Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-assumed

ATAR course examination 2018

Ratified Marking Key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2018/62746

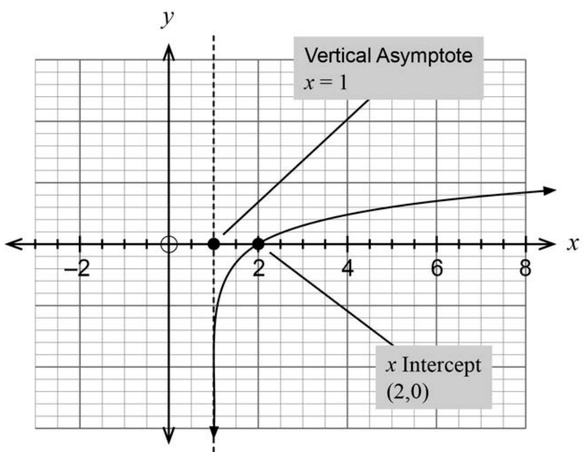
Published by the School Curriculum and Standards Authority of Western Australia
303 Sevenoaks Street
CANNINGTON WA 6107

Section Two: Calculator-assumed **65% (99 Marks)**

Question 8 **(8 marks)**

Consider the function $f(x) = \log_a(x-1)$ where $a > 1$.

(a) On the axes below, sketch the graph of $f(x)$, labelling important features. (3 marks)



Solution
See graph
Specific behaviours
✓ asymptote at $x=1$ ✓ gives correct shape ✓ x-int at $x=2$

(b) Determine the value of m if $f(m)=1$. (2 marks)

Solution
$1=\log_a(m-1)$ $m-1=a$ $m=a+1$
Specific behaviours
✓ equates $f(m)$ to 1 ✓ solves for m

The average sound intensity level for rainfall is 50 dB and for heavy traffic 85 dB.

(c) How many times more intense is the sound of traffic than that of rainfall? (3 marks)

Solution
$50=10\log\left(\frac{I_{rain}}{I_0}\right)\Rightarrow\frac{I_{rain}}{I_0}=10^5\Rightarrow I_{rain}=10^5 I_0$ $85=10\log\left(\frac{I_{traffic}}{I_0}\right)\Rightarrow\frac{I_{traffic}}{I_0}=10^{8.5}\Rightarrow I_{traffic}=10^{8.5} I_0$ $\therefore\frac{I_{traffic}}{I_{rain}}=\frac{10^{8.5}}{10^5}=10^{3.5}\approx 3200$
Specific behaviours
✓ rearranges logarithmic equations to exponentials ✓ writes ratio and cancels I_0 ✓ determines how many more times intense

Question 18 **(7 marks)**

The ear has the remarkable ability to handle an enormous range of sound levels. In order to express levels of sound meaningfully in numbers that are more manageable, a logarithmic scale is used, rather than a linear scale. This scale is the decibel (dB) scale.

The sound intensity level, L, is given by the formula below:

$L = 10\log\left(\frac{I}{I_0}\right)$ dB where I is the sound intensity and I_0 is the reference sound intensity.

I and I_0 are measured in watt/m^2.

(a) Listening to a sound intensity of 5 billion times that of the reference intensity ($I = 5\times10^9 I_0$) for more than 30 minutes is considered unsafe. To what sound intensity level does this correspond? (2 marks)

Solution
$L = 10\log\left(\frac{5\times10^9 I_0}{I_0}\right)$ ≈ 97 dB
Specific behaviours
✓ substitutes for L ✓ calculates level

(b) The reference sound intensity, I_0, has a sound intensity level of 0 dB. If a household vacuum cleaner has a sound intensity, $I = 1\times10^{-5}$ watt/m^2 and this corresponds to a sound intensity level $L = 70$ dB, determine I_0. (2 marks)

