Semester One Examination 2011
Question/answer booklet

YEAR 12 MATHEMATICS
3C/DMAT

Section Two
(Calculator-Assumed)

Student Name: **SOLUTIONS**

Circle your teacher's name

S. ROWDEN N. EDMUNDS

Time allowed for this section

Reading time before commencing work: 10 minutes
Working time for section: 100 minutes

Material required/recommended for this section

To be provided by the supervisor
Question/answer booklet for Section Two. Candidates may use the removable formula sheet from Section One.

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler
Special items: drawing instruments, templates, notes on up to two unfolded sheets of A4 paper, and up to three calculators, CAS, graphic or scientific, which satisfy the conditions set by the Curriculum Council for this course.

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this examination

	Number of questions available	Number of questions to be attempted	Suggested working time (minutes)	Marks available
Section One: Calculator—free	10	10	50 minutes	40
Section Two: Calculator—assumed	**15**	**15**	**100 minutes**	**80**
			Total marks	120

Instructions to candidates

1. Answer the questions in the spaces provided.

2. Spare answer pages are provided at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued i.e. give the page number.

3. **Show all your working clearly.**
   Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks.
   For any question or part question worth more than two marks, valid working or justification is required to receive full marks.
   If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

Section One: Calculator-assumed 80 Marks

This section has **Sixteen (16)** questions. Attempt **all** questions.

Question 11 (4 marks)

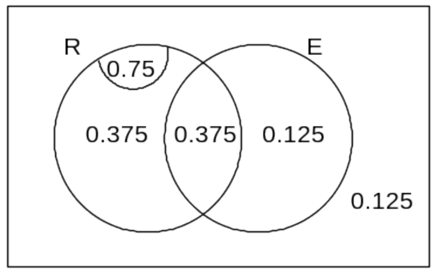
If R and E are independent events and $P(R) = 0.75$ and $P(R \cup E) = 0.875$, find

(a) $P(E)$ [2]

Solution

$P(R \cup E) = P(R) + P(E) - P(R)P(E)$
$0.875 = 0.75 + P(E) - 0.75\,P(E)$
$P(E) = 0.5$

OR



Specific behaviours

✓ Uses union rule substituting $P(R \cap E)$ with $P(R)P(E)$ or completes Venn diagram
✓ Solves correctly for $P(E)$

(b) $P(E \mid R)$ [1]

Solution

$$P(E \mid R) = \frac{0.375}{0.75}$$
$$= 0.5$$

Specific behaviours

✓ Correct value of $P(E \mid R)$

(c) P $R \mid \overline{E}$) [1]

Solution

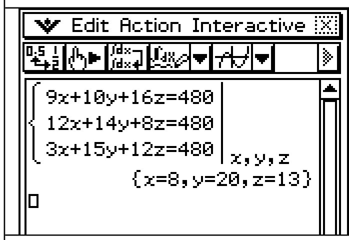
$$P(R \mid \overline{E}) = \frac{0.375}{0.5}$$
$$= 0.75$$

Specific behaviours

✓ Correct value of P ($R \mid \overline{E}$)

Question 12 (4 marks)

Three types of components, *X*, *Y*, and *Z*, must each be processed in three separate machines. The respective processing times in machine *A* are 9, 10 and 16 minutes. For machine *B*, the corresponding times are 12, 14 and 8 minutes, while, for machine C, the times are 3, 15 and 12. How many of each type of component should be manufactured in an 8-hour shift in order to keep all the machines fully occupied?

Solution

Specific behaviours
✓ correctly converts hours into minutes ✓✓ 3 correct equations ✓ 2 correct equations ✓ correct values for x, y and z

Question 13 (4 marks)

During an influenza epidemic, the proportion of the population in a particular suburb who are infected is denoted *p*(*t*) where *t* is time in weeks after the start of the epidemic.

Given that $p(t) = \frac{t}{4+t^2}$

(a) Describe the interpretation given to $\frac{dp}{dt}$.

[1]

Solution
Rate of change of population infected
Specific behaviours
✓ correct description

(b) Find using calculus when most of the population in the suburb is infected and the maximum population affected.

