Rossmoyne Senior High School

Semester One Examination, 2018

Question/Answer booklet

MATHEMATICS
METHODS
UNIT 3
Section Two:
Calculator-assumed

SOLUTIONS

Name: MARKING GUIDE.

Teacher's Name: ______________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	12	12	100	81	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Supplementary page

Question number: _________

Supplementary page

Question number: __________

Section Two: Calculator-assumed **65% (81 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(8 marks)**

The population of a city can be modelled by $P = P_0e^{kt}$, where P is the number of people living in the city, in millions, t years after the start of the year 2000. At the start of years 2007 and 2012 there were 2 245 000 and 2 521 000 people respectively living in the city.

(a) Determine the value of the constant k. (2 marks)

Solution
$2.521 = 2.245e^{5k}$ $k = 0.02319$
Specific behaviours
✓ equation   ✓ value of k to at least 3sf

Penalise if students put k to 2 significant figures. This would reduce their accuracy for later questions.

(b) Determine the value of the constant P_0. (2 marks)

Solution
$2.521 = P_0e^{0.02319(12)}$ $P_0 = 1.909$
Specific behaviours
✓ equation   ✓ value of P_0 (in millions)

If P_0 is written as 1908610 or similar (depending on k) accept it.

(c) Use the model to determine during which year the population of the city will first exceed 3 000 000. (2 marks)

Solution
$3 = 1.909e^{0.02319t}$ $t = 19.5 \Rightarrow \text{during } 2019$
Specific behaviours
✓ value of t ✓ correct year

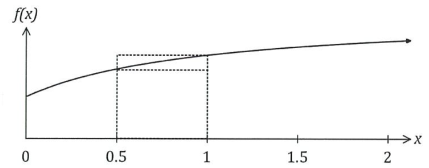
The 'during' is not necessary here, since the question is asking for the correct year which is 2019.

(d) Determine the rate of change of the city's population at the start of 2007. (2 marks)

Solution
$\frac{dP}{dt} = 0.02319 \times 2\,245\,000$ $= 52\,062$ people per year
Specific behaviours
✓ substitutes into rate of change   ✓ correct rate with units ← Units are deducted here.

Question 10 **(6 marks)**

The graph of $f(x) = \dfrac{6x+2}{x+1}$ is shown below.



Solution (a)
See table
Specific behaviours
✓ missing values

Two rectangles are also shown on the graph, with dotted lines, and they both have corners just touching the curve. The smaller is called the inscribed rectangle and the larger is called the circumscribed rectangle.

(a) Complete the missing values in the table below. (1 mark)

x	0	0.5	1	1.5	2
$f(x)$	2	$\frac{10}{3}$	4	$\frac{22}{5}$	$\frac{14}{3}$

(b) Complete the table of areas below and use the values to determine a lower and upper bound for $\int_0^2 f(x)\,dx$. (4 marks)

x interval	0 to 0.5	0.5 to 1	1 to 1.5	1.5 to 2
Area of inscribed rectangle	1	$\frac{5}{3}$	2	$\frac{11}{5}$
Area of circumscribed rectangle	$\frac{5}{3}$	2	$\frac{11}{5}$	$\frac{7}{3}$

Solution
Lower bound: $L = 1 + \frac{5}{3} + 2 + \frac{11}{5} = \frac{103}{15} \approx 6.867$
Upper bound: $U = \frac{5}{3} + 2 + \frac{11}{5} + \frac{7}{3} = \frac{41}{5} = 8.2$
Specific behaviours
✓ inscribed areas
✓ circumscribed areas
✓ states lower bound
✓ states upper bound

(c) Explain how the bounds you found in (b) would change if a smaller number of larger intervals were used. (1 mark)

Solution
The lower bound would decrease and the upper bound increase.
Specific behaviours
✓ describes changes to both bounds

'The bounds would be further apart' ← Accept this response. Students must talk about how the bounds change to get this mark.

Supplementary page

Question number: __________

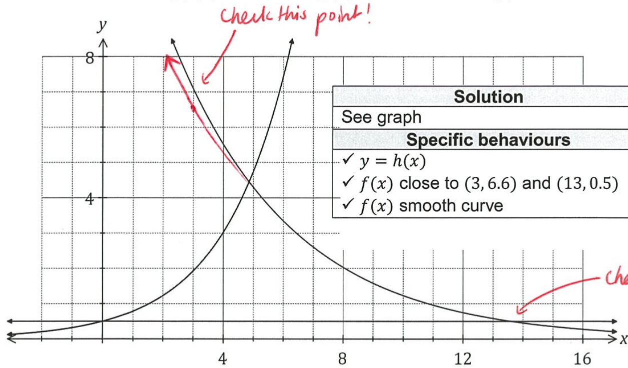
Supplementary page

Question number: _________

Question 11 **(8 marks)**

Three functions are defined by $f(x) = 14e^{-0.25x}$, $g(x) = 0.5e^{0.45x}$ and $h(x) = 0.5$.

