Carmel School
FAITH & KNOWLEDGE

Semester Two Examination, 2018

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section Two: Calculator-assumed

SOLUTIONS

Student number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(7 marks)**

(a) Given that $\dfrac{20\times19\times18}{19\times18\times17\times16}=\dfrac{{}^{a}P_{b}}{{}^{c}P_{4}}$, determine the values of a, b and c. (3 marks)

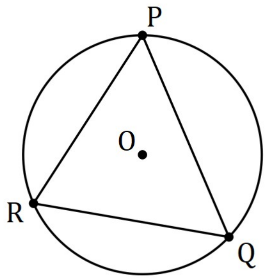
Solution
$\dfrac{20\times19\times18}{19\times18\times17\times16}=\dfrac{20!}{19!}\times\dfrac{15!}{17!}=\dfrac{20!}{17!}\times\dfrac{15!}{19!}$ $\dfrac{20!}{17!}={}^{20}P_{3},\ \dfrac{19!}{15!}={}^{19}P_{4}$ $a=20, b=3, c=19$
Specific behaviours
✓ expresses fraction with factorials   ✓ expresses as permutations   ✓ lists all values

(b) Determine how many integers between 1 and 100 inclusive are divisible by 2, 3 or 13. (4 marks)

Solution
$\lfloor 100\div2\rfloor+\lfloor 100\div3\rfloor+\lfloor 100\div13\rfloor=50+33+7=90$ $\lfloor 100\div6\rfloor+\lfloor 100\div26\rfloor+\lfloor 100\div39\rfloor=16+3+2=21$ $\lfloor 100\div78\rfloor=1$ $n=90-21+1=70$ integers
Specific behaviours
✓ correct number divisible singly   ✓ correct number divisible by pairs   ✓ correct number divisible by all three   ✓ correct total

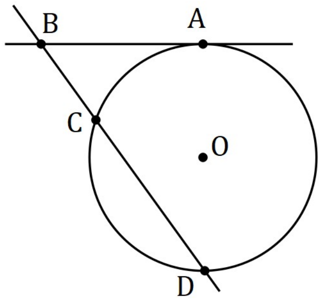
Question 10 **(6 marks)**

(a) In the circle shown below, minor arc PR subtends an angle of $120°$ at O, the centre of the circle, and the size of angle RPQ is $55°$. Determine the size of angle POQ. (2 marks)



Solution
$\angle ROQ = 2\times 55 = 110°$ $\angle POQ = 360 - 110 - 120 = 130°$
Specific behaviours
✓ indicates size of $\angle ROQ$ correct $\angle POR$

(b) In the diagram below, AB is tangent to the circle with centre O at A, secant BD intersects the circle at C and D, and the sizes of angles AOC and COD are $72°$ and $104°$ respectively. Determine the size of angle ABC. (4 marks)



Solution
$\angle ODC = \dfrac{180-104}{2} = 38°$ $\angle DOA = 72+104 = 176°$ $\angle OAB = 90°$ Using $OABD$: $\angle ABC = 360 - 38 - 176 - 90¿56°$
Specific behaviours
correct $\angle ODC$ correct $\angle DOA$ ✓ indicates $\angle OABA$ is right-angle correct $\angle ABC$

Question 11 **(8 marks)**

(a) Show how to express $0.\overline{23}$ as a rational number. (2 marks)

Solution
If $x=0.232323\ldots$ then $100x=23.232323\ldots$
Hence by subtraction $99x=23 \Rightarrow x=\frac{23}{99}$, which is rational.
Specific behaviours
✓ expresses as x and $100x$
□ uses subtraction to express as rational

(b) Prove that the sum of any three consecutive integers is always a multiple of three. (3 marks)

Solution
Let the integers be $n, n+1, n+2$ and their sum be S.
$S=n+n+1+n+2¿3n+3¿3(n+1)$
Hence S is always a multiple of 3.
Specific behaviours
✓ clearly indicates three consecutive integers
□ creates sum
□ factors out 3 and makes conclusion

