Papers written by
Australian Maths
Software

SEMESTER TWO

MATHEMATICS SPECIALIST

REVISION 2

UNIT 3-4

2016

SOLUTIONS

Section One

1. (6 marks)

(a) $z = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{1+2i+i^2}{1-i^2} = \frac{2i}{2} = i$ ✓

(i) $1 + \frac{1}{z} = 1 + \frac{1}{i} \times \frac{i}{i} = 1 - i$ ✓

(ii) $\frac{\bar{z}}{|z|^2} = \frac{-i}{|i|^2} = \frac{-i}{1} = -i$ ✓✓

(b) $\left\{ z : \ \frac{\pi}{4} \le arg(z) < \frac{3\pi}{4} \ \cap \ 1 < Im(z) \le 3 \right\}$ ✓✓

2. (6 marks)

(a) $sin(x-y) + cos(x+y) = 1$

$\left(1 - \frac{dy}{dx}\right) cos(x-y) - \left(1 + \frac{dy}{dx}\right) sin(x+y) = 0$ ✓✓

$\frac{dy}{dx}(-cos(x-y) - sin(x+y)) + cos(x-y) - sin(x+y) = 0$ ✓

$\frac{dy}{dx}(-cos(x-y) - sin(x+y)) = sin(x+y) - cos(x-y)$

$\frac{dy}{dx} = \frac{sin(x+y) - cos(x-y)}{-cos(x-y) - sin(x+y)}$ ✓

(b) At $\left(\frac{\pi}{2}, 0\right)$ $\frac{dy}{dx} = \frac{sin\left(\frac{\pi}{2}+0\right) - cos\left(\frac{\pi}{2}-0\right)}{-cos\left(\frac{\pi}{2}-0\right) - sin\left(\frac{\pi}{2}+0\right)}$ ✓

$\frac{dy}{dx} = \frac{1-0}{-1}$

$\frac{dy}{dx} = -1$ ✓

3. (10 marks)

(a) $\int tan(x)dx$

$= \int \frac{sin(x)}{cos(x)}dx$ put $u = cos(x)$

$\frac{du}{dx} = -sin(x)$

$-du = sin(x)dx$

$\equiv \int \frac{-du}{u}$

$= -ln(u) + c$

$\equiv -ln(cos(x)) + c$

(b) $\int_{e}^{e^2} \frac{ln(x)dx}{x}$ put $u = ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{dx}{x}$

If $u = e^2$, $u = ln(e^2) = 2$

If $u = e$, $u = ln(e) = 1$

$= \int_1^2 u\,du$

$= \frac{1}{2}\left[u^2\right]_1^2$

$= \frac{1}{2}(4-1)$

$= 1.5$

(c) $\int_0^{\frac{\pi}{3}} sin(3x)\times e^{cos(3x)}dx$

put $u = cos(3x)$

$$\frac{du}{dx} = -3sin(3x)$$

$$\frac{du}{-3} = sin(3x)dx$$

$$If\ x = \frac{\pi}{3},\ u = -1$$

$$If\ x = 0,\ u = 1$$

$$= \int_1^{-1} e^u \frac{du}{-3}$$

$$= -\frac{1}{3}\left[e^u\right]_1^{-1}$$

$$= -\frac{1}{3}\left(\frac{1}{e} - e\right)$$

$$= \frac{1}{3}\left(e - \frac{1}{e}\right)$$

4. (3 marks)

The other four roots are $-\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{2}$

$$z = cis\left(\frac{\pi}{6} + \frac{k2\pi}{6}\right)$$

$$z^6 = cis(\pi + 2k\pi)$$

$$z^6 = cos(\pi) + i\,sin(\pi + 2k\pi)$$

$$z^6 = -1$$

Check

$k = 0 \quad z = cis\left(\frac{\pi}{6}\right)$

$k = 1 \quad z = cis\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = cis\left(\frac{\pi}{2}\right)$ add multiples of $\frac{\pi}{3}$

$k = -1 \quad z = cis\left(\frac{\pi}{6} - \frac{\pi}{3}\right)$

$k = 2 \quad z = cis\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) = cis\left(\frac{5\pi}{6}\right)$

$k = -2 \quad z = cis\left(\frac{\pi}{6} - \frac{2\pi}{3}\right) = cis\left(-\frac{\pi}{2}\right)$

$k = -3 \quad z = cis\left(\frac{\pi}{6} - \frac{3\pi}{3}\right) = cis\left(-\frac{5\pi}{6}\right)$

5. (6 marks)

