Question 22 **(5 marks)**

The manufacturers of a new conical ice cream, the Piccoletto, are looking to save costs on packaging. The volume of the cone is to be $188cm^3$, and the manufacturers want to minimise the surface area.

The formulae for the volume and surface area of a cone of radius r and perpendicular height h are

$V = \frac{\pi r^2 h}{3}$ and $SA = \pi r^2 + \pi r\sqrt{r^2 + h^2}$ respectively.

a) Use this information to write a formula for the surface area of the cone in terms of its radius only. (2 marks)

$188 = \frac{\pi r^2 h}{3}$ $\qquad SA = \pi r^2 + \pi r\sqrt{r^2 + \left(\frac{564}{\pi r^2}\right)^2}$

$h = \frac{188 \times 3}{\pi r^2}$

$h = \frac{564}{\pi r^2}$

b) Determine the dimensions of the cone that will minimise the surface area. State this surface area. (3 marks)

min pt on graph $(3.99, 199.96)$

$\therefore$ radius $= 3.99cm$

height $= 11.28cm$

$SA = 199.96\ cm^2$.

---End Of Examination---

SCOTCH COLLEGE

Scotch College
Semester One Practice Examination 2, 2016

Question/Answer Booklet

Year 12 MATHEMATICS METHODS

Section One:
Calculator free

Teacher: ____J. Fletcher
____P. Newman
____S. Reyhani

Name: Solutions

Time allowed for this section
Reading time before commencing work: 5 minutes
Working time for this section: 50 minutes

Material required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters
Special items: nil

Important note to candidates
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Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available
Section One: Calculator-free	8	8	50	52
Section Two Calculator-assumed	14	14	100	91
				143

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3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

d) The ideal serving temperature for a cup of black coffee is 70°C. For how many minutes after the coffee is made should Barry wait before serving it? (1 mark)

$t = 4.05$ mins

e) One of Barry's customers had let their coffee get cold, and asked him to re-heat it.

The re-heating process is such that the rate of change of the temperature (°C/min) is proportional to the temperature, with the constant of proportionality being 0.686.

(i) If the coffee was at a temperature of 25°C when Barry began to re-heat it, write an expression for the temperature T (in °C) t minutes after the reheating process commences. (2 marks)

$$\frac{dT}{dt} = 0.686T$$

$$T = T_0 e^{0.686t} \qquad T_0 = 25$$

$$T = 25e^{0.686t}$$

(ii) Determine how long the reheating process would take to make the coffee reach ideal serving temperature of 70°C once more. (1 mark)

$$70 = 25e^{0.686t}$$

$t = 1.5$ mins

Question 21

(8 marks)

At his part-time job working in a cafe, mathematician Barry Easter noticed that, as cups of black coffee cooled, the temperature (T°C) t minutes after they had been made follows the exponential function

$$T = 75\,e^{-0.1t} + 20$$

a) Describe the transformations of the function $T = e^t$ required to produce this function. (2 marks)

- Dilation parallel to y-axis scale factor 75, translated up 20 units
- Dilation parallel to x-axis s.f. 10 and reflected through y-axis

b) What is the initial temperature of the coffee when it is made? (1 mark)

95°C

c) If left to cool, eventually the temperature of the coffee will be the same as the temperature of the cafe. What is this temperature? (1 mark)

20°C

Question 1. [4,4 = 8 marks]

a) Differentiate the following;

i) $y = 4x + \frac{3}{x^2}$

$y' = 4 - \frac{6}{x^3}$

ii) $y = (x+2)^2(4x+6)$

$y' = 2(x+2)(4x+6) + 4(x+2)^2$

b) Integrate the following;

i) $\int 24x^2(4x^3+8)^3 . dx$

$= \frac{24x^2(4x^3+8)^4}{4 \times 12x^2} + c$

$= \frac{(4x^3+8)^4}{2} + c$

ii) $\int (x^2 + \frac{3x^2}{x^5}) . dx$

$= \int x^2 + 3x^{-3}\ dx$

$= \frac{1}{3}x^3 - \frac{3}{2x^2} + c$

Question 2. [3 marks]

