Rossmoyne Senior High School

Semester One Examination, 2017

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section Two: Calculator-assumed

Teacher name Solutions.

Your name ______________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	11	11	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **eleven (11)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(7 marks)**

The voltage between the plates of a discharging capacitor can be modelled by the function $V(t) = 14e^{kt}$, where V is the voltage in volts, t is the time in seconds and k is a constant.

It was observed that after three minutes the voltage between the plates had decreased to 0.6 volts.

(a) State the initial voltage between the plates. (1 mark)

14 V ✓

✓ states value (units not required)

(b) Determine the value of k. (2 marks)

$0.6 = 14e^{180k}$ ✓

$k = -0.0175$ ✓

✓ writes equation
✓ solves,

(c) How long did it take for the initial voltage to halve? (2 marks)

$0.5 = e^{-0.0175t}$ ✓

$t = 39.6$ s ✓

✓ writes equation
✓ solves,

(d) At what rate was the voltage decreasing at the instant it reached 8 volts? (2 marks)

$V'(t) = kV$

$= -0.0175 \times 8 = -0.14$ ✓

Decreasing at 0.14 V/s ✓

✓ uses rate of change
✓ states decrease, dropping negative sign, must give rate v/s

Question 10 **(11 marks)**

The gradient function of f is given by $f'(x) = 12x^3 - 24x^2$.

(a) Show that the graph of $y = f(x)$ has two stationary points. (2 marks)

$12x^2(x-2) = 0$ ✓

$x = 0, \quad x = 2$ ✓

Must have at least both lines

✓ equates derivative to zero and factorises
✓ shows two solutions

(b) Determine the interval(s) for which the graph of the function is concave upward. (3 marks)

$f''(x) = 36x^2 - 48x$ ✓

$36x^2 - 48x > 0$ ✓

$x < 0, \quad x > \frac{4}{3}$ ✓

✓ shows condition for concave upwards
✓ uses second derivative
✓ states intervals

(c) Given that the graph of $y = f(x)$ passes through $(1, 0)$, determine $f(x)$. (2 marks)

$\int 12x^3 - 24x^2 \, dx$

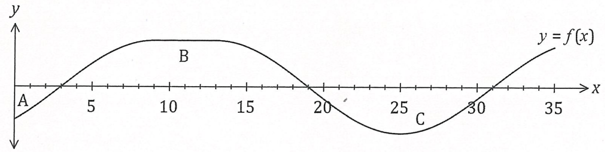
$= 3x^4 - 8x^3 + c$ ✓

$f(x) = 3x^4 - 8x^3 + 5$ ✓

✓ integrates $f'(x)$
✓ determines constant

Question 19 **(9 marks)**

The graph of $y = f(x)$ is shown below. The areas between the curve and the $x -$ axis for regions A, B and C are 3, 20 and 12 square units respectively.



(a) Evaluate

(i) $\int_0^{31} f(x)\,dx$. (1 mark)

5 ✓

(ii) $\int_{19}^{0} f(x)\,dx$. (2 marks)

$= -\int_0^{19} f(x)\,dx$ ✓ $= 17$ ✓

✓ reverses limits and negates
✓ sums signed areas

(iii) $\int_3^{31} 2 - 3f(x)\,dx$. (3 marks)

$\int_3^{31} 2 - 3\int_3^{31} f(x)\,dx$ ✓

$= 56 - 3 \times 8$ ✓ ✓

$= 32$

✓ splits integral and takes difference
✓ rectangle
✓ function

It is also known that $A(31) = 0$, where $A(x) = \int_{10}^{x} f(t)\,dt$.

(b) Evaluate

(i) $A(19)$. (1 mark)

12 ✓

(ii) $A(0)$. (2 marks)

$A(3) = -\int_3^{10} f(x)\,dx = -8$ ✓

$A(0) = -\int_3^{10} f(x)\,dx + -\int_0^{3} f(x)\,dx = -5$ ✓

Question 18 **(8 marks)**

A storage container of volume 36π cm^3 is to be made in the form of a right circular cylinder with one end open. The material for the circular end costs 12c per square centimetre and for the curved side costs 9c per square centimetre.

