

Mathematics: Specialist

Formula sheet Units 3C and 3D

Vectors

$$|(a_1, a_2, a_3)| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Vector equation of a line in space:

one point and the slope:

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$$

two points A and B:

$$\mathbf{r} = \mathbf{a} + \lambda (\mathbf{b} - \mathbf{a})$$

Cartesian equations of a line in space:

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

Parametric form of vector equation

$$x = a_1 + \lambda b_1 \dots \dots \dots (1)$$

of a line in space:

$$y = a_2 + \lambda b_2 \dots \dots \dots (2)$$

$$z = a_3 + \lambda b_3 \dots \dots \dots (3)$$

Vector equation of a plane in space:

$$\mathbf{r} \cdot \mathbf{n} = c \quad \text{or} \quad \mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$$

Trigonometry

In any triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$A = \frac{1}{2} ab \sin C$$

In a circle of radius r, for an arc subtending angle θ (radians) at the centre:

$$\text{Length of arc} = r\theta$$

$$\text{Area of sector} = \frac{1}{2} r^2 \theta$$

$$\text{Area of segment} = \frac{1}{2} r^2 (\theta - \sin \theta)$$

$$\cos(\theta \pm \phi) = \cos \theta \cos \phi \mp \sin \theta \sin \phi$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\sin(\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi$$

$$= 2\cos^2 \theta - 1$$

$$\sin 2\theta = 2\sin \theta \cos \theta$$

$$= 1 - 2\sin^2 \theta$$

$$\tan(\theta \pm \phi) = \frac{\tan \theta \pm \tan \phi}{1 \mp \tan \theta \tan \phi}$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

Simple Harmonic Motion: If $\frac{d^2x}{dt^2} = -k^2x$ then $x = A\sin(kt + \alpha)$ or $x = A\cos(kt + \beta)$

Exponentials and logarithms

For $a, b > 0$ and m, n real,

$$a^m a^n = a^{m+n}$$

$$a^0 = 1$$

$$(a^m)^n = a^{mn}$$

For m an integer and n a positive integer :

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

For $a, y > 0$ then

$$1 = a^0 \Leftrightarrow \log_a 1 = 0$$

$$\log_a (mn) = \log_a (m) + \log_a (n)$$

$$\log_a (m) = \frac{\log_b (m)}{\log_b (a)} \text{ (change of base)}$$

$$a^{m \cdot n} = \frac{a^m}{a^n}$$

$$a^{-n} = \frac{1}{a^n}$$

$$a^m b^m = (ab)^m$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

$$y = a^x \Leftrightarrow \log_a y = x$$

$$a = a^1 \Leftrightarrow \log_a a = 1$$

$$\log_a (m^n) = n \log_a (m)$$

$$\text{If } \frac{dp}{dt} = kP \quad \text{then } P = P_0 e^{kt}$$

Functions

$$\text{If } f(x) = y, \quad \text{then } f'(x) = \frac{dy}{dx}$$

$$\text{If } f(x) = e^x, \quad \text{then } f'(x) = e^x$$

$$\text{If } f(x) = \sin x \quad \text{then } f'(x) = \cos x$$

$$\text{If } f(x) = \tan x \quad \text{then } f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$$

Product rule:

$$\text{If } y = f(x) g(x) \quad \text{or}$$

$$\text{then } y' = f'(x) g(x) + f(x) g'(x)$$

$$\text{If } f(x) = x^n, \quad \text{then } f'(x) = n x^{n-1}$$

$$\text{If } f(x) = \ln x, \quad \text{then } f'(x) = \frac{1}{x}$$

$$\text{If } f(x) = \cos x \quad \text{then } f'(x) = -\sin x$$

$$\text{If } y = uv$$

$$\text{then } \frac{dy}{dx} = \frac{du}{dx} v + u \frac{dv}{dx}$$

Quotient rule:

$$\text{If } y = \frac{f(x)}{g(x)} \quad \text{or}$$

$$\text{then } y' = \frac{f'(x) g(x) - f(x) g'(x)}{(g(x))^2}$$

$$\text{If } y = \frac{u}{v}$$

$$\text{then } \frac{dy}{dx} = \frac{\frac{du}{dx} v - u \frac{dv}{dx}}{v^2}$$

$$\text{Incremental formula: } \delta y \simeq \frac{dy}{dx} \delta x \quad \text{or}$$

$$f(x+h) - f(x) \simeq f'(x)h$$

Chain rule:

If $y = f(g(x))$

or

If $y = f(u)$ and $u = g(x)$

then $y' = f'(g(x)) g'(x)$

then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Powers: $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

Exponentials: $\int e^x dx = e^x + c$

Logarithms: $\int \frac{1}{x} dx = \ln x + c$

Trigonometric:

$\int \sin x dx = -\cos x + c$

$\int \cos x dx = \sin x + c$

$\int \frac{1}{\cos^2 x} dx = \tan x + c$

Fundamental Theorem of Calculus:

$\frac{d}{dx} \int_a^x f(t) dt = f(x)$

and

$\int_a^b f'(x) dx = f(b) - f(a)$

Complex numbers

For $z = a + ib$, where $i^2 = -1$

Argument: $\arg z = \theta$, where $\tan \theta = \frac{b}{a}$ and $-\pi < \theta \leq \pi$

Modulus: $\text{mod } z = |z| = |a + ib| = \sqrt{a^2 + b^2} = r$

Product: $|z_1 z_2| = |z_1| |z_2|$

$\arg z_1 z_2 = \arg z_1 + \arg z_2$

Quotient: $\frac{|z_1|}{|z_2|} = \frac{|z_1|}{|z_2|}$

$\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2$

Polar form:

For $z = r \text{cis} \theta$ where $r = |z|$ and $\theta = \arg z$

$\text{cis} \theta = \cos \theta + i \sin \theta$

$\text{cis}(\theta + \phi) = \text{cis} \theta \text{cis} \phi$

$\text{cis}(0) = 1$

$\text{cis}(-\theta) = \frac{1}{\text{cis} \theta}$

$z_1 z_2 = r_1 r_2 \text{cis}(\theta + \phi)$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}(\theta - \phi)$

Exponential form:

$z = r e^{i\theta}$ where $r = |z|$ and $\theta = \arg z$

For complex conjugates:

$$z = a + bi$$

$$z = r \operatorname{cis} \theta$$

$$z = re^{i\theta}$$

$$z \bar{z} = |z|^2$$

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$\bar{z} = a - bi$$

$$\bar{z} = r \operatorname{cis}(-\theta)$$

$$\bar{z} = re^{-i\theta}$$

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

Mathematical reasoning

DeMoivre's theorem:

$$(\operatorname{cis} \theta)^n = (\cos \theta + i \sin \theta)^n$$

$$(\operatorname{cis} \theta)^n = \cos n\theta + i \sin n\theta$$

$$z^n = |z|^n \operatorname{cis}(n\theta)$$

$$z^{\frac{1}{q}} = |z|^{\frac{1}{q}} \left(\cos \left(\frac{\theta + 2\pi k}{q} \right) + i \sin \left(\frac{\theta + 2\pi k}{q} \right) \right) \text{ for } k \in \{\text{Integers}\}$$

Matrices

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A| = \det A = ad - bc$

$$A^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{Dilation} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

$$\text{Shear} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

$$\text{Rotation} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$\text{Reflection} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

Measurement

Circle :	$C = 2\pi r = \pi D$, where C is the circumference, r is the radius and D is the diameter $A = \pi r^2$, where A is the area
Triangle:	$A = \frac{1}{2} b h$, where b is the base and h is the perpendicular height
Parallelogram:	$A = b h$
Trapezium :	$A = \frac{1}{2} (a + b)h$ where a and b are the lengths of the parallel sides and h is the perpendicular height
Prism:	$V = Ah$, where V is the volume, A is the area of the base and h is the perpendicular height
Pyramid:	$V = \frac{1}{3} Ah$
Cylinder :	$S = 2\pi r h + 2\pi r^2$, where S is the total surface area $V = \pi r^2 h$
Cone :	$S = \pi r s + \pi r^2$ where s is the slant height $V = \frac{1}{3} \pi r^2 h$
Sphere :	$S = 4\pi r^2$ $V = \frac{4}{3} \pi r^3$

Note: Any additional formulas identified by the examination panel as necessary will be included in the body of the particular question.
