SCHOOL

Trial WACE Examination, 2012

Question/Answer Booklet

MATHEMATICS SPECIALIST 3C/3D

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination.

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	50	33
Section Two: Calculator-assumed	13	13	100	100	67
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2012*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

Section Two: Calculator-assumed (100 Marks)

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 (6 marks)

In two residential suburbs, A and B, from 1984 to 1999, the median house price, M dollars, increased at a rate given by $\frac{dM}{dt} = kM$, where t is the time, in years and k is a constant specific to each suburb.

For suburb A, the median price at the start of 1984 was \$55 300 and prices were observed to double every 9.5 years. For suburb B, the median price at the start of 1989 was \$74 100 and prices were observed to double every 8.5 years.

In which year was the median house price the same in both suburbs?

Suburb A, with $t = 0$ in 1989:

$e^{9.5k} = 2 \Rightarrow k = 0.072963$

$M = 55300e^{0.072963t}$

$M(5) = 79646$

$M_A = 79646e^{0.072963t}$

Suburb B, with $t = 0$ in 1989:

$e^{8.5k} = 2 \Rightarrow k = 0.081547$

$M_B = 74100e^{0.081547t}$

$$M_A = M_B$$

$$79646e^{0.072963t} = 74100e^{0.081547t}$$

$$t = 8.41 \text{ years}$$

Hence prices the same during 1989+8 = 1997.

Question 9 **(5 marks)**

The following Leslie matrix, L, applies to a population of beetles in which the female beetles in the population live for a maximum of 3 years and only propagate in their third year of life.

$$L = \begin{bmatrix} 0 & 0 & 5 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{5} & 0 \end{bmatrix}$$

(a) What is the probability that a newborn female beetle will survive to the 3rd year of its life? (1 mark)

$$\boxed{\frac{1}{2} \times \frac{2}{5} = \frac{1}{5}}$$

(b) Initially there are 500 females in each age group. How many females will there be altogether after 2 years? (2 marks)

$$L^2 \begin{bmatrix} 500 \\ 500 \\ 500 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1250 \\ 100 \end{bmatrix}$$

Total = 2350 beetles

(c) Comment on the long-term population of female beetles predicted by this model. (2 marks)

The total number of female beetles starts at 1500 initially, then nearly doubles after 1 year to 2950, falls back to 2350 after 2 years and then returns to 1500 after 3 years, with 500 in each age group as at the start of the cycle.

This cycle then repeats endlessly, according to the model.

Question 10 **(7 marks)**

(a) A triangle with vertices at $A(1,\ 1)$, $B(3,\ 1)$ and $C(3,\ 4)$ is reflected in the x-axis and then rotated 90° anticlockwise about the origin.

(i) Find the matrix T that will combine these two transformations in the order given. (3 marks)

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(ii) Find the coordinates of C after transformation by T. (1 mark)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$C'(4,\ 3)$

(b) Another transformation matrix is given by $R = \begin{bmatrix} -0.6 & 0 \\ -1.2 & -0.6 \end{bmatrix}$.

Determine the area of triangle ABC after transformation by T and then by R. (3 marks)

Original area of triangle ABC = 3 sq units.

Determinant of T is -1, so no change in area.

Determinant of R is 0.36, so final area $= 0.36 \times 3 = 1.08$ sq units.

Question 11 **(7 marks)**

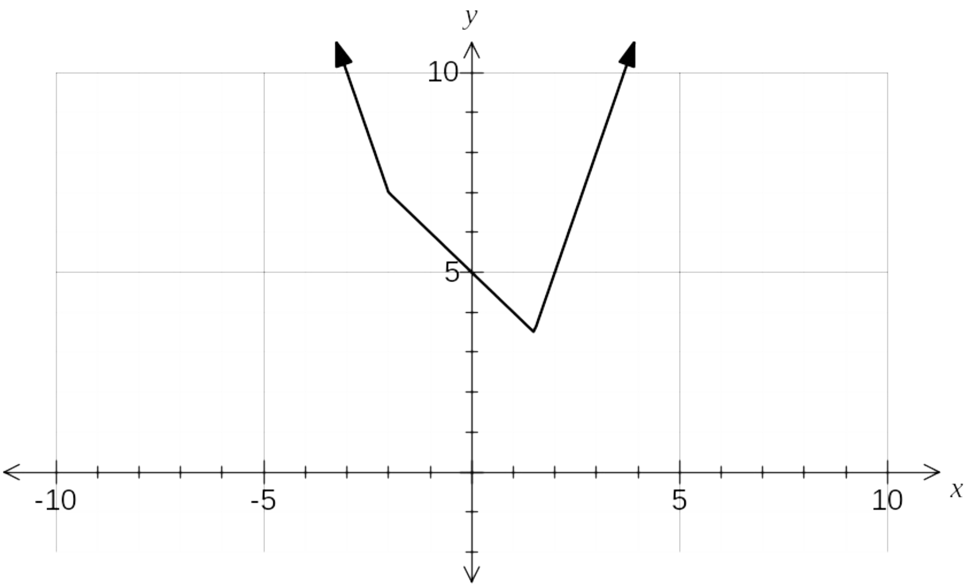
A function is defined as $f(x) = |x+2| + |3-2x|$.

