PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Test 1 Year 12

Student name: _______________ Teacher name: _______________

Task type: **Response/Investigation**

Reading time for this test : 5 mins

Working time allowed for this task: 40 mins

Number of questions: **7**

Materials required: No cals allowed!!

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, NO notes allowed!

Marks available: **41 marks**

Task weighting: **13%**

Formula sheet provided: **no, but formulae stated on page 2**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Useful formulae

Complex numbers

Cartesian form	
$z = a + bi$	$\bar{z} = a - bi$
$\text{Mod}(z) = \lvert z\rvert = \sqrt{a^2+b^2} = r$	$\text{Arg}(z) = \theta, \quad \tan\theta = \dfrac{b}{a}, \quad -\pi < \theta \le \pi$
$\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$	$\left\lvert \dfrac{z_1}{z_2}\right\rvert = \dfrac{\lvert z_1\rvert}{\lvert z_2\rvert}$
$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$	$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
$z\bar{z} = \lvert z\rvert^2$	$z^{-1} = \dfrac{1}{z} = \dfrac{\bar{z}}{\lvert z\rvert^2}$
$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$	$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
Polar form	
$z = a + bi = r(\cos\theta + i\sin\theta) = r\,cis\,\theta$	$\bar{z} = r\,cis(-\theta)$
$z_1 z_2 = r_1 r_2\,cis(\theta_1 + \theta_2)$	$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\,cis(\theta_1 - \theta_2)$
$cis(\theta_1 + \theta_2) = cis\,\theta_1\,cis\,\theta_2$	$cis(-\theta) = \dfrac{1}{cis\,\theta}$
De Moivre's theorem	
$z^n = \lvert z\rvert^n\,cis(n\theta)$	$(cis\,\theta)^n = \cos n\theta + i\sin n\theta$
$z^{\frac{1}{q}} = r^{\frac{1}{q}}\left(\cos\dfrac{\theta + 2\pi k}{q} + i\sin\dfrac{\theta+2\pi k}{q}\right)$, for k an integer	

$$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

No cals allowed!!

Q1 (2, 2, 2 & 2 = 8 marks)

If $z = 5 - 4i$ and $w = 2 + 3i$ determine the following:

a) zw

Solution
$(5-4i)(2+3i) = 10 + 12 - 8i + 15i$ $= 22 + 7i$
Specific behaviours
✓ real part ✓ Imaginary part

b) $\frac{1}{w}$

Solution
$\frac{1}{2+3i}\frac{2-3i}{2-3i} = \frac{2-3i}{13}$
Specific behaviours
✓ uses conjugate ✓ express answer

c) $\frac{\overline{z}}{w}$

Solution
$\frac{5+4i}{2+3i}\frac{2-3i}{2-3i} = \frac{22-7i}{13}$
Specific behaviours
✓ numerator ✓ denominator

d) $z^2\overline{w}$

Solution

$(5-4i)^2(2-3i)=(25-16-40i)(2-3i)$ $(9-40i)(2-3i)$ $=18-120-80i-27i$ $=-102-107i$
Specific behaviours
✓ evaluates square term ✓ determines answer

Q2 (2 & 3 = 5 marks)

a) Determine the complex roots of $3z^2+z+2=0$.

Solution
$3z^2+z+2=0$ $z=\dfrac{-1\pm\sqrt{1-24}}{6}$ $z=\dfrac{-1\pm\sqrt{23}i}{6}$
Specific behaviours
✓ uses quadratic formula ✓ has two complex roots

b) Use the quadratic equation to prove that if a quadratic equation with real coefficients has any non-real roots then it must have two complex roots and they must be conjugates of each other.

