Space for extra working

Question ....

WESLEY COLLEGE

Semester One Examination 2012
Question/Answer Booklet

MATHEMATICS 3CD

Section One
(Calculator Free)

SOLUTIONS

Time allowed for this section

Reading time before commencing work: 5 minutes
Working time for paper: 50 minutes

Material required/recommended for this section

To be provided by the supervisor
Question/answer booklet for Section One.
Formula sheet.

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler.

Important note to candidates

No other items may be taken into the examination room. It is your responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor before reading any further.

Structure of this examination

	Number of questions	Working time (minutes)	Marks available
This Section **Section One** **Calculator Free**	**9**	**50**	**50**
Section Two Calculator Assumed	13	100	100
		Total marks	150

Instructions:

1. Answer the questions in the spaces provided.

2. Spare answer pages are provided at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued i.e. give the page number.

3. Show all working clearly. Any question, or part question, worth more than 2 marks requires valid working or justification to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

Space for extra working

Question ....

Space for extra working

Question ....

Question 1. **(9 marks)**

Differentiate the following functions.

(You do not need to perform more than the most obvious algebraic simplifications)

(a) $y = \frac{2}{3}x^3 - 3x + \frac{1}{x}$

$$\frac{dy}{dx} = 2x^2 - 3 - \frac{1}{x^2}$$

[2]

(b) $y = e^{2x^3 - x}$

$$\frac{dy}{dx} = \left(6x^2 - 1\right)e^{2x^3 - x}$$

[2]

(c) $y = x^3 e^{-3x}$

$$\frac{dy}{dx} = x^3\left(-3e^{-3x}\right) + e^{-3x} \cdot 3x^2$$

[2]

(d) $y = \frac{\left(1 + x^4\right)^3}{e^x}$

$$\frac{dy}{dx} = \frac{e^x \cdot 3\left(1 + x^4\right)^2 \cdot 4x^3 - e^x\left(1 + x^4\right)^3}{e^{2x}}$$

[3]

Question 2. **(5 marks)**

Given $h(x) = e^x$ and $l(x) = \dfrac{1}{1-x}$

(a) State the natural domain for $l(x)$ $D_x : x \neq 1$

[1]

(b) State the natural range for $h(x)$ $R_y : y > 0$

[1]

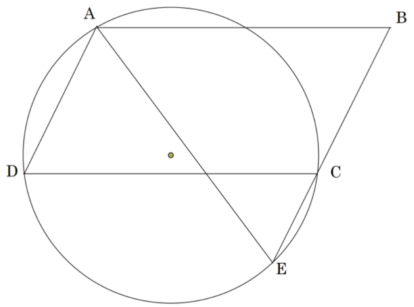
(c) Find the natural domain for the function $l \circ h(x)$

$$l \circ h(x) = \frac{1}{1-e^x}$$
$$1 - e^x \neq 0$$
$$e^x \neq 1$$
$$D_x : x \neq 0$$

[3]

Question 3. **(3 marks)**

Given $ABCD$ is a parallelogram, prove $\triangle ABE$ is isosceles.



In the major segment ADEC $\measuredangle ADC = \measuredangle AEC$ *(angles standing on the same arc)*

In parallelogram ABCD $\measuredangle ADC = \measuredangle ABC$ *(Property of par'm)*

Since $\measuredangle ABE = \measuredangle AEB$ then ΔABE is isosceles *(Property of isosceles triangle)*

[3]

Space for extra working

Question ....

(f) Prove your conjecture.

