Section One: Calculator-free (40 Marks)

This section has eight (8) questions. Answer all questions. Write your answers in the space provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
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Suggested working time for this section is 50 minutes.

Question 1 (6 marks)

For the functions $f(x) = e^{x-2}$ and $g(x) = \frac{1}{\sqrt{x}}$, determine

(a) $g \circ f(0)$, as a simplified exact value (2 marks)

$$g(e^{-2}) = \frac{1}{\sqrt{e^{-2}}} = e$$

(b) the domain of $g(x)$ (1 mark)

$$x > 0$$

(c) $f(g(x))$ (1 mark)

$$f\left(\frac{1}{\sqrt{x}}\right) = e^{\frac{1}{\sqrt{x}} - 2}$$

(d) the range of $f(g(x))$ (2 marks)

$$y > e^{-2} \quad ; \quad y > \frac{1}{e^2}$$

Question 2 (6 marks)

Differentiate the following:

(a) $y = e^{\sqrt{x}}$ (2 marks)

$$\frac{dy}{dx} = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

(b) $f(x) = \int_3^{x^2} \sqrt{5-2t}\, dt$ (1 mark)

$$f'(x) = \sqrt{5 - 2x^2} \cdot 2x$$

(c) $g(x) = x.e^x$ (1 mark)

$$g'(x) = 1.e^x + x.e^x$$

From your result for $g'(x)$ in part (c):

(d) find $\int x.e^x\, dx$ (2 marks)

$$\int g'(x) = \int e^x + \int xe^x$$

$$\therefore \int xe^x = g(x) - \int e^x = xe^x - e^x + c$$

Question 3 (5 marks)

A standard normal score of 1.28 is such that $P(0 < z < 1.28) = 0.4$
Use this information to determine:

(a) $P(0 < z < 1.28 \mid z < 1.28)$ (2 marks)

$$\frac{P(0 < Z < 1.28)}{P(Z < 1.28)} = \frac{0.4}{0.9} = \frac{4}{9}$$

(b) an 80% confidence interval for an observation from a normal population with mean 50 and standard deviation 10. (1 mark)

$\mu \pm z\sigma$ is 50 ± 12.8

i.e. $37.2 < \mu < 62.8$

(c) an 80% confidence interval for the mean of any sample of size 64 taken from any population of mean 50 and standard deviation 10. (2 marks)

$$\sigma = \frac{10}{8} = 1.25$$

$$\frac{5}{4} \times \cancel{1.28}^{\,0.32} = 1.6$$

50 ± 1.6 i.e. $48.4 < \mu < 51.6$

Question 4 (4 marks)

Determine the following integrals:

(a) $\int (e^{3x} - e^{-3x})^2\, dx$ (2 marks)

$$= \int e^{6x} - 2 + e^{-6x}\, dx$$

$$= \frac{e^{6x}}{6} - 2x - \frac{e^{-6x}}{6} + c$$

(b) $\int x\sqrt{4-x^2}\, dx$ (2 marks)

$$= \frac{(4-x^2)^{\frac{3}{2}}}{\frac{3}{2} \cdot -2} + c$$

$$= -\frac{(4-x^2)^{\frac{3}{2}}}{3} + c$$

Question 5 (5 marks)

Identify all the values of x for which $2-\frac{x}{2}\geq\frac{5}{x+3}$

Solve $2-\frac{x}{2}=\frac{5}{x+3}$

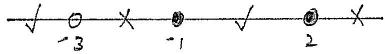
$4-x=\frac{10}{x+3}$

$(4-x)(x+3)=10$

$4x+12-x^2-3x=10$

$x^2-x-2=0$

$(x-2)(x+1)=0$; $x=2$ or -1



at $x=-4$ at $x=-2$ at $x=0$ at $x=3$

$4>-5$ $3<5$ $2\geq\frac{5}{3}$ $2-\frac{3}{2}<\frac{5}{6}$

$\therefore\ x<-3$ or $-1\leq x\leq 2$

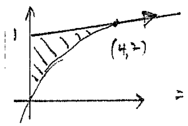
Question 6 (6 marks)

(a) A tangent is drawn to the curve $y=\sqrt{x}$ at the point $(4,2)$ (2 marks)
What is the equation of this tangent?

