Christ Church Grammar School

Semester Two
Examination, 2016

Question/Answer Booklet

MATHEMATICS
METHODS
UNITS 3 AND 4
Section One:
Calculator-free

Student Number:	In figures							

In words _____

Your name _____

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   • Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   • Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Booklet.

See next page

Section One: Calculator-free 35% (52 Marks)

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 (6 marks)

A particle leaves the origin when $t = 1$ and moves in a straight line with velocity $v(t)$, where $t \geq 1$, given by

$$v(t) = \frac{7}{t^2} + \frac{4}{t} - \frac{4}{4} \text{ ms}^{-1}$$

(a) Determine the time when the acceleration of the particle is zero. (2 marks)

(b) Determine the exact displacement of the particle from the origin when $t = 4$. (4 marks)

See next page

Question 2   **(7 marks)**

(a)   Calculate $f'(0)$ when $f(x) = e^{2x}(1 + 5x)^3$.   (3 marks)

(b)   Determine $\dfrac{d}{dx}\displaystyle\int_x^5 \sqrt{t^2 + 1}\, dt$.   (2 marks)

(c)   Given $f'(x) = (1 - 2x)^4$ and $f(1) = -1$, determine $f(x)$.   (2 marks)

See next page

Question 3 (7 marks)

(a) Find the exact value of $\int_0^{\ln 2} e^{5x}\,dx$. (3 marks)

(b) A curve has equation $y = 2x^5 - 5x^4 + 10$. Point A lies on the curve at $(-1, 3)$.

Use the increments formula $\delta y \approx \frac{dy}{dx} \times \delta x$ to estimate the y-coordinate of point B that has an x-coordinate of -0.99.

(4 marks)

See next page

Additional working space

Question number: _____

Question 4 (8 marks)

(a) Determine

(i) the equation of the asymptote of the graph of $y = \log_e(x - 3) - 2$. (1 mark)

(ii) the coordinates of the y-intercept of the graph of $y = \log_2(x + 8) - 5$. (2 marks)

(b) The graph of $y = e^{2x-1} - 4x$ has a single stationary point, as shown on the graph below.
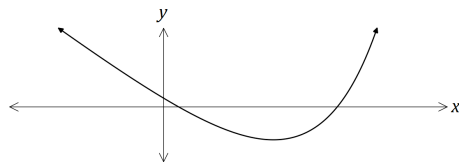
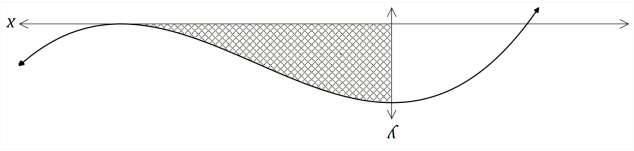Determine the exact coordinates of the stationary point. (5 marks)

Additional working space

Question number: _____

Question 5   (8 marks)

The diagram below shows the curve $y = x^3 - 3x^2 + k$, where k is a constant. The curve has a turning point on the x-axis.

(a)   Determine the value of k.   (3 marks)
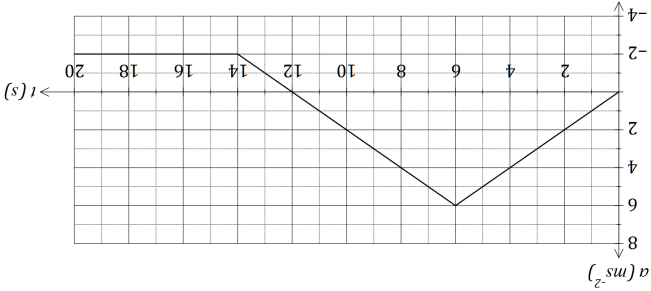

(b)   Determine the set of values of x for which $\frac{dy}{dx}$ is increasing.   (2 marks)

(c)   Calculate the area of the shaded region.   (3 marks)

Question 20   (9 marks)

A particle, initially stationary and at the origin, moves subject to an acceleration, a ms^{-2}, as shown in the graph below for $0 \leq t \leq 20$ seconds.

