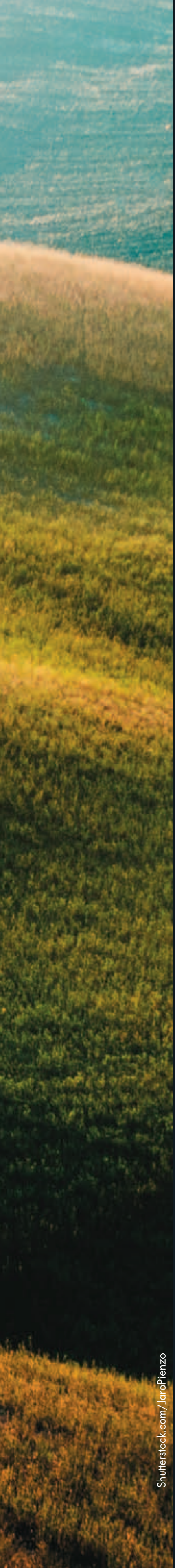

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4

TERMINOLOGY

algebraic area
antiderivative
constant of integration
definite integral
fundamental theorem of calculus
indefinite integral
integration
physical area

INTEGRALS

INTEGRATION AND AREAS

Prior learning

DEFINITE INTEGRALS

- examine the area problem, and use sums of the form $\sum_i f(x_i)\delta x_i$ to estimate the area under the curve $y = f(x)$ (ACMMM124)
- interpret the definite integral $\int_a^b f(x)dx$ as area under the curve $y = f(x)$ if $f(x) > 0$ (ACMMM125)
- recognise the definite integral $\int_a^b f(x)dx$ as a limit of sums of the form $\sum_i f(x_i)\delta x_i$ (ACMMM126)
- interpret $\int_a^b f(x)dx$ as a sum of signed areas (ACMMM127)
- recognise and use the additivity and linearity of definite integrals. (ACMMM128)

FUNDAMENTAL THEOREM

- understand the concept of the signed area function $F(x) = \int_a^x f(t)dt$ (ACMMM129)
- understand and use the theorem: $F'(x) = \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$, and illustrate its proof geometrically (ACMMM130)
- understand the formula $\int_a^b f'(x)dx = F(b) - F(a)$ and use it to calculate definite integrals. (ACMMM131)

APPLICATIONS OF INTEGRATION

- calculate the area under a curve (ACMMM132)

4.01 THE AREA UNDER A CURVE

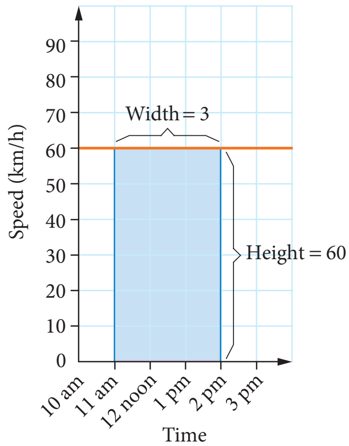
In many areas of knowledge such as surveying, physics and the social sciences, you must know how to find the area under a curve because it can give you important information. For example, the area under a speed graph shows the distance travelled.

Example 1

A car is travelling at a constant speed of 60 km/h.

a Find the distance it travels between 11 a.m. and 2 p.m. by using the graph below.

b Show that the area represents the distance that the car travels in that time.



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Solution

a Find the distance travelled at 60 km/h. The car travels 60 km in 1 hour so it travels 180 km in 3 hours.

b Find the area of the shaded rectangle.

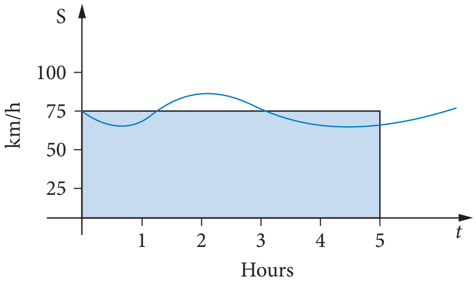
$$A = 3 \times 60$$
$$= 180$$

State the result. The area of the rectangle shows that the distance travelled is 180 km.

In real life, however, a car would rarely travel at a constant speed, so its graph would not be a straight line.

Example 2

The graph below shows the speed S of a car over t hours. Find the approximate distance it travels in 5 hours using the rectangle shown on the graph.



Solution

Find the area of the rectangle.

$$A = 75 \times 5$$
$$= 375$$

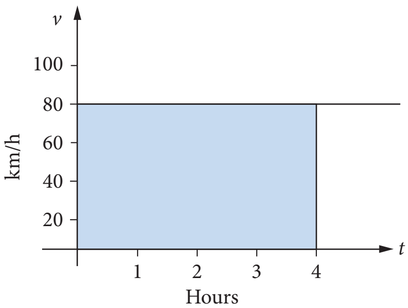
State the result. The distance travelled in 5 hours is approximately 375 km.

EXERCISE 4.01 The area under a curve

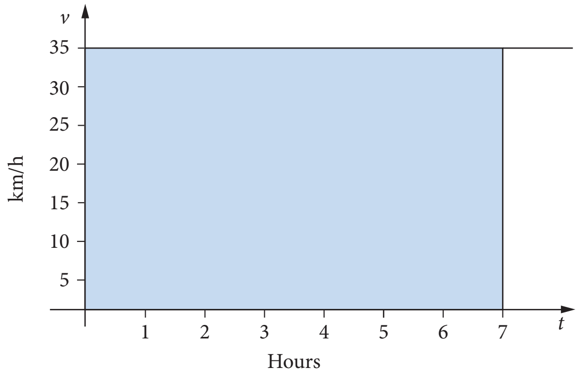
Concepts and techniques

1 Example 1 Find the distance travelled in each of the following by finding the area of the rectangle.

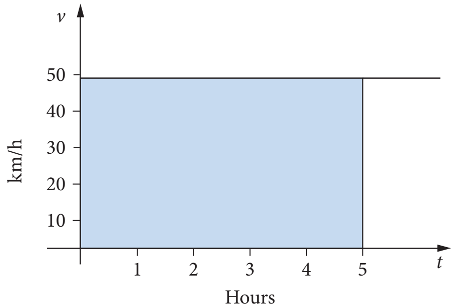
a



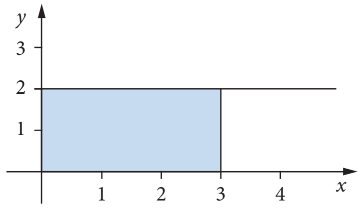
b



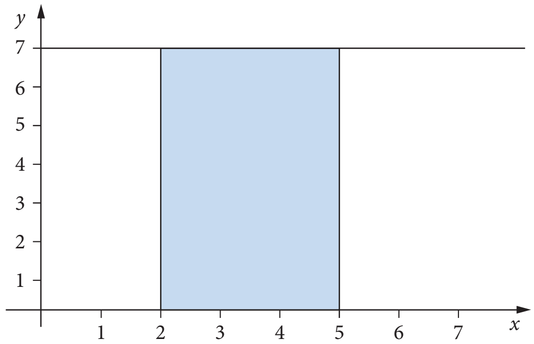
c



2 a Find the area under the graph between $x = 0$ and $x = 3$.

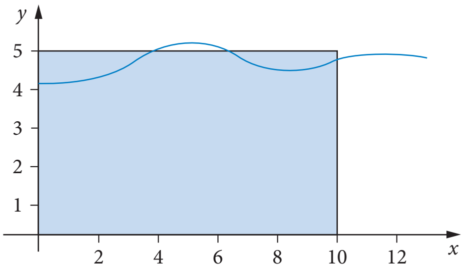


b Find the area under the graph bounded by $x = 2$ and $x = 5$.



3 Example 2 Find the approximate area under each curve by finding the shaded area of each rectangle.

a



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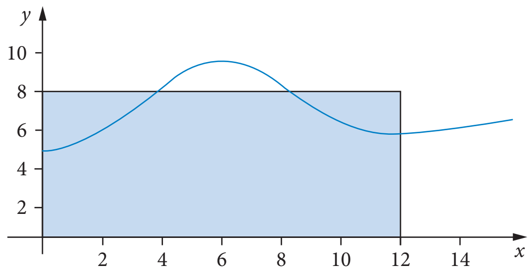
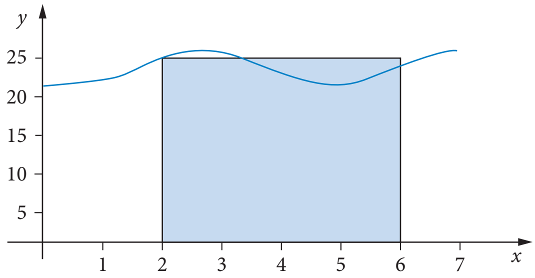
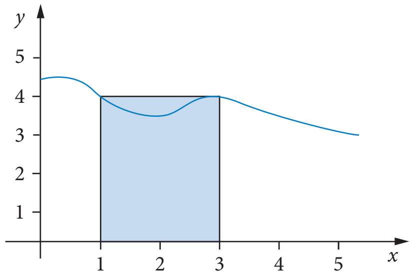
b



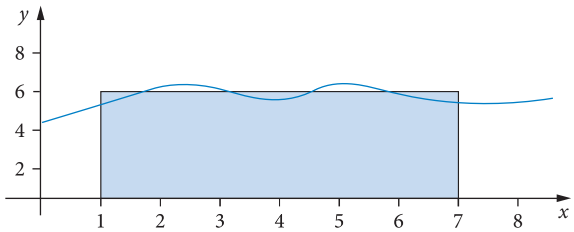
c



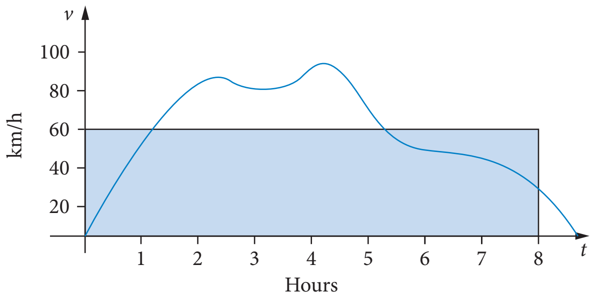
d

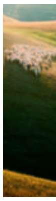


e



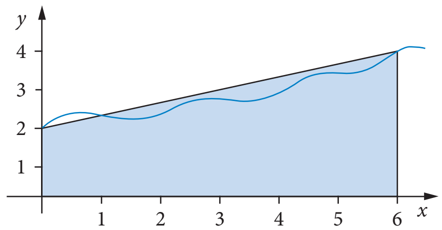
4 The travel graph shows the speed of a car as it travels along a road. Find the approximate distance travelled in 8 hours.



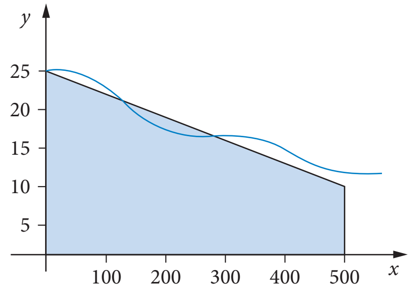


5 Find an approximation to each of the areas under a curve below by using the formula for the area of a trapezium.

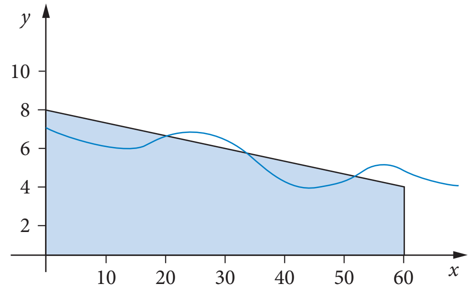
a



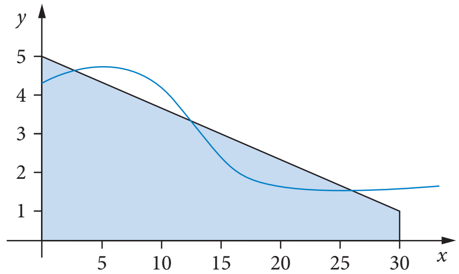
b



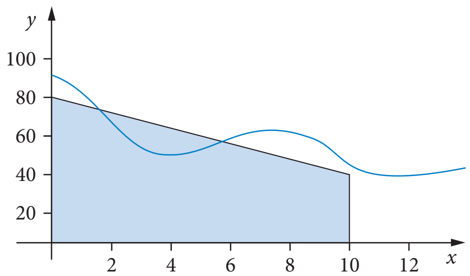
c



d



e



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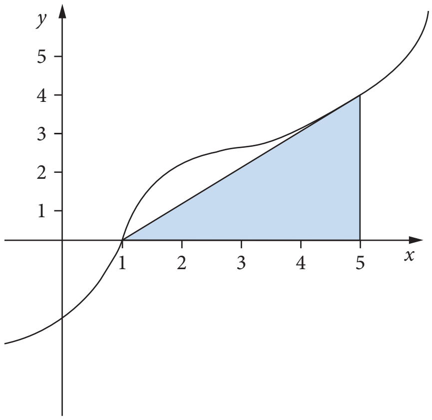
6 By finding the area of the shaded triangle, find an approximation to the area under the curve.

