Rossmoyne Senior High School

Semester One Examination, 2014

Question/Answer Booklet

MATHEMATICS SPECIALIST 3C

SOLUTIONS

Section Two: Calculator-assumed

Student Number: In figures

In words ________________________________

Your name ________________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination.

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	50	33⅓
Section Two: Calculator-assumed	13	13	100	100	66⅔
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2013*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

Section Two: Calculator-assumed **(100 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(5 marks)**

(a) Express the following in exact polar form $r \cdot cis\theta$:

(i) $5i$ (1 mark)

$\boxed{5cis\frac{\pi}{2}}$

(ii) $3+3\sqrt{3}i$ (1 mark)

$\boxed{6cis\frac{\pi}{3}}$

(b) Express the following in exact Cartesian form $a+bi$:

(i) $5(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})$ (1 mark)

$$\boxed{\frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i}$$

(ii) $\dfrac{6cis\frac{\pi}{4}}{3cis\frac{3\pi}{4}}$ (1 mark)

$\boxed{-2i}$

(iii) $\dfrac{1+i}{1-i}$ (1 mark)

$\boxed{i}$

Question 9 **(8 marks)**

Let $\mathbf{a} = \mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 4\mathbf{j} + \mathbf{k}$.

(a) Calculate the angle between $\mathbf{a}$ and $\mathbf{b}$, to the nearest degree. (2 marks)

$$\boxed{129.664 \approx 130^\circ}$$

(b) Determine vector $\mathbf{c}$, parallel to the resultant of $\mathbf{a}$ and $\mathbf{b}$ and with a magnitude of $\sqrt{10}$.
(2 marks)

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= \mathbf{i} + 4\mathbf{j} - 7\mathbf{k} + \mathbf{i} - 4\mathbf{j} + \mathbf{k}\\ &= 2\mathbf{i} - 6\mathbf{k}\end{aligned}$$

$$|2\mathbf{i} - 6\mathbf{k}| = 2\sqrt{10}$$

$$\mathbf{c} = \mathbf{i} - 3\mathbf{k}$$

(c) Determine a unit vector $\mathbf{d}$, perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. (4 marks)

Let $\lambda\mathbf{d} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$

$$\begin{bmatrix}1\\4\\-7\end{bmatrix} \bullet \begin{bmatrix}x\\y\\1\end{bmatrix} = 0 \text{ and } \begin{bmatrix}1\\-4\\1\end{bmatrix} \bullet \begin{bmatrix}x\\y\\1\end{bmatrix} = 0$$

$$x + 4y - 7 = 0 \text{ and } x - 4y + 1 = 0 \Rightarrow x = 3,\ y = 1$$

$$\lambda\mathbf{d} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\mathbf{d} = \tfrac{1}{\sqrt{11}}(3\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Question 10 **(5 marks)**

Let $f(x) = \int_1^x \frac{1}{t}\,dt$, $x > 0$.

(a) Write down a decimal approximation for $f(5)$, rounded to 2 decimal places. (2 marks)

$$\boxed{\begin{aligned} f(5) &= \ln 5 \\ &= 1.60943 \\ &= 1.61 \text{ (2dp)} \end{aligned}}$$

(b) Write down an exact value for $f'(5)$. (2 marks)

$$\boxed{\begin{aligned} f'(x) &= \frac{1}{x} \\ f'(5) &= \frac{1}{5} \end{aligned}}$$

(c) Write down a simplified expression for $f(x)$. (1 mark)

$$\boxed{f(x) = \log_e x}$$

Question 11 **(6 marks)**

A curve is defined implicitly by $\sqrt{y}+\sqrt{x}=3$.

(a) Determine an expression for $\frac{dy}{dx}$. (1 mark)

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx}+\frac{1}{2\sqrt{x}}=0$$
$$\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

(b) Describe the gradient of the curve for $0<x<9$

(i) as x approaches 9. (1 mark)

Gradient becomes very small and negative, tending to 0. $\frac{dy}{dx}\to 0^{-}$.

(ii) as y approaches 9. (1 mark)

Gradient becomes very large and negative. $\frac{dy}{dx}\to -\infty$.

