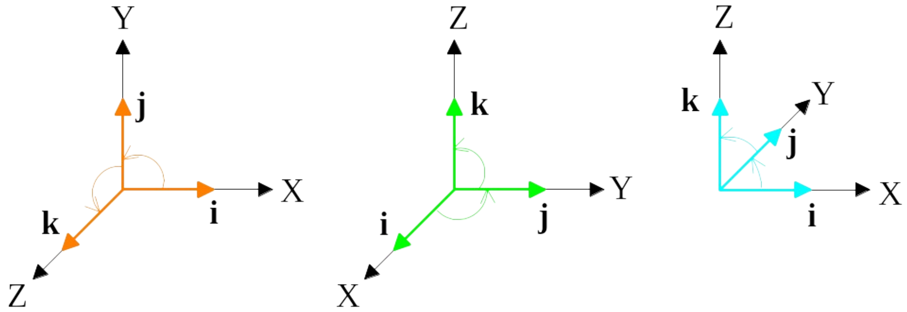
VECTORS IN THREE DIMENSIONS

1. THREE DIMENSIONAL VECTORS: The **unit vectors**, **i** and **j**, are used to express two dimensional vectors in component form, but we have **three dimensions**, so **unit vector k** is perpendicular to **i** and **j**, acting along the **Z-axis**. To determine the position of each axis/unit vector we use the "**right hand screw**" convention – starting at the **positive X-axis**, move in an **anti-clockwise** direction to the **positive Y-axis**, and then to the **positive Z-axis**, where each axis is **perpendicular** to the other two axes.



Hence, a 3-D vector, **x**, can be represented as a**i** + b**j** + c**k**. All vector concepts can be 'adjusted' for 3-D vectors, thus, **x** = a**i** + b**j** + c**k** can be written as $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, and

$|\mathbf{x}| = \sqrt{a^2 + b^2 + c^2}$. Likewise, if **x** = a_1**i** + a_2**j** + a_3**k** and **y** = b_1**i** + b_2**j** + b_3**k**, then **x.y** = $a_1b_1 + a_2b_2 + a_3b_3$ and **x.y** = | **x** || **y** | cos θ will determine the angle between the two vectors. In addition, the **projection** of **x** on **y** is given by $\dfrac{\mathbf{x.y}}{|\mathbf{y}|}$.

E.g.1. If **p** = 6**i** + 3**j** – 4**k** and **q** = 12**i** – 7**j** – 3**k**, then find:
- a) **p** + **q**
- b) 3**p** – 2**q**
- c) **p.q**
- d) | **q** |
- e) a unit vector parallel to **q**
- f) the angle between the two vectors
- g) the projection of **q** on **p**.

a) **p** + **q** = (6**i** + 3**j** – 4**k**) + (12**i** – 7**j** – 3**k**)
= 18**i** – 4**j** – 7**k**

b) 3**p** – 2**q** = 3(6**i** + 3**j** – 4**k**) – 2(12**i** – 7**j** – 3**k**)
= (18**i** + 9**j** – 12**k**) + (-24**i** + 14**j** + 6**k**)
= -6**i** + 23**j** – 6**k**

c) **p.q** = (6)(12) + (3)(-7) + (-4)(-3)
= 63

d) $|\,\mathbf{q}\,| = \sqrt{12^2 + (-7)^2 + (-3)^2}$
$= \sqrt{202}$ units

e) Thus, a unit vector parallel to **q** is $\dfrac{1}{\sqrt{202}}(12\mathbf{i} - 7\mathbf{j} - 3\mathbf{k})$.

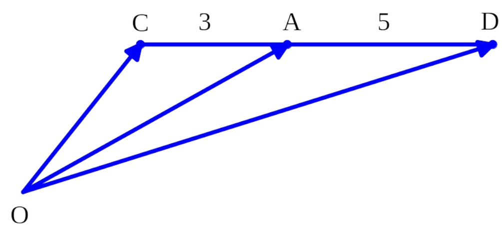
f) $|\,\mathbf{p}\,| = \sqrt{6^2 + 3^2 + (-4)^2}$
$= \sqrt{61}$

$$\cos\theta = \frac{63}{\sqrt{61} \times \sqrt{202}}$$
$\theta \approx 55.42^\circ$
$\therefore$ The angle between the vectors is approximately 55.42°.

g) Projection of **q** on **p** = $\dfrac{63}{\sqrt{61}}$

E.g.2. Point C has position vector $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and point D has position vector $\begin{pmatrix} 9 \\ -2 \\ -5 \end{pmatrix}$.

Find the position vector of the point A that divides CD internally in the ratio of 3:5.



$\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{CA}$

But $\overrightarrow{CA} = \frac{3}{8}\overrightarrow{CD}$

and $\overrightarrow{CD} = \overrightarrow{CO} + \overrightarrow{OD}$

$$= -\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 9 \\ -2 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ 0 \\ -8 \end{pmatrix}$$

$$\overrightarrow{CA} = \frac{3}{8}\begin{pmatrix} 8 \\ 0 \\ -8 \end{pmatrix}$$

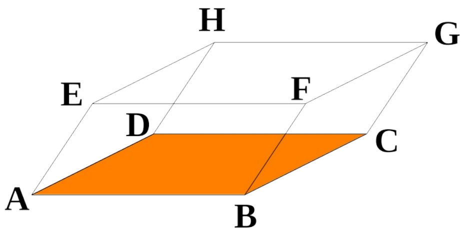
$$= \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$$

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$$

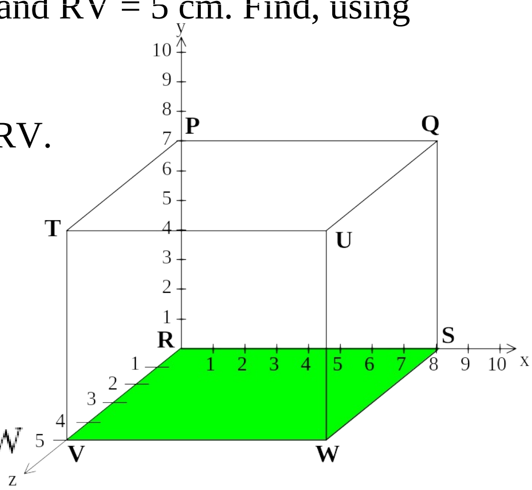
$$= \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$$

$\therefore$ A has position vector $\begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$.

2. **ANGLE BETWEEN LINES: Skew lines** are lines that are **not parallel** but do **not intersect**. This is only possible in three dimensions. The **angle between two skew lines** is the angle between **one skew line** and a **line parallel** to the other skew line which **intersects** the first line. For this diagram, the angle between the skew lines AB and EG is $\angle GEF$ or $\angle CAB$.



E.g.3. For this rectangular prism, PQ = 8 cm, PR = 7 cm and RV = 5 cm. Find, using vectors:
a) s∠URW
b) the acute angle between the skew lines PU and RV.



a) $\overrightarrow{RU} = 8\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$
$\overrightarrow{RW} = 8\mathbf{i} + 5\mathbf{k}$
$\therefore \overrightarrow{RU}.\overrightarrow{RW} = (8)(8) + (7)(0) + (5)(5)$
$= 89$
$\therefore 89 = \sqrt{8^2 + 7^2 + 5^2} \times \sqrt{8^2 + 0^2 + 5^2} \times \cos \angle URW$
s∠URW ≈ 37°

b) $\overrightarrow{PU}.\overrightarrow{RV} = (8\mathbf{i} + 5\mathbf{k}).(5\mathbf{k})$
$= 25$
$\therefore 25 = \sqrt{8^2 + 5^2} \times 5 \times \cos \angle UPT$
s∠UPT ≈ 58°
∴ The acute angle between the skew lines PU and RV is approximately 58°.

E.g.4. Given **p** = 14**i** – 5**j** + 2**k**, **q** = **i** + 2**j** + 3**k**, **r** = **i** + 5**j** + 4**k** and **s** = 12**i** – 3**j** – 2**k**, express **s** in the form of λ**p** + μ**q** + η**r**.

$$\begin{pmatrix} 12 \\ -3 \\ -2 \end{pmatrix} = \lambda \begin{pmatrix} 14 \\ -5 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \eta \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

⇒ 12 = 14λ + μ + η
-3 = -5λ + 2μ + 5η
-2 = 2λ + 3μ + 4η
From the CAS calc.,
λ = 1, μ = -4, η = 2
∴ **s** = **p** – 4**q** + 2**r**

Ref: Ex.3A Q.1-24 (even); 25-29
SCALAR PRODUCTS Q.1-28 (even)

3. VECTOR EQUATION OF A LINE: The vector equation of a line, in two dimensions, is also true for a line in three dimensions, i.e. **r** = **a** + λ**b**, where **a** is the position vector of a point on the line, **b** is a vector parallel to the line, and **λ** is some scalar value. Thus, if **r** = a**i** + b**j** + c**k** + λ(p**i** + q**j** + s**k**), then the **parametric equations** are **x** = **a** + λ**p**, **y** = **b** + λ**q** and **z** = **c** + λ**s**, and the **Cartesian equations** are $\frac{\mathbf{x - a}}{\mathbf{p}} = \frac{\mathbf{y - b}}{\mathbf{q}} = \frac{\mathbf{z - c}}{\mathbf{s}}$ for a line through (a,b,c) and parallel to p**i** + q**j** + s**k**.

E.g.5. Show that the lines L_1 and L_2 intersect and find the position vector of this point,

where $L_1: r = \begin{pmatrix} 2 \\ 7 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -5 \\ -7 \end{pmatrix}$ and $L_2: r = \begin{pmatrix} -2 \\ 10 \\ -10 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 4 \\ -6 \end{pmatrix}$.

If the lines intersect, then $L_1 = L_2$.

$$\begin{pmatrix} 2 \\ 7 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -5 \\ -7 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \\ -10 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 4 \\ -6 \end{pmatrix}$$

$\Rightarrow 2 + 8\lambda = -2 - 6\mu$ ……①

$7 - 5\lambda = 10 + 4\mu$ ……②

$9 - 7\lambda = -10 - 6\mu$ ……③

Solving ① & ②,

$\Rightarrow \lambda = 1, \mu = -2$

These values are consistent with ③.

Hence, L_1 intersects L_2 at the point with position vector $\begin{pmatrix} 10 \\ 2 \\ 2 \end{pmatrix}$.

The **shortest distance** between a line **r = a + λb** and a point with position vector **c** can be determined using **scalar products** and **projections** or by using **simultaneous equations**.

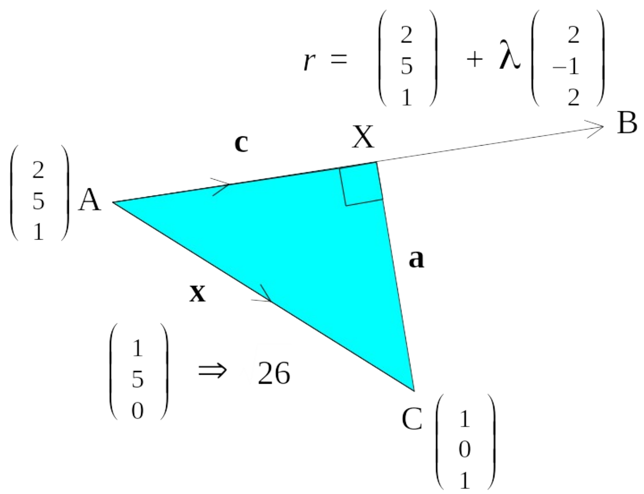
E.g.6. Find the shortest distance from the line $\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ to the point with position vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Projection of C on $\mathbf{r} = \left| \dfrac{\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}}{\left| \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \right|} \right|$

$= \left| \dfrac{2 - 5}{3} \right|$

$= 1$

$= |\mathbf{c}|$

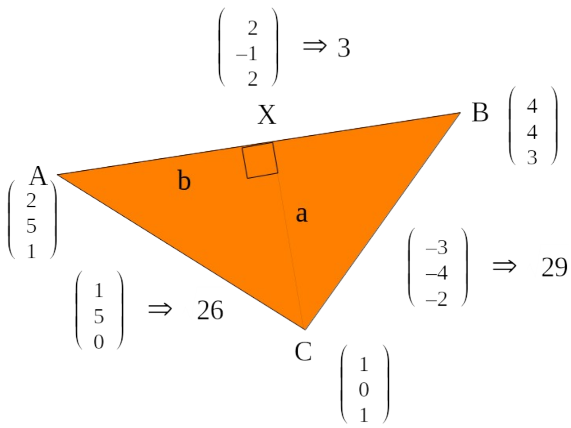


$\therefore$ Shortest distance $(|\mathbf{a}|) = \sqrt{26 - 1}$

$= 5$ units

Or

$a^2 = 26 - b^2$
$a^2 = 29 - (3 - b)^2$
$\Rightarrow a^2 = 29 - 9 + 6b - b^2$
$a^2 = 20 + 6b - b^2$
$26 - b^2 = 20 + 6b - b^2$
$b = 1$
& $a = \pm 5$
$\therefore$ The shortest distance is 5 units.



E.g.7. The points P, Q and R have position vectors $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, respectively.
a) Find the vector equation of the line passing through Q parallel to PR.
b) Find the vector equation of the line passing through Q perpendicular to PR.

a) $\overrightarrow{PR} = \langle -4, 3, -3 \rangle$
$\mathbf{r} = (2 - 4\lambda)\mathbf{i} + (4 + 3\lambda)\mathbf{j} + (-1 - 3\lambda)\mathbf{k}$
$\therefore$ $\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} + \lambda(-4\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})$ is the required vector equation.

b) Vector 'normal' to $\overrightarrow{PR}$: $\langle -4, 3, -3 \rangle \mathbf{.} \langle x, y, z \rangle = 0$
$\Rightarrow -4x + 3y - 3z = 0$
[Any (x, y, z) that make this true.]
[One possibility] $\langle 3, 5, 1 \rangle$
$\mathbf{r} \mathbf{.} \langle 3, 5, 1 \rangle = \langle 2, 4, -1 \rangle \mathbf{.} \langle 3, 5, 1 \rangle$
$\therefore$ $\mathbf{r} \mathbf{.} (3\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = 25$ is the required vector equation.

4. VECTOR EQUATION OF A PLANE: The **scalar product form** of the vector equation of a line, in two dimensions, describes a plane. Thus, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, is the **position vector** of a **plane** where **a** is the position vector of one point in the plane, and **b** and **c** are two non-parallel vectors that are parallel to the plane.

Alternatively, $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$ describes a plane where the position vector **a** is perpendicular to vector **n**, and hence $\mathbf{r}.\mathbf{n} = \mathbf{c}$, where $\mathbf{c} = \mathbf{a}.\mathbf{n}$. Thus, if $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{n} = p\mathbf{i} + q\mathbf{j} + s\mathbf{k}$, then $x\mathbf{p} + y\mathbf{q} + z\mathbf{s} = \mathbf{c}$ is the **Cartesian equation** of the plane.

Two planes (Π_1 and Π_2) in the form $\mathbf{r}.\mathbf{n} = \mathbf{c}$, are **parallel** if the **normals** (**n**) are the same or one normal is a **scalar product** of the other. To determine the **distance** between two parallel planes – let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, **form two equations** in terms of x, y and z, (usually) let **A (a,0,0)** be a point on plane Π_1, **substitute** x, y and z, **solve** for a, and use A and $\mathbf{n_2}$ to calculate the distance apart. $\text{Distance} = \left| \dfrac{(p)(a) + (q)(0) + (s)(0) - c_2}{\sqrt{p^2 + q^2 + s^2}} \right|$.

E.g.8. Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, the vector equation of the plane containing the line $r = \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and the point $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$.

r is parallel to $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and so the plane is parallel to $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

When $\lambda = 0$, $\begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix}$ is a point on the line, and hence is a point in the plane.

Thus, $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ -3 \end{pmatrix}$ is a vector parallel to the plane.

$\therefore r = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 1 \\ -3 \end{pmatrix}$ is the required equation.

E.g.9. Plane Π_1 has equation $\mathbf{r}.\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 3$ and plane Π_2 has the equation $\mathbf{r}.\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} = -5$.

a) Prove that $\Pi_1 \parallel \Pi_2$.
b) Find the distance the planes are apart.

Π_1: $\mathbf{r}.\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 3$ and Π_2: $\mathbf{r}.\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} = -5$

$$\Rightarrow \mathbf{r}.(-1)\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = -5$$

a) Since one normal is the opposite of the other normal, the planes are parallel.

b) Let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Π_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 3$ and Π_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} = -5$

$x - y + 3z = 3$ $\qquad$ $-x + y - 3z = -5$

Let A (a,0,0) be a point on the plane.

$a - 0 + 3(0) = 3$

$a = 3$

$\therefore$ A is (3,0,0).

$$\text{Distance} = \frac{|(-1)(3) + (1)(0) + (-3)(0) - (-5)|}{\sqrt{(-1)^2 + 1^2 + (-3)^2}} = \frac{2}{\sqrt{11}} \text{ units}$$

A plane can also be defined by any three **non-collinear** points that lie in the plane.

The vector equation of a line **r** = **a** + λ**b** by **t** (time), when solving **interception/collision** problems still holds true in three dimensions.

E.g.10. A plane contains the points F, G and H with position vectors –**i** + 4**j** – **k**, 2**i** + 2**j** + 3**k** and –**i** + 9**j** – 3**k**, respectively. Find the vector equation of the plane in the form:
a) ax + by + cz = d
b) **r.n** = c
c) **r** = **a** + λ**b** + μ**c**

a) –a + 4b – c = d
2a + 2b + 3c = d
–a + 9b – 3c = d
From the CAS calc.,
$$a = \frac{-16d}{25},\ b = \frac{6d}{25},\ c = \frac{3d}{5}$$
$$\Rightarrow \frac{-16}{25}dx + \frac{6}{25}dy + \frac{3}{5}dz = d$$
∴ -16x + 6y + 15z = 25 is the required equation.

b) Thus, **r.**(-16**i** + 6**j** + 15**k**) = 25 is the required equation.

c) Given the three points lie in the plane,
then (2**i** + 2**j** + 3**k**) – (-**i** + 4**j** – **k**) = 3**i** – 2**j** + 4**k**
and (2**i** + 2**j** + 3**k**) – (-**i** + 9**j** – 3**k**) = 3**i** – 7**j** + 6**k** also lie in the plane.
∴ **r** = -**i** + 4**j** – **k** + λ(3**i** – 2**j** + 4**k**) + μ(3**i** – 7**j** + 6**k**) is the required equation.

E.g.11. At 8 a.m., the position vectors (**r** m) of two hikers, Amber and Beryl, and their velocity vectors (**v** m/s) are: $\mathbf{r}_A = \begin{pmatrix} -4 \\ 9 \\ 5 \end{pmatrix}$ and $\mathbf{v}_A = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$, and $\mathbf{r}_B = \begin{pmatrix} 20 \\ -3 \\ -7 \end{pmatrix}$ and $\mathbf{v}_B = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$. Find the position vector and time of their meeting.

$\mathbf{r}_A(t) = \begin{pmatrix} -4 \\ 9 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{r}_B(t) = \begin{pmatrix} 20 \\ -3 \\ -7 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

For the hikers to meet, t hours after 8 a.m.:

$$\begin{pmatrix} -4 \\ 9 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 20 \\ -3 \\ -7 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$-4 + 3t = 20 + t \quad \Rightarrow t = 12$
$9 + t = -3 + 2t \quad \Rightarrow t = 12$
$5 + 3t = -7 + 4t \quad \Rightarrow t = 12$

$\therefore$ Amber and Beryl will meet at the point with position vector $\begin{pmatrix} 32 \\ 21 \\ 41 \end{pmatrix}$ at 8 p.m.

Ref: Ex.3B Q.1-20 (even); 21, 22

5. APPLICATIONS: The vector equation of a circle, centre (0,0) with radius **a** is **| r | = a**. Extending this concept to a three dimensional situation – the **vector equation** of a **sphere**, centre (0,0,0) and radius **a** is (likewise) **| r | = a** where **r** = x**i** + y**j** + z**k** and the **Cartesian equation** is $\mathbf{x^2 + y^2 + z^2 = a^2}$. Similarly, the **vector equation** of a **sphere** with its centre having position vector **d** and radius **a** is (likewise) **| r – d | = a** where **r** = x**i** + y**j** + z**k** and **d** = p**i** + q**j** + s**k** and hence, the **Cartesian equation** is $\mathbf{(x - p)^2 + (y - q)^2 + (z - s)^2 = a^2}$.

E.g.12. Find the centre and radius of the sphere $x^2 + y^2 + z^2 = 6x + 16$.

$$x^2 + y^2 + z^2 = 6x + 16$$
$$x^2 - 6x + y^2 + z^2 = 16$$
$$(x - 3)^2 - 9 + y^2 + z^2 = 16 \quad \leftarrow \text{Completing the square}$$
$$(x - 3)^2 + y^2 + z^2 = 16 + 9$$
$$(x - 3)^2 + y^2 + z^2 = 25$$
$$(x - 3)^2 + y^2 + z^2 = 5^2$$

$\therefore$ The sphere has its centre at (3,0,0) and a radius of 5.

E.g.13. Find the position vectors of the points where the line **r** = **i** – **j** + 2**k** + λ(2**i** + 9**j**) cuts the sphere **| r – (i** + 2**j** – **k**) **| =** 7.

Let point A lie on the line and the sphere.

$\Rightarrow \mathbf{r}_A = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 9\mathbf{j})$

& $|\, \mathbf{r}_A - (\mathbf{i} + 2\mathbf{j} - \mathbf{k})\, | = 7$

$\therefore |\, \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 9\mathbf{j}) - (\mathbf{i} + 2\mathbf{j} - \mathbf{k})\, | = 7$

$$|\, 2\lambda\mathbf{i} + (9\lambda - 3)\mathbf{j} + 3\mathbf{k}\, | = 7$$

$$4\lambda^2 + 81\lambda^2 - 54\lambda + 9 + 9 = 49$$

$$85\lambda^2 - 54\lambda - 31 = 0$$

From the CAS calc.,

$\lambda = 1$ or $\lambda = -\dfrac{31}{85}$

$\therefore$ The line cuts the sphere at position vectors $3\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}$ and $\dfrac{1}{85}(23\mathbf{i} - 364\mathbf{j} + 170\mathbf{k})$.

<u>Ref</u>: SPHERES Q.1-18 (even); 19-23

DIFFERENTIATING TRIGONOMETRIC FUNCTIONS

1. LIMITS: In order to differentiate **y = sin x**, **y = cos x** and **y = tan x**, **First Principles** must be used. There are two **trigonometric limits** which are required – $\lim\limits_{x \to 0} \frac{\sin x}{x}$ and $\lim\limits_{x \to 0} \frac{1 - \cos x}{x}$. The **value** of these limits can be determined by viewing the **graph**, using the **limiting chord process**, or by **geometric** or **algebraic proof**. [See p.82-86.] From any or all of these – $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$, and $\lim\limits_{x \to 0} \frac{1 - \cos x}{x} = 0$, where x is in radians.

2. DIFFERENTIATION: Using **First Principles**, if **y = sin x**, then –

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h}$$

$$= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

= sin x (0) + cos x (1)

= cos x

Likewise, if **y = cos x**, then $\frac{dy}{dx} = -\sin x$.

E.g.1. Differentiate:

a) $\sin x - x^3$

b) 3 cos x

c) (3 – 4 cos x)(5 + 3 sin x)

d) $\frac{\sin x}{\cos x}$

a) If $y = \sin x - x^3$

then $\frac{dy}{dx} = \cos x - 3x^2$

b) If y = 3 cos x

then $\frac{dy}{dx} = -3 \sin x$

c) If y = (3 – 4 cos x)(5 + 3 sin x)

then $\frac{dy}{dx} = (5 + 3\sin x)(4\sin x) + (3 - 4\cos x)(3\cos x)$

$= 20 \sin x + 12 \sin^2 x + 9 \cos x – 12 \cos^2 x$

d) If $y = \frac{\sin x}{\cos x}$

then $\frac{dy}{dx} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x}$

From the last example, if **y = tan x**, then $\mathbf{\frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x = 1 + \tan^2 x}$

E.g.2. Differentiate:
a) cos (2x + 1)
b) $\sin^3 x$
c) tan (3x + 2)
d) $3\tan^4 x$

a) If y = cos (2x + 1)

then $\frac{dy}{dx} = -\sin(2x + 1) \times 2$

$= -2 \sin (2x + 1)$

b) If $y = \sin^3 x$

then $\frac{dy}{dx} = 3\sin^2 x \times \cos x$

$= 3 \sin^2 x \cos x$

c) If y = tan (3x + 2)

then $\frac{dy}{dx} = \frac{1}{\cos^2(3x + 2)} \times 3$

$= \frac{3}{\cos^2(3x + 2)}$

$= 3 \sec^2 (3x + 2)$ All of these are acceptable answers

$= 3 [1 + \tan^2 (3x + 2)]$

$= 3 + 3 \tan^2 (3x + 2)$

d) If $y = 3\tan^4 x$

then $\dfrac{dy}{dx} = 12\tan^3 x \times \dfrac{1}{\cos^2 x}$

$= \dfrac{12\sin^3 x}{\cos^5 x}$

<u>Ref</u>: Ex.5A Q.1-84 (even)

DIFFERENTIATION

1. IMPLICITLY DEFINED FUNCTIONS: Functions of the form **y =** are said to be defined **explicitly**. The **subject** of the formula is clearly defined. Functions where there subject is not clearly indicated are said to be defined **implicitly**, e.g. $2xy - 3y + 5x = 1$ is defined implicitly.

