3CD MATHEMATICS SPECIALIST – INVESTIGATION 1 MARKING KEY

1. For n an integer, the product $(n-1).n.(n+1)$ ✓ is the product of three consecutive integers. In any 3 consecutive integers there will be at least one multiple of 2 and ✓ a multiple of 3. Any product which is a multiple of both 2 and 3 is a multiple of 6, ✓ and is therefore divisible by 6.

2. Prove: $1+4+7+...+(3n-2)=\frac{n(3n-1)}{2}$ for all positive integers n.

Step 1: Verify the statement is true when $n = 1$.

L.H.S. = 1

R.H.S. = $\frac{1(3\times1-1)}{2}$

$=\frac{1\times2}{2}$

$=1$ ✓

$\Rightarrow$ Statement is true for $n = 1$.

Step 2: Assume the statement is true for $n = k$.

That is, $1+4+7+...+(3k-2)=\frac{k(3k-1)}{2}$ ✓

Step 3: Prove the statement is true for $n = k + 1$

That is, prove $1+4+7+...+(3k-2)+(3[k+1]-2)=\frac{(k+1)(3[k+1]-1)}{2}$

$=\frac{(k+1)(3k+2)}{2}$

L.H.S.$=1+4+7+...+(3k-2)+(3[k+1]-2)$ ✓

$=\frac{k(3k-1)}{2}+(3k+1)$ (from step 2) ✓

$=\frac{k(3k-1)+2(3k+1)}{2}$ ✓

$=\frac{3k^2-k+6k+2}{2}$

$=\frac{3k^2+5k+2}{2}$ ✓

$=\frac{(3k+2)(k+1)}{2}$ ✓

= R.H.S.

$\Rightarrow$ The statement is true for $n = k + 1$ if it is true for $n = k$.

Step 4: As the statement is true for $n = 1$, it must be true for $n = 2$.

As the statement is true for n = 2, it must be true for n = 3 and so on.

Hence, $1 + 4 + 7 + ... + n = \frac{n(3n - 1)}{2}$ is true for all positive integers n. ✓

3. Prove $(|z|\text{cis}\theta)^n = |z|^n \text{cis}(n\theta)$ for all positive integers n.

Step 1 Verify the statement is true for n = 1

LHS = $(|z|\text{cis}(\theta))^1$

= $|z|\text{cis}(\theta)$

RHS = $|z|\text{cis}(1\theta)$

= $|z|\text{cis}(\theta)$

⇒ Statement true for n=1 ✓

Step 2 Assume it is true for n = k

That is, $(|z|\text{cis}\theta)^k = |z|^k \text{cis}(k\theta)$ ✓

Step 3 Prove the statement is true for n = k + 1

That is, prove $(|z|\text{cis}\theta)^{k+1} = |z|^{k+1}\text{cis}((k+1)\theta)$ ✓

LHS = $(|z|\text{cis}\theta)^k (|z|\text{cis}\theta)^1$

= $|z|^k (\text{cis}(k\theta))|z|(\text{cis}\theta)$ Step 1 and 2 ✓

= $|z|^{k+1} \text{cis}(k\theta)\, \text{cis}(\theta)$

= $|z|^{k+1}(\cos(k\theta) + i\sin(k\theta))(\cos\theta + i\sin\theta)$ ✓

= $|z|^{k+1}(\cos(k\theta)\cos\theta + i\cos(k\theta)\sin\theta + i\sin(k\theta)\cos\theta + i^2\sin(k\theta)\sin\theta)$ ✓

= $|z|^{k+1}(\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + i(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta$ ✓

= $|z|^{k+1}(\cos(k\theta + \theta) + i\sin(k\theta + \theta))$ ✓

= $|z|^{k+1}\text{cis}(k\theta + \theta)$

= $|z|^{k+1}\text{cis}((k+1)\theta)$

= RHS

Step 4: As the statement is true for n = 1, it must be true for n = 2.
As the statement is true for n = 2, it must be true for n = 3 and so on.

Hence, $(|z|\text{cis}\theta)^n = |z|^n \text{cis}(n\theta)$ is true for all positive integers n. ✓