Perth Modern School

PERTH MODERN SCHOOL
Exceptional schooling. Exceptional students.

Semester Two Examination, 2019

Question/Answer Booklet

MATHEMATICS
METHODS
UNIT 3 & 4
Section Two:
Calculator-assumed

Your name _____

Your Teacher's name_____

Time allowed for this section
Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction
                   fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper,
                   and up to three calculators approved for use in the WACE examinations

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that
you do not have any unauthorised notes or other items of a non-personal nature in the
examination room. If you have any unauthorised material with you, hand it to the supervisor
before reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	50	35
Section Two: Calculator-assumed	14	14	100	105	65
			Total	155	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section Two: Calculator-assumed **(105 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(4 marks)**

The area of a triangle can be found by the formula $Area = \frac{1}{2} ab \sin C$. Using the incremental formula, determine the approximate change in area of an equilateral triangle, with each side of 5 cm, when each side increases by 0.2 cm.

Solution
$A = \frac{1}{2} x^2 \sin 60$ $= \frac{\sqrt{3} x^2}{4}$ $\Delta A \approx \frac{2\sqrt{3} x}{4} \Delta x$ $= \frac{2\sqrt{3}(5)0.2}{4} \approx 0.866$
Specific behaviours
✓ obtains expression for area in terms of side length   ✓ uses incremental theorem   ✓ differentiates correctly   ✓ determines approx. change in area (no need for units)

See next page

Question 9 **(9 marks)**

(a) The probability of Australian Year 12 students obtaining their driver's licence is thought to be around 65%. How large a sample of students should be used to establish this within a margin of error of 5% at the 90% confidence level? (3 marks)

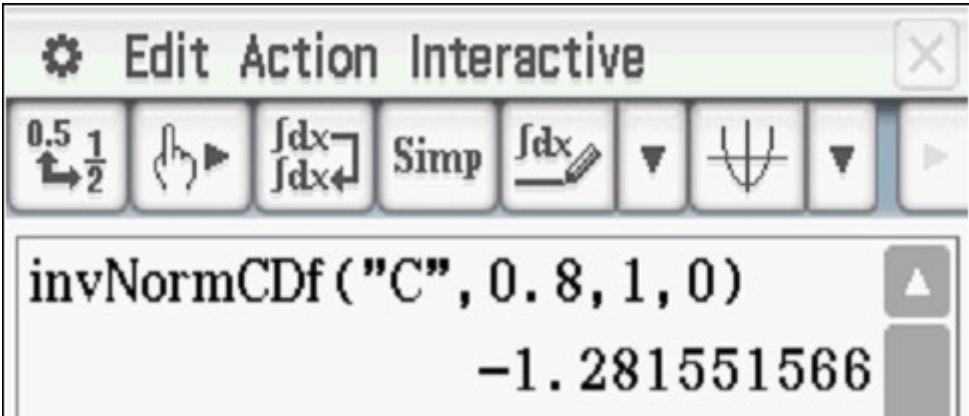| Solution |
|---|
| ☼ Edit Action Interactive ⌧

$\text{solve}\left(0.05 = 1.645 \cdot \sqrt{\dfrac{0.65 \cdot (1-0.65)}{n}}, n\right)$

$\{n = 246.248275\}$

n=247 |
| **Specific behaviours** |
| ✓ uses z percentile for 0.90
✓ states an equation for sample size
✓ rounds n up |

(b) A survey of 1000 Australians renting houses found that 35% of their income was paid in rent.

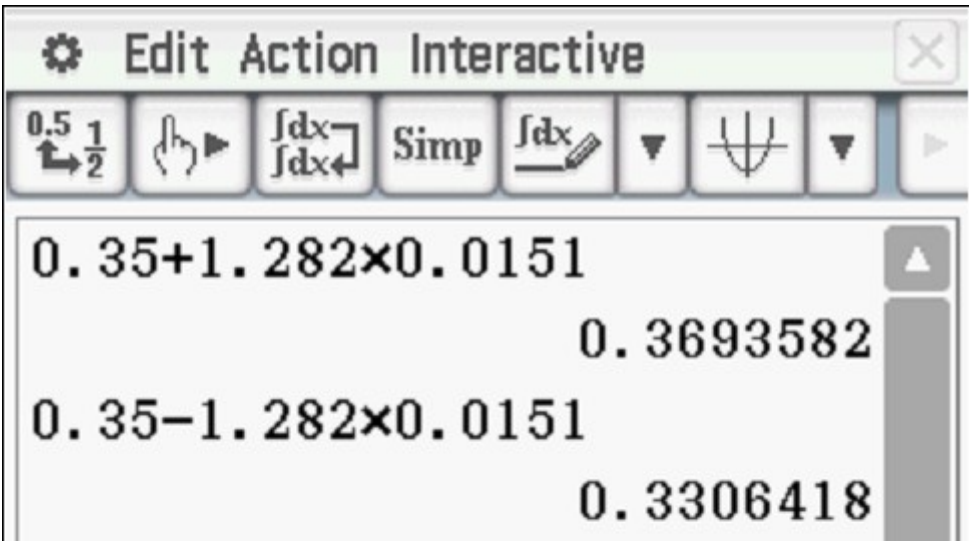(i) Estimate the standard deviation of the sampling distribution. (2 marks)

Solution
☼ Edit Action Interactive ⌧ $\sqrt{\dfrac{0.35(1-0.35)}{1000}}$ $0.0150831 0313$
Specific behaviours
✓ uses formula ✓ states standard deviation

See next page

(ii)   What z-score should be used for an 80% confidence interval?   (1 marks)

Solution
Edit Action Interactive
invNormCDf("C", 0.8, 1, 0)
−1.281551566
Z score 1.282
Specific behaviours
✓ states z score

(iii)   Estimate the 80% confidence interval for the proportion of income used for renting in Australia.   (3 marks)

Solution
Edit Action Interactive
0.35+1.282×0.0151
0.3693582
0.35−1.282×0.0151
0.3306418
Specific behaviours
✓ multiplies z score by standard deviation
✓ shows calculation for one endpoint
✓ states other endpoint (ignore number of dp)

See next page

NOTE: max of 2 marks for interval only

Question 10 **(6 marks)**

Let $f(x) = \dfrac{\ln(x+2)}{x}$.

a) Determine $f'(x)$ using the quotient rule. (3 marks)