$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$ gradient $=\frac{1}{4}$

$y=\frac{x}{4}+1$

(b) Calculate the area enclosed by this tangent, the curve $y=\sqrt{x}$ and the y-axis. (3 marks)



$\int_0^4 \frac{x}{4}+1-\sqrt{x}\ dx$

$=\left.\frac{x^2}{8}+x-\frac{2}{3}x^{\frac{3}{2}}\right|_0^4$

$=2+4-\frac{16}{3}$

$=\frac{2}{3}$ units^2

(c) Write down the integral, or integrals, that you would use to calculate the volume of the solid of revolution formed when the area in part (b) is revolved through 360° around the x-axis. (1 mark)

$V_x=\int_0^4 \pi\left(\frac{x}{4}+1\right)^2-\pi x\ dx$.

Question 7 (3 marks)

Solve the system of equations $\begin{cases}x+3y+z=6\\ x-y-z=0\\ 2x+6y+z=7\end{cases}$

$2x+6y+z=7$

$2x+6y+2z=12$

$\therefore\ z=5$

$x+3y=1$

$x-y=5$

$4y=-4$

$y=-1$

$x=4$

$\left.\begin{array}{l}x=4\\ y=-1\\ z=5\end{array}\right\}$

Question 8 (5 marks)

A function $f(x)$ is defined by $f(x)=\frac{ax+1}{x+b}$ for constants a and b.

(a) Write an expression for $f'(x)$ in terms of a and b and undertake any obvious simplifications. (2 marks)

$f'(x)=\frac{(x+b).a-(ax+1).1}{(x+b)^2}$

$=\frac{ax+ab-ax-1}{(x+b)^2}$

$=\frac{ab-1}{(x+b)^2}$

(b) Verify that $a=3$ and $b=1$ lead to the result $f(1)=f'(0)=2$. (1 mark)

$f(1)=\frac{3+1}{2}=2$ $f'(0)=\frac{3-1}{1}=2$

(c) Give two general observations about the slope of $y=f(x)$ when $a=3$ and $b=1$. (2 marks)

$f'(x)=\frac{2}{(x+1)^2}$

slope undefined at $x=-1$

slope positive (>0) elsewhere.

Section Two: Calculator-assumed **(80 Marks)**

This section has thirteen (13) questions. Answer all questions. Write your answers in the space provided.

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Suggested working time for this section is 100 minutes.

Question 9 **(5 marks)**

A Hilbert number H_n is an integer of the form $4n+1$ where n is a positive integer.

(a) Which Hilbert number corresponds to $n = 7$? (1 mark)

$4 \times 7 + 1 = 29$

(b) Is 49 a Hilbert number? Is 111? (1 mark)

$49 = 4 \times 12 + 1$; Yes $\quad 111 = 27 \times 4 + 3$ No.

(c) Prove that the product of any two (different) Hilbert numbers is itself a Hilbert number. (3 marks)

$H_n \times H_m = (4n+1)(4m+1)$

$= 16nm + 4n + 4m + 1$

$= 4(4nm + n + m) + 1$

which is Hilbert since $4nm+n+m$ is an integer.

Question 10 **(5 marks)**

Two events A and B are such that $P(A|B) = 0.6$ and $P(A \cap B) = 0.24$

Evaluate:

(a) $P(B)$ (1 mark)

$0.6 = \dfrac{0.24}{P(B)} \qquad P(B) = 0.4$

(b) $P(B|A)$ when A and B are independent events (1 mark)

$P(B|A) = P(B) = 0.4$

(c) $P(B|A)$ when $P(A \cup B) = 0.8$ (3 marks)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

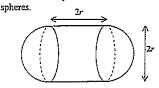
$0.8 = P(A) + 0.4 - 0.24$

$P(A) = 0.64$

$P(B|A) = \dfrac{0.24}{0.64} = \dfrac{3}{8}$ or 0.375

Question 11 **(8 marks)**

A pharmaceutical company is trialling a new anti-biotic tablet that is made in the shape of a cylinder with hemispherical ends. The cylindrical section has radius r and length equal to $2r$, the diameter of the hemispheres.



(a) Show clearly that the volume of this tablet is given by $V = \dfrac{10\pi r^3}{3}$ (2 marks)

$V = V_{cyl} + V_{sph}$

$= \pi r^2 . 2r + \frac{4}{3}\pi r^3$

$= \dfrac{6\pi r^3 + 4\pi r^3}{3} = \dfrac{10\pi r^3}{3}$

The tablet is designed to dissolve at a constant rate of 10 mm³ per minute.

