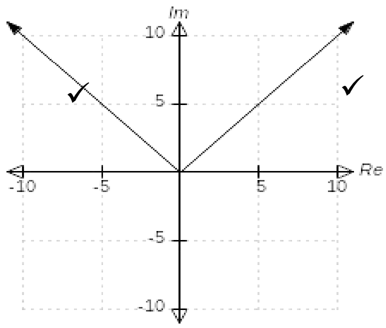
SPECIALIST 3 &4 PRACTICE

SOLUTIONS

1. (a)



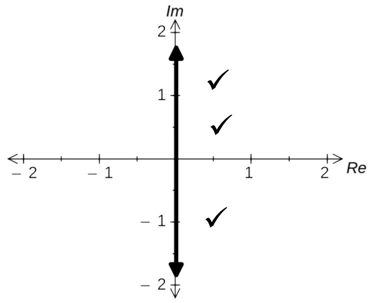
Let $z = x + y\,i$

Then Im(z) = y and Re(z) = x

Hence, Im(z) = |Re(z)| becomes

$y = |x|$.

(b)



Let $z = x + y\,i$.

$\Rightarrow$ Im(z) = y

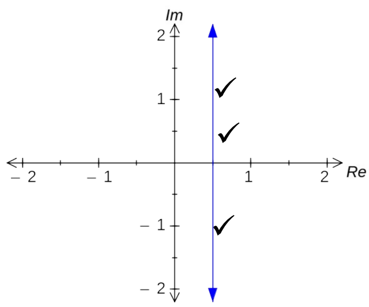
$|z| = \sqrt{(x^2 + y^2)}$

Hence Im(z) = $|z|$ becomes

$\sqrt{(x^2 + y^2)} = y$

$\Rightarrow \quad x = 0$

(c)



Let $z = x + y\,i$.

Then $z + \bar{z} = 1$ becomes

$x + y\,i + (x - y\,i) = 1$

$\Rightarrow \quad 2x = 1$

$x = ½$

[8]

2. (a) $z = \dfrac{-2\pi \pm \sqrt{4\pi^2 - 20\pi^2}}{2}$ ✓

$= -\pi \pm 2\pi i$ ✓✓

(b) $z^3 = e^3 \text{ cis}(\pi + 2n\pi)$ ✓

$z = e \text{ cis}(\frac{\pi}{3} + \frac{2n\pi}{3})$ ✓

$z = e\,(\frac{1}{2} + \frac{\sqrt{3}}{2} i)$, $-e$, $e\,(\frac{1}{2} - \frac{\sqrt{3}}{2} i)$ ✓✓

[7]

3. (a) $\{x : x > 0, x \neq \frac{1}{e}\}$ ✓✓

(b) $y \neq 0$ ✓✓

(c) Yes, as f is a one-to-one and onto function. ✓✓
(or the graph of $y = f(x)$ passes the horizontal line test.)

Write $x = \frac{1}{1+\ln y}$ ✓

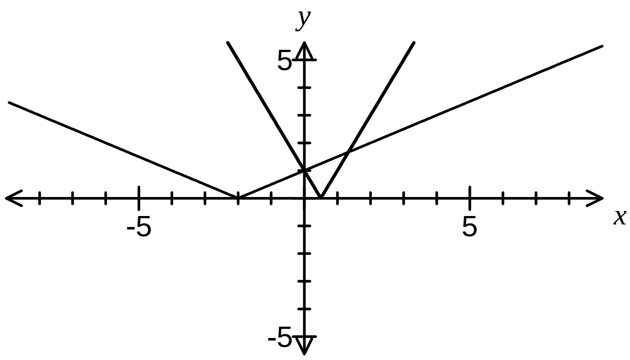
$1 + \ln y = \frac{1}{x}$

$y = e^{1/x-1}$ ✓

[8]

4. **(10 marks)**

The graph of $f(x) = |2x - 1|$ is shown below.



