MATHEMATICS SPECIALIST UNIT 3

Semester One

2018

SOLUTIONS

Calculator-free Solutions

1. Since $z=\pm 2$ are roots, then z^2-4 is a factor. ✓

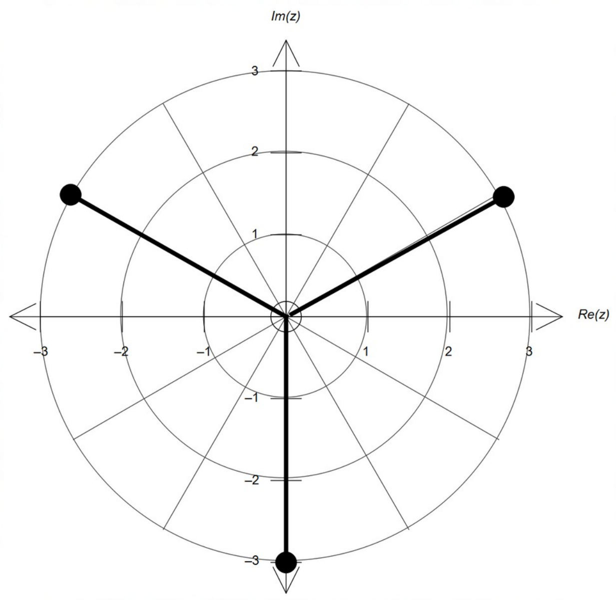
$\therefore$ dividing $P(z)$ by z^2-4 gives the quadratic factor $2z^2-4z+4$ ✓

and using the quadratic formula gives $z=1\pm i$ ✓✓

$\therefore P(z)=2(z+2)(z-2)(z-1-i)(z-1+i)$ ✓ [5]

2. (a) $w=z^3=\left[3\,cis\left(\frac{-\pi}{2}\right)\right]^3=27\,cis\left(\frac{-3\pi}{2}\right)=27\,cis\,\frac{\pi}{2}$ ✓✓

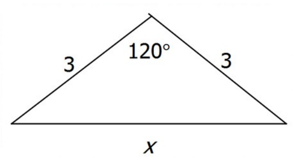
(b)



✓ magnitude = 3

✓ $\frac{2\pi}{3}$ radians apart

(c)



$x^2=9+9-2(9)\cos 120°$ ✓

$\therefore x=3\sqrt{3}$ ✓

$\therefore$ perimeter ¿ $9\sqrt{3}$ units, ✓ [7]

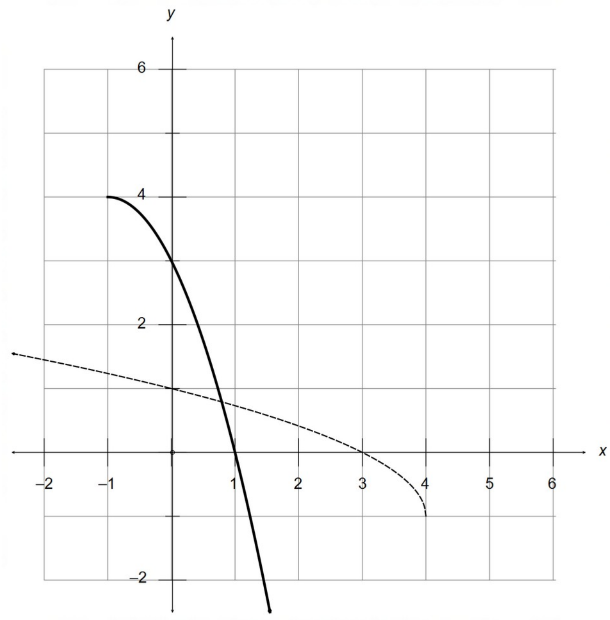
3. (a) $3a+7=0$ ✓

$\therefore a=\frac{-7}{3}$ ✓

(b) $a\neq-\frac{7}{3}$ ✓

$\therefore z = \dfrac{25}{3a+7}, y = \dfrac{23a+87}{3a+7}, x = \dfrac{5a+20}{3a+7}$ ✓✓✓ [6]

4. (a)



