Applecross Senior High School Name : SOLUTIONS.

Mathematics Method 3 Test 1, 2017 Time Allowed : 55 minutes.

Note : For both Section 1 and section 2, working out must be shown for full marks to be awarded.

Section 1 : [/ 17 marks] Section 2 : [/ 31 marks] Total : [/ 48 marks] = ____ %

Section 1 : Calculator and Resource Free Time : 20 minutes

1. [3,2 = 5 marks]

Differentiate the following with respect to x.

a) $f(x) = \frac{-x}{x^2+1}$ { Express numerator in simplest form }

$$f'(x) = \frac{-(x^2+1) - (-x)(2x)}{(x^2+1)^2}$$ ✓✓ ($-\frac{1}{2}$ mark per error / missing)

$$= \frac{-x^2-1+2x^2}{(x^2+1)^2}$$

$$= \frac{x^2-1}{(x^2+1)^2}$$ ✓

b) $y = (1-x)^3 \left(1+\frac{2}{x}\right)^2$ { Apply the product rule but do not simplify } ✓

$$\frac{dy}{dx} = \left[3(1-x)^2(-1)\left(1+\frac{2}{x}\right)^2\right] + \left[2\left(1+\frac{2}{x}\right)\left(-\frac{2}{x^2}\right)(1-x)^3\right]$$ ✓

$$-3(1-x)^2\left(1+\frac{2}{x}\right)^2 \qquad \frac{-4}{x^2}\left(1+\frac{2}{x}\right)(1-x)^3$$

2. [2,2= 4 marks]

A particle moves in a straight line such that its velocity, v m/s, depends upon displacement , x m , from some fixed point O according to the rule v = 5x-4

a) Find an expression in terms of x for the acceleration of the particle.

$a = \frac{dv}{dt}$, $V = \frac{dx}{dt}$, $\frac{dv}{dx} = 5$

$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$

$= 5(5x-4)$ ✓ ✓

$= 25x - 20.$

✓ only if $5ms^{-2}$

b) Determine the displacement and the acceleration of the particle when v = 6m/s.

$5x - 4 = 6$

$x = 2$

When $x = 2$ ✓

EL

2m.

$a = 25x - 20$

$= 25(2) - 20$

$a = 30 m/s^2$ ✓

$a = 5ms^{-2}$ ✓

3. [8 marks]

The equation of the tangent to the curve $y = ax^3 - bx^2 + 2$ where $x = -1$ is $y = 18x + c$.

The curve has a point of inflection when $x = 1$.

Find the values of a, b and c.

[Note : Working out must be shown]

$y = ax^3 - bx^2 + 2$

$\frac{dy}{dx} = 3ax^2 - 2bx$ ✓

Tangent is $y = 18x + c$ when $x = -1$

$\therefore \frac{dy}{dx} = 18$ when $x = -1$ ✓

when $x = -1$, $\frac{dy}{dx} = 18$

$3ax^2 - 2bx = 18$

$3a(-1)^2 - 2b(-1) = 18$

$3a + 2b = 18$ — ① ✓

Point of Inflection when $\frac{d^2y}{dx^2} = 0$.

$\frac{d^2y}{dx^2} = 6ax - 2b$.

when $x = 1$, $\frac{d^2y}{dx^2} = 0$.

$6ax - 2b = 0$.

$6a(1) - 2b = 0$ ✓

$6a - 2b = 0$ — ②.

Solve Simultaneously

$3a + 2b = 18$ — ①

$6a - 2b = 0$ — ②

① + ② $9a = 18$ ✓

$a = 2$

$6a - 2b = 0$

$12 - 2b = 0$

$b = 6$ ✓

when $x = -1$

$y = 2x^3 - 6x^2 + 2$.

$y = 2(-1)^3 - 6(-1)^2 + 2$

$y = -2 - 6 + 2$

$y = -6$ ✓

when $x = -1$, $y = -6$.

$y = 18x + c$

$-6 = -18 + c$ ✓

$c = 12$

$\therefore a = 2$

$b = 6$

$c = 12$.

Name : ______________________________ Marks: $\overline{\quad}$ 31

Section 2 : Calculator and Resource Assumed.	Time Allowed : 35 minutes
Note : Show working for full marks to be awarded.	

1. [2,2= 4 marks]
A company produces **n** items of a certain product.
The cost function \$C is given by C (n) = 1200 + $5n^{1/3}$
Each item sells for \$52.

