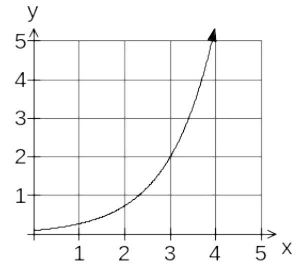
THE LOGISTIC FUNCTION

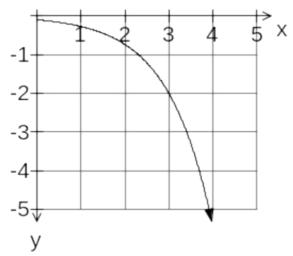
Exponential Functions

Consider the functions below:

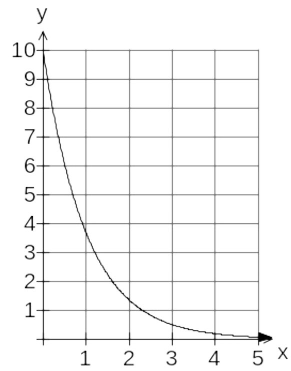
Unbounded exponential functions:



$$y = \frac{1}{10}\left(e^{x}\right)$$

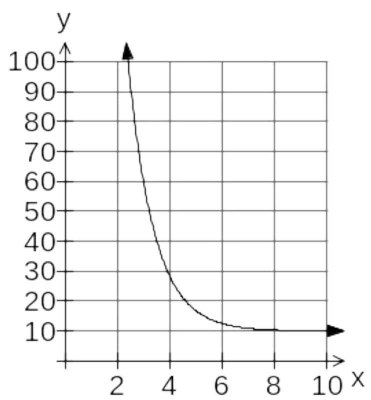


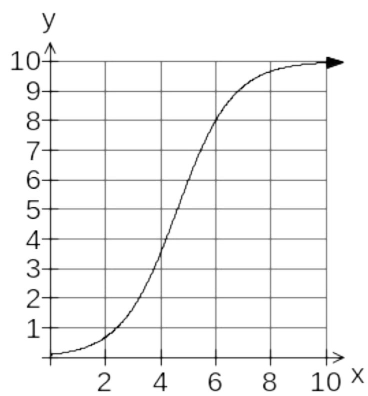
$$y = -\frac{1}{10}\left(e^{-x}\right)$$



$$y = 10e^{-x}$$

Bounded exponential function:



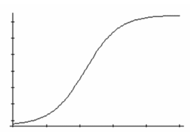


$y = 10 \times \left(1 + 100e^{-x}\right)$ but $y = 10 \times \frac{1}{\left(1 + 100e^{-x}\right)}$ seems to be bounded.

Use limits to find the upper limit.

The logistic function models a population that has a maximum carrying capacity. For example, If a number of rabbits are placed on an island, they thrive and populate until they reach the population that is sustainable by the amount of food available on the island. Likewise, if the number of rabbits put on the island exceeds the carrying capacity of the island, the population decreases until the population becomes stable given the food supply.

The number of bacteria in a petri dish doubles every few minutes. This is exponential growth, but the growth cannot continue indefinitely. The rate of increase slows as the dish “fills”.

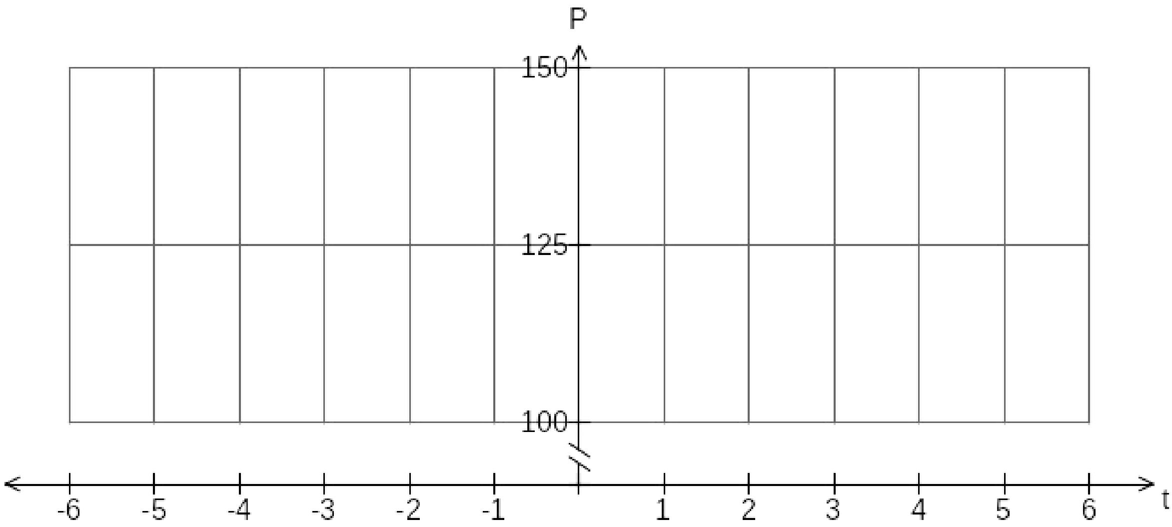


INVESTIGATING Exponential Functions

1. Consider the following functions

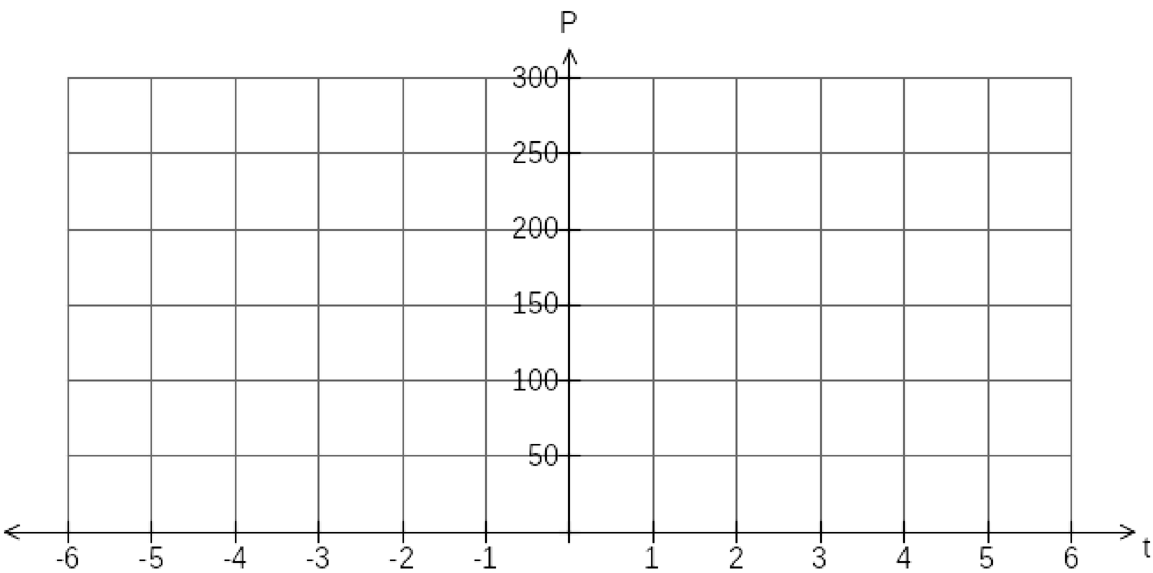
(a) $f(t) = \dfrac{600}{(4 + 11e^{-t})}$

