WESLEY COLLEGE
SOUTH PERTH

MATHEMATICS
3C/3D

Semester Two Examination 2012

Question/Answer Booklet

Section Two:
Calculator-Assumed

Student Name: Solutions / Marking Guide

Teacher Name ______________________

Time allowed for this section
Reading time before commencing work: Ten (10) minutes
Working time for this section: One Hundred (100) minutes

Material required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

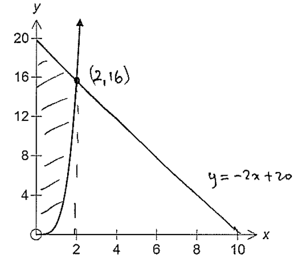
To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: Classpad, Scientific Calculator and 2 pages of back–to–back notes

Important note to candidates
No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question 1 (3, 4, 1, 2, 1 = 11 marks)

A water tank is obtained by revolving the curve $y = x^4$ about the y–axis.



(a) Express the exact volume of water in the tank as a function of its depth h

$V_y = \pi \int_0^h x^2 \, dy$

$= \pi \int_0^h \sqrt{y} \, dy$

$= \pi \left[\frac{2y^{3/2}}{3} \right]_0^h$

$V_y = \frac{2h^{3/2}}{3} \pi$

(b) Water drains through a small hole in its base at a rate according to:

$$\frac{dv}{dt} = -\sqrt{h} \quad litres / hr$$

Show that the water level falls at a constant rate.

$\frac{dv}{dh} = h^{1/2} \cdot \pi$

$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$

$= \frac{1}{\pi \cdot \cancel{\sqrt{h}}} \cdot -\cancel{\sqrt{h}}$

$= -\frac{1}{\pi}$ (independent of h)

∴ water level falls at a constant rate

(c) On the diagram above, draw in a second function $2x + y = 20$

(d) Write an expression for the area enclosed between the two functions and the **y–axis**

$A = \int_0^2 (-2x+20) \, dx - \int_0^2 x^4 \, dx$

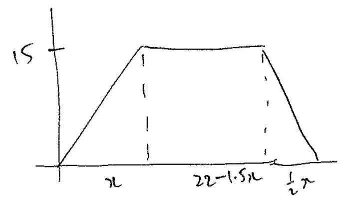
(e) Calculate this area.

$A = 29.6$ sq units.

Question 13 (2, 4 = 6 marks)

A train starts from rest and moves with a constant acceleration until it reaches a speed of 15 m/sec. It continues at this speed for a period of time, after which it is brought to rest with a constant retardation or de-acceleration. The total time taken is 22 seconds and the distance travelled is 240 m. If the time taken for the retardation is half that for the acceleration:

(a) Sketch a velocity – time graph



(b) Determine the amount of time the train takes to accelerate to its maximum speed.

Area under vel curve = displacement

ie $\frac{1}{2} \cdot x \cdot 15 + \frac{1}{2}(22 - 1.5x) \cdot 15 + \frac{1}{2} \cdot \frac{1}{2}x \cdot 15 = 240$

ie $22 - 0.75x = 16$

Solve: $x = 8$

∴ train takes 8 sec to accel to max speed.

END OF SECTION 2

Question 12 (2, 3 = 5 marks)

The safe lifetime of standard radial tyres under regular driving conditions are normally distributed with a mean 20 000 km and a standard deviation of 1500 km

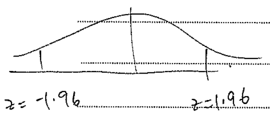
(a) What is the probability that a standard radial tyre will last longer than 21 000 km if it has already lasted 18 000 km

$P(X > 21000 \mid X > 18000) = \dfrac{P(X > 21000)}{P(X > 18000)}$

$= 0.2778$

The tyre company also produce the Extra Grip radial tyre, whose lifetimes are also normally distributed with the same standard deviation of 1500 km but with a possibly different mean μ hours. A quality control expert at the company wishes to estimate μ using the mean lifetime of a random sample of Extra Grip radial tyres.

(b) How large should the sample be in order to be at least 95% confident that the estimate will be no more than 200 km in error?