Check this point!



Check this point!

Solution
See graph
Specific behaviours
✓ $y = h(x)$ ✓ $f(x)$ close to $(3, 6.6)$ and $(13, 0.5)$ ✓ $f(x)$ smooth curve

(a) One of the functions is shown on the graph above. Add the graphs of the other two functions. (3 marks)

(b) Working to three decimal places throughout, determine the area of the region enclosed by all three functions. (5 marks)

Solution
$f(x) = g(x)$ when $x = 4.760$ $\int_0^{4.760} g(x) - h(x)\,dx = 5.972$ $g(x) = h(x)$ when $x = 13.329$ $\int_{4.760}^{13.329} f(x) - h(x)\,dx = 10.752$ Area $= 5.972 + 10.752 = 16.724$ sq units
Specific behaviours
✓ writes first integral ✓ evaluates first integral ✓ writes second integral ✓ evaluates second integral ✓ total area *(Rounding instruction supplied for guidance only)*

Do not penalise if the students did not state explicitly the 5.972 and 10.752.

Check that their integrals work and have the correct bounds.

Question 12 **(3 marks)**

The Richter magnitude, M, of an earthquake is determined from the logarithm of the amplitude, A, of waves recorded by seismographs.

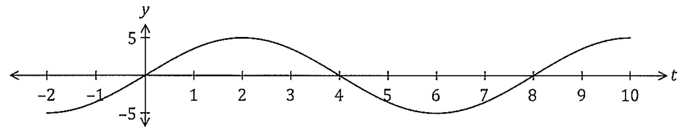
$$M = \log_{10}\frac{A}{A_0}, \text{ where } A_0 \text{ is a reference value.}$$

In January 1995, an earthquake in the city of Kobe, Japan was estimated at 7.2 on the Richter scale, while an earthquake in Chino Hills, U.S.A. measured 5.5 on the same scale in July 2008. How many times larger was the amplitude of the waves in Kobe compared to those at Chino Hills?

Solution
Let A_K be the amplitude of the earthquake in Kobe, and A_{CH} be the amplitude of the earthquake in Chino Hills. $M = \log_{10}\frac{A}{A_0}$ $A = A_0 \times 10^M$ $\frac{A_K}{A_{CH}} = \frac{10^{7.2}}{10^{5.5}} = 10^{1.7}$ ≈ 50.1187 (Decimal approximation is fine) ∴ The amplitude of the waves in Kobe was $10^{1.7}$ times greater than those at Chino Hills.
Specific behaviours
✓ converts log statement to index form ✓ subtracts Richter magnitudes ✓ determines the ratio of the amplitudes

Question 20 **(7 marks)**

The graph of $y = f(t)$ is shown below, where $f(t) = 5\sin\left(\frac{\pi t}{4}\right)$.



(a) Determine the exact area between the horizontal axis and the curve for $0 \le t \le 4$. (2 marks)

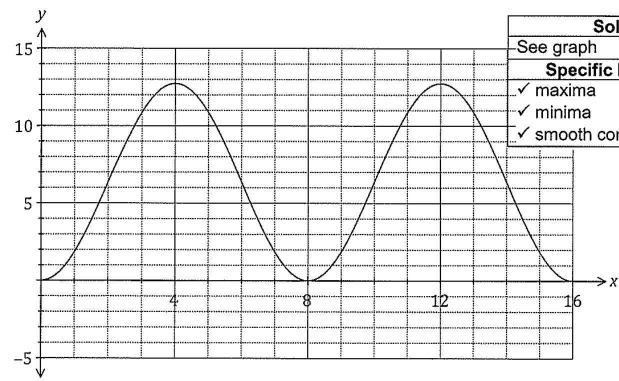
Solution
$\int_0^4 5\sin\left(\frac{\pi t}{4}\right) = \frac{40}{\pi}$ ← Must be exact.
Specific behaviours
✓ writes integral ✓ evaluates

Another function, F, is defined as $F(x) = \int_0^x f(t)\,dt$ over the domain $0 \le x \le 16$.

