Papers written by
Australian Maths
Software

SEMESTER TWO

REVISION 3

MATHEMATICS METHODS

UNITS 3-4

2016

SOLUTIONS

SECTION ONE

1. (9 marks)

(a) (i) After the first month.

(ii) $P(t) = ln(t)$

(b) $log_x 9 = -2$ for $x \geq 0$.

$$9 = x^{-2}$$

$$\frac{1}{x^2} = 9$$

$$x^2 = \frac{1}{9}$$

$x = \frac{1}{3}$ as $x \geq 0$

(c)
$$\frac{(log_a 2 + log_a 4) \times (log_a 3^2)}{2\, log_a 9 \times (log_a 2 - log_a 1)}$$

$$= \frac{(log_a 8) \times (log_a 3^2)}{2\, log_a 9 \times (log_a 2)}$$

$$= \frac{3\cancel{(log_a 2)} \times 2\cancel{(log_a 3)}}{4\cancel{log_a 3} \times \cancel{(log_a 2)}}$$

$$= \frac{3}{2}$$

2. (14 marks)

(a) (i)
$$f(x) = ln\left(\sqrt{e^{-2x}}\right) = ln\left(\left(e^{-2x}\right)^{\frac{1}{2}}\right) = ln\, e^{-x} = -x \times 1 = -x$$

$$f'(x) = -1$$

(ii)
$$g(x) = \frac{ln(x)}{x^2}$$

$$g'(x) = \frac{\frac{1}{x}(x^2) - 2x(ln(x))}{x^4}$$

$$g'(x) = \frac{x(1 - 2\,ln(x))}{x^4}$$

$$g'(x) = \frac{1 - 2\,ln(x)}{x^3}$$

(iii) $h(x) = (e^x)\cos(2x)$

$h'(x) = (e^x)\cos(2x) + (-2\sin(2x))(e^x)$

$h'(x) = e^x(\cos(2x) - 2\sin(2x))$

(b) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 3\,cm^3s^{-1} \qquad \frac{dr}{dt} = ? \quad at \quad r = 2\,cm$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$3 = 4\pi(2)^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{3}{16\pi}\,cm\,s^{-1}$

(c) (i) $k(x) = \sqrt{1+x^4}$ and $m(x) = \left(\frac{x}{2}\right)^2 - 4$

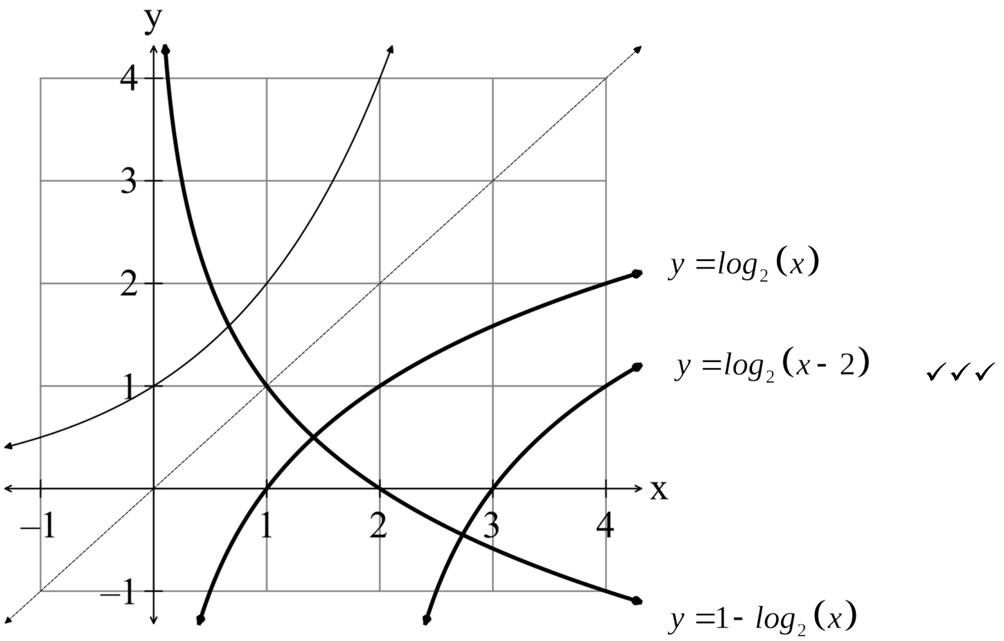
(ii) $k(x) = \sqrt{1+x^4}$ $\qquad$ $m(x) = \left(\frac{x}{2}\right)^2 - 4$