Solution
$f(x) = \dfrac{\ln(x+2)}{x}$ $f'(x) = \dfrac{x\dfrac{1}{x+2} - \ln(x+2)}{x^2}$
Specific behaviours
✓ uses quotient rule   ✓ diff ln correctly   ✓ states an un-simplified expression

b) Sketch $f(x)$ on the axes below showing any intercepts and asymptotes. (3 marks)

Solution
Specific behaviours
✓ shows two asymptotes   ✓ shows x intercept   ✓ correct shape shown only for x>-2

See next page

Question 11   **(9 marks)**

In a general population, 15% of people are left-handed. A class has 32 students and number of who are left-handed is counted. Let X = number of left-handed students in a class of 32 students.
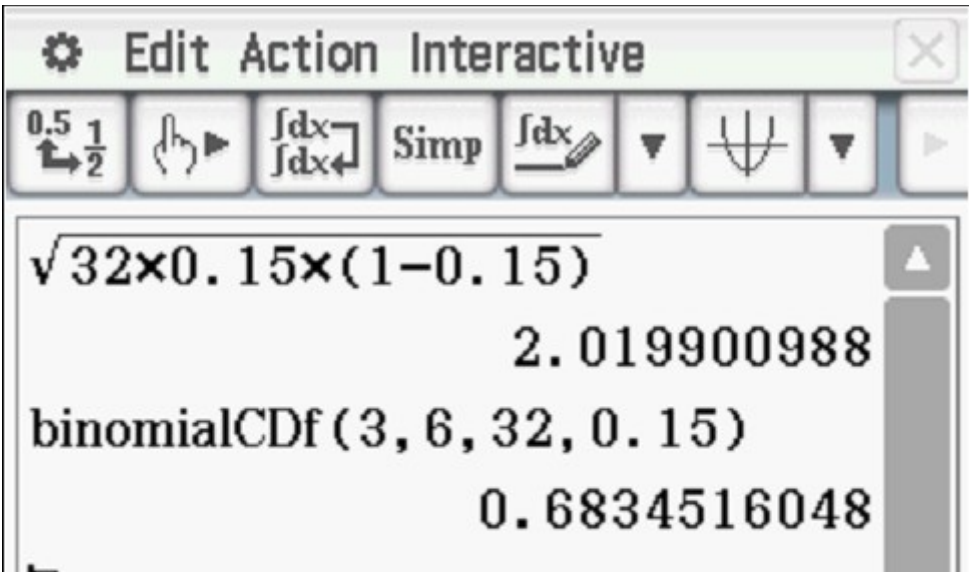
(a) State and describe the type of distribution of X.   (3 marks)

Solution
$X \sim Bin(32, 0.15)$
Specific behaviours
✓ states Binomial ✓ states n value ✓ states p value

(b) How many in the class would you expected to be left-handed?   (1 mark)

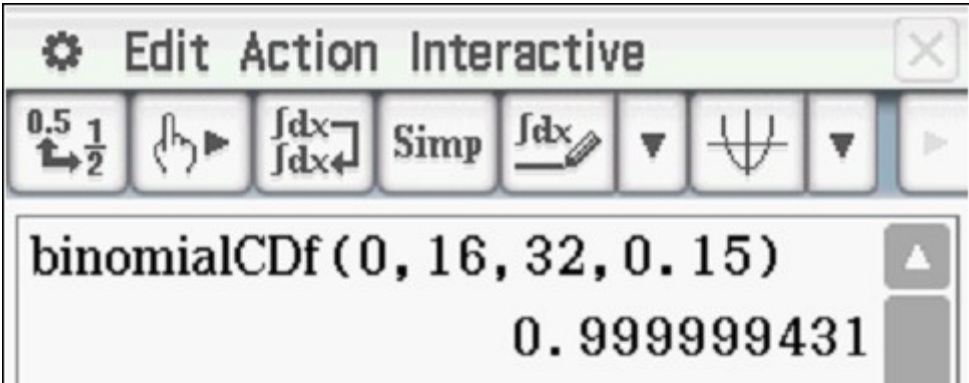| Solution |
|---|
| $\mu = 32 \times 0.15 = 4.8$ |
| **Specific behaviours** |
| ✓ states mean value |

(c) What is the probability that the number of left-handed students in the class is within one standard deviation of the mean?   (3 marks)

Solution
Specific behaviours
✓ calculates standard deviation

See next page

✓ states appropriate integer endpoints **within** one standard deviation
✓ states probability

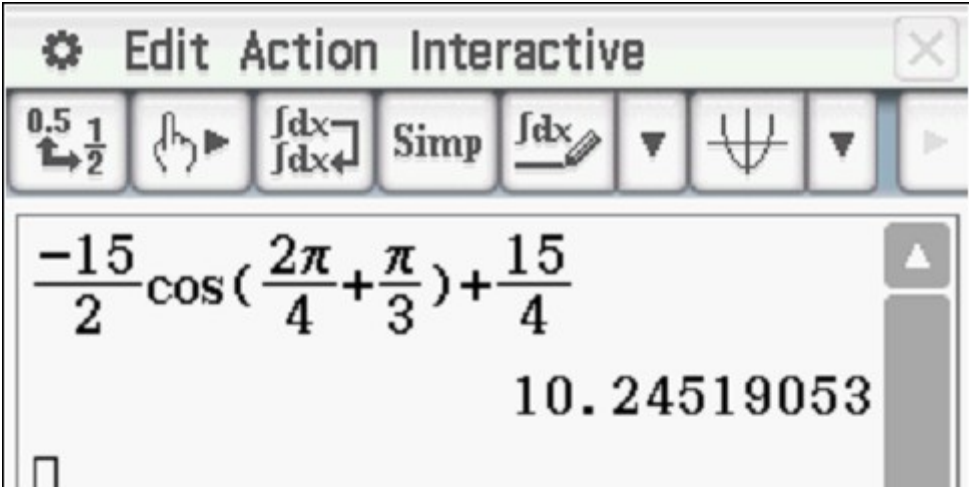(d) Find the probability that at most half of the class are left-handed. (2 marks)

Solution

```
⚙ Edit Action Interactive                    ✕

0.5 1/2  👆▶  ∫dx⌐/∫dx↵  Simp  ∫dx✎  ▼  ⊎  ▼      ▶

binomialCDf(0,16,32,0.15)           △
                        0.999999431
```

Specific behaviours

✓ uses correct limits
✓ states probability(accept 1.0)

Question 12   **(6 marks)**

The equation $a = 15\sin(2t + \frac{\pi}{3})$ is used to measure the acceleration, a metres per second squared at time t seconds, of a weight at the end of a spring. Initially the weight is at rest at the origin.

