Semester Two Examination, 2020

Question/Answer Booklet

MATHEMATICS METHODS
ATAR Year 12
Section One: Calculator-free

SOLUTIONS

Student Name: ____

Please circle your teacher's name

Teacher: Miss Long Miss Rowden Ms Stone

Time allowed for this paper
Reading time before commencing work: 5 minutes
Working time for paper: 50 minutes

Materials required/recommended for this paper
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

Number of additional answer booklets used (if applicable):	

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Suggested working time (minutes)	Marks available	Percentage of examination
Section One: Calculator free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the ATAR course examinations are detailed in the *Year 12 Information Handbook 2020*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Supplementary pages for the use planning/continuing your answer to a question have been provided at the end of the Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has eight (8) questions. Answer **all** questions. Write your answers in the spaces provided.

Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

Working time: 50 minutes.

Question 1 **(7 marks)**

(a) Determine an expression for $f'(x)$ when

(i) $f(x)=\ln(1-\cos 3x)$. (2 marks)

Solution
$f'(x)=\dfrac{3\sin 3x}{1-\cos 3x}$
Specific behaviours
✓ numerator
✓ denominator

(ii) $f(x)=e^{5x}(5-2x)^3$. (3 marks)

Solution
$f'(x)=5e^{5x}(5-2x)^3+e^{5x}\cdot 3(-2)(5-2x)^2$ $¿5e^{5x}(5-2x)^3-6e^{5x}(5-2x)^2$ *N.B. Simplifies to* $(19-10x)(5-2x)^2e^{5x}$
Specific behaviours
✓ derivative of e^{5x}
✓ derivative of $(5-2x)^3$
✓ correct expression using product rule

(b) For the positive number x, let $A(x)=\int_0^x (8-2^{t^2})\,dt$.

Determine the value(s) of x for which $\dfrac{dA}{dx}=0$. (2 marks)

Solution
$\dfrac{dA}{dx}=\dfrac{d}{dx}\int_0^x (8-2^{t2})dt¿8-2^{x^2}$ $\therefore 2^{x^2}=8=2^3\Rightarrow x^2=3\Rightarrow x=\sqrt{3}$
Specific behaviours
✓ expression for $A'(x)$
✓ correct value of x

See next page

Question 2 **(7 marks)**

The discrete random variable X is defined by

$$P(X=x)=\begin{cases} \dfrac{x+b}{5x+2} & x=0,1 \\ ¿ & \text{elsewhere}¿ \\ & ¿ \end{cases}$$

(a) Determine the value of the constant b. (2 marks)

Solution
$\frac{b}{2}+\frac{1+b}{7}=17b+2+2b=14b=\frac{4}{3}$
Specific behaviours
✓ forms equation using $x=0$ and $x=1$
✓ correct value

(b) Determine

(i) $P(X=0)$. (1 mark)

Solution
$P(X=0)=\frac{4}{3}\div 2=\frac{2}{3}$
Specific behaviours
✓ correct probability

(ii) $E(7X-3)$. (2 marks)

Solution
$E(X)=p=P(X=1)=1-\frac{2}{3}=\frac{1}{3}$ $E(7X-3)=7\times\frac{1}{3}-3=\frac{-2}{3}$
Specific behaviours
✓ indicates $E(X)$
✓ correct value

(iii) $Var(7X-3)$. (2 marks)

Solution
$Var(X)=p(1-p)=\frac{1}{3}\times\frac{2}{3}=\frac{2}{9}$ $Var(7X-3)=7^2\times\frac{2}{9}=\frac{98}{9}$
Specific behaviours
✓ indicates $Var(X)$
✓ correct value

Question 3 **(5 marks)**

The rate of change of pressure in an air tank is given by $P'(t) = -3e^{-0.05t}$, where t is the time in in minutes since it began to empty.

(a) Determine an expression for the pressure P in the tank at any time $t, t \geq 0$. (2 marks)

Solution
$P(t) = \frac{-3}{-0.05}e^{-0.05t} + c ¿ 60e^{-0.05t} + c$
$(0,70) \Rightarrow 70 = 60e^{0} + c c = 10$
$P(t) = 10 + 60e^{-0.05t}$
Specific behaviours
✓ correctly integrates $P'(t)$
✓ correct expression for $P(t)$

(b) Determine

(i) the time taken for the pressure in the tank to fall to 40 psi. (2 marks)

Solution
$10 + 60e^{-0.05t} = 40 e^{-0.05t} = 0.5$
$-0.05t = \ln 0.5 t = -20\ln 0.5 (¿ 20\ln 2)$
Specific behaviours
✓ simplifies equation to $e^{-0.05t} = k$
✓ correct time

(ii) the minimum pressure in the tank for $t \geq 0$. (1 mark)

Solution
$t \to \infty, P \to 10\,\text{psi}$
Specific behaviours
✓ correct pressure

Question 4 **(6 marks)**

The continuous random variable X takes values in the interval 3 to 8 and has **cumulative** distribution function $F(x)$ where

$$F(x)=P(X\le x)=\begin{cases} 0 & x<3 \\ \dfrac{x-3}{5} & 3\le x\le 8 \\ 1 & x>8. \end{cases}$$