Determine the equation of a tangent to the curve $y = 3x^3 + 4x$ at the point $(3,93)$

$y' = 9x^2 + 4$

$x = 3 \quad y' = 9(3)^2 + 4$
$= 85$

tangent $y = 85x + c$
$93 = 85(3) + c$
$93 = 255 + c$
$c = -162$

$\therefore y = 85x - 162$

Question 20 **(6 marks)**

$t = 0 \quad v = 0 \quad x = 0$

A particle, originally at rest at the origin, moves in such a way that its acceleration a m/s^2, t seconds later is given by the formula $a = 12t - 18$.

a) State expressions for the velocity and displacement of the particle in terms of t. (2 marks)

$v = 6t^2 - 18t + c$
$c = 0$
$v = 6t^2 - 18t$

$x = 2t^3 - 9t^2 + c$
$c = 0$
$x = 2t^3 - 9t^2$

b) State the time and position when the particle is next at rest. (2 marks)

$0 = 6t^2 - 18t$
$t = 3$

$x = 2(3)^3 - 9(3)^2$
$= -27$

$\therefore$ next at rest at 3 seconds and is 27m to left of origin.

c) How far does the particle travel in total in the first 10 seconds? (2 marks)

$\int_0^{10} |6t^2 - 18t| \, dt = 1154\text{m}$

Question 19. [2,2 =4 marks]

The marginal costs involved in printing x copies of a particular book follow the rule $C'(x) = \frac{2.5}{\sqrt[3]{x}} + 3$.

a) Write an expression involving integration for the extra cost incurred by producing 1000 copies rather than 500. Use this expression to determine the average cost per book of producing the second 500 books. (2 marks)

$$\text{Extra Cost} = \int_{500}^{1000} \frac{2.5}{\sqrt[3]{x}} + 3\,dx$$

$$\text{Average Cost} = \frac{\int_{500}^{1000} \frac{2.5}{\sqrt[3]{x}} + 3\,dx}{500}$$

$= \$3.28$ per book

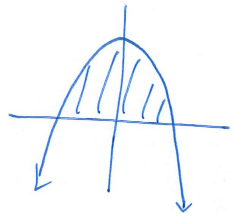
b) Use the marginal rate to estimate the cost of printing one more book at the stage in the printing when 1000 copies have been produced. Compare this cost with the average cost of producing the second 500 copies of the book. (2 marks)

$$C'(1000) = \frac{2.5}{\sqrt[3]{1000}} + 3 = \$3.25$$

The cost of the next book is 3cents cheaper than the average cost of the seond 500 books

Question 3. [4 marks]

Determine the exact area enclosed by the graph of $y = -2x^2 + 6$ and the x axis.



$0 = -2x^2 + 6$

$x^2 = 3$

$x = \pm\sqrt{3}$

$$\text{Area} = \int_{-\sqrt{3}}^{\sqrt{3}} -2x^2 + 6\,dx$$

$$= -\frac{2}{3}x^3 + 6x \Big]_{-\sqrt{3}}^{\sqrt{3}}$$

$$= \left[-\frac{2}{3}(3\sqrt{3}) + 6\sqrt{3}\right] - \left[-\frac{2}{3}(-3\sqrt{3}) - 6\sqrt{3}\right)$$

$$= -2\sqrt{3} + 6\sqrt{3} - 2\sqrt{3} + 6\sqrt{3}$$

$$= 8\sqrt{3} \text{ units}^2$$

Question 4. [12 marks]

(a) Find the derivative of each of these functions with respect to x, simplifying your answers where possible.

