SOLUTIONS PART A – MATHEMATICAL INDUCTION INVESTIGATION

1. Required to prove $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all positive integers n..

Step 1 Verify the statement is true when n = 1

L.H.S.= 1

R.H.S. = $\dfrac{1.2.3}{6}$ = 1

⇒ Statement is true for n = 1

Step 2 Assume the statement is true for n = k

That is, $1^2 + 2^2 + \ldots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$

Step 3 Prove statement true for n = k + 1

That is, prove $1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \dfrac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

LHS $= 1^2 + 2^2 + \ldots + k^2 + (k+1)^2$

$= \dfrac{k(k+1)(2k+1)}{6} + (k+1)^2$ from Step 2

$= \dfrac{k(k+1)(2k+1) + 6(k+1)^2}{6}$

$= \dfrac{(k+1)((k(2k+1) + 6(k+1))}{6}$

$= \dfrac{(k+1)(2k^2 + k + 6k + 6)}{6}$

$= \dfrac{(k+1)(2k^2 + 7k + 6)}{6}$

$= \dfrac{(k+1)(2k+3)(k+2)}{6}$

$= \dfrac{(k+1)(2k+3)(k+2)}{6}$

$= \dfrac{(k+1)((k+1)+1)(2(k+1)+1))}{6}$

=RHS

⇒ The statement is true for n = k + 1 if it is true for n = k.

Step 4 As the statement is true for n = 1 it must be true for n =2.

As the statement is true for n = 2 it must be true for n =3 and so on.

Hence, $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ is true for all positive integers n.

2. Required to prove $1^3 + 2^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for all positive integers n.

Step 1 Verify the statement is true when n = 1

L.H.S.= 1

R.H.S. = $\frac{1.2^2}{4}$ = 1

$\Rightarrow$ Statement is true for n = 1

Step 2 Assume the statement is true for n = k

That is, $1^3 + 2^3 + ... + k^3 = \frac{k^2(k+1)^2}{4}$

Step 3 Prove statement true for n = k + 1

That is, prove $1^3 + 2^3 + ... + k^3 + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$

LHS $= 1^3 + 2^3 + ... + k^3 + (k+1)^3$

$= \frac{k^2(k+1)^2}{4} + (k+1)^3$ from Step 2

$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$

$= \frac{(k+1)^2(k^2 + 4(k+1))}{4}$

$= \frac{(k+1)^2(k^2 + 4k + 4)}{4}$

$= \frac{(k+1)^2(k+2)^2}{4}$

$= \frac{(k+1)^2((k+1)+1)^2}{4}$

=RHS

$\Rightarrow$ The statement is true for n = k + 1 if it is true for n = k.

Step 4 As the statement is true for n = 1 it must be true for n =2.
As the statement is true for n = 2 it must be true for n =3 and so on.

Hence, $1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$ is true for all positive integers n.

3. Required to prove $1 \times 2 + 2 \times 3 + 3 \times 4 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for all positive integers n.

Step 1 Verify the statement is true when n = 1
L.H.S.= 2

R.H.S. = $\frac{1.2.3}{3}$ = 2

⇒ Statement is true for n = 1

Step 2 Assume the statement is true for n = k

That is, $1\times 2+2\times 3+3\times 4+...+k(k+1) = \frac{k(k+1)(k+2)}{3}$

Step 3 Prove statement true for n = k + 1

That is, prove $1\times 2+2\times 3+3\times 4+...+k(k+1)+(k+1)((k+1)+1)=$ $\frac{(k+1)((k+1)+1)((k+1)+2)}{3}$

LHS $= 1\times 2+2\times 3+3\times 4+...+k(k+1)+(k+1)((k+1)+1)$

$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$ from Step 2

$= \frac{k(k+1)(k+2)+3(k+1)(k+2)}{3}$

$= \frac{(k+1)(k+2)(k+3)}{3}$

$= \frac{(k+1)((k+1)+1)((k+1)+2)}{3}$

=RHS

⇒ The statement is true for n = k + 1 if it is true for n = k.

