PERTH MODERN SCHOOL

UNIT 3C/3D MAS – 2011

TEST 1 – POLAR COORDINATES & COMPLEX NUMBERS

NAME: SOLUTIONS **DATE: 9/10 February, 2011**

[To achieve full marks and to allow assessment of particular outcomes, working and reasoning should be shown.]
[A maximum of 2 marks will be deducted for incorrect rounding, units, etc.]

This is *Resource Rich – 50 minutes for 53 marks*:

1. [1, 1, 1, 1 = 4 marks]
   Convert:
   a) (4,-6) into polar coordinates with $-180^\circ < \theta \leq 180^\circ$.

   toPol([4,-6]) ⇒ [7.21, -56.3°] ✓

   b) $(3,-\sqrt{3})$ into ***exact*** polar coordinates with $-\pi < \theta \leq \pi$.

   toPol([3,-$\sqrt{3}$]) ⇒ [2$\sqrt{3}$, $-\frac{\pi}{6}$] ✓

   c) [4,35°] into Cartesian coordinates.

   toRect([4,∠35°]) ⇒ (3.28, 2.29) ✓

   d) [8,-¾π] into ***exact*** Cartesian coordinates.

   toRect([8,∠-¾π]) ⇒ ($-4\sqrt{2}$, $-4\sqrt{2}$) ✓

2. [3, 3 = 6 marks]
Clearly show how you obtain your answers, find:
a) the distance between $[20,-210^\circ]$ and $[\sqrt{5},-50^\circ]$.

toRect([20,∠(-210°)]) ⇒ (-17.32, -10) ✓
toRect([$\sqrt{5}$,∠(-50°)]) ⇒ (1.44, -1.71) ✓
norm([-17.32, -10] – [1.44, -1.71]) ⇒ 22.11 ✓

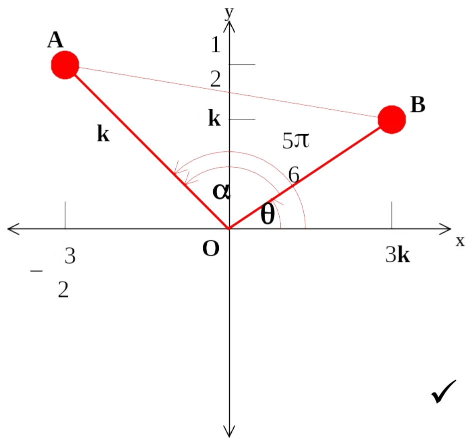
b) the ***exact*** distance between $\left[\frac{\sqrt{5}}{3}, -\frac{2\pi}{3}\right]$ and $\left[10, -\frac{7\pi}{6}\right]$.

toRect([$\frac{\sqrt{5}}{3}$,∠($-\frac{2\pi}{3}$)]) ⇒ ($-\frac{\sqrt{5}}{6}$, $-\frac{\sqrt{15}}{6}$) ✓
toRect([10,∠($-\frac{7\pi}{6}$)]) ⇒ ($-5\sqrt{3}$, 5) ✓
norm([$-\frac{\sqrt{5}}{6}$, $-\frac{\sqrt{15}}{6}$] – [$-5\sqrt{3}$, 5]) ⇒ $\frac{\sqrt{905}}{3}$ ✓

3. [8 marks]

Find the ***exact*** distance between $(\sqrt{3k}, \sqrt{k})$ and $\left[\sqrt{\mathbf{k}}, \frac{5\pi}{6}\right]$. Draw a diagram.

$$OB = \sqrt{3\mathbf{k} + \mathbf{k}}$$
$$= 2\sqrt{\mathbf{k}}$$ ✓
$$\tan\theta = \frac{\sqrt{\mathbf{k}}}{\sqrt{3\mathbf{k}}}$$ ✓
$$= \frac{\sqrt{3}}{3}$$
$$\theta = \frac{\pi}{6}$$ ✓
$$\therefore \alpha = \frac{2\pi}{3}$$ ✓



$AB^2 = OA^2 + OB^2 – 2(OA)(OB)\cos\alpha$ ✓
$= k + 4k – 2(\sqrt{\mathbf{k}})(2\sqrt{\mathbf{k}})(-½)$
$= 7k$ ✓
$\therefore AB = \sqrt{7\mathbf{k}}$
∴ The distance between the points is $\sqrt{7\mathbf{k}}$ units. ✓

4. [3, 2 = 5 marks]

a) Find, in ***exact*** form, the modulus and principal argument of $-\sqrt{3} + i$, and hence rewrite $-\sqrt{3} + i$ in ***exact*** polar (***cis***) form.

Modulus = $\sqrt{3 + 1} = 2$ ✓

Principal argument = $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$ ✓

$\therefore\ -\sqrt{3} + i = 2 \text{ cis } \frac{5\pi}{6}$ ✓

b) Convert $2\ cis\left(\frac{\pi}{4}\right)$ into ***exact*** algebraic Cartesian/rectangular form.