Solution
$ax^2+bx+c=0$ $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ $b^2-4ac=-n^2=i^2n^2$ $x=\dfrac{-b\pm\sqrt{i^2n^2}}{2a}=\dfrac{-b\pm in}{2a}$

Specific behaviours
✓ sets up equation with a negative discriminant ✓ uses $i^2 = -1$ with discriminant ✓ derives two complex roots which are conjugates of each other

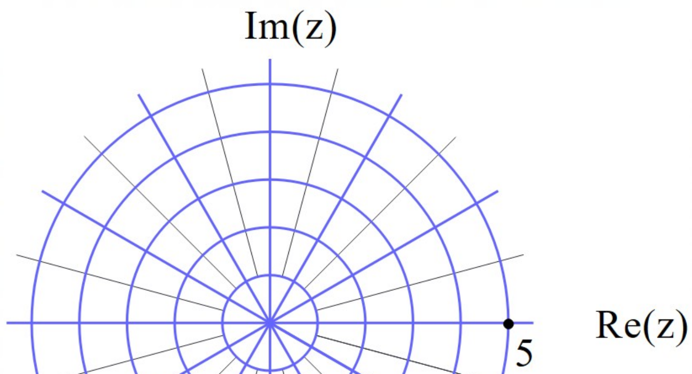
Q3 (4 marks)

Determine all possible real number pairs $a \& b$ such that $\frac{37+9i}{5+ai} = b - i$.

Solution
$\frac{37+9i}{5+ai} = b - i$ $37+9i = (5+ai)(b-i) = 5b + a + i(ab-5)$ $37 = 5b + a$ $9 = ab - 5, ab = 14, a = \frac{14}{b}$ $37 = 5b + \frac{14}{b}$ $37b = 5b^2 + 14$ $5b^2 - 37b + 14 = 0$ $(5b-2)(b-7) = 0$ $b = 7, a = 2$ $b = \frac{2}{5}, a = 35$
Specific behaviours
✓ sets up equation and equates real and imaginary ✓ obtains two simultaneous equations ✓ solves for one pair of values ✓ solves for two pairs of values

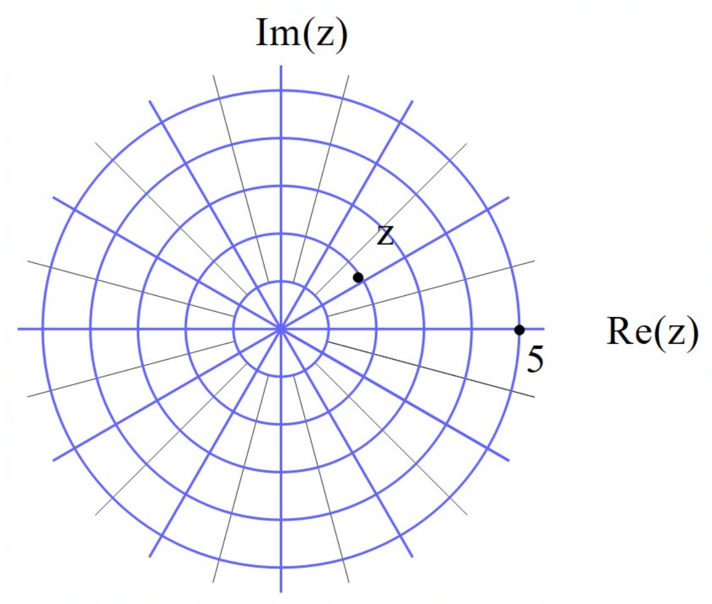
Q4 (2, 2, 2 & 2 = 8 marks)

Consider the complex number $z = \sqrt{3} + i$.