Determine the limit as $t \to 0$, and as $t \to \infty$. Sketch the graph.



(b) $f(t) = \dfrac{3000}{(10 + 40e^{-t})}$

Determine the limit as $t \to 0$, and as $t \to \infty$. Sketch the graph.



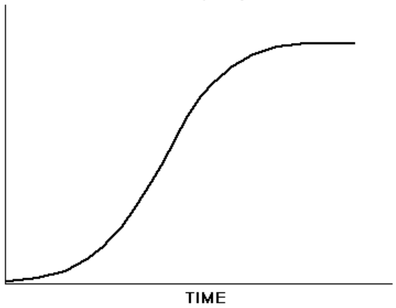
2. Sketch the following graphs

(a) $$P(t) = \frac{2500}{(200 + 50e^{-t})}$$

(b) $$P(t) = \frac{450}{(5 + 5e^{-t})}$$

(c) $$P(t) = \frac{200}{(20 + 30e^{-t})}$$

3. Experiment with the sketch the function $P(t) = \dfrac{200}{20 + be^{-t}}$ for b = 1, 10, 20, 40, 100, 1000.
Comment on the effect of changing the value of b.
Comment on the effect of changing the 200.
Consider the graphs you have drawn but using the domain $x \geq 0$.
Which of them models the graph below? What does this imply about b?



4. Experiment with the sketch the function $P(t) = \dfrac{200}{20 + 1000e^{-kt}}$ for k = 1, 2, 5.
Comment on the effect of changing the value of k.

5. Show that the function $P(t) = \dfrac{2500e^{t}}{(50e^{t} + 200)}$ can be expressed in the form $P(t) = \dfrac{c}{a + be^{-t}}$.

6. Given $P(t) = \dfrac{c}{a + be^{-t}}$, determine $\lim\limits_{t \to 0}\left(\dfrac{c}{a + be^{-t}}\right)$ and $\lim\limits_{t \to \infty}\left(\dfrac{c}{a + be^{-t}}\right)$.

Sketch the function.

Investigate the function for different values of a, b and c.

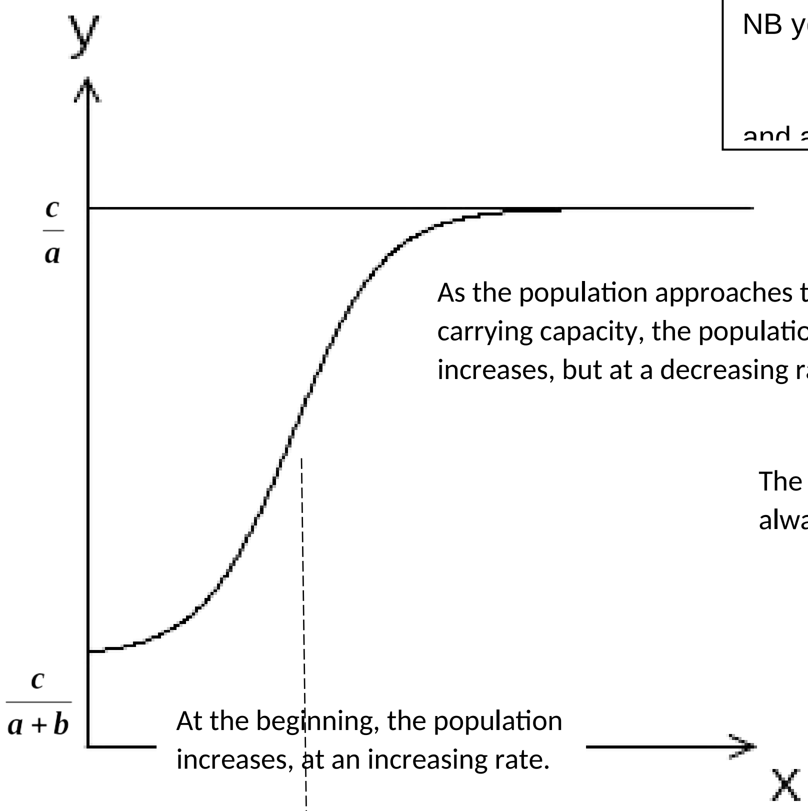
THE LOGISTIC FUNCTION

The logistic equation is in the form $y = \dfrac{c}{a + be^{-x}}$ where $0 < a < b$ and $c > 0$.

$y = \dfrac{c}{a + be^{-x}}$ where $0 < a < b$ and $c > 0$. has the graph

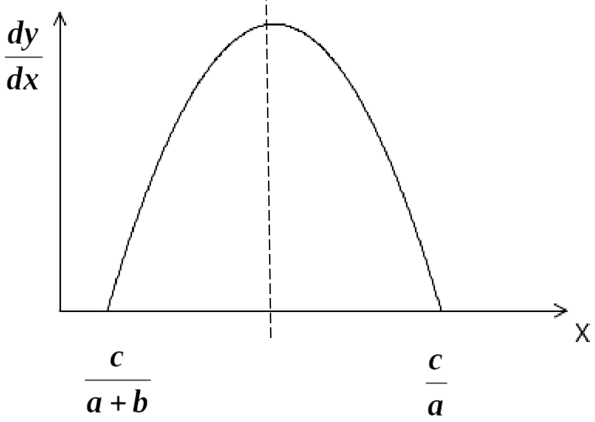
NB y(0)= $\dfrac{c}{a+b}$

and as $x \to \infty,\ e^{-x} \to 0$ so $y \to \dfrac{c}{a}$



NB $\dfrac{c}{a+b} < \dfrac{c}{a}$ for all positive a,b,c with $a < b$

The gradient function can been seen to be of the form



Since the logistic equation has an increasing component and a decreasing component we use

$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$ where P is the population and K is the carrying capacity.