✓ intersects f(x) along the line y=x

✓ correct shape and location

(b) $f^{-1}(x)=-(x+1)^2+4=-x^2-2x+3$ ✓✓

Domain ¿$\{x\in R: x\geq -1\}$ ✓

(c) $f(g)=\sqrt{4-g}-1=\sqrt{x^2}-1=|x|-1$ ✓

(d) Condition for composition to exist: $4-g(x)\geq 0$ ✓

$4-4+x^2\geq 0$

$x^2\geq 0$

$\therefore x\in R$ ✓

$\therefore$ no changes needed for the domain of $g(x)$ ✓

Range ¿$\{y\in R: y\geq -1\}$ ✓ [10]

5. (a) $\overrightarrow{FD}=\begin{pmatrix}-3\\3\\2\end{pmatrix}$ ✓

$\therefore r=\begin{pmatrix}3\\0\\0\end{pmatrix}\pm\lambda\begin{pmatrix}-3\\3\\2\end{pmatrix}$ OR ¿$\begin{pmatrix}0\\3\\2\end{pmatrix}\pm\lambda\begin{pmatrix}-3\\3\\2\end{pmatrix}$ ✓

(b) Use $\overrightarrow{FD}$ as the normal vector of the plane ✓

$\therefore k=\begin{pmatrix}-3\\3\\2\end{pmatrix}\cdot\begin{pmatrix}3\\3\\2\end{pmatrix}=-9+9+4=4$ ✓

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} = 4 \rightarrow -3x + 3y + 2z = 4$$ ✓

5. (c) $\begin{pmatrix} 3-3\lambda \\ 3\lambda \\ 2\lambda \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} = 4 \rightarrow \therefore \lambda = \frac{13}{22}$ ✓✓

$$\therefore \begin{pmatrix} 3-\frac{39}{22} \\ \frac{39}{22} \\ \frac{26}{22} \end{pmatrix} = \frac{1}{22}\begin{pmatrix} 27 \\ 39 \\ 26 \end{pmatrix} = \frac{27}{22}i + \frac{39}{22}j + \frac{13}{11}k$$ ✓

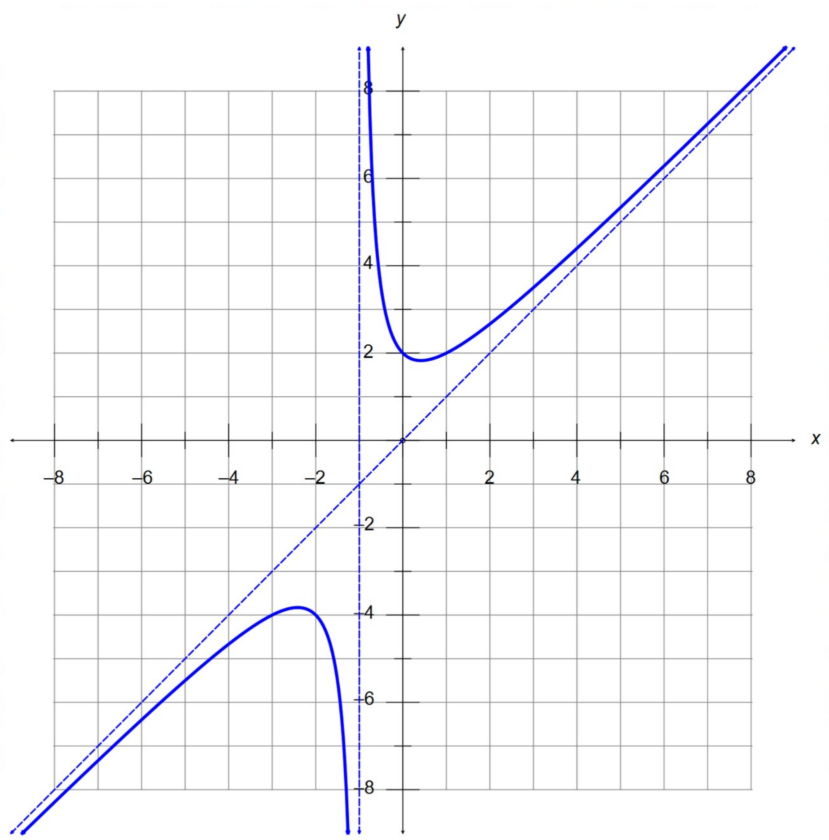
(d) Radius $\text{¿}\, p = \frac{1}{2}|FD| = \frac{1}{2}\left|\begin{matrix} -3 \\ 3 \\ 2 \end{matrix}\right| = \frac{\sqrt{22}}{2}$ ✓