[3]

Solution
$p(t) = t(4+t^2)^{-1}$ $p'(t) = \frac{-t^2+4}{(t^2+4)^2}$ $0 = \frac{-t^2+4}{(t^2+4)^2}$ $t = 2 \ \ or \ \ t = -2$ 25% is the maximum population infected at 2 weeks
Specific behaviours
✓ determines the derivative of p(t) ✓ solves p'(t) = 0 correctly ✓states max population and time when this occurs

Question 14 (6 marks)

The Australian Olympic Three Day Event Team must comprise of 4 elite event riders and their horses. There are 14 possible contenders and horses for this team, which will represent Australia at the 2012 London Olympics.

(a) How many different selections are possible?

[1]

Solution
$\binom{14}{4} = 1001$
Specific behaviours
✓ correct number of different selections

Ella and her horse Simmo, are the Australian champions and Emily, with her horse Nobby is the runner up champion.

(b) What is the probability that of the 4 event riders chosen at random:

(i) Ella is included?

[1]

Solution
$\dfrac{\binom{1}{1}\binom{13}{3}}{\binom{14}{4}} = \dfrac{287}{1001} = \dfrac{2}{7} = 0.2857$
Specific behaviours
✓ correct probability

(ii) Ella and Emily are included?

[1]

Solution
$\dfrac{\binom{2}{2}\binom{12}{2}}{\binom{14}{4}} = \dfrac{66}{1001} = \dfrac{6}{91} = 0.0659$
Specific behaviours
✓ correct probability

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(iii) Ella or Emily?

[1]

Solution
$1-\dfrac{\binom{1}{0}\binom{1}{0}\binom{12}{4}}{\binom{14}{4}}=1-\dfrac{495}{1001}=\dfrac{46}{91}=0.5055$
Specific behaviours
✓ correct probability

(c) If Ella is selected for the Olympic team, what is the probability that Emily is also selected?

[2]

Solution
$\dfrac{\binom{1}{1}\binom{1}{1}\binom{12}{2}}{\binom{1}{1}\binom{13}{3}}=\dfrac{66}{286}=\dfrac{3}{13}=0.2308$
Specific behaviours
✓ Uses conditional probability ✓ Correct probability

Question 15 (4 marks)

Given $f(x) = \sqrt{x} + 2$, $g(x) = \frac{1}{x+5}$ and $k(x) = \frac{1}{x} - 5$ determine:

(a) $g \circ f(1)$

[2]

Solution
$g \circ f(1) = g(3)$ $= \frac{1}{8}$
Specific behaviours
✓ correctly evaluates f(1) ✓ correctly evaluates $g \circ f(1)$

(b) the domain and range of $k \circ f(x)$

[2]

Solution
$D_{k \circ f} = D_f = \{x : x \geq 0, x \in R)$ $R_{k \circ f} = R_f = \{y : -5 < y \leq -4.5, y \in R)$
Specific behaviours
✓ correctly states domain ✓ correctly states range

Question 16 (7 marks)

Gas is escaping from a spherical balloon at the rate of 0.4 m^3/min.

(a) What is the change in volume during the first 10 minutes? [2]

Solution
$10 \times 0.4 = 4\ m^3$ Volume decreases by 4 m^3 in the first 10 minutes.
Specific behaviours
✓ correctly calculates the change in volume ✓ interprets change in volume as a decrease

(b) How fast is the surface area shrinking when the radius is 4 m? [5]

Solution

$S = 4\pi r^2, \qquad \dfrac{dS}{dr} = 8\pi r$

$V = \dfrac{4}{3}\pi r^3, \qquad \dfrac{dV}{dr} = 4\pi r^2$

$$\frac{dV}{dt} = -0.4$$

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$$

¿ $8\pi r \times \dfrac{1}{4\pi r^2} \times (-0.4)$

¿ $-\dfrac{0.8}{r}$

When $r = 4$, $\dfrac{dS}{dt} = -0.2$ m^2/min

The surface area is shrinking at the rate of 0.2 m^2/min when the radius is 4 m.

