Section Two: Calculator – 80 marks

This section has **eleven (11)** questions. Attempt **all** questions.

Suggested working time: **100 minutes**

Question 9: [4 marks]

Vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are as follows: $\underset{\sim}{a} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 3 \\ t \\ -6 \end{pmatrix}$. Determine the value of t if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are:

a) Parallel to each other.

Parallel if $\lambda \underset{\sim}{a} = \underset{\sim}{b}$

$$\lambda \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ t \\ -6 \end{pmatrix}$$

$\lambda = 3$

$\therefore t = -9$ ✓

[1]

b) Perpendicular to each other.

Perpendicular when: $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$$\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ t \\ -6 \end{pmatrix} = 0$$ ✓

$(1)(3) + (-3)(t) + (-2)(-6) = 0$

$3 - 3t + 12 = 0$ ✓

$3t = 15$

$\therefore t = 5$ ✓

[3]

Question 10: [6 marks]

Prove that the square of an integer NOT divisible by 5 leaves a remainder of 1 or 4 when divided by 5.

Let integer $n = 5x + a$, where $a = 0, 1, 2, 3, 4$. ✓

Case 1: $n = 5x$

$n^2 = (5x)^2$

$n^2 = 25x^2$

$n^2 = 5(5x^2)$

Multiple of 5 not considered ✓
(or if $n = 5x$ is not included)

Case 2: $n = 5x + 1$

$n^2 = (5x + 1)^2$

$n^2 = (25x^2 + 10x + 1)$

$n^2 = 5(5x^2 + 2x) + 1$

Remainder of 1, when divisible by 5

Case 3: $n = 5x + 2$

$n^2 = (5x + 2)^2$

$n^2 = 25x^2 + 20x + 4$

$n^2 = 5(5x^2 + 4x) + 4$

Remainder of 4, when divisible by 5

Case 4: $n = 5x + 3$

$n^2 = (5x + 3)^2$

$n^2 = (25x^2 + 30x + 9)$

$n^2 = 5(5x^2 + 6x + 1) + 4$

Remainder of 4, when divisible by 5

Case 5: $n = 5x + 4$

$n^2 = (5x + 4)^2$

$n^2 = 25x^2 + 40x + 16$

$n^2 = 5(5x^2 + 8x + 3) + 1$

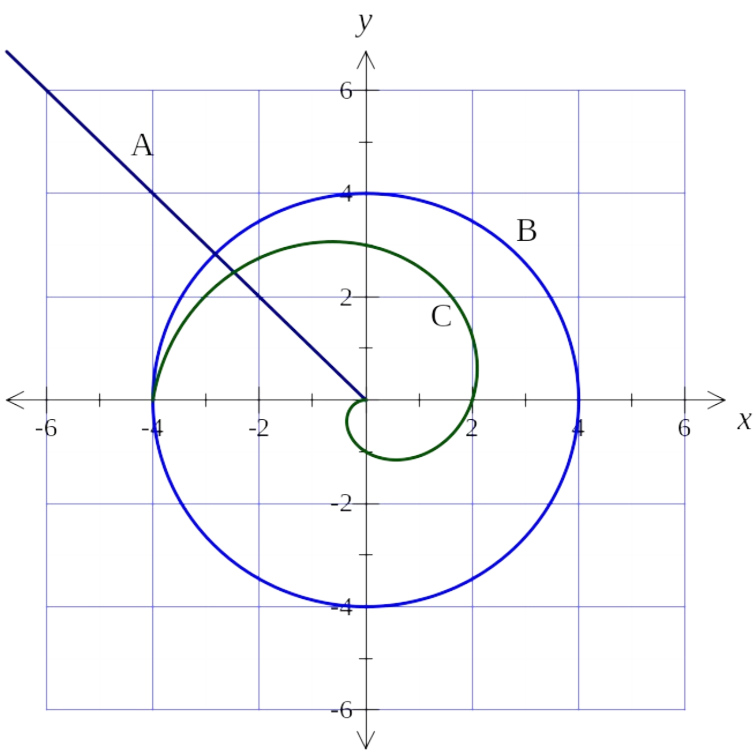
Remainder of 1, when divisible by 5

✓ Logical presentation
✓ All cases considered
✓ Working
✓ Statement about remainders and divisibility (at end of each or overall)

