PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Test 2 Year 12

Student name: _______________ Teacher name: _______________

Task type: **Response/Investigation**

Reading time for this test : 5 mins

Working time allowed for this task: 40 mins

Number of questions: 7

Materials required: Upto 3 classpads/calculators

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **41 marks**

Task weighting: **13%**

Formula sheet provided: no but formulae stated on page 2

Note: All part questions worth more than 2 marks require working to obtain full marks.

Useful formulae

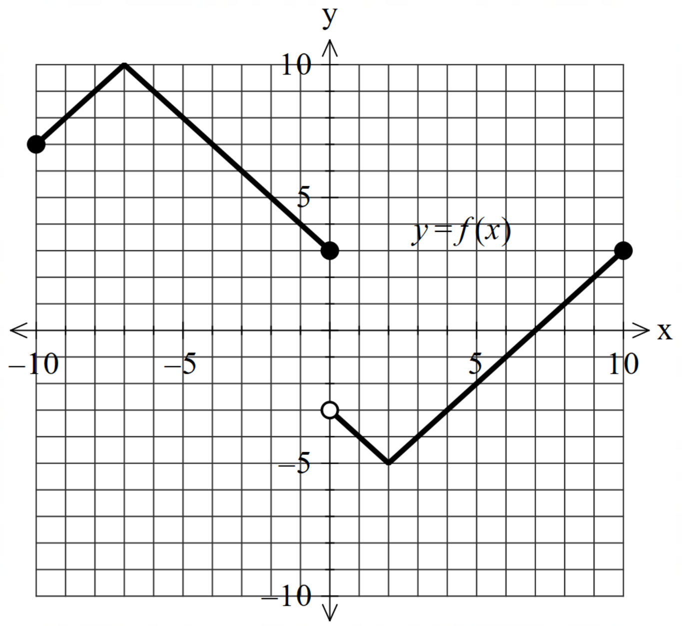
Complex numbers

Cartesian form	
$z = a + bi$	$\bar{z} = a - bi$
$\text{Mod}(z) = \lvert z\rvert = \sqrt{a^2 + b^2} = r$	$\text{Arg}(z) = \theta, \quad \tan\theta = \frac{b}{a}, \quad -\pi < \theta \le \pi$
$\lvert z_1 z_2\rvert = \lvert z_1\rvert \lvert z_2\rvert$	$\left\lvert \frac{z_1}{z_2} \right\rvert = \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$
$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$	$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
$z\bar{z} = \lvert z\rvert^2$	$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{\lvert z\rvert^2}$
$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$	$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
Polar form	
$z = a + bi = r(\cos\theta + i\sin\theta) = r\,cis\,\theta$	$\bar{z} = r\,cis(-\theta)$
$z_1 z_2 = r_1 r_2\,cis(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,cis(\theta_1 - \theta_2)$
$cis(\theta_1 + \theta_2) = cis\,\theta_1\,cis\,\theta_2$	$cis(-\theta) = \frac{1}{cis\,\theta}$
De Moivre's theorem	
$z^n = \lvert z\rvert^n cis(n\theta)$	$(cis\,\theta)^n = \cos n\theta + i\sin n\theta$
$z^{\frac{1}{q}} = r^{\frac{1}{q}}\left(\cos\frac{\theta + 2\pi k}{q} + i\sin\frac{\theta + 2\pi k}{q}\right)$, for k an integer	

$$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$$

Q1 (2 & 3 = 5 marks)

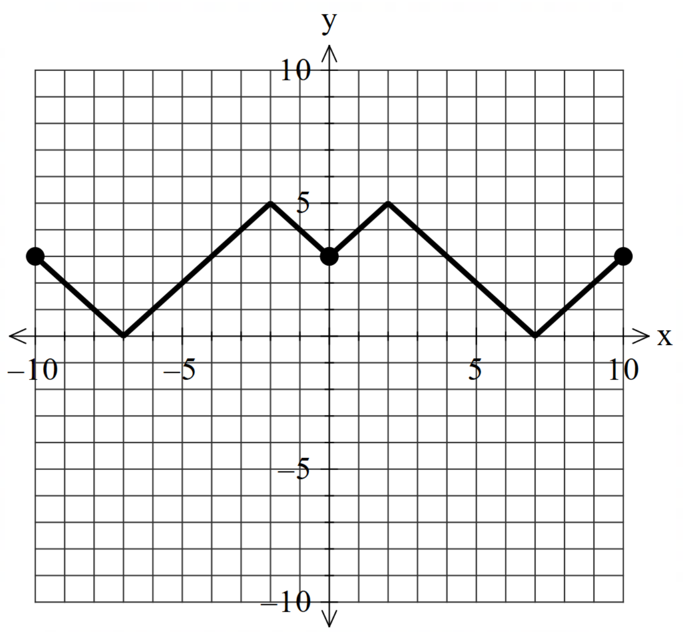
Consider the function $f(x)$ plotted below.