Find
a) An expression for the marginal profit $P'(n)$

$C(n) = 1200 + 5n^{1/3}$

$R(n) = 52n$

$P(n) = 52n - 1200 - 5n^{1/3}$ ✓

$P'(n) = 52 - \frac{5}{3}n^{-2/3}$

$P'(n) = 52 - \frac{5}{3n^{2/3}}$ ✓

b) A value for $P'(64)$ and comment on its meaning.

$P'(64) = 52 - \frac{5}{3(64^{2/3})}$

$= 51.90$ ✓

Profit of 65th Unit is \$51.90. ✓

must have both to get, mark.

c) Find the depth of the drinking trough to the nearest mm, if the amount of stainless steel is to be kept to a minimum. Justify your answer by using Calculus techniques.

$A = \frac{60}{h} - h + 2h\sqrt{2} + 120$

$A = 60h^{-1} - h + 2h\sqrt{2} + 120$.

$\frac{dA}{dh} = -60h^{-2} - 1 + 2\sqrt{2}$ ✓

For minimum value $\frac{dA}{dh} = 0$.

$-60h^{-2} - 1 + 2\sqrt{2} = 0$ ✓

$h = 5.72844 5657 \simeq 5.728\text{m}$ (3dp).

$\frac{d^2A}{dh^2} = 120h^{-3} = \frac{120}{h^3}$

$\left.\frac{d^2A}{dh^2}\right|_{h=5.728} > 0 \therefore$ Rel. Min ✓

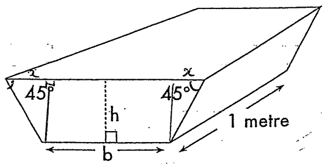
$\therefore$ Depth $= 5.728\text{ m}$.

End of Test

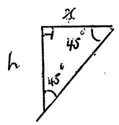
5. [3,2,3 = 8 marks]

An animal drinking trough is constructed from stainless steel in the shape of a trapezoidal prism, with height ‘h’ metres and length of 1 metre.

The cross section of the prism is an isosceles trapezium with acute angle of 45^0, base ‘b’ metres and area of 60 m^2.



a) Show that $b = \frac{60}{h} - h$



$Tan\ 45^o = \frac{h}{x}$

$h = x\ Tan\ 45^0$.

$h = x(1)$

$\underline{h = x}$. ✓

$h = x$ (isos Δ).

$Area = \frac{1}{2}(b + b + 2h) \times h = 60$ ✓

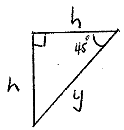
$\frac{2b + 2h}{2} \times h = 60$.

$\frac{\cancel{2}(b+h)h}{\cancel{2}} = 60$.

$b + h = \frac{60}{h}$.

$b = \underline{\frac{60}{h} - h}$ ✓

b) Show that the surface area ‘A’ in m^2 is : $A = \frac{60}{h} - h + 2h\sqrt{2} + 120$



$y^2 = h^2 + h^2$

$y^2 = 2h^2$

$\underline{y = \sqrt{2}\, h}$. ✓

$Area = A = 2 \times 60 + b(1) + 2\sqrt{2}\, h \times 1$.

$A = 120 + b + 2\sqrt{2}\, h$

$b = \frac{60}{h} - h$.

$A = 120 + \frac{60}{h} - h + 2\sqrt{2}\, h$. ✓

2. [2,4,4 = 10 marks]

a) It takes 12 hours to drain a storage tank by opening the valve at the bottom. The **depth ‘y’** of fluid in the tank ‘t’ **hours after** the valve is opened is given by

$y = 6\,(1 - \frac{t}{12})^2$ metres.

i) Show ,with full working out , the rate $\frac{dy}{dt}$ m/hour at which the tank is draining at time t is $\frac{t}{12} - 1$

$y = 6\left(1 - \frac{t}{12}\right)^2$ ✓

$\frac{dy}{dt} = 12\left(1 - \frac{t}{12}\right)^1\left(-\frac{1}{12}\right) = -\left(1 - \frac{t}{12}\right)$ ✓

$= \underline{\frac{t}{12} - 1}$

ii) **a) When** is the fluid in the tank falling **fastest** and **slowest** ?

Slowest : when $t = \underline{12h}$ ✓

Fastest : when $t = \underline{0\ h}$ ✓

b) **What** are the values of $\frac{dy}{dt}$ at these times?

Slowest: $\frac{dy}{dt} = 0$.

$\frac{t}{12} - 1 = 0$

$t = 12\ h$.

Fastest

$t = 0,\ \frac{dy}{dt} = \frac{0}{12} - 1$

$\frac{dy}{dt} = -1\,m/h$

Units not needed.