(c) Determine the equation of the tangent to the curve when $x=1$. (3 marks)

$$\sqrt{y}+\sqrt{1}=3 \Rightarrow y=4$$
$$\frac{dy}{dx}=-\frac{\sqrt{4}}{\sqrt{1}}=-2$$
$$y-4=-2(x-1)$$
$$y=-2x+6$$

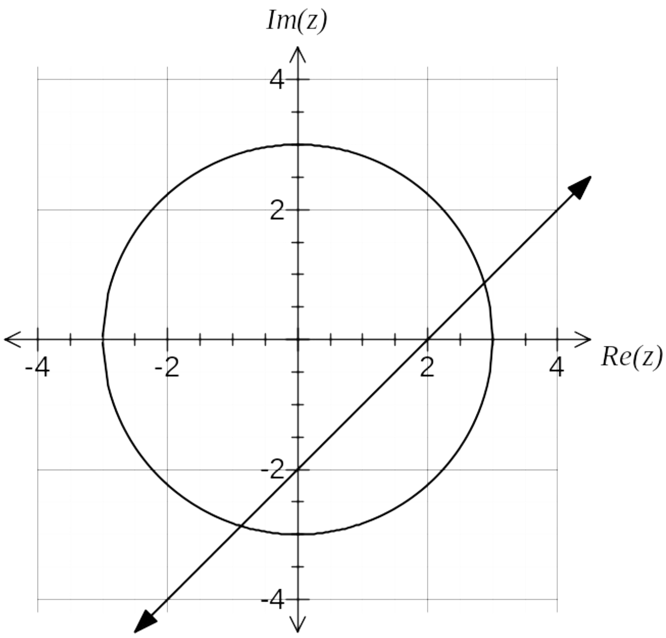
Question 12 **(8 marks)**

(a) On the Argand diagram below, sketch and label both of the following sets of points:

(i) $z \cdot \bar{z} = 9$ (2 marks)

$$\boxed{z \cdot \bar{z} = |z|^2 = 3^2}$$

(ii) $|z - 1 - i| = |z - 3 + i|$ (3 marks)



(b) Determine, to four decimal places, the largest possible argument, $-\pi \le \theta \le \pi$, of the complex number z that satisfies $\{ z \cdot \bar{z} = 9 \} \cap \{ |z - 1 - i| = |z - 3 + i| \}$. (3 marks)

$(x-1)^2 + (y-1)^2 = (x-3)^2 + (y+1)^2$

$x^2 + y^2 = 9$

Solve simultaneously:

$x = \dfrac{\sqrt{14}}{2} + 1,\ y = \dfrac{\sqrt{14}}{2} - 1$

$\theta = \tan^{-1}(y/x) = 0.2945$

Question 13 **(10 marks)**

The Cartesian equation of a plane is $x + 2y - z = 1$.

(a) State a vector equation of this plane. (1 mark)

$$\boxed{\mathbf{r} \bullet (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 1}$$

The parametric equations of line L are $\begin{cases} x = 4 + 2\lambda \\ y = -2 - \lambda \\ z = 3 + \lambda \end{cases}$.

(b) State a vector equation for line L. (1 mark)

$$\boxed{\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})}$$

(c) Determine the position vector of A, the point of intersection of line L and the plane. (4 marks)

$$\begin{bmatrix} 4 + 2\lambda \\ -2 - \lambda \\ 3 + \lambda \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 1$$

$$4 + 2\lambda - 4 - 2\lambda - 3 - \lambda = 1$$

$$\lambda = -4$$

$$\begin{bmatrix} 4 - 8 \\ -2 + 4 \\ 3 - 4 \end{bmatrix} = -4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

(d) The points B and C both lie on line L such that B divides AC internally in the ratio 1:2. Determine the position vector of C given that B has position vector $10\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$.