a) Solve for $|f(x)| = 5$.

x=-2,2
Specific behaviours
P one value P exactly two values

b) Sketch $y = |f(|x|)|$ on the axes below.



Specific behaviours
P y intercept P Absolute value used to reflect negative y values in x axis P reflection of right to give new left side

Q2 (2, 3 & 3 = 8 marks)

Consider the functions $f(x)=\dfrac{1}{\sqrt{2x-9}}$ and $g(x)=\dfrac{1}{3x-1}$.

a) Determine the natural domain and range of $g(x)$.

$d_g : x \neq \frac{1}{3}$ $r_g : y \neq 0$
Specific behaviours
P domain P range

b) Does $f \circ g(x)$ exist over the natural domain of $g(x)$? Explain.

$f \circ g(x)$ $to\,exist\ r_g \subseteq d_f$ $r_g : y \neq 0$ $d_f : x > \frac{9}{2}$ not $\geq$ $\therefore not$ $r_g \not\subset d_f$ "memorize" this
Specific behaviours
P states relevant domain and range P states reason to exist P states does not exist with a reason

c) Determine the largest possible domain for $f \circ g(x)$.

$f \circ g(x) = \dfrac{1}{\sqrt{2\frac{1}{3x-1} - 9}} = \sqrt{\dfrac{3x-1}{11-27x}}$ $3x - 1 > 0 \Rightarrow x > \dfrac{1}{3}$ $11 - 27x > 0 \Rightarrow x < \dfrac{11}{27}$ $d: \dfrac{1}{3} < x < \dfrac{11}{27}$
Specific behaviours
P states rule, no need to simplify OR gives reasoning P determines lower limit of domain (non inclusive) P determines upper limit of domain (non inclusive) Do not award if inequality incorrect

Q3 (3, 3, & 2 = 8 marks)

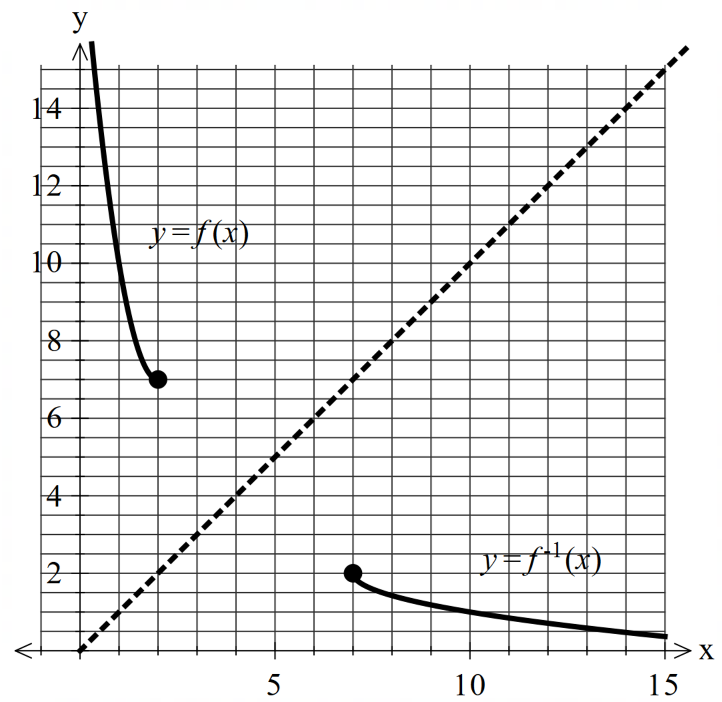
Consider the function $f(x) = 3x^2 - 12x + 19, \ x \le 2$.

a) Determine $f^{-1}(x)$ and state its domain.

