PERTH MODERN SCHOOL

Exceptional schooling. Exceptional students.

INDEPENDENT PUBLIC SCHOOL

WAEP Semester Two Examination, 2019

Question/Answer booklet

MATHEMATICS METHODS UNITS 3&4

Section One: Calculator-free

SOLUTIONS

Student number: In figures

In words ____________________

Your name ____________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

Supplementary page

Question number: _________

Question 8 **(6 marks)**

Let $f(x) = \frac{x}{x+1}$.

(a) Determine $f(x)$ and $f(x + \delta x)$ when $x = 70$ and $\delta x = 5$. (1 mark)

Solution
$f(70) = \frac{70}{71}, \quad f(75) = \frac{75}{76}$
Specific behaviours
✓ both correct fractions

(b) Use $f(x)$ and the increments formula to estimate the difference between $\frac{89}{90}$ and $\frac{92}{93}$. (5 marks)

Solution
$f'(x) = \frac{1(x+1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$ Find δy when $x = 89$ and $\delta x = 3$. $\delta y \approx f'(x).\delta x \approx \frac{1}{(x+1)^2} \times \delta x \approx \frac{3}{90^2} \approx \frac{1}{2700}$ Difference is approximately $\frac{1}{2700}$.
Specific behaviours
✓ use of quotient rule for $f'(x)$ ✓ correct $f'(x)$ ✓ indicates values of x and δx ✓ uses increments formula   ✓ substitutes, simplifies and states difference

End of questions

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(4 marks)**

Determine the following:

(a) $\int 12(2x+1)^2\,dx$. (2 marks)

Solution
$\frac{12}{2 \times 3}(2x+1)^3 = 2(2x+1)^3 + c$
Specific behaviours
✓ integrates correctly   ✓ includes constant

(b) $\frac{d}{dx}\cos(2x+1)$. (1 mark)

Solution
$-2\sin(2x+1)$
Specific behaviours
✓ correct derivative

(c) $\frac{d}{dx}\int_3^x (2t+1)\,dt$. (1 mark)

Solution
$2x+1$
Specific behaviours
✓ correct use of fundamental theorem

See next page

Question 2 **(7 marks)**

The velocity of a small body moving in a straight line at time t seconds is given by

$$v = \frac{8}{1+t} \text{ m/s}, \qquad t \geq 0.$$

(a) Determine the velocity of the body when its acceleration is -2 m/s^2. (4 marks)

Solution
$\frac{dv}{dt} = \frac{d}{dt}(8(1+t)^{-1})$ $= -8(1+t)^{-2}$ $-2 = -\frac{8}{(1+t)^2}$ $(1+t)^2 = 4$ $t = -1 \pm 2$ $t = 1$ $v(1) = 8 \div 2 = 4 \text{ m/s}$
Specific behaviours
✓ correctly differentiates   ✓ equates to required value and simplifies   ✓ indicates time   ✓ correct velocity

(b) Calculate the distance travelled by the body in the first 3 seconds. (3 marks)

Solution
$\int_0^3 \frac{8}{1+t} dt = [8\ln(1+t)]_0^3$ $= 8\ln 4 - 8\ln 1$ $= 16\ln 2 \text{ m}$
Specific behaviours
✓ writes definite integral   ✓ correct antiderivative   ✓ substitutes bounds and simplifies

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

Question 7 **(7 marks)**

In a class of 25 students, 20 are right-handed.

(a) One student is selected at random from the class and the random variable X is the number of right-handed students in the selection. Determine the mean and standard deviation of X. (3 marks)

Solution
$\mathrm{E}(X) = p = \frac{20}{25} = \frac{4}{5}$ $\mathrm{Var}(X) = p(1-p) = \frac{4}{5} \times \frac{1}{5} = \frac{4}{25}$ Standard deviation $= \sqrt{\frac{4}{25}} = \frac{2}{5}$
Specific behaviours
✓ mean   ✓ variance   ✓ standard deviation

(b) Two students are selected at random from the class without replacement and the random variable Y is the number of right-handed students in the selection.

(i) Complete the probability distribution table below. (3 marks)

y	0	1	2
$P(Y=y)$	**1/30**	**1/3**	**19/30**

Solution
$P(Y=2) = \frac{20}{25} \times \frac{19}{24} = \frac{4}{5} \times \frac{19}{24} = \frac{19}{30}$ $P(Y=0) = \frac{5}{25} \times \frac{4}{24} = \frac{1}{5} \times \frac{1}{6} = \frac{1}{30}$ $P(Y=1) = 1 - \frac{19}{30} - \frac{1}{30} = \frac{10}{30}$
Specific behaviours
✓✓✓ each correct probability

(ii) Determine $\mathrm{E}(Y)$. (1 mark)

Solution
$\mathrm{E}(Y) = 0 + \frac{10}{30} + \frac{2(19)}{30} = \frac{48}{30} = \frac{24}{15}$
Specific behaviours
✓ correct value

Question 6 **(8 marks)**

Let $f(x) = (1-x)e^{-2x}$.

(a) Determine the coordinates of the stationary point of the graph of $y = f(x)$ and use the second derivative test to determine its nature. (6 marks)

Solution

$$f'(x) = -e^{-2x} - 2(1-x)e^{-2x}$$

$$f'(x) = 0 \Rightarrow (2x-3)e^{-2x} = 0 \Rightarrow x = \frac{3}{2}, y = -\frac{1}{2e^3}$$

$$f''(x) = 2e^{-2x} - 2(2x-3)e^{-2x} = (8-4x)e^{-2x}$$

$$f''\left(\frac{3}{2}\right) = 2e^{-3} \Rightarrow \text{Min}$$

Stationary point is at $\left(\frac{3}{2}, -\frac{1}{2e^3}\right)$ and is a minimum.