(a) Let $P(z)=z^3-2z^2-3z+10$ and $Q(z)=z^3-7z^2+17z-15$

$P(-2)=-8-8+6+10=0$

$\therefore\ z=-2$

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -3 & 10 \\ & \downarrow & -2 & 8 & -10 \\ \hline & 1 & -4 & 5 & 0 \end{array}$$

$\therefore P(z)=(z+2)(z^2-4z+5)$

$z=-2$ or $(z^2-4z+5)=0$

$$z=\frac{4\pm\sqrt{16-20}}{2}$$

$z=-2$ or $z=\dfrac{4\pm 2i}{2}$

$z=-2$ or $z=2\pm i$

$P(3)=27-63+51-15=78-78=0$

$\therefore\ z=3$

$$\begin{array}{r|rrrr} 3 & 1 & -7 & 17 & -15 \\ & \downarrow & 3 & -12 & 15 \\ \hline & 1 & -4 & 5 & 0 \end{array}$$

$\therefore P(z)=(z-3)(z^2-4z+5)$

$z=-3$ or $(z^2-4z+5)=0$

which is the same quadratic factor so there are two common roots.

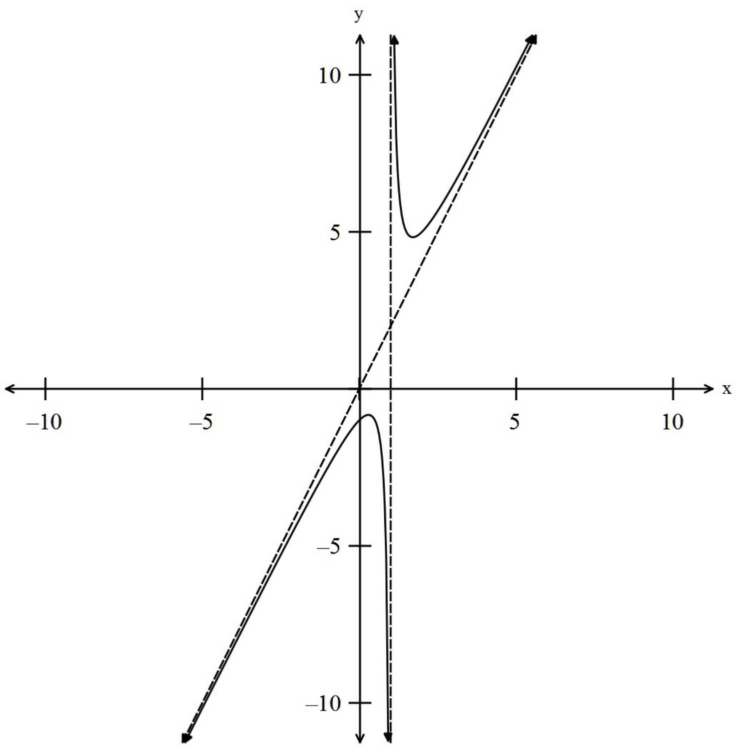
(b) $(z-i)(z-2+i)(z-3-i)$

$=(z-i)(z^2-5z+7-i)$

$=z^3+(-5-i)z^2+(7+4i)-1-7i$

6. (6 marks)

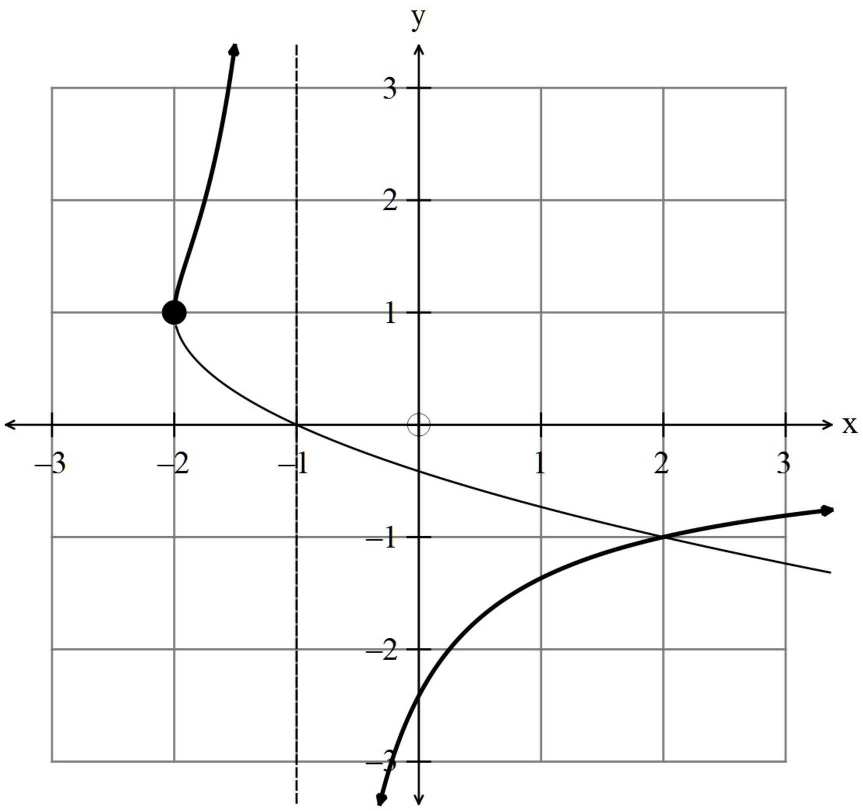
(a)