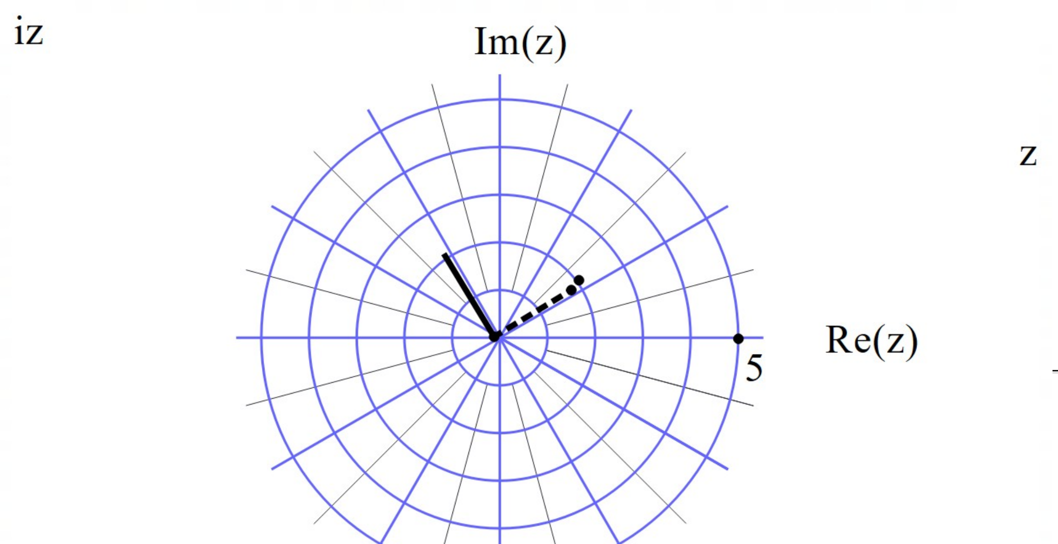


Plot the following on the axes above.