Centre $\text{¿}\, \overrightarrow{OF} + \frac{1}{2}\overrightarrow{FD}$ ✓

$$\text{¿} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{pmatrix}$$ ✓

$\therefore a = b = \frac{3}{2}\, c = 1$ ✓ [12]

6. $y = x + \frac{2}{x+1}$ (after long division or otherwise) ✓

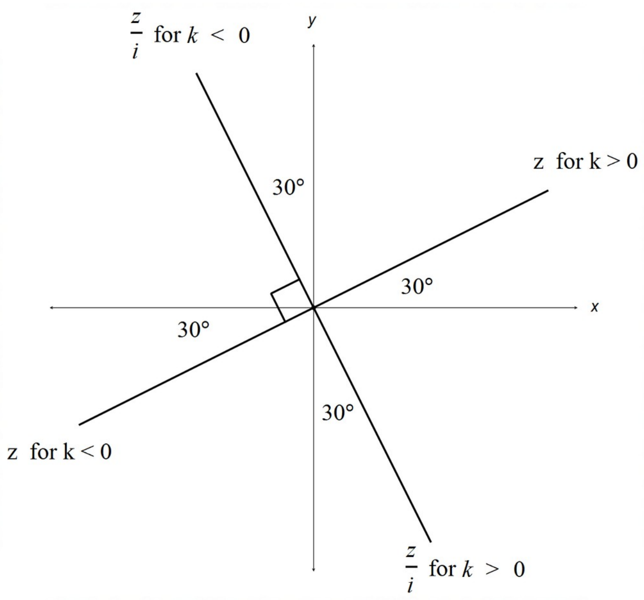


✓ Vertical asymptote x=-1
✓ Oblique asymptote y=x
✓ y intercept at (0,2)
✓ Shape and accuracy

[5]

7. (a) $|z|=\sqrt{3k^2+k^2}=2|k| \quad (k \neq 0)$ ✓

(b)



$arg(z)=\frac{\pi}{6}$ for $k>0$ ✓

$arg(z)=\frac{-5\pi}{6}$ for $k<0$ ✓

(c) $arg(z)=\frac{-\pi}{3}$ for $k>0$ ✓

$arg(z)=\frac{2\pi}{3}$ for $k<0$ ✓ [5]

Calculator-Assumed Solutions

8. For ABC to be collinear, AB // AC // BC (any two)

$\overrightarrow{AB}=\begin{pmatrix}4\\\alpha-1\\-2\end{pmatrix}\overrightarrow{AC}=\begin{pmatrix}8\\2\\\beta-1\end{pmatrix}$ ✓

$\therefore\begin{pmatrix}8\\2\\\beta-1\end{pmatrix}=k\begin{pmatrix}4\\\alpha-1\\-2\end{pmatrix}$ ✓

$\therefore k=2\rightarrow\alpha=2\beta=-3$ ✓✓ [4]

9. (a) $\omega=\frac{1}{8}\rightarrow T=\frac{2\pi}{\omega}=\frac{2\pi}{\frac{1}{8}}=16\pi=50.27\,seconds$ ✓✓

(b) $\frac{d}{dt}r(t)=\dot{r}(t)=\frac{5}{4}\cos\left(\frac{t}{8}\right)i+\frac{3}{4}\sin\left(\frac{t}{8}\right)j$ ✓✓

(c) speed ¿$|\dot{r}(t)|=\sqrt{\frac{25}{16}\cos^2\left(\frac{t}{8}\right)+\frac{9}{16}\sin^2\left(\frac{t}{8}\right)}$ ✓✓