$f(x) = 3x^2 - 12x + 19, \ x \le 2$ $x = 3y^2 - 12y + 19, y \le 2$ $0 = 3y^2 - 12y + 19 - x$ $y = \dfrac{12 \pm \sqrt{144 - 12(19 - x)}}{6} = \dfrac{12 \pm \sqrt{12x - 84}}{6}$ $f^{-1}(x) = \dfrac{12 - 2\sqrt{3x - 21}}{6} = \dfrac{6 - \sqrt{3x - 21}}{3}, x \ge 7$
Specific behaviours
P swaps x and y P states inverse rule with initially two possibilities P discards positive and states domain

Q3 continued

b) Sketch $f(x)\,\&\,f^{-1}(x)$ on the same set of axes below.



Specific behaviours
P sketches f with point (2,7) clearly plotted P sketches f(-1) with point (7,2) clearly plotted P both functions appear to be reflected in line y=x

c) Determine value(s) of x, if any, such that $f\circ f(x)=x$. Explain.

$f\circ f(x)=x$ results in $f(x)=f^{-1}(x)$ graphs overlapping at these points From graph above it is apparent that $f(x)\neq f^{-1}(x)$ therefore no solutions
Specific behaviours
P explains that $f\circ f(x)=x$ results in $f(x)=f^{-1}(x)$ P states no solution to equation with a reason

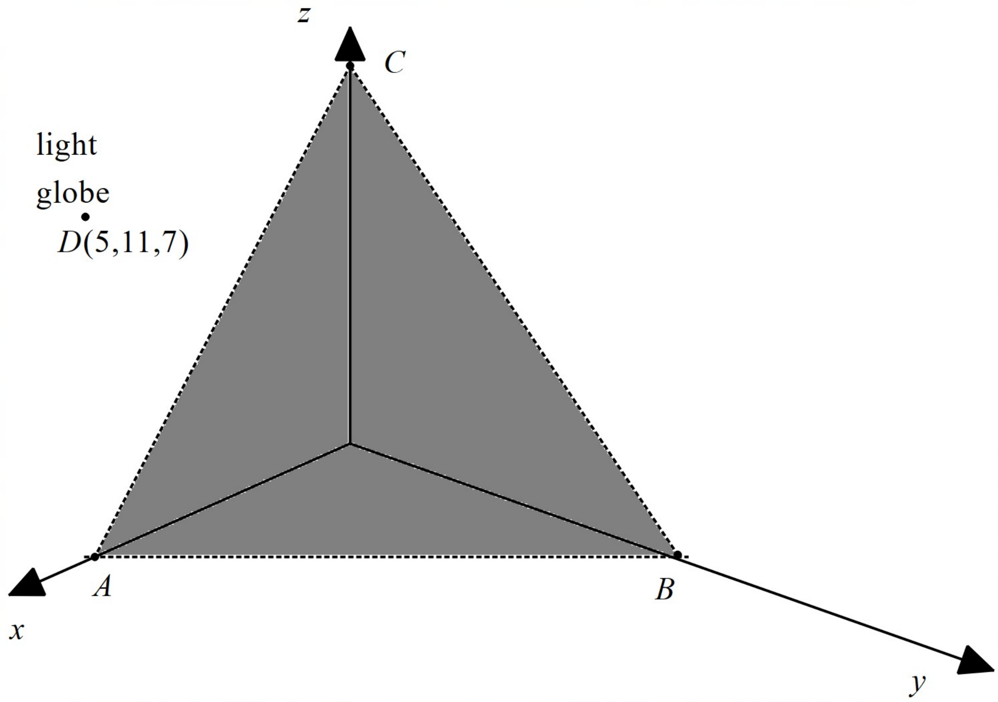
Q4 (3 marks)

If $z = 27 cis \frac{7\pi}{8}$ is a solution to the equation $z^n = ir$ where r is a positive real number and n is a positive integer, determine the smallest possible value for r in the form 3^p. **Justify** your answer.

$z^n = 3^{3n} cis \frac{7\pi n}{8} = ir = r cis \frac{k\pi}{2}, k = 1,5,9,13,17,21...$ $\frac{7\pi n}{8} = \frac{k\pi}{2}$ $n = \frac{4k}{7}$ $smallest\ k = 21, n = 12$ $r = 3^{36}$
Specific behaviours
P establishes a relationship between n and k algebraically P determines smallest value for n P expresses r as a power of 3.