(b) Determine the rate at which the radius r is changing when the radius is 2.5 mm. (3 mark)

$\dfrac{dV}{dt} = 10\pi r^2 . \dfrac{dr}{dt}$

$\therefore \dfrac{dr}{dt} = \dfrac{-10}{10\pi . 2.5^2} = \dfrac{-1}{6.25\pi}$ or $\dfrac{-4}{25\pi}$

decreasing at $\dfrac{-1}{6.25\pi} = 0.051$ mm/min

(c) What is the rate at which the surface area is changing when the radius is 2.5 mm? (3 marks)

$SA = 2\pi r \times 2r + 4\pi r^2$

$= 8\pi r^2$

$\dfrac{dSA}{dr} = 16\pi r \dfrac{dr}{dt} = 16\pi . 2 . \dfrac{-1}{6.25\pi} = -6.4$

decreasing at 6.4 mm² / min

Question 12 **(4 marks)**

An economics model once trialled by the Department of Treasury and Finance in Canberra calculated the annual inflation rate $I(x)$ based on the GST rate x%.

The *marginal* inflation rate was defined as $I'(x) = \dfrac{9}{2\sqrt{x}}$

(a) Use the incremental technique $\delta y \approx \dfrac{dy}{dx}\delta x$ to estimate the change in the annual inflation rate if the GST was increased from 9% to 9.5% (1 mark)

$\delta I = I'(x)\, \delta x$

$= \dfrac{9}{2.3} \times 0.5 = 0.75$

an increase of 0.75 % points

(b) Apply an integration method to calculate the predicted inflation rate for a GST of 16%, given that a 9% GST is associated with an inflation rate of 3.5% (2 marks)

$I(16) = I(9) + \displaystyle\int_9^{16} \frac{9}{2\sqrt{x}}\, dx$

$= 3.5 + 9\sqrt{x} \Big|_9^{16}$

$= 3.5 + (36 - 27)$

$= 12.5\%$

(c) Could the incremental technique be reliably used to predict the effect of a GST increase from 9% to 16%? Explain. (1 mark)

No ; increments must be small

Question 20 **(10 marks)**

The Wyvern Mining Company NL operates two small mines, both producing both copper and lead ores, which are transported to a nearby processing plant for refining into ore concentrates.

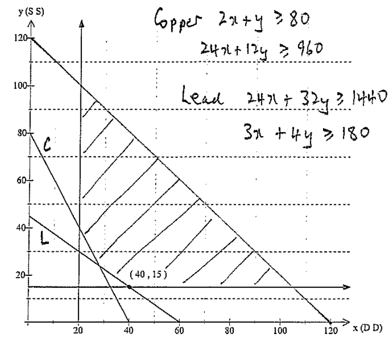
The mine known as David's Diggings costs \$20 000 per hour to operate in producing 24 tonnes of copper ore per hour and 24 tonnes of lead ore per hour.

Stephen's Shaft is now an open pit, costing \$15 000 per hour in producing ~~24~~ 12 tonnes of copper and 32 tonnes of lead each hour.

The company is contracted to produce at least 960 tonnes of copper ore and 1440 tonnes of lead ore each week.

David's Diggings must operate for at least 20 hours per week and Stephen's Shaft also has a minimum operating time requirement.

These constraints are graphed, with the operating hours per week as the variables: x for David's Diggings and y for Stephen's Shaft.



(a) Identify the minimum operating time per week for Stephen's Shaft (1 mark)

15 hours

(b) The total possible maximum operating hours for the two mines combined is 120 hours. Add this constraint to the graph and clearly mark the resulting feasible region. (2 marks)

$x + y \leq 120$

(c) For how many hours should each mine operate each week in order to minimise total costs? (3 marks)

$C = 20x + 15y$ (in K's)

(20,100) too high
(20,40) 1000
(28,24) 920
(40,15) 1025
(105,15) too high

DD : 28 hours
SS : 24 hours

(d) Determine the possible changes to the costs per hour for David's Diggings that would result in an alteration to the optimal solution found in (c). (4 marks)

K = new cost ; $C = Kx + 15y$

$(20,40) < (28,24)$ $\quad 20K + 600 < 28K + 360$

$8K > 240$

$K > 30$

$(40,15) < (28,24)$ $\quad 40K + 225 < 28K + 360$

$12K < 135$

$K < \$11.25$

alter if costs rise by \$10 000 per hour or more or fall by \$8750 or more.

Question 21 **(4 marks)**

Gas is leaking from a storage tank in a large industrial and processing facility. The rate of this leak is directly proportional to the pressure, P, of the gas remaining in the tank.

