-2 max for rounding.

Mercedes College

Semester One Examination, 2017

Question/Answer booklet

MATHEMATICS
METHODS
UNIT 3

Section Two:
Calculator-assumed

If required by your examination administrator, please place your student identification label in this box

Student Number: In figures

In words ______________________

Your name Solns - Marking keys

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	11	11	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **eleven (11)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(7 marks)**

The voltage between the plates of a discharging capacitor can be modelled by the function $V(t) = 14e^{kt}$, where V is the voltage in volts, t is the time in seconds and k is a constant.

It was observed that after three minutes the voltage between the plates had decreased to 0.6 volts.

(a) State the initial voltage between the plates. (1 mark)

$t=0: \; v_0 = 14$ volts ✓

(1)

(b) Determine the value of k (correct to 4 decimal places). (2 marks)

$0.6 = 14e^{k(180)}$ ✓ 3mins = 180 s

Solver: $k = -0.0175$ ✓

(2)

If $0.6 = 14e^{3k}$
$k = -1.0500$

(c) How long did it take for the initial voltage to halve (correct to 1 decimal place)? (2 marks)

$0.5 = e^{-0.0175t}$ ✓ or $7 = 14e^{-0.0175t}$

$t = 39.6$ s ✓

(2)

If use $k = -1.0500$
$t = 0.6601$

(d) At what rate was the voltage decreasing at the instant it reached 8 volts (correct to 2 decimal places)? (2 marks)

$\frac{dV}{dt} = kV$

$= -0.0175 \times 8$ ✓

$= -0.14$

Decreasing at 0.14 volts/s ✓

(2)

2 marks if -0.14 V/s

7

** OR $8 = 14e^{-0.0175t}$

$t = 31.9780$

$\frac{dV}{dt} = -0.2450e^{-0.0175t}$

$= -0.2450e^{-0.0175(31.9780)}$

$= -0.14$

If use $k = -1.0500$
Decreasing 8.4 V/s

Question 10 **(11 marks)**

The gradient function of f is given by $f'(x) = 12x^3 - 24x^2$.

(a) Show that the graph of $y = f(x)$ has two stationary points. (2 marks)

$f'(x) = 0$

$12x^3 - 24x^2 = 0$ ✓ or $12x^2(x-2) = 0$

$\therefore \quad x = 0$ or $x = 2$ ✓

Hence, $f(x)$ has two stationary points.

(2)

(b) Determine the interval(s) for which the graph of the function is concave upward. (3 marks)

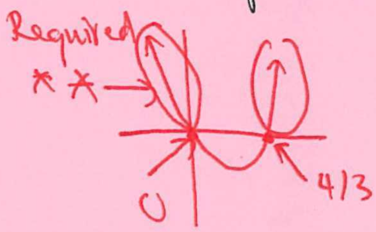
$f''(x) = 36x^2 - 48x$ ✓

$f''(x) > 0$ for the function to be concave upward.

$36x^2 - 48x > 0$ ✓

$x < 0$ or $x > \frac{4}{3}$

✓



(½) (½) (3)

(c) Given that the graph of $y = f(x)$ passes through $(1, 0)$, determine $f(x)$. (2 marks)

