# SEMESTER TWO

# MATHEMATICS METHODS UNITS 3 & 4

# 2020

# SOLUTIONS

## Calculator-free Solutions

1. (a)

| $a$ | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| P($A$ = $a$) | 0.2 | 0.3 | 0.4 | 0.1 |

✓✓

(b) $E(A) = (-1)(0.2) + (0)(0.3) + (1)(0.4) + (2)(0.1)$ ✓

$= -0.2 + 0.4 + 0.2 = 0.4$ or \$400 ✓

(c) $VAR(A) = (-1)^2(0.2) + (1)^2(0.4) + (2)^2(0.1) - (0.4)^2$ ✓

$= 0.2 + 0.4 + 0.4 - 0.16$ ✓

$= 0.84$ ✓ [7]

2. (a)

| $m$ | 0 | 1 |
|---|---|---|
| P($M$ = $m$) | $\dfrac{\binom{1}{0}\binom{4}{1}}{\binom{5}{1}} = \dfrac{4}{5}$ | $\dfrac{1}{5}$ |

✓✓

(b) It has two possibilities, independent events and is a DRV ✓

(c) $E(M) = (0)\left(\dfrac{4}{5}\right) + (1)\left(\dfrac{1}{5}\right) = \dfrac{1}{5}$ ✓

$VAR(M) = (0)^2\left(\dfrac{4}{5}\right) + (1)^2\left(\dfrac{1}{5}\right) - \left(\dfrac{1}{5}\right)^2 = \dfrac{4}{25}$

$\therefore \quad SD = \dfrac{2}{5}$ ✓

(d) $X \sim B(8, \dfrac{1}{5})$ ✓

(e) $P(X = 3) = \binom{8}{3}\left(\dfrac{1}{5}\right)^3\left(\dfrac{4}{5}\right)^5$ ✓ [7]

3. (a) $\frac{d}{dx}\left[(3x^2+5)^4\right] = 4(3x^2+5)^3(6x)$ ✓✓

(b) $g'(x) = e^{3x}(3)\cos(2x) + e^{3x}(-\sin(2x))(2)$ ✓

$g'(\pi) = 3e^{3\pi}\cos(2\pi) + 2e^{3\pi}(-2\sin(2\pi))$ ✓

$\therefore \quad = 3e^{3\pi}$ ✓

(c) (i) $-2\cos(3x) + c$ ✓✓

(ii) ¿ ✓

$= \ln(7) - \ln(1) = \ln 7$ ✓

(d) $y(x) = -\frac{2}{5}e^{-5x} + c$ ✓

Since $y(0) = -\frac{2}{5} + c = -3 \rightarrow c = -\frac{13}{5}$

$\therefore \quad y = -\frac{2}{5}e^{-5x} - \frac{13}{5}$ ✓ [11]

4. (a) $X \sim N(7,\ 1.5^2)$

$\therefore \quad 7 - 3(1.5) < X < 7 + 3(1.5)$ ✓

$\therefore \quad 2.5 < X < 11.5$ so 9 years ✓

(b) $P(X < 5{\cdot}5) = 0{\cdot}5 - 0{\cdot}34 = 0{\cdot}16$ ✓

$\therefore$ 16% ✓

(c) $P(X > h) = 0.975 \rightarrow h = 7 - 2(1.5) = 4$ years ✓✓ [6]

5. Width of CI = $2E$

Since $E = \sqrt{\frac{\hat{p}\left(\left(1-\hat{p}\right)\right)}{n}} \rightarrow E \propto \frac{1}{\sqrt{n}}$ ✓

$\therefore \quad n \propto \frac{1}{E^2}$ ✓

$\therefore$ New $n = \frac{1}{4}n = \frac{n}{4}$ ✓ [3]

6. (a) (i) $(2x)(\ln x^2) + (x^2)\left(\dfrac{2x}{x^2}\right) = 4x\ln x + 2x$ ✓✓

(ii) $\int (4x\ln x + 2x)\ dx = x^2\ln x^2 + c$ ✓

$\therefore \int (4x\ln x)\ dx + \int 2x\ dx = x^2\ln x^2 + c$ ✓

$\therefore 4\int (x\ \ln x)\ dx + x^2 = x^2\ln x^2 + c$ ✓

$\int (x\ \ln x)\ dx = \frac{1}{4}(x^2\ln x^2) - \frac{x^2}{4} + c$

$= \dfrac{x^2(2\ln x - 1)}{4} + c$ ✓

(b) Area $= \int_0^1 |\ x\ln x\ |\ dx$ ✓

$=$ -¿ ✓

$=$ - $\dfrac{1(2\ln 1 - 1)}{4} - 0 = \dfrac{1}{4}$ ✓✓ [10]