(a) Add the function $g(x) = \frac{|x+2|}{2}$ to the graph. (1 mark)

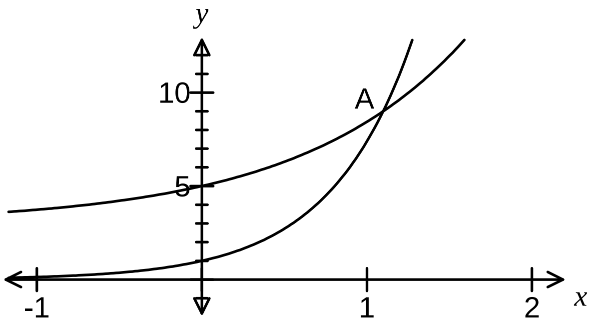
(b) Solve $f(x) - g(x) \geq 0$. (3 marks)

$f(x) = g(x)$ when $x = 0$ and when

$0.5x + 1 = 2x - 1 \Rightarrow x = \frac{4}{3}$

Hence $x \leq 0$ or $x \geq \frac{4}{3}$

The graphs of $y = e^{2x}$ and $y = 2e^x + 3$ intersect at the point A, shown on the graph below.



(c) Show that the x-coordinate of A is $\log_e 3$. (3 marks)

$$e^{2x} = 2e^x + 3$$
$$e^{2x} - 2e^x - 3 = 0$$
$$(e^x + 1)(e^x - 3) = 0$$
$$\therefore e^x = 3 \Rightarrow x = \ln 3$$

(d) Determine the exact area, in simplest form, of the region bounded by the two curves and the y-axis. (3 marks)

$$\int_0^{ln 3} 2e^x + 3 - e^{2x}\,dx$$
$$= \left[2e^x + 3x - \frac{e^{2x}}{2}\right]_0^{ln 3}$$
$$= \left[6 + 3\,ln 3 - \frac{9}{2}\right] - \left[2 + 0 - \frac{1}{2}\right]$$
$$= 3\,ln 3$$

5.

Question 2 **(7 marks)**

Two complex numbers are given by $z = 2cis\frac{\pi}{3}$ and $w = \sqrt{3} - i$.

(a) Determine $arg\frac{z}{w}$ (2 marks)

$$\boxed{\begin{aligned} arg\frac{z}{w} &= arg\frac{2cis\frac{\pi}{3}}{2cis\left(-\frac{\pi}{6}\right)} \\ &= \frac{\pi}{3} + \frac{\pi}{6} \\ &= \frac{\pi}{2} \end{aligned}}$$

(b) Evaluate $\left| w \times \overline{w \times z} \right|$. (3 marks)

$$\boxed{\begin{aligned} \left| w \times \overline{w \times z} \right| &= \left| w \times \bar{w} \times \bar{z} \right| \\ &= |w|^2 \times |\bar{z}| \\ &= 4 \times 2 \\ &= 8 \end{aligned}}$$

(c) Find the complex number u given that $\frac{z \times u}{2} = cis\left(\frac{3\pi}{4}\right)$. (2 marks)

$$\boxed{\begin{aligned} u &= \frac{2cis\frac{3\pi}{4}}{2cis\frac{\pi}{3}} \\ &= cis\frac{5\pi}{12} \end{aligned}}$$

6. (a) $I = \int x^2 - x^2(1+x^3)^{\frac{1}{2}}\,dx$ or $\int x^2(1-y^{\frac{1}{2}})\frac{dy}{3x^2}$ ✓

$= \frac{x^3}{3} - \frac{1}{3}\int 3x^2(1+x^3)^{\frac{1}{2}}\,dx$ $= \frac{1}{3}\int(1-y^{\frac{1}{2}})\,dy$ ✓

$= \frac{x^3}{3} - \frac{1}{3}\left[\frac{(1+x^3)^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$ $= \frac{1}{3}(y - \frac{2}{3}y^{\frac{3}{2}}) + C$ ✓