(a) Find the velocity of the weight after $\frac{\pi}{4}$ seconds.   (3 marks)

Solution
$a = 15\sin(2t + \frac{\pi}{3})$ $v = -\frac{15}{2}\cos(2t + \frac{\pi}{3}) + c$ $0 = \frac{-15}{4} + c$ $c = \frac{15}{4}$ ⚙ **Edit Action Interactive** ☒ $\frac{-15}{2}\cos(\frac{2\pi}{4} + \frac{\pi}{3}) + \frac{15}{4}$ 10.24519053
Specific behaviours
✓ integrates sine term   ✓ solves for constant   ✓ determines velocity(no need for units)

(b) Find its displacement from the origin after $\frac{\pi}{2}$ seconds.   (3 marks)

Solution

See next page

$$v = -\frac{15}{2}\cos(2t + \frac{\pi}{3}) + \frac{15}{4}$$

$$x = \frac{-15}{4}\sin(2t + \frac{\pi}{3}) + \frac{15}{4}t + c$$

$$c = \frac{15}{4}\sin\frac{\pi}{3} = 3.248$$

⚙ Edit Action Interactive   ✕

$$\frac{-15}{4}\sin(2(\frac{\pi}{2}) + \frac{\pi}{3}) + \frac{15}{4}(\frac{\pi}{2}) + \frac{15\sin(\frac{\pi}{3})}{4}$$

          12.38567675

Specific behaviours

✓ integrates cosine term
✓ solves for constant
✓ determines displacement (no need for units) NOTE: follow through from (a)above

Question 13   **(9 marks)**

A chocolate factory makes bars of chocolate such that the mass of these bars forms a normal distribution of mean mass of 35 grams and a standard deviation of 4 grams. Only those bars that have a mass within 1.5 standard deviations are usually sent to stores for sale.

   a) Determine the probability that a bar of chocolate produced, will be usually sent to a store
       for sale.   (3 marks)

Solution

⚙ Edit Action Interactive   ✕

normCDf(29, 41, 4, 35)

          0.8663855975

Specific behaviours

✓ states lower limit
✓ states upper limit
✓ determines probability
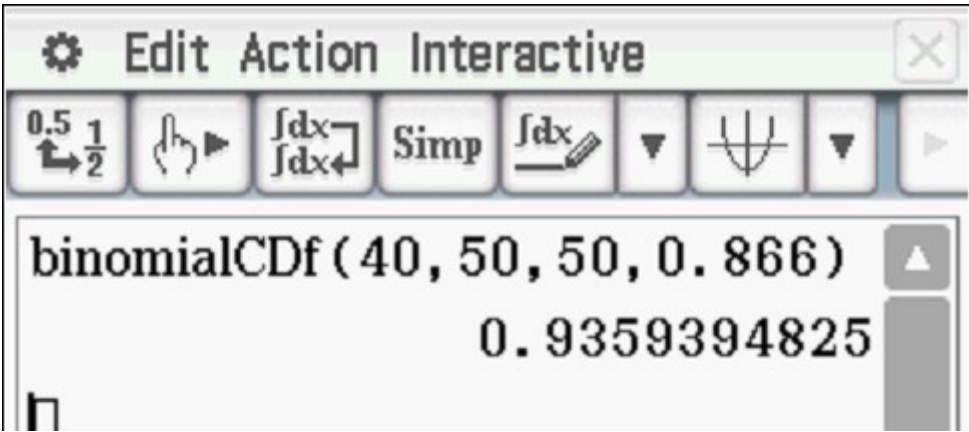Note: 2 marks for answer only

See next page

In December each year, the person who's job is to check the mass of bars is on holiday and no one replaces their role. The bars of chocolate are packed in boxes of 50.

Let X = number of bars suitable for sale in a box of 50 in December.

b) State the distribution for X giving the mean and standard deviation.   (3 marks)

Solution
$X \sim Bin(50, 0.866)$ $\mu = 43.3$ $\sigma = 2.410$
Specific behaviours
✓ states binomial   ✓ states mean   ✓ states standard deviation

c) Determine the probability that in a given box of 50 that at least 40 are suitable for sale in December.   (2 marks)

Solution
☼ Edit Action Interactive ✕   binomialCDf (40, 50, 50, 0.866)   0.9359394825
Specific behaviours
✓ uses correct parameters   ✓ states probability

See next page

d) It is decided to employ a part time worker in December to select boxes of 50 at random to check the number of bars that are suitable for sale. State a method for randomly selecting these boxes for inspection.   (1 mark)

Solution
Use a random number generator for selecting which order of boxes will be examined.
Specific behaviours
✓ reasonable random method of selecting boxes. (Do not accept every third box etc as not random)

Question 14   **(8 marks)**

For the following questions, show **using calculus** how to determine an optimal solution.

(a) Determine two positive numbers, x and y, whose product is 36 and such that $2x+y$ is a minimum.

                                                   (4 marks)

Solution
$xy = 36$ $y = \dfrac{36}{x}$ $S = 2x + y = 2x + 36x^{-1}$ $\dfrac{dS}{dx} = 2 - 36x^{-2} = 0$ $x^2 = 18$ $x = \sqrt{18} = 3\sqrt{2}$ $y = \dfrac{36}{\sqrt{18}} = \dfrac{12}{\sqrt{2}} = 6\sqrt{2}$
Specific behaviours
✓ obtains expression in terms of one variable only ✓ differentiates ✓ equates to zero and solves for exact value of one variable (no need to simplify) ✓ solves for other variable

(b) Find the area of the largest rectangle of lengths $x \& y$ that can be enclosed in a circle of radius r.

(4 marks)

Solution
$x^2 + y^2 = 4r^2$
$y = \sqrt{4r^2 - x^2}$
$A = xy = x\sqrt{4r^2 - x^2}$
$\dfrac{dA}{dx} = x\dfrac{1}{2}\left(4r^2 - x^2\right)^{\frac{-1}{2}}(-2x) + \sqrt{4r^2 - x^2} = 0$
$\dfrac{x^2}{\sqrt{4r^2 - x^2}} = \sqrt{4r^2 - x^2}$
$x^2 = 4r^2 - x^2$
$2x^2 = 4r^2$
$x = \sqrt{2}r$
$y = \sqrt{4r^2 - 2r^2} = \sqrt{2}r$
$A = xy = 2r^2$
Specific behaviours
✓ obtains y in terms of x and r (or vice-versa)
✓ differentiates area using the product rule
✓ equates to zero and solves for x in terms of r
✓ determines other variable in terms of r and states area in terms of r.