To **differentiate** an implicitly defined function, either **re-arrange** to form an explicitly defined function and **differentiate** as usual, or **differentiate directly** utilizing the **Chain Rule** and other appropriate rules. This direct method is valuable as **re-arranging** can sometimes be **difficult or impossible**. The ClassPad will differentiate implicitly by **impDiff(**.

E.g.1. Differentiate $2xy - 3y + 5x = 1$ –
a) by first re-arranging the equation,
b) directly.

a) $2xy - 3y + 5x = 1$

$y(2x - 3) = 1 - 5x$

$$y = \frac{1 - 5x}{2x - 3}$$

$$\frac{dy}{dx} = \frac{(2x - 3)(-5) - (1 - 5x)(2)}{(2x - 3)^2}$$

$$= \frac{-10x + 15 - 2 + 10x}{(2x - 3)^2}$$

$$= \frac{13}{(2x - 3)^2}$$

b) $2xy - 3y + 5x = 1$

$$\underbrace{\frac{d}{dx}(2xy)}_{\text{Product Rule}} - \frac{d}{dx}(3y) + \frac{d}{dx}(5x) = \frac{d}{dx}(1)$$

$$\left(y\frac{d}{dx}(2x) + 2x\frac{d}{dx}(y)\right) - 3\frac{d}{dx}(y) + 5 = 0$$

$$2y + 2x\frac{dy}{dx} - 3\frac{dy}{dx} = -5$$

$$\frac{dy}{dx}(2x - 3) = -(2y + 5)$$

$$\frac{dy}{dx} = -\frac{2y + 5}{2x - 3}$$

E.g.2. Determine the gradient of the curve $y^3 + 3x = x^2 - 4xy + 5$ at the point $(2,1)$.

Using the ClassPad:

impDiff ($y^3 + 3x = x^2 - 4xy + 5$, x, y)| x = 2 $\Rightarrow \dfrac{dy}{dx} = \dfrac{1-4y}{3y^2+8}$

$\dfrac{1-4y}{3y^2+8}\,|y = 1 \Rightarrow -\dfrac{3}{11}$

$\therefore$ The gradient of the curve $y^3 + 3x = x^2 - 4xy + 5$ at the point (2,1) is $-\dfrac{3}{11}$.

Ref: Ex.6A Q.1-26 (even)

2. PARAMETRIC EQUATIONS: Functions can be defined **parametrically**. This occurs when two variables are related by a third variable, called the **parameter**, e.g. $x = 4 - 3t$ and $t = ½y^2$ are **parametric equations**.

To **differentiate** parametrically defined functions, either **express** the functions directly for the required two variables and **differentiate** as usual, or **differentiate directly** utilizing the **Chain Rule** and differentiating **inverse functions**.

E.g.3. Differentiate $x = 4 - 3t$ and $y = ½t^2$ –
a) by first re-arranging the equations,
b) directly.

a) $x = 4 - 3t$ and $y = ½t^2$

$$t = \frac{4-x}{3}$$

$$y = \frac{(4-x)^2}{18}$$

$$\frac{dy}{dx} = \frac{(-2)(4-x)(1)}{18}$$

$$= -\frac{4-x}{9}$$

b) $\dfrac{dx}{dt} = -3$ and $\dfrac{dy}{dt} = t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= -\frac{t}{3}$$

E.g.4. Find the gradient of the curve defined parametrically as –

$$\begin{cases} x = \sin t \\ y = 2\cos 3t \end{cases}$$

at the point where $t = \frac{\pi}{6}$.

$x = \sin t$ and $y = 2\cos 3t$

$\frac{dx}{dt} = \cos t \quad \frac{dy}{dt} = -2\sin 3t \times 3 = -6\sin 3t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= -6\sin 3t \times \frac{1}{\cos t}$

When $t = \frac{\pi}{6}$, Gradient $= -6\sin\frac{\pi}{2} \times \frac{1}{\cos\frac{\pi}{6}}$

$= -4\sqrt{3}$

≈ -6.93

To find the second derivative, $\mathbf{\frac{d^2y}{dx^2}}$, of parametric equations – differentiate $\frac{dy}{dx}$ with respect to t, i.e. $\frac{d\left(\frac{dy}{dx}\right)}{dt}$ and multiply by $\frac{dt}{dx}$, i.e. $\mathbf{\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dt} \times \frac{dt}{dx}}$.

This is necessary because, whilst $\frac{dt}{dx} = \frac{1}{\left(\frac{dx}{dt}\right)}$, $\frac{d^2t}{dx^2} \neq \frac{1}{\left(\frac{d^2x}{dt^2}\right)}$.

E.g.5. If $x = t + \frac{4}{t}$ and $y = t^2 + 4t$, determine:

a) $\frac{dy}{dx}$

b) $\frac{d^2y}{dx^2}$

a) $x = t + \dfrac{4}{t}$ and $y = t^2 + 4t$

$$\frac{dx}{dt} = 1 - \frac{4}{t^2} \qquad \frac{dy}{dt} = 2t + 4$$

$$= \frac{t^2 - 4}{t^2}$$

$$\frac{dt}{dx} = \frac{t^2}{t^2 - 4}$$

$$\frac{dy}{dx} = (2t + 4) \times \frac{t^2}{t^2 - 4}$$

$$= \frac{2t^2\cancel{(t + 2)}}{\cancel{(t + 2)}(t - 2)}$$

$$= \frac{2t^2}{t - 2} \qquad (t + 2 \neq 0)$$

b) $$\frac{d^2 y}{dx^2} = \frac{4t(t - 2) - 2t^2(1)}{(t - 2)^2} \times \frac{t^2}{t^2 - 4}$$

$$= \frac{2t^2 - 8t}{(t - 2)^2} \times \frac{t^2}{t^2 - 4}$$

$$= \frac{2t^3(t - 4)}{(t - 2)^3(t + 2)}$$

<u>Ref</u>: Ex.6B Q.1-12 (even); 13, 14

MATRICES, BASIC IDEAS

1. ADDING AND SUBTRACTING OF MATRICES: A **matrix** is a rectangular array of numbers. The **size** of a matrix is given by the number of **rows** × the number of **columns**.

Thus, this matrix is a 3 × 4 (↓→), e.g. $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & -2 & 3 & 5 \\ 2 & 1 & -3 & 4 \end{bmatrix}$. A matrix consisting of just one column is called a **column matrix**, e.g. $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$. Likewise, a matrix consisting of just one row is called a **row matrix**. e.g. [3 2 -1]. A **diagonal matrix** is square matrix with zeros in all spaces **not** on the **leading diagonal**, e.g. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \text{[illegible]} \end{bmatrix}$ Leading diagonal. As seen in this example, a diagonal matrix may **also** have zeros in the leading diagonal.

A **matrix** is usually named with a capital letter, e.g. A, and hence each **element** can be correspondingly named using the same **lower case letter** with its **row** and **column** position given in a **subscript**, e.g. a_{32} is the element in the 3rd row and the 2nd column of matrix A.

Matrices can only be added to/subtracted from matrices of the same size. Two matrices are **equal** iff (if and only if) all corresponding elements are equal. A matrix can also be multiplied by a **scalar** (constant).

E.g.1. Given A = $\begin{bmatrix} 4 & 3 \\ -1 & 7 \\ 0 & -5 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & -7 \\ 4 & -2 \\ 3 & -1 \end{bmatrix}$ then find:

a) A + B
b) 5A
c) A – B
d) A + 3B
e) C if A + 2C = B

f) x, y and k if D = $\begin{bmatrix} x & 6 \\ -2 & y \\ 0 & -10 \end{bmatrix}$ and D = kA

a) A + B = $\begin{bmatrix} 4 & 3 \\ -1 & 7 \\ 0 & -5 \end{bmatrix} + \begin{bmatrix} 2 & -7 \\ 4 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 3 & 5 \\ 3 & -6 \end{bmatrix}$

b) 5A = 5 $\begin{bmatrix} 4 & 3 \\ -1 & 7 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 20 & 15 \\ -5 & 35 \\ 0 & -25 \end{bmatrix}$

c) $A - B = \begin{bmatrix} 4 & 3 \\ -1 & 7 \\ 0 & -5 \end{bmatrix} - \begin{bmatrix} 2 & -7 \\ 4 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 10 \\ -5 & 9 \\ -3 & -4 \end{bmatrix}$

d) $A + 3B = \begin{bmatrix} 4 & 3 \\ -1 & 7 \\ 0 & -5 \end{bmatrix} + 3\begin{bmatrix} 2 & -7 \\ 4 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -1 & 7 \\ 0 & -5 \end{bmatrix} + \begin{bmatrix} 6 & -21 \\ 12 & -6 \\ 9 & -3 \end{bmatrix}$

$= \begin{bmatrix} 10 & -18 \\ 11 & 1 \\ 9 & -8 \end{bmatrix}$

e) If A + 2C = B
then C = ½ (B – A)

i.e. $C = \dfrac{1}{2}\left(\begin{bmatrix} 2 & -7 \\ 4 & -2 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ -1 & 7 \\ 0 & -5 \end{bmatrix} \right)$

$= \dfrac{1}{2}\begin{bmatrix} -2 & -10 \\ 5 & -9 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} -1 & -5 \\ 2.5 & -4.5 \\ 1.5 & 2 \end{bmatrix}$

f) $\begin{bmatrix} x & 6 \\ -2 & y \\ 0 & -10 \end{bmatrix} = k\begin{bmatrix} 4 & 3 \\ -1 & 7 \\ 0 & -5 \end{bmatrix}$

By comparing corresponding elements, k = 2.
$\therefore$x = 8 and y = 14.

E.g.2. If $A = \begin{bmatrix} 1 & 6 \\ 3 & -2 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ -3 & 5 \\ -2 & 0 \end{bmatrix}$, then find i and j if:

a) $a_{ij} = b_{ij}$
b) $a_{ij} > b_{ij}$

a) By comparing corresponding elements,
$a_{12} = b_{12} = 6$ or $a_{32} = b_{32} = 0$
$\therefore$ i = 1 and j = 2 $\therefore$ i = 3 and j = 2

b) By comparing corresponding elements,
3 > -3 or 5 > -2
i.e. $a_{21} > b_{21}$ $a_{31} > b_{31}$
$\therefore$ i =2 and j = 1 $\therefore$i =3 and j = 1

E.g.3. In a class of 30 students, the ratio of males to females was 1:2. When asked if an Assignment was done, the number of "No" answers was the same for males and females. Construct a matrix to show this given the ratio of "Yes" to "No" was 3:2.

	Y	**N**	
M	4	6	**10**
F	14	6	**20**
	18	**12**	**30**

1:2

3:2

i.e. $\begin{array}{cc} & \begin{array}{cc} Y & N \end{array} \\ \begin{array}{c} M \\ F \end{array} & \begin{bmatrix} 4 & 6 \\ 14 & 6 \end{bmatrix} \end{array}$

<u>Ref</u>: *Ex.3A Q.1-14 (even)*

2. MATRIX MULTIPLICATION: Matrix multiplication is only possible if the "inside dimensions" are equal, i.e. $(a \times \mathbf{b}) \times (\mathbf{b} \times c) = (a \times c)$. To multiply matrices, the sums of the 1^{st} row $\times$ 1^{st} column, 2^{nd} row $\times$ 1^{st} column, etc. are found, i.e.

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \times \begin{bmatrix} g & h \\ i & j \\ k & l \end{bmatrix} = \begin{bmatrix} ag + bi + ck & ah + bj + cl \\ dg + ei + fk & dh + ej + fl \end{bmatrix}$$

$(2 \times 3) \times (3 \times 2) = (2 \times 2)$

E.g.4. Complete the matrix multiplication for each of the following:

a) $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \times \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} 5 & 4 \\ -2 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 4 & 2 \\ 0 & -1 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

d) $\begin{bmatrix} 3 & 0 & -1 \\ 4 & 2 & 1 \\ 0 & -1 & -2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ -3 & 0 \\ 0 & -1 \end{bmatrix}$

e) $\begin{bmatrix} 4 & 3 & 0 \\ 7 & 2 & 5 \\ 2 & -1 & 6 \end{bmatrix} \times \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

f) $\begin{bmatrix} 3 & -1 & 2 \\ 0 & 4 & -5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← This is called an **Identity Matrix**

a) $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \times \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 14 \\ 42 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} 5 & 4 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 6 \\ -6 & 6 \end{bmatrix}$

c) $\begin{bmatrix} 4 & 2 \\ 0 & -1 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \\ -1 \end{bmatrix}$

d) $\begin{bmatrix} 3 & 0 & -1 \\ 4 & 2 & 1 \\ 0 & -1 & -2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ -3 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 3 \\ 3 & 2 \end{bmatrix}$

e) $\begin{bmatrix} 4 & 3 & 0 \\ 7 & 2 & 5 \\ 2 & -1 & 6 \end{bmatrix} \times \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ ← This is not possible!

f) $\begin{bmatrix} 3 & -1 & 2 \\ 0 & 4 & -5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 4 & -5 \end{bmatrix}$

E.g.5. Given A = $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$, B = [3 0 -1 2] and C = $\begin{bmatrix} 6 & 2 \\ -1 & 0 \\ 3 & -4 \end{bmatrix}$, find the product of these three matrices by first arranging them in a suitable order.

$A(2 \times 1)$ $B(1 \times 4)$ $C(3 \times 2)$
$\therefore C \times A \times B$

$$\begin{bmatrix} 6 & 2 \\ -1 & 0 \\ 3 & -4 \end{bmatrix} \times \begin{bmatrix} 4 \\ -1 \end{bmatrix} \times [3 \;\; 0 \;\; -1 \;\; 2] = \begin{bmatrix} 22 \\ -4 \\ 16 \end{bmatrix} \times [3 \;\; 0 \;\; -1 \;\; 2]$$

$$= \begin{bmatrix} 66 & 0 & -22 & 44 \\ -12 & 0 & 4 & -8 \\ 48 & 0 & -16 & 32 \end{bmatrix}$$

E.g.6. THAM Company sells products X and Y at \$7 and \$4 each, respectively. Its 3 salesmen, Ashley, Brooke and Courtney, in the last week, have sold the products as shown in the table below:

	A	B	C	
X	15	18	25	58
Y	20	17	14	51

Use matrices to calculate the value/s of the sales.

$$[58 \;\; 51] \times \begin{bmatrix} 7 \\ 4 \end{bmatrix} = [610]$$

∴the total value of the sales is \$610.

Are the following properties **always** true for matrices, and if not, under what conditions would they be true:

AB = BA? – **No**
(AB)C = A(BC)? – **Yes**
A(B + C) = AB + AC? – **Yes**
$(A + B)^2 = A^2 + 2AB + B^2$? – **No**
$B^2 – A^2 = (B – A)(B + A)$? – **No**

CONDITIONS:

- A & B are the same size
- A & B are square
- AB = BA, i.e. they are commutative

Ref: *Ex.3B Q.1-18 (even); 19-35 (odd)*

3. IDENTITY MATRIX: Any matrix which has every entry as zero is called a **zero matrix**. e.g. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Unlike in arithmetic if AB = 0 then either A = 0 and/or B = 0, but for matrices if AB = 0 it is not necessarily true that A = 0 and/or B = 0.

E.g.7. Given M = $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ and N = $\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$, then find MN.

$$MN = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Likewise, in arithmetic if AB = CB, for B ≠ 0, then A = C, but for matrices again this is not necessarily true.

E.g.8. Given P = $\begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}$, Q = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and R = $\begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix}$ then find PQ and RQ.

$$PQ = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

$$RQ = \begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

Thus PQ = RQ, Q $\neq$ 0, but P $\neq$ R.

An identity matrix leaves all other matrices unchanged under multiplication, thus the matrix **I** is an **identity matrix** or **unit matrix** (for multiplication) if, for any **square matrix A**, **AI = IA = A**. The identity matrix for a 2 $\times$ 2 square matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

4. INVERSE MATRIX: For a **square matrix A**, the **inverse matrix** is denoted by $\mathbf{A^{-1}}$ and is defined by $\mathbf{A.A^{-1} = A^{-1}.A = I}$. In other words, matrix A needs to be multiplied by another matrix so that the product is I. If matrix A = $\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$ and a.d – b.c $\neq$ 0, then matrix A is invertible and $\mathbf{A}^{-1} = \dfrac{1}{\mathbf{a.d - b.c}} \begin{bmatrix} \mathbf{d} & -\mathbf{b} \\ -\mathbf{c} & \mathbf{a} \end{bmatrix}$. The expression **a.d – b.c** is known as the **determinant** of A. If the determinant is equal to zero then the matrix has no inverse and is said to be a **singular matrix**.

NOTE: The above formula only applies to 2 $\times$ 2 matrices.

E.g.9. Find the inverse of B = $\begin{bmatrix} 3 & 9 \\ -2 & 5 \end{bmatrix}$ if one exists and check if this inverse is correct by multiplying B and B^{-1}.

$B = \begin{bmatrix} 3 & 9 \\ -2 & 5 \end{bmatrix}$ $\qquad$ $a.d - b.c = 15 - (-18)$

$$= 33$$

$\therefore$.an inverse matrix does exist.

$$B^{-1} = \frac{1}{33}\begin{bmatrix} 5 & -9 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{33} & -\frac{3}{11} \\ \frac{2}{33} & \frac{1}{11} \end{bmatrix}$$

$$B.B^{-1} = \begin{bmatrix} 3 & 9 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} \frac{5}{33} & -\frac{3}{11} \\ \frac{2}{33} & \frac{1}{11} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{11} + \frac{6}{11} & -\frac{9}{11} + \frac{9}{11} \\ -\frac{10}{33} + \frac{10}{33} & \frac{6}{11} + \frac{5}{11} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

<u>Ref</u>: *Ex.3C Q.1-40 (even)*

5. **INVERSE MATRIX ALGEBRA:** If matrices **A, X** and **B** are square, A has an inverse and they are compatible for matrix multiplication i.e. **A.X = B** then,

$$\begin{aligned} A.X &= B \\ \text{i.e. } A^{-1}.A.X &= A^{-1}.B \\ \text{i.e. } I.X &= A^{-1}.B \\ \text{i.e. } \mathbf{X} &= \mathbf{A^{-1}.B} \end{aligned}$$

Likewise, if **X.A = B** then **X = B.A**$^{\mathbf{-1}}$

Simultaneous equations can also be expressed as a matrix equation in the form of **A.X = B** and hence solved using **X = A**$^{\mathbf{-1}}$**.B**, provided that the matrix of coefficients has an inverse.

E.g.10. Solve:

a) $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$

b) $3x + y = 32$
$5x + 7y = 64$

a) $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$

If $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ then a.d – b.c = 6 – 5 = 1,

and hence $A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

Thus $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 11 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

i.e. x = 1 and y = 3

b) $3x + \ \ y = 32$
$5x + 7y = 64$

$$\Rightarrow \begin{bmatrix} 3 & 1 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 32 \\ 64 \end{bmatrix}$$

a.d – b.c = 21 – 5 = 16

Hence, $A^{-1} = \frac{1}{16} \begin{bmatrix} 7 & -1 \\ -5 & 3 \end{bmatrix}$

$$\text{i.e.} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 7 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 32 \\ 64 \end{bmatrix}$$

$$\text{i.e.} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 160 \\ 32 \end{bmatrix}$$

i.e. $x = 10$ and $y = 2$

NOTE: When the determinant causes the values in the matrix to become fractions it may be easier to not multiply the determinant through the matrix (until the “end”).

<u>Ref</u>: *Ex.3D Q.1-11 (odd)*

TRANSFORMATION MATRICES

1. TRANSFORMATIONS: A **transformation** is a mapping of a geometric figure or a matching between points of the plane. Any **object**, say point **A**, can be transformed (changed) by pre-multiplying its co-ordinates in order to find its **image A'**. E.g. For the transformation equations x' = ax + by and y' = cx + dy, written in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

NOTE: The **co-ordinates** of the point in the Cartesian Plane are written as a **column matrix**.

Under these transformations, the origin (0,0) is an **invariant** point, i.e. it doesn't move.

NOTE: The **prime** used to indicate an **image** point/co-ordinate should not be confused with that used to indicate differentiation.

E.g.1. Transform the point A (2,3) by pre-multiplying it by the matrix $\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$, hence find its image A'.

$$\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

$\therefore$ A' is (-3,7).

There are many different types of transformations. The transformations considered here are in the Cartesian Plane and will be governed by 2$\times$2 matrices, thus **translations** will not be considered as the **origin** is **not invariant**. Only those transformations known as **linear transformations**, i.e. **reflection**, **rotation**, **dilation** and **shear**, in that the **origin is fixed** and all **lines map to lines** will be considered.

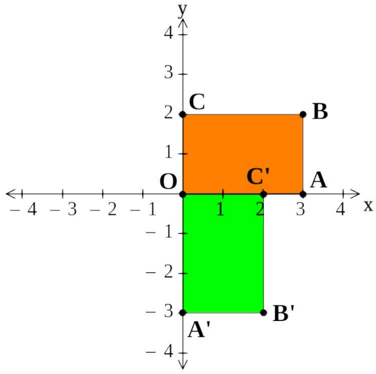
In a **shear** transformation – all points move **parallel** to an **invariant line** and the **further** a point is **from** the invariant line, the **further** it **moves**.

E.g.2. By considering the effect on the rectangle O(0,0), A(3,0), B(3,2) and C(0,2), determine the transformation represented by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

$$O': \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad A': \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

$$B': \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \qquad C': \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

By considering the graph on the right of OABC and O'A'B'C', we can see that the matrix represents a clockwise rotation about the origin of -90°.



Ref: *Ex.4A Q.1-13 (odd)*

2. **TRANSFORMATION MATRICES:** The transformation matrices for each transformation do not need to be memorized.

Reflection is a **congruence transformation** and can be in any line but we will only consider reflections in the –

X – axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Y – axis $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

The line $y = x$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The line $y = -x$ $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

The **enlargement transformation** is a transformation that **magnifies** (or **reduces**) in both the X- and Y-directions and is called a **dilation**. The matrix that dilates an object about the origin by a **factor** of b is $\begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$.

A **scaling** could be described as a **dilation** with two different factors – one for the X-direction and another for the Y-direction. The matrix that scales an object by factor ‘a’ in the X-direction and factor ‘b’ in the Y-direction is $\begin{bmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{b} \end{bmatrix}$.

NOTE: When a shape with an area of A square units is transformed by a matrix with a determinant of k then the area of the image shape A' is **|k|A** square units.

Rotation is another of the congruence transformations. In considering rotations about the origin, it is necessary to give the **direction** and the **angle of rotation** to specify the rotational matrix.

NOTE: An **anticlockwise** rotation has a **positive** angle and hence a **clockwise** rotation has a **negative** angle.

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation about the origin in an anticlockwise direction of angle θ.

NOTE: For the “rotation” matrix, **exact values** must be used.

θ	**0°**	**30°**	**45°**	**60°**	**90°**
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

A **shear** is a transformation in which **every** (image) **point** has the same **displacement** (as the object) in a given direction, **parallel** to a given line. The fixed line is called the **axis of the shear** and only shears parallel to the X-axis or the Y-axis will be considered.

$\begin{bmatrix} \mathbf{1} & \mathbf{k} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$ is a shear **parallel** to the **X-axis** with **factor k**.

i.e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$

$\therefore$ x' = x + ky and y' = y

$\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{m} & \mathbf{1} \end{bmatrix}$ is a shear **parallel** to the **Y-axis** with **factor m**.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ mx + y \end{bmatrix}$$

$\therefore$ x' = x and y' = y + mx

If a matrix **T transforms** point **A to** its image **A'** then $\mathbf{T}^{-1}$, the multiplicative **inverse of T**, will **transform A'** back **to A** provided that $\mathbf{T}^{-1}$ **exists**.

E.g.3. ΔCDE is transformed to ΔC'D'E' by matrix $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. ΔC'D'E' is transformed to ΔC"D"E" by matrix $\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$. Find the single matrix that will transform ΔCDE directly to ΔC"D"E".