$f(x) = \int 12x^3 - 24x^2 \, dx$

$f(x) = 3x^4 - 8x^3 + c$ ✓

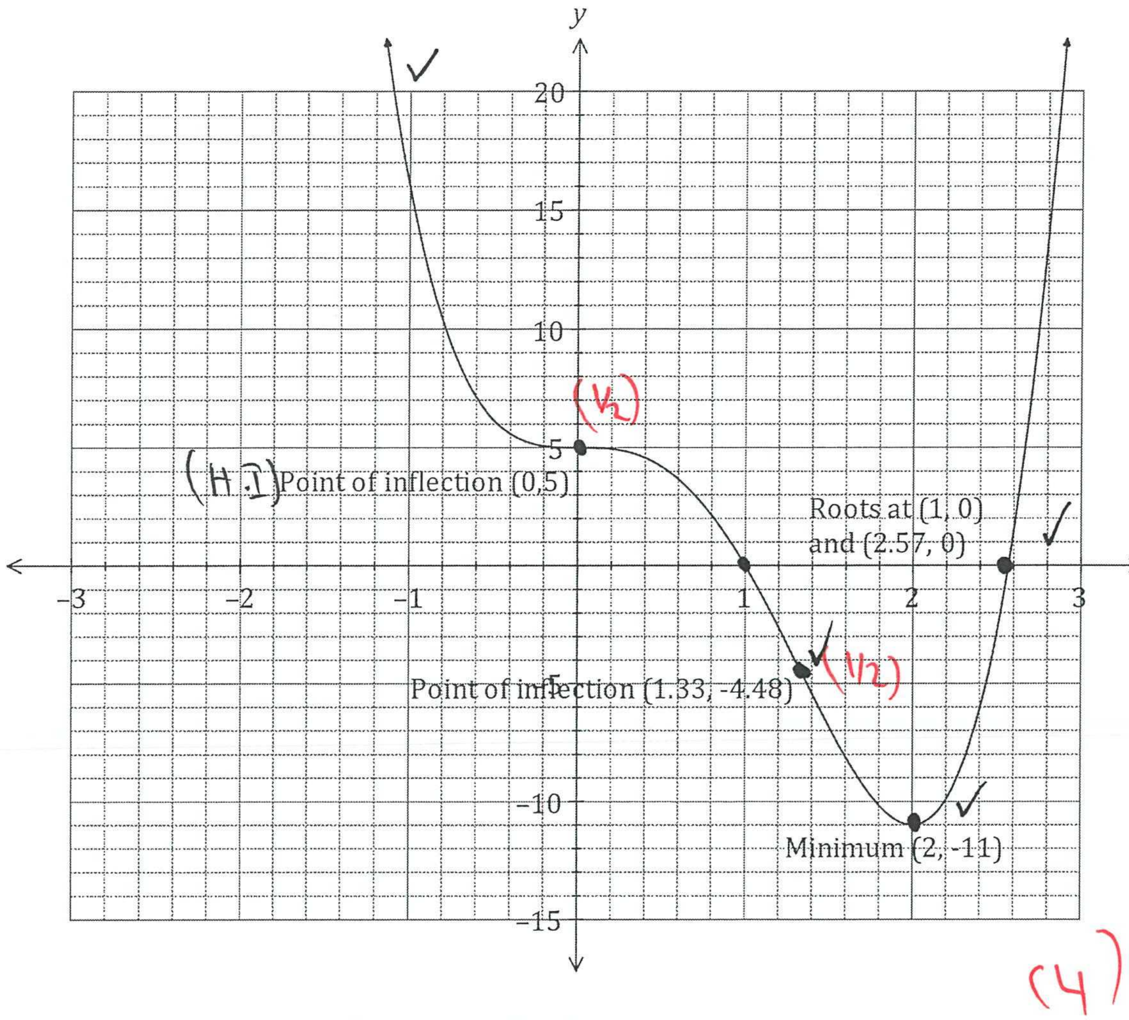
$(1,0): \quad 0 = 3(1)^4 - 8(1)^3 + c$

$c = 5$

Hence, $f(x) = 3x^4 - 8x^3 + 5$ ✓

(2)

(d) Sketch the graph of $y = f(x)$, indicating all key features. (4 marks)



Solution
See graph
Specific behaviours
✓ minimum ✓ roots ✓ points of inflection ✓ smooth curve

11

Question 11 **(7 marks)**

(a) Four random variables W, X, Y and Z are defined below. State whether the distribution of the random variable is Bernoulli, binomial, uniform or none of these.

(4 marks)

The dice referred to is a cube with faces numbered with the integers 1, 2, 3, 4, 5 and 6.

(i) W is the number of 3's rolled when the dice is thrown 8 times.

Binomial ✓

(ii) X is the score when a dice is thrown.

Uniform ✓

(iii) Y is the number of odd numbers showing when a dice is thrown.

Bernoulli ✓

(iv) Z is the total of the scores when two dice are thrown.

Neither ✓

(b) It is known that a certain proportion, p of the pegs produced by a manufacturer are defective (the probability of any one peg being defective is independent of any other pegs). The pegs are sold in bags of n for \$4.95. The random variable X is the number of faulty pegs in a bag.

