Functions Exponentials Logarithms

Questions are taken from VCE Secondary Papers

2009

Question 1

a. Differentiate $x\log_e(x)$ with respect to x.

b. For $f(x) = \dfrac{\cos(x)}{2x+2}$ find $f'(\pi)$.

Question 2

a. Find an anti-derivative of $\dfrac{1}{1-2x}$ with respect to x.

b. Evaluate $\displaystyle\int_1^4 \left(\sqrt{x}+1\right)dx$.

Question 6

Oil is leaking at a constant rate to form a circular puddle on the floor. The oil is being added to the puddle at the rate of 10 mm^3 per minute causing the puddle to spread out evenly, with constant depth of 2 mm.

When the radius of the puddle is r mm, the volume, V mm^3, of oil in the puddle is given by $V = 2\pi r^2$.

Find the rate of change of the radius of the puddle when the radius is 30 mm. Give an exact answer, with units of mm per minute.

Question 8

Let $f: R \rightarrow R, f(x) = e^x + k$, where k is a real number. The tangent to the graph of f at the point where $x = a$ passes through the point (0, 0). Find the value of k in terms of a.

Question 9

Solve the equation $2\log_e(x) - \log_e(x+3) = \log_e\left(\dfrac{1}{2}\right)$ for x.

Answers

1a. $x \times \frac{1}{x} + \log_e(x)$

$= 1 + \log_e(x)$

b. $f'(x) = \frac{(2x+2) \times -\sin(x) - \cos(x) \times 2}{(2x+2)^2}$

$f'(\pi) = \frac{1}{2(\pi+1)^2}$

2a. $-\frac{1}{2}\log_e(|1-2x|)$

b. $\int_1^4 (x^{\frac{1}{2}} + 1)dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + x\right]_1^4$

$= (\frac{2 \times 4^{\frac{3}{2}}}{3} + 4) - (\frac{2}{3} + 1)$

$= 7\frac{2}{3} = \frac{23}{3}$

6. $\frac{dV}{dr} = 4\pi r$

$\frac{dr}{dt} = \frac{dr}{dV}\frac{dV}{dt} = \frac{1}{4\pi r} \times 10$

$\frac{dr}{dt} = \frac{1}{12\pi}$ when $r = 30$

8. $f(a) = e^a + k$

$f'(x) = e^x, f'(a) = e^a$ so tangent is $y = e^a x$

Then solve by using common y-values $y_a = e^a a = f(a)$, or common gradients $\frac{e^a + k - 0}{a - 0} = e^a$

both lead to $e^a a = e^a + k$

$\Rightarrow k = e^a a - e^a = e^a(a-1)$

9. $\log_e\left(\frac{x^2}{x+3}\right) = \log_e\left(\frac{1}{2}\right)$

$\frac{x^2}{x+3} = \frac{1}{2} \Rightarrow 2x^2 = x+3$

$2x^2 - x - 3 = 0$

Solve (factorising or by formula) to get $x = \frac{3}{2}$ or $x = -1$.

Since x must be positive, the answer is

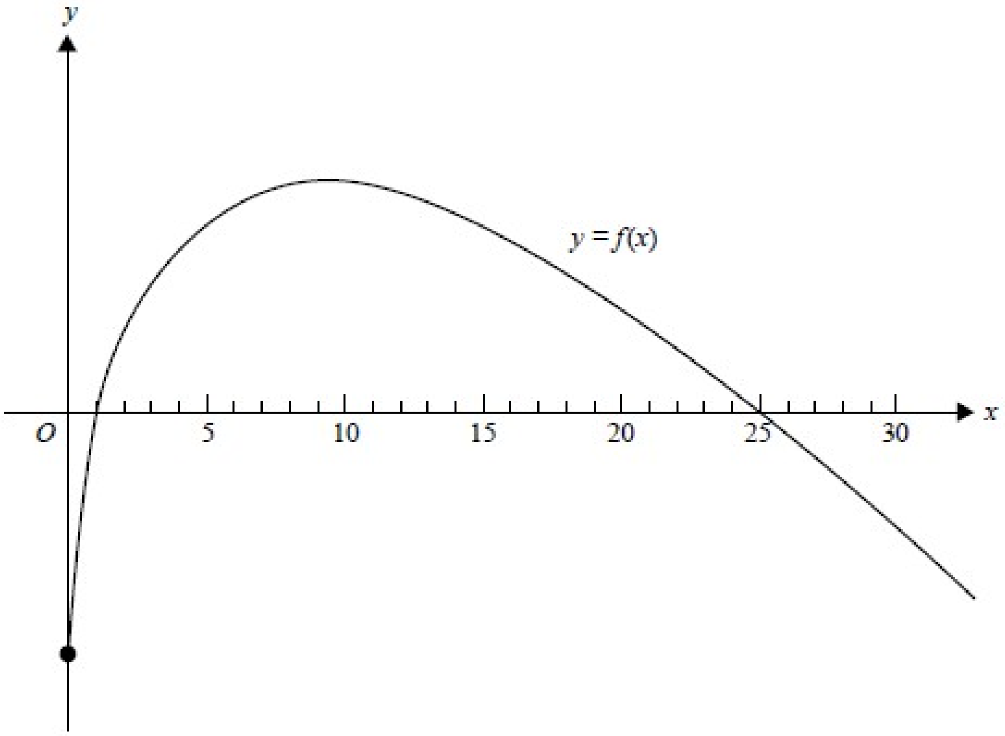
$x = \frac{3}{2}$.

More 2009

Question 1

Let $f: R^+ \cup \{0\} \to R,\ f(x) = 6\sqrt{x} - x - 5$.

The graph of $y = f(x)$ is shown below.

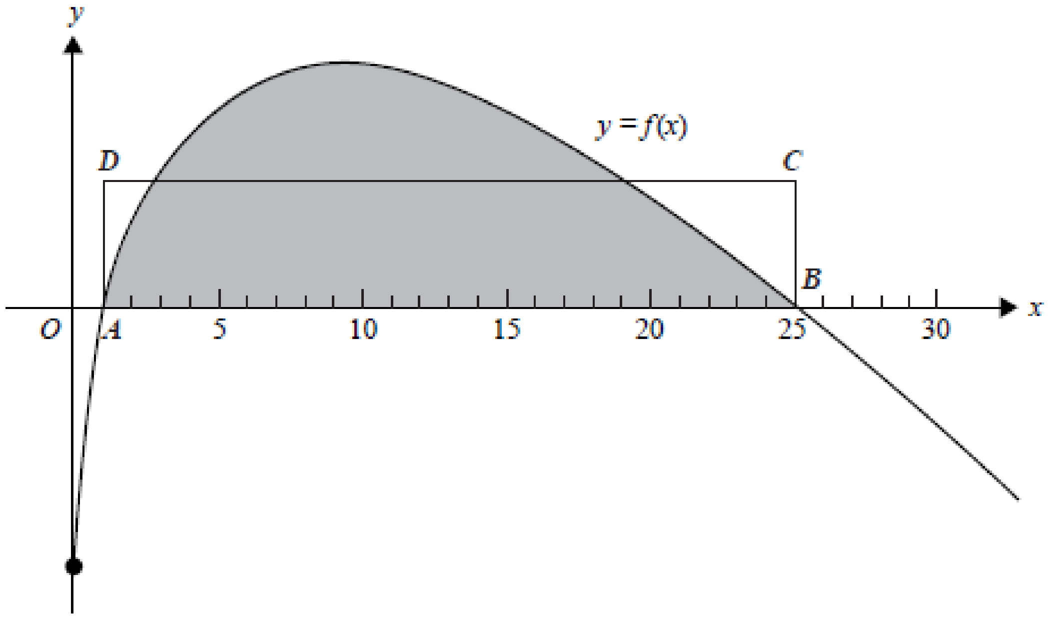


a. State the interval for which the graph of f is strictly decreasing.

