ALL SAINTS' COLLEGE

Semester 1 Examination, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNIT 3

Section One: Calculator-free

SOLUTIONS

Student Number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	48	35
Section Two: Calculator-assumed	13	13	100	101	65
			Total	149	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

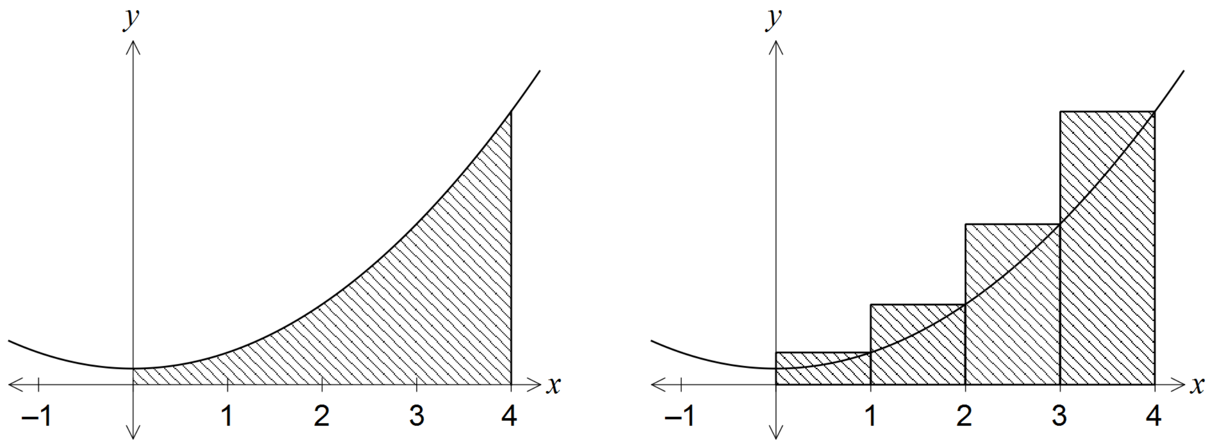
Section One: Calculator-free **35% (48 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(5 marks)**

Part of the graph of $y = x^2 + 1$ is shown in the diagrams below.



An approximation for the area beneath the curve between $x = 0$ and $x = 4$ is made using rectangles as shown in the right-hand diagram. Determine the exact amount by which the approximate area exceeds the exact area.

Solution
$A_1 = \int_0^4 x^2 + 1\,dx = \left[\dfrac{x^3}{3} + x\right]_0^4 = \dfrac{64}{3} + 4 = \dfrac{76}{3}$ $A_2 = 2 + 5 + 10 + 17 = 34$ $A_2 - A_1 = 34 - \dfrac{76}{3} = \dfrac{102 - 76}{3} = \dfrac{26}{3}$ sq units
Specific behaviours
✓ antidifferentiates correctly   ✓ evaluates exact area   ✓ calculates rectangle heights   ✓ evaluates approximate area   ✓ calculates difference

Question 2 **(9 marks)**

(a) Differentiate the following with respect to x, simplifying your answers.

(i) $y = \int_x^1 (t - t^3)\, dt$. (2 marks)

Solution
$\frac{d}{dx}\int_x^1 (t - t^3)\, dt = -\frac{d}{dx}\int_1^x (t - t^3)\, dt$ $= x^3 - x$
Specific behaviours
✓ adjusts limits of integral   ✓ simplifies derivative

(ii) $y = \sin^3(2x+1)$. (3 marks)

Solution
$y = u^3 \qquad u = \sin(2x+1)$ $\frac{dy}{du} = 3u^2 \qquad \frac{du}{dx} = 2\cos(2x+1)$ $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= 3\sin^2(2x+1) \times 2\cos(2x+1)$ $= 6\sin^2(2x+1)\cos(2x+1)$
Specific behaviours
✓ uses chain rule for $\sin(2x+1)$ ✓ uses chain rule for $\sin^3()$ ✓ simplifies result

(b) Determine the values of the constants a, b and c, given that $f''(x) = e^{3x}(ax^2 + bx + c)$ when $f(x) = x^2 e^{3x}$. (4 marks)

Solution
$f'(x) = 2xe^{3x} + 3x^2e^{3x}$ $f''(x) = 2e^{3x} + 6xe^{3x} + 3f'(x)$ $= 2e^{3x} + 6xe^{3x} + 3(2xe^{3x} + 3x^2e^{3x})$ $= 2e^{3x} + 12xe^{3x} + 9x^2e^{3x}$ $= e^{3x}(2 + 12x + 9x^2) \Rightarrow a = 9,\ b = 12,\ c = 2$
Specific behaviours
✓ uses product rule for first derivative   ✓ uses chain rule for first derivative   ✓ uses product and chain rules for second derivative   ✓ simplifies and states values

Question 3 **(6 marks)**

A function $P(x)$ is such that $\frac{dP}{dx} = ax^2 - 12x$, where a is a constant and the graph of $y = P(x)$ has a stationary point at (4, 8). Determine $P(10)$.

Solution
$P'(4) = 0 \Rightarrow x(ax - 12) = 0$ $4(4a - 12) = 0 \Rightarrow a = 3$ $P'(x) = 3x^2 - 12x$ $P(x) = x^3 - 6x^2 + c$ $P(4) = 8 \Rightarrow 64 - 96 + c = 8$ $c = 40$ $P(x) = x^3 - 6x^2 + 40$ $P(10) = 1000 - 600 + 40 = 440$
Specific behaviours
✓ substitutes $x = 4$ and equates to zero ✓ determines value of a ✓ antidifferentiates ✓ determines c ✓ states equation and substitutes $x = 10$

✓ states

Question 4 **(7 marks)**

Consider the function defined by $f(x)=\frac{x}{2}-\sqrt{x}$, $x\geq 0$.

