Papers written by Australian Maths Software

YEAR 11

2016

REVISION 2

MATHEMATICS SPECIALIST

UNITS 1 & 2

SEMESTER TWO

SOLUTIONS

SECTION 1 – Calculator-free

Question 1 **(7 marks)**

(a) Any matrix in the form $\begin{bmatrix} a & b \\ ka & kb \end{bmatrix}$ or $\begin{bmatrix} ka & kb \\ a & b \end{bmatrix}$ or $\begin{bmatrix} a & kb \\ a & kb \end{bmatrix}$ or $\begin{bmatrix} ka & b \\ ka & b \end{bmatrix}$. ✓✓

(b) $x + y = 2$ and $2x - 3y = 11 = 0$

$$\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$$

$$-\frac{1}{5} \times \begin{bmatrix} -3 & -1 \\ -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{5} \times \begin{bmatrix} -3 & -1 \\ -2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 11 \end{bmatrix} \quad ✓$$

$$-\frac{1}{5} \times \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{5} \times \begin{bmatrix} -17 \\ 7 \end{bmatrix} \quad ✓$$

$$I \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.4 \\ -1.4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.4 \\ -1.4 \end{bmatrix} \quad ✓$$

The point of intersection of the two lines $x + y = 2$ and $2x - 3y = 11 = 0$ is (3.4, -1.4). ✓

(c) $(A+B)^2 = A^2 + 2AB + B^2$ is not valid if the two matrices are not commutative (which is usually the case). i.e. if $AB \neq BA$ ✓✓

Question 2 **(17 marks)**

(a) (i) $z_1 + z_2 - z_3 = 1 + i + 1 - i - (2 - 3i) = 3i$ ✓

(ii) $\dfrac{z_1 \times z_2}{z_3} = \dfrac{(1+i)\times(1-i)}{(2-3i)} \times \dfrac{(2+3i)}{(2+3i)} = \dfrac{2(2+3i)}{4+9} = \dfrac{2}{13}(2+3i)$ ✓ ✓

(iii) $\left| \dfrac{(z_1)^2}{\overline{z_2}} \right| = \left| \dfrac{(1+i)^2}{1+i} \right| = |1+i| = \sqrt{2}$ ✓ ✓

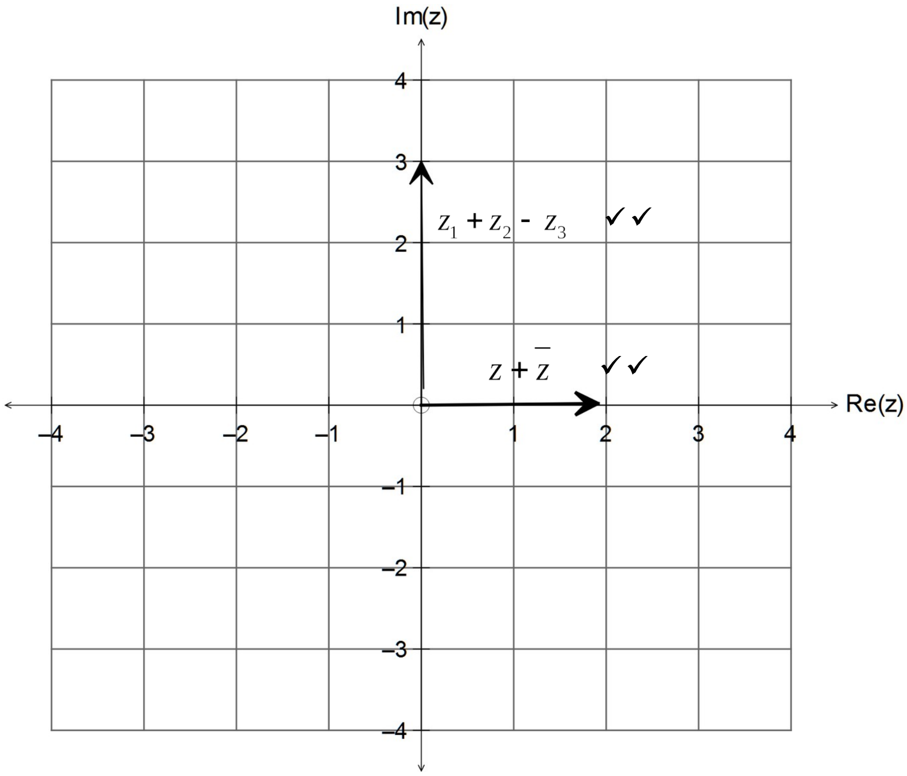
(iv) $(z_2)^2 + (z_1)^{-1} = (1-i)^2 + \dfrac{1}{(1-i)} \times \dfrac{(1+i)}{(1+i)}$ ✓

$$= \cancel{1} - 2i + \cancel{i^2} + \frac{(1+i)}{(1-i^2)} \quad ✓$$

$$= -2i + \frac{1+i}{2}$$

$$= \frac{1}{2} - \frac{3i}{2} \quad ✓$$

(b) (i)



