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SEMESTER TWO

REVISION 1

MATHEMATICS METHODS

UNITS 3-4

2016

SOLUTIONS

SECTION ONE

1. (8 marks)

(a) $\int (2x+4)^6\,dx = \dfrac{(2x+4)^7}{7\times 2} + c = \dfrac{(2x+4)^7}{14} + c$

(b) $\int_{\pi/4}^{\pi/2} 2\sin(x) - \cos(x)\,dx$

$= \left[-\cos(x) - \sin(x)\right]_{\pi/4}^{\pi/2}$

$= -\left(\left(\cos\left(\dfrac{\pi}{2}\right) + \sin\left(\dfrac{\pi}{2}\right)\right) - \left(\cos\left(\dfrac{\pi}{4}\right) + \sin\left(\dfrac{\pi}{4}\right)\right)\right)$

$= -1 + \sqrt{2}$

(c) $\int \left(x^4 + e^{2x} + \dfrac{2}{x}\right)dx = \dfrac{x^5}{5} + \dfrac{e^{2x}}{2} + 2\ln(x) + c$ ✓✓ -1/error

2. (16 marks)

(a) (i) $f(x) = \ln\left(\dfrac{x^2-3}{1+x}\right) = \ln(x^2-3) - \ln(1+x)$

$f'(x) = \dfrac{2x}{(x^2-3)} - \dfrac{1}{(1+x)}$

(ii) $g(x) = \dfrac{e^{\sin(x)}}{\cos(x)}$

$g'(x) = \dfrac{\left(e^{\sin(x)}\cos(x)\right)\cos(x) - (-\sin(x))e^{\sin(x)}}{(\cos(x))^2}$

$g'(x) = \dfrac{e^{\sin(x)}\left(\cos^2(x) + \sin(x)\right)}{\cos^2(x)}$

(iii) $h(x) = e^x \times \ln(x^2) = 2e^x \times \ln(x)$

$h'(x) = 2\left(e^x \times \ln(x) + \dfrac{e^x}{x}\right)$ ✓✓

$h'(x) = 2e^x\left(\ln(x) + \dfrac{1}{x}\right)$

(b) (i) Given $g(x)=\sqrt{sin(x)}$ show that $g'(x)=\dfrac{cos(x)}{2\sqrt{sin(x)}}$.

$$g'(x)=\frac{1}{2}(sin(x))^{\frac{-1}{2}}\times cos(x)$$
$$g'(x)=\frac{cos(x)}{2\sqrt{sin(x)}}$$

(ii)
$$\int -\frac{3cos(x)}{\sqrt{sin(x)}}dx = -3\times 2\int\frac{cos(x)}{2\sqrt{sin(x)}}dx$$
$$=-6\int\frac{cos(x)}{2\sqrt{sin(x)}}dx$$
$$=-6\sqrt{sin(x)}+c$$

(c) Given $\int_0^{10} f(x)dx=-6.4$ and $\int_4^{10} f(x)dx=2.3$.

$$2\int_0^4 \;=2\int_0^{10}\; -2\int_4^{10}$$
$$2\int_0^4(1-f(x))dx=2\left(\int_0^4 1dx-\int_0^4 f(x)dx\right)$$
$$=2\left([x]_0^4-\left(\int_0^{10} f(x)dx-\int_4^{10} f(x)dx\right)\right)$$
$$=2(4-(-6.4-2.3))$$
$$=2(4+8.7)$$
$$=2\times 12.7$$
$$=25.4$$

(d) $\dfrac{dr}{d\theta}=\dfrac{dr}{dt}\times\dfrac{dt}{dx}\times\dfrac{dx}{d\theta}$ $\qquad t=4x=4cos(\theta)$

$$\frac{dr}{d\theta}=\frac{1}{2}t^{-\frac{1}{2}}\times 4\times(-sin(\theta))$$
$$\frac{dr}{d\theta}=-\frac{2sin(\theta)}{\sqrt{t}}$$
$$\frac{dr}{d\theta}=-\frac{2sin(\theta)}{\sqrt{4cos(\theta)}}$$
$$\frac{dr}{d\theta}=-\frac{sin(\theta)}{\sqrt{cos(\theta)}}$$

