MATHEMATICS SPECIALIST

MAWA Year 12 Examination 2017

Calculator-free

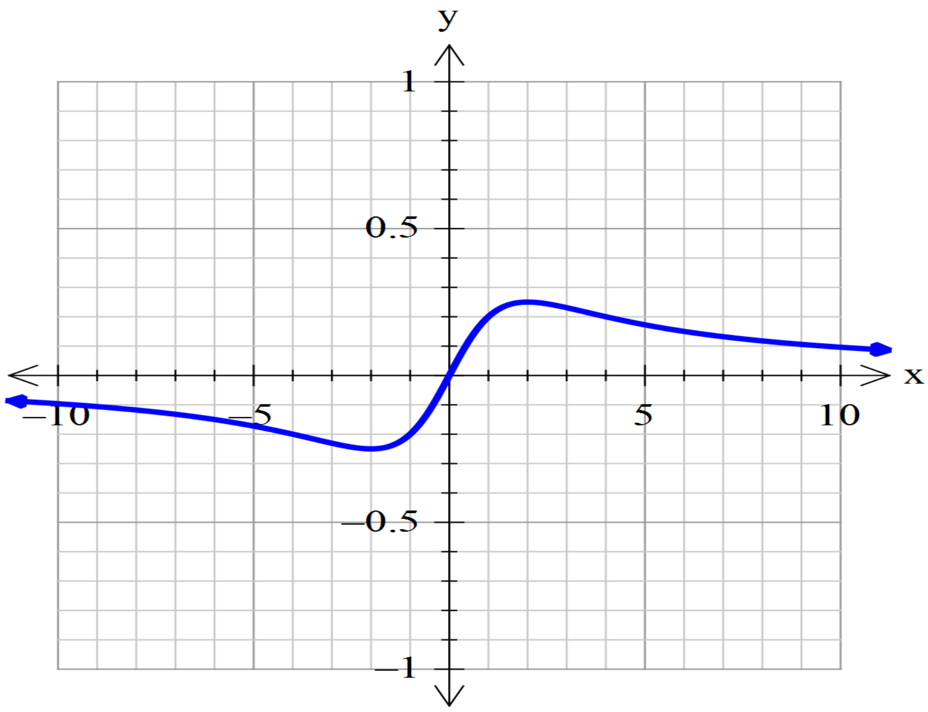
Marking Key

Question 1

Solution
Separating variables gives that $$-\int \frac{1}{y^2}\,dy = \int 2x\,dx$$ and so $\frac{1}{y} = x^2 + C$ Since $y = 2$ when $x = 0$ then $C = 2$. Hence $y = \frac{2}{1+x^2}$
Specific behaviours
✓ separates variables correctly ✓✓ integrates each side correctly ✓ evaluates the constant correctly

Question 2

Solution
The function is odd as $f(-x)=-f(x)$. Now as $x\to \pm\infty$ clearly have that $f(x)\to 0$. Also $$f'(x)=\frac{(x^2+4).1-x(2x)}{(x^2+4)^2}=\frac{4-x^2}{(x^2+4)^2}$$ which indicates turning points at $x=\pm 2$. Also $f(\pm 2)=\pm 1/4$.
Specific behaviours
✓ identifies correct behaviour for large values of x ✓✓ differentiates using the quotient rule ✓ identifies the turning points ✓✓draws a neat sketch with a function with a properly identified max/min and being odd in x



Question 3 (a)

Solution
Curve cuts the x axis where $y=0$ so $x^2=9 \Rightarrow x=\pm 3$ Hence P and Q are the points $(\pm 3, 0)$
Specific behaviours
✓ writes down the correct criterion for determining the points P and Q ✓ solves for x and hence the two points

