Mercedes College

WA Exams Practice Paper E, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ______________________________________

Your name ______________________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	12	12	100	98	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Additional working space

Question number: _________

Additional working space

Question number: __________

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(6 marks)**

Water flows into a tank, initially holding 125 L, at a rate given by $V'(t) = \dfrac{24t - t^2}{3}$ for $0 \le t \le 24$, where $V'(t)$ is measured in litres per hour and t is in hours.

(a) Determine how much water is in the tank after 24 hours. (3 marks)

$$V = 125 + \int_0^{24} \left(\frac{24t - t^2}{3} \right) dt$$

$$= 125 + 768$$

$$= 893 \text{ L}$$

(b) Determine the time it takes for the tank to fill to 500 L. (3 marks)

$$500 = 125 + \int_0^{k} \left(\frac{24t - t^2}{3} \right) dt$$

$$375 = 4k^2 - \frac{k^3}{9}$$

$$k = 11.81 \text{ h}$$

Question 9 **(8 marks)**

The temperature, T °C, of a bronze casting t seconds after being removed from an oven was modelled for $0 \le t \le 800$ by $T = T_0 e^{-0.0034t}$.

(a) How long, to the nearest second, did it take for the initial temperature of the casting to halve? (3 marks)

$$e^{-0.0034t} = 0.5$$
$$t = 203.867$$
$$\approx 204 \text{ seconds}$$

(b) Determine the initial temperature of the casting, given that it had cooled to 787°C after one minute. (2 marks)

1 minute = 60 seconds

$$787 = T_0 e^{-0.0034(60)}$$
$$T_0 = 965.097 \approx 965°C$$

(c) Can the above rate of change model be used to calculate how long it takes the temperature of the casting to fall below 40°C? Explain your answer. (3 marks)

No.

$$40 = 965.097 e^{-0.0034t}$$
$$t = 936 \text{ seconds}$$

The model states $0 \le t \le 800$, but the model predicts it will take 936 seconds which is outside this domain and so may be unreliable.

Additional working space

Question number: _________

Question 19 **(9 marks)**

The discrete random variable X has the probability distribution shown in the table.

x	1	2	3	4	5
$P(X = x)$	b	a^2	a	0.14	0.05

(a) Determine the values of a and b, if $E(X) = 2.58$. (4 marks)

$$b + a^2 + a + 0.14 + 0.05 = 1$$
$$b = 0.81 - a^2 - a$$

$$b + 2a^2 + 3a + 4(0.14) + 5(0.05) = 2.58$$
$$b + 2a^2 + 3a - 1.77 = 0$$

$$\left(0.81 - a^2 - a\right) + 2a^2 + 3a - 1.77 = 0$$
$$a^2 + 2a - 0.96 = 0$$
$$a = \cancel{-2.4}, \ 0.4$$

$$b = 0.25$$

(b) Determine $P(X \le 4 \mid X \ge 4)$. (1 mark)

$$\frac{0.14}{0.14 + 0.05} = \frac{14}{19}$$

(c) Calculate the values of

(i) $Var(X)$. (2 marks)

	list1	list2
1	1	0.25
2	2	0.16
3	3	0.4
4	4	0.14
5	5	0.05

sum((list1−2.58)^2×list2
1.3236

$Var(X) = 1.3236$

(ii) $Var(10X + 3)$. (1 mark)

$$1.3236 \times 10^2 = 132.36$$

(iii) $Var(3 - 2X)$. (1 mark)

$$1.3236 \times (-2)^2 = 5.2944$$

End of questions

Question 10 **(8 marks)**

A pottery produces souvenir coffee mugs, of which it is known that 5% are defective and the rest are good.