Assuming the initial population is less than the **carrying capacity** we have

$K > P$ so $\frac{P}{K} < 1 \Rightarrow \quad 0 < 1 - \frac{P}{K} < 1$

P increases as $t \to \infty$

$\frac{P}{K} \to 1$ as $t \to \infty$, so $\left(1 - \frac{P}{K}\right) \to 0$ as $t \to \infty$

The **carrying capacity** of a biological species in an environment is the maximum population size of the species that the environment can sustain indefinitely, given the food, habitat, water, and other necessities available in the environment.

Investigate the values of $\frac{dP}{dt}$, where $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$ at P close to P_0, P close to K and P greater than K. Assume $r = 0.5$ and $K = 500.$

At

$P = 10, \ \frac{dP}{dt} =$

$P = 250, \ \frac{dP}{dt} =$

$P = 490, \ \frac{dP}{dt} =$

$P = 600, \ \frac{dP}{dt} =$

Sketch the derivative function:

To find the logistic function from the derivative function:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

so $\dfrac{dt}{dP} = \dfrac{1}{rP\left(1 - \frac{P}{K}\right)}$ Reciprocate both sides

$$\Rightarrow rt = \int \frac{1}{P\left(1 - \frac{P}{K}\right)}\, dP$$

$$\frac{1 + 0 \times P}{P\left(1 - \frac{P}{K}\right)} = \frac{a}{P} + \frac{b}{\left(1 - \frac{P}{K}\right)}$$ Using partial fractions

$$= \frac{a\left(1 - \frac{P}{K}\right) + bP}{P\left(1 - \frac{P}{K}\right)}$$

$$\frac{0 \times P + 1}{P\left(1 - \frac{P}{K}\right)} = \frac{P\left(-\frac{a}{K} + b\right) + a}{P\left(1 - \frac{P}{K}\right)}$$

$\therefore a = 1$ and $0 = -\dfrac{1}{K} + b \Rightarrow b = \dfrac{1}{K}$

$$\therefore \frac{1}{P\left(1 - \frac{P}{K}\right)} = \frac{1}{P} + \frac{1/K}{1 - \frac{P}{K}}$$

$= \dfrac{1}{P} + \dfrac{1}{K - P}$ after multiplying by $\dfrac{K}{K}$

$$\therefore rt = \int\left(\frac{1}{P} + \frac{1}{K - P}\right) dP$$

$$rt = \ln P - \ln(K - P) + c$$

$$rt - c = \ln \frac{P}{K - P}$$

$$\therefore \frac{P}{K - P} = e^{rt - c}$$

$\dfrac{P}{K - P} = Ae^{rt}$ where $A = e^{-c}$

$$\therefore P = Ae^{rt}(K - P)$$

$$P = \frac{KAe^{rt}}{1 + Ae^{rt}}$$

$$P = \frac{KAe^{rt}}{1+Ae^{rt}} \times \frac{1/e^{rt}}{1/e^{rt}}$$

$$P = \frac{KA}{1/e^{rt} + A}$$

$$P = \frac{KA}{e^{-rt} + A}$$

$$P = \frac{KA}{A + e^{-rt}}$$

where K is the carrying capacity, t is the time and A and r are constants.

At t = 0, $P = P_0$

$$P_0 = \frac{KA}{A+1}$$

$$\Rightarrow P_0(A+1) = KA$$

$$A(K - P_0) = P_0$$

$$\therefore A = \frac{P_0}{K - P_0}$$

$$\therefore P = \frac{K \times \left(\frac{P_0}{K - P_0}\right)}{\frac{P_0}{K - P_0} + e^{-rt} \times \left(\frac{K - P_0}{K - P_0}\right)}$$

where $\left(\frac{K - P_0}{K - P_0}\right) = 1$

$$P = \frac{K \times \left(\frac{P_0}{K - P_0}\right)}{\frac{P_0}{K - P_0} + e^{-rt} \times \left(\frac{K - P_0}{K - P_0}\right)} \times \left(\frac{K - P_0}{K - P_0}\right)$$

$$P = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

Logistic growth can be modelled by the equation $P = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$ where P_0 is the initial population, and K is the carrying capacity, and r a constant related to the rate of growth.

EXAMPLES

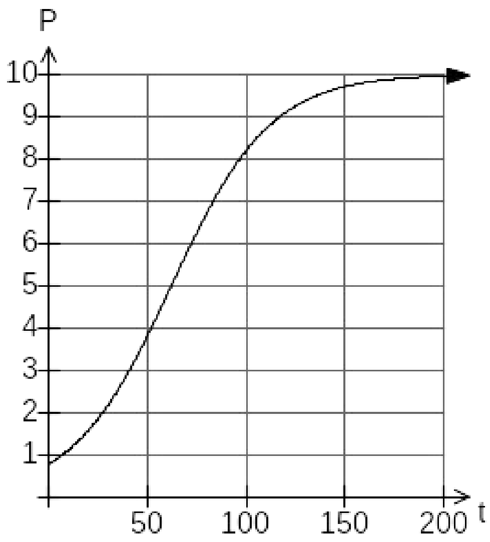
1. (a) Plot the following function $P(t)=\dfrac{10}{1+12e^{-0.04t}}$ on your calculator to confirm it has the S shape expected of the logistic function.

(b) Find an expression for $P'(t)$ to confirm slope of the function P is always positive.

(c) Use the expression $P(t)=\dfrac{10}{1+12e^{-0.04t}}$ to determine the limiting value of $P(t)$ as $t\to 0$ and $t\to\infty$.

Solution:

1. (a)



(b) $P(t)=\dfrac{10}{1+12e^{-0.04t}}$

$$P'(t)=\frac{-12e^{-0.04t}\times(-0.04)}{\left(1+12e^{-0.04t}\right)^2}$$

$$P'(t)=\frac{0.48e^{-0.04t}}{\left(1+12e^{-0.04t}\right)^2}$$

$e^{-0.04t}>0$ and $\left(1+12e^{-0.04t}\right)^2>0$

Therefore the slope is always increasing.