We want $1.96 \times \dfrac{\sigma}{\sqrt{n}} = 200$

ie $\sqrt{n} = \dfrac{1.96 \times 1500}{200}$

$\Rightarrow n \approx 216.09$

$\therefore$ 217 req to be at least 95% confident

Question 2 (3, 3, 2, 3 = 11 marks)

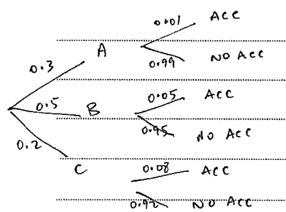
Unknown to its customers, a motor vehicle insurance company classifies its drivers as

CLASS A	–	good risks	
CLASS B	–	medium risks	
CLASS C	–	poor risks	based on factors such as age, previous history …

It believes that 30% of the drivers who apply for insurance are CLASS A risks, 50% are CLASS B risks and 20% are CLASS C risks.

The probability that a CLASS A driver will have 1 or more accidents in any 12 month period is 0.01. For a CLASS B driver the probability is 0.05 and for a CLASS C driver the probability is 0.08. Using a tree diagram or otherwise:

(a) Determine the probability that a driver will have at least one accident within 12 months of buying an insurance policy from the company



$P(Acc) = 0.3 \times 0.01 + 0.05 \times 0.5 + 0.2 \times 0.08$

$= 0.044$

(b) If the company sells someone an insurance policy and within 12 months that person has an accident, what is the probability that the driver is a CLASS C risk?

$P(C \mid Acc) = \dfrac{P(C \cap Acc)}{P(Acc)}$

$= \dfrac{0.2 \times 0.08}{0.044}$

$= 0.\overline{36}$

(c) On a certain day, 6 different people phone in and apply for insurance from the company. Determine the probability that of those 6

(i) less than three of them are good risks

$P(X \le 2) = 0.7443$

(ii) the 5th phone call turned out to be the 3rd medium risk driver to apply

$= P(2 \text{ from } 1^{st} 4 \text{ calls}) \times P(5^{th} \text{ is med risk})$

$= 0.375 \times 0.5$

$= 0.1875$

Question 3 (3, 6 = 9 marks)

(a) Prove, showing all working, that for any real number $x\ (x \neq 0)$ the sum of the reciprocals of $(1+x)$ and $\left(1+\dfrac{1}{x}\right)$ is always constant

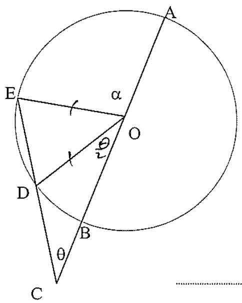
$$\frac{1}{1+x} + \frac{1}{1+\frac{1}{x}} = \frac{1}{1+x} + \frac{1}{\frac{x+1}{x}}$$

$$= \frac{1}{1+x} + \frac{x}{x+1}$$

$$= \frac{1+x}{1+x}$$

$$= 1 \qquad \therefore \text{ Always constant}$$

(b)



In the diagram O is the centre of the circle, AOBC and CDE are straight lines, $\angle AOE = \alpha$ and $\angle DCB = \theta$ In addition, $\angle DCB = 2 \times \angle BOD$

Prove that $2\alpha = 5\theta$

In ΔODC $\quad \angle ODC = 180 - \frac{3\theta}{2}$ $\quad$ Sum of Angles $\Delta = 180°$

$\therefore \angle EDO = \frac{3\theta}{2}$ $\quad$ Supplementary Angle

$\therefore \angle DEO = \frac{3\theta}{2}$ $\quad$ OE = OD Equal radii

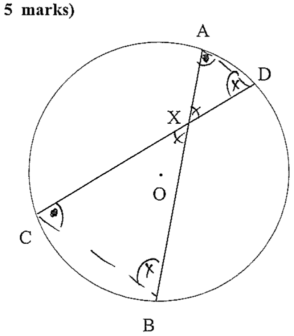
$\therefore \angle DOE = 180 - 3\theta$ $\quad$ Sum of angles $\Delta ODE = 180°$

$\alpha + \cancel{180} - 3\theta + \frac{\theta}{2} = \cancel{180°}$ $\quad$ AOB is a str angle = 180°

$\alpha - \frac{5\theta}{2} = 0$

$\Rightarrow \alpha = \frac{5\theta}{2}$ $\quad$ as req.

