Semester One Examination, 2015

Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 1

Section One: Calculator-free

SOLUTIONS

Student Number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2015*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

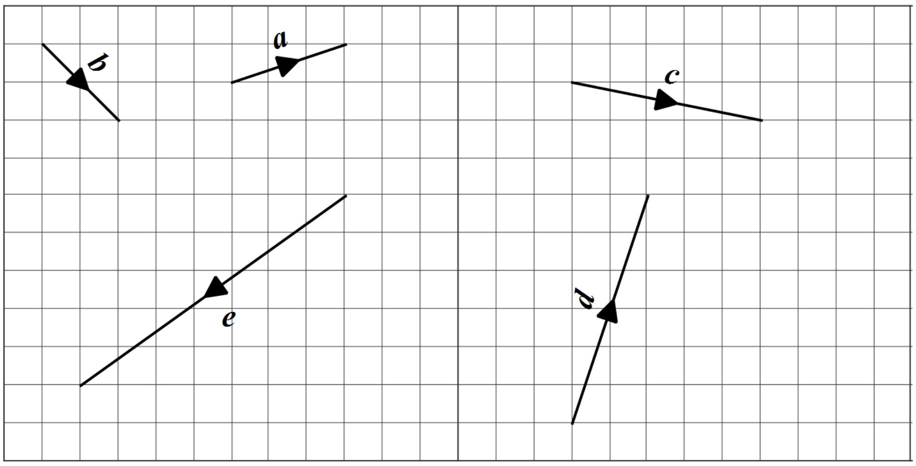
Section One: Calculator-free **(52 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(8 marks)**

(a) Two vectors, $\mathbf{a}$ and $\mathbf{b}$, are shown on the grid below.



Draw and label the vectors $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$ on the grid, where $\mathbf{c} = \mathbf{a} + \mathbf{b}$, $\mathbf{d} = 2\mathbf{a} - 2\mathbf{b}$ and $\mathbf{e} = \mathbf{b} - 3\mathbf{a}$. (3 marks)

(b) Determine a unit vector perpendicular to the vector $8\mathbf{i} - 6\mathbf{j}$. (2 marks)

$$|8\mathbf{i} - 6\mathbf{j}| = 10$$

$$\hat{\mathbf{u}} = \frac{1}{10}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix} \quad \Big(\text{or} \quad \Big[$$

(c) The point P divides the line segment from M(-3, 3) to N(13, -9) in the ratio 1:3. Determine the position vector of point P. (3 marks)

$$\mathbf{p} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} + \frac{1}{1+3}\left(\begin{bmatrix} 13 \\ -9 \end{bmatrix} - \begin{bmatrix} -3 \\ 3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -3 \\ 3 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 16 \\ -12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Question 2 **(6 marks)**

The statement 'if two rectangles are congruent then they have the same area' is true.

(a) Write the inverse of the statement and explain whether or not the inverse is also true. (2 marks)

> If two rectangles are not congruent then they do not have the same area.
>
> False – eg a 2x6 and a 3x4 rectangle.

(b) Write the contrapositive of the statement and explain whether or not the contrapositive is also true. (2 marks)

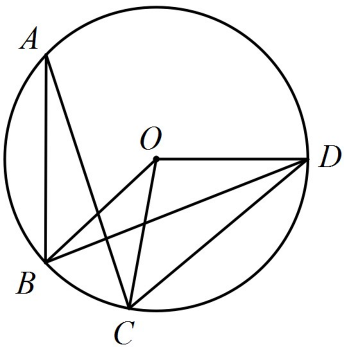
> If two rectangles do not have the same area then they are not congruent.
>
> True – contrapositive statements are always true.

(c) Write the converse of the statement and explain whether or not the converse is also true. (2 marks)

> If two rectangles have the same area then they are congruent.
>
> False – eg a 2x6 and a 3x4 rectangle.

Question 3 **(7 marks)**

(a) In the diagram below, $\angle OBD = 25^\circ$ and $\angle OCD = 40^\circ$.