Mark allocation:

$T_n = 1+2+\ldots+n+(\boldsymbol{n+1}) = \frac{(n+1)}{2}(n+2)$

Replace T_n with correct expression

$8T_n + 1 = 8 \times \frac{(n+1)}{2}(n+2)+1$

Expand and then simplify

$= 4(n^2+3n+2)+1$

$= 4n^2+12n+9$

$= (2n+3)(2n+3)$

Factorise as a perfect square

$= (2n+3)^2$

shown

[3]

Question 4. **(4 marks)**

Determine the gradient of $y = 2\sqrt[3]{x} + \frac{6}{x^3}$ at the point (1, 8)

$y = 2x^{\frac{1}{3}} + 6x^{-3}$

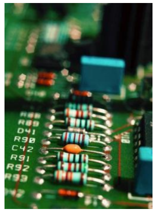
$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{2}{3}} - 18x^{-4} \quad \left(or \ \frac{2}{3x^{\frac{2}{3}}} - \frac{18}{x^4} \right)$

$\frac{dy}{dx}_{|x=1} = \frac{2}{3} - 18$

$= -17\frac{1}{3}$

[4]

Question 5. **(5 marks)**



When resistors are positioned in series, the total resistance, R, is given by

$$R = R_1 + R_2$$

Given $R = \frac{35}{(x+1)(x+2)}$, $R_1 = \frac{x}{x+1}$ and $R_2 = \frac{x}{x+2}$, and $x > 0$

find the value of x.

$i.e.\ \frac{x}{x+1} + \frac{x}{x+2} = \frac{35}{(x+1)(x+2)} \quad \Rightarrow \quad \frac{x(x+2)+x(x+1)}{(x+1)(x+2)} = \frac{35}{(x+1)(x+2)}$

$x^2+2x+x^2+x = 35$

$2x^2+3x-35 = 0$

$(2x-7)(x+5) = 0$

$\boldsymbol{x = \frac{7}{2}}$ $\quad or \quad x = -5$ (*reject this as* $x > 0$)

[5]

Question 6. **(6 marks)**

(a) If $y = kx^3$ for some constant k, use the incremental formula to estimate the percentage change in x required to yield a 15% increase in y.

$$\frac{dy}{dx} = 3kx^2$$

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} \quad \Rightarrow \quad 0.15y = 3kx^2 \cdot \Delta x$$

$$\therefore \Delta x = \frac{0.15kx^3}{3kx^2} = \frac{0.15x}{3} = 0.05x$$

A percentage <u>increase</u> of 5% in x is required.

[3]

(b) A company sells goods such that its revenue, in dollars, from selling x items is given by the equation

$$R(x) = 5x\left(20x - x^2\right)$$

(i) Determine the <u>marginal revenue</u>, when $x = 10$

$$R(x) = 100x^2 - 5x^3$$
$$R'(x) = 200x - 15x^2$$

$$\therefore R'(10) = 2000 - 1500$$
$$= \$500$$

[2]

(ii) What does this represent?

$R'(10)$ represents the revenue from selling the 11th item

[1]

[5]

Question 22. **(9 marks)**

The sequence of numbers **3, 6, 10, 15, 21, …** are known as triangular numbers.

(a) Show that the first three triangular numbers can each be written as the sum of the first n consecutive positive integers.

$$T_1 = 3 = 1 + 2$$
$$T_2 = 6 = 1 + 2 + 3$$
$$T_3 = 10 = 1 + 2 + 3 + 4$$

[1]

(b) Hence determine the 8th triangular number

$$T_8 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \mathbf{+ 9} = 45$$

[1]

The formula $\frac{n}{2}(n+1)$ can be used to determine the sum of the first n positive integers.

(c) Use this formula to determine the 79th triangular number

$$T_{79} = \frac{\mathbf{80}}{2}(\mathbf{80} + 1) = 40 \times 81$$
$$= 3240$$

[2]

(d) For each of the first three triangular numbers, multiply the number by 8 and then add 1

$$3 \times 8 + 1 = 25$$
$$6 \times 8 + 1 = 49$$
$$10 \times 8 + 1 = 81$$

[1]

(e) Based on your results from (d), write a conjecture relating to multiplying ***any*** triangular number by 8 and then adding 1

Multiplying ***any*** triangular number by 8 and then adding 1 produces a ***square number***

[1]

PLEASE TURN OVER→

[2]

Question 21. **(10 marks)**

A radio-active substance has a half-life of 16 months. After a year, only 700 g were left.
Assume the radioactive substance decays exponentially.