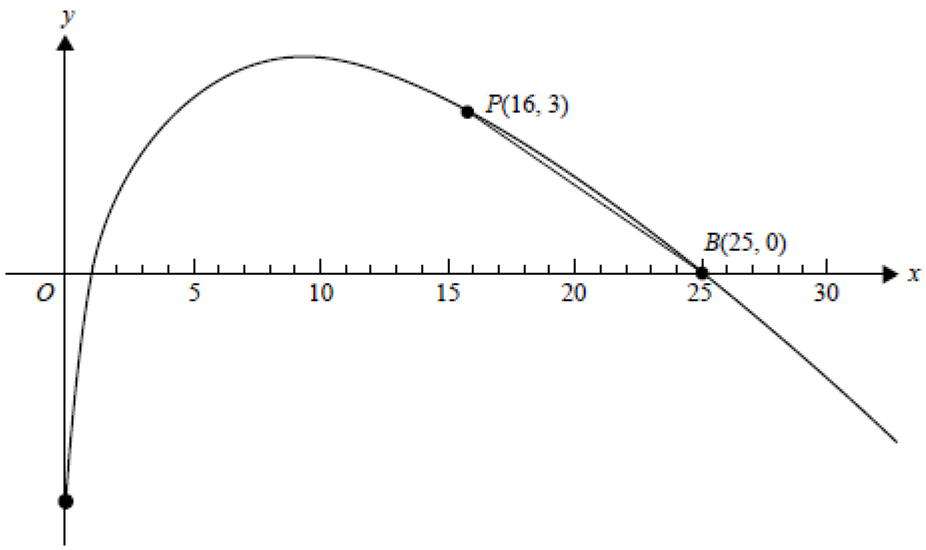
b. On the set of axes above, sketch the graph of $y = |f(x)|$.

c. Points A and B are the points of intersection of $y = f(x)$ with the x-axis. Point A has coordinates (1, 0) and point B has coordinates (25, 0).

Find the length of AD such that the area of rectangle $ABCD$ is equal to the area of the shaded region.



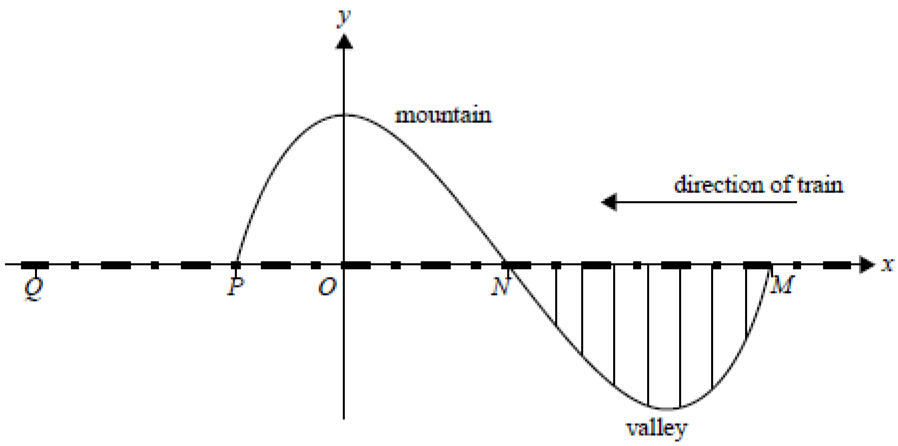
d. The points $P(16, 3)$ and $B(25, 0)$ are labelled on the diagram.



i. Find m, the gradient of the chord PB. (Exact value to be given.)

ii. Find $a \in [16, 25]$ such that $f'(a) = m$. (Exact value to be given.)

Question 2



A train is travelling at a constant speed of w km/h along a straight level track from M towards Q.
The train will travel along a section of track $MNPQ$.

Section MN passes along a bridge over a valley.
Section NP passes through a tunnel in a mountain.
Section PQ is 6.2 km long.

From M to P, the curve of the valley and the mountain, directly below and above the train track, is modelled by the graph of

$$y = \frac{1}{200}(ax^3 + bx^2 + c) \text{ where } a, b \text{ and } c \text{ are real numbers.}$$

All measurements are in kilometres.

a. The curve defined from M to P passes through $N(2, 0)$. The gradient of the curve at N is -0.06 and the curve has a turning point at $x = 4$.

  i. From this information write down three simultaneous equations in a, b and c.

  ii. Hence show that $a = 1$, $b = -6$ and $c = 16$.

b. Find, giving exact values

  i. the coordinates of M and P

  ii. the length of the tunnel

  iii. the maximum depth of the valley below the train track.

The driver sees a large rock on the track at a point Q, 6.2 km from P. The driver puts on the brakes at the instant that the front of the train comes out of the tunnel at P.
From its initial speed of w km/h, the train slows down from point P so that its speed v km/h is given by

$$v = k\log_e\left(\frac{(d+1)}{7}\right),$$

where d km is the distance of the front of the train from P and k is a real constant.