Q5 (3 & 3 = 6 marks)

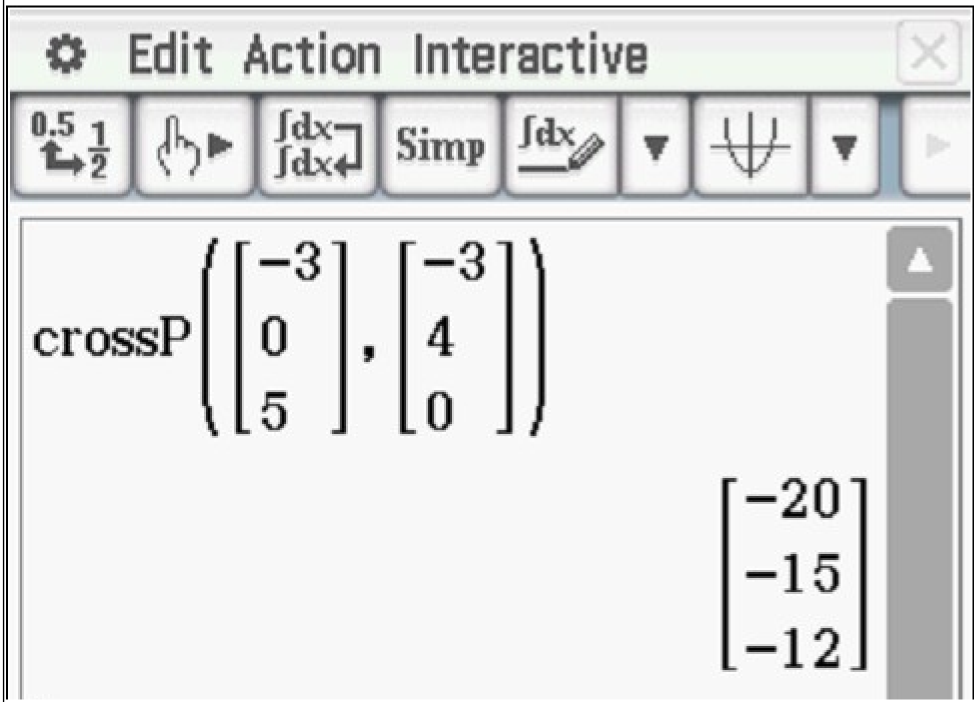
Consider a triangular plane with vertices $A(3,0,0), B(0,4,0)\ \&\ C(0,0,5)$ shaded as shown below. There is a light globe situated at point $D(5,11,7)$.



a) Determine the cartesian equation of the shaded plane ABC above.

$$\overrightarrow{AC} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 5 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix}$$



$$r.\begin{pmatrix} -20 \\ -15 \\ -12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}.\begin{pmatrix} -20 \\ -15 \\ -12 \end{pmatrix} = -60$$

