



PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Test 3 Year 12

Student name: _____ Teacher name: _____

Task type: Response

Time allowed for this task: ____40____ mins

Number of questions: ____7____

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: ____44____ marks

Task weighting: ____10____%

Formula sheet provided: Yes

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (6 marks)

a) Solve the following system of linear equations.

(3 marks)

$$x + 2y - 3z = -28$$

$$2x - 7y + 5z = 76$$

$$3x - 4y + 6z = 71$$

Solution				
$\begin{bmatrix} 1 & 2 & -3 & -28 \\ 2 & -7 & 5 & 76 \\ 3 & -4 & 6 & 71 \end{bmatrix}$				
$\begin{bmatrix} 1 & 2 & -3 & -28 \\ 0 & 11 & -11 & -132 \\ 0 & 10 & -15 & -155 \end{bmatrix}$				
$\begin{bmatrix} 1 & 2 & -3 & -28 \\ 0 & 11 & -11 & -132 \\ 0 & 0 & 55 & 385 \end{bmatrix}$				
$55z = 385$				
$z = 7$				
$y = -5$				
$x = 3$				
Specific behaviours				
<ul style="list-style-type: none"> ✓ eliminates one variable from two equations ✓ eliminates two variables ✓ solves for all variables 				

b) Determine all possible values of p & q for the three scenarios below.

(3 marks)

$$x + 2y - 3z = q$$

$$2x - 7y + 5z = 76$$

$$3x - 4y + pz = 71$$

- i) No solutions
- ii) One solution

iii) Infinite solutions

Solution	
$\begin{bmatrix} 1 & 2 & -3 & q \\ 2 & -7 & 5 & 76 \\ 3 & -4 & p & 71 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -3 & q \\ 0 & 11 & -11 & 2q-76 \\ 0 & 10 & -9-p & 3q-71 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -3 & -14 \\ 0 & 11 & -11 & 2q-76 \\ 0 & 0 & -11+11p & 21-13q \end{bmatrix}$ <p> i) $p=1 \& q \neq \frac{21}{13}$ ii) $p \neq 1$ iii) $p=1 \& q = \frac{21}{13}$ </p>	
Specific behaviours	
<ul style="list-style-type: none"> ✓ derives equation with two variables eliminated ✓ states values for uniqueness ✓ states values for no solution and infinite (follow through) 	

Q2 (9 marks)

A particle moves with acceleration $a = \begin{pmatrix} t^3 \\ \sqrt{t} \end{pmatrix} m/s^2$ at time t seconds. The initial velocity is $\begin{pmatrix} 3 \\ -2 \end{pmatrix} m/s$ and initial position $\begin{pmatrix} 4 \\ -1 \end{pmatrix} m$.

a) Determine the velocity at time t seconds.

(2 marks)

Solution	
$a = \begin{pmatrix} t^3 \\ \sqrt{t} \end{pmatrix}$ $v = \begin{pmatrix} \frac{t^4}{4} \\ \frac{2t^{\frac{3}{2}}}{3} \end{pmatrix} + c$ $\begin{pmatrix} 3 \\ -2 \end{pmatrix} = c$ $v = \begin{pmatrix} \frac{t^4}{4} + 3 \\ \frac{2t^{\frac{3}{2}}}{3} - 2 \end{pmatrix} m/s$	
Specific behaviours	
✓ anti-differentiates ✓ solves for constant	

b) Determine the position vector at time $t = 5$ seconds.

(2 marks)

Solution

$v = \begin{pmatrix} \frac{t^4}{4} + 3 \\ \frac{2t^2}{3} - 2 \end{pmatrix}$ $r = \begin{pmatrix} \frac{t^5}{20} + 3t \\ \frac{4t^2}{15} - 2t \end{pmatrix} + k$ $\begin{pmatrix} 4 \\ -1 \end{pmatrix} = k$ $r = \begin{pmatrix} \frac{t^5}{20} + 3t + 4 \\ \frac{4t^2}{15} - 2t - 1 \end{pmatrix}$ $r = \begin{pmatrix} \frac{701}{4} \\ \frac{20\sqrt{5}}{3} - 11 \end{pmatrix} \text{ or } \begin{pmatrix} 175.25 \\ 3.91 \end{pmatrix} m$
Specific behaviours
✓ determines r ✓ approx. at t=5 (maybe exact or 2 dp)