Question 11 (3, 2 = 5 marks)



AB and CD are two chords of a circle centre O, that intersect at point X

(a) Prove that $\Delta ADX \sim \Delta CBX$

In Δ's ADX, CBX

any two req

$\angle BCD \cong \angle DAX$ $\quad$ Angles in same arc

$\angle CBA \cong \angle CDA$ $\quad$ " " " "

$\angle AXD \cong \angle CXB$ $\quad$ Vert opp Angles

$\Delta ADX \cong \Delta CBX$ $\quad$ (A.A similarity)

(b) Hence or otherwise, find AX if BX = 12 CX = 9 and DX = 4

Since Δ's are similar

then $\dfrac{AX}{CX} = \dfrac{DX}{BX}$

$\dfrac{AX}{9} = \dfrac{4}{12}$

$\Rightarrow AX = 3$

Question 10 (1, 1, 2, 5 = 9 marks)

(a) In a game involving two players, the players take turns to toss three fair coins. They keep tossing the coins until they all appear as heads. The player who makes such a toss is the winner.

(i) Calculate p_1 the probability that the game is won on the first toss

$p_1 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

(ii) Calculate p_2 the probability that the game is won on the second toss

$p_2 = \frac{7}{8} \times \frac{1}{8}$

$= \frac{7}{64}$

(iii) Write a formula for p_n the probability that the game is won on the n[th] toss

$p_n = \left(\frac{7}{8}\right)^{n-1} \times \frac{1}{8}$ or $\frac{7^{n-1}}{8^n}$

(b) For two independent events A and B, $P(A \cup B) = 0.7$ and $P(A \cap B) = 0.15$

Determine $P(A)$ and $P(B)$

$0.7 = P(A) + P(B) - 0.15$

$\therefore \quad P(A) + P(B) = 0.85$

and $\quad P(A) \times P(B) = 0.15$

ie $\quad (0.85 - P(B)) \times P(B) = 0.15$

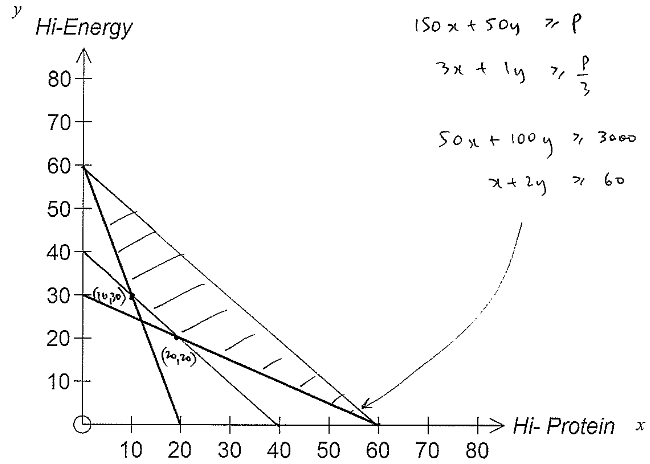
Solving (classpad) $\quad P(B) = 0.25$ or 0.6

$\therefore \quad P(A) = 0.6$ or 0.25

$(P(A), P(B))$ sol^ns are $(0.6, 0.25)$ or $(0.25, 0.6)$

Question 4 (2, 3, 3, 3 = 11 marks)

A health food shop packages and sells two different blends of nuts and sultanas.
A packet of Hi–Protein contains 150g of nuts and 50g of sultanas.
A packet of Hi–Energy contains 50g of nuts and 100g of sultanas.
The manager estimates that the shop would need to sell a total of at least 40 packets each day to be profitable but the shop would be unlikely to sell more than 60 packets in total.
Because of commitments to suppliers, the shop must package at least 'p' kg of nuts and at least 3 kg of sultanas each day. Let x represent the number of packets of Hi–Protein and y the number of packets of Hi–Energy sold each day.