If $E(X) = 1.8$ and $Var(X) = 1.728$, determine n and p. (3 marks)

$X \sim Bin(n, p)$

$E(X) = np = 1.8$ ✓

$Var(X) = np(1-p) = 1.728$

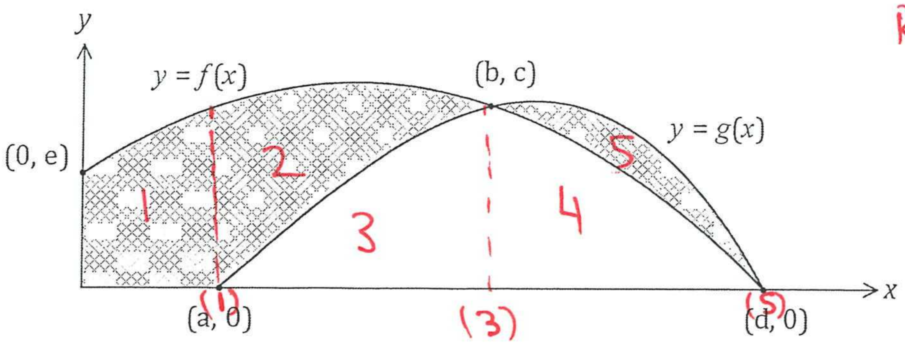
Solver: $n = 45$ ✓

$p = 0.04$ ✓ OR $\left(\frac{1}{25}\right)$

7

Question 12 **(7 marks)**

The graphs of the functions f and g are shown below, intersecting at the points (b, c) and $(d, 0)$.



Required Areas $= 1 + 2 + 5$

(a) Using definite integrals, write an expression for the area of the shaded region. (2 marks)

$$\text{Area} = \int_0^b f(x)\,dx - \int_a^b g(x)\,dx + \int_b^d [g(x) - f(x)]\,dx$$

$1 + 2 + \cancel{3}$ $\quad - \cancel{3}$ $\quad + \cancel{4} + 5$ $\quad - \cancel{4}$ ✓✓

$$\text{or Area} = \int_0^a f(x)\,dx + \int_a^b [f(x) - g(x)]\,dx + \int_b^d [g(x) - f(x)]\,dx$$

$1 \quad + \quad 2 \quad + \quad 5$

$$\text{or Area} = \int_0^d f(x)\,dx + \int_a^d g(x)\,dx - \left[2\int_a^b g(x)\,dx\right] - \left[2\int_b^d f(x)\,dx\right]$$

(b) Evaluate the area when $f(x) = 15 + 12x - 3x^2$ and $g(x) = -x^3 + 3x^2 + 13x - 15$. (5 marks)

$$15 + 12x - 3x^2 = -x^3 + 3x^2 + 13x - 15$$

$$x = -2, 3, 5$$

$$\Rightarrow b = 3, \; d = 5 \quad ✓$$

$$x^3 - 3x^2 + 13x - 15 = 0 \quad ✓$$

$$\Rightarrow a = 1$$

$$\text{Area} = \int_0^3 f(x)\,dx - \int_1^3 g(x)\,dx + \int_3^5 [g(x) - f(x)]\,dx$$

$$= 72 - 28 + 8 \quad ✓✓$$

$$= 52 \text{ units}^2 \quad ✓$$

7

$$\text{OR Area} = \int_0^1 f(x)\,dx + \int_1^3 [f(x) - g(x)]\,dx + \int_3^5 [g(x) - f(x)]\,dx$$

$$= 20 + 24 + 8$$

$$= 52 \text{ units}^2$$

Question 13 **(9 marks)**

75% of the avocados produced by a farm are known to be first grade, the rest being second grade. Trays of 24 avocados are filled at random in a packing shed and sent to market.

Let the random variable X be the number of first grade avocados in a single tray.