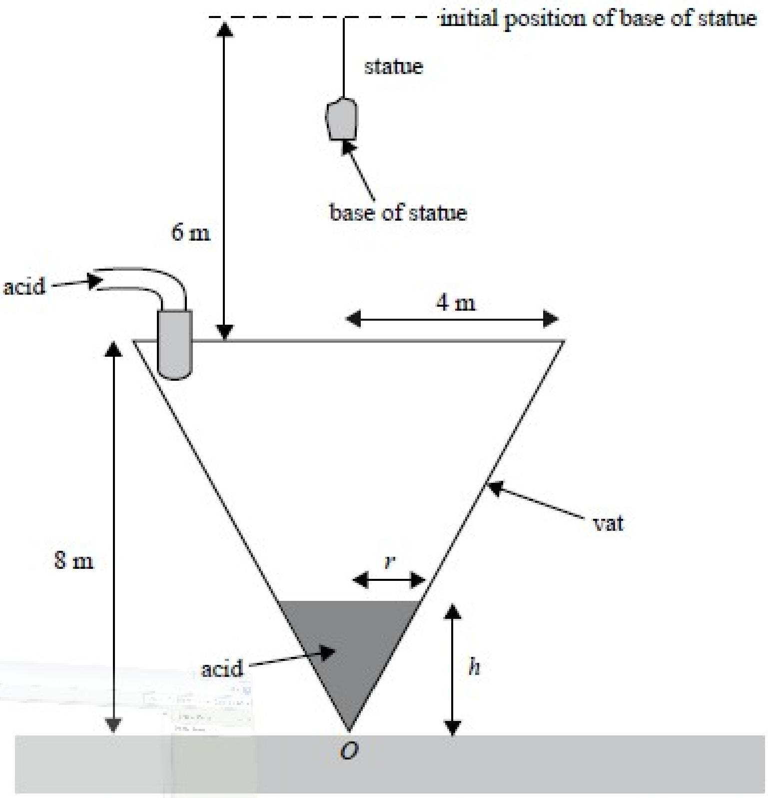
c. Find the value of k in terms of w.

d. If $v = \dfrac{120\log_e(2)}{\log_e(7)}$ when $d = 2.5$, find the value of w.

e. Find the exact distance from the front of the train to the large rock when the train finally stops.

Question 4

A Zambeji tribe has stolen a precious marble statue from Tasmania Jones. The statue has been tied to a rope and is suspended so that its base is initially 6 metres above the top of a vat. The vat is an inverted right circular cone with base radius 4 metres and height 8 metres.



At 9.00 am the tribe starts to lower the marble statue towards the vat at a rate of 1 metre per hour. At the same time acid begins to be poured into the vat at a constant rate of $\frac{9\pi}{4}$ m^3 per hour. The vat is initially empty. When the statue touches the acid, it will start to dissolve.

At time t hours after 9.00 am, the height of acid in the vat is h metres and the radius of the surface of the acid in the vat is r metres.

a. **i.** Show that $h = 2r$.

ii. Hence find an expression for the volume of acid in the vat at time t, in terms of h.

b. Show that $\frac{dh}{dt}$, the rate at which the height of the acid is increasing, is $\frac{9}{h^2}$ metres per hour.

c. Find, giving exact values

i. the rate at which the height of the acid is increasing when its height is 2 metres

ii. the height of the acid when it is increasing at half the rate found in **c. i.**

d. **i.** Write an expression for $\frac{dt}{dh}$ in terms of h.

ii. Hence find an expression for the height of the acid in terms of t.

Tasmania Jones will try to save the statue.

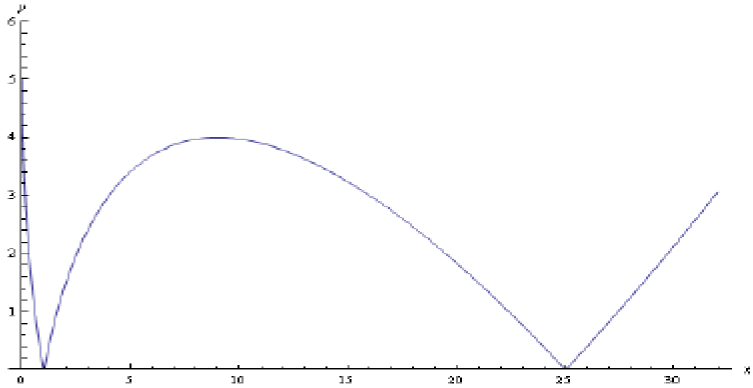
e. **i.** Write an expression for the distance of the base of the statue above ground level t hours after 9.00 am. (The vertex of the cone, O, is at ground level.)

ii. At what time would the statue first touch the acid?

Answers

1a. $f'(x)=0$, $x=9$, $[9,\ \infty)$ The interval for which the graph is strictly decreasing is when $f(b)<f(a)$ for $b>a$ and $a,\ b\in R$.

1b.



1c. $\int_1^{25}\left(f(x)\right)dx=64$, $AD=\frac{64}{24}=\frac{8}{3}$ or average value $=\frac{1}{25-1}\int_1^{25}\left(f(x)\right)dx=\frac{8}{3}$

1d. i. $m=\frac{3-0}{16-25}=-\frac{1}{3}$ ii. $f'(a)=-\frac{1}{3}$, $a=\frac{81}{4}$

$f(g(x))=6\sqrt{x^2}-x^2-5$ or $f(g(x))=6|x|-x^2-5$ or $f(g(x))=\begin{cases}6x-x^2-5, & \text{for } x\geq 0\\ -6x-x^2-5, & \text{for } x<0\end{cases}$ were acceptable equivalent forms.

2. a. i. $0=\frac{1}{200}(8a+4b+c)\ldots(1)$, $-0.06=\frac{1}{200}(12a+4b)\ldots(2)$, $0=\frac{1}{200}(48a+8b)\ldots(3)$

ii. (3) − (2), $3a=3$, $a=1$. Substitute $a=1$ into (2), $3+b=-3, b=-6$.
Substitute $a=1$ and $b=-6$ into (1), $0=8-24+c$, $c=16$.

b. i. $P\ (2-2\sqrt{3},0)$, $M\ (2+2\sqrt{3},0)$

ii. $2\sqrt{3}$ km

iii. $\frac{2}{25}=0.08$ km $=80$ m

c. $k=\frac{w}{\log_e\left(\frac{1}{7}\right)}=-\frac{w}{\log_e(7)}$

d. $\frac{120\log_e(2)}{\log_e(7)}=\frac{w}{\log_e\left(\frac{1}{7}\right)}\times\log_e\left(\frac{3.5}{7}\right)$, $w=120$

e. $v = 0 \Rightarrow 0 = \dfrac{120}{\log_e\left(\frac{1}{7}\right)}$, $\log_e\left(\dfrac{d+1}{7}\right) \Rightarrow d = 6$ km

Distance from $Q = 6.2 - 6 = 0.2$ km

4. a. i. By similar triangles, $\dfrac{h}{r} = \dfrac{8}{4} \Rightarrow h = 2r$

ii. $V = \dfrac{1}{3}\pi r^2 h,\ r = \dfrac{h}{2} \Rightarrow V = \dfrac{\pi h^3}{12}$

b. $\dfrac{dh}{dt} = \dfrac{dh}{dV} \times \dfrac{dV}{dt} = \dfrac{9\pi}{4} \times \dfrac{4}{\pi h^2} = \dfrac{9}{h^2}$

c. i. When $h = 2$, $\dfrac{dh}{dt} = \dfrac{9}{4}$ m/h

ii. When $\dfrac{dh}{dt} = \dfrac{9}{8}$, $\dfrac{9}{h^2} = \dfrac{9}{8} \Rightarrow h = 2\sqrt{2}$ m

d. i. $\dfrac{dt}{dh} = \dfrac{h^2}{9}$

ii. $t = \displaystyle\int \frac{h^2}{9}\,dh = \frac{h^3}{27} + c$, when $h = 0$, $t = 0$, so $t = \dfrac{h^3}{27}$, $h = 3t^{\frac{1}{3}}$

e. i. Height of statue above ground level $= 14 - t$

ii. $14 - t = 3t^{1/3}, \Rightarrow t = 8$ or 5.00 pm

2008

Question 1

a. Let $y = (3x^2 - 5x)^5$. Find $\dfrac{dy}{dx}$.

b. Let $f(x) = xe^{3x}$. Evaluate $f'(0)$.