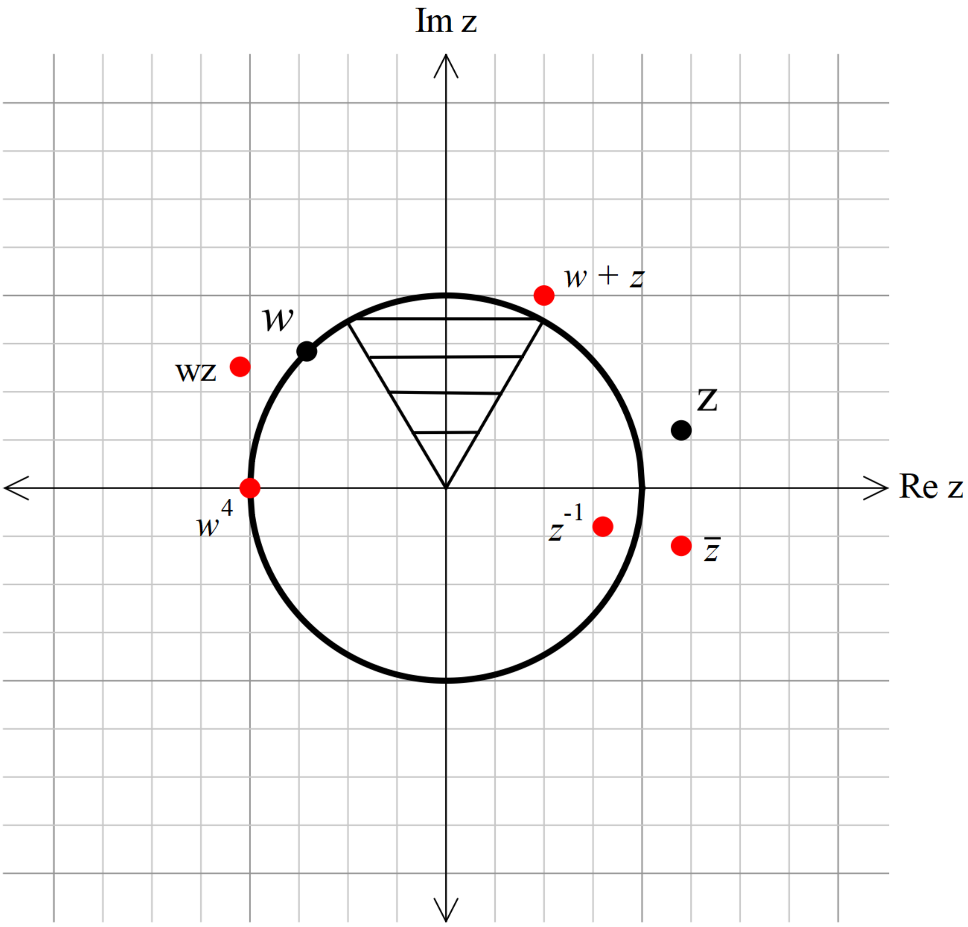
Question 3 (b)

Solution
Differentiating implicitly gives $$2x + x\frac{dy}{dx} + y + 3y^2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x+y}{x+3y^2}$$ Where $y=0$ have $\frac{dy}{dx} = -2$ so the two tangents are parallel.
Specific behaviours
✓✓ differentiates correctly (one mark for each implicit term) ✓ shows the gradients are the same at the two points P and Q ✓ deduces that the two tangents are parallel

Question 3 (c)

Solution
If P=(3,0) then the y co-ordinate of R is approximately $y_R \approx \left(\frac{dy}{dx}\right)_P (x_R - x_P) \approx -2(-0.01) = 0.02$
Specific behaviours
✓ applies increments formula correctly ✓ deduces approximate value of the required co-ordinate

Question 4

Solution

Specific behaviours
✓✓✓✓✓ locates each of $\bar{z}$, wz, z^{-1}, $w+z$, w^4 in correct position ✓ shows the correct boundary rays ✓ shows the arc of the unit circle on the boundary

Question 5 (a)

Solution
$\int \sin^2 x \cos^2 x \; dx = \frac{1}{4} \int \sin^2 2x \; dx$ $= \frac{1}{8} \int (1 - \cos 4x) \, dx$ $= \frac{1}{8} x - \frac{1}{32} \sin 4x + c$
Specific behaviours
✓ converts the integrand into the form involving $\sin 2x$ ✓ uses the trigonometric identity to write in terms of $\cos 4x$ ✓ integrates correctly (with no penalty for omitting the constant)

Question 5 (b)

Solution
$\int_0^{\pi/4} \tan^2 x \; dx = \int_0^{\pi/4} \sec^2 x - 1 \, dx$ $= \left[\tan x - x \right]_0^{\pi/4} = 1 - \frac{\pi}{4}$
Specific behaviours
✓ expresses $\tan^2 x$ in terms of $\sec^2 x$ ✓ integrates correctly ✓ evaluates the limits correctly

Question 5 (c)

Solution
$u = x^5 + 4 \rightarrow \dfrac{du}{dx} = 5x^4.$ Then $$\int_0^2 \frac{x^4}{\sqrt{x^5+4}}\,dx = \frac{1}{5}\int_4^{36} u^{-1/2}\,du = \frac{2}{5}\left[u^{1/2}\right]_4^{36} = \frac{2}{5}(6-2) = \frac{8}{5}$$
Specific behaviours
✓ changes variable to u in integral ✓ anti-differentiates with respect to u ✓ evaluates correctly

Question 5 (d)

