1. Consider the function $f(x) = x^2$

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a. By filling in the table of values, complete the limiting chord process for $f(x) = x^2$ at the point $x = 1$.

a	b	$h = b - a$	$\frac{f(b) - f(a)}{b - a}$
1	2	1	☐
1	1.5	☐	☐
1	1.1	☐	☐
1	1.05	☐	☐
1	1.01	☐	☐
1	1.001	☐	☐
1	1.000 1	☐	☐

b. The instantaneous rate of change of $f(x)$ at $x = 1$ is:

a. The values for the column labelled $h = b - a$ are found by calculating $b - a$ for each relevant row.
For example, for row 2, the calculation will be
$1.5 - 1 = 0.5$

The values for the final column are found by calculating the value on each line to the expression

$$\frac{f(b)-f(a)}{b-a} = \frac{f(\square)-f(\square)}{\square - \square}$$

to calculate $f(a)$ for any value, substitute the value a, into the function $f(x)$.
Do the same to calculate $f(b)$.

By filling in the table of values, complete the limiting chord process for $f(x) = x^2$ at the point $x = 1$.

a	b	$h = b - a$	$\frac{f(b)-f(a)}{b-a}$
1	2	1	3
1	1.5	0.5	2.5
1	1.1	0.1	2.1
1	1.05	0.05	2.05
1	1.01	0.01	2.01
1	1.001	0.001	2.001
1	$1.000\,1$	$0.000\,1$	$2.000\,1$

b. The instantaneous rate of change at a point is the limiting value that $\frac{f(b)-f(a)}{b-a}$ takes as we approach 1.
Looking at the last row of the table, what value does it appear to be approaching?

= 2

2. The daily net profit of an upmarket restaurant can be modelled by the equation $y = -16x^2 + 304x$, where x is the number of customers.

a. Find the value of y at $x = 0$.

b. Find the value of y at $x = 9$.

c. Hence find the average rate of change in net profit over the interval $[0,\ 9]$.

a. Substitute $x = 0$ into $y = -16x^2 + 304x$.

$y = -16 \times 0^2 + 304 \times 0$

Evaluate.

$y = 0$

b. Substitute $x = 9$ into $y = -16x^2 + 304x$.

$y = -16 \times 9^2 + 304 \times 9$

Evaluate the products.

$y = -1\,296 + 2\,736$

Evaluate.

$y = 1\,440$

c. The average rate of change of the function is looking for how much the net profit changes, on average, for every additional customer.
How can the two values found in the previous parts be used to find this rate?

Over an increase of 9 customers, the net profit changes by $(1\,440 - 0)$. How can we use this to find the change in the net profit for an increase of 1 customer?
Note that if the net profit decreases as the number of customers increases, the rate of change is negative.

$= \dfrac{1\,440 - 0}{9 - 0}$

Simplify the expression.

$= 160$

3. Differentiate the function $f(x) = (3x-2)\left(4x^2-5\right)$.

You may use the substitution $u = 3x - 2$ and $v = 4x^2 - 5$ in your working.

Notice that $f(x) = (3x-2)\left(4x^2-5\right)$ is the product of two functions and so we can express it in the form $f(x) = u \times v$. We can use the product rule $f'(x) = u'v + v'u$ to find the derivative of such a function.

For the given function, we can let $u = 3x - 2$ and $v = 4x^2 - 5$. What are the derivatives of u and v?

To find u' and v', use the fact that $\frac{d}{dx}x^n = nx^{n-1}$

In the case of v, we get $v' = 4 \times 2x = 8x$.

$$f'(x) = (3x-2) \times 8x + \left(4x^2-5\right) \times 3$$

Expand the brackets.

$$f'(x) = 24x^2 - 16x + 12x^2 - 15$$

Combine like terms.

$$f'(x) = 36x^2 - 16x - 15$$

4. Differentiate $f(x) = \left(x^2 + 3x - 2\right)\left(x^2 - 3x - 2\right)$.

You may use the substitution $u = x^2 + 3x - 2$ and $v = x^2 - 3x - 2$ in your working.

