Question 22 **(3 marks)**

In a game at the Perth Royal Show, a person can win a prize by guessing which one of 5 identical boxes contains the prize. After each guess, if the prize has been won, a new prize is randomly placed in one of the 5 boxes. If the prize has not been won, then the prize is again randomly placed in one of the 5 boxes.

Janey makes 4 guesses.

(a) What is the probability that Janey wins a prize exactly twice? [2]

Solution
X: number of prizes won out of 4 games X ~ b(4,0.2) P(X = 2) = 0.1536
Specific behaviours
✓ states the binomial distribution and its parameters ✓ correctly determines probability

(b) On average, how many prizes could Janey expect to win from her 4 guesses? [1]

Solution
Expect to win 0.2 x 4 = 0.8 prizes
Specific behaviours
✓ correctly calculates number of prizes

End of questions

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Student Name: ____________________

Methodist Ladies' College Semester 2, 2010

3CD MATHEMATICS

Question/Answer Booklet – Section 2 – Calculator-assumed

Teacher's Name: __________SOLUTIONS__________

Time allowed for this paper

Section	Reading	Working
~~**Calculator-free**~~	~~5 minutes~~	~~50 minutes~~
Calculator-assumed	10 minutes	100 minutes

Materials required/recommended for this paper

Section One (Calculator-assumed): 80 marks

To be provided by the supervisor
Section Two Question/Answer booklet Formula sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this course.

Important Note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. **All** questions should be attempted.
2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare answer pages may be found at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued (i.e. give the page number).
3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.
4. It is recommended that you **do not use pencil** except in diagrams.

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Question 21 **(6 marks)**

The lengths of individual shellfish in a population of 10 000 shellfish are approximately normally distributed with mean 10 cm and standard deviation 0.2 cm.

A random sample of 25 shellfish is taken.

(a) Determine the probability that the sample mean is less than 9.95 cm. [4]

Solution
$\overline{X}$: the average length of shellfish in cm in a sample of size 25 By the Central Limit Theorem $\overline{X}$ is approximately normally distributed with a mean of 10 cm and standard deviation of $\frac{0.2}{5}$ cm $P(\overline{X}<9.95)\approx 0.10565$
Specific behaviours
✓ recognises that $\overline{X}$ is a random variable with a mean of 10 cm ✓ determines the standard deviation ✓ recognises that $\overline{X}$ is normally distributed ✓ correctly calculates probability

(b) Determine the probability that the sample mean is more than 10.1 cm given it is more than 10 cm. [2]

Solution
$P(\overline{X}>10.1\mid\overline{X}>10)=\frac{P(\overline{X}>10.1)}{P(\overline{X}>10)}\approx\frac{0.0062097}{0.5}\approx 0.01242$
Specific behaviours
✓ correctly identifies appropriate conditional probability ✓ calculates probability correctly

Question 20 **(7 marks)**

A test engineer wants to estimate the mean petrol mileage μ (in km per litre) for a particular model of car. A random sample of 49 of these cars is subjected to a road test, and their mileage is computed for each car. The mean and standard deviation of these values are 8.5 km/L and 1.6 km/L, respectively.

(a) Determine a 95% confidence interval for the mean mileage for this model of car. [3]

Solution
X: the number of km per litre for a car $n=49, \bar{x}=8.5\text{ km/L}, s_x=1.6\text{ km/L}$ A 95% confidence interval for μ is given by $\bar{x}-1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x}+1.96\frac{\sigma}{\sqrt{n}}$ Since σ is unknown, assume $\sigma = s_x = 1.6$ Hence, $8.052 \le \mu \le 8.948$ km/L
Specific behaviours
✓ uses correct interval for 95% ✓ assumes $\sigma = s_x$ and correctly substitutes mean and standard deviation ✓ correctly calculates confidence interval

(b) Explain what the phrase '95% confidence' means as used in the answer to (a). [1]

Solution
95% certain that μ lies between 8.052 and 8.948
Specific behaviours
✓ gives correct explanation

(c) If the test engineer wants to be 99% certain that the mean of the sample taken differs from the population mean by less than 1 km/L, what size sample would be needed? [3]

Solution
Require $2.576 \times \frac{\sigma}{\sqrt{n}} < 1$ i.e. $2.576 \times \frac{1.6}{\sqrt{n}} < 1$ $\quad \sqrt{n} > 2.576 \times 1.6$ $\quad n > 16.99$ The sample size must be at least 17.
Specific behaviours
✓ chooses correct value of z ✓ correctly forms and expression for n or $\sqrt{n}$ ✓ correctly calculates sample size

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Section Two: Calculator-assumed **(80 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the space provided.

