ALL SAINTS' COLLEGE

Semester One Examination, 2020

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section One: Calculator-free

SOLUTIONS

WA student number: In figures ☐☐ ☐☐☐ ☐☐☐

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable): ☐

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

General comments for Calculator free section:

- *It is clear that when students are practicing and doing class exercises that they are saying "close enough is good enough" As such they do not have the rigour or setting out that is acceptable in an assessment. It is this lack of finishing properly, introducing a process in the middle and the lack of clear and logical working that may incur penalties. If you make an arithmetic error, and there is no logic for the marker to follow - you will lose all marks rather than 1 at the point of error. This is NOT a skill you can pull out of your hat in an assessment – practice this EVERYtime you do a question. Look carefully at the setting out in the solutions…is yours as good? If not FIX it and practice it.*
- *And let's talk about arithmetic errors…there are WAY too many careless errors – (here are some examples: 8x8=16, 8x8=81, 5x5=5…..etc..) each one will carry a penalty and can be very costly. If time was an issue, this is understandable, but it there was plenty of time to check – then recalculate – this was especially evident in definite integrals with fractions. At your level we expect you to be able to do simple fractions and to keep track of negative signs!!!!*
- *Don't half differentiate a function in one line and finish differentiating it in the other line….this is incorrect.*
-

Section One: Calculator-free **35% (52 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(5 marks)**

Determine the area bounded by the line $y=-2x$ and the parabola $y=x^2-6x$.

Solution
Intersect when $-2x-(x^2-6x)=0 \quad 4x-x^2=0 \quad x(4-x)=0 \quad x=0,4$ Bounded area $A=\int_0^4 4x-x^2\,dx ¿ \left[2x^2-\frac{x^3}{3}\right]_0^4 ¿ \left(32-\frac{64}{3}\right)-(0) ¿ 32-21.\overline{3}$ $¿ 10.\overline{6}=10\frac{2}{3}$ square units
Specific behaviours
✓ equates functions, simplifies and solves. correct order or correct use of || antidifferentiates substitutes correctly follow through area (-1 if no units^2)

)

I find it hard to believe how badly done this question was. The marksbreakdown said there was an area question in CF – so you should have practiced doing this by hand – at least a few times.

The arithmetic errors were unbelievable!!! Students who thought that they could do $\int_0^4 |4x-x^2|dx$ without a calculator!

Common error:

- $-2x=(x^2-6x) \Rightarrow 4x=x^2 \Rightarrow x=4$ *!! !!*
  REALLY ? ? !! ! I thought we sorted this out LAST year !! !
- The higher function was clearly unknown, !!! For this question you either had to draw a sketch or find the integral, knowing that if it was negative that you needed the positive (absolute value). It is OK to get a negative integral and then say $\therefore\ the\ area=\left|-10\frac{2}{3}\right|=10\frac{2}{3}\ units^2$ what is NOT OK is to just "misplace" the negative whilst still using the = sign.
  Too many times students did it the wrong way around then tried to fudge their solutions. Did you think we wouldn't read your working? – this then became 2 mistakes

Supplementary page

Question number: _________

ignoring the order of integration. i.e. $\int_a^b f(x)dx = F(b) - F(a)$ NOT $F(a) - F(b)$ and yes that DOES make a difference.
Very few students did this in a clear and logical way showing the logical steps..as I said LEARN IT.

Question 2 **(5 marks)**

A curve, defined for $x>0$, passes through the point $P(2,3)$ and its gradient is given by

$$\frac{dy}{dx} = 6x^2 - \frac{4}{x^2} - 23$$

(a) Verify that P is a stationary point, determine the value of the second derivative at P and hence describe the nature of the stationary point. (3 marks)

Solution
$f'(x) = 6x^2 - \frac{4}{x^2} - 23 \Rightarrow f'(2) = 24 - 1 - 23 = 0$ $f'(2) = 0$, so P is a stationary point. $f''(x) = 12x + \frac{8}{x^3} \Rightarrow f''(2) = 24 + 1 = 25$ $f''(2) > 0$, so P is a local minimum.
Specific behaviours
✓ simplifies $f'(2)$ to three integers that sum to zero   correct value of second derivative

The key word here was VERIFY – that means show it….no shortcuts. It is NOT enough to say f'(x) = 0 so x =2, unless you actually did it (which of course is NOT the way to do this question – easy way is shown above). Find f"(2) don't just say it is positive – again this is a process. This is a <u>LOCAL</u> minimum. I did not penalise either of these things but the next marker probably will.