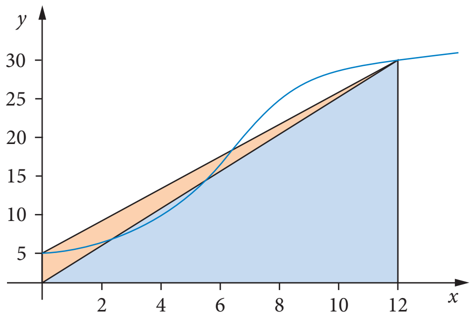


Reasoning and communication

7 Find an approximate area under the curve below by finding the area of a

a triangle

b trapezium

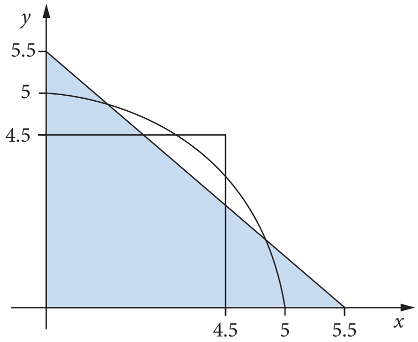


8 a Use the formula for the area of a circle to find the exact area of the quarter circle below correct to 2 decimal places.

b Find an approximation of the area under the curve by finding the area of a

i square with side 4.5 units

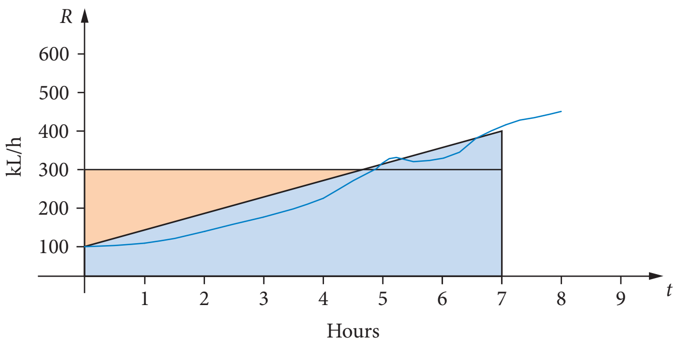
ii right-angled triangle with base and height of 5.5 units



9 The rate at which a dam is filling with water is shown in the graph below. Find the approximate volume of water in the dam after 7 hours by finding the area of a

a rectangle

b trapezium



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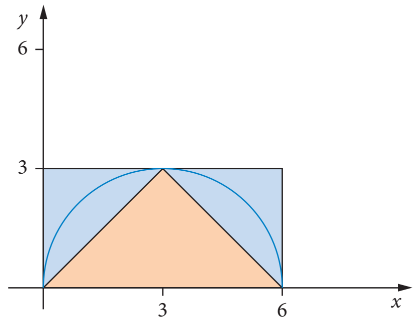
10 a Find the area of the semicircle below correct to 2 decimal places.

b Find the approximate area of the semicircle by finding the area of

i the shaded rectangle

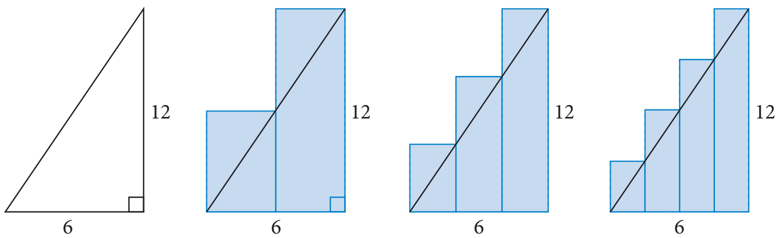
ii the shaded triangle

iii the average of these



4.02 AREA APPROXIMATIONS

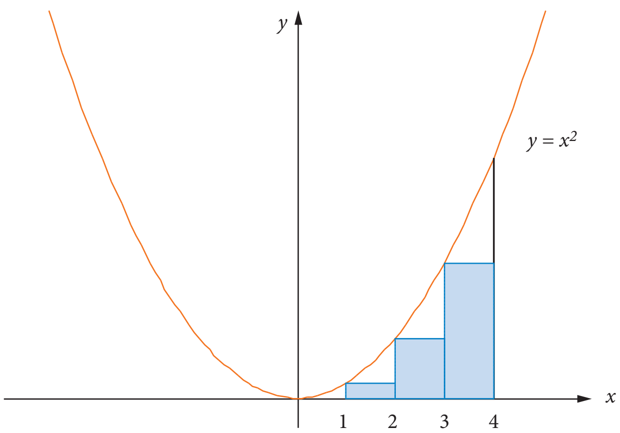
You can use more than one rectangle to find the area of an irregular shape. The more rectangles you use, the more accurate the area will be. For example, the triangle below has an area of 36 units^2. On the right, it has been approximated by 2, 3 and 4 rectangles.



For the rectangle approximations you get the following.
For two rectangles, $A = 3 \times 6 + 3 \times 12 = 54$ units^2.
For three rectangles, $A = 2 \times 4 + 2 \times 8 + 2 \times 12 = 48$ units^2.
For four rectangles, $A = 1.5 \times 3 + 1.5 \times 6 + 1.5 \times 9 + 1.5 \times 12 = 45$ units^2.
As the number of rectangles increases, the area gets closer to the true area of the triangle.

Example 3

Find an approximation to the area under the curve $y = x^2$ between $x = 1$ and $x = 4$ using the sum of the three rectangles shown below.



Solution

Find the height of each rectangle.

$f(1) = 1^2 = 1$
$f(2) = 2^2 = 4$
$f(3) = 3^2 = 9$

Find the area of each rectangle.

$A_1 = 1 \times 1 = 1$
$A_2 = 1 \times 4 = 4$
$A_3 = 1 \times 9 = 9$

Find the sum of the rectangles.

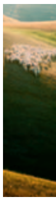
$A = 1 + 4 + 9 = 14$

State the result.

The area under the curve is approximately 14 units^2.

In Example **3**, the rectangles have been drawn so that their tops touch the function on their left-hand sides. They are called left rectangles. You can also draw the rectangles so their tops touch the curve on the right-hand side. Obviously, they are called right rectangles.

You can use your CAS calculator to find the approximate area under a curve for large numbers of rectangles.



Example 4

Find an approximation to the area under the curve $y = 12 - x^3$ between $x = 0$ and $x = 2$ using

a 4 left rectangles

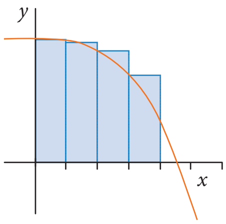
b 4 right rectangles

c the average of parts **a** and **b**

d CAS 50 left rectangles.

Solution

a Sketch the graph.
Draw the rectangles so that each touches the curve at the top left.



State the left value for each rectangle.

The values are at 0, 0.5, 1 and 1.5.

Find the height of each rectangle.

$f(0) = 12 - 0^3 = 12$
$f(0.5) = 12 - 0.5^3 = 11.875$
$f(1) = 12 - 1^3 = 11$
$f(1.5) = 12 - 1.5^3 = 8.625$

Find the area of each rectangle.

$A_1 = 0.5 \times 12 = 6$
$A_2 = 0.5 \times 11.875 = 5.9375$
$A_3 = 0.5 \times 11 = 5.5$
$A_4 = 0.5 \times 8.625 = 4.3125$

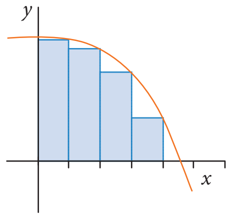
Find the sum of the rectangles.

$A = 6 + 5.9375 + 5.5 + 4.3125$
$= 21.75$

State the result.

The area is approximately 21.75 units^2.

b Sketch the graph.
Draw the rectangles so that each touches the curve at the top right.



State the right value for each rectangle.

The values are at 0.5, 1, 1.5 and 2.

Find the height of each rectangle.

$f(0.5) = 12 - 0.5^3 = 11.875$
$f(1) = 12 - 1^3 = 11$
$f(1.5) = 12 - 1.5^3 = 8.625$
$f(2) = 12 - 2^3 = 4$

Find the area of each rectangle.

$A_1 = 0.5 \times 11.875 = 5.9375$
$A_2 = 0.5 \times 11 = 5.5$
$A_3 = 0.5 \times 8.625 = 4.3125$
$A_4 = 0.5 \times 4 = 2$

Find the sum of the rectangles.

$A = 5.9375 + 5.5 + 4.3125 + 2$
$= 17.75$

State the result.

The area is approximately 17.75 units^2.

c Find the average.

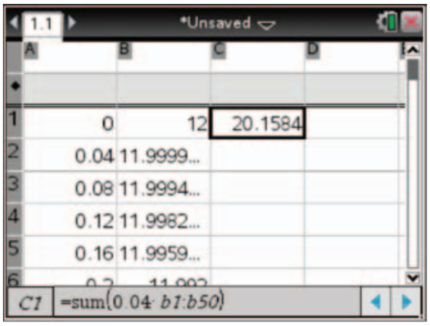
$$\text{Average area} = \frac{21.75 + 17.75}{2}$$
$$= 19.75 \text{ units}^2$$

d Find the widths of the rectangles.

Width $= 2 \div 50 = 0.04$

TI-Nspire CAS

Use the Lists and Spreadsheet page.
Enter 0 into cell A1 and “=a1+0.04” into cell A2.
Copy cell A2 using ctrl C, mark Cells A3 to A50 and copy using ctrl V.
Type “=12–a^3” into B1 and copy down to B50.
Finally type “=sum(0.04´B1:B50)” into C1.



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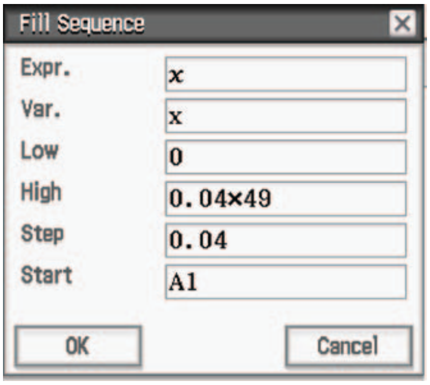
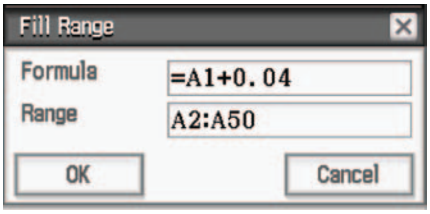
ClassPad

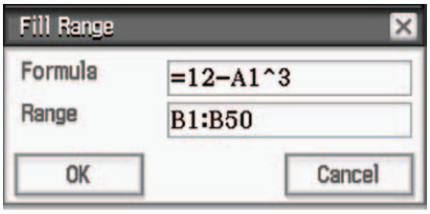
Use the Spreadsheet menu.
Tap cell A1.
Tap Edit, then Fill, then Fill Sequence.
Enter values as on the right. Most are self-explanatory. The expression is the function of x, which in this case is just the x values themselves. High will be the value in A50, which will be 49 lots of 0.04.

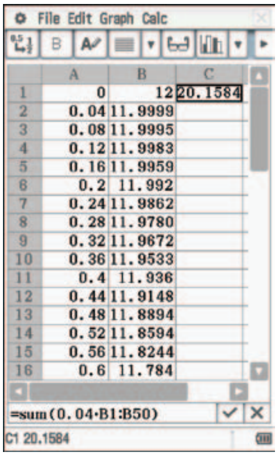


Alternate method

Enter 0 in A1. A2 should be selected.
Tap Edit, Fill and Fill Range.
Formula is =A1+1 and Range is A2:A50.
Tap B1, then Edit, Fill and Fill Range.
Formula is =12 – A1^3 and Range is B1:B50.
The widths are 0.04, and the heights are from B1 to B50.
Tap C1 (or any blank cell) and enter =sum(0.04´B1:B50).







Write the answer. The area is approximately 20.158 units^2.

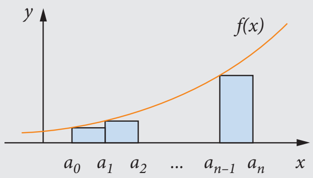
You can use a normal computer spreadsheet such as Excel in the same way as your CAS calculator in Example **4**.

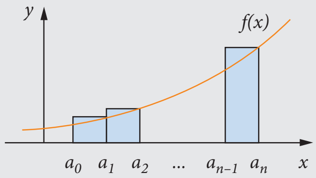
The area in Example **4** is clearly overestimated by the left rectangles and underestimated by the right rectangles. You can use the values in the centres of the rectangles to get a better estimate.

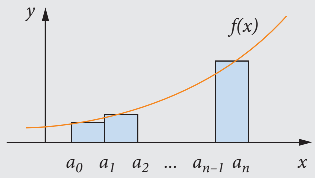
IMPORTANT

The rectangles used to find the approximate area under a curve can use function values on the left, right or in the centre of the rectangles.

'Left' rectangles **'Right' rectangles** **'Centred rectangles'**







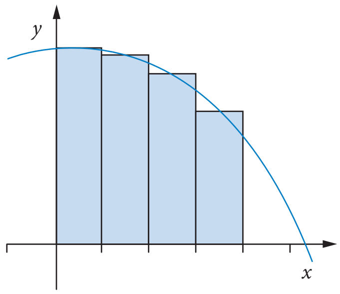
Centred rectangles can give a closer approximation to the area under the curve.

Example 5

Find an approximation to the area under the curve $y = 12 - x^3$ between $x = 0$ and $x = 2$ using 4 centred rectangles.