- c) Determine $\frac{dy}{dx}$ on the cartesian path at time $t=5$ seconds. (2 marks)

Solution
$v = \begin{pmatrix} \frac{t^4}{4} + 3 \\ \frac{2t^2}{3} - 2 \end{pmatrix} = \begin{pmatrix} 159.25 \\ 5.45 \end{pmatrix}$ $\frac{dy}{dx} \approx \frac{5.45}{159.25} \approx 0.03$

Specific behaviours
<ul style="list-style-type: none"> ✓ uses v at t=5 ✓ determines rate

- d) Determine $\frac{d^2y}{dx^2}$ on the cartesian path at time $t=5$ seconds. (3 marks)

Solution
$v = \begin{pmatrix} \frac{t^4}{4} + 3 \\ \frac{2t^{\frac{3}{2}}}{3} - 2 \end{pmatrix}$ $\frac{dy}{dx} = \frac{\frac{2t^{\frac{3}{2}}}{3} - 2}{\frac{t^4}{4} + 3}$
<p>The screenshot shows a TI-Nspire calculator window titled "Edit Action Interactive". The main display shows the derivative of the velocity vector components with respect to time, evaluated at t=5:</p> $\frac{d}{dt} \begin{pmatrix} \frac{2 \cdot t^{\frac{3}{2}}}{3} - 2 \\ \frac{t^4}{4} + 3 \end{pmatrix} \bigg _{t=5}$ <p>The result is displayed as:</p> $\frac{-0.01283887789}{\frac{t^4}{4} + 3} \bigg _{t=5}$ <p>The final result is:</p> $-8.062089727E-5$

Rate =0.00

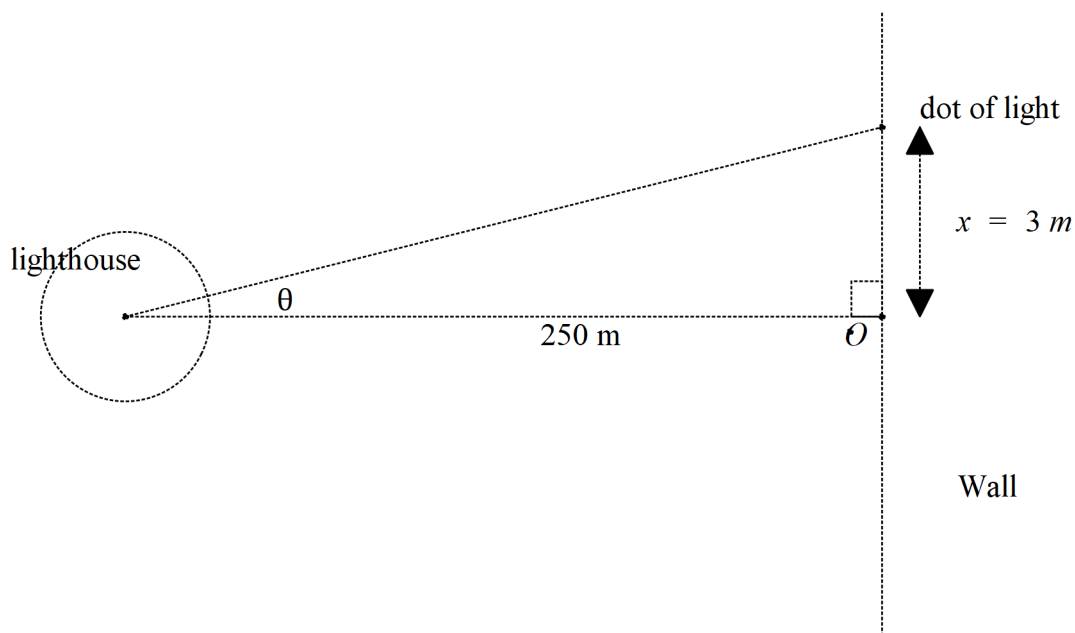
Specific behaviours

- ✓ time diff dy/dx
- ✓ divides by dx/dt
- ✓ determines approx. rate (do not penalise if not 2dp)

Q3 (7 marks)

Consider an artificial island that contains a revolving light that is 250 metres from shore. There is a long wall on the shore and the light from the lighthouse can be seen as a moving dot of light on the

wall. The angular speed of the light is 24 radians/second, ($\frac{d\theta}{dt} = 24$).