✓✓✓✓

y intercept, asymptotes, no x intercept

(b)



7. (11 marks)

(a) $\boldsymbol{r}(t) = (10cos(t))\boldsymbol{i} + (6sin(t))\boldsymbol{j}$

$\boldsymbol{v}(t) = (-10sin(t))\boldsymbol{i} + (6cos(t))\boldsymbol{j}$

$|\boldsymbol{v}(t)| = \sqrt{100sin^2(t) + 36cos^2(t)}$

$|\boldsymbol{v}(t)| = \sqrt{64sin^2(t) + 36}$

(b) Max speed occurs when $sin(t) = 1$ and is 10 m/sec

Min speed occurs when $sin(t) = 0$ and is 6 m/sec

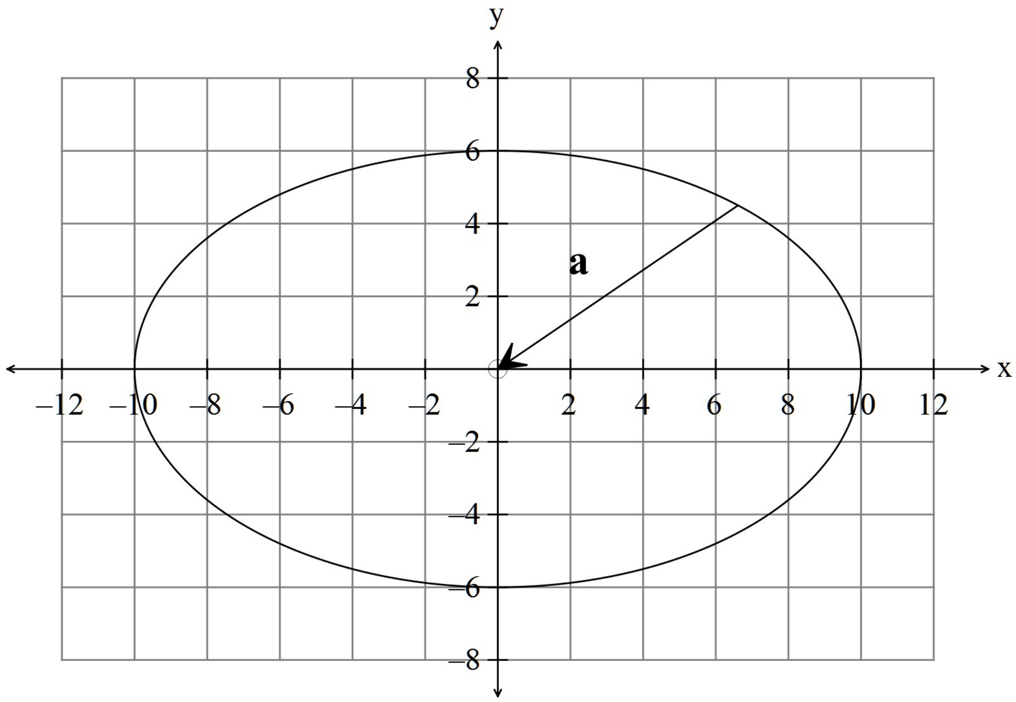
(c) Max : $sin(t) = \pm 1$ when $t = \frac{\pi}{2}, \frac{3\pi}{2}$ at $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$. ✓

Min : $sin(t) = 0$ when $t = 0, \pi$ at $\begin{pmatrix} 10 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -10 \\ 0 \end{pmatrix}$.