Question 15   **(8 marks)**

A game consists of a wheel divided into sixteen equal sectors numbered 1 through 16. You may place a $2 bet on any number. If your number comes up, you get $20 back. That is, you win $18 after getting your $2 back. If you lose then you have lost $2.

(a) Let X be the amount of profit won on a bet by the player. Find the amount the player
$$E(x)$$
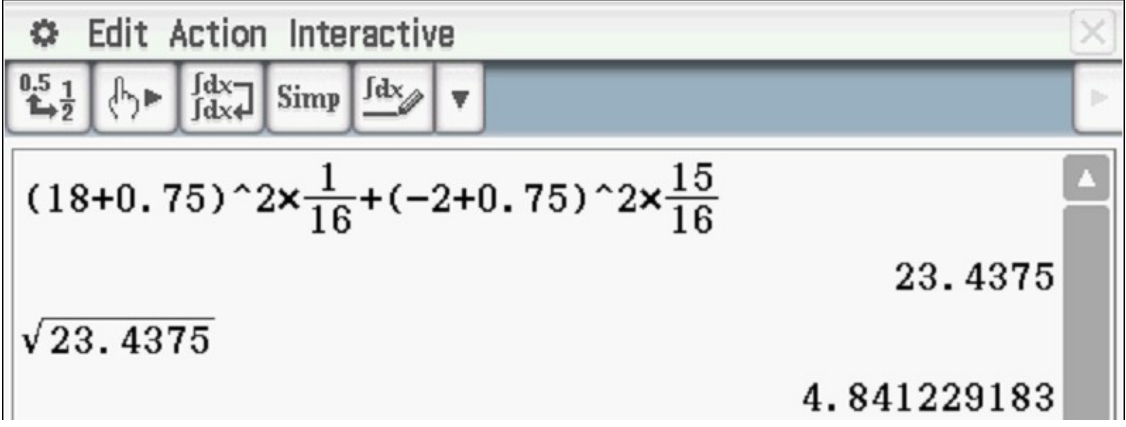expects to profit on a bet. i.e   , explain what this means.
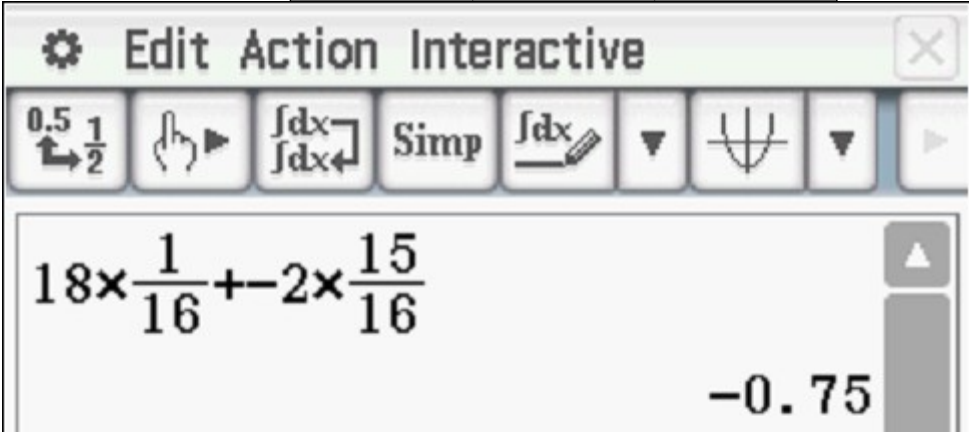
(3 marks)

	Solution	
Solution		

x	$18	-$2
$P(x)$	1/16	15/16

$$18 \times \frac{1}{16} + -2 \times \frac{15}{16}$$

$$-0.75$$

Specific behaviours

✓ shows calculation of mean
✓ states mean
✓ explains that the player expects to lose on a play
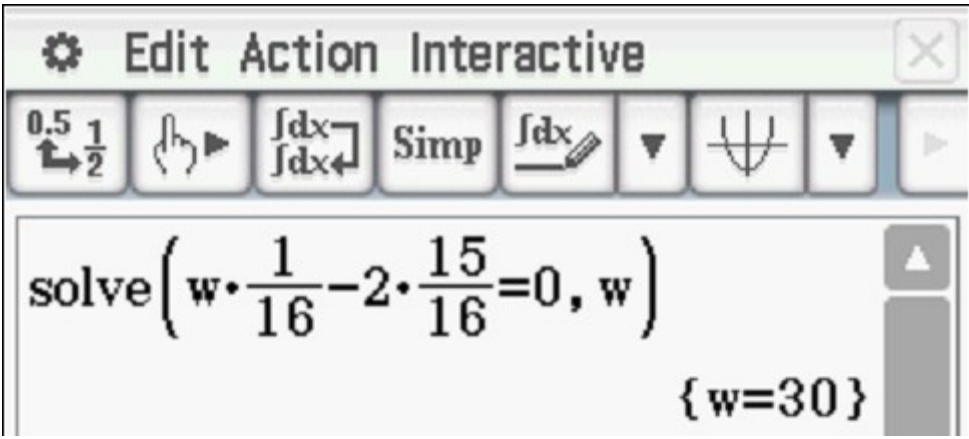
(b) Determine the standard deviation of X , amount of profit won on a bet by the player
(3 marks)

Solution

☼ Edit Action Interactive ⊠ ▷
$(18+0.75)^{\wedge}2 \times \dfrac{1}{16}+(-2+0.75)^{\wedge}2 \times \dfrac{15}{16}$ 23.4375 $\sqrt{23.4375}$ 4.841229183
Specific behaviours
✓ shows calculation ✓ determines variance(uses square root in calculation) ✓ determines standard deviation

(c) How much should you win for the game to be fair? ($E(x)=0$)   (2 marks)

Solution
☼ Edit Action Interactive ⊠ $\text{solve}\left(w \cdot \dfrac{1}{16}-2 \cdot \dfrac{15}{16}=0, w\right)$ {w=30}
WIN $30 i.e profit of $30 Or you get back $32
Specific behaviours
✓ sets up equation for profit on win ✓ states winning profit or cash back