Determine the sizes of

(i) $\angle BDC$. (1 mark)

$\boxed{40 - 25 = 15^\circ}$

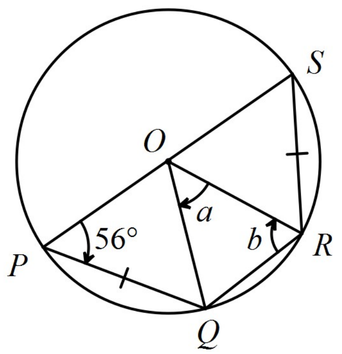
(ii) $\angle BOC$. (1 mark)

$\boxed{2 \times 15 = 30^\circ}$

(iii) $\angle CAB$. (1 mark)

$\boxed{15^\circ}$

(b) Determine, with reasons, the sizes of the angles marked a and b in the diagram below. (4 marks)



$\angle OPQ = \angle OQP = 56^\circ$ (isosceles triangle)

$\angle POQ = 180 - 56 - 56$ (angle sum of triangle)
$= 68^\circ$

$\angle SOR = \angle POQ$ (angle on equal length chord)

$a = 180 - 68 - 68$ (angle on straight line)
$= 44^\circ$

$b = \dfrac{180 - 44}{2}$ (isosceles triangle)
$= 68^\circ$

Question 4 **(8 marks)**

(a) Simplify $\frac{28! \times 7!}{10! \times 26!}$. (2 marks)

$$\frac{28\times 27\times 26!\times 7!}{10\times 9\times 8\times 7!\times 26!} = \frac{28\times}{10\times}$$
$$= \frac{7\times 3}{10\times}$$
$$= \frac{21}{20}$$

(b) Prove that ${}^{n}P_{r} = n \times {}^{n-1}P_{r-1}$. (3 marks)

$$\begin{aligned}
{}^{n}P_{r} &= \frac{n!}{(n-r)!} \\
&= \frac{n\times(n-1)!}{(n-1-r+1)!} \\
&= n\times\frac{(n-1)!}{((n-1)-(r-1))!} \\
&= n\times {}^{n-1}P_{r-1}
\end{aligned}$$

(c) If ${}^{9}P_{3} = 504$ and ${}^{10}P_{6} = 151200$, determine

(i) ${}^{9}P_{5}$. (1 mark)

$$\begin{aligned} {}^{10}P_{6} &= 10 \times {}^{9}P_{5} \\ 151200 &= 10 \times {}^{9}P_{5} \\ {}^{9}P_{5} &= 15120 \end{aligned}$$

(ii) ${}^{11}P_{5}$. (2 marks)

$$\begin{aligned} {}^{11}P_{5} &= 11 \times {}^{10}P_{4} \\ &= 11 \times 10 \times {}^{9}P_{3} \\ &= 11 \times 10 \times 504 \\ &= 55440 \end{aligned}$$

Question 5 **(8 marks)**

The vectors $\mathbf{a}$ and $\mathbf{b}$ are given by $\mathbf{a} = (5, 12)$ and $\mathbf{b} = (2, -1)$.

(a) Determine

(i) $\mathbf{a} - 3\mathbf{b}$. (1 mark)

$$\begin{bmatrix} 5 \\ 12 \end{bmatrix} - 3\begin{bmatrix} 2 \\ -1 \end{bmatrix} =$$

(ii) $|\mathbf{a}| \times |\mathbf{b}|$. (1 mark)

$$\begin{aligned} |\mathbf{a}| \times |\mathbf{b}| &= 13 \times \sqrt{5} \\ &= 13\sqrt{5} \end{aligned}$$

(iii) the vector projection of $\mathbf{a}$ onto $\mathbf{b}$. (2 marks)

$$\begin{aligned} \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \times \hat{\mathbf{b}} &= \frac{5 \times 2 + 12(-1)}{\sqrt{5}} \times \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \frac{-2}{5}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -0.8 \\ 0.4 \end{bmatrix} \end{aligned}$$

