Rossmoyne Senior High School

Semester Two Examination, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section One: Calculator-free

SOLUTIONS

Student Number: In figures

In words

Your name

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	12	12	100	97	65
			Total	149	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Booklet.

Section One: Calculator-free 35% (52 Marks)

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 (5 marks)

The polynomial $h(z)=z^4-6z^3+3az^2-30z+10a$, where a is a real constant, has a zero of $3-i$. Determine the value of a and all other zeros of $h(z)$.

Solution
$h(z)=(z-3+i)(z-3-i)\times f(z)$ $z^4-6z^3+3az^2-30z+10a=(z^2-6z+10)\times f(z)$ By inspection, $f(z)=(z^2+bz+a)$ $z^4-6z^3+3az^2-30z+10a=(z^2-6z+10)(z^2+bz+a)$ Comparing z^3 coefficients, $-6=-6+b\Rightarrow b=0$ Comparing z^2 coefficients, $3a=10+a\Rightarrow a=5$ Hence $h(z)=(z^2-6z+10)(z^2+5)$ Other zeroes of $h(z)$ are $3+i$, $\sqrt{5}i$ and $-\sqrt{5}i$.
Specific behaviours
✓ deduces another zero of $3+i$ ✓ uses two zeroes to obtain quadratic factor with real coefficients ✓ deduces other factor must be of form z^2+a

✓ determines value of

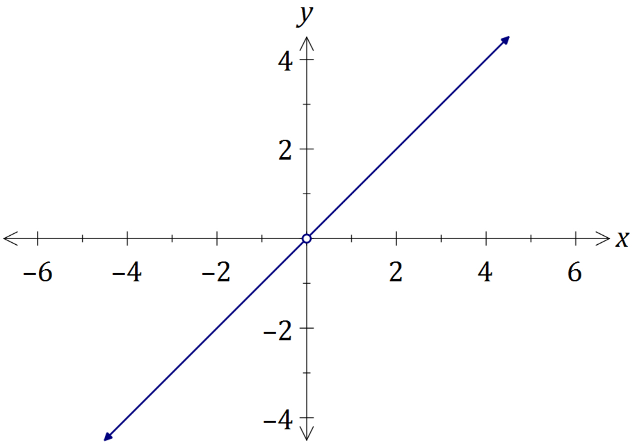
Question 2 **(8 marks)**

Two functions are defined by $f(x)=\sqrt{3x-1}$ and $g(x)=\dfrac{1}{x}$.

(a) Determine the composite function $f(g(x))$ and the domain over which it is defined. (3 marks)

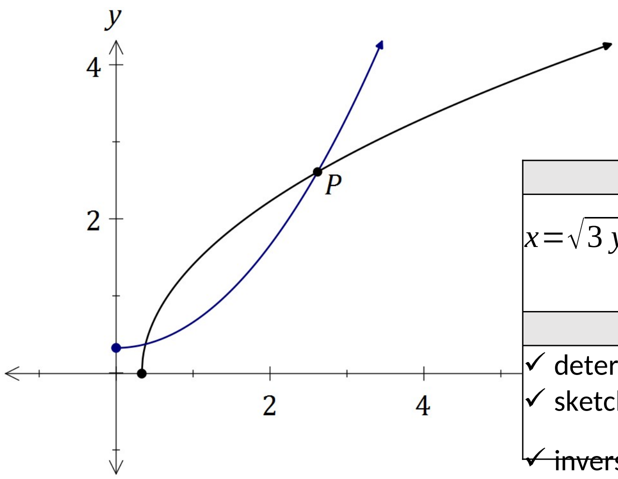
Solution
$f(g(x))=\sqrt{\dfrac{3}{x}-1}$ Domain: $\dfrac{1}{x}\geq\dfrac{1}{3}\Rightarrow 0<x\leq 3$.
Specific behaviours
✓ writes composite function   ✓ states upper limit of domain as $x\leq 3$ ✓ states lower limit of domain as