Specific behaviours

✓ correctly determines $\dfrac{dS}{dr}$ and $\dfrac{dV}{dr}$
✓correctly determines $\dfrac{dV}{dt}$
✓ correctly applies the chain rule
✓ substitutes to find $\dfrac{dS}{dt}$
✓ states the rate at which the surface area is shrinking, with correct units

Question 17 (7 marks)

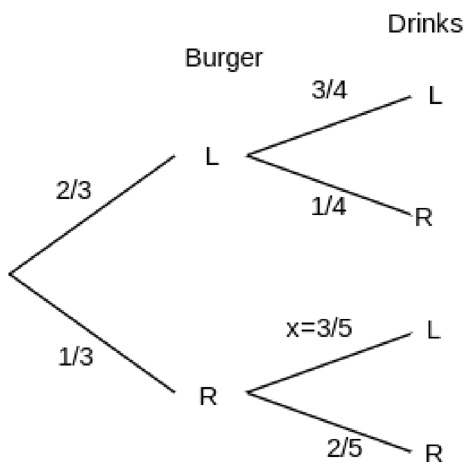
A fast food restaurant has a deal that when a customer buys either a regular or large burger they can choose either a regular or a large drink at a discount price.

Records show that two out of every three customers buy a large burger and, of these customers, one quarter of them choose a regular drink. Also, after choosing a burger, seven out of every ten customers choose a large drink.

(a) Draw a probability tree showing the possible burger and drink choices.

[3]

Solution



$$\frac{2}{3}\times\frac{3}{4}=\frac{1}{2}$$
$$\frac{1}{3}x+\frac{1}{2}=\frac{7}{10}$$
$$x=\frac{3}{5}$$

Specific behaviours
✓ correct probabilities for first branch of tree diagram ✓ correct probabilities for drinks branches from large burger ✓ calculates correctly probabilities for drink branches from regular burger

(b) Determine the probability that a randomly selected customer:

(i) will choose both a large burger and a large drink?

[1]

Solution
$P(LB \cap LD) = 0.5$
Specific behaviours
✓ correct probability

(ii) will choose a regular drink given that they have chosen a regular burger

[1]

Solution
$P(RD \mid RB) = \dfrac{\frac{2}{15}}{\frac{1}{3}} = \dfrac{2}{5}$
Specific behaviours
✓ uses conditional probability to determine probability correctly

(iii) will choose a large burger given that they have chosen a regular drink?

[2]

Question 18 (8 marks)

A toy train sits at the

Solution
$P(LB\|RD) = \dfrac{\frac{2}{3}\times\frac{1}{4}}{\frac{1}{3}\times\frac{2}{5}+\frac{2}{3}\times\frac{1}{4}} = \frac{5}{9}$
Specific behaviours
✓ uses conditional probability rule ✓ correct probability

centre of a length of track. The displacement, s, of the train from the central position, O, after t seconds is given by

$$s = 0.8t^2 - 6.4t \quad \text{cm}$$

(a) Determine the displacement of the train after 3 seconds. [1]

Solution
s = -12 cm
Specific behaviours
✓ correct displacement

(b) What speed is the train travelling after 3 seconds? [2]

Solution
$v = 1.6t - 6.4$ $v(3) = -1.6$m/s Speed is 1.6 m/s
Specific behaviours
✓ determines velocity at 3 seconds ✓ identifies correctly the speed as the magnitude of the velocity

(c) What is the acceleration after 3 seconds? [1]

Solution
Acceleration is 1.6 m/s^2
Specific behaviours
✓ interprets acceleration as the derivative of velocity and derives correctly

(d) At what time(s) does the train stop in its first 20 seconds of motion? [1]

SEE NEXT PAGE

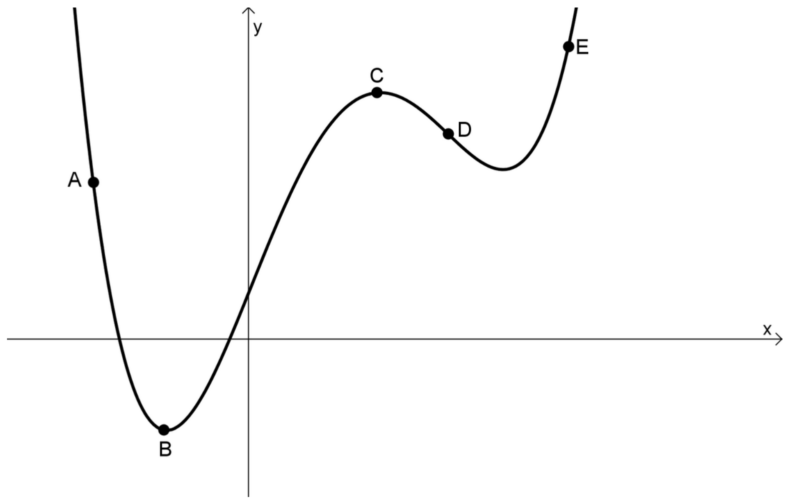
Solution
Train stop when v = 0   t = 4 seconds
Specific behaviours
✓ solves velocity equal to 0 correct for time