(a) Find the initial amount of the substance

$A = A_0 e^{-kt}$

when $t = 16$ months $A = \frac{1}{2} A_0$

$\frac{1}{2} A_0 = A_0 e^{-16k} \quad \Rightarrow k = 0.0433$

when $t = 12$ months $A = 700$

$700 = A_0 e^{-0.0433 \times 12} \quad \Rightarrow A_0 = 1177.25\,g$

[5]

(b) Find the instantaneous rate of decay when 75% of the original amount has decayed.

$A = 1177.25 e^{-0.0433t}$

Rate of decay $\frac{dA}{dt} = kA$

$= -0.0433A$

For $A = 0.25A_0 = 0.25 \times 1177.25 = 294.31$

$\frac{dA}{dt} = -0.0433 \times 294.31$

$= -12.75\ grams\ /\ month$

Question 7. **(7 marks)**

The points $P(-4,3)$, $Q(6,3)$ and $R(-2,-1)$ all lie on the graph $f(x) = ax^2 + bx + c$.

Calculate the values of a, b and c.

Substituting :

$(-4,3)$: $16a - 4b + c = 3$ (1)

$(6,3)$: $36a + 6b + c = 3$ (2)

$(-2,-1)$: $4a - 2b + c = -1$ (3)

$(2)-(1)$: $20a + 10b = 0$ *or* $2a + b = 0$ (4)

$(2)-(3)$: $32a + 8b = 4$ *or* $8a + 2b = 1$ (5)

$(5) - 2\times(4)$: $8a + 2b = 1$

$4a + 2b = 0$

$4a = 1 \quad \Rightarrow a = \frac{1}{4}$

$in(4)$: $\frac{1}{2} + b = 0 \quad \Rightarrow b = -\frac{1}{2}$

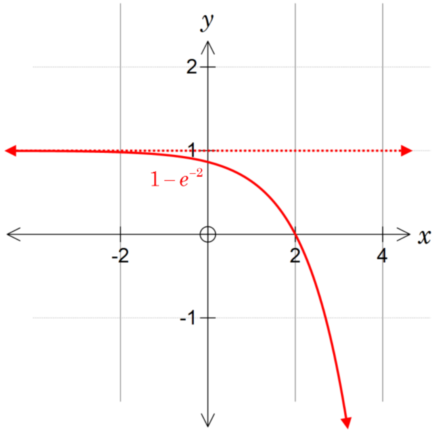
$in(3)$: $1 + 1 + c = -1 \quad \Rightarrow c = -3$

[7]

Question 8. **(5 marks)**

(a) Sketch the graph of $y = 1 - e^{x-2}$ on the axes provided.

Indicate clearly the intercept(s) and asymptote(s)



[3]

(b) Find $g(x)$ if the curve $y = e^x$ is mapped to $y = g(x)$ by the following sequence of transformations
A reflection about the x-axis followed by a dilation in the direction of the positive x-axis by a factor of 4 followed by a reflection about the y-axis

$y = e^x$

$\Rightarrow \quad y = -e^x$

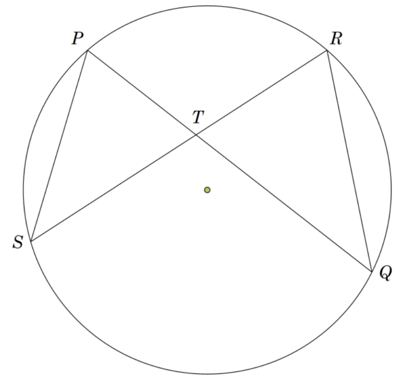
$\Rightarrow \quad y = -e^{\frac{x}{4}}$

$\Rightarrow \quad y = -e^{\frac{-x}{4}}$

[2]

Question 20. **(8 marks)**

In the diagram below the chords PQ and RS intersect at the point T.
The area of $\Delta\ TPS$ is $17.5cm^2$



(a) Explain why $\measuredangle TPS = \measuredangle TRQ$

Angles in the same major segment are equal – they are subtended by the same arc

[1]