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 9 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the required matrix is $\begin{bmatrix} 9 & 9 \\ 2 & 6 \end{bmatrix}$.

NOTE: If a transformation T maps A to B and another transformation R maps B to C then **RT** will map directly from A to C.

Ref: *Ex.4B Q.1-21 (odd)*

3. GENERAL REFLECTIONS: A **reflection** in the line **y = mx** where **m = tan θ**, is given by the matrix $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

NOTE: For this "reflection" matrix, **exact values** must be used.

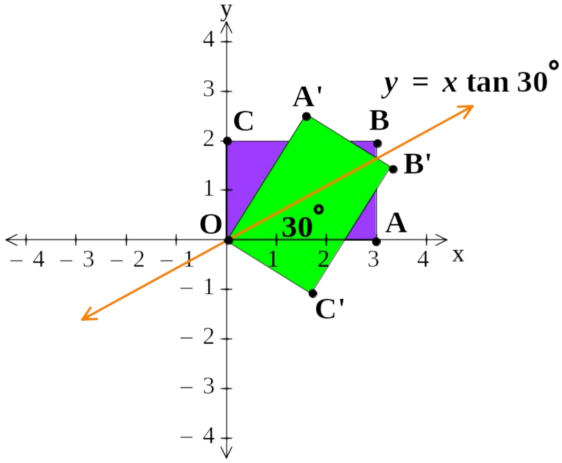
E.g.4. Determine the co-ordinates of the rectangle O'A'B'C' given the original rectangle has co-ordinates of O(0,0), A(3,0), B(3,2) and C(0,2) and is reflected in the line y = x tan 30°.

$$O': \begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A': \begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\frac{1}{2} \\ \frac{3\sqrt{3}}{2} \end{bmatrix}$$

$$B': \begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} + 1\frac{1}{2} \\ \frac{3\sqrt{3}}{2} - 1 \end{bmatrix}$$

$$C': \begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$$



<u>Ref</u>: *Ex.4C Q.1-5*

TRANSITION MATRICES

1. TRANSITION MATRICES: A **transition matrix** shows the results of **changing From** one situation **To** another situation **one stage later**. **From** is usually written at the top of the matrix, and **To** is written on the left side. If the entries are **probabilities**, then the matrix is called a **probability transition matrix**, with **0 ≤ entry ≤ 1**, and the **sum** of each **column** is **1**.

If changing from one situation to another **n stages later**, then raise the **transition matrix** to the **power** of **n**. It is **assumed** that the **changes remain unchanged** at each stage. Systems for which the next stage of the system depends only on the current state are called **Markov Chains**.

E.g.1. There are 3 brands of juice that dominate the market – Brand A, Brand B and Brand C. People switch from one brand to another all the time. If they use Brand A this week, 70% will continue to use it next week, 20% will switch to Brand B and 10% will switch to Brand C. If they use Brand B this week, 40% will switch to Brand A, 30% will continue to use it next week and 30% will switch to Brand C. If they use Brand C this week, 20% will switch to Brand A, 30% will switch to Brand B and 50% will continue to use it next week.

a) If a family is using Brand B, what is the probability they will be using Brand A in 3 weeks time?

The local supermarket sold 500 litres of Brand A, 350 litres of Brand B and 430 litres of Brand C this week.

b) Calculate the expected sales of each brand next week.

a)

$$\begin{array}{cc} & \text{From} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{To} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.7 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{bmatrix} \end{array} = T$$

$$T^3 = \begin{array}{cc} & \text{From} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{To} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.541 & 0.482 & 0.436 \\ 0.241 & 0.254 & 0.264 \\ 0.218 & 0.264 & 0.300 \end{bmatrix} \end{array}$$

∴ If a family was using Brand B then there is a 0.482 probability of it using Brand A in 3 weeks time.

b)

$$T = \begin{array}{c c} & \text{From} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{To} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.7 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{bmatrix} \end{array}$$

$$M = \begin{array}{cc} \begin{bmatrix} 500 \\ 350 \\ 430 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \end{array} \end{array}$$

$$T \times M = \begin{array}{cc} \begin{bmatrix} 576 \\ 334 \\ 370 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \end{array} \end{array}$$

$\therefore$ The expected sales of Brand A are 576 L, Brand B are 334 L and Brand C are 370 L next week.

E.g.2. Three friends, Jody, Kelly and Lane, throw a ball to each other. There is a probability of ⅓ that Jody will throw the ball to Kelly, ½ that Kelly will throw the ball to Lane, and ¼ that Lane will throw the ball to Jody. If Kelly starts with the ball, what is the probability that he will have the ball back after 2 throws?

$$\begin{array}{c c} & \text{From} \\ & \begin{array}{ccc} J & K & L \end{array} \\ \text{To} \begin{array}{c} J \\ K \\ L \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{3}{4} \\ \frac{2}{3} & \frac{1}{2} & 0 \end{bmatrix} \end{array} = T$$

$$T^2 = \begin{array}{c c} & \text{From} \\ & \begin{array}{ccc} J & K & L \end{array} \\ \text{To} \begin{array}{c} J \\ K \\ L \end{array} & \begin{bmatrix} \frac{1}{3} & \frac{1}{8} & \frac{3}{8} \\ \frac{1}{2} & \frac{13}{24} & \frac{1}{12} \\ \frac{1}{6} & \frac{1}{3} & \frac{13}{24} \end{bmatrix} \end{array}$$

$\therefore$ If Kelly starts with the ball, he has a probability of $\frac{13}{24}$ of having it back after 2 throws.

NOTE: When the **leading diagonal** consists of **zeros** it is not possible **initially** to go from one situation to itself.

Ref: *Ex.5A Q.1-6 (even)*

The matrix regarding the number of litres sold in Example 1 is called the **initial state matrix**, S_0. A **steady state** is achieved when the **probability matrix**^{n} $\times$ **initial state matrix** results in the same values for different values of **n** usually $\geq$ **15**, and/or different **initial state values**. These **long term** proportions are **independent** of the initial distribution and only **dependent** on the transition matrix.

The **steady state proportions** can be determined algebraically

$$\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$ with proportions of $\dfrac{\mathbf{a}}{\mathbf{a+b}}$ and $\dfrac{\mathbf{b}}{\mathbf{a+b}}$.

E.g.3. A car hire company has three depots in Perth – the International Airport (I), the City Centre (C), and the Domestic Airport (D). Customers can pick up a car at any of these depots and return it to any depot. A study of weekly rentals revealed a car picked up at I has a probability of 0.6 of being returned to I, 0.2 of being returned to C and 0.2 of being returned to D; a car picked up at C has a probability of 0.3 of being returned to I, 0.5 of being returned to C and 0.2 of being returned to D; and a car picked up at D has a probability of 0.4 of being returned to I, 0.1 of being returned to C and 0.5 of being returned to D.

a) Define the transition matrix showing this information.

Given that in Week 1, the number of cars at each depot was 150 at I, 120 at C and 140 at D,

b) calculate the number of cars at each depot after two weeks of operation.

c) Show that in the long term, the number of cars in each depot will stabilise.

a)

$$\begin{array}{cc} & \text{From} \\ & \begin{array}{ccc} \text{I} & \text{C} & \text{D} \end{array} \\ T = \text{To} \begin{array}{c} \text{I} \\ \text{C} \\ \text{D} \end{array} & \begin{bmatrix} 0.6 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.1 \\ 0.2 & 0.2 & 0.5 \end{bmatrix} \end{array}$$

b) $$M = \begin{bmatrix} 150 \\ 120 \\ 140 \end{bmatrix} \begin{array}{c} \text{I} \\ \text{C} \\ \text{D} \end{array}$$

$$T \times M = \begin{bmatrix} 182 \\ 104 \\ 124 \end{bmatrix} \begin{array}{c} \text{I} \\ \text{C} \\ \text{D} \end{array}$$

c)

$$T = \begin{array}{cc} & \text{From} \\ & \begin{array}{ccc} I & C & D \end{array} \\ \text{To} \begin{array}{c} I \\ C \\ D \end{array} & \begin{bmatrix} 0.6 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.1 \\ 0.2 & 0.2 & 0.5 \end{bmatrix} \end{array}$$

$$M = \begin{bmatrix} 150 \\ 120 \\ 140 \end{bmatrix} \begin{array}{c} I \\ C \\ D \end{array}$$

$$T^{20} \times M = \begin{bmatrix} 193 \\ 100 \\ 117 \end{bmatrix} \begin{array}{c} I \\ C \\ D \end{array}$$

$$T^{30} \times M = \begin{bmatrix} 193 \\ 100 \\ 117 \end{bmatrix} \begin{array}{c} I \\ C \\ D \end{array}$$

$\therefore$ In the long term, the number of cars at each depot will stabilise.

Ref: *Ex.5B Q.1-8 (even)*

2. **LESLIE MATRICES:** A **Leslie matrix** is used to model population growth where a **life expectancy** is known. For generation **n** – the **reproduction rate r_n** and the **survival rate s_n** are required. The matrix takes the form –

$$\begin{bmatrix} r_1 & r_2 & r_3 & r_4 \\ s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \end{bmatrix}$$

Knowing the **initial population matrix**, $P_{initial}$ and the appropriate **Leslie matrix**, then the **population** in **n-generations** time can be determined.

NOTE: Leslie matrices are sometimes used with survival rates, reproduction rates and population numbers for **females only** as these directly relate to the number of newborns.

E.g.4. A colony of bats living in a cave has the following population statistics recorded:

AGE (months)	**0-6**	**6-12**	**12-18**	**18-24**
POPULATION	4 500	1 800	900	130
BIRTH RATE	0	1.9	1.5	0.7
DEATH RATE	0.5	0.2	0.6	1

The rates are relative to the whole population and not just females.
a) Calculate the survival rates for the group.
b) Set up Leslie and population matrices.

Calculate the population for each age group after:
c) 6 months,
d) 12 months,
e) 18 months, and
f) 5 years.
g) What is the long term % change in the total population on a 6-monthly basis?

a)

AGE (months)	**0-6**	**6-12**	**12-18**	**18-24**
SURVIVAL RATE	0.5	0.8	0.4	0

b)

$$L = \begin{bmatrix} 0 & 1.9 & 1.5 & 0.7 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 4\,500 \\ 1\,800 \\ 900 \\ 130 \end{bmatrix}$$

c)

$$L \times P = \begin{bmatrix} 0 & 1.9 & 1.5 & 0.7 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} 4\,500 \\ 1\,800 \\ 900 \\ 130 \end{bmatrix}$$

$$= \begin{bmatrix} 4\,861 \\ 2\,250 \\ 1\,440 \\ 360 \end{bmatrix}$$

d)

$$L^2 \times P = \begin{bmatrix} 6\,687 \\ 2\,431 \\ 1\,800 \\ 576 \end{bmatrix}$$

e)

$$L^3 \times P = \begin{bmatrix} 7\,721 \\ 3\,344 \\ 1\,944 \\ 720 \end{bmatrix}$$

f)

$$L^{10} \times P = \begin{bmatrix} 33\,512 \\ 13\,628 \\ 8\,865 \\ 2\,884 \end{bmatrix}$$

g) [1 1 1 1] = H

	TOTAL
$\mathbf{H \times L}$	7 330
$\mathbf{H \times L \times P}$	8 911
$\mathbf{H \times L^2 \times P}$	11 494
$\mathbf{H \times L^3 \times P}$	13 729
$\mathbf{H \times L^4 \times P}$	17 086
$\mathbf{H \times L^5 \times P}$	20 936
$\mathbf{H \times L^6 \times P}$	25 756
$\mathbf{H \times L^7 \times P}$	31 679
$\mathbf{H \times L^8 \times P}$	38 944
$\mathbf{H \times L^9 \times P}$	47 894
$\mathbf{H \times L^{10} \times P}$	58 890

$$\text{Growth rate} = \frac{\text{New population}}{\text{Previous population}}$$

From the table:

$$\text{Growth rate} \approx \frac{47\,894}{38\,944} \approx 1.23$$

$$\text{Growth rate} \approx \frac{58\,890}{47\,894} \approx 1.23$$

$\therefore$ Long term growth rate is approximately 23%.

Ref: *Ex.5C Q.1-3*

E.g.5. An island has a population of Koala Bears with the following statistics:

AGE (years)	**0-2**	**2-4**	**4-6**	**6-8**	**8-10**
POPULATION	3 400	2 500	2 300	1 750	650
BREEDING RATE	0	0	3.9	2.7	0.9
SURVIVAL RATE	0.5	0.8	0.7	0.4	0

a) Set up Leslie and population matrices.
b) Calculate the long term population growth per two-year cycle.
c) Calculate a culling rate on a two-year cycle that will stabilise the population.

a)

$$L = \begin{bmatrix} 0 & 0 & 3.9 & 2.7 & 0.9 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 3\,400 \\ 2\,500 \\ 2\,300 \\ 1\,750 \\ 650 \end{bmatrix}$$

b) $[1 \;\; 1 \;\; 1 \;\; 1] = H$

20 years: $H \times L^{10} \times P \approx 175\,313$
22 years: $H \times L^{11} \times P \approx 246\,427$
24 years: $H \times L^{12} \times P \approx 299\,698$

$$\text{Growth rate} \approx \frac{264\,427}{175\,313} \approx 1.41$$

$$\text{Growth rate} \approx \frac{299\,698}{264\,427} \approx 1.21$$

$\therefore$ Average long term growth rate is approximately 1.31.

c) For a stable population: $P_2 = P_1$
but $P_2 = 1.31 \times P_1$
Let $1.31k = 1$

$$k = \frac{1}{1.31} \approx 0.77$$

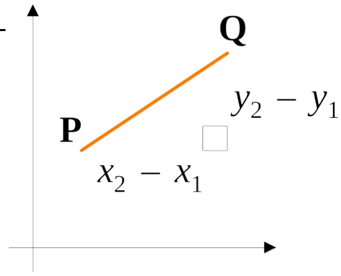
$\therefore$ The culling rate should be approximately 23% per two-year cycle for a stable population.

<u>Ref</u>: *Ex.5D Q.1-3*

POLAR CO-ORDINATES

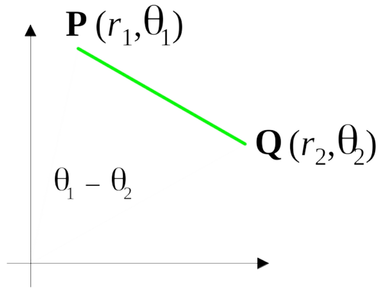
1. **DISTANCE BETWEEN TWO POINTS:** In the **Cartesian** plane –

   Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.



   In the **Polar** plane –

   Distance $= \sqrt{r_1^{\,2} + r_2^{\,2} - 2r_1 r_2 \cos(\theta_1 - \theta_2)}$.



   E.g.1. Find the exact length of XY:

   a) X (5, 25°), Y (8, 85°)

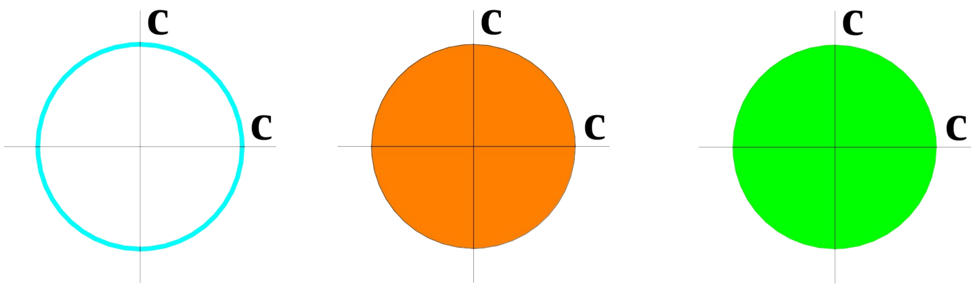
   b) $X\left(12, \frac{\pi}{3}\right)$, $Y\left(5, -\frac{\pi}{2}\right)$

   a) $XY = \sqrt{5^2 + 8^2 - 2(5)(8)\cos(25^\circ - 85^\circ)}$
   $= 7$ units

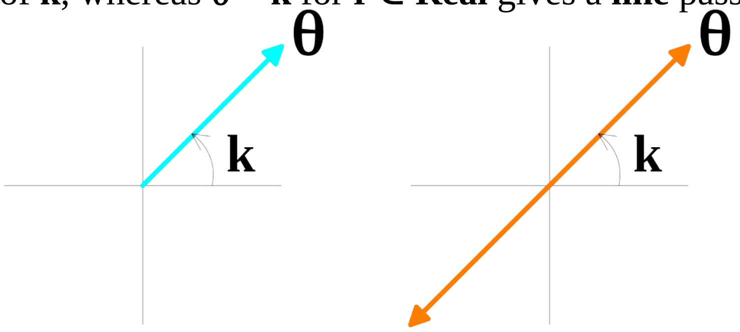
   b) $XY = \sqrt{12^2 + 5^2 - 2(12)(5)\cos(\frac{\pi}{3} - (-\frac{\pi}{2}))}$
   $= \sqrt{169 + 60\sqrt{3}}$

   Ref: Ex.1A Q.1, 2 (R.H.S.); 3-5

2. **GRAPHS OF POLAR EQUATIONS:** In the Polar plane, if r is constant, then the equation **r = c** gives a **circle** of **radius c units**, whereas **r ≤ c** gives a **circular region**, and **r < c** gives a **circular region** but the circle **edge** is '**broken**' indicating that it is not included in the region.

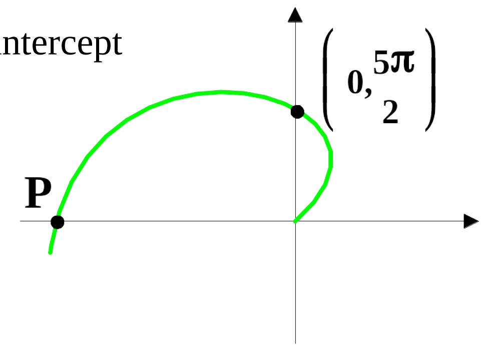


If θ is constant, then the equation **$\theta = k$** for **$r \geq 0$**, gives a **ray** from the **origin** at an angle of **k**, whereas **$\theta = k$** for **$r \in$ Real** gives a **line** passing through the origin at an angle of k.



Equations of the form **$r = k\theta$** describe graphs that **spiral** ‘out’ from the origin [See p.22], where the **polar co-ordinates** of the ‘first’ **y-intercept** = $\left(\frac{\pi}{2}, \frac{k\pi}{2}\right)$, and the ‘first’ **x-intercept = (kπ, π)**; and the **Cartesian co-ordinates** of the ‘first’ **y-intercept** = $\left(0, \frac{k\pi}{2}\right)$, and the ‘first’ **x-intercept = (-kπ, 0)**.

E.g.2. This graph shows the Cartesian co-ordinates of one intercept for $r = k\theta$ with $r \geq 0$ and θ in radians.
a) Determine the value of k,
b) the Cartesian co-ordinates, and
c) the polar co-ordinates of P.



a) k = 5

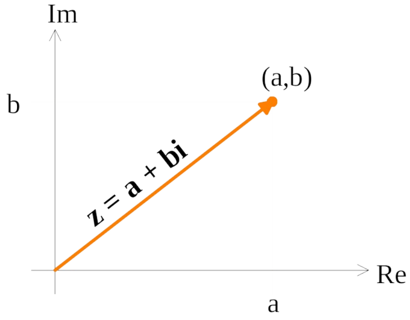
b) P = (-5π, 0)

c) P = (5π, π)

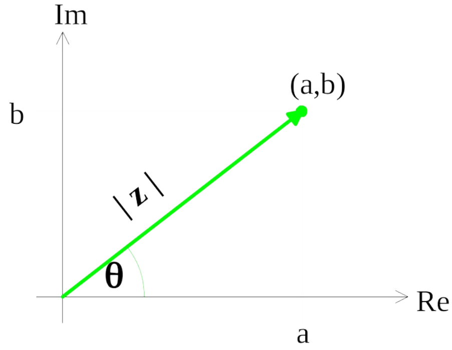
Ref: Ex.1B Q.1-11 (odd); 12

COMPLEX NUMBERS

1. **POLAR FORM: z = a + bi** is the **rectangular** or **Cartesian** form of a complex number, and may be represented on an **Argand diagram**, as a point **(a,b)**, or as a **position vector** from the origin to (a,b).

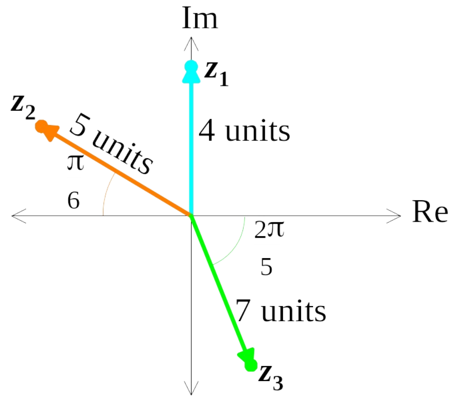


If **z** makes an angle **θ** with the positive Real axis and has a **magnitude** of **| z |**, then the vector can be given as **z = | z | (cos θ + i sin θ)**. This is the **polar form** of a complex number **z**, where **| z | ≥ 0**, and $|z| = \sqrt{a^2 + b^2}$. **| z |** is also called the **modulus** (**mod**) of **z**, and the **argument** of the complex number (**arg z**) is **θ**, usually given in radians, with **θ** measured in an anti-clockwise direction from the positive Real axis. The **principal argument** of a complex number is for **-π < θ ≤ π**. The polar form is sometimes written as **[r,θ]**.



Some calculators have the capacity to determine **mod z** and **arg z**, and/or to convert **Cartesian** co-ordinates to **polar** co-ordinates **directly**.

E.g.1. For each vector in this Argand diagram, state the modulus, and principal argument.



For z_1: modulus = 4

principal argument = $\frac{\pi}{2}$

For z_2: modulus = 5

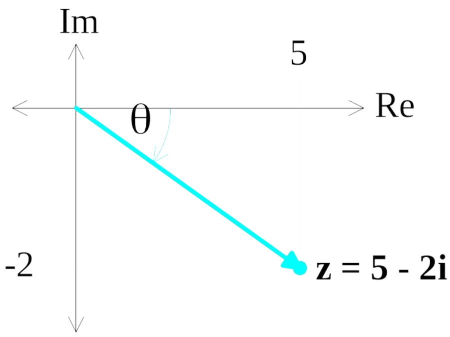
principal argument = $\frac{5\pi}{6}$

For z_3: modulus = 7

principal argument = $-\frac{2\pi}{5}$

E.g.2. Express the complex number 5 – 2i in the form **r(cos θ + i sin θ)**, for $-\pi < \theta \leq \pi$, giving θ correct to 2 decimal places.

$$\text{mod } z = \sqrt{5^2 + (-2)^2}$$
$$= \sqrt{29}$$



$$\tan\theta = \frac{2}{5}$$

$\Rightarrow$ arg z $\approx 0.38^R$

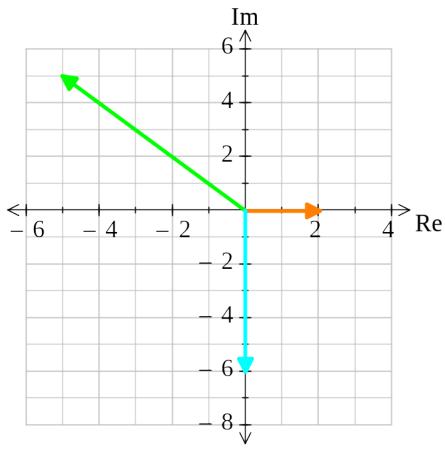
$$\therefore z = \sqrt{29}\,[\cos(-0.38) + i\sin(-0.38)]$$

<u>Ref</u>: Ex.2A Q.1-5 (R.H.S.)

The 'trigonometric' portion of the polar form of a complex number cos θ + **i** sin θ can be abbreviated to **cis θ** and is read as "**sis**" theta, and hence **r(cos θ + i sin θ)** = **r cis θ**.

E.g.3. Express each of the following in the form **r cis θ**, with **r ≥ 0** and **$-\pi < \theta \leq \pi$**.
a) 2
b) -5 + 5i
c) -6i

a) 2 = 2 cis 0^R



b) $\text{mod}(-5 + 5i) = \sqrt{(-5)^2 + 5^2}$
$$= 5\sqrt{2}$$

$$\arg(-5 + 5i) = \frac{3\pi}{4}$$

$$\therefore -5 + 5\mathbf{i} = 5\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}$$

c) $-6\mathbf{i} = 6\,\text{cis}\left(-\frac{\pi}{2}\right)$

<u>Ref</u>: Ex.2B Q.1; 2-30 (even)

2. MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS IN POLAR FORM: To **multiply** two **complex numbers** – **multiply** the **moduli**, and **add** the **arguments**, **adding/subtracting** multiples of 2π, so that **$-\pi <$ arg $\leq \pi$**. Thus, if **$z = r_1$ cis α** and **$w = r_2$ cis β**, then **$zw = r_1r_2$ cis (α + β)**, or **$[r_1,\alpha][r_2,\beta] = [r_1r_2,\alpha + \beta]$**.