(e) What distance does the train travel in the first 20 seconds of motion?
Show your working.

[3]

Solution		
When t = 0, x = 0 m When t = 4, x = -12.8 m When t = 20, x = 192 m	OR	$\int_0^{20} \lvert 1.6t - 6.4 \rvert dt$
Distance travelled in the first 20 seconds [illegible]		
Specific behaviours		
✓ determines displacement at time 0 and 20 ✓ determines displacement at time 4 ✓ correct distance	or	✓✓ definite integral of absolute value of function between 0 and 20 ✓ correct distance

Question 19 (3 marks)



In each part, list the points (A-E) on the graph of $f(x)$ that satisfy the given conditions.

(a) $f'(x) > 0$ and $f''(x) > 0$

(b) $f'(x) < 0$ and $f''(x) > 0$

(c) $f'(x) = 0$ and $f''(x) < 0$

(d) $f'(x) = 0$ and $f''(x) > 0$

(e) $f'(x) < 0$ and $f''(x) = 0$

Solution
(a) E, (b) A, (c) C, (d) B, (e) D
Specific behaviours
✓ ✓✓ correctly identifies all 5 points ✓✓ correctly identifies 3 or 4 points ✓ correctly identifies up to 2 points

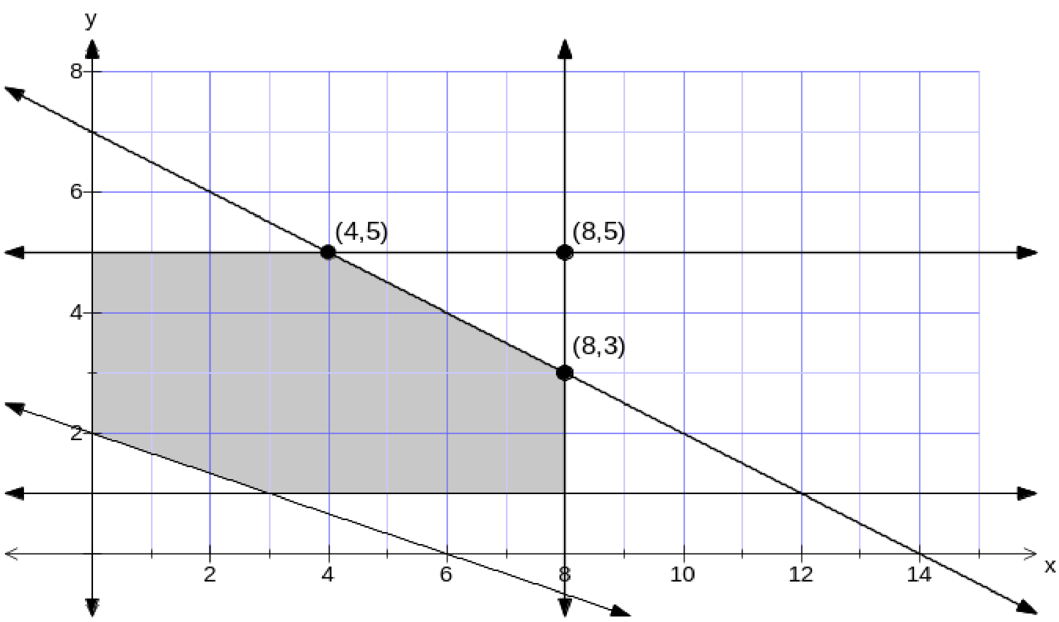
Question 20 (13 marks)

A dance teacher is writing up a business plan for a dance studio he is intends to start. He plans to teach jazz dance and ballet.
Each jazz dance class will involve 3 hours of tuition a week and produce an income of $120. Each class of ballet will involve 6 hours of tuition a week and produce an income of $360.
The teacher plans to work for up to 42 hours a week and would incur fixed costs (rent, power, etc) of $720.
He plans to teach a least 1 ballet class per week but also decides to teach no more than 5 ballet classes and no more than 8 jazz dance classes per week.
The teacher must make a profit (income – costs) or he will not set up his business.

