



Rossmoyne Senior High School

Semester One Examination, 2016

Question/Answer Booklet

MATHEMATICS  
METHODS  
UNIT 3

Section Two:

Calculator-assumed

SOLUTIONS

Student Number: In figures

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In words

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Your name

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Time allowed for this section

Reading time before commencing work: ten minutes

Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet

Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	48	35
Section Two: Calculator-assumed	13	13	100	101	65
Total			149	100	

Instructions to candidates

- The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.
- Write your answers in this Question/Answer Booklet.
- You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.
- Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
  - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
  - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.
- Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.
- It is recommended that you **do not use pencil**, except in diagrams.
- The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Additional working space

Question number: \_\_\_\_\_

Question 9 (5 marks)

A recent news report said that it took 34 months for the population of Australia to increase from 23 to 24 million people.

- (a) Assuming that the rate of growth of the population can be modelled by the equation  $\frac{dP}{dt} = kP$ , where  $P$  is the population of Australia at time  $t$  months, determine the value of the constant  $k$ . (3 marks)

<b>Solution</b>
$P = P_0 e^{kt}$
$24 = 23e^{34k}$
$k = 0.001252$
<b>Specific behaviours</b>
✓ uses growth and decay equation ✓ substitutes correctly ✓ solves for $k$

- (b) Assuming the current rate of growth continues, how long will it take for the population to increase from 24 million to 25 million people? (2 marks)

<b>Solution</b>
$25 = 24e^{0.001252t}$
$t = 32.6 \text{ months}$
<b>Specific behaviours</b>
✓ sets up correct values in equation, using $k$ to at least 3sf ✓ solves for $t$

<b>Solution</b>
$k = \frac{1}{t} \ln \left( \frac{P}{P_0} \right) = \frac{1}{34} \ln \left( \frac{24}{23} \right)$
$k = 0.001252$
<b>Specific behaviours</b>
✓ separates into two integrals ✓ determines $k$

(b) Calculate  $k$ . (2 marks)

<b>Solution</b>
Integral will calculate change in displacement, but since particle turned around after one second, this will not be the same as distance travelled.
<b>Specific behaviours</b>
✓ explains change in displacement not distance travelled in this instance

- (i) Explain why  $k \neq \int_0^1 v(t) dt$ . (1 mark)

(d) After five seconds, the particle has moved a distance of  $k$  metres. (2 marks)

<b>Solution</b>
$\frac{dP}{dt} = kP$
$\ln P = kt + C$
$\ln 24 = 34k + C$
$\ln 23 = C$
$\ln 24 - \ln 23 = 34k$
$k = \frac{\ln 24 - \ln 23}{34}$
<b>Specific behaviours</b>
✓ determines acceleration function ✓ shows that acceleration always positive for $t \geq 0$

(c) Show that the acceleration of the particle is always positive. (2 marks)

<b>Solution</b>
$v(t) = 0 \Rightarrow t^2 + 2t - 3 = 0 \Rightarrow t = -3, 1$
$x(1) = 2 \text{ m}$
<b>Specific behaviours</b>
✓ solves $v=0$ over domain ✓ determines displacement

(b) Determine the displacement of the particle at the instant it is stationary. (2 marks)

<b>Solution</b>
$v(t) = \frac{dP}{dt}$
$v(t) = t^2 + 2t - 3$
$\frac{d^2P}{dt^2} = 2t + 2$
<b>Specific behaviours</b>
✓ relates velocity to first derivative ✓ determines the first derivative

(a) Determine the velocity function,  $v(t)$ , for the particle. (2 marks)

seconds and  $x$  is the displacement of the particle from a fixed point  $O$ , in metres.

A particle moves in a straight line according to the function  $x(t) = \frac{t^3 + 3}{t + 1}$ ,  $t \geq 0$ , where  $t$  is in

Question 15 (9 marks)

Question 10 (7 marks)

A small object is moving in a straight line with acceleration  $a = 6t + k \text{ ms}^{-2}$ , where  $t$  is the time in seconds and  $k$  is a constant. When  $t = 1$  the object was stationary and had a displacement of 4 metres relative to a fixed point  $O$  on the line. When  $t = 2$  the object had a velocity of  $1 \text{ ms}^{-1}$ .