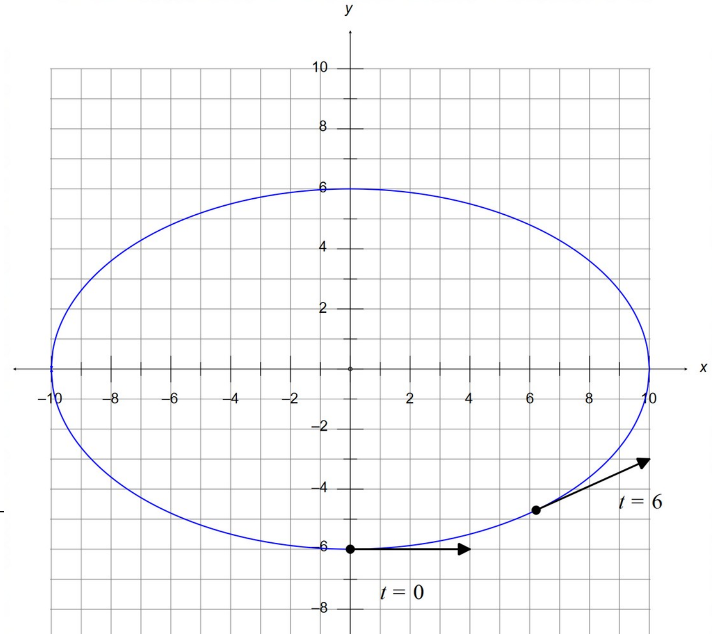
¿$\sqrt{\left[\frac{9}{16}\cos^2\left(\frac{t}{8}\right)+\frac{9}{16}\sin^2\left(\frac{t}{8}\right)\right]+\cos^2\left(\frac{t}{8}\right)}=\sqrt{\frac{9}{16}+\cos^2\left(\frac{t}{8}\right)}$

max speed at $\cos^2\left(\frac{t}{8}\right)=1$ or $\cos\left(\frac{t}{8}\right)=\pm1$

$\therefore max\,speed=\frac{5}{4}\,for\,t=8n\pi,n=0,1,2,\ldots$ ✓

$\therefore r(8n\pi)=\pm6j$ ✓✓

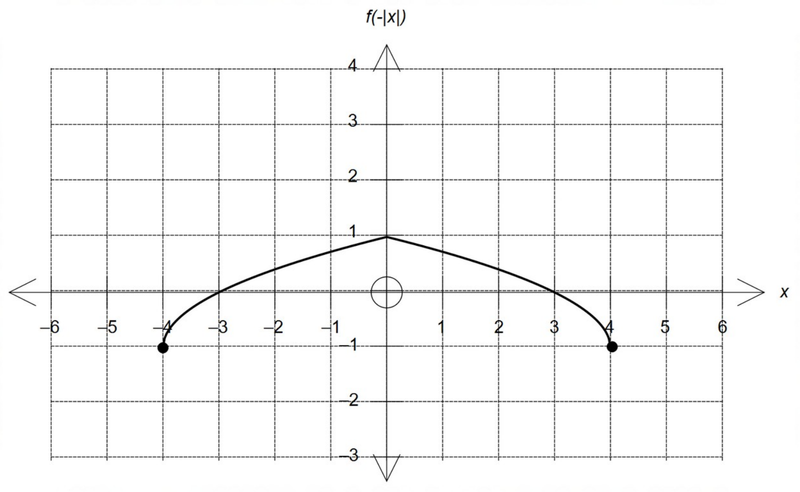
(d) Elliptical path with intercepts $(\pm10,0)$ and $(0,\pm6)$ ✓✓



$r(0)=-6j$	✓
$\dot{r}(0)=\frac{5}{4}i$	✓
$r(6)=6.82i-4.39j$	✓
$\dot{r}(6)=0.91i+0.51j$	✓

[15]

