PERTH MODERN SCHOOL
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Semester One Examination, 2018

Question/Answer booklet

MATHEMATICS SPECIALIST
UNIT 3
Section One:
Calculator-free

Marking Key

Time allowed for this section
Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Mark	Question	Mark
1		5	
2		6	
3		7	
4		8	

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	50	35
Section Two: Calculator-assumed	13	13	100	95	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

See next page

Acknowledgements

Section One: Calculator-free **(50 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 50 minutes.

Question 1 **(8 marks)**

Consider the polynomial $f(z)=z^2+3z+3$ where $z=x+iy$

(a) Determine the roots of $f(z)=0$ and label them $\alpha \& \beta$ (2 marks)

Solution
$z=\frac{-3\pm\sqrt{9-4(1)(3)}}{2}$ $=\frac{-3\pm i\sqrt{3}}{2}$
Specific behaviours
✓ uses quadratic formula ✓ obtains two complex roots

(b) Determine $\alpha\overline{\beta}+\overline{\alpha}\beta$ (3 marks)

Solution
$\frac{-3+i\sqrt{3}}{2}\frac{-3+i\sqrt{3}}{2}+\frac{-3-i\sqrt{3}}{2}\frac{-3-i\sqrt{3}}{2}$ $=\frac{9-6\sqrt{3}i-3}{4}+\frac{9+6\sqrt{3}i-3}{4}$ $=3$
Specific behaviours
✓ uses conjugates correctly ✓multiplies correctly ✓ obtains simplified sum

See next page

(c) Determine $f(\alpha+\beta)+3$ (3 marks)

Solution
$(\alpha+\beta)^2+3(\alpha+\beta)+3+3$ $\alpha^2+2\alpha\beta+\beta^2+3\alpha+3\beta+3+3$ $\alpha^2+3\alpha+3+\beta^2+3\beta+3+2\alpha\beta$ $0+0+2\frac{-3+i\sqrt{3}}{2}\frac{-3-i\sqrt{3}}{2}$ $\frac{9+3}{2}$ 6
Specific behaviours
✓ expands expression in terms of variables ✓separates sums that give zero ✓ obtains simplified real result

Question 2 **(7 marks)**

Let $w=2a-bi$ where $a\&b$ are real numbers.

(a) Show that $|w^6|=(4a^2+b^2)^3$ (2 marks)

Solution
$\lvert w\rvert=\sqrt{4a^2+b^2}$ $\lvert w^6\rvert=\lvert w\rvert^6=(4a^2+b^2)^{\frac{6}{2}}=(4a^2+b^2)^3$
Specific behaviours
✓ obtains expression for modulus of w ✓ raises modulus to power of 6

See next page

Additional working space

Question number: _____

Additional working space

Question number: _____

(b) The expression $w+\frac{1}{w}$ can be written in the form $\frac{c+di}{4a^2+b^2}$, determine expressions for the real constants $c \& d$ in terms of $a \& b$. (3 marks)

Solution
$2a - bi + \frac{1}{2a - bi}\frac{2a+bi}{2a+bi}$ $(2a - bi)\frac{4a^2+b^2}{4a^2+b^2} + \frac{2a+bi}{4a^2+b^2}$ $\frac{8a^3+2ab^2 - i(4a^2b+b^3)+2a+bi}{4a^2+b^2}$ $\frac{8a^3+2ab^2+2a - i(4a^2b+b^3 - b)}{4a^2+b^2}$ $=\frac{c+di}{4a^2+b^2}$
Specific behaviours
✓ uses conjugate to obtain real denominator ✓changes both terms to have the same denominator ✓ obtains final expressions for c & d

(c) Given that the $Arg(w^6)=Arg(w)$ and $0<Arg(w)<\frac{\pi}{2}$, determine the $Arg(w)$. (2 marks)

Solution
$Arg(w)=\theta$ $Arg(w^6)=6\theta=\theta+2n\pi$ $5\theta=2n\pi \quad n\in J$ $\theta=\frac{2n\pi}{5}=\frac{2\pi}{5}$
Specific behaviours
✓ multiplies Arg(w) by 6

