MATHEMATICS METHODS

MAWA Semester 1 (Unit 3) Examination 2019
Calculator-assumed

Marking Key

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The release date for this exam and marking scheme is 14th June.

Section Two: Calculator-assumed (100 Marks)

Question 8(a) (4 marks)

Solution	
$\int_2^6 \frac{f(x)}{3}dx = 4$ (i) $\int_2^6 f(x)dx = 3\times 4 = 12$ (ii) $\int_2^6 \frac{3f(x)-1}{2}dx = \int_2^6 \frac{3f(x)}{2}dx - \int_2^6 \frac{1}{2}dx$ $= \frac{3}{2}\int_2^6 f(x)dx - \int_2^6 \frac{1}{2}dx$ $= \left(\frac{3}{2}\times 12\right) - \left[\frac{1}{2}x\right]_2^6$ $= 18 - [3-1]$ $= 16$	
Mathematical behaviours	**Marks**
• states $\int_2^6 f(x)dx = 12$	1
• uses linearity and additivity to deduce $\int_2^6 \frac{3f(x)-1}{2}dx = \int_2^6 \frac{3f(x)}{2}dx - \int_2^6 \frac{1}{2}dx$	1
• anti-differentiates $\frac{1}{2}$	1
• determines correct result of 16	1

Question 8(b) (3 marks)

Solution	
$\int_{-\frac{1}{4}}^{0} e^{4x+1}dx = \frac{1}{4}\int_{-\frac{1}{4}}^{0} 4e^{4x+1}dx$ $= \frac{1}{4}\left[e^{4x+1}\right]_{-\frac{1}{4}}^{0}$ $= \frac{1}{4}\left[e^{1} - e^{0}\right]$ $= \frac{1}{4}[e-1]$	
Mathematical behaviours	**Marks**
• anti-differentiates correctly	1
• substitutes limits of integration correctly	1
• determines exact result	1

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Question 9(a) **(1 mark)**

Solution	
$f'(x) = (x-1)^2(4x-1) = 4x^3 + bx^2 + cx + d + e$ $hence\ f(x) = x^4 + ...\qquad ie\ \ a > 0$	
Mathematical behaviours	Mark
• states $a > 0$ justifies answer using anti-differentiation	1

Question 9(b) **(1 mark)**

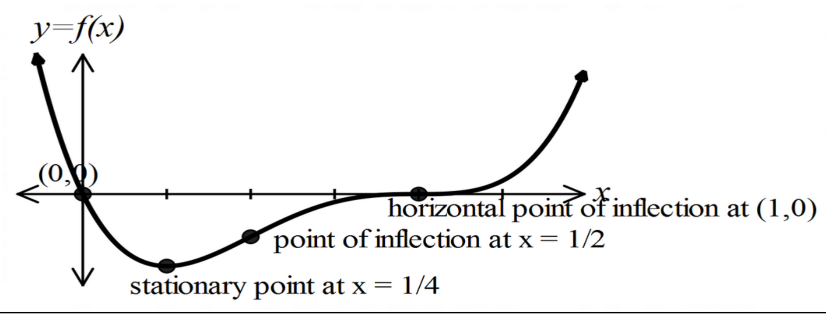
Solution	
For stationary points, $f'(x) = 0$ $ie\,(x-1)^2(4x-1) = 0 \Rightarrow x = 1, \frac{1}{4}$	
Mathematical behaviours	Marks
• states x coordinates of stationary points	1

Question 9(c) **(3 marks)**

Solution	
$f'(1) = 0$ and $f''(1) = 0$ $f''(x) = 6(x-1)(2x-1)$ $f''(1^-) = -ve \times +ve = -ve$ $f''(1^+) = +ve \times +ve = +ve$ Hence there is a change in concavity at $x = 1$ and $f'(1) = 0$ so there is a horizontal point of inflection at $x = 1$. Hence *m* = 1.	
Mathematical behaviours	Marks
• states $f'(1) = 0$ and $f''(1) = 0$ • demonstrates change in concavity at $x = 1$ • states that horizontal point of inflection occurs at *m* = 1.	1 1 1