Question 5

The area of the region bounded by the y-axis, the x-axis, the curve $y = e^{2x}$ and the line $x = C$, where C is a positive real constant, is $\dfrac{5}{2}$. Find C.

Answers

1. a. then $f^{1}(x) = 5(6x - 5)(3x^2 - 5x)^4$

   b. $f^{1}(x) = x.3e^{3x} + e^{3x}$

   $f^{1}(0) = 1$

5. $\int_0^c e^{2x}dx = \frac{5}{2}$

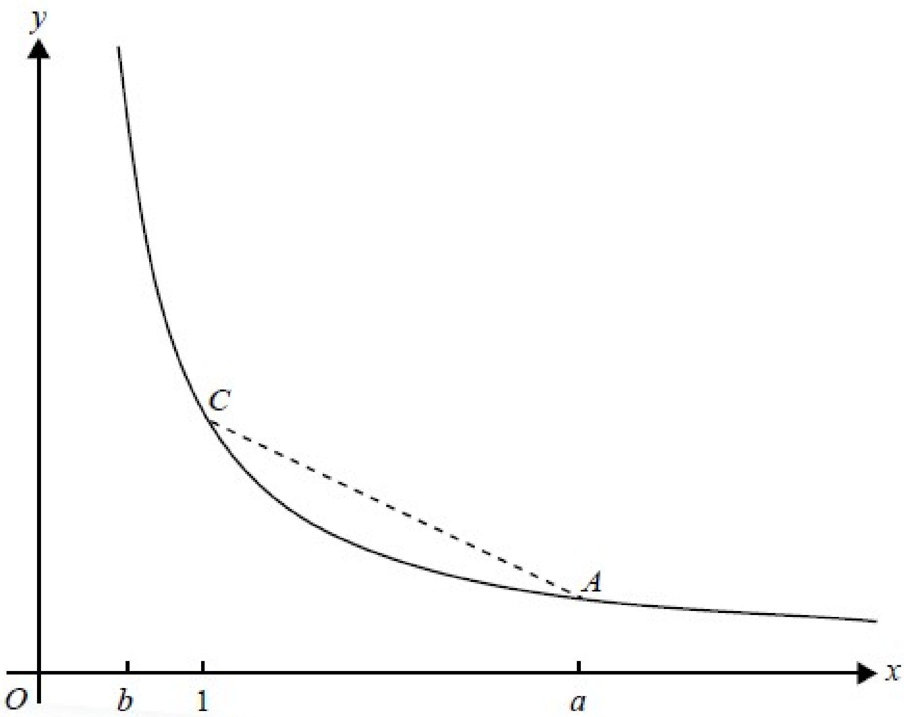
$\left[\frac{1}{2}e^{2x}\right]_0^c = \frac{5}{2}$

$e^{2c} - 1 = 5$

$c = \frac{1}{2}\log_e(6)$ or $\log_e(\sqrt{6})$

Question 2

The diagram below shows part of the graph of the function $f: R^+ \to R,\quad f(x) = \frac{7}{x}$.



The line segment CA is drawn from the point $C(1,\ f(1))$ to the point $A(a,\ f(a))$ where $a > 1$.

a. **i.** Calculate the gradient of CA in terms of a.

ii. At what value of x between 1 and a does the tangent to the graph of f have the same gradient as CA?

b. **i.** Calculate $\int_{1}^{e} f(x)dx$.

ii. Let b be a positive real number less than one. Find the exact value of b such that $\int_{b}^{1} f(x)dx$ is equal to 7.

c. **i.** Express the area of the region bounded by the line segment CA, the x-axis, the line $x = 1$ and the line $x = a$ in terms of a.

ii. For what exact value of a does this area equal 7?

iii. Using the value for a determined in **c.ii.**, explain in words, without evaluating the integral, why $\int_{1}^{a} f(x)dx < 7$.

Use this result to explain why $a < e$.

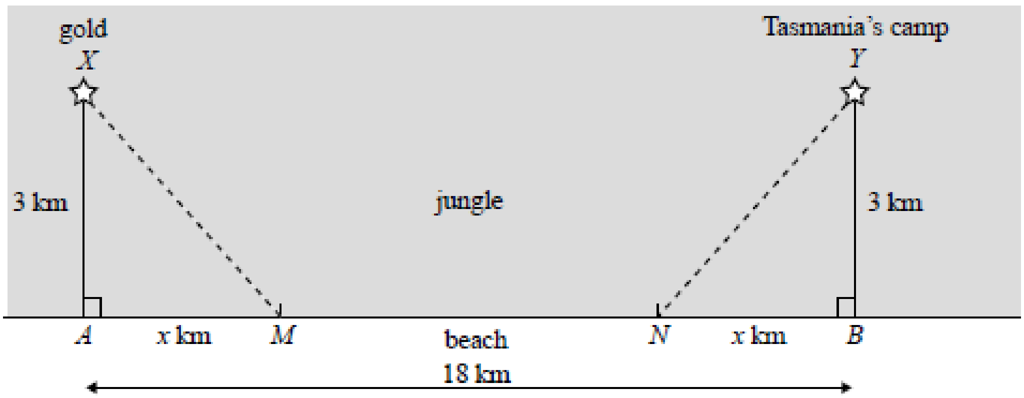
d. Find the exact values of m and n such that $\int_{1}^{mn} f(x)dx = 3$ and $\int_{1}^{\frac{m}{n}} f(x)dx = 2$.

Question 3

Tasmania Jones is in the jungle, digging for gold. He finds the gold at X which is 3 km from a point A. Point A is on a straight beach.

Tasmania's camp is at Y which is 3 km from a point B. Point B is also on the straight beach.

$AB = 18$ km and $AM = NB = x$ km and $AX = BY = 3$ km.



While he is digging up the gold, Tasmania is bitten by a snake which injects toxin into his blood. After he is bitten, the concentration of the toxin in his bloodstream increases over time according to the equation

$$y = 50\log_e(1 + 2t)$$

where y is the concentration, and t is the time in hours after the snake bites him.

The toxin will kill him if its concentration reaches 100.

a. Find the time, to the nearest minute, that Tasmania has to find an antidote (that is, a cure for the toxin).