Solution

Sketch the graph.
Draw the rectangles so that each touches the curve at the top centre.



State the right value for each rectangle. — The values are at 0.25, 0.75, 1.25 and 1.75.

Find the height of each rectangle.

$f(0.25) = 12 - 0.25^3 \approx 11.98$
$f(0.75) = 12 - 0.75^3 \approx 11.58$
$f(1.25) = 12 - 1.25^3 \approx 10.05$
$f(1.75) = 12 - 1.75^3 = 6.64$

Find the area of each rectangle.

$A_1 \approx 0.5 \times 11.98 = 5.99$
$A_2 \approx 0.5 \times 11.58 = 5.79$
$A_3 \approx 0.5 \times 10.05 = 5.02$
$A_4 \approx 0.5 \times 6.64 = 3.32$

Find the sum of the rectangles. — $A = 5.99 + 5.79 + 5.02 + 3.32 = 20.12$

State the result. — The area is approximately 20.12 units^2.

Notice that using 4 centred rectangles gives almost the same approximation as using 50 left rectangles.

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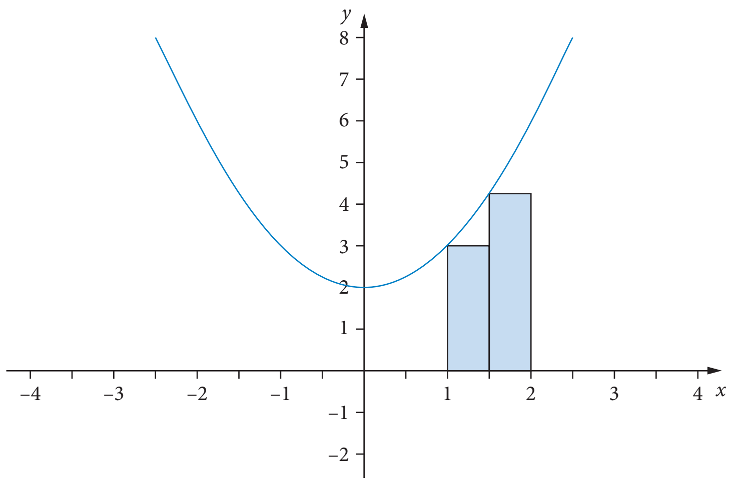
EXERCISE 4.02 Area approximations

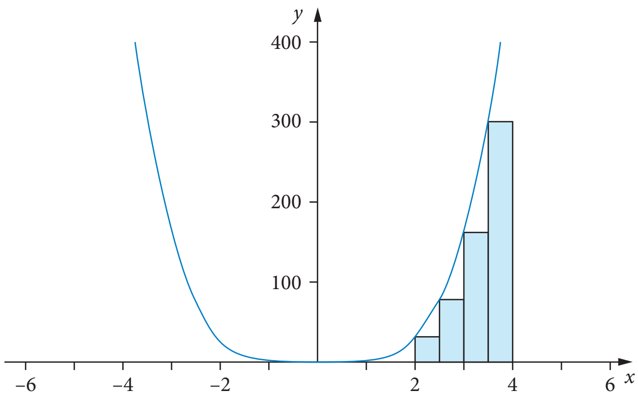
Concepts and techniques

1 Example 3 Find the approximate area of each of the following.

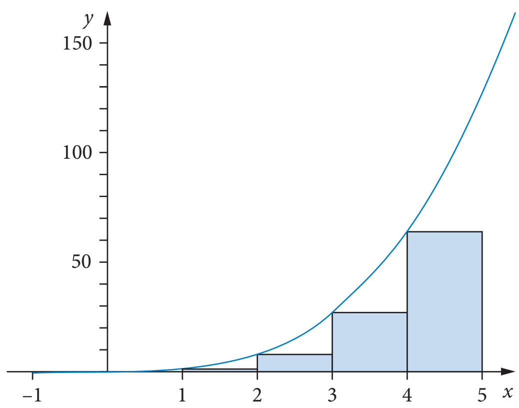
a The area under the curve $y = x^2 + 2$ between $x = 1$ and $x = 2$, using two rectangles as shown.



b The area under the curve $y = 2x^4$ between $x = 2$ and $x = 4$, using four rectangles as shown.



c The area under the curve $y = x^3$ between $x = 1$ and $x = 5$, using four rectangles as shown.



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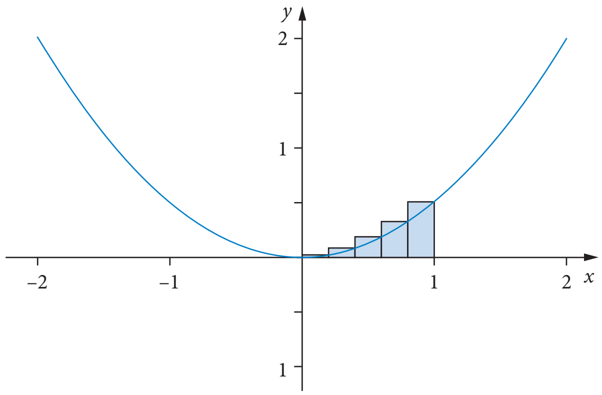
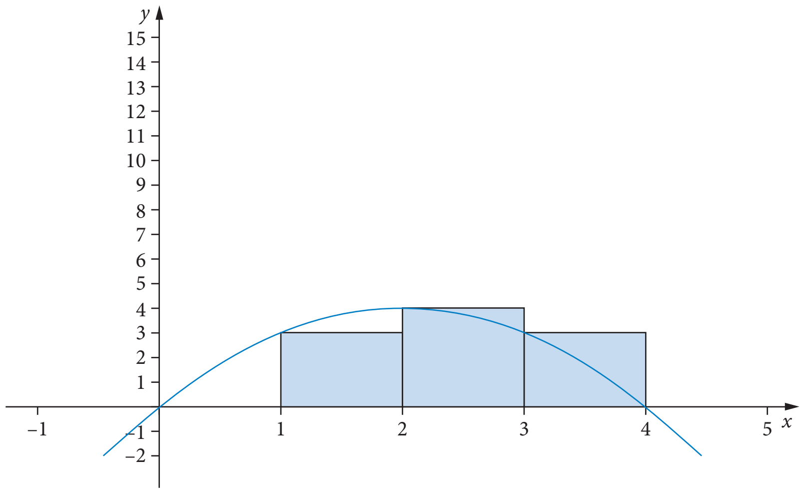
d The area under the curve $y = x^2$ between $x = 0$ and $x = 1$, using five right rectangles as shown. The first rectangle is very low, so doesn't show up very well on this diagram.

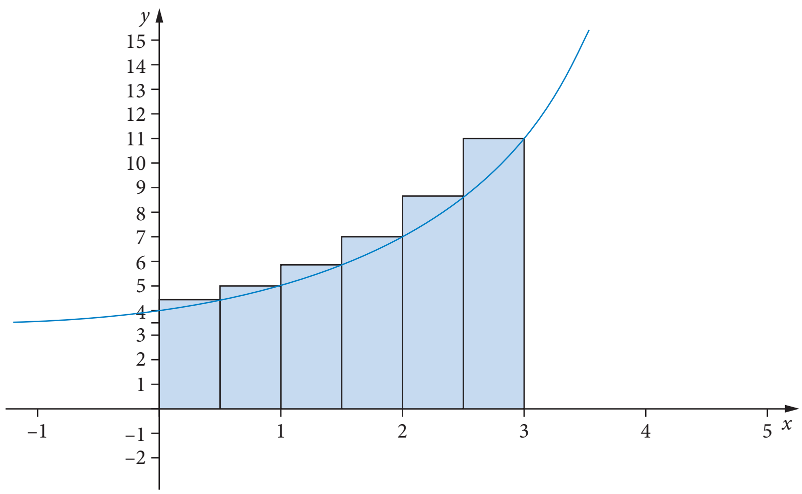


e The area under the curve $y = 4x - x^2$ between $x = 1$ and $x = 4$, using three rectangles as shown.



2 Find an approximation to each area.

a The area under the curve $y = 2^x + 3$ between $x = 0$ and $x = 3$, using six rectangles as shown.



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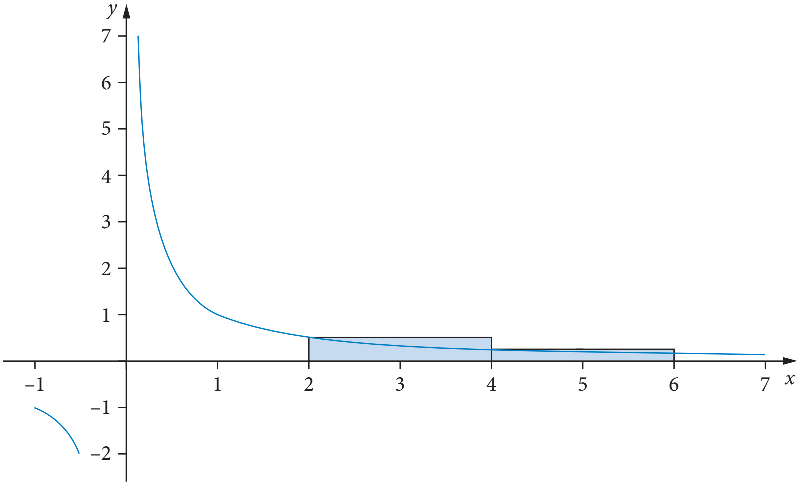
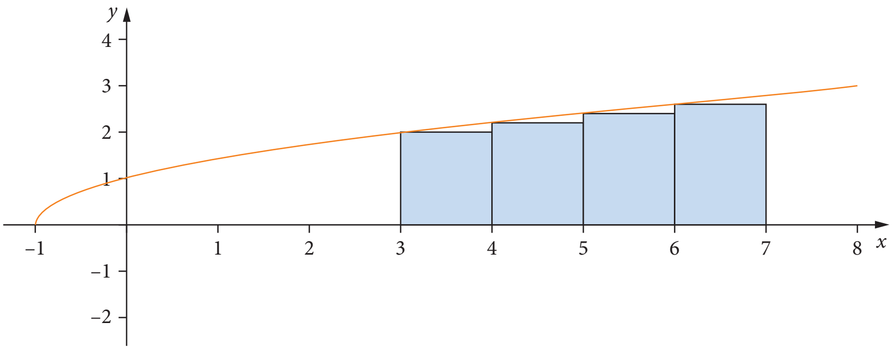
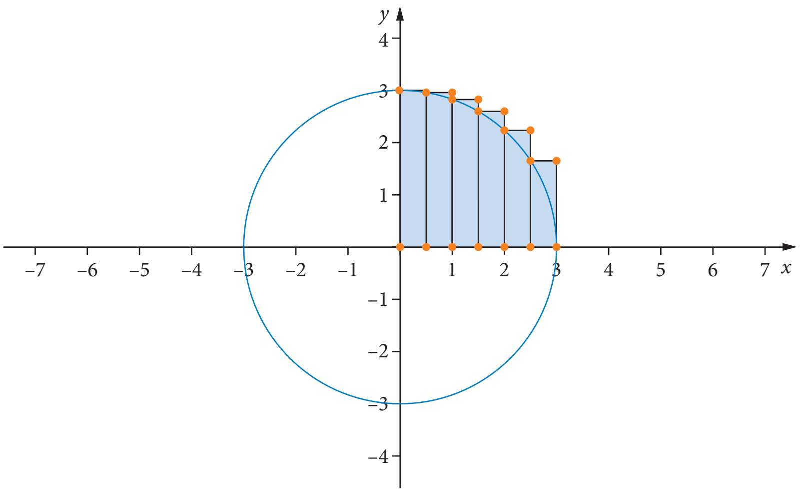
b The area under the curve $y = \dfrac{1}{x}$ between $x = 2$ and $x = 6$, using two rectangles as shown.

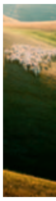


c The area under the curve $y = \sqrt{x+1}$ between $x = 3$ and $x = 7$, using four rectangles as shown.

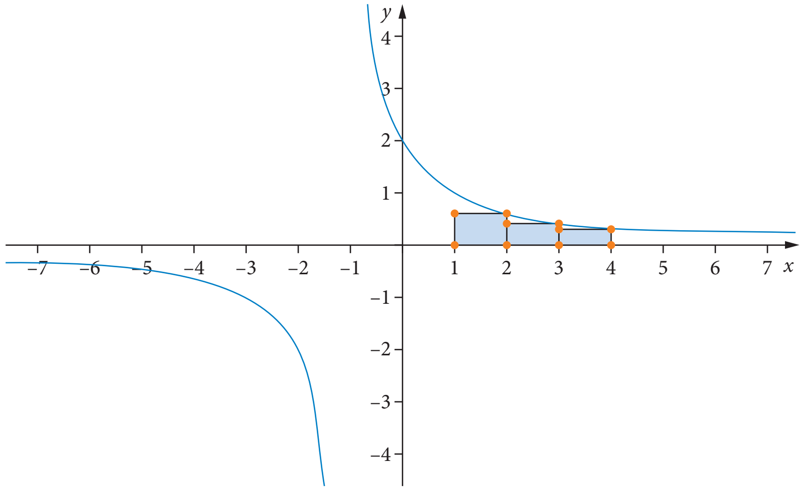


d The area under the curve $y = \sqrt{9 - x^2}$ between $x = 0$ and $x = 3$, using six rectangles as shown.