$k'(x) = \frac{1}{2}(1+x^4)^{-\frac{1}{2}}(4x^3)$ $\qquad$ $m(x) = \frac{x^2}{4} - 4$

$k'(x) = \frac{2x^3}{\sqrt{1+x^4}}$ $\qquad$ $m'(x) = \frac{x}{2}$

3. (7 marks)

(a) (i) (ii) (iii)



4. (9 marks)

(a) (i) $\int (3y-5)^{-2}\,dy = \dfrac{(3y-5)^{-1}}{-1\times 3} + c = -\dfrac{1}{3(3y-5)} + c$

(ii) $\int_{\pi/4}^{\pi/6} cos^{-2}(x)dx = \int_{\pi/4}^{\pi/6} sec^2(x)dx = \left[tan(x)\right]_{\pi/4}^{\pi/6} = tan\left(\pi/6\right) - tan\left(\pi/4\right) = \dfrac{1}{\sqrt{3}} - 1$

(iii) $\int_2^3 \left(x^2 + x + 1 + \dfrac{1}{x}\right)dx$

$= \left[\dfrac{x^3}{3} + \dfrac{x^2}{2} + x + ln(x)\right]_2^3$

$= (9 + 4.5 + 3 + ln(3)) - \left(\dfrac{8}{3} + 2 + 2 + ln(2)\right)$

$= 16.5 - 6\dfrac{2}{3} + ln\left(\dfrac{3}{2}\right)$

$= 9\dfrac{5}{6} + ln\left(\dfrac{3}{2}\right)$

(b) $g'(x) = e^{-x}$

$g(x) = \int e^{-x}dx$

$g(x) = \dfrac{e^{-x}}{-1} + c$

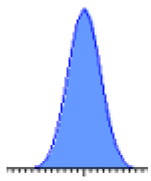
Given $g(0) = -1$

$-1 = -e^0 + c \;\rightarrow\; c = 0$

$g(x) = -e^{-x}$

5. (11 marks)

(a)

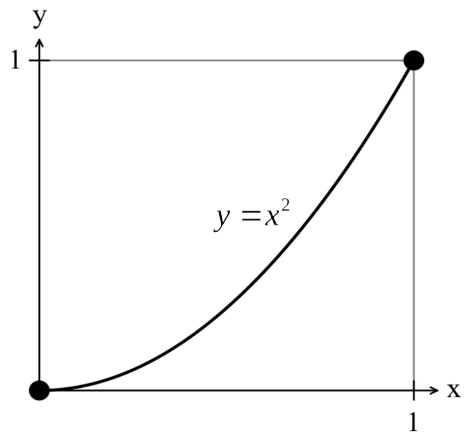


(b) $f(x) = \begin{cases} x^2 \text{ for } 0 \le x \le 1 \\ 0 \quad \text{otherwise} \end{cases}$

This function is NOT a continuous probability density function as it is contained inside a square of area one.
Therefore the area under the curve is less than one.

Alternatively you can show that

$\int_0^1 x^2 dx = \frac{1}{3}$ ✓✓✓



(c) (i) Each sample has a mean which is the average of the sample. The average will be closer to the population mean than independent samples as it will not include outliers and be the average of the highs and lows.
Similarly, the mean of the means will be an even closer approximation to the population mean. ✓✓

(ii) The means of the samples are closer to the population mean than the individual scores. The mean of the means will be even more closely clustered. This means that the standard deviation of the sampling distribution will be very small.✓✓

(iii) Different types of bias when conducting a survey include
Selection bias (not a random sample)
Design flaw bias (no loaded questions) ✓✓ for any 3
Interviewer bias
Recall/reporting bias
Completion (lack of) bias
Non reponse bias (do not take part when selected)

END OF SECTION ONE

SECTION TWO

6. (4 marks)

(a) $A = \int_1^3 \frac{1}{x}\,dx = 1.099$

(b) Need the equation of the line.

