MATHEMATICS METHODS UNITS 3 & 4

Semester Two

2019

SOLUTIONS

Calculator-free Solutions

1. (a) $-\left[3x\right]_0^2 + 4a$ ✓

$= 4a - 6$ ✓

(b) $2xe^{\ln x} = 2x^2$ ✓

$\therefore \int_0^2 2x^2 \, dx = \frac{16}{3}$ ✓ [4]

2. (a) $\frac{\cos x}{\sin(x) + 1} = 0$ (where $\sin(x) \neq -1$ therefore $x \neq -\frac{\pi}{2}$) ✓

$\cos x = 0$

$x = \frac{\pi}{2}$ ✓

Stationary point at ($\frac{\pi}{2}$, ln 2) ✓

(b) $f''(x) = \frac{(\sin(x) + 1)(-\sin x) - (\cos x)(\cos x)}{(\sin(x) + 1)^2}$ ✓

$= \frac{-\cos^2(x) - \sin^2(x) - \sin(x)}{(\sin(x) + 1)^2} \Big|_{x = \frac{\pi}{2}}$ ✓

$= -\frac{1}{2}$ therefore maximum turning point. ✓ [6]

3. $h'(x) = 3ax^2 + 2bx + c$ and $h''(x) = 6ax + 2b$ ✓

$6ax + 2b = 0 \therefore x = -\frac{2b}{6a}$ ✓

$x = m \therefore m = -\frac{b}{3a}$ ✓ [3]

4. (a) (i) $(\ln x)^2 + 2\ln x - 3 = 0$

Let $\ln x = m$

$(m + 3)(m - 1) = 0$ ✓

$\ln x = -3$ or $\ln x = 1$ ✓

$x = \frac{1}{e^3}$ or $x = e$ ✓

(ii) $2^5 = 3x - 4$ ✓

$36 = 3x$

$x = 12$ ✓

(b) $g(x) = \ln 5 - 2\ln x$ ✓

$g'(x) = -\frac{2}{x}$ therefore $m = -\frac{2}{e}$ at $(e, \ln 5 - 2)$ ✓

$\ln 5 - 2 = -\frac{2}{e}(e) + c$

$y = -\frac{2}{e}x + \ln 5$ ✓ [8]

5. (a) $\frac{1-a}{1-b}$ ✓✓

(b) (i) False ✓
In a very large number of samples, 95% of those confidence intervals would contain *p.* The endpoints of the confidence interval refer to the confidence and not the probability. *p* is not a random variable and is either in the CI or not. ✓

(ii) False ✓
The confidence interval is about the population proportion and not about the individual Australian households ✓

(iii) True ✓

(iv) True ✓ [8]
(the centre of the interval contains the sample proportion)

6. (a) $2 - 4\cos t = 0 \therefore \cos t = \frac{1}{2}$

$t = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ ✓✓

(b) $a(t) = 4\sin t = 0$

$t = 0$ or π or 2π ✓

$a'(0) > 0$ Min $\quad a'(\pi) < 0$ Max $\quad a'(2\pi) > 0$ Min ✓

$v(\pi) = 2 - 4(-1) = 6$

Max velocity is 6m/s ✓

(c) $-\int_0^{\frac{\pi}{3}} 2 - 4\cos t \; dt + \int_{\frac{\pi}{3}}^{\pi} 2 - 4\cos t \; dt$ ✓

$= \left[4\sin t - 2t\right]_0^{\frac{\pi}{3}} + \left[4\sin t - 2t\right]_{\frac{\pi}{3}}^{\pi}$ ✓

$= -\frac{2\pi}{3} + 2\sqrt{3} + \frac{4\pi}{3} + 2\sqrt{3}$ ✓

$= \frac{2\pi}{3} + 4\sqrt{3}$ m. ✓ [9]

7. (a) $\int_{-3}^{-1} f(x)dx = 0$ and $\int_{2}^{3} f(x)dx = 0$

$\int_{-1}^{2} f(x)dx = 3$ ✓

$\therefore \int_{-p}^{-3} f(x)dx + \int_{3}^{p} f(x)dx = -3$ ✓

$1 \times 1.5 + 1 \times 1.5 = 3 \therefore p = 3 + 1.5$

$p = 4.5$ ✓

(b) $\frac{1}{3}\int_0^1 \frac{3x^2}{x^3 + 1} dx = \frac{1}{3}\left[\ln\left(x^3 + 1\right)\right]_0^1$ ✓

$= \frac{1}{3}(\ln 2 - \ln 1)$ ✓

$= \dfrac{1}{3}\ln 2$ ✓ [6]

