PERTH MODERN SCHOOL Exceptional schooling. Exceptional students. Independent Public School	Year 12 Specialist TEST 1 Friday 9 February 2018 TIME: 5 mins reading 40 minutes working Classpads **allowed!** 37 marks 7 Questions

Name: SOLUTIONS

Teacher: ____________________

Note: All part questions worth more than 2 marks require working to obtain full marks.

Some useful Formulae

Cartesian form	
$z = a + bi$	$\bar{z} = a - bi$
$\text{Mod}(z) = \lvert z\rvert = \sqrt{a^2+b^2} = r$	$\text{Arg}(z) = \theta,\quad \tan\theta = \frac{b}{a},\quad -\pi < \theta \le \pi$
$\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$	$\left\lvert\frac{z_1}{z_2}\right\rvert = \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$
$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$	$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
$z\bar{z} = \lvert z\rvert^2$	$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{\lvert z\rvert^2}$
$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$	$\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$
Polar form	
$z = a + bi = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$	$\bar{z} = r\,\text{cis}(-\theta)$
$z_1 z_2 = r_1 r_2\,\text{cis}(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\text{cis}(\theta_1 - \theta_2)$
$\text{cis}(\theta_1 + \theta_2) = \text{cis}\,\theta_1\,\text{cis}\,\theta_2$	$\text{cis}(-\theta) = \frac{1}{\text{cis}\,\theta}$
De Moivres theorem	
$z^n = \lvert z\rvert^n\,\text{cis}(n\theta)$	$(\text{cis}\,\theta)^n = \cos n\theta + i\sin n\theta$
$z^{\frac{1}{q}} = r^{\frac{1}{q}}\left(\cos\frac{\theta+2\pi k}{q} + i\sin\frac{\theta+2\pi k}{q}\right)$, for k an integer	

$\cos^2 x + \sin^2 x = 1$	$1 + \tan^2 x = \sec^2 x$
$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$	$\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$
$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$	$\sin 2x = 2\sin x\cos x$
$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$	$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$
$\cos A\cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$	$\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$
$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$	$\cos A\sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1) (2, 2, 2, 2 & 1 = 9 marks)
If $w = 2 - 2i$ and $z = 9 - 5i$ determine exactly:

a) wz $\quad 8 - 28i$ ✓ Real term ✓ Imaginary

b) $\frac{w}{z}$ $\quad \frac{2-2i}{9-5i}\frac{(9+5i)}{(9+5i)} = \frac{28-8i}{106}$ ✓ numerator ✓ denominator

c) $z\bar{w}$ $\quad 28 + 8i$ ✓ Real ✓ Imaginary

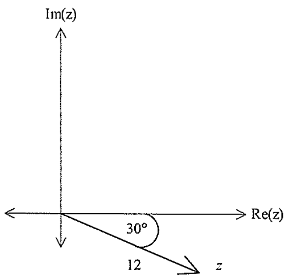
d) $w\bar{z}$ $\quad 28 - 8i$ ✓ Real ✓ Imaginary

e) What do you notice about (c) and (d)?
Conjugates of each other ✓ mentions conjugates

Q2 (2 & 2 = 4 marks)
Express each of the following into Cartesian form, $a + bi$

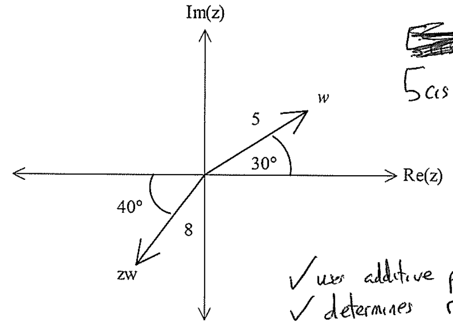
a) $7cis\left(-\frac{2\pi}{3}\right) = 7\left(\cos -\frac{2\pi}{3} + i\sin -\frac{2\pi}{3}\right) = -\frac{7}{2} - \frac{7\sqrt{3}}{2}i$
✓ expands cis
✓ evaluates Re + Im parts

b)



$12\cos 30^\circ - 12\sin 30^\circ i = 6\sqrt{3} - 6i$
✓ real part
✓ Imaginary part.

Q4 (3 marks)
Determine z in polar form given that w and zw have been drawn below.



$5\,\text{cis}\,30^\circ \; r\,\text{cis}\,\theta = 8\,\text{cis}(220^\circ)$

$r = \frac{8}{5} \qquad \theta = 190^\circ$

$z = \frac{8}{5}\,\text{cis}\,190^\circ$ or $\frac{8}{5}\,\text{cis}(-170^\circ)$

✓ uses additive property of cis θ
✓ determines r
✓ determines θ

Q5 (5, 3 & 3 =11 marks)

a) Determine all the roots of the equation $z^5 = 1 - i$, expressing them all in polar form with $r \geq 0$ and $-\pi < Arg z \leq \pi$

$z^5 = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4} + 2n\pi\right) \quad n = 0, \pm1, \pm2 \ldots$

