

Name:

CHENG / Version 2

Date: Wed 2nd May

You may have a formula sheet for this section of the test.  
 Classpad Calculators  
 1 page of Notes

Total /41

45 minutes +5 minutes READING

(5 marks)

Question 1

The discrete random variable  $X$  has the probability distribution shown in the table below.

$x$	0	1	2	3
$P(X = x)$	$\frac{2a^2}{3}$	$\frac{1-3a}{3}$	$\frac{1+2a}{3}$	$\frac{4a^2}{3}$

Determine the value of the constant  $a$ .

$$\frac{2a^2}{3} + \frac{1-3a}{3} + \frac{1+2a}{3} + \frac{4a^2}{3} = 1$$

$$6a^2 - a - 1 = 0$$

$$(2a-1)(3a+1) = 0$$

$$\therefore a = \frac{1}{2} \text{ or } a = -\frac{1}{3}$$

$$\text{Check: } a = \frac{1}{2}, \quad \frac{1-3a}{3} < 0$$

$$\therefore a = -\frac{1}{3}$$

If  $a = \frac{1}{2}$  or  $a = -\frac{1}{3}$   
 only

reject  $a = \frac{1}{2}$

## Question 2

(8 marks)

- (a) Differentiate
- $e^{-3x} \sin(2x)$
- with respect to
- $x$
- , showing full working.

(2 marks)

$$\begin{aligned} \frac{d}{dx} e^{-3x} \sin(2x) &= -3e^{-3x} \sin(2x) + e^{-3x} \cos(2x) \times 2 \\ &= e^{-3x} (-3 \sin(2x) + 2 \cos(2x)) \end{aligned}$$

- (b) Hence find the following indefinite integral.

(3 marks)

$$-3 \int e^{-3x} \sin(2x) dx + 2 \int e^{-3x} \cos(2x) dx.$$

And using a similar process as part (a), find the indefinite integral for

$$-3 \int e^{-3x} \cos(2x) dx - 2 \int e^{-3x} \sin(2x) dx.$$

$$\text{By (a), } \int -3e^{-3x} \sin(2x) + 2e^{-3x} \cos(2x) dx$$

$$= e^{-3x} \sin(2x) + C_1$$

$$\frac{d}{dx} e^{-3x} \cos(2x) = -3e^{-3x} \cos(2x) - e^{-3x} \sin(2x) \times 2$$

$$\therefore \int -3e^{-3x} \cos(2x) dx - \int 2e^{-3x} \sin(2x) dx$$

$$= e^{-3x} \cos(2x) + C_2$$

If  $\frac{3(2 \cos(2x) + 3 \sin(2x)) e^{-3x}}{13} - \frac{2(3 \cos(2x) - 2 \sin(2x)) e^{-3x}}{13}$

\* Should use Interactive Simplify answer  
 $\Rightarrow \sin(2x) e^{-3x} + C$

## Question 6

(9 marks)

- (a) A sample of six objects is to be drawn from a large population in which 20% of the objects are defective. Find the probability that the sample contains:

- (i) three defectives.

(2 marks)

$$X \sim \text{Bin}(6, 0.2)$$

$$P(X=3) = \text{binomial PDF}(3, 6, 0.2) = 0.08192$$

- (ii) fewer than three defectives.

(2 marks)

$$X \sim \text{Bin}(6, 0.2)$$

$$P(X \leq 2) = \text{binomial CDF}(0, 2, 6, 0.2) = 0.90112$$

- (b) Another large population contains a proportion
- $p$
- of defective items.

- (i) Write down an expression in terms of
- $p$
- for
- $P$
- , the probability that a sample of six items contains exactly two defectives.

(2 marks)

$$P = {}^6C_2 p^2 (1-p)^4 = 15 p^2 (1-p)^4$$

- (ii) By differentiating to find
- $\frac{dP}{dp}$
- , show that
- $P$
- is greatest when
- $p = \frac{1}{3}$
- .

(3 marks)

$$\frac{dP}{dp} = 15 \times 2p \times (1-p)^4 + 15p^2 \times 4(1-p)^3 \times (-1)$$

$$= 30p(1-p)^4 - 60p^2(1-p)^3$$

$$= 30p(1-p)^3((1-p) - 2p)$$

$$= 30p(1-p)^3(1-3p) = 0$$

$$p = 0, 1, \frac{1}{3}$$

reject  $p = 0, 1$  as  $p$  is proportion

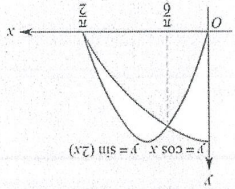
$$\therefore p = \frac{1}{3}$$

$$\begin{aligned} \frac{d^2P}{dp^2}(0) &= 30 \text{ +ve } \therefore \text{min} \\ \frac{d^2P}{dp^2}(1) &= 0 \therefore p, 0, 1 \\ \frac{d^2P}{dp^2}\left(\frac{1}{3}\right) &= -\frac{80}{9} \text{ -ve } \therefore \text{MAX} \end{aligned}$$

$$\rightarrow 90p^5 - 300p^4 + 360p^3 - 180p^2 + 30p$$

Question 5

Find the area between the two curves from  $0 \leq x \leq \frac{\pi}{2}$ , showing full algebraic reasoning.