(a)   Determine the velocity of the object when

        (i) $t = 6$.   (1 mark)

        (ii) $t = 20$.   (2 marks)

(b)   At what time is the velocity of the body a maximum, and what is the maximum velocity?   (2 marks)

(c)   Determine the distance of the particle from the origin after 3 seconds.   (4 marks)
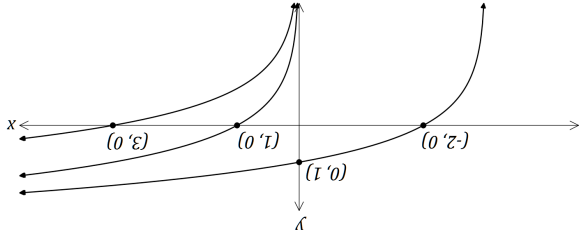
End of questions

Question 6   **(8 marks)**

The discrete random variable X is defined by $P(X = x) = k \log x$ for $x = 2, 5$ and 10.

(a)   Determine the value of k, giving your answer as a fraction.   (3 marks)

(b)   Determine $P(X = 2 \mid X < 10)$.   (2 marks)

(c) $E(X) = a\left(b + \log \sqrt{c}\right)$, where the constants a, b and c are prime numbers. Determine the values of a, b and c.   (3 marks)

Question 19   **(7 marks)**

The moment magnitude scale M_w is used by seismologists to measure the size of earthquakes in terms of the energy released. It was developed to succeed the 1930's-era Richter magnitude scale.

The moment magnitude has no units and is defined as $M_w = \frac{2}{3}\log_{10}(M_0) - 10.7$, where M_0 is the total amount of energy that is transformed during an earthquake, measured in dyn·cm.

(a)   On 28 June 2016, an estimated 2.82×10^{21} dyn·cm of energy was transformed during an earthquake near Norseman, WA. Calculate the moment magnitude for this earthquake.   (1 mark)

(b)   A few days later, on 8 July 2016, there was another earthquake with moment magnitude 5.2 just north of Norseman. Calculate how much energy was transformed during this earthquake.   (2 marks)

(c)   Show that an increase of 2 on the moment magnitude scale corresponds to the transformation of 1000 times more energy during an earthquake.   (4 marks)

Question 18 (7 marks)

A trough for holding water is to be formed by taking a length of metal sheet 70 cm wide and folding 15 cm on either end, up through an angle of θ. The following diagram shows the cross-section of the trough with the cross-sectional area, A, shaded.

(a) Determine A in terms of w and h. (1 mark)

(b) Show that $A = 600 \sin \theta + 225 \sin \theta \cos \theta$. (2 marks)

(c) Use calculus to determine the maximum possible cross-sectional area. (4 marks)

Question 7 (8 marks)

(a) The function f is defined by $f(x) = \log_a x$, $x > 0$, where a is a constant, $a > 1$.

The graphs shown below have equations $y = f(x)$, $y = f(x + b)$ and $y = f(x) + c$, where b and c are constants.

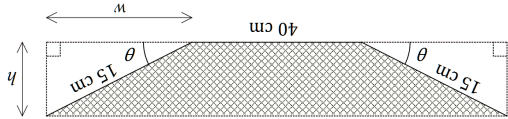Determine the values of the constants a, b and c. (4 marks)

Question 7 (continued)

(b)   Find $\lim_{h\to 0}(1 + h)^{\frac{1}{h}}$.   (2 marks)

(c)   Find $\int \frac{1}{2 + e^{-x}}\, dx$.   (2 marks)

Question 17   **(8 marks)**

(a)   Using rectangles or trapezia of width 1 unit, find an approximate value to $\int_0^5 \frac{1}{x+1}\, dx$.