(a) Show clearly how the equation $P(t) = P_0 e^{kt}$ models this situation. (2 marks)

$\dfrac{dP}{dt} \propto P$

$P(t) = P_0 e^{kt} \Rightarrow P'(t) = k . P_0 e^{kt}$

$= k . P(t)$

i.e. $\dfrac{dP}{dt} \propto P$

(b) Determine the instantaneous rate of the loss of pressure, as a percentage of the remaining pressure, given that the pressure dropped by 50% in the first 4 hours after the leak developed. (2 marks)

$0.5 P_0 = P_0 e^{4k}$

$e^{4k} = 0.5$

$k = -0.1733$

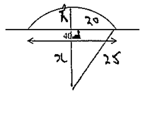
i.e. 17.33% (per hour)

End of questions

Question 13 (3 marks)

An engineer designed a bridge with the profile of the Sydney Harbour bridge, with a circular arch of radius 25 metre above a horizontal roadway that is a 40 metre long chord of the circle.

Calculate the maximum height of the arch above the roadway.



$x^2 = 25^2 - 20^2$

$x = 15$

$\therefore$ height $= 25 - 15$

$= 10$ m.

Question 14 (3 marks)

Two kangaroo shooters, Wayne and Clint, have respective probabilities of 0.75 and 0.6 of hitting any target, independent of any other event.

They both fired at a kangaroo.

What is the probability Wayne fired the bullet that hit the kangaroo, if it was hit by (exactly) one bullet?

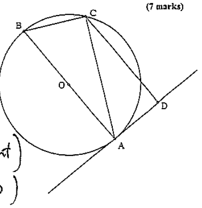
$P(\text{Wayne} \mid 1 \text{ hit}) = \dfrac{P(\text{W hits, C misses})}{P(1 \text{ hit})}$

$= \dfrac{0.75 \times 0.4}{0.75 \times 0.4 + 0.25 \times 0.6}$

$= \dfrac{2}{3}$

Question 15 (7 marks)

In this diagram, AOB is the diameter of a circle, AC is a chord of the circle and CD is perpendicular to the tangent AD.



(a) Prove that ΔABC is similar to ΔCAD (3 marks)

$\angle ACB = \angle ADC$ (both $90°$)

$\angle ABC = \angle DAC$) (alt segment)

or $\angle BAC = \angle ACD$ ($AB \parallel CD$)

then 3rd angle

$\therefore \Delta ABC \sim \Delta CAD$ (AAA)

(b) Hence show that $AC^2 = AB.CD$ (2 marks)

Corresp. sides proportional

$\dfrac{AC}{CD} = \dfrac{AB}{AC}$

$\therefore AC^2 = AB.CD$

(c) Determine the radius of the circle when AC = 15 cm and AD = 12 cm. (2 marks)

$CD = 9$ (Pyth: $15^2 - 12^2 = 81$)

$\therefore AB = \dfrac{15^2}{9} = 25$

radius $= 12.5$ cm.

Question 16 (8 marks)

A charged sub-atomic particle enters a variable magnetic field with an initial velocity of 4 cm sec^{-1} and an acceleration at time t defined by $a(t) = t - 3$ cm sec^{-2}.

(a) Write an expression for the velocity of this particle at time t. (2 marks)

$v = \int t - 3 \; dt$

$= \dfrac{t^2}{2} - 3t + 4$ cm sec^{-1}

(b) What is the position of the particle, relative to the edge of the magnetic field, at time $t = 6$ seconds? (2 marks)

$s(t) = \int v(t) = \dfrac{t^3}{6} - \dfrac{3t^2}{2} + 4t$

$s(6) = 36 - 54 + 24$

$= 6$ cm

(c) Calculate the distance travelled by the particle between $t = 0$ and $t = 6$. (2 marks)

Distance $= \int_0^6 |v(t)| \, dt$

$= 7\frac{1}{3}$ cm

(d) Identify the minimum velocity for $0 \le t \le 6$ (2 marks)

$v'(t) = 0 \Rightarrow t = 3$

$v''(t) = 1 > 0 \quad \therefore \min v$

$v(3) = \frac{9}{2} - 9 + 4 = -0.5$ cm sec^{-1}

is req'd min

Question 17 (6 marks)

A horse trainer is working with 5 colts and 4 fillies and he randomly selects five of these horses to enter the 5 events at a small country race meeting.

