Hale School
2011

Question/Answer Booklet

MATHEMATICS 3CD
SEMESTER 2
Section One
(Calculator Free)

Booklet 1 of 3

Circle your teacher's initials

GJ JJB BAH VMU

Your name ______________________

Time allowed for this section

Reading time before commencing work: 5 minutes
Working time for paper: 50 minutes

Material required/recommended for this section

To be provided by the supervisor
This Question/answer booklet for Section One.
Formula sheet.

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler.

Important note to candidates

No other items may be taken into the examination room. It is your responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor before reading any further.

Structure of this examination

		Number of questions	Working time (minutes)	Marks available
Booklet 1 This Booklet (Section 1)	**Calculator Free**	**8**	**50**	**40**
Booklet 2 (Section 2)	Calculator Assumed	7	100	40
Booklet 3 (Section 2)		5		40
			Total marks	120

Instructions to candidates

1. The rules for the conduct of WACE external examinations are detailed in the booklet *WACE Examinations Handbook*. Sitting this examination implies that you agree to abide by these rules.
2. Answer the questions in the spaces provided.
3. Spare answer pages are provided at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued i.e. give the page number.
4. Show all working clearly. Any question, or part question, worth more than 2 marks requires valid working or justification to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

Question 1 **(5 marks)**

Consider the system of equations below;

$$\begin{aligned} x + 3y + 2z &= 7a-2 \quad (1)\\ 2x + 4y + z &= 6a-3 \quad (2)\\ 2x + 5y + 3z &= 11a-2 \quad (3)\end{aligned}$$

✓ labels equations

Find algebraically, showing full working, the solution to the equations above, giving your answers in terms of a where necessary. [5]

$2\times(1)-(2)$: $2y + 3z = 8a - 1$ (4) ✓ eliminates x

$(3)-(2)$: $y + 2z = 5a + 1$ (5) ✓ accuracy

$2\times(5)-(4)$: $z = 2a + 3$ ✓ solves for z

$y = 5a + 1 - 2(2a+3)$

$= a - 5$

$x = 7a - 2 - 3(a-5) - 2(2a+3)$

$x = 7$

✓ finds x, y

$\therefore$ $\underline{x = 7}$, $\underline{y = a - 5}$, $\underline{z = 2a + 3}$

5

Question 2 **(7 marks)**

a) Find the exact value of the definite integral $\int_5^{13}(2x-1)^{-3/2}dx$; [3]

$\int_5^{13}(2x-1)^{-3/2}dx = \left[-(2x-1)^{-\frac{1}{2}}\right]_5^{13}$ ✓ integrates

$= -25^{-\frac{1}{2}} + 9^{-\frac{1}{2}}$ ✓ substitutes

$= \frac{1}{3} - \frac{1}{5}$

$= \frac{2}{15}$ ✓ evaluates

b) Differentiate x^2e^{2x} and hence find $\int_0^1 x(1+x)e^{2x}dx$ [4]

$\frac{d}{dx}\left(x^2e^{2x}\right) = 2xe^{2x} + x^2.2e^{2x}$ ✓ either correct

$= 2x(1+x)e^{2x}$

$\int_0^1 x(1+x)e^{2x}\,dx = \left[\frac{1}{2}x^2e^{2x}\right]_0^1$ ✓ integrates ✓ factor of $\frac{1}{2}$

$= \frac{1}{2}e^2 - 0$

$= \frac{1}{2}e^2$ ✓ evaluates

7

Question 3 **(5 marks)**

The table below shows the **cumulative probability distribution** for a random variable, X.

x	1	2	3	4	5
$P(X \le x)$	0.1	0.2	0.4	p	1
$P(X=x)$	0.1	0.1	0.2	$p-0.4$	$1-p$

Given that the expected value for the probability distribution is 3.5,

✓ Finds probabilities

a) Find the value of p. [3]

$1\times0.1 + 2\times0.1 + 3\times0.2 + 4\times(p-0.4) + 5(1-p)$
$= 3.5$

$0.9 + 4p - 1.6 + 5 - 5p = 3.5$ ✓ sets up equation

$4.3 - p = 3.5$

$\underline{p = 0.8}$ ✓ solves for p

b) Find $P(X<4|X<5)$ [2]

$P(x<4|x<5) = \dfrac{0.4}{0.8}$ ✓ given set

$= 0.5$ ✓ answer

5

Question 4 **(6 marks)**

Two positive numbers x and y add up to 10. Use calculus to find the values of x and y so that the product x^3y^2 is maximised.