(c) $\lim\limits_{t\to 0}\dfrac{10}{1+12e^{-0.04t}}=\dfrac{10}{1+12}=\dfrac{10}{13}$ and $\lim\limits_{t\to\infty}\dfrac{10}{1+12e^{-0.04t}}=\dfrac{10}{1+0}=10$

2. A fish farmer stocks a large pond with 800 fish. He estimated that the carrying capacity of the pond is about 9,000 fish. From experience he expects that that the fish population will double during the first year.

   (a) Use the logistic equation to determine an expression for the fish population after t years.

   (b) How long will it take the fish population to reach 6000?

Solution:

(a) Using $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$ where P is the population of fish at time t, r is the constant related to the rate of growth and K is the carrying capacity, we have

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

$$= \frac{9,000 \times 800}{800 + (9,000 - 800)e^{-rt}} \times \frac{1/800}{1/800}$$

$$= \frac{9,000}{1 + \left(\frac{9,000}{800} - 1\right)e^{-rt}}$$

$$= \frac{9,000}{1 + (11.25 - 1)e^{-rt}}$$

$$P(t) = \frac{9,000}{1 + (10.25)e^{-rt}}$$

At the end of the first year, the number of fish will have doubled. i.e. $P(1) = 2 \times 800$

We can now solve for r.

$$At \ \ t = 1$$

$$1600 = \frac{9000}{1 + (10.25)e^{-r \times 1}}$$

$$1 + (10.25)e^{-r} = \frac{9000}{1600}$$

$$(10.25)e^{-r} = 5.625 - 1$$

$$e^{-r} = \frac{4.625}{10.25}$$

$$-r = \ln\left(\frac{4.625}{10.25}\right)$$

$$-r = -0.7958013345$$

$$r = 0.7958013345$$

Therefore we have

$$P(t) = \frac{9{,}000}{1 + (10.25)e^{-0.7958013345t}}$$

(b) Find t for $P(t)$ = 6000.

If using Solve on the calculator. You may find it quicker to rearrange the equation so that there is no denominator (to avoid the calculator freezing).

$$6000 = \frac{9000}{1 + (10.25)e^{-0.7958013345t}}$$

$$1 + (10.25)e^{-0.7958013345t} = \frac{9000}{6000}$$

$$(10.25)e^{-0.7958013345t} = \frac{3}{2} - 1$$

$$e^{-0.7958013345t} = \frac{0.5}{10.25}$$

$$-0.7958013345t = \ln\left(\frac{0.5}{10.25}\right)$$

$$t = \frac{-3.020424886}{-0.7958013345}$$

$$t = 3.795$$

i.e. three years 9.5 months for the population of fish to reach 6000.

3. In a small village of 2 000 people, a new and dangerous flu is spreading slowly.

Today, 5 people have been diagnosed with the flu.

The rate of number of people catching the flu is given by $\frac{dN}{dt} = 0.00005N(2000 - N)$ where N represents the number of people with the flu and t represents weeks.

(a) Find an expression for the number of people infected.

(b) Determine the expected number of people to be infected in 5 weeks time.

(c) Determine how many weeks it will take for the number of infected people to double.

Solution

(a) $$\frac{dN}{dt} = 0.00005N(2000 - N)$$

$$\therefore \frac{dt}{dN} = \frac{20\,000}{N(2000 - N)}$$

$$\frac{t}{20000} = \int \frac{1}{N(2000 - N)}\, dN$$

http://summaryonarticle.blogspot.com.au/2013/01/mount-sinai-researchers-discover-how.html

Using partial fractions,

$$\frac{1}{N(2000 - N)} = \frac{a}{N} + \frac{b}{(2000 - N)}$$

$$= \frac{a(2000 - N) + bN}{N(2000 - N)}$$

$$\frac{0 \times N + 1}{N(2000 - N)} = \frac{N(b - a) + 2000a}{N(2000 - N)}$$

$$0 = b - a \text{ and } 1 = (2000)a$$

$$a = b \text{ and } a = \frac{1}{2000} = 0.0005$$

$$\frac{1}{N(2000 - N)} = \frac{1}{2000N} + \frac{1}{(2000)(2000 - N)}$$

$$\frac{t}{20000} = \int \frac{1}{2000N} + \frac{1}{(2000)(2000 - N)}\, dN$$

$$= \frac{1}{2000}\left[\int \frac{1}{N}\, dN + \int \frac{1}{(2000 - N)}\, dN\right]$$

$$\frac{2000t}{20000} = \ln N + (-1)\ln(2000 - N) + C$$

$$0.1t - C = \ln\left(\frac{N}{2000 - N}\right)$$

$$\frac{N}{2000-N}=e^{0.1t-C}$$
$$\frac{N}{2000-N}=Ae^{0.1t} \quad where\ A=e^{-C}$$

Rearrange the formula to get N=

$$N=Ae^{0.1t}(2000-N)$$
$$N\left(1+Ae^{0.1t}\right)=2000Ae^{0.1t}$$
$$N=\frac{2000Ae^{0.1t}}{\left(1+Ae^{0.1t}\right)}\times\frac{1/e^{0.1t}}{1/e^{0.1t}}$$
$$N=\frac{2000A}{e^{-0.1t}+A}$$
$$N=\frac{2000A}{A+e^{-0.1t}}$$

At $t=0, N=5.$ Solve for A

$$5=\frac{2000A}{A+e^{0}}$$
$$5(A+1)=2000A$$
$$A(2000-5)=5$$
$$A=\frac{5}{(1995)}$$
$$A=0.002506265664$$
$$\text{but }N=\frac{2000A}{A+e^{-0.1t}}$$
$$\therefore N=\frac{5.012531328}{0.002506265664+e^{-0.1t}}$$

(b) In five weeks time, $t = 5, N = ?$

$$N_5=\frac{5.012531328}{0.002506265664+e^{-0.1\times 5}}$$
$$N_5=8.23925$$

i.e. in 5 weeks time, expect 8 people to have the flu.