(c) $\dfrac{(1-3i)^2}{2-i} = a + bi$

$$\frac{(1-3i)^2}{2-i} = \frac{1-6i+9i^2}{2-i} \times \frac{2+i}{2+i} \quad ✓$$

$$= \frac{(-8-6i)(2+i)}{4-i^2} \quad ✓$$

$$= \frac{1}{5}\left(-16-12i-8i-6i^2\right)$$

$$= \frac{-10-20i}{5}$$

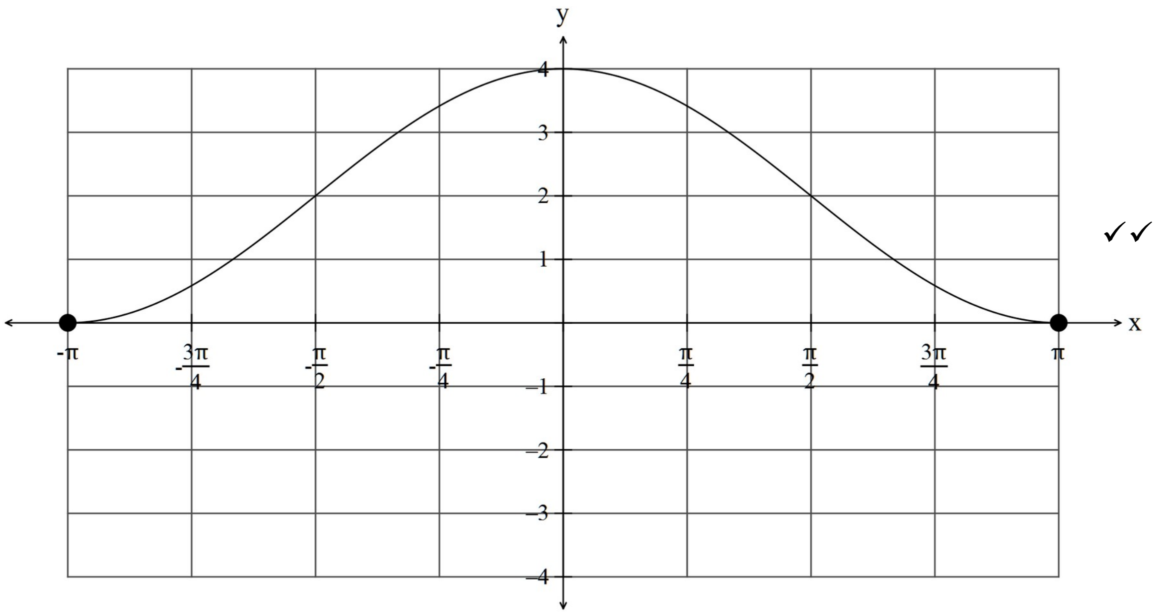
$$= -2-4i$$

$$a = -2, \qquad b = -4$$

✓ ✓

Question 3 **(5 marks)**

(a) $y = 2 + 2\cos(x)$



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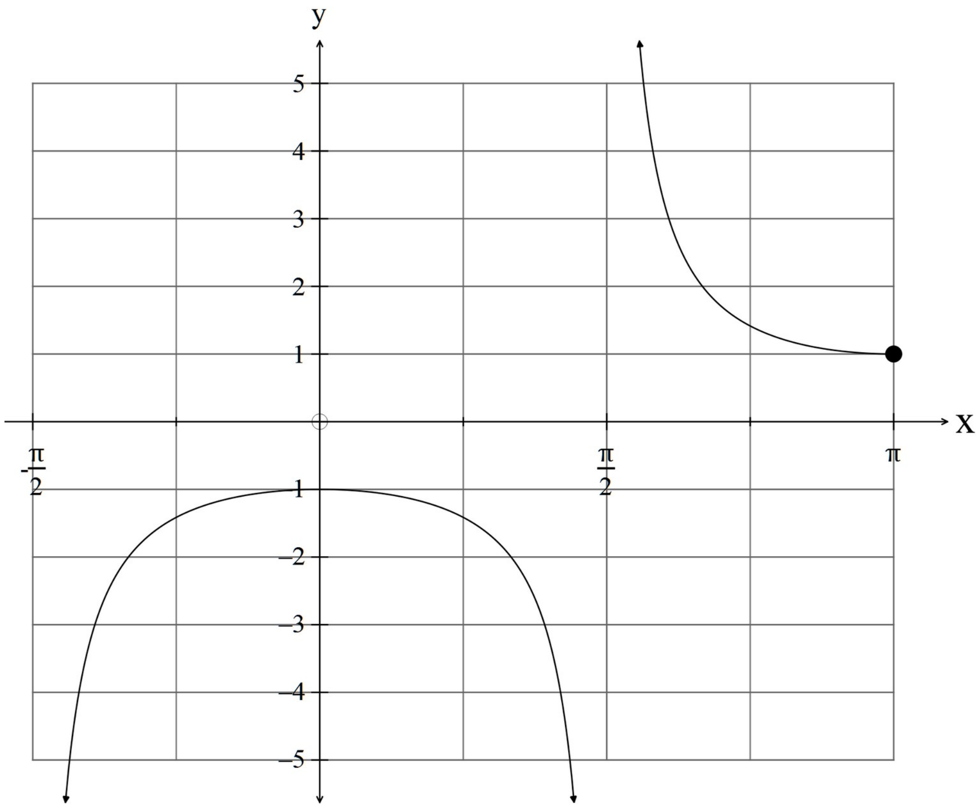
(b) $y = -\sec(x)$



