Penrhos College
Semester 2 Examination, 2011

Question/Answer Booklet

MATHEMATICS SPECIALIST: 3C/3DMAS

Section One: Calculator-free

Student Name:___SOLUTIONS____

Time allowed for this section

Reading time before commencing work: 5 minutes
Working time for this section: 50 minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be used in this section of the examination. It is your responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor before reading any further.

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Instructions to candidates

1. **All** questions should be attempted.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare answer pages may be found at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued (i.e. give the page number).

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

Structure of this paper

Questions	Marks available	Your score
1	2	
2	6	
3	5	
4	6	
5	4	
6	7	
7	5	
8	5	
Total:	**40**	
9	4	
10	9	
11	6	
12	8	
13	6	
14	5	
15	5	
16	6	
17	6	
18	10	
19	5	
20	10	
Total:	**80**	
Total marks = 120		
		%

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See next page

Section One: Calculator-free **(40 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the space provided.

Suggested working time for this section is 50 minutes.

Question 1 **(2 marks)**

Given the matrix $\boldsymbol{A} = \begin{bmatrix} x & 12 \\ 3 & 9 \end{bmatrix}$ find all values of *x* such that the matrix is singular.

Solution
For $\boldsymbol{A}^{-1}$ to exist it cannot be singular. Hence $\lvert \boldsymbol{A} \rvert \neq 0$ $9x - 36 \neq 0$ $x \neq 4$
Specific behaviours
✓ determines an expression for the determinant equal to 0 ✓solves correctly x

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Question 2 **(6 marks)**

If $z_1 = 2 + 2\sqrt{3}i$ and $z_2 = 6cis\left(-\frac{\pi}{3}\right)$, determine in simplest form:

(a) iz_1

[1]

Solution
$iz_1 = -2\sqrt{3} + 2i$
Specific behaviours
✓correct value for iz_1

(b) $\frac{1}{z_2}$

[1]

Solution
$\frac{1}{z_2} = \frac{1}{6}cis\left(\frac{2\pi}{3}\right)$
Specific behaviours
✓ correct value for $\frac{1}{z_2}$

(c) $z_1 z_2$

[2]

Solution
$z_1 z_2 = 4cis\left(\frac{\pi}{3}\right).6cis\left(\frac{-\pi}{3}\right)$ $= 24$
Specific behaviours
✓ convert z_1 to polar form ✓correct value for $z_1 z_2$

(d) $z_1 + z_2$

[2]

Solution
$z_1 + z_2 = 2 + 2\sqrt{3}i + 3 - 3\sqrt{3}i$ $= 5 - \sqrt{3}i$
Specific behaviours
✓ convert z_2 to Cartesian form ✓ correct value of $z_1 + z_2$

Question 3 **(5 marks)**

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(a) Prove the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

[3]

Solution
LHS = $\cos(\theta + 2\theta)$ $= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta$ $= \cos\theta\left(2\cos^2\theta - 1\right) - 2\sin^2\theta\cos\theta$ $= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$ $= 4\cos^3\theta - 3\cos\theta$ = RHS
Specific behaviours
✓uses compound angle identity ✓uses $2\cos^2\theta - 1 = \cos 2\theta$ identity ✓uses $2\sin\theta\cos\theta = \sin 2\theta$ identity

(b) Hence or otherwise determine the indefinite integral $\int 2\cos^3\theta\, d\theta$

[2]

Solution
$\int 2\cos^3\theta\, d\theta = \frac{1}{2}\int (\cos 3\theta + 3\cos\theta)\, d\theta$ $= \frac{1}{2}\left(\frac{\sin 3\theta}{3} + (3\sin\theta)\right) + c$ $= \frac{\sin 3\theta}{6} + \frac{3\sin\theta}{2} + c$
Specific behaviours
✓rewrite $\int 2\cos^3\theta\, d\theta$ to the form $\frac{1}{2}\int(\cos 3\theta + 3\cos\theta)d\theta$ ✓integrates expression correctly

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Question 4 **(6 marks)**

Determine the following integrals, writing your answers in simplified form.

(a) $\int \cos 2t \sin^5 2t \ dt$

[2]

Solution
$\int \cos 2t \sin^5 2t$ $= \dfrac{\sin^6 2t}{6} + c$
Specific behaviours
✓ applies the chain rule ✓ correct solution including the constant

(b) $\int \dfrac{4 + 4\cos x}{x + \sin x} \, dx$

[2]

Solution
$\int \dfrac{4 + 4\cos x}{x + \sin x} \, dx$ $= 4\ln\lvert x + \sin x\rvert + c$
Specific behaviours
✓ recognizes $\dfrac{f'(x)}{f(x)}$ ✓ correct solution

(c) $\int_1^e \dfrac{(1 + \ln x)^2}{x} dx$

[2]

Solution
$\int_1^e \dfrac{(1+\ln x)^2}{x} \, dx$ $= \left[\dfrac{(1+\ln x)^3}{3}\right]_1^e$ $= \dfrac{7}{3}$
Specific behaviours
✓ correct antiderivative ✓ correct answer

Question 5 **(4 marks)**

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Prove $\left(1-i\sqrt{3}\right)^n+\left(1+i\sqrt{3}\right)^n=2^{n+1}\cos\left(\frac{\pi n}{3}\right)$ where n = 1, 2, 3, …

