Papers written by
Australian Maths
Software

SEMESTER TWO

REVISION 2

MATHEMATICS METHODS

UNITS 3-4

2016

SOLUTIONS

SECTION ONE

1. (6 marks)

(a) (i) $f'(x)=2e^{2x}\tan(x)+e^{2x}\sec^2(x)=e^{2x}\left[2\tan(x)+\sec^2(x)\right]$

(ii) $g(x)=ln(1+x)(1-x)$

$g(x)=ln(1+x)+ln(1-x)$

$g'(x)=\frac{1}{(1+x)}-\frac{1}{(1-x)}=-\frac{2x}{(1-x^2)}$

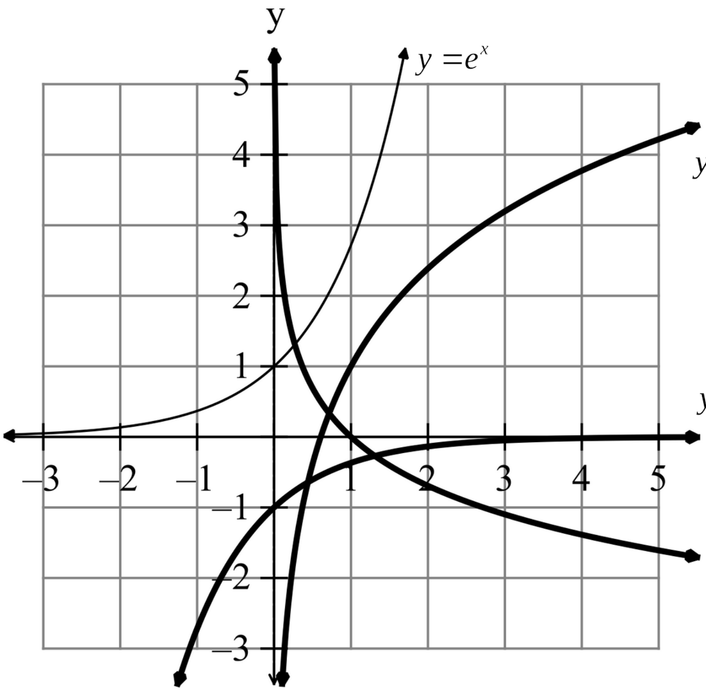
(iii) $h(x)=\frac{\sin(x)}{\cos(3x)}$

$h'(x)=\frac{\cos(x)\cos(3x)-3(-\sin(3x))\sin(x)}{\cos^2(3x)}$ ✓✓ -1/error

$h'(x)=\frac{\cos(x)\cos(3x)+3\sin(3x)\sin(x)}{\cos^2(3x)}$

2. (16 marks)

(a) (i) (ii) (iii)



$y=1+2f^{-1}(x)=1+2\ln(x)$

✓✓✓

$y=g(x)=-e^{-x}$

$y=-f^{-1}(x)=-\ln(x)$

(b) (i) $log_x(36) = 2$

$36 = x^2$

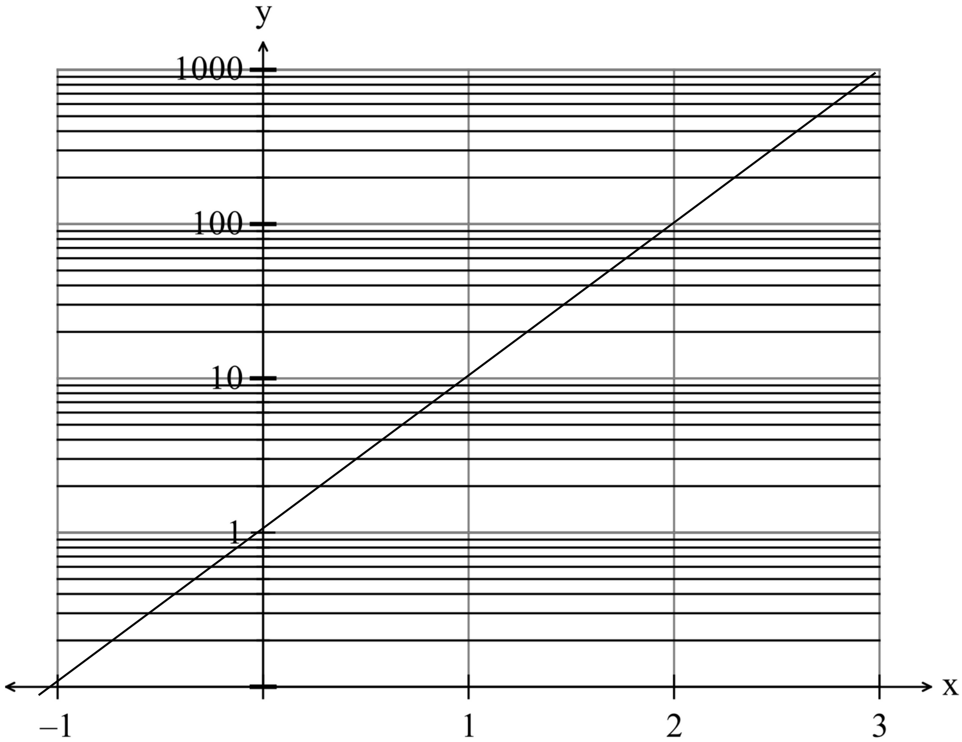
$x = \pm 6$ but $x > 0$ as a base cannot be negative.