Tasmania has an antidote to the toxin at his camp. He can run through the jungle at 5 km/h and he can run along the beach at 13 km/h.

b. Show that he will not get the antidote in time if he runs directly to his camp through the jungle.

In order to get the antidote, Tasmania runs through the jungle to M on the beach, runs along the beach to N and then runs through the jungle to the camp at Y. M is x km from A and N is x km from B. (See diagram.)

c. Show that the time taken to reach the camp, T hours, is given by

$$T = 2\left(\frac{\sqrt{9+x^2}}{5} + \frac{9-x}{13}\right)$$

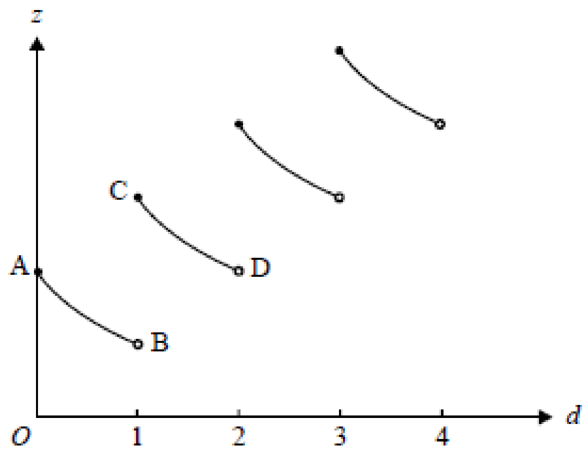
d. Find the value of x which allows Tasmania to get to his camp in the minimum time.

e. Show that he gets to his camp in time to get the antidote.

At his camp, Tasmania Jones takes a capsule containing 16 units of antidote to the toxin. After taking the capsule the quantity of antidote in his body decreases over time.

At exactly the same time on successive days, he takes another capsule containing 16 units of antidote and again the quantity of antidote decreases in his body.

The graph of the quantity of antidote z units in his body at time d days after taking the first capsule looks like this. Each section of the curve has exactly the same shape as curve AB.



The equation of the curve AB is $z = \dfrac{16}{d+1}$

f. Write down the coordinates of the points A and C.

g. Find the equation of the curve CD.

Tasmania will no longer be affected by the snake toxin when he first has 50 units of the antidote in his body.

h. Assuming he takes a capsule at the same time each day, on how many days does he need to take a capsule so that he will no longer be affected by the snake toxin?

Answers

2. a. i. $\dfrac{f(a)-f(1)}{a-1} = \dfrac{\frac{7}{a}-7}{a-1} = -\dfrac{7}{a}$

ii. $\dfrac{dy}{dx} = -\dfrac{7}{x^2} = -\dfrac{7}{a}$, $x = \sqrt{a}$, as $x > 1$

$\int_1^e f(x)dx = 7$

b. i.

ii. $\int_b^1 \left(\frac{7}{x}\right)dx = 7,\ b = e^{-1}$

c. i. $Area = A_{\text{trapezium}} = \frac{1}{2}(7+\frac{7}{a})(a-1) = \frac{7a}{2} - \frac{7}{2a} = \frac{7(a^2-1)}{2a}$

or

$Area = A_{\text{triangle}} + A_{\text{rectangle}} = \frac{1}{2}(a-1)(7-\frac{7}{a}) + \frac{7}{a}(a-1)$

or

$Area = \int_1^a \left(-\frac{7}{a}x + \frac{7}{a} + 7\right)dx = \frac{7(a^2-1)}{2a}$

ii. $\frac{7(a^2-1)}{2a} = 7,\ a = \sqrt{2}+1$

2. c. iii. The area under the curve is less than the area of the trapezium. Hence $\int_1^a f(x)dx < 7$. From b.i. $\int_1^e f(x)dx = 7$ but $\int_1^a f(x)dx < 7$, so $a < e$.

d. $m = e^{\frac{5}{14}}$ and $n = e^{\frac{1}{14}}$ OR $m = -e^{\frac{5}{14}}$ and $n = -e^{\frac{1}{14}}$

3. a. $100 = 50\log_e(1+2t)$, $t \approx 3.1945$ h = 192 minutes, to the nearest minute

b. 18 km at 5 km/h = 3.6 h, 3.6 h > 3.1945 h, therefore he will not get the antidote in time.

c. Time$(AM) = \frac{\sqrt{(9+x^2)}}{5}$ = Time(NY); Time$(MN) = \frac{18-2x}{13}$

$T = 2\left(\frac{\sqrt{(9+x^2)}}{5}\right) + \frac{18-2x}{13} = 2\left(\frac{\sqrt{9+x^2}}{5} + \frac{9-x}{13}\right)$

d. $\frac{dT}{dx} = 0,\ x = 1.25$ km

e. When $x = 1.25$, $T \approx 2.492$, 2.492 h < 3.19 h. Therefore, he gets the antidote in time.

f. $A = (0, 16)$ $C = (1, 24)$

g. $z = \frac{16}{d} + 8$

h. 6 days

2009

Question 5

Consider the family of curves defined by the relation $3x^3 - y^2 + kx + 5y - 2xy = 4$ where $k \in R$.

a. Verify that every curve in the family passes through the point (0, 4), and find the other point of intersection with the y-axis.

b. Find an expression for $\frac{dy}{dx}$ in terms of x, y and k.

c. Hence evaluate the gradient of the curve at the point (1, 1).

Question 7

A mass has acceleration a ms^{-2} given by $a = v^2 - 3$, where v ms^{-1} is the velocity of the mass when it has a displacement of x metres from the origin.

Find v in terms of x given that $v = -2$ where $x = 1$.