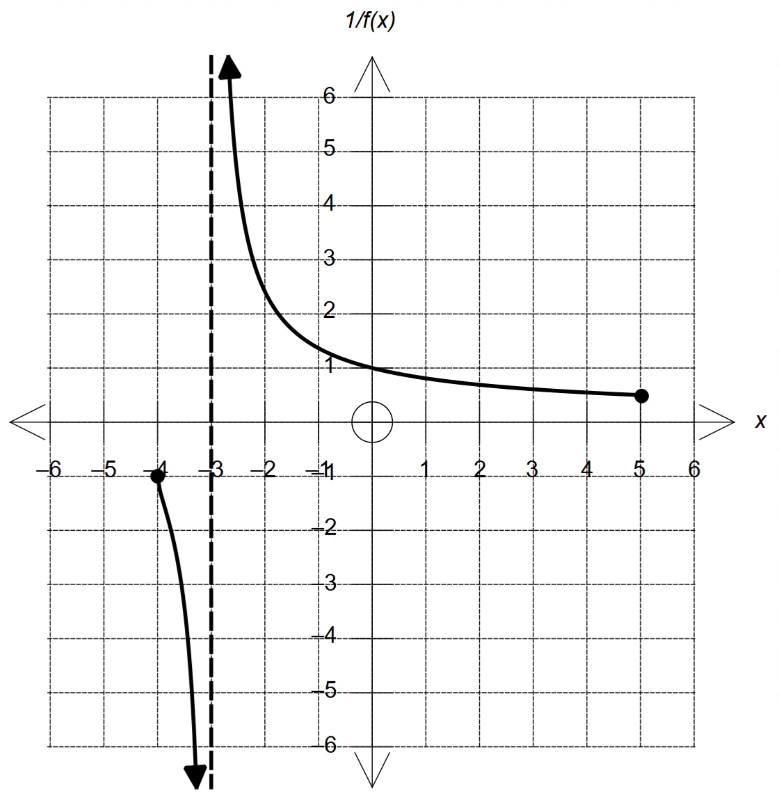
10. (a)



✓ Mirror image curve over the y axis

✓ Location and accuracy

(b)



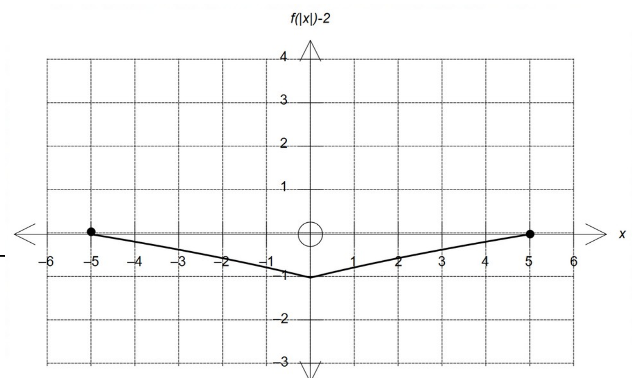
✓ Vertical asymptote x = -3

✓ Indicates y →|∞| as |x| → -3

✓ Correct domain.

✓ crosses f(x) when y=1 or -1

(c) from the graph below $x = \pm 5$ ✓✓ [8]



11. (a) (i) $\Im(z) \geq -2$ ✓

$\Im(z) \leq 2\Re(z) + 1$ ✓✓

(i) $2 \leq |z| \leq 4$ ✓

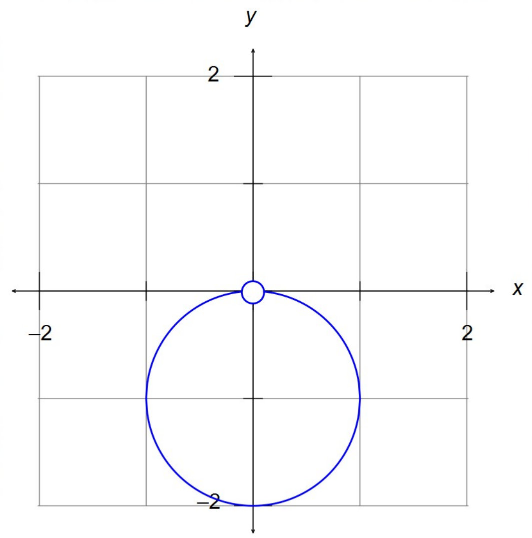
$\frac{-\pi}{2} \leq arg(z) \leq \frac{\pi}{10}$ ✓✓