$x = 6$

(ii) $x = e^{ln(2)}$ $log_e(x) = ln(2)$ $x = 2$

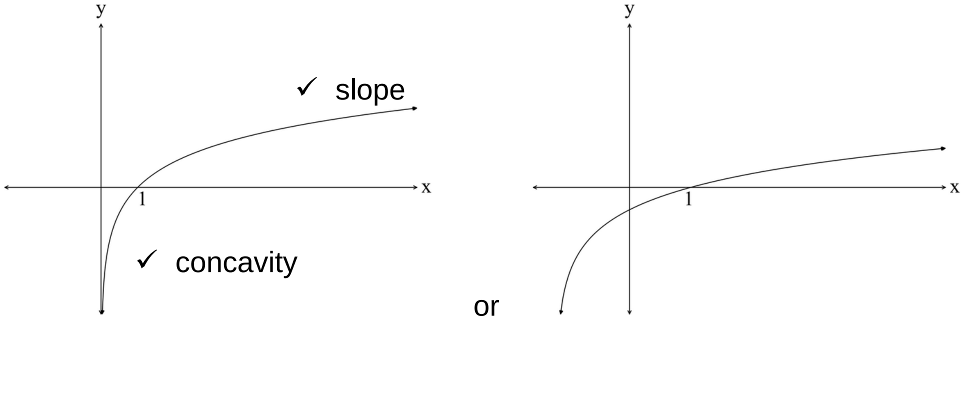
(c) $\dfrac{(log_6 3 + log_6 2)^2}{-log_6 36} = \dfrac{(log_6 6)^2}{-log_6 6^2} = \dfrac{1^2}{-2(1)} = -\dfrac{1}{2}$

(d) $y = 10^x$



3. (7 marks)

(a) (i)



(b) (i) $\frac{d}{dx}(ln(x-2)) = \frac{1}{x-2}$

(ii) $\int \frac{2}{(x-2)}\,dx = 2\,ln(x-2) \quad for\ x > 2$

4. (7 marks)

(a) $\int \frac{3}{x^2} - 2x + \sqrt{x}\ dx$

$$= \int 3x^{-2} - 2x + x^{1/2}\ dx$$

$$= -\frac{3}{x} - x^2 + \frac{1}{2}x^{-1/2} + c$$

$$= -\frac{3}{x} - x^2 + \frac{1}{2\sqrt{x}} + c$$

(b) $\int \frac{dx}{(1-4x)^5} = \int (1-4x)^{-5}dx = \frac{(1-4x)^{-4}}{(-4)(-4)} + c \ = \ \frac{1}{16(1-4x)^4} + c$

(c) $\int_0^{\pi/2} sin(2x) + cos(2x)dx$

$$= \left[-\frac{cos(2x)}{2} + \frac{sin(2x)}{2}\right]_0^{\pi/2} \quad \checkmark$$

$$= -\frac{1}{2}\left[cos(2x) - sin(2x)\right]_0^{\pi/2}$$

$$= -\frac{1}{2}\left((cos(\pi) - sin(\pi)) - (cos(0) - sin(0))\right)$$

$$= -\frac{1}{2}((-1) - (1))$$

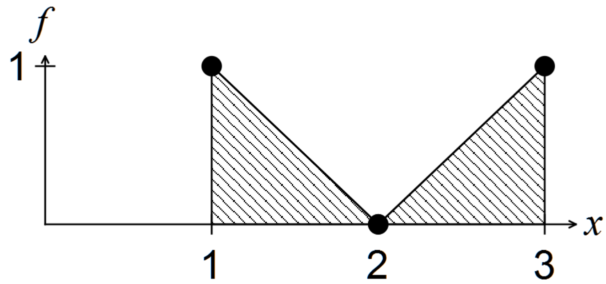
$$= 1$$

5. (14 marks)