(b) Prove that ΔTPS is similar to ΔTRQ

$\measuredangle TPS = \measuredangle TRQ$ *Stand on arc QS*

$\measuredangle TSP = \measuredangle TQR$ *Stand on arc PR*

$\measuredangle PTS = \measuredangle RTQ$ *Vertically opposite*

$\Delta TPS \approx \Delta TRQ$ *(AAA)*

[3]

(c) Use your result from (b) to show that $PT \times QT = ST \times RT$

$\Delta TPS \approx \Delta TRQ \qquad \dfrac{PT}{ST} = \dfrac{RT}{QT}$

[2]

$\Rightarrow \quad PT \times QT = ST \times RT$

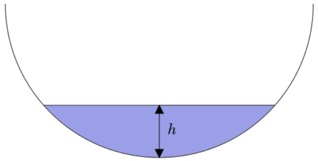
(d) Find the area of ΔTRQ if $RT = 1 \cdot 4 \times PT$

If $RT = 1.4 \times PT$ then $ST = 1.4 \times QT$

Area of $\Delta TRQ = 1.4 \times 1.4 \times 17.5 = 34.3cm^2$

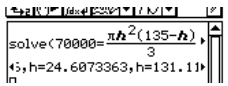
Question 19. **(6 marks)**

When fluid rests in the bottom of a hemisphere of radius r, the volume of fluid V, can be calculated using the formula $V = \dfrac{\pi h^2(3r-h)}{3}$, where h is the depth of the fluid.



If water is poured into a hemisphere of radius 45 cm at a constant rate of 2 litres per minute, how fast is the depth of water increasing at the instant that the hemisphere contains 70 L of water? Give your answer to 3 s.f.
$(1L = 1000cm^3)$

Given: $V = 70\times1000 = 70000cm^3$
$r = 45cm$
(ignore other invalid solutions)



$h = 24.607cm$ **2 mk**

$V = \dfrac{\pi h^2(135-h)}{3} \quad \Rightarrow \dfrac{dV}{dh} = 90\pi h - \pi h^2$ **1 mk**

$\dfrac{dV}{dt} = 2000$ **1 mk**

$\Rightarrow \dfrac{dh}{dt} = \dfrac{dh}{dV}\cdot\dfrac{dV}{dt}$

$\Rightarrow \dfrac{dh}{dt} = \left(\dfrac{1}{90\pi h - \pi h^2}\times2000\right)_{|h=24.607}$ **1mk**

$= 0.396 cm/\min(3 s.f.)$ **1mk**

[6]

Question 9. **(6 marks)**

Functions $f(x)$ and $g(x)$ are defined as $f(x)=\sqrt{x-3}$ and $g(x)=\dfrac{2x-7}{x}$

(a) Evaluate $gf(7)$ $= g(2) = \dfrac{-3}{2}$

[1]

(b) To find the domain of $f\circ g(x)$, it is necessary to solve the inequality

$\dfrac{2x-7}{x}\geq 3$

(i) Explain why

Since $\sqrt{x-3}$ only exists for $x\geq3$, when x is replaced by $\dfrac{2x-7}{x}$ ***it*** will need to be ≥ 3

[1]

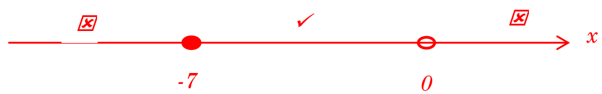
(ii) Find the domain of $f\circ g(x)$

$\dfrac{2x-7}{x}\geq 3 \qquad x\neq 0$

Solve $\quad 2x-7=3x$

i.e. $\quad x=-7$

Test:



$-7\leq x<0$

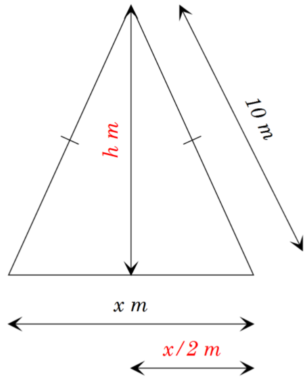
[4]

Space for extra working

Question ....