Answers

5. a. $(0, 1)$

   b. $\frac{dy}{dx} = \frac{2y - 9x^2 - k}{5 - 2y - 2x}$

   c. $\frac{dy}{dx} = -6$

7. $v = -\sqrt{e^{2x-2} + 3}$

2009

Question 1

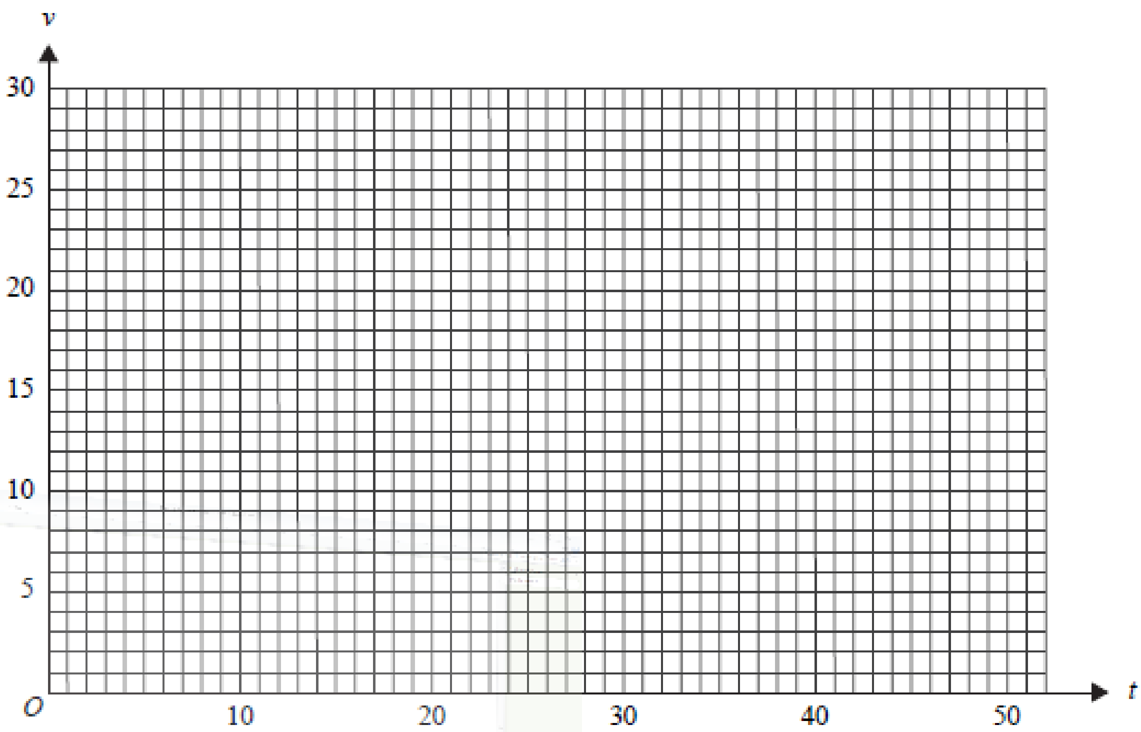
A car accelerates from rest at traffic light A to a velocity of 27 ms^{-1} in nine seconds. During this period of acceleration its velocity v ms^{-1} after t seconds, is given by

$$v = t^{\frac{3}{2}} \quad \text{for} \quad 0 \le t \le 9$$

The car then travels at a constant velocity of 27 ms^{-1} for another thirty seconds, and finally decelerates until it comes to rest at traffic light B. During deceleration its velocity v ms^{-1} is given by

$$v = 27\cos\left(\frac{\pi}{24}(t-39)\right) \quad \text{for} \quad 39 \le t \le 51$$

a. On the axes below, draw a velocity–time graph which shows the motion of the car as it travels from traffic light A to traffic light B.



b. Calculate the distance travelled by the car during the first nine seconds of its motion.

c. Calculate, correct to the nearest 0.1 m, the distance travelled by the car while it is decelerating.

d. Calculate, correct to the nearest 0.1 ms^{-1}, the average speed of the car as it travels from traffic light A to traffic light B.

The speed limit on this road is $\frac{200}{9}$ ms^{-1} (80 kilometres per hour).

e. Find the time interval $t_1 < t < t_2$ for which the car exceeds the speed limit.
Give your answers for t_1 and t_2 correct to the nearest 0.1 seconds.

Just as the car begins to accelerate away from traffic light A, a motorcycle travelling at a constant 20 ms^{-1} passes the car.

f. Find the time, correct to the nearest 0.1 seconds, and the distance, correct to the nearest metre, for the car to overtake the motorcycle.

Question 5

Scientists use a pressure sensitive device which measures depths as it sinks toward the sea bed. The device of mass 2 kg is released from rest at the ocean's surface and as it sinks in a vertical line, the water exerts a resistance of $4v$ newtons to its motion, where v ms^{-1} is the velocity of the device t seconds after release.

a. Draw a diagram showing the forces acting on the device, and show that $a = g - 2v$, where a ms^{-2} is the acceleration of the device when its velocity is v ms^{-1}.

b. Hence use calculus to show that $t = 0.5\log_e\left(\dfrac{g}{g-2v}\right)$

c. Write down the limiting (terminal) velocity of the device.

d. How many seconds after its release is the velocity of the device $\dfrac{g}{4}$ ms^{-1}?
Give your answer in the exact form $\log_e(a)$ where a is a positive real number.

At a particular location, the device is released from rest at the surface of the ocean.

The relation given in **part b.** may be rearranged to $v = \dfrac{g}{2}\left(1 - e^{-2t}\right)$.

e. If the device takes 180 seconds to hit the sea bed, how deep is the ocean at that location?
Give your answer correct to the nearest metre.

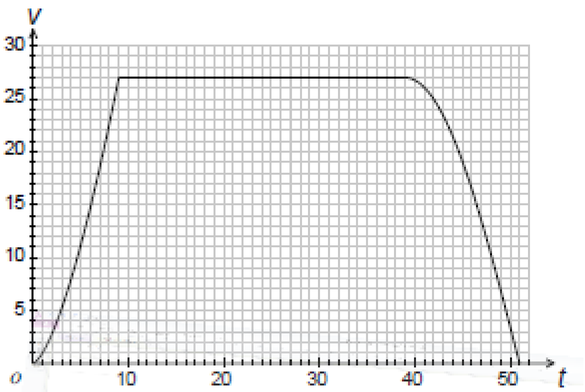
f. Determine the depth at which the velocity of the device is $\dfrac{g}{3}$ ms^{-1}.
Give your answer correct to the nearest tenth of a metre.

The device is released from a boat at a different location. At the instant of release, the boat begins to move away from the device in a horizontal straight line at a constant velocity of 2 ms^{-1}. The device falls 1200 m vertically and hits the sea bed.

g. Find the distance from the boat to the device when it hits the sea bed.
Give your answer correct to the nearest metre.

Answers

1. a.



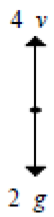
b. $\displaystyle\int_0^9 t^{\frac{3}{2}}\,dt = 97.2$ m

1. c. $\displaystyle\int_{39}^{51} 27\cos\left(\frac{\pi}{24}(t-39)\right)dt = 206.3$ m, correct to the nearest 0.1 m

d. 21.8 ms^{-1}, correct to the nearest 0.1 ms^{-1}

e. $t_1 = 7.9$, $t_2 = 43.6$, correct to the nearest 0.1 s

f. $97.2 + (t-9) \times 27 = 20t \Rightarrow t = 20.8$ s, distance = 417 m

5. a. 

