PERTH MODERN SCHOOL
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Semester 1 Examination 2012

Question/Answer Booklet

MATHEMATICS:
SPECIALIST 3CD

Section Two:
Calculator-assumed

Name of Student: ________________________Marking Key____________________

Time allowed for this section
Reading time before commencing work: 10 minutes
Working time for this section: 100 minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the student
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination

Important note to students
No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One Calculator-free	6	6	50	50	
Section Two Calculator-assumed	11	11	100	100	
			Total	150	100

Instructions to students

1 Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer. If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued. i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

2 **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

3 It is recommended that you **do not use pencil**, except in diagrams.

4 You must be careful to confine your responses to the specific questions asked and to follow any instructions that are specific to a particular question.

Section Two: Calculator-assumed **(100 marks)**

This section has **eleven (11)** questions. Answer all questions. Write your answers in the spaces provided.

Working time: 100 minutes

Question 7 **(7 marks)**

(a) Use **proof by exhaustion** to prove that all values of 2^n end in 2, 4, 6, or 8,

n>0 and n is an integer. (3)

Solution
Case 1: $2^{4k} = 16^k = 10m + 6$ where m is an arbitrary positive integer Case 2: $2^{4k+1} = 2.2^{4k} = 10m + 2$ Case 3: $2^{4k+2} = 2^2.2^{4k} = 10m + 4$ Case 4: $2^{4k+3} = 2^3.2^{4k} = 10m + 8$ All values of 2^n end in 2, 4, 6, or 8 OR $2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,$ $2^5 = 32,\ 2^6 = 64,\ 2^7 = 128,\ 2^8 = 256$, $2^9 = 512, 2^{10} = 1024$, etc Last digit ends in 2, 4, 6, or 8
Specific behaviours
✓✓ states all possible cases ✓ unit digit is always a 2, 4, 6 or 8 OR ✓✓ calculates 2^n from n = 1 to 4, n = 5 to 8 and compares last digit ✓ correctly explains why pattern continues

(b) Prove that $\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$. (4)

Solution
$LHS = \cos^4\theta$ $= (\cos^2\theta)^2$ $= \left(\frac{\cos 2\theta + 1}{2}\right)^2$ $= \frac{1}{4}(\cos^2 2\theta + 2\cos 2\theta + 1)$ $= \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{\cos 4\theta + 1}{2}\right)$ $= \frac{1}{4}\left(\frac{2 + 4\cos 2\theta + \cos 4\theta + 1}{2}\right)$ $= \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$ $RHS = \frac{1}{8}(3 + 4\cos 2\theta + 2\cos^2 2\theta - 1)$ $= \frac{1}{8}(2 + 4\cos 2\theta + 2\cos^2 2\theta)$ $= \frac{1}{8}\left(2 + 4(2\cos^2\theta - 1) + 2(2\cos^2\theta - 1)^2\right)$ $= \frac{1}{8}(2 + 8\cos^2\theta - 4 + 8\cos^4\theta - 8\cos^2\theta + 2)$ $= \cos^4\theta$
Specific behaviours
✓ expresses in terms of $\cos 2\theta$ ✓ expand correctly ✓ expresses in terms of $\cos 4\theta$ ✓ correctly simplifies to RHS

Question 8 **(6 marks)**

A water tank has vertical sides of height *h* and is initially full. Through a small hole in the base of the tank, water leaks out at a rate, which, at any time *t*, is proportional to the depth *x* of the remaining water in the tank at that instant. That is $\frac{dx}{dt} = -kx$. The tank is exactly half empty in 2 hours.