- (a) Determine the value of  $k$  and hence an equation for the velocity of the object at time  $t$ . (4 marks)

Solution
$v = 3t^2 + kt + c$ $t = 1, 3 + k + c = 0$ $t = 2, 12 + 2k + c = 1$ $k = -8$ $c = 5$ $v = 3t^2 - 8t + 5$
Specific behaviours
✓ antidifferentiates acceleration, adding constant ✓ derives simultaneous equations from information ✓ solves equations ✓ writes velocity equation

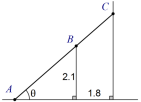
- (b) Determine the displacement of the object when  $t = 2$ . (3 marks)

Solution
$s = t^3 - 4t^2 + 5t + c$ $t = 1, 4 = 1 - 4 + 5 + c$ $c = 2$ $s = t^3 - 4t^2 + 5t + 2$ $s(2) = 8 - 16 + 10 + 2$ $= 4 \text{ m}$
Specific behaviours
✓ antidifferentiates velocity ✓ determines constant ✓ evaluates displacement

See next page

Question 21 (7 marks)

A vertical wall, 2.1 metres tall, stands on level ground and 1.8 metres away from the wall of a house. A ladder, of negligible width, leans at an angle of  $\theta$  to the ground and just touches the ground, wall and house, as shown in the diagram.



- (a) Show that the length of the ladder,  $L$ , is given by  $L = \frac{2.1}{\sin \theta} + \frac{1.8}{\cos \theta}$ . (3 marks)

Solution
$\sin \theta = \frac{2.1}{AB} \Rightarrow AB = \frac{2.1}{\sin \theta}, \cos \theta = \frac{1.8}{BC} \Rightarrow BC = \frac{1.8}{\cos \theta}$  $L = AB + BC$ $= \frac{2.1}{\sin \theta} + \frac{1.8}{\cos \theta}$
Specific behaviours
✓ shows on diagram and writes length of AB ✓ shows on diagram and writes length of BC ✓ sums lengths to obtain total, using labels added to diagram

- (b) Use a calculus method to determine the length of the shortest ladder that can touch the ground, wall and house at the same time. (4 marks)

Solution
$L = 2.1(\sin \theta)^{-1} + 1.8(\cos \theta)^{-1}$ $\frac{dL}{d\theta} = -2.1 \cos \theta (\sin \theta)^{-2} - 1.8(-\sin \theta)(\cos \theta)^{-2}$ $= \frac{1.8 \sin^3 \theta - 2.1 \cos^3 \theta}{\sin^2 \theta \cos^2 \theta}$ $\frac{dL}{d\theta} = 0 \Rightarrow 1.8 \sin^3 \theta - 2.1 \cos^3 \theta = 0$ $\tan^3 \theta = \frac{2.1}{1.8} \Rightarrow \theta = \tan^{-1} \sqrt[3]{\frac{2.1}{1.8}} = 0.8111$ $L(0.8111) = 5.51 \text{ metres}$
Specific behaviours
✓ shows first derivative of $L$ (may use CAS, but must show key results) ✓ solves derivative equal to 0 to find stationary point ✓ obtains acute angle solution ✓ substitutes into original equation to obtain minimum length

End of questions

See next page

Solution
$P = 0.2 \times P(Y = 2) \text{ where } Y \sim B(4, 0.2)$ $P(Y = 2) = 0.1536$ $P = 0.03072$
Specific behaviours
✓ identifies binomial situation for other two days ✓ evaluates probability of missing bus on two other days ✓ determines probability

- (iii) mases the bus on Tuesday and on two other days? (3 marks)

Solution
$P(X \geq 2) = 0.26272$ $X \sim B(5, 0.2)$
Specific behaviours
✓ identifies binomial situation ✓ sums cumulative probability

- (ii) mases the bus at least twice? (2 marks)

Solution
$0.2 \times 0.8^4 = 0.08192$
Specific behaviours
✓ uses 0.8 for not catching bus ✓ determines probability

- (i) only misses the bus on Tuesday? (2 marks)

Over five consecutive weekdays, what is the probability that the student miss the bus on any day is independent of whether they missed it on the previous day.