Let x denote the number of jazz dance classes and y the number of ballet classes.
Three of the constraints on x and y have been drawn on the diagram below.

(a) Determine two more constraints, draw them on the above diagram and shade the feasible region.

[4]

Solution
 $y \geq 1$ $x + 3y \geq 6$
Specific behaviours
✓ correctly identifies one more constraint ✓✓ correctly identifies both constraints ✓ both lines drawn correctly ✓ correct feasible region

Question 20 Continued

(b) How many jazz and ballet classes should the dance teacher aim to have pupils for so that he can maximise his profit per week? Show all working and state clearly you optimal solution and maximum value.

[3]

Solution

(x, y)	$P = 120x + 360y - 720$
(0, 2)	\$0
(0, 5)	\$1080
(4, 5)	\$1560
(8, 3)	\$1320
(8, 1)	\$600
(3, 1)	\$0

4 Jazz dance and 5 ballet classes
With a maximum profit of \$1560

Specific behaviours

✓ Determines profit for all critical points
✓ Correctly identifies optimal solution
✓ States maximum profit

(c) Consider the vertex of the feasible region that is your solution to part (b) and the two adjacent vertices (one on either side). For each of these three vertices calculate the profit per week and per hour and comment upon your values.

[3]

Solution

Vertex	Profit/week	Hours	Profit/hour
(0, 5)	1080	30	36.00
(4, 5)	1560	42	37.14
(8, 3)	1320	42	31.42

Whilst (4, 5) still gives the maximum profit per hour, (0, 5) gives a better profit per hour than (8,3), even though the profit per week is better

Specific behaviours

✓correct profit per week for the vertices
✓ correct profit per hour for the vertices
✓ appropriate comment on values

(d) There are no dance classes in the school holidays and throughout the year some classes may need to be cancelled for other reasons. Consequently, the dance teacher realizes that if he is considering his average profit over the whole year, the more realist income from jazz dance classes is $100 per class. What is the lowest income per ballet class that the teacher can receive so that the solution from part (b) of this question is the only combination of classes that gives maximum profit per week? [3]

Solution

100x + ky   0   4   8
            5   5   3
            5k   400 + 5k   800 + 3k   OR $-\frac{1}{2} < -\frac{100}{b} < 0$
                                             $b > 200$
400 + 5k > 800 + 3k
k > 200
Lowest income for ballet per class is $200.01

Specific behaviours
✓ determines inequalities that bound the maximum vertex ✓ determines k/b > 200 ✓ correct lowest income

Question 21 (4 marks)

Hershey's are famous for their chocolate kisses and have decided to design a new size Hershey kiss and package in a special gift box.

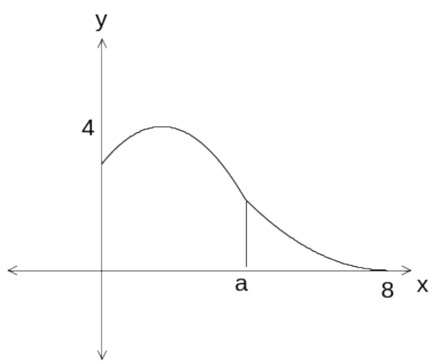
The kiss is to be modeled on the following equations that are rotated about the x axis for the given domains.

$f(x) = -0.375x^2 + 1.25x + 3$ for $0 \le x \le a$

and

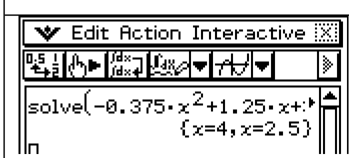
$g(x) = 0.125x^2 - 2x + 8$ for $a < x \le 8$

Each unit on the axes represents 1 cm.