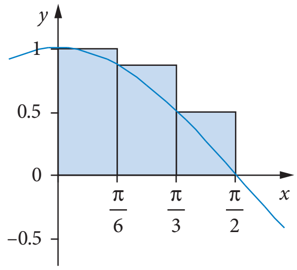




e The area under the curve $y = \dfrac{2}{x+1}$ between $x = 1$ and $x = 4$, using three rectangles as shown.



3 a Find an approximation for the area under the curve $y = \cos(x)$ between $x = 0$ and $x = \dfrac{\pi}{2}$ using the rectangles below.



b Use a CAS calculator or computer application to find the area using 20 rectangles.

4 Example 4 For each question below,

i find an approximate area under the curve using the given number of left rectangles

ii find an approximate area under the curve using the given number of right rectangles

iii CAS find the area using 20 left rectangles.

a $y = x^2$ between $x = 1$ and $x = 2$ using two rectangles.

b $y = x^2 + 2x$ between $x = 0$ and $x = 4$ using four rectangles.

c $y = x^3 + 1$ between $x = 0$ and $x = 2$ using two rectangles.

d $y = x^2 - x - 2$ between $x = 2$ and $x = 4$ using four rectangles.

e $y = e^x$ between $x = 0$ and $x = 5$ using five rectangles.

5 Example 5 Use centred rectangles to find an approximation to each area.

a $y = x^2$ between $x = 1$ and $x = 2$ with 4 rectangles.

b $y = x^3$ between $x = 0$ and $x = 1$ with 5 rectangles.

c $y = 2x^2 + 3$ between $x = 0$ and $x = 2$ with 4 rectangles.

d $y = x^2 - 1$ between $x = 2$ and $x = 6$ with 8 rectangles.

e $y = \sin(x)$ between $x = 0$ and $x = 1$ with 10 rectangles.

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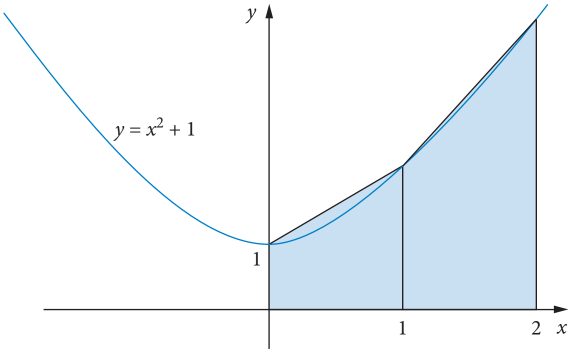
6 a Find the approximate area under the line $y = x - 1$ between $x = 1$ and $x = 4$ by using 3 centred rectangles.

b Find the exact area using geometry.

Reasoning and communication

7 The trapezoidal rule uses trapeziums rather than rectangles to find approximate areas under a curve. Find an approximation to the area under the curve $y = x^2 + 1$ between $x = 0$ and $x = 2$ by using the sum of each trapezium.



8 a Find the approximate area under the curve $y = \frac{1}{x+2}$ between $x = 1$ and $x = 2$ by using

i 4 left rectangles   ii 4 right rectangles   iii 4 centred rectangles.

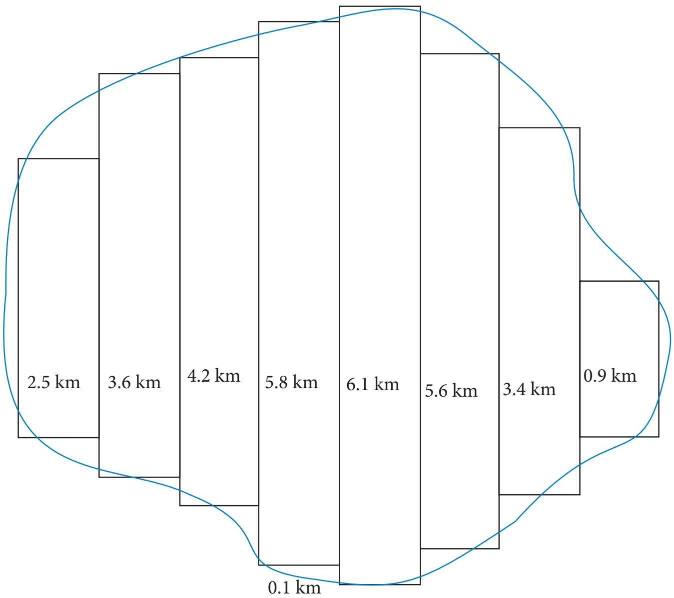
b Find the approximate area under the curve by using a trapezium with sides $f(1)$ and $f(2)$.

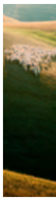
c Use a CAS calculator or computer application to find the area using 50 centred rectangles.

9 A lake has an irregular surface as shown below and an average depth of 850 metres.

a Find an approximation to the area of the surface of the lake using the rectangles shown with width 0.1 km.

b Find an approximation to the volume of water in the lake in km^3.



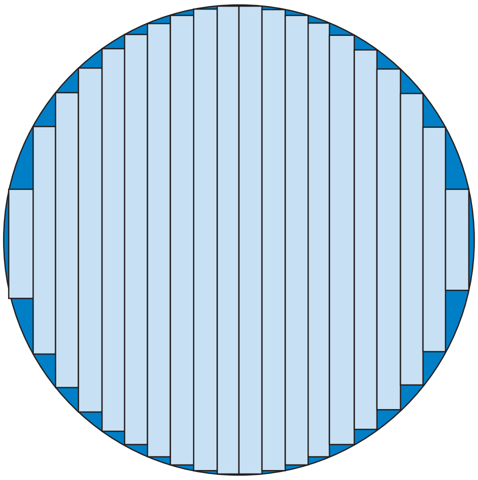


4.03 THE DEFINITE INTEGRAL

Integration is a process used to find the exact value of the area under a curve.

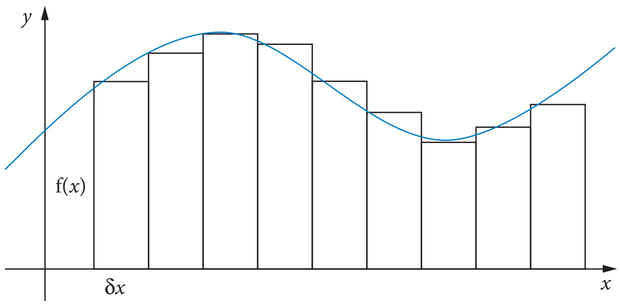
For a function $y = f(x)$, where $f(x) > 0$ between $x = a$ and $x = b$, the **definite integral** is the area under the curve.

Archimedes (287–212 BCE) found the area of a circle by cutting it into very thin layers and finding the sum of the areas of these rectangles.



You can find an approximation for the definite integral in the same way.

The area of each rectangle is given by $f(x)\delta x$, where $f(x)$ is the height of each rectangle and δx is the width.



IMPORTANT

The area under a curve can be approximated by a sum of rectangle areas. This can be written as $\sum_{i=1}^{n} f(x_i)\delta x_i$ or $\sum_{x=a}^{b} f(x_i)\delta x_i$. If the widths are all the same, this can be changed to $\delta x \sum_{x=a}^{b} f(x)$.

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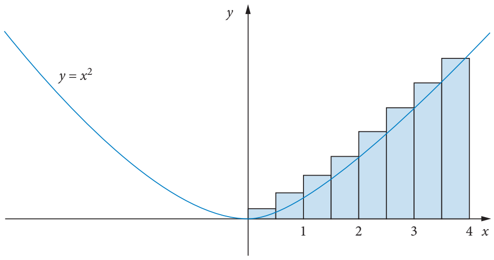
Remember that the capital Greek letter sigma means 'the sum of', so the first sum is read as 'the sum of $f(x_i)$ times delta x_i for $i = 1$ to n' and the second as 'the sum of $f(x_i)$ times delta x_i for $x = a$ to $x = b$'. The first case is for n rectangles and the second is for an unspecified number of rectangles between a and b.

Example 6

Find the approximate area under the curve $y = x^2$ between $x = 0$ to $x = 4$ using 8 right rectangles.

Solution

Draw the graph showing 8 right rectangles.



Find the value of δx.

$$\delta x = \frac{4-0}{8} = 0.5$$

Find the height of each rectangle.

$f(0.5) = 0.5^2 = 0.25$
$f(1) = 1^2 = 1$
$f(1.5) = 1.5^2 = 2.25$
$f(2) = 2^2 = 4$
$f(2.5) = 2.5^2 = 6.25$
$f(3) = 3^2 = 9$
$f(3.5) = 3.5^2 = 12.25$
$f(4) = 4^2 = 16$

Find the sum of the areas of the rectangles.

$$\begin{aligned} A &= 0.5 \times 0.25 + 0.5 \times 1 + 0.5 \times 2.25 + 0.5 \times 4 \\ &\quad + 0.5 \times 6.25 + 0.5 \times 9 + 0.5 \times 12.25 \\ &\quad + 0.5 \times 16 \\ &= 0.5 \times (0.25 + 1 + 2.25 + 4 + 6.25 + 9 \\ &\quad + 12.25 + 16) \\ &= 25.5 \end{aligned}$$

State the result.

The area is approximately 25.5 units^2.

You can also use a CAS calculator as shown previously in Example **4**. The more rectangles you use, the more accurate the approximation becomes. The limit of the sum as the number of rectangles increases, or as the widths decrease, will be the exact area.

IMPORTANT

The **definite integral** of the function $f(x)$ from a to b, $\int_a^b f(x)dx$, is the exact area under the curve $y = f(x)$ between $x = a$ and $x = b$. This is given by the limit of the sum of the areas of the rectangles:

$$\int_a^b f(x)dx = \lim_{\delta x \to 0} \sum f(x)\delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\delta x_i$$

For $f(x) > 0$, the definite integral is the area under the curve. However, a definite integral is usually considered to be a value, not an area. Obviously, when $f(x) < 0$, it will be negative and when $f(x)$ varies in sign, it could be either positive or negative.

Example 7

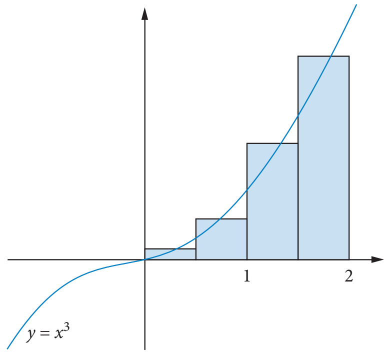
Use 4 centred rectangles to find an approximation to $\int_0^2 x^3 dx$.

Solution

Find the widths.

$\delta x = \dfrac{2-0}{4} = 0.5$

Draw the graph showing 4 centred rectangles.



Find the height of each rectangle.

$f(0.25) = 0.25^3 \approx 0.016$
$f(0.75) = 0.75^3 \approx 0.42$
$f(1.25) = 1.25^3 \approx 1.95$
$f(1.75) = 1.75^3 \approx 5.36$

Find the sum of the areas of the rectangles.

$$\begin{aligned} A &= 0.5 \times 0.016 + 0.5 \times 0.42 + 0.5 \times 1.95 + 0.5 \\ &\quad \times 5.36 \\ &= 3.875 \end{aligned}$$

State the result.

$\int_0^2 x^3 dx \approx 3.875$

As the number of rectangles increases, the approximate area under a curve becomes more accurate.

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Example 8

CAS Use 20 left rectangles to find an approximation to $\int_0^5 e^x\,dx$.

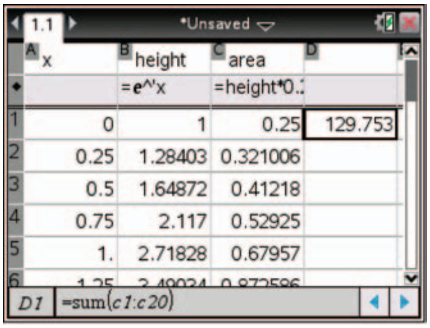
Solution

Find the widths.

$$\delta x = \frac{5-0}{20} = 0.25$$

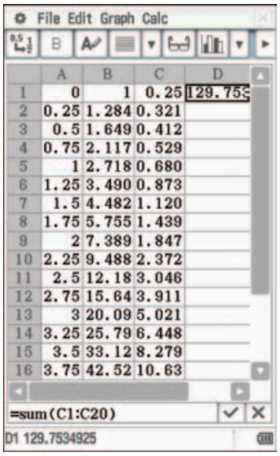
TI-Nspire CAS

Use the Lists and Spreadsheet page.
Enter 0 into cell A1 and type = a1 + 0.25 in A2 and fill down to A20.
Use the headings x, height and area for columns A, B and C
Type =e^x into the formula cell for column B and =height*0.25 into the formula cell for column C.
Finally type =sum(C1:C50)" into D1.



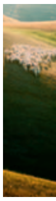
ClassPad

Use the Spreadsheet menu.
Enter 0 into cell A1 and "=a1+0.25" into cell A2 and fill down to A20 using Edit, Fill and Fill Range A2:A20.
Type =e^A1 into B1 and fill down to B20.
Type =0.25´B1 into C1 and fill down to C20.
Then type =sum(C1:C20) into D1.