Solution
$$v = \ln x \rightarrow \frac{dv}{dx} = \frac{1}{x}$$ Then $$\int \frac{dx}{x\ln x} = \int \frac{dv}{v} = \ln v + c = \ln(\ln x) + c$$ Hence $$\int_e^Q \frac{dx}{x\ln x} = \ln(\ln Q) - \ln(\ln e) = \ln(\ln Q) - \ln(1) = \ln(\ln(Q)).$$ Thus the integral equals 1 if $\ln(Q) = e \Rightarrow Q = e^e \ (= \exp(e))$
Specific behaviours
✓ changes variable to v in integral ✓ evaluates the integral correctly ✓ deduces the correct value of Q

Question 6 (a)

Solution
From the formula sheet $E = \dfrac{z_\alpha \sigma}{\sqrt{n}}$
Specific behaviours
✓ obtains correct answer

Question 6 (b)(i)

Solution
Since $E = \dfrac{z_\alpha \sigma}{\sqrt{n}}$, E decreases by a factor of $\sqrt{2}$ if n is doubled.
Specific behaviours
✓ draws the correct conclusion

Question 6 (b)(ii)

Solution
Since $E = \dfrac{z_\alpha \sigma}{\sqrt{n}}$ and z_α increases as the level of confidence increases, E increases if the level of confidence increases.
Specific behaviours
✓ draws the correct conclusion

Question 6 (b)(iii)

Solution
Since $E = \dfrac{z_\alpha \sigma}{\sqrt{n}}$, E doubles if σ doubles.
Specific behaviours
✓ draws the correct conclusion

Question 6 (c)(i)

Solution
This statement is false. Reason: it is possible that all ten confidence intervals contain μ (or none even!)
Specific behaviours
✓ obtains correct answer ✓ gives a valid reason

Question 6 (c)(ii)

Solution
This statement is false Reason: If the underlying population is normal and the sample size is large enough, the confidence interval will be smaller than any interval that contains 95% of the underlying population

Specific behaviours
✓ obtains correct answer ✓ gives a valid reason

Question 7 (a)

Solution
$\mathbf{a}+\lambda\mathbf{b} \perp \mathbf{c} \Rightarrow (\mathbf{a}+\lambda\mathbf{b})\bullet\mathbf{c} = 0$ $\mathbf{a}+\lambda\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3+\lambda \\ -2+2\lambda \\ 1-3\lambda \end{pmatrix}$ $(\mathbf{a}+\lambda\mathbf{b})\bullet\mathbf{c} = 0 \Rightarrow \begin{pmatrix} 3+\lambda \\ -2+2\lambda \\ 1-3\lambda \end{pmatrix}\bullet\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = 0 \Rightarrow -3-\lambda-2+2\lambda+2-6\lambda = 0$ $\Rightarrow \lambda = -\frac{3}{5}$
Specific behaviours
✓ determines $\mathbf{a}+\lambda\mathbf{b}$ in terms of λ ✓ uses $(\mathbf{a}+\lambda\mathbf{b})\bullet\mathbf{c} = 0$ ✓ evaluates λ

Question 7 (b)

Solution
Vector equation of a line: $r = t\,\mathbf{c}$ As line is parallel to $\mathbf{c}$ $\Rightarrow$ $\mathbf{r} = <-t, t, 2t>$ (or $\mathbf{r} = -t\,\mathbf{i} + t\,\mathbf{j} + 2t\,\mathbf{k}$) In parametric form: $x = -t \quad y = t \quad z = 2t$
Specific behaviours
✓ states the equation in vector form ✓ states the equation in parametric form

Question 7 (c)

Solution
$\mathbf{a}\times\mathbf{b} = (3,-2,1)\times(1,2,-3) = (4,10,8)$
Specific behaviours
✓✓ calculates the vector product correctly (one mark if one component is incorrect)

Question 7 (d)

Solution

Vector equation of a plane: $\mathbf{n} \bullet (\mathbf{r} - \mathbf{r}_0) = \mathbf{0}$ We can use $\mathbf{a} \times \mathbf{b} = 4\mathbf{i} + 10\mathbf{j} + 8\mathbf{k}$ and $\mathbf{r}_0 = < 0, 0, 0 >$ $\Rightarrow (4\mathbf{i} + 10\mathbf{j} + 8\mathbf{k}) \bullet (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 0$ $\Rightarrow 4x + 10y + 8z = 0$ or $2x + 5y + 4z = 0$ is the equation of the plane
Specific behaviours
✓ uses vector $\mathbf{a} \times \mathbf{b}$ as the normal ✓ states the correct equation of the plane