(c) Prove by contradiction that $\sqrt{7}$ is irrational. (3 marks)

Solution
Assume that $\sqrt{7}$ is rational and can be expressed in the form $\frac{a}{b}$, where *a* and *b* are integers with **no common factor** greater than 1.
$\sqrt{7}=\frac{a}{b} \Rightarrow a^2=7b^2$, so that a^2 and hence *a* must be a multiple of 7.
Since $a=7k$ (*k* an integer) then $(7k)^2=7b^2 \Rightarrow 7k^2=b^2$, so that b^2 and hence *b* must be a multiple of 7.
Since *a* and *b* are both multiples of 7, the assumption they have no common factor is contradicted and so $\sqrt{7}$ must be irrational.
Specific behaviours
✓ makes rational assumption including bolded condition

Question 12 **(8 marks)**

Let vector $a=4i-6j$.

(a) Determine the angle between a and $-7i-10j$. (1 mark)

Solution
Using CAS $\theta=68.7°$
Specific behaviours
correct angle

(b) Let vector $b=14i+tj$. Determine the value of t so that a is

(i) parallel to b. (2 marks)

Solution
$\frac{4}{14}=\frac{-6}{t}\Rightarrow t=-6\times\frac{14}{4}=-21$
Specific behaviours
✓ indicates method correct value

(ii) perpendicular to b. (2 marks)

Solution
$a\cdot b=0\Rightarrow(4)(14)+(-6)(t)=0$ $t=\frac{28}{3}=9.\overline{3}$
Specific behaviours
✓ indicates method correct value

(c) Determine the vector projection of a on $-6i+8j$. (3 marks)

Solution
Let $c=-6i+8j$. Then $\hat{c}=-0.6i+0.8j$. Using CAS, $(a\cdot\hat{c})\hat{c}=\frac{108}{25}i-\frac{144}{25}j=4.32i-5.76j$
Specific behaviours
✓ indicates unit vector $\hat{c}$ indicates method correct projection

Question 13 **(8 marks)**

Two matrices are given by $P=\begin{bmatrix} 4 & 7 \\ -8 & 3 \end{bmatrix}$ and $Q=\begin{bmatrix} 3 & -7 \\ 8 & 4 \end{bmatrix}$.

(a) Determine PQ. (1 mark)

Solution
$PQ=\begin{bmatrix} 68 & 0 \\ 0 & 68 \end{bmatrix}$
Specific behaviours
✓ correct product

(b) Given that $Q^{-1}=kP$, determine the exact value of the constant k. (2 marks)

Solution
$Q^{-1}Q=kPQ \Rightarrow I=kPQ$ $k=\frac{1}{68}$
Specific behaviours
✓ uses matrix algebra or states Q^{-1} ✓ correct value

The system of equations $3a=7b+102$ and $8a+4b+34=0$ can be expressed as a matrix equation in the form $QX=R$.

(c) Determine matrices X and R. (2 marks)

Solution
$\begin{bmatrix} 3 & -7 \\ 8 & 4 \end{bmatrix} \times \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 102 \\ -34 \end{bmatrix}$ $X=\begin{bmatrix} a \\ b \end{bmatrix}, R=\begin{bmatrix} 102 \\ -34 \end{bmatrix}$
Specific behaviours
✓ correct matrix X ✓ correct matrix

(d) Express matrix X in terms of matrices P and R. (2 marks)

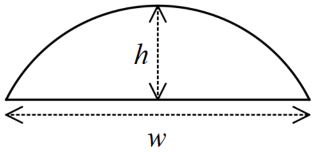
Solution
$QX=R$ $Q^{-1}QX=Q^{-1}R$ $X=\frac{1}{68}PR$
Specific behaviours
✓ pre-multiplies by Q^{-1} ✓ correct expression

(e) Solve the system of equations. (1 mark)

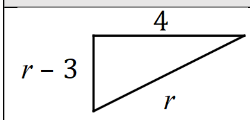
Solution
$a=2.5, b=-13.5$
Specific behaviours
✓ correct solution

Question 14 **(6 marks)**

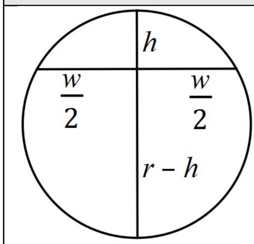
A segment of a circle has a perpendicular height of h and width w.