Notice that $f(x) = \left(x^2 + 3x - 2\right)\left(x^2 - 3x - 2\right)$ is the product of two functions and so we can express it in the form $f(x) = u \times v$.

We can use the product rule $f'(x) = u'v + v'u$ to find the derivative of such a function.

For the given function, we can let $u = x^2 + 3x - 2$ and $v = x^2 - 3x - 2$. What are the derivatives of u and v?

To find u' and v', use the fact that $\frac{d}{dx}x^n = nx^{n-1}$

In the case of v, we get $v' = 2x - 3$.

$$f'(x) = \left(x^2 + 3x - 2\right)(2x - 3) + \left(x^2 - 3x - 2\right)(2x + 3)$$

Expand the brackets.

$$f'(x) = 2x^3 - 3x^2 + 6x^2 - 9x - 4x + 6 + 2x^3 + 3x^2 - 6x^2 - 9x - 4x - 6$$

Combine like terms.

$$f'(x) = 4x^3 - 26x$$

5. Suppose we want to differentiate $y = \frac{9x}{8x - 5}$ using the Quotient Rule.

a. Identify the function u.

b. Identify the function v.

c. Find u'.

d. Find v'.

e. Hence find y'.

f. Is it possible for the derivative of this function to be zero?

A No

B Yes

a. The Quotient Rule states that $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$.

Notice that u is the numerator of the rational function we are differentiating. What is the numerator of $y = \frac{9x}{8x-5}$?

$u = 9x$

b. The Quotient Rule states that $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$.

Notice that v is the denominator of the rational function we are differentiating. What is the denominator of $y = \frac{9x}{8x-5}$?

$v = 8x - 5$

c. We found that $u = 9x$.
To find the derivative of $u = 9x$, we can use the power rule $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

This says that to differentiate a power x^n, we have to bring down the exponent in front of the expression and then decrease the exponent by 1 to get nx^{n-1}.

$u' = 1 \times 9x^{1-1}$

Find the value of the difference in the exponent.

$u' = 9x^0$

We can rewrite x^0 using the zero-exponent property $A^0 = 1$.

$u' = 9$

d. We found that $v = 8x - 5$.
To find the derivative of $v = 8x - 5$, which is a sum of terms, we have to find the derivative of each term separately.
To find the derivative of each term, we can use the Power Rule $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

This says that to differentiate a power x^n, we have to bring down the exponent in front of the expression and then decrease the exponent by 1 to get nx^{n-1}.

$$v' = 1 \times 8x^{1-1}$$

Find the value of the difference in the exponent.

$$v' = 8x^0$$

We can rewrite x^0 using the zero-exponent property $A^0 = 1$.

$$v' = 8$$

e. Using the derivatives u' and v', we can use the quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$ to find y'.

What are u' and v'?

We found that:

- $u = 9x$
- $v = 8x - 5$
- $u' = 9$
- $v' = 8$

Substitute these expressions into $\frac{u'v - v'u}{v^2}$.

$$y' = \frac{9(8x-5) - 8 \times 9x}{(8x-5)^2}$$

To simplify $9x \times 8$, we have to multiply the integers.

$$y' = \frac{9(8x-5) - 72x}{(8x-5)^2}$$

To expand the brackets in the numerator, we have to use the distributive law $A(B-C) = AB - AC$.

This states that we have to multiply the term outside the brackets by each of the terms inside the brackets.

$$y' = \frac{72x - 45 - 72x}{(8x-5)^2}$$

We can simplify the expression in the numerator by combining like terms. That is, by combining the terms that have the same variable parts.

Since $72x$ and $-72x$ are like terms, we can combine them by adding them together.

$$y' = \frac{-45}{(8x-5)^2}$$

f. Are there any values of x that will make $y' = \frac{-45}{(8x-5)^2}$ equal to 0?