$m = e - 1$

$y = (e - 1)x + 1$

Area $= \int_0^1 \left((e-1)x + 1 - e^x\right)dx$ ✓

7. (10 marks)

(a) (i) $f'(x) = e^x$ and $f''(x) = e^x$

(ii) There are no turning points on this graph, so $f'(x) \neq 0$.

The function f is a continuous function which given $f'(x) \neq 0$ implies the gradient of f is either always positive or always negative.

It can be seen that the function is always increasing, so it is always positive.

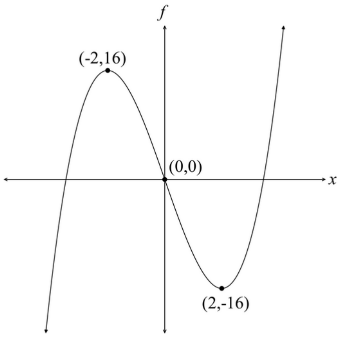
The graph of $y = f'(x)$ is also always positive which confirms the gradient of f is either always positive.

The graph of $y = f''(x)$ is also always positive which means the concavity of f is constant and is concave upwards. There are no points of inflection which require $f''(x) = 0$.

So, with $y = f'(x)$ always positive we have an increasing function, together with $y = f''(x)$ always positive, the function f is concave upwards.

✓✓✓

(b) (i)



✓✓✓ -1/error

(ii)

$$f'(x) = (x-2)(x+2) = x^2 - 4$$

$$f(x) = k\left(\frac{x^3}{3} - 4x + c\right) \quad but \;\; c = 0 \;\; as \; (0,0) \in f$$

$$(2,-16) \Rightarrow \quad -16 = k\left(\frac{8}{3} - 8\right) \quad \Rightarrow \;\; k = 3 \qquad \therefore \;\; f(x) = x^3 - 12x$$

8. (6 marks)

(a) $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dr} \approx \frac{\delta V}{\delta r}$$

$$\delta V \approx 4\pi r^2 \times \delta r$$

$$At \;\; r = 1, \quad \delta r = 0.05$$

$$\delta V \approx 4\pi(1)^2 \times 0.05$$

$$\delta V \approx \frac{\pi}{5} cm^3$$

(b) 1.38629 ✓ (= ln(4))

9. (5 marks)

(a) $$\frac{d}{dx}\int_1^{\sqrt{x}} \frac{2}{(1-t^4)^2}\,dt = \frac{2}{\left(1-(\sqrt{x})^4\right)^2} = \frac{2}{(1-x^2)^2}$$

✓✓

(b) (i) $f(x) = \sqrt{x} \;\rightarrow\; f'(x) = \frac{1}{2\sqrt{x}}$

(ii) $$\int_1^4 \left(\frac{1}{2\sqrt{x}}\right) dx = \left[\sqrt{x}\right]_1^4 = 2 - 1 = 1$$

10. (6 marks)

(a) $23 = P_0 e^{k18}$ and $46.9 = P_0 e^{k75}$

$\therefore \dfrac{23}{e^{k18}} = \dfrac{46.9}{e^{k75}} \qquad = P_0$

$\dfrac{e^{k75}}{e^{k18}} = \dfrac{46.9}{23}$

$e^{k57} = \dfrac{46.9}{23}$

$k = 0.0125$

$P_0 = ?$

$23 = P_0 e^{18\times 0.0125}$

$P_0 = 18.3657$

$P = 18.3657 e^{0.0125a}$

(b) $P = 18.3657 e^{0.0125a} \quad \rightarrow \quad P = 44\%$

11. (6 marks)

(a) $x = 3t^2 - 6t\, m,$

$v = 6t - 6$

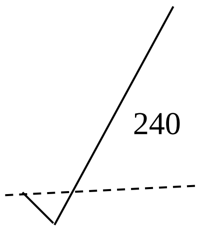
Changes direction at $v = 0$ i.e. at $t = 1$

$x = -3$

(b) $a = 6\, ms^{-2}$

(c) $x_0 = 0 \qquad x_1 = -3 \qquad x_{10} = 300 - 60 = 240$

$\therefore$ Distance travelled $= 3 + 3 + 240 = 246\, m$



12. (8 marks)

(a) $d = 2\sin\left(\dfrac{2\pi}{3}t\right)$

(i) 2 cm up and 2 cm down so 4 cm.

