Rossmoyne Senior High School

Semester Two Examination, 2017

Question/Answer booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section One: Calculator-free

SOLUTIONS

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	97	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

The discrete random variable X is defined by

$$P(X=x)=\begin{cases}\dfrac{k}{3-x} & x=0,1\\ 0 & \text{elsewhere.}\end{cases}$$

(a) Determine the value of the constant k. (2 marks)

Solution
$\dfrac{k}{3}+\dfrac{k}{2}=1$ $k=\dfrac{6}{5}$
Specific behaviours
✓ sums probabilities to 1   ✓ states value

(b) Determine

(i) $\mathrm{E}(6-5X)$. (2 marks)

Solution
Bernoulli distribution, $p=P(X=1)=\frac{3}{5}$ $E(X)=p=\dfrac{3}{5}$ $E(6-5X)=6-5\left(\dfrac{3}{5}\right)=3$
Specific behaviours
✓ uses $E(X)=p=P(X=1)$ ✓ determines expected value

(ii) $\mathrm{Var}(2+5X)$. (2 marks)

Solution
$Var(X)=\dfrac{3}{5}\times\dfrac{2}{5}=\dfrac{6}{25}$ $Var(2+5X)=5^2\times\dfrac{6}{25}=6$
Specific behaviours
✓ uses $Var(X)=p(1-p)$ ✓ determines required variance

Question 2 **(6 marks)**

(a) Determine c, if $\log_5 8 - 2\log_5 3 - 1 = \log_5 c$. (3 marks)

Solution
$\text{LHS} = \log_5 8 - \log_5 3^2 - \log_5 5$ $= \log_4\left(\frac{8}{9 \times 5}\right)$ $c = \frac{8}{45}$
Specific behaviours
✓ writes $2\log_5 3$ as $\log 3^2$ ✓ writes 1 as $\log_5 5$ ✓ combines as single log and states value of c

(b) Determine the exact solution to $2(3)^{x+2} = 10$. (3 marks)

Solution
$\log 3^{x+2} = \log 5$ $(x+2)\log 3 = \log 5$ $x = \frac{\log 5}{\log 3} - 2$
Specific behaviours
✓ divides both sides by 2   ✓ logs both sides to any base   ✓ solves for x

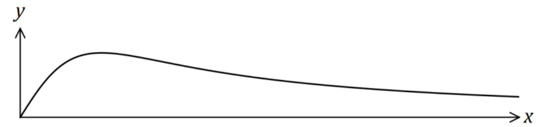
Alternative solution
$3^{x+2} = 5$ $x + 2 = \log_3 5$ $x = \log_3 5 - 2$
Specific behaviours
✓ divides both sides by 2   ✓ logs to base 3   ✓ solves for x

Additional working space

Question number: __________

Question 3 **(7 marks)**

The graph of $y = f(x),\ x \geq 0$, is shown below, where $f(x) = \dfrac{4x}{x^2+3}$.



(a) Determine the gradient of the curve when $x = 2$. (3 marks)

Solution
$f'(x) = \dfrac{4(x^2+3) - 4x(2x)}{(x^2+3)^2}$ $f'(2) = \dfrac{4(7) - 8(4)}{(7)^2} = -\dfrac{4}{49}$
Specific behaviours
✓ uses quotient rule   ✓ correct $f'(x)$ ✓ correct gradient

(b) Determine the exact area bounded by the curve $y = f(x)$ and the lines $y = 0$ and $x = 2$, simplifying your answer. (4 marks)

Solution
$A = \int_0^2 f(x)\,dx$ $= [2\ln(x^2+3)]_0^2$ $= 2\ln 7 - 2\ln 3$ $= 2\ln\dfrac{7}{3}$
Specific behaviours
✓ writes integral   ✓ antidifferentiates   ✓ substitutes   ✓ simplifies

Question 4 **(7 marks)**

The rate of change of displacement of a particle moving in a straight line at any time t seconds is given by

$$\frac{dx}{dt} = 5 + 2e^{0.2t} \text{ cm/s.}$$

Initially, when $t = 0$, the particle is at P, a fixed point on the line.