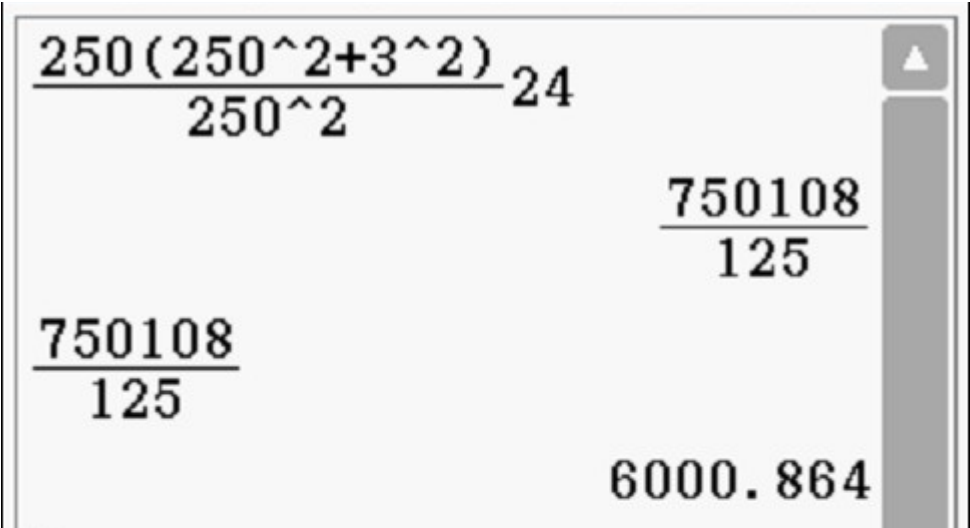
- a) Determine the speed of the dot of light on the wall when the dot is 3 metres away from the closest point to the lighthouse, pt O, see diagram above. (4 marks)

Solution

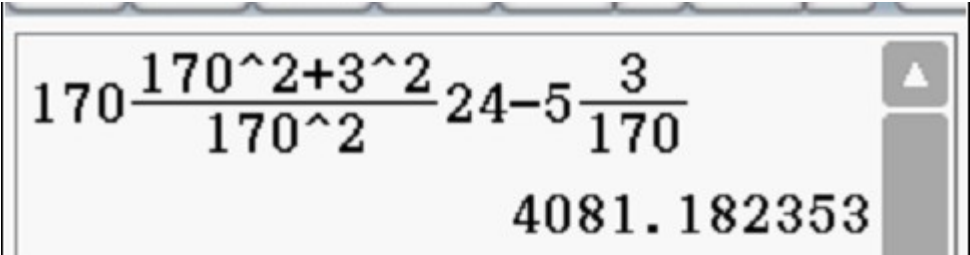
$$\tan \theta = \frac{x}{250}$$

$$250 \sec^2 \theta \frac{d\theta}{dt} = \frac{dx}{dt}$$

$$\cos \theta = \frac{250}{\sqrt{250^2 + 3^2}}$$

	
Specific behaviours	
<ul style="list-style-type: none"> ✓ states relationship between x and angle ✓ implicit diff wrt time (or related rates) ✓ subs values and uses derivative of angle ✓ determines approx. speed (no need for units) 	

- b) If the artificial island containing the lighthouse is moving towards the shore, pt O, at a speed of 5 metres per second, determine the speed of the dot when 3 metres away from pt O and the lighthouse being 170 metres from the shore, pt O. (3 marks)

Solution	
$\tan \theta = \frac{x}{L}, \dot{L} = -5 \text{ m/s}$ $x = L \tan \theta$ $\dot{x} = L \sec^2 \theta (\dot{\theta}) + \dot{L} \tan \theta$ $\tan \theta = \frac{3}{170} \quad \cos \theta = \frac{170}{\sqrt{170^2 + 3^2}}$	
	
Specific behaviours	
<ul style="list-style-type: none"> ✓ uses three variables ✓ uses time implicit with product or quotient and all time rates with correct signs ✓ determines speed 	

Q4 (3 marks)

Show using logarithmic differentiation how to differentiate $y = x^{\sin(2x)}$.