3. (6 marks)

(a) $$\frac{log_{10}\left(4\times 3^2\right) - log_{10}(3\times 6) - 3log_{10}2}{-2log_{10}2}$$

$$=\frac{log_{10}\left(\frac{4\times 3^2}{3\times 6\times 8}\right)}{-2log_{10}2}$$

$$=\frac{log_{10}\left(\frac{1}{4}\right)}{-2log_{10}2}$$

$$=\frac{-2log_{10}2}{-2log_{10}2}$$

$$=1$$

(b) $\left(log_3(x)-1\right)\left(ln(x)-1\right)=0$

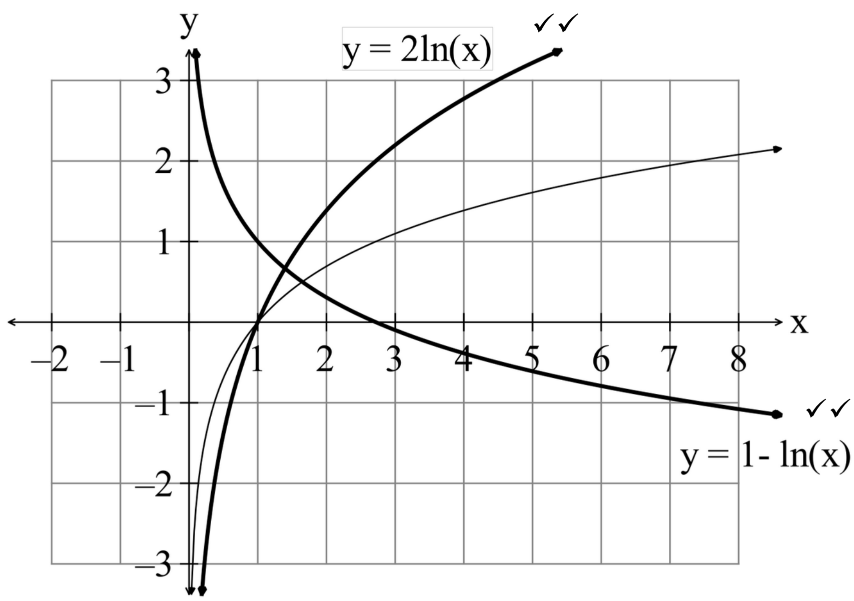
$log_3(x)-1=0 \quad or \quad ln(x)-1=0$

$log_3(x)=1 \quad or \quad ln(x)=1$

$x=3 \quad or \quad x=e$

4. (6 marks)

(a) (i)



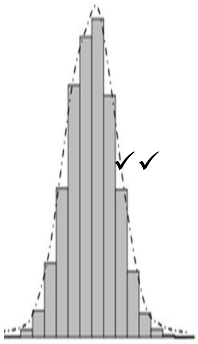
(ii) $f(x)=ln(x) \Rightarrow f^{-1}(x)=e^{x} \quad for\ x\in Re$

5. (16 marks)

(a) (i) Not a probability density functions as you cannot have negative probabilities.

(ii) Is a probability density functions as the probabilities add to one.

(b) The shape will be a tightly clustered histogram, close to a normal curve.



✓✓

(c) (i) $\int_1^e \frac{1}{x} dx = \left[ln(x) \right]_1^e = ln(e) - ln(1) = 1 - 0 = 1$

(ii) $P(1 \le x \le 2) = \int_1^2 \frac{1}{x} dx = ln(2) - ln(1) = ln(2)$

(d) $F(x) = \int_0^x \frac{x}{2} dx = \frac{1}{4}\left[x^2 \right]_0^x = \frac{x^2}{4}$

(e) $p(x) = \begin{cases} 0.2 & \text{for } 5 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$

$E(x) = \int_a^b x \times p(x) dx$

$E(x) = \int_5^{10} x \times 0.2\, dx = 0.2\left[\frac{x^2}{2} \right]_5^{10} = 0.1(100 - 25) = 7.5$