State the result to reasonable accuracy.

$$\int_0^5 e^x\,dx \approx 130$$



WS

Areas using rectangles

EXERCISE 4.03 The definite integral

Concepts and techniques

1 Example 6 Find the approximate area under each of the curves below.

a $y = x^2 + x$ between $x = 0$ and $x = 3$ using 6 left rectangles.

b $y = x^3 + 1$ between $x = 0$ and $x = 5$ using 10 right rectangles.

c $y = x^2 - 1$ between $x = 1$ and $x = 3$ using 8 left rectangles.

d $y = x^4$ between $x = 0$ and $x = 6$ using 6 left rectangles.

e $y = \sin(x)$ between $x = 0$ and $x = 3$ using 6 right rectangles.

2 Example 7 Find the approximate value of each of the following using 8 centred rectangles.

a $\int_1^2 (x^2+2)dx$ b $\int_2^4 2x^4 dx$ c $\int_1^5 x^3 dx$

d $\int_3^7 \sqrt{x+1}\, dx$ e $\int_1^9 (x^2+4x)dx$

3 Find an approximation to each integral using 6 right rectangles.

a $\int_0^3 (2^x+3)dx$ b $\int_2^5 \frac{1}{x}dx$ c $\int_0^3 \sqrt{9-x^2}\,dx$

d $\int_1^7 \frac{2}{x+1}dx$ e $\int_0^6 (x^3+2)\,dx$

4 Find an approximation to $\int_0^{\frac{\pi}{2}} \cos(x)dx$ in exact form by using 2 left rectangles.

5 CAS Example 8 Use a CAS calculator or computer application with 15 left rectangles to find an approximation for each definite integral.

a $\int_0^3 2x^3 dx$ b $\int_1^4 (x^2+2)dx$

6 CAS Use 20 left rectangles to find an approximation to $\int_2^{12} \frac{x-2}{x+1}dx$.

7 Use 50 right rectangles to find an approximation to $\int_1^6 (x^2-1)\,dx$.

Reasoning and communication

8 a Find an approximation to $\int_0^2 x^3 dx$ using 8 centred rectangles.

b CAS Find $\int_{-2}^2 x^3 dx$ using 100 centred rectangles.

c Draw the graph of $y = x^3$ and explain the result in part **b**.

9 CAS Evaluate $\int_1^3 (x^2-4x)dx$ using 40 centred rectangles and explain the result with a graph.

10 The velocity of an object is given by $v = 6t - t^2$ m/s and the initial position is at $x = 0$.

a Find the approximate distance travelled in the first 4 seconds using 8 centred rectangles.

b Find the exact distance travelled in the first 4 seconds.

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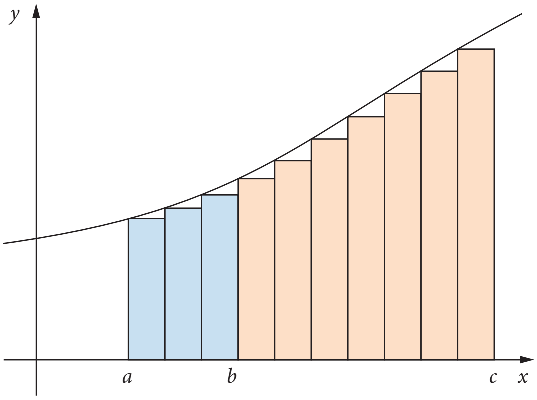
4.04 PROPERTIES OF THE DEFINITE INTEGRAL

You can work out properties of definite integrals using the definition as the limit of a sum or demonstrate them using sums of rectangle areas. The first property is shown below.

IMPORTANT

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

$\int_a^b f(x)dx = \lim_{\delta x \to 0} \sum f(x)\delta x$, $\int_b^c f(x)dx = \lim_{\delta x \to 0} \sum f(x)\delta x$ and $\int_a^c f(x)dx = \lim_{\delta x \to 0} \sum f(x)\delta x$ for appropriate rectangles. Consider the situation where the widths are all the same before the limit is worked out. It could look like the following for left rectangles.



In this case there are, say, 3 rectangles from a to b and 7 from b to c, making 10 from a to c.

For these rectangles,

$$\begin{aligned}
\int_a^b f(x)dx + \int_b^c f(x)dx &\approx \sum_{i=1}^{3} f(x_i)\delta x + \sum_{i=4}^{10} f(x_i)\delta x \\
&= \sum_{i=1}^{10} f(x_i)\delta x \\
&\approx \int_a^c f(x)dx
\end{aligned}$$

When limits are taken, the approximations become exact.

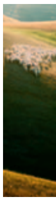
You can demonstrate this using the approximations you used previously.

Example 9

a Use left rectangles of width 0.5 units to find approximations for the following.

i $\int_0^2 x^2 dx$ ii $\int_2^3 x^2 dx$ iii $\int_0^3 x^2 dx$

b Show that, for these sums, $\int_0^2 x^2 dx + \int_2^3 x^2 dx = \int_0^3 x^2 dx$



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Solution

a **i** Write the values of x_i. $x_1 = 0, x_2 = 0.5, x_3 = 1$ and $x_4 = 1.5$

Find the values of $f(x_i)$. $f(x_1) = 0, f(x_2) = 0.25, f(x_3) = 1$ and $f(x_4) = 2.25$

Find the sum. $\sum_{i=1}^{4} f(x_i)\delta x = 1.75$

State the result. $\int_0^2 x^2 dx = 1.75$

ii Write the values of x_i. $x_5 = 2, x_6 = 2.5$

Find the values of $f(x_i)$. $f(x_5) = 4, f(x_6) = 6.25$

Find the sum. $\sum_{i=1}^{4} f(x_i)\delta x = 5.125$

State the result. $\int_2^3 x^2 dx = 5.125$

iii Write the values of x_i. $x_1 = 0, x_2 = 0.5, x_3 = 1, x_4 = 1.5, x_5 = 2, x_6 = 2.5$

Find the values of $f(x_i)$. $f(x_1) = 0, f(x_2) = 0.25, f(x_3) = 1, f(x_4) = 2.25,$
$f(x_5) = 4, f(x_6) = 6.25$

Find the sum. $\sum_{i=1}^{6} f(x_i)\delta x = 6.875$

State the result. $\int_0^3 x^2 dx = 6.875$

b Check the LHS.

$$\begin{aligned}\text{LHS} &= \int_0^2 x^2 dx + \int_2^3 x^2 dx \\ &= 1.75 + 5.125 \\ &= 6.875\end{aligned}$$

Check the RHS.

$$\begin{aligned}\text{RHS} &= \int_0^3 x^2 dx \\ &= 6.875\end{aligned}$$

State the result. LHS = RHS, so $\int_0^2 x^2 dx + \int_2^3 x^2 dx = \int_0^3 x^2 dx$

IMPORTANT

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

This is easily shown using the definition.

$$\begin{aligned}\int_a^b kf(x)dx &= \lim_{\delta x \to 0} \sum kf(x)\delta x \\ &= \lim_{\delta x \to 0} k\sum f(x)\delta x \\ &= k\lim_{\delta x \to 0} \sum f(x)\delta x \\ &= k\int_a^b f(x)dx\end{aligned}$$

Considering the rectangles for the sums, each rectangle for $kf(x_i)$ is k times as high as the corresponding rectangle for $f(x_i)$, so the whole sum and limit must also be k times as big.

Example 10

CAS **a** Find approximations for the following using 100 left rectangles.

i $\int_1^3 5x^3 dx$ ii $\int_1^3 x^3 dx$

b Show that, for these sums, $\int_1^3 5x^3 dx = 5\int_1^3 x^3 dx$

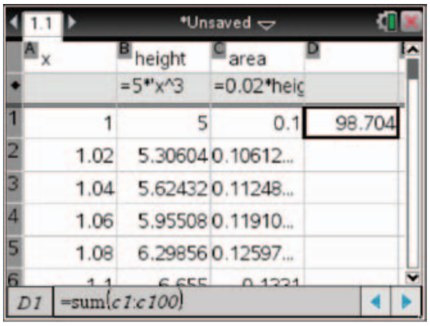
Solution

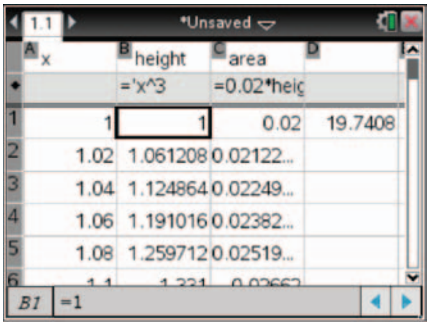
a Find the widths.

$$dx = \frac{3-1}{100} = 0.02$$

TI-Nspire CAS

Follow the same steps as in Example **8** on page 173 for parts **i** and **ii**.





ClassPad

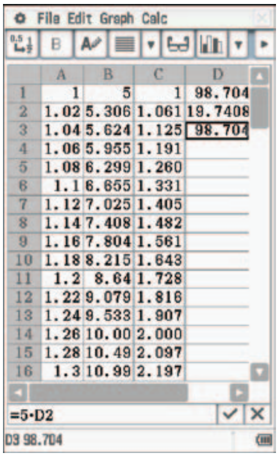
Follow the same steps as in Example **8** on page 173 for parts **i** and **ii**., but put the values for **i** in column B and the values for **ii** in column C. Add the areas by filling cells as follows.

D1: =sum(0.02×B1:B100)

D2: =sum(0.02×C1:C100)

D3: =5×D2

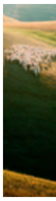
The columns have been narrowed to show the complete result.



Write the results.

i $\int_1^3 5x^3 dx$» 98.704

ii $\int_1^3 x^3 dx$» 19.7408

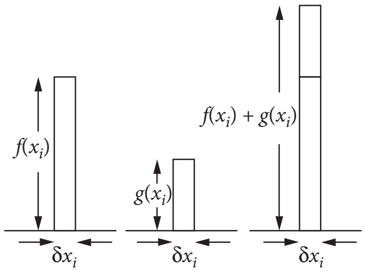


b Check the sides.

$$\begin{aligned}\text{LHS} &= \int_1^3 5x^3\,dx\\ &= 98.704\\ &= 5\times 19.7408\\ &= 5\int_1^3 x^3\,dx\\ &= \text{RHS}\end{aligned}$$

When you work out the value of $f(x) + g(x)$, you just add the values of the two functions. This means that the height of the rectangle for $f(x_i) + g(x_i)$ is just the sum of the heights of the individual rectangles, as shown in the diagram on the right.

Since this is true for every rectangle, it is true for the sum and true for the limit.



IMPORTANT

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Example 11

a Use centred rectangles with width 1 unit to find the approximate value of the following definite integrals.

i $\int_3^8 x^2dx$ ii $\int_3^8 2x\,dx$ iii $\int_3^8 (x^2+2x)dx$

b Show that $\int_0^5 (x^2+2x)dx = \int_0^5 x^2dx + \int_0^5 2x\,dx$ for the values you have obtained.

Solution

a Write the values of x_i. $x_1 = 3.5, x_2 = 4.5, x_3 = 5.5, x_4 = 6.5, x_5 = 7.5$

i Find the values of $f(x_i)$. $f(x_1) = 12.25, f(x_2) = 20.25, f(x_3) = 30.25, f(x_4) = 42.25, f(x_5) = 56.25$

Find the sum. $\sum_{i=1}^{5} f(x_i)\delta x = 161.25$

Write the result. $\int_3^8 x^2dx \approx 161.25$

ii Find the values of $f(x_i)$. $f(x_1) = 7, f(x_2) = 9, f(x_3) = 11, f(x_4) = 13, f(x_5) = 15$

Find the sum. $\sum_{i=1}^{5} f(x_i)\delta x = 55$

Write the result. $\int_3^8 2x\,dx \approx 55$

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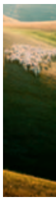
iii Find the values of $f(x_i)$.	$f(x_1) = 19.25, f(x_2) = 29.25, f(x_3) = 41.25, f(x_4) = 55.25, f(x_5) = 71.25$
Find the sum.	$\sum_{i=1}^{5} f(x_i)\delta x = 216.25$
Write the result.	$\int_3^8 (x^2+2x)dx \approx 216.25$
b Check the LHS.	$\text{LHS} = \int_0^5 (x^2+2x)dx$ $= 216.25$
Check the RHS.	$\text{RHS} = \int_0^5 x^2dx + \int_0^5 2x\,dx$ $= 161.25 + 55$ $= 216.25$
State the result.	LHS = RHS, so $\int_0^5 (x^2+2x)dx = \int_0^5 x^2dx + \int_0^5 2x\,dx$



You can combine the results of this section to show that the definite integral of a linear combination of functions preserves the linear sum. This is sometimes called the **linearity** property.

IMPORTANT

$\int_a^b [cf(x)+dg(x)]dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx$, for any functions and constants.

This is easily shown as follows.

$$\begin{aligned}\int_a^b [cf(x)+dg(x)]dx &= \int_a^b cf(x)dx + \int_a^b dg(x)dx \\ &= c\int_a^b f(x)dx + d\int_a^b g(x)dx\end{aligned}$$

This includes the case where $c = 1$ and $d = -1$, so it follows that

$$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

EXERCISE 4.04 Properties of the definite integral

Concepts and techniques

1 Example 9 **a** Use left rectangles with width 1 unit to approximate

i $\int_1^4 x^3dx$ ii $\int_4^6 x^3dx$ iii $\int_1^6 x^3dx$

b Show that $\int_1^6 x^3dx = \int_1^4 x^3dx + \int_4^6 x^3dx$ for these sums.