$2g - 4v = 2a \Rightarrow a = g - 2v$

b. $\frac{dt}{dv} = \frac{1}{g-2v}$, $t = -0.5\log_e(g-2v) + c$, $t = 0, v = 0 \Rightarrow c = 0.5\log_e g$ and so $t = 0.5\log_e g - 0.5\log_e(g-2v)$ which gives the stated result.

c. $v = \frac{g}{2}$

d. $\log_e \sqrt{2}$ s

e. $\int_0^{180} \frac{g}{2}\left(1 - e^{-2t}\right) d = 880$ m, correct to the nearest m

f. $v\frac{dv}{dx} = g - 2v$, $x = \int_0^{g/3} \frac{v}{g-2v} dv = 1.1$ m, correct to the nearest 0.1 m

g. $1200 = \int_0^T \frac{g}{2}\left(1 - e^{-2t}\right) dt$ or $1200 = \frac{g}{2}\left(t + 0.5e^{-2t}\right) - \frac{g}{4}$ gives 245.4 seconds for the device to fall 1200 m.

distance = $\sqrt{1200^2 + (245.4 \times 2)^2} = 1296$ m, correct to the nearest m

2008

Question 2

Given the relation $3x^2 + 2xy + y^2 = 11$, find the gradient of the **normal** to the graph of the relation at the point in the first quadrant where $x = 1$.

Question 5

A particle moves in a straight line so that at time t seconds, it has acceleration a m/s^2, velocity v m/s and position x m relative to a fixed point on the line. The velocity and position of the particle at any time t seconds are related by $v = -x^2$. Initially $x = 1$.

a. Find the initial acceleration of the particle.

b. Express x in terms of t.

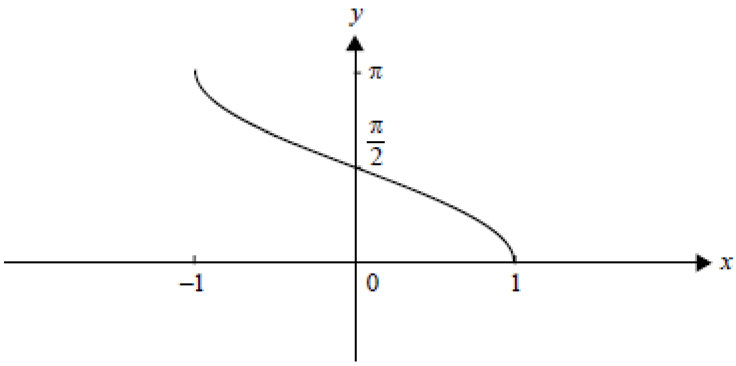
Question 6

The curve with equation $y = f(x)$ passes through the point $P\left(\frac{\pi}{8}, 2\right)$ and has a gradient of -1 at this point.

Find the exact gradient of the curve at $x = \frac{\pi}{12}$ given that $f''(x) = -\sec^2(2x)$.

Question 9

The graph of $y = \cos^{-1}(x)$, $x \in [-1, 1]$ is shown below.



Find the area bounded by the graph shown above, the x-axis and the line with equation $x = -1$.

Answers

2. $\frac{3}{5}$

5. a. 2

   b. $x = \frac{1}{t+1}$

6. $-\frac{\sqrt{3}}{6} - \frac{1}{2}$

9. π

2008

Question 4

An island has a population of rabbits and a population of foxes. The foxes eat rabbits as their food source and if the rabbit population decreases, then after some time, so will the fox population. Also, if the rabbit population increases, then after some time, so too will the fox population.

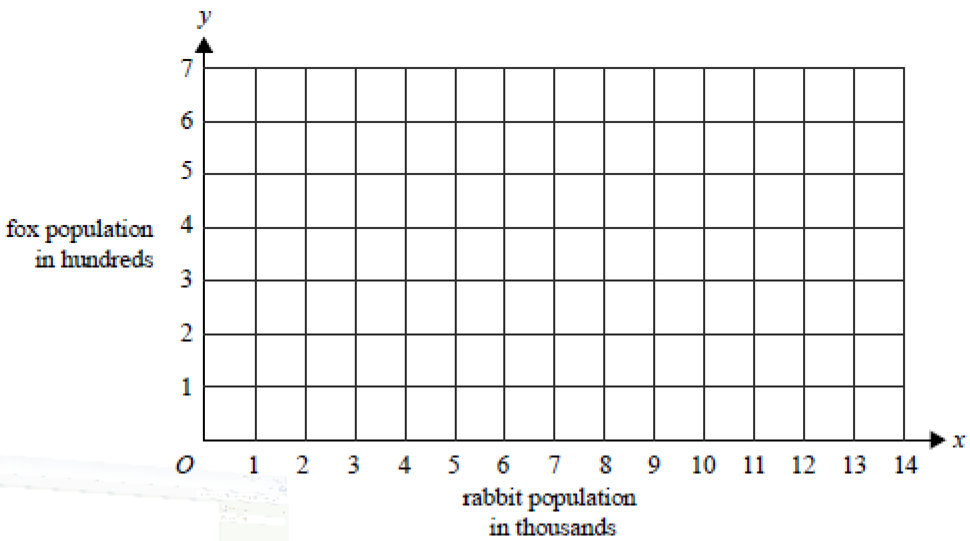
At time t months from the start of the year there are x thousand rabbits and y hundred foxes. A model for the two populations is given by the parametric equations

Rabbits $\quad x = 10 + 3\cos\left(\frac{\pi t}{6}\right),\ t \geq 0$

Foxes $\quad y = 5 + \sin\left(\frac{\pi t}{6}\right),\ t \geq 0.$

a. Find the cartesian equation relating x and y according to this model.

b. Sketch the relationship between x and y on the axes below.



c. **i.** After how many months from the start of the year is the population of rabbits, x thousand, a minimum?

ii. How many foxes are on the island at this time?

An alternative model for the interaction of the two populations, which more accurately allows for the dependency of the foxes on the rabbits as a food source, is given by the pair of differential equations

Rabbits $\quad \frac{dx}{dt} = 0.5x - 0.1xy,\ t \geq 0$

Foxes $\quad \frac{dy}{dt} = -0.2y + 0.02xy,\ t \geq 0$

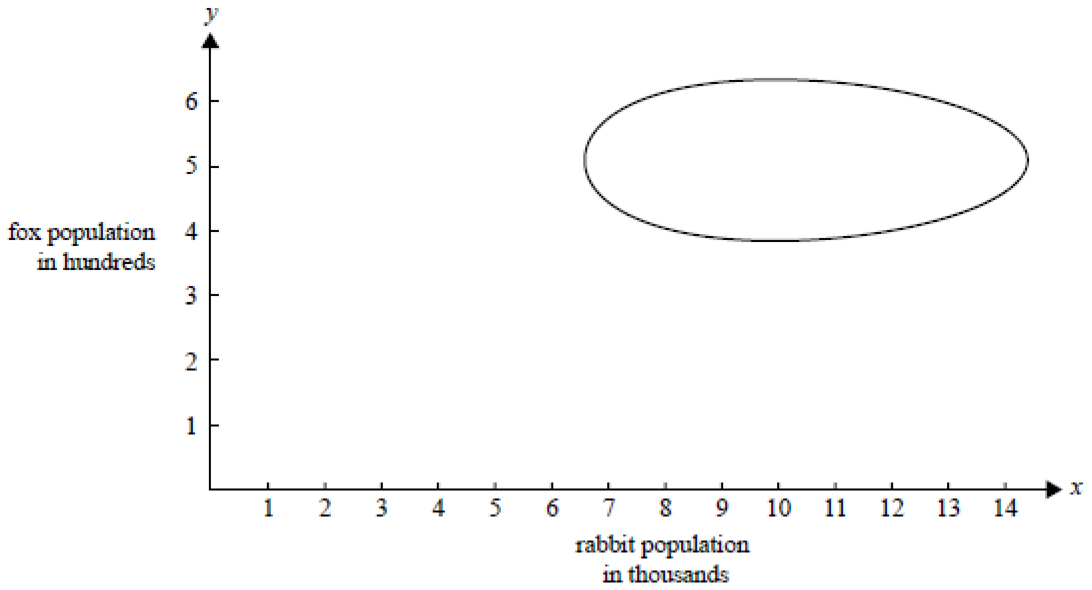
d. **i.** Show that $\frac{dy}{dx} = \frac{xy - 10y}{25x - 5xy}$.

