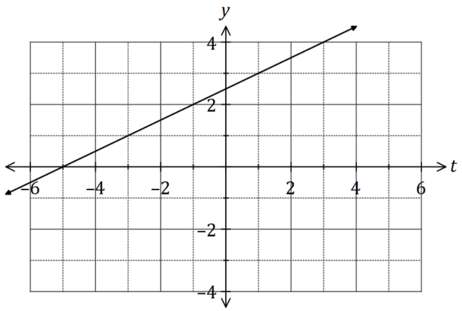
1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.
2. Write your answers in this Question/Answer booklet.
3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.
4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.
5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.
6. It is recommended that you do not use pencil, except in diagrams.
7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Additional working space

Question number: _________

Question 8 **(5 marks)**

Part of the graph of the linear function $y = f(t)$ is shown below.



Another function $A(x)$ is given by

$$A(x) = \int_{-1}^{x} f(t)\, dt\,.$$

Use the increments formula to estimate the change in A as x increases from 7 to 7.1.

Solution
$\frac{dA}{dx} = \frac{d}{dx}\int_{-1}^{x} f(t)\, dt = f(x)$ $f(x) = 0.5x + 2.5$ $\delta A \approx \frac{dA}{dx}\delta x \approx (0.5(7) + 2.5)(0.1)$ ≈ 0.6
Specific behaviours
✓ indicates $A'(x)$ ✓ uses $x = 7,\ \delta x = 0.1$ ✓ determines $f(x)$ ✓ uses increments formula ✓ determines change

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

The discrete random variable X is defined by

$$P(X = x) = \begin{cases} \dfrac{k}{x+1} & x = 0, 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Determine the value of the constant k. (2 marks)

Solution
$\frac{k}{1} + \frac{k}{2} = 1$ $k = \frac{2}{3}$
Specific behaviours
✓ sums probabilities to 1 ✓ states value

(b) Determine

(i) $\mathrm{E}(5 - 3X)$. (2 marks)

Solution
Bernoulli distribution, $p = P(X = 1) = \frac{1}{3}$ $E(X) = p = \frac{1}{3}$ $E(5 - 3X) = 5 - 3\left(\frac{1}{3}\right) = 4$
Specific behaviours
✓ uses $E(X) = p = P(X = 1)$ ✓ determines expected value

(ii) $\mathrm{Var}(1 + 6X)$. (2 marks)

Solution
$Var(X) = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$ $Var(1 + 6X) = 6^2 \times \frac{2}{9} = 8$
Specific behaviours
✓ uses $Var(X) = p(1 - p)$ ✓ determines required variance

Question 2 **(6 marks)**

(a) Determine k, if $2\log_4 6 - \log_4 3 + 1 = \log_4 k$. (3 marks)

Solution
$\text{LHS} = \log_4 6^2 - \log_4 3 + \log_4 4$ $= \log_4\left(\frac{36\times 4}{3}\right)$ $k = 48$
Specific behaviours
✓ writes $2\log_4 6$ as $\log_4 6^2$ ✓ writes 1 as $\log_4 4$ ✓ combines as single log and states value of k

(b) Determine the exact solution to $3(4)^{x-1} = 18$. (3 marks)

Solution
$\log 4^{x-1} = \log 6$ $(x-1)\log 4 = \log 6$ $x = \frac{\log 6}{\log 4} + 1$
Specific behaviours
✓ divides both sides by 3   ✓ logs both sides to any base   ✓ solves for x

Alternative solution
$4^{x-1} = 6$ $x - 1 = \log_4 6$ $x = \log_4 6 + 1$
Specific behaviours
✓ divides both sides by 3   ✓ logs to base 4   ✓ solves for x

Question 7 **(7 marks)**

A function is defined by $f(x) = \frac{1+\ln x}{-2x}$.

(a) State the natural domain of f. (1 mark)

Solution
$x > 0$
Specific behaviours
✓ states domain

(b) Show that $f'(1) = 0$. (3 marks)

Solution
$f'(x) = \frac{\left(\frac{1}{x}\right)(-2x) - (1+\ln x)(-2)}{(-2x)^2}$ $f'(1) = \frac{-2-(-2)}{(-2)^2} = 0$
Specific behaviours
✓ uses quotient rule   ✓ $u'v$ and uv' expressions   ✓ substitutes $x = 1$, showing numerator is 0

(c) Use the second derivative test to determine the nature of the stationary point of the function at $x = 1$. (3 marks)

Solution
$f'(x) = \frac{\ln x}{2x^2}$ $f''(x) = \frac{\left(\frac{1}{x}\right)(2x^2) - (\ln x)(4x)}{(2x^2)^2}$ $f''(1) = \frac{2-0}{2^2} = +ve$ Since $f''(1) > 0$, then point is a minimum.
Specific behaviours
✓ simplifies $f'(x)$ and differentiates with quotient rule   ✓ differentiates correctly   ✓ indicates and interprets sign of $f''(1)$

Question 6 **(6 marks)**

The functions f and g intersect at the point $(-1, 7)$.

The first derivatives of the functions are $f'(x) = 30(5x+7)^2$ and $g'(x) = 10\pi\sin(\pi(1-2x))$.

Determine an expression for each function.