$2 \text{ cis}\left(\frac{\pi}{4}\right) = 2\left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right]$ ✓

$= \sqrt{2} + i\sqrt{2}$ ✓

5. [3, 3 = 6 marks]

Evaluate, giving answers in ***exact*** form:

a) $4\ cis\frac{\pi}{3} \times 2\ cis\frac{3\pi}{4} = 8 \text{ cis}\left(\frac{\pi}{3} + \frac{3\pi}{4}\right)$ ✓

$= 8 \text{ cis } \frac{13\pi}{12}$ ✓

$= 8 \text{ cis}\left(-\frac{11\pi}{12}\right)$ ✓

b) $\dfrac{4\ cis\left(-\frac{5\pi}{6}\right)}{2\ cis\left(\frac{5\pi}{6}\right)} = 2 \text{ cis}\left(-\frac{5\pi}{6} - \frac{5\pi}{6}\right)$ ✓

$= 2 \text{ cis}\left(-\frac{5\pi}{3}\right)$ ✓

$= 2 \text{ cis}\left(\frac{\pi}{3}\right)$ ✓

6. [4 marks]

Given $z = 2\ cis\ \frac{\pi}{4}$, express z^{-1} and $\bar{z}$ in ***exact*** polar and rectangular form.

$$z^{-1} = \left[2\ \text{cis}\ \frac{\pi}{4} \right]^{-1}$$

$$= \tfrac{1}{2}\ \text{cis} \left(-\frac{\pi}{4} \right) \quad \leftarrow \text{Polar form} \ \checkmark$$

$$= \frac{\sqrt{2}}{4}(1 - \mathbf{i}) \quad \leftarrow \text{Rectangular form} \ \checkmark$$

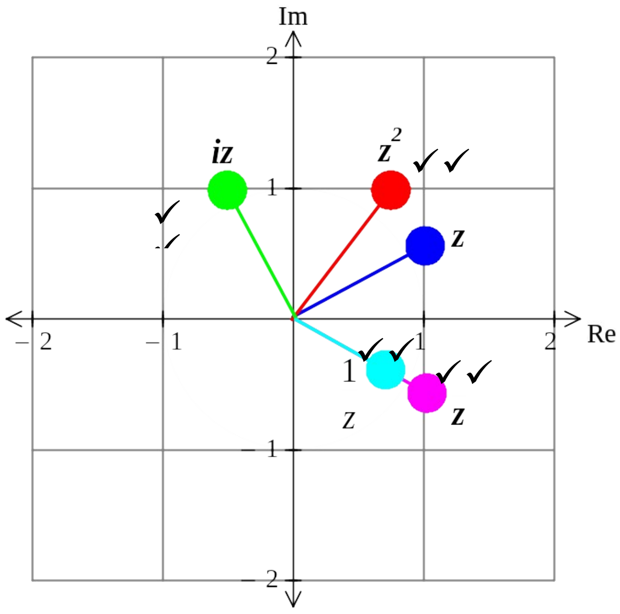
$$\bar{z} = 2\ \text{cis} \left(-\frac{\pi}{4} \right) \quad \leftarrow \text{Polar form} \ \checkmark$$

$$= \sqrt{2}(1 - \mathbf{i}) \quad \leftarrow \text{Rectangular form} \ \checkmark$$

7. [2, 2, 2, 2 = 8 marks]

The Argand diagram below shows the point representing the complex number z where $|\,z\,| > 1$. Plot on the same diagram, the points representing the complex numbers:

a) $\bar{z}$ b) iz c) z^2 d) $\frac{1}{z}$



If $z = 1 + \tfrac{1}{2}i$

$\Rightarrow \bar{z} = 1 - \tfrac{1}{2}i$

$iz = -\tfrac{1}{2} + i$

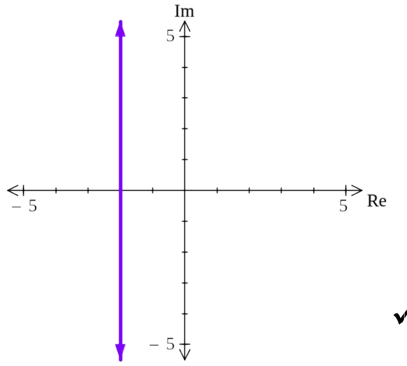
$z^2 = \tfrac{3}{4} + i$

$\frac{1}{z} = \frac{4}{5}(1 - \tfrac{1}{2}i)$

8. [3, 3, 3, 3 = 12 marks]

Sketch on an Argand diagram, the locus of the point $z = x + iy$, satisfying each of the following conditions. In each case, give the Cartesian equation or inequality of the locus.

a) Re(z) = -2 ⇒ x = -2 ✓



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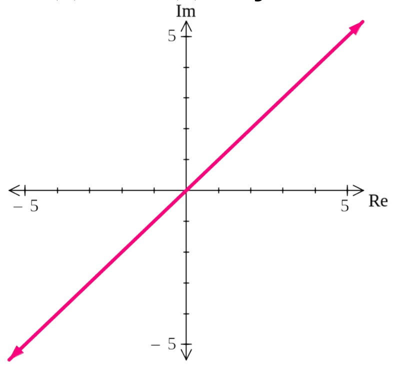
b) Im(z) = Re(z) ⇒ y = x ✓



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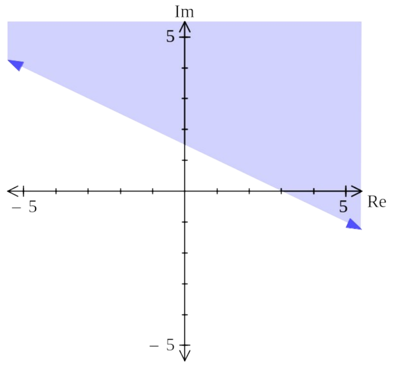
c) Re(z) + 2Im(z) > 3 ⇒ x + 2y > 3 ✓



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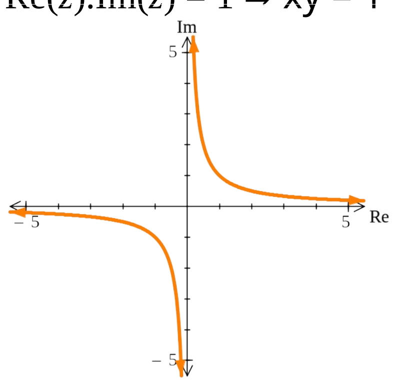
d) Re(z).Im(z) = 1 ⇒ xy = 1 ✓



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