[6]

Question 11: [6 marks]

Determine the equations of A, B and C from the graph below:



A = $\theta = \frac{3\pi}{4}$ ✓✓ or $y = -x$ ✓ for $x \leq 0$ ✓

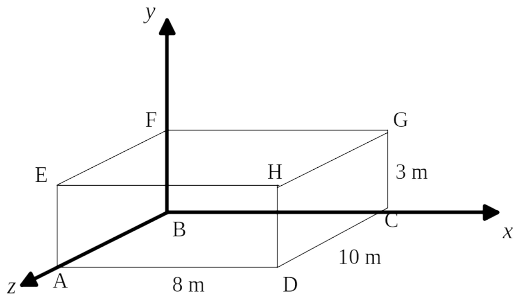
B = $r = 4$ ✓✓ or ✓ $x^2 + y^2 = 4^2$ ✓

C = $r = \frac{-2\theta}{\pi}$ ✓✓

[6]

Question 12: **[9 marks]**

The diagram on the right shows a rectangular prism.



Determine:

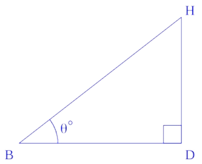
a) $\overrightarrow{BH} = \langle 8, 3, 10 \rangle$

[1]

b) $\overrightarrow{BD} = \langle 8, 0, 10 \rangle$

[1]

c) $\angle HBD$ (in degrees, to 1 decimal place).



$$\cos\theta = \frac{\overrightarrow{BH}.\overrightarrow{BD}}{|\overrightarrow{BH}|\,|\overrightarrow{BD}|}$$

$$= \frac{8^2 + (3 \times 0) + 10^2}{\sqrt{8^2 + 3^2 + 10^2}\,\sqrt{8^2 + 0^2 + 10^2}}$$

$$= \frac{164}{\sqrt{173}\,\sqrt{164}}$$

$$\theta \approx 13{\cdot}2^\circ$$

[3]

d) The acute angle between the skew lines BD and FE.

$\overrightarrow{FE} = \overrightarrow{BA}$

$\overrightarrow{BA} = \langle 0, 0, 10 \rangle$

$$\cos\theta = \frac{\overrightarrow{BD}.\overrightarrow{BA}}{|\overrightarrow{BD}|\,|\overrightarrow{BA}|}$$

$$= \frac{(8)(0) + 0^2 + 10^2}{\sqrt{164}\,\sqrt{100}}$$

$$= \frac{100}{20\sqrt{41}}$$

$$\theta \approx 38{\cdot}7^\circ$$

[4]

Question 13: [4 marks]

Prove the following:

$$\frac{\sin A}{\cos B} + \frac{\cos A}{\sin B} = \frac{2\cos(A - B)}{\sin 2B}.$$

$$\text{RHS} = \frac{2\cos(A-B)}{\sin 2B}$$

$$= \frac{2(\cos A\cos B + \sin A\sin B)}{2\sin B\cos B} \quad ✓✓$$

$$= \frac{\cos A\cancel{\cos B}}{\sin B\cancel{\cos B}} + \frac{\sin A\cancel{\sin B}}{\cancel{\sin B}\cos B} \quad ✓$$

$$= \frac{\cos A}{\sin B} + \frac{\sin A}{\cos B}$$

$$= \frac{\sin A}{\cos B} + \frac{\cos A}{\sin B} \quad ✓$$

= LHS

∴ Proved

[4]

Question 14: [4 marks]

Use calculus techniques to determine: $\int 5^x\,dx$.