a) z

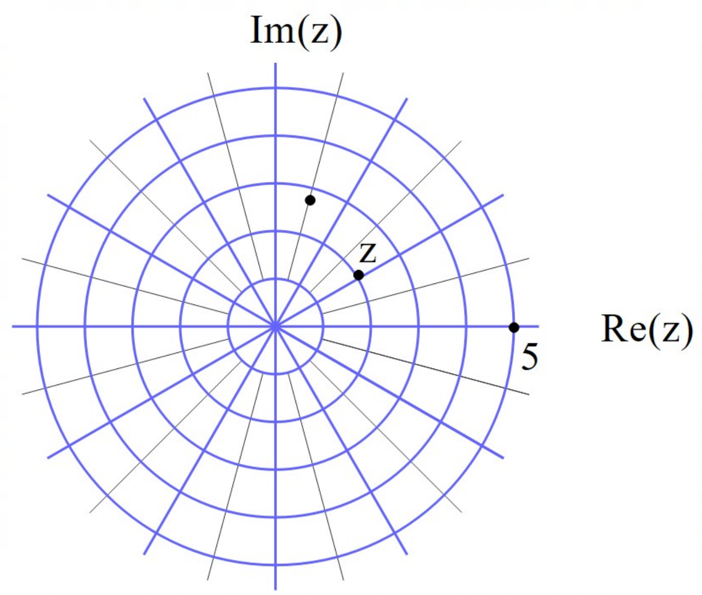
Solution

Specific behaviours
✓ argument ✓ length of 2 units

b) iz

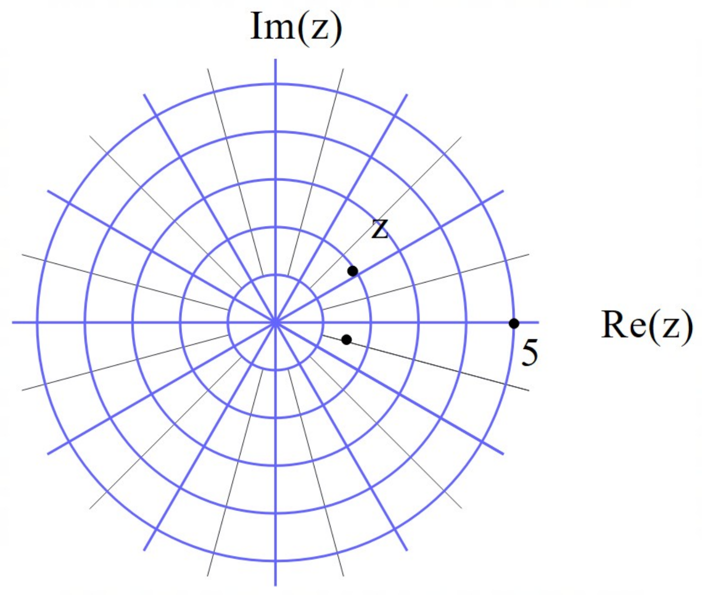
Solution


Specific behaviours
✓ uses right angle ✓ rotates anticlockwise with unchanged length

c) $(1+i)z$

Solution

Specific behaviours
✓ argument ✓ modulus

d) $\dfrac{z}{(1+i)}$

Solution


Specific behaviours
✓ argument ✓ modulus

Q5 (5 marks)

Consider the polynomial $f(z)=az^4+bz^3+cz^2+dz+e$ where $a,b,c,d\,\&\,e$ are real numbers.

Given that $f(2+i)=0=f(5-2i)$

and $f(0)=-290$

Determine the values of $a,b,c,d\,\&\,e$.

(Note: answers without working will receive zero marks)

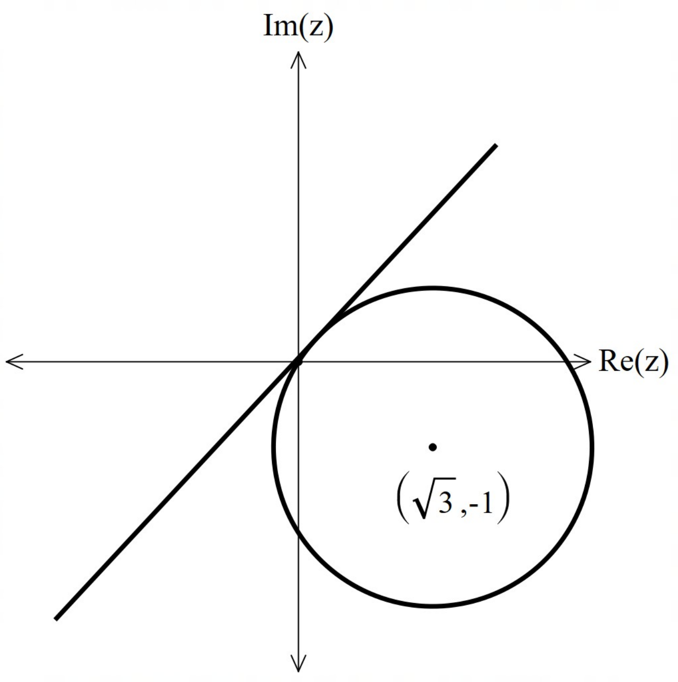
Solution

$(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta$ $-(\alpha+\beta)=-2\,\text{Re}\,al,\ \alpha\beta=\lvert z\rvert^2$ $f(z)=a(z^2-4z+5)(z^2-10z+29)$ $z=0,\ f(z)=-290\ \therefore a=-2$ $f(z)=-2(z^4-14z^3+74z^2-166z+145)$ $a=-2$ $b=28$ $c=-148$ $d=332$ $e=-290$
Specific behaviours
✓*shows reasoning for determining value of a* ✓*uses ONE quadratic factor* ✓*uses two quadratic factors* ✓*shows reasoning in determining quadratic factors (i.e roots in brackets)* ✓*shows reasoning on how to determine quartic polynomial.* *Note: Any statement of values without reasoning will NOT receive any marks!*

Q6 (2, 1, 2 & 2 = 7 marks)

Consider the locus of complex numbers z that satisfy $\left|z-\sqrt{3}+i\right|=2$.

a) Sketch the locus on the axes below.