(a) Show that the cost of materials for the container is $12\pi r^2 + \dfrac{648\pi}{r}$ cents, where r is the radius of the cylinder. (4 marks)

$$36\pi = \pi r^2 h$$

$$h = \frac{36}{r^2}$$

$$C = 12 \times \pi r^2 + 9(2\pi r h)$$

$$= 12 \times \pi r^2 + 9\left(2\pi r\left(\frac{36}{r^2}\right)\right)$$

$$= 12\pi r^2 + \frac{648\pi}{r}$$

✓ uses volume formula ✓ expression for h in terms of r ✓ uses area formula adjusted for one end and cost ✓ substitutes for h in cost formula

(b) Use calculus techniques to determine the dimensions of the container that minimise its material costs and state this minimum cost. (4 marks)

$$\frac{dC}{dr} = \frac{24\pi r^3 - 648\pi}{r^2}$$

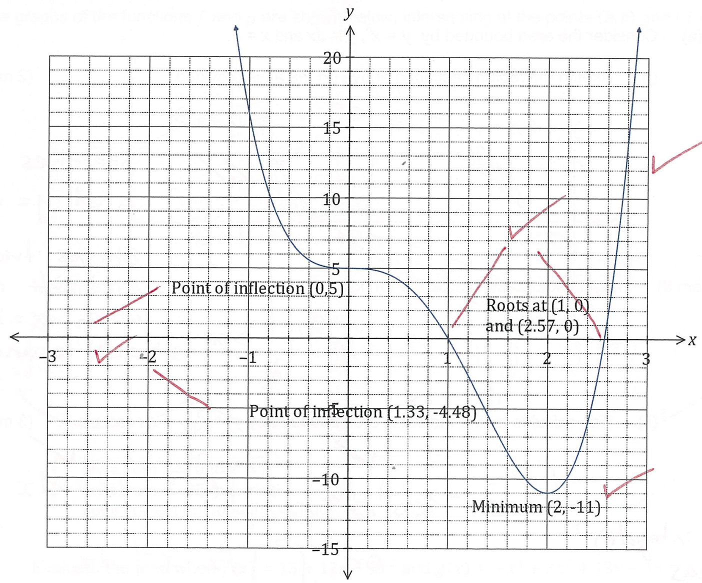
$$\frac{24\pi r^3 - 648\pi}{r^2} = 0, \quad r = 3\text{ cm}$$

$C(3) = 324\pi$ ¢ ie $10.18

Min cost $10.18, when $r = 3$cm $h = 4$cm

✓ differentiates ✓ equates $C'(r) = 0$ and solves for r ✓ determines min cost ✓ states dimensions

(d) Sketch the graph of $y = f(x)$, indicating all key features. (4 marks)

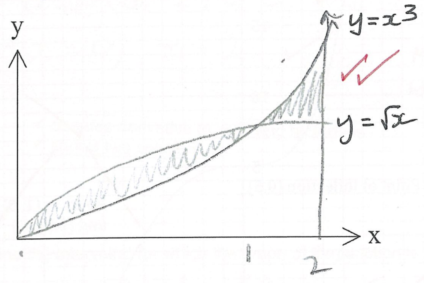


Solution
See graph
Specific behaviours
✓ minimum ✓ roots ✓ points of inflection ✓ smooth curve

Question 11 **(8 marks)**

(a) Consider the area bounded by $y = x^3$, $y = \sqrt{x}$ and $x = 2$.

(i) Sketch the region described above on the axes provided. (2 marks)



Sketches $y=x^3$ and $y=\sqrt{x}$ ✓

Shows two distinct regions with $x=2$ as one boundary ✓

(ii) Use calculus to find the exact area bounded by $y = x^3$, $y = \sqrt{x}$ and $x = 2$. (3 marks)

2.948

$$\int_0^1 (\sqrt{x} - x^3)\,dx + \int_1^2 (x^3 - \sqrt{x})\,dx$$

$$\text{or } \int_0^2 |\sqrt{x} - x^3|\,dx$$

✓✓ Gives integral(s) for areas

✓ evaluates exactly

$$= \frac{5}{12} + \frac{53}{12} - \frac{4\sqrt{2}}{3} = \frac{29}{6} - \frac{4\sqrt{2}}{3}$$

(b) The marginal cost function for producing x electronic components per day is $M_c(x) = \frac{100}{\sqrt{x}} + 150$.

Determine the cost of increasing production from 100 components per day to 400 components per day. (3 marks)

$$\int_{100}^{400} \left(\frac{100}{\sqrt{x}} + 150\right) dx$$

$$= 47\,000$$

✓ writes integral

✓ gives bounds

✓ evaluates

(e) Let Y be a Bernoulli random variable with parameter $p = P(A)$. Determine the mean and standard deviation of Y. (2 marks)

$$\mu = \frac{5}{42} \doteq 0.119$$

$$\sigma = \sqrt{\frac{5}{42} \times \frac{37}{42}} = 0.324$$

(f) Determine the probability that A occurs no more than twice in ten random selections of four letters from those in the word LOGARITHM. (2 marks)

$$W \sim B\left(10, \frac{5}{42}\right)$$

$$P(0 \le W \le 2) = 0.8933$$

Question 17 **(10 marks)**

Let the random variable X be the number of vowels in a random selection of four letters from those in the word LOGARITHM, with no letter to be chosen more than once.