(a) (i) Not a probability density function as the probabilities do not add to 1. ✓

(i) Yes, a probability density function as the probabilities add to 1. ✓

(b) (i) $f(x) = \begin{cases} |x-2| & \text{for } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$



The area $= 2 \times \left(\frac{1}{2} \times 1 \times 1\right) = 1 \quad \therefore \ pdf$

(ii) $P(2.5 \le x \le 3) = \int_{2.5}^{3} (x-2)\,dx = \left[\frac{x^2}{2} - 2x\right]_{2.5}^{3} = \left(\frac{9}{2} - 6\right) - \left(\frac{6.25}{2} - 5\right) = \frac{3}{8}$

Probably easier to use areas.

(c) (i) $a = \frac{1}{6}$

(ii) $P(3 \le x \le 5) = \frac{2}{6}$

(iii) $P(x \le 3 \mid x \le 7) = \frac{1}{5}$ ✓✓

(iv) $F(x) = \int_{2}^{x} \frac{1}{6}\,dx$

$= \frac{1}{6} \times [x]_{2}^{x}$

$= \frac{1}{6} \times (x-2)$

$F(x) = \frac{(x-2)}{6}$

END OF SECTION ONE

SECTION TWO

6. (6 marks)

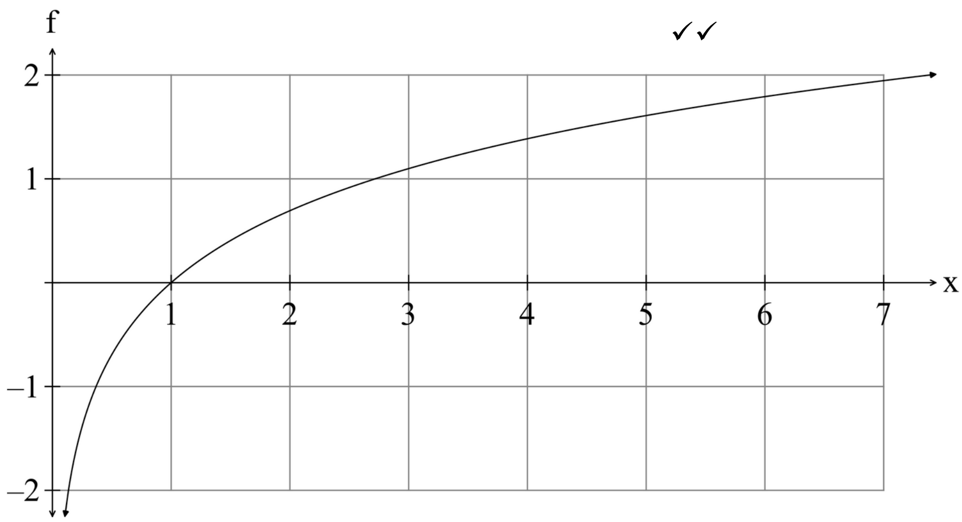
(a)

x	$f(x)$
0.5	-0.69
1	0
2	1.69
3	1.099
4	1.386
7	1.946

✓✓✓ -1/error

(b)

$y = f(x)$



(c) $y = ln(x)$

7. (9 marks)

(a) (i) $f(x) = tan(x)$ ✓ (c = 0)

(ii) $f'(x) = sec^2(x) = (cos(x))^{-2}$

$f''(x) = -2(cos(x))^{-3}(-sin(x))$ ✓

$f''(x) = \dfrac{2\,sin(x)}{cos^3(x)}$

(b) There are no turning points in f so $f' \neq 0$.

f has a positive gradient at all defined points,

f is concave up for all $f > 0$ and concave down for all $f < 0$.

$f' > 0$ at all defined points as the gradient of f is always positive.

For $0 < x < \frac{\pi}{2}$ function f' is increasing so concave up and $f'' > 0$.

For $\frac{\pi}{2} < x < \pi$ function f' is decreasing so concave down and $f'' < 0$..

f' has turning points so f'' has x intercepts.

At $x = 0,\ \pi,\ 2\pi$ f' has turning points so $f'' = 0$.