(ii) Period = $\dfrac{2\pi}{2\pi/3} = 3$ seconds

(iii) $1.5 = \dfrac{2\pi}{n} \quad \rightarrow \quad n = \dfrac{4\pi}{3} \quad \rightarrow \quad d = 2\sin\left(\dfrac{4\pi}{3}t\right)$

✓✓

(b) (i) $V = ln(10+3t) m^3$

$V_2 = ln(16) m^3 \approx 2.77 m^3$ ✓

(ii) $t = 3.36184$

12.22 $p.m.$

13. (7 marks)

(a) $2^2 = x^2 + r^2$

$x = \sqrt{4 - r^2}$

$h = x + 2$

$h = \sqrt{4 - r^2} + 2$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi r^2 \left(\sqrt{4 - r^2} + 2\right)$

(b) To determine the dimensions of the cone of maximum volume:

Find expressions for $\frac{dV}{dr}$ and $\frac{d^2V}{dr^2}$ as maximum volume occurs when

$\frac{dV}{dr} = 0$ and $\frac{d^2V}{dr^2} < 0$.

Solve $\frac{dV}{dr} = 0$ and exclude any values of r that are negative or greater than 2.

Test the solution for $\frac{d^2V}{dr^2} < 0$ to ensure you have the maximum volume.

You need the dimensions of the maximum cone so calculate the value of h.

Write a concluding statement giving the dimensions, h and r, that are required for the maximum volume of the cone.

-1 if mention of final statement.

14. (12 marks)

(a) (i)

Score when added	2	3	4	5	6	7	8
P(score)	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{3}{15}$	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$

✓✓✓

(ii) P(the score is odd) = $\frac{7}{15}$

(iii) P(there is at least one odd number) = $\frac{13}{15}$

(iv) P(a score of 6 or 7) = $\frac{5}{15} = \frac{1}{3}$

(b) (i) $E(X)=2\times\frac{1}{15}+3\times\frac{2}{15}+4\times\frac{3}{15}+5\times\frac{3}{15}+6\times\frac{3}{15}+7\times\frac{2}{15}+8\times\frac{1}{15}$ ✓

$E(X)=\frac{1}{15}(2+6+12+15+18+14+8)$

$E(X)=\frac{75}{15}=5$

(ii) $Var(X)=E(X^2)-(E(X))^2$

$Var(X)=2^2\times\frac{1}{15}+3^2\times\frac{2}{15}+4^2\times\frac{3}{15}+5^2\times\frac{3}{15}+6^2\times\frac{3}{15}+7^2\times\frac{2}{15}+8^2\times\frac{1}{15}-5^2$

$Var(X)=\frac{1}{15}(4+18+48+75+108+98+64)-25$

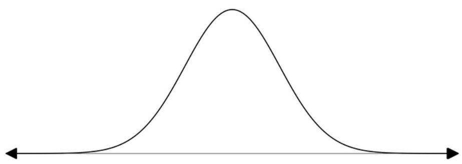
$Var(X)=27\frac{2}{3}-25$

$Var(X)=2\frac{2}{3}$

(c) (i) Not a probability density function as the probabilities do not add to 1. ✓

(ii) Not a probability density function as one of the probabilities is negative. ✓

15. (8 marks)



$\mu=300,\ \sigma=10$ grams

(a) $P(x<280)=0.02275$ ✓✓

(b) $P(x\geq 300\mid x>280)=\frac{P(x\geq 300)}{P(x>280)}=\frac{0.5}{0.97725}=0.51164$

✓✓

(c) B(4, 0,5)

$P(x\geq 3)=0.25+0.0625=0.3125$

16. (6 marks)

(a) $P(5.5 \le x \le 6.5) = 2\int_{5.5}^{6}(x-5)dx = 0.75$

(b) By symmetry, $E(x) = 6$

$$Var(x) = \int_5^6 (x-5)(x-6)^2\,dx + \int_6^7 (-x+7)(x-6)^2 dx$$
$$= \frac{1}{12} + \frac{1}{12}$$
$$= \frac{1}{6}$$

17. (9 marks)

(a) P(on the next 6 Mondays Bill manages to buy a sultana bun) = $\left(\frac{2}{3}\right)^6 = 0.08779$ ✓✓