(a) Determine

(i) $P(X\le 4.5)$. (1 mark)

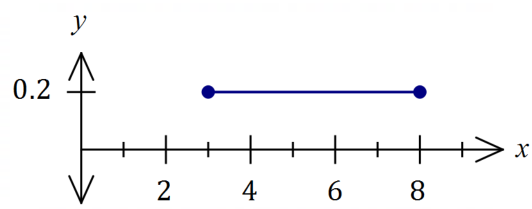
Solution
$P(X\le 4.4)=\dfrac{4.5-3}{5}=\dfrac{1.5}{5}=\dfrac{3}{10}=0.3$
Specific behaviours
✓ correct probability

(ii) the value of k, if $P(X>k)=0.75$. (2 marks)

Solution
$P(X\le k)=1-P(X>k)=1-0.75=0.25$ $\dfrac{k-3}{5}=0.25\Rightarrow k=4.25$
Specific behaviours
✓ indicates $P(X\le k)$ ✓ correct value

(b) Determine $f(x)$, the probability density function of X, and sketch the graph of $y=f(x)$.

(3 marks)

Solution
$f(x)=F'(x)=\begin{cases} \dfrac{1}{5} & 3\le x\le 8 \\ 0 & \text{elsewhere} \end{cases}$ 
Specific behaviours
✓ $f(x)$ (no penalty for just $f(x)=\dfrac{1}{5}$ if sketch correct) ✓ draws $y=0.2$ between endpoints

Question 5 **(7 marks)**

The function f is defined by $f(x)=\frac{x^2-7}{4-x}, x\neq 4$.

The second derivative of f is $f''(x)=18(4-x)^{-3}$.

Determine the coordinates and nature of all stationary points of the graph of $y=f(x)$.

Solution
$f'(x)=\frac{2x(4-x)-(-1)(x^2-7)}{(4-x)^2}$ $f'(x)=0\Rightarrow 8x-2x^2+x^2-7=0(x^2-8x+7)=0(x-1)(x-7)=0$ $x=1,7$ $f''(1)=\frac{18}{3^3}>0\Rightarrow \text{Min}, f''(7)=\frac{18}{-3^3}<0\Rightarrow \text{Max}$ $f(1)=\frac{-6}{3}=-2, f(7)=\frac{42}{-3}=-14$ $f(x)$ has a minimum at $(1,-2)$ and a maximum at $(7,-14)$.
Specific behaviours
✓ indicates correct use of quotient rule ✓ correct $f'(x)$ ✓ equates numerator to zero ✓ determines x-coordinates of stationary points ✓ indicates correct use of second derivative for nature ✓ correct minimum ✓ correct maximum

Question 6 **(7 marks)**

(a) Simplify $3\log 20 - \log 4 + \log 5$. (2 marks)

Solution
$\log 20^3 - \log 4 + \log 5 = \log(8000 \div 4 \times 5)$
$\text{¿}\log 10^4 \text{¿} 4$
Specific behaviours
✓ expresses as single log
✓ simplifies to number

(b) Given that $\log_a x = 0.82$, determine the value of $\log_a\left(\frac{\sqrt{x}}{x}\right)$. (2 marks)

Solution
$\log_a \frac{\sqrt{x}}{x} = \log_a \sqrt{x} - \log_a x \text{¿} 0.5\log_a x - \log_a x$
$\text{¿} -0.5\log_a x \text{¿} -0.5 \times 0.82 \text{¿} -0.41$
Specific behaviours
✓ obtains multiple of $\log_a x$
✓ correct value

(c) Determine the solution to the equation $2^{3x} = 5^{2-x}$ in the form $x = \frac{\log a}{\log b}$. (3 marks)

Solution
$3x\log 2 = (2-x)\log 5$
$3x\log 2 = 2\log 5 - x\log 5$
$3x\log 2 + x\log 5 = 2\log 5$
$x(3\log 2 + \log 5) = 2\log 5 x = \frac{\log 5^2}{\log(2 \text{¿} \text{¿} 3 \times 5) \text{¿}}$
$x = \frac{\log 25}{\log 40}$
Specific behaviours
✓ writes as log equation
✓ factors out x
✓ solves and simplifies into required form

Question 7 **(6 marks)**

The acceleration at time t seconds of a small body travelling in a straight line is given by

$$a(t)=\frac{-27}{\sqrt{3t+1}}\,\text{cm/s}^2,\, t\geq 0.$$

When $t=1$ the body was at the origin and 7 seconds later its displacement was 22 cm.

Determine the velocity of the body when $t=5$.