Question 9(d) **(3 marks)**

Solution



Mathematical behaviours	Marks
• sketch shows $x \to \pm\infty,\ f(x) \to \infty$ and roots • clearly shows x coordinate of minimum turning point • graphs correct shape and clearly labels points of inflection	1 1 1

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Question 10(a) **(1 mark)**

Solution	
X has a binomial distribution with parameters n and $p=0.5$ ie $X \sim Bin(n, 0.5)$	
Mathematical behaviours	Marks
• identifies binomial distribution and states parameters	1

Question 10(b) **(1 mark)**

Solution	
$E(X)=\mu=np=\dfrac{n}{2}$	
Mathematical behaviours	Marks
• states correct answer	1

Question 10(c) **(3 marks)**

Solution	
$n=20:\ P_1=P(5\le X\le 15)\cong 0.988$ $n=1000:\ P_1=P(495\le X\le 505)\cong 0.272$ $n=10\,000:\ P_1=P(4995\le X\le 5005)\cong 0.088$ (from calculator)	
Mathematical behaviours	Marks
• states a probability inequality relevant to one of the n values • calculates one probability correctly • calculates all probabilities correctly	1 1 1

Question 10(d) **(1 mark)**

Solution	
$P_1 \to 0$ as $n \to \infty$	
Mathematical behaviours	Marks
• obtains correct answer	1

Question 10(e) **(3 marks)**

Solution	
$n=20:\ P_2=P(9.5\le X\le 10.5)=P(X=10)\cong 0.176$ $n=200:\ P_2=P(95\le X\le 105)\cong 0.563$ $n=1000:\ P_2=P(475\le X\le 525)\cong 0.893$	
Mathematical behaviours	Marks
• states probability inequality relevant to $n=20$ • calculates one probability correctly • calculates all probabilities correctly	1 1 1

Question 10(f) **(1 mark)**

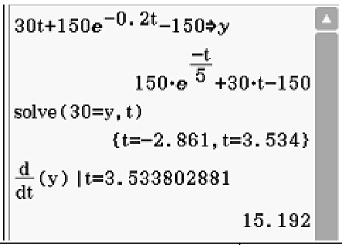
Solution	
$P_2 \to 1$ as $n \to \infty$	
Mathematical behaviours	Marks

• obtains correct answer	1

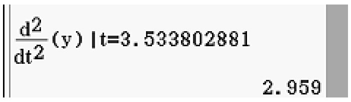
Question 11(a) **(1 mark)**

Solution	
$y=30t+150e^{-0.2t}+k$ $t=0, y=0 \Rightarrow 0=150+k \Rightarrow k=-150$	
Mathematical behaviours	**Mark**
• evaluates k	1

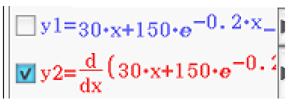
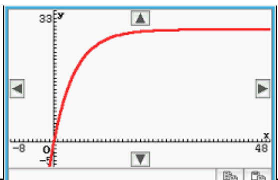
Question 11(b) **(3 marks)**

Solution	
$y=30t+150e^{-0.2t}-150$ $y=30 \Rightarrow t=3.53s$ $v=30-30e^{-0.2t}$ $v\Big\|_{t=3.53}=15.19m/s$	
Mathematical behaviours	**Marks**
• equates $y=30$ and determines time taken to hit the ground • differentiates to obtain v • calculates the speed	1 1 1

Question 11(c) **(2 marks)**

Solution	
$v=30-30e^{-0.2t}$ $\Rightarrow a=6e^{-0.2t}m/s^2>0$ Since $v>0$ and $a>0$ the ball is speeding up.	
Mathematical behaviours	**Marks**
• differentiates v to determine a and states $a>0$ • draws conclusion noting the same sign of both v and a.	1 1

Question 11(d) **(1 mark)**

Solution	
$v=30-30e^{-0.2t}$, $a=6e^{-0.2t}$ $t\to\infty, v\to 30, a\to 0$ Hence constant speed is attained.	 
Mathematical behaviours	**Marks**
• states $v\to 30m/s$ ie is constant	1

Question 11(e) **(1 mark)**

Solution	
A restriction on the domain is needed. ie $0\le t\le 3.53$	
Mathematical behaviours	**Marks**