Suggested working time for this section is 100 minutes.

Question 10 **(5 marks)**

(a) The population {2, 3, 5, 7} has mean $\mu = 4.25$ and standard deviation $\sigma = 1.92$.

(i) When sampling **with** replacement, how many different samples of size 2 can be selected from this population? [1]

Solution
16 different samples
Specific behaviours
✓ correct number of samples

The mean of each of these samples is computed.

(ii) What is the mean and standard deviation of the distribution of sample means? [2]

Solution
mean = 4.25 standard deviation = $\frac{1.92}{\sqrt{2}} \approx 1.36$
Specific behaviours
✓ correct value for mean ✓ correct value for standard deviation

See next page

(b) An urn contains exactly three balls numbered 1, 2 and 3 respectively. Random samples of two balls are drawn from the urn **without** replacement. The average, $\overline{X}$, of the selected balls is recorded after each drawing.

Write down the probability distribution for $\overline{X}$. [2]

Solution

Sample	mean
1,2	1.5
1,3	2
2,3	2.5

Probability distribution for $\overline{X}$ is

$\bar{x}$	1.5	2	2.5
$P(\overline{X}=\bar{x})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Specific behaviours

✓ states values for $\overline{X}$
✓ states associated probabilities

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Question 19 **(5 marks)**

The metabolic rate of a person who has just eaten a meal tends to go up and then, after some time has passed, returns to a resting metabolic rate. This phenomenon is known as the *thermic effect of food*. Researchers have indicated that the thermic effect of food (in kJ/h) for a particular person is

$$f(t) = -10.28 + 175.9t\,e^{\frac{-t}{1.3}},$$

where t is the number of hours that have elapsed since eating a meal.*

(a) Find the average rate of change of the thermic effect of food during the first hour of eating. [1]

Solution
$\dfrac{f(1)-f(0)}{1} = 175.9e^{\frac{-1}{1.3}} \approx 81.5$ kJ/h per hour
Specific behaviours
✓ correctly calculates average rate of change

(b) Determine the instantaneous rate of change of the thermic effect of food one hour after eating. [1]

Solution
Using calculator, $f'(1) \approx 18.8$ kJ/h per hour
Specific behaviours
correctly states $f'(1)$

(c) When is the thermic effect of food a maximum? What is this maximum value? [3]

Solution
Using calculator, $f_{max} \approx 73.8$ kJ/h when $t \approx 1.3$ i.e. the thermic effect of food is a maximum approx 1.3 hours after eating the meal.
Specific behaviours
✓ correctly states f_{max} correctly states t ✓ concludes when the maximum occurs

*Reed, G. amd J.Hill, "Measuring the Thermic Effect of Food," *American Journal of Clinical Nutrition*, Vol.63, 1996, pp. 164-169.

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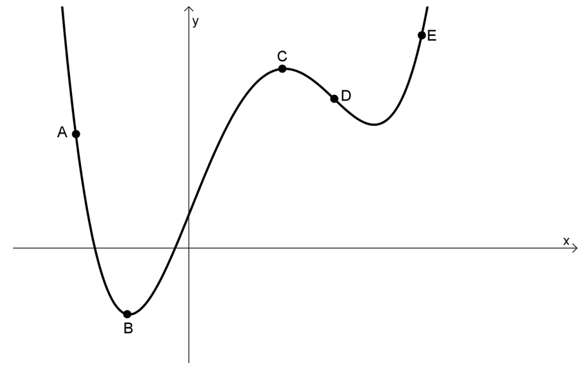
Question 11 **(3 marks)**



In each part, list the points (A-E) on the graph of f that satisfy the given conditions.

(a) $f'(x) > 0$ and $f(x) >$

(b) $f'(x) < 0$ and $f(x) >$

(c) $f'(x) = 0$ and $f(x) <$

(d) $f'(x) = 0$ and $f(x) >$

(e) $f'(x) < 0$ and $f(x) =$

Solution
(a) E, (b) A, (c) C, (d) B, (e) D
Specific behaviours
✓✓✓ correctly identifies all 5 points ✓✓ correctly identifies 3 points ✓ correctly identifies up to 2 points

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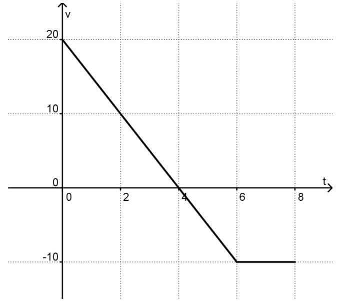
Question 12 **(5 marks)**

The diagram below shows the $v-t$ graph for a particle which moves in a horizontal straight line for $0 \le t \le 8$ seconds. At time $t=0$ the particle is at a point O on the line; the initial velocity is 20 ms^{-1}.