(b) Sketch the graph of $y=g(g(x))$ on the axes below. (2 marks)



Solution
$g(g(x))=x, x\neq 0$
Specific behaviours
✓ sketches graph of $y=x$ ✓ indicates discontinuity at $x=0$

(c) The graph of $y=f(x)$ is shown below, passing through point P with coordinates $(2.62, 2.62)$. Determine $f^{-1}(x)$, the inverse of $f(x)$, and sketch the graph of $y=f^{-1}(x)$ on the same axes. (3 marks)



Solution
$x=\sqrt{3y-1}\Rightarrow y=f^{-1}(x)=\dfrac{x^2+1}{3}$
Specific behaviours
✓ determines rule for inverse   ✓ sketches inverse using reflection in $y=x$ ✓ inverse starts at $\left(0,\frac{1}{3}\right)$ and passes thru P

Question 3 **(5 marks)**

An object, initially at rest, is dropped from the top of tall building so that after t seconds it has velocity v meters per second.

The air resistance encountered by the object is proportional to its velocity, so that the velocity satisfies the equation $\frac{dv}{dt}=10-kv$, where k is a constant.

(a) Express the velocity of the object in terms of t and k. (4 marks)

Solution
$\int \frac{dv}{10-kv}=\int dt$ $\frac{-1}{k}\ln\lvert 10-kv\rvert=t+c$ $\ln\lvert 10-kv\rvert=-kt-kc$ $10-kv=ae^{-kt} t=0, v=0 \Rightarrow a=10$ $v=\frac{10-10e^{-kt}}{k}$
Specific behaviours
✓ separates variables ✓ integrates both sides ✓ evaluates constant *a* ✓ expresses velocity as required

(b) Sensors on the object indicate that its velocity will never exceed 55 metres per second. Determine the value of the constant k. (1 mark)

Solution
As $t\to\infty, 55=\frac{10}{k}\Rightarrow k=\frac{10}{55}=\frac{2}{11}$
Specific behaviours
✓ evaluates constant

Question 4 **(5 marks)**

Let $v=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$.

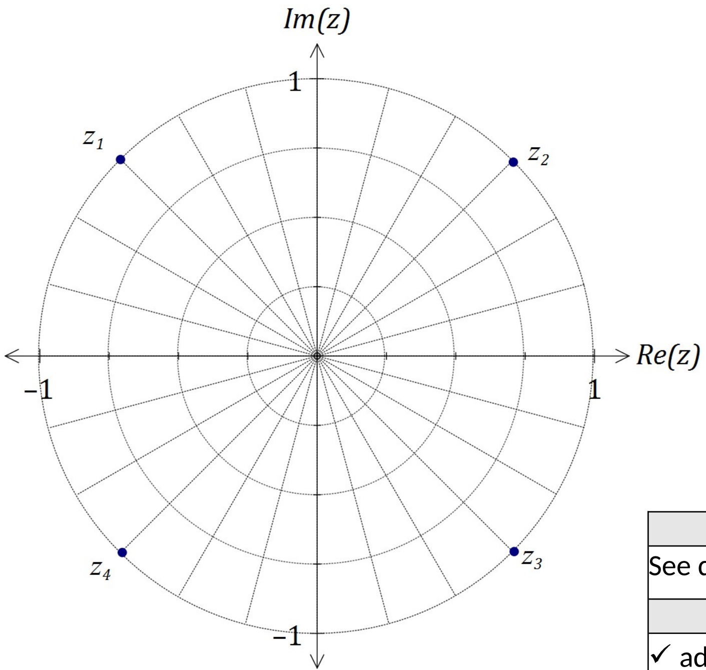
(a) Express v in polar form. (2 marks)

Solution
$\lvert v\rvert=1,\ arg\,v=\frac{-\pi}{4},\ v=cis\left(\frac{-\pi}{4}\right)$
Specific behaviours
✓ determines correct modulus ✓ determines correct argument

(b) Show that $v^4=-1$. (1 mark)

