PERTH MODERN SCHOOL
Exceptional schooling. Exceptional students.

Semester 1 Examination 2012

Question/Answer Booklet

MATHEMATICS:
SPECIALIST 3CD

Section Two:
Calculator-assumed

Name of Student: ________________________Marking Key____________________

Time allowed for this section

Reading time before commencing work:	10 minutes
Working time for this section:	100 minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the student

Standard items:	pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters
Special items:	drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination

Important note to students

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One Calculator-free	6	6	50	50	
Section Two Calculator-assumed	11	11	100	100	
			Total	150	100

Instructions to students

1 Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer. If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued. i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

2 **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

3 It is recommended that you **do not use pencil**, except in diagrams.

4 You must be careful to confine your responses to the specific questions asked and to follow any instructions that are specific to a particular question.

Section Two: Calculator-assumed **(100 marks)**

This section has **eleven (11)** questions. Answer all questions. Write your answers in the spaces provided.

Working time: 100 minutes

Question 7 **(7 marks)**

(a) Use **proof by exhaustion** to prove that all values of 2^n end in 2, 4, 6, or 8,

n>0 and n is an integer. (3)

(b) Prove that $\cos^4\theta = \dfrac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$. (4)

Question 8 **(6 marks)**

A water tank has vertical sides of height *h* and is initially full. Through a small hole in the base of the tank, water leaks out at a rate, which, at any time *t*, is proportional to the depth *x* of the remaining water in the tank at that instant. That is $\frac{dx}{dt} = -kx$. The tank is exactly half empty in 2 hours.

(a) Show that the exact value of *k* is $\frac{1}{2}\ln 2$. (4)

(b) Determine the depth of water in $\frac{1}{2}$ hour giving your answer in terms of *h* (2)

Question 9 **(10 marks)**

(a) Find the equation of the plane passing through (1, -1, 3) and parallel to the plane
$\underset{\sim}{r} \bullet \left(3\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right) = 7$ (2)

(b) Find the **obtuse** angle between the two planes defined by

Plane I: $\underset{\sim}{r} \cdot (\underset{\sim}{i} + \underset{\sim}{j}) = 1$

Plane II: $\underset{\sim}{r} \cdot (2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}) = 2$ (2)

(c) Find the shortest distance from the point P(2, -3, 4) to the plane $\underset{\sim}{r} \cdot (\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}) = 13$

(6)

Question 10 **(9 marks)**

(a) If $y = \ln\left(\dfrac{1+\sin x}{\cos x}\right)$, show that $\dfrac{dy}{dx} = \dfrac{1}{\cos x}$ (4)

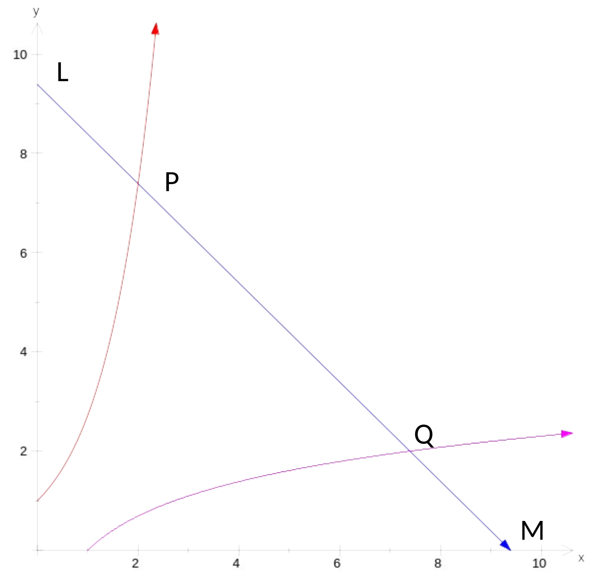
(b) The length, *l*, of an arc of a curve *y* = *f*(*x*) from *x* = *a* to *x* = *b* is given by

$$l = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$.

Find the **exact** length of the curve $y = \ln(\cos x)$ from *x* = 0 to *x* = $\dfrac{\pi}{3}$ showing sufficient steps how you use your answer from part (a) to find*l*. (5)

Question 11 **(9 marks)**

The graphs of $y=e^x$ and $y=\ln x$ for $x \geq 0$ are shown. The line segment LM with equation $y=-x+e^2+2$ meets these graphs at $P(2,e^2)$ and $Q(e^2,2)$.