(a) Explain why X is a discrete random variable, and identify its probability distribution. (2 marks)

X is a discrete random variable as it can only take integer values from 0 to 24. ✓

$X \sim Bin(24, 0.75)$ (2) ✓

(b) Calculate the mean and standard deviation of X. (2 marks)

Mean $= \bar{X} = 24 \times 0.75 = 18$ ✓

S.D. $= \sigma_X = \sqrt{24 \times 0.75 \times 0.25}$

$= \frac{3\sqrt{2}}{2} \approx 2.1213$ ✓ (2)

(c) Determine the probability (correct to 4 dp) that a randomly chosen tray contains

(i) 18 first grade avocados. (1 mark)

$P(X=18) = 0.1853$ ✓ binomialPDF(18,24,0.75)

(1)

(ii) more than 15 but less than 20 first grade avocados. (2 marks)

$P(15 < X < 20) = P(16 \leq X \leq 19)$ ✓ binomialCDF(16,19,24,0.75)

$= 0.6320$ ✓

(2)

(d) In a random sample of 1000 trays, how many trays are likely to have fewer first grade than second grade avocados. (2 marks)

$\left.\begin{array}{l}75\% \text{ is } 1^{st} \text{ Grade} = 0.75 \times 24 = 18 \\ 25\% \text{ is } 2^{nd} \text{ Grade} = 0.25 \times 24 = 6\end{array}\right\}$ Fewer 1^{st} grade than 2^{nd} grade $\Rightarrow$ less than 12

$P(X<12) = P(X \leq 11) = 0.0021$ ✓ binomial CDF (0,11,24,0.75)

No. of trays $= 0.0021 \times 1000 \approx 2$ trays ✓ (2) 9

Question 14 (8 marks)

(a) The rate of decay of a radioactive compound is defined by:

$$\frac{dA}{dt} = -12e^{-0.03t}$$

where A is the mass (in grams) remaining after t days.

or $A = 400e^{-0.03t}$

$A = 400e^{-0.03(10)} = 296.33$

Decay $= 400 - 296.33 = 103.67$ g

(i) How many grams decayed during the first 10 days (correct to 2 decimal places)? (2 marks)

Amount decayed $= \int_0^{10} |-12e^{-0.03t}| dt = 103.67$ g

Hence, 103.67 grams decayed. ✓✓ (2)

OR $A = \int_0^{10} -12e^{-0.03t} dt$

$= -103.67$

Hence, 103.67 grams decayed

(ii) How many more days (after $t = 10$) are required for the same amount (as calculated above) to decay again (correct to 2 decimal places)? (3 marks)

$\int_0^{x} (-12e^{-0.03t}) dt = -103.67$ ✓

Solver : $t = 24.35$ days ✓

Hence, 14.35 more days ✓
(after 10 days)

(3)

Classpad

solve$(\int_0^{y} -12e^{-0.03x} dx = -103.67, y))$

~~$296.33 - 103.67 = 192.66$~~

or $192.66 = 400e^{-0.03t}$

$t = 24.35$

$24.35 - 10 = 14.35$ days

(b) Use the Fundamental Theorem of Calculus to find (3 marks)

$$\frac{d}{dx}\left[\int_2^{e^{4x}} \cos 3t \, dt + \int_{\pi x}^{1} \sin 2t \, dt\right]$$

$= \frac{d}{dx}\int_2^{e^{4x}} \cos 3t \, dt - \frac{d}{dx}\int_1^{\pi x} \sin 2t \, dt$ ✓

$= 4e^{4x} \cdot \cos(3e^{4x}) - \pi \sin(2\pi x)$

✓ ✓

(3)

8

Question 15 **(10 marks)**

A slot machine is programmed to operate at random, making various payouts after patrons pay \$2 and press a start button. The random variable X is the amount of the payout, in dollars, in one play of the machine. Each payout can be assumed to be independent of other payouts.

The probability, P, that the machine makes a certain payout, x, is shown in the table below.

