Working out space

PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Test 3 Year 12

Student name: ______________ Teacher name: ______________

Task type: **Response**

Reading time for this test : 5 mins

Working time allowed for this task: 40 mins

Number of questions: **_____6______**

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **___39___ marks**

Task weighting: **_14___%**

Formula sheet provided: no but formulae given on page 2

Note: All part questions worth more than 2 marks require working to obtain full marks.

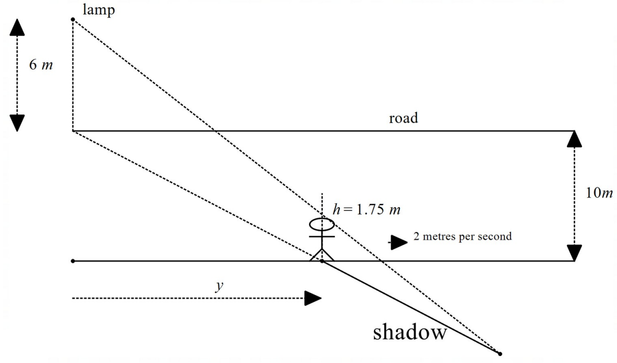
Useful formulae

$\frac{d}{dx}\ln x = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c$
$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$	$\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert + c$
$\frac{d}{dx}\sin f(x) = f'(x)\cos f(x)$	$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\cos f(x) = -f'(x)\sin f(x)$	$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\tan f(x) = f'(x)\sec^2 f(x) = \frac{f'(x)}{\cos^2 f(x)}$	$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$

Volumes of solids of revolution	
About the x-axis	$V = \pi\int_a^b [f(x)]^2 dx$
About the y-axis	$V = \pi\int_c^d [f(y)]^2 dy$

Prism	$V = Ah$, where A is the area of the cross section	
Pyramid	$V = \frac{1}{3}Ah$, where A is the area of the base	
Cylinder	$V = \pi r^2 h$	$TSA = 2\pi rh + 2\pi r^2$
Cone	$V = \frac{1}{3}\pi r^2 h$	$TSA = \pi rs + \pi r^2$, where s is the slant height
Sphere	$V = \frac{4}{3}\pi r^3$	$TSA = 4\pi r^2$

Consider a woman of height 1.75 m, travelling at 2 m/s along the edge of a road of width 10 m. A lamp of height 6 m on the other side of the road, casts a shadow of the woman as shown below. Determine the time rate of change of the length of the shadow when $y = 20$ m.



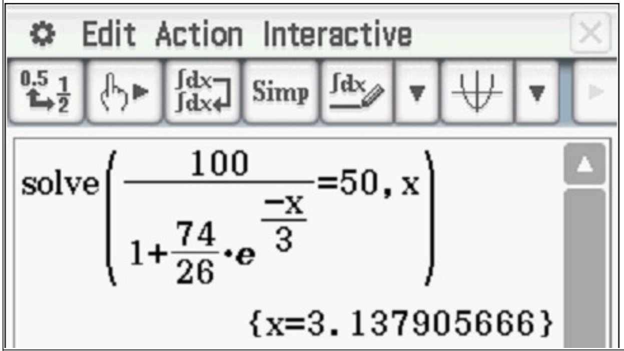
c
$\frac{l}{1.75} = \frac{l + \sqrt{100 + y^2}}{6}$ $6l = 1.75l + 1.75\sqrt{100 + y^2}$ $4.25l = 1.75\sqrt{100 + y^2}$ $4.25\dot{l} = \frac{1.75y\dot{y}}{\left(\sqrt{100 + y^2}\right)}$ $\dot{l} = \frac{1.75(20)2}{4.25\left(\sqrt{100 + (20)^2}\right)} = \frac{28\sqrt{5}}{85}\ m/s$
Specific behaviours
✓ uses similar triangles ✓ obtains expression between y and length of shadow ✓ uses implicit diff ✓ obtains expression for time rate of length of shadow ✓expresses exact simplified rate, no need for units

b) Determine the limiting value of the population of kangaroos.

c
$t \to \infty$ Limiting value =100 kangaroos
Specific behaviours
✓ uses time approaches infinity ✓ states limit (no need for units)

Q5 cont-

c) Determine the time taken for the maximum growth rate.

c
$\frac{100}{1+\frac{74}{26}e^{-\frac{t}{3}}}=50$ 
Specific behaviours
✓ subs N= half of limiting value ✓ approx. value of time, no need for units (accept exact)

Q6 (5 marks)

Q1 (2, 3 & 3 = 8 marks)

An object starts from rest at the origin and moves with a velocity $v=\begin{pmatrix}-5\sin 2t\\ 3\sin t\end{pmatrix}$ m/s at time t seconds.