$$= \int e^{(\ln 5)x}\,dx \quad ✓✓$$

$$= \frac{1}{\ln 5} \times e^{(\ln 5)x} + c \quad ✓$$

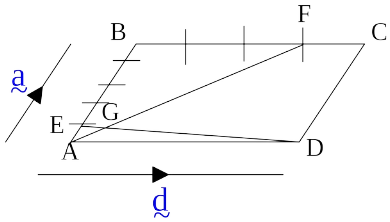
$$= \frac{5^x}{\ln 5} + c \quad ✓$$

[4]

Question 15: [11 marks]

ABCD is a parallelogram with points E and F such that $\overrightarrow{AE} : \overrightarrow{EB} = 1 : 4$ and $\overrightarrow{BF} : \overrightarrow{FC} = 3 : 1$. $\overrightarrow{ED}$ and $\overrightarrow{AF}$ intersect each other at G. Let: $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{AD} = \underset{\sim}{d}$.

a) Complete the diagram below with the information given above.



✓✓ **(-1 each mistake)**

[2]

b) Determine the ratios in which $\overrightarrow{AF}$ and $\overrightarrow{ED}$ intersect each other, if the intersection point is at G.

$$\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF}$$
$$= \underset{\sim}{a} + \frac{3}{4}\overrightarrow{BC}$$
$$= \underset{\sim}{a} + \frac{3}{4}\underset{\sim}{d} \quad ✓$$

$$\overrightarrow{ED} = \overrightarrow{EA} + \overrightarrow{AD}$$
$$= -\frac{1}{5}\underset{\sim}{a} + \underset{\sim}{d} \quad ✓$$

$$\overrightarrow{AG} = h\,\overrightarrow{AF}$$
$$= h\left(\underset{\sim}{a} + \frac{3}{4}\underset{\sim}{d}\right) \quad ✓$$

$$\overrightarrow{EG} = \underset{\sim}{k}\,\overrightarrow{ED}$$
$$= \underset{\sim}{k}\left(-\frac{1}{5}\underset{\sim}{a} + \underset{\sim}{d}\right) \quad ✓$$

$$\overrightarrow{AG} = \overrightarrow{AE} + \overrightarrow{EG}$$
$$h\underset{\sim}{a} + \frac{3}{4}h\underset{\sim}{d} = \frac{1}{5}\underset{\sim}{a} - \frac{1}{5}\underset{\sim}{k}\,\underset{\sim}{a} + \underset{\sim}{k}\,\underset{\sim}{d} \quad ✓$$

$\underset{\sim}{a}$: $h = \frac{1}{5} - \frac{1}{5}\underset{\sim}{k}$ (Equation 1) ✓

$\underset{\sim}{d}$: $\frac{3}{4}h = \underset{\sim}{k}$ (Equation 2)

$\therefore h = \frac{4}{23}$ and $\underset{\sim}{k} = \frac{3}{23}$ ✓

$\therefore$ *G* divides $\overrightarrow{AF}$ in the ratio 4 : 19 ✓

$\therefore$ *G* divides $\overrightarrow{ED}$ in the ratio 3 : 20 ✓

[9]

Question 16: [15 marks]

If: **a** = <8, -6, 0> and **b** = <-2, 4, -1>, determine:

a) 2**b** – **a**

$= 2 < -2,\ 4,\ -1 > \ - \ < 8,\ -6, 0 >$

$= \ < -12,\ 14,\ -2 >$ ✓

[1]

b) A vector in the same direction as **a** but equal in magnitude to **b**.

$= |\underset{\sim}{b}| \times \hat{\underset{\sim}{a}}$

$= \sqrt{2^2 + 4^2 + 1^2} \times \dfrac{1}{\sqrt{8^2 + 6^2 + 0^2}} < 8,\ -6, 0 >$

✓ $= \dfrac{\sqrt{21}}{10} < 8,\ -6, 0 >$ ✓ ✓

[3]

c) The acute angle that vector **a** makes with the *y*-axis.