(a) Determine the radius of the arc of the segment when $h=3$ cm and $w=8$ cm. (3 marks)

Solution
 $$r^2=(r-3)^2+4^2$$ $$6r=9+16$$ $$r=\frac{25}{6}=4.1\overline{6}\,\text{cm}$$
Specific behaviours
✓ relevant sketch ✓ uses Pythagoras' Theorem ✓ correct radius

(b) Use the intersecting chord theorem to derive a formula for the radius of the arc of a segment of width w and height h, where the chords are the straight edge of the segment and the diameter of the circle. (3 marks)

Solution
 $$h(2r-h)=\left(\frac{w}{2}\right)^2$$ $$2rh=h^2+\frac{w^2}{4}$$ $$r=\frac{h}{2}+\frac{w^2}{8h}$$
Specific behaviours
✓ labelled sketch of intersecting chords ✓ uses theorem to form equation ✓ correct formula

Question 15 **(8 marks)**

Circle C has equation $(x-2)^2+(y+6)^2=16$.

(a) Circle C is transformed by the matrix $M=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ to circle C'. Describe transformation M and state the equation of circle C'. (3 marks)

Solution
M is a reflection in the line $y=x$.
Centre: $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2\\-6\end{bmatrix}=\begin{bmatrix}-6\\2\end{bmatrix}$
Equation: $(x+6)^2+(y-2)^2=4^2=16$
Specific behaviours
✓ states reflection with equation of line
✓ identifies new centre
✓ correct equation

(b) Circle C' is then transformed by the matrix $N=\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$ to circle C''. Describe transformation N and state the equation of circle C''. (3 marks)

Solution
N is a dilation about $(0,0)$ of scale factor 3.
Centre: $\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}-6\\2\end{bmatrix}=\begin{bmatrix}-18\\6\end{bmatrix}$
Equation: $(x+18)^2+(y-6)^2=(4\times 3)^2=12^2=144$
Specific behaviours
✓ states dilation with scale factor (*dilation centre not required*)
✓ identifies new centre
✓ correct equation

(c) Determine the single matrix P that will transform circle C'' back to circle C. (2 marks)

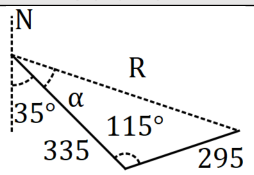
Solution
$(NM)^{-1}=\begin{bmatrix}0 & \frac{1}{3}\\ \frac{1}{3} & 0\end{bmatrix}$
Specific behaviours
✓ indicates correct method

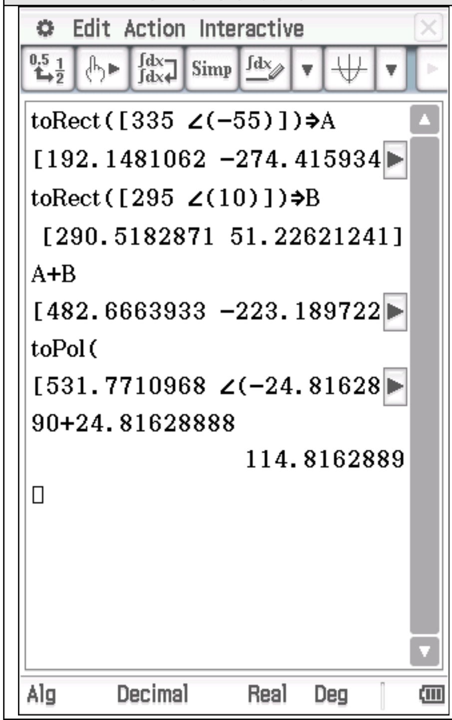
Question 16 **(11 marks)**

Two forces act on a body. F_1 has a magnitude of 335 N and acts on a bearing of 145. F_2 has a magnitude of 295 N and acts on a bearing of 080.