State whether the approximate value found is an under-estimate or over-estimate of the true value of the integral, giving a reason for your answer.   (4 marks)

(b)   The graphs of $y = \cos^2\left(x + \frac{\pi}{6}\right)$ and $y = \frac{3}{4}$ are shown below. Determine the exact area of the shaded region they enclose.   (4 marks)
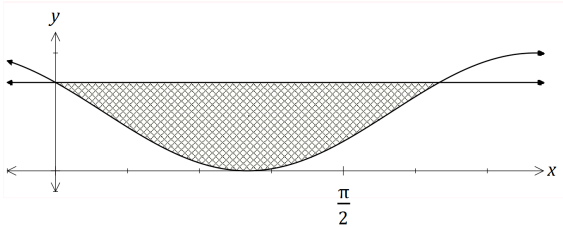
Question 16 (continued)

(c) Find the lower quartile of X. (2 marks)

(d) Let $Y = aX + b$, where a and b are constants with $a > 0$.

Find the values of a and b for which $E(Y) = 0$ and $Var(Y) = 1$. (3 marks)

Additional working space

Question number: _____

Additional working space

Question number: _____

Question 16 (10 marks)

The continuous random variable X has probability density function $f(x)$ given by

$$f(x) = \begin{cases} k(x+3) & -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(a)   <u>Show</u> that $k = \frac{1}{18}$. (2 marks)

(b)   Find $E(X)$ and $Var(X)$. (3 marks)

Question 15 (continued)

(b)   The stationery company that supplies pens to the conference centre claim that no more than 3 in 50 pens fail to write. Use your previous working to comment on the validity of this claim.   (2 marks)

(c)   Comment on how the margin of error would change in (a) (ii) if

(i)   the quality of the pens had been better.   (1 mark)

(ii)   the required level of confidence decreased.   (1 mark)

Christ Church Grammar School

**Semester Two
Examination, 2016**

Question/Answer Booklet

MATHEMATICS METHODS UNITS 3 AND 4

**Section Two:
Calculator-assumed**

Student Number: In figures   ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐

         In words   _____

         Your name   _____

Time allowed for this section
Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction
                fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper,
               and up to three calculators approved for use in the WACE examinations

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Question 15   **(9 marks)**

The management at a conference centre was concerned about the quality of the free pens that it provided in its meeting rooms. A staff member tested a random sample of 150 pens and found that 18 of them fail to write.

(a)   If p is the true proportion of pens that fail to write and $\hat{p}$ is the corresponding sample proportion, use the above sample to determine

     (i) $\hat{p}$.   (1 mark)

     (ii)   the approximate margin of error for a 98% confidence interval for p.   (3 marks)

     (iii)   an approximate 98% confidence interval for p.   (1 mark)

See next page

Question 14 (8 marks)

An analysis of the number of dogs registered by each household within a suburb resulted in the following information:

Number of dogs registered	0	1	2	3 or more
Percentage of households	21	44	27	8

(a) A council worker selects households at random to visit. What is the probability that the first five households visited all have at least one dog registered? (2 marks)

(b) A random sample of 40 households within the suburb is selected.

Use a binomial distribution with $n = 40$, together with relevant information from the table in each case, to determine the probability that the sample contains:

(i) exactly 6 households with no dogs registered. (2 marks)

(ii) no more than 15 households with at least two dogs registered. (2 marks)

(c) A random sample of 25 households within the city is to be selected. If X is the number of households in the sample that have exactly one dog registered, determine the mean and variance of X. (2 marks)

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Section Two: Calculator-assumed 65% (98 Marks)

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 (5 marks)

Zebra mussels are an invasive species of shellfish recently discovered in some North American waterways. The mussel density, D, in shellfish per square metre, observed in a power station water supply pipe t days after a colony began, was modelled by the following equation, where k is a positive constant:

$$D = 200e^{kt}$$

(a) What was the mussel density in the colony when observations began? (1 mark)

The mussel density was observed to double every eight days.