$Var(x) = \int_a^b p(x)(x - \mu)^2 dx$

$Var(x) = \int_5^{10} 0.2(x - 7.5)^2 dx$ ✓✓

END OF SECTION ONE

SECTION TWO

6. (7 marks)

(a) $A = \int_0^a e^x dx = \left[e^x \right]_0^a = e^a - e^0 = e^a - 1$

(b) $A = e^a - 1$

$\frac{dA}{da} = e^a$

$\frac{\delta A}{\delta a} \approx \frac{dA}{da}$

$\delta A \approx \frac{dA}{da} \times \delta a$

$At \quad a = 3,\ \delta a = 0.1$

$\delta A \approx e^3 \times 0.1$

$\delta A \approx 2.0086$

7. (6 marks)

(a) $v = 10t - 1 ms^{-1}.$

$a = 10\ ms^{-2}$

$x = \int (10t - 1) dt$

$x = 5t^2 - t + c$

$At\ t = 0, x = 3$

$x = 5t^2 - t + 3$

(b) Changes direction when $v = 0$ i.e. at t = 0.1

(c) $At\ t = 0,\ x = 3$

$At\ t = 0.1,\ x = 0.05 - 0.1 + 3 = 2.95$

$At\ t = 5,\ x = 123$

Distance travelled $= 123 - 2.95 + 0.05 = 120.1\ m$

8. (7 marks)

(a) (i) $f(t)=\sqrt{sin(\pi t)}$

$$f'(t)=\frac{1}{2}(sin(\pi t))^{\frac{-1}{2}}\pi cos(\pi t) \quad \checkmark$$

$$f'(t)=\frac{\pi cos(\pi t)}{2\sqrt{sin(\pi t)}}$$

(ii) $$\int_{1/6}^{1/2}\frac{\pi cos(\pi t)}{2\sqrt{sin(\pi t)}}dt=\left[\sqrt{sin(\pi t)}\right]_{1/6}^{1/2}$$

$$=\sqrt{sin\left(\frac{\pi}{2}\right)}-\sqrt{sin\left(\frac{\pi}{6}\right)}$$

$$=1-\frac{1}{\sqrt{2}}$$

(b) $$F(x)=\frac{d}{dx}\int_1^x\left(\frac{1}{t}\right)dt=\frac{1}{x}$$

$$\int_1^2 F(x)dx=\int_1^2\frac{1}{x}dx=\left[ln(x)\right]_1^2=ln(2)-ln(1)=ln(2)$$

9. (8 marks)

(a) Turning points occur when $f'(x)=0$.

There are no points where this occurs so there are no turning points.

Likewise, there are no points where $f''(x)=0,$ so there are no points of inflection.

(b) $y=f''(x)>0$ which suggests that the concavity is concave upwards for all x values. ✓✓

(c) $f(x)=ln(x),\quad f'(x)=\frac{1}{x},\quad f''(x)=\frac{-1}{x^2}$

10. (7 marks)

(a) 0.693 ✓✓ (= ln(2))

(b) (i) True as +C can have any value

(ii) No, this is bounded and has only one solution.

$$\int_1^2 3x^2dx = \left[x^3\right]_1^2 = 8 - 1 = 7$$

(iii) True.

(iv) $ln\left(f(x)\right) = ln(3) + 2\,ln(x)$

Not valid as initially, x can be negative as it is squared.

Need $2\,ln\left|(x)\right|$ ✓✓

11. (8 marks)

(a) $t = 0, \quad P = 23$

$78.9 = 23e^{60(k)}$

$k = 0.02054478$

(b) $t = 66, \quad P = 89.25$ (million)

(c) Te actual population is smaller than 89.25 so the growth rate is slowing down (minimally!)

(d) $2010 \qquad 78.9 \times 10^6 \quad t = 0$

$2016 \qquad 87\,238\,973$

$P = 78.9$

$At \quad t = 6, \qquad 87\,238\,973 = 78.9 \times 10^6 e^{6k}$

$k = 0.01674499021$

$P = 78.9 \times 10^6 \times e^{0.01674499021 \times t}$

$P = 100 \times 10^6, t = ?$

$100 = 78.9 \times e^{0.01674499021 \times t}$

$t = 14.15$

The population is expected to reach 100 million just into 2025.