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Question 4 **(6 marks)**

(a) $\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)$

Let $A=x, B=2x$

$\sin(x+2x)=\sin(x)\cos(2x)+\cos(x)\sin(2x)$

$\sin(3x)=\sin(x)\left(1-2\sin^2(x)\right)+\cos(x)\left(2\sin(x)\cos(x)\right)$ ✓ ✓

$\sin(3x)=\sin(x)-2\sin^3(x)+2\sin(x)\left(1-\sin^2(x)\right)$

$\sin(3x)=\sin(x)-2\sin^3(x)+2\sin(x)-2\sin^3(x)$

$\sin(3x)=3\sin(x)-4\sin^3(x)$ ✓

(b) Solve $\sin\left(60^o+\theta\right)-\sin\left(60^o-\theta\right)=\frac{1}{2}$ for $0^o\le\theta\le 180^o$.

$\sin\left(60^o+\theta\right)-\sin\left(60^o-\theta\right)=\cancel{\sin\left(60^o\right)\cos(\theta)}+\cos\left(60^o\right)\sin(\theta)-\left(\cancel{\sin\left(60^o\right)\cos(\theta)}-\cos\left(60^o\right)\sin(\theta)\right)$ ✓

$=2\cos\left(60^o\right)\sin(\theta)$

$=\sin(\theta)$ ✓

$\sin\left(60^o+\theta\right)-\sin\left(60^o-\theta\right)=\frac{1}{2}$

$\sin(\theta)=\frac{1}{2}$

$\theta=30^o$ or $\theta=150^o$

✓

Question 5 **(9 marks)**

Given the points $A(2,2)$, $B(5,5)$, $C(-4,4)$, $D(0,8)$ and $E(3,5)$.

(a) (i) $\mathbf{AB}=\begin{pmatrix}3\\3\end{pmatrix}$ $\therefore|\mathbf{AB}|=\sqrt{9+9}=3\sqrt{2}$ ✓

(ii) $\mathbf{AB}=\begin{pmatrix}3\\3\end{pmatrix}$, $\mathbf{CD}=\begin{pmatrix}4\\4\end{pmatrix}=\frac{4}{3}\mathbf{AB}$

$\therefore\mathbf{AB}\,||\,\mathbf{CD}$ ✓

(iii) $\mathbf{AC}=\begin{pmatrix}-6\\2\end{pmatrix}$, $\mathbf{AE}=\begin{pmatrix}1\\3\end{pmatrix}$

$\mathbf{AC}\cdot\mathbf{AE}=-6+6=0$

$|\mathbf{AC}|\ne 0,\ |\mathbf{AE}|\ne 0,\ \therefore\cos(\theta)=0$

$\therefore\ \theta=\frac{\pi}{2}$ ✓

(b) (i) $\mathbf{AB} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ $E(3,5)$

$\mathbf{r}(t) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

$\mathbf{r}(t) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - 1\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$P_1(6,8), P_1(0,2)$
✓ ✓

(ii) $\mathbf{AB} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ $\mathbf{AE} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\cos\angle BAE = \frac{\mathbf{AB}\cdot\mathbf{AE}}{|\mathbf{AB}||\mathbf{AE}|}$ ✓

$\cos\angle BAE = \frac{3+9}{\sqrt{18}\sqrt{10}} = \frac{12}{6\sqrt{5}}$

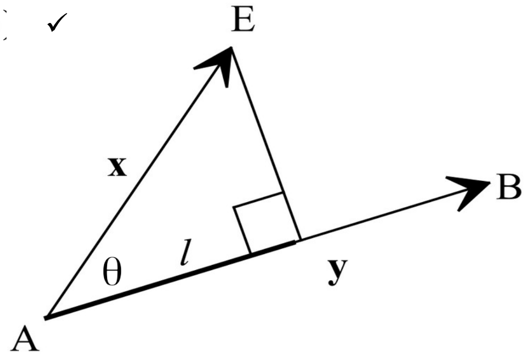
$\cos\angle BAE = \frac{2}{\sqrt{5}}$

✓

(iii) $\cos(\theta) = \frac{l}{|\boldsymbol{x}|} \Rightarrow l = |\boldsymbol{x}|\cos(\theta)$ ✓

