Question 8 **(6 marks)**

The variables k and m are both integers such that $m^2+3=2k$.

(a) Use counter-examples to disprove any two of the three conjectures listed below. (2 marks)

- m can be any even integer.

Let m be 2 $2^2+3=7$ $2k=7$ means $k=3.5$ (Not an integer)

$\therefore$ Statement false. ✓

- m can be any odd integer.

Statement always true.

- m must be a positive odd integer.

Let $m=-3$ $m^2+3=9+3=12$

$2k=12 \therefore k=6$ ✓

Statement can be true for negative integers $\therefore$ false.

(b) Using the fact that any odd integer can be written in the form $2n+1$ or otherwise, prove that k is always the sum of three square numbers. (4 marks)

Let m be any odd integer $2n+1$

$$m^2+3=(2n+1)^2+3$$
$$=4n^2+4n+1+3$$
$$=4n^2+4n+4$$
$$=2k$$
$$\therefore k=2n^2+2n+2$$
$$=n^2+\underbrace{n^2+2n+1}+1$$
$$=n^2+(n+1)^2+1$$

QED

End of questions

2011 TRIAL WACE
CHURCHLANDS

Section One: Calculator-free **(40 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(4 marks)**

Find the minimum and maximum values of $f(x)=2x^3-3x^2-12x+27$ over the interval $-3\le x\le 3$.

$$f'(x)=6x^2-6x-12$$
$$=6(x^2-x-2)$$
$$=6(x-2)(x+1)$$
$$x=-1 \text{ or } 2 \quad ✓$$
$$f(-1)=2(-1)^3-3(-1)^2-12(-1)+27$$
$$=-2-3+12+27=\underline{34}$$
$$f(2)=2(2)^3-3(2)^2-12(2)+27$$
$$=16-12-24+27=\underline{7}$$
$$f(3)=2(3)^3-3(3)^2-12(3)+27$$
$$=54-27-36+27=\underline{18}$$
$$f(-3)=2(-3)^3-3(-3)^2-12(-3)+27$$
$$=-54-27+36+27=\underline{-18}$$

✓✓✓

−1 / mistake
−2 if endpoints not tested.

Max value 34
Min value −18.

See next page

Question 2 **(5 marks)**

Find $\frac{dy}{dx}$ in terms of x for each of the following.

(a) $y = x(1+2e^{3x})$ (2 marks)

$$\frac{dy}{dx} = (1+2e^{3x})1 + x(6e^{3x})$$
$$= 1 + 2e^{3x} + 6xe^{3x}$$

−1 if mistake in simplifying

(b) $y = \int_1^x t^2 + t - 1 \; dt$ (1 mark)

$$\frac{dy}{dx} = \frac{d}{dx}\int_1^x t^2+t-1 \, dt$$
$$= x^2+x-1$$

(c) $y = z^3 - z$ and $z = x^2 - 9$ (2 marks)

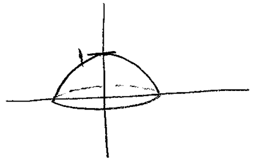
$$\frac{dy}{dz} = 3z^2-1 \qquad \frac{dz}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$
$$= (3z^2-1)\,2x$$
$$= 2x(3(x^2-9)^2-1)$$
$$= 6x(x^2-9)^2 - 2x$$

Question 7 **(4 marks)**

The region in the first quadrant bounded by $x=0$, $y=0$ and $y = 1-\frac{x^2}{9}$ is rotated 360° about the y-axis. If x and y are distances measured in centimetres, find the volume of the solid formed.

$$y = -\frac{x^2}{9} + 1$$



$$V = \pi\int x^2 \, dy$$
$$V = \pi\int_0^1 -9(y-1)\, dy$$
$$V = \pi\int_0^1 9 - 9y \, . \, dy$$
$$= \pi\left[9y - \frac{9y^2}{2}\right]_0^1$$
$$= \pi\left(9 - \frac{9}{2}\right)$$
$$= \frac{9\pi}{2} \text{ cm}^3$$

Question 6 **(5 marks)**

(a) Determine $\int \frac{2e^{-0.2y}}{5}\,dy$. (1 mark)

$= -2\int\left(-\frac{2}{10}\right) e^{-0.2y}\,dy$

$= -2e^{-0.2y} + C$ ✓

(b) Determine $\int (t-1)(1-2t+t^2)^3\,dt$. (2 marks)

$\frac{1}{2}\int (1-2t+t^3)^3(2t-2)\,dt$

$= \frac{1}{2}\frac{(1-2t+t^3)^4}{4}$ ✓

$= \frac{(1-2t+t^3)^4}{8} + C$ ✓

(c) Evaluate $\int_1^6 \frac{3}{x^2}\,dx$. (2 marks)

$\int_1^6 3x^{-2}\,dx = \left[\frac{3x^{-1}}{-1}\right]_1^6$ ✓

$= \left.-\frac{3}{x}\right]_1^6$

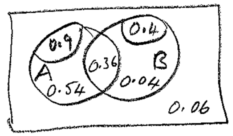
$= -\frac{1}{2} - (-3)$

$= 2\frac{1}{2}$ ✓

Question 3 **(5 marks)**

Two independent events A and B are such that P(A) = 0.9 and P(B) = 0.4.