12. (7 marks)

(a) Area $= \int_{0.05}^{1.47} \left(ln(x) - (e^x - 4)\right) = 1.68$ units^2

(b) (i) $P(0) = 22\,ln(3) = 24.169 \approx 24$

(ii) $100 = 22\,ln(t+3)$

$t = 91.203$

$2002 + 91 = 2093$

The population will reach 100 in 2094 or just into 2094.

13. (7 marks)

(a) $A = x \times y \qquad y^2 = 1 - x^2$

$A = x\sqrt{1 - x^2}$

(b) Maximum area when $\dfrac{dA}{dx} = 0$ and $\dfrac{d^2A}{dx^2} < 0$

$$\frac{dA}{dx} = -\frac{2x^2 - 1}{\sqrt{1 - x^2}}$$

$$\frac{d^2A}{dx^2} = -\frac{2x^3\sqrt{1-x^2} + 4x\sqrt{(1-x^2)^3} - x\sqrt{1-x^2}}{(x^2-1)^2}$$

$$If\ \frac{dA}{dx} = 0, \quad 2x^2 - 1 = 0$$

$x^2 = 0.5$

$x = \dfrac{1}{\sqrt{2}} \quad x > 0$

Max or min?

$$\frac{d^2A}{dx^2} = -\frac{4\left(\frac{1}{2\sqrt{2}}\right)\sqrt{0.5} + 4\left(\frac{1}{2\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\left(\sqrt{0.5}\right)}{2} = -\frac{\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)\sqrt{0.5} + 4\left(\frac{1}{2\sqrt{2}}\right)}{2} < 0$$

$\therefore$ max

$$x = \frac{1}{\sqrt{2}}, \qquad y = \sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}}$$

Therefore $P\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

14. (6 marks)

(a) (i)

x	0	1	2	3	4	5	≥ 6 ✓✓
$P(X=x)$	0.1465	0.366	0.3945	0.068	0.015	0.0075	0.0025

(ii) 0.366+0.3945 = 0.7605
✓ ✓

(iii) $E(x)=\sum x_i \times P(x_i)$

$=0+1\times0.366+2\times0.3945+3\times0.068+4\times0.015+5\times0.0075+6\times0.0025$

$E(x)=1.4715$

15. (7 marks)

(a) P(a given battery will last at least 400 hours) $=0.006209665326\approx0.0062$ ✓ ✓

(b) P(a given battery will last between 320 and 380 hours) $=0.8663855975\approx0.866$ ✓ ✓

(c) P(Jenny’s battery will run out in a 3 hour exam if it has been used for 340 hours)

$$=\frac{P(340\leq x\leq343)}{P(x\geq340)}=\frac{0.0546318101}{0.6914624613}=0.07900907592$$

Therefore P(Jenny’s battery will run out in a 3 hour exam if it has been used for 340 hours already) ≈0.08

16. (6 marks)

(a) P(there are 5 girls and 5 boys) = 0.24609375 ✓ ✓

(b) P(there are no more than 4 boys) = 0.376953125 ✓ ✓

(c) BB(8 more births) Binomial with n = 8

P(exactly 4 girls) = 0.2734375

17. (22 marks)

(a) (i) A biased sample is one in which not every sample point has equal chance of being selected. ✓ ✓

(ii) A sample that is chosen in such a way that each point has an equal chance of being selected. The process requires each possible selection point being numbered and then the sample is generated using random numbers. ✓ ✓

(iii) A skewed distribution is a set of data where a large proportion of the data is clustered at one end. ✓ ✓

$p = 0.2 \;\; q = 0.98 \Rightarrow \;\; np = 100 \times 0.2 = 20 > 5$

$nq = 100 \times 0.8 = 80 > 5$ so can use normal distribution.