(i) $y = \ln\left(\left(x^2-3\right)^3\right)$ (2)

$$\frac{dy}{dx} = \frac{3(x^2-3)^2(2x)}{(x^2-3)^3} = \frac{6x}{x^2-3}$$

(ii) $y = x^2 e^{2x}$ (2)

$$y' = 2xe^{2x} + 2e^{2x}x^2$$
$$= 2xe^{2x}(1+x)$$

(iii) $y = \sin^3\left(\frac{3\pi x}{4}\right)$ (2)

$$y' = \frac{3\pi}{4} \times 3\sin^2\left(\frac{3\pi x}{4}\right)\cos\left(\frac{3\pi x}{4}\right)$$
$$= \frac{9\pi}{4}\sin^2\left(\frac{3\pi x}{4}\right)\cos\left(\frac{3\pi x}{4}\right)$$

(iv) $y = \frac{1}{\sqrt{\tan x}}$ $\sqrt{\tan x}$. $= (\tan x)^{1/2} = \left(\frac{\sin x}{\cos x}\right)^{1/2}$ (3)

$$y' = \frac{1}{2}\left(\frac{\sin x}{\cos x}\right)^{-1/2} \times \cos x\cos x + \sin x \sin x$$
$$= \frac{\cos^2 x + \sin^2 x}{2\sqrt{\tan x}}$$

The manufacturers decided each flower was too small and did not weigh enough and were therefore blown around by the wind. It was decided to double all the measurements of the flowers and to make them 1.5cm thick.

c) Determine the volume of plastic used to produce one of the new flowers.

S.A originally $9 \times 25 = 225\text{cm}^2$

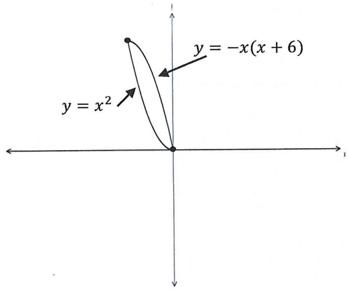
double all measurements $= 225 \times 4$
$= 900\text{cm}^2$

$V = 900 \times 1.5$
$= 1350\text{ cm}^3$

1350×4
$= 5400\text{cm}^3$

Question 18. [2,5,4 = 11 marks]

A manufacturing company has been hired to make flowers from coloured plastic sheeting. The flowers have **four** identical petals each. One such petal is shown below.



a) Show that the two equations that make up the petal intersect at $(0,0)$ and $(-3,9)$.

$x^2 = -x(x+6)$

$x^2 = -x^2 - 6x$

$2x^2 + 6x = 0$

$2x(x+3) = 0$

$x = 0 \quad x = -3$

$\therefore (0,0)$

$(-3,9)$

Note: Each unit of the graph represent **5cm** in real life.

b) Determine the amount of plastic used to create 50 entire flowers. Give your answer to the nearest mm^2.

$\int_{-3}^{0} -x^2 - 6x - x^2 \, dx = 9 \text{units}^2$ for 1.

$9 \times 50 \times 25 \times 100 \times 4$

$= 4{,}500{,}000 \text{ mm}^2$

(b) Evaluate the following integral:

$\int \cos^3 x \sin x \; dx$ (3)

$= -\frac{\cos^4 x}{4} + c$

Question 5. [6 marks]

(a) Given $y = \frac{1}{4}\sin(2t) + \frac{t}{2} + \sin(t)$, $0 \le t \le 2\pi$, determine $\frac{dy}{dt}$ (3)

$\frac{dy}{dt} = \frac{1}{2}\cos(2t) + \frac{1}{2} + \cos t$

(b) If $x = \sin 3t$ and $y = 3e^{2t}$, find $\frac{dy}{dx}$ when $t = 0$. (3)

$\frac{dx}{dt} = 3\cos 3t \qquad \frac{dy}{dt} = 6e^{2t}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

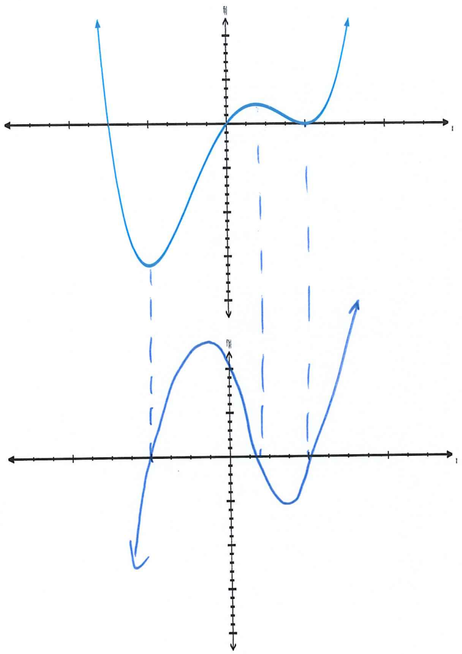
$= \frac{6e^{2t}}{3\cos 3t}$

$t=0 \qquad = \frac{6}{3}$

$= 2$

Question 6. [4 marks]

Given the graph of $f(x)$ draw the graph of it's derivative on the axis given.