(a) Express $f(x)$ without the use of absolute value bars. (3 marks)

$$f(x) = \begin{cases} -(x+2)+(3-2x) & x < -2 \\ (x+2)+(3-2x) & -2 \le x \le 1.5 \\ (x+2)-(3-2x) & x > 1.5 \end{cases}$$

$$f(x) = \begin{cases} 1-3x & x < -2 \\ 5-x & -2 \le x \le 1.5 \\ 3x-1 & x > 1.5 \end{cases}$$

(b) Sketch the graph of $f(x)$ (2 marks)



(c) Solve $f(x) \le x+5$. (2 marks)

$x+5 = 5-x \Rightarrow x = 0$

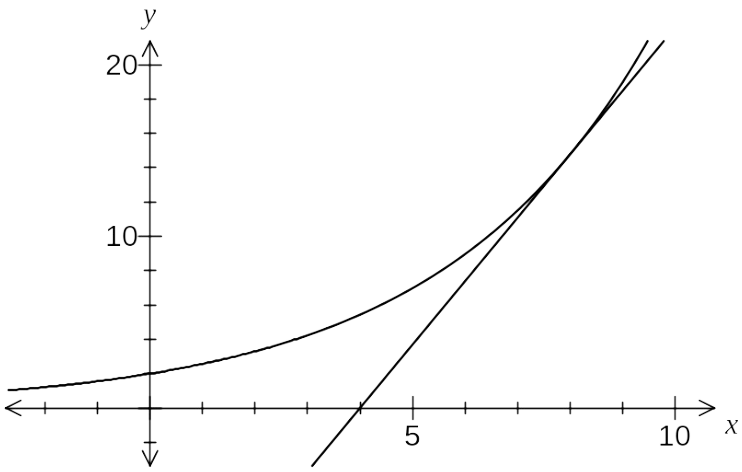
$x+5 = 3x-1 \Rightarrow x = 3$

Solution:

$0 \le x \le 3$

Question 12 **(6 marks)**

Find the exact area bounded by the x-axis, the y-axis, the function $f(x) = 2e^{0.25x}$ and the tangent to $f(x)$ when $x = 8$.



$f(8) = 2e^2$

$f'(x) = \frac{e^{0.25x}}{2}$

$f'(8) = \frac{e^2}{2}$

Tangent:

$y - 2e^2 = \frac{e^2}{2}(x - 8)$

$y = \frac{e^2}{2}x - 2e^2$

Area:

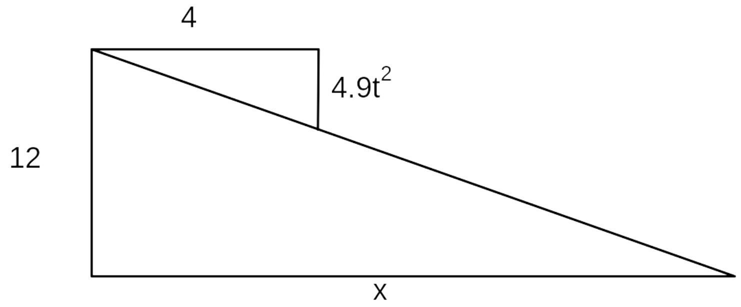
$\int_0^8 f(x)dx - \int_4^8 \frac{e^2}{2}x - 2e^2 dx$

$= \left(8e^2 - 8\right) - \left(4e^2\right)$

$= 4e^2 - 8$

Question 13 **(6 marks)**

A light is positioned at the top of a vertical post 12 m high. A small ball is dropped from the same height as the light but at a point 4 m away.
If the distance travelled by the ball t seconds after release is given by $4.9t^2$, how fast is the shadow of the ball moving along the horizontal ground half a second after the ball is dropped?



Find $\frac{dx}{dt}$ when $t = 0.5$.