(a) Show that the exact value of *k* is $\frac{1}{2}\ln 2$. (4)

Solution
$x(t) = he^{-kt}$ $t = 2, x = \frac{1}{2}h$ $\frac{1}{2}h = he^{-k(2)}$ $\frac{1}{2} = e^{-k(2)}$ $\ln\frac{1}{2} = -2k$ $\ln 1 - \ln 2 = -2k$ $\ln 2 = 2k$ $k = \frac{1}{2}\ln 2$
Specific behaviours
✓ expresses as an exponential equation ✓writes as exponential equation using t=2, x = $\frac{1}{2}h$ ✓✓ solves for k exactly

(b) Determine the depth of water in $\frac{1}{2}$ hour giving your answer in terms of *h* (2)

Solution
$t = \frac{1}{2},\ \ x = he^{(-\frac{1}{2}\ln 2)(\frac{1}{2})}$ $x = he^{(-\frac{1}{4}\ln 2)}$ $x = 2^{-\frac{1}{4}}h$ $x = 0.8409h$
Specific behaviours
✓ substitutes $t = \frac{1}{2}$ ✓ answer in terms of h

Question 9 **(10 marks)**

(a) Find the equation of the plane passing through (1, -1, 3) and parallel to the plane
$\underset{\sim}{r} \bullet (3\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) = 7$ (2)

Solution
$\underset{\sim}{r}\cdot(3,1,1) = (1,-1,3)\cdot(3,1,1)$ $\underset{\sim}{r}\cdot(3\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) = 5$
Specific behaviours
✓ use the rule $\underset{\sim}{r}\cdot\underset{\sim}{n} = \underset{\sim}{a}\cdot\underset{\sim}{n}$ ✓ correct answer of $\underset{\sim}{r}\cdot(3\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) = 5$

(b) Find the **obtuse** angle between the two planes defined by

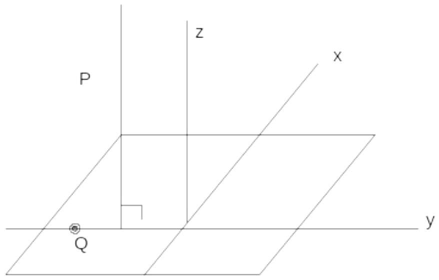
Plane I: $\underset{\sim}{r}\cdot(\underset{\sim}{i} + \underset{\sim}{j}) = 1$

Plane II: $\underset{\sim}{r}\cdot(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}) = 2$ (2)

Solution
$(\underset{\sim}{i} + \underset{\sim}{j})\cdot(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}) = \lvert(\underset{\sim}{i} + \underset{\sim}{j})\rvert \lvert 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\rvert \cos\theta$ $\theta = \frac{\pi}{4}$ OR Using CAS angle([1,1,0],[2,1,-2]) results in $\theta = \frac{\pi}{4}$ $\therefore$ obtuse angle between the two planes is $\frac{3\pi}{4}$ or 135^{0}
Specific behaviours
✓ calculates angle between the two normal using CAS ✓ states the obtuse angle

(c) Find the shortest distance from the point P(2, -3, 4) to the plane $\underset{\sim}{r}\cdot(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}) = 13$ (6)

Solution



Let Q(x,y,z) be any point on the plane
P must be perpendicular at closest approach.
Therefore, P lies on line $r=\langle 2, -3, 4\rangle + \lambda\langle 1,2,3\rangle$
This line intersects plane $r.\langle 1,2,3\rangle = 13$
Therefore $\langle 2+\lambda, -3+2\lambda, 4+3, 4+3\lambda\rangle.\langle 1,2,3\rangle = 13$
Using CAS, $\lambda = 5/14$
Therefore Q is at $\langle 33/14, -16/7, 71/14\rangle$
$|PQ| = |P-Q|$
$= 5/\sqrt{14}$ units or 1.34 units (2 d.p.)