$150x + 50y \geq p$

$3x + 1y \geq \frac{p}{3}$

$50x + 100y \geq 3000$

$x + 2y \geq 60$

(a) In the above graph two of the lines determining the boundaries of the feasible region have been drawn. Determine the value of 'p'

$p = 3$

$150x + 50y \geq p$

$50(3x + y) \geq 60 \times 50$

(b) Write down and then draw in the other two boundary lines and indicate clearly the feasible region.

$x + y \geq 40$

$x + y \leq 60$

The profit on each packet of Hi–Protein is $1.20 and on each packet of Hi–Energy $1.60.

(c) Using the feasible region from (b), determine the **least** possible profit and the number of packets of each blend that would need to be sold to achieve this profit.

Test Pts	$P = 1.20x + 1.60y$
(0,60)	\$96
(10,30)	\$60
(20,20)	\$56
(60,0)	\$72

∴ least profit of \$56 with 20 of each packet.

(d) If the profit on each packet of Hi–Energy remains at $1.60, to what value must the profit on each packet of Hi–Protein fall in order that your solution to (c) is not unique

Gradient approach : if profit on Hi-Prot falls gradient of Profit line decreases

Profit $= mx + 1.60y$

∴ Sol^n ⇒ (60,0)

gradient (20,20) → (60,0) $= -\frac{1}{2}$

∴ $-\frac{m}{1.6} = -\frac{1}{2}$

⇒ $m = 0.8$

∴ Profit must fall by 80c in order for sol^n (c) is not unique.

Question 9 (1, 3, 3 = 7 marks)

The amount A of a drug in the bloodstream will decline at a rate proportional to the current amount. That is $\frac{dA}{dt} = -\left(\frac{1}{E}\right)A$ where E hours is a constant called the elimination time.

(a) write down the formula for $A(t)$ the amount of the drug in the bloodstream after t hours, in terms of t, E and the initial amount A_0

$A(t) = A_0 e^{-\frac{1}{E} \cdot t}$

(b) what percentage of the drug (correct to two decimal places), remains after E hours?

when $t = E$ $\quad A = A_0 e^{-\frac{E}{E}}$

$\frac{A}{A_0} = e^{-1}$

$= 0.3679$

⇒ 36.79% of drug remains

The drug sodium pentobarbitol can be used to tranquillize animals. A dog is tranquillized if its bloodstream contains at least 45 milligrams of the drug for each kilogram of the dog's weight. The elimination time for the drug is 6 hours.

(c) what single dose of this drug should be given in order to tranquillize a 12 kg dog for 1 hour?

Given $E = 6$ + $A(t) = A_0 e^{-\frac{1}{E}t}$

Amt of drug req when $t = 1$ is 12×45 mg

∴ $12 \times 45 \text{ mg} = A_0 e^{-\frac{1}{6}}$

∴ $A_0 = 12 \times 45 \times e^{\frac{1}{6}}$

$= 638$ mg

∴ Initial dose of 638 mg req.

Question 8 (1, 1, 2, 3 = 7 marks)

650 g labelled cartons of free-range farm eggs are filled automatically according to the uniform probability density function:

$$P(X = x) = \begin{cases} \frac{1}{12} & 644 \le x \le 656 \\ 0 & elsewhere \end{cases}$$

(a) determine the probability that the carton will be filled with eggs weighing less than 647.5 g

$\frac{3.5}{12} = \frac{7}{24}$

If the standard deviation of a uniform distribution

$$P(X = x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & elsewhere \end{cases}$$ is given by $st\ dev = \sqrt{\frac{(b-a)^2}{12}}$

(b) Calculate the standard deviation for this distribution

$\sqrt{12}$

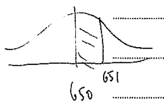
Several samples of 40 cartons each were examined by Government authorities and the mean weight of each sample calculated.