At the points of inflection in f , f' has a turning point, $f'' = 0$, the concavity of f changes so the gradient of f' changes from increasing to decreasing and vice versa.

8. (7 marks)

(a) $\frac{dh}{dt} = 10\, cm\ /\ min$

$\frac{dV}{dt} = ?$ at $h = 20\, cm$

$V = \frac{\pi}{3} r^2 h$ but $r = \frac{4h}{3}$

$V = \frac{\pi}{3}\left(\frac{4h}{3}\right)^2 h$

$V = \frac{16\pi}{27} h^3$

$\frac{dV}{dh} = \frac{16\pi}{9} h^2$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$\frac{dV}{dt} = \frac{16\pi}{9} h^2 \times 10$

$At\ \ h = 20\ \ \frac{dV}{dt} = \frac{16\pi}{9} \times 20^2 \times 10$

$\frac{dV}{dt} = 22340.21 cm^3.min$

When the height is 20 cm, the rate of change of volume is 22 340.21 cm^3/min.

(b) $V = \frac{16\pi}{27} h^3$

$\frac{dV}{dh} = \frac{16\pi}{9} h^2$

$\frac{dV}{dh} \approx \frac{\delta V}{\delta r}$

$\delta V \approx \frac{16\pi}{9} h^2 \times \delta r$

$At \;\; h = 2$ and $\delta r = 0.01$

$\delta v \approx \frac{16\pi}{9} \times 15^2 \times 0.01$

$\delta V \approx 4\pi m^3 \qquad \left(= 12.77 m^3\right)$

9. (4 marks)

(a) Area = 0.5 x 1 + 0.5 x 0.67 + 0.5 x 0.5 + 0.5 x 0.4 + 0.5 x 0.33
= 0.5 x [1 + 0.67 + 0.5 + 0.4 + 0.33]
= 1.45 units^2

(b) $\int_{0.5}^{3} \frac{1}{x} dx \approx 1.79$

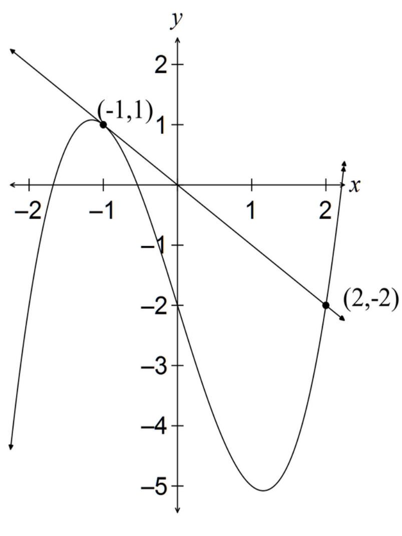
10. (4 marks)

The points of intersection are (-1,1) and (2,-2).

The area between the curves is found by

$A = \int_{-1}^{2} \left(-x - \left(x^3 - 4x - 2\right)\right) dx$

$A = 6.75 \;\; m^2$



11. (6 marks)

(a) $a = 4\,ms^{-2},\ v_0 = 2\,ms^{-1}, x_0 = 1\,m$

$v = \int 4dt = 4t + c_1$

$At\ t = 0, v_0 = 2\,ms^{-1} \rightarrow \quad c_1 = 2$

$\therefore\ v = 4t + 2$

$x = \int (4t + 2)dt = 2t^2 + 2t + c_2$

$At\ t = 0, x_0 = 1\,m \rightarrow \quad c_2 = 1$

$\therefore\ x = 2t^2 + 2t + 1$

$At\, t = 2 \quad v = 4(2) + 2 = 10\,ms^{-1}$

$x = 2(2)^2 + 2(2) + 1 = 13\,m$

(b) $v = 0\ \ at\, 4t + 2 = 0 \quad t = -0.5$ but $t \geq 0$.

There are no changes of direction on the defined domain. 0/2 if not checked.