Specific behaviours

✓ $r=\langle 2, -3, 4\rangle + \lambda\langle 1,2,3\rangle$
✓✓ solves $r=\langle 2, -3, 4\rangle + \lambda\langle 1,2,3\rangle$, $\lambda = 5/14$
✓ Q is at $\langle 33/14, -16/7, 71/14\rangle$
✓ $|PQ| = |P-Q|$
✓ states shortest distance

Question 10 **(9 marks)**

(a) If $y = \ln\left(\frac{1+\sin x}{\cos x}\right)$, show that $\frac{dy}{dx} = \frac{1}{\cos x}$ (4)

Solution
$y = \ln(1+\sin x) - \ln(\cos x)$ OR $y = \ln\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)$ $\frac{dy}{dx} = \frac{\cos x}{1+\sin x} - \frac{(-\sin x)}{\cos x}$ $\quad \frac{dy}{dx} = \frac{\frac{\sin x}{\cos^2 x} + \frac{1}{\cos^2 x}}{\frac{1+\sin x}{\cos x}}$ $\frac{dy}{dx} = \frac{\cos^2 x + \sin x + \sin^2 x}{\cos x(1+\sin x)}$ $\quad \frac{dy}{dx} = \frac{\sin x + 1}{\cos^2 x} \times \frac{\cos x}{1+\sin x}$ $\frac{dy}{dx} = \frac{1+\sin x}{\cos x(1+\sin x)} = \frac{1}{\cos x}$ $\quad \frac{dy}{dx} = \frac{1}{\cos x}$
Specific behaviours
✓ express in terms of “ln” or differentiate “ln” and use quotient rule   ✓✓ differentiate each part correctly   ✓ simplify to $\frac{1}{\cos x}$

(b) The length, *l*, of an arc of a curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$l = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$.

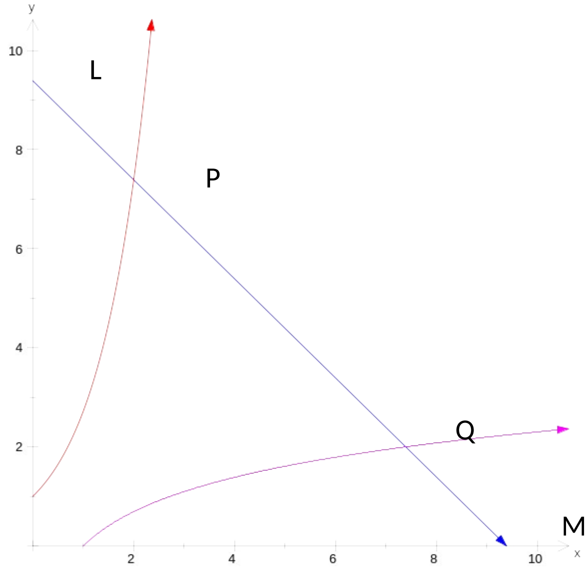
Find the **exact** length of the curve $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{3}$ showing sufficient steps how you use your answer from part (a) to find *l*. (5)

Solution
$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$ $l = \int_0^{\frac{\pi}{3}} \sqrt{1+(-\tan x)^2}\, dx$ $l = \int_0^{\frac{\pi}{3}} \sec x\, dx$

$I = \int_0^{\frac{\pi}{3}} \frac{1}{\cos x}\, dx$ $I = \ln\left(\frac{1+\sin x}{\cos x}\right)\Big\|_0^{\frac{\pi}{3}}$ $I = \ln\left(\frac{1+\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}\right) - \ln\left(\frac{1+\sin 0}{\cos 0}\right)$ $I = \ln\left(\frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) - \ln 1$ $I = \ln\left(\frac{2+\sqrt{3}}{2}\times\frac{2}{1}\right)$ $I = \ln\left(2+\sqrt{3}\right)$ $I = 1.317$
Specific behaviours
✓ $\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$ ✓ substitutes into "I" and simplify to $\frac{1}{\cos x}$ ✓ use answer to part (a) $\int \frac{1}{\cos x} dx = \ln\left\|\frac{1+\sin x}{\cos x}\right\|$ ✓✓use limits of integration to solve for value of "I"

Question 11 **(9 marks)**

The graphs of $y=e^x$ and $y=\ln x$ for x $\geq$0 are shown. The line segment LM with equation $y=-x+e^2+2$ meets these graphs at $P(2,e^2)$ and $Q(e^2,2)$.