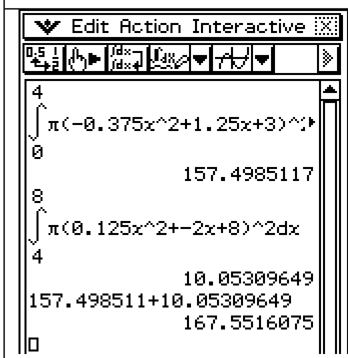


(a) Determine the value of a given $a > 3$.

[1]

Solution
 $a = 4$
Specific behaviours
✓ solves $f(x) = g(x)$ and states correct value of a

(b) Determine the volume of chocolate needed to make the kisses if the special gift box contains 3 kisses. [3]

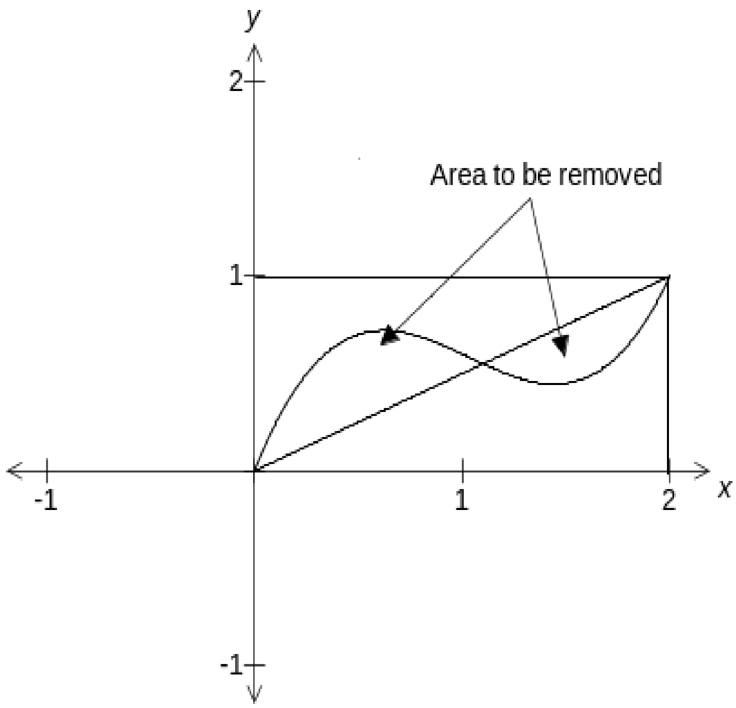
Solution
   167.55 x 3 = 502.65 cm^3
Specific behaviours
✓determines volume f(x) over correct domain ✓ determines volume g(x) over correct domain ✓ correctly determines total volume of chocolate for gift box

Question 22 (4 marks)

A dressmaker wishes to cut a section of cloth from a piece of material measuring two metres by one metre. The curved edges of the piece of cloth to be removed are defined as being between the following equations:

$$y_1 = 0.5x \text{ and } y_2 = x^3 - 3.1x^2 + 2.7x$$

Determine, using calculus, the area of the cloth removed correct to 2 decimal places.



Solution
$A = \int_0^{1.1} (x^3 - 3.1x^2 + 2.7x - 0.5x)dx + \int_{1.1}^{2} (0.5x - (x^3 - 3.1x^2 + 2.7x))dx$ $= \left[\frac{x^4}{4} - \frac{3.1x^3}{3} + \frac{1.2x^2}{2}\right]_0^{1.1} + \left[-\frac{1.2x^2}{2} - \frac{x^4}{4} + \frac{3.1x^3}{3}\right]_{1.1}^{2}$ $= 0.51 \text{ cm}^2$
Specific behaviours
✓ correct definite integral expression for area   ✓ integrates linear equation correctly   ✓ integrates cubic equation correctly   ✓ correct area

Question 23 (2 marks)

Which of the following statements is true for two events, each with probability greater than 0?
Justify your answer.

A: If the events are mutually exclusive, they must be independent.

B: If the events are independent, they must be mutually exclusive.

C: If the events are not mutually exclusive, they must be independent.

