Semester One Examination, 2020

Question/Answer Booklet

MATHEMATICS METHODS

ATAR Year 12
Section One: Calculator-free

SOLUTIONS

Student Name: ____

Please circle your teacher's name

Teacher: Miss Long Miss Rowden Ms Stone

Time allowed for this paper

Reading time before commencing work:	5 minutes
Working time for paper:	50 minutes

Materials required/recommended for this paper

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

Number of additional answer booklets used (if applicable): ☐

To be provided by the candidate

Standard items:	pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters
Special items:	nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Suggested working time (minutes)	Marks available	Percentage of examination
Section One: Calculator free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the ATAR course examinations are detailed in the *Year 12 Information Handbook 2020*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Supplementary pages for the use planning/continuing your answer to a question have been provided at the end of the Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has eight (8) questions. Answer **all** questions. Write your answers in the spaces provided.

Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

Working time: 50 minutes.

Question 1 **(5 marks)**

Determine the area bounded by the line $y=x$ and the parabola $y=x^2+4x$.

Solution
Intersect when
$x-(x^2+4x)=0-3x-x^2=0-x(3+x)=0x=0,-3$
Bounded area
$A=\int_{-3}^{0}-3x-x^2\,dx\text{¿}\left[\frac{-3x^2}{2}-\frac{x^3}{3}\right]_{-3}^{0}\text{¿}(0)-\left(\frac{-27}{2}-(-9)\right)$
$\text{¿}13.5-9\text{¿}4.5\text{ square units}$
Specific behaviours
ü equates functions and simplifies
✓ bounds of integral
✓ writes definite integral
✓ antidifferentiates
✓ correct area

Question 2 **(7 marks)**

Determine the following

(a) $\int 4\sin 2x + \frac{1}{e^{2x}}\,dx$

(2 marks)

Solution
$\int 4\sin 2x + \frac{1}{e^{2x}}\,dx = -2\cos 2x - \frac{1}{2e^{2x}} + c$
Specific behaviours
ü correct integration of trig term
correct integration of term involving e
- if no + *c* (penalise once only – either a or b or for both)

(b) $\int 6x^3(3x^4 - 8)^4\,dx$

(2 marks)

Solution
$\frac{(3x^4 - 8)^5}{10} + c$
Specific behaviours
ü correct numerator
correct denominator

(c) Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2(x)}\,dx$

(3 marks)

Solution
$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2(x)}\,dx = [\tan(x)]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $= \tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)$ $= \sqrt{3} - 1$
Specific behaviours
ü correct antiderivative
evaluates constant and writes equation

Question 3 **(8 marks)**

Determine

(a) $f'(x)$ when $f(x)=\sqrt{4x-3}$.

(2 marks)

Solution
$f'(x)=\frac{1}{2}(4)(4x-3)^{\frac{-1}{2}}¿\frac{2}{\sqrt{4x-3}}$
Specific behaviours
ü indicates correct use of chain rule
✓ correct derivative (any form)

(b) $\frac{d}{dx}\left(x^3e^{4x}\right)$ when $x=2$.

(3 marks)

Solution
$¿3x^2e^{4x}+4x^3e^{4x}\vert_{x=2}$ $¿12e^8+32e^8=44e^8$
Specific behaviours
✓ u' or v' correct
✓ correct derivative in terms of x
✓ correct value

(c) $f'\left(\frac{\pi}{4}\right)$ when $f(t)=\frac{1+\cos t}{\sin t}$.