Solution
$v^4=\left(cis\left(\frac{-\pi}{4}\right)\right)^4=cis(-\pi)=-1$
Specific behaviours
✓ states ^{4} in polar form

(c) Plot the roots of $z^4+1=0$ on the following Argand diagram. (2 marks)



Solution
See diagram
Specific behaviours
✓ adds scale and plots $v(¿z_1)$ ✓ plots other three roots

Question 5 **(8 marks)**

(a) Using partial fractions, or otherwise, determine $\int \frac{x-19}{(x+1)(x-4)}\,dx$. (4 marks)

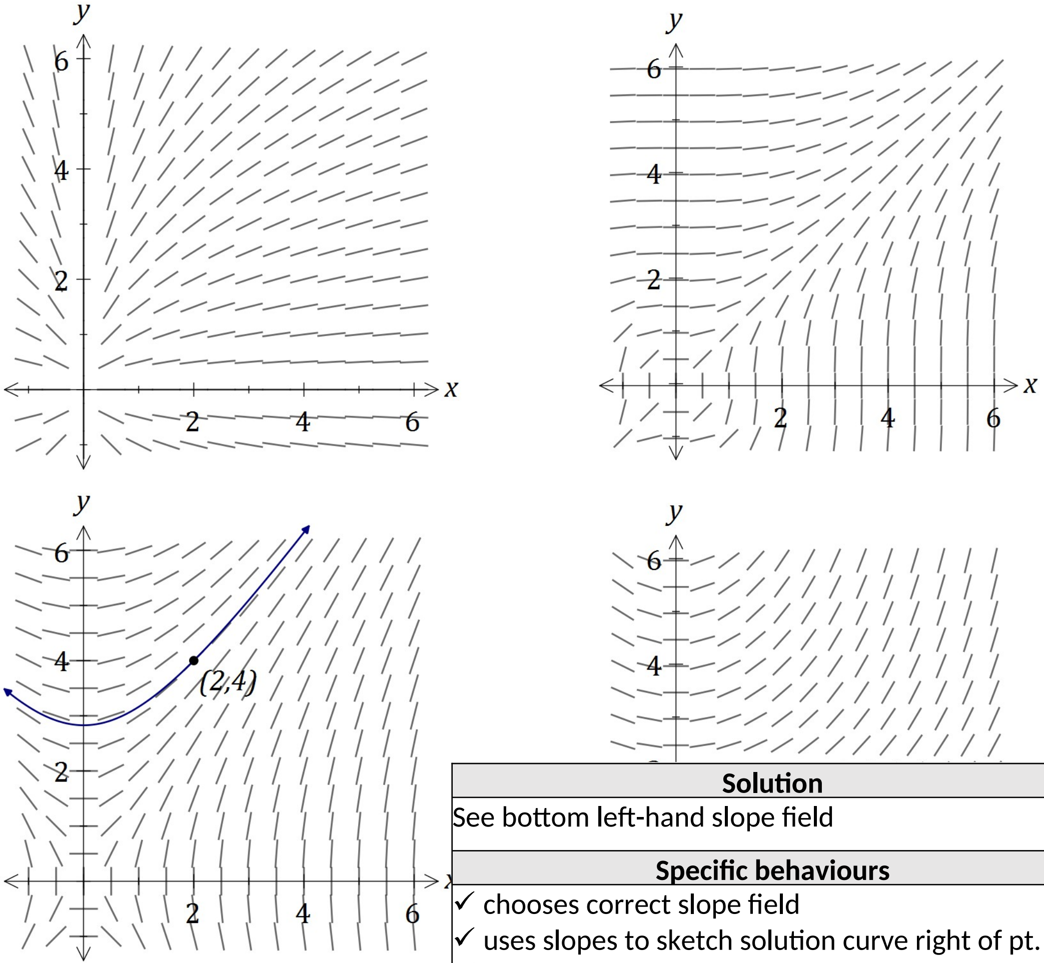
Solution
$A(x-4)+B(x+1)=x-19 \Rightarrow A+B=1,\ B-4A=-19$ Solving gives $A=4,\ B=-3$ $\int \frac{x-19}{(x+1)(x-4)}\,dx=\int \frac{4}{x+1}-\frac{3}{x-4}\,dx$ ¿ $4\ln\lvert x+1\rvert-3\ln\lvert x-4\rvert+c$
Specific behaviours
✓ writes equations for A and B   ✓ determines A and B   ✓ integrates both fractions correctly   ✓ includes constant of integration