(a) In a box of 12 mugs, what is the probability that there are no defectives? (2 marks)

$Y \sim B(10, 0.05)$

$P(Y = 0) = 0.5404$

(b) In a box of 24 mugs, what is the probability that there are at least 4 defectives? (2 marks)

$X \sim B(24, 0.05)$

$P(X \ge 4) = 0.0298$

(c) What is the probability that in 10 boxes, each containing 12 mugs, that either two or three of the boxes contain no defectives? (2 marks)

$W \sim B(10, 0.54036)$

$P(2 \le W \le 3) = 0.1082$

(d) The pottery decides to pack n mugs per box for wholesale clients, so that the chance of there being at least one defective mug in a box is no more than 75%. Find the largest value of n. (2 marks)

$(1 - 0.05)^n > 1 - 0.75$

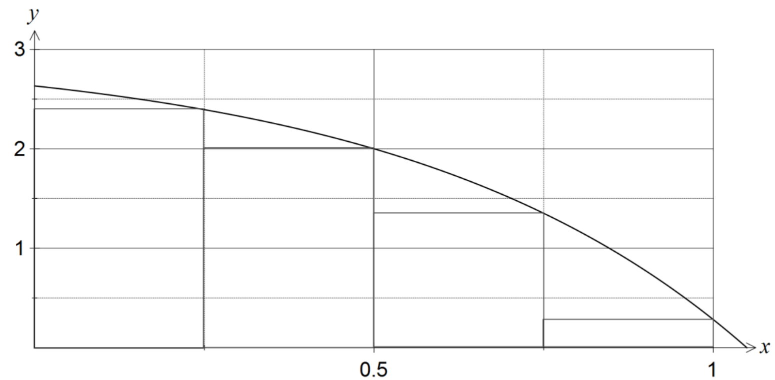
$n < 27.03$

Hence, no more than 27 mugs per box.

See next page

Question 11 **(7 marks)**

The graph below shows the function $f(x) = 3 - e^{2x-1}$.



An estimate is required for the area under the curve between $x = 0$ and $x = 1$, using the average of inscribed rectangles (shown above) and circumscribed rectangles (not shown).

(a) Complete the table below, rounding values to two decimal places. (2 marks)

x	0	0.25	0.5	0.75	1
$f(x)$	**2.63**	**2.39**	2.00	**1.35**	**0.28**

(b) Use the right-rectangles shown to calculate an under-estimate for the area. (2 marks)

$$A = 0.25 \times (2.39 + 2 + 1.35 + 0.28)$$
$$= 1.505 \text{ sq u}$$

(c) Use four left-rectangles to calculate an over-estimate for the area. (2 marks)

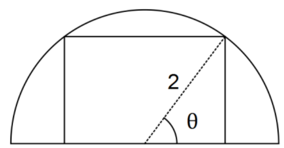
$$A = 0.25 \times (2.63 + 2.39 + 2 + 1.35)$$
$$= 2.0925 \text{ sq u}$$

(d) Use your over- and under- estimates to calculate an estimate for the area under the curve between $x = 0$ and $x = 1$. (1 mark)

$$A = \frac{1.505 + 2.0925}{2} = 1.79875 \approx 1.8 \text{ sq u}$$

Question 18 **(8 marks)**

A rectangle is inscribed in a semicircle of radius 2 metres, as shown in the diagram.



(a) Show that the perimeter of the rectangle is given by $4\sin\theta + 8\cos\theta$. (3 marks)

$$h = 2\sin\theta$$

$$w = 2 \times 2\cos\theta = 4\cos\theta$$

$$P = 2h + 2w$$
$$= 4\sin\theta + 8\cos\theta$$

(b) Use calculus methods to determine the maximum perimeter of the rectangle, and state the dimensions of the rectangle to achieve this maximum. (5 marks)

$$\frac{dP}{d\theta} = 4\cos\theta - 8\sin\theta$$

$$4\cos\theta - 8\sin\theta = 0 \Rightarrow \theta = \tan^{-1}\frac{1}{2} \approx 0.4636$$

$$h = 2\sin(\tan^{-1}\frac{1}{2}) = \frac{2\sqrt{5}}{5} \approx 3.58$$
$$w = 4\cos(\tan^{-1}\frac{1}{2}) = \frac{8\sqrt{5}}{5} \approx 0.89$$

$$P_{MAX} = 2h + 2w = 4\sqrt{5} \approx 8.94 \text{ m}$$

Question 17 **(9 marks)**

In the mailroom of a large company it is known that 20% of incoming letters contain a cheque. Let X be the number of randomly chosen letters that are opened until a cheque is discovered.