To **divide** two **complex numbers** – **divide** the **moduli**, and **subtract** the **arguments**, **adding/subtracting** multiples of 2π, so that **$-\pi < \arg \leq \pi$**. Thus, if **$z = r_1 \text{ cis } \alpha$** and **$w = r_2 \text{ cis } \beta$**, then $\frac{z}{w} = \frac{r_1}{r_2} \text{cis } (\alpha - \beta)$.

Geometrically, zw **increases** the length of z by a factor of | w | and **rotates** z **anticlockwise** about the **origin** by β.

E.g.4. Given $z = 2 \text{ cis } \frac{5\pi}{6}$ and $w = 3 \text{ cis } \frac{2\pi}{5}$, express each of the following in the form $r \text{ cis } \theta$ with $r \geq 0$ and $-\pi < \theta \leq \pi$:

a) zw

b) $\frac{w}{z}$

c) 3w

d) iz

a) $zw = 2 \times 3 \text{ cis } \left(\frac{5\pi}{6} + \frac{2\pi}{5} \right)$

$= 6 \text{ cis } \frac{37\pi}{30}$

$= 6 \text{ cis } \left(-\frac{23\pi}{30} \right)$

b) $\frac{w}{z} = \frac{3}{2} \text{cis } \left(\frac{2\pi}{5} - \frac{5\pi}{6} \right)$

$= 1.5 \text{ cis } \left(-\frac{13\pi}{30} \right)$

c) $3w = 3 \text{ cis } 0 \, . \, 3 \text{ cis } \frac{2\pi}{5}$

$= 9 \text{ cis } \frac{2\pi}{5}$

d) $iz = 1 \text{ cis } \frac{\pi}{2} . 2 \text{ cis } \frac{5\pi}{6}$

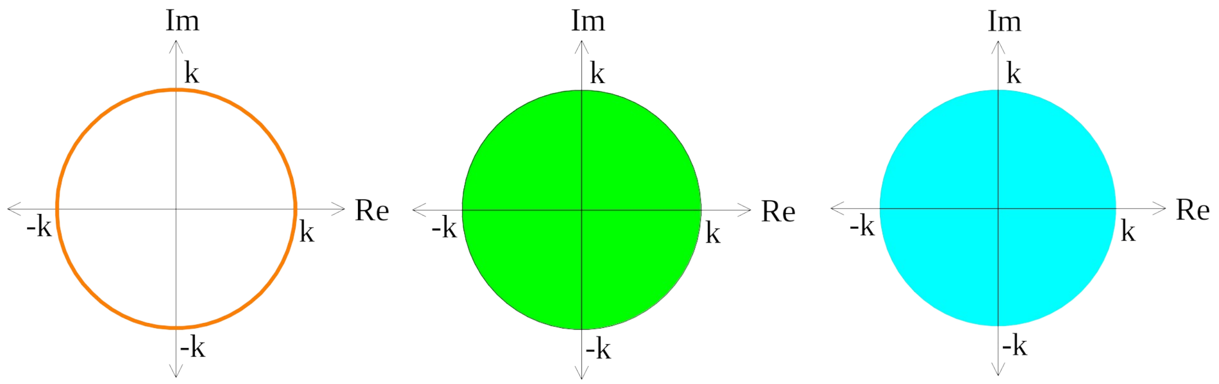
$$= 2 \text{ cis } \left(\frac{\pi}{2} + \frac{5\pi}{6} \right)$$

$$= 2 \text{ cis } \frac{4\pi}{3}$$

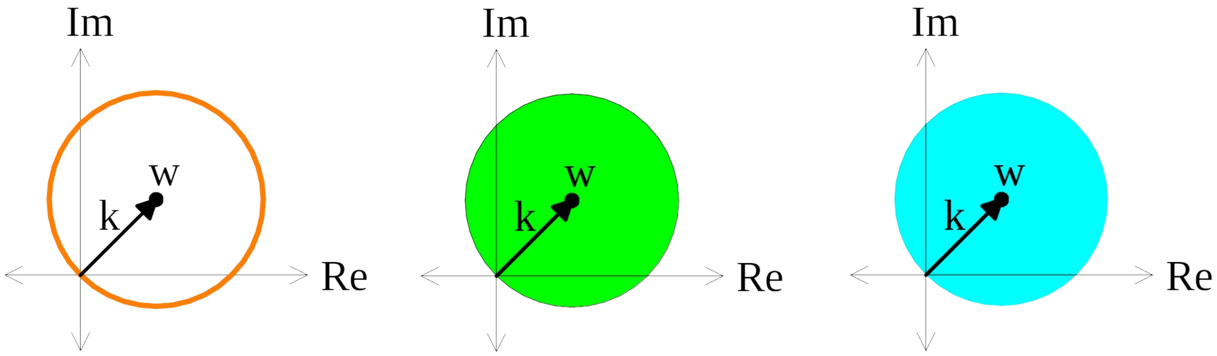
$$= 2 \text{ cis } \left(-\frac{2\pi}{3} \right)$$

<u>Ref</u>: Ex.2C Q.1-29 (odd)

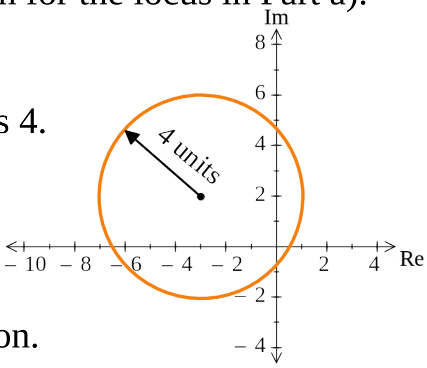
3. REGIONS IN THE COMPLEX PLANE: The **locus** of all points satisfying **| z | = k** is a **circle**, **centre** at the **origin**, with a **radius** of **k**. This is written as **{z: | z | = k}**, and is read as “**the set of all z such that mod z equals k**”. Likewise, **{z: | z | ≤ k}** forms a **circular region** centre at the origin and of radius k, and **{z: | z | < k}** forms a **circular region** centre at the origin and of radius k but with a ‘**broken**’ edge.



For numbers **r** and **d** – **| r – d |** is the distance **r** is from **d**. Similarly, for complex numbers **z** and **w** – **| z – w |** is the distance between **z** and **w** in the complex plane. Thus, the locus of all z numbers satisfying **| z – w | = k** would form a **circle**, **centre** at **w**, with **radius** of **k**. Similarly, **| z – w | ≤ k** is a **circular region** centre at **w** and of radius **k**, and **| z – w | < k** forms a **circular region** centre at **w** and of radius **k** but with a ‘**broken**’ edge.



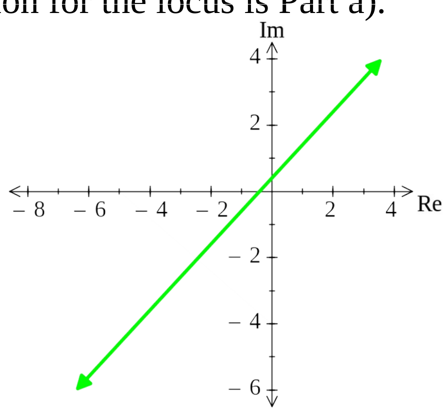
E.g.5. a) Represent $\{z: | z + 3 - 2i | = 4\}$ on an Argand diagram.
b) If $z = x + iy$, then determine the Cartesian equation for the locus in Part a).



a) $| z + 3 - 2i | = 4$
$\therefore | z - (-3 + 2i) | = 4$ is a circle, centre $(-3 + 2i)$ and radius 4.

b) If $z = x + iy$,
then $| x + iy + 3 - 2i | = 4$
$| (x + 3) + i(y - 2) | = 4$
$\therefore (x + 3)^2 + (y - 2)^2 = 16$ is the required Cartesian equation.

E.g.6. a) Represent $\{z: | z + 4i | = | z + 5 |\}$ on an Argand diagram.
b) If $z = x + iy$, then determine the Cartesian equation for the locus is Part a).



a) $| z + 4i |$ is the distance from z to $0 - 4i$,
and $| z + 5 |$ is the distance from z to $-5 + 0i$.
Thus, $\{z: | z + 4i | = | z + 5 |\}$ is the set of all points equidistant from $-4i$ and -5.

b) If $z = x + iy$,
then
$$| x + iy + 4i | = | x + iy + 5 |$$
$$| x + i(y + 4) | = | (x + 5) + iy |$$
$$x^2 + (y + 4)^2 = (x + 5)^2 + y^2$$
$$x^2 + y^2 + 8y + 16 = x^2 + 10x + 25 + y^2$$
$$8y + 16 = 10x + 25$$
$$8y = 10x + 9$$
$\therefore y = \frac{5x}{4} + \frac{9}{8}$ is the required Cartesian equation.

<u>Ref</u>: Ex.2D Q.1-8; 9-24 (even)

4. CONJUGATES: For a complex number, z, $\mathbf{z = x + iy = (x,y)}$ is the **Cartesian** form,
$\mathbf{z = r\ cis\ \theta = [r,\theta]}$ is the **Polar** form, where $r = \sqrt{x^2 + y^2}$, and $\tan\theta = \frac{y}{x}$ for $-\pi < \theta \le \pi$.
The **conjugate**, in polar form, is $\bar{z} = r\ cis\ (-\theta) = [r, -\theta]$ and it can also be shown that $z \times \bar{z} = |z|^2$, $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ and $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$. Additionally, $\frac{\bar{z}}{|z|^2}$ is the **reciprocal** of any **non-zero** complex number.

E.g.7. If $z = 2\ cis\ \frac{\pi}{4}$, then prove that $z \times \bar{z} = |z|^2$.

$\bar{z} = 2 \operatorname{cis}\left(-\frac{\pi}{4}\right)$ & $|z|^2 = r^2 = 4$

$z \times \bar{z} = 2 \operatorname{cis}\frac{\pi}{4} \times 2 \operatorname{cis}\left(-\frac{\pi}{4}\right)$

$= 4 \operatorname{cis}\left(\frac{\pi}{4} + \left(-\frac{\pi}{4}\right)\right)$

$= 4 \operatorname{cis} 0$

$= 4$

$= |z|^2$ (Q.E.D.)

$\therefore z \times \bar{z} = |z|^2$.

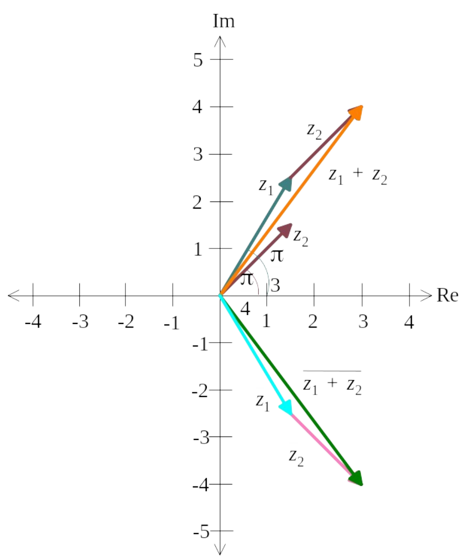
E.g.8. If $z_1 = 3 \operatorname{cis}\frac{\pi}{3}$ and $z_2 = 2 \operatorname{cis}\frac{\pi}{4}$, then:

a) use an Argand diagram to show that $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$,

b) prove that $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$, and

c) prove that $\frac{\bar{z}_1}{|z_1|^2} = \frac{1}{z_1}$.

a)



b) $z_1 \times z_2 = \left(3 \text{ cis } \frac{\pi}{3}\right) \times \left(2 \text{ cis } \frac{\pi}{4}\right)$

$$= 6 \text{ cis } \left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$

$$= 6 \text{ cis } \frac{7\pi}{12}$$

$\therefore \overline{z_1 \times z_2} = 6 \text{ cis } \left(-\frac{7\pi}{12}\right)$

$\bar{z}_1 \times \bar{z}_2 = \left[3 \text{ cis } \left(-\frac{\pi}{3}\right)\right] \times \left[2 \text{ cis } \left(-\frac{\pi}{4}\right)\right]$

$$= 6 \text{ cis } \left(-\frac{7\pi}{12}\right)$$

$= \overline{z_1 \times z_2}$ (Q.E.D.)

$\therefore \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

c) $\frac{\bar{z}_1}{|z_1|^2} = \frac{3 \text{ cis } \left(-\frac{\pi}{3}\right)}{9}$

$$= \frac{1}{3} \text{ cis } \left(-\frac{\pi}{3}\right)$$

$\frac{1}{z_1} = 1 \div z_1$

$$= (1 \text{ cis } 0) \div \left(3 \text{ cis } \frac{\pi}{3}\right)$$

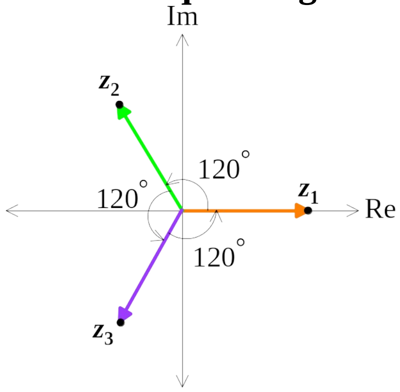
$= \frac{1}{3} \text{ cis } \left(-\frac{\pi}{3}\right)$ (Q.E.D.)

$\therefore \frac{\bar{z}_1}{|z_1|^2} = \frac{1}{z_1}$

<u>Ref</u>: CONJUGATES

COMPLEX NUMBERS

1. ROOTS OF COMPLEX NUMBERS: The equation $\mathbf{x^3 = 1}$ has **one real solution** x = 1. However, there are also **two complex solutions** to this equation, $-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}$ and $-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}$. Alternatively, the solutions are 1 cis 0, 1 cis ⅔π and 1 cis (-⅔π). If these solutions are drawn on an Argand diagram – the three **roots** are of **equal length** and divide the complex plane into **three equal size regions**.

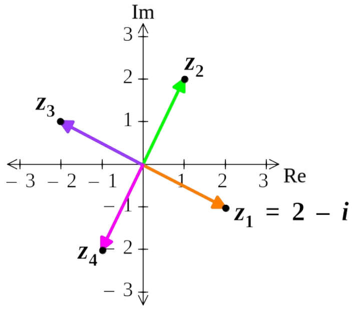


Thus, to determine the $\mathbf{n^{th}}$ **roots** of a **non-zero complex number**, if **one root** is **known** or is **calculable**, then all **other roots** are located by **dividing** the **complex plane** into **n equal regions**.

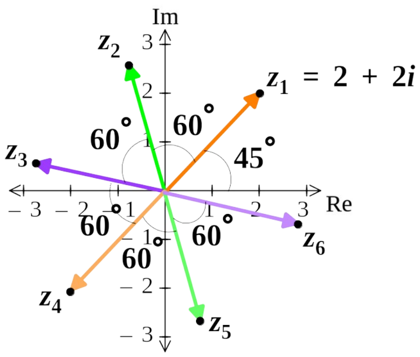
E.g.1. a) As $(2 - \mathbf{i})^4 = -7 - 24\mathbf{i}$, show the four roots of $-7 - 24\mathbf{i}$ on an Argand diagram in the form of $a + b\mathbf{i}$.

b) As $(2 + 2\mathbf{i})^6 = -512\mathbf{i}$, determine the six roots of $-512\mathbf{i}$ expressed in the form of r cis θ° with $r \geq 0$ and $-180° < \theta \leq 180°$, displaying these on an Argand diagram.

a) $z_1 = 2 - \mathbf{i}$
$z_2 = 1 + 2\mathbf{i}$
$z_3 = -2 + \mathbf{i}$
$z_4 = -1 - 2\mathbf{i}$



b) $z_1 = 2\sqrt{2}\text{ cis } 45°$
$z_2 = 2\sqrt{2}\text{ cis } 105°$
$z_3 = 2\sqrt{2}\text{ cis } 165°$
$z_4 = 2\sqrt{2}\text{ cis } (-135°)$
$z_5 = 2\sqrt{2}\text{ cis } (-75°)$
$z_6 = 2\sqrt{2}\text{ cis } (-15°)$



Ref: *Ex.2A Q.1-9 (odd)*

2. DE MOIVRE'S THEOREM: If we consider –

$(r_1 \text{ cis } \theta)(r_2 \text{ cis } \alpha) = r_1 r_2 \text{ cis } (\theta + \alpha)$

$\Rightarrow \quad \text{cis } \theta \text{ cis } \alpha = \text{cis } (\theta + \alpha)$

$\Rightarrow \quad (\text{cis } \theta)^2 = \text{cis } (\theta + \theta) = \text{cis } (2\theta)$

Hence, $\quad (\text{cis } \theta)^3 = \text{cis } (3\theta)$

and $\quad (\text{cis } \theta)^4 = \text{cis } (4\theta)$

and $\quad \mathbf{(cis\ \theta)^n = cis\ (n\theta)}$.

Thus, $\mathbf{(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta}$, and $\mathbf{z^n = (|z| \cos\theta)^n = |z|^n \cos (n\theta)}$. These are alternate forms of **de Moivre's theorem** which holds for all **rational values** of **n**.

De Moivre's theorem can be used to **express** cos nθ and sin nθ in terms of cos θ and sin θ, **find powers** of a complex number, and **find** the **nth roots** of a complex number. $\mathbf{z = r \text{ cis } (\theta + 2k\pi)}$ for $\boldsymbol{r \geq 0}$ and **k** an integer is an alternative polar form of a complex number.

E.g.2. Use de Moivre's theorem to express cos 3θ and sin 3θ in terms of sin θ and cos θ. Hence determine cos 3θ in terms of cos θ.

$$\begin{aligned}\cos 3\theta + \mathbf{i} \sin 3\theta &= (\cos\theta + \mathbf{i} \sin\theta)^3 \\ &= \cos^3\theta + 3\cos^2\theta\,(\mathbf{i}\sin\theta) + 3\cos\theta\,(\mathbf{i}\sin\theta)^2 + (\mathbf{i}\sin\theta)^3 \\ &= \cos^3\theta + 3\cos^2\theta\,(\mathbf{i}\sin\theta) - 3\cos\theta\sin^2\theta - \mathbf{i}\sin^3\theta \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + \mathbf{i}\,(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

$\therefore \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\&\ \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

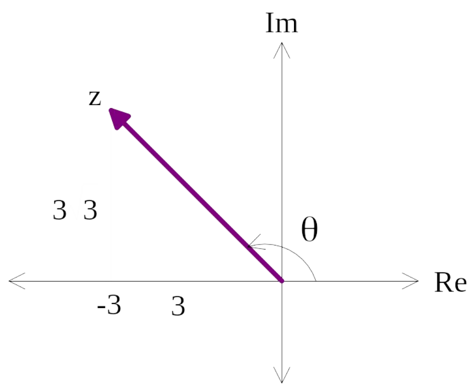
$$\begin{aligned}\text{Hence, } \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta\,(1 - \cos^2\theta) \\ &= \cos^3\theta - 3\cos\theta - 3\cos^3\theta \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

E.g.3. Use de Moivre's theorem to determine $(-3 + 3\sqrt{3}\,\mathbf{i})^4$, giving your answer in exact polar form.

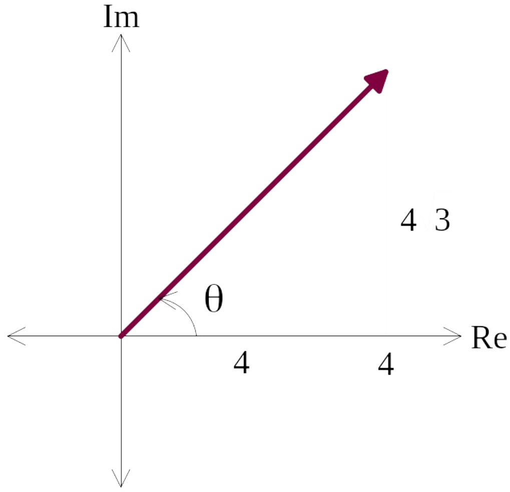
$|z| = 6 \ \& \ \theta = \dfrac{2\pi}{3}$

$\therefore \ -3 + 3\sqrt{3}\,i = 6 \text{ cis } \dfrac{2\pi}{3}$

$$\begin{aligned}(-3 + 3\sqrt{3}\,i)^4 &= \left(6 \text{ cis } \frac{2\pi}{3}\right)^4 \\ &= 6^4 \text{ cis } \frac{8\pi}{3} \\ &= 1\,296 \text{ cis } \frac{2\pi}{3}\end{aligned}$$



E.g.4. Use de Moivre's theorem to determine the three cube roots of $(4 + 4\sqrt{3}\,\mathbf{i})$, giving your answer in exact polar form.



$$|z| = 8 \ \& \ \theta = \frac{\pi}{3}$$

$$4 + 4\sqrt{3}\,i = 8 \text{ cis } \left(\frac{\pi}{3} + 2k\pi \right)$$

$$\therefore z^3 = 8 \text{ cis } \left(\frac{\pi}{3} + 2k\pi \right)$$

$$z = \left[8 \text{ cis } \left(\frac{\pi}{3} + 2k\pi \right) \right]^{\frac{1}{3}}$$

$$= \sqrt[3]{8} \text{ cis } \left(\frac{\pi}{9} + \frac{2k\pi}{3} \right)$$

$$= 2 \text{ cis } \left(\frac{\pi}{9} + \frac{2k\pi}{3} \right)$$

The **principal** root is when **k = 0**: $2 \text{ cis } \frac{\pi}{9}$

When k = 1: $2 \text{ cis } \frac{7\pi}{9}$

When k = 2: $2 \text{ cis } \frac{13\pi}{9} \Rightarrow 2 \text{ cis } \left(-\frac{5\pi}{9} \right)$

$\therefore$ The three cube roots of $4 + 4\sqrt{3}\,\mathbf{i}$ are $2 \text{ cis } \frac{\pi}{9}$, $2 \text{ cis } \frac{7\pi}{9}$ and $2 \text{ cis } \left(-\frac{5\pi}{9} \right)$.

<u>Ref</u>: *Ex.2B Q.1-15 (odd)*

3. EXPONENTIAL FORM: If a complex number can be differentiated, then for $z = \cos\theta + \mathbf{i}\sin\theta$,

$$\frac{dx}{d\theta} = -\sin\theta + i\cos\theta$$
$$= \mathbf{i}^2\sin\theta + \mathbf{i}\cos\theta$$
$$= \mathbf{i}(\cos\theta + \mathbf{i}\sin\theta)$$
$$= \mathbf{i}z$$
$$\Rightarrow \frac{d}{d\theta}\text{cis}\,\theta = i\,\text{cis}\,\theta$$

Euler's formula states $\mathbf{\cos\theta + i\sin\theta = cis\,\theta = e^{i\theta}}$. Thus, a complex number, $\mathbf{z} = \boldsymbol{r}\,\mathbf{cis}\,\boldsymbol{\theta}$ can be expressed in **Cartesian** form, $\mathbf{x + iy}$; **polar** (or **trigonometric**) form, $\mathbf{r\,cis\,\theta}$; and/or **exponential** form, $\mathbf{re^{i\theta}}$. This leads us to two important relationships $\mathbf{e^{i\pi} = -1}$ or $\mathbf{e^{i\pi} + 1 = 0}$, and $\mathbf{\ln(-1) = i\pi}$.

The **exponential form** allows us to **differentiate** and **integrate complex numbers**, i.e.

$$\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{e}^{i\pi} = \mathbf{i}\,\mathbf{e}^{i\pi} \text{ and } \int \mathbf{e}^{i\pi}\,\mathbf{dx} = \frac{\mathbf{e}^{i\pi}}{\mathbf{i}} + \mathbf{c} = -\,\mathbf{i}\,\mathbf{e}^{i\pi} + \mathbf{c}.$$

E.g.5. Express the complex number $z = 4\,\text{cis}\,\frac{\pi}{3}$ in:

a) exact exponential form, and
b) exact Cartesian form.