Solution
$70 = 10\log\left(\frac{1\times10^{-5}}{I_0}\right)$ $I_0 = \frac{1\times10^{-5}}{10^7} = 1\times10^{-12}$ watt/m^2
Specific behaviours
✓ substitutes for L and I ✓ determines I_0 including units

(c) Determine the coordinates of the x – intercept of $f(x+b)+c$, where b and c are positive real constants. (3 marks)

Solution
$0 = \log_a(x-1+b)+c$ $-c = \log_a(x-1+b)$ $a^{-c} = x-1+b$ $x = a^{-c}+1-b$ coordinates are: $\left(a^{-c}+1-b, 0\right)$
Specific behaviours
✓ equates new function to zero   ✓ solves for x ✓ states coordinates

Question 9 **(8 marks)**

The concentration, C, of a drug in the blood of a patient t hours after the initial dose can be modelled by the equation below.

$$C = 4e^{-0.05t} \text{ mg/L}$$

Patients requiring this drug are said to be in crisis if the concentration of the drug in their blood falls below 2.5 mg/L.

A patient is given a dose of the drug at 9 am.

(a) What was the concentration in the patient's blood immediately following the initial dose? (1 mark)

Solution
Initial dose when $t = 0$ $C(0) = 4$ mg/L
Specific behaviours
✓ determines concentration, including the unit

(b) What is the concentration of the drug in the patient's blood at 11.30 am? (2 marks)

Solution
$C = 4e^{-0.05(2.5)}$ $C = 3.53 \text{ mg/L}$
Specific behaviours
✓ substitutes $t = 2.5$ ✓ calculates concentration

(c) Find the rate of change of C at 1 pm. (2 marks)

Solution
$\frac{dC}{dt} = -0.2e^{-0.05t}$ $\left.\frac{dC}{dt}\right\vert_{t=4} = -0.164 \quad \text{mg/L/hour}$
Specific behaviours
✓ finds derivative of C wrt t ✓ calculates rate of change when $t = 4$

(e) Calculate a 95% confidence interval for the proportion of large mangoes produced on the farm, rounded to four decimal places. (3 marks)

Solution
95% confidence interval $= \left(0.5 - 1.96 \times \sqrt{\frac{0.5\times0.5}{500}}, 0.5 + 1.96 \times \sqrt{\frac{0.5\times0.5}{500}}\right)$ $= (0.5 - 0.04383, 0.5 + 0.04383)$ $= (0.4562, 0.5438)$
Specific behaviours
✓ uses the correct value for the standard error ✓ uses the correct z-value interval ✓ calculates the confidence interval to 4 decimal places

(f) On the basis of your calculations, how would you respond to Tina's belief that the proportion of large mangoes produced is at least 60%? Justify your response. (2 marks)

Solution
Since 0.6 is not contained in the 95% confidence interval, it is unlikely that Tina is correct.
Specific behaviours
✓ refers to 0.6 not being in the interval ✓ concluding that it is unlikely that Tina is correct

(g) What can Tina do to further test her belief? (1 mark)

Solution
Tina should take another random sample and obtain another 95% confidence interval.
Specific behaviours
✓ states answer

Question 17 **(14 marks)**

Tina believes that approximately 60% of the mangoes she produces on her farm are large. She takes a random sample of 500 mangoes from a day's picking.

(a) Assuming Tina is correct and 60% of the mangoes her farm produces are large, what is the approximate probability distribution of the sample proportion of large mangoes in her sample? (3 marks)

Solution
$\hat{p} \sim N\left(0.6, \frac{0.6 \times 0.4}{500}\right)$ That is, $\hat{p} \sim N(0.6, 0.02191^2)$
Specific behaviours
✓ states the distribution as normal ✓ gives the correct value of the mean ✓ gives the correct value of the variance (or standard deviation)

(b) What is the probability that the sample proportion of large mangoes is less than 0.58? (2 marks)