That is, are there any solutions to $\frac{-45}{(8x-5)^2} = 0$?

We can try to solve the equation by clearing the fraction by multiplying each side by $(8x-5)^2$.

If we do this, we get $-45 = 0$. Are there any solutions to this equation?

A No

6. Suppose we want to differentiate $y = \frac{3x}{2x^2 - 5}$ using the quotient rule.

a. Identify the function u.

b. Identify the function v.

c. Find u'.

d. Find v'.

e. Hence find y', giving your answer in factorised form.

a. The quotient rule states that $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$.

Notice that u is the numerator of the rational function we are differentiating. What is the numerator of $y = \frac{3x}{2x^2 - 5}$?

$u = 3x$

b. The quotient rule states that $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$.

Notice that v is the denominator of the rational function we are differentiating. What is the denominator of $y = \frac{3x}{2x^2 - 5}$?

$v = 2x^2 - 5$

c. We found that $u = 3x$.
To find the derivative of $u = 3x$, we can use the power rule $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

This says that to differentiate a power x^n, we have to bring down the exponent in front of the expression and then decrease the exponent by 1 to get nx^{n-1}.

$u' = 1 \times 3x^{1-1}$

Find the value of the difference in the exponent.

$u' = 3x^0$

We can rewrite x^0 using the zero-exponent property $A^0 = 1$.

$u' = 3$

d. We found that $v = 2x^2 - 5$.

To find the derivative of $v = 2x^2 - 5$, which is a sum of terms, we have to find the derivative of each term separately.
To find the derivative of each term, we can use the power rule $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

This says that to differentiate a power x^n, we have to bring down the exponent in front of the expression and then decrease the exponent by 1 to get nx^{n-1}.

$$v' = 2 \times 2x^{2-1}$$

Find the value of the difference in the exponent.

$$v' = 2 \times 2x$$

Find the product of the integers.

$$v' = 4x$$

e. Using the derivatives u' and v', we can use the quotient rule $y' = \frac{u'v - v'u}{v^2}$ to find y'.

What are u' and v'?

We found that:

- $u = 3x$
- $v = 2x^2 - 5$
- $u' = 3$
- $v' = 4x$

Substitute these expressions into $\frac{u'v - v'u}{v^2}$.

$$y' = \frac{\left(2x^2 - 5\right) \times 3 - 3x \times 4x}{\left(2x^2 - 5\right)^2}$$

To simplify $3x \times 4x$, we have to multiply the numerical coefficients 3 and 4, and we have to multiply the variables. To multiply the variables, we have to use the product rule of exponents $A^m \times A^n = A^{m+n}$. This rule says that to multiply exponential expressions with the same base, we have to add the exponents and keep the same base.

$$y' = \frac{\left(2x^2 - 5\right) \times 3 - 12x^2}{\left(2x^2 - 5\right)^2}$$

To expand the brackets in the numerator, we have to use the distributive law $A\left(B - C\right) = AB - AC$.
This states that we have to multiply the term outside the brackets by each of the terms inside the brackets.

$$y' = \frac{6x^2 - 15 - 12x^2}{\left(2x^2 - 5\right)^2}$$

We can simplify the expression in the numerator by combining like terms. That is, by combining the terms that have the same variable parts.

Since $6x^2$ and $-12x^2$ are like terms, we can combine them by adding them together to make one term.

$$y' = \frac{-6x^2 - 15}{\left(2x^2 - 5\right)^2}$$

Since the coefficient is negative, we have to factor out the negative of the highest common factor from each term of $-6x^2 - 15$.

What is the highest common factor of $6x^2$ and 15?

$6x^2$ and 15 have a highest common factor of 3.