Step 4 As the statement is true for n = 1 it must be true for n =2.
As the statement is true for n = 2 it must be true for n =3 and so on.

Hence, $1\times 2+2\times 3+3\times 4+...+n(n+1)=\frac{n(n+1)(n+2)}{3}$ is true for all positive integers n.

4. Required to prove $\cos(n\pi + x) = (-1)^n \cos x$ for all positive integers n.

Step 1 Verify the statement is true when n = 1

L.H.S $= \cos(\pi + x)$

$= \cos\pi\cos x - \sin\pi\sin x$

$= -1.\cos x - 0.\sin x$

$= -\cos x$

R.H.S. $= (-1)^1 \cos x$

$= -\cos x$

⇒ Statement is true for n = 1

Step 2 Assume the statement is true for n = k

That is, $\cos(k\pi + x) = (-1)^k \cos x$

Step 3 Prove statement true for n = k + 1

That is, prove $\cos((k+1)\pi + x) = (-1)^{k+1} \cos x$

LHS $= \cos((k+1)\pi + x)$

$= \cos(k\pi + \pi + x)$

$= \cos((k\pi + x) + \pi)$

$= \cos(k\pi + x)\cos\pi - \sin(kx + \pi)\sin\pi$

$= (-1)^k \cos x.(-1) - \sin(kx + \pi).0$ from Step 2

$= (-1)^{k+1} \cos x$

=RHS

⇒ The statement is true for n = k + 1 if it is true for n = k.

Step 4 As the statement is true for n = 1 it must be true for n =2.

As the statement is true for n = 2 it must be true for n =3 and so on.

Hence, $\cos(n\pi + x) = (-1)^n \cos x$ is true for all positive integers n.

5. Required to prove $n(n^2 + 5)$ is divisible by 6 for all positive integers n.

Step 1 Verify the statement is true when n = 1

$1(1+5)$ = 6

Divisible by 6

⇒ Statement is true for n = 1

Step 2 Assume the statement is true for n = k

That is, $k(k^2 + 5)$ is divisible by 6

Step 3 Prove statement true for n = k + 1

That is, prove $(k+1)((k+1)^2+5)$ is divisible by 6

$(k+1)((k+1)^2+5)$

$= (k+1)(k^2+2k+1+5)$

$=(k+1)(k^2+2k+6)$

$=k^3+2k^2+6k+k^2+2k+6$

$=k^3+3k^2+8k+6$

$=k^3+5k+3k^2+3k+6$

$=k(k^2+5)+3(k^2+k+2)$

$k(k^2+5)$ is divisible by 6 from Step 2

$3(k^2+k+2)$ is divisible by 6 if (k^2+k+2) is even

If k is an odd number k^2 is odd so (k^2+k+2) = odd + odd +2 = even

If k is an even number k^2 is even so (k^2+k+2) = even + even +2 = even

So (k^2+k+2) is even for all positive integers k so $3(k^2+k+2)$ is divisible by 6

$k(k^2+5)+3(k^2+k+2)$ is divisible by 6

$\Rightarrow$ The statement is true for n = k + 1 if it is true for n = k.

Step 4 As the statement is true for n = 1 it must be true for n =2.

As the statement is true for n = 2 it must be true for n =3 and so on.

Hence, $n(n^2+5)$ is divisible by 6 is true for all positive integers n.

6. Required to prove that the nth derivative of $y = x\ln x, x>0$ is $\dfrac{d^n y}{dx^n} = \dfrac{(-1)^n(n-2)!}{x^{n-1}}$ for all positive integers n≥ 2.