Question 18. **(9 marks)**

As part of their community service, the Wesley College senior prefects designed and built a new garden bed for the local hospice according to the following sketch:



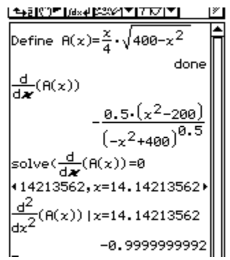
(a) Show that the area of the garden bed, A, as a function of x is given by $A = \frac{1}{4}x\sqrt{400 - x^2}$

$$h = \sqrt{100 - \left(\frac{x}{2}\right)^2}$$

$$\begin{aligned} A &= \frac{1}{2} \cdot x \cdot \sqrt{100 - \left(\frac{x}{2}\right)^2} \\ &= \frac{x}{2} \cdot \sqrt{\frac{400 - x^2}{4}} \\ &= \frac{x}{2} \cdot \frac{1}{2} \cdot \sqrt{400 - x^2} \\ &= \frac{x}{4} \cdot \sqrt{400 - x^2} \qquad \textit{shown} \end{aligned}$$

[3]

(b) Use calculus methods, showing full reasoning, to find the value of x that will maximise the area of the garden bed.



Mark allocation:

Determine $\frac{dA}{dx}$ **(1mk)**

Set $\frac{dA}{dx} = 0$ and find solve for x (ignore negative)

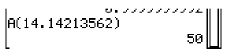
(1 mk each)

Demonstrate that $\left.\frac{d^2A}{dx^2}\right|_{x=14.14...} < 0 \quad \Rightarrow$ maximum

(1 mk each)

[5]

(c) What would this maximum area be? $A = 50m^2$



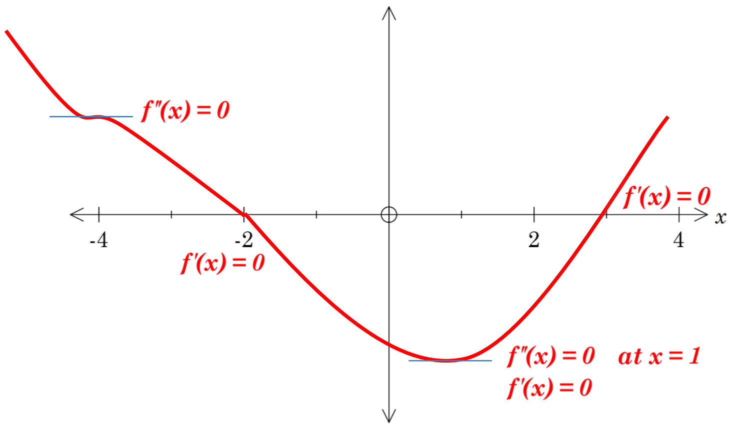
[1]

Question 17. **(8 marks)**

The graph of $y = f'(x)$ has the following properties:

- *exactly* two roots at $x = -2, x = 3$
- two stationary points at $x = -4$, $x = 1$
- a positive gradient at $x = 2$
- a negative gradient at $x = -2$

(a) Sketch the graph of $y = f'(x)$



[4]

(b) Use your answer to (a), or otherwise, to determine the value(s) of x at which $f(x)$ has

(i) a local minimum $x = 3$ $\left(f''(3) > 0 \quad \therefore \text{min}\right)$

[1]

(ii) a local maximum $x = -2$ $\left(f''(-2) < 0 \quad \therefore \text{max}\right)$

[1]

(iii) a point of inflection $x = 1$ *Pay follow through if max at* $x = -4$

[2]

Space for extra working

Question ....

Space for extra working

Question ....

Question 16. **(6 marks)**

Air pressure decreases exponentially (approximately) with the height in metres above sea level h by the rule

$$P = P_0 e^{-1.35\times10^{-4} h}$$

(a) What does P_0 represent?

The air pressure at sea level

[1]

(b) Mt. Kosciusko is 2230 metres above sea level.
Determine the percentage **decrease** in air pressure from a point at sea level to a point on top of the mountain.