• states restriction required on the domain	1

Question 12(a) **(2 marks)**

Solution	
$\mu = \dfrac{49 \times 63.3 + 38 \times 54.1}{87} = 59.28$	
Mathematical behaviours	**Marks**
• uses correct expression • obtains correct answer	1 1

Question 12(b) **(4 marks)**

Solution	
If $Y = aX + b$, then $E(Y) = aE(X) + b$ and $St.Dev(Y) = aSt.Dev(X)$ So $59.28 = a \times 63.3 + b$ and $9 = a \times 7.6$ So $a = 1.18$ and $b = -15.68$	
Mathematical behaviours	**Marks**
• expresses $E(Y)$ in terms of $E(X)$ • expresses $Std\ Dev\ (Y)$ in terms of $Std\ Dev(X)$ • calculates a • calculates b	1 1 1 1

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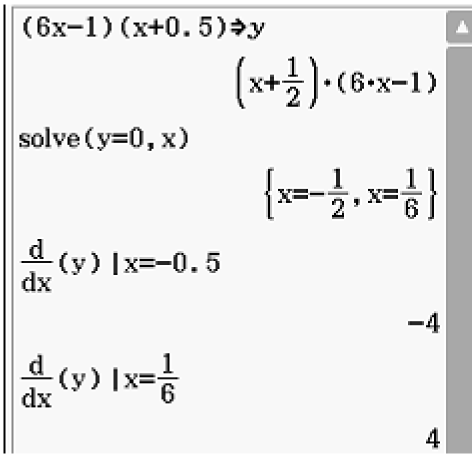
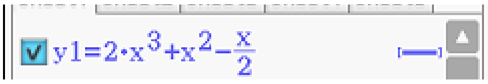
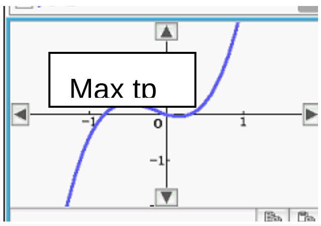
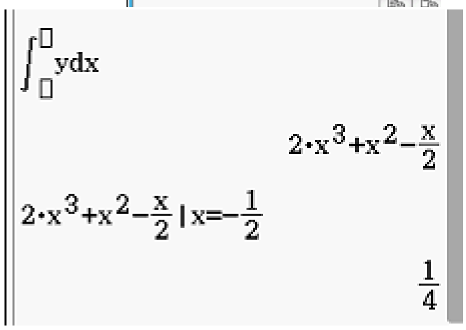
Question 13(a) **(5 marks)**

Solution
Stationary Points: $\frac{dy}{dx}=0$ i.e. $(6x-1)\left(x+\frac{1}{2}\right)=0$ $x=\frac{1}{6}$ or $x=-\frac{1}{2}$ Now $\frac{dy}{dx}=(6x-1)\left(x+\frac{1}{2}\right)$ $=6x^2+2x-\frac{1}{2}$ $\frac{d^2y}{dx^2}=12x+2$ At $x=\frac{1}{6}$, $\frac{d^2y}{dx^2}=4\Rightarrow \text{min}$ At $x=-\frac{1}{2}$, $\frac{d^2y}{dx^2}=-4\Rightarrow \text{max}$ $\therefore \text{max turning pt at } \left(-\frac{1}{2},1\right)$ Now $y=2x^3+x^2-\frac{1}{2}x+c$ $\left(-\frac{1}{2},1\right)\Rightarrow 1=2\left(-\frac{1}{2}\right)^3+\left(-\frac{1}{2}\right)^2-\frac{1}{2}\cdot\left(-\frac{1}{2}\right)+c$ $1=\frac{1}{4}+c\Rightarrow c=\frac{3}{4}$ ∴ equation of the function is $y=2x^3+x^2-\frac{1}{2}x+\frac{3}{4}$









Mathematical behaviours	Marks
• uses $\frac{dy}{dx}=0$ to find stationary points	1
• substitutes into $\frac{d^2y}{dx^2}$, $x=\frac{1}{6}$ and $x=-\frac{1}{2}$ to find which x value gives a local maximum turning point or clearly shows on sketch location of maximum and confirms maximum using 2nd derivative test	1 1
• integrates the derivative function correctly	1
	1