Step 1 Verify the statement is true when n = 2

L.H.S. $y = x\ln x$

$$\frac{dy}{dx} = 1.\ln x + \frac{1}{x}.x$$

$$= \ln x + 1$$

$$\frac{d^2x}{dy^2} = \frac{1}{x}$$

R.H.S. = $\dfrac{(-1)^2 0!}{x} = \dfrac{1}{x}$ =LHS

$\Rightarrow$ Statement is true for n = 2

Step 2 Assume the statement is true for n = k

That is, $\frac{d^k y}{dx^k} = \frac{(-1)^k (k-2)!}{x^{k-1}}$

Step 3 Prove statement true for n = k + 1

That is, prove $\frac{d^{k+1} y}{dx^{k+1}} = \frac{(-1)^{k+1}((k+1)-2)!}{x^{(k+1)-1}}$

LHS $= \frac{d^{k+1}y}{dx^{k+1}}$

$= \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$

$= \frac{d}{dx}\left(\frac{(-1)^k (k-2)!}{x^{k-1}}\right)$ from Step 2

$= \frac{d}{dx}\left((-1)^k (k-2)! x^{-(k-1)}\right)$

$= \left((-1)^k (k-2)!(-(k-1))x^{-(k-1)-1}\right)$

$= \left((-1)^k (k-2)!(-(k-1))x^{-(k-1)-1}\right)$

$= \left((-1)^k (k-2)!(-1)(k-1)x^{-k}\right)$

$= \left((-1)^{k+1} (k-1)(k-2)! x^{-k}\right)$

$= \frac{(-1)^{k+1}(k-1)!}{x^k}$

$= \frac{(-1)^{k+1}((k+1)-2)!}{x^{(k+1)-1}}$

=RHS

$\Rightarrow$ The statement is true for n = k + 1 if it is true for n = k.

Step 4 As the statement is true for n = 2 it must be true for n =3.
As the statement is true for n = 3 it must be true for n =4 and so on.

Hence, the nth derivative of $y = x\ln x, x > 0$ is $\frac{d^n y}{dx^n} = \frac{(-1)^n (n-2)!}{x^{n-1}}$ for all positive integers n ≥ 2.

7.

Required to prove $\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\dots\dots\dots\dots+\frac{1}{(n+1)(n+2)}=\frac{n}{2(n+2)}$ for $n\geq 1$

Step 1 Conjecture is true for $n=1 \Leftrightarrow \frac{1}{(2)\times(3)}=\frac{1}{2(3)}$ True

Step 2 Assume conjecture is true for $n=k$

$$\Leftrightarrow \frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\dots\dots\dots\dots+\frac{1}{(k+1)(k+2)}=\frac{k}{2(k+2)}$$

Step 3 Conjecture is true for $n=k+1$

$$\Leftrightarrow \frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\dots\dots\dots\dots+\frac{1}{(k+2)(k+3)}=\frac{k+1}{2(k+3)}$$

$$\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\dots\dots\dots\dots+\frac{1}{(k+2)(k+3)}=$$

$$=\left(\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\dots\dots\dots\dots+\frac{1}{(k+1)(k+2)}\right)+\frac{1}{(k+2)(k+3)}$$......Associative property

$$=\left(\frac{k}{2(k+2)}\right)+\left(\frac{1}{(k+2)(k+3)}\right)$$..Assumption in step 2

$$=\frac{k(k+3)}{2(k+2)(k+3)}+\frac{2}{2(k+2)(k+3)}$$..Common denominators

$$=\frac{k^2+3k+2}{2(k+2)(k+3)}$$...Adding fractions

$$=\frac{(k+1)(k+2)}{2(k+2)(k+3)}$$...Factorising numerator

$$=\frac{k+1}{2(k+3)}$$...Simplifying factors

$$\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\dots\dots\dots\dots+\frac{1}{(k+2)(k+3)}=\frac{k+1}{2(k+3)}$$......................QED

Step 4 As the statement is true for n = 1 it must be true for n =2.
As the statement is true for n = 2 it must be true for n =3 and so on.

Hence $\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\dots+\frac{1}{(n+1)(n+2)}=\frac{n}{2(n+2)}$ is true for all positive integers n.