Find:

(a) the distance of the particle from O when $t=8$. [3]

Solution
Distance travelled in first 4 seconds = $\frac{1}{2}\times 4\times 20=40$ m. Distance travelled in next 4 seconds = $\frac{1}{2}\times(4+2)\times 10=30$ m. At $t=8$, particle is 10 m to the right of 0.
Specific behaviours
✓ calculates correctly distance travelled forward ✓ calculates correctly distance travelled backwards ✓ calculates correctly position relative to 0.

(b) the maximum distance of the particle from O. [1]

Solution
Maximum distance from 0 is 40 m.
Specific behaviours
✓ Calculates correct maximum distance

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Question 18 **(7 marks)**

Gas is escaping from a spherical balloon at the rate of 0.4 m^3/min.

(a) What is the change in volume during the first 10 minutes? [2]

Solution
$10 \times 0.4 = 4\ m^3$ Volume decreases by 4 m^3 in the first 10 minutes.
Specific behaviours
✓ correctly calculates the change in volume   ✓ interprets change in volume as a decrease

(b) How fast is the surface area shrinking when the radius is 4 m? [5]

Solution
$S = 4\pi r^2, \quad \frac{dS}{dr} = 8\pi r$ $V = \frac{4}{3}\pi r^3, \quad \frac{dV}{dr} = 4\pi r^2$ $\frac{dV}{dt} = -0.4$ $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$ ¿ $8\pi r \times \frac{1}{4\pi r^2} \times (-0.4)$ ¿ $-\frac{0.8}{r}$ When $r = 4$, $\frac{dS}{dt} = -0.2$ m^2/min   The surface area is shrinking at the rate of 0.2 m^2/min when the radius is 4 m.
Specific behaviours
✓ correctly determines $\frac{dS}{dr}$ and $\frac{dV}{dr}$ ✓ correctly determines $\frac{dV}{dt}$ ✓ correctly applies the chain rule   ✓ substitutes to find $\frac{dS}{dt}$ ✓ states the rate at which the surface area is shrinking, with correct units

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(c) the acceleration of the particle when $t = 2$. [1]

Solution
$a(2) = \frac{-20}{4} = -5$
Specific behaviours
✓ determines gradient of line segment when $t = 2$.

Question 13 **(8 marks)**

The Hiltonia group is planning a new hotel which is to be built on the Perth Esplanade. The hotel will have no more than 650 bedrooms, comprising single bedrooms and double bedrooms. To meet anticipated demand, there should be at most 14 double bedrooms for every 5 single bedrooms. It takes 30 minutes to clean a single bedroom and 50 minutes to clean a double bedroom. There are no more than 425 man hours available each day for cleaning the bedrooms.

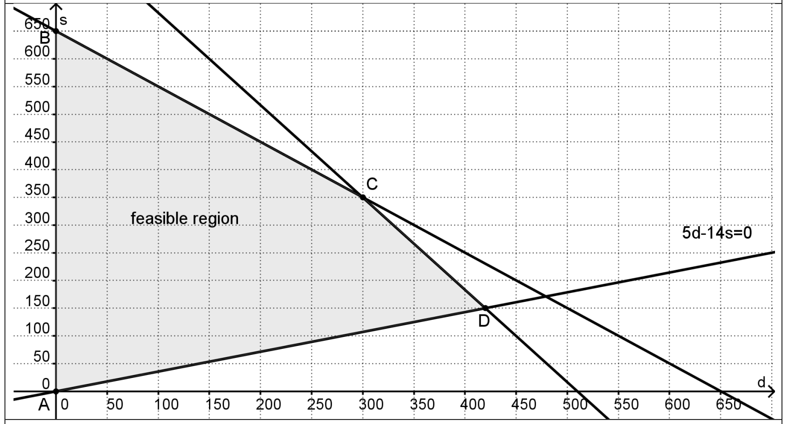
Each occupied single bedroom provides a daily profit of $60 and each occupied double bedroom provides a daily profit of $80.