(4 marks)

$$OC = OA + 3AB$$

$$= \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 10-(-4) \\ -5-2 \\ 6-(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 42 \\ -21 \\ 21 \end{bmatrix}$$

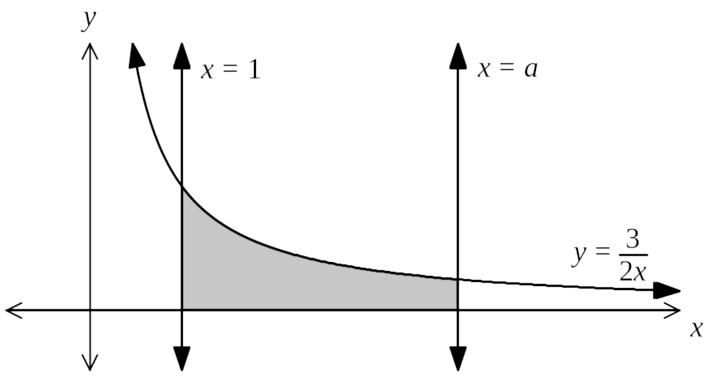
$$= 38\mathbf{i} - 19\mathbf{j} + 20\mathbf{k}$$

Question 14 **(7 marks)**

(a) Show that the gradient of the curve $y = x\log_e(2 - e^x)$ at the origin is 0. (3 marks)

$$\frac{dy}{dx} = 1\cdot\log_e(2-e^x) + x\cdot\frac{-e^x}{2-e^x}\bigg|_{x=0}$$
$$= \log_e(2-e^0) + 0$$
$$= \log_e 1$$
$$= 0$$

(b) The graphs of $y = \frac{3}{2x}$, $x = 1$ and $x = a$ are shown below.



(i) Determine the exact value of a, $a > 1$, given that the shaded region has an area of 6 square units. (3 marks)

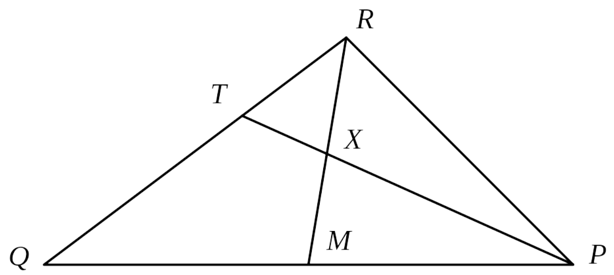
$$\int_1^a \frac{3}{2x}\,dx = \tfrac{3}{2}\ln a$$
$$\ln a = \tfrac{3}{2}\times 6 \Rightarrow a = e^4$$

(ii) Determine the exact area between $y = \frac{3}{2x}$ and the x-axis from $x = 1$ to $x = 2a$. (1 mark)

$$\int_1^{2e^4} \frac{3}{2x}\,dx = \tfrac{3}{2}\ln(2) + 6$$

Question 15 **(7 marks)**

In the diagram of triangle PQR below, M is the midpoint of side PQ, T lies on the side QR such that $RQ = 3RT$ and X is the point of intersection of lines PT and RM such that $TP = 4TX$.



Let $\mathbf{a} = RT$ and $\mathbf{b} = QM$.

(a) Express the following vectors in terms of $\mathbf{a} = RT$ and/or $\mathbf{b} = QM$.

(i) TQ (1 mark)

$$\boxed{\begin{aligned} TQ &= TR + RQ \\ &= -\mathbf{a} + 3\mathbf{a} \\ &= 2\mathbf{a} \end{aligned}}$$

(ii) TP (1 mark)

$$\boxed{\begin{aligned} TP &= TQ + QP \\ &= 2\mathbf{a} + 2\mathbf{b} \end{aligned}}$$

(b) Show that $RX = \frac{1}{2}(3\mathbf{a} + \mathbf{b})$. (2 marks)

$$\boxed{\begin{aligned} RX &= RT + TX \\ &= \mathbf{a} + \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) \\ &= \tfrac{1}{2}(3\mathbf{a} + \mathbf{b}) \end{aligned}}$$