(b) Use the substitution $u=\sin x$ to evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx$. (4 marks)

Solution
$du=\cos x\,dx,\ x=\frac{\pi}{2}\Rightarrow u=1,\ x=\frac{\pi}{6}\Rightarrow u=\frac{1}{2}$ $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx=\int_{\frac{1}{2}}^{1}\frac{1}{\sqrt{u}}\,du$ ¿ $\left[2\sqrt{u}\right]_{\frac{1}{2}}^{1}$ ¿ $2-\sqrt{2}$
Specific behaviours
✓ uses substitution correctly   ✓ determines new bounds of integration   ✓ integrates correctly   ✓ evaluates integral

Question 6 **(5 marks)**

The differential equation $y' = \dfrac{2x}{y}$ is shown in just one of the four slope fields below.



Solution
See bottom left-hand slope field
Specific behaviours
✓ chooses correct slope field ✓ uses slopes to sketch solution curve right of pt. ✓ uses slopes to sketch solution curve left of pt.

(a) On the slope field for $y' = \dfrac{2x}{y}$, sketch the solution of the differential equation that passes through the point $(2, 4)$. (3 marks)

(b) Another solution to the differential equation passes through the point $(6, -3)$. Use the incremental formula $\delta y \approx \dfrac{dy}{dx} \times \delta x$, with $\delta x = \dfrac{1}{10}$, to estimate the y-coordinate of this curve when $x = 6.1$. (2 marks)

Solution
$\delta y \approx \dfrac{2(6)}{-3} \times \dfrac{1}{10} = -0.4$ $y \approx -3 - 0.4 \approx -3.4$
Specific behaviours
✓ calculates change in y ✓ calculates new y-coordinate

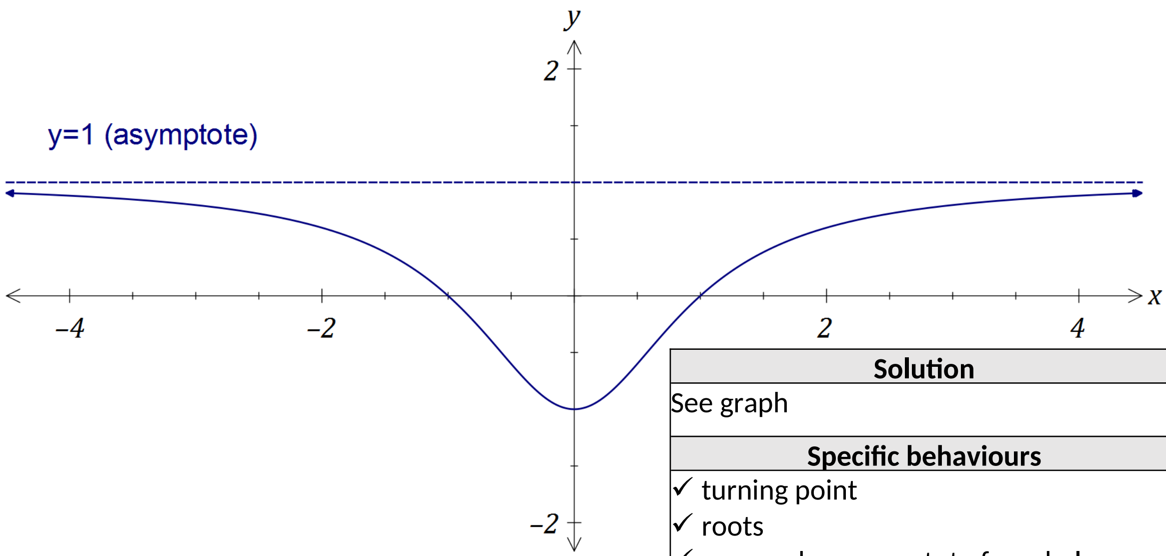
Question 7 **(8 marks)**

The function f is defined as $f(x)=\frac{x^2-1}{x^2+1}$.