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• uses the point $\left(-\frac{1}{2},1\right)$ to determine the value of c • states the correct equation of the function	1

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Question 13(b) **(5 marks)**

Solution
(i) $V = \frac{\pi h}{3}(R^2 + r^2 + Rr)$ $V = \frac{\pi(15)}{3}(5^2 + 3^2 + 5\times 3)$ $\approx 769.69\,cm^3 \approx 770\,cm^3$ (ii) $V = \frac{\pi 15}{3}(R^2 + 3^2 + 3R)$ $\frac{dV}{dR} = 5\pi(2R + 3)$ $\frac{dV}{dR} \approx \frac{\delta V}{\delta R}, R = 5, \delta R = -0.2$ $\delta V \approx 5\pi(2\times 5 + 3)(-0.2)$ $\delta V \approx -40.84\ cm^3 \approx -41\ cm^3$ *ie a decrease in capacity of approximately* 41 *millilitres*

Mathematical behaviours	Mark
(i)	
• states correct volume to the nearest cubic centimetre	1
(ii)	
• states V in terms of R	1
• uses incremental formula to obtain expression for small change in V	1
• substitutes, $R = 5$ and $\delta R = -0.2$	1
• states the decrease in capacity	1

Question 14(a) **(3 marks)**

Solution

Total number of cars in sample is $27+13+11+4+14=69$

Proportions of the various colours, and rounded to a whole multiple of 0.05:

White	Black	Red	Blue	Other
$\frac{27}{69} \cong 0.391 \cong$	$\frac{13}{69} \cong 0.188 \cong$	$11/69 \cong 0.15$	$4/69 \cong 0.058$ 7	$14/69 \cong 0.20$

Mathematical behaviours	Marks
• obtains total sample size	1
• calculates all fractions correctly	1
• rounds all answers correctly	1

Question 14(b) **(3 marks)**

Solution	
Expected number of points per car ¿$2\times 0.4+4\times 0.2+7\times 0.15+9\times 0.05+5\times 0.2=4.1$ So expected number of points per 100 cars ¿$100\times 4.1=410$	
Mathematical behaviours	**Marks**
• obtains correct expression for expected value • calculates expected value (per car) correctly • obtains correct answer	1 1 1

Question 14(c) **(2 marks)**

Solution

Expected number of points per car (by colour)

White	Black	Red	Blue	Other
$2\times 0.4=0.8$	$4\times 0.2=0.8$	$7\times 0.15=1.05$	$9\times 0.05=0.45$	$5\times 0.2=1$

Since the expected points per car is greatest for Rodney's red cars, Rodney is most likely to accumulate points fastest.

Mathematical behaviours	Marks
• evaluates expected values correctly • correct answer	1 1

Question 14(d) **(2 marks)**

Solution	
$P=0.4^2+0.2^2+0.15^2+0.05^2+0.2^2=0.265$	
Mathematical behaviours	**Marks**
• uses correct formula • evaluates correctly	1 1

Question 15(a) **(4 marks)**

Solution	
(i) none (consecutive selections are not independent so not binomial) (ii) uniform (iii) binomial (iv) binomial	
Mathematical behaviours	**Marks**
i) • states none (ii) • states uniform (iii) • states binomial (iv) • states binomial	 1 1 1 1

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Question 15(b) **(4 marks)**

Solution	
(i) $f(x)=\frac{x}{6}$, where $x=-1,1,2,4.$ No, since $f(-1)=\frac{-1}{6}$ represents a probability and probability cannot be negative (ii) <table><tr><td>x</td><td>4</td><td>6</td><td>8</td><td>10</td></tr><tr><td>$f(x)$</td><td>0.05</td><td>0.30</td><td>0.25</td><td>0.4</td></tr></table> Yes as $0 \le f(x) \le 1 \ \forall \ x$ and the sum of the probabilities is 1.	
Mathematical behaviours	**Marks**
(i) • states no • recognises negative probability (ii) • states yes • states both reasons	 1 1 1 1