$l = \sqrt{10} \times \frac{2}{\sqrt{5}}$

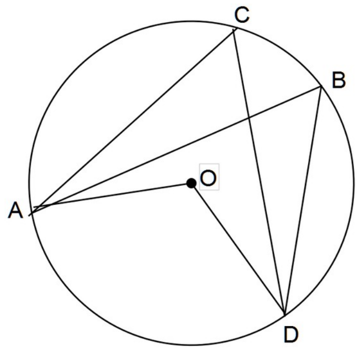
$l = 2\sqrt{2}$ ✓



Question 6 **(8 marks)**

(a) Deductive reasoning is when a conclusion follows as a direct consequence of the previous statements. ✓

(b) Prove that "*Angles at the circumference of a circle subtended by the same arc are equal*". (3)



Join AO, DO.

a) $\angle AOD = 2\angle ACD$
b) $\angle AOD = 2\angle ABD$ ✓
An angle at the circumference of a circle is half the angle at the centre subtended by the same arc. ✓ for reason

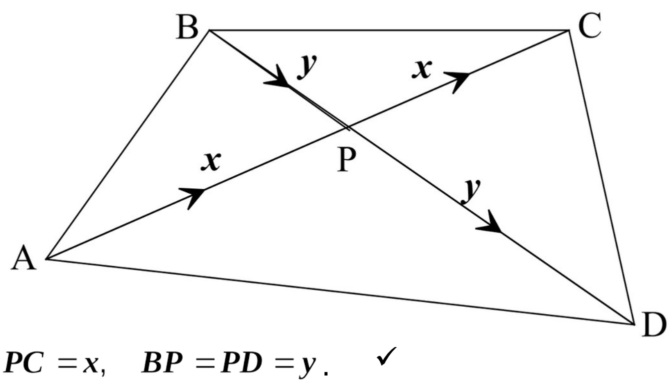
$\therefore \angle ACD = \angle ABD$ ✓

Therefore
"*Angles at the circumference of a circle subtended by the same arc are equal*".

(c) Prove the following using vectors:

"*If the diagonals of a quadrilateral bisect each other,*

then the quadrilateral is a parallelogram."



Let $\boldsymbol{AP} = \boldsymbol{PC} = \boldsymbol{x}, \quad \boldsymbol{BP} = \boldsymbol{PD} = \boldsymbol{y}$. ✓
$\boldsymbol{AB} = \boldsymbol{x} - \boldsymbol{y}, \quad \boldsymbol{DC} = -\boldsymbol{y} + \boldsymbol{x}$ ✓
$\therefore \ \boldsymbol{AB} = \boldsymbol{DC}$ ✓
$\boldsymbol{BC} = \boldsymbol{y} + \boldsymbol{x}, \quad \boldsymbol{AD} = \boldsymbol{x} + \boldsymbol{y}$
$\therefore \ \boldsymbol{BC} = \boldsymbol{DC}$
Two pairs of equal and parallel sides as equal vectors. (One is enough!)
Therefore ABCD is a parallelogram. ✓ for reason

SECTION 2 – Calculator-assumed

Question 7 **(7 marks)**

(a) If you pick five cards from a standard deck of 52 cards, then at least two will be of the same suit. ✓

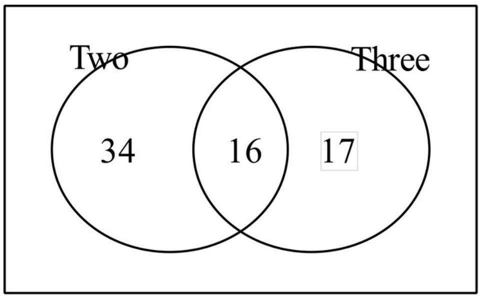
Each of the first four cards can belong to four different suits. The remaining card must be the same suit as one of the other four. ✓

(b) (i) ${}^{20}C_6 = 38760$ ✓

(ii) ${}^{10}C_6 = 210$ ✓

(c) $1,2,3,....100$

$n(2) = 50 \quad n(3) = 33 \quad n(6) = 16$



✓✓

$34 + 17 = 51$ ✓

Therefore 51 of the counting numbers from 1 to 100 have both a factor of 2 or 3 but not a factor of 6.