(b) Determine the value(s) of x for which $F(x)$ has a maximum and state the value of $F(x)$ at this location. (2 marks)

Solution
$x = 4, x = 12, \quad F(4) = F(12) = \frac{40}{\pi}$
Specific behaviours
✓ values of x ✓ value of $F(x)$

(c) Sketch the graph of $y = F(x)$ on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ maxima ✓ minima ✓ smooth continuous curve

Question 19 **(5 marks)**

An isosceles triangle has an area K, given by the equation $K = \frac{1}{2}r^2 \sin\theta$, where r is the length of each equal side and θ is the angle between these two equal sides.

(a) Use the incremental formula to approximate the increase in K, as θ changes from $\frac{\pi}{4}$ to 0.3π in a triangle with side length of r = 4 cm. (3 marks)

Solution

$$K = \frac{1}{2}r^2 \sin\theta, \quad \frac{dK}{d\theta} = 8\cos\theta \ \ when\ r = 4$$

$$\delta K = \frac{dK}{d\theta} \times \delta\theta$$

$$\delta K = 8\cos\frac{\pi}{4} \times 0.05\pi \ when\ \theta = \frac{\pi}{4}$$

$$\delta K = 0.889 \textbf{ OR } \frac{\sqrt{2}\pi}{5}\ (Exact)$$

Specific behaviours

✓ differentiates K
✓ substitutes into the increments formula
✓ determines δK

(b) Determine the exact increase in K and hence determine the percentage error in your approximation from (a). Give your answer to one decimal place. (2 marks)

Solution

$$\Delta K = \frac{1}{2} \times 4^2 \times (\sin 0.3\pi - \sin\frac{\pi}{4})$$

$$\Delta K = 8\left(\sin 0.3\pi - \frac{\sqrt{2}}{2}\right) \textbf{ OR } 2\sqrt{5} + 2 - 4\sqrt{2}$$ ← Must be exact

$$Error = \frac{(2\sqrt{5} + 2 - 4\sqrt{2}) - \frac{\sqrt{2}\pi}{5}}{2\sqrt{5} + 2 - 4\sqrt{2}} \times 100 = 9.0\%$$ ← Rounding penalty here.

Specific behaviours

✓ shows the increase in K
✓ determines the percentage error

Question 13 **(7 marks)**

A fuel storage tank, initially containing 430 L, is being filled at a rate given by

$$\frac{dV}{dt} = \frac{t^2(120 - 3t)}{200}, \quad 0 \le t \le 40$$

where V is the volume of fuel in the tank in litres and t is the time in minutes since filling began. The tank will be completely full after 40 minutes.

(a) Calculate the volume of fuel in the tank after 20 minutes. (3 marks)

Units can be deducted here.

Solution

$$\Delta V = \int_0^{20} V'(t)\,dt$$
$$= 1000$$

$$V = 430 + 1000 = 1\,430 \text{ L}$$

Specific behaviours

✓ indicates use of integral of rate of change
✓ calculates increase
✓ states volume

(b) Determine the time taken for the tank to fill to one-quarter of its maximum capacity. (4 marks)

Solution

$$V = 430 + \int_0^{40} V'(t)\,dt$$
$$= 430 + 3200 = 3630$$

$$V(T) = \int_0^T V'(t)\,dt = \frac{T^3}{5} - \frac{3T^4}{800} + 430$$

$$\frac{T^3}{5} - \frac{3T^4}{800} + 430 = \frac{3630}{4}$$

$$T = 14.9 \text{ minutes}$$

Specific behaviours

✓ calculates V_{MAX}
✓ indicates $V(T)$
✓ indicates equation
✓ solves for time

Question 14 **(7 marks)**

The monthly profit, P thousand dollars, of a retail store is modelled by $P = t\ln\left(\frac{t}{2}\right)$ for $0 < t \leq 24$ where t is the time in months after establishing the store.

a) Find the instantaneous rate of change of profit with respect to time when $t = 2$. (2 marks)

Solution

$$\frac{dP}{dt} = \ln\frac{t}{2} + 1$$
$$= \ln\frac{2}{2} + 1$$
$$= 1 \ (\$1000\ per\ month)$$

If left as 1, that is fine.

Specific behaviours

✓ differentiates to find the rate of change of P with respect to t
✓ determines the instantaneous rate of change at $t = 2$

b) Determine the maximum rate of change of profit with respect to time. (2 marks)

Solution

$$\frac{dP}{dt} = \ln\frac{t}{2} + 1$$
$$= \ln\frac{24}{2} + 1$$

∴ The maximum rate of change will be \$3484.91 *per month*

Specific behaviours

✓ uses the right endpoint to find the maximum rate of change
✓ states the maximum rate of change

c) Find the largest loss that the store experienced and when it occurred. (3 marks)

Solution

$$\frac{dP}{dt} = \ln\frac{t}{2} + 1$$
$$0 = \ln\frac{t}{2} + 1$$
$$t = 2e^{-1}$$

$$\frac{d^2P}{dt^2} = \frac{1}{t} = \frac{1}{2e^{-1}}$$

$$P = 2e^{-1} \times \ln\frac{2e^{-1}}{2} = -0.735759$$

0.73576 months is fine.