(a) State the **exact** coordinates of points L and M, the axis intercepts of the line segment LM. (2)

(b) Calculate the **exact** value of the area of the region between $y=-x+e^2+2$ and $y=e^x$, from $x=0$ and $x=2$. (4)

Question 11 (continued)

(c) Give a reason why the area of the region bounded bybetween y=-x+e^2+2 and y=e^x, from x=0 and x=2 is equal to the area of the region enclosed by the graph of $y = \ln x$, the line segment LM, and the x-axis. (1)

(d) **Hence** calculate the **exact**area of the region bounded by $y = e^x$, y=-x+e^2+2, $y = \ln x$, x-axis and y-axis. (2)

Question 12 **(10 marks)**

Determine

(a) $\int \cos^2 2x \ dx$ (2)

(b) $\int \cos^3 2x \ dx$ (3)

Question 12 (continued)

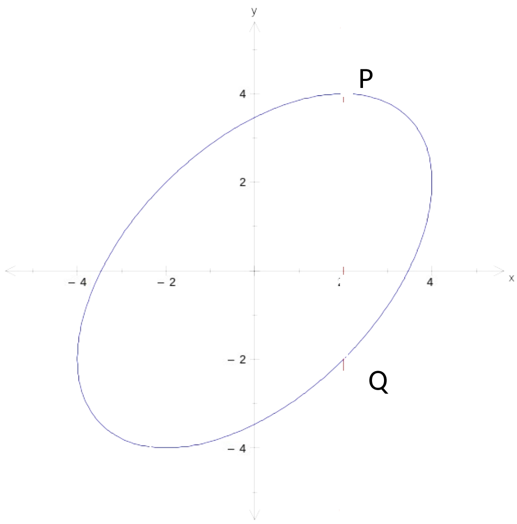
(c) **Hence**, using your answer to parts (a) & (b), determine

$$\int \sin^2 x \cos^4 x \, dx$$

(5)

Question 13 **(12 marks)**

The graph of x^2-xy+y^2=12 is drawn below.



(a) Draw the line x=2 and hence find the **coordinates** of the points of intersection, P and Q where P lies in the 1st quadrant and Q in the 4th quadrant. Show these points on the diagram.

(2)

(b) Show that $\frac{dy}{dx} = \frac{y - 2x}{2y - x}$ (3)

Question 13 (continued)

(c) Determine the equation of the tangent to the curve at

(i) P (2)

(ii) Q (2)

(d) These two tangents intersect at point T. Show that Δ PQT is an isosceles triangle. (3)

Question 14 **(13 marks)**

The position vectors of the points A and B relative to the origin, are given by $\underset{\sim}{i} - 7\underset{\sim}{j} + 5\underset{\sim}{k}$ and $-2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$ respectively. The line L_1 passes through A and is parallel to $9\underset{\sim}{i} + 3\underset{\sim}{j} - 9\underset{\sim}{k} + c\,\underset{\sim}{j}$. The line L_2 passes through B and is parallel to $\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$

(i) Show that $c = -43\frac{1}{2}$ if the lines intersect. (5)

(ii) Hence state the coordinates of the point of intersection, P. (2)

(iii) Determine the angle between L_1 and L_2. (2)

Question 14 (continued)

(iv) Hence determine the shortest distance from Q(0, 5, 10) which lies on L_2 to the line L_1.

(4)

Question 15 **(10 marks)**

If $N = \dfrac{10\ 000}{1+99e^{-\frac{1}{2}t}}$

(i) Express $e^{-\frac{1}{2}t}$ in terms of N (2)

(ii) Hence using implicit differentiation, show that $\dfrac{dN}{dt} = \dfrac{1}{2}N\left(\dfrac{10\ 000 - N}{10\ 000}\right)$ (5)

Question 15 (continued)

(iii) Find the value of t, correct to 3 significant figures when $\frac{dN}{dt}$ is a maximum. (3)

Question 16 **(9 marks)**

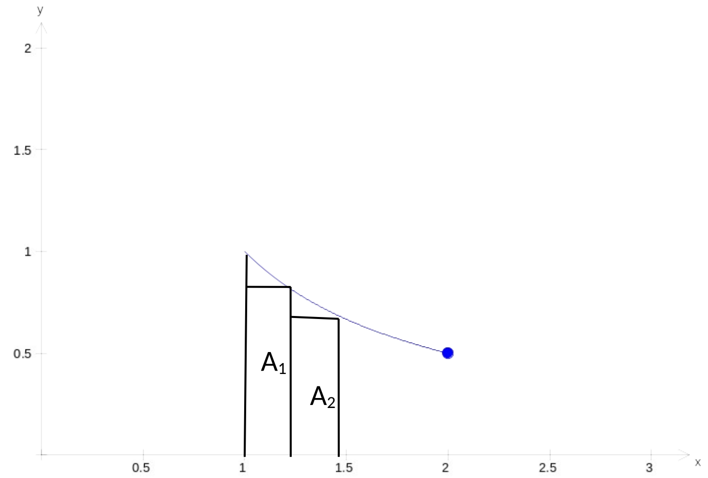
(a) Use First Principles to determine the derivative of $y = \sin^2 x$. (5)

Question 16 (continued)

(b) Given that $\ln y = \sqrt{1+8e^x}$ prove that $\ln y \dfrac{dy}{dx} = 4ye^x$ (4)

Question 17 **(5 marks)**

The area of the region between the curve $y = \frac{1}{x}$ and the *x*-axis, for $1 \le x \le 2$, is estimated using *n* rectangles of equal widths $\frac{1}{n}$ as shown in the diagram.



(i) Show that $\int_1^2 \frac{1}{x}\,dx$ is approximately equal to $\sum_{r=1}^{n} \frac{1}{n+r}$. (3)

(ii) Deduce the value of $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r}$. (2)