(a) Find $P(\overline{A \cup B})$. (2 marks)



$P(A \cap B) = P(A) \times P(B)$

$= 0.36$ ✓

$P(\overline{A \cup B}) = 0.06$ ✓

(b) Find $P(\overline{B} \mid \overline{A} \cup B)$. (1 mark)

$\frac{P(\overline{B} \cap \overline{A} \cup B)}{P(\overline{A} \cup B)} = \frac{0.06}{0.46}$

$= \frac{6}{46}$ or $\frac{3}{23}$ ✓

(c) Show that $\overline{A}$ and $\overline{B}$ are also independent. (2 marks)

$P(\overline{A}) = 0.1 \quad P(\overline{B}) = 0.6$

$P(\overline{A} \cap \overline{B}) = 0.06$ from Venn diagram ✓

NB $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$

Hence

$P(\overline{A} \cap \overline{B}) = P(\overline{A}) \times P(\overline{B})$

$0.06 = 0.1 \times 0.6$ ✓

∴ Independent

Question 4 **(7 marks)**

Two functions are defined as $f(x)=\sqrt{x-1}$ and $g(x)=\dfrac{1}{x-1}$.

(a) Evaluate $g\circ f\left(\dfrac{13}{9}\right)$. (2 marks)

$$g\circ f(x)=\frac{1}{\sqrt{x-1}-1}\qquad g\circ f\left(\frac{13}{9}\right)=\frac{1}{\sqrt{\frac{13}{9}-1}-1}\ \checkmark$$

$$=\frac{1}{\sqrt{\frac{4}{9}}-1}$$

$$=\frac{1}{\frac{2}{3}-1}=\frac{1}{-\frac{1}{3}}=-3\ \checkmark$$

(b) Find in simplified form $g\circ g(x)$. (2 marks)

$$g\circ g(x)=\frac{1}{\frac{1}{x-1}-1}\ \checkmark$$

$$=\frac{1}{\frac{1-(x-1)}{x-1}}$$

$$=\frac{1}{\frac{2-x}{x-1}}=\frac{x-1}{2-x}\ \checkmark$$

(c) Determine the domain of $f(g(x))$. (3 marks)

$$f(g(x))=\sqrt{\frac{1}{x-1}-1}$$

$$\frac{1}{x-1}-1\geqslant 0$$

$$\frac{1}{x-1}\geqslant 1$$

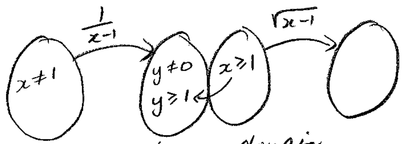
$x>1$ $\quad 1\geqslant x+1$

$2\geqslant x$

x cannot be <1 because $\frac{1}{x-1}$ cannot be negative

$\therefore\ 1<x\leq 2$

OR.



Need to change domain

$\frac{1}{x-1}\geqslant 1\quad \therefore\ 1<x\leq 2$

OR $\quad \frac{1}{x-1}-1\geqslant 0$

$$\frac{1-(x-1)}{x-1}\geqslant 0$$

$$\frac{2-x}{x-1}\geqslant 0$$

Hence $1<x\leq 2$

Question 5 **(4 marks)**

Solve the system of equations
$$c+2a=3+4b$$
$$a+2b+2c=4$$
$$5a+3c=5+2b$$

$$\begin{bmatrix}1 & 2 & 2 & 4\\ 2 & -4 & 1 & 3\\ 5 & -2 & 3 & 5\end{bmatrix}$$

$$\begin{array}{l}\begin{bmatrix}1 & 2 & 2 & 4\\ 0 & -8 & -3 & -5\\ 0 & -12 & -7 & -15\end{bmatrix}\end{array}\begin{array}{l}\\ R_2-2R_1\\ R_3-5R_1\end{array}$$

$$\begin{bmatrix}1 & 2 & 2 & 4\\ 0 & +8 & +3 & +5\\ 0 & 0 & 5 & 15\end{bmatrix}\ 3R_2-2R_3$$

$5c=15\quad \therefore\ c=3$

$8b+9=5\quad \therefore\ b=-\frac{1}{2}$

$a-1+6=4\quad \therefore\ a=-1$