Solution
$P(\hat{p} < 0.58) = P\left(Z < \frac{0.58 - 0.6}{\sqrt{0.6 \times 0.4/500}}\right) = P(Z < -0.9129) = 0.18066$
Specific behaviours
✓ calculates the z-value correctly ✓ obtains the correct probability

(c) Tina decides to select the mangoes for her sample as they pass along the conveyor belt to be sorted. Describe briefly how Tina should select her sample. (2 marks)

Solution
She should use a random number generator and pick the sample using the numbers she obtains.
Specific behaviours
✓ indicates some random mechanism ✓ indicates that the mangoes are selected accordingly

A random sample of 500 contains 250 large mangoes.

(d) On the basis of this data, estimate the proportion of large mangoes produced on the farm. (1 mark)

Solution
$\hat{p} = \frac{250}{500} = 0.5$
Specific behaviours
✓ calculates the correct sample proportion

(d) What is the latest time the patient can receive another dose of the drug if they are to avoid being in crisis? (3 marks)

Solution
$2.5 = 4e^{-0.05t}$ $t = 9.4$ hours Latest time = 6:24 pm (6:25 too late)
Specific behaviours
✓ substitutes $C = 2.5$ ✓ solves for t ✓ states latest time

Question 10 **(7 marks)**

The following function is a probability density function on the given interval:

$$f(x) = \begin{cases} ax^2(x-2) & \text{for } 0 \le x \le 2. \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of a. (3 marks)

Solution
If pdf on domain then $\int_0^2 f(x)dx = 1$ $\int_0^2 f(x)\,dx = 1$ $\int_0^2 ax^2(x-2)\,dx = -\frac{4a}{3}$ $\therefore -\frac{4a}{3} = 1$ $\therefore a = -\frac{3}{4}$
Specific behaviours
✓ uses integration for domain =1   ✓ calculates integration   ✓ finds a

(b) Find the probability that $x \ge 1{\cdot}2$. (2 marks)

Solution
$\int_{1.2}^{2} \frac{-3x^2(x-2)}{4}\,dx$ $= 0.5248$
Specific behaviours
✓ uses correct integral   ✓ calculates probability

(c) Find the median of the distribution. (2 marks)

Solution
Solve $\int_0^m f(x)\,dx = 0.5$ over domain $0 \le x \le 2$ $\int_0^m f(x)dx = -\frac{3m^4}{16} + \frac{m^3}{2}$ for median: $-\frac{3m^4}{16} + \frac{m^3}{2} = 0.5$ $m = 1.2285$
Specific behaviours
✓ uses correct integral   ✓ determines $m = 1.2285$

Question 16 **(8 marks)**

Let $f(x)$ be a function such that $f(-2) = 4, f(-1) = 0, f(0) = -1, f(1) = 0$ and $f(3) = 2$. Further, $f'(x) < 0$ for $-2 \le x < 0, f'(0) = 0$ and $f'(x) > 0$ for $0 < x \le 3$.

(a) Evaluate the following definite integrals:

(i) $\int_0^3 f'(x)\,dx.$ (2 marks)

Solution
By the fundamental theorem of calculus $\int_0^3 f'(x)\,dx = \left[f(x)\right]_0^3 = f(3) - f(0) = 2-(-1) = 3.$
Specific behaviours
✓ uses the fundamental theorem of calculus   ✓ obtains the correct value for the integral

(ii) $\int_{-2}^3 f'(x)\,dx.$ (2 marks)

Solution
By the fundamental theorem of calculus $\int_{-2}^3 f'(x)\,dx = \left[f(x)\right]_{-2}^3 = f(3) - f(-2) = 2-4 = -2.$
Specific behaviours
✓ uses the fundamental theorem of calculus   ✓ obtains the correct value for the integral

(b) What is the area bounded by the graph of $f'(x)$ and the x axis between $x = -2$ and $x = 3$? Justify your answer. (4 marks)