Therefore, we can factor out -3 from each term of the expression.

$$y' = \frac{-3\left(2x^2 + 5\right)}{\left(2x^2 - 5\right)^2}$$

7. Consider the function $f(x) = \left(5x^3 + 8x^2 - 3x - 5\right)^6$.
Redefine the function as composite functions $f(u)$ and $u(x)$, where $u(x)$ is a polynomial.

$u(x) = \square$

$f(u) = (\square)^{\square}$

We can see that $f(x)$ is the result of raising a polynomial to a power.
By substituting $u(x)$ for the polynomial, we can rewrite $f(x)$ as a function of u.

Substituting $u(x) = 5x^2 + 8x - 3$, we can now define f as a function of u.
Replacing $5x^2 + 8x - 3$ with u in $f(x) = \left(5x^2 + 8x - 3\right)^6$, we get $f(u) = (\square)^6$.

$u(x) = 5x^3 + 8x^2 - 3x - 5$

$f(u) = u^6$

y is the primitive function of $\frac{dy}{dx} = (x+2)^2$. To find the primitive y, we need to reverse the differentiation.
Notice that $\frac{dy}{dx}$ consists of a function of x, raised to a power.
How can we find the antiderivative of this type of equation?

An equation of the form $\frac{dy}{dx} = \left(f(x)\right)^n$ has a primitive function given by $y = \frac{1}{n+1} \times \left(f(x)\right)^{n+1} \times \frac{1}{f'(x)} + C$.
This says that we add 1 to the power, divide by the new power, and divide by the derivative of the function within the brackets. Finally, we add the constant of integration, C.

$y = \frac{1}{3} \times (x+2)^{2+1} + C$

Evaluate the addition in the exponent.

$y = \frac{1}{3}(x+2)^3 + C$

To find the value of C, we can use the fact the curve passes through the point $(-5,\ -7)$ by substituting $x = -5$ and $y = -7$ into the equation.

$-7 = \frac{1}{3}(-5+2)^3 + C$

Evaluate the expression in the brackets.

$-7 = \frac{1}{3}(-3)^3 + C$

Evaluate the cube term.

$-7 = \frac{1}{3} \times (-27) + C$

Evaluate the multiplication.

$-7 = -9 + C$

Collect the constant terms to one side of the equation to isolate C.

$C = 2$

Substitute $C = 2$ back into $y = \frac{1}{3}(x+2)^3 + C$.

$y = \frac{(x+2)^3}{3} + 2$

11. The position (in metres) of an object along a straight line after t seconds is modelled by $s(t) = 3t^2 + 7t + 4$.
We want to find the velocity of the object after 4 seconds.

a. Determine $v(t)$, the velocity function.

b. What is the velocity of the object after 4 seconds?

a. Velocity is the rate of change of displacement over time.
So velocity is the derivative of the displacement function.

Since $s(t)$ is a function whose terms are powers of t, we can differentiate each term using the rule:
$\frac{d}{dx}x^n = nx^{n-1}$

$v(t) = 6t + 7$

b. In part (a), we found that the velocity of the particle at any time t is given by $v(t) = 6t + 7$.
How can we use this to find the velocity of the particle after 4 seconds?

Substitute $t = 4$ into $v(t) = 6t + 7$

velocity= $6 \times 4 + 7$ m/s

Evaluate the product.

velocity= $24 + 7$ m/s

Evaluate the sum.

velocity= 31 m/s

12. Find the equation of a curve given that the gradient at any point $(x,\ y)$ is given by $\frac{dy}{dx} = (x+2)^2$, and that the point $(-5,\ -7)$ lies on the curve.
Use C as the constant of integration.

8. Find the primitive function of $9x^2 - 8x - 2$.
Use C as the constant of integration.

To find the primitive, we need to reverse the differentiation.
The primitive of x^n is $\frac{x^{n+1}}{n+1} + C$,
where $n \neq -1$ and C is some constant.
This says that to undo differentiation, we increase the power by 1 and then divide by the new power.
We can apply this to each term.