(d) $\boldsymbol{a}(t) = (-10\cos(t))\boldsymbol{i} - (6\sin(t))\boldsymbol{j}$

$\boldsymbol{a}\left(\frac{\pi}{4}\right) = \left(-10\cos\left(\frac{\pi}{4}\right)\right)\boldsymbol{i} - \left(6\sin\left(\frac{\pi}{4}\right)\right)\boldsymbol{j} = -5\sqrt{2}\,\boldsymbol{i} - 3\sqrt{2}\,\boldsymbol{j}$

$\boldsymbol{r}\left(\frac{\pi}{4}\right) = \left(10\cos\left(\frac{\pi}{4}\right)\right)\boldsymbol{i} + \left(6\sin\left(\frac{\pi}{4}\right)\right)\boldsymbol{j} = 5\sqrt{2}\,\boldsymbol{i} + 3\sqrt{2}\,\boldsymbol{j}$



(e) $x = 10\cos(t),\ y = 6\sin(t)$

$\sin^2(t) + \cos^2(t) = 1$

$\left(\frac{x}{10}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$

8 (4 marks)

$$\int \frac{-3dx}{(x-2)(x+1)} = -3\int \frac{dx}{(x-2)(x+1)}$$

$$\frac{1}{(x-2)(x+1)} = \frac{a}{(x-2)} + \frac{b}{(x+1)}$$

$$\frac{0x+1}{(x-2)(x+1)} = \frac{a(x+1)+b(x-2)}{(x-2)(x+1)}$$

$$= \frac{x(a+b)+(a-2b)}{(x-2)(x+1)}$$

Equating coefficients

$a+b=0 \qquad a-2b=1$

$a=-b \qquad -3b=1$

$a=\frac{1}{3} \qquad b=-\frac{1}{3}$

$$\int \frac{-3dx}{(x-2)(x+1)} = -3\left(\int \left(\frac{1/3}{(x-2)} - \frac{1/3}{(x+1)}\right) dx\right)$$

$$= \int \left(\frac{1}{(x+1)} - \frac{1}{(x-2)}\right) dx$$

$$= ln(x+1) - ln(x-2) + c$$

$$\int \frac{-3dx}{(x-2)(x+1)} = ln\frac{(x+1)}{(x-2)} + c$$

END OF SECTION ONE

Section Two

9. (4 marks)

(a) $(x-1)^2+(y-2)^2+z^2=4$

$At \ \ y=3, \quad (x-1)^2+z^2=3$

which is a circle with centre $(1,3,0)$ with radius $\sqrt{3}$ and lying in the plane $y=3$.

(b) $x=1-t$ $\qquad$ $x=-s$

$y=1+2t$ $\qquad$ $y=1+4s$

$z=2+3t$ $\qquad$ $z=3+5s$

$If \ \ x=x$ $\qquad$ $If \ \ y=y$

$1-t=-s$ $\qquad$ $1+2t=1+4s$

$t=1+s$ $\qquad$ $2t=4s$

$t=2s$

$\therefore 1+s=2s$

$s=1, t=2$

$If \ s=1, z=2+6=8$

$If \ t=2, z=2+3(2)=8$ Same z value

Therefore the two lines do meet, and they meet at the point P(-1,5,8)

10. (10 marks)

Given $f(x)=x^2$, $g(x)=\frac{1}{x}$, $h(x)=ln(x), j(x)=2x-1$ and $k(x)=\sqrt{x}$

(a) $A \ \ h(f(x))=h(x^2)=ln(x^2)$ ✓✓

$B \ \ j(g(x))=j\left(\frac{1}{x}\right)=\frac{2}{x}-1$ ✓✓

$C \ \ f(g(x))=f\left(\frac{1}{x}\right)=\frac{1}{x^2} \quad OR \quad g(f(x))=g(x^2)=\frac{1}{x^2}$ ✓✓