$when\ h = 2230 \qquad P = P_0 e^{-1.35\times10^{-4}\times2230}$

$= 0.74P_0$

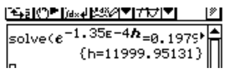
$\therefore 26\%\ decrease$

[2]

(c) When a commercial jet is at a maximum cruising speed the percentage decrease in air pressure from sea level is 80.21%
Determine the height of the jet to the nearest metre.

$\%\ decrease = 80.21 \qquad \therefore P = 0.1979P_0$

$e^{-1.35\times10^{-4}h} = 0.1979$



Height of 12 000 m

[3]

Question 15. **(11 marks)**

A particle is initially at an origin O. It is then projected away from O and moves in a straight line such that its displacement from O, t seconds later is x metres where $x = t^3 - 6t^2 + 9t$.

Determine:

(a) the initial speed of projection

$\dot{x} = 3t^2 - 12t + 9 \qquad \dot{x}(0) = 9$

Initial speed is 9m/s

[2]

(b) when the particle is at rest and how far it is from the origin at these times

$\dot{x} = 0 \text{ when } \quad 3t^2 - 12t + 9 = 0$

$\Rightarrow t = 1 \text{ or } t = 3$

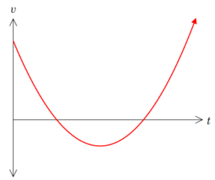
$x(1) = 4$

$x(3) = 0$

The particle is at rest after $t = 1$, 4 metres from the origin and again at $t = 3$ at the origin

[4]

(c) when the particle is moving in a positive direction



$\dot{x} > 0 \Rightarrow \quad 0 < t < 1 \quad or \quad t > 3$

[2]

(d) the total distance travelled in the first 5 seconds

$x(0) = 0;\ x(1) = 4;\ x(3) = 0;\ x(5) = 20$

$Dist \ = 4 + 4 + 20$

$28m$

[3]

WESLEY COLLEGE

Semester One Examination 2012
Question/Answer Booklet

MATHEMATICS 3CD

Section Two
(Calculator Assumed)

SOLUTIONS

Time allowed for this section

Reading time before commencing work: 10 minutes
Working time for paper: 100 minutes

Material required/recommended for this section

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Formula sheet.

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Special items: drawing instruments, templates, notes on up to two unfolded sheets of A4 paper, and up to three calculators, CAS, graphic or scientific, which satisfy the conditions set by the Curriculum Council for this course.

Important note to candidates

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This Section Section Two Calculator Assumed	**13**	**100**	**100**
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Question 14. **(8 marks)**

(a) Use an **algebraic** method to find the natural domain and range for $f(x)=\dfrac{1}{\sqrt{1+x^2}}$.

Answers must be supported with appropriate working.

Since $1+x^2 \geq 0$ for all values of x, $D_x : x \in \mathbb{R}$

$1+x^2$ has a minimum value of 1 (at $x=0$)

$\therefore \dfrac{1}{\sqrt{1+x^2}}$ has a maximum value of 1 (at $x=0$)

As x gets very large

$(x \to \pm\infty)$ $1+x^2 \to \infty$

so $\dfrac{1}{\sqrt{1+x^2}} \to 0$

$R_y : \quad 0 < y \leq 1$

[4]

(b) Given that $f \circ g(x)=\dfrac{x}{x-1}$ and $f(x)=3x+1$, find the rule for g.

Answers must be supported with appropriate working.

Full working must be shown for full marks

$$3(g(x))+1 = \frac{x}{x-1}$$

$$3g(x) = \frac{x}{x-1}-1 = \frac{x-(x-1)}{x-1} = \frac{1}{x-1}$$

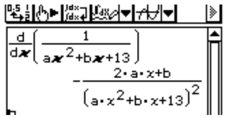
$$\therefore g(x)=\frac{1}{3(x-1)}$$

[4]

Question 13. **(6 marks)**

The gradient of the curve with equation $y = \dfrac{1}{ax^2 + bx + 13}$ at the point $\left(2, \dfrac{1}{5}\right)$ is zero.