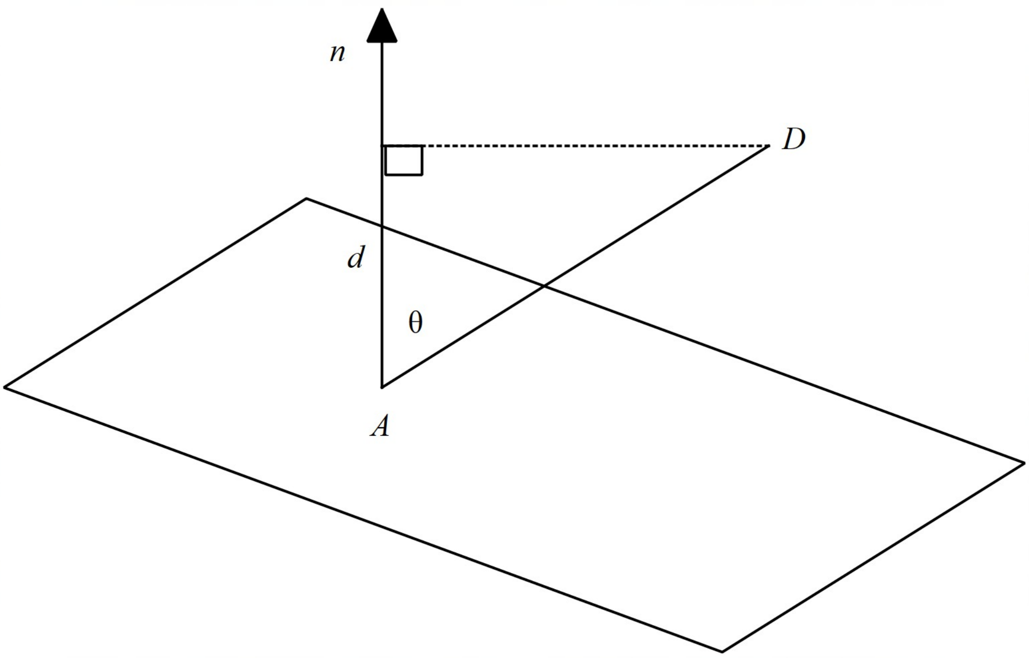
$$-20x - 15y - 12z = -60$$

Specific behaviours

P uses cross product using any two sides of triangle
P sets up vector equation of plane
P derives cartesian equation (or any multiple)

Q5 continued

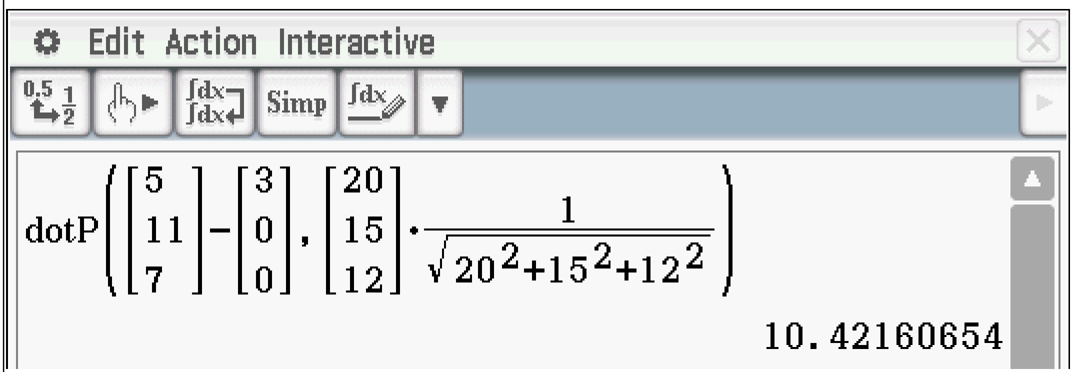
b) Determine the distance of the globe to the shaded plane ABC.



$d = |\overrightarrow{AD}| \cos\theta$

$= |\overrightarrow{AD} \cdot \hat{n}|$

$(\hat{n} = \frac{n}{|n|})$



Specific behaviours
P determines AD vector or line through pt D
P uses scalar dot product with normal or line meeting plane
P determines approx. distance

Q6 (5 marks)

Consider the line A $r=\begin{pmatrix}-3\\0\\11\end{pmatrix}+\lambda\begin{pmatrix}5\\-1\\2\end{pmatrix}$ and the sphere B $\left|r-\begin{pmatrix}-3\\\alpha\\1\end{pmatrix}\right|=10$ where α is a real constant.

Determine all possible values of α ,to one decimal place such that:

i) the line misses the sphere.

ii) the line just touches the sphere.

iii) the line pierces the sphere at two points.

$$r=\begin{pmatrix}-3\\0\\11\end{pmatrix}+\lambda\begin{pmatrix}5\\-1\\2\end{pmatrix}=\begin{pmatrix}-3+5\lambda\\-\lambda\\11+2\lambda\end{pmatrix}$$

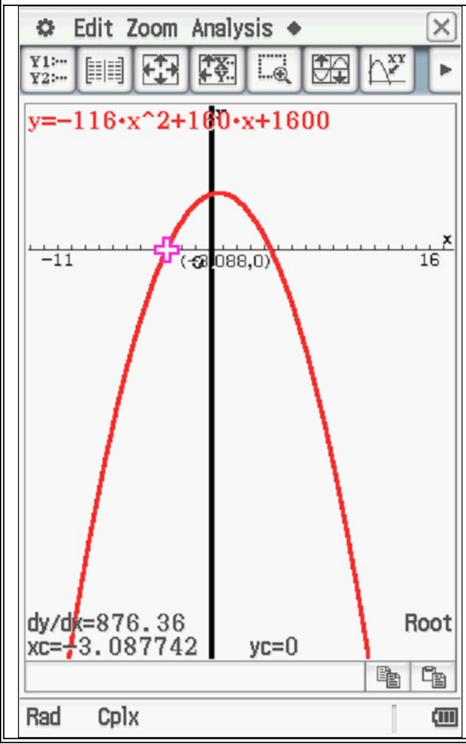
$$\left|\begin{pmatrix}-3+5\lambda\\-\lambda\\11+2\lambda\end{pmatrix}-\begin{pmatrix}-3\\\alpha\\1\end{pmatrix}\right|=10$$