Solution
$\text{LHS}=\left[2cis\left(-\frac{\pi}{3}\right)\right]^n+\left[2cis\left(\frac{\pi}{3}\right)\right]^n$ $=2^n cis\left(-\frac{\pi n}{3}\right)+2^n cis\left(\frac{\pi n}{3}\right)$ $=2^n\left[\cos\left(-\frac{\pi n}{3}\right)+i\sin\left(-\frac{\pi n}{3}\right)+\cos\left(\frac{\pi n}{3}\right)+i\sin\left(\frac{\pi n}{3}\right)\right]$ $=2^n\left[\cos\left(\frac{\pi n}{3}\right)-i\sin\left(\frac{\pi n}{3}\right)+\cos\left(\frac{\pi n}{3}\right)+i\sin\left(\frac{\pi n}{3}\right)\right]$ $=2^n\left[2\cos\left(\frac{\pi n}{3}\right)\right]$ $=2^{n+1}\cos\left(\frac{\pi n}{3}\right)$ $=\text{RHS}$
Specific behaviours
✓expresses complex numbers on LHS in polar form ✓applies de Moivre's theorem ✓uses sin θ = -sin θ and cos θ = cos $(-\theta)$ ✓establish correct conclusion

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Question 6 **(7 marks)**

Given the curve $\sin(xy) + y^2 - \left(\frac{4}{\pi}\right)x = \frac{4}{\pi}$ find:

(a) $\frac{dy}{dx}$

[3]

Solution
$\cos(xy)y + \cos(xy)x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{4}{\pi} = 0$ $\frac{dy}{dx}(\cos(xy)x + 2y) = \frac{4}{\pi} - \cos(xy)y$ $\frac{dy}{dx} = \frac{\frac{4}{\pi} - \cos(xy)y}{\cos(xy)x + 2y}$
Specific behaviours
✓ correctly differentiates implicitly   ✓ correctly applies product rule   ✓correctly rearranges equation for $\frac{dy}{dx}$

(b) the value of x when $y = 0$.

[1]

Solution
$y = 0 \quad \sin(x(0)) + 0^2 - \left(\frac{4}{\pi}\right)x = \frac{4}{\pi}$ $-\left(\frac{4}{\pi}\right)x = \frac{4}{\pi}$ $x = -1$
Specific behaviours
✓ correct value of x

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(c) using the incremental formula, find the approximate change in *x* when *y* changes from 0 to 0.1. Give your answer in exact form.

[3]

Solution
$\delta x \approx \frac{dx}{dy}\delta y$ $\approx \frac{\cos(xy)x + 2y}{\frac{4}{\pi} - \cos(xy)y}\delta y$ $\approx \frac{\cos(-1\times 0)(-1) + 2(0)}{\frac{4}{\pi} - \cos(-1\times 0)(0)}0.1$ $= -\frac{\pi}{40}$
Specific behaviours
✓ chooses increment formula in correct form correct substitutions for x, y and δy ✓ correct value of δx

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Question 7 **(5 marks)**

A particle in simple harmonic motion has acceleration $\ddot{x}$ such that $\ddot{x}+\pi^2 x=0$ where x is the displacement from the origin O.
The particle is instantaneously at rest at time $t=0$ seconds and at position $x=4$.

(a) Find the displacement of x as a function of t.

[3]

Solution
Let $x=a\cos(nt+\alpha)$ Since $\ddot{x}=-\pi x^2$ $n=\pi$ $x=a\cos(\pi t+\alpha)$ $\dot{x}=-a\pi\sin(\pi t+\alpha)$ When $t=0$ $\dot{x}=0$ $0=-a\pi\sin(\alpha)$ $\alpha=0$ When $t=0$ $x=4$ $4=a\cos(0)$ $a=4$ $\therefore x=4\cos(\pi t)$
Specific behaviours
✓correct period ✓correct amplitude ✓correctly states displacement of x as a function of t

(b) Find the maximum velocity of the particle as an exact value.

[2]

Solution
$\dot{x}=-4\pi\sin(\pi t)$ Maximum velocity occurs at t=0.5, 1.5, 2.5, etc $\therefore$ maximum velocity is 4m/s
Specific behaviours
✓correctly differentiates ✓correct maximum velocity as an exact value

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Question 8 **(5 marks)**

Determine $\int_0^{\ln 3} (4e^{2x})\sqrt[3]{e^{2x} - 1}\ dx$ Let $u = e^{2x} - 1$

Solution	
$\int_0^{\ln 3} (4e^{2x})\sqrt[3]{e^{2x} - 1}\ dx$ $= \int_0^8 2u^{\frac{1}{3}}\ du$ $= \left[\frac{3u^{\frac{4}{3}}}{2}\right]_0^8$ $= 24$	$u = e^{2x} - 1$ $dx = \frac{du}{2e^{2x}}$ $x = 0 \quad u = 0$ $x = \ln 3 \quad u = 8$
Specific behaviours	
✓ Substitutes for *x* ✓ Substitutes for *dx* ✓ Changes upper and lower limits ✓ Simplifies and finds antiderivative ✓ Evaluates	

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End of questions

Additional working space

Question number(s):_______________

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Additional working space

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