Question 17. [4,5 = 9 marks]

An internet marketing company has found that important news spreads through a university population according to the formula:

$\frac{dN}{dt} = k(P - N)$ where N is the number of people who know the important news t hours after the important news is announced. P and k are constants where P is the total population of the university and k is the growth factor.

(a) Let A_0 be the initial number of people not knowing the news. Use integration to show that $N = P - A_0 e^{-kt}$ is the solution of the differential equation. **[4]**

$\int \frac{1}{P-N}\, dN = \int k\, dt$

$-\ln(P-N) = kt + c$

$P - N = e^{-kt-c}$

$N = P - e^{-kt-c}$ $\quad t=0 \quad N=0 \quad \therefore e^{-c} = A_0 = P$

$\therefore N = P - A_0 e^{-kt}$

(b) Challenger University has a student population of 18 000 students. News that boy band, No Direction, would be visiting the campus was announced at noon. Three hours later, 60% of the student population knew about the news. At what time, to the nearest hour, will 95% of the population know the news? **[5]**

$N = 18\,000 - 18000e^{-3k}$

$10800 = 18000 - 18000e^{-3t}$

$k = 0.305$

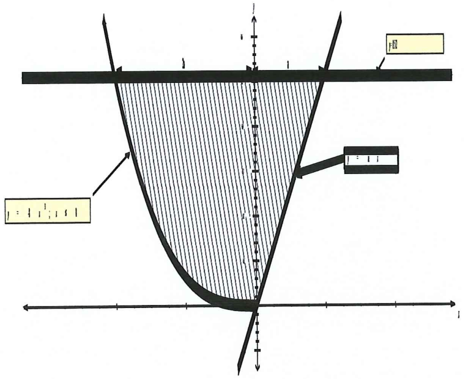
$0.05 = e^{-0.305t}$

$t = 9.808$

$\therefore$ 10pm to nearest hr.

Question 16. [3,4 = 7 marks]

The cross section of the blade of a paddle is shown below:



The blade is determined by the curved edge $y = -8x^3$ and the straight edges, $y = mx$ and $y = 0 \cdot 512$.

a) Determine the value of m and a.

$0.512 = -8x^3$

$-0.4 = x$

$\therefore 2a = 0.4$

$a = 0.2$

$m = \frac{0.512}{0.2}$

$= 2.56$

b) Using calculus, determine the area of the shaded region.

$\int_{-0.4}^{0} (0.512 + 8x^3)\,dx + \frac{1}{2} \times 0.512 \times 0.2$

$= 0.1536 + 0.0512$

$= 0.2048$ units^2

Question 7. [3,5 = 8 marks]

(a) Solve the following equations for x.

(i) $\log_2 x = -3$ (1)

$2^{-3} = x$

$x = 1/8$

(ii) $2(3^{1-x}) = \cancel{54}\,50$ (2)

$3^{1-x} = 25$

$(1-x)\log 3 = \log 25$

$1-x = \frac{\log 25}{\log 3} \rightarrow x = 1 - \frac{\log 25}{\log 3}$

(b) For the equation $2(4^x) = 5(2^x) - 2$

(i) Let $y = 2^x$, and show that $(y-2)(2y-1) = 0$ (3)

$2(2^x)^2 + 5(2^x) + 2 = 0$

$2y^2 + 5y + 2 = 0$

$(y-2)(2y-1) = 0$

(ii) Solve the equation for x. (2)

$y = 2 \qquad y = \frac{1}{2}$

$2^x = 2 \qquad 2^y = \frac{1}{2}$

$x = 1 \qquad y = -1$

Question 8. [2,2,3 = 7 marks]

Consider the general form of a parabolic graph $y = ax^2 + bx + c$.

a) Using calculus, show that the turning point of any parabola is where the x value is $\frac{-b}{2a}$.