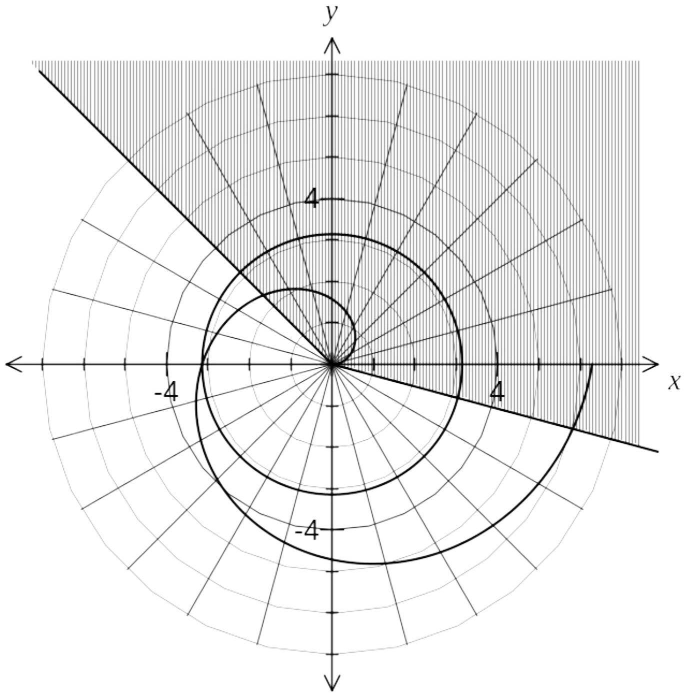
(c) Prove that X bisects RM. (3 marks)

$$RM = RQ + QM$$
$$= 3\mathbf{a} + \mathbf{b}$$

Since $RX = \frac{1}{2}RM$ then X must bisect RM.

Question 16 **(8 marks)**

(a) Use polar inequalities to describe the shaded region shown below. (2 marks)



$$-\frac{\pi}{12} \le \theta \le \frac{3\pi}{4}$$

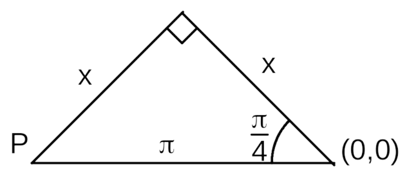
(b) Sketch the polar graph $r = \pi$ on the axes above. (1 mark)

(c) Sketch the polar graph $r = \theta$ on the axes above. (2 marks)

(d) Show that the shortest distance from the point of intersection of $r = \pi$ and $r = \theta$ to the shaded region shown above is $\frac{\pi}{\sqrt{2}}$. (3 marks)

Intersect at P, when $r = \theta = \pi$.

Shortest distance is through P and perpendicular to $\theta = \frac{3\pi}{4}$.

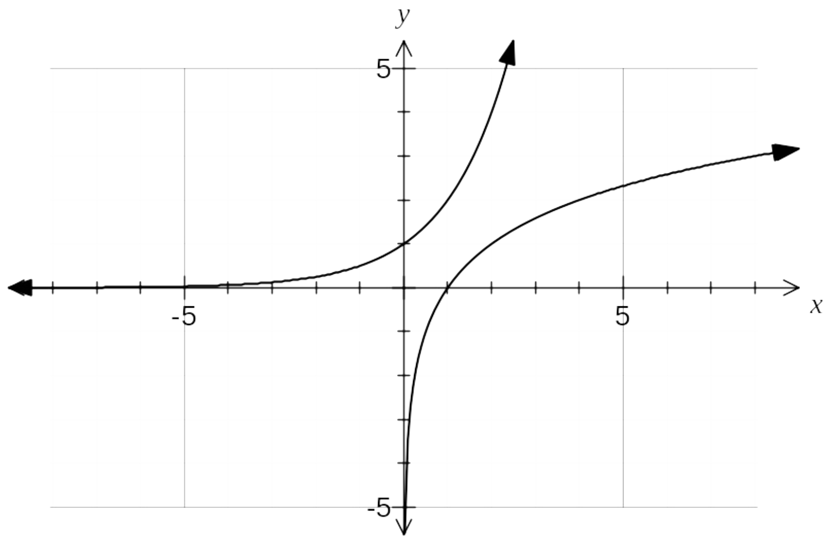


Using the sketch and Pythagoras' theorem: $2x^2 = \pi^2 \Rightarrow x = \frac{\pi}{\sqrt{2}}$

Question 17 **(8 marks)**

Let $f(x) = 2^x$ and $g(x) = \log_2 x$.