8. (a) $\log_m ba^2 = \log_m 5$ ✓

$b = \dfrac{5}{a^2}$ ✓

(b) $\log_3 9 + \log_3 5 + 2\log_7 7 = 2 + \log_3 5 + 2$ ✓

$= 4 + \log_3 5$ ✓ [4]

9. $4x^3 - 4x^2 + 3x = 2x$

$x(4x^2 - 4x + 1) = 0$

$x(2x - 1)^2 = 0$

$x = 0 \text{ or } \dfrac{1}{2}$ ✓

$$\int_0^{\frac{1}{2}} 4x^3 - 4x^2 + x \ dx = \left[x^4 - \frac{4x^3}{3} + \frac{x^2}{2} \right]_0^{\frac{1}{2}}$$ ✓

$= \dfrac{1}{16} - \dfrac{1}{6} + \dfrac{1}{8} = \dfrac{3 - 8 + 6}{48}$ ✓

$= \dfrac{1}{48}$ units^2 ✓ [4]

Calculator-assumed Solutions

10. (a) $\hat{p} - z\sqrt{\left(\frac{\hat{p}\left(1-\hat{p}\right)}{500}\right)} = 0.2278$ and $\hat{p} + z\sqrt{\left(\frac{\hat{p}\left(1-\hat{p}\right)}{500}\right)} = 0.2922$ ✓

$\hat{p} = 0.26$ ✓

0.26 of 500 = 130. ✓

(b) $z = 1.641$ Therefore 90 % confidence level. ✓✓ [5]

11. (a) (i) $n = 20$ $p = 0.63$
P (X ≥ 14) = 0.3453 ✓

(ii) P (X = 20 | X ≥ 14) = $\frac{0.000097}{0.3453} = 0.00028$ ✓✓

(b) P(X≥1) ≥0.95

$1 - P(X = 0) \geq 0.95$

$P(X = 0) \leq 0.05$ ✓

$0.37^{n} \leq 0.05$ ✓

$n = 3.01$
4 people would need to be selected ✓

(c) E(X)= np
= 300 x 0.63
= 189 ✓

$\sigma = \sqrt{npq} = \sqrt{300 \times 0.63 \times 0.37} = 8.36$ ✓

(d) $0.05 = 1.645\sqrt{\frac{0.63 \cdot (1 - 0.63)}{n}}$ ✓

n = 253 credit card holders ✓ [10]

12. $\int_{5}^{10} 10 + 10e^{-x} - (5 - 0.02x)\ dx$ ✓✓

= 25.81693
The profit earned on sales between 500 and 1000 litres is $25.82. ✓ [3]

13. (a) (i) $\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right) = \frac{27}{256}$ or 0.1055 ✓

(ii) $P(X = 4) + P(X = 5) + P(X = 6)$ ✓

= 0.1055 + 0.079101 + 0.059326

= 0.243927 ✓

(b) (i) Discrete, independent probability ✓

two outcomes: Success or fail ✓ ✓

(ii) E(X) = 0.25 ✓

Standard deviation = $\frac{\sqrt{3}}{4}$ = 0.43301 ✓ [7]

14. (a) $\mu = \frac{6.675 \cdot 10^5}{75}$ = 8900 litres ✓

(b) Variance = $E(X^2) - (E(X))^2 = \frac{5.978 \cdot 10^9}{75} - 8900^2$

= 496 666.67 ✓

Standard deviation = 704.75 ≈705 litres ✓

(c) P (X ≤ 8550 | 7000 < X < 10 000) = $\frac{0.306268}{0.937135}$ = 0.3268 ✓✓

(d) $-2.6 = \frac{x - 8900}{705}$ ✓

x = 7067 litres

Garage must sell 7068 litres to be profitable ✓ [7]

15. Underestimate: 1 x (0 + 10 + 15 + 11 + 4) = 40 ✓

Over estimate: 1 x (10 + 15 + 16 + 16 + 11) = 68 ✓

Average = 54

Distance is therefore approximately 54 m. ✓ [3]

16. (a) Self-selective sample and therefore not representative of the population. ✓

Only radio listeners were part of the survey, therefore not representative. ✓

(b) Logical answer for example:

a randomly selected sample from the general public is used ✓

(c) 0.03 ✓

(d) $\hat{p} = 0.15$ Therefore point estimate = 0.15 or $\frac{45}{300}$ ✓

(e)

$0.03 = z\sqrt{\frac{0.15(1-0.15)}{300}}$ ✓

$z = 1.45521$ ✓

$P(-1.45521 < z < 1.45521) = 0.85439$

85.4% confidence level ✓ [8]