$At\ t = 0,\ x = 1$

$At\ t = 4,\ x = 41$

The distance travelled is 40 m. ✓ ✓

12. (6 marks)

(a) $\frac{d}{dx}\int_1^x (tan(2t) - 2)dt = tan(2x) - 2$ ✓ ✓

(b) (i) $f(t) = cos(ln(t))$

$f'(t) = (-sin(ln(t))) \times \frac{1}{t} = \frac{-sin(ln(t))}{t}$

(ii) $\int_1^e -\frac{sin(ln(t))}{t}dt = \left[cos(ln(t))\right]_1^e$

$= cos(ln(e)) - cos(ln(1))$

$= cos(1) - cos(0)$

$= cos(1) - 1$

13. (6 marks)

(a) $t=0 \quad 1990 \quad P=2635(r)^t$

$t=20 \quad 2010 \quad 2075=2635(r)^{20}$

$r=0.98812497$

The annual rate of decrease of the mountain pygmy possum population from 1990 to 2010 was1.1875 %

(b) $t=25 \quad 2015 \quad P=2635(0.98812497)^{25}$

$P=1955$

Given the actual population in 2015 was 1830, the culling was not effective. The population is a lot less than expected.

14. (6 marks)

(a) $V=$area of end $\times$ height

$V=\pi r^2 \times w$

$l=2\pi r \;\; and \;\; l=10-w$

$2\pi r=10-w$

$w=10-2\pi r$

$V=\pi r^2 \times (10-2\pi r)$

$V=10\pi r^2-2\pi^2 r^3$

(b) $V=10\pi r^2-2\pi^2 r^3$

$\frac{dV}{dr}=20\pi r-6\pi^2 r^2$

$\frac{d^2V}{dr^2}=20\pi-12\pi^2 r$

Maximum volume when $\frac{dV}{dr}=0$ and $\frac{d^2V}{dr^2}<0$

$If \; \frac{dV}{dr}=0, \quad 0=20\pi r-6\pi^2 r^2$

$0=2\pi r(10-3\pi r) \quad r\neq 0$

$r=\frac{10}{3\pi}$

Max or min?

$\frac{d^2V}{dr^2}=20\pi-12\pi^2\left(\frac{10}{3\pi}\right)=-20\pi<0 \quad \therefore$ max

Need dimensions of rectangle.

$w=10-2\pi r=10-2\pi\left(\frac{10}{3\pi}\right)=3\frac{1}{3}$

$l=10-w=6\frac{2}{3}$

15. (4 marks)

(a) $B\left(5, \frac{1}{6}\right)$

$P(x=2) = {}^5C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = 0.1607510288 \approx 0.16$ ✓ ✓

(b) $P(2 \text{ or } 3 \text{ or } 4 \text{ or } 5) = 0.196244856 \approx 0.20$ ✓ ✓

16. (9 marks)

(a)

y	0	1	2	3
$P(Y=y)$	0.5787	0.3472	0.0694	0.0046

✓ ✓ -1/error

(b) (i) $P(X \geq 2) = 0.3 + 0.4 = 0.7$ ✓

(ii) $E(X) = 0 \times 0.1 + 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.4 = 2 \quad \therefore \mu = 2$ ✓ ✓

$Var(X) = E(X^2) - \mu^2$

$Var(X) = 0^2 \times 0.1 + 1^2 \times 0.2 + 2^2 \times 0.3 + 3^2 \times 0.4 - 2^2$

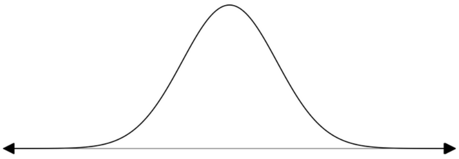
$Var(X) = 1$ ✓

(ii) $\mu_{2X+1} = 2(2) + 1 = 5$

Only mulitplying changes the variance, not adding.

$Var(2X+1) = 2^2 = 4$

17. (11 marks)



$\mu = \$1\,800\,000, \ \ \sigma = \$150\,000$

(a) $P(x > 2\,000\,000) = 0.09121121973 \approx 0.09$ ✓ ✓

(b) $P(x > 2\,000\,000 \mid x > 1650\,000) = \dfrac{P(x > 2\,000\,000)}{P(x > 1650\,000)} = \dfrac{0.09121112197}{0.8413447461} \approx 0.1084$

(c) $P(1\,200\,000 < x < 1\,900\,000) = 0.7474757912 \approx 0.75$ ✓ ✓

(d) $P(x > 2\,000\,000) = 0.09121121973 \approx 0.09$ B(4, 0.09)

P(she had sales worth over \$2 000 000 in two of the months and sales worth less than \$2 000 000 in the other two months)

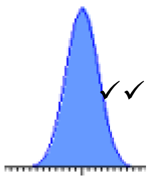
$= {}^4C_2\,(0.09121121973)^2\,(1 - 0.09121121973)^2$

$= 0.04122623$ ✓✓

≈ 0.04

18. (22 marks)

(a)



✓✓

(b) (i) Bias occurs because buyers are there possibly to buy the specials or they may be in a financial position where they need to buy goods on special. ✓✓

(ii) Assuming the survey only needs to speak to Woolworth's customers, you could randomly sample the people on the electoral role, then ask if they are Woolworth's customers. If so, you have a sample point. If not, try someone else. ✓✓

(c) $p=0.4,\ q=0.6\ \Rightarrow\ np=80\times0.4=32>5$

$nq=80\times0.6=48>5$ so can use normal distribution.