(a) Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the axes below. (2 marks)



(b) Describe the relationship between the functions $f(x)$ and $g(x)$. (1 mark)

f and g are inverses of each other.

(c) Determine an expression for $g^{-1} \circ f^{-1}(x)$, $x > 0$. (1 mark)

$g^{-1} \circ f^{-1}(x) = x$

(d) Show that for $x > 0$, $\frac{d}{dx}(g(2x)) = \frac{1}{x \log_e 2}$. (2 marks)

$$\begin{aligned} g(2x) &= \log_2 2x \\ &= \frac{\log_e 2x}{\log_e 2} \\ \frac{d}{dx} g(2x) &= \frac{2}{2x \log_e 2} \\ &= \frac{1}{x \log_e 2} \end{aligned}$$

(e) If $g(a) = b$ and $g'(a) = c$, where a, b and c are positive real numbers, determine an expression for $f'(b)$. (2 marks)

Since g is the inverse of f, then $f(b) = a$ and so $f'(b)$ will be the reciprocal of the slope of g when $x = a$.

$f'(b) = \frac{1}{g'(a)} = \frac{1}{c}$

Question 18 **(9 marks)**

In this question you may assume that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$ and the properties

$\lim\limits_{x\to a}\big(f(x)\cdot g(x)\big)=\lim\limits_{x\to a}f(x)\cdot\lim\limits_{x\to a}g(x)$ and $\lim\limits_{x\to a}\big(f(x)\pm g(x)\big)=\lim\limits_{x\to a}f(x)\pm\lim\limits_{x\to a}g(x)$.

(a) Continue to simplify the limit below, to establish that $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}=0$. (3 marks)

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}$$

$=$

$$\begin{aligned}&=\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}\\&=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}\\&=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{1+\cos x}\\&=1\cdot\frac{0}{2}=0\end{aligned}$$

(b) If $y=\sin x$, then from first principles, $\dfrac{d}{dx}\sin x=\lim\limits_{h\to 0}\dfrac{\sin(x+h)-\sin x}{h}$. Continue to simplify this limit, to show that $\dfrac{d}{dx}\sin x=\cos x$. (3 marks)

$$\begin{aligned}\frac{d}{dx}\sin x&=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\&=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\&=\lim_{h\to 0}\frac{\sin x\cos h-\sin x}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}\\&=\sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}\\&=\sin x(0)+\cos x(1)\\&=\cos x\end{aligned}$$

(c) Show that $\frac{d^2}{dx^2}\left(\sin(\log_e x)\right) = \frac{\cos(\log_e x) + \sin(\log_e x)}{-x^2}$. (3 marks)

$$\frac{dy}{dx} = \frac{1}{x}\cos(\ln x)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\cdot\cos(\ln x) + \frac{1}{x}\cdot\frac{1}{x}(-\sin(\ln x))$$
$$= \frac{\cos(\ln x) + \sin(\ln x)}{-x^2}$$

Question 19 **(14 marks)**

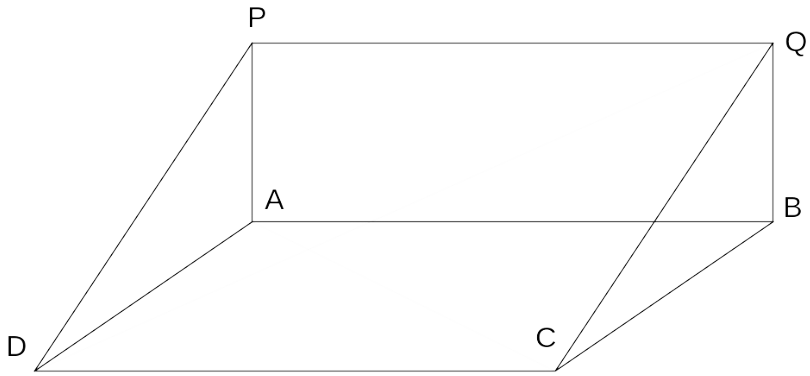
A right triangular prism $ABCDPQ$ is shown (not to scale). All lengths are in metres.

