PERTH MODERN SCHOOL

Exceptional schooling. Exceptional students.

Independent Public School

Course _______Specialist______ Year __12_____

Student name: _____________ Teacher name: _____________

Date: 24 Feb

Task type: **Response**

Time allowed for this task: **_____45______ mins**

Number of questions: **_____8______**

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **__50____ marks**

Task weighting: **_10___%**

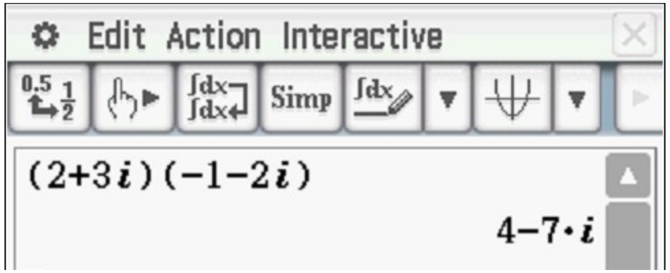
Formula sheet provided: **Yes/No**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (3.1.1, 3.1.2, 3.1.3) (2, 2, 3 & 3 = 10 marks)

If $z = 2+3i$ and $w = -1+2i$ determine exactly the following. (Simplify)

a) $z\overline{w}$

b) Solution

Specific behaviours
✓ determines conjugate of w ✓ determines product

b) $w\overline{w}$

Solution
$1^2 + 2^2 = 5$
Specific behaviours
✓ gives a real result ✓ determines product

c) $w \div \overline{w}$

Solution
$\frac{-1+2i}{-1-2i} \times \frac{-1+2i}{-1+2i} = \frac{1-4-4i}{1+4} = \frac{-3-4i}{5}$
Specific behaviours
✓ multiplies by conjugate over conjugate ✓ evaluates numerator ✓evaluates denominator

d) $\frac{1}{z} + \frac{1}{w}$

Working out space

b) Given that $f(x) = -2x^2 + 12x - 13, \quad x \le 3$, determine the defining rule for $y = f^{-1}(x)$.

Solution
$y \le 3$ $x = -2y^2 + 12y - 13$ $-2y^2 + 12y - 13 - x = 0$ $y = \frac{-12 \pm \sqrt{144 - 4(-2)(-13 - x)}}{-4} = \frac{-12 \pm \sqrt{144 - 104 - 8x}}{-4}$ $= \frac{-12 \pm 2\sqrt{10 - 2x}}{-4} = \frac{-6 \pm \sqrt{2(5 - x)}}{-2}$ $f^{-1}(x) = 3 - \sqrt{\frac{5 - x}{2}}$
Specific behaviours
✓ interchanges x and y to solve for inverse   ✓ states possible rules for inverse   ✓states correct rule with negative only

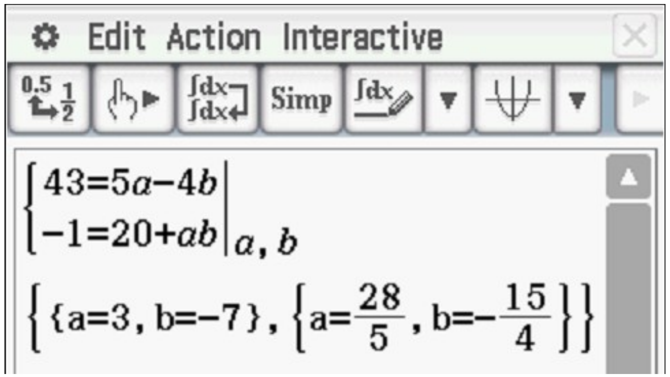
c) Consider the function $h(x) = ax^3$ where a is a positive constant. Solve in terms of a, the solution(s) to $h(x) = h^{-1}(x)$.

c) Solution
$h(x) = h^{-1}(x) = x \quad or\ ax^3 = \sqrt[3]{\frac{x}{a}} \quad \Rightarrow a^3x^9 = \frac{x}{a} \Rightarrow a^4x^9 - x = 0 \Rightarrow x(a^4x^8 - 1) = 0$ $ax^3 = x$ $x(ax^2 - 1) = 0$ $x = 0, \frac{1}{\sqrt{a}}, -\frac{1}{\sqrt{a}}$
Specific behaviours
✓ sets up equation in terms of x & a   ✓factorises equation   ✓states all three x values in terms of a.