Question 16 **(8 marks)**
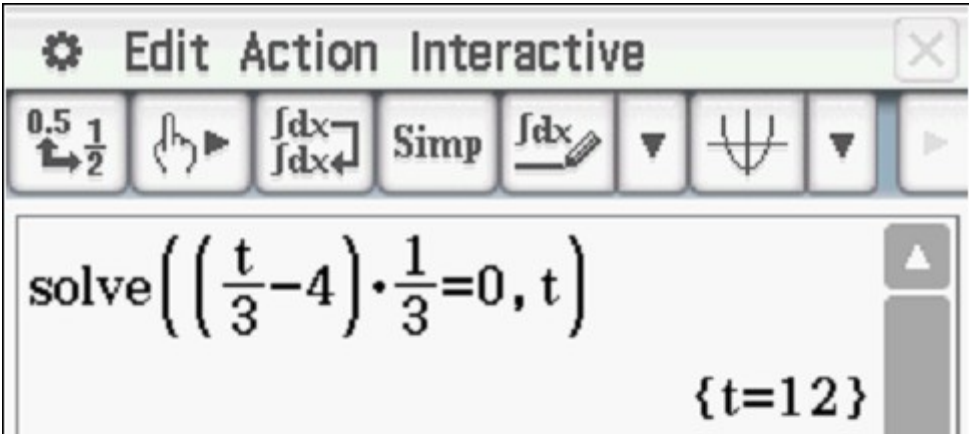
The displacement, x , in centimetres of a particle from the point O in a straight line is given by

$$x(t)=\frac{1}{2}\left(\frac{t}{3}-4\right)^2-2 \text{ for } 0\leq t\leq 12, \text{ where } t \text{ is measured in seconds.}$$
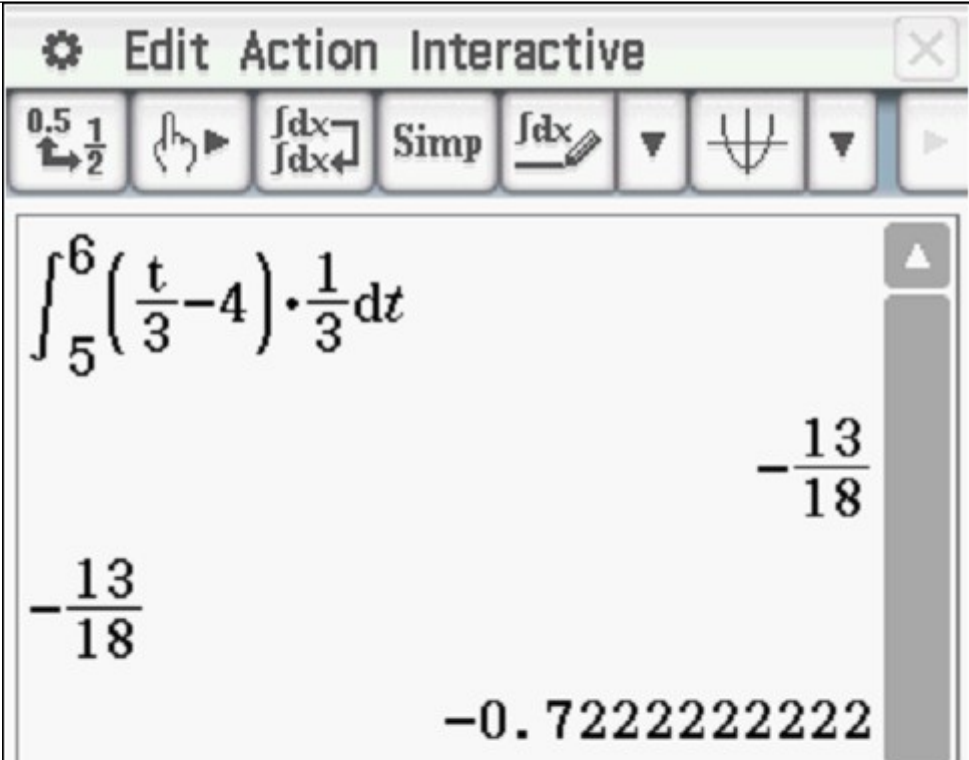
Calculate the:

(a) Time(s) that the particle is at rest. (2 marks)

Solution
☼ Edit Action Interactive ✕
solve$\left(\left(\frac{t}{3}-4\right)\cdot\frac{1}{3}=0, t\right)$
{t=12}
Specific behaviours
✓ differentiates   ✓ solves for t (no need for units)

(b) Displacement of the particle during the sixth second. (2 marks)
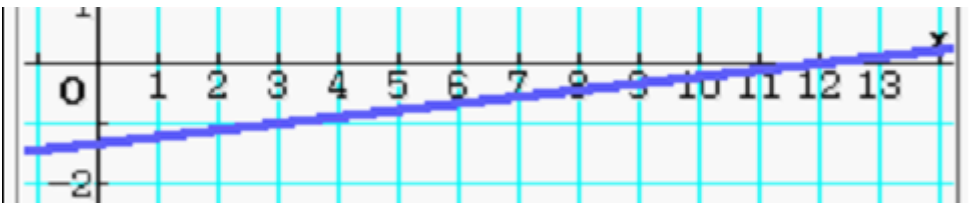
Solution

Specific behaviours
✓ integrates velocity over sixth second or uses displacement for 5 & 6 sec
✓ determines displacement
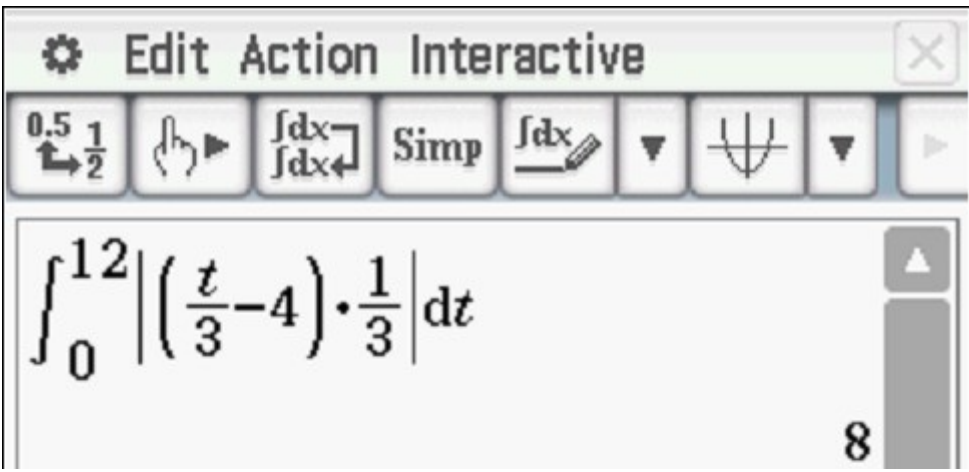
(c) Maximum speed of the particle and the time when this occurs.   (2 marks)