Solution
Required area is A. $A = \int_{-2}^3 \lvert f'(x)\rvert\,dx$ Since $f'(x)$ is positive for $x > 0$ and negative for $x < 0$, the area is   A= $\left\lvert\int_0^3 f'(x)\,dx\right\rvert + \left\lvert\int_{-2}^0 f'(x)\,dx\right\rvert = (2-(-1) + \lvert -1-4\rvert = 8$.
Specific behaviours
✓ writes the expression for area in terms of absolute value   ✓ uses the intervals where $f'(x)$ is positive and negative   ✓ breaks the integral over the correct intervals   ✓ calculates the correct value of the area

Question 15 **(5 marks)**

The population of mosquitos, P (in thousands), in an artificial lake in a housing estate is measured at the beginning of the year. The population after t months is given by the function, $P(t)=t^3+at^2+bt+2,\ 0\le t\le 12$.

The rate of growth of the population is initially increasing. It then slows to be momentarily stationary in mid-winter (at $t=6$), then continues to increase again in the last half of the year.

Determine the values of a and b.

Solution
For HPI: $\begin{cases} P'(6)=0 \\ P''(6)=0 \end{cases}$ $P'(t)=3t^2+2at+b$ $P''(t)=6t+2a$ $0=108+12a+b$ $0=36+2a$ solving gives: $a=-18$ $b=108$
Specific behaviours
✓ determines first derivative ✓ determines second derivative ✓ equates first and second derivatives to zero when $t=6$ ✓ determines the value of a ✓ determines the value of b

Question 11 **(8 marks)**

Ava is flying a drone in a large open space at a constant height of 5 metres above the ground. She flies the drone due north so that it passes directly over her head and then, sometime later, reverses it direction and flies the drone due south so it passes directly over her again. With $t=0$ defined as the moment when the drone first flies directly above Ava's head, the velocity of the drone, at time t seconds, is given by:

$$v=2\sin\left(\frac{t}{3}+\frac{\pi}{6}\right)\ \text{m/s}\quad 0\le t\le 16.$$

(a) Determine $x(t)$, the displacement of the drone at t seconds, where $x(0)=0$. (3 marks)

Solution
$\int 2\sin\left(\frac{t}{3}+\frac{\pi}{6}\right)dt$ $=-6\cos\left(\frac{t}{3}+\frac{\pi}{6}\right)+C$ Solve: $-6\cos\left(\frac{0}{3}+\frac{\pi}{6}\right)+C=0$ $C=3\sqrt{3}$ OR 5.196152423 $\therefore x(t)=-6\cos\left(\frac{t}{3}+\frac{\pi}{6}\right)+5.196$ OR $x(t)=-6\cos\left(\frac{t}{3}+\frac{\pi}{6}\right)+3\sqrt{3}$
Specific behaviours
✓ integrates $v(t)$ to determine cosine expression ✓ recognises $x(t)$ involves a constant term and equates $x(0)$ to 0 ✓ solves for C and states $x(t)$

(b) Where is the drone in relation to the pilot after 16 seconds? (2 marks)

Solution
$x(16)=-6\cos\left(\frac{16}{3}+\frac{\pi}{6}\right)+3\sqrt{3}$ $=-0.266975$ The drone is 0.27 m (27 cm) due south of the pilot.
Specific behaviours
✓ evaluates displacement at $t=16$ ✓ interprets solution

Question 11 (continued)

(c) At a particular time, the drone is heading due south and it is decelerating at 0.5 m/s^2. How far has the drone travelled from its initial position directly above Ava's head until this particular time? (3 marks)

Solution
$a(t)=\frac{2}{3}\cos\left(\frac{t}{3}+\frac{\pi}{6}\right)$ $-0.5=\frac{2}{3}\cos\left(\frac{t}{3}+\frac{\pi}{6}\right)$ $t=5.6858$ or 10.0222 heading south at $t=10.0222$ distance travelled $=\int_0^{10.0222}\left\lvert 2\sin\left(\frac{t}{3}+\frac{\pi}{6}\right)\right\rvert dt$ $=12.696$ The drone has travelled 12.696 metres.
Specific behaviours
✓ equates derivative to –0.5 m/s^2 ✓ recognises 10.02 s is when the drone is heading south   ✓ determines distance travelled