2 CAS **a** Use left rectangles of width 0.04 to find the value of

i $\int_0^2 (x^2+3)dx$ ii $\int_2^4 (x^2+3)dx$ iii $\int_0^4 (x^2+3)dx$

b Show that $\int_0^4 (x^2+3)dx = \int_0^2 (x^2+3)dx + \int_2^4 (x^2+3)dx$ for these sums.

3 Write the following as single integrals.

a $\int_0^1 x^2 dx + \int_1^5 x^2 dx$

b $\int_1^4 (x+1)dx + \int_4^7 (x+1)dx$

c $\int_{-2}^0 (x^3 - x - 1)dx + \int_0^2 (x^3 - x - 1)dx$

d $\int_0^2 (2x+1)dx + \int_2^3 (2x+1)dx$

e $\int_1^2 6x^3 dx + \int_2^3 6x^3 dx$

f $\int_{-1}^1 (3x^2 - 4x - 1)dx + \int_1^3 (3x^2 - 4x - 1)dx$

g $\int_{-2}^0 (x^2 - 2)dx + \int_0^2 (x^2 - 2)dx$

h $\int_0^3 3dx + \int_3^7 3dx$

i $\int_1^2 5x^4 dx + \int_2^3 5x^4 dx$

j $\int_0^4 (2x-3)dx + \int_4^6 (2x-3)dx$

4 Example 10 a Using 10 centred rectangles, find the approximate value of

i $\int_0^{10} x^2 dx$ ii $\int_0^{10} 3x^2 dx$

b Show that $\int_0^{10} 3x^2 dx = 3\int_0^{10} x^2 dx$ for these sums.

5 CAS a Use 100 centred rectangles to evaluate

i $\int_2^5 x^5 dx$ ii $\int_2^5 2x^5 dx$

b Show that $\int_2^5 2x^5 dx = 2\int_2^5 x^5 dx$ for these sums.

6 Example 11 a Use 4 left rectangles to find an approximation to

i $\int_1^2 3x\,dx$ ii $\int_1^2 2x^2 dx$ iii $\int_1^2 (2x^2 + 3x)dx$

b Show that $\int_1^2 (2x^2 + 3x)dx = \int_1^2 2x^2 dx + \int_1^2 3x\,dx$ for these sums.

7 Simplify each of the following.

a $\int_0^2 (3x^2 + 2)dx + \int_0^2 2x\,dx$

b $\int_1^2 x^3 dx + \int_1^2 (2x^3 - 3x + 1)dx$

c $\int_{-1}^1 (2x^4 + 3)dx + \int_{-1}^1 (x^3 - x^2 - 4)dx$

d $\int_0^3 (x^2 + 4x - 3)dx + \int_0^3 (x^2 - x - 1)dx$

e $\int_1^5 2x\,dx + \int_1^5 7dx$

Reasoning and communication

8 a Use 8 left rectangles to find an approximate value of

i $\int_2^6 x^3 dx$ ii $\int_2^6 x^2 dx$ iii $\int_2^6 (x^3 - x^2)dx$

b Show that $\int_2^6 (x^3 - x^2)dx = \int_2^6 x^3 dx - \int_2^6 x^2 dx$ for these sums.

9 CAS a Use 100 right rectangles to evaluate $\int_1^3 x^3 dx$.

b Use a lower boundary of 3 and upper boundary 1 to evaluate $\int_3^1 x^3 dx$.

c Find a relationship between $\int_1^3 x^3 dx$ and $\int_3^1 x^3 dx$.

d Prove the relationship between $\int_a^b f(x)dx$ and $\int_b^a f(x)dx$.

10 The velocity of a particle is given by $v = 6t - t^2$ m/s. Find the distance that the particle travels between 1 and 5 seconds by using 8 centred rectangles.

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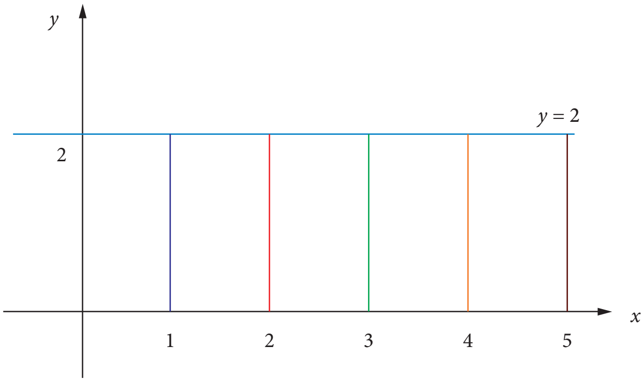
4.05 THE FUNDAMENTAL THEOREM OF CALCULUS

You can find the area under a straight line such as $y = 2$ and draw the graph of the area function.



From the graph you can see area A under the line $y = 2$ for different values of x:

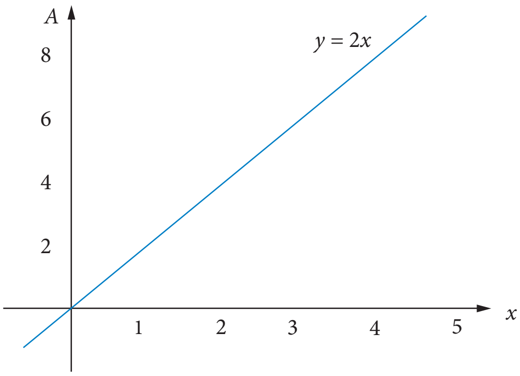
When $x = 1$, $A = 2$

When $x = 2$, $A = 4$

When $x = 3$, $A = 6$

When $x = 4$, $A = 8$ and so on.

You can draw the graph of these areas by plotting the points (1, 2), (2, 4), (3, 6) and (4, 8).



This is the graph of $y = 2x$, so the area function for the line $y = 2$ is $A(x) = 2x$.

INVESTIGATION Areas under a curve

For each of the following,

1 draw the graph of the function
2 find the area A under the curve at different values of x
3 draw the graph of the area A as a function of x
4 find the equation of the area function.
5 Can you find a relationship between the original function and its area function?

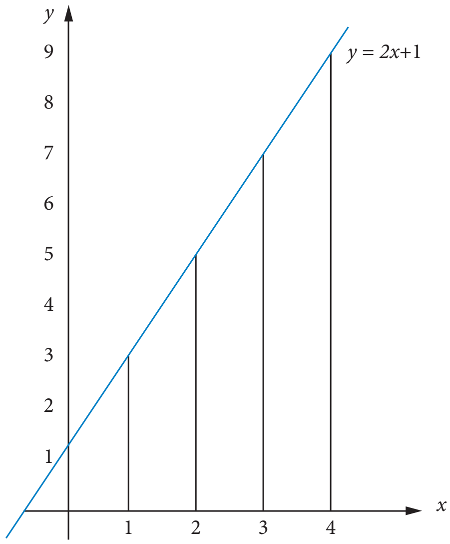
a $y = 1$ b $y = 3$ c $y = 4$
d $y = x$ e $y = 2x$ f $y = 3x$

Example 12

a Draw the graph of $f(x) = 2x + 1$ for $x = 0$ to 4.
b Find the area under the function from 0 to 0, 1, 2, 3 and 4 and draw the graph of the area function $A(x)$.
c CAS Find the equation of the function $y = A(x)$.

Solution

a Draw the graph of $y = 2x + 1$.



b Find the area under the curve for values of x using the area of a trapezium:

$A = \frac{1}{2}h(a+b)$.

When $x = 0$, $A = 0$

When $x = 1$: $A = \frac{1}{2} \times 1 \times (1+3) = 2$

When $x = 2$: $A = 2 + \frac{1}{2} \times 1 \times (3+5) = 6$

When $x = 3$: $A = 6 + \frac{1}{2} \times 1 \times (5+7) = 12$

When $x = 4$: $A = 12 + \frac{1}{2} \times 1 \times (7+9) = 20$

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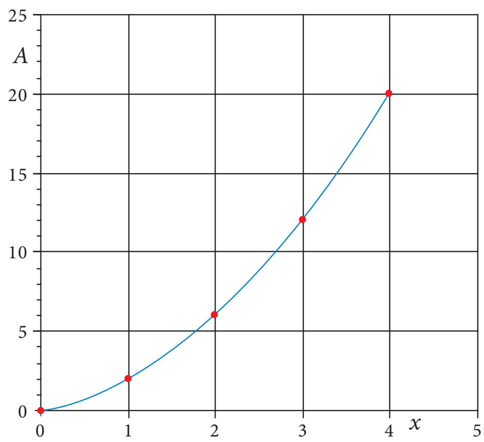
Draw the area function using points (0, 0), (1, 2), (2, 6), (3, 12) and (4, 20).

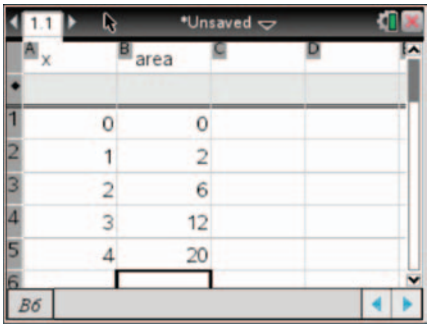


c Take a guess at the function.

The function looks like a quadratic.

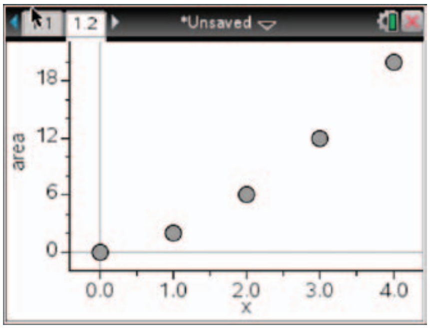
TI-Nspire CAS

Use Lists and Spreadsheet.
Put the headings x and area and type in the data.

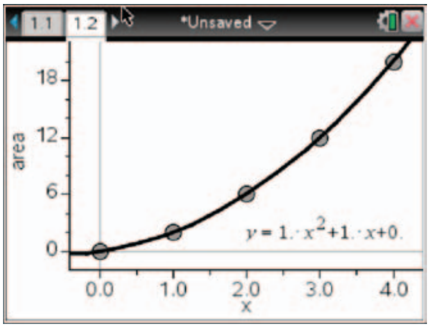


Add a Data and Statistics page and the data will be shown somewhat randomly.

Click to add variable at the bottom, and select x. Click to add variable on the left and select area. You can also press menu, 2: Plot properties and 8: Add Y Variable and choose area.

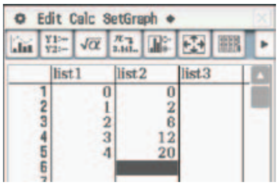


Now press menu, 4 Analyse, 6 Regression and 4 Show Quadratic to see the equation of the graph.

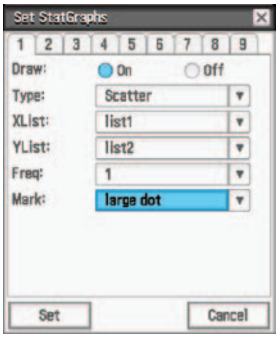


ClassPad

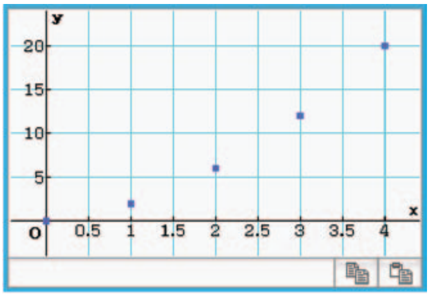
Use the Statistics menu.
Enter the x values in list1 and the corresponding areas in list2.



Tap SetGraph and make sure that only StatGraph1 is ticked. Tap Setting and define a Scatter graph as on the right. The kind of mark chosen is not important. Click Set.



Use View Window to make sure that your graph fits the values. If you want a grid, choose suitable scales for x and y.
Tap [icon] to draw the graph.



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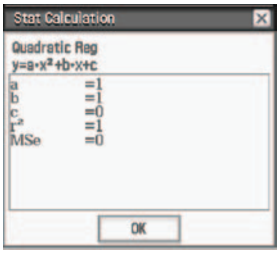
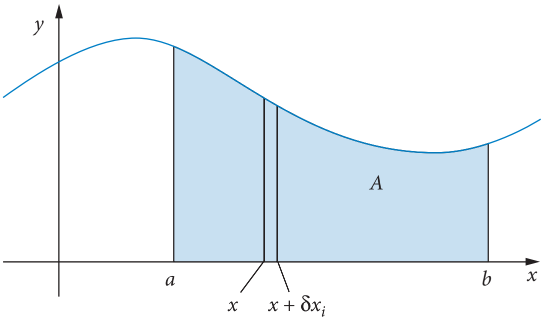
Tap anywhere in the top half of the screen, preferably on an empty cell.
Tap Calc, Regression and Quadratic Reg.
Make sure XList is list1, YList is list2 and Freq is 1.



Write the result.

The equation of the area function is $A(x) = x^2 + x$

You may notice that the area function in each of the examples and the investigation above are primitive functions (antiderivatives) of the function. You can see how this works by looking at the area from a to b for $f(x)$. Consider a rectangle $f(x_i)$ by δx_i in the sum $\sum_{i=1}^{n} f(x_i)\delta x_i$.