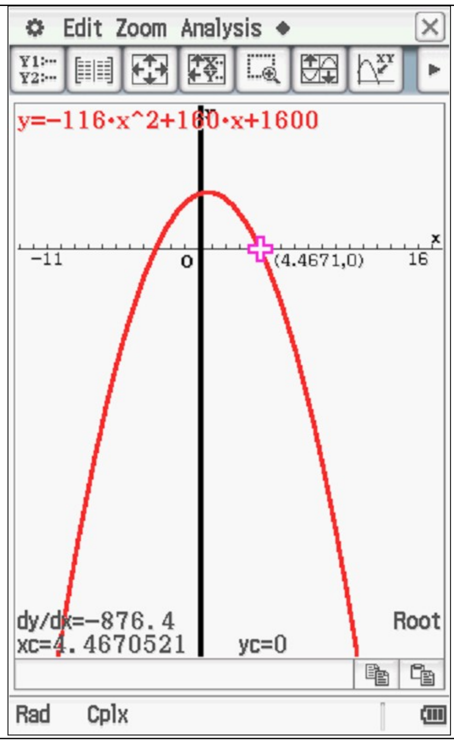
$$\left|\begin{pmatrix}5\lambda\\-\lambda-\alpha\\10+2\lambda\end{pmatrix}\right|=10$$

$$\sqrt{25\lambda^2+\lambda^2+2\lambda\alpha+\alpha^2+100+40\lambda+4\lambda^2}=10$$

$$30\lambda^2+(2\alpha+40)\lambda+\alpha^2=0$$

$$\Delta=(2\alpha+40)^2-4(30)(\alpha^2)=-116\alpha^2+160\alpha+1600$$





$\Delta < 0$ *misses* $\alpha < -3.1, \alpha > 4.5$ $\Delta = 0$ *touches* $\alpha = -3.1, \alpha = 4.5$ $\Delta > 0$ *pierces* $-3.1 < \alpha < 4.5$
Specific behaviours
P sets up an equation for $\lambda \& \alpha$ P states a quadratic equation or uses shortest distance approach P uses discriminant expression or compares distances to radius P states values of α for all three scenarios Pstates a condition for **each** of the three scenarios to determine values

Q7 (3 & 3 = 6 marks)

Consider two rockets $A \& B$ that are ignited at the same time from different positions and move with constant velocities as shown below.

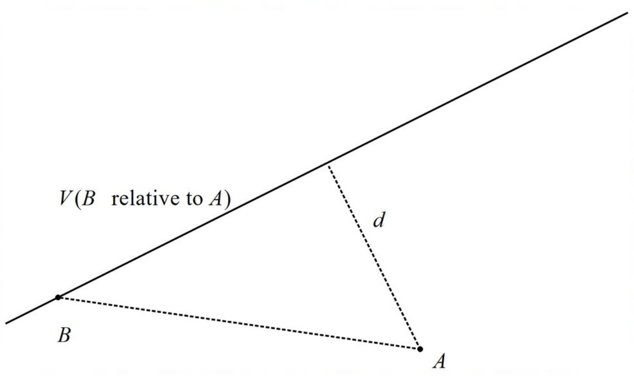
$$r_A = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} km \quad , v_A = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} km/h$$