$x + y = 10$

$P = x^3y^2 = x^3(10-x)^2 = x^3(100 - 20x + x^2)$ ✓ substitutes

$= x^5 - 20x^4 + 100x^3$

$\dfrac{dP}{dx} = 5x^4 - 80x^3 + 300x^2$ ✓ Finds $\dfrac{dP}{dx}$

$= 5x^2(x^2 - 16x + 60)$

$= 5x^2(x-6)(x-10)$ ✓ Factorises

At $x=0$, $x=10$ $P=0$ $\therefore$ not a maximum

$\dfrac{d^2P}{dx^2} = 20x^3 - 240x^2 + 600x$ ✓ $\dfrac{d^2P}{dx^2}$

$= 20x(x^2 - 12x + 30)$

At $x=6$ $\dfrac{d^2P}{dx^2} = 120(36 - 72 + 30) < 0$ ✓ Justifies MAX @ $x=6$

$\therefore$ local MAXIMUM

$\therefore$ Maximum value occurs when $\underline{x=6}$, $\underline{y=4}$

✓ States values.

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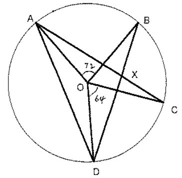
Question 5 **(5 marks)**



The diagram shows four points A, B, C and D on the circumference of a circle, centre O. X is the point of intersection of the chords AC and BD.

It is known that s< DOC = 64° and s< AOB = 72° .

Find, with full reasoning,

i) the size of angle DAC.

$s\angle DAC = 32^\circ$ (angle subtended at circumference = $\frac{1}{2}$ angle at centre on the same arc)

✓ value ✓ reason

ii) the size of angle AXB.

$s\angle ADB = 36^\circ$ (angle at circ = $\frac{1}{2}$ angle at centre)

$s\angle AXD = 180^\circ - s\angle ADX - s\angle DAX$ (angle sum of ΔADX)

$= 180^\circ - 36^\circ - 32^\circ$

$= 112^\circ$

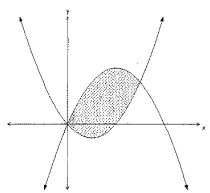
$s\angle AXB = 180^\circ - 112^\circ$

$= 68^\circ$ (angles on a straight line add up to 180°)

✓ answer
✓ some reasoning
✓ full and complete reasoning

Question 6 **(5 marks)**

The diagram below shows graphs of $y = 4x - x^2$ and $y = x^2 - 2x$. Find the shaded area.



Intersect when $4x - x^2 = x^2 - 2x$

$0 = 2x^2 - 6x$

$0 = 2x(x-3)$

$x = 0$ or $x = 3$ ✓ finds intersection

Area $= \int_0^3 4x - x^2 - (x^2 - 2x)\, dx$ ✓ sets up integral

$= \int_0^3 6x - 2x^2\, dx$ ✓ simplifies

$= \left[3x^2 - \dfrac{2x^3}{3}\right]_0^3$ ✓ integrates

$= 27 - 18$ ✓ evaluates

$= \underline{\underline{9 \text{ sq. units}}}$

5

Question 18 (8 marks)

The temperature of a metal bar at time t minutes after it is taken out of a fire is given by $T = A + Be^{-kt}$ where, T is the temperature of the rod and A, B and k are positive constants.

a) Show that $\frac{dT}{dt} = -k(T-A)$ [2]

$T = A + Be^{-kt}$

$\Rightarrow \frac{dT}{dt} = -kBe^{-kt}$ ✓

$= -k(T-A)$ ✓

b) Given that the initial temperature of the rod is 300°C and eventually the temperature drops towards a lowest value of 30°C, determine the values of A and B. [2]