Question 16(a) **(3 marks)**

Solution	
$v=\int 8\,dt$ $v=\frac{ds}{dt}=8t+c$ $v(0)=p \Rightarrow \quad p=c$ $\therefore v=\frac{ds}{dt}=8t+p$ $s=4t^2+pt+k$ $s(1)=q \Rightarrow \quad q=4+p+k \ \Rightarrow \quad k=q-4-p$ $\therefore s=4t^2+pt+q-4-p$ or $s=4t^2+pt+q-p-4$ as required	
Mathematical behaviours	**Marks**
• anti-differentiates $a(t)$ to obtain $v(t)$ and uses $v(0)=p$ to get correct expression for c. • anti-differentiates $v(t)$ to obtain $s(t)$ and uses $s(1)=q$ to get correct expression for k • states required answer	1 1 1

Question 16(b) **(1 mark)**

Solution	
Distance travelled = $\int_0^3 \lvert 8t + p \rvert \, dt$	
Mathematical behaviours	Marks
• states the integral of the **absolute** velocity function from $t = 0$ to $t = 3$	1

Question 17(a) **(2 marks)**

Solution	
Define the random variable, X as the number of batteries that last for less than 2000 hours. Hence, $X \sim Bin(120, 0.1)$ $P(X = 15) \approx 0.0742$	
Mathematical behaviours	Marks
• recognizes Binomial nature	1
• obtains correct answer	1

Question 17(b) **(2 marks)**

Solution	
$X \sim Bin(120, 0.1)$ $P(X \le 15) \approx 0.8560$	
Mathematical behaviours	Marks
• recognizes binomial nature	1
• obtains correct answer	1

Question 17(c) **(2 marks)**

Solution	
From part (b) we can conclude that there is an 85.6% chance that no more than 15 batteries out of 120 last less than 2000hrs. This would imply that there is only a 14.4% chance that more than 15 out of 120 batteries last less than 2000hrs. Hence the test does not imply compelling evidence that the manufacturer's claim is false.	
Mathematical behaviours	Marks
• obtains correct answer	1
• gives valid reason	1

Question 18(a) **(3 marks)**

Solution

Outcome	Death	Permanent Disability	No payout
Profit	-49000	-9000	1000
Probability	0.01	0.02	0.97

Mathematical behaviours	Marks
• completes Probability row of table correctly	1

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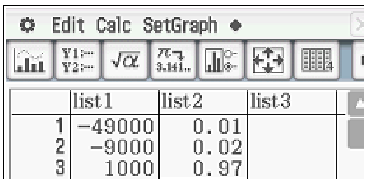
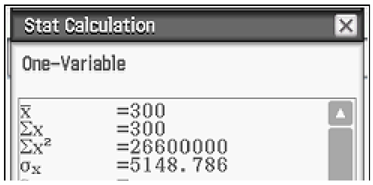
• completes exactly 2 entries of Profit row of table correctly	1
• completes table correctly	1

Question 18(b) **(2 marks)**

Solution
$E(X) = 0.01 \times (-49000) + 0.02 \times (-9000) + 0.97 \times 1000 = 300$ Hence the expected profit is \$300    

Mathematical behaviours	Marks
• states correct formula for $E(X)$	1
• obtains correct answer	1

Question 18(c) **(3 marks)**

Solution
$Var(X) = (-49000-300)^2 \times 0.01 + (-9000-300)^2 \times 0.02 + (1000-300)^2 \times 0.97 = 26510000$ $Std\ Dev = \sqrt{26510000} \approx 5149$ or $E(X^2) = (-49000)^2 \times 0.01 + (-9000)^2 \times 0.02 + (1000)^2 \times 0.97 = 26600000$ $Var(X) = E(X^2) - (E(X))^2$ $= 26600000 - (300)^2 = 26510000$ $Std\ Dev(X) = \sqrt{26510000} \approx 5149$ Note: CAS screen above shows $E(X^2) = 26600000$

Mathematical behaviours	Marks
• demonstrates calculations required to obtain variance	1
• obtains variance	1
• obtains standard deviation	1

Question 19(a) **(1 mark)**

Solution
$\int_{-1}^{2} f(x)\,dx = 5 + 16 = 21$

Mathematical behaviours	Marks
• states correct answer	1

Question 19(b) **(1 mark)**

Solution
$\int_{-1}^{4} f(x)\,dx = 5 + 16 + 11 - 27 = 5$

Mathematical behaviours	Marks
• states correct answer	1

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Question 19(c) **(1 mark)**

Solution	
$A = \int_{-1}^{4} \lvert f(x) \rvert dx = 5 + 16 + 11 + 27 = 59$	
Mathematical behaviours	**Marks**
• states correct answer	1