(b) Determine the equation of the curve. (2 marks)

Solution
$f(x) = 2x^3 + \frac{4}{x} - 23x + c$ $f(2) = 16 + 2 - 46 + c = 3 \Rightarrow c = 31$ $y = 2x^3 + \frac{4}{x} - 23x + 31$
Specific behaviours
✓ correct antiderivative   evaluates constant and writes equation

A number of students misread this to mean the equation of the tangent – then tried to find one. Read the question – highlight requirements. Most people who DID do this did it well. Those who still can't integrate this function, then find the constant of integration, should seriously think about why they are doing this course.

Question 3 **(7 marks)**

A bag contains 40 counters, 15 marked with 0 and the remainder marked with 1. The random variable X is the number on a randomly selected counter from the bag.

(a) Explain why X is a Bernoulli random variable and determine the mean and variance of X. (3 marks)

Solution

X is a Bernoulli random variable as it can only take on two values, 0 and 1.

$$E(X)=p=\frac{40-15}{40}=\frac{5}{8}$$

$$\sigma^2=\frac{5}{8}\times\frac{3}{8}=\frac{15}{64}$$

Specific behaviours

✓ states X can only take on two values
✓ mean
✓ variance

Each of the 32 students in a class randomly select a counter from the bag, note the number on the counter and then replace it back in the bag. The random variable Y is the number of students in the class who select a counter marked with 0.

(b) Define the distribution of Y and determine the mean and variance of Y. (3 marks)

Solution

$$Y \quad B\left(32,\frac{3}{8}\right)$$

$$E(Y)=np=32\times\frac{3}{8}=12$$

$$\sigma^2=12\times\frac{5}{8}=\frac{15}{2}=7.5$$

Specific behaviours

✓ states binomial with parameters
✓ mean
✓ variance

(c) Explain why it is important that the students replace their counters for the distribution of Y in part (b) to be valid. (1 mark)

Solution

If counters not replaced, the probability of a success (selecting a counter marked with 0) would not remain constant.

Specific behaviours

✓ indicates that probability of success must be constant

This was generally well done. The mistakes were as a result of poor arithmetic skills when multiplying fractions (i.e. not simplifying them first – unsimplified answers get full marks but ones stated as a product incur penalty)

(c) I was generous but the explanations were just not precise enough. The best: “because replacing counters means that each trial is independent, so the probabilities remained constant”

The worse were:

“so it (whatever it is) remains constant”…. “ if they didn’t it wouldn’t be binomial” (WHY??)…. “because n will change each time”(AND that means what???) “otherwise it would affect the probability (And that’s a problem because????)

power of a half so $5^{2\times3\times0.5}=5^3=125$ Question 8 **(7 marks)**

(a) Determine an expression for $\frac{d}{dt}\left(6t\cos\left(\frac{\pi t}{6}\right)\right)$. (2 marks)

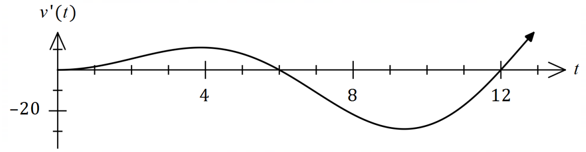
Solution

$$\frac{d}{dt}\left(6t\cos\left(\frac{\pi t}{6}\right)\right)=6\cos\left(\frac{\pi t}{6}\right)-\pi t\sin\left(\frac{\pi t}{6}\right)$$

Specific behaviours

✓ correct use of product rule
✓ correct derivative

The volume of water in a tank, v litres, is changing at a rate given by $v'(t)=\pi t\sin\left(\frac{\pi t}{6}\right)$, where t is the time in hours. The rate of change is shown in the graph below.



(b) Using the result from part (a) or otherwise, determine the change in volume of water in the tank between $t=0$ and $t=12$ hours. (5 marks)

Solution

$$\Delta v=\int_0^{12} v'(t)\,dt=\int_0^{12}\pi t\sin\left(\frac{\pi t}{6}\right)dt$$

1. Using (a):

$$\int\frac{d}{dt}\left(6t\cos\left(\frac{\pi t}{6}\right)\right)dt=\int 6\cos\left(\frac{\pi t}{6}\right)dt-\int\pi t\sin\left(\frac{\pi t}{6}\right)dt$$

2. And so:

$$\int\pi t\sin\left(\frac{\pi t}{6}\right)dt=\int 6\cos\left(\frac{\pi t}{6}\right)dt-6t\cos\left(\frac{\pi t}{6}\right)$$

3. Hence:

$$\int_0^{12}\pi t\sin\left(\frac{\pi t}{6}\right)dt=\left[\frac{36}{\pi}\sin\left(\frac{\pi t}{6}\right)\right]_0^{12}-\left[6t\cos\left(\frac{\pi t}{6}\right)\right]_0^{12} ¿[0-0]-[72-0]\Delta v=-72\text{ L}$$

Specific behaviours

✓ indicates required definite integral
✓ line 1 - uses part (a)
✓ line 2 - expression to evaluate integral
✓ line 3 - antidifferentiates ready for substitution
✓ correct change in volume, with units

(a) in this was really rather well done, and if it wasn’t (b) just fell apart and it was very hard to get any marks as students made the question easier. Too many students just igbnored one of the integrals – this cannot gain marks – the correct answer with incorrect working just doesn’t get you anything. This is a pretty standard ‘tough’ question – learn it. One thing that I noticed was people

Question 7 **(8 marks)**

Initially, particle P is stationary and at the origin. Particle P moves in a straight line so that at time t seconds its acceleration a cms^{-2} is given by $a=8-3\sqrt{t}$ where $t\geq 0$.