Solution
$f(x) = \dfrac{30(5x+7)^3}{3\times 5} + c$ $= 2(5x+7)^3 + c$ $c = 7 - 2(-5+7)^3 = 7 - 16 = -9$ $f(x) = 2(5x+7)^3 - 9$ $g(x) = \dfrac{-10\pi\cos(\pi(1-2x))}{-2\pi} + c$ $= 5\cos(\pi(1-2x)) + c$ $c = 7 - 5\cos 3\pi = 12$ $g(x) = 5\cos(\pi(1-2x)) + 12$
Specific behaviours
✓ antidifferentiates f ✓ evaluates constant ✓ states f in simplified form ✓ antidifferentiates g ✓ evaluates constant ✓ states g in simplified form

Question 3 **(7 marks)**

The rate of change of displacement of a particle moving in a straight line at any time t seconds is given by

$$\frac{dx}{dt} = 3 + 2e^{0.1t}\ \text{cm/s}.$$

Initially, when $t = 0$, the particle is at A, a fixed point on the line.

(a) Calculate the initial velocity of the particle. (1 mark)

Solution
$v(0) = 3 + 2e^0 = 5\ \text{cm/s}$
Specific behaviours
✓ velocity

(b) Determine the distance of the particle from A after 20 s. (3 marks)

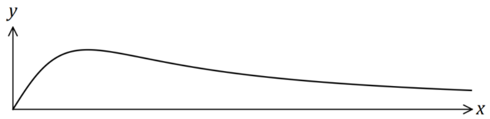
Solution
$x = 3t + 20e^{0.1t} + c$ $c = 0 - 20e^0 = -20$ $x(20) = 3(20) + 20e^2 - 20$ $= 40 + 20e^2\ \text{cm}$
Specific behaviours
✓ integrates ✓ evaluates constant ✓ substitutes to obtain distance

(c) Determine when the acceleration of the particle is $7\ \text{cm/s}^2$. (3 marks)

Solution
$a = 0.2e^{0.1t}$ $0.2e^{0.1t} = 7 \Rightarrow 0.1t = \ln 35$ $t = 10\ln 35\ \text{s}$
Specific behaviours
✓ differentiates for acceleration ✓ eliminates e ✓ solves for t

Question 4 **(7 marks)**

The graph of $y = f(x)$, $x \geq 0$, is shown below, where $f(x) = \dfrac{4x}{x^2+3}$.



(a) Determine the gradient of the curve when $x = 2$. (3 marks)

Solution
$f'(x) = \dfrac{4(x^2+3) - 4x(2x)}{(x^2+3)^2}$ $f'(2) = \dfrac{4(7) - 8(4)}{(7)^2} = -\dfrac{4}{49}$
Specific behaviours
✓ uses quotient rule ✓ correct $f'(x)$ ✓ correct gradient

(b) Determine the exact area bounded by the curve $y = f(x)$ and the lines $y = 0$ and $x = 2$, simplifying your answer. (4 marks)

Solution
$A = \int_0^2 f(x)\,dx$ $= [2\ln(x^2+3)]_0^2$ $= 2\ln 7 - 2\ln 3$ $= 2\ln\dfrac{7}{3}$
Specific behaviours
✓ writes integral ✓ antidifferentiates ✓ substitutes ✓ simplifies

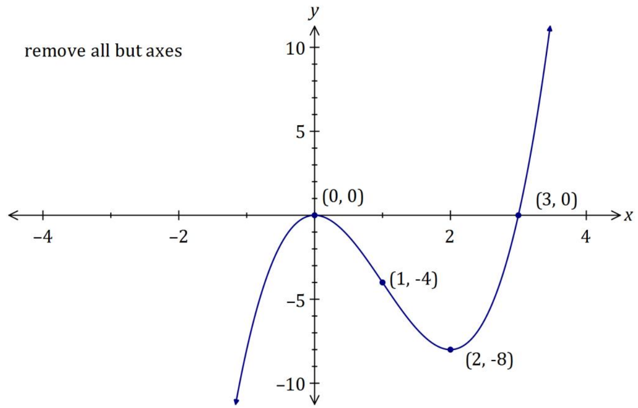
Question 5 **(8 marks)**

A curve has first derivative $\dfrac{dy}{dx} = 6x(x-2)$ and passes through the point $P(-1, -8)$.

(a) Determine the value(s) of x for which $\dfrac{d^2y}{dx^2} = 0$. (2 marks)

Solution
$\dfrac{d^2y}{dx^2} = 12x - 12$ $12x - 12 = 0 \Rightarrow x = 1$
Specific behaviours
✓ differentiates ✓ states value

(b) Sketch the curve on the axes below, clearly indicating the location of all axes intercepts, stationary points and points of inflection. (6 marks)



Solution
$y' = 0 \Rightarrow x = 0, 2$ $y' = 6x^2 - 12x \Rightarrow y = 2x^3 - 6x^2 + c$ $c = -8 - 2(-1)^3 + 6(-1)^2 = 0$ $y = 2x^2(x-3) \Rightarrow$ zeroes at $x = 0, 3$ $x = 1, y = -4;\ \ x = 2, y = -8$ See graph
Specific behaviours
✓ obtains expression for y ✓ obtains zeroes of y ✓ indicates coordinates of minimum and point of inflection ✓ indicates scales ✓ indicates location on graph of min, max, pt infl, roots ✓ single smooth continuous curve