Mean $= p = 0.2$

$$sd_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.2 \times 0.8}{100}}$$

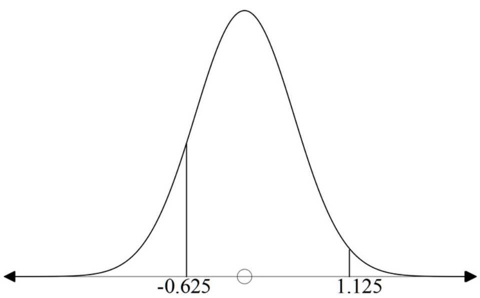
$$sd_{\hat{p}} = 0.04$$

Standardised score (using 17.5 to 24.5)

$$z = \frac{X - \mu}{\sigma}$$

$$z_{0.075} = \frac{\frac{17.5}{100} - 0.2}{0.04} = -0.625$$

$$z_{0.145} = \frac{\frac{24.5}{100} - 0.2}{0.04} = 1.125$$



$$P\left(-0.625 \le z \le 1.125\right) = 0.6037199538$$

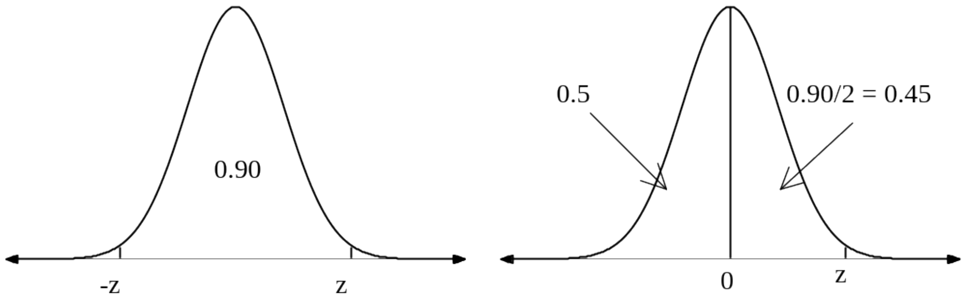
The probability that between 18 and 24 of the wine tasters should not drive is 0.604.

(c) (i) $\hat{p} = \frac{180}{200} = 0.9$

(ii) $\hat{p} = 0.9$

$$sd_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.9 \times 0.1}{200}}$$
$$sd_{\hat{p}} = 0.0212132$$

(iii)



P(X< z) = 0.95

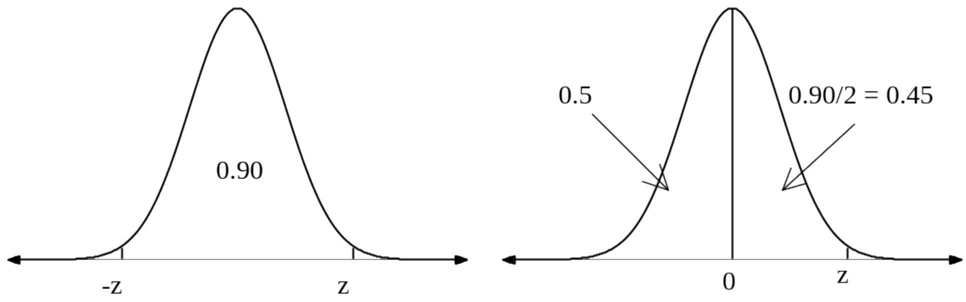
z = 1.645

$sd_{\hat{p}} = 0.0212132$

$$E = z \times s = 1.645 \times 0.0212132$$
$$E \approx 0.0349$$

The 90% confidence limit are 0.9 ± 0.03 *i.e.* $(0.90, 0.96)$

(d)



P(X< z) = 0.95
z = 1.645

Use $p = 0.5$ as the maximum value as p is unknown.

So with $p = 0.5$ $\quad sd = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\dfrac{0.25}{n}}$

$E = z \times s$ but $E = 0.05$

Therefore

$0.05 = 1.645 \times \sqrt{\dfrac{0.25}{n}}$

$n = 270.6$

You need to survey 271 people to have an error margin of 5% at a confidence level of 90%

END OF SECTION TWO