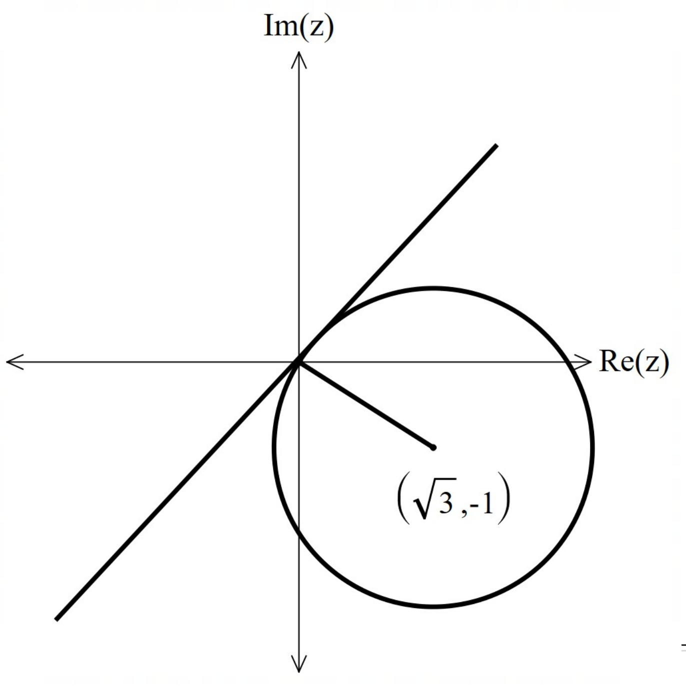
Solution


Specific behaviours
✓ circle with centre coordinates stated ✓ goes through origin

b) State the maximum value of $|z|$

Solution
$|z| = 4$
Specific behaviours
✓ states maximum

c) State the minimum value of $Arg(z)$

Solution


$m \frac{-1}{\sqrt{3}} = -1$ $m = \frac{\sqrt{3}}{1} = \tan\theta$ $\theta = \frac{\pi}{3}, \frac{-2\pi}{3}$
Specific behaviours
✓ determines gradient of tangent ✓ determines min argument

d) State the maximum value of $Arg(z)$

Solution
Max = $\frac{\pi}{3}$ See above
Specific behaviours
✓ determines gradient of tangent ✓ determines max argument

Q7 (4 marks)

Consider the roots of the equation $z^n = a$ with z being a complex variable with a as a complex constant and n being an integer $n > 3$. A root is defined to be in the first quadrant if the Argument lies in $0 < Arg(z) < \frac{\pi}{2}$.

Determine **all** the allowable values of n such that there will be **exactly** 3 roots in the first quadrant and the smallest argument of these 3 roots will be $\frac{\pi}{10}$.

Solution

$Arg(z_1) = \frac{\pi}{10}$ $Arg(z_2) = \frac{\pi}{10} + \frac{2\pi}{n}$ $Arg(z_3) = \frac{\pi}{10} + \frac{4\pi}{n}$ $Arg(z_4) = \frac{\pi}{10} + \frac{6\pi}{n}$ $\frac{\pi}{10} + \frac{4\pi}{n} < \frac{\pi}{2}, \frac{4}{n} < \frac{1}{2} - \frac{1}{10} = \frac{2}{5}$ $\frac{n}{4} > \frac{5}{2}, n > 10$ $\frac{\pi}{10} + \frac{6\pi}{n} > \frac{\pi}{2}, \frac{6}{n} > \frac{1}{2} - \frac{1}{10} = \frac{2}{5}$ $\frac{n}{6} < \frac{5}{2}, n < 15$ $10 < n < 15$ Note; accept n=15 though point out that SCSA would not!
Specific behaviours
✓ uses correct difference in arguments ✓ sets up inequality for lower n value using 3rd root ✓ sets up inequality for upper n value using 4th root ✓ solves for interval of n values NOTE: any statement that is not supported receives zero marks)