(a) State the **exact** coordinates of points L and M, the axis intercepts of the line segment LM. (2)

Solution
$L=(0,e^2+2)$, $M=(e^2+2,0)$
Specific behaviours
✓✓ I mark for each of the points as an ordered pair

(b) Calculate the **exact** value of the area of the region between $y=-x+e^2+2$ and $y=e^x$, from $x=0$ and $x=2$. (4)

Solution
$Area=\int_0^2 -x+e^2+2\,dx-\int_0^2 e^x\,dx$ OR $Area=\left[\frac{1}{2}(e^2+2+e^2)\times 2\right]-\int_0^2 e^x\,dx$ $=\left[\frac{-x^2}{2}+(e^2+2)x\right]_0^2-\left[e^x\right]_0^2$ $= 2+2e^2-\left[e^x\right]_0^2$

$=\left[\frac{-4}{2}+(e^2+2)(2)-0\right]-\left[e^2-e^0\right]$ $= 2+2e^2-e^2+e^0$ $= -2+2e^2+4-e^2+1$ $= 2+2e^2-e^2+1$ $=e^2+3=e^2+3$
Specific behaviours
$Area=\int_0^2 -x+e^2+2\,dx-\int_0^2 e^x\,dx$ ✓✓ area of trapezium – area under y = e^x from x=0 to x=2 or   ✓✓ simplify to correct exact answer

Question 11 (continued)

(c) Give a reason why the area of the region bounded bybetween $y=-x+e^2+2$ and $y=e^x$, from $x=0$ and $x=2$ is equal to the area of the region enclosed by the graph of $y=\ln x$, the line segment LM, and the x-axis. (1)

Solution
$y=e^x$ is the inverse function of $y=\ln x$ The two regions are symmetrical about the line y = x   Triangle LOM is a right isosceles triangle and as the functions are symmetrical about y=x, the two regions are congruent
Specific behaviours
✓ any one of the reasons

(d) **Hence** calculate the **exact**area of the region bounded by $y=e^x$, $y=-x+e^2+2$, $y=\ln x$, x-axis and y-axis. (2)

Solution
Area = $\frac{1}{2}\times(e^2+2)(e^2+2)-2(3+e^2)$ $=\frac{1}{2}e^4+2e^2+2-6-2e^2$ $=\frac{1}{2}e^4-4$
Specific behaviours
✓ calculation   ✓ correct answer of $\frac{1}{2}e^4-4$

Question 12 **(10 marks)**

Determine

(a) $\int \cos^2 2x \, dx$ (2)

Solution
$= \int \frac{\cos 4x + 1}{2} dx$ $= \frac{1}{2}\left[\frac{\sin 4x}{4} + x\right] + c$
Specific behaviours
✓ expresses in terms of cos 4x   ✓ integrates correctly

(b) $\int \cos^3 2x \, dx$ (3)

Solution
$= \int \cos 2x\left(1 - \sin^2 2x\right) dx$ $= \int \cos 2x - \cos 2x \sin^2 2x \, dx$ $= \frac{\sin 2x}{2} - \frac{1}{2}\frac{\sin^3 2x}{3} + c$ $= \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6} + c$
Specific behaviours
✓ express with sin 2x   ✓ uses $\int f'(x)\left[f(x)\right]^n dx = \frac{\left[f(x)\right]^{n+1}}{n+1} + c$ ✓ integrates correctly

Question 12 (continued)