(a) Complete the table below for the values of $x = 1,\ 2,\ 3$ and 4. (2 marks)

x	1	2	3	4
$P(X = x)$	**0.2**	**0.16**	**0.128**	**0.1024**

(b) Determine a rule for $P(X = x)$ for any integer value of x greater than 0. (2 marks)

$$P(X = x) = 0.2(0.8)^{x-1}$$

(c) Calculate

(i) $P(X = 10)$. (1 mark)

$$P(X = 10) = 0.2(0.8)^9 = 0.0268$$

(ii) $P(3 \le X \le 6)$. (2 marks)

$$P(X = 5) = 0.2(0.8)^4 \approx 0.0819$$
$$P(X = 6) = 0.2(0.8)^5 \approx 0.0655$$
$$P(3 \le X \le 6) \approx 0.128 + 0.1024 + 0.0819 + 0.0655$$
$$\approx 0.3778$$

Alternatively, use CAS GeoCDf function:

```
geoCDf(3,6,0.2)
                0.37785
```

(iii) the smallest value of k, so that $P(X = k) < 0.001$. (2 marks)

$$0.2(0.8)^{x-1} < 0.001$$
$$x < 24.74 \Rightarrow k = 25$$

Question 12 **(8 marks)**

A small body travels in a straight line with acceleration given by $a = 2e^{-0.1t}$ ms^{-2}.

The body had an initial displacement of 250 metres relative to a fixed point O, at which time its velocity was 10 ms^{-1}.

(a) Determine the velocity of the body after one second, rounded to three significant figures. (3 marks)

$$v = \int 2e^{-0.1t}\,dt$$
$$= -20e^{-0.1t} + c$$
$$t = 0, v = 10 \Rightarrow c = 30$$
$$v = 30 - 20e^{-0.1t}$$
$$v(1) = 30 - 20e^{-0.1} \approx 11.9 \text{ m/s}$$

Using CAS:

$$10+\int_0^1 2e^{-0.1t}dt$$
11.90325164

(b) Determine an equation for the displacement of the body at time t. (2 marks)

$$x = \int \left(30 - 20e^{-0.1t}\right) dt$$
$$= 30t + 200e^{-0.1t} + c$$
$$t = 0,\ x = 250 \Rightarrow c = 50$$
$$x = 30t + 200e^{-0.1t} + 50$$

(c) Calculate the change in displacement of the body during the sixth second. (3 marks)

From $t = 5$ to $t = 6$.

$$\Delta x = \int_5^6 \left(30 - 20e^{-0.1t}\right) dt$$
$$\approx 18.46 \text{ m}$$

Question 13 **(7 marks)**

A cubical six-sided dice is known to be biased. It is thrown 3 times and the total number of sixes is noted. This experiment is then repeated 200 times, with the results shown in this table.

Number of sixes	0	1	2	3
Frequency	67	93	33	7

(a) What is the mean number of sixes per experiment? (2 marks)

$$\bar{x} = \frac{0\times 67 + 1\times 93 + 2\times 33 + 3\times 7}{200}$$

$$\bar{x} = 0.9$$

(b) Name a suitable discrete probability distribution to model the number of sixes obtained in one experiment. (1 mark)

Binomial distribution.

(c) What is the probability of obtaining a six when this dice is thrown? (2 marks)

X is number of sixes in 3 throws of the dice.

$X \sim \mathrm{B}(3,\ p),\ E(X) = np$

$3p = 0.9 \Rightarrow p = 0.3$

(d) Use the distribution from (b) to calculate the expected number of times that no sixes would occur in 200 experiments and comment on how well your answer agrees with the experimental result above. (2 marks)

If $X \sim \mathrm{B}(3,\ 0.3)$ then

$200 \times P(X = 0) = 200 \times 0.343 \approx 69$.

The expected number of 69 is very close to the experimental result of 67, and so the binomial model looks to be appropriate.