The value x_i could be at the left end, the right end or somewhere in between. The area of the strip is given by $\delta A_i = f(x_i)\delta x_i$. Rearranging gives $f(x_i) = \dfrac{\delta A_i}{\delta x_i}$. It doesn't matter which strip you pick, you will always get this result. But $\lim\limits_{x \to 0} \dfrac{\delta A}{\delta x} = \dfrac{dA}{dx} = A'(x)$, so $f(x)$ is the derivative of the area function. From your work last year with indefinite integrals, you can write $A(x) = F(x) + c$, where $F(x)$ is a primitive of $f(x)$ and c is a constant.

The integral from a to a is zero, so $A(0) = 0$.

Thus $0 = F(a) + c$ so $c = -F(a)$.

Substituting back in $A(x) = F(x) + c$, you get $A(x) = F(x) - F(a)$, so $A(b) = F(b) - F(a)$.

Using integral notation gives $\int_a^b f(x)dx = F(b) - F(a)$.

IMPORTANT

The **fundamental theorem of calculus** states that $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is an antiderivative (primitive) of $f(x)$.

Calculation of a definite integral using an indefinite integral is normally shown as follows.

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

where $F(x)$ is a primitive, antiderivative or indefinite integral $\left(\int f(x)dx\right)$.

The 'dx' indicates what variable the integral refers to. It is *not multiplied* by the function; it is just part of the formal notation.

Example 13

Evaluate the following.

a $\int_0^5 x^2 dx$ b $\int_2^6 x^3 dx$

Solution

a Find an antiderivative of x^2. $F(x) = \frac{x^3}{3}$ is an antiderivative of $f(x) = x^2$

Use the antiderivative. $\int_0^5 x^2 dx = \left[\frac{x^3}{3}\right]_0^5$

Apply the theorem. $= \frac{5^3}{3} - \frac{0^3}{3}$
$= \frac{125}{3}$
$= 41\frac{2}{3}$

State the result. $\int_0^5 x^2 dx = 41\frac{2}{3}$

b Find a primitive of x^3. $F(x) = \frac{x^4}{4}$ is a primitive of $f(x) = x^3$

Use the primitive. $\int_2^6 x^3 dx = \left[\frac{x^4}{4}\right]_2^6$

Apply the theorem. $= \frac{6^4}{4} - \frac{2^4}{4}$
$= 324 - 4$
$= 320$

State the result. $\int_2^6 x^3 dx = 320$

EXERCISE 4.05 The fundamental theorem of calculus

Concepts and techniques

1 Example 12 a By drawing the graph of $y = 9$, find the area under the graph for different values of x and draw the graph of the area function $A(x)$.
b Find the equation of the function $y = A(x)$.

2 a Draw the graph of $f(x) = 6x$.
b Find the area under the function for different values of x and draw the graph of the area function $A(x)$.
c Find the equation of the function $y = A(x)$.

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3 CAS a Draw the graph of $f(x) = 4x + 3$.

b Find the area under the function for different values of x and draw the graph of the area function $A(x)$.

c Find the equation of the function $y = A(x)$.

4 Example 13 Evaluate each definite integral.

a $\int_0^6 x^2 dx$ b $\int_0^3 x^3 dx$ c $\int_0^2 x^5 dx$

d $\int_0^4 x^7 dx$ e $\int_0^5 x^4 dx$

5 Evaluate each of the following.

a $\int_1^3 x^2 dx$ b $\int_2^8 x\,dx$ c $\int_3^5 x^4 dx$

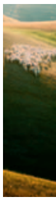
d $\int_3^4 x^3 dx$ e $\int_1^6 x^2 dx$

6 Find the value of each definite integral.

a $\int_2^6 x^5 dx$ b $\int_1^4 x^9 dx$ c $\int_4^6 x\,dx$ d $\int_1^2 x^5 dx$

e $\int_2^3 x^3 dx$ f $\int_1^4 x^4 dx$ g $\int_2^5 x\,dx$ h $\int_3^5 x^7 dx$

i $\int_1^2 x^9 dx$ j $\int_3^6 x^5 dx$



Reasoning and communication

7 The speed of a particle is given by $S = t^2$ m/s.

a Find the speed after

i 5 seconds ii 10 seconds.

b Find the distance travelled in the first 5 seconds.

c Find the distance travelled between 5 and 10 seconds.

8 a Draw the graph of $y = x^3$ and shade the area under the curve between $x = 2$ and $x = 4$.

b Write this area as a definite integral.

c Find this area under the curve.

9 CAS a Use a CAS calculator to evaluate $\int_0^5 (2t^2 + 5t + 9)dt$.

b Find the distance travelled by an object after 5 seconds if its speed is given by $v = 2t^2 + 5t + 9$ m/s.

c Find the distance that the object travels between 3 and 5 seconds.

10 The acceleration of a particle is given by $a = t^3$ m/s^2.

a Find the acceleration after 2 seconds.

b Find the speed at which the particle is moving after 2 seconds.

c Find the speed at which the particle moves in the next 2 seconds.

4.06 CALCULATION OF DEFINITE INTEGRALS

WS Integration of power functions

You can use and combine the properties of definite integrals to help evaluate them directly.

Example 14

Evaluate each definite integral.

a $\int_1^3 5x^2 dx$

b $\int_2^6 (x^3 - x^2)dx$

c $\int_3^4 (3x^2 - 2x + 4)dx$

Solution

a Use the property $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ and integrate.

$$\int_1^3 5x^2 dx = 5\int_1^3 x^2 dx$$

$$= 5\left[\frac{x^3}{3}\right]_1^3$$

Use the fundamental theorem.

$$= 5\left[\left(\frac{3^3}{3}\right) - \left(\frac{1^3}{3}\right)\right]$$

$$= 43\frac{1}{3}$$

Write the answer.

$$\int_1^3 5x^2 dx = 43\frac{1}{3}$$

b Use the property $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$ to integrate.

$$\int_2^6 (x^3 - x^2)dx = \int_2^6 x^3 dx - \int_2^6 x^2 dx$$

$$= \left[\frac{x^4}{4}\right]_2^6 - \left[\frac{x^3}{3}\right]_2^6$$

Use the fundamental theorem.

$$= \left(\frac{6^4}{4} - \frac{2^4}{4}\right) - \left(\frac{6^3}{3} - \frac{2^3}{3}\right)$$

$$= 320 - 69\frac{1}{3}$$

$$= 250\frac{2}{3}$$

Write the result.

$$\int_2^6 (x^3 - x^2)dx = 250\frac{2}{3}$$

c Write the integral.

$$\int_3^4 (3x^2 - 2x + 4)dx$$

Use the antiderivative of $3x^2 - 2x + 4$.

$$= \left[x^3 - x^2 + 4x\right]_3^4$$

Use the fundamental theorem.

$$= (64 - 16 + 16) - (27 - 9 + 12)$$

$$= 64 - 30$$

$$= 34$$

Write the answer.

$$\int_3^4 (3x^2 - 2x + 4)dx = 34$$

In Chapter 1, you learnt that $\frac{d}{dx}(e^x) = e^x$. You can use this to find some integrals.

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Example 15

a Find the exact value of $\int_0^3 2e^x dx$

b Evaluate $\int_1^2 e^x dx + \int_2^5 e^x dx$

Solution

a Try the derivative of $2e^x$.

$$\frac{d}{dx}2e^x = 2\frac{d}{dx}e^x$$
$$= 2e^x$$

Write the primitive.

$2e^x$ is a primitive of $2e^x$.

Use the fundamental theorem.

$$\int_0^3 2e^x dx = \left[2e^x\right]_0^3$$

Evaluate.

$$= 2e^3 - 2e^0$$
$$= 2e^3 - 2$$

b Simplify using

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

$$\int_1^2 e^x dx + \int_2^5 e^x dx = \int_1^5 e^x dx$$

Use that fact that $\frac{d}{dx}(e^x) = e^x$.

$$= \left[e^x\right]_1^5$$

Substitute and evaluate.

$$= e^5 - e^1$$
$$= e^5 - e$$
$$= e(e^4 - 1)$$

You found these derivatives in Chapter 1: $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$

Example 16

Find the following definite integrals

a $\int_0^{\frac{\pi}{3}} \sin(x)dx$ b $\int_0^{\frac{\pi}{4}} 2\cos(x)dx$

Solution

a Write the integral so you can use the derivative of $\cos(x)$.

$$\int_0^{\frac{\pi}{3}} \sin(x)dx = -\int_0^{\frac{\pi}{3}} -\sin(x)dx$$

Use the fundamental theorem and $\frac{d}{dx}\cos(x) = -\sin(x)$.

$$= -\left[\cos(x)\right]_0^{\frac{\pi}{3}}$$

Substitute values.

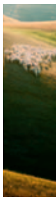
$$= -\left[\cos\left(\frac{\pi}{3}\right) - \cos(0)\right]$$

Use exact values.

$$= -\left(\frac{1}{2} - 1\right)$$

Write the answer.

$$\int_0^{\frac{\pi}{3}} \sin(x)dx = \frac{1}{2}$$



b Use definite integral properties.

$$\int_0^{\frac{\pi}{4}} 2\cos(x)dx = 2\int_0^{\frac{\pi}{4}} \cos(x)dx$$

Use the fundamental theorem and $\frac{d}{dx}\sin(x) = \cos(x)$.

$$= [\sin(x)]_0^{\frac{\pi}{4}}$$

Substitute values.

$$= \sin\left(\frac{\pi}{4}\right) - \sin(0)$$

Use exact values.

$$= \frac{\sqrt{2}}{2} - 0$$

Write the answer.

$$\int_0^{\frac{\pi}{4}} 2\cos(x)dx = \frac{\sqrt{2}}{2} \approx 0.7071$$

EXERCISE 4.06 Calculation of definite integrals

Concepts and techniques

1 Example 14 Find the following definite integrals.

a $\int_1^3 4x\,dx$ b $\int_0^2 7x^6\,dx$ c $\int_1^2 4x^3\,dx$ d $\int_2^3 (2x-1)dx$

e $\int_0^4 (x+2)dx$ f $\int_1^5 (6x-5)dx$ g $\int_0^1 (x^3-3x^2+1)dx$ h $\int_0^3 (x^2-x-2)dx$

i $\int_1^2 (8x^3-5)dx$ j $\int_0^1 (x^4-x^2+1)dx$

2 Evaluate each of the following definite integrals.

a $\int_0^2 \frac{x^2}{2}dx$ b $\int_{-1}^1 (3x^2+4x)dx$ c $\int_{-1}^2 (x^2+1)dx$

d $\int_{-2}^3 (4x^3-3)dx$ e $\int_{-1}^0 (x^2+3x+5)dx$

3 Example 15 Evaluate each of the following definite integrals.

a $\int_0^4 e^x\,dx$ b $\int_1^3 5e^x\,dx$ c $\int_0^2 (2e^x+x)dx$

d $\int_1^5 (e^x-1)dx$ e $\int_2^4 (x^3-e^x)dx$

4 Example 16 Evaluate each of the following.

a $\int_0^{\frac{\pi}{4}} \cos(x)dx$ b $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin(x)dx$ c $\int_0^{\pi} 3\sin(x)dx$

d $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cos(x)dx$ e $\int_0^{\frac{\pi}{2}} 7\sin(x)dx$

5 Find the values of the definite integrals below.

a $\int_0^{\pi} [x+\sin(x)]dx$ b $\int_0^{\frac{\pi}{4}} [\cos(x)-\sin(x)]dx$

c $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} [\cos(x)+1]dx$ d $\int_0^{\frac{\pi}{3}} [2\sin(x)+3\cos(x)]dx$

e $\int_1^3 \left[\sin(x)+3x^2\right]dx$ (answer correct to 2 decimal places)

6 Evaluate the following.

a $\int_0^{\pi} 3\cos(x)dx$ b $\int_{\pi}^{\frac{4\pi}{3}} \cos(x)dx$ c $\int_{\frac{2\pi}{3}}^{\frac{3\pi}{2}} 3\sin(x)dx$

d $\int_{\pi}^{\frac{5\pi}{4}} \sqrt{2}\cos(x)dx$ e $\int_{\pi}^{\frac{11\pi}{6}} 2\sin(x)dx$

Reasoning and communication

7 i Simplify and ii evaluate each of the definite integrals below.

a $\int_0^3 (2x-1)dx + \int_3^5 (2x-1)dx$

b $\int_0^4 e^x dx + \int_0^4 x\,dx$

c $\int_0^{\frac{\pi}{6}} \cos(x)dx - \int_0^{\frac{\pi}{6}} 2\sin(x)dx$

8 a What is the derivative of $\tan(x)$?

b Find $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2(x)}dx$.

9 a Differentiate e^{4x}

b Find $\int_0^3 4e^{4x}dx$

c Find $\int_0^3 e^{4x}dx$

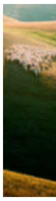
10 The velocity of an object moving in a straight line is given by $\frac{dx}{dt} = 3t^2 + 2t - 5$ cm/s and its displacement is 3 cm after 2 seconds. Find

a its initial velocity

b the equation of displacement x in terms of t

c its displacement after 5 seconds

d its acceleration after 3 seconds.