(c) **Hence**, using your answer to parts (a) & (b), determine

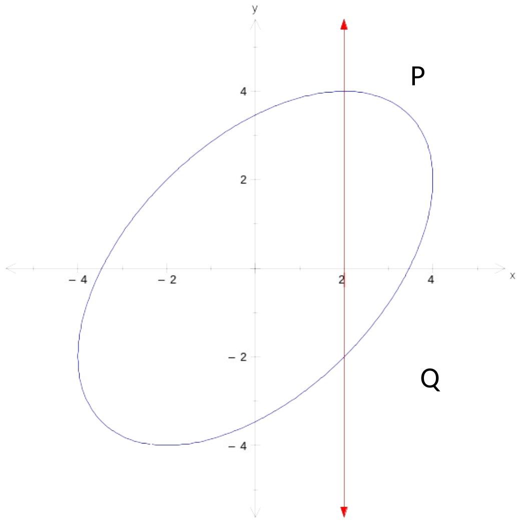
$$\int \sin^2 x \cos^4 x \, dx$$

(5)

Solution
$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{\cos 2x+1}{2}\right)^2 dx$ $=\frac{1}{8}\int (1-\cos 2x)(\cos^2 2x + 2\cos 2x + 1)dx$ $=\frac{1}{8}\int \cos^2 2x + 2\cos 2x + 1 - \cos^3 2x - 2\cos^2 2x - \cos 2x \, dx$ $=\frac{1}{8}\int 1 + \cos 2x - \cos^2 2x - \cos^3 2x \, dx$ $=\frac{1}{8}\left[x+\frac{\sin 2x}{2}-\frac{\sin 4x}{8}-\frac{x}{2}-\frac{\sin 2x}{2}+\frac{\sin^3 2x}{6}\right]+c$ $=\frac{1}{8}\left[\frac{x}{2}-\frac{\sin 4x}{8}+\frac{\sin^3 2x}{6}\right]+c$ $=\frac{1}{16}\left[x-\frac{\sin 4x}{4}+\frac{\sin^3 2x}{3}\right]+c$
Specific behaviours
✓ express in terms of cos 2x ✓✓ expand and simplify ✓ uses parts (a) and (b) ✓ simplify to correct answer

Question 13 **(12 marks)**

The graph of $x^2 - xy + y^2 = 12$ is drawn below.



(a) Draw the line $x=2$ and hence find the **coordinates** of the points of intersection, P and Q where P lies in the 1st quadrant and Q in the 4th quadrant. Show these points on the diagram.

(2)

Solution
Using CAS coordinates of P = (2, 4) Q = (2, -2)
Specific behaviours
✓✓ 1 mark each for P and Q

(b) Show that $\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$ (3)

Solution
Differentiate implicitly with respect to x results in $2x - y - x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$ $(2y - x)\dfrac{dy}{dx} = y - 2x$ $\dfrac{dy}{dx} = \dfrac{y - 2x}{(2y - x)}$
Specific behaviours
✓✓ implicit differentiate with respect to x

✓ rearrange and isolate $\frac{dy}{dx}$

Question 13 (continued)

(c) Determine the equation of the tangent to the curve at

(i) P (2)

Solution
$\left.\frac{dy}{dx}\right\|_{(2,4)} = 0$ Equation of tangent at P is y = 4
Specific behaviours
✓ gradient of zero ✓ Equation of tangent at P

(ii) Q (2)

Solution
$\left.\frac{dy}{dx}\right\|_{(2,-2)} = 1$ Equation of tangent at Q is $y+2=1(x-2)$ or $y=x-4$
Specific behaviours
✓ gradient ✓ equation of tangent at Q

(d) These two tangents intersect at point T. Show that Δ PQT is an isosceles triangle. (3)

Solution
Coordinates of T = (8, 4) PT = 6, PQ = 6 $\therefore \Delta QPT$ is a right isosceles triangle
Specific behaviours
✓ coordinates of T ✓ PT = PQ = 6 ✓ states triangle is isosceles with two sides congruent

Question 14 **(13 marks)**

The position vectors of the points A and B relative to the origin, are given by $\underset{\sim}{i} - 7\underset{\sim}{j} + 5\underset{\sim}{k}$ and $-2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$ respectively. The line L_1 passes through A and is parallel to $9\underset{\sim}{i} + 3\underset{\sim}{j} - 9\underset{\sim}{k} + c\underset{\sim}{j}$. The line L_2 passes through B and is parallel to $\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$