$= \frac{9x^{2+1}}{2+1} - \frac{8x^{1+1}}{1+1} - \frac{2x^{0+1}}{0+1} + C$

Find the values of the sums.

$= \frac{9x^3}{3} - \frac{8x^2}{2} - 2x + C$

To simplify the fractions, we have to cancel out the common factors from the numerators and the denominators.
What is the highest common factor of $9x^3$ and 3?
What is the highest common factor of $8x^2$ and 2?

$9x^3$ and 3 have a highest common factor of 3.
$8x^2$ and 2 have a highest common factor of 2.

$= 3x^3 - 4x^2 - 2x + C$

9. Let $y = (x+3)^5$ be defined as a composition of the functions $y = u^5$ and $u = x + 3$.

a. Determine $\frac{dy}{du}$.

b. Determine $\frac{du}{dx}$.

c. Hence determine $\frac{dy}{dx}$.

a. To find the derivative of a term of the form u^n, we can use the power rule:
$\frac{d}{du}\left(u^n\right) = nu^{n-1}$.

That is, to differentiate a power term u^n, we bring down the exponent to multiply in front of the expression, and then decrease the exponent by 1.

$$\frac{dy}{du} = 5u^{5-1}$$

Evaluate the subtraction in the exponent.

$$\frac{dy}{du} = 5u^4$$

b. To find the derivative of $u = x + 3$, which is a sum of terms, we can find the derivative of each term separately.
To find the derivative of each term of a polynomial, we can use the power rule $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

That is, to differentiate a power term x^n, we bring down the exponent to multiply in front of the expression, and then decrease the exponent by 1.

Note that the constant term 3 can be written as $3x^0$. So when we differentiate, the constant term will become 0.

$$\frac{du}{dx} = 1$$

c. Use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.
Replace $\frac{dy}{du}$ and $\frac{du}{dx}$ with the expressions found in the previous parts.

In part (a) we found that $\frac{dy}{du} = 5u^4$.
In part (b) we found that $\frac{du}{dx} = 1$.

$$\frac{dy}{dx} = 5u^4 \times 1$$

Replace u with $x + 3$.

$$\frac{dy}{dx} = 5(x+3)^4 \times 1$$

Evaluate the multiplication.

$$\frac{dy}{dx} = 5(x+3)^4$$

10.

Find y if $\frac{dy}{dx} = \frac{1}{(4x+9)^6}$.
Use C as the constant of integration.

y is the primitive function of $\frac{dy}{dx} = \frac{1}{(4x+9)^6}$. To find the primitive y, we first need to rewrite the expression without a fraction.

We will need to write $\frac{dy}{dx}$ using a negative exponent.
Recall that $\frac{1}{a^n} = a^{-n}$.

$$\frac{dy}{dx} = (4x+9)^{-6}$$

Notice that $\frac{dy}{dx} = (4x+9)^{-6}$ consists of a function of x, raised to a power. How can we find the antiderivative of this type of equation?

An equation of the form $\frac{dy}{dx} = \left(f(x)\right)^n$ has a primitive function given by $y = \frac{1}{n+1} \times \left(f(x)\right)^{n+1} \times \frac{1}{f'(x)} + C$.
This says that we add 1 to the power, divide by the new power, and divide by the derivative of the function within the brackets. Finally, we add the constant of integration, C.

$$y = \frac{1}{-6+1}(4x+9)^{-6+1} \times \frac{1}{4} + C$$

Evaluate addition in the exponent.

$$y = \frac{1}{-6+1}(4x+9)^{-5} \times \frac{1}{4} + C$$

Evaluate the sum in the denominator.

$$y = \frac{1}{-5} \times \frac{1}{4}(4x+9)^{-5} + C$$

Evaluate the multiplication.

$$y = \frac{1}{-20}(4x+9)^{-5} + C$$

Rewrite the expression with positive indices.
Use the fact that $a^{-n} = \frac{1}{a^n}$.

$$y = \frac{1}{-20(4x+9)^5} + C$$