(a) Determine the speed of P after 1 second. (3 marks)

Solution
$v=\int 8-3t^{0.5}\,dt \text{¿} 8t-2t^{1.5}+c\; v(0)=0\Rightarrow c=0$
$v=8t-2t^{1.5}$
$v(1)=8(1)-2(1)^{1.5}=6$
Hence speed is 6 cm/s.
Specific behaviours
✓ indicates v is integral of a
✓ expression for velocity v with c explained
✓ correct speed, with units

(b) Determine the speed of P when it returns to the origin. (5 marks)

Solution
Require 0 change in displacement for $0\leq t\leq T$
$\Delta x=\int_0^T 8t-2t^{1.5}\,dt0=\left[4t^2-\frac{4}{5}t^{2.5}\right]_0^T 4T^2-\frac{4}{5}T^{2.5}=0$
$4T^2\left(1-\frac{1}{5}\sqrt{T}\right)=0\sqrt{T}=5T=25$
$v(25)=8(25)-2(25)^{\frac{3}{2}}\text{¿}200-250=-50$
Hence speed is 50 cm/s.
Specific behaviours
✓ obtains expression for Δx in terms of T
✓✓ equates $\Delta x=0$ and solves for T
✓ obtains velocity
✓ correct speed, with units

Slightly different Solution
$x(t)=4t^2-\frac{4}{5}t^{2.5}+k$
$x(0)=0\therefore k=0$
$x(t)=0\Rightarrow 4t^2-\frac{4}{5}t^{2.5}=04t^2\left(1-\frac{1}{5}\sqrt{t}\right)=0t=0,\sqrt{t}=5$
$t=25$
$v(25)=8(25)-2(25)^{\frac{3}{2}}\text{¿}200-2(5)^3$
$\text{¿}200-250=-50\,cm/s$
Hence speed is 50 cm/s.
Specific behaviours
✓ obtains expression for $x(t)$, *equating constant*.
✓✓ equates $\Delta x=0$ and solves for T
✓ obtains velocity
✓ correct states speed, with units

Firstly, If you are going to use the second method and you have 2 constants of integration, name them differently c and k or c_1 and c_2.
Secondly – read the question – the units were cm/s…a sad loss of a mark…yes kick yourself!!!
As it wanted speed and you found velocity…STATE the speed.

Students who got off the ground with this one, kind of followed a correct method – until it came to solving $4t^2-\frac{4}{5}t^{2.5}=0$…oh the horror!!!! Remember $t^{2.5}=t^2\times t^{0.5}$ so….

$$t^2\left(4-\frac{4}{5}t^{0.5}\right)=0\Rightarrow 4=\frac{4}{5}\sqrt{t}\Rightarrow 5=\sqrt{t}\wedge here's\,the\,kicker\,so\,t=25\,not\,\sqrt{5}$$

And then you had to substitute it into the velocity:

$(25)^{\frac{3}{2}}\,no\,do\,not\,do\,\sqrt{25^3}\,by\,hand\ldots simplfy\,index\,i.e.\,25\,is\,5^2,the\,square\,root\,is\,the$

Question 4 **(8 marks)**

Determine

(a) $f'(x)$ when $f(x)=\sqrt{4x-3}$. (2 marks)

Solution
$f'(x)=\frac{1}{2}(4)(4x-3)^{\frac{-1}{2}}\text{¿}\frac{2}{\sqrt{4x-3}}$
Specific behaviours
✓ indicates correct use of chain rule
✓ correct derivative (any form)

(b) $\frac{d}{d\theta}\left(\theta^3e^{4\theta}\right)$ when $\theta=2$. (3 marks)

Solution
$\text{¿}3\theta^2e^{4\theta}+4\theta^3e^{4\theta}\vert_{\theta=2}$
$\text{¿}12e^8+32e^8=44e^8$
Specific behaviours
✓✓ correct derivative in terms of θ
✓ correct value

(c) $f'\left(\frac{\pi}{4}\right)$ when $f(t)=\frac{1+\cos t}{\sin t}$. (3 marks)

Solution
$f'(t)=\frac{-\sin t\cdot\sin t-(1+\cos t)\cdot\cos t}{\sin^2 t}$
$\text{¿}\frac{-\cos t-\sin^2 t-\cos^2 t}{\sin^2 t}\text{¿}\frac{-1-\cos t}{\sin^2 t}$
$f'\left(\frac{\pi}{4}\right)=\left(-1-\frac{1}{\sqrt{2}}\right)\div\frac{1}{2}\text{¿}-2-\sqrt{2}$
Specific behaviours
✓✓ correct derivative
✓ correct value, simplified

This question was fairly well done as far as the derivatives were concerned (as so they should be) The problem was with the substitution and simplification of answers.