∴ The largest loss will occur during the first month and will be a loss of \$735.76

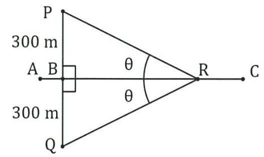
A final statement must be written. Students should not leave their value as −\$735.76.

Specific behaviours

✓ determines the stationary point
✓ substitutes the stationary point into the profit equation
✓ states the largest loss in profit and when it occurs

Question 18 **(7 marks)**

Two houses, P and Q, are 600 m apart on either side of a straight railway line AC. AC is the perpendicular bisector of PQ and the midpoint of PQ is B. A small train, R, leaves station C and travels towards B, 1000 m from C.



Let $\angle PRB = \angle QRB = \theta$, where $0 < \theta < 90°$, and $X = PR + QR + CR$, the sum of the distances of the train from the houses and station.

(a) By forming expressions for PR, BR and CR, show that $X = 1000 + \dfrac{300(2-\cos\theta)}{\sin\theta}$. (3 marks)

Solution

$$PR = \frac{300}{\sin\theta}, \quad BR = PR\cos\theta = \frac{300\cos\theta}{\sin\theta}, \quad CR = 1000 - BR = 1000 - \frac{300\cos\theta}{\sin\theta}$$

$$X = 2\times\frac{300}{\sin\theta} + 1000 - \frac{300\cos\theta}{\sin\theta} = 1000 + \frac{300(2-\cos\theta)}{\sin\theta}$$

$\frac{300}{\tan\theta}$ is fine. However they must convert it to $\frac{\sin\theta}{\cos\theta}$ for the final mark.

Specific behaviours

✓ expression for PR in terms of θ
✓ expressions for BR and CR in terms of θ
✓ expression for X in terms of θ

(b) Use a calculus method to determine the minimum value of X. (4 marks)

Solution

$$\frac{dX}{d\theta} = 300\left(\frac{\sin\theta\times\sin\theta - (2-\cos\theta)(\cos\theta)}{\sin^2\theta}\right)$$
$$= 300\left(\frac{\sin^2\theta + \cos^2\theta - 2\cos\theta}{\sin^2\theta}\right)$$
$$= 300\left(\frac{1-2\cos\theta}{\sin^2\theta}\right)$$

$$\frac{dX}{d\theta} = 0 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$$

$$X_{MIN} = 1000 + \frac{300(2-\cos\theta)}{\sin\theta} = 1000 + 300\left(\frac{3}{2}\right)\times\frac{2}{\sqrt{3}} = 1000 + 300\sqrt{3}\text{ m}$$

≈ 1519.6152 to 4 d.p.

Specific behaviours

✓ uses quotient rule
✓ simplifies derivative
✓ roots of derivative
✓ minimum value of X_{MIN}

The decimal approximation is fine here.

(d) Determine $\frac{dh}{dt}$ when the height of the balloon is 17.92 km. (3 marks)

Solution

$h(t) = 17.92 \Rightarrow t = 48$

$$\frac{dh}{dt} = \frac{180t - 3t^2}{5400}$$

$$= \frac{180(48) - 3(48)^2}{5400} = \frac{8}{25} = 0.32 \text{ km/min}$$

Specific behaviours

✓ determines time
✓ indicates derivative
✓ determines rate of change

(e) Determine $\frac{dP}{dt}$ when the height of the balloon is 17.92 km. (3 marks)

Solution

$$\frac{dP}{dh} = -0.128 \times 101.3e^{-0.128(17.92)}$$
$$= -1.308$$

$$\frac{dP}{dt} = \frac{dP}{dh} \times \frac{dh}{dt}$$
$$= -1.308 \times 0.32$$
$$= -0.4186 \text{ kPa/min}$$

Specific behaviours

✓ rate of change of P wrt h
✓ indicates use of chain rule
✓ correct rate of change

Question 15 **(9 marks)**

A particle starts from rest at O and travels in a straight line.

Its velocity v ms^{-1}, at time t s, is given by $v = 14t - 3t^2$ for $0 \le t \le 4$ and $v = 128t^{-2}$ for $t > 4$.

