WESLEY COLLEGE
By daring & by doing

YEAR 12 MATHEMATICS METHODS
SEMESTER ONE 2018 TEST 3
DIFFERENTIAL CALCULUS APPLICATIONS, DISCRETE RANDOM VARIABLES, BERNOULLI TRIALS AND BINOMIAL DISTRIBUTIONS

Thursday 12th April

Name: SOLUTIONS

Time: 50 minutes **Part A:** $\frac{\quad}{20}$ **Part B:** $\frac{\quad}{30}$ **Total:** $\frac{\quad}{50}$ **%**

- Answer all questions neatly in the spaces provided. **Show all working.**
- You are permitted to use the Formula Sheet for both sections, and an A4 page of notes, plus up to 3 permitted calculators in the Calculator Allowed section.

Topic	Confidence
Further differentiations and applications • The second derivative and applications of differentiation	Low Moderate High
Discrete random variables • General discrete random variables	Low Moderate High
• Bernoulli distributions	Low Moderate High
• Binomial distributions	Low Moderate High

Self reflection (eg. comparison to target, content gaps, study and work habits etc)

Calculator Free Section **20 minutes** **/20**

1. [8 marks]

The displacement, $x\text{ cm}$, of a particle at time t seconds, moving along a horizontal track is described by the function $x = 5\cos(3t)$.

a) Determine the initial position and velocity of the particle.

$$x(0) = 5\text{ cm}$$
$$\dot{x} = -15\sin(3t)$$
$$\dot{x}(0) = 0\text{ cm/s}$$

[3]

b) Determine the exact time when the particle first turns around.

$$\text{Let } \dot{x} = 0$$
$$-15\sin(3t) = 0$$
$$\sin(3t) = 0$$
$$3t = 0, \pi, 2\pi, \ldots$$
$$t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots$$

First turns around when $t = \dfrac{\pi}{3}\text{ s}$

[2]

c) Determine the exact rate of change of speed of the particle when $t = \dfrac{\pi}{4}$ seconds.

$$\ddot{x} = -45\cos(3t)$$
$$\ddot{x}\left(\frac{\pi}{4}\right) = -45\cos\left(\frac{3\pi}{4}\right)$$
$$= \frac{45\sqrt{2}}{2}\text{ cm/s}^2$$

[3]

2. **[7 marks]**

Jack was investigating the variance of binomial distributions for different probabilities and exploring the connection to calculus.

a) For a random variable Y, where $Y \sim \mathrm{Bin}(5, 0.4)$, calculate the variance, $\mathrm{Var}(Y)$.

$$\begin{aligned}\sigma^2 &= np(1-p)\\ &= 5 \times 0.4 \times 0.6\\ &= 1.2\end{aligned}$$

[2]

b) For the general random variable X, where $X \sim \mathrm{Bin}(n, p)$,

i) Determine a function in terms of the probability p, for the variance, $\mathrm{Var}(X)$.

$$\begin{aligned}\sigma^2 &= np(1-p)\\ &= np - np^2\end{aligned}$$

ii) Use calculus techniques to show that the maximum variance is achieved when $p = 0.5$. Justify that your result is a maximum.

$$\text{Let } V = np - np^2 \qquad 0 \le p \le 1$$

$$\frac{dV}{dp} = n - 2np$$

$$\text{Let } 0 = n - 2np$$

$$p = \frac{1}{2}$$

$$\frac{d^2V}{dp^2} = -2n$$

$$< 0 \ \forall n \in ¢^{+}$$

Hence max

[5]

3. [5 marks]

A discrete random variable X has the following properties:

- the expected value $E(X) = 18$
- the standard deviation $\sigma = \dfrac{3\sqrt{5}}{2}$.

a) If the random variable is binomial, determine the number of trials and probability of success.

$$np = 18 \ldots [1]$$

$$np(1-p) = \left(\frac{3\sqrt{5}}{2}\right)^2 = \frac{45}{4} \ldots [2]$$

sub [1] into [2]

$$18(1-p) = \frac{45}{4}$$

$$1-p = \frac{5}{8}$$

$$p = \frac{3}{8}$$

$$n = 48$$

[3]

b) Determine the expected value $E(Y)$ and variance $\mathrm{Var}(Y)$ if Y is a random variable such that $Y = 5 - 2X$.