(i) Show that c=- $43\frac{1}{2}$ if the lines intersect. (5)

Solution
L_1: $\underset{\sim}{r} = (\underset{\sim}{i} - 7\underset{\sim}{j} + 5\underset{\sim}{k}) + \lambda(9\underset{\sim}{i} + 3\underset{\sim}{j} - 9\underset{\sim}{k} + c\underset{\sim}{j})$ L_2: $\underset{\sim}{s} = (-2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}) + \mu(\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k})$ For intersection L_1=L_2 $(1+9\lambda) = -2 + \mu$ $-7 + 3\lambda + c\lambda = -1 + 3\mu$ $5 - 9\lambda = 4 + 3\mu$ Solve using CAS $\lambda = \frac{-2}{9},\ \mu = 1,\ c = \frac{-87}{2}$ Hence $c = -43\frac{1}{2}$
Specific behaviours
✓✓ equations of the two lines ✓✓ equates the **i**, **j** and **k** components ✓ solve using CAS for values of $\lambda = \frac{-2}{9},\ \mu = 1,\ c = \frac{-87}{2}$

(ii) Hence state the coordinates of the point of intersection, P. (2)

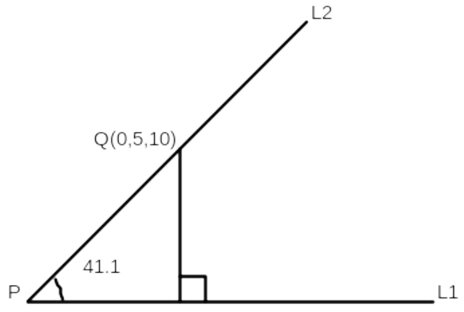
Solution
$OP = (-2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}) + 1(\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k})$ $OP = -\underset{\sim}{i} + 2\underset{\sim}{j} + 7\underset{\sim}{k}$, coordinates of P = (-1, 2, 7)
Specific behaviours
✓✓ coordinates of P

(iii) Determine the angle between L_1 and L_2. (2)

Solution
Direction of L_1 is $\frac{-2}{9}\left(9\underset{\sim}{i}+3\underset{\sim}{j}-9\underset{\sim}{k}-\frac{87}{2}\underset{\sim}{j}\right)=\left(-2\underset{\sim}{i}+9\underset{\sim}{j}+2\underset{\sim}{k}\right)$ Using CAS: Angle([-2,9,2],[1,3,3]) results in angle being 41.1^0(acute)or 138.9^0(obtuse)
Specific behaviours
✓ direction of L_1 ✓ correct size of angle

Question 14 (continued)

(iv) Hence determine the shortest distance from Q(0, 5, 10) which lies on L_2 to the line L_1.
(4)

Solution
$QL_1=(1+9\lambda)\underset{\sim}{i}+(-12-40.5\lambda)\underset{\sim}{j}+(-5-9\lambda)\underset{\sim}{k}$ $QL_1 \cdot L_1=0$ $(1+9\lambda)9+(-12-40.5\lambda)(-40.5)+(-5-9\lambda)(-9)=0$ $\lambda=-0.3$ $QL_1=-1.7\underset{\sim}{i}+0.15\underset{\sim}{j}-2.3\underset{\sim}{k}$ $\therefore \lvert QL_1\rvert=2.86$ Or  $PQ=(1,-2,-7)+(0,5,10)=(1,3,3)$ $\lvert PQ\rvert=\sqrt{19}$ Let shortest distance be x $\sin 41.1^0=\frac{x}{\sqrt{19}}$, x = 2.86 Hence shortest distance from Q to line L_1 is 2.86 units
Specific behaviours

✓ determines PQ ✓ determines $\lvert PQ \rvert$ ✓✓ identifies shortest and calculates it

Question 15 **(10 marks)**

If $N = \dfrac{10\ 000}{1 + 99e^{-\frac{1}{2}t}}$

(i) Express $e^{-\frac{1}{2}t}$ in terms of *N* (2)