(c) If $N = 10, \; t = ?$

$$10 = \frac{5.012531328}{0.002506265664 + e^{-0.1t}}$$

$$0.002506265664 + e^{-0.1t} = \frac{5.012531328}{10}$$

$$e^{-0.1t} = 0.5012531328 - 0.002506265664$$

$$e^{-0.1t} = 0.4987468671$$

$$t = \frac{\ln(0.4987468671)}{-0.1}$$

$t = 6.957$ weeks

The number of people with the flu will double in about 7 weeks.

Verhulst published in 1838 the equation:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

when $N(t)$ represents number of individuals at time t, r the intrinsic growth rate and K is the carrying capacity, or the maximum number of individuals that the environment can support.

In a paper published in 1845 he called the solution to this the logistic function.

This model was rediscovered in 1920 by Raymond Pearl and Lowell Reed, who promoted its use.

https://en.wikipedia.org/wiki/Pierre_Fran%C3%A7ois_Verhulst

http://wmueller.com/precalculus/families/1_80.html

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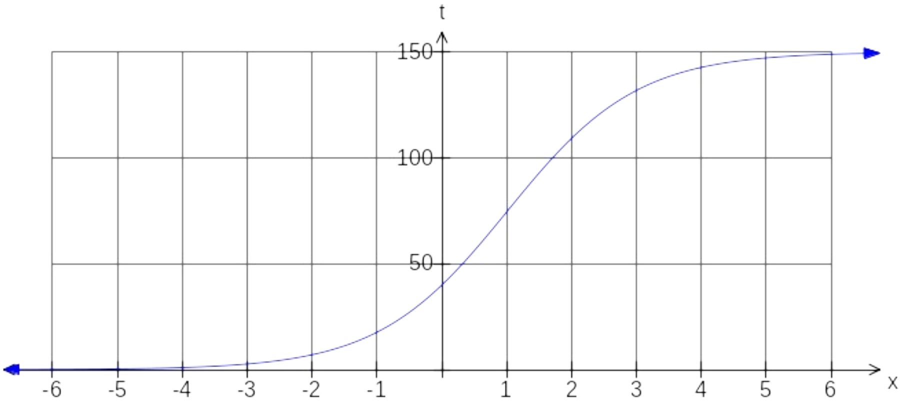
Solutions to the investigation

1. (a) $$f(t) = \frac{600}{(4 + 11e^{-t})}$$

$$\lim_{t \to 0} \left(\frac{600}{(4 + 11e^{-t})} \right) = \frac{600}{15} = 40$$

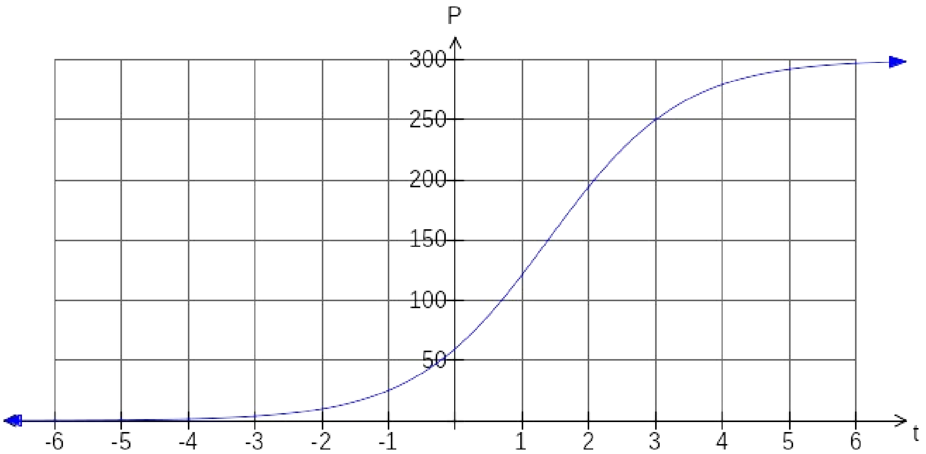
$$\lim_{t \to \infty} \left(\frac{600}{(4 + 11e^{-t})} \right) = \frac{600}{4 + 0} = 150$$



(b) $$f(t) = \frac{3000}{(10 + 40e^{-t})}$$

$$\lim_{t \to 0} \left(\frac{3000}{(10 + 40e^{-t})} \right) = \frac{3000}{50} = 60$$

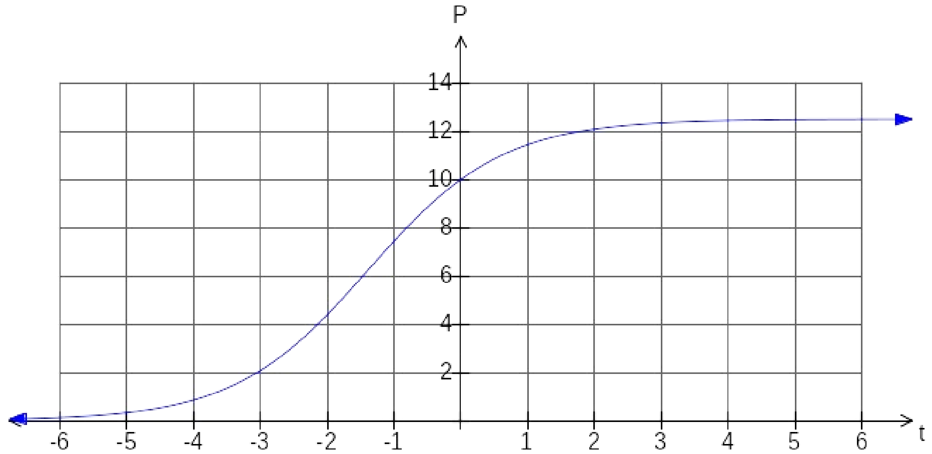
$$\lim_{t \to \infty} \left(\frac{3000}{(10 + 40e^{-t})} \right) = \frac{3000}{10} = 300$$



2. (a) $$P(t) = \frac{2500}{(200 + 50e^{-t})}$$

$$\lim_{t \to 0} \left(\frac{2500}{(200 + 50e^{-t})} \right) = \frac{2500}{250} = 10$$