$$E(Y) = 5 - 2 \times 18$$
$$= -31$$

$$Var(Y) = (-2)^2 \times \frac{45}{4}$$
$$= 45$$

[2]

Name: ______________________________

Calculator Allowed Section 30 minutes /30

4. [11 marks]

Consider the function $y = \dfrac{10\ln(x)}{x^2}$.

a) Determine $\dfrac{dy}{dx}$ and its associated domain. Hence determine the exact location and nature of the stationary point(s).

$$\frac{dy}{dx} = \frac{-(20\ln(x)-10)}{x^3},\ x>0$$

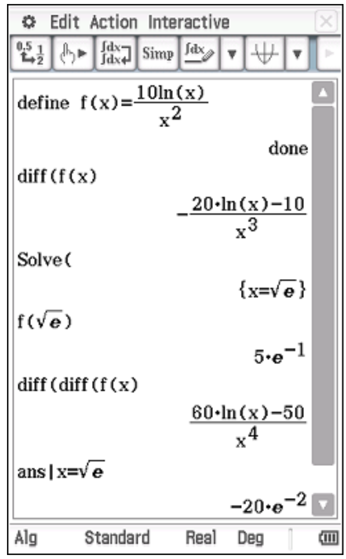
$$\text{Let } 0 = \frac{-(20\ln(x)-10)}{x^3}$$

$$x = \sqrt{e}$$

$$\frac{d^2y}{dx^2} = \frac{60\ln(x)-50}{x^4}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\sqrt{e}} < 0 \text{ Hence max TP}$$

$$\left(\sqrt{e}, \frac{5}{e}\right)$$



[5]

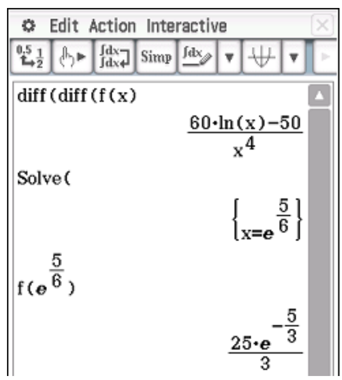
b) Determine the exact location of any inflection points.

$$\frac{d^2y}{dx^2} = \frac{60\ln(x)-50}{x^4},\ x>0$$

$$\text{Let } 0 = \frac{60\ln(x)-50}{x^4}$$

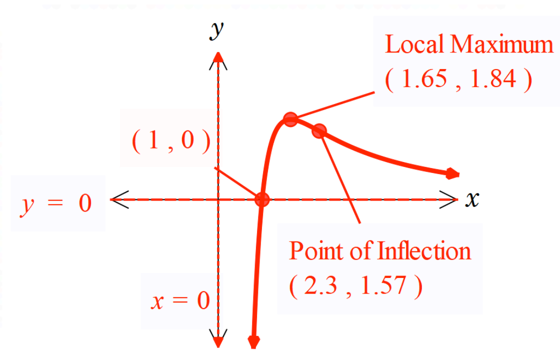
$$x = e^{\frac{5}{6}}$$

$$\left(e^{\frac{5}{6}}, \frac{25e^{-\frac{5}{3}}}{3}\right)$$



[3]

c) Sketch the graph of the function labelling key features (to 2 decimal places).



[3]

5. **[9 marks]**

Aaron and Brad are playing a tennis match. The match continues until one player wins a total of two (2) sets. Aaron estimates from past experience that his chance of winning any set against Brad, independent from any previous sets, is $\frac{3}{10}$.