$\therefore$ Slowest: $\frac{dy}{dt} = \underline{0\,m/h}$ ✓ Fastest: $\frac{dy}{dt} = \underline{-1\,m/h}$. ✓

b) If the volume of a cylinder is given by $V = 2\pi r^3$, find the approximate percentage change in V when r changes by $\frac{1}{2}$%.

$\frac{\delta r}{r} = \frac{1}{2}\% = 0.005$

$V = 2\pi r^3$

$\frac{dV}{dr} = 6\pi r^2$ ✓

$\frac{\delta V}{\delta r} \approx \frac{dV}{dr}$

$\delta V \approx \frac{dV}{dr} \times \delta r$.

$\delta V \approx 6\pi r^2 \times \delta r$.

$\frac{\delta V}{V} \approx \frac{6\pi r^2 \delta r}{2\pi r^3}$ ✓

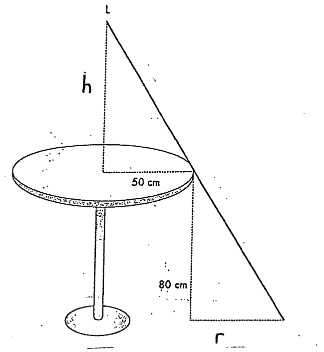
$\approx 3\frac{\delta r}{r}$ ✓

$\approx 3 \times 0.005$

≈ 0.015

$= \underline{1.5\%}$. ✓

3. [1,3 = 4 marks]



A table has a radius of 50 cm and a height of 80 cm.

A light (L) is lowered vertically downwards from a point above the centre of the table at a constant rate of 0.2 cm per second.

When the light is h cm above the table it casts a shadow that extends r cm from the edge of the table.

a) Show that $r = \frac{4000}{h}$

As triangles are similar, corresponding sides are in proportion.

$\frac{r}{50} = \frac{80}{h}$ ✓

$\therefore r = \frac{4000}{h}$

b) Find the rate at which r is changing when h = 60

$\frac{dr}{dt} = \frac{dr}{dh} \cdot \frac{dh}{dt}$

$= \boxed{\frac{-4000}{h^2}} \times \frac{dh}{dt}$ ✓

$\frac{dh}{dt} = -0,2$ ✓

When $h = 60$

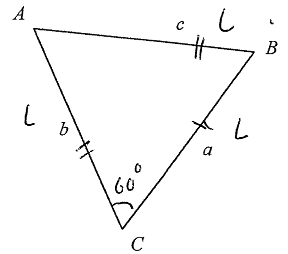
$\frac{dr}{dt} = \frac{-4000}{60^2} \times (-0,2)$

$= \frac{2}{9}$ or $0.\dot{2}$ cm/sec ✓

$\therefore$ Radius is increasing at $2/9$ cm/sec.

4. [5 marks]

The area of a triangle can be found by the formula : Area $= \frac{ab \ Sin \ C}{2}$



Using the **incremental formula**, determine the approximate change in area (to 3 decimal places) of **an equilateral triangle** with each side of 10 cm, when each side increases by 0.1cm.

[Hint : Use exact value for 60^0]

$\triangle ABC$ is an equilateral $\triangle$

Let $a = b = c = L$ cm.

$\angle C = 60^\circ = \frac{\sqrt{3}}{2}$

$$\left.\begin{aligned} \text{Area} &= \tfrac{1}{2} ab \sin\theta \\ &= \tfrac{1}{2}(L)(L)\sin\tfrac{\pi}{3} \\ &= \tfrac{1}{2} L^2 \cdot \tfrac{\sqrt{3}}{2} \end{aligned}\right\} \checkmark$$

$A = \frac{\sqrt{3}}{4} L^2$

$\frac{dA}{dL} = \frac{2L \cdot \sqrt{3}}{4}$ ✓

$L = 10$ cm

$\delta L = 0.1$ cm.

$\frac{\delta A}{\delta L} \approx \frac{dA}{dL} \approx \frac{2L \cdot \sqrt{3}}{4}$

$\delta A \approx 2L\frac{\sqrt{3}}{4} \cdot \delta L$ ✓

$\delta A \simeq 2(10) \cdot \frac{\sqrt{3}}{4} \cdot (0.1)$ ✓

$\delta A \simeq 0.8660254038$

$\delta A \simeq 0.866$ sq cm (3 dp) ✓

$\therefore$ Approximate Change in area of 0.866 sq cm.