Using similar triangles,

$$\frac{12}{x} = \frac{4.9t^2}{4}$$

$$x = \frac{48}{4.9t^2}$$

$$\frac{dx}{dt} = \frac{-2(48)}{4.9t^3}$$

$$t = 0.5$$

$$\frac{dx}{dt} = -\frac{7680}{49} \approx -156.7$$

Hence speed of shadow is 156.7 m/s.

Question 14 **(8 marks)**

(a) Use proof by contradiction to prove that $\sqrt{2}$ is irrational. (4 marks)

Assume that $\sqrt{2}$ is rational and can be written as the simplified fraction $\frac{a}{b}$,
where a and b are integers with no common factors.

$\frac{a}{b} = \sqrt{2} \Rightarrow a^2 = 2b^2 \Rightarrow a$ is an even integer.

If a is even then it can be written as $a = 2n$.

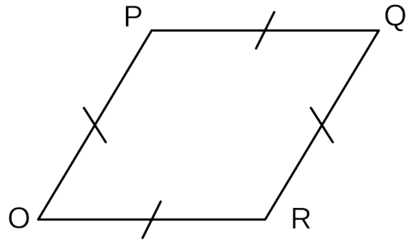
Hence $(2n)^2 = 2b^2 \Rightarrow 4n^2 = 2b^2$

Hence $b^2 = 2n^2 \Rightarrow b$ is an even integer.

But if a and b are both even integers then they have 2 as a common factor, which contradicts the assumption that they have no common factors.

Thus the assumption is incorrect, and $\sqrt{2}$ must be irrational.

(b) Use a vector method to prove that the diagonals of the rhombus OPQR are perpendicular. (4 marks)



Let $OP = RQ =$ **a** and $OR = PQ =$ **b** .

Then $OQ = \mathbf{a} + \mathbf{b}$ and $RP = \mathbf{a} - \mathbf{b}$.

So dot product of diagonals is

$$OQ \bullet RP = (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b})$$
$$= |\mathbf{a}|^2 - |\mathbf{b}|^2$$
$$= 0 \text{ since } |\mathbf{a}| = |\mathbf{b}|$$

Hence the diagonals are perpendicular.

Question 15 **(12 marks)**

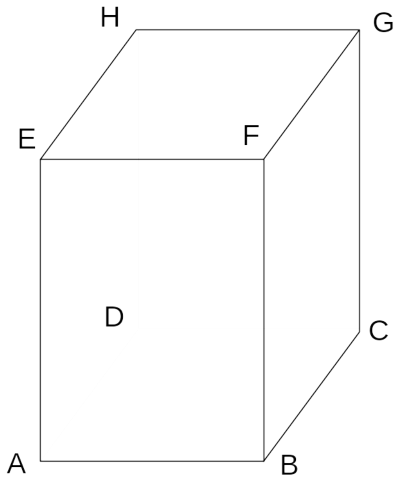
ABCDEFGH is a rectangular prism with

$OA = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$OB = 4\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$

$OD = -7\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$

$OE = 13\mathbf{i} + 27\mathbf{j} - 6\mathbf{k}$



(a) Find a vector equation for the plane EFGH in the form $\mathbf{r} \bullet \mathbf{n} = c$. (3 marks)

$$\mathbf{n} = \overrightarrow{AE} = \begin{bmatrix} 13 \\ 27 \\ -6 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 25 \\ -9 \end{bmatrix}$$

$$c = \mathbf{n} \bullet \overrightarrow{OE} = \begin{bmatrix} 12 \\ 25 \\ -9 \end{bmatrix} \bullet \begin{bmatrix} 13 \\ 27 \\ -6 \end{bmatrix} = 885$$

$$\mathbf{r} \bullet \begin{bmatrix} 12 \\ 25 \\ -9 \end{bmatrix} = 885$$

(b) Find a vector equation for the line passing through A and E. (2 marks)

$$\mathbf{r} = OA + \lambda AE$$
$$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(12\mathbf{i} + 25\mathbf{j} - 9\mathbf{k})$$

(c) The point P lies on the line through AE so that the size of $\angle HPD$ is 90°. Find the shortest possible distance from A to P.