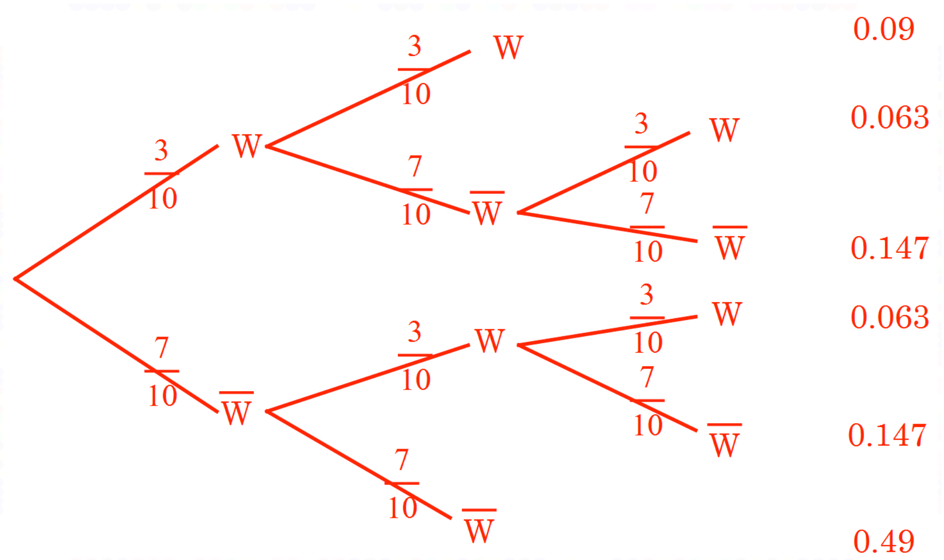
Let the random variable X be the number of sets won by Aaron in the match.

a) Give a reason as to why X cannot be modelled by a binomial distribution.

Number of trials is not fixed.

[1]

b) Draw a tree diagram to show the possible outcomes of the match and the associated probabilities. Hence complete the probability density function for X in the table below, stating answers as fractions.



x	0	1	2
$P(X = x)$	$0.49 = \frac{49}{100}$	$0.294 = \frac{147}{500}$	$0.216 = \frac{27}{125}$

[4]

c) Determine the probability that Aaron wins the match, given he wins the first set.

$$\frac{0.09 + 0.063}{0.3} = 0.51$$

[2]

d) Calculate the expected value of X as a decimal, and explain its meaning in the context of the question.

$$E(X) = 0 \times 0.49 + 1 \times 0.294 + 2 \times 0.216$$
$$= 0.726$$

Aaron can expect to win, on average, ~0.73 sets in each match he plays against Brad.

[2]

6. [7 marks]

Based on shipments of mobile phones to Australia in the last quarter of 2017, the Apple iPhone has a market share of around 37%[i]. Assume that every Australian has exactly one mobile phone.

A random survey of 20 people was conducted on mobile phone type. Showing appropriate probability notation, determine the probability, to three decimal places, that

$X \sim Bin(20, 0.37)$

a) Exactly six respondents had an iPhone.

$P(X=6) \approx 0.154$

[2]

b) At least six respondents had an iPhone.

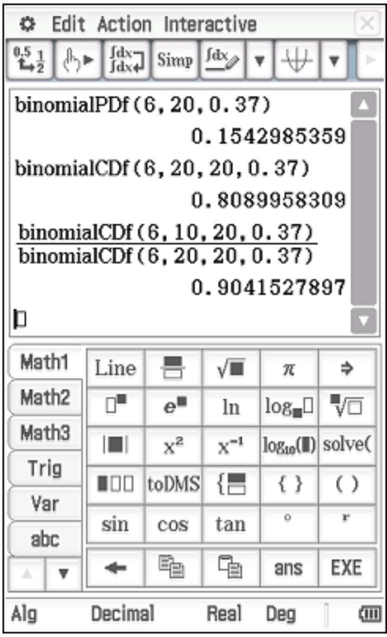
$P(X \geq 6) \approx 0.809$

[2]

c) No more than ten respondents had an iPhone, if it is known at least six had an iPhone.

$$P(X \leq 10 \mid X \geq 6) = \frac{P(6 \leq X \leq 10)}{P(X \geq 6)}$$

$$\approx 0.904$$



[3]

7. [3 marks]

How many times should a fair die be rolled so that the probability of rolling exactly one six is the same as the probability of not rolling a six at all?

$$X \sim Bin\left(n, \frac{1}{6}\right)$$

$$P(X=0) = P(X=1)$$

$$\binom{n}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^n = \binom{n}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{n-1}$$

$$\left(\frac{5}{6}\right)^n = n \times \left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{n-1}$$

$$\frac{5}{6} = \frac{n}{6}$$

$$n = 5$$

[3]

[i] https://www.statista.com/statistics/436033/australia-smartphone-shipments-vendor-market-share/