(b) P(on the next 6 Mondays Bill only manages to buy a sultana bun 4 times out of 6) = 0.32922 ✓✓

(c) P(Bill can buy a sultana bun if on the last three Mondays the shop had run out of buns.) = $\frac{2}{3}$ ✓✓

(d) $P\left(B \cap \overline{B} \cap B \cap \overline{B} \cap B \cap \overline{B}\right) + P\left(\overline{B} \cap B \cap \overline{B} \cap B \cap \overline{B} \cap B\right) = \left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3 \times 2 = 0.021948$

18. (14 marks)

(a) $p = 0.1, q = 0.9 \Rightarrow np = 60 \times 0.1 = 6 > 5$

$nq = 60 \times 0.9 = 54 > 5$ so can use normal distribution.

Mean $= p = 0.1$

$$sd_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.1 \times 0.9}{60}}$$

$$sd_{\hat{p}} = 0.03873$$

Standardised score (using 4.5 to 9.5)

$$z = \frac{X - \mu}{\sigma}$$

$$z_1 = \frac{\frac{4.5}{60} - 0.1}{0.03873} = -0.6455$$

$$z_2 = \frac{\frac{9.5}{60} - 0.1}{0.03873} = 1.5062$$

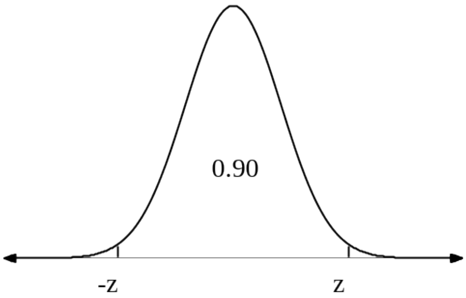
The probability is 0.675

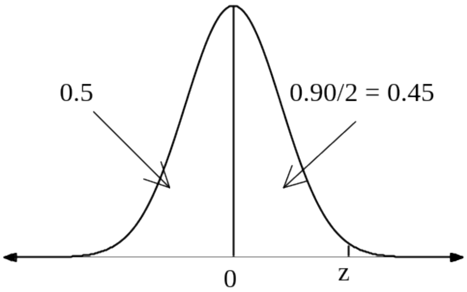
(b) (i) $\hat{p} = \frac{85}{100} = 0.85$

(ii) $\hat{p} = 0.85$

$$sd_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.85 \times 0.15}{100}}$$

$$sd_{\hat{p}} = 0.035707$$





P(X< z) = 0.95

z = 1.645

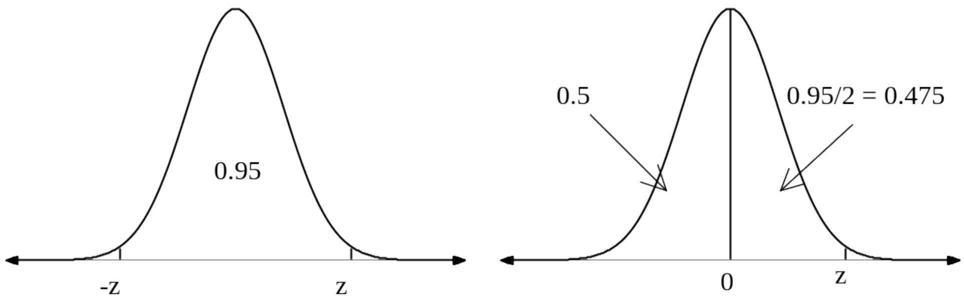
s = 0.035707

$$E = z \times s = 1.645 \times 0.035707$$

$$E \approx 0.059$$

The confidence limit is 0.85 ± 0.0587 *i.e.* $(0.79, 0.91)$

(c)



P(X< z) = 0.975

So, 95% confidence level means z = 1.96

Use $p = 0.5$ as the maximum value as p is unknown.

So with $p = 0.5$ $sd = \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} = \sqrt{\frac{0.25}{n}}$

$E = z \times s$ with $E = 0.10$

Therefore

$0.10 = 1.96 \times \sqrt{\frac{0.25}{n}}$

n = 96.04

$n \approx 96$

Should use a sample size of 96 people to have a confidence level of 95% with an error margin of 10%

END OF SECTION TWO