Question 8 **(9 marks)**

(a) $\dfrac{\sqrt{(1-3i)^2(1+3i)^2}}{2i}$

$= \dfrac{\sqrt{(1-9i^2)^2}}{2i} \times \dfrac{i}{i}$ ✓

$= \dfrac{10i}{-2}$ ✓

$= -5i$ ✓

(b) $u = 1 + i$ and $v = 1 - 2i$

(i) $$\frac{(u+v)v}{u} = \frac{(1+i+1-2i)(1-2i)}{(1+i)}$$

$$= \frac{(2-i)(1-2i)}{(1+i)} \times \frac{(1-i)}{(1-i)}$$

$$= \frac{(-5i)(1-i)}{2} \quad ✓$$

$$= \frac{-5-5i}{2} \quad ✓$$

(ii) $$\frac{u}{v} + \frac{v}{u} = \frac{u^2+v^2}{uv} \quad ✓$$

$$= \frac{(1+i)^2 + (1-2i)^2}{(1+i)(1-2i)}$$

$$= \frac{2i+1-4i-4}{3-i} \times \frac{3+i}{3+i} \quad ✓$$

$$= \frac{(3-2i)(3+i)}{10}$$

$$= \frac{-7-9i}{10} \quad ✓$$

Question 9 **(4 marks)**

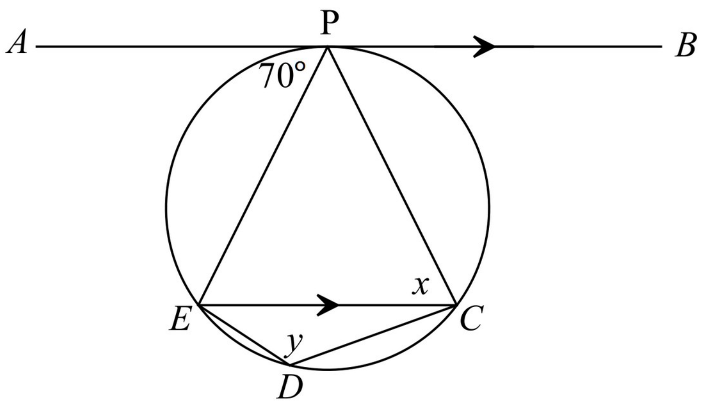
(a) $$\begin{pmatrix} 3 \\ 8 \end{pmatrix} + \begin{pmatrix} 0 \\ 10 \end{pmatrix} + \begin{pmatrix} -2 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 25 \end{pmatrix} \quad ✓✓$$

(b) Magnitude $\sqrt{1^2 + 25^2} = \sqrt{126} = 25.02$ ✓

Direction $tan^{-1}\left(\frac{25}{1}\right) = 87.7^o$ i.e. direction is 088^o. ✓

Question 10 **(6 marks)**

(a) Solve for x and y



✓ $x = 70^o$ An angle between a chord and a tangent is equal to the angle in the alternate segment.

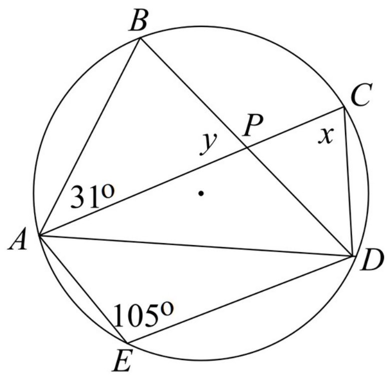
$AB \parallel EC$

$\angle PEC = 70^o$ Alternate angles ✓ Reasons

$\angle EPC = 40^o$ Angles in a triangle add to 180^o

✓ $y = 140^o$ The opposite angles of a cyclic quadrilateral are supplementary

(b) Solve for x and y



$\angle ABP = 75^o$ The opposite angles of a cyclic quadrilateral are supplementary

$\angle ACD = 75^o = x$ Two angles at the circumference subtended by the same arc are equal.

$y = 74^o$ Angles in a triangle add to 180^o ✓ Reasons

$x = 75^o$ $y = 74^o$
✓ ✓

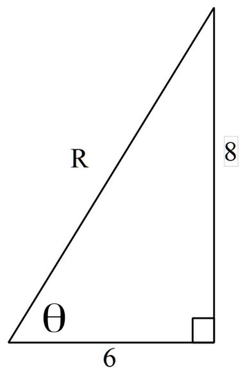
Question 11 **(13 marks)**

(a) (i) $R\cos(x+\theta) = \underline{6\cos(x)} - \underline{\underline{8\sin(x)}}$

$R\cos(x+\theta) = R\underline{\cos(x)}\cos(\theta) - R\underline{\underline{\sin(x)}}\sin(\theta)$ ✓