Payout (\$) x	0	1	2	5	10	20	50	100
Probability $P(X = x)$	0.25	0.45	0.2125	0.0625	0.0125	0.005	0.005	0.0025

(a) Determine the probability (correct to 4 decimal places) that

(i) in one play of the machine, a payout of more than \$1 is made. (1 mark)

$P(X>1) = 1 - [P(X=0) + P(X=1)]$

$= 1 - (0.25 + 0.45)$

$= 0.3000$ ✓ (1)

(ii) in ten plays of the machine, it makes a payout of \$5 no more than once. (2 marks)

$Y \sim Bin(10, 0.0625)$ ✓

$P(Y \le 1) = 0.8741$ ✓

binomial CDF (0, 1, 10, 0.0625)

(2)

(iii) in five plays of the machine, the second payout of \$1 occurs on the fifth play. (3 marks)

- 1^{st} payout in one of four plays: $W \sim Bin(4, 0.45)$

  $P(W=1) = 0.2995$ ✓

  binomial PDF (1, 4, 0.45)

- 2^{nd} payout on the 5^{th} play $p = 0.45$ ✓

Hence, The second payout of \$1 occurs on the 5^{th} play

$= 0.2995 \times 0.45$

$= 0.1348$ ✓ (3)

(b) Calculate the mean and standard deviation of X. (2 marks)

Mean $= \bar{x} = 1.9125$ ✓ Using G.C.

S.D. $= \sigma_x = 6.321$ ✓

(2)

(c) In the long run, what percentage of the patron's money is returned to them? (2 marks)

% patron's money returned

$= \frac{1.9125}{2} \times 100$ ✓

$= 95.63\%$ ✓

(2)

10

Question 16 **(12 marks)**

Particle P leaves point A at time $t = 0$ seconds and moves in a straight line with acceleration given by

$$a = \frac{16}{(2t+1)^3}\ \mathrm{ms}^{-2}.$$

Particle P has an initial velocity of $-3\ \mathrm{ms}^{-1}$ and point A has a displacement of 4 metres from the origin.

(a) Calculate the initial acceleration of P. (1 mark)

$$t=0 \ : \quad a = \frac{16}{(2(0)+1)^3} = 16\ \mathrm{m/s^2} \quad ✓$$

(1)

(b) Is P ever stationary? If your answer is yes, determine the time(s) when this happens. If your answer is no, explain why. (3 marks)

$$v = \int a\,dt = \int \frac{16}{(2t+1)^3}\,dt$$

$$v = -\frac{4}{(2t+1)^2} + c$$

$$t=0, v=-3 : \ -3 = -\frac{4}{(2(0)+1)^2} + c$$

$$c = -3+4 = 1$$

Hence, $v = -\dfrac{4}{(2t+1)^2} + 1$ ✓

Let $v=0$: $-\dfrac{4}{(2t+1)^2} + 1 = 0$ (3)

✓ $\Rightarrow t = 0.5\,s$ ✓ (ignored $t=-1.5$)

YES. P is stationary when $t = 0.5\,s$

(c) Calculate the displacement of P from the origin when $t = 12$ seconds. (2 marks)

$$\Delta x = \int_0^{12} v\,dt = \int_0^{12} -\frac{4}{(2t+1)^2} + 1\,dt = 10.08\ \mathrm{m} \ ✓$$

$\therefore$ Displacement $= 10.08 + 4 = 14.08\,\mathrm{m}$ ✓

OR $x(t) = \int -\dfrac{4}{(2t+1)^2} + 1\,dt = t + \dfrac{2}{2t+1} + c$

$t=0, x=4 : \ 4 = 0 + \dfrac{2}{2(0)+1} + c \Rightarrow c = 2$

$x(t) = t + \dfrac{2}{2t+1} + 2$

$t=12 : x = 12 + \dfrac{2}{2(12)+1} + 2 = 14.08\,\mathrm{m}$

(2)

(d) Calculate the change of displacement of P during the third second. (2 marks)

Change of displacement

$\Delta x = \int_2^3 v\,dt = \int_2^3 -\frac{4}{(2t+1)^2}+1\,dt$ ✓

$= \frac{31}{35} \approx 0.8857$ m ✓

OR

$\Delta x = x(3) - x(2)$

$\approx 5.2857 - 4.4$

≈ 0.8857 m

1 mark for upper + lower bounds

(2)

(e) Determine the maximum speed of P during the first three seconds and the time when this occurs. (2 marks)

$a = 0 \Rightarrow$ No solution $\Rightarrow$ speed decreases

$t = 0: \quad v = -3$ m/s $\Rightarrow$ speed $= |v| = 3$ m/s ✓

$t = 3: \quad v = 0.9184$ m/s $\Rightarrow$ speed $= 0.9184$ m/s

Hence, the maximum speed is 3 m/s. ✓

(2)