$t=0, \quad T = A+B = 300$

$t \to \infty \quad T \to A \quad \therefore \quad A = 30$

$\underline{A = 30}, \quad \underline{B = 270}$ ✓✓

c) After 5 minutes the temperature of the metal bar has fallen to 250°C. Use this fact and the answers from part b) to determine

i) the value of k accurate to 4 decimal places. [2]

$250 = 30 + 270e^{-5k}$ ✓ eqn

$k = 0.0410 \quad (4dp)$ ✓ answer

ii) the time it takes for the temperature of the metal bar to fall to a value of 100°C. [2]

$100 = 30 + 270e^{-0.0410t}$ ✓ eqn

$t = 32.96 \quad \approx \quad \underline{33 \text{ mins}}$ ✓ answer

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Question 19 (8 marks)

A continuous random variable, X has the following probability density function;

$$f(x) = \begin{cases} Ae^{-kx} & 0 \le x \le 5 \\ 0 & otherwise \end{cases}$$

a) Determine a relationship between A and k to ensure that f(x) is a probability density function. [2]

$\int_0^5 Ae^{-kx}\,dx = 1$ ✓ integral equation

$\frac{-Ae^{-5k}}{k} + \frac{A}{k} = 1$ ✓ A, k equation

b) Find a relationship between A and k given that the median of the probability distribution is 2. [2]

$\int_0^2 Ae^{-kx}\,dx = 0.5$ ✓ integral equation

$\frac{-Ae^{-2k}}{k} + \frac{A}{k} = \frac{1}{2}$ ✓ A, k equation

c) Determine the values of A and k [2]

$A = 0.2934 \quad (4dp)$ ✓

$k = 0.1644 \quad (4dp)$ ✓ values

d) Find $P(X \le 4 | X \ge 2)$ [2]

$\frac{\int_2^4 Ae^{-kx}\,dx}{\int_2^5 Ae^{-kx}\,dx} = \frac{0.3599}{0.5} = 0.7197 \quad (4dp)$ ✓ expression ✓ answer

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Question 20 (8 marks)

Consider the functions $f(x) = 1 + \sqrt{x-2}$ and $g(x) = \frac{1}{x-5}$

a) Write down the natural domain and corresponding range for f(x). [2]

$D_f = \{x : x \ge 2, x \in \mathbb{R}\}$ ✓ domain

$R_f = \{f : f \ge 1, f \in \mathbb{R}\}$ ✓ range

b) Find i) $g \circ f(6) = g(3) = -\frac{1}{2}$ ✓[1] value

ii) x such that $f \circ f(x) = 4$ [2]

$1 + \sqrt{1 + \sqrt{x-2} - 2} = 4$ ✓ equation

$x = 102$ ✓ solution

c) State the domain and range of $g \circ f(x)$ [3]

$g \circ f(x) = \frac{1}{1 + \sqrt{x-2} - 5} = \frac{1}{\sqrt{x-2} - 4}$

$D_{g \circ f} = \{x : x \ge 2, x \ne 18, x \in \mathbb{R}\}$

$R_{g \circ f} = \{g \circ f : g \circ f \le -\frac{1}{4} \text{ or } g \circ f > 0\}$

✓ $x \ge 2$ in domain and $y > 0$ in range ✓ excludes $x = 18$ ✓ complete range

End of Booklet 3

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Question 7 (4 marks)

Solve algebraically the inequality $\frac{2x+1}{x-3} \le \frac{x+2}{x-1}$

Method 1 : Equal when $(2x+1)(x-1) = (x-3)(x+2)$

$\Rightarrow 2x^2 - x - 1 = x^2 - x - 6$

$\Rightarrow x^2 = -5$

Which has no solution

$\therefore$ Critical values are 3 and 1

Testing : $x = 0 \quad -\frac{1}{3} \le -2$ ✗

$x = 2 \quad -5 \le 4$ ✓

$x = 4 \quad 9 \le 8/3$ ✗

$\therefore \quad \underline{1 < x < 3}$

✓ correct method
✓ algebraic accuracy
✓ answer
✓ logical explanation or set-out

Method 2 : $\frac{2x+1}{x-3} - \frac{x+2}{x-1} \le 0$

$\Rightarrow \frac{(2x+1)(x-1) - (x+2)(x-3)}{(x-3)(x-1)} \le 0$

$\Rightarrow \frac{2x^2 - x - 1 - (x^2 - x - 6)}{(x-3)(x-1)} \le 0$

$\Rightarrow \frac{x^2 + 5}{(x-3)(x-1)} \le 0$

True for $\underline{1 < x < 3}$

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Question 8 (3 marks)