Solution
$\ln y = \ln x^{\sin(2x)} = \sin 2x \ln x$ $\frac{1}{y} y' = \sin 2x \frac{1}{x} + 2 \cos 2x \ln x$ $y' = \left(\sin 2x \frac{1}{x} + 2 \cos 2x \ln x \right) y$
Specific behaviours
<ul style="list-style-type: none"> ✓ takes natural log of both sides ✓ implicit diff of both sides and uses product rule ✓ expresses in terms of x&y

Q5 (8 marks)

Show how to evaluate the following without any use of the classpad. Show all working.

a) $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx$

(4 marks)

Solution
$\int_0^{\frac{\pi}{2}} \sin x \sin^2 x \, dx$ $\int_0^{\frac{\pi}{2}} \sin x (1 - \cos^2 x) \, dx$ $\int_0^{\frac{\pi}{2}} \sin x - \sin x \cos^2 x \, dx$ $\left[-\cos x + \frac{1}{3} \cos^3 x \right]_0^{\frac{\pi}{2}} = (0) - \left(-1 + \frac{1}{3} \right) = \frac{2}{3}$
Specific behaviours

- ✓ uses Pythagorean identity
- ✓ breaks into two terms with $\sin x$
- ✓ anti- diffs both terms
- ✓ subs both limits to give final result

Q5 cont-

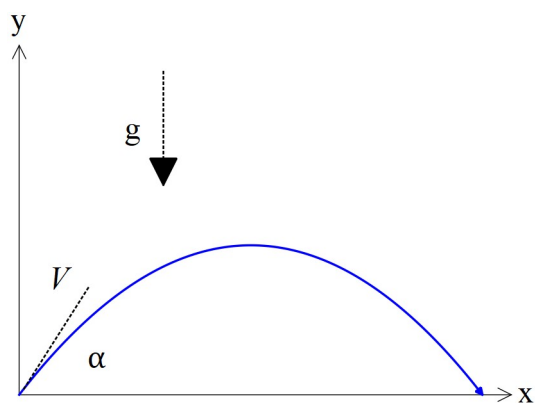
b) $\int \frac{2x+1}{(x-3)(x+5)} dx$

(4 marks)

Solution
$\frac{2x+1}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$ $2x+1 = A(x+5) + B(x-3)$ $x=3$ $7 = 8A \quad A = \frac{7}{8}$ $x=-5$ $-9 = -8B \quad B = \frac{9}{8}$ $\int \frac{2x+1}{(x-3)(x+5)} dx = \frac{7}{8} \ln x-3 + \frac{9}{8} \ln x+5 + c$
Specific behaviours
<ul style="list-style-type: none"> ✓ uses partial fractions ✓ solves for constants ✓ integrates using logs ✓ states answer with a plus constant

Q6 (7 marks).

Consider a projectile that leaves with speed $V \text{ m/s}$ at an angle α to the horizontal, see diagram.
 Assume that the constant acceleration is $-g \text{ m/s}^2$.