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Solution
$Va(Y) = 2.45$ $Va(Y) = (0.4)(-1.5)^2 + (0.1)(0.5)^2 + (0.1)(1.5)^2 + (0.2)(2.5)^2$
Specific behaviours
✓ states variance ✓ applies correct formula

- (ii)  $Va(1 - 2Y)$ . (1 mark)

Solution
$Va(Y) = 2.45$ $Va(Y) = (0.4)(-1.5)^2 + (0.1)(0.5)^2 + (0.1)(1.5)^2 + (0.2)(2.5)^2$
Specific behaviours
✓ uses correct formula ✓ applies both linear changes to obtain expected value

- (i)  $Va(Y)$ . (2 marks)

Solution
$E(Y) = 1.2$ $E(Y) = (0.4)(-1.5) + (0.1)(0.5) + (0.1)(1.5) + (0.2)(2.5)$ $= -0.5$
Specific behaviours
✓ forms products to obtain expected value ✓ sums products to obtain expected value

- (ii)  $E(Y)$ . (1 mark)

Solution
$P(Y \geq 0   Y \leq 1) = \frac{0.2}{1}$
Specific behaviours
✓ correct denominator ✓ correct numerator and simplification

- (a) Determine  $P(Y \geq 0 | Y \leq 1)$ . (2 marks)

$y$	0.4	0.2	0.1	0.1	0.2
$x$	-2	-1	0	1	2

The discrete random variable  $Y$  has the probability distribution shown in the table below.

Question 16 (8 marks)

Question 20 (7 marks)

Consider the function  $f(t) = \frac{t-4}{2}$  and the function  $A(x) = \int_0^x f(t) dt$ .

- (a) Complete the table below. (2 marks)

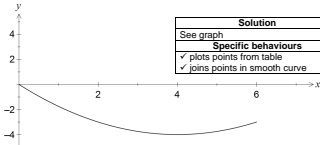
$x$	0	1	2	3	4	5	6
$A(x)$	0	-1.75	-3	-3.75	-4	-3.75	-3

<b>Solution</b>
See table
<b>Specific behaviours</b>
✓ calculates at least 3 correct values
✓ calculates all values correctly

- (b) For what value(s) of  $x$  is the function  $A(x)$  increasing? (1 mark)

<b>Solution</b>
$x > 4$
<b>Specific behaviours</b>
✓ correct inequality

- (c) On the axes below, sketch the graph of  $y = A(x)$  for  $0 \leq x \leq 6$ . (2 marks)



- (d) Determine (1 mark)

(i) when  $A(x) = 0$ . (1 mark)

<b>Solution</b>
$x = 4$
<b>Specific behaviours</b>
✓ states value

- (ii) the function  $A(x)$  in terms of  $x$ . (1 mark)

<b>Solution</b>
$A(x) = \int_0^x \frac{t}{2} - 2 dt = \frac{x^2}{4} - 2x$
<b>Specific behaviours</b>
✓ states function in terms of $x$

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Question 11 (7 marks)

It is known that 15% of Year 12 students in a large country study advanced mathematics.

A random sample of  $n$  students is selected from all Year 12's in this country, and the random variable  $X$  is the number of those in the sample who study advanced mathematics.

- (a) Describe the distribution of  $X$ . (2 marks)

<b>Solution</b>
$X \sim B(n, 0.15)$ - binomial distribution with $n$ trials and $p = 0.15$ .
<b>Specific behaviours</b>
✓ states binomial distribution
✓ states parameters of binomial distribution

- (b) If  $n = 22$ , determine the probability that (1 mark)

(i) three of the students in the sample study advanced mathematics. (1 mark)

<b>Solution</b>
$P(X = 3) = 0.2370$
<b>Specific behaviours</b>
✓ evaluates probability

- (ii) more than three of the students in the sample study advanced mathematics. (1 mark)

<b>Solution</b>
$P(X \geq 4) = 0.4248$
<b>Specific behaviours</b>
✓ evaluates probability

- (iii) none of the students in the sample study advanced mathematics. (1 mark)

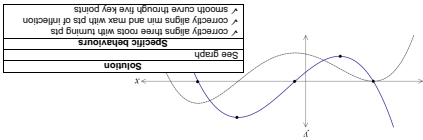
<b>Solution</b>
$P(X = 0) = 0.0280$
<b>Specific behaviours</b>
✓ evaluates probability

- (c) If ten random samples of 22 students are selected, determine the probability that at least one of these samples has no students who study advanced mathematics. (2 marks)

<b>Solution</b>
$Y \sim B(10, 0.028)$
$P(Y \geq 1) = 0.247$
<b>Specific behaviours</b>
✓ states binomial distribution with parameters
✓ evaluates probability