See next page

✓ obtains expression for Arg(w) in terms of pi

See next page

Additional working space

Question number: _____

Additional working space

Question number: _____

Question 3 **(7 marks)**

Consider the following functions f & g.

$$f(x)=\frac{1}{\sqrt{x-5}} \qquad g(x)=(2x-1)^3$$

(a) State the natural domain and range of f. (2 marks)

Solution
Domain $x>5$ Range $y>0$
Specific behaviours
✓ states domain ✓ states range

(b) State the natural domain and range of $g \circ f(x)$ (3 marks)

Solution
$g \circ f(x)=\left(\frac{2}{\sqrt{x-5}}-1\right)^3$ $domain: x>5$ $range: -1<y<\infty$
Specific behaviours
✓ uses domain of f(x) ✓uses rule to find range ✓ states range

(c) Does $f \circ g(x)$ exist over the natural domain of g? Explain. (2 marks)

Solution
$r_g: R$ $d_f: x>5$ $r_g \not\subset d_f \therefore does\ not\ exist$

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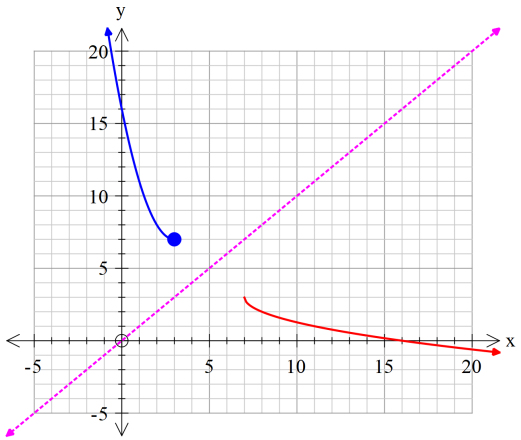
Specific behaviours
✓ states condition for existence ✓ states relevant domain and ranges

Question 4 **(7 marks)**

Consider the function $f(x) = x^2 - 6x + 16$, $x \leq 3$ which is plotted on the axes below.

(a) Sketch the inverse function $f^{-1}(x)$ on the axes above. (3 marks)

Solution



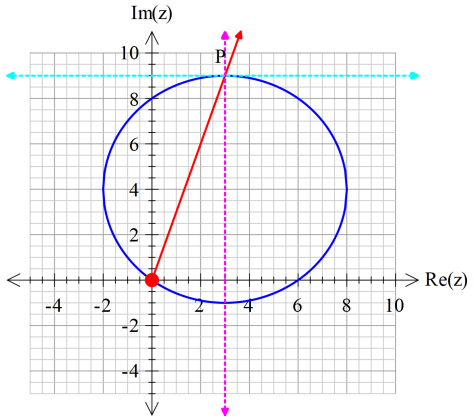
Specific behaviours
✓ vertex of (7,3) plotted ✓reflected in line y=x ✓ x intercept of (16,0) plotted

See next page

Additional working space

Question number: _____

(b) Sketch this circle and point P in the complex plane below showing all major features.
(2 marks)

Solution

Specific behaviours
✓ sketches circle with correct centre and radius going through (0,0)
✓ shows point P to be on circle at (3,9)

End of questions

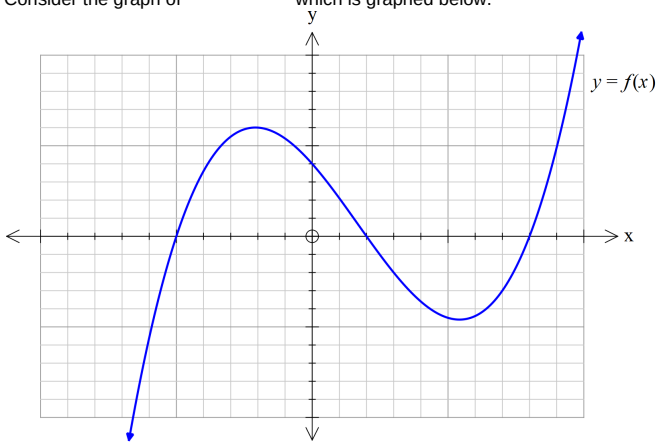
(b) Determine the rule for the inverse function $f^{-1}(x)$ and state the domain and range.
(4 marks)