$\underset{\sim}{a} \cdot \underset{\sim}{j} \ = \ < 8,\ -6, 0 > \ . \ < 0,\ -6, 0 >$ ✓

$= (8)(0) \ + \ (-6)^2 + 0^2$

$= 36$ ✓

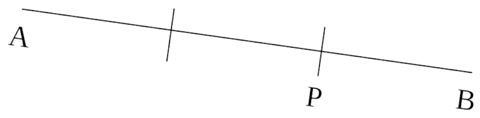
$\cos\theta \ = \ \dfrac{36}{10 \times 6}$ ✓

$\therefore \theta \ \approx 53^\circ$ ✓

[4]

Question 16 cont…

d) Point P lies on the line AB (from **a** to **b**) and it is known that: $\overrightarrow{AP} : \overrightarrow{AB} = 2:3$. Determine the position vector of the point P.



$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$
$$= -<8,\ -6,0> + <-2,\ 4,\ -1>$$
$$= <-10,\ 10,\ -1>$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$
$$= \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$$
$$= <8,\ -6,0> + \frac{2}{3} <-10,\ 10,\ -1>$$
$$= <8,\ -6,0> + <-\frac{20}{3},\ \frac{20}{3},\ -\frac{2}{3}>$$
$$= <\frac{4}{3},\ \frac{2}{3},\ -\frac{2}{3}>$$

[4]

e) If: **c** = <6, 5, -2> and **d** = <-10, -38, 11>, express **d** in the form: λ**a** + μ**b** + η**c** , and hence determine $\lambda, \mu,$ and η. (Hint: Use the vectors **a** and **b** from the start of the question).

$$\begin{pmatrix} 8 \\ -6 \\ 0 \end{pmatrix}\lambda + \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}\mu + \begin{pmatrix} 6 \\ 5 \\ -2 \end{pmatrix}\eta = \begin{pmatrix} -10 \\ -38 \\ 11 \end{pmatrix}$$

Simultaneous equations:

$$8\lambda - 2\mu + 6\eta = -10$$
$$-6\lambda + 4\mu + 5\eta = -38$$
$$-\mu - 2\eta = 11$$

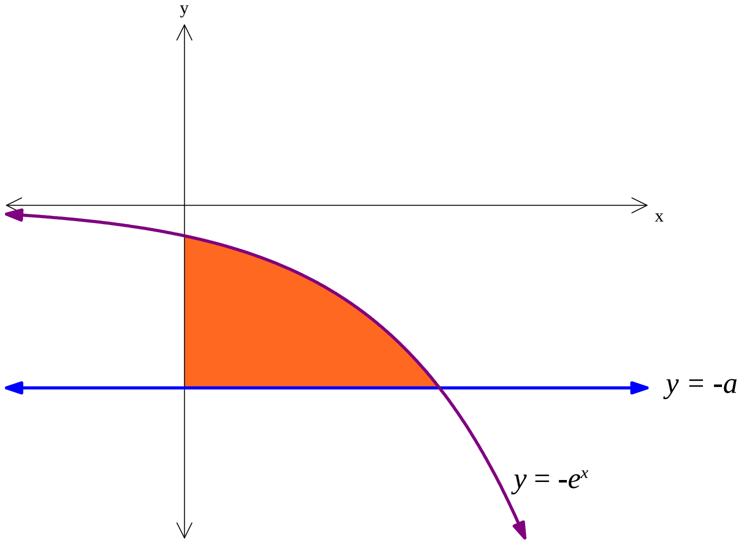
$$\mu = 1,\ \mu = -3,\ \eta = -4$$

$$\therefore \underset{\sim}{d} = \underset{\sim}{a} - 3\underset{\sim}{b} - 4\underset{\sim}{c}$$

[3]

Question 17: [7 marks]

The graphs of $y = -a$ and $y = -e^x$ are shown below:



a) Write an expression for the shaded area (above), in the form: $\int_0^b c \ dx$.

b: $-e^x = -a$

$e^x = a$

$x \cancel{\ln e} = \ln a$

$x = \ln a$

$\therefore \int_0^{\ln a} (-e^x + a)\, dx$

[3]

b) Determine the values of a and b, if it is known that: $-e^x = -a$, when $x = 1 \cdot 7918$.