Solution
$e^{-\frac{1}{2}t} = \dfrac{10000 - N}{99N}$
Specific behaviours
✓✓ rearranges and isolate $e^{-\frac{1}{2}t}$ correctly

(ii) Hence using implicit differentiation, show that $\dfrac{dN}{dt} = \dfrac{1}{2}N\left(\dfrac{10\ 000 - N}{10\ 000}\right)$ (5)

Solution

Differentiate implicitly

$$-\frac{1}{2}e^{-\frac{1}{2}t} = -\frac{10000}{99N^2}\cdot\frac{dN}{dt}$$

$$\frac{dN}{dt} = \frac{1}{2}e^{-\frac{1}{2}t}\cdot\frac{99N^2}{10000}$$

$$\frac{dN}{dt} = \frac{1}{2}\cdot\frac{99N^2}{10000}\cdot\left(\frac{10000}{99N} - \frac{1}{99}\right)$$

$$\frac{dN}{dt} = \frac{1}{2}\cdot\left(N - \frac{N^2}{10000}\right)$$

$$\frac{dN}{dt} = \frac{1}{2}N\cdot\left(1 - \frac{N}{10000}\right)$$

$$\frac{dN}{dt} = \frac{1}{2}N\cdot\left(\frac{10000 - N}{10000}\right)$$

Specific behaviours

✓✓ differentiate implicitly and rearrange to isolate $\frac{dN}{dt}$ ✓ substitutes $e^{-\frac{1}{2}t}$ ✓✓ simplifies expression

Question 15 (continued)

(iii) Find the value of *t*,correct to 3 significant figures when $\frac{dN}{dt}$ is a maximum. (3)

Solution
From CAS, $\frac{dN}{dt}$ is maximum when N = 5000 $5000 = \frac{10000}{1+99e^{-\frac{1}{2}t}}$ Using CAS , t = 9.19 to 3 sig figures
Specific behaviours
✓ N value when $\frac{dN}{dt}$ is maximum ✓Substitute into equation ✓ solves correctly for “t” to 3 significant figures

Question 16 **(9 marks)**

(a) Use First Principles to determine the derivative of $y=\sin^2 x$. (5)

Solution
$f(x)=(\sin x)^2$, $f(x+h)=[\sin(x+h)]^2=(\sin x\cos h+\cos x\sin h)^2$ $f(x+h)=\sin^2 x\cos^2 h+2\sin x\cos x\cos h\sin h+\cos^2 x\sin^2 h$ $\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ $\frac{dy}{dx}=\lim_{h\to 0}\frac{\sin^2 x\cos^2 h+2\sin x\cos x\cos h\sin h+\cos^2 x\sin^2 h-\sin^2 x}{h}$ $\frac{dy}{dx}=\lim_{h\to 0}\frac{\sin^2 x(\cos^2 h-1)}{h}+\lim_{h\to 0}\frac{2\sin x\cos x\cos h\sin h}{h}+\lim_{h\to 0}\frac{\cos^2 x\sin^2 h}{h}$ $\frac{dy}{dx}=\lim_{h\to 0}\frac{\sin^2 x(\cos h-1)(\cos h+1)}{h}+2\sin x\cos x\lim_{h\to 0}\frac{\cos h\sin h}{h}+\cos^2 x\lim_{h\to 0}\frac{\sin h\cdot\sin h}{h}$ $\frac{dy}{dx}=\sin^2 x[0\cdot(2)]+2\sin x\cos x.(1.1)+\cos^2 x(1\cdot 0)$ $\frac{dy}{dx}=2\sin x\cos x$
Specific behaviours
✓✓ f(x+h) and expand sin(x+h) ✓✓ group and evident use of $\lim_{h\to 0}\frac{\cos h-1}{h}=0, \lim_{h\to 0}\frac{\sin h}{h}=1$ ✓ gets to $2\sin x\cos x$ with no shortcuts