(a) Determine the coordinates of the stationary point of $f(x)$. (3 marks)

Solution
$f'(x)=\frac{1}{2}-\frac{1}{2\sqrt{x}}$ $\frac{1}{2}-\frac{1}{2\sqrt{x}}=0 \Rightarrow x=1$ $f(1)=\frac{1}{2}-\sqrt{1}=-\frac{1}{2} \Rightarrow$ stationary point at $\left(1,\ -\frac{1}{2}\right)$
Specific behaviours
✓ differentiates function ✓ solves $f'(x)=0$ ✓ states coordinates of point

(b) Use the second derivative test to determine the nature of the stationary point found in (a). (3 marks)

Solution
$f''(x)=\frac{1}{4\sqrt{x^3}}$ $f''(1)=\frac{1}{4}$ $f''(1)>0 \Rightarrow$ local minimum
Specific behaviours
✓ determines second derivative ✓ shows $f''(1)>0$ ✓ states conclusion that point is local minimum

(c) State the global minimum of $f(x)$. (1 mark)

Solution
$-\frac{1}{2}$
Specific behaviours
✓ states correct value of global minimum

Question 5 **(5 marks)**

The area of a segment with central angle θ in a circle of radius r is given by $A = \frac{r^2}{2}(\theta - \sin\theta)$.
Use the increments formula to approximate the increase in area of a segment in a circle of radius 10 cm as the central angle increases from $\frac{\pi}{3}$ to $\frac{11\pi}{30}$.

Solution
$A = 50(\theta - \sin\theta)$ $\frac{dA}{d\theta} = 50(1 - \cos\theta)$ $\theta = \frac{\pi}{3} \Rightarrow \frac{dA}{d\theta} = 50(1 - 0.5) = 25$ $\delta\theta = \frac{11\pi}{30} - \frac{10\pi}{30} = \frac{\pi}{30}$ $\delta A \approx \frac{dA}{d\theta} \times \delta\theta$ $\approx 25 \times \frac{\pi}{30} \approx \frac{5\pi}{6}$ cm^2
Specific behaviours
✓ differentiates A wrt θ ✓ determines $\frac{dA}{d\theta}$ for $\theta = \frac{\pi}{3}$ and $r = 10$ ✓ determines small change in angle ✓ shows use of increments formula ✓ determines increase in area

Question 6 **(5 marks)**

(a) Differentiate $y=\dfrac{2x+1}{e^x}$, simplifying your answer. (3 marks)

Solution
$\dfrac{dy}{dx}=\dfrac{(2)(e^x)-(2x+1)(e^x)}{(e^x)(e^x)}$ $=\dfrac{(e^x)(1-2x)}{(e^x)(e^x)}$ $=\dfrac{1-2x}{e^x}$
Specific behaviours
✓ uses quotient rule   ✓ factors out exponential term   ✓ simplifies

(b) Evaluate $\displaystyle\int_1^2\left(\frac{1-2x}{e^x}\right)dx$. (2 marks)

Solution
$\displaystyle\int_1^2\left(\frac{1-2x}{e^x}\right)dx=\left[\frac{2x+1}{e^x}\right]_1^2$ $=\dfrac{5}{e^2}-\dfrac{3}{e}$ $=\dfrac{5-3e}{e^2}$
Specific behaviours
✓ recognises antiderivative will be function from (a)   ✓ substitutes limits

Question 7 **(6 marks)**

The discrete random variable X has the probability distribution shown in the table below.

x	0	1	2	3
$P(X=x)$	$\frac{2a^2}{3}$	$\frac{1-3a}{3}$	$\frac{1+2a}{3}$	$\frac{4a^2}{3}$

Determine the value of the constant a.

Solution
$\frac{2a^2+1-3a+1+2a+4a^2}{3}=1$ $6a^2-a-1=0$ $(3a+1)(2a-1)=0$ $a=-\frac{1}{3},\ \frac{1}{2}$ Check $a=-\frac{1}{3}$ $\left[\frac{2}{27}\quad\frac{2}{3}\quad\frac{1}{9}\quad\frac{4}{27}\right]$ Check $a=\frac{1}{2}$ $\left[\frac{1}{6}\quad\cancel{-\frac{1}{6}}\quad\frac{2}{3}\quad\frac{1}{3}\right]$ $a=-\frac{1}{3}$
Specific behaviours
✓ sums distribution to 1 ✓ simplifies equation ✓ solves equation for two values of a ✓ checks first value for valid probabilities ✓ checks second value for valid probabilities ✓ states only valid value of a

Question 8 **(5 marks)**

The area bounded by the curve $y = e^{2-x}$ and the lines $y = 0$, $x = 1$ and $x = k$ is exactly $e - 1$ square units. Determine the value of the constant *k*, given that $k > 1$.

Solution
$\int_1^k e^{2-x}dx = \left[-e^{2-x}\right]_1^k$ $= \left(-e^{2-k}\right) - \left(-e^1\right)$ $= e - e^{2-k}$ $e - e^{2-k} = e - 1$ $e^{2-k} = 1$ $k = 2$
Specific behaviours
✓ writes integral to determine area ✓ antidifferentiates exponential function ✓ substitutes 1 and *k* and simplifies ✓ equates to area ✓ solves for *k*

End of questions

Additional working space

Question number: _________