Question 14 **(5 marks)**

(a) The table below examines the values of $\frac{a^h-1}{h}$ for various values of a as h approaches zero. Complete the table, rounding your values to five decimal places. (2 marks)

h	$a=2.60$	$a=2.70$	$a=2.72$	$a=2.80$
0.1	1.00265	**1.04425**	1.05241	1.08449
0.001	0.95597	0.99375	**1.00113**	**1.03015**
0.00001	0.95552	**0.99326**	**1.00064**	1.02962

Solution
See table
Specific behaviours
✓ correctly completes three table values   ✓ correctly completes all entries and rounds to 5dp

It can be shown that $\frac{d}{dx}(a^x)=a^x\lim\limits_{h\to0}\left(\frac{a^h-1}{h}\right)$.

(b) What is the exact value of a for which $\frac{d}{dx}(a^x)=a^x$? Explain how the above definition and the table in part (a) support your answer. (3 marks)

Solution
$a=e\approx2.71828$ When $a=e$ the table shows that the value of $\lim\limits_{h\to0}\left(\frac{a^h-1}{h}\right)$ is 1.   It follows then from the definition that $\frac{d}{dx}(e^x)=e^x\times1$ $=e^x$.
Specific behaviours
✓ states $a=e$ or 2.71828   ✓ explains table result   ✓ explains significance of table result for part (b)

Question 13 (continued)

(c) Six months later, the consulting firm carries out a random sampling of towing vehicles. A 99% confidence interval calculated for the proportion of vehicles with incorrect towing capacity is (0.342, 0.558). Determine the number of vehicles in the sample that have an incorrect towing capacity. (4 marks)

Solution
$p=\dfrac{0.342+0.558}{2}=0.45$ $E=0.558-0.45=0.108$ $E=z\sqrt{\dfrac{p(1-p)}{n}}$ $0.108=2.576\sqrt{\dfrac{0.45(1-0.45)}{n}}$ $n=141$ Number of vehicles with incorrect towing capacity $=np$ $=141\times 0.45$ ≈ 63
Specific behaviours
✓ finds correct p ✓ finds correct E ✓ finds number in sample ✓ finds number of vehicles with incorrect towing capacity

Question 12 **(19 marks)**

The manager of the mail distribution centre in an organisation estimates that the weight, x (kg), of parcels that are posted is normally distributed, with mean 3 kg and standard deviation 1 kg.

(a) What percentage of parcels weigh more than 3.7 kg? (2 marks)

Solution
$X \sim N(3,1)$ $P(X > 3.7) = 0.24196$ 24.2% are greater than 3.7 kg.
Specific behaviours
✓ states weight required greater than 3.7 kg ✓ obtains the correct percentage

(b) Twenty parcels are received for posting. What is the probability that at least half of them weigh more than 3.7 kg? (3 marks)

Solution
Let the random variable M denote the number of parcels that weigh more than 3.7 kg. Then $M \sim Bin(20, 0.24196)$. $P(M \geq 10) = 0.01095$
Specific behaviours
✓ states the distribution as binomial ✓ determines the correct parameters of the distribution ✓ obtains the correct probability

The cost of postage, ($) y, depends on the weight of a parcel as follows:
- a cost of $5 for parcels below 1 kg
- a variable cost of $1.50 for every kilogram or part thereof above 1 kg to a maximum of 4 kg
- a cost of $12 for parcels above 4 kg.