A spherical cloud is expanding at a constant rate of 5000π m³ per second. Find the radius of the cloud at the instant when the radius of the cloud is expanding at the rate of 2 m per second.

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2 \cdot \frac{dr}{dt}$ ✓ Differentiates to find $\frac{dV}{dt}$

$5000\pi = 4\pi r^2 \cdot 2$ ✓ establishes equation in r

$r^2 = \frac{5000}{8}$

$r^2 = 625$

$\underline{r = 25 \text{ m}}$ ✓ solves for r

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End of paper

Question 9 (7 marks)

A bank is considering the passwords that are allowed for their customers to enter personal accounts on the bank's website.

a) Currently, the password is composed of 4 different characters chosen from the 26 lower case letters of the alphabet and the 10 digits, 0, 1, 2, . . . ,9.

The bank's IT manager has calculated that the number of available passwords is 36 x 35 x 34 x 33 = 1413720.

By what factor will the number of available passwords increase if,

i) the letters used can be upper or lower case? [2]

$\frac{62 \times 61 \times 60 \times 59}{36 \times 35 \times 34 \times 33} = 9.47 \quad (2dp)$ ✓ numerator ✓ divide by ...

ii) repetition is allowed (letters can only be lower case) [2]

$\frac{36^4}{36.35.34.33} = 1.18 \quad (2dp)$ ✓ numerator ✓ denominator

b) The bank decides to introduce a new password system which has more structure and is therefore easy to remember. 3 different upper case letters will be followed by a number of different digits excluding 0. How many digits are required to ensure that there are at least 10^9 passwords available? [3]

$26 \times 25 \times 24 \times {}^9P_n > 10^9$ ✓ inequality

${}^9P_n > 64102.6$ ✓ working

${}^9P_6 = 60480 \quad \therefore$ Need 7 digits at least

${}^9P_7 = 181440$ ✓ answer

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Question 10 (7 marks)

The weights of a supply of ball bearings are normally distributed with a mean weight of 0.62 N and standard deviation of 0.01 N.

a) Find the probability that any one ball bearing has a weight of between 0.61 N and 0.62 N. [1]

$P(0.61 \le X \le 0.62) = 0.3413 \quad (4dp)$

b) If 10 ball bearings are selected at random find the probability that at least 4 of them have a weight between 0.61 N and 0.62 N [2]

$Y \sim bin(10, 0.3413)$

$P(Y \ge 4) = 0.4625 \quad (4dp)$

c) If a sample of 100 ball bearings are measured and recorded to the nearest 0.01 N, find the number of ball bearings that you would expect to be measured as 0.61 N. [2]

$0.605 \longleftrightarrow 0.615$

$100 \times P(0.605 \le X \le 0.615) = 24.17$

Expect 24 ball bearings

d) If a sample of 100 ball bearings is taken, find the probability that the mean value of the sample is less than 0.618 N. [2]

$\bar{X} \sim N\left(0.62, \left(\frac{0.01}{\sqrt{100}}\right)^2\right)$

$P(\bar{X} < 0.618) = 0.0228 \quad (4dp)$

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Question 11 (3 marks)

Two normal six sided dice are thrown and the total of the uppermost faces recorded. This is repeated a number of times.