Mean $=p=0.4$

$$sd_{\hat{p}}=\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}=\sqrt{\frac{0.4\times0.6}{80}}$$

$$sd_{\hat{p}}=0.05477$$

Standardised score (using 19.5 to 25.5)

$$z=\frac{X-\mu}{\sigma}$$

$$z_1=\frac{\frac{19.5}{80}-0.4}{0.05477}=-2.85284$$

$$z_2=\frac{\frac{25.5}{80}-0.4}{0.05477}=-1.48348$$

The probability is 0.07.

(d) (i) $\hat{p}=\dfrac{20}{120}=0.1\dot{6}$

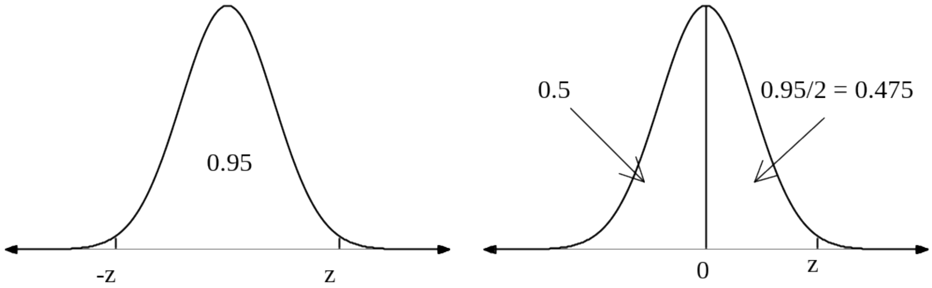
(ii) $\hat{p}=0.1\dot{6}$

$$Var_{\hat{p}}=\frac{\hat{p}(1-\hat{p})}{n}$$

$$Var_{\hat{p}}=\frac{0.1\dot{6}\times0.8\dot{3}}{120}$$

$$Var_{\hat{p}}\approx0.00116$$

(iii) $sd_{\hat{p}} = 0.03402$ $\qquad sd_{\hat{p}} = \sqrt{\dfrac{0.1\dot{6} \times 0.8\dot{3}}{120}}$



P(X< z) = 0.975

So, 95% confidence level means z = 1.96

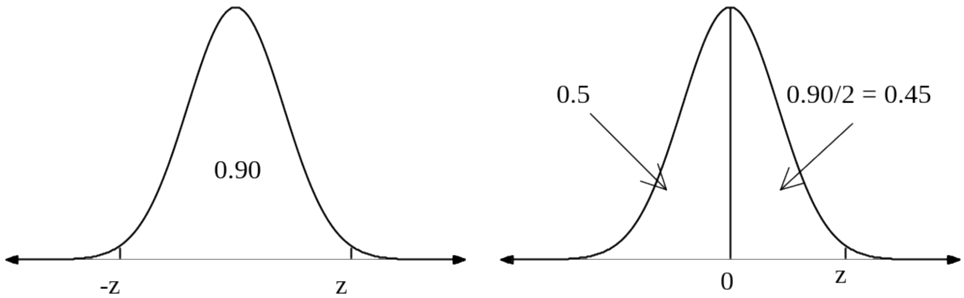
s = 0.03402

$$E = z \times s = 1.96 \times 0.03402$$

$$E = 0.06668$$

The confidence limit is $0.16667 \pm 0.06668 \, i.e. \; (0.10, 0.23)$

(e)



P(X< z) = 0.95

z = 1.645

Use $p = 0.5$ as the maximum value as p is unknown.

So with $p = 0.5$ $\quad sd = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\dfrac{0.25}{n}}$

$E = z \times s$ but $E = 0.10$

Therefore

$$0.10 = 1.645 \times \sqrt{\frac{0.25}{n}}$$

$$n = 67.65$$

$$n \approx 68$$

Should use a sample size of 68 people to have a confidence level of 90% with an error margin of 10%.

END OF SECTION TWO