$z = 2^{\frac{1}{10}}\,\text{cis}\left(-\frac{\pi}{20} + \frac{2n\pi}{5}\right)$

$= 2^{\frac{1}{10}}\,\text{cis}\left(-\frac{\pi}{20} + \frac{8n\pi}{20}\right)$

$z_1 = 2^{\frac{1}{10}}\,\text{cis}\left(-\frac{\pi}{20}\right)$ ✓ uses $2^{\frac{1}{10}}$

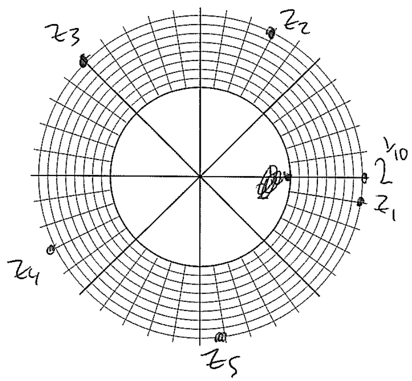
$z_2 = 2^{\frac{1}{10}}\,\text{cis}\left(\frac{7\pi}{20}\right)$ ✓ identifies $\frac{-\pi}{20}$

$z_3 = 2^{\frac{1}{10}}\,\text{cis}\left(-\frac{9\pi}{20}\right)$ ✓ determines 5 different arguments

$z_4 = 2^{\frac{1}{10}}\,\text{cis}\left(\frac{15\pi}{20}\right)$ ✓ converts to principal Arg

$z_5 = 2^{\frac{1}{10}}\,\text{cis}\left(-\frac{17\pi}{20}\right)$ ✓ states all 5 roots

b) Plot the roots on the diagram below. (Note: each minor angle is $\frac{\pi}{20}$ radians.)

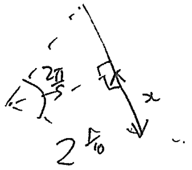


✓ shows scale ($r = 2^{\frac{1}{10}}$)

✓ Five equally spaced points

✓ all 5 pts have correct angle.

c) The roots form the vertices of a pentagon. Determine the value for the perimeter of the pentagon to two decimal places.



$\sin\frac{\pi}{5} = \frac{x}{2^{\frac{1}{10}}}$ $\quad x = 2^{\frac{1}{10}}\sin\frac{\pi}{5}$

Perimeter $= 10\left(2^{\frac{1}{10}}\sin\frac{\pi}{5}\right)$

$= 6.30$ units

✓ using correct angle

✓ solving opposite side of triangle.

✓ determines perimeter to 2 d.p

Q6 (5 marks)
Determine, **using de Moivre's theorem**, an expression for $\sin 3\theta$ in terms of $\sin\theta$ only.
{Hint: start with $(\cos\theta + i\sin\theta)^3$}

$$(\cos\theta + i\sin\theta)^3 = \operatorname{cis}3\theta$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta \quad -(\sin^3\theta - 3\cos^2\theta\sin\theta)i$$

$$= \cos 3\theta + i\sin 3\theta$$

$$\sin 3\theta = -\sin^3\theta + 3\cos^2\theta\sin\theta$$
$$= -\sin^3\theta + 3(1-\sin^2\theta)\sin\theta$$
$$= -\sin^3\theta + 3\sin\theta - 3\sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

✓ equates $(\cos\theta + i\sin\theta)^3$ to $\operatorname{cis}3\theta$

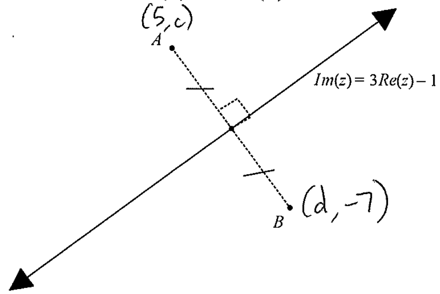
✓ expands $(\cos\theta + i\sin\theta)^3$

✓ equates Im part to $\sin 3\theta$

✓ replaces $\cos^2\theta$ with $1-\sin^2\theta$ f/t

✓ obtains final expression in terms of $\sin\theta$

Q7 (5 marks)
Consider the points A and B in the complex plane. The perpendicular bisector of the line AB is represented by $\operatorname{Im}(z) = 3\operatorname{Re}(z) - 1$



If point A is $5 + ci$ and point B is $d - 7i$ in the complex plane, determine the values of the constants c and d.

Midpoint AB $= \left(\frac{5+d}{2}, \frac{c-7}{2}\right)$ $\quad \frac{c-7}{2} = 3\left(\frac{5+d}{2}\right) - 1$

$m_{AB} = \frac{c+7}{5-d} = -\frac{1}{3}$

Use simultaneous $\quad c = -12\frac{1}{4}$

$d = -10\frac{3}{4}$

✓ determines midpoint in terms of c & d

✓ determines gradient in terms of c & d.

✓ obtains one equation c & d (ie midpoint into line eqn)

✓ obtains two equations c & d (ie $m_1 \times m_2 = -1$)

✓ Solves for c & d.