Find the probability of getting

i) a score of at least 11 on the first throw. [1]

$P(11 \text{ or } 12) = P((5,6), (6,5) \text{ or } (6,6))$

$= \frac{3}{36}$

$= \frac{1}{12}$ ✓

ii) a score of at least 11 on exactly 2 of the first 3 throws. [2]

Y = no of times at least 11 is scored

$Y \sim \text{bin}(3, 1/12)$

$P(Y=2) = {}^3C_2 \times \left(\frac{1}{12}\right)^2 \times \left(\frac{11}{12}\right)$ ✓

$= \frac{11}{576}$

$= 0.0191$ (4dp) ✓

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Question 12 (6 marks)

a) Write down, in the correct order, the transformations that are needed to change the graph of $y = 2e^{x-1}$ into the graph of $y = e^{0.5x+3}$. [3]

$2e^{x-1} \longrightarrow e^{x-1}$ vertical dilation of factor $\frac{1}{2}$

$e^{x-1} \longrightarrow e^{x+3}$ translation 4 units left

$e^{x+3} \longrightarrow e^{0.5x+3}$ horizontal dilation factor 2

✓✓✓ −1 each error

b) Find the equation of the new graph when the graph of the function $y = 4 - 5e^{3(x-4)}$ is subject to the following sequence of transformations, in the order shown; [3]

- Dilation of factor 6 horizontally
- Translation of 12 units to the left
- Reflection in the x – axis

$4 - 5e^{3(x-4)} \longrightarrow 4 - 5e^{3(\frac{x}{6}-4)}$

$4 - 5e^{3(\frac{x}{6}-4)} \longrightarrow 4 - 5e^{3(\frac{x+12}{6}-4)}$

$4 - 5e^{3(\frac{x+12}{6}-4)} \longrightarrow -4 + 5e^{3(\frac{x+12}{6}-4)}$

$\therefore \quad y = -4 + 5e^{0.5x-6}$

✓✓✓

−1 each error

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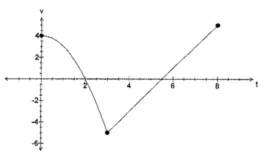
Question 13 (8 marks)

The velocity-time graph for the motion of a particle, P is shown in the diagram below. V is measured in ms^{-1} and t is measured in seconds.



The formula for the velocity at time t is given by

$v(t) = \begin{cases} a - t^2 & \text{for } 0 \le t \le 3 \\ 2t - b & \text{for } 3 < t \le 8 \end{cases}$

a) Determine the values of a and b. [2]

$a = 4$ ✓, $b = 11$ ✓ values

b) Find the acceleration of the particle at time t = 2s. [2]

$\frac{dv}{dt} = -2t$ ✓ at $t = 2$ $a = -4 \text{ ms}^{-2}$ ✓

c) Find the distance travelled in the 8 seconds shown. [2]

$\int_0^3 |4 - t^2| dt + \int_3^8 |2t - 11| dt = 20\frac{1}{6}$ m ✓ integral

$7\frac{2}{3} + 12\frac{1}{2}$ ✓ answer

d) Find the average velocity during the first 8 seconds. [2]

Displacement $= \int_0^3 4 - t^2 dt + \int_3^8 2t - 11 dt$

$= 3 + 0$ ✓ displacement

Av. velocity $= \frac{3}{8} = 0.375 \text{ ms}^{-1}$ ✓ answer

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Question 14 (5 marks)

The waiting times at a doctor's surgery are distributed with a mean value of μ and a standard deviation of σ.

a) The waiting times of 200 patients were recorded and found to have a mean value of 25 minutes with a standard deviation of 8 minutes.

Find a 95% confidence interval for the value of μ, accurate to 2 decimal places. [2]

$25 - \frac{1.96 \times 8}{\sqrt{200}} \le \mu \le 25 + \frac{1.96 \times 8}{\sqrt{200}}$ ✓

$23.89 \le \mu \le 26.11$ (2dp) ✓

is the 95% confidence interval

b) In another sample of 200 patients the mean value was $\bar{x}$ and the standard deviation s.

From these observations a 95% confidence interval was found to be $20.45 \le \mu \le 21.95$. Find the values of $\bar{x}$ and s. [3]

$\bar{x} - \frac{1.96 \times s}{\sqrt{200}} = 20.45$

$\bar{x} + \frac{1.96 \times s}{\sqrt{200}} = 21.95$ ✓ equations

$\therefore \quad \bar{x} = 21.2$ ✓ mean

$s = 5.41$ ✓ s.d.