(a) Calculate the probability he selects more colts than fillies in his selection. (3 marks)

$P(3 \text{ or } 4 \text{ or } 5 \text{ colts})$

$= \dfrac{{}^5C_5\,{}^4C_0 + {}^5C_4\,{}^4C_1 + {}^5C_3\,{}^4C_2}{{}^9C_5}$

$= \dfrac{1 + 20 + 60}{126} = \dfrac{9}{14}$ or 0.643

(b) If he actually selects 3 colts and 2 fillies and then randomly allocated each horse to a different race, what are the chances the fillies do not compete in consecutive events (3 marks)

CCCFF

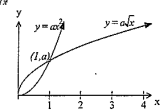
$1 - P(\text{Are in adjacent races})$

$= 1 - \dfrac{2 \times 4!}{5!}$

$= \dfrac{3}{5}$

Question 18 (5 marks)

The curves $y = a\,x^2$ and $y = a\sqrt{x}$ intersect at the point $(1, a)$, as shown.



(a) Determine the value of a which makes the shaded area = 1 unit^2 for $0 \le x \le 4$. (3 marks)

$\int_0^1 ax^2 \, dx + \int_1^4 a\sqrt{x} \, dx = 1$

$\dfrac{ax^3}{3}\Big|_0^1 + \dfrac{2}{3}ax^{\frac{3}{2}}\Big|_1^4 = 1$

$\dfrac{a}{3} + \dfrac{16a}{3} - \dfrac{2a}{3} = 1$

$a = \dfrac{1}{5}$ or 0.2

(b) Write down, but do not evaluate, an integral expression to find the volume generated when the unshaded area enclosed between $y = a\,x^2$ and $y = a\sqrt{x}$ for $0 \le x \le 1$ is rotated around the y axis. (2 marks)

$V_y = \int_0^a \pi x^2 \, dy$

$= \pi \int_0^{0.2} \dfrac{y}{0.2} - \dfrac{y^4}{0.2^4} \, dx$

$= \pi \int_0^{0.2} 5y - 625y^4 \, dx$

Question 19 (12 marks)

A botanist has found that 75% of the seeds of Eucalyptus Barrettii planted actually germinate and that the germination of each seed is statistically independent of any other event.

(a) For a packet of 20 seeds, determine the probability of at most 16 germinations, given that at least 14 seeds germinated. (3 marks)

Binomial $n = 20$ $p = 0.75$

$P(X \le 16 \mid X \ge 14) = \dfrac{\text{Bin CD}(14, 16, 20, 0.75)}{\text{Bin CD}(14, 20, 20, 0.75)}$

$= \dfrac{0.5606}{0.7858} = 0.7135$

(b) How many seeds should he plant before his chances of at least one seed not germinating exceed 0.99? (2 marks)

$P(\text{all germinate}) \le 0.01$

$0.75^n \le 0.01$

$n \ge 16.007$

ie 17 seeds or more

(c) The botanist has sent boxes containing 200 such packets, each containing 20 Eucalyptus Barrettii seeds, all around the world. For these boxes, describe the distribution of the average number of germinations per packet within each box, assuming a constant germination rate of 75%. Specify the type of distribution, its mean and its standard deviation. (2 marks)

Normal $\mu = 15$

$\sigma = \dfrac{\sqrt{np(1-p)}}{\sqrt{n_2}} = \dfrac{\sqrt{15 \times 0.25}}{\sqrt{200}} = \dfrac{1.9365}{14.142}$

$= 0.137$

(d) How many packets are needed per box so that the botanist can be 95% confident that the mean number of germinations is within 0.5 of the expected or overall average number. (3 marks)

95% C.I. $\Rightarrow z = 1.96$

$\dfrac{1.96 \times \sigma}{\sqrt{n}} \le 0.5$

$\dfrac{1.96 \times 1.9365}{\sqrt{n}} \le 0.5$

$n \ge 57.6$

ie 58 packets or more.

(e) Another supplier of Eucalyptus Barrettii seeds finds that his overall average number of germinations from packets of 20 seeds when packed in boxes of 200 packets is 15.3 By calculating the probability that his mean exceeds 15.3 and assuming the same standard deviation, decide how likely is it that the mean germination rates are the same. (2 marks)

$P(\bar{X} > 15.3 \mid \mu = 15) = \text{Norm}(15.3, \infty, \mu = 15, \sigma = 0.137)$

$= 0.0143$

only $1\frac{1}{2}$% chance they are the same

$\therefore 98\frac{1}{2}$% chance they are different.