Solution
$\frac{1}{2+3i}\frac{2-3i}{2-3i} + \frac{1}{-1+2i}\frac{-1-2i}{-1-2i}$ $\frac{2-3i}{13} + \frac{-1-2i}{5}$ $\frac{10-15i}{65} + \frac{-13-26i}{65} = \frac{-3-41i}{65}$
Specific behaviours
✓ uses idea of conjugates   ✓ expresses both reciprocals with real denominator   ✓determines correct simplified sum

Q2 (3.1.3) (3 marks)

Determine all possible real values of $a \,\&\, b$ such that $\frac{43 - i}{a + 4i} = 5 + bi$

Solution
$(43 - i) = (5 + bi)(a + 4i) = 5a - 4b + i(20 + ab)$ $43 = 5a - 4b$ $-1 = 20 + ab$ 
Specific behaviours
✓ sets up one equation involving a or b   ✓ sets up two equations for a & b

✓solves for two pairs of real values for a & b

Consider the function $h(x)=x-8$.

c) Does the function $\left[f(x)\right]^2=h(x)$? Justify your answer.

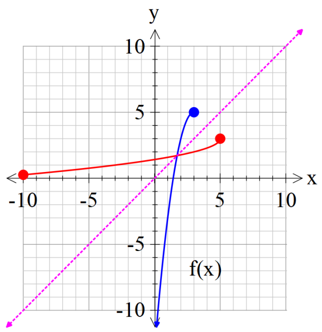
Solution
$\left[f(x)\right]^2$ $d: x\geq 8$ $h(x)$ $d: R$ $not\ equal\ as\ different\ domains$
Specific behaviours
✓ states not equal with a reason ✓ shows that domains are different

Q8 (3.2.3, 3.2.4) (2 & 3, 3 = 8 marks)

Consider the function $f(x)$ drawn below.

a) Sketch $y=f^{-1}(x)$ on the axes above.

b) Solution



Specific behaviours
✓ reflects in line y=x ✓ inverse contains pt (5,3)

Q3 (3.1.14, 3.1.15) (3& 3 = 6 marks)

Consider the quadratic equation $x^2+bx+c=0$ where $b\&c$ are real.

a) If one root of the above equation is $x=4-2i$, determine $b\&c$.

Solution
$(4-2i)^2+b(4-2i)+c=0$ $16-4-16i+4b-2bi+c=0$ $12+4b+c=0$ $-16-2b=0$ $b=-8$ $c=-12+32=20$
Specific behaviours
✓ sets up one equation involving c or b ✓ sets up two equations for c & b ✓solves for one pair of real values for c & b

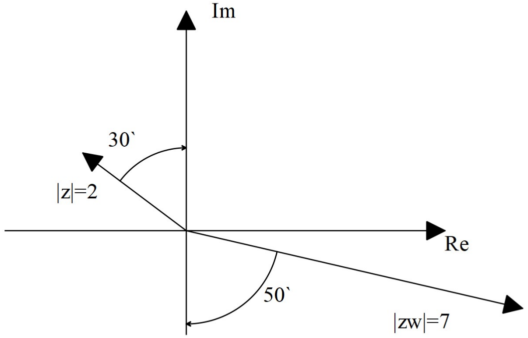
Consider the equation $x^3+px^2+qx+w=0$ where $p,q\&w$ are real.

b) If the cubic equation above has roots $x=2\&x=\sqrt{3}i$, determine $p,q\&w$.

Solution
$(x-2)(x-\sqrt{3}i)(x+\sqrt{3}i)$ $(x-2)(x^2+3)$ $x^3+3x-2x^2-6$ x^3-2x^2+3x-6 $p=-2,q=3\&w=-6$
Specific behaviours
✓ recognises that conjugate also a root ✓ expands at least two linear factors ✓solves for p,q&w

Q4 (3.1.3, 3.1.3, 3.1.3) (2 marks)

Determine $z \& w$ in the form $r cis\theta$ with $-\pi < \theta \le \pi$. (Note: diagram not drawn to scale)



Solution
$z = 2cis120 \ or \ 2cis\frac{2\pi}{3}$ $w = \frac{7}{2}cis(-160) \ or \ \frac{7}{2}cis\left(\frac{-8\pi}{9}\right)$ Accept radians or degrees
Specific behaviours
✓ determines z with principal argument ✓ determines w with principal argument

Q7 (3.2.1, 3.2.2) (1, 2, 2 & 2 = 7 marks)

Consider the functions $f(x) = \sqrt{x-8}$ & $g(x) = x^3$.

a) Give the defining rule for $f \circ g(x)$.

Solution
$f \circ g(x) = \sqrt{x^3 - 8}$
Specific behaviours
✓ states rule

a) Does $f \circ g(x)$ exist over the natural domain of $g(x)$? Explain

Solution
$r_g : R$ $d_f : x \ge 8$ $r_g \not\subset d_f \quad \therefore does \ not \ exist$
Specific behaviours
✓ determine appropriate domain and range ✓ shows that condition not meet for natural domain of g

b) State the natural domain and range for $f \circ g(x)$.