$y' = 2ax + b$

$0 = 2ax + b$

$-b = 2ax$

$x = \frac{-b}{2a}$

b) Determine the y coordinate of this turning point in terms of a, b and c.

$y = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c$

$= \frac{b^2}{4a} - \frac{b^2}{2a} + c$

c) If it is known that the turning point of a particular parabola is also the y intercept and that the 'a' value is $\frac{1}{2}$, determine all possible values of b.

$c = \frac{b^2}{4(\frac{1}{2})} - \frac{b^2}{2(\frac{1}{2})} + c$

$0 = \frac{b^2}{2} - b^2$

$0 = -\frac{1}{2}b^2$

$b = 0$

--End Of Section 1--

Question 15. [2,2 = 4 marks]

Find the following:

(a) $\frac{d}{dt}\int_1^{t^3}\left[\frac{\sqrt{x^2+1}}{x}\right]dx$ $= \frac{\sqrt{t^6+1}}{t^3} \times 3t^2$ [2]

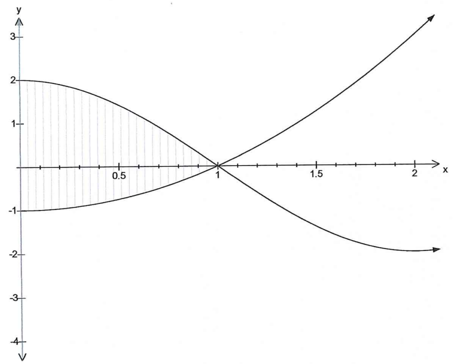
(b) $\int_1^{t^3}\frac{d}{dx}\left(\frac{\sqrt{x^2+1}}{x}\right)dx$ [2]

$= \left[\frac{\sqrt{x^2+1}}{x}\right]_1^{t^3}$

$= \frac{\sqrt{t^6+1}}{t^3} - \sqrt{2}$

Question 14. [4 marks]

The shaded region in the diagram below is bounded by the y-axis, and the curves with equations $y = x^2 - 1$ and $y = 2\cos\left(\frac{\pi x}{2}\right)$. Find the EXACT value of the area of the shaded region.



$$\text{Area} = \int_0^1 \left(2\cos\left(\frac{\pi x}{2}\right) - (x^2-1)\right) dx$$

$$= \left[\frac{4}{\pi}\sin\left(\frac{\pi x}{2}\right) - \frac{x^3}{3} + x\right]_0^1$$

$$= \left(\frac{4}{\pi} + \frac{2}{3}\right) \text{ units}^2$$

SCOTCH COLLEGE

Scotch College
Semester One Practice Examination 2, 2016
Question/Answer Booklet

Year 12 MATHEMATICS METHODS

Section Two:
Calculator-assumed

Teacher: ____ J. Fletcher ____ P. Newman ____ S. Reyhani

Name: Solutions.

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Question 13. [4,2 = 6 marks]

(a) If $y=\sin\left(x^2\right)$, show that $\frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}+4x^2y=0$ [4]

$\frac{dy}{dx} = 2x\cdot\cos(x^2)$

$\frac{d^2y}{dx^2} = 2\cos(x^2) - 2x\sin(x^2)\,2x$

$= 2\cos(x^2) - 4x^2\sin(x^2)$

$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = 2\cos(x^2) - 4x^2\sin(x^2) - \frac{1}{x}(2x\cos(x^2)) + 4x^2\sin(x^2)$

$= 2\cos(x^2) - 4x^2\sin(x^2) - 2\cos(x^2) + 4x^2\sin(x^2)$

$= 0$

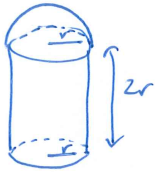
(b) Find the derivatives of $f(x)=\int_{\pi}^{x^2}\frac{1}{\sin^3 t}\,dt$ [2]

$f'(x) = \frac{1}{\sin^3(x^2)} \times 2x.$

Question 12. [3,5 = 8 marks]

A dome shaped tent is being erected which is in the shape of a cylinder with a hemisphere sitting on the top. The radius of the cylinder (r) and the radius of the hemisphere are equal. The height of the cylinder is twice as long as the radius.

a) Show that the volume (v) of air inside the dome is $v = \frac{8\pi r^3}{3}$



$V = \pi r^2(2r) + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$= 2\pi r^3 + \frac{2}{3}\pi r^3$

$= \frac{8}{3}\pi r^3$

b) Using differentiation determine the change in volume of the tent if the radius was increased by 10cm when the volume was $576\pi m^3$.