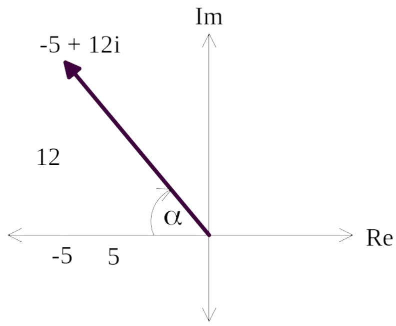
a) $z = 4\,\text{cis}\,\frac{\pi}{3}$

$$= 4\,e^{\frac{i\pi}{3}}$$

b) $z = 4\,\text{cis}\,\frac{\pi}{3}$

$$= 4\left(\cos\frac{\pi}{3} + \mathbf{i}\sin\frac{\pi}{3}\right)$$
$$= 2 + 2\mathbf{i}\sqrt{3}$$

E.g.6. Express $-5 + 12\mathbf{i}$ as a complex exponential $re^{i\theta}$ with r an exact value, and θ correct to 2 d.p.



$\tan\alpha = \dfrac{12}{5}$

$\therefore\ \alpha \approx 1.18$

$\Rightarrow\ \theta = \pi - \alpha$

≈ 1.97

If -5 + 12**i** = *r* cis θ

$r = \sqrt{(-5)^2 + 12^2}$

$= 13$

$\therefore$ -5 + 12**i** ≈ 13 cis 1.97

$= 13e^{1.97\mathbf{i}}$

E.g.7. a) If $y = e^{0.5i\pi x}$, determine $\dfrac{dy}{dx}$.

b) Determine $\int e^{2i\pi x}\,dx$.

a) If $y = e^{0.5i\pi x}$

then $\dfrac{dy}{dx} = 0.5i\pi e^{0.5i\pi x}$

b) $\displaystyle\int e^{2i\pi x}\,dx = \frac{e^{2i\pi x}}{2i\pi} + c$

$\displaystyle = -\frac{ie^{2i\pi x}}{2\pi} + c$

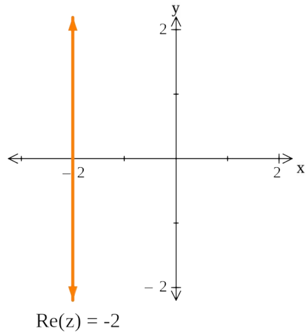
<u>Ref</u>: *Ex.2C Q.1-22 (even); 23-25*

LOCUS

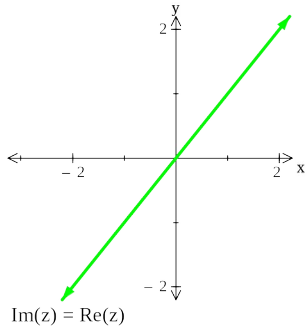
1. LOCI IN THE COMPLEX PLANE: A **locus** is the set of all points that satisfy one or more conditions/**constraints**. A locus can be **described** in **words**, or by **drawing** a **diagram**. **Loci** involving **complex numbers** are drawn on an **Argand diagram**. Most loci are best drawn after rewriting the constraints in **Cartesian** form.

E.g.1. Sketch on an Argand diagram the locus of the point **z** = x + y**i** satisfying each of the following conditions:
a) Re(z) = -2
b) Im(z) = Re(z)
c) Re(z) + 2Im(z) > 3
d) Re(z).Im(z) = 1

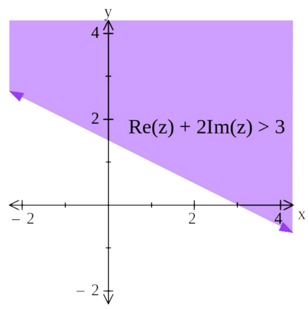
a) Re(z) = -2 $\Rightarrow$ Cartesian equation is x = -2



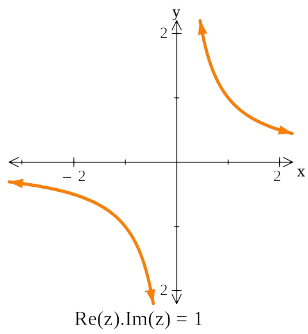
b) Im(z) = Re(z) $\Rightarrow$ Cartesian equation is y = x



c) Re(z) + 2Im(z) > 3 $\Rightarrow$ Cartesian equation is x + 2y > 3



d) Re(z).Im(z) = 1 $\Rightarrow$ Cartesian equation is xy = 1

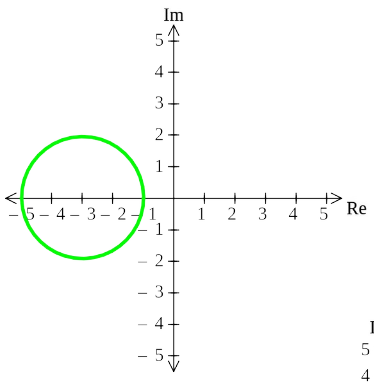


2. LOCUS INVOLVING THE MODULUS: For **$| \mathbf{z} - \mathbf{z_1} | = \mathbf{k}$**, $| z - z_1 |$ is interpreted **geometrically** as the '**distance**' **between** the points **z and $\mathbf{z_1}$**, and hence, $| z - z_1 | = k$ represents the locus of all points where the **distance to $\mathbf{z_1}$ is constant**. Thus, $| z - z_1 | = k$ represents a **circle centre $\mathbf{z_1}$** with a **radius** of **k**. The Cartesian form of the locus is $\mathbf{(x - a)^2 + (y - b)^2 = k^2}$, where $\mathbf{z} = x + y\mathbf{i}$ and $\mathbf{z_1} = a + b\mathbf{i}$. This represents a circle, centre (a,b) with a radius of k.

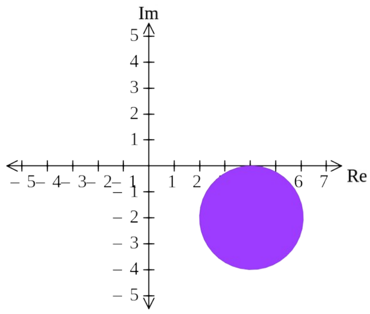
E.g.2. Sketch, on an Argand diagram, the locus of the point $\mathbf{z} = x + y\mathbf{i}$ satisfying each of the following conditions:

a) $| \mathbf{z} + 3 | = 2$
b) $| \mathbf{z} - 4 + 2\mathbf{i} | < 2$
c) $| \mathbf{z} - 4 + 3\mathbf{i} | \leq 2$
d) $| \mathbf{z} - 4 + 2\mathbf{i} | < 2$ and $| \mathbf{z} - 4 + 3\mathbf{i} | \leq 2$

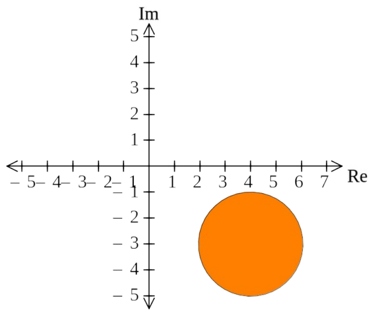
a) $| \mathbf{z} + 3 | = 2$
$\Rightarrow | \mathbf{z} - (-3 + 0\mathbf{i}) | = 2$
$\Rightarrow$ Circle centre (-3,0) and radius 2.



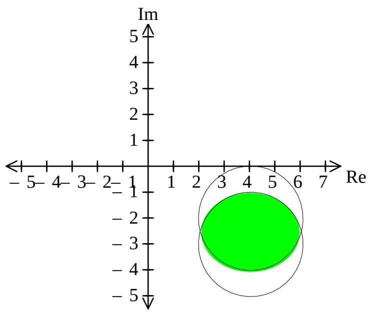
b) $| \mathbf{z} - 4 + 2\mathbf{i} | < 2$
$\Rightarrow | \mathbf{z} - (4 - 2\mathbf{i}) | < 2$
$\Rightarrow$ Circular region centre (4,-2) and radius 2.



c) $| \mathbf{z} - 4 + 3\mathbf{i} | \leq 2$
$\Rightarrow | \mathbf{z} - (4 - 3\mathbf{i}) | \leq 2$
$\Rightarrow$ Circular region centre (4,-3) and radius 2.



d) $| \mathbf{z} - 4 + 2\mathbf{i} | < 2$ and $| \mathbf{z} - 4 + 3\mathbf{i} | \leq 2$
The locus is the intersection between the two circular regions in Parts b) and c).



Thus, **$|z - z_1| = |z - z_2|$** is the locus of all points where the distance between z and z_1 is equal to the distance between z and z_2. Hence, the locus is the **perpendicular bisector** of the line segment joining z_1 to z_2.

E.g.3. Sketch, on an Argand diagram, the locus of the point $\mathbf{z} = x + y\mathbf{i}$ satisfying the following conditions:

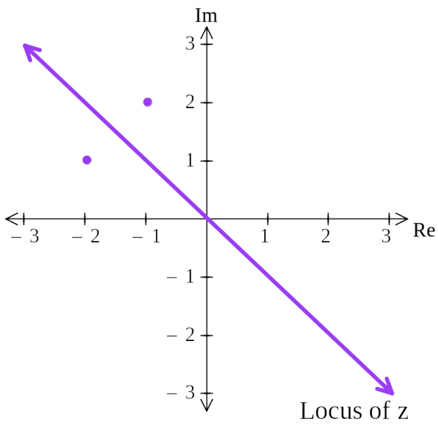
a) $|\mathbf{z} + 2 - \mathbf{i}| = |\mathbf{z} + 1 - 2\mathbf{i}|$

b) $|\mathbf{z} + 2 - \mathbf{i}| \geq |\mathbf{z} + 1 - 2\mathbf{i}|$ and $\text{Im}(z) \leq 2$

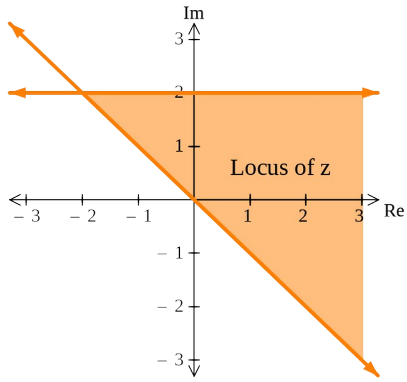
a) $|\mathbf{z} + 2 - \mathbf{i}| = |\mathbf{z} + 1 - 2\mathbf{i}|$

$\Rightarrow |\mathbf{z} - (-2 + \mathbf{i})| = |\mathbf{z} - (-1 + 2\mathbf{i})|$

$\Rightarrow$ The locus is the perpendicular bisector of the line segment joining (-2,1) to (-1,2).



b) $|\mathbf{z} + 2 - \mathbf{i}| \geq |\mathbf{z} + 1 - 2\mathbf{i}|$ and $\text{Im}(z) \leq 2$



E.g.4. Given that $\mathbf{a} = 1 + \mathbf{i}$ and $\mathbf{b} = 5 + 7\mathbf{i}$, sketch the locus of **z**, defined by:

a) $|\mathbf{z} - \mathbf{a}| + |\mathbf{z} - \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$

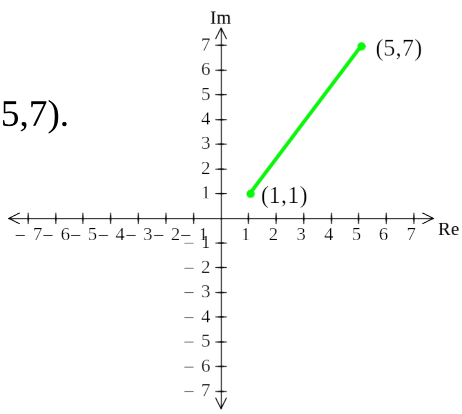
b) $|\mathbf{z} - \mathbf{a}| + |\mathbf{a} - \mathbf{b}| = |\mathbf{z} - \mathbf{b}|$

c) $|\mathbf{z} + \mathbf{a}| + |\mathbf{z} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$

a) $|\mathbf{z} - \mathbf{a}| + |\mathbf{z} - \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$

$\Rightarrow |\mathbf{z} - (1 + \mathbf{i})| + |\mathbf{z} - (5 + 7\mathbf{i})| = |(1 + \mathbf{i}) - (5 + 7\mathbf{i})|$

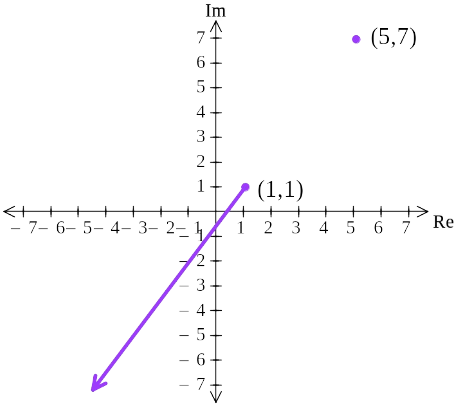
$\Rightarrow$ **z** must be a point on the line segment joining (1,1) and (5,7).



b) | **z – a** | + | **a – b** | = | **z – b** |

⇒ | **z** – (1 + **i**) | + | (1 + **i**) – (5 + 7**i**) | = | **z** – (5 + 7**i**) |

⇒ **z** must be a point on the ray from (1,1) in the opposite direction to the line segment joining (1,1) and (5,7).

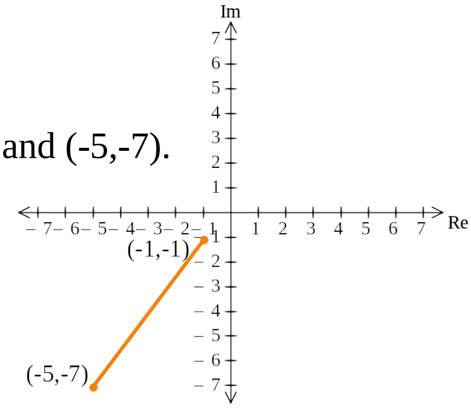


c) | **z + a** | + | **z + b** | = | **a – b** |

⇒ | **z** + (1 + **i**) | + | **z** + (5 + 7**i**) | = | (1 + **i**) – (5 + 7**i**) |

⇒ | **z** – (-1 – **i**) | + | **z** – (-5 – 7**i**) | = | (1 + **i**) – (5 + 7**i**) |

⇒ **z** must be a point on the line segment joining (-1,-1) and (-5,-7).



NOTE: The distance between (1,1) and (5,7) is identical to the distance between (-1,-1) and (-5,-7).

3. LOCUS INVOLVING THE ARGUMENT: For **z** = x + y**i** with **arg(z) = θ** where **θ is constant**, then $\tan\theta = \frac{y}{x}$ and hence, **y = x tan θ**. As (x,y) is only in one specific quadrant, the **locus** is a **ray** with equation **y = x tan θ** in the quadrant **containing (x,y)**. Hence, **arg(z) = θ** gives a **ray** with endpoint **(0,0)** at an angle of **θ** to the **positive Real axis**. Likewise, **arg(z – z_1) = θ** gives **ray** with endpoint at z_1 at an angle of **θ** to the **positive Real axis**.

E.g.5. Sketch, on an Argand diagram, the locus of the point **z** = x + y**i** satisfying the following conditions:

a) $\arg(z) = -\frac{5\pi}{6}$

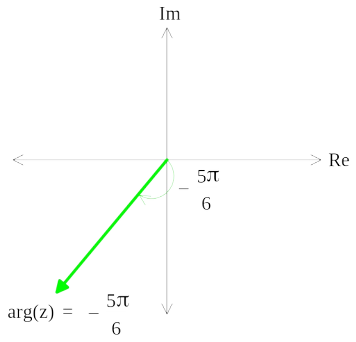
b) $\frac{\pi}{6} \le \arg(z) \le \frac{\pi}{3}$

c) $\frac{\pi}{6} \le \arg(z) \le \frac{\pi}{3}$ and $2 \le |z| \le 4$

d) $\arg(z + 1) \le \frac{\pi}{3}$

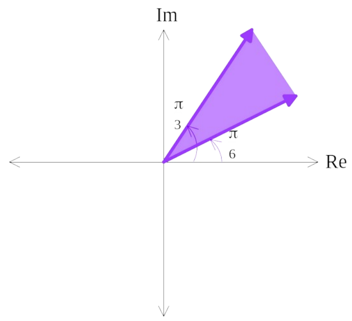
a) $\arg(z) = -\dfrac{5\pi}{6}$

$\Rightarrow$ A ray, endpoint (0,0) and at an angle of $-\dfrac{5\pi}{6}$.



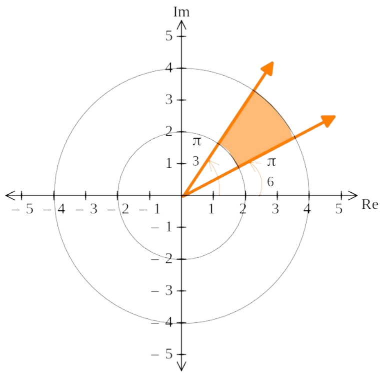
b) $\dfrac{\pi}{6} \le \arg(z) \le \dfrac{\pi}{3}$

$\Rightarrow$ The region between rays, endpoints (0,0) and at angles of $\dfrac{\pi}{6}$ and $\dfrac{\pi}{3}$.



c) $\dfrac{\pi}{6} \le \arg(z) \le \dfrac{\pi}{3}$ and $2 \le |\, z \,| \le 4$

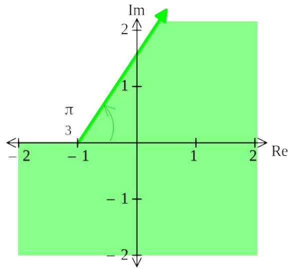
$\Rightarrow$ The region between rays, endpoints (0,0) and at angles of $\dfrac{\pi}{6}$ and $\dfrac{\pi}{3}$, and the circles, centre (0,0) of radii 2 and 4.



d) $\arg(z + 1) \le \dfrac{\pi}{3}$

$\Rightarrow \arg[z - (-1 + 0i)] \le \dfrac{\pi}{3}$

$\Rightarrow$ The region is to the ‘right’ of the ray, endpoint (-1,0) and at an angle of $\dfrac{\pi}{3}$ to the Real axis, and ‘below’ the Real axis.



E.g.6. Rewrite in Cartesian form, and hence sketch the locus of $\mathbf{z} = x + y\mathbf{i}$ for each of the following:

a) $\operatorname{Im}\left[\dfrac{\mathbf{z} - \mathbf{i}}{\mathbf{z} + \mathbf{i}}\right] = 1$

b) $\mathbf{z} + \overline{\mathbf{z}} = \mathbf{z}\overline{\mathbf{z}}$

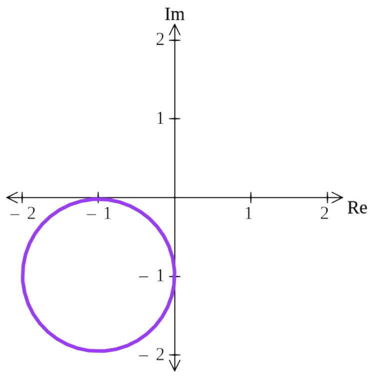
a) $$\text{Im}\left[\frac{\mathbf{z} - \mathbf{i}}{\mathbf{z} + \mathbf{i}}\right] = 1$$

$$\text{Im}\left[\frac{x + (y - 1)\mathbf{i}}{x + (y + 1)\mathbf{i}}\right] = 1$$

$$\text{Im}\left[\frac{x + (y - 1)\mathbf{i}}{x + (y + 1)\mathbf{i}} \times \frac{x - (y + 1)\mathbf{i}}{x - (y + 1)\mathbf{i}}\right] = 1$$

$$\text{Im}\left[\frac{x^2 + y^2 - 1 - 2x\mathbf{i}}{x^2 + (y + 1)^2}\right] = 1$$

$$\text{Im}\left[\frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2} + \frac{-2x\mathbf{i}}{x^2 + (y + 1)^2}\right] = 1$$



$$\Rightarrow \quad \frac{-2x}{x^2 + (y + 1)^2} = 1$$

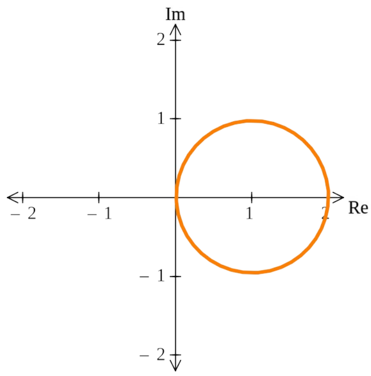
$$x^2 + 2x + (y + 1)^2 = 0$$

$$(x + 1)^2 - 1 + (y + 1)^2 = 0$$

$$(x + 1)^2 + (y + 1)^2 = 1$$

$\Rightarrow$ Circle, centre (-1,-1) and radius 1.

b) $\mathbf{z} + \overline{\mathbf{z}} = \mathbf{z}\overline{\mathbf{z}}$



$\Rightarrow (x + y\mathbf{i}) + (x - y\mathbf{i}) = (x + y\mathbf{i})(x - y\mathbf{i})$

$$2x = x^2 + y^2$$

$$x^2 - 2x + y^2 = 0$$

$$(x - 1)^2 + y^2 = 1$$

$\Rightarrow$ Circle, centre (1,0) and radius 1.

<u>Ref</u>: *LOCI Q.1-17 (odd)*

MATHEMATICAL REASONING

1. DEDUCTIVE PROOFS USING ALGEBRA: In Mathematics, we try to discover **fundamental truths** (statements which are always true) and apply **logical processes** to **verify** the certainty of these truths beyond any doubt. Some statements are **always true**, **always false**, or **sometimes true** and **sometimes false**. Some statements **do not** need to be **proven** because they are **true by definition**. Other statements are **simply accepted as being true without** the need for a **proof**. These are called **axioms**. **Axioms** serve as **starting points** from which other statements are logically derived.

To **prove** a **statement is true** involves **defining** the set of **conditions**, then making **statements** with **reasons** in a **logical sequence** and, finally, making a **concluding statement**. **Theorems** are statements that are proved true using **accepted definitions**, **axioms** and/or any other **proven theorems**. Thus, a **theorem** is **deduced by reasoning** from other accepted truths. This is called a **proof by deduction** and its **validity** depends on the correctness of the axioms and theorems used to deduce it.

A **conjecture** is a mathematical statement which appears likely to be true, but has not been **formally proven** to be true under the rules of mathematical logic. To **disprove** a statement only takes one **counter example**. Mathematical **conjectures** can often be **proved** using **algebra**.

2. GEOMETRIC PROOFS: A **geometric deductive proof** consists of a list of **statements** and **reasons** that are **known** to be true which rely on the **properties** of geometric shapes such as **triangles**, **parallel lines**, **angles** and **tangents to circles**.

The structure of a **geometric proof** requires a **drawing** to illustrate what is to be proven with any **other facts** that can be **deduced** and/or any required **extensions** to the diagram **marked on the diagram**, the '**required to prove**' statement, a **list** of the **given** conditions or **deduced** statements, and the **conclusion** that was to be proven.

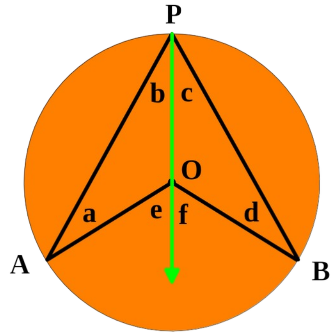
E.g.1. **Conjecture:** The product of two odd numbers is odd. Prove this using algebra.

Let $2m + 1$
and $2p + 1$ be the two odd numbers.

$$\begin{aligned}(2m + 1)(2p + 1) &= 4mp + 2m + 2p + 1\\ &= 2(2mp + m + p) + 1\end{aligned}$$

$2(2mp + m + p)$ is an even number and so $2(2mp + m + p) + 1$ is an odd number.
$\therefore$ The product of two odd numbers is always odd.

E.g.2. This diagram illustrates the **Central Angle Theorem I** which states that any angle in a major arc is half the size of the central angle. Thus, prove that s∠AOB = 2 × s∠APB.



R.T.P.: s∠AOB = 2 × s∠APB

Extension To Diagram: $\overrightarrow{PO}$

Proof:

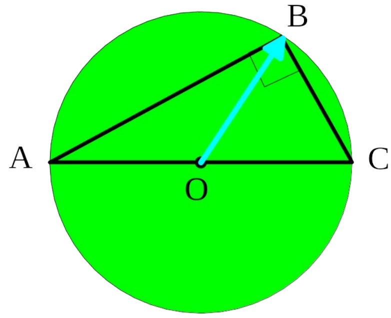
STATEMENT	**REASON**
OA = OP = OB	Radii of the same circle
e = a + b	Triangle Exterior Angle Theorem
f = c + d	Triangle Exterior Angle Theorem
But a = b	ΔAPO is isosceles
and c = d	ΔBPO is isosceles
∴ e + f = 2b + 2c	Equality axiom
e + f = 2(b + c)	Equality axiom
i.e. s∠AOB = 2 × s∠APB	Q.E.D.

Conclusion: Any angle in a major arc is half the size of the central angle.