$= \frac{x^3}{3} - \frac{2}{9}(1+x^3)^{\frac{3}{2}} + C$ $= \frac{1}{3}[(1+x^3) - \frac{2}{3}(1+x^3)^{\frac{3}{2}}] + C$ ✓

(b) $I = \int \frac{e^{-x}}{1-e^{-x}}\,dx$ ✓

$= \ln|1 - e^{-x}| + C$ ✓

(c) $I = \int \cos x + i\sin x\,dx$

$= \sin x - i\cos x + C$ ✓

$= -i(\cos x + i\sin x) + C$

$= -i\,\mathrm{cis}\,x + C$ ✓ [8]

7. (a)

$$\frac{d}{dx}\,[\ \ln\cos^2 2x\ ] = \frac{d}{dx}[\ 2\ln\cos 2x\ ]$$
$$= 2 \times -\frac{2\sin 2x}{\cos 2x}$$

✓

$$= -4\tan 2x$$ ✓

(b) Since,

$$\frac{d}{dx}[\ \ln\ \cos^2 2x\ ] = -4\tan 2x$$
$$\int -4\tan 2x\ dx = \ln\cos^2 2x + c_1$$

✓

$$\therefore\ \int \tan 2x\ dx = -\frac{1}{4}\ln\cos^2 2x + c_2$$

✓ [4]

8. (a)

(6 marks)

(a) Find the exact gradient of the curve $y = 4^{3x-5}$ at the point (2, 4). (3 marks)

✓ ✓ ✓

$$\boxed{\begin{aligned} &\ln y = (3x-5)\ln 4 \\ &\frac{1}{y}\frac{dy}{dx} = 3\ln 4 \\ &\frac{dy}{dx} = 3y\ln 4\Big|_{y=4} \\ &= 12\ln 4 \end{aligned}}$$

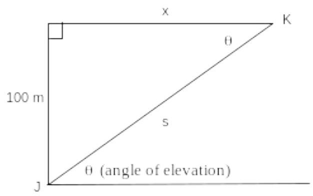
(b) Evaluate $\int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$. (3 marks)

NB If $y = e^{\sqrt{x}}$ then $y' = \frac{1}{2\sqrt{x}} e^{\sqrt{x}}$

$$2\int_0^4 \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx = 2\left[e^{\sqrt{x}}\right]_0^4$$

$$= 2\left(e^2 - 1\right)$$

8. (a)



$$\tan\theta = \frac{100}{x}$$

✓

$$\therefore \quad \frac{1}{\cos^2\theta}\frac{d\theta}{dt} = -\frac{100}{x^2}\frac{dx}{dt}$$

✓

When $t = 0$, $x = \sqrt{(120^2 - 100^2)} = \sqrt{4400}$ ✓

When $t = 5$, horizontal distance covered by kite = $10 \times 5 = 50$ m.

Hence, when $t = 5$, $x = \sqrt{4400} + 50 = 116.3325$ m.

$$\therefore \quad \tan\theta = \frac{100}{116{\cdot}3365} \rightarrow \theta = 40{\cdot}68^\circ$$ ✓

$\therefore$ when $t = 5$,

$$\frac{1}{\cos^2 40{\cdot}68^\circ}\frac{d\theta}{dt} = -\frac{100}{116{\cdot}3325^2} \times 10$$

$= -0.042$ radians per second ✓

(b) $s^2 = 100^2 + x^2$ ✓

$$\therefore 2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

✓

At $t = 5$, $x = 116.3325$, $s = \sqrt{(100^2 + 116.3325^2)} = 153.4055$: ✓

$$\therefore \frac{ds}{dt} = 7{\cdot}58 \text{ m/sec}$$

✓ [9]

9. (a) (i)

$$\sqrt{2}\, cis\frac{\pi}{4} \times 2\, cis\frac{\pi}{3} = 2\sqrt{2}\, cis\frac{7\pi}{12}$$