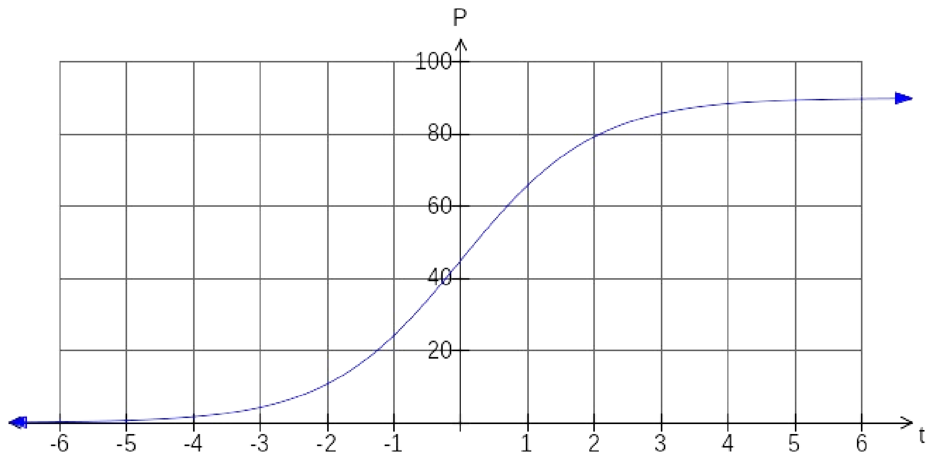
$$\lim_{t \to \infty} \left(\frac{2500}{(200 + 50e^{-t})} \right) = \frac{2500}{200} = 12.5$$



(b) $$P(t) = \frac{450}{(5 + 5e^{-t})}$$

$$\lim_{t \to 0} \left(\frac{450}{(5 + 5e^{-t})} \right) = \frac{450}{10} = 45$$

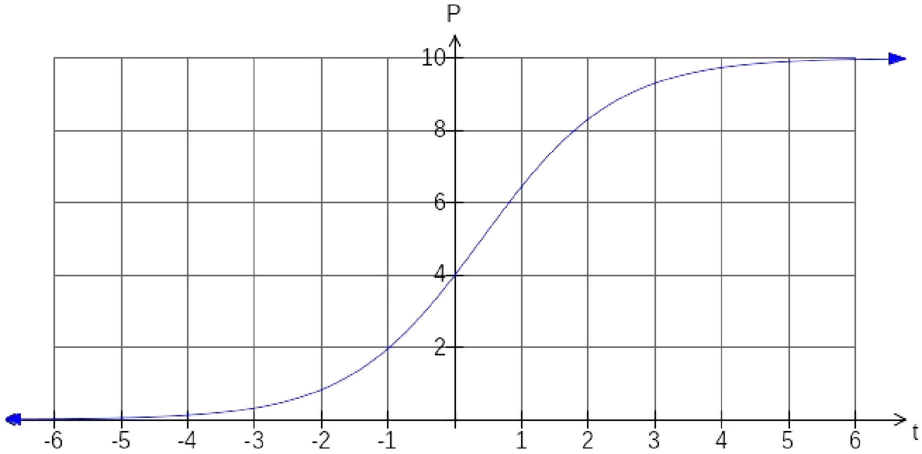
$$\lim_{t \to \infty} \left(\frac{450}{(5 + 5e^{-t})} \right) = \frac{450}{5} = 90$$



(c) $$P(t) = \frac{200}{\left(20 + 30e^{-t}\right)}$$

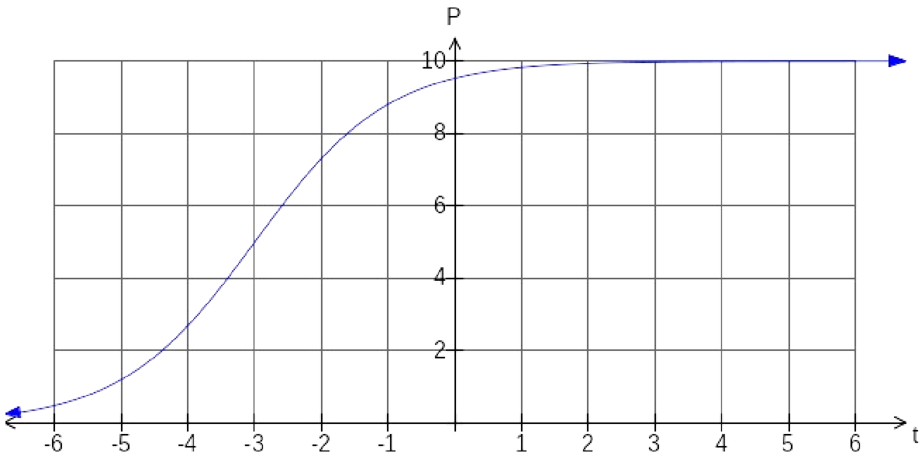
$$\lim_{t \to 0}\left(\frac{200}{\left(20 + 30e^{-t}\right)}\right) = \frac{200}{50} = 4$$

$$\lim_{t \to \infty}\left(\frac{200}{\left(20 + 30e^{-t}\right)}\right) = \frac{200}{20} = 10$$