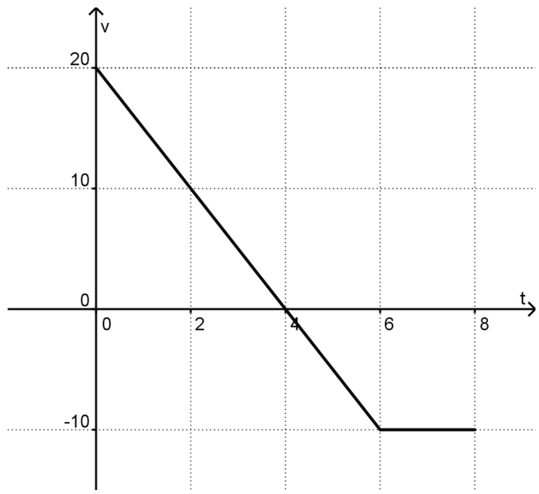
D: If the events are not independent, they must be mutually exclusive.

E: If the events are mutually exclusive, they cannot be independent.

Solution
Statement E is true If events X and Y are mutually exclusive then $P(X \cap Y) = 0$ If events X and Y are independent, then $P(X \cap Y) = P(X).P(Y) > 0$ because both $P(X) > 0$ and $P(Y) > 0$ Hence, if events X and Y are mutually exclusive, then they cannot be independent
Specific behaviours
✓ identifies E as the only true statement ✓ justifies choice of event E

Question 24 (5 marks)

The diagram below shows the 'velocity/time' graph for a particle which moves in a horizontal straight line for $0 \le t \le 8$ seconds. At time t = 0 seconds the particle is at a point O on the line; the initial velocity is 20 ms^{-1}.



Find:

(a) the distance of the particle from O when t = 8.

[3]

Solution
Distance travelled in first 4 seconds = $\frac{1}{2} \times 4 \times 20 = 40$ m. Distance travelled in next 4 seconds = $\frac{1}{2} \times (4+2) \times 10 = 30$ m. At $t = 8$, particle is 10 m to the right of 0.
Specific behaviours
✓ calculates correctly distance travelled forward ✓ calculates correctly distance travelled backwards ✓ calculates correctly position relative to 0.

(b) the maximum distance of the particle from O.

[1]

Solution
Maximum distance from 0 is 40 m.
Specific behaviours
✓ Calculates correct maximum distance

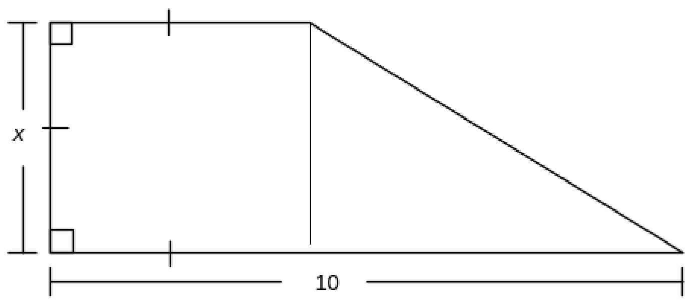
(c) the acceleration of the particle when $t = 2$.

[1]

Solution
$a(2)=\frac{-20}{4}=-5$
Specific behaviours
✓ determines gradient of line segment when $t=2$.

Question 25 (5 marks)

Consider the diagram below.



Let *A* denote the area of the above figure.

(a) Show that $A = \dfrac{x^2}{2} + 5x$

[2]

Solution
$A = x \times x + \frac{1}{2}(10 - x)x$ OR $A = \frac{1}{2}x(x + 10)$ $= \frac{x^2}{2} + 5x$ $\quad\quad$ $= \frac{x^2}{2} + 5x$
Specific behaviours
✓ Calculates area of figure ✓ Simplifies area expression

(b) Use differentiation to find the approximate percentage change in *A*, if the percentage change in *x* is 5%, given that *x* = 4.

[3]

Solution
$\delta A \approx \frac{dA}{dx}\delta x$ $\approx (x + 5)\delta x$ $\delta A\% = \frac{(x+5)4 \times 0.05}{0.5x^2 + 5x} \times 100$ $= \frac{(4+5)4 \times 0.05}{0.5(4)^2 + 5(4)} \times 100$ $= 6.4\%$ When x changes by 5%, A changes by 6.4%
Specific behaviours
✓ Correct formula for δA% ✓ Substitutes x and δx into equation ✓ Correct percentage change in A

END OF QUESTIONS

Additional working space

Question number: _______

Additional working space

Question number: _______