$\therefore R\cos(\theta) = 6 \quad R\sin(\theta) = 8$

$\cos(\theta) = \dfrac{6}{R} \qquad \sin(\theta) = \dfrac{8}{R}$ ✓



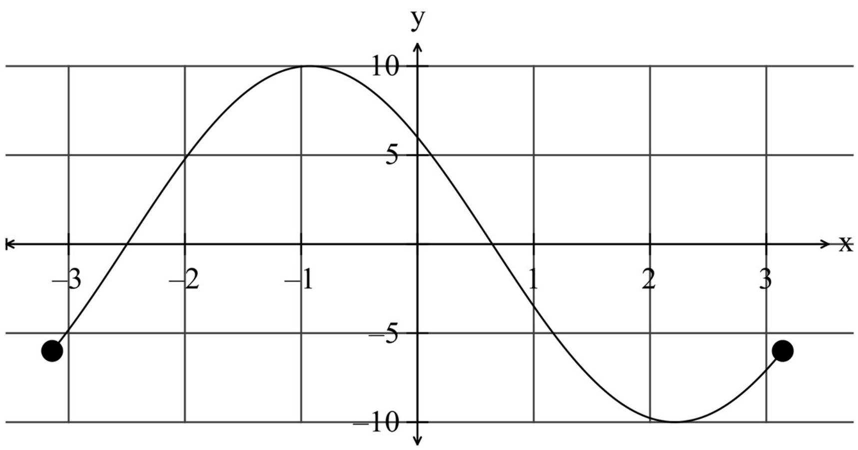
Pythagoras $R^2 = 6^2 + 8^2 \rightarrow R = 10$

$\tan(\theta) = \dfrac{8}{6}$

$\theta = 0.927$

$\therefore R = 10 \quad and \quad \theta = 0.927$
✓ ✓

(ii) $y = 6\cos(x) - 8\sin(x) = 10\cos(x + 0.927)$



✓✓

(b) $y = 1 - 3\sin\left(x + \dfrac{\pi}{4}\right)$
✓ ✓ ✓

(c) $\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{b} = -3\boldsymbol{i} + 4\boldsymbol{j}$

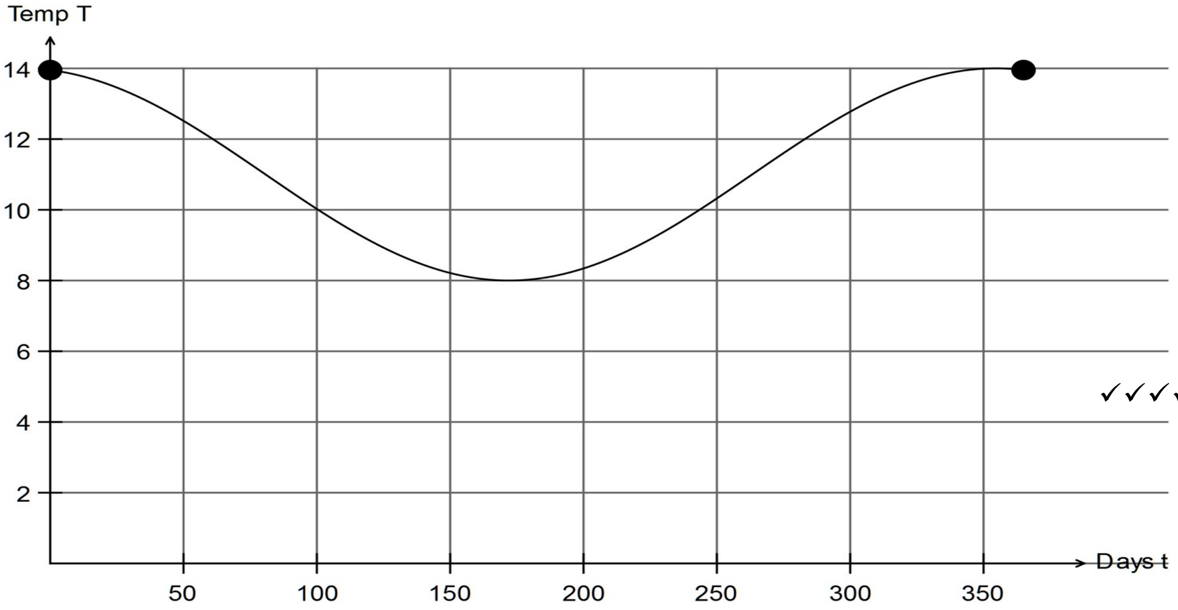
(i) $\boldsymbol{a} \bullet \boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 4 \end{pmatrix} = -6 + 4 = -2$. ✓

(ii) $2\boldsymbol{a} - 3\boldsymbol{b} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 3(-3\boldsymbol{i} + 4\boldsymbol{j}) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 9 \\ -12 \end{pmatrix} = \begin{pmatrix} 13 \\ -10 \end{pmatrix}$ ✓✓

(iii) $|\boldsymbol{b}| = |-3\boldsymbol{i} + 4\boldsymbol{j}| = 5 \quad \therefore \pm(-6\boldsymbol{i} + 8\boldsymbol{j})$
✓ ✓

Question 12 **(8 marks)**

(a)