(a) Show that the **only** stationary point of the function occurs when $x=0$. (2 marks)

Solution
$f'(x)=\frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$ Hence $f'(x)=0$ only when $4x=0\Rightarrow x=0$.
Specific behaviours
✓ differentiates function   ✓ simplifies and makes conclusion

(b) Sketch the graph of $y=f(x)$ on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ turning point   ✓ roots   ✓ approaches asymptote from below

(c) Using your graph, or otherwise, determine all solutions to

(i) $f(x)=|f(x)|$. (1 mark)

Solution
$x\le -1 \cup x\ge 1$
Specific behaviours
✓ states both inequalities

(ii) $f(x)=f(|x|)$. (1 mark)

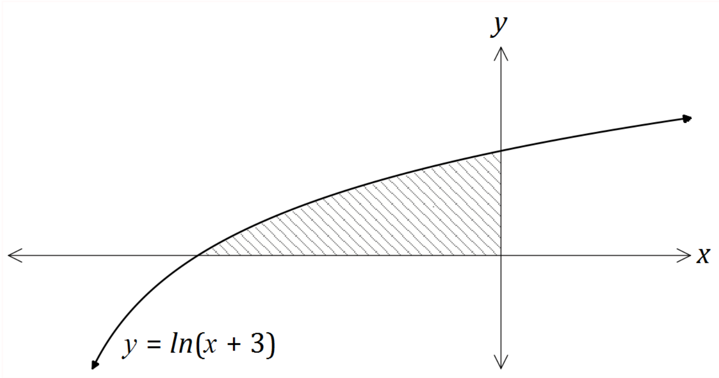
Solution
$x\in R$
Specific behaviours
✓ states all reals

(iii) $f(x)=\frac{1}{f(x)}$. (1 mark)

Solution
$x=0$
Specific behaviours
✓ states solution

Question 8 **(8 marks)**

A region is bounded by $x=0$, $y=0$ and $y=\ln(x+3)$ as shown in the graph below.



(a) Show that the area of the region is given by $\int_0^{\ln 3}(3-e^y)\,dy$. (3 marks)

(Do not evaluate the integral).

Solution
$y=\ln(x+3) \Rightarrow x=e^y-3,\ x=0 \Rightarrow y=\ln 3$ Region is left of *y*-axis, so integral will be negative - need to multiply by *-1* for area $A=-\int_0^{\ln 3}(e^y-3)\,dy=\int_0^{\ln 3}(3-e^y)\,dy$
Specific behaviours
✓ expresses *x* in terms of *y*   ✓ derives upper limit of integration   ✓ explains need to multiply integral by *-1*,

(b) Determine the volume of the solid generated when the region is rotated through 2π about the *y*-axis. (5 marks)

Solution
$V=\pi\int_0^{\ln 3}(e^y-3)^2\,dy\text{¿}\pi\int_0^{\ln 3}e^{2y}-6e^y+9\,dy$ $\text{¿}\pi\left[\frac{e^{2y}}{2}-6e^y+9y\right]_0^{\ln 3}$ $\text{¿}\pi\left[\frac{9}{2}-18+9\ln 3-\frac{1}{2}+6\right]\text{¿}\pi(9\ln 3-8)$
Specific behaviours
✓ writes correct integral   ✓ expands $(e^y-3)^2$ ✓ integrates correctly   ✓ substitutes upper and lower limits   ✓ simplifies completely

End of questions

Additional working space

Question number: _________