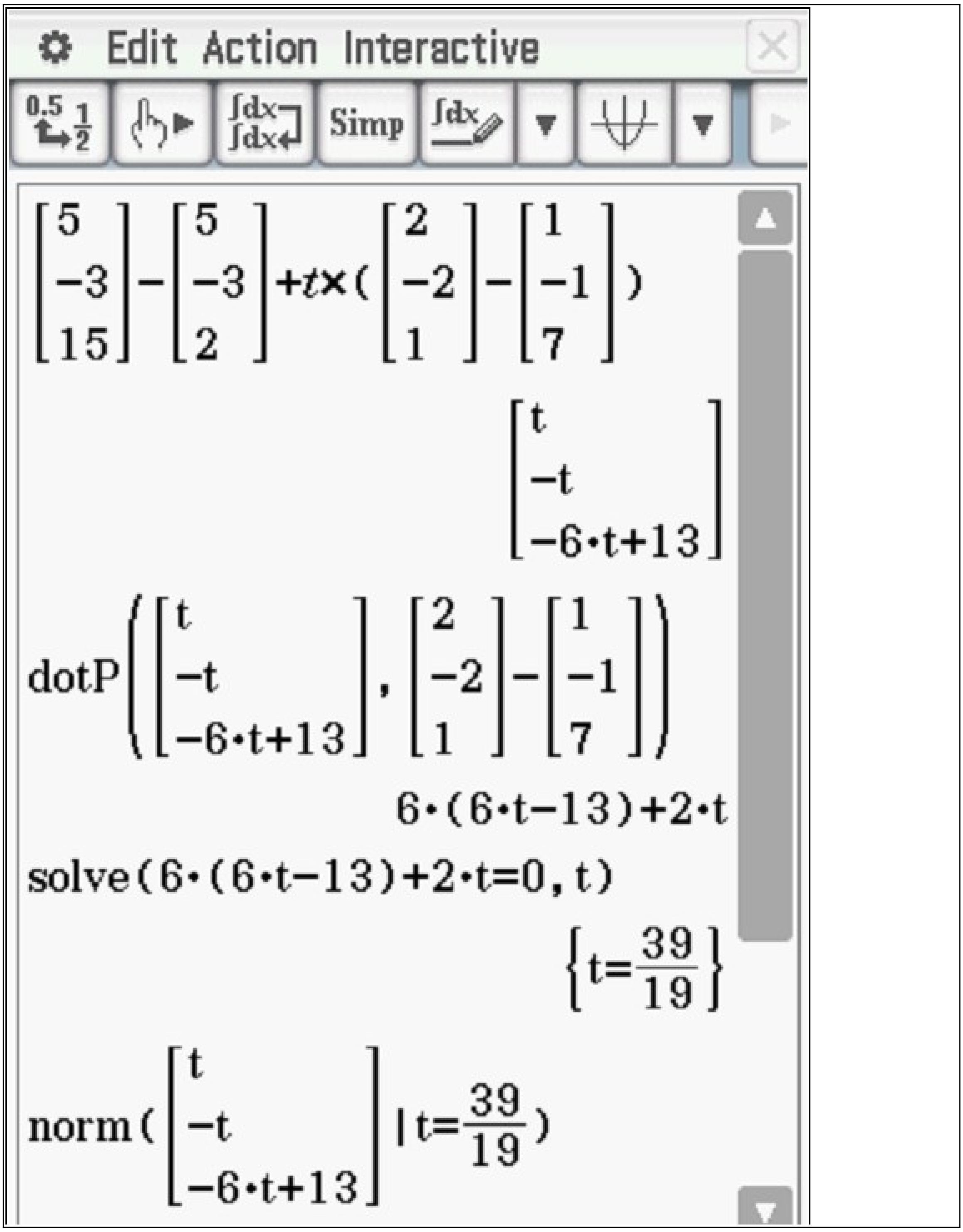
$$r_B = \begin{pmatrix} 5 \\ -3 \\ 15 \end{pmatrix} km \quad , v_B = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} km/h$$

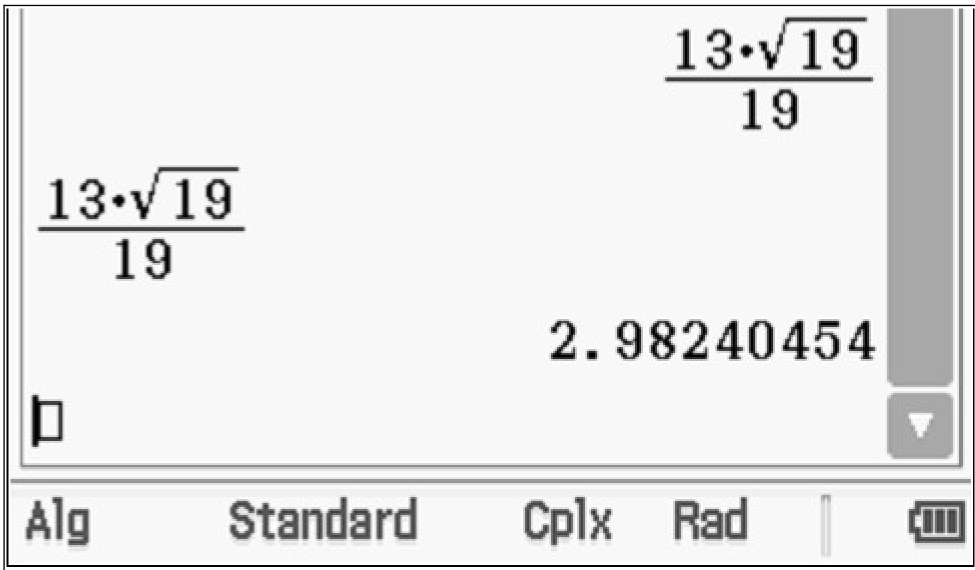
Both rockets leave a smoke trail that stays in the air for at least 6 hours.