$\vec{OA} = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$

$\vec{AB} = -\mathbf{i} + 10\mathbf{j} - 2\mathbf{k}$

$\vec{AP} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

$\vec{AD} = 2\mathbf{i} - \mathbf{k}$



(a) Show that AB, AP and AD are all perpendicular to each other. (2 marks)

$$\begin{bmatrix}-1\\10\\-2\end{bmatrix} \bullet \begin{bmatrix}2\\1\\4\end{bmatrix} = -2+10-8=0 \qquad \begin{bmatrix}-1\\10\\-2\end{bmatrix} \bullet \begin{bmatrix}2\\0\\-1\end{bmatrix} = -2+0+2=0 \qquad \begin{bmatrix}2\\1\\4\end{bmatrix} \bullet \begin{bmatrix}2\\0\\-1\end{bmatrix} = 4+0-1=0$$

(b) Show that $|DQ| = \sqrt{131}$. (2 marks)

$$\vec{DQ} = \vec{AB} - \vec{AD} + \vec{AP} = \begin{bmatrix}-1\\11\\3\end{bmatrix} \Rightarrow |\vec{DQ}| = \sqrt{1^2 + 11^2 + 3^2} = \sqrt{131}$$

(c) At time $t = 0$ seconds, a small body, X, leaves D and moves directly towards Q with a constant velocity of 1.5 m/s. Write down an equation for the position vector of X at any time t. (3 marks)

$$\begin{aligned} OX &= OD + t \cdot \lambda \cdot DQ \\ &= \begin{bmatrix}4\\4\\4\end{bmatrix} + \begin{bmatrix}2\\0\\-1\end{bmatrix} + t \cdot \frac{1.5}{\sqrt{131}} \begin{bmatrix}-1\\11\\3\end{bmatrix} \\ &= \begin{bmatrix}6\\4\\3\end{bmatrix} + \frac{3t}{2\sqrt{131}} \begin{bmatrix}-1\\11\\3\end{bmatrix} \end{aligned}$$

(d) At the same instant that X leaves D, a second small body, Y, leaves A and moves directly towards C with a constant velocity of 1 m/s. Determine the minimum distance between X and Y during this motion, to the nearest centimetre. (7 marks)

$$\vec{AC} = \vec{AD} + \vec{AB} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 10 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -3 \end{bmatrix} \Rightarrow |AC| = \sqrt{110}$$

$$\vec{OY} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} + t \cdot \frac{1}{\sqrt{110}} \begin{bmatrix} 1 \\ 10 \\ -3 \end{bmatrix}$$

$${}_x r_y = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$${}_x v_y = \frac{3}{2\sqrt{131}} \begin{bmatrix} -1 \\ 11 \\ 3 \end{bmatrix} - \frac{1}{\sqrt{110}} \begin{bmatrix} 1 \\ 10 \\ -3 \end{bmatrix}$$

Minimum distance when ${}_x v_y \bullet (t\,{}_x v_y + {}_x r_y) = 0$

$0.7508676t - 1.1320093 = 0 \Rightarrow t = 1.5076$ seconds

$1.5076\,{}_x v_y + {}_x r_y = \begin{bmatrix} 1.6587 \\ 0.7359 \\ 0.0240 \end{bmatrix} \Rightarrow d = 1.8148 \approx 1.81$ m to nearest cm.

Question 20 **(5 marks)**

Prove that the last digit of n^5, where n is a positive integer, is always the last digit of n.

n can be written in the form $n = 10a + b$, where a and b are both integers, $a > 0$ and $0 \le b \le 9$.

The last digit of n^5 will be the same as the last digit of b^5 since
$(10a + b)^5 = 10(10000a^5 + 5000a^4b + 1000a^3b^2 + 100a^2b^3 + 5ab^4) + b^5$

Since $0 \le b \le 9$, there are 10 cases to consider:

$$0^5 = 0$$
$$1^5 = 1$$
$$2^5 = 32$$
$$3^5 = 243$$
$$4^5 = 1024$$
$$5^5 = 3125$$
$$6^5 = 7776$$
$$7^5 = 16807$$
$$8^5 = 32768$$
$$9^5 = 59149$$

In each case, it can be seen that the last digit of b^5 is the last digit of b, and so the last digit of n^5 will always be n.

End of questions

Additional working space

Question number: _________

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