$a = e^{1 \cdot 7818}$

$a \approx 6$

$b = \ln a$

$= \ln 6$

$\therefore b = 1 \cdot 791\,8$

[3]

c) Determine the area of the shaded region (to two decimal places).

$\therefore \int_0^{1 \cdot 7918} (-e^x + 6)\, dx$

$\approx 5 \cdot 75 \text{ units}^2$ **(-1 overall if no units)**

[1]

Question 18: **[14 marks]**

The two countries Zedlandia and Xenutia are at war. Zedlandia (z) fired a ground to air missile in order to intercept a missile coming in from Xenutia (x). When z was launched, the position vectors (in metres), relative to the army base were:

$$r_z = \begin{pmatrix} 720 \\ 0 \\ 0 \end{pmatrix} \text{ and } r_x = \begin{pmatrix} 3000 \\ 4800 \\ 650 \end{pmatrix}.$$

Both z and x have constant velocities (m/s):

$$v_z = \begin{pmatrix} -200 \\ 208 \\ 40 \end{pmatrix} \text{ and } v_x = \begin{pmatrix} -255 \\ 127 \\ 3 \end{pmatrix}.$$

a) Prove that the two missiles *did not* intercept.

$$\underset{\sim}{r}_z(t) = \begin{pmatrix} 720 - 200\lambda \\ 208\lambda \\ 40\lambda \end{pmatrix} \quad ✓$$

$$\underset{\sim}{r}_x(t) = \begin{pmatrix} 3\,000 - 255\mu \\ 4800 + 127\mu \\ 650 + 3\mu \end{pmatrix}$$

$\underset{\sim}{i}$: $720 - 200\lambda = 3\,000 - 255\mu$ ✓

$\underset{\sim}{j}$: $208\lambda = 4\,800 + 127\mu$

$$\lambda = \frac{37\,839}{691} \text{ and } \mu = \frac{35\,856}{691} \quad ✓$$

$\underset{\sim}{k}$: $40\lambda = 650 + 3\mu$

$$\frac{15\,135\,600}{691} \neq \frac{556\,718}{691} \quad ✓$$

$\therefore$ Missiles do not intersect

[4]

Question 18 cont…

b) Determine by how much (distance and time) the two missiles missed each other (Note: Enough working must be shown in order to gain full marks).

$${}_zV_x = V_z - V_x$$
$$= \begin{pmatrix} -200 \\ 208 \\ 40 \end{pmatrix} - \begin{pmatrix} -255 \\ 127 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 55 \\ 81 \\ 37 \end{pmatrix} \checkmark$$

$$\overrightarrow{ZX} = \overrightarrow{ZO} + \overrightarrow{OX}$$
$$= -\begin{pmatrix} 720 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3\,000 \\ 4\,800 \\ 650 \end{pmatrix}$$
$$= \begin{pmatrix} 2\,280 \\ 4\,800 \\ 650 \end{pmatrix} \checkmark$$

Let P be the point that is on Z and forms the shortest distance between the missiles.

$$\overrightarrow{PX} = -V_x + \overrightarrow{ZX}$$
$$= -t\begin{pmatrix} 55 \\ 81 \\ 37 \end{pmatrix} + \begin{pmatrix} 2\,280 \\ 4\,800 \\ 650 \end{pmatrix}$$
$$= \begin{pmatrix} 2\,280 - 55t \\ 4\,800 - 81t \\ 650 - 37t \end{pmatrix} \checkmark$$