4.07 AREAS UNDER CURVES

One of the main uses of definite integrals is to calculate the exact area under a curve. You have already seen that for $f(x) > 0$, the definite integral is the area under a curve. You can use the fundamental theorem to find these areas exactly.

Example 17

Find the area under $y = 6x + 2$ from $x = 3$ to 5.

Solution

Make sure that the conditions are correct.	$6x + 2 > 0$ for $x = 3$ to $x = 5$
Write the area as an integral.	Area $= \int_3^5 (6x+2)dx$
Use the fundamental theorem.	$= \left[3x^2 + 2x\right]_3^5$
Substitute values.	$= 85 - 33 = 52$ square units

EXERCISE 4.07 Areas under curves

Concepts and techniques

1 Example 17 Find the area under $y = 4x + 1$ from $x = 6$ to 9.

2 Find the area under $f(x) = x^2$ from 4 to 7.

3 Find the area under $y = x^3$ from $x = 1$ to 5.

4 Find the area under $f(x) = x^2 + 3$ from 2 to 5.

5 Find the area under $y = x^3 + 9$ from $x = -2$ to 4.

6 Find the area under $y = 7x - x^2 - 1$ from $x = 1$ to 4.

7 Find the area under $y = 6x^3 + 2x^2 + 3$ from 2 to 8.

8 Find the area under $y = x^2 - 5x + 2$ from $x = 6$ to 10.

Reasoning and communication

9 a Sketch the function $y = x^3$ from $x = -3$ to $x = 3$.

b Find $\int_{-2}^{0} x^3 dx$.

c Find $\int_{0}^{2} x^3 dx$.

d Find $\int_{-2}^{2} x^3 dx$.

e What is the area between $y = x^3$ and the x-axis from −2 and 2?

f Why are the answers for parts **d** and **e** different?

10 a What is the sign of $f(x) = x^2 - 10x + 16$ from $x = 3$ to $x = 7$?

b Find $\int_{3}^{7} (x^2 - 10x + 16) dx$.

c What is the area between $f(x) = x^2 - 10x + 16$ and the x-axis from $x = 3$ to $x = 7$?

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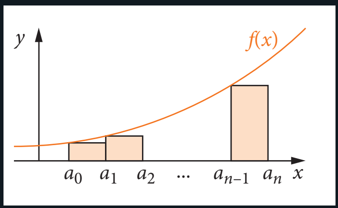
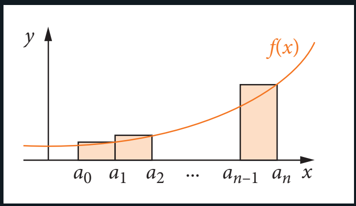
CHAPTER SUMMARY
INTEGRATION AND AREAS

- Rectangles are used to find the approximate area under a curve. The rectangles can use function values on the left, right or in the centre of the rectangles, with the area written as $\sum_{i=1}^{n} f(x_i)\delta x_i$ or $\sum_{x=a}^{b} f(x_i)\delta x_i$.

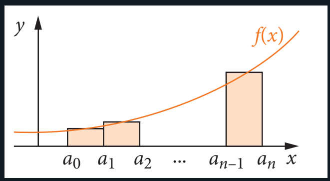
'Left' rectangles



'Right' rectangles



'Centred rectangles'



- Centred rectangles can give a closer approximation to the area under the curve.
- **Integration** is the process used to find the exact value of the area under a curve.
- The **definite integral** of the function $f(x)$ from a to b, $\int_a^b f(x)dx$, is the exact area under the curve $y = f(x)$ between $x = a$ and $x = b$. This is given by the limit of the sum of the areas of rectangles:

$$\int_a^b f(x)dx = \lim_{\delta x \to 0} \sum f(x)\delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\delta x_i$$

- The **fundamental theorem of calculus** states that

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative (primitive) of $f(x)$.

- The calculation is shown as

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

- Some of the properties of definite integrals are:

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx,$$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx \text{ and}$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

- The **linearity** property,

$$\int_a^b [cf(x)+dg(x)]dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx$$

preserves linear combinations of functions.

4 CHAPTER REVIEW
INTEGRATION AND AREAS

Multiple choice

1 Example 6 Find the approximate value of $\int_1^3 x^2 dx$ using right rectangles with width 0.5 units.

A $0.5(1^2 + 1.5^2 + 2^2 + 2.5^2)$
B $0.5(1.5^2 + 2^2 + 2.5^2 + 3^2)$
C $0.5(1^2 + 1.5^2 + 2^2 + 2.5^2 + 3^2)$
D $0.5(1.5^2 + 2^2 + 2.5^2)$
E $0.5(1.25^2 + 1.75^2 + 2.25^2 + 2.75^2)$

2 Example 7 Find the approximate value of the definite integral $\int_0^5 (x^3 + 1)dx$ by using 5 centred rectangles.

A $0.5(0.5^3 + 1 + 1.5^3 + 1 + 2.5^3 + 1 + 3.5^3 + 1 + 4.5^3 + 1)$
B $0^3 + 1 + 1^3 + 1 + 2^3 + 1 + 3^3 + 1 + 4^3 + 1$
C $1^3 + 1 + 2^3 + 1 + 3^3 + 1 + 4^3 + 1 + 5^3 + 1$
D $0.5^3 + 1 + 1.5^3 + 1 + 2.5^3 + 1 + 3.5^3 + 1 + 4.5^3 + 1$
E $1.5^3 + 1 + 2.5^3 + 1 + 3.5^3 + 1 + 4.5^3 + 1 + 5.5^3 + 1$

3 Examples 9–11 Which of the following is true?

A $\int_2^4 (3x^3 - 5x^2 + 4x + 1)dx - \int_2^4 (x^3 + x^2 - 5x - 3)dx = \int_2^4 (2x^3 - 6x^2 - x + 4)dx$
B $\int_2^4 (3x^3 - 5x^2 + 4x + 1)dx - \int_2^4 (x^3 + x^2 - 5x - 3)dx = \int_2^4 (2x^3 - 4x^2 - x - 2)dx$
C $\int_2^4 (3x^3 - 5x^2 + 4x + 1)dx - \int_2^4 (x^3 + x^2 - 5x - 3)dx = \int_2^4 (2x^3 - 6x^2 + 9x - 2)dx$
D $\int_2^4 (3x^3 - 5x^2 + 4x + 1)dx - \int_2^4 (x^3 + x^2 - 5x - 3)dx = \int_2^4 (2x^3 - 6x^2 + 9x + 4)dx$
E $\int_2^4 (3x^3 - 5x^2 + 4x + 1)dx - \int_2^4 (x^3 + x^2 - 5x - 3)dx = \int_2^4 (4x^3 - 6x^2 + 9x - 2)dx$

4 Example 14 Find $\int_{-2}^2 (12x^2 - 6x + 5)dx$.

A 0 B 54 C 60 D 84 E 108

5 Example 17 Find the area under the function $f(x) = x^2 - 1$ from $x = 2$ to $x = 4$.

A $7\frac{2}{3}$ B 14 C $16\frac{2}{3}$ D 56 E $56\frac{1}{3}$

Short answer

6 Example 1 **a** If a car is travelling at 75 km/h, find the distance it travels in 8 hours.

b By drawing the graph of the speed of the car, show that the distance it travels after 8 hours is equal to the area under the graph between 0 and 8 hours.

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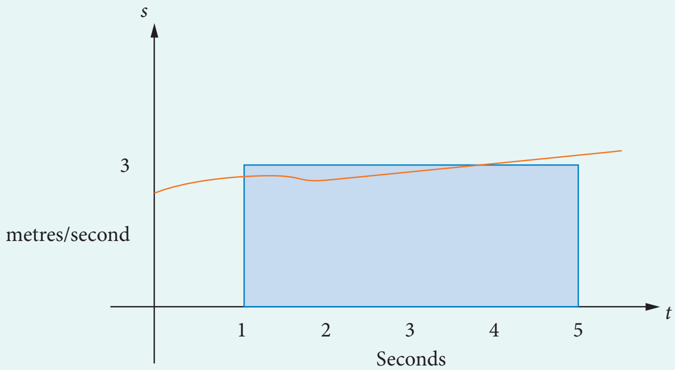
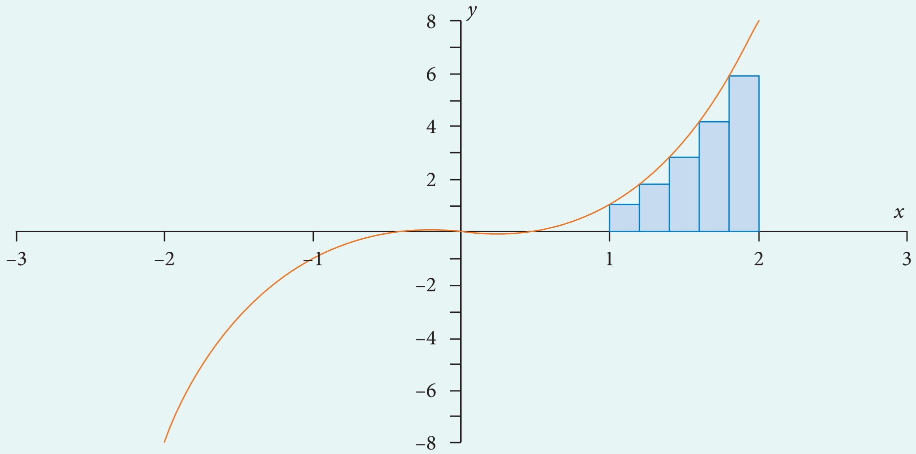
7 Example 2 Find the approximate distance travelled by a particle between 1 and 5 seconds by finding the shaded area below.



8 Example 3 Find the approximate area under the curve $y = x^3$ from $x = 1$ to $x = 2$ by finding the areas of the rectangles below.



9 Example 4 **a** Find the approximate area under the curve $y = x^2 - 2x + 1$ between $x = 1$ and $x = 3$ by using

i 4 left rectangles **ii** 4 right rectangles

b CAS 50 left rectangles.

10 Examples 5–7 Find an approximate area under the curve $y = x^2$ between $x = 0$ and $x = 2$ by using

a 8 left rectangles **b** 8 right rectangles **c** 8 centred rectangles.

11 CAS Example 8 Find the approximate area under the curve $y = x^3$ between $x = 0$ and $x = 4$ using 40 left rectangles.

12 CAS Example 8 Find an approximation to $\int_0^{10} (x^2 + 2x)dx$ by using 20 left rectangles.

13 CAS Example 9 a Sketch the graph of $y = 3x + 2$ for $x = 0$ to 5.
b Find the area under the function from 0 to 0, 1, 2, 3 and 4 and draw the graph of the area function $A(x)$.
c CAS Find the equation of the function $y = A(x)$.

14 Example 10 a Use right rectangles with width 0.5 units to find an approximation to
i $\int_1^2 (2x^2+1)dx$ ii $\int_2^4 (2x^2+1)dx$ iii $\int_1^4 (2x^2+1)dx$
b Show that $\int_1^4 (2x^2+1)dx = \int_1^2 (2x^2+1)dx + \int_2^4 (2x^2+1)dx$

15 CAS Example 11 a Use 100 left rectangles to find approximations for
i $\int_2^8 x^2 dx$ ii $\int_2^8 6x^2 dx$
b Show that $\int_2^8 6x^2 dx = 6\int_2^8 x^2 dx$ using the sums above.

16 Example 12 a Find an approximation to each of the following definite integrals using left rectangles with width 0.25 units.
i $\int_1^2 x^3 dx$ ii $\int_1^2 2x\,dx$ iii $\int_1^2 (x^3+2x)dx$
b Show that $\int_1^2 (x^3+2x)dx = \int_1^2 x^3 dx + \int_1^2 2x\,dx$ for the sums calculated.

17 Example 14 Evaluate the following definite integrals.
a $\int_0^2 x^3 dx$ b $\int_1^3 x\,dx$ c $\int_0^3 (x^2+3x-4)dx$ d $\int_1^2 (3x-2)dx$

18 Example 15 Evaluate $\int_0^7 3e^x dx$.

19 Example 16 Evaluate the following integrals.
a $\int_0^{\frac{\pi}{4}} \sin(x)dx$ b $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos(x)dx$

20 Example 17 a Find the area under the curve $f(x) = x^2 - 9$ between $x = 4$ and $x = 6$.
b Find the area under the curve $f(x) = 3x^2 - 2x - 1$ between $x = 2$ and $x = 5$.

Application

21 The velocity of an object is given by $v = 3\cos(t)$ cm/s. Find
a the velocity after 1 second
b the distance travelled in the first second
c the distance travelled between 0.6 and 0.9 seconds.

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22 a Differentiate $\frac{x}{e^x}$

b Find the definite integral $\int_0^1 \frac{1-x}{e^x}dx$.

23 a What is the sign of $f(x) = x^3 - 6x^2 + 12x - 8$ from $x = 0$ to $x = 2$?

b What is the sign of $f(x) = x^3 - 6x^2 + 12x - 8$ from $x = 2$ to $x = 4$?

c Find $\int_0^4 (x^3 - 6x^2 + 12x - 8)dx$.

d Find $\int_0^2 (x^3 - 6x^2 + 12x - 8)dx$.

e What is the area between $f(x) = x^3 - 6x^2 + 12x - 8$ and the x-axis from $x = 0$ to $x = 4$?

f Explain why the answers to **c** and **e** are different.

Qz Practice quiz