(a) Determine the initial acceleration of the particle. (2 marks)

Solution

$$a = \frac{dv}{dt} = 14 - 6t \Rightarrow a(0) = 14 \text{ ms}^{-2}$$

Specific behaviours

✓ differentiates velocity
✓ acceleration

Units can be deducted in this question

(b) Calculate the change in displacement of the particle during the first four seconds. (2 marks)

Solution

$$x = \int_0^4 14t - 3t^2 \, dt = 48 \text{ m}$$

Specific behaviours

✓ integrates velocity
✓ change in displacement

(c) Determine, in terms of t, an expression for the displacement, x m, of the particle from O for $t > 4$. (2 marks)

Solution

$$x = \int \frac{128}{t^2} dt = -\frac{128}{t} + c$$

$$x(4) = 48 = -\frac{128}{4} + c \Rightarrow c = 80$$

$$x = -\frac{128}{t} + 80$$

Specific behaviours

✓ integrates velocity
✓ evaluates c

(d) Determine the distance of the particle from O when its acceleration is -0.5 ms^{-2} and $t > 4$. (3 marks)

Solution

$$a = -\frac{256}{t^3}$$

$$-\frac{256}{t^3} = -0.5 \Rightarrow t = 8$$

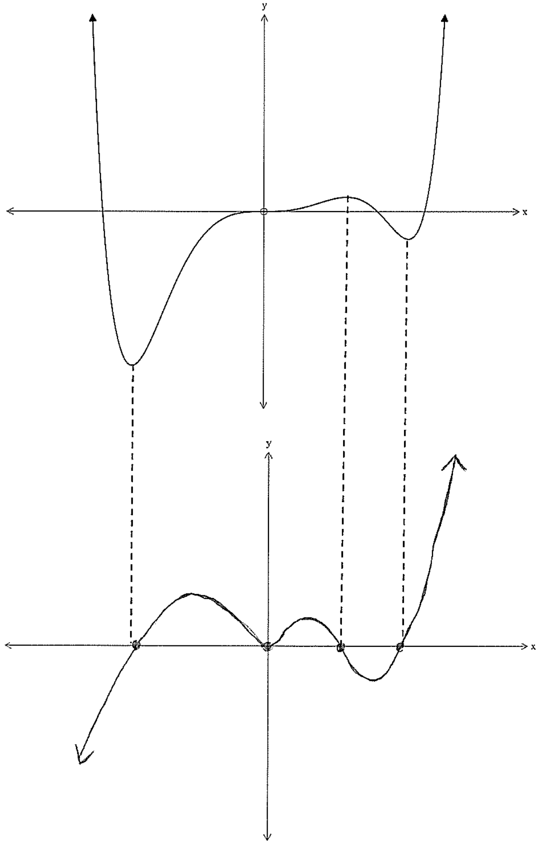
$$x(8) = 64 \Rightarrow \text{Distance from } O = 64 \text{ m}$$

Specific behaviours

✓ acceleration for $t > 4$
✓ solves for time
✓ calculates distance

Question 16 **(3 marks)**

Below is the graph of $f(x)$. Using the axes below, construct the graph of $f'(x)$.



Solution
See graph
Specific behaviours
✓ Roots match the stationary points from f(x) ✓ Turning points match the points of inflection from f(x) ✓ Smooth curve joining all points

Question 17 **(11 marks)**

The air pressure, $P(h)$ in kPa, experienced by a weather balloon varies with its height above sea level h km and is given by

$$P(h) = 101.3e^{-0.128h}, 0 \leq h \leq 20 .$$

(a) Determine $\frac{dP}{dh}$ when the height of the balloon is 1.8 km. (2 marks)

Solution
$\frac{dP}{dh} = -0.128 \times 101.3e^{-0.128(1.8)}$ $= -10.3$ kPa/km
Specific behaviours
✓ uses derivative ✓ correct rate of change

Units can be deducted here for this entire question.

(b) What is the meaning of your answer to (a). (1 mark)

Solution
The rate of change of pressure with respect to height when the height is 1.8 km.
Specific behaviours
✓ meaning *(must include wrt h and refer to height)*

This statement must be written to get this mark.

The height of the balloon above sea level varies with time t minutes and is given by

$$h(t) = \frac{t^2(90-t)}{5400}, 0 \leq t \leq 60 .$$

(c) Determine the air pressure experienced by the balloon when $t = 42$. (2 marks)

Solution
$h(48) = 15.68$ km $P(15.68) = 13.61$ kPa
Specific behaviours
✓ determines height ✓ determines pressure