$${}_zV_x \,.\, \overrightarrow{PX} = 0$$
$$= \begin{pmatrix} 55 \\ 81 \\ 37 \end{pmatrix} . \begin{pmatrix} 2\,280 - 55t \\ 4\,800 - 81t \\ 650 - 37t \end{pmatrix} = 0$$

$$538\,250 - 10\,955\,t = 0$$

$$t \approx \frac{538\,250}{10\,955}$$
$$t \approx 49{\cdot}13 \text{ secs} \checkmark$$

$$\overrightarrow{PX} \approx \begin{pmatrix} -422{\cdot}3 \\ 820{\cdot}2 \\ -1167{\cdot}9 \end{pmatrix} \checkmark$$

$$\left|\overrightarrow{PX}\right| \approx 1\,488{\cdot}3 \text{ m} \checkmark$$

(-1 overall if no units)

[6]

Question 18 cont…

c) In order for the two missiles to collide, the *z* missile changed its flight plan after 15 seconds. The *x* continued on its path and then interception occurred 25 seconds later. Determine the constant velocity that *z* maintained in this second stage for interception to occur.

Method 1:

$$\underset{\sim}{r}_z(15) = \begin{pmatrix} 20 - 20 \times 15 \\ 208 \times 15 \\ 40 \times 15 \end{pmatrix} = \begin{pmatrix} -2\,280 \\ 3\,120 \\ 600 \end{pmatrix} \checkmark$$

$$\underset{\sim}{r}_x(40) = \begin{pmatrix} 3\,000 - 255 \times 40 \\ 4\,800 + 127 \times 40 \\ 650 + 3 \times 40 \end{pmatrix} = \begin{pmatrix} -7\,200 \\ 9\,880 \\ 770 \end{pmatrix} \checkmark$$

$$\overrightarrow{ZX} = \overrightarrow{ZO} + \overrightarrow{OX}$$

$$= -\underset{\sim}{r}_z(15) + \underset{\sim}{r}_x(40)$$

$$= -\begin{pmatrix} -2\,280 \\ 3\,120 \\ 600 \end{pmatrix} + \begin{pmatrix} -7\,200 \\ 9\,880 \\ 770 \end{pmatrix}$$

$$= \begin{pmatrix} -4\,920 \\ 6\,760 \\ 170 \end{pmatrix} \checkmark$$

$$V = \frac{s}{t}$$

$$= \frac{< -4\,920,\ 6\,760,\ 170 >}{25}$$

$$\approx \begin{pmatrix} -196{\cdot}8 \\ 270{\cdot}4 \\ 6{\cdot}8 \end{pmatrix} \text{ m/s} \checkmark$$

Method 2:

$$\underset{\sim}{r}_z(40) = \begin{pmatrix} 720 \\ 0 \\ 0 \end{pmatrix} + 15\begin{pmatrix} -200 \\ 208 \\ 40 \end{pmatrix} + 25\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} -2\,280 \\ 3\,120 \\ 600 \end{pmatrix} + 25\begin{pmatrix} x \\ y \\ z \end{pmatrix} \checkmark$$

$$\underset{\sim}{r}_x(40) = \begin{pmatrix} 3\,000 \\ 4\,800 \\ 650 \end{pmatrix} + 40\begin{pmatrix} -255 \\ 127 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -7\,200 \\ 9\,880 \\ 770 \end{pmatrix} \checkmark$$

At collision $\underset{\sim}{r}_z(40) = \underset{\sim}{r}_x(40)$

$$\begin{pmatrix} -2\,280 \\ 3\,120 \\ 600 \end{pmatrix} + 25\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7\,200 \\ 9\,880 \\ 770 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -196{\cdot}8 \\ 270{\cdot}4 \\ 6{\cdot}8 \end{pmatrix}$$

$$\therefore \underset{\sim}{v}_z = \begin{pmatrix} -196{\cdot}8 \\ 270{\cdot}4 \\ 6{\cdot}8 \end{pmatrix} \text{ m/s} \checkmark$$

(-1 overall if no units)

[4]

END OF EXAM