- a) Using vector calculus and starting with the acceleration, show how to derive the cartesian equation of the path in terms of V, g & α . (4 marks)

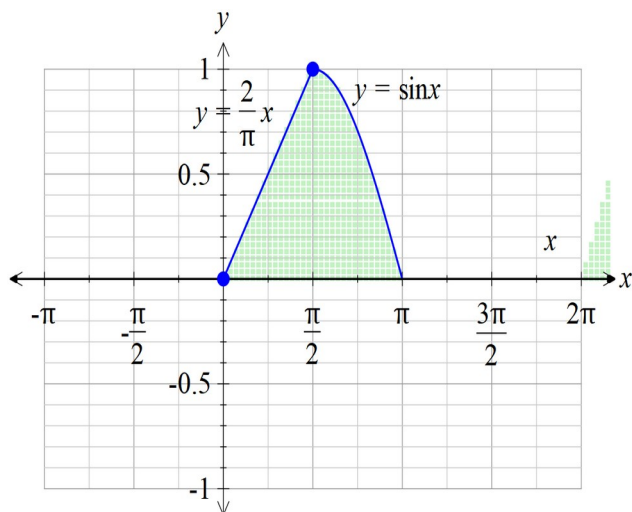
Solution	
$\ddot{r} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$	
$\dot{r} = \begin{pmatrix} 0 \\ -gt \end{pmatrix} + c$	
$\begin{pmatrix} V \cos \alpha \\ V \sin \alpha \end{pmatrix} = c$	
$\dot{r} = \begin{pmatrix} V \cos \alpha \\ V \sin \alpha - gt \end{pmatrix}$	
$r = \begin{pmatrix} Vt \cos \alpha \\ Vt \sin \alpha - \frac{gt^2}{2} \end{pmatrix} + c$	
$c = 0$	
$t = \frac{x}{V \cos \alpha}$	
$y = \frac{xV \sin \alpha}{V \cos \alpha} - \frac{gx^2}{2V^2 \cos^2 \alpha} = x \tan \alpha - \frac{gx^2}{2V^2 \cos^2 \alpha}$	
Specific behaviours	
<ul style="list-style-type: none"> ✓ integrates to find velocity and solves for constant ✓ integrates to find r ✓ subs x expression into y by eliminating t ✓ obtains cartesian expression in terms of constants 	


- b) Given that $V = 50 \text{ m/s}$ and that $y = 44 \text{ m}$ when $x = 38 \text{ m}$, determine possible value(s) for α .
(3 marks)

Solution
$y = x \tan \alpha - \frac{gx^2}{2V^2}(1 + \tan^2 \alpha)$ $44 = 38 \tan \alpha - \frac{10(38^2)}{2(50^2)}(1 + \tan^2 \alpha)$
<p style="text-align: center;">Specific behaviours</p> <ul style="list-style-type: none"> ✓ subs all knowns into cartesian equation ✓ solves for one angle ✓ solves for two angles

Q7 (4 marks)

Consider the area between $y = \sin x$, $y = \frac{2}{\pi}x$ and the x axis with $0 \leq x \leq \pi$, as shown below.



<p>Solution</p> $\int_0^1 \pi x^2 dy = \int_{y=0}^{y=1} \pi x^2 \frac{dy}{dx} dx, y = \sin x$ $\int_{\pi}^{\frac{\pi}{2}} \pi x^2 \cos x dx - \frac{1}{3} \pi \left(\frac{\pi}{2} \right)^2$	
 <p>The screenshot shows a TI-Nspire calculator screen. At the top, there's a title bar that says "Edit Action Interactive". Below it is a toolbar with various icons: a zoom icon (0.5, 1/2), a hand icon, a left and right arrow icon, a "Simp" button, a pencil icon, a dropdown arrow, a parabola icon, another dropdown arrow, and a right arrow icon. The main display area shows the integral expression: $\int_{\pi}^{\pi/2} \pi x^2 \cos(x) dx - \frac{1}{3} \pi \left(\frac{\pi}{2} \right)^2$. Below the expression, the numerical result is displayed: 18.62373628.</p>	
<p>(Note- use of inverse sine function without a translation is incorrect as sine x is a many to one function over 0 to pi domain)</p>	
<p>Specific behaviours</p>	
<p>✓ uses correct integral with appropriate limits for area under sin x</p>	
<p>✓ uses change of variable with correct order of limits</p>	
<p>✓ determines volume of cone</p>	

✓ obtains correct volume -must be numeric
(no need to round)
(Max 1 out of 4 if revolved around x axis- too easy)

r