(a) Determine

(i) the magnitude and direction of the sum of the two forces. (6 marks)

Solution

$R^2 = 345^2 + 295^2 - 2(345)(295)\cos 115$
$R = 531.8\,\text{N}$
$\dfrac{531.8}{\sin 115} = \dfrac{295}{\sin\alpha}$
$\alpha = 30.2\,°$
$\text{Bearing}: 180 - 35 - 30.2 \approx 115$
Specific behaviours
✓ triangle showing sum of two forces
□ use of cosine rule with correct angle
□ correct magnitude
□ indicates use of sine rule
□ correct value of α
□ correct bearing

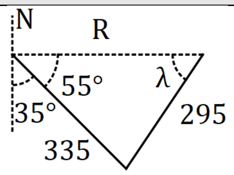
CAS Solution

Specific behaviours
□ correct polar angles
✓ converts F_1 to Cartesian
✓ converts F_2 to Cartesian
□ adds forces
□ converts back to polar
□ states magnitude and bearing

(ii) the magnitude and direction of a third force that would keep the body in equilibrium. (1 mark)

Solution
$115 + 180 = 295$
$F_3 = 531.8\,\text{N on bearing } 295$
Specific behaviours
✓ correct magnitude and bearing

(b) The bearing F_2 acts on is changed so that the direction of $F_1 + F_2$ is due east. Determine the new bearing of F_2. (4 marks)

Solution



$$\frac{335}{\sin\lambda} = \frac{295}{\sin 55}$$

$$\lambda = 68.5° \text{ or } 111.5°$$

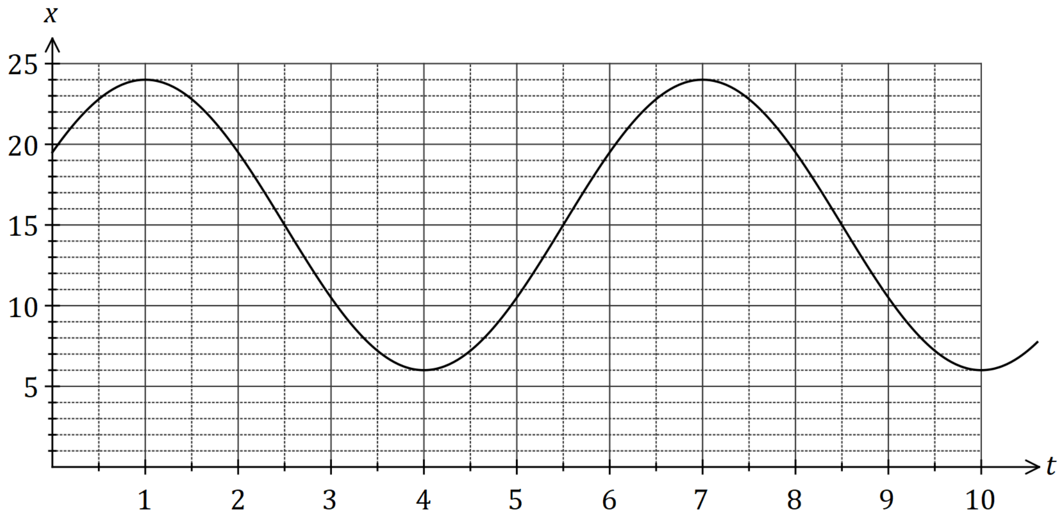
$$\text{Bearings}: 90 - 68.5 = 021.5$$
$$\text{or}: 360 - (111.5 - 90) = 338.5$$

Specific behaviours

✓ diagram
✓ indicates use of sine rule
✓ correct values of λ
✓ both possible bearings

Question 17 **(8 marks)**

A small body P moves in a straight line. The displacement of the body from a fixed point O is given by $x = a\sin(b(t+c)) + d$, where x is in centimetres, t is the time in seconds. The graph of x against t is shown below.



