SEMESTER TWO

MATHEMATICS SPECIALIST

REVISION 3

UNIT 3-4

2016

SOLUTIONS

Section One

1.   (3 marks)
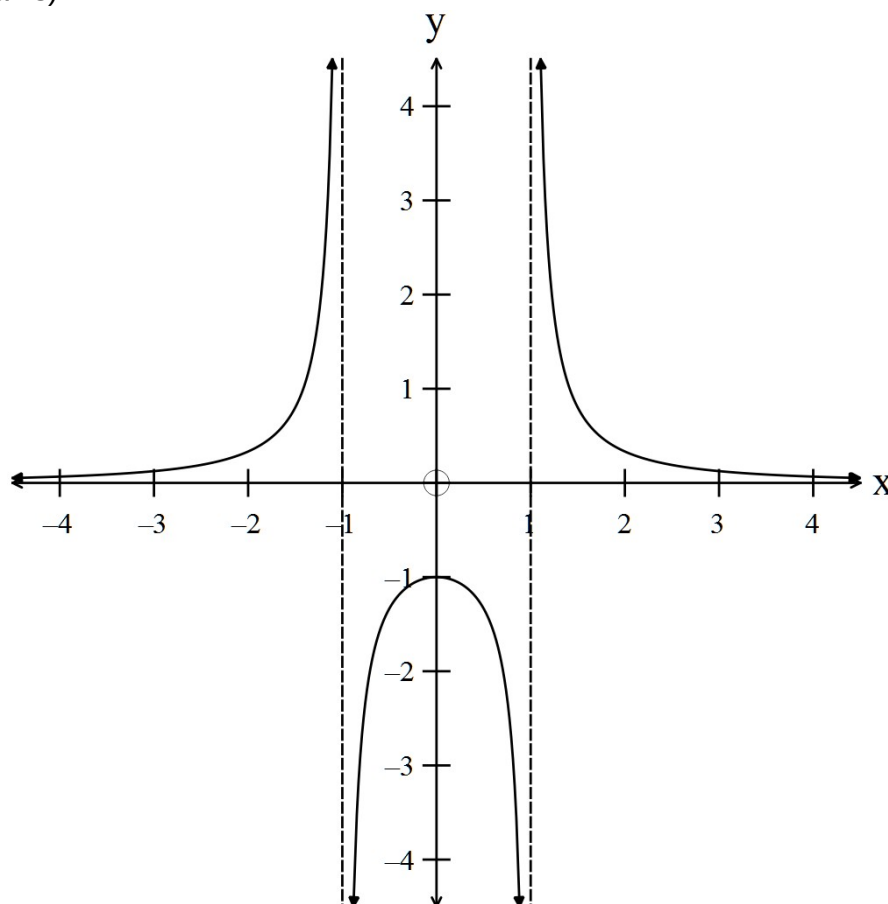
2.   (5 marks)

(a) $x^4 + 5x^2 + 4 = 0.$

$$\left(x^2 + 1\right)\left(x^2 + 4\right) = 0$$

$$x^2 = -1 \quad or \quad x^2 = -4$$

$$x = \pm i \quad or \quad x = \pm 2i$$

(b) $z = 2cis\left(\dfrac{\pi}{4} + n\dfrac{\pi}{2}\right)$

$$z^4 = 16\left(cis\left(\dfrac{\pi}{4} + n\dfrac{\pi}{2}\right)\right)^4$$

$$z^4 = 16\,cis\left(\pi + n2\pi\right)$$

$$= 16\left(cos\left(\pi\right) + i\,sin\left(\pi\right)\right)$$

$$z^4 = -16$$

3. (7 marks)

(a) $\dfrac{2x+3y}{x^2-y^2}=4$

$2x+3y=4x^2-4y^2 \quad for \quad x \neq \pm y$

$2+3\dfrac{dy}{dx}=8x-8y\dfrac{dy}{dx} \qquad$ ✔✔

$\dfrac{dy}{dx}(3+8y)=8x-2$

$\dfrac{dy}{dx}=\dfrac{8x-2}{3+8y}$

(b) $If \quad x=0, \quad \dfrac{3y}{-y^2}=4$

$4y^2+3y=0$

$y(4y+3)=0$

$y=0 \ \ or \ \ y=-\dfrac{3}{4}$

but $x \neq y$, so $y=-\dfrac{3}{4}$ only

$\dfrac{dy}{dx}=\dfrac{8x-2}{3+8y}$

At $\left(0,-\dfrac{3}{4}\right) \quad \dfrac{dy}{dx}=\dfrac{-2}{3+8\left(-\dfrac{3}{4}\right)}$

$\qquad\qquad\qquad \dfrac{dy}{dx}=\dfrac{2}{3}$

4   (9 marks)