Determine the following.

a) Acceleration at time t.

c
$v=\begin{pmatrix}-5\sin 2t\\ 3\sin t\end{pmatrix}$ $a=\begin{pmatrix}-10\cos 2t\\ 3\cos t\end{pmatrix}$
Specific behaviours
✓ diffs velocity ✓ obtains acceleration function

b) The cartesian equation of the path of the object.

c
$v=\begin{pmatrix}-5\sin 2t\\ 3\sin t\end{pmatrix}$ $r=\begin{pmatrix}\frac{5}{2}\cos 2t\\ -3\cos t\end{pmatrix}+c$ $0=\begin{pmatrix}\frac{5}{2}\\ -3\end{pmatrix}+c$ $c=\begin{pmatrix}-\frac{5}{2}\\ +3\end{pmatrix}$ $r=\begin{pmatrix}\frac{5}{2}\cos 2t-\frac{5}{2}\\ -3\cos t+3\end{pmatrix}$ $x=5\left(\cos^2 t-\frac{1}{2}\right)-\frac{5}{2}$ $y=-3\cos t+3\to \cos t=\frac{3-y}{3}$ $x=5\left(\left[\frac{3-y}{3}\right]^2-\frac{1}{2}\right)-\frac{5}{2}, x\le 0$ *or* $2x=\frac{10}{9}y^2-\frac{20}{3}y$

Specific behaviours
✓ integrates and solves for constant ✓ uses double angle formula for cosine ✓ obtains expression in cartesian form (unsimplfied)

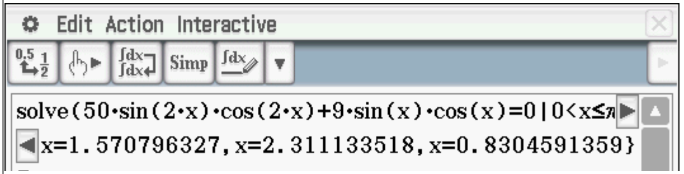
c) Determine to the nearest second the first time for $t > 0$ that the acceleration and velocity are perpendicular.

c

$v = \begin{pmatrix} -5\sin 2t \\ 3\sin t \end{pmatrix}$

$a = \begin{pmatrix} -10\cos 2t \\ 3\cos t \end{pmatrix}$

$v.a = 50\sin 2t\cos 2t + 9\sin t\cos t = 0$



Time = 1 second.

Specific behaviours
✓ sets up dot equation with v and a ✓ equates to zero and solves for time ✓ selects first time greater than zero and rounds to nearest second with units

Q2 (5 marks)

If $\frac{dy}{dx} = xy^2$ find an expression for $\frac{d^2y}{dx^2}$ in terms of $x \& y$.

c

$\frac{dy}{dx} = xy^2$

$\frac{d^2y}{dx^2} = y^2 + x2y\frac{dy}{dx} = y^2 + 2x^2y^3$

Specific behaviours
✓ implicit diff used ✓ product rule used correctly ✓ chain rule used correctly ✓ subs derivative

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$\frac{dN}{dt} = \frac{1}{300}N(100 - N)$

$\frac{dN}{dt} = 0, N < 100$

$300\int \frac{dN}{N(100-N)} = \int dt$

$\frac{1}{N(100-N)} = \frac{a}{N} + \frac{b}{100-N}$

$1 = a(100 - N) + bN$

$N = 0$

$1 = 100a \to a = \frac{1}{100}$

$N = 100$

$1 = 100b \to b = \frac{1}{100}$

$3\int \frac{1}{N} + \frac{1}{100-N}dN = \int dt$

$3\ln N - 3\ln|100 - N| = t + c, \quad N < 100 \therefore no\ need\ absolute$

$\ln\frac{N}{100-N} = \frac{t}{3} + c$

$Ae^{\frac{t}{3}} = \frac{N}{100-N}$

$Ae^{-\frac{t}{3}} = \frac{100-N}{N}$

$ANe^{-\frac{t}{3}} = 100 - N$

$N = \frac{100}{1 + Ae^{-\frac{t}{3}}}$

$26 = \frac{100}{1+A}$

$A = \frac{74}{26}$

$N = \frac{100}{1 + \frac{74}{26}e^{-\frac{t}{3}}}$

Specific behaviours
✓ separates variables ✓ uses partial fractions with correct coefficients ✓ integrates correctly AND shows that absolute value not needed ✓ rearranges for N(t) ✓solves for constant exactly