✓✓✓✓

(b) January 31

February 28

March 31 90 ✓

At t = 90, L = ? L = 10.524 hours of daylight on March 31st ✓

(c) $12 = 11 + 3\sin\left(\frac{2\pi}{365}(t - 263.25)\right)$

Intersection at t = 61 and t = 283 ✓

$\frac{61 + (365 - 283)}{365} \times 100 = 39.2\%$ ✓

About 39% of days have

Question 13 **(3 marks)**

Prove $\left(\tan(x)+\sec(x)\right)^2 = \dfrac{1+\sin(x)}{1-\sin(x)}$

$$\left(\tan(x)+\sec(x)\right)^2 = \left(\frac{\sin(x)}{\cos(x)}+\frac{1}{\cos(x)}\right)^2 \quad ✓$$

$$= \frac{\left(1+\sin(x)\right)^2}{\cos^2(x)}$$

$$= \frac{\left(1+\sin(x)\right)\left(1+\sin(x)\right)}{1-\sin^2(x)} \quad ✓$$

$$= \frac{\left(1+\sin(x)\right)\cancel{\left(1+\sin(x)\right)}}{\left(1-\sin(x)\right)\cancel{\left(1+\sin(x)\right)}} \quad ✓$$

$$= \frac{1+\sin(x)}{1-\sin(x)}$$

Question 14 **(10 marks)**

(a) $z^2-2z+5=0$.

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

✓ $$z=\frac{2\pm\sqrt{4-20}}{2}$$

$$z=\frac{2\pm\sqrt{-16}}{2} \quad ✓$$

✓ ✓ $$z=\frac{2\pm4i}{2} \quad where\ \sqrt{-1}=\sqrt{i^2}=i$$

$$z=1\pm2i$$

✓ ✓✓

(b) $z^3 - 3z^2 + 5z - 3 = 0$

$Let \quad P(z) = z^3 - 3z^2 + 5z - 3$

$P1 = 1 - 3 + 5 - 3 = 0$

$\therefore$ 1 is a root ✓

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 5 & -3 \\ & \downarrow & 1 & -2 & 3 \\ \hline & 1 & -2 & 3 & 0 \end{array}$$

$P(z) = (z-1)(z^2 - 2z + 3)$ ✓

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{2 \pm \sqrt{4 - 12}}{2}$$

✓✓ $$z = \frac{2 \pm \sqrt{8i^2}}{2}$$

✓ $$z = \frac{2 \pm i2\sqrt{2}}{2}$$

$z = 1 \pm \sqrt{2}\, i$ ✓

(c) (i) Given $z = a + bi$, prove that $z \times \bar{z} = |z|^2$.

$z \times \bar{z} = (a + bi)(a - bi)$ ✓

$= a^2 - b^2 i^2$

$= a^2 + b^2$ ✓

$= \left(\sqrt{a^2 + b^2}\right)^2$ ✓

$= |z|^2$

(ii) $z \times \bar{z} = |z|^2 = 8^2 + (-15)^2 = 17^2 = 289$ ✓

Question 15 **(9 marks)**

(a) "*If Susan does not buy some avocados, then they do not cost less than* \$4." ✓

(b) The converse is "*If Terry sits outside with his coffee, then today is a sunny day*." ✓

(c) Prove $\binom{n}{n}+\binom{n}{n-1}+\binom{n}{n-2}+.....+\binom{n}{1}+\binom{n}{0}=2^n$

$$(a+b)^n=\binom{n}{n}a^n+\binom{n}{n-1}a^{n-1}b+\binom{n}{n-2}a^{n-2}b^2+.....+\binom{n}{1}a^1b^{n-1}+\binom{n}{0}b^n \quad ✓ ✓$$

$Let \ \ a=b=1$

$$2^n=\binom{n}{n}1^n+\binom{n}{n-1}1^{n-1}1+\binom{n}{n-2}1^{n-2}1^2+.....+\binom{n}{1}1^11^{n-1}+\binom{n}{0}1^n \quad ✓$$

$$\therefore \ \ 2^n=\binom{n}{n}+\binom{n}{n-1}+\binom{n}{n-2}+.....+\binom{n}{1}+\binom{n}{0}$$

(d) Use the method of proof by contradiction to prove that there are no positive integer solutions to the Diophantine equation $x^2-y^2=1$.
NB A Diophantine equation is an equation for which you seek integer solutions.

. Assume that there is a solution (x,y) where x and y are <u>positive</u> integers. ✓

Factorise the left side: $x^2-y^2=(x-y)(x+y)=1$

Since x and y are integers, it follows that either
$x-y=1$ and $x+y=1$ or $x-y=-1$ and $x+y=-1$.

If $x-y=1$ and $x+y=1$ then we can add the two equations to get x = 1 and y = 0, contradicting our assumption that x and y are positive. ✓

If $x-y=-1$ and $x+y=-1$, we get x = -1 and y = 0, again contradicting our assumption. ✓

Therefore our assumption that there is a solution (x, y) where x and y are positive integers is invalid. ✓ ✓

Therefore there are no positive integer solutions to the Diophantine equation $x^2-y^2=1$.