(c) Describe the probability distribution that best models this distribution of sample means

$\sim$ Normal Distⁿ $\quad \mu = 650 \quad \sigma = \frac{\sqrt{12}}{\sqrt{40}}$

(d) Find the probability that a randomly chosen sample has a mean no less than 650 g given it is no more than 651 g

$P(x > 650 \mid x < 651) = \frac{P(650 < x < 651)}{P(x < 651)}$



$= \frac{0.466}{0.966}$

$= 0.482$ (to 3 dec pl)

Question 5 (5, 2 = 7 marks)

After clearing all the trees from a river valley, it has been found that the rate of increase in the concentration of salt in the topsoil at a given distance from the riverbank, d metres, is given by $\frac{dS}{dt} = \frac{e^{bt}}{d+1}$ for $0 < d < 100$

where b = a real constant
S = salt concentration in parts per million
t = time in years after clearing

Immediately after the clearing of the valley, the following measurements were taken: at $d = 1$ m S was 100 parts per million of salt and at $d = 9$ m, S was 90 parts per million of salt.

(a) Determine the expression for the concentration of S in terms of t and d using the initial measurements given.

$S = \int \frac{e^{bt}}{d+1}\, dt$

$= \frac{e^{bt}}{b\cdot(d+1)} + c$

$t = 0$
$d = 1$
$S = 100$

$100 = \frac{1}{2b} + c$
$90 = \frac{1}{10b} + c$
$b = \frac{1}{25}$
$c = 87.5$

$\therefore S = \frac{e^{0.04t}}{0.04(d+1)} + 87.5$

(b) Find the concentration of salt expected 50 years after clearing and at a position of 30 m from the riverbank.

$S = \frac{e^{2}}{(0.04) \times 31} + 87.5$

$= 93.46$

Question 6 (1, 1, 2, 2 = 6 marks)

A television director for the closing ceremony of the Olympic Games has two different commercials for each of 6 products. These are to be shown during three commercial breaks. Each break has four commercials and each commercial is shown only once. It is possible for both commercials for the same product to be shown during the same break.

The director places the commercials into groups of four, choosing the commercials for the first break, then choosing the four commercials for the second break and finally the four for the last break.

(a) How many groups could the director choose for the

(i) 1st break

$\binom{12}{4} = 495$

(ii) 2nd break

$\binom{8}{4} = 70$

Now consider the first group of four commercials.

(b) In how many different orders can the commercials be shown if:

(i) they are all different products

$12 \times 10 \times 8 \times 6 = 5760$

(ii) there are only two products advertised and the two advertisements for each product are not shown consecutively

$12 \times 10 \times 1 \times 1 = 120$

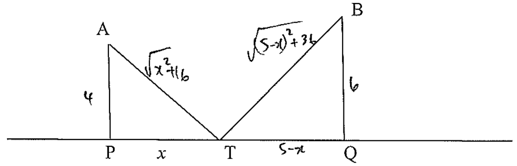
or $\binom{6}{2} \times \underbrace{ABAB}_{2\times1\times2\times1} \times 2$

$= 15 \times 8$

$= 120$

Question 7 (2, 3, 1 = 6 marks)

Two houses A and B are respectively 4 and 6 km from two points P and Q on a straight road as shown in the diagram below.



Eastern Power Company are to erect one power pole (T) between P and Q so that it can serve both houses. Once erected, power lines AT and BT will be put in place. Given PQ = 5 km, and x the distance between P and T:

(i) show that the distance TA + TB can be represented by:

$$TA + TB = \sqrt{x^2+16} + \sqrt{x^2-10x+61}$$

$TA = \sqrt{x^2+16}$ $\qquad$ $TB = \sqrt{(5-x)^2+36}$

$\therefore Dist = \sqrt{x^2+16} + \sqrt{x^2-10x+25+36}$

$= \sqrt{x^2+16} + \sqrt{x^2-10x+61}$

Eastern Power Company wish to minimise the distance TA + TB.

(ii) Use calculus to determine how far from P the pole should be erected in order that the distance TA + TB is a minimum.

let $TA + TB = Dist$

$\frac{dD}{dt} = \frac{1}{2}(x^2+16)^{-\frac{1}{2}} \cdot 2x + \frac{1}{2}(x^2-10x+61)^{-\frac{1}{2}} \cdot (2x-10)$

$\frac{dD}{dt} = 0 \Rightarrow x = 2$

(iii) verify that the distance found in (ii) is a minimum.

$\left.\frac{d^2D}{dx^2}\right|_{x=2} > 0 \quad \therefore$ min