(7 marks)

$$\overrightarrow{OH} = \overrightarrow{OE} + \overrightarrow{EH} = \begin{bmatrix} 13 \\ 27 \\ -6 \end{bmatrix} + \begin{bmatrix} -8 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 33 \\ 0 \end{bmatrix}$$

$$\overrightarrow{HP} = - \begin{bmatrix} 5 \\ 33 \\ 0 \end{bmatrix} = \begin{bmatrix} 12\lambda - 4 \\ 25\lambda - 31 \\ -9\lambda + 3 \end{bmatrix}$$

$$\overrightarrow{DP} = \begin{bmatrix} 1+12\lambda \\ 2+25\lambda \\ 3-9\lambda \end{bmatrix} - \begin{bmatrix} -7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 12\lambda + 8 \\ 25\lambda - 6 \\ -9\lambda - 6 \end{bmatrix}$$

$$\overrightarrow{HP} \bullet \overrightarrow{DP} = 0$$

$$(12\lambda - 4)(12\lambda + 8) + (25\lambda - 31)(25\lambda - 6) + (-9\lambda + 3)(-9\lambda - 6) = 0$$

$$850\lambda^2 - 850\lambda + 136 = 0$$

$$\lambda = 0.2,\ \lambda = 0.8$$

$$\overrightarrow{AP} = 0.2 \times \begin{bmatrix} 12 \\ 25 \\ 9 \end{bmatrix} = \begin{bmatrix} 2.4 \\ 5 \\ 1.8 \end{bmatrix}$$

$$\left|\overrightarrow{AP}\right| = \sqrt{34} \approx 5.83 \text{ units}$$

Question 16 **(9 marks)**

Let $w = cos\theta + i\sin\theta$ and $z = cos\phi + i\sin\phi$.

(a) Use Euler's formula to express the product wz in exponential form. (1 mark)

$$\boxed{wz = e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}}$$

(b) Use w and z to show that $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$. (4 marks)

$$e^{i(\theta+\phi)} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$

$$\cos(\theta+\phi) + i\sin(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \sin\theta\cos\phi)$$

Equating imaginary parts gives

$$\sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi$$

as required.

(c) Hence show that $\int 3\cos\left(\theta+\frac{\pi}{4}\right)\left(\frac{\cos\theta}{\sqrt{2}} + \frac{\sin\theta}{\sqrt{2}}\right)^2 d\theta = \sin^3\left(\theta+\frac{\pi}{4}\right) + c$. (4 marks)

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\int 3\cos\left(\theta+\frac{\pi}{4}\right)\left(\cos\theta\sin\frac{\pi}{4} + \sin\theta\cos\frac{\pi}{4}\right)^2 d\theta$$

$$= \int 3\cos\left(\theta+\frac{\pi}{4}\right)\sin^2\left(\theta+\frac{\pi}{4}\right) d\theta$$

$$= \sin^3\left(\theta+\frac{\pi}{4}\right) + c$$

Question 17 **(8 marks)**

A particle moves along the x-axis, with displacement x cm from the origin, after t seconds, given by $x = a\cos\left(\frac{\pi t}{3}\right)$, where a is a positive constant. After 1 second, the particle is 12 cm from the origin.

(a) Find the value of a. (1 mark)

$$12 = a\cos\frac{\pi}{3} \Rightarrow a = 24$$

(b) Show that the motion of the particle is simple harmonic. (2 marks)

$$x = 24\cos\left(\frac{\pi}{3}\right)$$

$$\dot{x} = -\frac{\pi}{3}(24)\sin\left(\frac{\pi}{3}\right)$$

$$\ddot{x} = -\left(\frac{\pi}{3}\right)^2 (24)\cos\left(\frac{\pi}{3}\right)$$

$$\ddot{x} = -\left(\frac{\pi}{3}\right)^2 x \Rightarrow \text{SHM}$$

(c) Find the speed of the particle as it passes through the origin. (2 marks)

Maximum speed when passing through origin, so

$$\dot{x} = -\frac{\pi}{3}(24)\sin\left(\frac{\pi}{3}\right)\bigg|_{\sin\left(\frac{\pi}{3}\right)=1}$$

$\dot{x} = -8\pi \Rightarrow$ speed $= 8\pi$ cm/s

(d) Find the distance travelled by the particle during the first minute of its motion. (3 marks)

Period is $\frac{2\pi}{\pi/3} = 6$ seconds, so in 60 seconds will move through exactly 10 cycles.