Question 16 **(13 marks)**

ABCD is a parallelogram. A linear transformation given by $(x, y) \to (x, x - y)$ that

(a) (i) $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ ✓✓

(ii) B(2, 0) C(3, 1)

✓

A(0, 0) D(1, 1)

$m_{AD} = 1$ $m_{BC} = 1$

$m_{AB} = 0$ $m_{DC} = 0$

Two pairs of parallel sides so parallelogram. ✓

(iii) $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{-1}\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ ✓

$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x - y \end{pmatrix}$

✓ Yes, A'B'C'D' can be transformed back to ABCD using $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

(b) (i) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} cos(120^o) & -sin(120^o) \\ sin(120^o) & cos(120^o) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$

✓ ✓ ✓

(ii) $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1.5 + \sqrt{3} \\ 1.5\sqrt{3} - 1 \end{pmatrix}$ ✓ ✓

$\therefore \; C'\left(1.5 + \sqrt{3},\ 1.5\sqrt{3} - 1\right)$

(iii) $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}^{-1} = \frac{1}{2} \times \frac{1}{-1-3}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

✓✓ -1/error

Question 17 **(8 marks)**

It is known that the matrix M performs a reflection about the line $y = mx$

where $M = \begin{bmatrix} p & q \\ q & -p \end{bmatrix}$, $p = \dfrac{1-m^2}{1+m^2}$, $q = \dfrac{2m}{1+m^2}$ and m is the gradient of the line.

(a) $p = \dfrac{1-9}{1+9} = -0.8 \qquad q = \dfrac{6}{1+9} = 0.6$ ✓

$M = \begin{pmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}$ ✓

(b) $\begin{pmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -0.8a + 0.6b \\ 0.6a + 0.8b \end{pmatrix}$ ✓

$\therefore\ P'(-0.8a + 0.6b,\ 0.6a + 0.8b)$

(c) Let M be the matrix that reflects a set of points about a line that contains the origin.

$$M = \begin{pmatrix} p & q \\ q & -p \end{pmatrix}$$

$$M^2 = \begin{pmatrix} p & q \\ q & -p \end{pmatrix} \times \begin{pmatrix} p & q \\ q & -p \end{pmatrix} = \begin{pmatrix} p^2 + q^2 & pq - pq \\ pq - pq & q^2 + p^2 \end{pmatrix} = \begin{pmatrix} p^2 + q^2 & 0 \\ 0 & p^2 + q^2 \end{pmatrix}$$

$$p^2 + q^2 = \left(\frac{1-m^2}{1+m^2}\right)^2 + \left(\frac{2m}{1+m^2}\right)^2$$

$$= \frac{1}{(1+m^2)^2}\left((1-m^2)^2 + (2m)^2\right)$$

$$= \frac{1}{(1+m^2)^2}\left(1 - 2m^2 + m^4 + 4m^2\right)$$

$$= \frac{1}{(1+m^2)^2}\left(1 + 2m^2 + m^4\right)$$

$$= \frac{(1+m^2)^2}{(1+m^2)^2}$$

$$p^2 + q^2 = 1$$

Therefore $M^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so the points end back where it started.

i.e. a reflection of a reflection is itself (about a line containing the origin) .

Question 18 **(8 marks)**

(a) $x^2 - x - 1 - 0$

$\Delta = b^2 - 4ac$

$\Delta = (-1)^2 - 4(1)(-1)$

$\Delta = 5$

The roots are irrational. ✓

(b) Prove using mathematical induction, that $4^{n+1} + 5^{2n-1}$ is divisible by 21

Test for $n = 1$.

$4^{1+1} + 5^{2-1} = 16 + 5 = 21$

So divisible by 21 for $n = 1$. ✓

Assume valid for n i.e. that $4^{n+1} + 5^{2n-1} = 21k$ ✓

Test for $"n" = "n+1"$

$4^{n+1} + 5^{2n-1} \rightarrow 4^{n+1+1} + 5^{2(n+1)-1}$ ✓

$$5^{2n-1} = 21k - 4^{n+1}$$

$$4^{n+2} + 5^{2n+1} = 4^{n+2} + 5^2 \times 5^{2n-1}$$

$= 4^{n+2} + 25\left(21k - 4^{n+1}\right)$ ✓

$= 25 \times 21k + 4^{n+2} - 25 \times 4^{n+1}$

$= 21(25k) + 4^{n+1}(4 - 25)$ ✓

$= 21(25k) - 4^{n+1}(21)$

$= 21\left(25k - 4^{n+1}\right)$ which is a multiple of 21. ✓

So, if valid for *n*, then valid for *n*+1.

✓

Valid for *n* = 1, so valid for *n* = 2 etc.

Therefore $4^{n+1} + 5^{2n-1}$ is divisible by 21.

End of solutions