Solution
$f \circ g(x) = \sqrt{x^3 - 8}$ $x^3 \ge 8$ $x \ge 2$ $y \ge 0$
Specific behaviours
✓ states natural domain ✓ states range

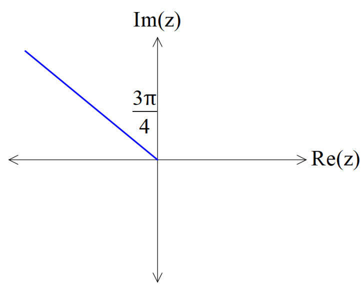
✓all five roots equally spaced

Q5 (3.1.10) (2, 2 & 3 = 7 marks)

Sketch the following regions in the complex plane showing major features.

a) $Arg(z) = \dfrac{3\pi}{4}$

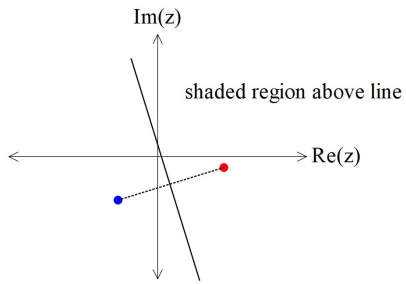
Solution



Specific behaviours
✓ uses line y=-x
✓ only drawn for Re(z)<0 (allow zero)

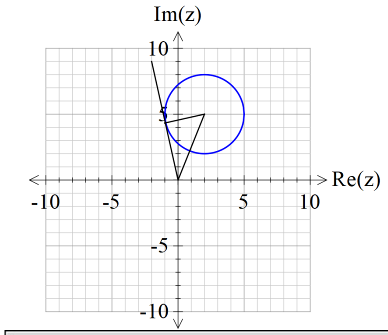
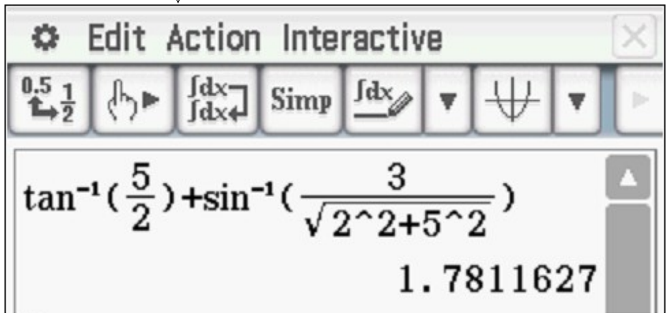
b) $|z+3+4i| \geq |z-5+i|$

Solution



Specific behaviours
✓ plots correct points
✓ shades above perpendicular bisector

c) Consider all the complex numbers z that satisfy $|z-(2+5i)|=3$, determine the maximum possible value of $Arg(z)$, giving your answer in radians correct to two decimal places.

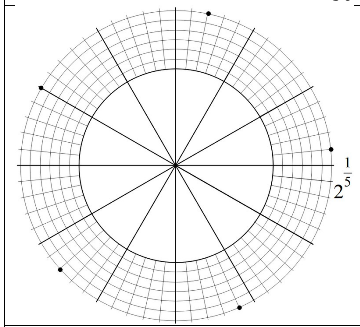
Solution
 
Specific behaviours
✓ determines argument of centre of circle ✓ uses left tangent idea for max argument ✓solves for max argument in radians(no need to round to 2 dp)

Q6 (3.1.7, 3.1.12) (4 & 3=7 marks)

a) Determine all the roots of $z^5=\sqrt{3}+i$ expressing in the form $rcis\theta$ with $-\pi<\theta\le\pi$.

Solution
$z^5=\sqrt{3}+i=2cis\left(\frac{\pi}{6}+2n\pi\right)$, $n=0,\pm1,\pm2...$ $z=2^{\frac{1}{5}}cis\left(\frac{\pi}{30}+\frac{2n\pi}{5}\right)=2^{\frac{1}{5}}cis\left(\frac{\pi}{30}+\frac{12n\pi}{30}\right)$ $z_1=2^{\frac{1}{5}}cis\left(\frac{\pi}{30}\right)$ $z_2=2^{\frac{1}{5}}cis\left(\frac{13\pi}{30}\right)$ $z_3=2^{\frac{1}{5}}cis\left(-\frac{11\pi}{30}\right)$ $z_4=2^{\frac{1}{5}}cis\left(\frac{-23\pi}{30}\right)$ $z_5=2^{\frac{1}{5}}cis\left(\frac{25\pi}{30}\right) or\ 2^{\frac{1}{5}}cis\left(\frac{5\pi}{6}\right)$
Specific behaviours
✓ expresses right hand side into polar form ✓ uses De Moivre's theorem ✓ obtains five distinct roots in polar form ✓uses principal arguments for all roots

b) Plot all of these roots on the diagram below.

Solution

Specific behaviours
✓ shows scale ✓ plots one root correctly