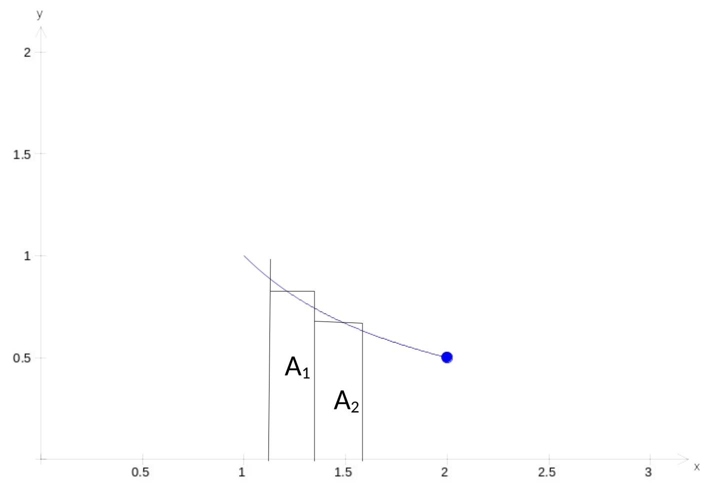
Question 16 (continued)

(b) Given that $\ln y = \sqrt{1+8e^x}$ prove that $\ln y \frac{dy}{dx} = 4ye^x$ (4)

Solution
$\ln y = (1+8e^x)^{\frac{1}{2}}$ Implicit differentiate results in $\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}(1+8e^x)^{-\frac{1}{2}} \cdot 8e^x$ $\frac{1}{y}\frac{dy}{dx} = \frac{4e^x}{(1+8e^x)^{\frac{1}{2}}}$ $\frac{dy}{dx} = \frac{4e^x y}{(1+8e^x)^{\frac{1}{2}}}$ $(1+8e^x)^{\frac{1}{2}} \frac{dy}{dx} = 4ye^x$ $\ln y \frac{dy}{dx} = 4ye^x$
Specific behaviours
✓✓ implicit differentiate correctly ✓ simplify ✓ rearrange to get $\ln y \frac{dy}{dx} = 4ye^x$

Question 17 **(5 marks)**

The area of the region between the curve $y = \frac{1}{x}$ and the *x*-axis, for $1 \le x \le 2$, is estimated using *n* rectangles of equal widths $\frac{1}{n}$ as shown in the diagram.



(i) Show that $\int_1^2 \frac{1}{x}\,dx$ is approximately equal to $\sum_{r=1}^{n} \frac{1}{n+r}$. (3)

Solution
Let width of rectangles be $h = \frac{1}{n}$ For the first rectangle, A_1, $x = 1 + h = 1 + \frac{1}{n} = \frac{n+1}{n}$ $y = \frac{n}{n+1}$ Area of A_1 = $\frac{1}{n} \cdot \frac{n}{n+1} = \frac{1}{n+1}$ For the second rectangle, A_2, $x = 1 + 2h = 1 + \frac{2}{n} = \frac{n+2}{n}$ y= $\frac{n}{n+2}$

Area of A2 = $\frac{1}{n}\cdot\frac{n}{n+2}=\frac{1}{n+2}$ Hence for the ith rectangle, A_i = $\frac{1}{n+i}$ $\therefore \int_1^2 \frac{1}{x}dx \approx \sum_{r=1}^{n} A_r = \sum_{r=1}^{n}\frac{1}{n+r}$
Specific behaviours
✓ areas of rectangles 1,2,..i ✓✓ $\int_1^2 \frac{1}{x}dx$ is the sum of the rectangles

(ii) Deduce the value of $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}$. (2)

Solution
$\sum_{r=1}^{n}\frac{1}{n+r} = \int_1^2 \frac{1}{x}dx$ $=\left[\ln\lvert x\rvert\right]_1^2 = \ln 2 - \ln 1 = \ln 2 = 0.6931$
Specific behaviours
✓ integrates to get ln x ✓ numerical value