(a) Use your ClassPad to find an expression for $\dfrac{dy}{dx}$ in terms of a and b



[1]

(b) Form two equations and hence find the values of a and b.

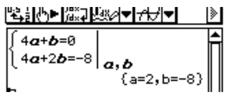
$\dfrac{dy}{dx}_{|x=2} = 0 \quad \Rightarrow \quad \dfrac{-1(4a+b)}{(4a+2b+13)^2} = 0$

$\Rightarrow \quad 4a + b = 0 \qquad (1)$

$\left(2, \dfrac{1}{5}\right)$ *lies on curve* $\quad \Rightarrow \quad \dfrac{1}{5} = \dfrac{1}{4a+2b+13}$

$\Rightarrow \quad 4a + 2b + 13 = 5$

or $\quad 4a + 2b = -8 \qquad (2)$



[5]

Question 10. **(9 marks)**

(a) A conjecture is true **only** if it is always true. State whether the following is false or true. If it is false, give a counter-example, otherwise give one example of when it is true.

(i) Every factor of an even number is even

FALSE: e.g. 6 is even, but has a factor of 3 (odd)

[2]

(ii) The sum of three counting numbers in an arithmetic progression is a multiple of 3

TRUE: e.g. 4, 7, 10, … is in A.P.
$4+7+10=21$ which is a multiple of 3

[2]

(iii) If a and b are odd counting numbers with $a > b$, then $a^2 - b^2$ is a multiple of 8

TRUE: e.g. $a = 15, b = 11$
then $225 - 121 = 104 = 13 \times 8$
i.e. a multiple of 8

[2]

(b) Explain why the sum of 3 consecutive even integers is always a multiple of 6. *Don't just give examples; your answer must be supported by reasoning.*

Let the consecutive even integers be $2n, 2n+2, 2n+4$
Then $sum = 6n + 6 = 6(n+1) = 6 \times integer$
$\therefore$ sum is a multiple of 6

[3]

Question 11. **(4 marks)**

As part of a university teaching project, a group of first-year students is brought together with a group made up of final-year and mature-age students, so that each first-year student is paired with an older student. No student remains without a partner. There are a total of 30 students in the project.

There are

x first-year students, aged 17 years
y final-year students, aged 21 years
z mature-age students, aged 27 years

The mean age of all the students is 20 years.

(a) Write down three equations that can be used to solve for x, y and z.

$x = y + z$

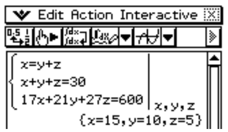
$x + y + z = 30$

$\dfrac{17x + 21y + 27z}{30} = 20$ *or* $17x + 21y + 27z = 600$

[3]

(b) How many final-year students are involved in the project?

Solving using ClassPad:



There were 10 final-year students

[1]

Question 12. **(6 marks)**

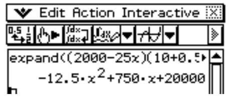
Organisers of the **"*Plains to Peaks*"** cycling race are assuming that they will get 2000 entrants if the entry fee is $10. If the entry fee is increased by 50 cents, they predict they will lose 25 competitors. Before they take any entrants they must raise $24 000 to cover costs for running the event.

Let x represent each 50 cent increase.

(a) Show that the revenue can be expressed as $20000 + 750x - 12{\cdot}5x^2$

Revenue $=$ Number of competitors $\times$ entry fee

$= (2000 - 25x)(10 + 0.5x)$



$= 20000 + 750x - 12.5x^2$

[3]

(b) Find the expression for profit, in terms of x.

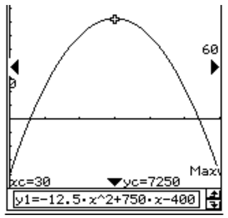
Profit $=$ Revenue $-$ Costs

$= 20000 + 750x - 12.5x^2 - 24000$

$= -12.5x^2 + 750x - 4000$

[1]

(c) How many entries are required to achieve the maximum profit?



Using ClassPad
$x = 30$ produces maximum profit

Number of entries $= 2000 - 25 \times 30 = 1250$

[2]