✓✓

(ii)

$$2(1 + i)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = (1 - \sqrt{3}) + (1 + \sqrt{3})\, i$$

✓✓

(iii)

$$2\sqrt{2}\, cis\, \frac{7\pi}{12} = (1 - \sqrt{3}) + (1 + \sqrt{3})\, i$$

✓

$$2\sqrt{2}\left(\frac{\cos 7\pi}{12} + i\,\frac{\sin 7\pi}{12}\right) = (1 - \sqrt{3}) + (1 + \sqrt{3})\, i$$

Hence, $\frac{\cos 7\pi}{12} = \frac{1}{2\sqrt{2}}(1 - \sqrt{3})$ or $\frac{\sqrt{2}}{4}(1 - \sqrt{3})$ ✓

. (b) (i) $2 + i$ and $-2i$. ✓✓

(ii) $f(z) = k(z - 2i)(z + 2i)(z - 2 + i)(z - 2 - i)$ ✓

$= k(z^4 - 4z^3 + 9z^2 - 16z + 20)$ ✓

$f(1) = 20 \Rightarrow 20 = k(10)$

Hence, $k = 2$ ✓

$f(z) = 2z^4 - 8z^3 + 18z^2 - 32z + 40$

$\therefore \quad a = 2,\ b = -8,\ c = 18,\ d = -32,\ e = 40$ ✓✓ [13]

10. (a) At the y-axis $y = -\pi$ ✓

$$\frac{dy}{dx} = \cos 2xy \left[2x\frac{dy}{dx} + 2y \right] + \pi(2\cos x \sin x)$$ ✓✓

$$\therefore \frac{dy}{dx} = (1) \times [-2\pi] - 0 = -2\pi$$ ✓

(b) If perpendicular $m = \dfrac{1}{2\pi}$ ✓

$$\therefore y = \frac{1}{2\pi}x - \pi$$ ✓ [6]

11.

$$z = \frac{2(a - i)(1 + i) + (3 - i)(1 - i)}{(1 - i)(1 + i)}$$

✓✓

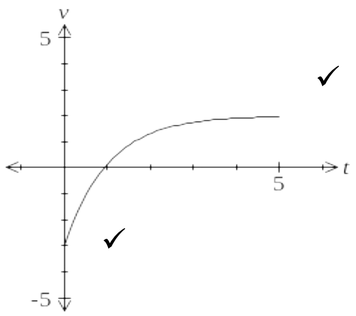
$= (a + 2) + (a - 3)i$ ✓

$|z| = 5 \Rightarrow (a + 2)^2 + (a - 3)^2 = 25$ ✓

$a^2 - a - 6 = 0$ ✓

$a = -2, 3$ ✓✓ [7]

12. (a)



(b) Moves with constant velocity of 2 metres per second. ✓

(c) $t = 0.91629 = 0.92$ seconds (Use GC, root) ✓

(d) $$\left|\int_0^{0.91625} v \, dt\right| = \int_{0.91625}^{k} v \, dt$$ ✓

$$\therefore \left[5e^{-t} + 2t\right]_{0.91625}^{k} = 1.16742$$ ✓

$1.16742 = 5e^{-k} + 2k - 3.83258$ ✓

$k = 2.23$ (use GC Solver) ✓

(e)

$$\text{Distance travelled} = \int_0^5 |v| \, dt$$ ✓

$= 7.3685 = 7.4$ metres ✓✓ [11]

13. (a) $x^3 - 5x^2 + 6x = 0$ ✓

$x(x - 2)(x - 3) = 0$ ✓

$\Rightarrow x = 0, 2, 3$ ✓

(b)

$$V = \pi \int_0^2 x^3 - 5x^2 + 6x \ dx + \pi \int_3^4 x^3 - 5x^2 + 6x \ dx$$

✓✓

$$= \pi \left[\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2 \right]_0^2 + \pi \left[\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2 \right]_3^4$$ ✓

$$= \frac{23\pi}{4}$$

✓ [7]