Question 19(d) **(2 marks)**

Solution	
Shaded area marked M $= (16 \times 3) - 21 = 27$	
Marking key/mathematical behaviours	**Marks**
• recognises area of rectangle subtract $\int_{-1}^{2} f(x) dx$ • states correct answer	1 1

Question 19(e) **(2 marks)**

(i) Correct statement is $\int_{k}^{4} f(x) dx = 48$ (ii) Use CAS and solve for k: Solve ($\left(\int_{k}^{4} (12x^2 - 4x^3) dx = 48, k\right)$ $\Rightarrow$ $k = -2$	
Mathematical behaviours	**Marks**
(i) • chooses correct statement (ii) • solves for k	 1 1

Question 20(a) **(3 marks)**

Solution	
For the circle, $l = 2\pi r \Rightarrow r = \frac{l}{2\pi}$ $\therefore A_c = \pi \left(\frac{l}{2\pi}\right)^2$ Hence, $A = \pi \left(\frac{l}{2\pi}\right)^2 + \left(\frac{100 - l}{4}\right)^2$	For the square, $x = \left(\frac{100 - l}{4}\right)$ $\therefore A_s = \left(\frac{100 - l}{4}\right)^2$
Mathematical behaviours	**Mark**
• demonstrates that $r = \frac{1}{2\pi}$ and states expression for the area of the circle • demonstrates that side length = $\frac{100 - l}{4}$ and states expression for the area of the square • concludes formula for A	1 1 1

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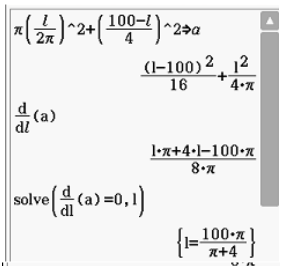
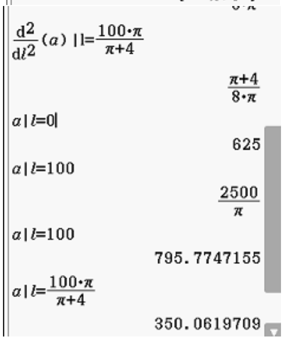
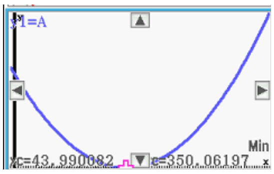
Question 20(b) **(5 marks)**

Solution
$A=\pi\left(\frac{l}{2\pi}\right)^2+\left(\frac{100-l}{4}\right)^2$ $\frac{dA}{dl}=\frac{2\pi l}{4\pi^2}-\frac{1}{8}(100-l)=\frac{l}{2\pi}-\frac{1}{8}(100-l)=l\left(\frac{1}{2\pi}+\frac{1}{8}\right)-\frac{25}{2}=l\left(\frac{4+\pi}{8\pi}\right)-\frac{25}{2}$ $\frac{dA}{dl}=0\Rightarrow l=\frac{25}{2}\left(\frac{8\pi}{4+\pi}\right)\approx 43.99\,cm$ $\frac{d^2A}{dl^2}=\frac{4+\pi}{8\pi}>0\Rightarrow min$ $A\big\|_{l=43.99}=350.06$ $A\big\|_{l=0}=25^2=625$ $A\big\|_{l=100}=\pi\left(\frac{100}{2\pi}\right)^2\approx 795.77$ Or, to establish minimum has been achieved at $l=43.99\,cm$, states coefficient of l^2 is positive, hence minimum turning point or demonstrates with graph Hence the minimum total area is obtained when $l=43.99\,cm$







Mathematical behaviours	Marks
• determines $\frac{dA}{dl}$	1
• equates $\frac{dA}{dl}=0$ and solves	1
• establishes $\left.\frac{d^2A}{dl^2}\right\|_{l=43.99}>0$ hence a minimum	1
• determines A for $l=0$ and $l=100$ OR demonstrates through graph or coefficient of l^2 that A is a quadratic with a minimum turning point	1
• concludes minimum area is when $l=43.99\,cm$	1

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