(3 marks)

Solution
$f'(t)=\frac{-\sin t\cdot\sin t-(1+\cos t)\cdot\cos t}{\sin^2 t}$ $¿\frac{-\cos t-\sin^2 t-\cos^2 t}{\sin^2 t}¿\frac{-1-\cos t}{\sin^2 t}$ $f'\left(\frac{\pi}{4}\right)=\left(-1-\frac{1}{\sqrt{2}}\right)\div\frac{1}{2}¿-2-\sqrt{2}$
Specific behaviours
ü indicates correct use of quotient rule
✓ correct derivative
✓ correct value, simplified

Question 4 **(7 marks)**

(a) Find x if:

(i) $\log_4 128 = x$

(2 marks)

Solution
$4^x = 128$
$2^{2x} = 2^7$
$2x = 7$
$x = \frac{7}{2}$
Specific behaviours
ü rewrites logarithmic equation as exponential with prime bases
ü solves correctly for x

(ii) $\log 125 - x = \log \frac{1}{8}$

(3 marks)

Solution
$\log 125 + \log 8 = x$
$\log 1000 = x$
$x = 3$
Specific behaviours
ü rearranges to make x the subject
ü applies the logarithm laws to simplify to a single logarithm
ü solves correctly for x

(b) Simplify $\ln(8x)^{\frac{1}{2}} + \ln(4x)^2 - \ln(16x)^{\frac{1}{2}}$

(2 marks)

Solution
$\frac{1}{2}(\ln(8) + \ln(x)) + 2(\ln(4) + \ln(x)) - \frac{1}{2}(\ln(16) + \ln x)$
$= \frac{3}{2}\ln(2) + \frac{1}{2}\ln(x) + 4\ln(2) + 2\ln(x) - 2\ln 2 - \frac{1}{2}\ln x$
$= \frac{7}{2}\ln(2) + 2\ln(x)$
Specific behaviours
ü expands each term
ü final answer

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

Question 5 **(5 marks)**

The graph of $y=f(x)$ has a stationary point at $(4,-3)$ and $f'(x)=ax^2+6x+8$, where a is a constant.

Determine the interval over which $f'(x)>0$ and $f''(x)>0$.

Solution
$f'(4)=16a+24+8=0 \quad a=-2$ Concave up: $f'(x)=-2x^2+6x+8 \quad f''(x)=-4x+6$ $f''(x)>0 \Rightarrow x<1.5$ Other stationary point: $-2x^2+6x+8=0 \quad -2(x+1)(x-4)=0 \quad x=-1$ Hence $f'(x)>0$ when $-1<x<4$. Required interval: $-1<x<1.5$.
Specific behaviours
ü value of a ✓ interval where $f''(x)>0$ ✓ second stationary point ✓ interval where $f'(x)>0$ ✓ correct interval

Question 6 **(5 marks)**

A curve, defined for $x>0$, passes through the point $B(2,5)$ and its gradient is given by

$$\frac{dy}{dx}=3x^2-\frac{12}{x^2}-9$$

(a) Verify that B is a stationary point, determine the value of the second derivative at B and hence describe the nature of the stationary point. (3 marks)

Solution
$f'(x)=3x^2-\frac{12}{x^2}-9 \Rightarrow f'(2)=12-3-9=0$ $f'(2)=0$, so B is a stationary point. $f''(x)=6x+\frac{24}{x^3} \Rightarrow f''(2)=12+3=15$ $f''(2)>0$, so B is a local minimum.
Specific behaviours
ü simplifies $f'(2)$ to three integers that sum to zero ü correct value of second derivative

(b) Determine the equation of the curve. (2 marks)

Solution
$f(x)=x^3+\frac{12}{x}-9x+c$ $f(2)=8+6-18+c=5 \Rightarrow c=9$ $y=x^3+\frac{12}{x}-9x+9$
Specific behaviours
ü correct antiderivative ü evaluates constant and writes equation

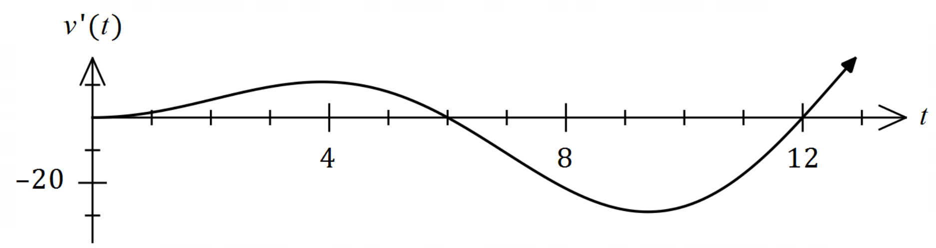
Question 7 **(7 marks)**

(a) Determine an expression for $\frac{d}{dt}\left(6t\cos\left(\frac{\pi t}{6}\right)\right)$.