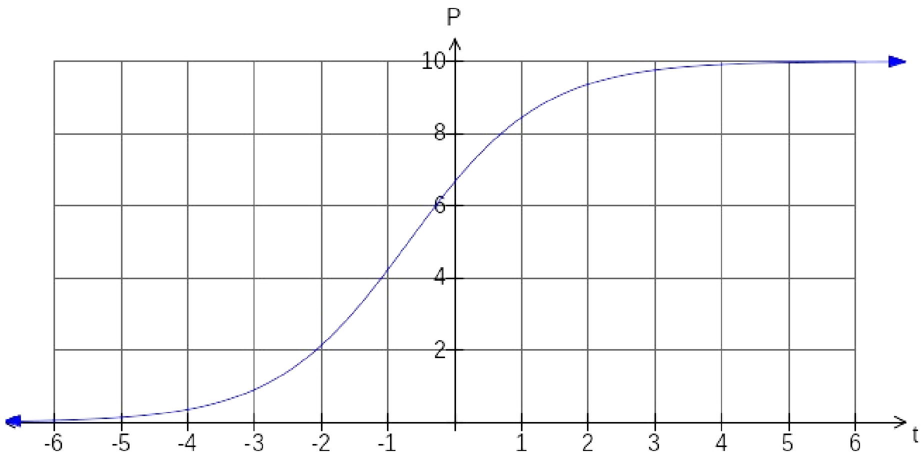


3. $$P(t) = \frac{200}{20 + be^{-t}}$$ b = 1, 10, 20, 40

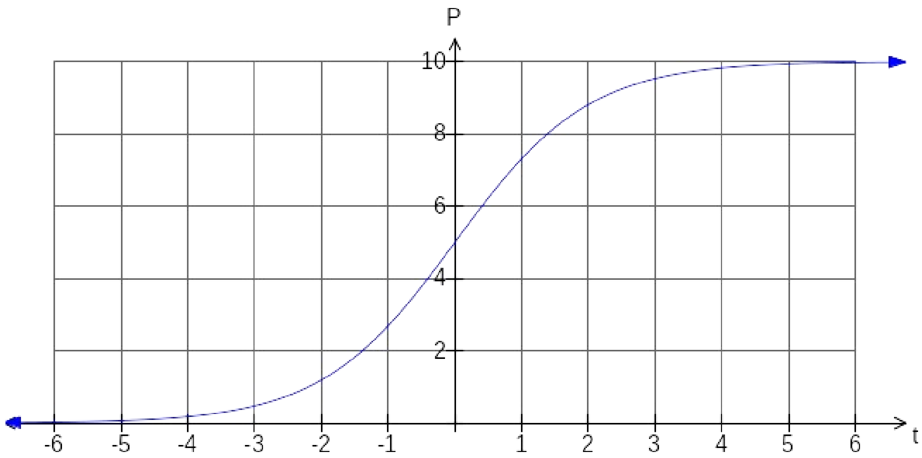
b = 1



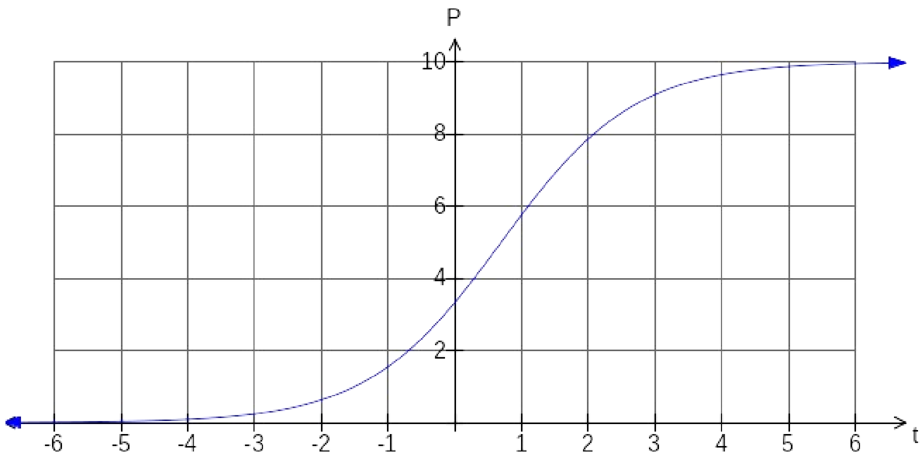
b = 10


P
10
8
6
4
2
-6 -5 -4 -3 -2 -1 1 2 3 4 5 6
t

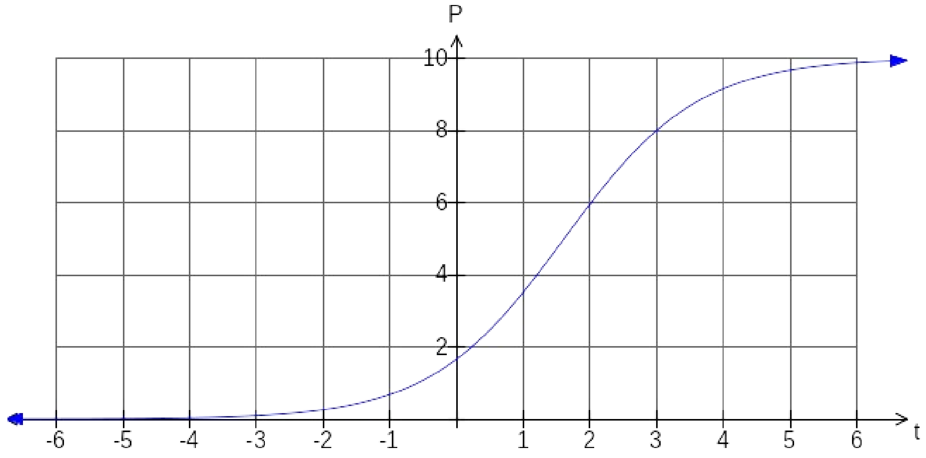
b = 20


P
10
8
6
4
2
-6 -5 -4 -3 -2 -1 1 2 3 4 5 6
t

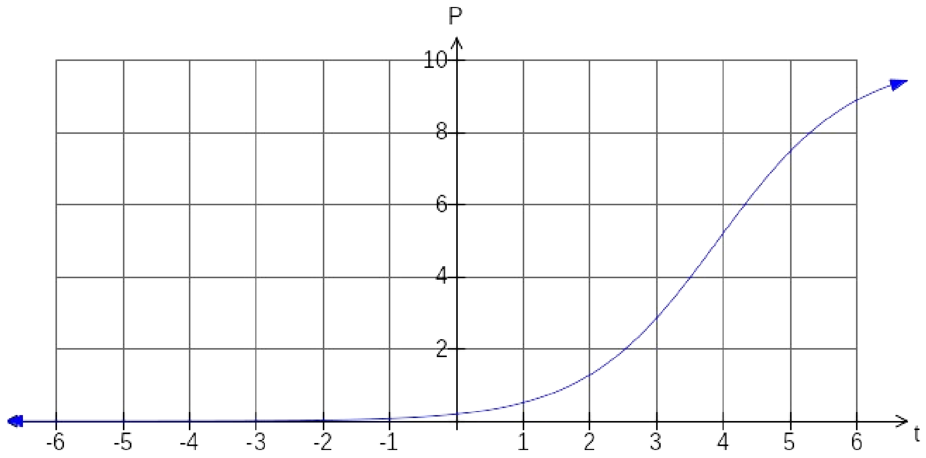
b = 40


P
10
8
6
4
2
-6 -5 -4 -3 -2 -1 1 2 3 4 5 6
t

b = 100



b = 1000

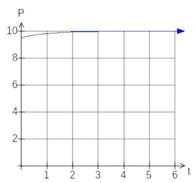


As b increases, the graph moves to the right.