(b) Let $z = x + yi$ and $z \neq 0$

$\frac{1}{x+yi} - \frac{1}{x-yi} = \frac{-2yi}{x^2+y^2} = i$ ✓

$-2y = x^2 + y^2$

$\therefore x^2 + (y+1)^2 = 1$ ✓



✓ circle centred at (0,-1) with radius = 1

✓ discontinuity at the origin

[10]

12. (a) $45\,min \rightarrow t = \frac{3}{4}$

$\begin{pmatrix} -6 \\ 12 \\ -0.6 \end{pmatrix} + \frac{3}{4}v = 0$ ✓

$\therefore v = \frac{4}{3}\begin{pmatrix} 6 \\ -12 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 8 \\ -16 \\ 0.8 \end{pmatrix} = 8i - 16j + 0.8k$ km/h ✓

12. (b) $20\,min \to t=\dfrac{1}{3}$

$\overrightarrow{OA}=O+\dfrac{1}{3}\begin{pmatrix}12\\-12\\0\end{pmatrix}=\begin{pmatrix}4\\-4\\0\end{pmatrix}$ ✓

$\overrightarrow{OS}=\begin{pmatrix}-6\\9\\-0.6\end{pmatrix}+\dfrac{1}{3}\begin{pmatrix}6\\-9\\0.6\end{pmatrix}=\begin{pmatrix}-4\\6\\-0.4\end{pmatrix}$ ✓

$\therefore\ \overrightarrow{AS}=\begin{pmatrix}-4\\6\\-0.4\end{pmatrix}-\begin{pmatrix}4\\-4\\0\end{pmatrix}=\begin{pmatrix}-8\\10\\-0.4\end{pmatrix}$ ✓

$|AS|=\begin{vmatrix}-8\\10\\-0.4\end{vmatrix}=12.81\,km$ ✓

(c) From $t_0=1500$:

$\overrightarrow{OT}=\begin{pmatrix}8\\-8\\0\end{pmatrix}+t\begin{pmatrix}x\\y\\z\end{pmatrix}$ where $\begin{vmatrix}x\\y\\z\end{vmatrix}=500$

Condition for collision:

$\overrightarrow{OT}=\begin{pmatrix}8\\-8\\0\end{pmatrix}+t\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-2\\3\\-0.2\end{pmatrix}+t\begin{pmatrix}6\\-9\\0.6\end{pmatrix}=\overrightarrow{OS}$ ✓

$\therefore\ x=6-\dfrac{10}{t}\ y=\dfrac{11}{t}-9\ z=0.6-\dfrac{0.2}{t}$ ✓

and $\left(6-\dfrac{10}{t}\right)^2+\left(\dfrac{11}{t}-9\right)^2+\left(0.6-\dfrac{0.2}{t}\right)^2=500^2$ ✓

CAS: $t=0.029112$ or 1.75 ~ 2 min (1502hrs) ✓

and $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-337.50\\368.85\\-6.27\end{pmatrix}$ km/h ✓ [11]