(b) was Ok but I noticed a really poor use of notation again. $3\theta^2e^{4\theta}$ does not equal $12e^8$ – it just DOESn't…indicate that you are substituting in a value. Again I did not penalise this – but it doesn't mean you are correct.

(c) was pretty poor. Lucky for you this was only worth 1 mark – which was often lost.

Question 5 **(7 marks)**

Functions f and g are such that

$$f(4)=2, f'(x)=18(3x-10)^{-2}$$
$$g(-4)=2, g'(x)=18(3x+10)^{-2}$$

(a) Determine $f(6)$. (3 marks)

Solution
$f(6)=f(4)+\int_4^6 18(3x-10)^{-2}dx$
¿ $2+\left[\frac{-6}{3x-10}\right]_4^6$ ¿ $2+\left(\frac{-3}{4}-(-3)\right)$ ¿ $\frac{17}{4}=4\frac{1}{4}$
Specific behaviours
✓ integrates rate of change
✓ determines change
✓ correct value

Alternate Solution
$\int 18(3x-10)^{-2}dx=\frac{-6}{3x-10}+c$
$f(4)=2 \; so \; 2=\frac{-6}{2}+c \Rightarrow c=5$
$f(x)=\frac{-6}{3x-10}+5$
$f(6)=\frac{-6}{8}+5=4\frac{1}{4}$
Specific behaviours
✓ integrates
✓ determines c

(b) Use the increments formula to determine an approximation for $g(-3.98)$. (3 marks)

Solution
$x=-4, \delta x=0.02$
$\delta y \approx \frac{18}{(3x+10)^2}\times \delta x \approx \frac{18}{4}\times 0.02 \approx 0.09$
$g(-3.98)\approx 2+0.09 \approx 2.09$
Specific behaviours
✓ values of x and δx
✓ use of increments formula
✓ correct approximation

(c) Briefly discuss whether using the information given about f and the increments formula would yield a reasonable approximation for $f(6)$. (1 mark)

Solution
No, approximation wouldn't - the change $\delta x=2$ is not a small change.
Specific behaviours
✓ states no with correct reason

This question was reasonably done. 99% of students used 2nd method for (a) and although more work, most did correctly although once again too many arithmetic errors.

(b) was actually not badly done although many students did not find the approximation.

(c) very well done.

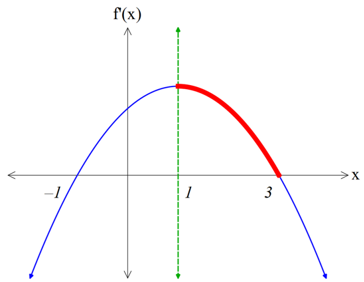
Question 6 **(5 marks)**

The graph of $y=f(x)$ has a stationary point at $(-1,2)$ and $f'(x)=ax^2+4x+6$, where a is a constant.

Determine the interval over which $f'(x)>0$ and $f''(x)<0$.

Solution
$f'(-1)=a-4+6=0 a=-2$
Concave down:
$f'(x)=-2x^2+4x+6 f''(x)=-4x+4 f''(x)<0 \Rightarrow x>1$
Other stationary point:
$-2x^2+4x+6=0 -2(x+1)(x-3)=0 x=3$
Hence $f'(x)>0$ when $-1<x<3$.
Required interval: $1<x<3$.
Specific behaviours
✓ value of a
✓ interval where $f''(x)<0$
✓ second stationary point
✓ interval where $f'(x)>0$
✓ correct interval

How I would have done it
$f'(-1)=a-4+6=0 a=-2$
$i.e. -2x^2+4x+6=0 -2(x+1)(x-3)=0 x=-1 \vee x=3$
So from here I would have sketched the graph of $y=f'(x)$
Hence $f'(x)>0$ when $-1<x<3$.
$f''(x)<0 (graph \, decreasing) \Rightarrow x>1$
So BOTH:
Required interval: $1<x<3$.
Specific behaviours
✓ value of a
✓ solve
✓ interval where $f''(x)<0$
✓ interval where $f'(x)>0$
✓ correct interval



This was not done well. The students who were on the right track often made too many mistakes to get there. Many students could not even make a start, the 2 methods above are not the only ways to do this. A number of students did all the work, but did not complete it by stating the interval where the BOTH conditions were met.