<u>Ref</u>: Ex.4A Q.1-12 (even); 7

3. **VECTOR PROOFS:** The **axioms** used as **starting points** from which other statements about vectors are logically derived are – **equal vectors** have the same magnitude and direction, if **a** = λ**b**, then λ > 0 where **a** and **b** are like parallel vectors, if **a** = λ**b**, then λ < 0 where **a** and **b** are unlike parallel vectors, vectors are **added/subtracted** using the 'nose-to-tail' triangle or the parallelogram law, if h**a** = k**b**, then **a** and **b** are parallel vectors or h = k = 0, **a.b** = 0 when **a** and **b** are non-zero, perpendicular vectors, **a.b** = **b.a**, **a.**(λ**b**) = (λ**a**)**.b** = λ(**a.b**), **a.a** = a^2, **a.**(**b** + **c**) = **a.b** + **a.c**, (**a** + **b**)**.c** = **a.c** + **b.c**, and (**a** + **b**)**.**(**c** + **d**) = **a.c** + **a.d** + **b.c** + **b.d**.

E.g.3. This diagram illustrates the **Angle In A Semi-circle Theorem** which states that any angle in a semi-circle is a right angle. Thus prove that s∠ABC = 90°.



R.T.P.: $s\angle ABC = 90^\circ$

Extension To Diagram: $\overrightarrow{OB}$

Proof:

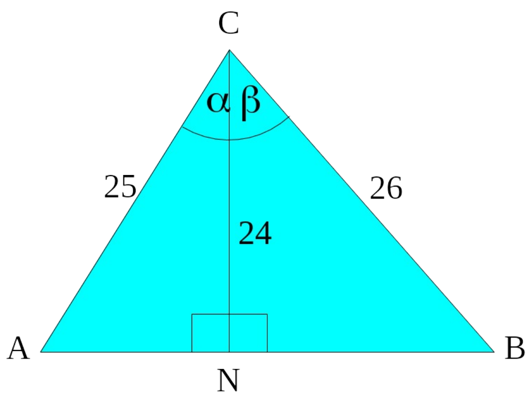
STATEMENT	REASON
Let $\overrightarrow{OC} = \mathbf{c}$ & $\overrightarrow{OB} = \mathbf{b}$	Definitions
$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$	Equality axiom
But $\overrightarrow{AO} = \overrightarrow{OC}$	Radii of same circle
$\therefore \overrightarrow{AB} = \mathbf{c} + \mathbf{b}$	Equality axiom
& $\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OB}$	Equality axiom
$\therefore \overrightarrow{CB} = \mathbf{b} - \mathbf{c}$	Equality axiom
$\overrightarrow{AB} \cdot \overrightarrow{CB} = (\mathbf{c} + \mathbf{b}).(\mathbf{b} - \mathbf{c})$	Equality axiom
$= \lvert \mathbf{b}^2 \rvert - \lvert \mathbf{c}^2 \rvert$	Equality axiom
But $\lvert \mathbf{b} \rvert = \lvert \mathbf{c} \rvert$	Radii of same circle
$\therefore \overrightarrow{AB} \cdot \overrightarrow{CB} = 0$	Equality axiom
$\Leftrightarrow \overrightarrow{AB} \perp \overrightarrow{CB}$	By definition
$\Leftrightarrow s\angle ABC = 90^\circ$	By definition

Conclusion: Any angle in a semi-circle is a right angle.

<u>Ref</u>: Ex.4B Q.1-15 (odd)

4. TRIGONOMETRIC PROOFS: Previously, the trigonometric identities **$\sin^2 \theta + \cos^2 \theta = 1$**, **$\sin (A \pm B) = \sin A \cos B \pm \sin B \cos A$**, and **$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$** have been used to prove the further identities – **$\sin 2A$**, **$\cos 2A$**, **$\tan (A \pm B)$**, etc.

E.g.4. Use this diagram to prove $\cos(\alpha + \beta) = \dfrac{506}{650}$.



R.T.P.: $\cos(\alpha + \beta) = \dfrac{506}{650}$

Proof:

STATEMENT	**REASON**
In ΔANC,	
$\lvert\overrightarrow{AN}\rvert^2 = 25^2 - 24^2$	Pythagorean theorem
$\lvert\overrightarrow{AN}\rvert = 7$	Equality axiom
$\Rightarrow \sin\alpha = \dfrac{7}{25}$ and $\cos\alpha = \dfrac{24}{25}$	Definitions
In ΔBNC,	
$\lvert\overrightarrow{BN}\rvert^2 = 26^2 - 24^2$	Pythagorean theorem
$\lvert\overrightarrow{BN}\rvert = 10$	Equality axiom
$\Rightarrow \sin\beta = \dfrac{10}{26}$ and $\cos\beta = \dfrac{24}{26}$	Definitions
But $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$	Trig identity
Then $\cos(\alpha + \beta) = \dfrac{24}{25} \times \dfrac{24}{26} - \dfrac{7}{25} \times \dfrac{10}{26}$	Equality axiom
$= \dfrac{506}{650}$	Q.E.D.

5. **PROOF BY EXHAUSTION:** The previous proofs have all been forms of **deductive proofs**. There are three other methods of proof – **proof by exhaustion**, **proof by contradiction**, and **proof by induction** (covered in Unit 3D).

A **proof by exhaustion** – **exhausts** all possibilities by considering (completely) **all possible options**.

E.g.5. Prove that every integer that is a perfect cube is either a multiple of 9, or 1 more than a multiple of 9, or 1 less than a multiple of 9.

All integers, n, can be described by: $3p - 1$, $3p$, $3p + 1$ for $p \in$ integers.

Case 1: If $n = (3p - 1) \Leftrightarrow (3p - 1)^3 = 27p^3 - 27p^2 + 9p - 1$

$= 9(3p^3 - 3p^2 + p) - 1 \Leftarrow$ is 1 less than a multiple of 9

e.g. If $n = 5 \Rightarrow 5^3 = 9 \times 14 - 1$

Case 2: If $n = 3p \Leftrightarrow (3p)^3 = 27p^3$

$= 9(3p^3) \Leftarrow$ is a multiple of 9

e.g. If $n = 6 \Rightarrow 6^3 = 9 \times 24$

Case 3: If $n = (3p + 1) \Leftrightarrow (3p + 1)^3 = 27p^3 + 27p^2 + 9p + 1$

$= 9(3p^3 + 3p^2 + p) + 1$ $\Leftarrow$ is 1 more than a multiple of 9

e.g. If $n = 7 \Rightarrow 7^3 = 9 \times 38 + 1$

or If $n = 1 \Rightarrow 1^3 = 1$

$= 9 \times 0 + 1$ $\Leftarrow$ is 1 more than a multiple of 9

Ref: Ex.4C Q.1-6 (even)

6. **PROOF BY CONTRADICTION: Proofs by contradiction** assumes that the 'opposite' or **contradiction** is true, then **proves** this is **false**, and hence, the **original statement** must be **true**. This method is also known as **reduction to the absurd**.

E.g.6. **Proposition:** For every squared number, $\mathbf{q}^2$, which is odd, then q must also be odd.

Assumption: $\mathbf{q}$ is not odd $\Rightarrow \mathbf{q}^2$ is odd

Let $\mathbf{q} = 2n$

$\Leftrightarrow \mathbf{q}^2 = (2n)^2$

$\Leftrightarrow \mathbf{q}^2 = 2(2n^2)$

$\Leftrightarrow \mathbf{q}^2$ is even

Hence, '**q** is not odd' is a false proposition.

Hence, '**q** is odd'.

Ref: Ex.4D Q.1-8 (even)

EXPONENTIALS AND LOGARITHMS

1. EXPONENTIAL FUNCTIONS: The **exponential number**, **e** = 2.71828…, where the **derivatives** of the **exponential function** are – $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(e^{kx}) = k \times e^{kx}$ and $\frac{d}{dx}(e^{f(x)}) = f'(x)\, e^{f(x)}$.

E.g.1. Differentiate:
a) $\tan x + e^{2x}$
b) e^{x^2-2x+5}
c) $e^{\sqrt{x}}$
d) $e^{3\cos 2x}$

a) If $y = \tan x + e^{2x}$

then $\frac{dy}{dx} = 1 + \tan^2 x + 2e^{2x}$

b) If $y = e^{x^2-2x+5}$

then $\frac{dy}{dx} = (2x - 2)e^{x^2 - 2x + 5}$

c) If $y = e^{\sqrt{x}}$

then $\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

d) If $y = e^{3\cos 2x}$

then $\frac{dy}{dx} = -6\sin 2x\, e^{3\cos 2x}$

<u>Ref</u>: Ex.8A Q.1-15 (R.H.S.); 16-18

Since, $\frac{d}{dx}(e^x) = e^x$, then $\int e^x\, dx = e^x + c$, and since $\frac{d}{dx}(e^{kx}) = k \times e^{kx}$, then $\int (k \times e^{kx})\, dx = e^{kx} + c$, and since $\frac{d}{dx}(e^{f(x)}) = f'(x)\, e^{f(x)}$, then $\int f'(x)\, e^{f(x)}\, dx = e^{f(x)} + c$.

E.g.2. Find:

a) $\int e^{7x}\,dx$

b) $\int \left(6e^{3x} - 8x + \frac{1}{e^{2x}} \right) dx$

c) $\int 4e^{-x}(3 + e^{-x})^2\,dx$

d) $\int \frac{e^{3x} - 1}{e^{x}}\,dx$

a) $\int e^{7x}\,dx = \frac{1}{7}e^{7x} + c$

b) $\int \left(6e^{3x} - 8x + \frac{1}{e^{2x}} \right) dx = \int (6e^{3x} - 8x + e^{-2x})\,dx$

$= \frac{6e^{3x}}{3} - \frac{8x^2}{2} + \frac{e^{-2x}}{-2} + c$

$= 2e^{3x} - 4x^2 - \frac{1}{2e^{2x}} + c$

c) $\int 4e^{-x}(3 + e^{-x})^2\,dx = -4 \int -e^{-x}(3 + e^{-x})\,dx$

$= -4 \times \frac{(3 + e^{-x})^3}{3} + c$

$= -\frac{4(3 + e^{-x})^3}{3} + c$

Or

$\int 4e^{-x}(3 + e^{-x})^2\,dx = \int 4e^{-x}(9 + 6e^{-x} + e^{-2x})\,dx$

$= \int (36e^{-x} + 24e^{-2x} + 4e^{-3x})\,dx$

$= -36e^{-x} - 12e^{-2x} - \frac{4}{3}e^{-3x} + k$

But $-\frac{4(3 + e^{-x})^3}{3} + c = -36 - 36e^{-x} - 12e^{-2x} - \frac{4}{3}e^{-3x} + c$

Hence, $c \neq k$

However, both solutions are correct.

d) $\displaystyle\int \frac{e^{3x} - 1}{e^x}\,dx = \int (e^{2x} - e^{-x})\,dx$

$$= \frac{e^{2x}}{2} - \frac{e^{-x}}{-1} + c$$

$$= \frac{e^{2x}}{2} + \frac{1}{e^x} + c$$

Ref: Ex.8B Q.1-20 (R.H.S.); 21-27

2. **LOGARITHMS:** The **inverse** of an **exponential** function is a **logarithmic function**. Thus, $\mathbf{y = a^x \Leftrightarrow x = \log_a y}$, and $\mathbf{y = e^x \Leftrightarrow x = \ln y}$. The base of any logarithm can be changed using $\displaystyle\mathbf{\log_a b = \frac{\log_c b}{\log_c a}}$. Hence, $\displaystyle\mathbf{\frac{d}{dx}(\ln x) = \frac{1}{x}}$, $\displaystyle\mathbf{\frac{d}{dx}(\log_b x) = \frac{1}{\ln b} \times \frac{1}{x}}$, and $\displaystyle\mathbf{\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}}$.

Ref: Ex.8C Q.1-48 (R.H.S.)

E.g.3. Differentiate:
a) $y = 3 \ln x + 2x$
b) $y = \ln 5x^2$
c) $y = -4\ln\frac{2}{x}$
d) $y = \log_7 x$
e) $y = \log_{10}(x^3 - 4x)$
f) $y = \ln(\ln 2x^3)$

a) $y = 3 \ln x + 2x$

$$\frac{dy}{dx} = 3 \times \frac{1}{x} + 2$$

$$= \frac{3}{x} + 2$$

b) $y = \ln 5x^2$

$$\frac{dy}{dx} = \frac{10x}{5x^2}$$

$$= \frac{2}{x}$$

c) $y = -4\ln\dfrac{2}{x}$

$$\frac{dy}{dx} = -4 \times \frac{2(-1)x^{-2}}{2x^{-1}}$$

$$= \frac{8x}{2x^2}$$

$$= \frac{4}{x}$$

d) $y = \log_7 x$

$$\frac{dy}{dx} = \frac{1}{\ln 7} \times \frac{1}{x}$$

$$= \frac{1}{x\ln 7}$$

e) $y = \log_{10}(x^3 - 4x)$

$$\frac{dy}{dx} = \frac{1}{\ln 10} \times \frac{3x^2 - 4}{x^3 - 4x}$$

$$= \frac{3x^2 - 4}{(x^3 - 4x)\ln 10}$$

f) $y = \ln(\ln 2x^3)$

$$\frac{dy}{dx} = \frac{1}{\ln 2x^3} \times \frac{6x^2}{2x^3}$$

$$= \frac{3}{x\ln 2x^3}$$

<u>Ref</u>: Ex.8D Q.1-21 (R.H.S.); 22-28

Since, $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$, then $\displaystyle\int \frac{1}{x}\,dx = \int \frac{dx}{x} = \ln|x| + c$ for **$x \neq 0$**, and since, $\dfrac{d}{dx}[\ln f(x)] = \dfrac{f'(x)}{f(x)}$, then $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$.

If f is **continuous** on **[a;b]**, and F is any **antiderivative** of f, then the **area under the curve** between a and b is given by $\displaystyle\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$. This is known as the **Fundamental Theorem of Calculus**.

E.g.4. Find:

a) $\int \frac{4x}{x^2 - 6}\, dx$

b) $\int \frac{3}{\tan x \cos^2 x}\, dx$

c) $\int \frac{6\cos 2x}{\sin 2x}\, dx$

a) $\int \frac{4x}{x^2 - 6}\, dx = 2 \int \frac{2x}{x^2 - 6}\, dx$

$= 2 \ln | x^2 - 6 | + c$

b) $\int \frac{3}{\tan x \cos^2 x}\, dx = 3 \int \frac{1}{\cos^2 x} \times \frac{1}{\tan x}\, dx$

$= 3 \ln | \tan x | + c$

c) $\int \frac{6\cos 2x}{\sin 2x}\, dx = 3 \int \frac{2\cos 2x}{\sin 2x}\, dx$

$= 3 \ln | \sin 2x | + c$

E.g.5. Evaluate:

a) $\int_1^2 \frac{3x^2}{x^3}\, dx$

b) $\int_1^3 (2e^x + 3)\, dx$

a) $\int_1^2 \frac{3x^2}{x^3}\, dx = [\ln | x^3 |]_1^2$

$= \ln 2^3 - \ln 1^3$

$= \ln 8$

b) $\int_1^3 (2e^x + 3)\, dx = [2e^x + 3x]_1^3$

$= 2e^3 + 3(3) - [2e^1 - 3(1)]$

$= 2e^3 - 2e + 6$

<u>Ref</u>: Ex.8E Q.1-21 (R.H.S.); 22-30 (even); 25

ANTIDIFFERENTIATION AND INTEGRATION

1. ANTIDIFFERENTIATION: Antidifferentiation, as the name implies, is the reverse of differentiation. E.g. If $\mathbf{f'(x) = 2x}$, then $\mathbf{f(x) = x^2}$ or $\mathbf{f(x) = x^2 + 1}$ or $\mathbf{f(x) = x^2 - 3}$ etc. All of these, and more, have $\mathbf{f'(x) = 2x}$. These are known as a **Family of Curves**. Thus, we say that the **antiderivative** (or **primitive**) of **2x** is $\mathbf{x^2 + c}$, where **c** is a constant. Hence, any function has an **infinite number of antiderivatives** which differ from each other by a constant only. **Additional information** may make it possible to determine the value of **c**. **Antidifferentiation** is also known as **integration**. An **integral** that has '**+ c**' is called an **Indefinite Integral** or a **Primitive Integral**.

Previously, to differentiate $\mathbf{ax^n}$, we "**multiplied by the power and decreased the power by one**". So, to reverse the process, we "**increase the power by one and divide by the (new) power**". Thus, if $\frac{dy}{dx} = ax^n$, then $y = \frac{ax^{n+1}}{n+1} + c$ for $\mathbf{n \neq -1}$.

NOTE: The restriction placed on n for the above rule is because, so far, we do not know any polynomial, y, for which $\frac{dy}{dx} = x^{-1}$.

To differentiate $\mathbf{(ax + b)^n}$, we used the chain rule or its modified version. We "**multiplied by the power, reduced the power by one and then multiplied by the derivative of the 'bracket term'**". So, to reverse the process, i.e. to antidifferentiate, we "**divide by the derivative of the 'bracket term', increase the power by one and then divide by the (new) power**". Thus, in general, if $\mathbf{f'(x) = (ax + b)^n}$, then

$$f(x) = \frac{(ax + b)^{n+1}}{(n + 1) \times a} + c .$$

E.g.1. Find the antiderivative of:

a) $2x^2$

b) $x^2 + 1$

c) $(x + 2)(x - 1)$

d) $2x^3 - \frac{1}{\sqrt{x}}$

e) $x^4 - \frac{5}{x^4}$ given $x = -1, y = -1$

f) $\left(\sqrt{x} - \frac{1}{x}\right)^2$

a) $\dfrac{dy}{dx} = 2x^2$

$$y = 2 \times \frac{x^3}{3} + c$$

$$= \tfrac{2}{3}x^3 + c$$

b) $\dfrac{dy}{dx} = (x^2 + 1)$

$$y = \frac{x^3}{3} + x + c$$

$$= \tfrac{1}{3}x^3 + x + c$$

c) $\dfrac{dy}{dx} = (x + 2)(x - 1) = x^2 + x - 2$

$$y = \frac{x^3}{3} + \frac{x^2}{2} - 2x + c$$

$$= \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 - 2x + c$$

d) $\dfrac{dy}{dx} = \left(2x^3 - \dfrac{1}{\sqrt{x}} \right) = \left(2x^3 - x^{-\frac{1}{2}} \right)$

$$y = \frac{2x^4}{4} - \frac{x^{\frac{1}{2}}}{\tfrac{1}{2}} + c$$

$$= \frac{1}{2}x^4 - 2\sqrt{x} + c$$

e) $\dfrac{dy}{dx} = \left(x^4 - \dfrac{5}{x^4} \right) = (x^4 - 5x^{-4})$

$$y = \frac{x^5}{5} - \frac{5x^{-3}}{-3} + c$$

$$= \frac{x^5}{5} + \frac{5}{3x^3} + c$$

Given x = -1, y = -1

Then $-1 = \dfrac{(-1)^5}{5} + \dfrac{5}{3(-1)^3} + c$

$$-1 = -\frac{1}{5} - \frac{5}{3} + c$$

$$c = \frac{13}{15}$$

$$\therefore y = \frac{x^5}{5} + \frac{5}{3x^3} + \frac{13}{15}$$

f) $\dfrac{dy}{dx} = \left(\sqrt{x} - \dfrac{1}{x} \right)^2$

$= (x^{½} – x^{-1})^2$
$= (x – 2x^{-½} + x^{-2})$

$$y = \frac{x^2}{2} - \frac{2x^{\frac{1}{2}}}{½} + \frac{x^{-1}}{-1} + c$$

$$= \frac{1}{2}x^2 - 4\sqrt{x} - \frac{1}{x} + c$$

Similarly, if **y = f ′(x) [f(x)]ⁿ**, then $\displaystyle\int \mathbf{f\,'(x)[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + c}$, and if **y = [f(x)]ⁿ**, then $\displaystyle\int \mathbf{[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{f\,'(x) \times (n+1)} + c}$.

E.g.2. Integrate:

a) $(x + 4)^3$

b) $\sqrt{4x - 7}$

c) $\dfrac{1}{(3x + 1)^2}$

d) $\dfrac{7}{\sqrt{2x + 3}}$

a) $\dfrac{dy}{dx} = (x + 4)^3$

$$y = \frac{(x + 4)^4}{4 \times 1} + c$$

$$= ¼(x + 4)^4 + c$$

b) $\dfrac{dy}{dx} = \sqrt{4x - 7} = (4x - 7)^{\frac{1}{2}}$

$$y = \frac{(4x - 7)}{\times 4} + c$$

$$y = \frac{(\sqrt{4x - 7})^3}{6} + c$$

c) $\dfrac{dy}{dx} = \dfrac{1}{(3x + 1)^2} = (3x + 1)^{-2}$

$$y = \frac{(3x + 1)^{-1}}{-1 \times 3} + c$$

$$= -\frac{1}{3(3x + 1)} + c$$

$$= -\frac{1}{9x + 3} + c$$

d) $\dfrac{dy}{dx} = \dfrac{7}{\sqrt{2x + 3}} = 7(2x + 3)^{-\frac{1}{2}}$

$$y = \frac{7(2x + 3)^{\frac{1}{2}}}{½ \times 2} + c$$

$$= 7\sqrt{2x + 3} + c$$

<u>Ref</u>: Ex.7A Q.1-27 (odd), 28, 29

As **trigonometric functions** can be differentiated, then they can also be **integrated**. Thus **$\int \cos x \, dx = \sin x + c$**, **$\int \sin x \, dx = -\cos x + c$**, **$\int f'(x) \cos f(x) \, dx = \sin f(x) + c$**, and **$\int f'(x) \sin f(x) \, dx = -\cos f(x) + c$**.

For more '**complicated**' **trigonometric expressions** – make an '**intelligent**' **first guess** based on your **knowledge of trigonometric identities** and/or **compound angles**, then **test** your 'guess' **by differentiating** and make any suitable/necessary **adjustments**.

E.g.3. Find:

a) $\int \sin x \cos x \, dx$

b) $\int \tan 2x \, dx$

c) $\int \sin x \cos^5 x \, dx$

a) $\int \sin x \cos x \, dx$

As $\sin 2x = 2 \sin x \cos x$

$\int \sin x \cos x \, dx = \frac{1}{2} \int \sin 2x \, dx$

$$= \frac{1}{2}\left[\frac{-\cos 2x}{2}\right] + c$$

$$= -\frac{\cos 2x}{4} + c$$

b) $\int \tan 2x \, dx = \int \frac{\sin 2x}{\cos 2x} dx$

As $\frac{d}{dx} \cos 2x = -2 \sin 2x$

$\int \tan 2x \, dx = -\frac{1}{2} \int \frac{-2 \sin 2x}{\cos 2x} dx$

$= -\frac{1}{2} \ln | \cos 2x | + c$

c) $\int \sin x \cos^5 x \, dx$

Try: $y = \cos^6 x \Rightarrow \frac{dy}{dx} = 6 \cos^5 x \times (-\sin x)$

$= -6 \sin x \cos^5 x$

$\therefore \int \sin x \cos^5 x \, dx = - \int (-\sin x) \cos^5 x \, dx$

$$= -\frac{\cos^6 x}{6} + c$$

<u>Ref</u>: Ex.7B Q.1-49 (odd); 50-59

2. INTEGRATION: An **alternative method of integration** involves **changing the variable** by making a **suitable substitution**. This is **similar to** the (initial) differentiation by the **chain rule** but the **final answer** is in terms of the **given variable**, **not** the **substituted variable**.