13. (a) $y=\dfrac{-1}{2}(x-2)^2+8=\dfrac{-x^2}{2}+2x+6$

$\therefore\ a=\dfrac{-1}{2}\ b=2\ c=6$ ✓✓

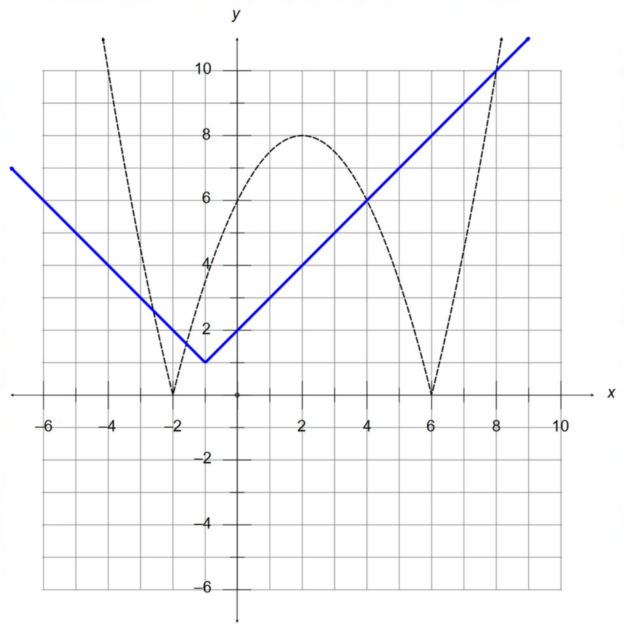
OR

$$y=\frac{1}{2}(x-2)^2-8=\frac{x^2}{2}-2x-6$$

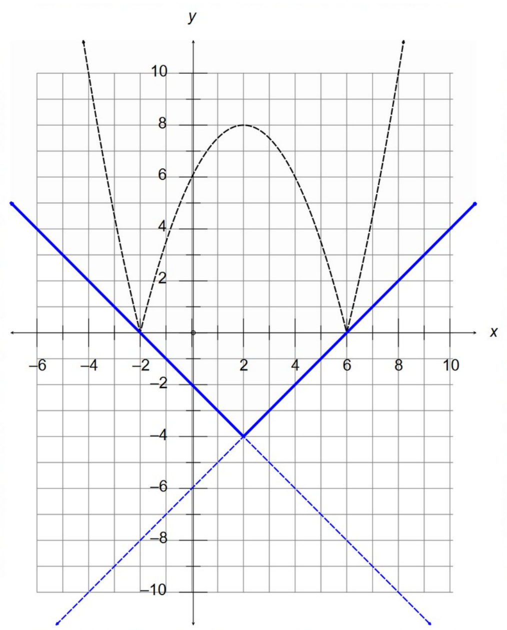
$$\therefore a=\frac{1}{2}\ b=-2\ c=-6$$ ✓

(BOTH solutions must be given)

13. (b) ✓✓



(c)



From the diagram above:

(i) $n \geq m - 6$ ✓✓

OR $n \geq -m - 2$ ✓

(both solutions must be given)

(ii) $n < m - 6$ ✓

AND $n < -m - 2$ ✓ [10]

14. (a) $(x-1)^2+(y-1)^2+z^2=2$ ✓

$\therefore$ centre at $(1,1,0)$ and radius ¿$\sqrt{2}$ ✓✓

(b) $(x-1)^2+(y-1)^2+1^2=2$

$\therefore (x-1)^2+(y-1)^2=1$

centre at $(1,1,1)$ and radius ¿1 ✓✓

(c) $d=\overrightarrow{CP}=\begin{pmatrix}1\\0\\1\end{pmatrix}$ ✓

$r=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$ ✓

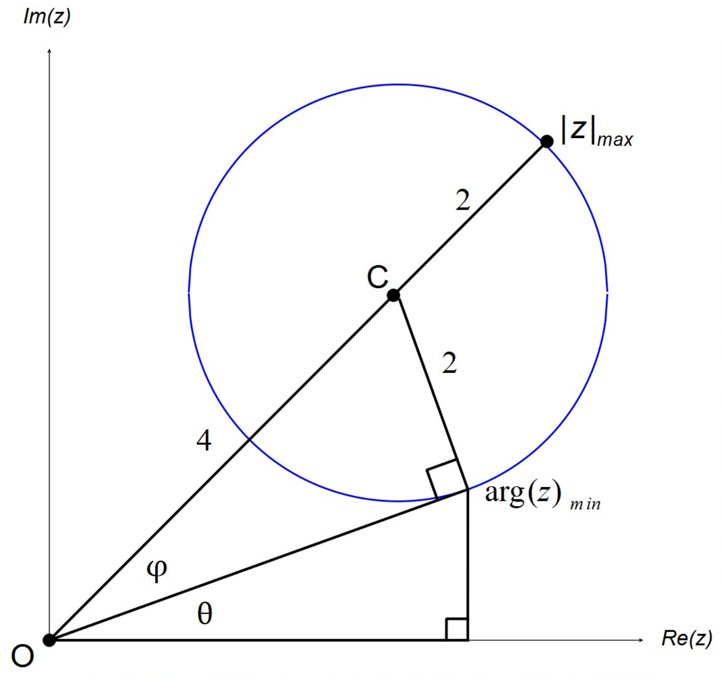
$y=1$ and $x-1=z$ ✓

(d) $n=d=\begin{pmatrix}1\\0\\1\end{pmatrix}$

$k=n\cdot\overrightarrow{DP}=\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}=2+0+1=3$ ✓

$\therefore r\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=3$ ✓ [10]

15. (a) See diagram below.