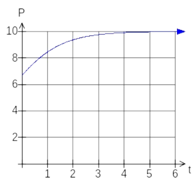
The 200 is the stretch factor with respect to the y axis.

With b = 1000, the model is like the logistic function for x ≥ 0.

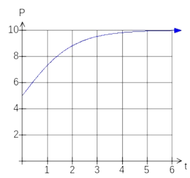
b =1



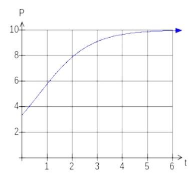
b =10



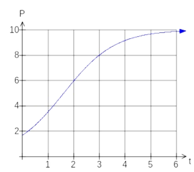
b =20



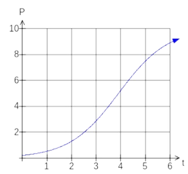
b = 40



b = 100



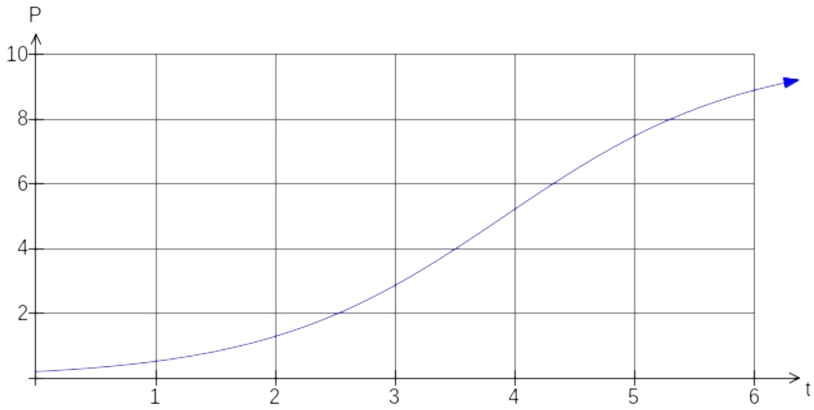
b = 1000



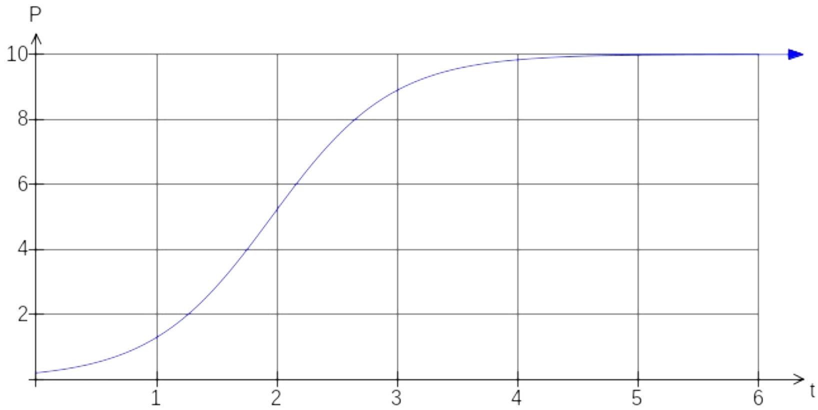
For b = 1000 the graph mostly resembles the logistic function.

If b is large compared to the value of 20 in the denominator, the graph resembles the logistic function for x ≥ 0.

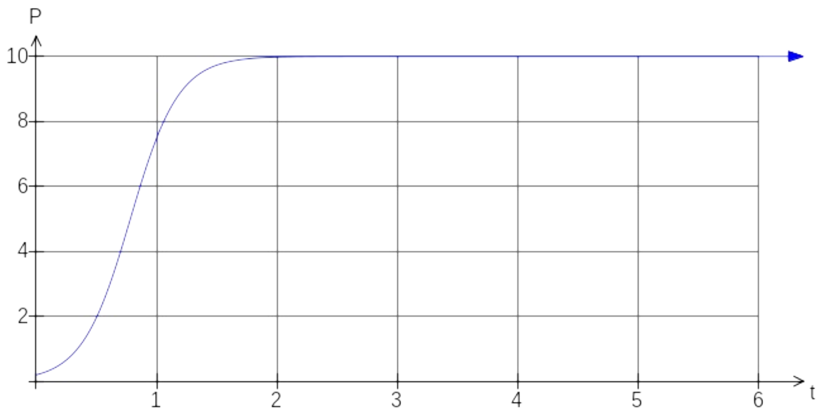
4. Experiment with the sketch the function $P(t) = \dfrac{200}{20 + 1000e^{-kt}}$ for k = 1, 2, 5, 0.1, 0.001.
Comment on the effect of changing the value of k.
k = 1



k = 2



k = 5



As k increases, the upper limit is reached at a faster rate.

5. Show that the function $P(t) = \frac{2500e^{t}}{(50e^{t}+200)}$ can be expressed in the form $P(t) = \frac{c}{a+be^{-t}}$.

$$\frac{2500e^{t}}{50e^{t}+200} = \frac{2500e^{t}}{50e^{t}+200} \times \frac{1/e^{t}}{1/e^{t}}$$

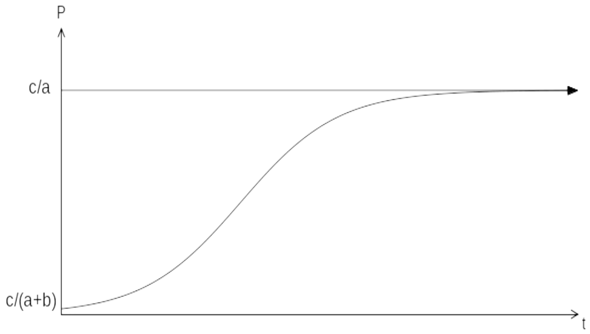
$$= \frac{2500}{50+200e^{-t}}$$

$$\therefore \quad P(t) = \frac{2500}{50+200e^{-t}}$$

6. Given $P(t) = \frac{c}{a+be^{-t}}$,

$$\lim_{t\to 0}\left(\frac{c}{a+be^{-t}}\right) = \frac{c}{a+b}$$

$$\lim_{t\to \infty}\left(\frac{c}{a+be^{-t}}\right) = \frac{c}{a}$$



As b decreases, the graph translates to the left.

a and b need to be examined together.

c is the vertical multiplying factor.