(c) Determine the acceleration of the body when $t > 0$ and it has a velocity of 27 cm/s. (5 marks)

$$\begin{aligned} v(t) &= 2t^2 - 5t + 2 + (t-5)(4t-5) \\ &= 2t^2 - 5t + 2 + 4t^2 - 25t + 25 \\ &= 6t^2 - 30t + 27 \end{aligned}$$

$6t^2 - 30t + 27 = 27$

$6t(t-5) = 0 \Rightarrow t = \cancel{0},\ 5$

$$\begin{aligned} a(t) &= v'(t) \\ &= 12t - 30 \end{aligned}$$

$a(5) = 12(5) - 30 = 30$ cm/s^2

Question 16 **(10 marks)**

The displacement, in centimetres, of a small body from a fixed point O after t seconds is given by $x(t)$, where $x(t)=(t-a)(2t^2-5t+2)$, where a is a constant.

(a) Show that the body is at O when $t=\frac{1}{2}$. (1 mark)

$$x\left(\frac{1}{2}\right)=\left(\frac{1}{2}-a\right)\left(2\left(\frac{1}{2}\right)^2-5\left(\frac{1}{2}\right)+2\right)$$
$$=\left(\frac{1}{2}-a\right)\times 0=0$$

(b) Given that the body has a velocity of 3 cm/s when $t=1$, determine the value of the constant a. (4 marks)

$$v(t)=x'(t)$$
$$=(1)\left(2t^2-5t+2\right)+(t-a)(4t-5)$$

$$v(1)=3$$
$$3=(2-5+2)+(1-a)(-1)$$

$$3=-1-1+a$$
$$a=5$$

Question 14 **(9 marks)**

A closed cylindrical can of radius r cm has a volume of 250π cm^3.

(a) Show that the total surface area, A cm^2, of this can is given by $A=\frac{500\pi}{r}+2\pi r^2$. (3 marks)

$$V=\pi r^2 h$$
$$250\pi=\pi r^2 h \Rightarrow h=\frac{250}{r^2}$$

$$A=2\pi r^2+2\pi rh$$
$$=2\pi r^2+2\pi r\frac{250}{r^2}$$
$$=\frac{500\pi}{r}+2\pi r^2$$

(b) Use derivatives to determine the minimum possible surface area of the can, justifying that it is a minimum, and state the radius and height required to achieve this minimum area. (6 marks)

$$\frac{dA}{dr}=-\frac{500\pi}{r^2}+4\pi r$$

$$-\frac{500\pi}{r^2}+4\pi r=0$$
$$r^3=125 \Rightarrow \underline{r=5\text{ cm}}$$

$$\frac{d^2A}{dr^2}=\frac{1000\pi}{r^3}\bigg|_{r=5}>0$$

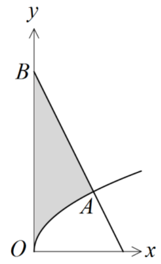
$\Rightarrow$ concave up, so a minimum

$$h=\frac{250}{5^2}=\underline{10\text{ cm}}$$

$$A=\frac{500\pi}{5}+2\pi\times 5^2$$
$$\underline{A=150\pi}$$

Question 15 **(9 marks)**

The diagram below shows the graph of the function $y=\sqrt{x}$ and the straight line AB that is perpendicular to the curve at A, where $x=4$.



(a) Determine the equation of AB. (3 marks)

$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$

$\frac{1}{2\sqrt{4}}=\frac{1}{4} \Rightarrow m_{AB}=-4$

$y-2=-4(x-4) \Rightarrow y=-4x+18$

(b) Determine the shaded area in the diagram, enclosed by the curve $y=\sqrt{x}$, the straight line AB and the y-axis. (3 marks)

$A=\int_0^4\left((18-4x)-\sqrt{x}\right)\,dx$

$=\frac{104}{3}$

$=34.\overline{6}$ sq u

(c) Determine the area enclosed by the curve $y=\sqrt{x}$, the straight line AB and the x-axis. (3 marks)

AB cuts x-axis at $x=4.5$.

Area of triangle is $\frac{1}{2}\times 4.5\times 18=\frac{81}{2}$.

Enclosed area is $\frac{81}{2}-\frac{104}{3}=\frac{35}{6}=5.8\overline{3}$