(i) $|z|_{max} = |\overrightarrow{OC}| + r = 2\sqrt{2} \times \sqrt{2} + 2 = 6$ units ✓✓

(ii) $\sin\phi = \frac{2}{4} \rightarrow \phi = \frac{\pi}{6}$ ✓

$\tan(\phi+\theta) = \frac{2\sqrt{2}}{2\sqrt{2}} = 1 \rightarrow \phi+\theta = \frac{\pi}{4}$ ✓

$\therefore \theta = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$ ✓

(b) (i) $\cos(n\theta) = \frac{z^n + z^{-n}}{2}$ ✓

$\sin(n\theta) = \frac{z^n - z^{-n}}{2i}$ ✓

(ii) $\cos^3\theta = \left(\frac{z^1 + z^{-1}}{2}\right)^3$ ✓

¿ $\frac{1}{8}\left[z^3 + 3z^2 z^{-1} + 3z^1 z^{-2} + z^{-3}\right]$ ✓

¿ $\frac{1}{4}\left[\frac{z^3 + z^{-3}}{2}\right] + \frac{3}{4}\left[\frac{z^1 + z^{-1}}{2}\right]$ ✓✓

¿ $\frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos\theta$ as required [11]

16. (a) $f(f)=\dfrac{f+2}{f-1}=\dfrac{\dfrac{x+2}{x-1}+2}{\dfrac{x+2}{x-1}-1}=x$ ✓✓

$\therefore f(x)$ is its own inverse ✓

(b) $f(x)=\dfrac{x+2}{x-1}=1+\dfrac{3}{x-1}$

Domain ¿ $\{x \in R : x \neq 1\}$ ✓

Range ¿ $\{y \in R : y \neq 1\}$ ✓

(c) Since $y=x$ and $y=-x$ are perpendicular, a reflection

of $y=x$ does not affect the symmetry of $g(x)$ over $y=-x$ ✓

therefore, the reflected function continues to be its own inverse ✓

(d) $f^{-1}(x)=f(x) \rightarrow h(x)=f^{-1}(f(h))$ ✓

¿ $\dfrac{f(h)+2}{f(h)-1}=\dfrac{\dfrac{1}{x}+2}{x}$ ✓

$\therefore h(x)=3x+1$ ✓ [10]

17. $P\left(\dfrac{1}{w}\right)=\left(\dfrac{1}{w}\right)^{8n}-\left(\dfrac{1}{w}\right)^{4n}+1=\dfrac{1}{w^{8n}}-\dfrac{1}{w^{4n}}+1$ ✓

¿ $\dfrac{1}{w^{8n}}\left(1-w^{4n}+w^{8n}\right)=\dfrac{1}{w^{8n}}\times P(w)$ ✓

$\dfrac{1}{w^{8n}}\times 0=0 \quad \therefore \dfrac{1}{w}$ is also a root of $P(w)$ ✓

$P(iw)=(iw)^{8n}-(iw)^{4n}+1=i^{8n}w^{8n}-i^{4n}w^{4n}+1$ ✓

¿ $1\times w^{8n}-1\times w^{4n}+1=P(w)=0$ ✓

$\therefore iw$ is also a root of $P(w)$ ✓ [6]

18. $\cos\theta=\dfrac{x}{2}$ ✓

$y=\cos 2\theta=2\cos^2\theta-1$ ✓

$\therefore y=2\left(\dfrac{x}{2}\right)^2-1=\dfrac{x^2}{2}-1$ ✓

$x = 2\cos\theta, \therefore -2 \le x \leftarrow 2$ Domain ¿ $\{x \in R : -2 \le x \le 2\}$ ✓

$y = \cos(2\theta)$ Range ¿ $\{y \in R : -1 \le y \le 1\}$ ✓ [5]