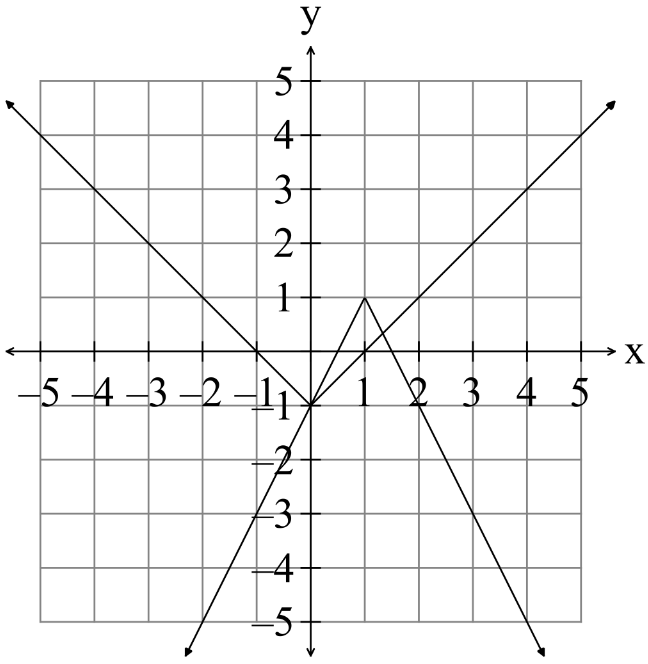
(b) $D \ \ g^{-1}(x)=\frac{1}{x}$

$E \ \ h^{-1}(x)=e^x$

$F \ \ k^{-1}(x)=x^2$ for $x\geq 0$

$G \ \ j^{-1}(x)=\frac{1+x}{2}$

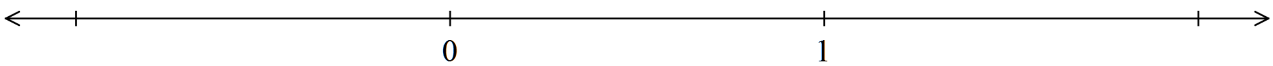
11. (5 marks)



Not a good method to find fractions

$$|x|-1=\begin{cases} x-1 & for \quad x\ge 0 \\ 1-x & for \quad x<0 \end{cases}$$

$$1-2|x-1|=\begin{cases} 3-2x & for \quad x\ge 1 \\ -1+2x & for \quad x<1 \end{cases}$$



$1-x=-1+2x$

$2=3x$

$x=\frac{2}{3}$

Not in given domain

$x-1=-1+2x$

$x=0$

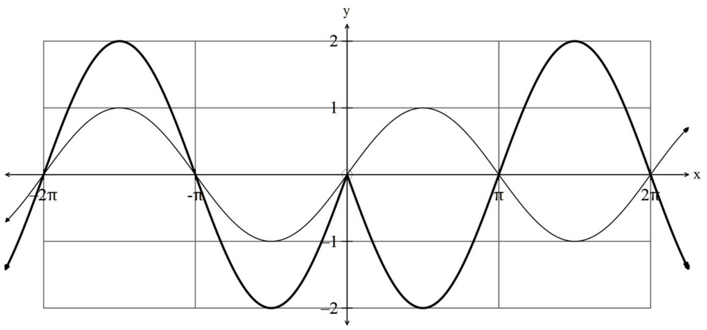
$x-1=3-2x$

$3x=4$

$x=\frac{4}{3}$

$$x=0 \quad OR \quad x=\frac{4}{3}$$

(b)



12. (8 marks)

(a) $z^5 - 1 = 0$

$z^5 = 1 = \left(cis\left(0 + n\left(2\pi\right)\right)\right)$

$z = \left(cis\left(2n\pi\right)\right)^{\frac{1}{5}}$

$z = \left(cis\left(\frac{2n\pi}{5}\right)\right)$

$n = 0 \qquad z = \left(cis\left(0\right)\right)$

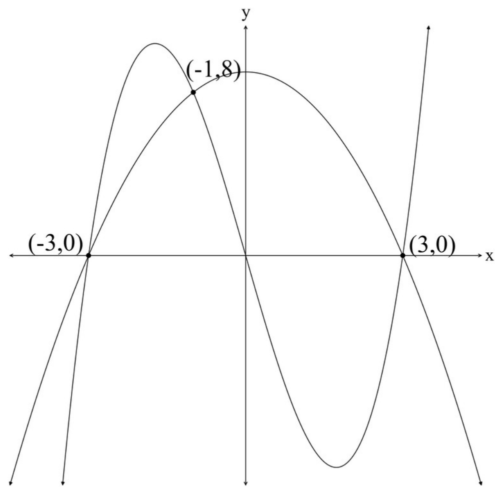
$n = 1 \qquad z = cis\left(\frac{2\pi}{5}\right)$

$n = 2 \qquad z = cis\left(\frac{4\pi}{5}\right)$

$n = -1 \qquad z = cis\left(-\frac{2\pi}{5}\right)$

$n = -2 \qquad z = cis\left(-\frac{4\pi}{5}\right)$ ✓✓✓ -1/ error

(b) $f\left(x\right) = x\left(x - 3\right)\left(x + 3\right)$ and $g\left(x\right) = 9 - x^2$.