E.g.4. Integrate the following with respect to the appropriate variable using the suggested substitution:
a) $x(1 - x^2)^{½}$ $u = (1 - x^2)$
b) $(4b - 3)^5$ $u = 4b - 3$
c) $\sqrt{1 - 2t}$ $u = 1 - 2t$

a) $x(1 - x^2)^{½}$ $u = (1 - x^2) \Rightarrow \dfrac{du}{dx} = -2x \Rightarrow \dfrac{dx}{du} = -\dfrac{1}{2x}$

$\therefore \int x(1 - x^2)^{½}\, dx = \int x.u^{½} \dfrac{dx}{du}\, du$

$$= \int \cancel{x}.u^{\frac{1}{2}} \left(-\frac{1}{2\cancel{x}} \right) du$$

$= -½ \int u^{½}\, du$

$$= -\frac{1}{2}\underline{u^{\frac{3}{2}}} + c$$

$$= -\frac{u^{\frac{3}{2}}}{3} + c$$

$$= -\frac{(1 - x^2)^{\frac{3}{2}}}{3} + c$$

b) $(4b - 3)^5$ $u = 4b - 3 \Rightarrow \dfrac{du}{db} = 4 \Rightarrow \dfrac{db}{du} = \dfrac{1}{4}$

$\int (4b - 3)^5\, db = \int u^5 \dfrac{db}{du}\, du$

$= ¼ \int u^5\, du$

$$= \frac{1}{4}\frac{u^6}{6} + c$$

$$= \frac{(4b - 3)^6}{24} + c$$

c) $\sqrt{1-2t}$ $\quad u = 1 - 2t \Rightarrow \frac{du}{dt} = -2 \Rightarrow \frac{dt}{du} = -\frac{1}{2}$

$\int \sqrt{1-2t}\, dt = \int (1-2t)^{½}\, dt$

$$= \int u^{\frac{1}{2}} \frac{dt}{du}\, du$$

$$= -½ \int u^{½}\, du$$

$$= -\frac{1}{2}\underline{u^{\frac{3}{2}}} + c$$

$$= -\frac{u^{\frac{3}{2}}}{3} + c$$

$$= -\frac{\sqrt{(1-2t)^3}}{3} + c$$

<u>Ref</u>: Ex.7C Q.1-16 (even)
Ex.7D Q.1-33 (odd)

3. DIFFERENTIAL EQUATIONS: Any equation involving one or more derivatives is called a **differential equation**, e.g. $\frac{dv}{dt} = 4t^3 + 3t^2 - 5t + 6$. To **solve** a differential equation – **integrate** both sides of the equation, **only one** integration constant required, unless solving a **second** (or higher) **order** differential equation.

The **differential equation** must be in the form of $\mathbf{f(y)\frac{dy}{dx} = g(x)}$ or be able to be expressed in this form. This allows the **variables** to be ‘**separated**’. Additional information will allow the **particular** solution to be obtained, instead of the **general** solution.

E.g.5. Solve:

a) $y\frac{dy}{dx} = 3x + 1$

b) $\frac{dy}{dx} = \frac{6x^2 - 14}{y^3 - 7y}$

c) $(y^2 - 2y + 5)\, dy = (14 - 6x^2)\, dx$, given that y = 3 when x = 1.

a) $y\dfrac{dy}{dx} = 3x + 1$

$$\int y \frac{dy}{dx}\, dx = \int (3x + 1)\, dx$$

$$\int y\, dy = \int (3x + 1)\, dx$$

$$\frac{y^2}{2} = \frac{3x^2}{2} + x + c$$

b) $$\frac{dy}{dx} = \frac{6x^2 - 14}{y^3 - 7y}$$

$$(y^3 - 7y)\, dy = (6x^2 - 14)\, dx$$

$$\int (y^3 - 7y)\, dy = \int (6x^2 - 14)\, dx$$

$$\frac{y^4}{4} - \frac{7y^2}{2} = \frac{6x^3}{3} - 14x + c$$

$$\frac{y^4}{4} - \frac{7y^2}{2} = 2x^3 - 14x + c$$

c) $(y^2 - 2y + 5)\, dy = (14 - 6x^2)\, dx$

$$\int (y^2 - 2y + 5)\, dy = \int (14 - 6x^2)\, dx$$

$$\frac{y^3}{3} - \frac{2y^2}{2} + 5y = 14x - 2x^3 + c$$

$$\frac{y^3}{3} - y^2 + 5y = 14x - 2x^3 + c$$

Given y = 3 when x = 1,

$$\frac{3^3}{3} - 3^2 + 5(3) = 14(1) - 2(1^3) + c$$

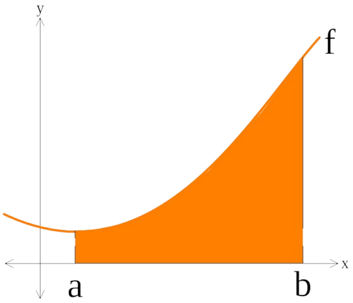
$$15 = 12 + c$$

$$c = 3$$

$$\therefore \frac{y^3}{3} - y^2 + 5y = 14x - 2x^3 + 3$$

Ref: Ex.7E Q.1-14 (even); 15-17

4. AREA UNDER A CURVE: Areas of **regular figures** such as squares, circles, etc. can be readily found by **common formulae**. However, the **area** of **non-regular figures** such as the curve **y = f(x)** are not so readily found. We need to set some "**boundaries**" for the area being considered. It is usually the area bounded by the curve, the **X-axis** and the lines **x = a** and **x = b** for an interval **[a;b]**. This is known as the **area under the curve**.



One method of **approximating the area** is to draw an **accurate graph** on grid paper and then simply **count the** “**squares**”. Another method is to **divide the interval** [a;b] into a number of equal parts, **construct** rectangles or trapezia, both upper and lower, and then **sum** the areas of these shapes (**S_U** and **S_L**). The “**true**” **area** lies somewhere between the two. Thus, $\mathbf{A} \approx \dfrac{S_U + S_L}{2}$. The more sub-intervals taken, the smaller the difference between S_U and S_L and the better the approximation to the true area. Hence,

$$\mathbf{A} = \lim_{n \to \infty} S_U = \lim_{n \to \infty} S_L .$$

If the width of each rectangle was **δx**, then $\mathbf{A} = \lim\limits_{n \to \infty} \sum\limits_{x=a}^{x=b} y\delta x$. This limit of a sum process is called **integration** and is denoted by the **integral sign ∫**. Hence, the area could be written as $\mathbf{A} = \int_a^b y\,dx$.

From comparing the results of the area under a curve over many different intervals, it appeared that the **area** was **equivalent to** the **antiderivative** of the function, **evaluated** over the given interval. It is written as **$\int f'(x)\,dx = f(x) + c$**. Thus, if **f** is **continuous** on **[a;b]**, and **F** is any **antiderivative** of f, then the **area under the curve** between a and b is $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$. This is known as the **Fundamental Theorem of Calculus**. $\int_a^b f(x)\,dx$ is called the **Definite Integral** of the function from a to b.

E.g.6. Evaluate:

a) $\int_1^2 x^3\,dx$

b) $\int_1^3 (2x + 3)^2\,dx$

a) $\int_1^2 x^3\,dx = \left[\dfrac{x^4}{4}\right]_1^2$

$= \dfrac{2^4}{4} - \dfrac{1^4}{4}$

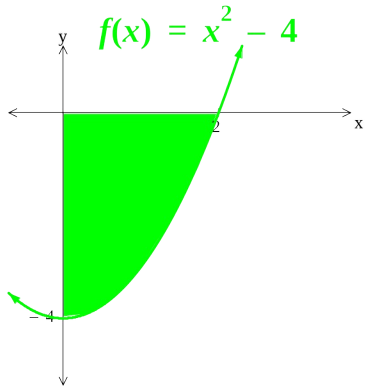
$= 3\frac{3}{4}$

b) $\int_1^3 (2x + 3)^2\,dx = \left[\dfrac{(2x + 3)^3}{3 \times 2}\right]_1^3$

$= \dfrac{9^3}{6} - \dfrac{5^3}{6}$

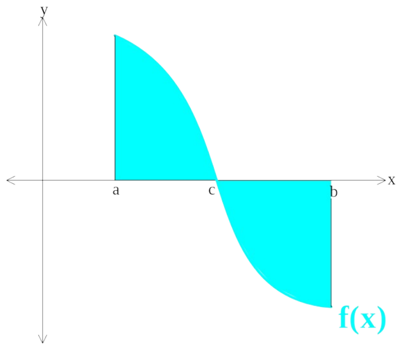
$= 100\frac{2}{3}$

On the interval [0;2], the function $\mathbf{f(x) = x^2 - 4}$ is negative, i.e. the graph is **under** the **X-axis**. Hence the **area** is also "**negative**". Therefore, if a function is negative on the interval [a;b], then the area is given by $\mathbf{A} = -\int_a^b \mathbf{f(x)\,dx} = \int_b^a \mathbf{f(x)\,dx}$.

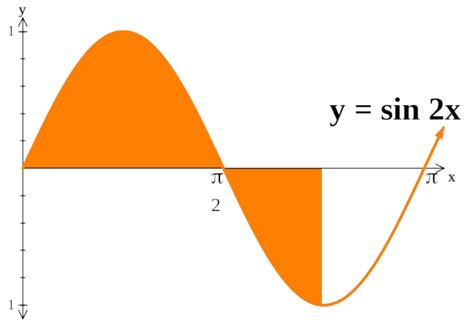


Consider the graph of this function. In order to determine the **area** between the function and the X-axis – first **calculate** the **x-intercepts** of the function and hence, **split** the **integrals** into appropriate parts, i.e.

$\mathbf{A} = \int_a^c \mathbf{f(x)\,dx} - \int_c^b \mathbf{f(x)\,dx}$ or **calculate** the **area** of the **absolute value** of the function, i.e. $\int_a^b \mathbf{|f(x)|\,dx}$.



E.g.7. Find the total area enclosed by the curve y = sin 2x and the X-axis from x = 0 to x = ¾π.



There is one root at $\frac{\pi}{2}$ between 0 and ¾π.

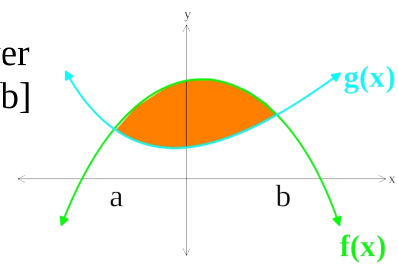
$\therefore A = \int_0^{\frac{3}{4}\pi} \sin 2x\,dx$

$= \int_0 \sin 2x\,dx - \int^{\frac{3}{4}\pi} \sin 2x\,dx$

$= 1\frac{1}{2}$ units^2

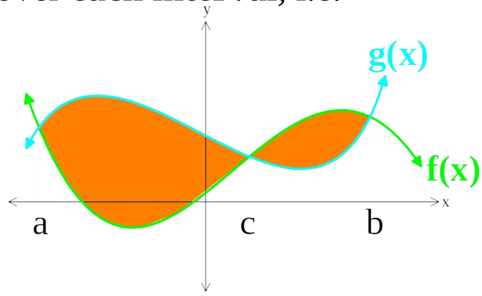
5. AREA BETWEEN CURVES: The **area between the curves** $\mathbf{y = f(x)}$ and $\mathbf{y = g(x)}$ over the interval [a;b] (where $\mathbf{f(x) > g(x)}$ over [a;b]) is the **area under f(x) minus the area under g(x)** over [a;b] provided the **curves do not intersect** between a and b.

$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \Rightarrow \mathbf{A} = \int_a^b \mathbf{[f(x) - g(x)]\,dx}$



If the **curves** do **intersect**, then the **areas** are **calculated** over **each interval**, i.e.

$$\mathbf{A} = \int_a^c [g(x) - f(x)]\,dx + \int_c^b [f(x) - g(x)]\,dx\,.$$

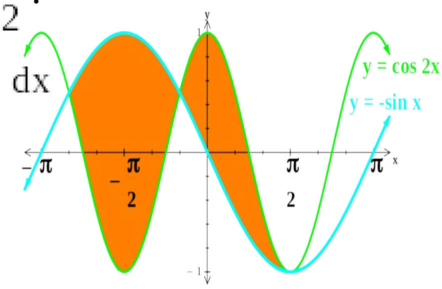


E.g.8. Calculate the area between y = cos 2x and y = -sin x for $-\frac{5\pi}{6} \le x \le \frac{\pi}{2}$.

There is one point of intersection at $-\frac{\pi}{6}$ between $-\frac{5\pi}{6}$ and $\frac{\pi}{2}$.

$$A = \int_{-\frac{5\pi}{6}}^{-\frac{\pi}{6}} (-\sin x - \cos 2x)\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} [\cos 2x - (-\sin x)]\,dx$$

$$= \frac{9\sqrt{3}}{4} \text{ units}^2$$



6. DEFINITE INTEGRALS: When evaluating **definite integrals** using the **change of variable** method, care must be taken to **recalculate** the **upper** and **lower limits**.

E.g.9. Use the substitution u = 5 – 2x to determine $\int_0^2 \frac{4x}{\sqrt{5 - 2x}}\,dx$.

$$u = 5 - 2x \Rightarrow \frac{du}{dx} = -2$$

$$x = \frac{5 - u}{2} \Rightarrow \frac{dx}{du} = -\frac{1}{2}$$

$$\int_0^2 \frac{4x}{\sqrt{5 - 2x}}\,dx = \int_{x=0}^{x=2} \frac{4x}{\sqrt{5 - 2x}} \frac{dx}{du}\,du$$

$$= \int_{u=5}^{u=1} \frac{2(5 - u)}{\sqrt{u}} \left(-\frac{1}{2} \right) du$$

$$= -\int_1^5 \left(u^{\frac{1}{2}} - 5u^{-\frac{1}{2}} \right) du$$

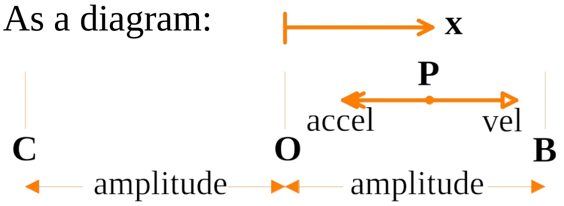
$$= \frac{4}{3}(5\sqrt{5} - 7)$$

<u>Ref</u>: Ex.7F Q.1-24 (even)
Ex.7G Q.1, 9; 2-16 (even); 17-22

SIMPLE HARMONIC MOTIION

1. **RECTILINEAR MOTION:** A body exhibits **simple harmonic motion** (**SHM**) if it **oscillates** about a fixed point called the **mean position** (**O**) where its **acceleration** (**a**) is **proportional** to its **displacement** (**x**) and the acceleration acts **towards** the **mean position**. Thus, $\frac{d^2 x}{dt^2} \equiv \ddot{x} \propto x \Rightarrow \ddot{x} = -k^2 x$. The '–' is because the acceleration is always directed towards O. Remember: **velocity** (**v**), $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2 x}{dt^2}$.

The **amplitude** (**A**) is the distance between the mean position and an extreme position. As a diagram:



$\ddot{x} = -k^2 x$ is a **differential equation**, and from this

$$\frac{dv}{dt} = -k^2 x$$

$$\frac{dv}{dx} \cdot \frac{dx}{dt} = -k^2 x$$

$$v\frac{dv}{dx} = -k^2 x$$

$$\int v\,dv = \int -k^2 x\,dx$$

$$\frac{v^2}{2} = \frac{-k^2 x^2}{2} + c$$

If A is the amplitude, then v = 0 when x = A.

$$\therefore 0 = -\frac{k^2 A^2}{2} + c$$

$$c = \frac{k^2 A^2}{2}$$

$$\Rightarrow \qquad v^2 = k^2 (A^2 - x^2)$$

$$v = k\sqrt{(A^2 - x^2)}$$

$$\int \frac{dx}{\sqrt{(A^2 - x^2)}} = \int k\,dt$$

Using the substitution $x = A \sin u \Rightarrow \frac{dx}{du} = a \cos u$, then …

$$\int \frac{1}{\sqrt{(A^2 - A^2 \sin^2 u)}} \frac{dx}{du} du = \int k\, dt$$

$$\int \frac{1}{\sqrt{(A^2(1 - \sin^2 u))}} \times A \cos u\, du = \int k\, dt$$

$$\int \frac{A \cos u}{\sqrt{(A^2 \cos^2 u)}} du = \int k\, dt$$

$$\int \frac{A \cos u}{A \cos u} du = \int k\, dt$$

$$\int 1\, du = \int k\, dt$$

$$u = kt + \alpha \quad \leftarrow \text{where } \alpha \text{ is the integration constant}$$

$$\Rightarrow \mathbf{x = A \sin (kt + \alpha)}$$

This indicates that the motion is **periodic**. Hence, if $\mathbf{x_1}$ is the position of the body at some time $\mathbf{t_1}$, then the body will be at $\mathbf{x_1}$ again at time $\mathbf{t_1 + \frac{2\pi}{k}}$. Thus, the **time period** (**T**) is $\mathbf{T = \frac{2\pi}{k}}$.

The constant **α** is the **phase angle**, and depends on the position of the body when **timing begins**. If the body starts from the mean position, then $\mathbf{x = \pm A \sin (kt)}$; if the body starts from an extreme position, then $\mathbf{x = \pm A \cos (kt)}$; and if the body starts from any other point, then $\mathbf{x = A \sin (kt + \alpha)}$ or $\mathbf{x = A \cos (kt + \alpha)}$. The choice of solution depends on the **initial position** of the body.

For each solution – the **amplitude** is **| A |**, the **angular velocity** is **k** radians per unit time with a **phase shift** of **α** radians, and the **period** is $\mathbf{T = \frac{2\pi}{k} = \frac{1}{f}}$ where **f** is the frequency of the motion, and the **velocity**, **v**, at time, **t**, is $\mathbf{v = Ak \cos (kt + \alpha)}$ or $\mathbf{v^2 = k^2(A^2 - x^2)}$, and $\mathbf{v_{max} = | kA |}$.

E.g.1. Determine the period of the simple harmonic motion defined by $\ddot{x} = -9x$.

$\ddot{x} = -9x \;\Rightarrow\; k = 3$

$$T = \frac{2\pi}{3}$$

$\therefore$ The period of the simple harmonic motion is $\frac{2\pi}{3}$ seconds.

E.g.2. A body moves such that its displacement from an origin O at time t seconds is x cm where $x = 2 \sin 3t$.

a) Prove that the motion is simple harmonic.

b) Determine the period and amplitude of the motion.

c) How far does the body move in the first second?

a) For SHM, $\ddot{x} = -k^2 x$

$x = 2 \sin 3t$

$\dot{x} = 6 \cos 3t$

$\ddot{x} = -18 \sin 3t$

$\quad = -9x$

This is in the form of $\ddot{x} = -k^2 x$

$\therefore$ The motion is simple harmonic.

b) $x = 2 \sin 3t \Rightarrow k = 3$ & $A = 2$

$$T = \frac{2\pi}{3}$$

$\therefore$ The time period is $\frac{2\pi}{3}$ seconds and the amplitude of the motion is 2 cm.

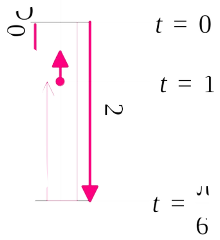
c) For the sine function, the extreme positions are at ¼ and ¾ of the complete curve, i.e. $t = \frac{2\pi}{3} \times \frac{1}{4} = \frac{\pi}{6}$ and $t = \frac{\pi}{2}$.

When $t = 0$, $x = 0$

When $t = 1$, $x = 2 \sin 3$

$\qquad\qquad \approx 0.28$ cm

But when $t = \frac{\pi}{6}$, $x = 2$ cm



Distance $\approx 2 + (2 - 0.28)$

$\qquad \approx 3.72$ cm

$\therefore$ The body travels approximately 3.72 cm in the first second.

E.g.3. A body experiences SHM at a rate of 5 oscillations per minute with an amplitude of 2 m. Find an expression for the displacement x (m) of the body from a fixed point O at time t minutes if the body starts from:

a) O,

b) an extreme position, and

c) the point where $x = 1$ m.

a) $k = 2\pi f$

$\quad = 2\pi \times 5$

$\quad = 10\pi$

b) $x = \pm\, 2 \cos (10\pi t)$

c) $x = 2\sin(10\pi t + \alpha)$

When $t = 0$, $x = 1$

$\Rightarrow 2\sin\alpha = 1$

$\therefore \alpha = \frac{\pi}{6}$ or $\alpha = \frac{5\pi}{6}$

Hence, $x = 2\sin\left(10\pi t + \frac{\pi}{6}\right)$ or $x = 2\sin\left(10\pi t + \frac{5\pi}{6}\right)$

Ref: *Ex.8A Q.1-24 (even)*

2. INTEGRATION BY PARTS: According to the **Product Rule** –

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} \Rightarrow uv = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$\Rightarrow \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

This is known as **integration by parts**.

E.g.4. Use integration by parts to determine $\int 2x\cos 3x\,dx$.

Let $u = 2x \Rightarrow \frac{du}{dx} = 2$

& $\frac{dv}{dx} = \cos 3x \Rightarrow v = \frac{1}{3}\sin 3x$

$$\begin{aligned}\therefore \int 2x\cos 3x\,dx &= 2x \times \tfrac{1}{3}\sin 3x - \int \tfrac{1}{3}\sin 3x.\ 2\,dx \\ &= \tfrac{2}{3}x\sin 3x - \tfrac{2}{3} \times \int \sin 3x\,dx \\ &= \tfrac{2}{3}x\sin 3x - \tfrac{2}{3} \times (-\tfrac{1}{3}\cos 3x) + c \\ &= \frac{2x}{3}\sin 3x + \frac{2}{9}\cos 3x + c \\ &= \tfrac{2}{3}[x\sin 3x + \tfrac{1}{3}\cos 3x] + c\end{aligned}$$

Ref: *Exercise Q.1-10 (even); 11-13 p.161*

APPLICATIONS OF CALCULUS

1. **SMALL CHANGES:** The study of **calculus** involves various rules and techniques for **differentiation** and **integration** associated with the concepts of – area under a curve, area between curves, locating turning points and points of inflection, optimization, **small changes**, **marginal rates of change**, **total change from rate of change**, **rectilinear motion**, **related rates**, **the fundamental theorem of calculus** and **volumes of revolution**.

Previously we obtained a rule to determine the **instantaneous rate of change** in one variable given a change in another related variable, i.e. $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ - where **h** is a small change in x values and **f(x + h) – f(x)** is the corresponding small change in y values. If we call **δx** the small change in x and **δy** the small change in y, then the rule becomes $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$. Thus if **δx** is very small, then $\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$.

E.g.1. Find the approximate change in volume when a tent, in the shape of a cone, with a base radius of 4 m, increases the height of the tent pole from 2.9 m to 2.95 m.

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dh} = \frac{1}{3}\pi r^2$

$\therefore \frac{\delta V}{\delta h} \approx \frac{1}{3}\pi r^2$

$\delta V \approx \frac{1}{3}\pi r^2 \times \delta h$

$\approx \frac{1}{3}\pi (4)^2(0.05)$

$\approx 0.84 \text{ m}^3$

∴ When the height of the tent pole changes from 2.9 m to 2.95 m the approximate change in the volume is 0.84 m³.

2. **MARGINAL RATES OF CHANGE:** In **economics** (business) there are three important functions, of a particular **commodity x**, that will be of interest – the **cost** function, **C(x)**, the **revenue** function, **R(x)** and the **profit** function, **P(x) = R(x) – C(x)**. Clearly C(x), R(x) and P(x) only have meaning for $\mathbf{x \geq 0}$ and for many commodities these functions only make sense for integer values of **x**.

The graphs should then be a set of **distinct points**. However, it is usual to **draw a smooth line** through these points and to assume that the functions are defined for all **non-negative x values**. The **gradient** of each function (graph) gives the **rate of change** for each function **with respect to x**. Economists call this gradient the **marginal cost**, **revenue** or **profit**, as appropriate.

When the cost, revenue and/or profit functions are **linear** then the corresponding marginal function(s) is **independent of x**. The **marginal cost**, **C(x)**, gives the **instantaneous cost per unit** when the $\mathbf{x^{th}}$ **unit** is being produced and/or the **approximate cost** of producing the **next unit**, i.e. the $\mathbf{(x + 1)^{th}}$ **unit**. The **average cost per unit** at any stage of the production is $\dfrac{\mathbf{C(x)}}{\mathbf{x}}$.

E.g.2. The revenue **R**, in dollars, obtained by making and selling **x** units of a certain article is given by **R(x) = 85x** and the cost **C**, in dollars, to make the article is given by $\mathbf{C(x) = 15\,000 + 285x - 2x^2}$. Find:
a) an expression for **P(x)**, the profit function,
b) how many articles must be made and sold for the manufacturer to break even with this product, and
c) the cost of making the 70th article.

a) $P(x) = R(x) - C(x)$
$= 85x - (15\,000 + 285x - 2x^2)$
$= 2x^2 - 200x - 15\,000$
$\therefore$ The profit function is $P(x) = 2x^2 - 200x - 15\,000$

b) For the manufacturer to break even R(x) = C(x), i.e. P(x) = 0
$2x^2 - 200x - 15\,000 = 0$
$\Rightarrow x = \cancel{-50}$ or $x = 150$
$\therefore$ To break even the manufacturer must make and sell 150 articles.

c) $C'(x) = 285 - 4x$
$C'(69) = 285 - 4(69)$
$= 9$
$\therefore$ It costs approximately \$9 to make the 70th article.