(a) If s and d represent the number of single bedrooms and double bedrooms respectively, then $s+d\le 650$ and $0.5s+\frac{5}{6}d\le 425$ are two of the constraint inequalities. Write down the other inequality, apart from $s\ge 0$ and $d\ge 0$. [1]

Solution
$d:s\le 14:5$ i.e. $5d-14s\le 0$
Specific behaviours
✓ correct inequality

(b) Sketch the remaining constraint and indicate the feasible region on the axes below. [2]

Solution



Specific behaviours
✓ correctly draws line ✓ correctly shades feasible region

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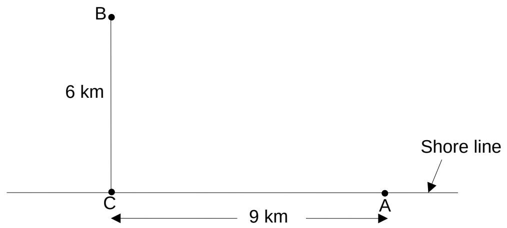
Question 17 **(6 marks)**

A company wishes to run a utility cable from point A on the shore to an installation at point B on an island. The island is 6 km from the shore (at point C) and point A is 9 km from point C. It costs $400 per km to run the cable on land and $500 per km underwater.

Assume that the cable starts at A and runs along the shoreline and then turns and runs under the water towards the island.

Use Calculus to determine the point at which the cable should turn in order to yield the minimum total cost.

Solution



Total cost, $T=400(9-x)+500(x^2+36)^{\frac{1}{2}}, 0\le x\le 9$

$$\frac{dT}{dx}=-400+250(x^2+36)^{\frac{-1}{2}}.2x$$

If $\frac{dT}{dx}=0$, then $\frac{500x}{(x^2+36)^{\frac{1}{2}}}=400$

Using calculator, $x=8$

If $x<8, \frac{dT}{dx}<0$. If $x>8, \frac{dT}{dx}>0$

Hence T is a minimum when $x=8$.

The cable should run 1 km along the shoreline from A and then turn.

Specific behaviours

- ✓ establishes distances cable runs along shoreline and under the water
- ✓ establishes expression for total cost
- ✓ correctly differentiates total cost
- ✓ correctly determines when the derivative is equal to zero
- ✓ checks that a minimum has been located
- ✓ correctly concludes where cable should turn

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(c) Assuming the hotel achieves 80% occupancy for each type of room, determine the maximum daily profit. [3]

Solution

Vertex (d,s)	$P=0.8(80d+60s)$
A(0,0)	0
B(0,650)	31 200
C(300,350)	36 000
D(420,150)	34 080

Maximum daily profit is $36 000.

Specific behaviours

- ✓ identifies the vertices of the feasible region
- ✓ calculates the value of the profit at each point
- ✓ identifies the maximum profit

(d) By how much can the daily profit on a single bedroom fall without affecting the optimal solution? Assume that the daily profit on a double bedroom does not change and that occupancy remains at 80%. [2]

Solution

Let $P=0.8(80d+ks)$

$$P(300,350)>P(420,150)$$
$$24000+350k>33600+150k$$
$$200k>9600$$
$$k>48$$

The daily profit on a single bedroom can fall by up to $12 without affecting the optimal solution.

Specific behaviours

- ✓ restates profit equation
- ✓ correctly states amount daily profit can fall

Question 14 **(10 marks)**

A bag contains 4 red balls and 6 green balls. Four balls are drawn at random from the bag without replacement.

(a) Calculate the probability that:

(i) all the balls drawn are green, [2]

Solution
$\frac{\binom{6}{4}\binom{4}{0}}{\binom{10}{4}}=\frac{15}{210}$ or $\frac{1}{14}$
Specific behaviours
✓ determines number of ways of selecting 4 green balls ✓ calculates probability

(ii) at least one ball of each colour is drawn, [2]

Solution
Either $1-\frac{15}{210}-\frac{1}{210}=\frac{194}{210}$ or $\frac{97}{105}$ Or $\frac{\binom{4}{1}\binom{6}{3}+\binom{4}{2}\binom{6}{2}+\binom{4}{3}\binom{6}{1}}{210}=\frac{80+90+24}{210}=\frac{194}{210}$ or $\frac{97}{105}$
Specific behaviours
✓ determines number of ways of selecting at least one ball of each colour ✓ calculates probability

(iii) at least two green balls are drawn, given that at least one of each colour is drawn. [3]

Solution
$\frac{\binom{6}{2}\binom{4}{2}+\binom{6}{3}\binom{4}{1}}{194}=\frac{90+80}{194}=\frac{170}{194}$ or $\frac{85}{97}$
Specific behaviours
✓ applies conditional probability rule ✓ determines number of ways of selecting at least 2 green balls and at least one of each colour ✓ calculates probability