The x values of the intersection points are -3, -1 and 3 ✓

Area =

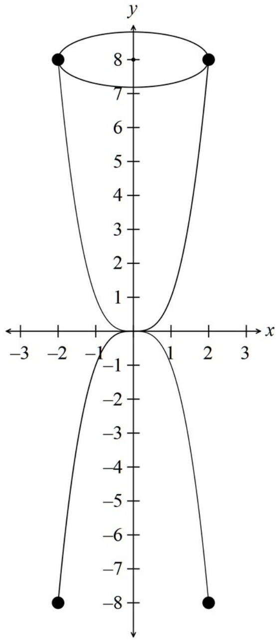
$\int_{3}^{-1}\left(x^3 - 9x^2\right) - \left(9 - x^2\right)dx + \int_{1}^{3}\left(9 - x^2\right) - \left(x^3 - 9x^2\right)dx = 6\frac{2}{3} + 42\frac{2}{3} = 49\frac{1}{3}\text{units}^2$

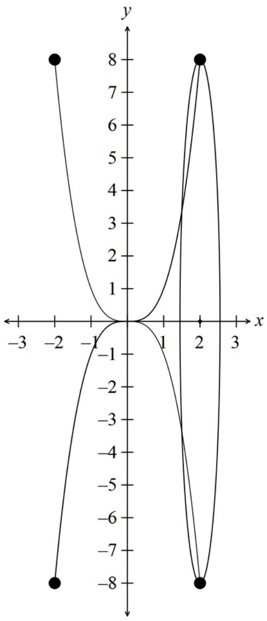
✓ ✓

OR $Area = \int_{3}^{3}\left|\left(x^3 - 9x^2\right) - \left(9 - x^2\right)\right|dx = 49\frac{1}{3}\text{units}^2$

13. (9 marks)

(a) About the x axis: About the y axis





✓

✓✓ $V_{y\ axis} = 2\int_0^8 \pi y^{\frac{2}{3}}\, dy \quad = 120.64\text{units}^3$ $V_{x\ axis} = 2\int_0^2 \pi x^6\, dx \quad = 114.89\text{units}^3$ ✓✓

There is a different shape generated so the volumes will be different.

(b) The graphs shade the same area above and below the x axis.

$$V = 2\int_0^8 \pi y^{\frac{2}{3}}\, dy \; - \; 2\int_0^8 \pi\left(\frac{y}{4}\right)^2 dy$$

✓ ✓✓

14. (12 marks)

(a) (i) $v = \sqrt{4 - 2x}$

$$a = \frac{d}{dx}\left(\frac{v^2}{2}\right)$$

$$a = \frac{d}{dx}\left(\frac{4 - 2x}{2}\right) = \frac{d}{dx}(2 - x)$$

$$a = -1 cm\, s^{-2}$$

$$At\ t = 2, a = -1 cm\, s^{-2}$$

(ii) $v = \sqrt{4 - 2x}$

$$At\ \ v = 3, \qquad 9 = 4 - 2x$$

$$2x = -5$$

$$x = -2.5$$

(b) $\dfrac{dy}{dx} = 2xy - y$

$\displaystyle\int \frac{dy}{y} = \int (2x - 1)\,dx$

$ln(y) = x^2 - x + c$

$(1, e) \quad ln\,e = 1 - 1 + c \ \rightarrow \ c = 1$

$ln(y) = x^2 - x + 1$

$y = e^{x^2 - x + 1}$

(c) $V = \dfrac{4}{3}\pi r^3$

$\dfrac{dV}{dt} = 2cm^3 / s \qquad \dfrac{dr}{dt} = ?\ at\ r = 10\,cm$

$\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$ ✓✓

$2 = 4\pi 10^2 \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{1}{200\pi} cm / s$

15. (7 marks)

(a) $\boldsymbol{r}(t) = \boldsymbol{AB} + t\,\boldsymbol{AB} + s\,\boldsymbol{AC}$

$\boldsymbol{r}(t) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 6 \\ 3 \\ 10 \end{pmatrix}$ ✓✓

(b) (i) No solution where we have $2p = 0$ and $1 - q \neq 0$ $\quad i.e.\ p = 0,\ q \neq 1$ ✓✓

(ii) An infinite number of solutions if $2p = 0$ and $1 - q = 0$ $\ i.e.\ p = 0,\ q = 1$ ✓✓

(iii) Exactly one solution if $2p \neq 0$ $\ i.e.\ p \neq 0$ ✓

16. (8 marks)

(a) $E\left(\overline{X}\right) = \mu = 15, \quad \sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{3}{\sqrt{16}} = \dfrac{3}{4}$