(a) Calculate the initial velocity of the particle. (1 mark)

Solution
$v(0) = 5 + 2e^0 = 7$ cm/s
Specific behaviours
✓ velocity

(b) Determine the distance of the particle from P after 5 s. (3 marks)

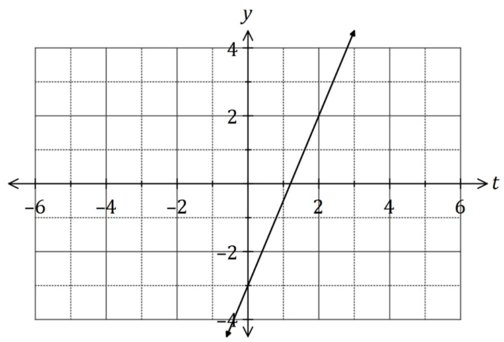
Solution
$x = 5t + 10e^{0.2t} + c$ $c = 0 - 10e^0 = -10$ $x(5) = 5(5) + 10e^1 - 10$ $= 15 + 10e$ cm
Specific behaviours
✓ integrates ✓ evaluates constant ✓ substitutes to obtain distance

(c) Determine when the acceleration of the particle is $20\ \text{cm/s}^2$. (3 marks)

Solution
$a = 0.4e^{0.2t}$ $0.4e^{0.2t} = 20 \Rightarrow 0.2t = \ln 50$ $t = 5 \ln 50$ s
Specific behaviours
✓ differentiates for acceleration ✓ eliminates e ✓ solves for t

Question 8 **(5 marks)**

Part of the graph of the linear function $y = f(t)$ is shown below.



Another function $A(x)$ is given by

$$A(x) = \int_{-2}^{x} f(t)\, dt\ .$$

Use the increments formula to estimate the change in A as x increases from 10 to 10.2.

Solution
$\frac{dA}{dx} = \frac{d}{dx}\int_{-2}^{x} f(t)\, dt = f(x)$ $f(x) = 2.5x - 3$ $\delta A \approx \frac{dA}{dx}\delta x \approx (2.5(10) - 3)(0.2)$ ≈ 4.4
Specific behaviours
✓ indicates $A'(x)$ ✓ uses $x = 10,\ \delta x = 0.2$ ✓ determines $f(x)$ ✓ uses increments formula ✓ determines change

(c) Use the second derivative to show that in order to maximise the area of OABC
sin 2x + x cos 2x > 0 (3 marks)

Solution
$A' = 3\cos 2x - 6x\sin 2x$ $A'' = -12\sin 2x - 12x\cos 2x$ $A'' = -12(\sin 2x + x\cos 2x)$ To achieve a maximum A''<0 so sin 2x + x cos 2x must be positive.
Specific behaviours
✓ Differentiates the first derivative ✓ Factorises the common factor of -12 ✓ Justifies that sin 2x + x cos 2x must be positive as A''<0

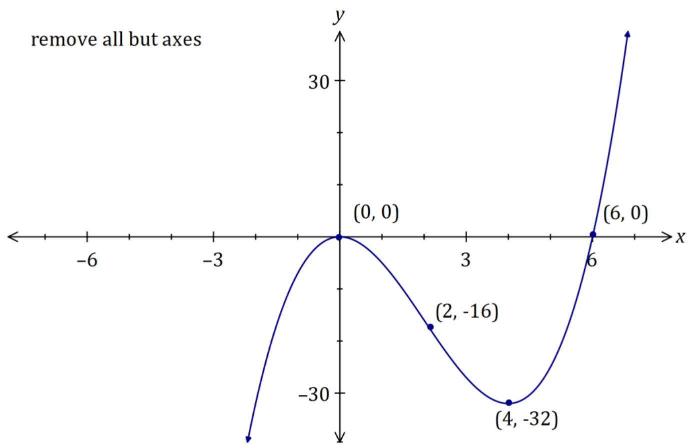
Question 5 **(7 marks)**

A curve has first derivative $\frac{dy}{dx} = 3x(x-4)$ and passes through the point $P(1,-5)$.