Solution
Initial speed of 4/3 m/3

Specific behaviours
✓ examines first 12 secs only
✓ states initial speed of 4/3 (no need for units) must be positive

See next page

(d) Total distance travelled in the first 12 seconds. (2 marks)

Solution
⚙ Edit Action Interactive
$$\int_0^{12}\left|\left(\frac{t}{3}-4\right)\cdot\frac{1}{3}\right|dt$$
8

Specific behaviours
✓ uses appropriate integral or number line diagram
✓ states distance of 8 metres (no need for units)

Question 17   **(7 marks)**

A school has analysed the exam scores for all Year 12 Methods students and found that the mean was 76 with a standard deviation of 21. Let X be the exam scores out of a 100. Determine the following.

(a) $E(X-6)$ (1 mark)

Solution
$76 - 6 = 70$

Specific behaviours
✓ states new expected value

(b) $Var(10-2X)$ (2 marks)

Solution
$Var = 2^2 \times 21^2 = 1764$

Specific behaviours
✓ squares standard deviation
✓ multiplies by positive four only.

The school has decided to scale the results down to mean of 65 and standard deviation of 15, using the transformation $Y=aX+b$ where a and b are constants and Y is the scaled percentage scores.

(c) Determine the value(s) of a and b.   (4 marks)

Solution

☼ Edit Action Interactive

[0.5 1/2] [☞▶] [∫dx⌐ / ∫dx↵] [Simp] [∫dx✎] [▼]

$\begin{cases} 21a=15 \\ 76a+b=65 \end{cases} \Big| a,b$

$\{a=0.7142857143, b=10.71428571\}$

$\begin{cases} 21a=-15 \\ 76a+b=65 \end{cases} \Big| a,b$

$\{a=-0.7142857143, b=119.2857143\}$

□

Alg   Decimal   Cplx   Rad
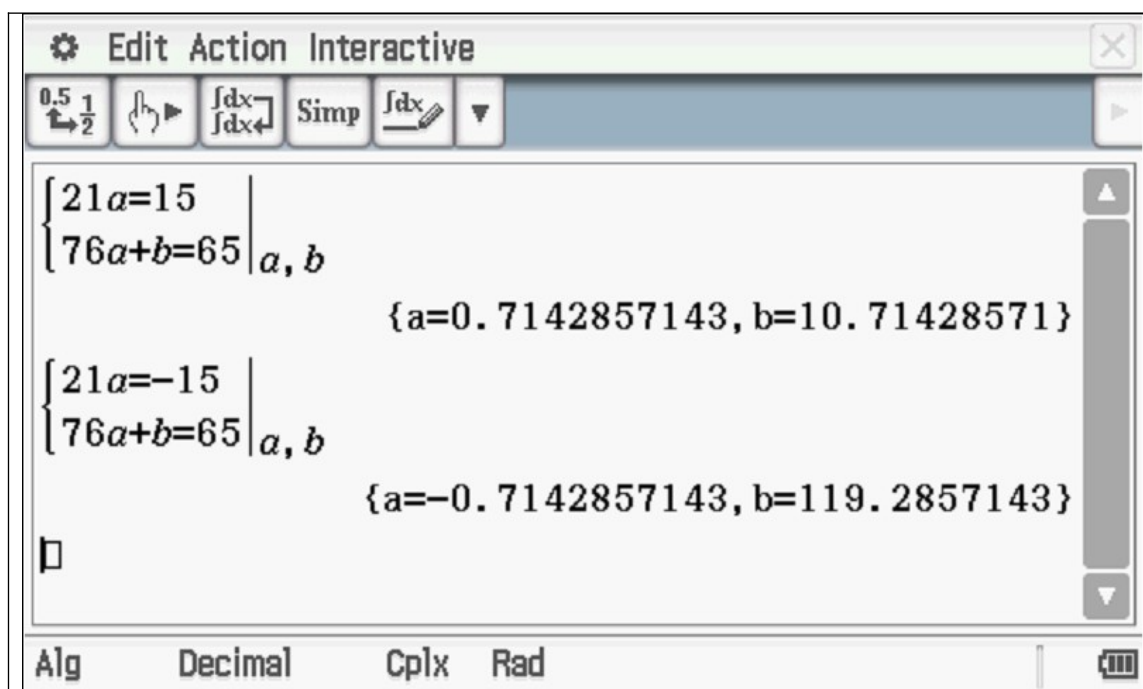
Specific behaviours
✓ sets up one equation for a only
✓ sets up second equation with a and b
✓ solves for one pair of values
✓ solves for second pair of values
Max of 3 out of 4 marks if multiplying by a is not first
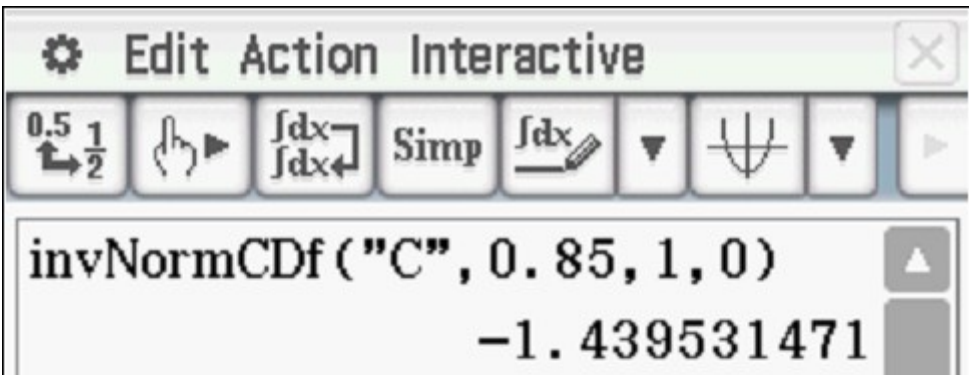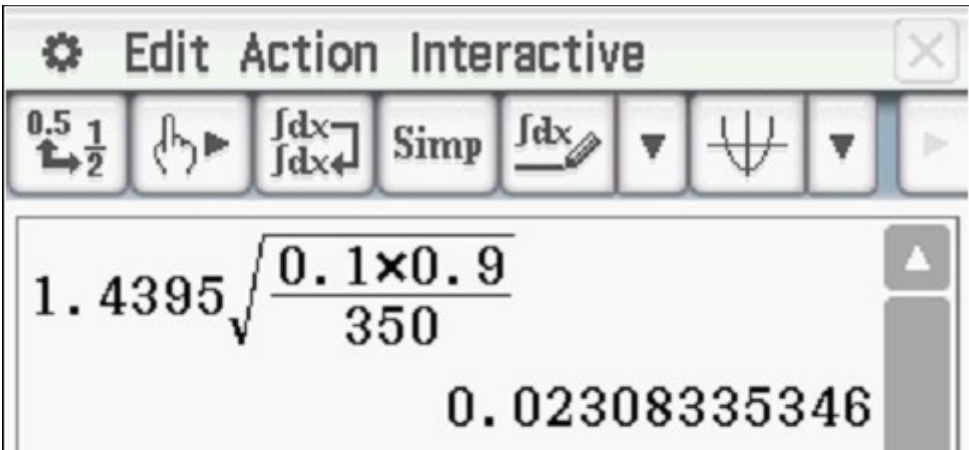
Question 18   **(10 marks)**
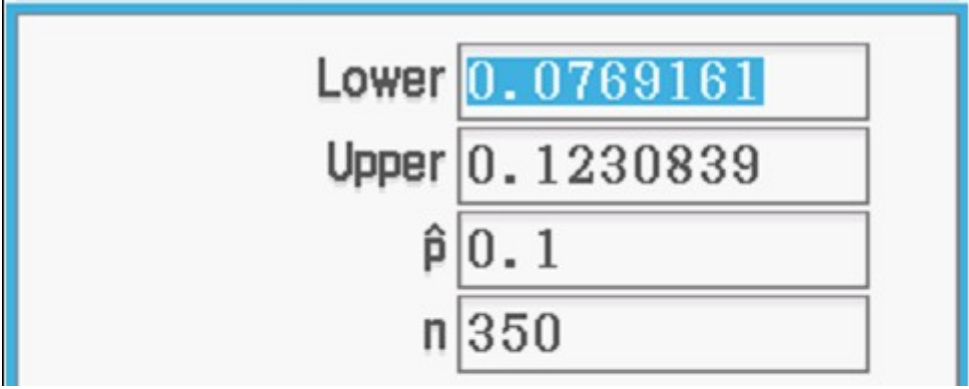
A government agency is investigating the number of motor vehicles in Perth that are producing an illegal amount of pollution. In a sample of 350 motor vehicles, 35 where found to produce an illegal amount of pollution.