(a) Determine the values of the **positive** constants a, b, c and d. (4 marks)

Solution
$a = (24-6) \div 2 = 9 \quad b = \dfrac{2\pi}{6} = \dfrac{\pi}{3}$
$c = \dfrac{1}{2}\,(\text{or } 6.5, 12.5, \ldots) \quad d = 24 - 9 = 15$
Specific behaviours
✓ each correct value

(b) Express the relationship between x and t as a cosine function. (2 marks)

Solution
$c = \dfrac{1}{2} - \dfrac{1}{4}(6) = -1\,(\text{or } -7, -1, 5, \ldots)$
$x = 9\cos\left(\dfrac{\pi}{3}(t-1)\right) + 15$
Specific behaviours
✓ only changes value of c

(c) Determine the first time that P is 18 cm from O after 150 seconds, giving your answer to two decimal places. (2 marks)

Solution
$9\sin\left(\dfrac{\pi}{3}\left(t - \dfrac{1}{2}\right)\right) + 15 = 18$
$t = 152.18\text{ s}$
Specific behaviours
✓ method
✓ correct time

Question 18 **(7 marks)**

Let $N=\{1,2,3,4,5,6,7,8\}$.

(a) Three or four-digit codes are to be formed using integers selected from N, such as 287 or 1381.

Determine the number of codes that can be formed if

(i) there are no restrictions. (2 marks)

Solution
$8^3+8^4=512+4096$¿4608 codes
Specific behaviours
✓ indicates number of 3- and 4-digit codes
✓ correct total

(ii) no integer may be used more than once in a code. (2 marks)

Solution
${}^8P_3+{}^8P_4=336+1680$¿2016 codes
Specific behaviours
✓ uses permutations for 3- and 4-digit codes
✓ correct total

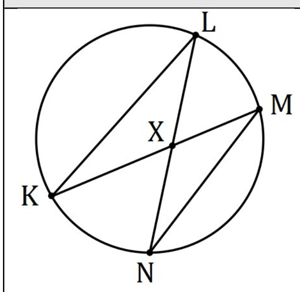
(b) Using the pigeon-hole principle or otherwise, prove that when five integers are selected from N, at least one pair of the integers will have a sum of 9. (3 marks)

Solution
Partition N into 4 pigeon-holes with sums of 9: $\{1,8\},\{2,7\},\{3,6\},\{4,5\}$ If 5 integers (pigeons) are selected from N then by the pigeon-hole principle, at least 2 must be in the same pigeon-hole. Hence at least one pair of the integers will have a sum of 9.
Specific behaviours
✓ lists pigeonholes
✓ uses pigeonhole principle

Question 19 **(8 marks)**

(a) The four points K, L, M and N lie in that order on the circumference of a circle. Chords KM and ln intersect at X. Prove that $\Delta KXL \quad \Delta NXM$. (4 marks)

Example solution



$\angle KXL = \angle NXM$ - vertically opposite

$\angle KLN = \angle KMN$ - stand on same arc

$\therefore \Delta KXL \quad \Delta NXM$ - AAA

Specific behaviours

✓ labelled diagram

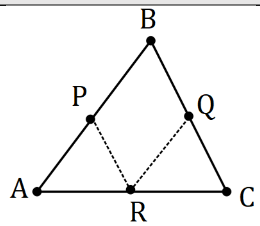
□ one pair of equal angles with reason

□ second pair of equal angles with reason

□ states similarity with reason

(b) In triangle ABC, P, Q and R are the midpoints of AB, AC and BC respectively. If $\overrightarrow{AB} = b$ and $\overrightarrow{AC} = c$, use a vector method to prove that $PBRQ$ is a parallelogram. (4 marks)