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(e) Based on your results from (d), write a conjecture relating to multiplying any triangular number by 8 and then adding 1. [1]

Solution
The result is a perfect square.
Specific behaviours
✓ States conjecture

(f) Prove your conjecture. [3]

Solution
$T_n=1+2+3+\ldots+n+(n+1)=\frac{(n+1)}{2}(n+2)$ $T_n\times 8+1=\frac{(n+1)}{2}(n+2)\times 8+1\text{¿}4(n^2+3n+2)+1\text{¿}4n^2+12n+9\text{¿}(2n+3)^2$ Hence, when a triangular number is multiplied by 8 and then 1 is added, the result is a perfect square.
Specific behaviours
✓ Replaces T_n by $\frac{(n+1)}{2}(n+2)$ ✓ Expands and simplifies expression ✓ Factorises as a perfect square

Question 16 **(8 marks)**

The sequence of numbers 3, 6, 10, 15, 21, … are known as triangular numbers.

(a) Show that the first three triangular numbers can each be written as the sum of the first n consecutive positive integers. [1]

Solution
$T_1 = 3 = 1+2$ $T_2 = 6 = 1+2+3$ $T_3 = 10 = 1+2+3+4$
Specific behaviours
✓ Correctly writes the first three triangular numbers as the sum of the first n positive integers

(b) Hence, determine the 10th triangular number. [1]

Solution
$T_{10} = 1+2+3+\ldots+9+10+11 = 66$
Specific behaviours
✓ Correctly calculates $T_{10} = 66$

The formula $\frac{n}{2}(1+n)$ can be used to determine the sum of the first n positive integers.

(c) Use this formula to determine the 99th triangular number. [1]

Solution
$T_{99} = 1+2+3+\ldots+99+100 = \frac{100}{2}(1+100) = 5050$
Specific behaviours
✓ Correctly calculates $T_{99} = 5050$

(d) For each of the first three triangular numbers, multiply the number by 8 and then add 1. [1]

Solution
$3 \times 8+1 = 256 \times 8+1 = 4910 \times 8+1 = 81$
Specific behaviours
✓ Correct calculations

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(b) Are the events 'at least 2 green balls are drawn' and 'at least one ball of each colour is drawn' independent? Justify your answer. [3]

Solution
P(at least 2 green balls) = $\frac{\binom{6}{2}\binom{4}{2}+\binom{6}{3}\binom{4}{1}+\binom{6}{4}\binom{4}{0}}{210} = \frac{185}{210}$ P(at least one ball of each colour) = $\frac{194}{210}$ P(at least 2 green balls and at least one of each colour) = $\frac{170}{210}$ Since $\frac{185}{210} \times \frac{194}{210} = \frac{3007}{3675} \neq \frac{170}{210}$, the events are not independent
Specific behaviours
✓ calculates probability of at least 2 green balls ✓ multiplies $\frac{185}{210} \times \frac{194}{210}$ and compares with $\frac{170}{210}$ ✓ concludes events are not independent

Question 15 **(7 marks)**

The continuous random variable X has probability density function f given by:

$$f(x) = \begin{cases} k(4-x^2) & -2 \le x \le 2 \\ 0 & otherwise \end{cases}$$

(a) Show that $k = \frac{3}{32}$. [3]

Solution
Using calculator, $\int_{-2}^{2} (4-x^2)\,dx = \frac{32}{3}$ Since, $\int_{-2}^{2} k(4-x^2)\,dx = 1$, $k \cdot \frac{32}{3} = 1$ Hence, $k = \frac{3}{32}$
Specific behaviours
✓ correctly evaluates $\int_{-2}^{2} (4-x^2)\,dx$

See next page

✓ applies $\int_{-2}^{2} k(4-x^2)\,dx=1$ ✓ correctly deduces value of k

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(b) Find $P(X<1)$. [2]

Solution
$P(X<1)=\int_{-2}^{1} f(x)\,dx=\frac{27}{32}$
Specific behaviours
✓ equates probability with definite integral ✓ correct calculation

(c) Determine the median value of X. [2]

Solution
$P(X\le m)=0.5$ $\int_{-2}^{m} f(x)\,dx=0.5$ Using calculator, $m=0$ OR, since the graph of f is symmetrical about the vertical axis, $m=0$.
Specific behaviours
✓ identifies the value of the median is such that $P(X\le m)=0.5$ ✓ correctly determines the value of m