E.g.3. An engineer finds that her profit from manufacturing a certain machine is given by $\mathbf{P(x) = 500 + 240x - 2x^2}$, where x is the number of machines manufactured per day. The machines sell for \$1 500 each.
a) Write expressions for the revenue function, R(x), and the cost function, C(x).
b) Evaluate C'(55), R'(55), P'(55) and explain what information your answers give.
c) Evaluate C'(60), R'(60), P'(60), C'(65), R'(65), P'(65), compare your answers with b) and comment.

a) $R(x) = 1500x$
$C(x) = R(x) - P(x)$
$= 1500x - (500 + 240x - 2x^2)$
$= 2x^2 + 1260x - 500$

b) $C'(55) = 4(55) + 1260 = \$1480$ per machine
$R'(55) = \$1500$ per machine
$P'(55) = 240 – 4(55) = \$20$ per machine
The revenue is constant per each new machine manufactured. When the manufacturing level reaches 55 machines per day it will then cost approximately \$1480 to produce one more machine. This extra machine will bring in a profit of approximately \$20 (\$1500 – \$1480).

c) $C'(60) = \$1500$ per machine
$R'(60) = \$1500$ per machine
$P'(60) = \$0$ per machine

$C'(65) = \$1520$ per machine
$R'(65) = \$1500$ per machine
$P'(65) = -\$20$ per machine

The profit per machine is dropping from \$20 per each new machine manufactured to \$0, to -\$20, as production levels rise from 55 to 60 to 65 machines per day. When 65 machines are produced per day then one extra machine costs more to manufacture than it sells for. It would appear that the maximum profit would be achieved for 60 machines manufactured per day.

3. **TOTAL CHANGE:** The **derivative** measures the **rate** at which any dependent variable changes as the independent variable changes. Hence, the **definite integral** will determine the **total change** from a rate of change.

E.g.4. The cost, **\$C**, of producing **x** pairs of shoes where $\mathbf{C'(x) = 750 – 12x + 0.1x^2}$.
Find the cost of producing 50 pairs rather than 40 pairs.

$C'(x) = 7\,50 – 12x + 0.1x^2$

$\therefore\ C(x) = \int_{40}^{50} (750 - 12x + 0.1x^2)\,dx$

$\approx \$2\,303.33$

$\therefore$ It costs approximately \$2 303.33 more to produce 50 pairs of shoes than 40 pairs.

Ref: *Ex.7A Q.1-16 (even)*

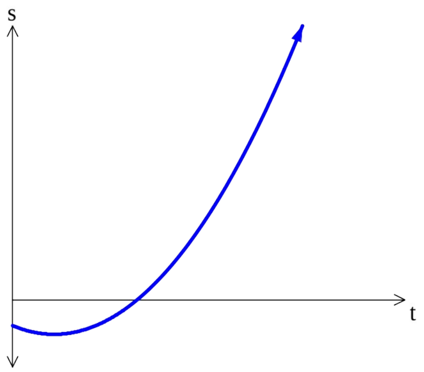
4. **RECTILINEAR MOTION:** When a particle moves in a straight line, i.e. **rectilinear motion** its change of position (or **displacement**) (**x**, in metres) from its starting position (or origin) may be described as a function of time (t), i.e. **f(t)**. **Velocity** (**v**, in m/s) is the rate of change of displacement with respect to time $\mathbf{v(t) = \frac{dx}{dt}}$. Hence, velocity can be equated to the first derivative of the displacement equation, i.e. **v(t) = f '(t)**. **Acceleration** (**a**, in m/s²) is the rate of change of velocity with respect to time $\mathbf{a(t) = \frac{dv}{dt}}$. Hence, acceleration can be equated to the second derivative of the displacement equation, i.e. $\mathbf{a(t) = f''(t) = \frac{d^2x}{dt^2}}$.

Likewise, $\mathbf{v = \int a\, dt}$ and $\mathbf{x = \int v\, dt}$. Displacement, velocity and acceleration are **vector quantities** in that they have **size** and **direction**.

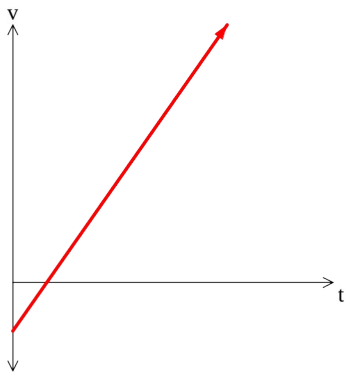
NOTE: For differentiation with respect to time, $\dot{y}$ and $\ddot{y}$ are generally used instead of y' and y''.

E.g.5. A particle P moves in a straight line such that its displacement, x metres, from an origin at time, t seconds, is $\mathbf{x(t) = 3t^2 - 2t - 1}$. Sketch a graph (each) of the displacement, velocity and acceleration of the particle P. Hence determine:
a) the initial position of particle P,
b) the initial velocity of P,
c) the acceleration at t = 3,
d) when the particle is momentarily at rest,
e) the displacement at t = 0, t = ⅓ and t = 4, and
f) the total distance covered between t = 0 and t = 4?

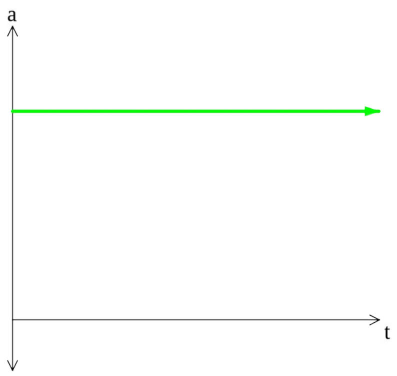
Displacement: $s(t) = 3t^2 - 2t - 1$



Velocity: $v(t) = s'(t) = 6t - 2$



Acceleration: $a(t) = v'(t) = s''(t) = 6$



a) Initial position is when $t = 0$,
$\therefore\ s(0) = 3(0)^2 - 2(0) - 1 = -1$ metres

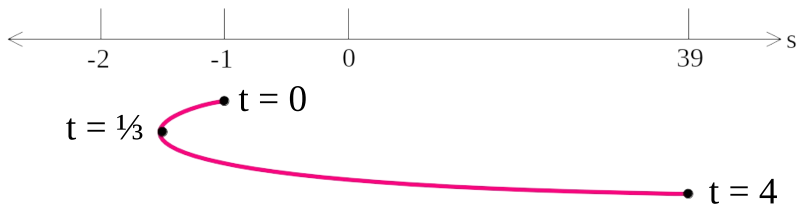
b) Initial velocity is when $t = 0$,
$\therefore\ v(0) = 6(0) - 2 = -2$ m/s

c) When $t = 3$, $a(3) = 6$ m/s^2 [Acceleration is constant!]

d) The particle is momentarily at rest when $v(t) = 0$,
$6t - 2 = 0$
$6t = 2$
$\therefore$ At $t = \frac{1}{3}$ seconds the particle is momentarily at rest.

e) Displacement at $t = 0$, $t = \frac{1}{3}$ and $t = 4$,
$s(0) = -1$ m
$s(\frac{1}{3}) = 3(\frac{1}{3})^2 - 2(\frac{1}{3}) - 1 = -1\frac{1}{3}$ m
$s(4) = 3(4)^2 - 2(4) - 1 = 39$ m

f) Total distance covered between $t = 0$ and $t = 4$,



$\therefore$ Total distance $= \frac{1}{3} + 40\frac{1}{3} = 40\frac{2}{3}$ m

E.g.6. A body moves in a straight line so that at any time **t** seconds its acceleration is **4t + 5** m/s².
a) Find the displacement in t seconds given that after 2 seconds the velocity is 5 m/s and that initially **s** = -6 m.
b) Find the distance the body moves in the first 3 seconds.

a) $a = 4t + 5$

$v = \int (4t + 5)\,dt$

$= \dfrac{4t^2}{2} + 5t + c$

$= 2t^2 + 5t + c$

But when $t = 2$, $v = 5$

$2(2)^2 + 5(2) + c = 5$

$18 + c = 5$

$c = -13$

$\therefore v = 2t^2 + 5t - 13$

$s = \int (2t^2 + 5t - 13)\,dt$

$= \dfrac{2t^3}{3} + \dfrac{5t^2}{2} - 13t + k$

But when $t = 0$, $s = -6$

$k = -6$

$\therefore s = \dfrac{2t^3}{3} + \dfrac{5t^2}{2} - 13t - 6$

b) [Check for change of direction in the first 3 seconds by graphing s(t).]

$s(0) = -6$ m

From the CAS calc.,

$s(1.6) \approx -17.7$ m and $s(3) = -4.5$ m

Distance travelled $\approx [-6 - (-17.7)] + [-4.5 - (-17.7)]$

≈ 24.8 m

OR

$s(t) = \displaystyle\int_0^3 |2t^2 + 5t - 13|\,dt$

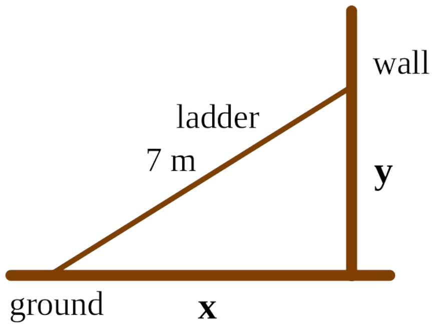
≈ 24.8 m

$\therefore$ the body moves approximately 24.8 m in the first 3 seconds.

Ref: *Ex.7B Q.1-33 (odd)*

5. RELATED RATES: Knowing how one quantity changes with respect to time, e.g. $\mathbf{x = g(t)}$, and how another **related** quantity changes with respect to the first quantity, e.g. $\mathbf{y = f(x)}$, then how the second quantity changes with respect to time can be determined, e.g. $\dfrac{dy}{dt} = f'(x)\dfrac{dx}{dt}$.

E.g.7. A ladder, 7 m long, rests against a vertical wall. The ladder is standing on flat ground. The bottom of the ladder is being pulled, along the ground, towards the wall at a steady rate of $0.1\ ms^{-1}$. How fast is the top of the ladder sliding up the wall when the bottom is 2 m out from the wall?



$x^2 + y^2 = 49$

$\Rightarrow\ 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$

But $\dfrac{dx}{dt} = -0.1$ ← constant

When x = 2, y = $\sqrt{45} = 3\sqrt{5}$

$\Rightarrow\ 2(2)(-0.1) + 2(3\sqrt{5})\dfrac{dy}{dt} = 0$

$$\frac{dy}{dt} = \frac{\sqrt{5}}{75}$$

∴ The top of the ladder is moving up the wall at a rate of $\dfrac{\sqrt{5}}{75}$ ms^{-1}.

E.g.8. For $y = 10e^{0.05x}$, where x = f(t), find $\dfrac{dy}{dt}$ given $\dfrac{dx}{dt} = 1.2$ when x = 0.

$y = 10e^{0.05x}$

$\dfrac{dy}{dt} = 0.5e^{0.05x}\dfrac{dx}{dt}$

When x = 0, $\dfrac{dx}{dt} = 1.2$

Hence, $\dfrac{dy}{dt} = 0.5e^{0.05(0)}(1.2)$

$= 0.6$

Ref: *Ex.7C Q.1-21 (odd)*

6. **THE FUNDAMENTAL THEOREM OF CALCULUS:** Previously, we established that the **fundamental theorem of calculus** is $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$ or. if $\mathbf{F(x)} = \displaystyle\int_a^x f(t)\,dt$, then **F'(x) = f(x)**. However, there is an alternative form of this theorem – $\dfrac{d}{dx}\left(\displaystyle\int_a^x f(t)\,dt\right) = f(x)$. Thus, the first **integrates the derivative** to obtain the function and the second **differentiates the integral** to obtain the function.

There is a third version of the fundamental theorem which states that if $\mathbf{F(x)} = \displaystyle\int_a^{g(x)} f(t)\,dt$, then **F'(x) = f[g(x)].g'(x)**.

E.g.9. Find the derivative of F(x) given that:

a) $F(x) = \int_1^x [\sin(t)]\, dt$

b) $F(x) = \int_1^{x^2+1} [2t + 1]\, dt$

a) $F(x) = \int_1^x [\sin(t)]\, dt$

$\therefore F'(x) = \sin x$

b) $F(x) = \int_1^{x^2+1} [2t + 1]\, dt$

Let $f(x) = 2t + 1$ and $g(x) = x^2 + 1$

$F'(x) = f[g(x)].g'(x)$

$= [2(x^2 + 1) + 1].2x$

$= 2x(2x^2 + 3)$

$= 4x^3 + 6x$

E.g.10. Find:

a) $\frac{d}{dx} \int_0^x [4t + 1]\, dt$

b) $\frac{d}{dx} \int_0^{\sin x} [3t - 1]\, dt$

c) $\int_0^3 6x\, dx$

d) $\int_0^1 \frac{d}{dx} \left[\frac{1 - \sqrt{x}}{1 + \sqrt{x}} \right] dx$

a) $\frac{d}{dx} \int_0^x [4t + 1]\, dt = 4x + 1$

b) $\frac{d}{dx} \int_0^{\sin x} [3t - 1]\, dt = [3 \sin x - 1].\cos x$

c) $\int_0^3 6x\, dx = [3x^2]_0^3$

$= 27$

d) $$\int_0^1 \frac{d}{dx}\left[\frac{1-\sqrt{x}}{1+\sqrt{x}}\right] dx = \left[\frac{1-\sqrt{x}}{1+\sqrt{x}}\right]_0^1$$
$$= -1$$

7. VOLUME OF REVOLUTION: If the curve **y = f(x)** is **non-negative** over the interval **[a,b]** and if this region (**area under the curve**) is **rotated 2π** (**360°**) about the **X-axis**, then a **solid of revolution** is formed. The **volume** of this solid is $\mathbf{V = \pi \int_a^b [f(x)]^2\, dx}$.

Likewise, the **volume** of the solid formed by the curve **x = f(y)**, the **Y-axis** and the lines **y = a** and **y = b** is $\mathbf{V = \pi \int_a^b [f(y)]^2\, dy}$.

The **volume** generated by the **region**(**s**) trapped **between two curves**, e.g. **y = f(x)** and **y = g(x)**, **rotated 2π** (**360°**) about the **X-axis** is determined by locating the **points of intersection**, and **calculating** the **volumes** over **each interval** and then **adding** these **volumes**, or **calculating** the **volume** directly by $\mathbf{V = \pi \int_a^b \left|\, [f(x)]^2 - [g(x)]^2 \,\right| dx}$.

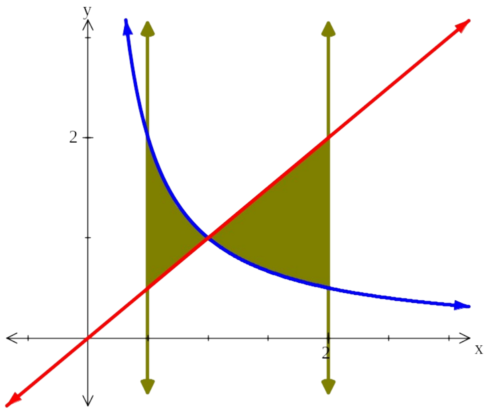
E.g.11. Find the **exact** volume of the solid formed by one revolution:
a) about the X-axis, for $y = x^2$ over [1,2],
b) about the Y-axis, for $y = x^2$; $y = 0.5$ and $y = 3$, and
c) about the X-axis, for $y = \frac{1}{x}$ and $y = x$ over [½,2].

a) $$V = \pi \int_1^2 [(x^2)]^2\, dx$$
$$= \frac{31\pi}{5} \text{ units}^3$$

b) $y = x^2 \Rightarrow x = \sqrt{y}$
$$V = \pi \int_{0.5}^3 [\sqrt{y}]^2\, dy$$
$$= \frac{35\pi}{8}$$

c) From the CAS calc., over [½,2]

$y = \frac{1}{x}$ and y = x intersect at (1,1).



$$V_1 = \pi \int_{0.5}^{1} \left[\frac{1}{x} \right]^2 dx - \pi \int_{0.5}^{1} x^2\,dx$$

$$= \frac{17\pi}{24}$$

$$V_2 = \pi \int_{1}^{2} x^2\,dx - \pi \int_{1}^{2} \left[\frac{1}{x} \right]^2 dx$$

$$= \frac{11\pi}{6}$$

$$\therefore V = \frac{61\pi}{24}$$

OR

$$V = \pi \int_{0.5}^{2} \left| \left[\left(\frac{1}{x} \right)^2 - x^2 \right] \right| dx$$

$$= \frac{61\pi}{24}$$

<u>Ref</u>: *Ex.7D Q.1-11 (odd)*
VOLUMES OF REVOLUTION

EXPONENTIALS AND LOGARITHMS

1. LOGARITHMIC SCALES: Logarithmic scales have a constant distance between consecutive powers of a (given) base.

E.g.

1	10	100	1 000	10 000
10^0	10^1	10^2	10^3	10^4
$\log_{10} 1 = 0$	$\log_{10} 10 = 1$	$\log_{10} 100 = 2$	$\log_{10} 1\,000 = 3$	$\log_{10} 10\,000 = 4$

Before calculators were invented, a **slide rule** used **logarithmic scales** for multiplication.

Logarithms, and natural logarithms, are often used to "convert" exponential functions/graphs to linear functions/graphs. A **log-linear** function/graph is achieved by –

- taking the log/ln of both sides of the equation,
- letting an associated variable equal the log/ln, e.g. Y = log y,
- finding the log/ln of the values in the original table,
- graphing the new table of values,
- drawing in the line of best fit, and hence
- estimating the gradient and y-intercept; or by
- using a calculator to determine the equation of the regression line for the new (linear) table of values.

Some **graph paper** has a logarithmic scale on the vertical axis. A logarithmic scale is also useful to display **data** with a **large range**, e.g. a timeline of the world's population.

There are some well known logarithmic scales –

- the **Richter scale** for seismic events, i.e. earthquakes, where $\mathbf{R = \log\left(\frac{I}{I_0}\right)}$ for **R**, the Richter scale reading, **I**, the intensity of the earthquake, and $\mathbf{I_0}$, a minimum intensity level used for comparison;
- the **pH scale** measuring the acidity or alkalinity of a solution where **pH** stands for the potential of hydrogen, and **pH = -log$_{10}$ (hydrogen ion concentration)** with $0 \leq \text{pH} \leq 14$ and a pH of 7 is **neutral** ⇔ **pure water**;
- **scale of loudness** which measures the intensity of a sound compared to the intensity of a sound just detectable by the human ear, where $\mathbf{L = 10\log_{10}\left(\frac{I}{I_0}\right)}$ for **L**, the loudness of the sound in decibels (**dB**), **I**, the intensity of the sound, and $\mathbf{I_0}$, the intensity of sound just audible to the human ear; and
- the **Music scales** where each time the frequency **f** doubles this is equivalent to one **octave**, thus $\frac{f_2}{f_1} = 2^x \Rightarrow \log\left(\frac{f_2}{f_1}\right) = x\log 2 \Rightarrow x \approx 3.32\log\left(\frac{f_2}{f_1}\right)$.

E.g.1. A student reads the following co-ordinates (to the nearest whole number) from a graph of a power function.

x	2	5	10	12
y	14	60	158	208

Determine the equivalent linear function for this data if it takes the form of $y = kx^n$ where k and n are constants and hence give the new table of values.

$y = kx^n$
$\log y = \log k + n \log x$
Let $Y = \log y$ and $X = \log x$
$\therefore\ Y = \log k + nX$ ← This is now a linear function.

X	0.3	0.7	1.0	1.08
Y	1.15	1.78	2.2	2.32

Ref: *Ex.6A Q.1-5*

2. GROWTH AND DECAY: If the rate of change of some quantity **A** is proprotional to the quanitity itself, then $\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA \Rightarrow \mathbf{A = A_0.e^{-kt}}$ where $\mathbf{A_0}$ is the value of A when $\mathbf{t = 0}$, and if $\mathbf{k > 0}$, then **A** is **growing** exponentially or if $\mathbf{k < 0}$, then **A** is **decaying** exponentially.

NOTE: For exponential growth/decay, a formula of the type $A = A_0.e^{-kt}$ is used and hence the natural log is the most appropriate to use.

E.g.2. The value of a car is said to decay exponentially against time. The table below shows the value of Blair's car for various ages.

Age t (years)	2	5	9	15
Value v ($1000's)	11.5	8.0	4.7	2.2

Estimate the value of the car as new and how long it would take for its value to reduce by ⅓.

$v = v_0.e^{-kt}$
$\ln v = \ln v_0 + \ln e^{-kt}$
$\ln v = \ln v_0 - kt.\ln e$
$\ln v = \ln v_0 - kt$
Let $V = \ln v$
$\therefore V = -kt + \ln v_0$ ← This is now a linear function.

t	2	5	9	15
V = ln v	2.44	2.08	1.55	0.79

From a graph or CAS calc.,
$k \approx 0.128$ and $\ln v_0 \approx 2.7$
$e^{2.7} = v_0$
$v_0 \approx 14.9$
∴ The new value of the car is approximately $14 900.

Reduce the value by ⅓ ⇒ $\frac{2}{3}v_0$ so $v \approx 9.933$
$v = v_0.e^{-kt}$
$9.933 = 14.9e^{-0.128t}$
$\frac{9.933}{14.9} = e^{-0.128t}$
$\frac{2}{3} = e^{-0.128t}$
$\ln(\frac{2}{3}) = \ln e^{-0.128t}$
$\ln(\frac{2}{3}) = -0.128t \times \ln e$
$t = \frac{\ln(\frac{2}{3})}{-0.128}$
$t \approx 3.2$ years
∴ It takes approximately 3.2 years to reduce the value of Blair's car by ⅓.

<u>Ref</u>: *Ex.6B Q.1-28 (even)*

3. **LOGARITHMIC DIFFERENTIATION: Logarithmic differentiation** can make differentiation of some functions easier. It involves taking the **logarithm/natural logarithm** of both sides of an equation, **implicitly differentiating** the equation, and then "**converting**" to the original format.

E.g.3. Use logarithmic differentiation to determine $\frac{dy}{dx}$ given that $y = x^{\cos 2x}$.

$$y = x^{\cos 2x}$$
$$\ln y = \ln x^{\cos 2x}$$
$$= \cos 2x \,.\, \ln x$$
$$\frac{d}{dx} \ln y = \frac{d}{dx}(\cos 2x \,.\, \ln x)$$
$$\frac{d}{dy}(\ln y)\frac{dy}{dx} = \frac{d}{dx}(\cos 2x \,.\, \ln x)$$
$$\frac{1}{y}\frac{dy}{dx} = -2\sin 2x \,.\, \ln x + \frac{1}{x}\cos 2x$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{-2x\sin 2x \,.\, \ln x + \cos 2x}{x}$$
$$\frac{dy}{dx} = y.x^{-1}(-2x\sin 2x \,.\, \ln x + \cos 2x)$$
$$= x^{\cos 2x} \,.\, x^{-1}(-2x \sin 2x \,.\, \ln x + \cos 2x)$$
$$= x^{\cos 2x - 1}(-2x \sin 2x \,.\, \ln x + \cos 2x)$$

Ref: *Ex.6C Q.1-4*

MATHEMATICAL REASONING

1. PROOF BY INDUCTION: A **proof by induction** has two steps – first, prove that **if** the statement is true for some general value of n, e.g. **n = k**, then it must be true for the next value of n, i.e. **n = k + 1**; and second, prove that the statement is true for a specific value of n, usually **n = 1**. Thus, a **proof by induction** proves that **if any value exists**, then the **next value** must also **exist**, and as at least **one value** does **exist**, therefore, an **infinite number** of values **exist**.

E.g.1. Prove that the sum $\mathbf{S_n}$ of **n** consecutive odd numbers is given by $\mathbf{S_n = n^2}$, i.e.
$\mathbf{S_n = \{1 + 3 + 5 + 7 + \ldots + 2n - 1\} = n^2}$.

If $S_n = \{1 + 3 + 5 + 7 + \ldots + 2n - 1\}$,
then $S_{n+1} = S_n + (2n + 1)$

If n = 1, then $S_1 = 1 = 1^2$
⇒ A value exists that is true.

If n = k, then $S_k = k^2$
If n = k + 1, then $S_{k+1} = (k + 1)^2$
$= k^2 + 2k + 1$
$= S_k + (2k + 1)$ (Q.E.D.)
∴ The sum of n consecutive odd numbers is given by $S_n = n^2$.

Ref: *Ex.1A Q.1-13 (odd)*

2. CONJECTURES: Remember – A **conjecture** is a mathematical statement which appears likely to be true, but has not been **formally proven** to be true under the rules of mathematical logic. A conjecture may not have been **proven true** but it may not have been **proven false**, and hence, may be **considered true**, though **unproven**. A **conjecture** that is later proven to be true then becomes a **theorem**.

There are a number of famous conjectures – **Goldbach's** conjecture and the related versions, the **twin prime** conjecture, **Fermat's** conjecture, and the **four colour** conjecture.

ACTIVITY 1:
Investigate these conjectures, by considering –
- What does each conjecture claim?
- Give some examples of these claims.
- What is the history of each conjecture?
- Who made each conjecture? When and where?
- Have any of the conjectures since been proven true? or false?