✓ ✓ ✓

(b) $\mu = E\left(\overline{X}\right) = 20$

$\sigma_{\bar{x}}^{\ 2} = 10, \qquad \sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$

$\sqrt{10} \times \sqrt{40} = \sigma$

$\sigma = 20$

(c) $\mu = 182, \ \sigma = 10$

$E\left(X\right) = 182, \qquad \sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{10}{\sqrt{9}} = \dfrac{10}{3}$

$P\left(X > 182\right) = 0.184$ $\qquad normCDf\,(185, \infty, \dfrac{10}{3}, 182) = 0.18406$

17. (7 marks)

(a) 99% confidence limits $\Rightarrow z = 2.5758$

$10 \pm 2.5758 \times \dfrac{2}{\sqrt{100}}$

$9.4848 \le \mu \le 10.5152$

(b) $\mu = \dfrac{8.5 + 14.5}{2} = 11.5$ ✓✓

95% confidence limits $\Rightarrow z = 1.96$

$11.5 + 1.96 \times \dfrac{\sigma}{\sqrt{n}} = 14.5$ (Assume n=100 from (a))

$\sigma = 15.306$

18. (12 marks)

(a) $N(2) = \dfrac{8}{1+128.866e^{-3.529\times 2}} = 7.20144 \approx 7$ ✓✓

(b) (i) $N = \dfrac{8}{1+128.866e^{-3.529t}}$

$$\frac{dN}{dx} = -8\left(1+128.866e^{-3.529t}\right)^{-2} \times \left(0+128.866e^{-3.529t} \times (-3.529)\right)$$

$$\frac{dN}{dx} = \frac{3638.144912e^{-3.529t}}{\left(1+128.866e^{-3.529t}\right)^2}$$

$At\ t = 5$

$$\frac{dN}{dx} = 0.000079 \approx 0$$

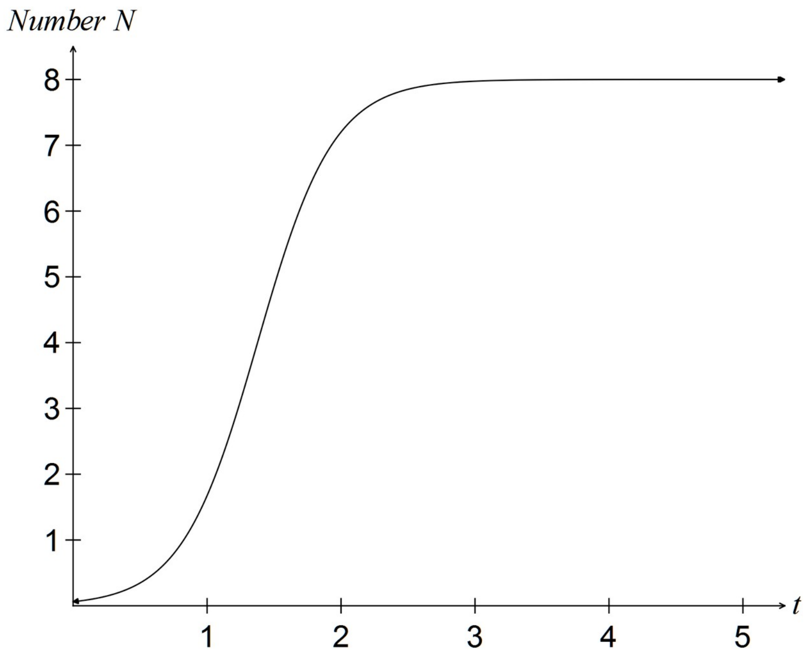
(ii) $$\frac{dN}{dx} = \frac{3638.144912e^{-3.529t}}{\left(1+128.866e^{-3.529t}\right)^2}$$