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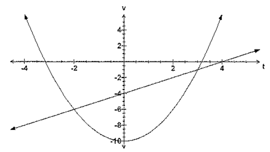
Question 15 (4 marks)

The diagram below shows the graph of the curve $y = x^2 - 10$ and the line y = x – 4.



The area trapped between the curve and the line is rotated through 360° about the x – axis.

i) Write down an integral calculation to determine the volume generated. [3]

$\text{Volume} = \int_{-2}^{3} (x^2 - 10)^2 - (x-4)^2 \, dx$

✓ limits
✓ uses $\int y^2 dx$
✓ correct

ii) Find the **exact** volume. [1]

$V = 250\pi$ units^3 ✓ answer

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End of Booklet 2

Question 16 (9 marks)

A gourmet delicatessen produces two types of gift basket, A and B.

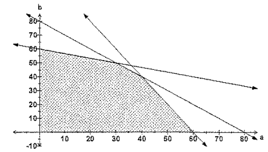
The following inequalities describe the constraints of production where a is the number of baskets of type A and b is the number of baskets of type B produced in a week.

$a + b \le 80$ $\quad a \ge 0$

$2a + b \le 120$ $\quad b \ge 0$

$a + 3b \le 180$

The graph below shows the lines equating to the inequalities above.



If each type A basket gives a profit of $12 and each type B basket produces a profit of $10,

a) Find the number of each type of basket that should be produced for maximum profit. Show your working. [3]

Vertex	12a+10b
(0, 60)	600
(30, 50)	860
(40, 40)	880
(60, 0)	720
(0, 0)	0

Should produce 40 of each type for a maximum profit of $880.

✓ statement
✓ vertices
✓ values

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b) If the profit on each type A basket remains as $12, by how much does the profit on each type B basket need to rise so that there is more than one option for producing maximum profit. State the options available. [3]

New : Profit = $12a + kb$

P at (30, 50) = P at (40, 40) $\Rightarrow$ $360 + 50k = 480 + 40k$

$k = 12$ ✓

Profit on B must rise by $2. ✓

Possible options are

(30, 50) (31, 49) (32, 48) (33, 47) (34, 46) (35, 45) (36, 44) (37, 43) (38, 42) (39, 41) (40, 40) ✓

c) If each type A basket gives a profit of $m and each type B basket produces a profit of $n, where m and n are positive constants, find conditions on m and n that will ensure that producing 30 type A baskets and 50 type B baskets is the only way to maximise the profit. [3]

$P(30, 50) > P(40, 40)$

$\Rightarrow \quad 30m + 50n > 40m + 40n$

$n > m$ ✓

$P(30, 50) > P(0, 60)$

$\Rightarrow \quad 30m + 50n > 60n$

$3m > n$ ✓

$\therefore \quad m < n < 3m$ ✓

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Question 17 (7 marks)

a) If A and B are independent, $P(A|B) = 0.8$ and $P(B|A) = 0.4$, find

i) P(A) $= P(A|B) = 0.8$ [1] ✓ answer

ii) $P(A \cup B)$ [2]

$= P(A) + P(B) - P(A \cap B)$ ✓ equation

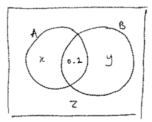
$= 0.8 + 0.4 - 0.8 \times 0.4$

$= 0.88$ ✓ answer

iii) $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$ [1]

$= 0.12$ ✓ complement of ii)

b) If $P(A|B) = 0.8$, $P(B|A) = 0.4$ and $P(A \cap B) = 0.2$ find $P(A \cup B)$ [3]



$P(A|B) = 0.8 \Rightarrow \frac{0.2}{0.2 + y} = 0.8$

$y = 0.05$

$P(B|A) = 0.4 \Rightarrow \frac{0.2}{0.2 + x} = 0.4$

$x = 0.3$

✓ equations
✓ evaluation
✓ $P(A \cup B)$

$P(A \cup B) = x + y + 0.2$

$= 0.55$

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