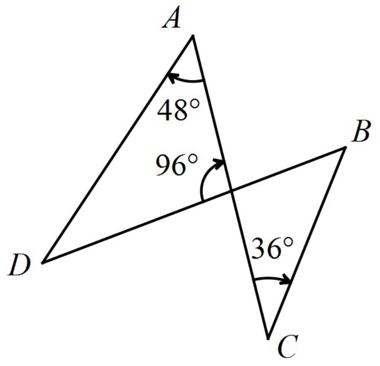
(b) Determine the vectors $\mathbf{c}$ and $\mathbf{d}$ if $2\mathbf{c} - 3\mathbf{d} = \mathbf{a}$ and $\mathbf{c} - 2\mathbf{d} = 2\mathbf{b}$. (4 marks)

$$\begin{aligned} 2\mathbf{c} - 3\mathbf{d} &= \mathbf{a} \\ -2\mathbf{c} + 4\mathbf{d} &= -4\mathbf{b} \\ \mathbf{d} &= \mathbf{a} - 4\mathbf{b} \\ &= \begin{bmatrix} 5 \\ 12 \end{bmatrix} - 4\begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 16 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{c} &= 2\mathbf{b} + 2\mathbf{d} \\ &= 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -3 \\ 16 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 30 \end{bmatrix} \end{aligned}$$

Question 6 **(7 marks)**

(a) Prove that it is possible to draw a circle through the points A, B, C and D shown below. (3 marks)



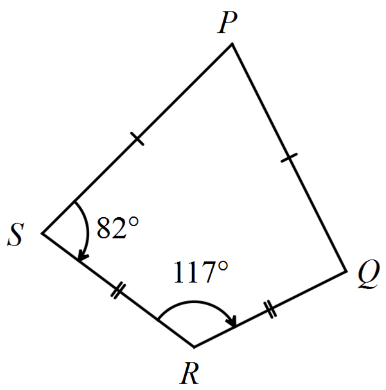
$\angle D = 180 - 48 - 96$
$= 36^\circ$
$= \angle C$

Let A and B be two points on a circle.

Since two angles at the circumference (C and D) subtended by the same arc (AB) are equal, then C and D must also lie on the circle.

Hence all four points lie on the same circle.

(b) Prove by contradiction that it is impossible to draw a circle through the vertices of the quadrilateral shown below. (4 marks)



Assume quadrilateral $PQRS$ to be cyclic, $\angle S + \angle Q = 180^\circ$.

Triangles PRS and PRQ are congruent (SSS).

Hence $\angle Q = \angle S = 82^\circ$ and so $\angle Q + \angle S = 164^\circ$.

This contradicts our original assumption that the quadrilateral is cyclic, and hence it is not, and so it is impossible to draw a circle through the vertices.

Question 7 **(8 marks)**

(a) A bag contains 17 identical cubes except for their colour, with four coloured orange, six coloured blue and seven coloured white.

(i) How many different arrangements of coloured cubes are possible when three cubes are drawn from the bag and placed in a line? (1 mark)

$3 \times 3 \times 3 = 27$

(ii) How many different combinations of coloured cubes are possible when three cubes are drawn from the bag? (2 marks)

All different colour: 1, Two same, 1 diff: $3 \times 2 = 6$, All same: $3 \times 1 = 3$.

Total: 10 combinations

(iii) Determine the least number of cubes that should be removed from the bag to ensure that the resulting selection contains at least three cubes of one colour. Justify your answer. (2 marks)

7 balls.

A maximum of 6 cubes (2 of each colour) can be taken without exceeding more than two of any one colour. So by taking 7 cubes, there must be at least three of one of the colours.

(b) Show that if 50 different integers are selected from the set {1, 2, 3, ..., 98, 99}, there will be at least two integers whose sum is 100. (3 marks)

Create pigeonholes using the sets {1,99}, {2,98}, . . . , {49,50}. There are 49 of these sets. Since there are 50 numbers (pigeons), by the pigeonhole principle there must be at least two numbers in the same pigeonhole, and each pair of numbers adds up to 100.

End of questions

Additional working space

Question number: _________