Solution
$y = x^2 - 6x + 16 = (x - 3)^2 - 7$ $x = (y - 3)^2 - 7$ $\pm\sqrt{x - 7} = y - 3$ $f^{-1}(x) = 3 - \sqrt{x - 7}$ $d : x \geq 7$ $r : y \leq 3$
Specific behaviours
✓ interchanges x and y
✓completes the square or uses quadratic formula
✓ obtains expression for inverse with minus sign
✓states domain and range of inverse

See next page

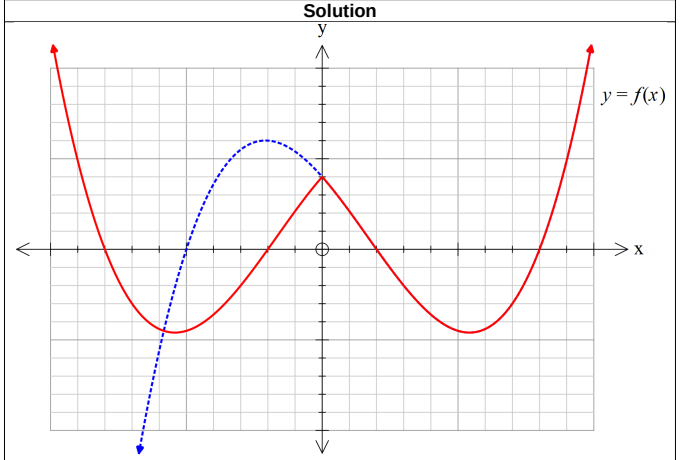
Question 5 **(5 marks)**

Consider the graph of $y = f(x)$ which is graphed below.



(a) On the axes below, sketch $y = f(|x|)$ (2 marks)

Solution



Specific behaviours

✓ right side reflected in y axis
✓ correct x , y intercepts and turning point on LHS

Question 8 **(5 marks)**

Consider a circle in the complex plane where the centre is given by $3+4i$ and a radius of 5 units. Let P be a point on this circle where $P = rcis\theta$ with $|P| \neq 0$ and $Arg(P) = \tan^{-1} 3$

(a) Determine P in exact cartesian form $x + iy$. (3 marks)

Solution

$$(x-3)^2 + (y-4)^2 = 25$$
$$y = 3x$$
$$x^2 - 6x + 9 + (3x-4)^2 = 25$$
$$x^2 - 6x + 9 + 9x^2 - 24x + 16 = 25$$
$$10x^2 - 30x = 0$$
$$10x(x-3) = 0$$
$$x = 0, 3$$
$$y = 0, 9$$
$$P = 3 + 9i \text{ as } |P| \neq 0$$

Specific behaviours

✓ uses y=3x for point P
✓subs y=3x into cartesian equation of circle
✓ identifies 3+9i as only solution for P

(b) Determine the cartesian equation of the plane that contains points $A, B \& C$. (3 marks)

Solution

$$\overrightarrow{AB} = \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix} \times \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 8(6) - 0 \\ 0 - -24 \\ 0 - -32 \end{pmatrix} = 8\begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

$$\underline{r}.\begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}.\begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} = 24$$

$$6x + 3y + 4z = 24$$

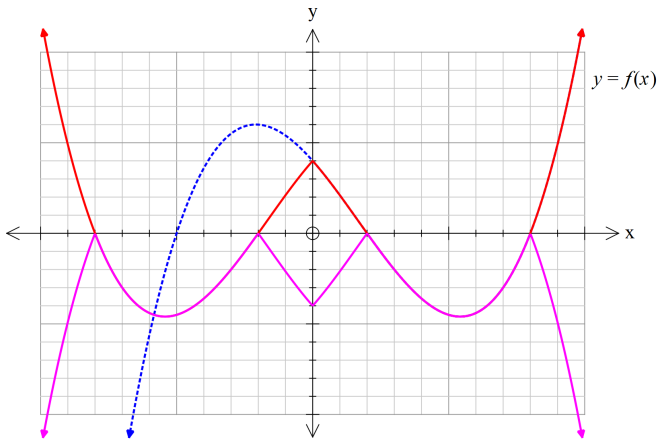
Specific behaviours

✓ determines two vectors in plane
✓uses cross product to obtain normal vector
✓ determines cartesian equation from vector equation of plane