See next page

See next page



- (d) On the axes below, sketch a possible graph of  $y = f'(x)$ . The graph of  $y = f(x)$  is shown with a broken line for your reference. (3 marks)

<b>Solution</b>
Yes, as $x$ increases through $x = a$ , the gradient of $f$ changes from negative to zero to positive, indicating a horizontal pt of inflection.
<b>Specific behaviours</b>
✓ responds yes, indicating when $x = a$
✓ explains reason

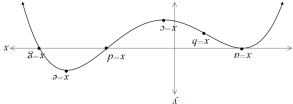
- (c) Does the graph of  $y = f(x)$  have a horizontal point of inflection? Justify your answer. (2 marks)

<b>Solution</b>
Yes, as $x$ increases through $x = a$ , the gradient of $f$ changes from positive to zero to negative, indicating a local maximum.
<b>Specific behaviours</b>
✓ responds yes, indicating when $x = a$
✓ explains reason

- (b) Does the graph of  $y = f(x)$  have a local maximum? Justify your answer. (2 marks)

<b>Solution</b>
$x = a, c, e$
<b>Specific behaviours</b>
✓ states all three values

- (a) For what value(s) of  $x$  does the graph of  $y = f(x)$  have a point of inflection? (1 mark)



and  $x = d$ , and roots when  $x = a, x = d$  and  $x = g$ , where  $a < b < c < d < e < g$ .

Question 17 (8 marks)

The graph of  $y = f(x)$ , the derivative of a polynomial function  $f$ , is shown below. The graph of  $y = f'(x)$  has stationary points when  $x = a, x = c$  and  $x = e$ , points of inflection when  $x = b$

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- (b) A Bernoulli trial, with probability of success  $p$ , is repeated  $n$  times. The resulting distribution of the number of successes has an expected value of 5.76 and a standard deviation of 1.92. Determine  $n$  and  $p$ . (4 marks)

<b>Solution</b>
$x \sim B(n, p)$
$np = 5.76$
$1 - p = 1.92^2 / 5.76 = 0.64$
$p = 0.36$
$n = 16$
<b>Specific behaviours</b>
✓ identifies distribution of successes as binomial
✓ states equation for mean
✓ states equation for variance (or standard deviation)
✓ solves equations for $n$ and $p$

- (a) Determine the mean of a Bernoulli distribution with variance of 0.24. (3 marks)

<b>Solution</b>
$p(1-p) = 0.24$
$p = 0.4, 0.6 \Rightarrow$ mean is either 0.4 or 0.6
<b>Specific behaviours</b>
✓ writes variance equation
✓ states both values of $p$ are possible means

Question 14 (14 marks)

Determine the mean of a Bernoulli distribution with variance of 0.24. (3 marks)

8

CALCULATOR-ASSUMED

Question 12 (8 marks)

The height of grain in a silo, initially 0.4 m, is increasing at a rate given by  $h'(t) = 0.55t - 0.05t^2$  for  $0 \leq t \leq 11$ , where  $h$  is the height of grain in metres and  $t$  is in hours.

- (a) At what time is the height of grain rising the fastest? (2 marks)

<b>Solution</b>
$h'(t) = 0.55 - 0.1t$
$0.55 - 0.1t = 0 \Rightarrow t = 5.5$ hours
<b>Specific behaviours</b>
✓ differentiates rate of change
✓ solves derivative equal to zero to obtain time.

- (b) Determine the height of grain in the silo after 11 hours. (3 marks)

<b>Solution</b>
$\Delta h = \int_0^{11} 0.55t - 0.05t^2 \, dt$
$\approx 11.09$
$h = 11.09 + 0.4$
$\approx 11.49$ m
<b>Specific behaviours</b>
✓ shows integral of rate of change
✓ evaluates integral to obtain change in height
✓ adds initial height

- (c) Calculate the time taken for the grain to reach a height of 4.45 m. (3 marks)

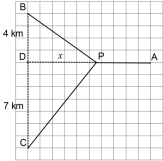
<b>Solution</b>
$\Delta h = 4.45 - 0.4 = 4.05$
$\Delta h = \int_0^k 0.55t - 0.05t^2 \, dt$
$= \frac{11k}{40} - \frac{k^3}{60}$
$\frac{11k}{40} - \frac{k^3}{60} = 4.05 \Rightarrow k = 4.5$ hours
<b>Specific behaviours</b>
✓ determines change in height
✓ evaluates integral using constant
✓ solves equation, ignoring solutions outside domain

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Question 19 (7 marks)

Three telecommunication towers, A, B and C, each need underground power cable connections directly to a new power station, P, that is to be built  $x$  km from depot D on a 10 km road running east-west between D and A.