(f) Calculate the total distance travelled by P during the first three seconds. (2 marks)

Total distance travelled $= \int_0^3 |v|\,dt$

$= \int_0^3 \left| -\frac{4}{(2t+1)^2}+1 \right| dt$ ✓

$= 2.2857$ m ✓

(2)

[or $d = -\int_0^{0.5} v\,dt + \int_{0.5}^3 v\,dt$]

$= 0.5 + 1.7857$

$= 2.2857$

12

Question 17 **(10 marks)**

Let the random variable X be the number of vowels in a random selection of four letters from those in the word LOGARITHM, with no letter to be chosen more than once.

(a) Complete the probability distribution of X below. (1 mark)

x	0	1	2	3
$P(X = x)$	$\frac{5}{42}$	$\frac{10}{21}$	$\frac{5}{14}$ ✓	$\frac{1}{21}$

$$1 - \left(\frac{5}{42} + \frac{10}{21} + \frac{1}{21}\right)$$

(1)

(b) Show how the probability for $P(X = 1)$ was calculated. (2 marks)

$$P(X=1) = \frac{\binom{3}{1}\cdot\binom{6}{3}}{\binom{9}{4}} \quad ✓$$

$$= \frac{3 \times 20}{126}$$

$$= \frac{60}{126} = \frac{10}{21} \quad ✓ \quad (0.4762)$$

$$4 \times \left[\frac{3}{9} \times \frac{6}{8} \times \frac{5}{7} \times \frac{4}{6}\right] = \frac{10}{21}$$

(2)

(c) Determine $P(X \geq 1 | X \leq 2)$. (2 marks)

$$P(X \geqslant 1 | X \leqslant 2) = \frac{P(1 \leqslant X \leqslant 2)}{P(X \leqslant 2)} = \frac{\frac{10}{21} + \frac{5}{14}}{1 - \frac{1}{21}} \quad ✓$$

$$= \frac{\frac{5}{6}}{\frac{20}{21}} = \frac{7}{8} \quad ✓$$

(2)

(0.875)

Let event A occur when no vowels are chosen in random selection of four letters from those in the word LOGARITHM.

(d) State $P(\bar{A})$. (1 mark)

$$= 1 - P(A)$$

$$= 1 - \frac{5}{42}$$

$$= \frac{37}{42} \quad ✓$$

(1)

(0.8810)

(e) Let Y be a Bernoulli random variable with parameter $p = P(A)$. Determine the mean and variance of Y. (2 marks)

Y is a Bernoulli random variable:

mean $= p = \frac{5}{42}$ ✓ (≈ 0.1190)

Variance $= p(1-p) = \frac{5}{42}\left(1 - \frac{5}{42}\right) = \frac{185}{1764}$ ✓ (≈ 0.1049)

(2)

(f) Determine the probability that A occurs no more than twice in ten random selections of four letters from those in the word LOGARITHM. (2 marks)

$W \sim Bin\left(10, \frac{5}{42}\right)$ ✓

$P(W \leq 2) = 0.8933$ ✓

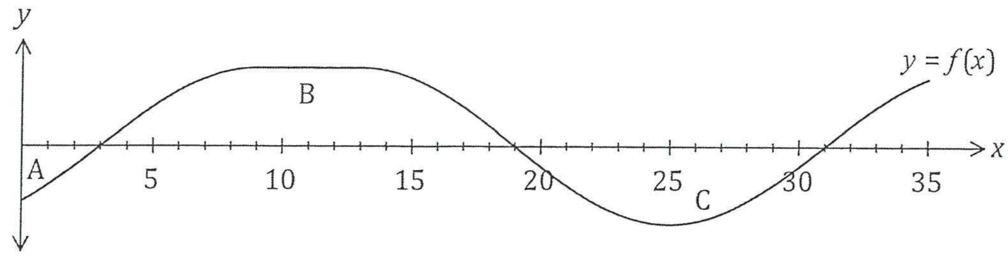
binomial CDf $\left(0, 2, 10, \frac{5}{42}\right)$

(2)

10

Question 18 **(9 marks)**

The graph of $y = f(x)$ is shown below. The areas between the curve and the $x-$ axis for regions A, B and C are 3, 20 and 12 square units respectively.