(a) Complete the probability distribution of X below. (1 mark)

x	0	1	2	3
$P(X = x)$	$\frac{5}{42}$	$\frac{10}{21}$	$\frac{5}{14}$	$\frac{1}{21}$

(b) Show how the probability for $P(X = 1)$ was calculated. (2 marks)

$$\frac{{}^3C_1 \times {}^6C_3}{{}^9C_4} = \frac{60}{126} = \frac{10}{21}$$

✓ uses combinations for numerator
✓ uses combinations for denominator and simplifies

(c) Determine $P(X \geq 1 | X \leq 2)$. (2 marks)

$$\frac{P(1 \leq x \leq 2)}{P(x \leq 2)} = \frac{\frac{10}{21} + \frac{5}{14}}{\frac{20}{21}} = \frac{7}{8}$$

✓ obtains numerator
✓ obtains denominator and simplifies

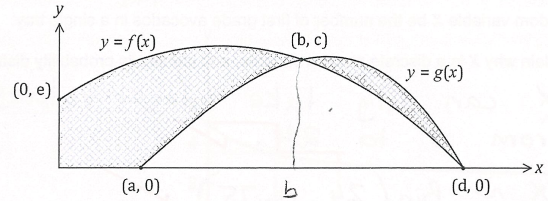
Let event A occur when no vowels are chosen in random selection of four letters from those in the word LOGARITHM.

(d) Find $P(\bar{A})$. (1 mark)

$$\frac{37}{42}$$

Question 12 **(7 marks)**

The graphs of the functions f and g are shown below, intersecting at the points (b, c) and $(d, 0)$.



(a) Using definite integrals, write an expression for the area of the shaded region. (3 marks)

$$\int_0^b f(x)\,dx - \int_a^b g(x)\,dx + \int_b^d g(x) - f(x)\,dx$$

✓ area from $x = 0$ to $x = b$
✓ area from $x = b$ to $x = d$
✓ uses correct notation throughout eg dx

(b) Evaluate the area when $f(x) = 15 + 12x - 3x^2$ and $g(x) = -x^3 + 3x^2 + 13x - 15$. (4 marks)

Solution
$a = 1, b = 3, d = 5$
$\int_0^3 f(x)\,dx - \int_1^3 g(x)\,dx = 72 - 28 = 44$
$\int_3^5 (g(x) - f(x))\,dx = 8$
Total area $= 44 + 8 = 52$ sq units
Specific behaviours
✓ determines values of a, b and d ✓ area from $x = 0$ to $x = b$ ✓ area from $x = b$ to $x = d$ ✓ correct area

Question 13 **(9 marks)**

75% of the avocados produced by a farm are known to be first grade, the rest being second grade. Trays of 24 avocados are filled at random in a packing shed and sent to market.

Let the random variable X be the number of first grade avocados in a single tray.

(a) Explain why X is a discrete random variable, and identify its probability distribution. (2 marks)

X can only take integer values from 0 to 24. ✓

$X \sim Bin(24, 0.75)$ ✓

(b) Calculate the mean and standard deviation of X. (2 marks)

$\bar{x} = 18$ ✓

$\sigma = \frac{3\sqrt{2}}{2}$ or 2.12 ✓

(c) Determine the probability that a randomly chosen tray contains

(i) 18 first grade avocados. (1 mark)

$P(x=18) = 0.1853$ ✓

(ii) more than 15 but less than 20 first grade avocados. (2 marks)

$P(16 \leq x \leq 19)$ ✓

$= 0.6320$ ✓

(d) In a random sample of 1000 trays, how many trays are likely to have fewer first grade than second grade avocados. (2 marks)

$P(x \leq 11) = 0.0021$ ✓

$0.0021 \times 1000 \approx 2$ trays ✓

(d) Calculate the change of displacement of P during the third second. (2 marks)

$$\int_2^3 \left(\frac{-4}{(2t+1)^2} + 1\right) dt$$ ✓

$= 31/35 \sim 0.886\text{m}$ ✓

✓ uses correct bounds
✓ integrates to find change in displacement

(e) Determine the maximum speed of P during the first three seconds and the time when this occurs. (2 marks)

$|v(0)| = 3\text{ m/s}$

$|v(3)| = \frac{45}{49} \approx 0.9\text{ m/s}$ ✓

Max Speed 3 m/s ✓

✓ examines v at endpoints
✓ determines maximum speed

(f) Calculate the total distance travelled by P during the first three seconds. (2 marks)

$$\int_0^3 \left|\frac{-4}{(2t+1)^2} + 1\right| dt$$

$= \frac{16}{7} \sim 2.286\text{ m}$

$x(0) = 4$

$x(0.5) = 3.5$

$x(3) = 5\frac{2}{7}$

$\frac{1}{2} + (5\frac{2}{7} - 3\frac{1}{2})$

$= 16/7$

✓ uses integral(s) to determine distance
✓ evaluates distance

Question 16 **(12 marks)**

Particle P leaves point A at time $t = 0$ seconds and moves in a straight line with acceleration given by

$$a = \frac{16}{(2t+1)^3} \text{ ms}^{-2}.$$

Particle P has an initial velocity of -3 ms^{-1} and point A has a displacement of 4 metres from the origin.