(4 marks)

$$= \frac{1}{2}$$

$$= \frac{1}{4}$$

$$+ \frac{1}{4}$$

$$= \left[ \left( \frac{1}{2} + \frac{1}{4} \right) - \left( 0 + \frac{1}{2} \right) \right] + \left[ \left( \frac{1}{2} - 1 \right) - \left( -\frac{1}{4} - \frac{1}{2} \right) \right]$$

$$= \left[ \sin x + \frac{\cos(2x)}{2} \right]_0^{\frac{\pi}{2}} + \left[ -\frac{\cos 2x}{2} - \sin x \right]_{\frac{\pi}{2}}^{\frac{\pi}{4}}$$

$$\int_{\frac{\pi}{2}}^0 (\cos x - \sin 2x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin 2x - \cos x) dx$$

(c) Use the two equations from (b) to determine  $\int e^{-3x} \sin(2x) dx$ .

(3 marks)

$$e^{-3x} \sin(2x) + C_1 = -3 \int e^{-3x} \sin(2x) dx + 2 \int e^{-3x} \cos(2x) dx \quad (1)$$

$$e^{-3x} \cos(2x) + C_2 = -3 \int e^{-3x} \cos(2x) dx - 2 \int e^{-3x} \sin(2x) dx \quad (2)$$

$$(1) \times 3: 3e^{-3x} \sin(2x) + 3C_1$$

$$= -9 \int e^{-3x} \sin(2x) dx + 6 \int e^{-3x} \cos(2x) dx \quad (3)$$

$$(2) \times 2: 2e^{-3x} \cos(2x) + 2C_2$$

$$= -6 \int e^{-3x} \cos(2x) dx - 4 \int e^{-3x} \sin(2x) dx \quad (4)$$

$$(3) + (4): 3e^{-3x} \sin(2x) + 2e^{-3x} \cos(2x) + C$$

$$= -13 \int e^{-3x} \sin(2x) dx$$

$$\therefore \int e^{-3x} \sin(2x) dx = -\frac{1}{13} \left( 3e^{-3x} \sin(2x) + 2e^{-3x} \cos(2x) \right)$$

\* If they don't "Use the two equations from (b)"  
 1 mark only

## Question 3

(6 marks)

Differentiate with respect to  $x$ , (show full working)

(a)  $y = \sin^3(2x+1)$ .

(3 marks)

$$\begin{aligned}\frac{dy}{dx} &= \frac{3 \sin^2(2x+1) \cos(2x+1) \times 2}{1} \\ &= 6 \sin^2(2x+1) \cos(2x+1)\end{aligned}$$

Evaluate the following, showing full working.

(b)  $\int_{\pi/6}^{\pi/2} \cos(2x) dx$

(3 marks)

$$\begin{aligned}&= \left[ \frac{1}{2} \sin(2x) \right]_{\pi/6}^{\pi/2} \quad \checkmark \quad (\text{Find anti-derivative}) \\ &= \frac{1}{2} \left( \sin \pi - \sin \frac{\pi}{3} \right) \quad \checkmark \quad (\text{Evaluate by substitution}) \\ &= \frac{1}{2} \left( 0 - \frac{\sqrt{3}}{2} \right) \\ &= -\frac{\sqrt{3}}{4} \quad \checkmark \\ &= -0.4330 \quad \checkmark \quad (\text{Final answer})\end{aligned}$$

## Question 4

(9 marks)

75% of the avocados produced by a farm are known to be first grade, the rest being second grade. Trays of 24 avocados are filled at random in a packing shed and sent to market.

Let the random variable  $X$  be the number of first grade avocados in a single tray.

(a) Explain why  $X$  is a discrete random variable, and identify its probability distribution.

$X$  is a DRV as it can only take integer values from 0 to 24.  $\checkmark$  (2 marks)

$X \sim B(24, 0.75)$  which is a binomial distribution.  $\checkmark$

(b) Calculate the mean and standard deviation of  $X$ .

(2 marks)

$$\bar{X} = 24 \times 0.75 = 18 \quad \checkmark$$

$$\sigma_X = \sqrt{18 \times 0.25} = \frac{3\sqrt{2}}{2} \approx 2.12 \quad \checkmark$$

(c) Determine the probability that a randomly chosen tray contains

(i) 18 first grade avocados.

(1 mark)

$$P(X=18) = 0.1853 \quad \checkmark$$

(ii) more than 15 but less than 20 first grade avocados.

(2 marks)

$$\begin{aligned}P(15 < X < 20) \\ = P(16 \leq X \leq 19) = 0.6320 \quad \checkmark\end{aligned}$$

(d) In a random sample of 1000 trays, how many trays are likely to have fewer first grade than second grade avocados.  $\checkmark$  (2 marks)

$$\begin{aligned}P(X \leq 11) &= 0.0021 \quad \checkmark \\ 0.0021 \times 1000 &= 2.1 \approx 2 \text{ trays.} \quad \checkmark\end{aligned}$$

*If trays of 24 then  $X$  must be  $\leq 12$  i.e.  $P(X \leq 11)$*