(a) Evaluate

(i) $\int_0^{31} f(x)\,dx$. (1 mark)

$= (-3) + 20 + (-12)$

$= 5$ ✓

(1)

(ii) $\int_{19}^{0} f(x)\,dx$. (2 marks)

$= -\int_0^{19} f(x)\,dx$ ✓

$= -[(-3)+20]$

$= -17$ ✓

(2)

(iii) $\int_3^{31} 2 - 3f(x)\,dx$. (3 marks)

$= \int_3^{31} 2\,dx - 3\int_3^{31} f(x)\,dx$ ✓

$\int = 2x\Big]_3^{31}$
$= 2(31-3)$
$= 2(28)$
$= 56$

$= 56 - 3(20-12)$ ✓

$= 56 - 24$

$= 32$ ✓

(3)

It is also known that $A(31) = 0$, where $A(x) = \int_{10}^{x} f(t)\, dt$.

(b) Evaluate

(i) $A(19)$. (1 mark)

$$A(31) = \int_{10}^{31} f(t)dt = \int_{10}^{19} f(t)\,dt + \int_{19}^{31} f(t)\,dt = 0$$

$$\Rightarrow A(19) = \int_{10}^{19} f(t)\,dt = -\int_{19}^{31} f(t)\,dt = -(-12) = 12$$

(1)

(ii) $A(0)$. (2 marks)

$$\int_{3}^{10} f(t)\,dt + \int_{10}^{19} f(t)\,dt = \int_{3}^{19} f(t)\,dt$$

$$\int_{3}^{10} f(t)\,dt + 12 = 20$$

$$\int_{3}^{10} f(t)\,dt = 20 - 12 = 8$$

$$A(3) = \int_{10}^{3} f(t)\,dt = -\int_{3}^{10} f(t)\,dt = -8$$

$$A(0) = \int_{10}^{0} f(t)\,dt = -\int_{0}^{10} f(t)\,dt$$

$$= -\left[-\int_{0}^{3} f(t)\,dt + \int_{3}^{10} f(t)\,dt\right]$$

$$= -(-3 + 8)$$

$$= -5$$

(2)

9

Question 19 **(8 marks)**

A storage container of volume 36π cm^3 is to be made in the form of a right circular cylinder with one end open. The material for the circular end costs 12c per square centimetre and for the curved side costs 9c per square centimetre.

(a) Show that the cost of materials for the container is $12\pi r^2 + \dfrac{648\pi}{r}$ cents, where r is the radius of the cylinder. (3 marks)

$V = \pi r^2 h \Rightarrow h = \dfrac{V}{\pi r^2} = \dfrac{36\cancel{\pi}}{\cancel{\pi} r^2} = \dfrac{36}{r^2}$ ✓

$A = \pi r^2 + 2\pi r h$

$\text{Cost} = 12(\pi r^2) + 9(2\pi r h)$ ✓

$C(r) = 12\pi r^2 + 18\pi r h$

$= 12\pi r^2 + 18\pi \cancel{r}\left(\dfrac{36}{r^{\cancel{2}}}\right)$ ✓

$= 12\pi r^2 + \dfrac{648\pi}{r}$

(3)

(b) Use calculus techniques to determine the dimensions of the container that minimise its material costs and state this minimum cost. (5 marks)

$C'(r) = \dfrac{24\pi r^3 - 648\pi}{r^2}$ ✓ OR $C'(r) = 24\pi r - \dfrac{648\pi}{r^2}$ diff

$C'(r) = 0 \Rightarrow 24\pi r^3 - 648\pi = 0$ solve(

$r = 3$ cm ✓

$C(3) = 324\pi = 1017.88$ cents $= \$10.18$

$h = \dfrac{36}{r^2} = \dfrac{36}{3^2} = 4$ cm ✓

$C''(r) = \dfrac{24\pi r^3 + 1296\pi}{r^3}$ ✓ (5)

$C''(3) = 72\pi > 0$ ∴ Minimum cost is \$10.18 ✓

∴ The minimum cost is \$10.18 when $r = 3$cm and $h = 4$ cm

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END OF QUESTIONS