$\frac{\delta V}{\delta r} = 8\pi r^2$

$\delta r = 0.1$

$576\pi = \frac{8}{3}\pi r^3$

$r = 6$

$\delta V = 8\pi r^2 \, \delta r$

$\delta V = 8\pi(6)^2(0.1)$

$= 90.478 m^3.$

Question 9. [1,3 = 4 marks]

The decibel scale can be used to describe how loud a sound is.

If P_1 is the intensity of the sound, then its loudness L, in decibels (dB), is given by the formula:

$$L = 10\log_{10}\frac{P_1}{P_0}$$

where P_0 represents the intensity of the quietest sound that is audible to the human ear.

(a) If a sound has 100 times the intensity of P_0, so that $P_1 = 100P_0$, show that it would register as 20dB on this scale. (1)

$L = 10\log_{10}\left(\frac{100P_0}{P_0}\right)$

$= 10\log_{10}(100)$

$= 10 \times 2$

$= 20$

(b) The sound of a lawnmower is measured as 85dB, and a conversation as 60dB.

How many times more intense is the lawnmower than the conversation? (3)

$8.5 = \log_{10}\left(\frac{P_1}{P_0}\right)$ $\quad 6.0 = \log_{10}\left(\frac{P_2}{P_0}\right)$

$\frac{P_1}{P_0} = 10^{8.5}$ $\quad \frac{P_2}{P_0} = 10^6$

$\frac{P_1}{P_2} = 10^{2.5} \approx 316$ times.

Question 10. [1,2,2,2 = 7 marks]

A common model of population growth is given by the formula

$$P(t)=\frac{KP_0e^{rt}}{K+P_0\left(e^{rt}-1\right)}$$

In this equation, $P(t)$ is the population at time t, P_0 is the initial population, K is the maximum population which the environment can sustain (the population ceiling), and r is the rate at which the population would grow in the absence of a population ceiling.

Suppose that an ant hill can sustain a population of 50 000 ants. Initially the population is 20 000, with the growth rate $r=0.05$, and time measured in weeks.

(a) What is the population after 10 weeks? (1)

$P(10)\approx 26181$

(b) By what percentage does the population increase in the first week? (2)

$P(1) = 20603$

Approximately 3% increase.

(c) After how many weeks will the population reach 30 000? (2)

$$30\,000 = \frac{50\,000 \times 20\,000e^{0.05t}}{50\,000 + 20\,000(e^{.05t}-1)}$$

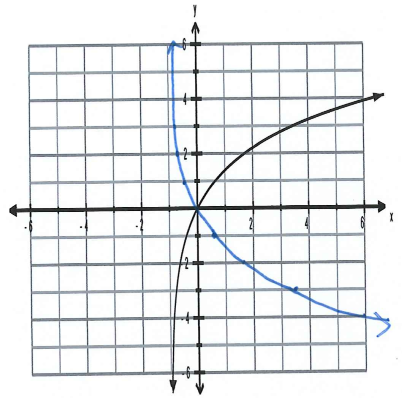
$t = 16.21$

(d) How does the size of the population change in the long term? (2)

Increases towards value of 50 000

Question 11. [2,2,4 = 8 marks]

The following diagram shows the graph of $f(x)=2\ln(x+a)$.



(a) Determine the value of a. (2)

$a = 1$

(b) Sketch the graph of $-f(x)$ on the diagram. (2)

(c) For what exact value of x is $f(x)=2$? (4)

$2 = 2\ln(x+1)$

$1 = \ln(x+1)$

$e^1 = x+1$

$x = e-1$