Tower B lies 4 km due north of depot D and tower C lies 7 km south of the depot, as shown in the diagram.



- (a) Determine an expression for the total length of underground cable required to connect A, B and C directly to P. (2 marks)

<b>Solution</b>
$L = \sqrt{16 + x^2} + \sqrt{49 + x^2} + 10 - x$
<b>Specific behaviours</b>
✓ uses Pythagoras' theorem for BP and CP
✓ determines correct expression

- (b) Show that the minimum length of cable occurs when  $\frac{x}{\sqrt{16 + x^2}} + \frac{x}{\sqrt{49 + x^2}} = 1$ . (3 marks)

<b>Solution</b>
$\frac{dL}{dx} = \frac{1}{2}(2x)(16 + x^2)^{-1/2} + \frac{1}{2}(2x)(49 + x^2)^{-1/2} - 1$
$\frac{dL}{dx} = 0 \Rightarrow \frac{x}{\sqrt{16 + x^2}} + \frac{x}{\sqrt{49 + x^2}} = 1$
<b>Specific behaviours</b>
✓ uses chain rule to determine derivative
✓ shows that derivative must equal zero
✓ simplifies equation to required result

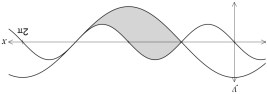
- (c) Determine the minimum length of cable required, rounding your answer to the nearest hundred metres. (2 marks)

<b>Solution</b>
$x \approx 3.025536$
$L = \sqrt{16 + x^2} + \sqrt{49 + x^2} + 10 - x \approx 19.6$ km
<b>Specific behaviours</b>
✓ solves
✓ substitutes to find length

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See next page

<b>Solution</b>
$2 \cos x + 2 \sin x \cos x = 0$
$\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$
$\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$
$A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-\sin(2x) - 2 \cos x) \, dx$
$= \left[ \frac{1}{2} \cos(2x) - 2 \sin x \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$
$= \left( -\frac{1}{2} + 2 \right) - \left( -\frac{1}{2} - 2 \right)$
$= 4$ sq units
<b>Specific behaviours</b>
✓ equates equations
✓ solves equations over interval (may choose to use CAS)
✓ substitutes limits
✓ shows antiderivative
✓ writes integral
✓ evaluates integral



The shaded region on the graph below is enclosed by the curves  $y = -\sin(2x)$  and  $y = 2 \cos x$ .

Question 13 (6 marks)  
METHODS UNIT 3

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<b>Solution</b>
$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} f(x) \, dx = \frac{\pi}{2}(-12 + 7 - 5 + 32) = 11$
<b>Specific behaviours</b>
✓ factors fraction out of integral
✓ evaluates integral and halves

- (b) (2 marks)

<b>Solution</b>
$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} f(x) \, dx = (-5 + 32) + (9) = 36$
<b>Specific behaviours</b>
✓ splits integral into two parts
✓ evaluates and sums each part

- (c) (2 marks)

<b>Solution</b>
$5 + 32 + 21 = 58$ sq units
<b>Specific behaviours</b>
✓ chooses regions C, D and E
✓ adds assigned values

- (b) (2 marks)

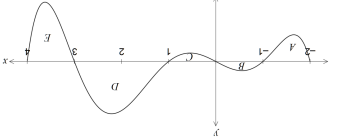
<b>Solution</b>
$-12 + 7 - 5 + 32 - 21 = 1$
<b>Specific behaviours</b>
✓ assigns sign to all areas
✓ adds signed values

- (a) (2 marks)

Determine the value of  $\int_0^{\frac{\pi}{2}} f(x) \, dx$ .

21 square units respectively.

The area of regions enclosed by the x-axis and the curve, A, B, C, D and E, are 12, 7, 5, 32 and



Question 18 (8 marks)  
CALCULATOR-ASSUMED

The graph of the function  $y = f(x)$  is shown below for  $-2 \leq x \leq 4$ .