7. $\dfrac{dy}{dx} = 3x^2 + 2ax + b$ ✓

$\dfrac{d^2y}{dx^2} = 6x + 2a$ ✓

Since HPOI $\dfrac{d^2y}{dx^2} = 0$ when $x = 2$ ✓

$\therefore$ $12 + 2a = 0 \rightarrow a = -6$ ✓

And $\dfrac{dy}{dx} = 0$ when $x = 2$

$\therefore$ $12 - 24 + b = 0 \rightarrow b = 12$ ✓

(2, 8) $\rightarrow 8 = 8 - 24 + 24 + c \rightarrow c = 0$ ✓ [6]

**<u>Calculator-Assumed Solutions</u>**

8. (a) $\int_0^2 ax^3\ dx = 1 \rightarrow \left.\dfrac{ax^4}{4}\right]_0^2 = 1$ ✓

$\therefore\quad 4a = 1 \rightarrow a = \dfrac{1}{4}$ ✓

(b) $P(x < 1) = \int_0^1 \dfrac{x^3}{4}\ dx$ ✓

$= \dfrac{1}{16}(1)^4 = \dfrac{1}{16}$ ✓

(c) $\int_q^2 \dfrac{x^3}{4} = \dfrac{1}{4}$ ✓

$\therefore\quad \dfrac{2^4}{16} - \dfrac{q^4}{16} = \dfrac{1}{4}$ ✓

$\therefore\quad q = 1.8612$ ✓ [7]

9. (a) (i) $f(2) = e^2 = 7.389$ ✓

(ii) $f(2) = 1 + \dfrac{2}{1} + \dfrac{4}{2} + \dfrac{8}{6} + \dfrac{16}{24} = 7$ ✓

(b) $f(-1) = 1 + \dfrac{(-1)}{1} + \dfrac{1}{2} + \dfrac{(-1)}{6} + \dfrac{1}{24} = 0.375$ ✓✓

(c) $\dfrac{d}{dx}(e^x) = \dfrac{d}{dx}\left(1 + \dfrac{x}{1} + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \ \ldots\right)$ ✓

$= 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \ \ldots = e^x$ ✓ [6]

10. (a) $S = 120 + 20 = 140°$ ✓

(b) $S = 120e^{-0.02(120)} + 20 = 30.89°$ ✓✓

(c) $25 = 120e^{-0.02t} + 20$ ✓

$\therefore\quad t = 159$ seconds $\approx 3$ mins ✓✓

(d) $\dfrac{d}{dt}\left(120e^{-0.02t} + 20\right) = -2.4e^{-0.02t}$ ✓

When $t = 30$, rate = -1.3°/sec ✓ [8]

11. (a) $X \sim N(85, 20^2)$

$P(X < 110) = 0.8944$ ✓

$\therefore$ 89.44% ✓

(b) $Y \sim Bin(15, 0.8944)$

(i) $P(Y = 10) = 0.01292$ ✓✓

(ii) $(0.8944)^3 \times Bin(6,12,12,0.8944)$ ✓✓

$= 0.7154$ ✓

(c) \$10 $\dfrac{0.0388}{0.1056} = 0.3677$ ✓

\$20 $\dfrac{0.0267}{0.1056} = 0.2532$ ✓

\$30 $\dfrac{0.0401}{0.1056} = 0.3792$ ✓

(d) E(Y) = $10 \times 0.3677 + 20 \times 0.2532 + 30 \times 0.3792$ ✓

= \$20.12 ✓

(e) VAR(Y) = $10^2(0.3677) + 20^2(0.2532) + 30^2(0.3792) - (20.12)^2$ ✓

= 74.516 ✓

St. Dev. = 8.63 ✓

(f) New Mean = $1.2 \times 20.12 - 1 = \$23.14$ ✓

New St. Dev. = $1.2 \times 8.63 = \$10.36$ ✓ [17]

12. (a) (i) 17.32 m/min ✓

(ii) 20 m/min ✓

(b) $x(t) = \displaystyle\int 20\cos\left(\frac{t}{3} + \frac{\pi}{6}\right) dt = 60\sin\left(\frac{t}{3} + \frac{\pi}{6}\right) + c$ ✓

$x(0) = 0 \rightarrow c = -30$ ✓

$\therefore$ $x(t) = 60\sin\left(\frac{t}{3} + \frac{\pi}{6}\right) - 30$ ✓

(c) (i) $x(4) = 27.56\ m$ ✓

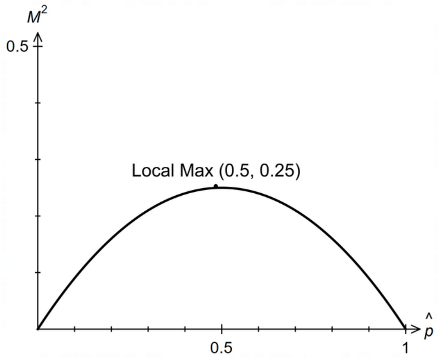
(ii) $v(4) = -5.6 \therefore$ returning to start. ✓