14.

$$\tan\theta = \frac{x}{2} \rightarrow x = 2\tan\theta \quad \text{and} \quad \delta\theta = 0{\cdot}01 \times \frac{\pi}{180}$$

✓

$$\therefore \frac{dx}{d\theta} = \frac{2}{\cos^2\theta}$$

✓

$$\therefore \delta x = \frac{2}{\cos^2\theta}\,\delta\theta$$

✓

$$= \frac{2}{\cos^2 2^\circ} \times 0{\cdot}01 \times \frac{\pi}{180}$$

✓

$$= 3.4949 \times 10^{-4} \text{ km}$$

$$= 0.349 \text{ metres}$$ ✓ [5]

.

15. A li

(a) F

Intersect when

$$\begin{bmatrix} -1 \\ 3-2\lambda \\ 2+4\lambda \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 33$$

$$\therefore \lambda = 3$$

Hence point of intersection at $\begin{bmatrix} -1 \\ -3 \\ 14 \end{bmatrix}$

(b) Find the acute angle between the line and plane. (3 marks)

Find angle between line and normal to plane:

$$\cos\theta = \frac{\begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} \right\| \times \left\| \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\|}$$

$\theta = 26.565$

Hence angle between line and plane $= 90 - 26.6 = 63.4^{\circ}$

16.

(10 marks)

A body, A, has an initial position of $\begin{bmatrix} -7 \\ 21 \\ 6 \end{bmatrix}$ metres and is moving with a constant velocity of $\begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$ metres per second.

(a) A second body, B, is moving with constant velocity of $\begin{bmatrix} 8 \\ 1 \\ -1 \end{bmatrix}$ metres per second and collides with body A after six seconds.

Determine the initial distance apart of body A and body B. (4 marks)

A and B collide at

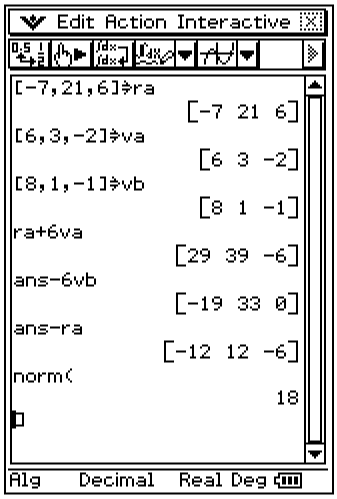
$$\begin{bmatrix} 7 \\ -21 \\ 6 \end{bmatrix} + 6\begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 29 \\ 39 \\ -6 \end{bmatrix}$$

Hence initial position of B is at

$$r_B + 6\begin{bmatrix} 8 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 29 \\ 39 \\ -6 \end{bmatrix} \Rightarrow r_B = \begin{bmatrix} -19 \\ 33 \\ 0 \end{bmatrix}$$

Distance apart of A and B is

$$\left\| \begin{bmatrix} -19 \\ 33 \\ 0 \end{bmatrix} - \begin{bmatrix} -7 \\ 21 \\ 6 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -12 \\ 12 \\ -6 \end{bmatrix} \right\| = 18$$



(b) A third body, C, is initially located at $\begin{bmatrix} 5 \\ -10 \\ 1 \end{bmatrix}$ metres and is also moving with a constant velocity $\begin{bmatrix} 2 \\ y \\ -3 \end{bmatrix}$. After five seconds, the distance between bodies A and C is a minimum.

Find the value of y for which the speed of C is also a minimum. (6 marks)

Let $\mathbf{v}$ and $\mathbf{r}$ be velocity and displacement of A relative to C

Then minimum distance apart when $\mathbf{v}$ and $\mathbf{r}+5\mathbf{v}$ are perpendicular, ie the dot product of $\mathbf{v}$ and $\mathbf{r}+5\mathbf{v}$ is zero.