(a) Determine the value(s) of x for which $\frac{d^2y}{dx^2} = 0$. (2 marks)

Solution
$\frac{d^2y}{dx^2} = 6x - 12$ $6x - 12 = 0 \Rightarrow x = 2$
Specific behaviours
✓ differentiates ✓ states value

(b) Sketch the curve on the axes below, clearly indicating the location of all axes intercepts, stationary points and points of inflection. (5 marks)



Solution
$y' = 0 \Rightarrow x = 0, 4$ $y' = 3x^2 - 12x \Rightarrow y = x^3 - 6x^2 + c$ $c = -5 - 1 + 6 = 0$ $y = x^2(x-6) \Rightarrow$ zeroes at $x = 0, 6$ $x = 2, y = -16;\ x = 4, y = -32$ See graph
Specific behaviours
✓ obtains expression for y ✓ obtains zeroes of y ✓ indicates coordinates of point of inflection ✓ indicates coordinates of the maximum and minimum T.P. ✓ single smooth continuous curve with an appropriate scale

Question 6 **(7 marks)**

Determine the following, giving your answers in exact form.

(a) $\int (5x - \cos 5x)\ dx$ (2 marks)

Solution
$\frac{5x^2}{2} - \frac{\sin 5x}{5} + c$
Specific behaviours
✓ integrates cos 5x ✓ obtains the correct expression with a constant

(b) $\int_0^4 (e^{2x} - \sqrt{x})\ dx$ (3 marks)

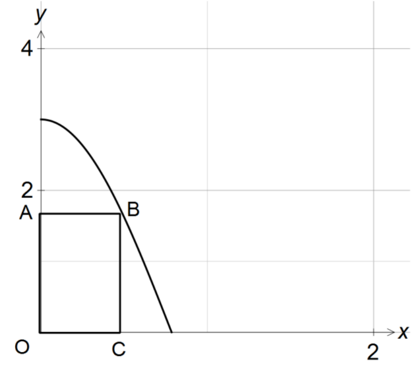
Solution
$\left[\frac{e^{2x}}{2} - \frac{2x^{\frac{3}{2}}}{3}\right]_0^4$ $= \left(\frac{e^8}{2} - \frac{2\sqrt{4^3}}{3}\right) - \left(\frac{e^0}{2} - \frac{2\sqrt{0^3}}{3}\right)$ $= \frac{e^8}{2} - \frac{35}{6}$ $= \frac{3e^8 - 35}{6}$
Specific behaviours
✓ integrates the function ✓ substitutes limits ✓ obtains the correct expression

(c) $\frac{d}{dx}\left(\int_x^{\pi} \sin(t)dt\right)$ (2 marks)

Solution
$-\sin(x)$
Specific behaviours
✓ applies Fundamental Theorem ✓ obtains the correct expression

Question 7 **(7 marks)**

The first quadrant of $y = 3\cos 2x$ is shown.



(a) Show that the area of rectangle OABC = $3x.\cos 2x$. (1 mark)

Solution
Length = $3\ cos\ 2x$ Width = x Area = Length × Width = $3x\ cos\ 2x$
Specific behaviours
✓ integrates cos 5x ✓ obtains the correct expression with a constant

(b) Show that for the area of OABC to be a maximum, $2x.\tan 2x - 1 = 0$. (3 marks)

Solution
$A = 3x\ cos\ 2x$ $A' = 3\cos 2x - 6x\sin 2x$ $0 = 3\cos 2x - 6x\sin 2x$ $6x\sin 2x = 3\cos 2x$ $\frac{6x\sin 2x}{3\cos 2x} = \frac{3\cos 2x}{3\cos 2x}$ $2x\tan 2x = 1$ $2x\tan 2x - 1 = 0$
Specific behaviours
✓ Differentiates the expression for area ✓ Equates the first derivative to zero ✓ Divides both sides by 3 cos 2x