(d) $a(t) = -\dfrac{20}{3}\sin\left(\frac{t}{3} + \frac{\pi}{6}\right)$ ✓

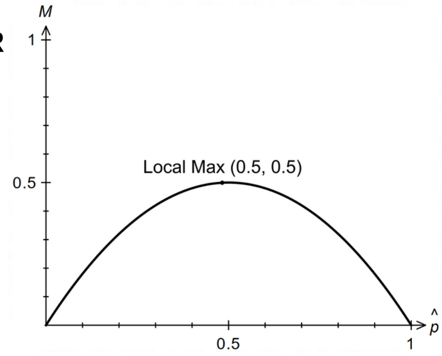
$\therefore$ $-\dfrac{20}{3}\sin\left(\frac{t}{3} + \frac{\pi}{6}\right) = -5 \rightarrow t = 5.30$ ✓

$\therefore$ Distance = $\displaystyle\int_0^{5.30} \left| 20\cos\left(\frac{t}{3} + \frac{\pi}{6}\right) \right| dt = 44.9\ m$ ✓ [10]

13. (a)




**OR**




✓✓

(b) $M = z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ where $\hat{p} = 0.5$

$\therefore \quad 0.06 = 1.96\sqrt{\dfrac{(0.5)(0.5)}{n}}$ ✓✓

$\therefore \quad n = 266.8$ ✓

Hence 267 trainers should be tested ✓

*(c)* $0.08 = 1.96\sqrt{\dfrac{(0.8)(0.2)}{n}}$ ✓✓

$\therefore \quad n = 96.04$ ✓

$\therefore \quad 96 \times 0.8 = 76.8$

Hence 76 or 77 trainers would be qualified. ✓ [10]

14. (a) $\hat{p} \sim N\left(0.04, \left(\sqrt{\dfrac{0.04(0.96)}{300}}\right)^2\right)$ ✓✓

i.e. $\hat{p} \sim N(0.04, 0.0113^2)$ ✓

(b) $P(X < 0.045) = 0.6709$ ✓✓

(c) $(0.04 \pm 1.96 \times 0.0113)$ ✓✓

$\therefore \quad (0.0179, 0.0621)$ ✓

(d) $\dfrac{24}{300} = 0.08$ ✓

Since 0.08 is not in the 95% CI ✓

it would be unlikely to happen. ✓

(e) (i) *n* should be larger ✓

Since the formula for standard error has *n* in the denominator, increasing n will decrease the standard error. ✓

(ii) From the standard normal distn, P $(-k < p < k) = 0.98$ ✓

$k = 2.326$ ✓ [15]

15. (a) ln 0 doesn't exist. ✓

(b) $f'(x) = 2x - \frac{1}{x}$ ✓

$f'(x) = 0$ when $2x = \frac{1}{x} \to 2x^2 = 1$ ✓

$\therefore \quad x = \pm\frac{1}{\sqrt{2}}$ ✓

$f''(x) = 2 + \frac{1}{x^2}$ ✓

Since $f''\left(-\frac{1}{\sqrt{2}}\right)$ doesn't exist

and $f''\left(\frac{1}{\sqrt{2}}\right) > 0$, $f\left(\frac{1}{\sqrt{2}}\right)$ is the minimum

$\therefore \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{2} + \frac{\ln 2}{2}\right)$ is the minimum point. ✓ [6]

16. (a) (i) $x = 2$ ✓

(ii) $a = 2$ ✓

(iii) We require another set of co-ordinates, as $\log_p 1 = 0$ for all $p$. ✓

(b) $f(x) + 1 = \log_p(x-2) + 1$ since $f(x)$ is translated 1 up. ✓

$\therefore \quad \log_p(x-2) + 1 = 0 \to x - 2 = \frac{1}{p}$ ✓

$\therefore \quad x = \frac{1}{p} + 2$ ✓

(c) $f(27) = \log_p(25) = 2 \to p^2 = 25$ ✓

$\therefore \quad p = 5$ ✓ [8]

17. (a) $\int_1^5 f'(x)\,dx = \left[f(x)\right]_1^5$ ✓

$= \quad f(5) - f(1) = -1 - (-2) = 1$ ✓

(b) Area $= \int_1^5 |f'(x)|\,dx$ ✓

$= \int_1^3 f'(x)\,dx - \int_3^5 f'(x)\,dx$ ✓

$= \quad \left[f(x)\right]_1^3 - \left[f(x)\right]_5^3 = 2 - (-2) + 2 - (-1) = 7$ ✓✓ [6]

18. (a) $\log w = 0.5 + 0.4(4) = 2.1$ ✓

$\therefore \quad w = 125.9\ kg$ ✓

(b) $\log 180 = 0.5 + 0.4h \rightarrow h = 4.39\ m$ ✓✓

(c) $\log 200 = 0.5 + 0.4h \rightarrow h = 4.50\ m$ ✓

$\log 100 = 0.5 + 0.4h \rightarrow h = 3.75\ m$ ✓

$\therefore \quad \dfrac{4.50}{3.75} = 1.2 \rightarrow 20\%$ taller ✓ [7]