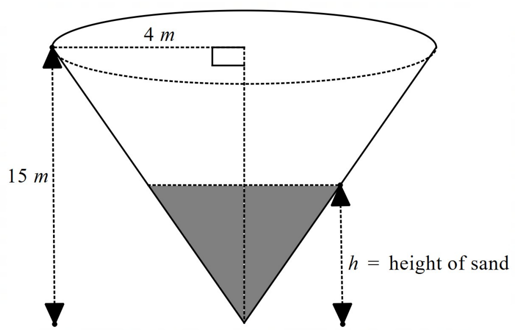
Q5 (5, 2 & 2 = 9 marks)
At time $t = 0$ years, 26 kangaroos are placed in an isolated habitat such that the number of kangaroos, N can be modelled by the differential equation $\frac{dN}{dt} = \frac{1}{3}N - \frac{1}{300}N^2$.

a) Using separation of variables and partial fractions determine $N(t)$ **without** the use of a classpad.

c

✓ express second derivative in terms of x and y only

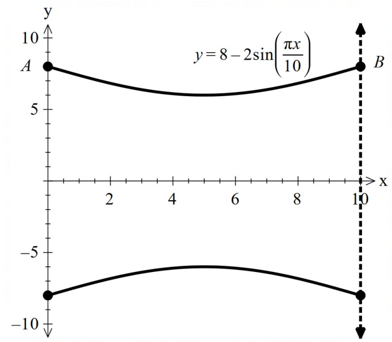
Q3 (6 marks)
Sand is poured into a gigantic metal cone of height 15 m and a radius of 4 m at a rate of 120 cubic metres per minute, as shown below.



Determine the time rate of change of the height, h metres, of the sand when the height is 5 m.

c
$V = \frac{1}{3}\pi r^2 h$ $\frac{r}{h} = \frac{4}{15}$ $V = \frac{1}{3}\pi \frac{16}{225}h^3 = \frac{16}{675}\pi h^3$ $\dot{V} = \frac{48}{675}\pi h^2 \dot{h}$ $120 = \frac{48}{675}\pi 5^2 \dot{h}$ $\dot{h} = \frac{135}{2\pi} m/\min$ $or\ 21.49 m/\min$
Specific behaviours
✓ uses volume of cone formula ✓ determines ratio of radius to height ✓ obtains expression for volume in terms of one variable ✓ uses given rate of volume ✓ obtains equation for height rate ✓gives approx. or exact height rate with units

Q4 (6 marks)
A water pipe of length 10 metres can be modelled by a cross-section AB where $y = 8 - 2\sin\left(\frac{\pi x}{10}\right)$, $0 \le x \le 10$ and this curve is revolved about the x axis.



Determine the volume of water that this length of pipe will hold. Show all working **without** the use of a classpad.

c
$y = 8 - 2\sin\left(\frac{\pi x}{10}\right)$ $\int_0^{10} \pi\left(8 - 2\sin\left(\frac{\pi x}{10}\right)\right)^2 dx$ $\int_0^{10} \pi\left(64 - 32\sin\left(\frac{\pi x}{10}\right) + 4\sin^2\left(\frac{\pi x}{10}\right)\right)dx$ $\int_0^{10} \pi\left(64 - 32\sin\left(\frac{\pi x}{10}\right) + 2 - 2\cos\left(\frac{\pi x}{5}\right)\right)dx$ $\pi\left[66x + \frac{320}{\pi}\cos\left(\frac{\pi x}{10}\right) - \frac{10}{\pi}\sin\left(\frac{\pi x}{5}\right)\right]_0^{10}$ $(660\pi - 320) - (320)$ $660\pi - 640$
Specific behaviours
✓ uses correct integral ✓ expands the squared brackets ✓ uses double angle formula ✓ integrates correctly ✓ subs both limits ✓simplifies