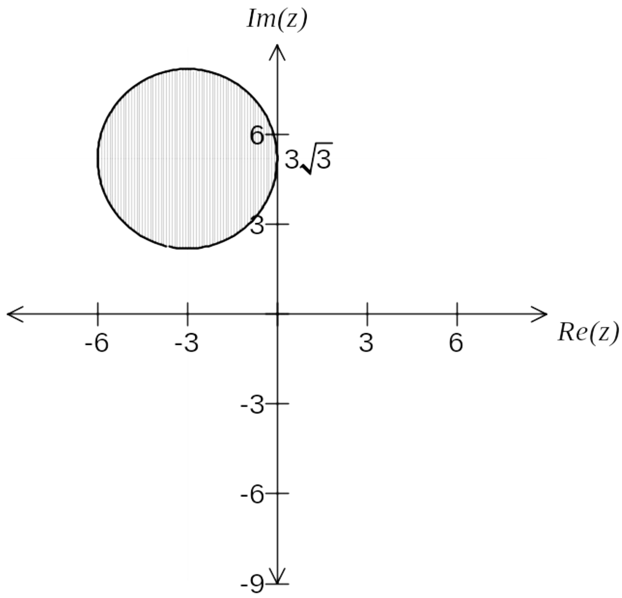
Amplitude is 24 cm, so in one complete cycle moves $24 \times 4 = 96$ cm.

Hence will travel $10 \times 96 = 960$ cm.

Question 18 **(9 marks)**

A complex inequality is given by $\left| z+3-3\sqrt{3}i \right| \le 3$.

(a) Sketch the region in the complex plane defined by this inequality. (3 marks)



(b) Find the minimum and maximum values of $\left| z \right|$. (3 marks)

Radius of circle is 3.

Distance from (0, 0) to circle centre is $\sqrt{3^2+(3\sqrt{3})^2}=6$.

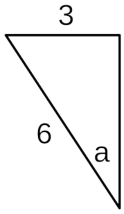
Min $\left| z \right|$ is $6-3=3$

Max $\left| z \right|$ is $6+3=9$

(c) Find the minimum and maximum values of $\arg z$. (3 marks)

Minimum value is $\dfrac{\pi}{2}$.

Maximum value is $\dfrac{\pi}{2}+2\times\dfrac{\pi}{6}=\dfrac{5\pi}{6}$.



sin a = 3/6

a = $\dfrac{\pi}{6}$

Question 19 **(10 marks)**

The velocity of a body moving in a straight line is given by $\frac{dx}{dt} = 3 + 4x$, where x is the displacement, in metres, from a fixed reference point at time t seconds. When $t = 1$, $x = 2$.

(a) Find an expression for x in terms of t. (5 marks)

$$\int \frac{1}{3+4x}\,dx = \int dt$$
$$\frac{1}{4}\ln(3+4x) = t + c$$
$$3 + 4x = ae^{4t}$$
$$t = 1,\ x = 2 \Rightarrow a = 11e^{-4}$$
$$3 + 4x = 11e^{-4}e^{4t}$$
$$x = \frac{11e^{4t-4} - 3}{4}$$

(b) What is the exact velocity of the body when

(i) $x = 3$? (1 mark)

$$v = 3 + 4(3) = 15 \text{ m/s}$$

(ii) $t = 3$? (2 marks)

$$x = \frac{11e^{4(3)-4} - 3}{4} = \frac{11e^{8}}{4}$$
$$v = 3 + 4 \times \frac{11e^{8}}{4} = 3 + 11e^{8} \text{ m/s}$$

(c) What is the acceleration of the body when $t = 1$? (2 marks)

$$v = 3 + 4x$$
$$\frac{dv}{dt} = 4\frac{dx}{dt}$$
$$= 4 \times (3 + 4(2))$$
$$= 44 \text{ m/s}^2$$

Question 20 **(7 marks)**

Let $P(n) = \frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$.

(a) Evaluate $P(1)$ and $P(4)$. (1 mark)

$P(1) = 1$

$P(4) = 228$

(b) Prove by induction that $P(n)$ is always an integer, when n is a positive integer. (6 marks)

$P(1) = 1 \Rightarrow P(n)$ is an integer when $n = 1$

Assume that $P(k)$ is an integer, where k is a positive integer, so that $\frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15} = I$

k+1 is the next consecutive integer after k.

$$P(k+1) = \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{7(k+1)}{15}$$
$$P(k+1) = \frac{k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1}{5} + \frac{k^3 + 3k^2 + 3k + 1}{3} + \frac{7k + 7}{15}$$
$$P(k+1) = \frac{k^5}{5} + k^4 + \frac{7k^3}{3} + 3k^2 + \frac{37k}{15} + 1$$
$$P(k+1) = \frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15} + k^4 + 2k^3 + 3k^2 + 2k + 1$$
$$P(k+1) = I + k^4 + 2k^3 + 3k^2 + 2k + 1$$

Hence, if $P(k)$ is an integer, then $P(k+1)$ is also an integer, as both I and k are integers.

Since $P(1)$ is an integer, then $P(2)$ must be an integer, and so on for all positive integer n.

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

Additional working space

Question number: _________

Published by WA Examination Papers
PO Box 445 Claremont WA 6910