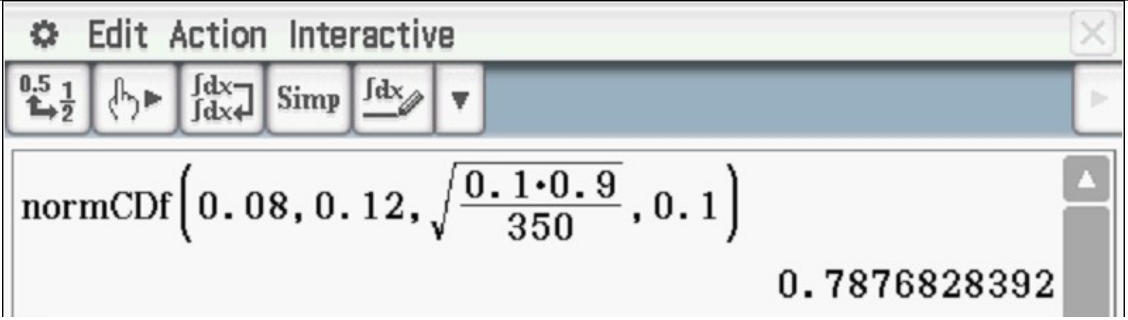a) Calculate the sample proportion of vehicles that are illegally polluting.   (1 mark)

Solution
$35/350 = 0.10$
Specific behaviours
✓ states proportion

b) Determine an 85% confidence interval of the true proportion of vehicles illegally polluting to four decimal places and state the margin of error,   (4 marks)

Solution

Lower 0.0769161
Upper 0.1230839
$\hat{p}$ 0.1
n 350

0.0769- 0.1231
Margin of error =0.0231

Specific behaviours
✓ states z percentile
✓ determines confidence interval
✓ determines margin of error
✓ rounds to 4 dp

c) Determine the probability that in a random sample of 350 vehicles, the proportion of polluting vehicles lies between 0.08 and 0.12. (2 marks)

Solution
☼ Edit Action Interactive
$\text{normCDf}\left(0.08, 0.12, \sqrt{\dfrac{0.1 \cdot 0.9}{350}}, 0.1\right)$
0.7876828392

Specific behaviours
✓ uses normal with correct parameters
✓ states probability (no need to give 4 dp)

d) Below are three samples of vehicles, one of them is actually from another country. Which one is most likely not to be in Perth and give reasons. (3 marks)
(Continue with a confidence of 85%)

Sample A	Sample B	Sample C
44 polluting vehicles out of a sample size of 340	20 polluting vehicles out of a sample size of 280	50 polluting vehicles out of a sample size of 420

Solution

Lower	0.1032073	Lower	0.0492729	Lower	0.0963001
Upper	0.1556162	Upper	0.0935843	Upper	0.1417951
$\hat{p}$	0.1294118	$\hat{p}$	0.0714286	$\hat{p}$	0.1190476
n	340	n	280	n	420
<< Back	☐ Help	<< Back	☐ Help	<< Back	☐ Help
OnePropZInt		nePropZInt		OnePropZInt	

Sample B' confidence interval does not overlap with the other two, therefore most likely.

OR
Not all intervals contain true value of p therefore unable to tell.