(a) Calculate the initial acceleration of P. (1 mark)

16 m/s

(b) Is P ever stationary? If your answer is yes, determine the time(s) when this happens. If your answer is no, explain why. (3 marks)

$a = 16(2t+1)^{-3}$

$v = -4(2t+1)^{-2} + c$

$-3 = \frac{-4}{(2(0)+1)^2} + c$

$c = 1$

$v = \frac{-4}{(2t+1)^2} + 1 = 0$

$\therefore t = 0.5$ Yes

✓ integrates to find velocity
✓ correct constant
✓ solves for zero

(c) Calculate the displacement of P when $t = 12$ seconds. (2 marks)

$x = \frac{2}{(2t+1)} + t + c$

$4 = 2 + c$

$c = 2$

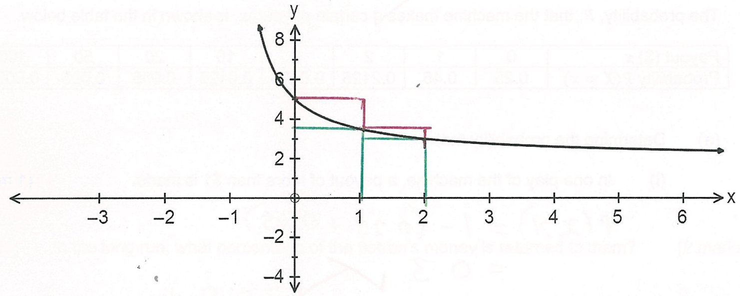
$x = \frac{2}{(2t+1)} + t + 2$

$= 14\ 2/25$

Solution
$\Delta x = \int_0^{12} v\,dt = 10.08$ $x(12) = 4 + 10.08 = 14.08$ m
Specific behaviours
✓ integrates to find change in displacement   ✓ calculates actual displacement

Question 14 **(7 marks)**

Consider the function $y = \frac{3}{x+1} + 2$ graphed below.



(a) Complete the table of values: (1 mark)

x	y
0	5
1	3.5
2	3
3	2.75
4	2.6

(b) Use 4 upper rectangles and 4 lower rectangles to approximate the area under the curve from $0 \le x \le 4$. (5 marks)

Lower

$A = 1 \times (f(1) + f(2) + f(3) + f(4))$

$= 3.5 + 3 + 2.75 + 2.6$

$= 11.85$

Upper

$A = 1 \times (f(0) + f(1) + f(2) + f(3))$

$= 5 + 3.5 + 3 + 2.75$

$= 14.25$

$\therefore A \sim \frac{14.25 + 11.85}{2}$

~ 13.05 units^2

(c) Suggest one change to the above procedure to improve the accuracy of the estimate. (1 mark)

* More rectangles by using smaller rectangles.

Question 15 **(10 marks)**

A slot machine is programmed to operate at random, making various payouts after patrons pay $2 and press a start button. The random variable X is the amount of the payout, in dollars, in one play of the machine. Each payout can be assumed to be independent of other payouts.

The probability, P, that the machine makes a certain payout, x, is shown in the table below.

Payout ($) x	0	1	2	5	10	20	50	100
Probability $P(X = x)$	0.25	0.45	0.2125	0.0625	0.0125	0.005	0.005	0.0025

(a) Determine the probability that

(i) in one play of the machine, a payout of more than $1 is made. (1 mark)

$P(x>1) = 1-(0.25+0.45)$
$= 0.3$ ✓

(ii) in ten plays of the machine, it makes a payout of $5 no more than once. (2 marks)

$X \sim Bin(10, 0.0625)$ ✓
$P(0 \le x \le 1) = 0.8741$ ✓

(iii) in five plays of the machine, the second payout of $1 occurs on the fifth play. (3 marks)

$X \sim Bin(4, 0.45)$
$P(x=1) = 0.2995$ ✓
$P = 0.45 \times 0.2995$ ✓
$= 0.1348$ ✓

✓ uses first and second event
✓ calculates P for first event
✓ calculates P for both events

(b) Calculate the mean and standard deviation of X. (2 marks)

$\mu = 1.9125$ ✓
$\sigma = 6.321$ ✓

(c) In the long run, what percentage of the patron's money is returned to them? (2 marks)

$\frac{1.9125}{2} \times 100$ ✓
$= 95.625\%$ ✓

✓ uses mean and payment
✓ calculates percentage