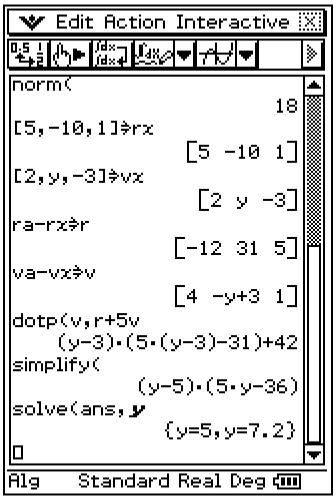
$$\mathbf{v}=\begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}-\begin{bmatrix} 2 \\ y \\ -3 \end{bmatrix}=\begin{bmatrix} 4 \\ 3-y \\ 1 \end{bmatrix}$$

$$\mathbf{r}=\begin{bmatrix} -7 \\ 21 \\ 6 \end{bmatrix}-\begin{bmatrix} 5 \\ -10 \\ 1 \end{bmatrix}=\begin{bmatrix} -12 \\ 31 \\ 5 \end{bmatrix}$$

$$\mathbf{r}+5\mathbf{v}=\begin{bmatrix} 8 \\ 46-5y \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 3-y \\ 1 \end{bmatrix}\bullet\begin{bmatrix} 8 \\ 46-5y \\ 10 \end{bmatrix}=5y^2-61y+180$$

$(y-5)(5y-36)=0$ when $y=5$ or $y=7.2$



17. (a)

$$v = -\pi \sin \frac{\pi t}{2} + \pi \sin \left(1 + \frac{\pi t}{2} \right)$$

✓

$$a = -\frac{\pi^2}{2} \cos \frac{\pi t}{2} + \frac{\pi^2}{2} \cos \left(1 + \frac{\pi t}{2} \right)$$

$$= -\frac{\pi^2}{4} x \quad \therefore \text{ SHM}$$

✓

(b)

$$v = 0 \text{ when } -\pi \sin \frac{\pi t}{2} + \pi \sin \left(1 + \frac{\pi t}{2} \right) = 0$$

$$\therefore \sin \frac{\pi t}{2} = \sin \left(1 + \frac{\pi t}{2} \right)$$

✓

$\therefore \quad t = 0.68$ ✓

(c) Maximum velocity = 3.01 ✓✓

(d) $a = -\frac{\pi^2}{4}(2) = -\frac{\pi^2}{2}$

✓✓ [8]

18. (a) $2(-0.5) + A = 0 \rightarrow A = 1$ ✓

$1 - B = 0 \rightarrow B = 1$ ✓

$f(0) = -\frac{C}{-B} = 2 \therefore C = 2$

✓

(b) Using GC

D(-0.90, -0.97), E(2.15, 0), F(-0.18, 1.80) ✓✓✓ [6]

19.

Question 14 **(8 marks)**

The temperature in a restaurant cool room is set to 4°C. One day, the refrigerator unit was turned back on after the temperature in the cool room had risen to 27°C due to cleaning and maintenance work. After 15 minutes, the temperature in the cool room had dropped to 11°C, with the temperature, T, falling according to the model

$$\frac{dT}{dt} = k(T - 4)$$

where t is the time in minutes since the refrigerator unit was turned back on.

(a) Find the value of k and express T as a function of t. (5 marks)

$$\int \frac{1}{T-4}\,dT = \int k\,dt$$

$$\ln(T-4) = kt + c$$

$$T = ae^{kt} + 4$$

$$T(0) = 27 \Rightarrow a = 23$$

$$T(15) = 11 \Rightarrow 11 = 23e^{15k} + 4 \Rightarrow k = -0.0793$$

$$T = 23e^{-0.0793t} + 4$$

(b) If the temperature continues to fall in this way, how long before the temperature in the cool room registers 4°C, to the nearest degree? (3 marks)

Temperature must drop to 4.5°C

$4.5 = 23e^{-0.0793t} + 4$

$t = 48.3$

After 48.3 minutes.