Example solution



$$\overrightarrow{PR} = \overrightarrow{PA} + \overrightarrow{AR} \text{¿} -\frac{1}{2}b + \frac{1}{2}c$$

$$\overrightarrow{BQ} = \frac{1}{2}\overrightarrow{BC} \text{¿} \frac{1}{2}\left(\overrightarrow{BA} + \overrightarrow{AC}\right)$$

$$\text{¿} \frac{1}{2}(-b+c) \text{¿} -\frac{1}{2}b + \frac{1}{2}c$$

$$\text{¿} \overrightarrow{PR}$$

Hence $PBRQ$ is a parallelogram since it has a pair of opposite sides that are parallel and equal in length.

Specific behaviours

✓ labelled diagram

□ derives vector for one side of parallelogram

□ derives second vector for opposite side

□ shows vectors are equal and makes conclusion

Question 20 **(6 marks)**

Use mathematical induction to prove that for all positive integers n

$$1\times 5+2\times 6+3\times 7+\ldots+n(n+4)=\frac{n}{6}(n+1)(2n+13).$$

Solution

Let $\text{Claim}(n)$ be the statement

$$1\times 5+2\times 6+3\times 7+\ldots+n(n+4)=\frac{n}{6}(n+1)(2n+13)$$

$\text{Claim}(1)$ is the statement $1\times 5=\frac{1}{6}(2)(15)$ and so $\text{Claim}(1)$ is shown to be true.

Assume $\text{Claim}(k)$ is true so that

$$1\times 5+2\times 6+3\times 7+\ldots+k(k+4)=\frac{k}{6}(k+1)(2k+13)$$

$$LHS\text{ of }\ \text{Claim}(k+1)=1\times 5+2\times 6+\ldots+k(k+4)+(k+1)(k+1+4)$$

$$\text{¿}\frac{k}{6}(k+1)(2k+13)+(k+1)(k+1+4)\text{ using Claim}(k)\text{¿}\frac{k+1}{6}(2k^2+13k+6k+30)$$

$$\text{¿}\frac{k+1}{6}(k+2)(2k+15)\text{¿}RHS\text{ of Claim}(k+1)$$

We have shown that $\text{Claim}(1)$ is true and that $\text{Claim}(k)\Rightarrow\text{Claim}(k+1)$ and so by the principle of mathematical induction it follows that Claim(n) is true.

Specific behaviours

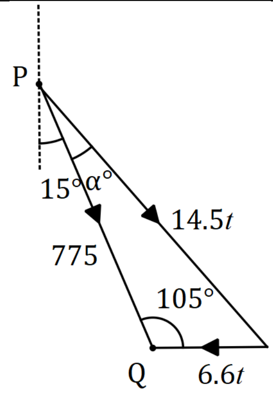
✓ shows truth of initial case
□ clearly states assumption
□ adds $k+1$ term to statement, using $\text{Claim}(k)$
□ factors out $(k+1)$
□ completes factorisation
□ closing statement

Question 21 **(7 marks)**

A small drone is to fly in a straight line and at a constant altitude from P to Q. Q lies 775 m away from P on a bearing of $165°$ and a steady wind of 6.6 ms^{-1} is blowing in the area from due east.

If the speed of the drone is set to 14.5 ms^{-1}, determine the bearing it should steer and the time that it will take to reach Q.

Solution



$$\frac{\sin 105}{14.5t}=\frac{\sin \alpha}{6.6t}$$

$$\alpha=26.1°$$

$$180-105-26.1=48.9°$$

$$\frac{\sin 105}{14.5t}=\frac{\sin 48.9}{775}$$

$$t=68.5\text{ seconds}$$

$$\text{Bearing: } 165-26.1=138.9°$$

Specific behaviours

- ✓ diagram with key elements
- ✓ angle between wind and PQ
- ✓ equation using sin rule for α
- ✓ solves for α
- ✓ equation using sin rule for t
- ✓ correct time
- ✓ correct bearing

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Supplementary page

Question number: _________