ii. Use calculus to verify that the curve with equation

$$25\log_e(y) - 5y - x + 10\log_e(x) = c$$

where c is a constant of integration, satisfies the differential equation given in **part d. i.**

For certain populations of foxes and rabbits $c = 27.5$. The graph of the solution curve from **part d. ii.** for this value of c is shown below.

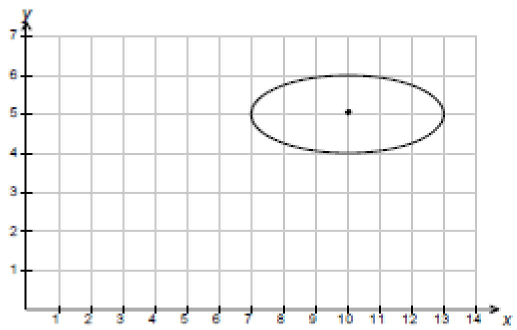


e. Determine the **minimum** and **maximum** numbers of rabbits possible according to the alternative model, correct to the nearest ten rabbits.

Answers

4. a. $\dfrac{(x-10)^2}{3^2} + (y-5)^2 = 1$

b.



c. i. 6 months

ii. 500 foxes

d. i. $\dfrac{dy}{dt} \times \dfrac{dt}{dx} = \dfrac{-0.2y + 0.02xy}{0.5x - 0.1xy} \times \dfrac{50}{50} = \dfrac{xy - 10y}{25x - 5xy}$

ii. $25 \times \dfrac{1}{y}\dfrac{dy}{dx} - 5\dfrac{dy}{dx} - 1 + \dfrac{10}{x} = 0\,,\ \dfrac{dy}{dx} = \dfrac{\left(1 - \dfrac{10}{x}\right)}{\left(\dfrac{25}{y} - 5\right)} \times \dfrac{xy}{xy} = \dfrac{xy - 10y}{25x - 5xy}$

e. 6590 and 14 430 rabbits

2007

Question 3

Find the equation of the tangent to the curve $x^3 - 2x^2y + 2y^2 = 2$ at the point $P(2, 3)$.

Question 5

A car travelling at 20 ms^{-1} passes a stationary police car, and then decelerates so that its velocity v ms^{-1}, at time t seconds after passing the police car, is given by $v = 20 - 2\tan^{-1}(t)$.

a. After how many seconds will the car's speed be 17 ms^{-1}? Give your answer correct to one decimal place.

b. Explain why v will never equal 16.

c. Write down a definite integral which gives the distance, x metres, travelled by the car after T seconds.

Three seconds later the police car starts to chase the passing car which has a polluting exhaust pipe. The police car accelerates so that its velocity v ms^{-1} at time t seconds after the polluting car passed it is given by

$$v = 13\cos^{-1}\left(\frac{13-2t}{7}\right) \text{ for } t \in [3, 8].$$

d. Write down an expression which gives how far the polluting car is ahead of the police car when $t = 8$ seconds.
Find this distance in metres correct to one decimal place.

After accelerating for five seconds the police car continues at a constant velocity.

e. At time $t = T_c$ the police car catches the polluting car. Write an equation which, when solved, gives the value of T_c.

f. Find T_c correct to the nearest second.

Answers

3. $y = 3x - 3$

5. a. $17 = 20 - 2\tan^{-1}(t)$, $t = \tan(1.5)$, $t = 14.1$, correct to one decimal place.

b. $\tan^{-1}(t) < \frac{\pi}{2}$, $V \to 20 - \pi \simeq 16.858$

c. $\int_0^T \left(20 - 2\tan^{-1}(t)\right) dt$

d. $\int_0^8 \left(20 - 2\tan^{-1}(t)\right) dt - \int_3^8 13\cos^{-1}\left(\frac{13-2t}{7}\right) dt = 60.7$ m, correct to one decimal place.

e. When $t = 8$, $V_{\text{police}} = 26.178\,(\text{m/s})$ so $60.7 + \int_8^{T_c} \left(20 - 2\tan^{-1}(t)\right) dt = 26.178(T_c - 8)$ or equivalent.

f. $T_c = 14.6 \simeq 15$ s, correct to the nearest second.

2005

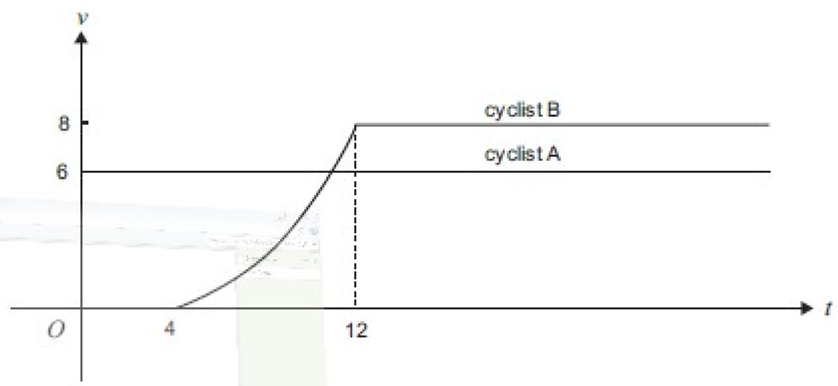
Question 4

At time $t = 0$, cyclist A, travelling at a speed of 6 m/s along a straight bicycle path, passes cyclist B who is stationary.

Four seconds later, at $t = 4$, cyclist B accelerates in the direction of cyclist A for 8 seconds in such a way that her speed, v m/s, is given by $v = (t-4)\tan\left(\frac{\pi}{48}t\right)$.

a. Show that cyclist B accelerates to a speed of 8 m/s.

Cyclist B then maintains her speed of 8 m/s. The velocity-time graph that represents this situation is shown below.



b. Find the time at which cyclist B passes cyclist A, correct to the nearest tenth of a second.

Answer

4. b. 36.6 sec