$At\ t = 3$

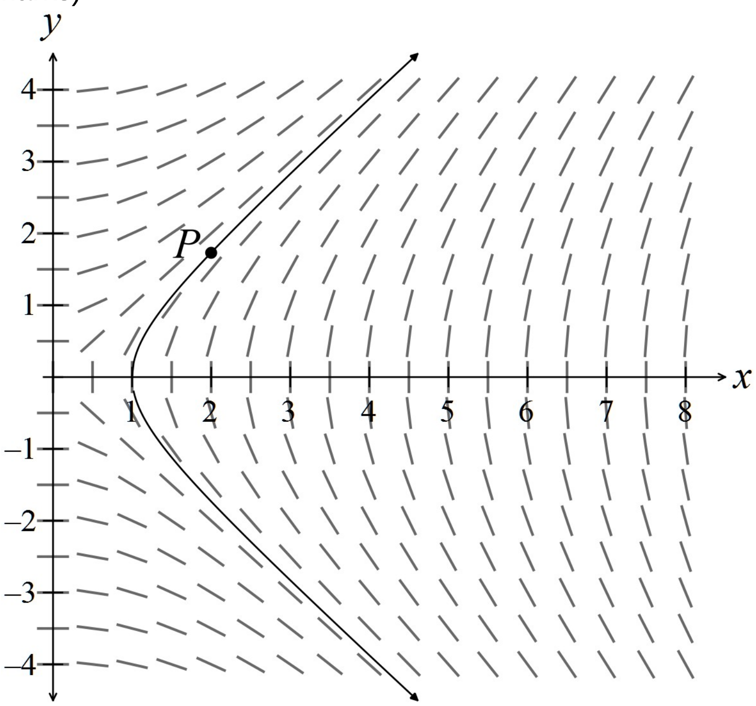
$$\frac{dN}{dx} = 0.09123$$

Positive so increasing.

(c)



19. (4 marks)



(a)

(b) The shape of the graph changes as the starting point changes.
At (4,0), the gradient is not defined. At (4,1), the gradient is steep, but becoming less steep as the y value increases; at (4,3) the gradient is close to 1
As the y value decreases from y = 0, the slope changes from undefined to close to -1. ✓✓

20. (11 marks)

(a) $$\frac{\left(cis\left(\frac{3\pi}{4}\right)\right)^{-4}\times\left(\frac{1+i}{1-i}\right)^{2}}{\sqrt{cis(2\pi)}}$$

$$=\frac{(cis(-3\pi))\times\left(\frac{2i}{-2i}\right)}{(1)^{\frac{1}{2}}}$$

$$=-1\times cis(-3\pi)$$

$$=-cis(\pi)$$

$$=1$$

(b) Prove that $cos^4(\theta)sin^3(\theta) = -\frac{1}{64}sin(7\theta) - \frac{1}{64}sin(5\theta) + \frac{3}{64}sin(3\theta) + \frac{3}{64}sin(\theta)$

$$Let \quad z = cos(\theta) + i\,sin(\theta)$$

$$\frac{1}{z} = cos(\theta) - i\,sin(\theta)$$

$$Add \quad z + \frac{1}{z} = 2\,cos(\theta) \quad Subtract \quad z - \frac{1}{z} = 2i\,sin(\theta)$$

$$z^n = (cos(\theta) + i\,sin(\theta))^n = cos(n\theta) + i\,sin(n\theta)$$

$$\frac{1}{z^n} = (cos(\theta) + i\,sin(\theta))^{-n} = cos(-n\theta) + i\,sin(-n\theta)$$

$$\therefore \frac{1}{z^n} = cos(n\theta) - i\,sin(n\theta)$$

$$z^n + \frac{1}{z^n} = 2\,cos(n\theta) \quad and \quad z^n - \frac{1}{z^n} = 2i\,sin(n\theta)$$

$$\begin{aligned}
cos^4(\theta)sin^3(\theta) &= \left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)^4 \left(\frac{1}{2i}\left(z - \frac{1}{z}\right)\right)^3 \\
&= \frac{1}{16} \times \left(-\frac{1}{8i}\right)\left(z^2 - \frac{1}{z^2}\right)^3 \left(z + \frac{1}{z}\right) \\
&= -\frac{1}{128i}\left(z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6}\right)\left(z + \frac{1}{z}\right) \\
&= -\frac{1}{128i}\left(z^7 - 3z^3 + \frac{3}{z} - \frac{1}{z^5} + z^5 - 3z + \frac{3}{z^3} - \frac{1}{z^7}\right) \\
&= -\frac{1}{128i}\left(\left(z^7 - \frac{1}{z^7}\right) + \left(z^5 - \frac{1}{z^5}\right) - 3\left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right)\right) \\
&= -\frac{1}{128i}\left(2i\,sin(7\theta) + 2i\,sin(5\theta) - 3 \times 2i\,sin(3\theta) - 3 \times 2i\,sin(\theta)\right)
\end{aligned}$$

$$cos^4(\theta)sin^3(\theta) = -\frac{1}{64} \times sin(7\theta) - \frac{1}{64} \times sin(5\theta) + \frac{3}{64} \times sin(3\theta) + \frac{3}{64} \times sin(\theta)$$

END OF SECTION TWO