(c) Complete the probability distribution table for Y. (4 marks)

x	≤ 1	$1 < x \leq 2$	$2 < x \leq 3$	$3 < x \leq 4$	$x > 4$
y	$5	**$6.50**	**$8**	**$9.50**	**$12**
$P(Y = y)$	**0.02275 (accept 0.02140)**	**0.13591**	**0.34134**	**0.34134**	**0.15866**

Solution
See table
Specific behaviours
✓ obtains two correct values of y ✓ obtains the other two correct values of y ✓ obtains two correct probabilities ✓ obtains the remaining correct probabilities

Question 12 (continued)

(d) Calculate the mean cost of postage per parcel. (2 marks)

Solution
$E(Y) = 5\times0.02275 + 6.5\times0.13591 + 8\times0.34134 + 9.50\times0.34134 + 12\times0.15866$ $= 8.874535$ That is, \$8.87 is the mean cost of postage per parcel.
Specific behaviours
✓ obtains the correct expression for the mean   ✓ obtains the correct value of the mean

(e) Calculate the standard deviation of the cost of postage per parcel. (3 marks)

Solution
$\sigma^2 = (5-8.87)^2\times0.02275 + (6.5-8.87)^2\times0.13591 + (8-8.87)^2\times0.34134$ $+\ (9.5-8.87)^2\times0.34134 + (12-8.87)^2\times0.15866$ $= 3.052310889$ $\therefore \sigma = 1.7470864$
Specific behaviours
✓ substitutes into variance formula correctly   ✓ calculates the variance correctly   ✓ calculates the standard deviation correctly

(f) If the cost of postage is increased by 20% and a surcharge of \$1 is added for all parcels, what will be the mean and standard deviation of the new cost? (3 marks)

Solution
The mean will increase by 20% to $1.2\times8.874535+1=11.64944$.   The standard deviation increases by 20% to $1.2\times1.747086=2.096504$.
Specific behaviours
✓ states new values will need to be multiplied by 1.2   ✓ correctly determines mean   ✓ correctly determines standard deviation

(g) Show one reason why the given normal distribution is not a good model for the weight of the parcels? (2 marks)

Solution
$P(Y<0)=0.001349898$ There is a non-zero (small) probability that the weight can be negative, which is not possible.
Specific behaviours
✓ calculates the probability of a weight below 0   ✓ explains that negative weights are not possible here

Question 13 **(10 marks)**

The proportion of caravans on the road being towed by vehicles that have the incorrect towing capacity is p.

(a) Show, using calculus, that to maximise the margin of error a value of $\hat{p}=0.5$ should be used. Note: As z and n are constants, the standard error formula can be reduced to $E=\sqrt{\hat{p}(1-\hat{p})}$. (3 marks)

Solution
$E=\sqrt{\hat{p}(1-\hat{p})}$ $\dfrac{dE}{d\hat{p}}=\dfrac{(1-2\hat{p})}{2\sqrt{\hat{p}(1-\hat{p})}}$ $0=1-2\hat{p}$ $\hat{p}=0.5$ $\left.\dfrac{d^2E}{d\hat{p}^2}\right\vert_{\hat{p}=0.5}=-2\Rightarrow \text{maximum}$
Specific behaviours
✓ differentiates E wrt $\hat{p}$ ✓ equates derivative to zero and solves for $\hat{p}$ ✓ uses second derivative or sign test to confirm maximum

(b) A consulting firm wants to determine p within 8% with 99% confidence. How many towing vehicles should be tested at a random check? (3 marks)

Solution
Use $\hat{p}$ = 0.5 z value for 99% = 2.576 E for sample proportion $E=z\sqrt{\dfrac{p(1-p)}{n}}$ and E = 0.08 $0.08=2.576\sqrt{\dfrac{0.5(1-0.5)}{n}}$ $n=259.21$ Hence 260 vehicles should be tested.
Specific behaviours
✓ uses $\hat{p}$ = 0.5 and z value   ✓ equates standard error to 0.08   ✓ solves for n and rounds up to 260