17. (a) At time t, the distance from B to S is $100 - 50t$. ✓

At time t, the distance from A to S is $80t$. ✓

Cosine rule $$r^2 = (100 - 50t)^2 + (80t)^2 - 2(100 - 50t)(80t)\cos\frac{\pi}{3}$$ ✓

Therefore $r^2 = 12900t^2 - 18000t + 10000$

(b) $\dfrac{d(r^2)}{dt} = 25800t - 18000 = 0$ or $\dfrac{dr}{dt} = \dfrac{5(258t - 180)}{\sqrt{129t^2 - 180t + 100}} = 0$ ✓

$t = \dfrac{30}{43} \approx 0.698$ ✓

$r^2(0.698) = 3720.932 \therefore r = 60.999 \approx 61\text{km}$ ✓

$r''(0.698) > 0 \therefore$ minimum ✓

The minimum distance is 61 km. [7]

18. (a) $p(x) = \dfrac{1}{15}$ for $25 \le x \le 40$ and 0 otherwise ✓

(b) 32.5 min ✓

(c) $P(X < 30) = \dfrac{1}{3}$ ✓

(d) (i) $P(X > 30) = \dfrac{2}{3}$ ✓

(ii) $\left(\dfrac{2}{3}\right)^3 \times 3 = \dfrac{8}{9}$ ✓

(e) E(X) = 32.5 Var(X)= 18.75 ✓

$a^2(18.75) = 12.25$ ✓

$a(32.5) + b = 26$ ✓

$a = -0.808$ $b = 52.27$ or $a = 0.808$ $b = -0.27$ ✓ [9]

19. (a) $f(t) = \frac{1}{20} e^{-\frac{1}{20}t}$ for t > 0 ✓✓

(b) F(20) = 0.6321 ✓

(c) $P(t \le 40 | t \ge 25) = \frac{P(25 \le t \le 40)}{P(t \ge 25)}$

$= \frac{0.151169}{0.2865047} = 0.5276$ ✓✓

or

$P(t < 15) = 1 - e^{-0.75} = 0.5276$

(d) E(t) = $\int_0^{\infty} t \times f(t) \text{ dt} = 20$

20 months is the mean time (or $k = 20 = \mu$) ✓

(e) $-e^{-\frac{q}{20}} + 1 = 0.096$

∴ q = 2.02 therefore just over 2 months ✓ [7]

20. (a) $1 = \int_0^x \frac{1}{8} t^3 e^{-1} \text{ dt.}$ ✓

x = 3.0539 ✓

(b) $\frac{dP}{dx} = \frac{1}{8} x^3 e^{-1} \mid x = 3$ ✓

=1.2416 ✓

(c) $P(x) = \frac{x^4 e^{-1}}{32} + c$

P(3) - P(1) = 0.93119 – 0.011496 = 0.9197

or $\int_1^3 \frac{1}{8} x^3 e^{-1} \, dx = 0.9197$ ✓✓

(d) $\delta x = \frac{1}{365}$ ✓

$\delta P = \frac{dP}{dx} \delta x$

$= 1.2416 \times \frac{1}{365} \approx 0.0034$ ✓✓ [9]

21. (a) 0.02275 x 3000 = 68.25 $\therefore$ 68 people ✓
(b) (i) \$75 538.04 ✓
✓✓
(ii) B ~ (3, 0.08) $\therefore$ P(X = 1) = 0.2031
(c) New mean = \$66 875 ✓
Standard deviation = \$7687.50 ✓
(d) $15 \times \dfrac{104}{366} = 4.26$ ✓
Therefore 4 interns ✓ [8]

22. (a) 384 carp ✓
(b) $385 - e^{0.04t} = 0$
t = 148.8 months ✓
In the 149th month
(c) $B(t) = 10e^{0.02t}$ ✓
$385 - e^{0.04t} = 10e^{0.02t}$ ✓
t = 136.2 months after the intro of bass (the 137th month) ✓ [5]

23. (a) $np = 6$ Therefore the distribution cannot be determined. ✓
(b) (i) Normal distribution ✓
$\hat{p} =$ N(0.06, 0.013711) ✓✓
(ii) 0.9999 ✓ [5]

24. (a) $f(x) = \ln(x + 1) + 2$ ✓✓
(b) (i) Whisper: I = 10^{13} W/m^2
Conversation: I = 10^{18} W/m^2 ✓
The intensity of the conversation is 10^5 times ✓
more than the whisper.
(ii) The loudness is an additional 3.0103 dB (10log 2) ✓ [5]