Specific behaviours

✓ gives two confidence intervals
✓ gives three confidence intervals
✓ gives an explanation which mentions confidence intervals

Question 19   **(6 marks)**

The diagram below shows the graph of $y=(x+1)(x-1)(x-d)$ with constant d where $d>1$. The graph intercepts the y-axis at the point A.
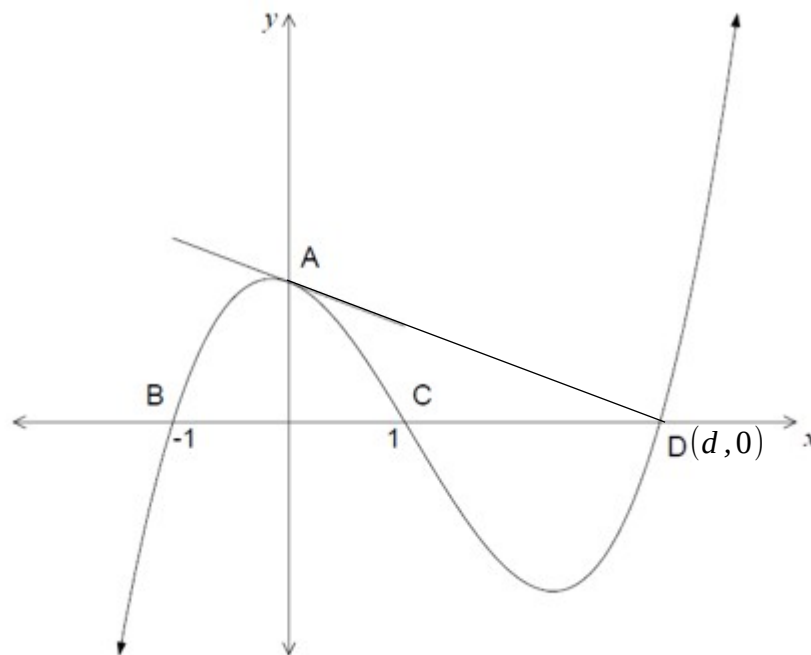
(a) Find the equation of line AD in terms of d.   (2 marks)

Solution
$A(0,d) \; D(d,0)$ $gradient = \dfrac{0-d}{d-0} = -1$ $y = -x + c$ $d = c$ $y = -x + d$
Specific behaviours
✓ determines gradient ✓ determines equation

(b) Show that a line that passes through A and D is tangent to the graph at A.   (4 marks)

See next page

Solution
⚙ Edit Action Interactive   ✕
$\dfrac{d}{dx}\left((x+1)\cdot(x-1)\cdot(x-d)\right)$
$3\cdot x^2 - 2\cdot d \cdot x - 1$
$3\cdot x^2 - 2\cdot d \cdot x - 1 \mid x=0$
-1

As both line and curve have a common point A at x=0 and same gradient at this point then line is a tangent

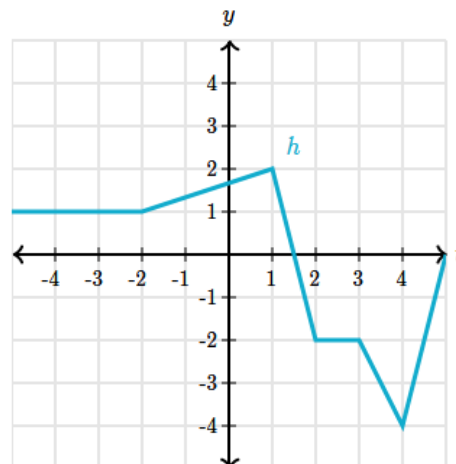| Specific behaviours |
|---|
| ✓ differentiates curve |
| ✓ determines derivative at x=0 |
| ✓ states common point A |
| ✓ states gradients both -1 at pt A |

Question 20 **(10 marks)**

The graph of $h(x)$ is shown on the right.

(a) Evaluate the following definite integrals

(i) $\int_{-2}^{1} h'(x)\,dx$ (2 marks)

Solution
$\int_{-2}^{1} h'(x)dx = \left[h(x) \right]_{-2}^{1} = h(1) - h(-2) = 2 - 1 = 1$
Specific behaviours
✓ uses fundamental theorem
✓ determines integral

(ii) $\int_{2}^{3} 10\,h'(x)\,dx$ (2 marks)

Solution
$10 \int_{2}^{3} h'(x)dx = 10\left[h(x) \right]_{2}^{3} = 10[h(3) - h(2)] = 10(0) = 0$
Specific behaviours
✓ uses fundamental theorem with factor 10
✓ determines integral

(iii) $\int_{-2}^{4} \left(h¿¿'(x) + 1 \right) dx¿$ (2 marks)

Solution
$\int_{-2}^{4} h'(x) + 1\,dx = \left[h(x) + x \right]_{-2}^{4} = (h(4) + 4) - (h(-2) - 2)$ $= 0 - 1 + 2 = 1$

See next page

Specific behaviours
✓ integrates both terms correctly
✓ uses limits to determine integral

(b) Find the area bounded by the graph of $h'(x)$ and the x axis between $x=-2$ and $x=4$. Justify your answer.   (4 marks)

Solution
$Area = \int_{-2}^{1} h'(x)dx + \left| \int_{1}^{4} h'(x)dx \right|$ $= 1 + |h(4) - h(1)| = 1 + |-4 - 2| = 7$
Specific behaviours
✓ recognises that some parts have a negative integral
✓ determines these sections separately
✓ uses absolute value
✓ determines total area

Question 21   **(5 marks)**

Let $g(x) = \int_{-2}^{x} f(t)\,dt$ with $g(2) = 20$ and $\dfrac{d^2 g}{dx^2} = 6x$. Determine the function $f(x)$.

Solution

$$g(x) = \int_{-2}^{x} f(t)\,dt$$

$$g'(x) = f(x)$$

$$g''(x) = f'(x)$$

$$f'(x) = 6x$$

$$f(x) = 3x^2 + c$$

$$g(2) = \int_{-2}^{2} (3x^2 + c)\,dt = 20 = \left[t^3 + ct\right]_{-2}^{2} = (8 + 2c) - (-8 - 2c)$$

$$20 = 16 + 4c$$

$$c = 1$$

$$f(x) = 3x^2 + 1$$

Specific behaviours
✓ uses fundamental theorem
✓ obtains expression for second derivative of g
✓ integrates to find f with constant
✓ uses g integral to set up 1 equation in terms of c OR 2 equations in terms of 2 constants
✓determined function f in terms of x

End of Questions

Additional working space

Question number: _____

Additional working space

Question number: _____