(2 marks)

Solution
$\frac{d}{dt}\left(6t\cos\left(\frac{\pi t}{6}\right)\right)=6\cos\left(\frac{\pi t}{6}\right)-\pi t\sin\left(\frac{\pi t}{6}\right)$
Specific behaviours
ü correct use of product rule
ü correct derivative

The volume of water in a tank, v litres, is changing at a rate given by $v'(t)=\pi t\sin\left(\frac{\pi t}{6}\right)$, where t is the time in hours. The rate of change is shown in the graph below.



(b) Using the result from part (a) or otherwise, determine the change in volume of water in the tank between $t=0$ and $t=12$ hours.

(5 marks)

Solution
$\Delta v=\int_0^{12} v'(t)\,dt¿\int_0^{12}\pi t\sin\left(\frac{\pi t}{6}\right)dt$
1. Using (a):
$\int\frac{d}{dt}\left(6t\cos\left(\frac{\pi t}{6}\right)\right)dt=\int 6\cos\left(\frac{\pi t}{6}\right)dt-\int\pi t\sin\left(\frac{\pi t}{6}\right)dt$
2. And so:
$\int\pi t\sin\left(\frac{\pi t}{6}\right)dt=\int 6\cos\left(\frac{\pi t}{6}\right)dt-6t\cos\left(\frac{\pi t}{6}\right)$
3. Hence:
$\int_0^{12}\pi t\sin\left(\frac{\pi t}{6}\right)dt=\left[\frac{36}{\pi}\sin\left(\frac{\pi t}{6}\right)\right]_0^{12}-\left[6t\cos\left(\frac{\pi t}{6}\right)\right]_0^{12}¿[0-0]-[72-0]$
$\Delta v=-72\,\mathrm{L}$
Specific behaviours
ü indicates required definite integral
ü line 1 - uses part (a)
ü line 2 - expression to evaluate integral
ü line 3 - antidifferentiates ready for substitution
ü correct change in volume, with units

Question 8 **(8 marks)**

Initially, particle P is stationary and at the origin. Particle P moves in a straight line so that at time t seconds its acceleration a cms^{-2} is given by $a=16-15\sqrt{t}$ where $t\geq 0$.

(a) Determine the speed of P after 1 second.

(3 marks)

Solution
$v=\int 16-15t^{0.5}\,dt¿16t-10t^{1.5}+c$ $v(0)=0\Rightarrow c=0v=16t-10t^{1.5}$ $v(1)=16(1)-10(1)^{1.5}=6\,\text{cm/s}$
Specific behaviours
✓ indicates v is integral of a ü expression for velocity v ✓ correct speed

(b) Determine the speed of P when it returns to the origin.

(5 marks)

Solution
Require 0 change in displacement for $0\leq t\leq T$ $\Delta x=\int_0^T 16t-10t^{1.5}\,dt0=\left[8t^2-4t^{2.5}\right]_0^T 8T^2-4T^{2.5}=0$ $4T^2\left(2-\sqrt{T}\right)=0\sqrt{T}=2T=4$ $v(4)=16(4)-10(4)^{\frac{3}{2}}¿64-10(2)^3¿64-80=-16$ Hence speed is 16 cm/s.
Specific behaviours
✓ obtains expression for Δx in terms of T ✓ equates $\Delta x=0$ ✓ solves for T ✓ obtains velocity ✓ correct speed, with units

End of questions

Supplementary page

Question number: ____________________

Supplementary page

Question number: ______________________