(b) Determine the value of k, rounded to four decimal places. (2 marks)

(c) The water supply pipe was seriously compromised when the mussel density reached 85 thousand shellfish per square metre. After how many days from the commencement of observations did this happen? (2 marks)

Question 13 (7 marks)

From a random sample of n people, it was found that 54 of them subscribe to a streaming music service. A symmetric confidence interval for the true population proportion who subscribe is $0.1842 < p < 0.2958$.

(a) Determine the value of n, by first finding the mid-point of the interval. (3 marks)

(b) Determine the confidence level of the interval. (4 marks)

Question 12 (7 marks)

A hardware store sells stakes, of nominal length 1.8 metres, to be used for supporting newly planted trees. The length, X metres, of the stakes can be modelled by a normal distribution with mean 1.85 and standard deviation σ.

(a) If $\sigma = 0.035$, determine

(i) the probability that a randomly chosen stake is shorter than 1.8 metres. (1 mark)

(ii) the probability that a randomly chosen stake is longer than 1.79 m given that it is shorter than 1.8 metres. (2 marks)

(iii) the value of k, if the longest 15% of stakes exceed k metres in length. (1 mark)

(b) A large number of stakes were measured and it was found that 97% of them were longer than their nominal length. Show how to use this information to deduce that the value of σ is 0.027 when rounded to three decimal places. (3 marks)

Question 9 (6 marks)

The speeds of 250 vehicles, on a section of freeway undergoing roadworks with a speed limit of 60 kmh⁻¹, had a mean and standard deviation of 56.9 kmh⁻¹ and 3.6 kmh⁻¹ respectively. A summary of the data is shown in the table below.

Speed (x kmh⁻¹)	$45 \leq x < 50$	$50 \leq x < 55$	$55 \leq x < 60$	$60 \leq x < 65$	$65 \leq x < 70$
Relative frequency	0.024	0.272	0.504	0.188	0.012

(a) Use the table of relative frequencies to estimate the probability that the next vehicle to pass the roadworks

(i) was not exceeding the speed limit. (1 mark)

(ii) had a speed of less than 65 kmh⁻¹, given they were exceeding the speed limit. (1 mark)

(b) Subsequent tests on the measuring equipment discovered that it had been wrongly calibrated. The correct speed of each vehicle, v, could be calculated from the measured speed, x, by increasing x by 6% and then adding 1.7.

(i) Calculate the adjusted mean and standard deviation of the vehicle speeds. (2 marks)

(ii) Determine the correct proportion of vehicles that were speeding. (2 marks)

Question 10 (7 marks)

A student planned to investigate what proportion of the 1260 students at their school had access to more than one computer at home.

(a) The student thought of the following three ways to select a sample from the population. Briefly discuss the main source of bias in each method.

(i) Wait at the bus-bay after school and ask the first 50 students who show up.

(1 mark)

(ii) Advertise the survey in a whole school assembly and ask the first 50 students who volunteer to stay behind.

(1 mark)

(iii) Select and ask every 100th student from the school roll. (1 mark)

(b) Assuming that 80% of students had access to more than one computer at home, the student carried out 100 simulations in which a sample proportion was calculated from a random sample of 64 students.

(i) Explain why it is reasonable to expect that the distribution of the sample proportions would approximate normality.

(2 marks)

(ii) Determine the mean and standard deviation of the normal distribution that the sample proportions would approximate.

(2 marks)

Question 11 (8 marks)

A box contains a large number of packets of buttons. The number of buttons in a packet may be modelled by the random variable X, with the probability distribution shown below. It is also known that $E(X) = 6.25$.

x	3 or fewer	4	5	6	7	8	9 or more
$P(X = x)$	0	0.05	a	b	0.25	0.15	0

(a) Two packets are randomly chosen from the box. Determine the probability that there are at least 15 buttons altogether in the two packets.

(2 marks)

(b) Determine the values of a and b.

(3 marks)

(c) Calculate $\text{Var}(X)$. (1 mark)

(d) As part of a fundraiser, patrons pay 75 cents to select a packet at random and then win back 10 cents for each button in the packet. If the random variable W represents the net gain per game for a patron in cents, determine the mean and variance of W.

(2 marks)