Solution
$v(t)=\int -27(3t+1)^{\frac{-1}{2}}\,dt ¿ \frac{-27}{\frac{1}{2}\times 3}(3t+1)^{\frac{1}{2}}+c$ $¿-18(3t+1)^{\frac{1}{2}}+c$ $\Delta x=\int_1^{1+7} -18(3t+1)^{\frac{1}{2}}+c\,dt ¿ \left[\frac{-18}{\frac{3}{2}\times 3}(3t+1)^{\frac{3}{2}}+ct\right]_1^8$ $¿\left[-4(3t+1)^{\frac{3}{2}}+ct\right]_1^8 ¿ [-4(125)+8c]-[-4(27)+c]$ $¿ 7c-468$ But $\Delta x=22$ $7c-468=227c=490c=70$ $v(5)=-18(3(5)+1)^{\frac{1}{2}}+70 ¿ -72+70 ¿ -2\,\text{cm/s}$

Specific behaviours
✓ antiderivative of $a(t)$ ✓ integral for change in displacement Δx ✓ antiderivative of $v(t)$ ✓ simplifies equation for c ✓ uses given Δx to determine value of c ✓ correct velocity

Question 7 – alternative method

Solution

$$v(t)=\int -27(3t+1)^{\frac{-1}{2}}\,dt \text{¿} \frac{-27}{\frac{1}{2}\times 3}(3t+1)^{\frac{1}{2}}+c \text{¿} -18(3t+1)^{\frac{1}{2}}+c$$

$$x(t)=\int -18(3t+1)^{\frac{1}{2}}dt \text{¿} \frac{-18}{\frac{3}{2}\times 3}(3t+1)^{\frac{3}{2}}+ct+k \text{¿} -4(3t+1)^{\frac{3}{2}}+ct+k$$

$$0=-4(3\times 1+1)^{\frac{3}{2}}+1c+k32=c+k \text{ eqn}$$

$$22=-4(3\times 8+1)^{\frac{3}{2}}+8c+k522=8c+k$$

$$7c=490c=70$$

$$v(5)=-18(3(5)+1)^{\frac{1}{2}}+70\text{¿}-72+70\text{¿}-2\,\text{cm/s}$$

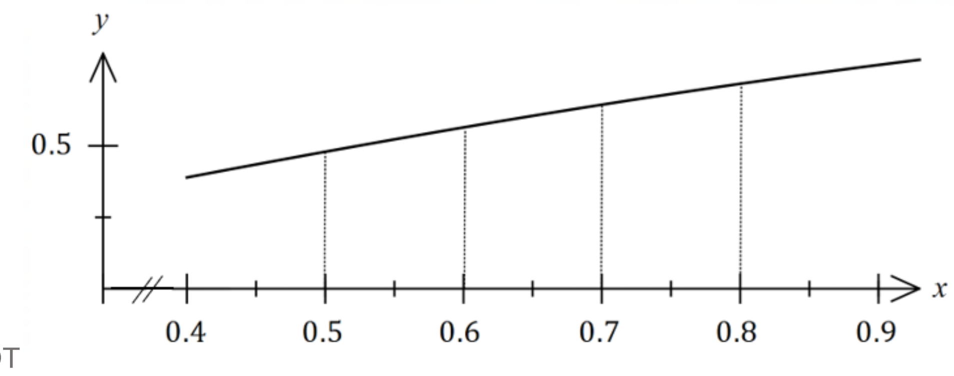
Specific behaviours

✓ antiderivative of $a(t)$
antiderivative of $v(t)$
$x(t)$ equation for $t=1$
✓ $x(t)$ equation for $t=8$
solves simultaneously to determine value of c
correct velocity

Question 8 **(7 marks)**

The graph and a table of values for $y=f(x)$ is shown below, where $f(x)=\sin x$.

End of questions



x	y
0.4	0.39
0.5	0.48
0.6	0.56
0.7	0.64
0.8	0.72
0.9	0.78

Let $I=\int_{0.5}^{0.8} \sin x\, dx$.

(a) By using the information shown and considering sums of the form $\sum_i f(x_i)\delta x_i$ explain why $I<0.192$.

(3 marks)

Solution
With $\delta x=0.1, x_1=0.6, x_2=0.7$ and $x_3=0.8$ then $\Sigma_i f(x_i)\delta x_i=0.1(0.56+0.64+0.72)$¿$0.1(1.92)$¿$0.192$ Hence I must be less than this value as it is the area of circumscribed rectangles that overestimate the area under the curve.
Specific behaviours
✓ indicates x-ordinates for circumscribed rectangles ✓ shows sum of $f(x_i)\delta x_i$ ✓ explains inequality

(b) In a similar manner to (a), determine the lower estimate, L, for the value of I. That is, the value of L for $I>L$.

(2 marks)

Solution
With $\delta x=0.1, x_1=0.5, x_2=0.6$ and $x_3=0.7$ then $L=\Sigma_i f(x_i)\delta x_i=0.1(0.48+0.56+0.64)$¿$0.1(1.68)$ ¿0.168
Specific behaviours
✓ indicates x-ordinates for inscribed rectangles ✓ value of L

(c) Use your previous answers to determine a numerical estimate for I and explain whether your estimate is smaller or larger than the exact value of I. (2 marks)

Solution
$I=\frac{0.168+0.192}{2}=0.18$ This value slightly underestimates the exact value of I as the curve is concave downwards.
Specific behaviours
✓ correctly averages values ✓ states underestimate with reason

End of questions