Specific behaviours

- ✓ correct $f'(x)$
- ✓ equates $f'(x)$ to zero and obtains x-coordinate
- ✓ obtains y-coordinate
- ✓ obtains $f''(x)$
- ✓ indicates sign of $f''(x)$ at point
- ✓ coordinates of point and nature

(b) Determine the coordinates of the point of inflection of the graph of $y = f(x)$. (2 marks)

Solution

$$(8-4x)e^{-2x} = 0 \Rightarrow x = 2$$

$$f(2) = -\frac{1}{e^4}$$

Point of inflection at $\left(2, -\frac{1}{e^4}\right)$

Specific behaviours

- ✓ solves $f''(x) = 0$
- ✓ coordinates

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

Question 3 **(7 marks)**

(a) Write $1 + \log_5 3 - 2\log_5 7$ in the form $\log_5 k$. (3 marks)

Solution

$$1 + \log_5 3 - 2\log_5 7 = \log_5 5 + \log_5 3 - 2\log_5 7 = \log_5 15 - \log_5 7^2 = \log_5 \frac{15}{49}$$

Specific behaviours

- ✓ uses $\log_a a = 1$
- ✓ uses $x \log_a y = \log_a y^x$
- ✓ uses $\log_a x \pm \log_a y$

(b) Solve for x the equation $e^{x-2} = \sqrt{3}$. (2 marks)

Solution

$$x - 2 = \ln\sqrt{3}$$

$$x = \frac{1}{2}\ln(3) + 2$$

Specific behaviours

- ✓ expresses using natural log
- ✓ simplifies

(c) Determine $\frac{d}{dx}\left(\log_e\left(\frac{1}{5x^2+1}\right)\right)$. (2 marks)

Solution

$$\log_e\left(\frac{1}{5x^2+1}\right) = -\ln(5x^2+1)$$

$$\frac{d}{dx}(-\ln(5x^2+1)) = -\frac{10x}{5x^2+1}$$

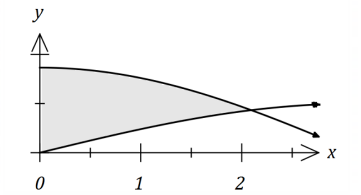
Specific behaviours

- ✓ uses log law
- ✓ correct derivative

Question 4 **(6 marks)**

Let $f(x) = \sqrt{3}\cos\left(\frac{x}{2}\right)$ and $g(x) = \sin\left(\frac{x}{2}\right)$.

The shaded region on the graph below is enclosed by $x = 0$, $y = f(x)$ and $y = g(x)$.



(a) Show that $f\left(\frac{2\pi}{3}\right) = g\left(\frac{2\pi}{3}\right)$. (2 marks)

Solution

$$f\left(\frac{2\pi}{3}\right) = \sqrt{3}\cos\left(\frac{\pi}{3}\right) = \sqrt{3} \times \frac{1}{2} = \frac{\sqrt{3}}{2}$$

$$g\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Hence $f\left(\frac{2\pi}{3}\right) = g\left(\frac{2\pi}{3}\right)$.

Specific behaviours

✓ evaluates $f(x)$
✓ evaluates $g(x)$, stating same as $f(x)$

(b) Determine the area of the shaded region. (4 marks)

Solution

$$\int_0^{\frac{2\pi}{3}} \sqrt{3}\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) dx$$

$$= \left[2\sqrt{3}\sin\left(\frac{x}{2}\right) + 2\cos\left(\frac{x}{2}\right)\right]_0^{\frac{2\pi}{3}}$$

$$= \left[2\sqrt{3}\sin\left(\frac{\pi}{3}\right) + 2\cos\left(\frac{\pi}{3}\right)\right] - \left[2\sqrt{3}\sin(0) + 2\cos(0)\right]$$

$$= \left(2\sqrt{3} \times \frac{\sqrt{3}}{2} + 2 \times \frac{1}{2}\right) - 2$$

$$= 3 + 1 - 2 = 2 \text{ sq units}$$

Specific behaviours

✓ writes correct integral
✓ integrates correctly
✓ substitutes correctly
✓ correct area

DO NOT WRITE IN THIS AREA AS IT WILL BE CUT OFF

Question 5 **(7 marks)**

The random variable X has probability density function $f(x)$ shown below, where k is a positive constant.

$$f(x) = \begin{cases} kx + \frac{1}{20} & 0 \le x \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Deduce that $k = \frac{1}{10}$. (3 marks)

Solution

$$\int_0^4 kx + \frac{1}{20}\,dx = \left[\frac{kx^2}{2} + \frac{x}{20}\right]_0^4$$

$$= 8k + \frac{1}{5}$$

$$8k + \frac{1}{5} = 1$$

$$8k = \frac{4}{5} \Rightarrow k = \frac{1}{10}$$

Specific behaviours

✓ integrates $f(x)$
✓ evaluates definite integral
✓ equates to 1 and shows steps to solve for k

(b) Determine the value of a if $P(1 < X < a) = \frac{1}{5}$. (4 marks)

Solution

$$\int_1^a \frac{x}{10} + \frac{1}{20}\,dx = \left[\frac{x^2}{20} + \frac{x}{20}\right]_1^a$$

$$= \frac{1}{20}(a^2 + a) - \frac{2}{20}$$

$$\frac{1}{20}(a^2 + a - 2) = \frac{1}{5}$$

$$a^2 + a - 6 = 0$$

$$(a + 3)(a - 2) = 0$$

$$a = 2$$

Specific behaviours

✓ integrates $f(x)$
✓ evaluates definite integral
✓ equates to probability and simplifies quadratic
✓ factorises and states the only valid value of a