a) Determine the distance of the closest approach between the rockets using scalar dot product

(3 marks)

 $d = AB + t\ {}_BV_A$ $d \cdot {}_BV_A = 0$

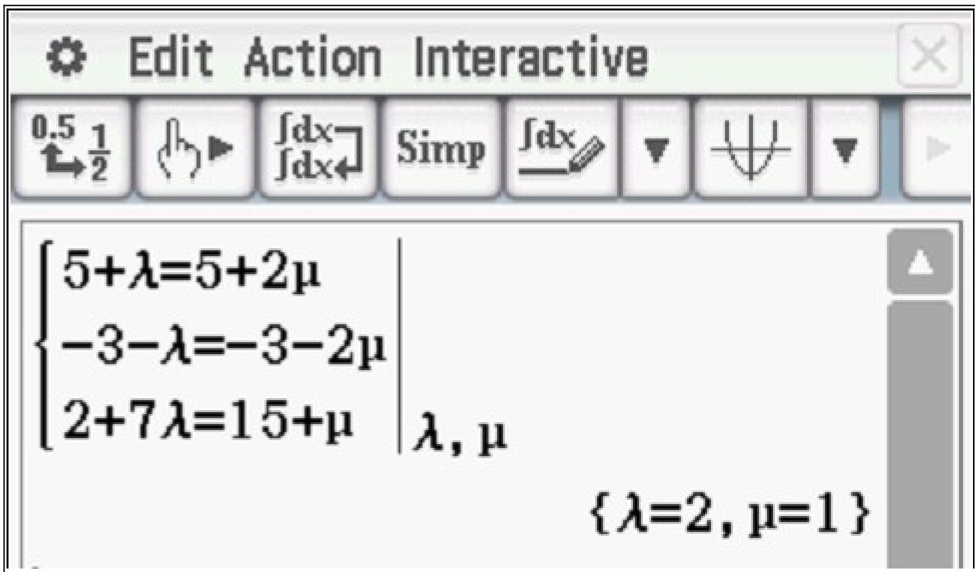

Edit Action Interactive
Simp
$\begin{bmatrix}5\\-3\\15\end{bmatrix}-\begin{bmatrix}5\\-3\\2\end{bmatrix}+t\times\left(\begin{bmatrix}2\\-2\\1\end{bmatrix}-\begin{bmatrix}1\\-1\\7\end{bmatrix}\right)$
$\begin{bmatrix}t\\-t\\-6\cdot t+13\end{bmatrix}$
$\text{dotP}\left(\begin{bmatrix}t\\-t\\-6\cdot t+13\end{bmatrix},\begin{bmatrix}2\\-2\\1\end{bmatrix}-\begin{bmatrix}1\\-1\\7\end{bmatrix}\right)$
$6\cdot(6\cdot t-13)+2\cdot t$
$\text{solve}(6\cdot(6\cdot t-13)+2\cdot t=0,t)$
$\left\{t=\frac{39}{19}\right\}$
$\text{norm}\left(\begin{bmatrix}t\\-t\\-6\cdot t+13\end{bmatrix}\middle|\,t=\frac{39}{19}\right)$



Specific behaviours
P determines expression for separation vector P uses scalar dot product P states approx. distance, no need for units

b) Determine the exact point in space, if any, where the smoke trails overlap at some time in the first 6 hours. (3 marks)

c
$r_A = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$ $r_B = \begin{pmatrix} 5 \\ -3 \\ 15 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ $r_A = r_B$



Smoke trails meet at (7,-5,16)km

Specific behaviours
P uses vector equation of lines P uses two different parameter variables P states exact point in space, no need for units (Max 1 mark if only one parameter used)

Working out space

Working out space

Working out space

End of test