End of questions

(b) On the axes below, sketch $y = -\left| f\left(|x|\right) \right|$ (3 marks)

Solution



Specific behaviours

✓lateral symmetry
✓only plotted on and below x axis
✓ correct x, y intercepts and turning points

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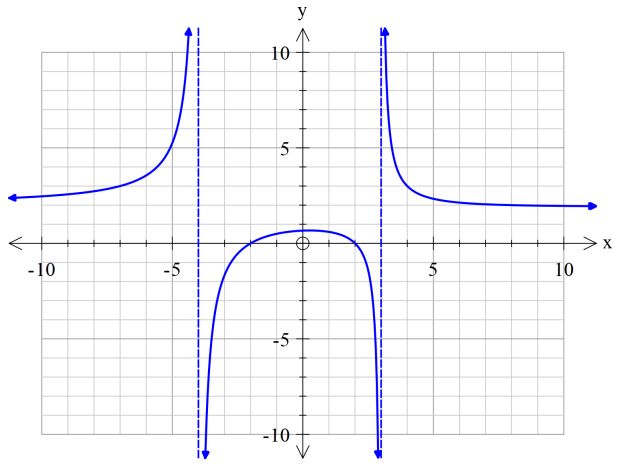
Question 6 **(6 marks)**

$$f(x) = \frac{2(x^2 - 4)}{(x^2 + x - 12)}$$

Sketch the graph of on the axes below.

Solution

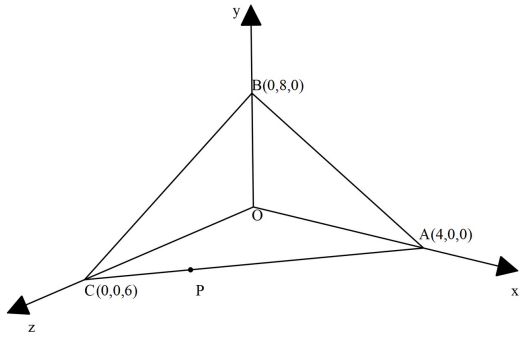


Specific behaviours

✓ vertical asymptotes at x =-4 & 3
✓horizontal asymptote ay y=2
✓ y intercept near y=2/3
✓x intercepts at x=-2,2
✓ correct shape between vertical asymptotes
✓correct shape outside asymptotes

Question 7 **(5 marks)**

A triangular prism OACB is shown below with O as the origin and points A, B & C have respective position vectors $\begin{pmatrix}4\\0\\0\end{pmatrix}, \begin{pmatrix}0\\8\\0\end{pmatrix}, \begin{pmatrix}0\\0\\6\end{pmatrix}$ Point P lies on the line $\overline{CA}$ in the ratio $1:2$.



(a) Determine the vector equation of the line that passes through points $B \& P$ (2 marks)

Solution

$$OP = \begin{pmatrix}0\\0\\6\end{pmatrix} + \frac{1}{3}\left\{\begin{pmatrix}4\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\6\end{pmatrix}\right\} = \begin{pmatrix}\frac{4}{3}\\0\\4\end{pmatrix}$$

$$\underset{\sim}{r} = \begin{pmatrix}0\\8\\0\end{pmatrix} + \lambda\left\{\begin{pmatrix}\frac{4}{3}\\0\\4\end{pmatrix} - \begin{pmatrix}0\\8\\0\end{pmatrix}\right\} = \begin{pmatrix}0\\8\\0\end{pmatrix} + \lambda\begin{pmatrix}\frac{4}{3}\\-8\\4\end{pmatrix}$$

Specific behaviours

✓ determines point P
✓ Determines vector equation of line