(a) $\int \dfrac{dx}{\cos^2(3x)} = \int \sec^2(3x)dx = \dfrac{\tan(3x)}{3} + c$

✓✓

(b) $\int \sin^5(x)\cos^3(x)dx$

$= \int \sin^5(x)\left(1 - \sin^2(x)\right)\cos(x)dx$

$= \int \left(\sin^5(x) - \sin^7(x)\right)\cos(x)dx$ put $u = \sin(x)$

$\dfrac{du}{dx} = \cos(x)$

$du = \cos(x)dx$

$\equiv \int \left(u^5 - u^7\right)du$

$= \dfrac{u^6}{6} - \dfrac{u^8}{8} + c$

$= \dfrac{\sin^6(x)}{6} - \dfrac{\sin^8(x)}{8} + c$

(c) $\int_1^2 \left(\dfrac{1}{1+x} - e^x\right)dx = \left[\ln(1+x) - e^x\right]_1^2$

$= \left(\ln(3) - e^2\right) - \left(\ln(2) - e^1\right)$

$= \ln\left(\dfrac{3}{2}\right) - e + e$

5.   (5 marks)

(a) $z = 2 - 3i$ and $z = 2 + 3i$

$\left(z - (2 - 3i)\right)\left(z - (2 + 3i)\right) = 0$

$z^2 - z\left(2 - \cancel{3i} + 2 + \cancel{3i}\right) + (2 - 3i)(2 + 3i) = 0$

$z^2 - 4z + 4 - 9i^2 = 0$

$z^2 - 4z + 13 = 0$

(b) $\left\{z : \ -\dfrac{\pi}{4} < arg(z) < \dfrac{\pi}{4} \ \cap \ r \leq 4\right\}$ ✓✓✓

6.   (10 marks)

(a) $r(t) = (10\cos(2t))\boldsymbol{i} \; + \; 6(\sin(t))\boldsymbol{j}$

$x = 10\cos(2t)$ and $y = 6\sin(t)$

$\cos(2t) = 1 - 2\sin^2(t)$

$\dfrac{x}{10} = 1 - 2\left(\dfrac{y}{6}\right)^2$

$\therefore x = 10\left(1 - \dfrac{y^2}{18}\right)$

which is parabolic

(b)   At end points when $y = \pm 6$.

$y = 6\sin(t)$

$6\sin(t) = \pm 6$ *when* $t = \pm\dfrac{\pi}{2}$ ✓✓

(c) $r(t) = (10\cos(2t))\boldsymbol{i} \; + \; (6\sin(t))\boldsymbol{j}$

$v(t) = (-20\sin(2t))\boldsymbol{i} \; + \; (6\cos(t))\boldsymbol{j}$

$a(t) = (-40\cos(2t))\boldsymbol{i} \; + \; (-6\sin(t))\boldsymbol{j}$

If $r(t) = -a(t)$

then $\begin{pmatrix} 10\cos(2t) \\ 6\sin(t) \end{pmatrix} = -\begin{pmatrix} -40\cos(2t) \\ -6\sin(t) \end{pmatrix}$

i.e. $\begin{pmatrix} 10\cos(2t) \\ 6\sin(t) \end{pmatrix} = \begin{pmatrix} 40\cos(2t) \\ 6\sin(t) \end{pmatrix}$

i.e. $\cos(2t) = 0$

$2t = \pm\dfrac{\pi}{2} + n\pi$

$t = \pm\dfrac{\pi}{4} + \dfrac{n\pi}{2}$

i.e. $t = \pm\dfrac{\pi}{4}, \; \pm\dfrac{3\pi}{4}, \; \pm\dfrac{5\pi}{4}$ .....

Where? $r(t) = (10\cos(2t))\boldsymbol{i} \; + \; (6\sin(t))\boldsymbol{j}$

$\left(0, 3\sqrt{2}\right), \left(0, -3\sqrt{2}\right)$

7.   (5 marks)

Since one of the direction vectors of the plane is the same as the direction vetor
of the line, the line is either parallel ti the plane or contained IN the plane.   ✓
Need to determine if the point P(2, 0, 1) belongs to the plane.

Plane:

$$r = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$$

$x = 1 - s - 3t,$ $y = 1 + s + 2t,$ $z = 2 - s$

$If\ z = 1$ $1 = 2 - s$

                                                $s = 1$

$\Rightarrow x = -3t$ and $y = 2 + 2t$

If $x = 2$, then $t = -\dfrac{2}{3} \Rightarrow y = 2 + 2\left(-\dfrac{2}{3}\right) = 2 - \dfrac{4}{3}$

$$y = \dfrac{2}{3}$$

But $y = 0$

Therefore the point does NOT belong to the plane.
Therefore the line is parallel to the plane.

8.   (3 marks)

$$\frac{\left(cis\left(\dfrac{\pi}{6}\right)\right)^{-6} cis\left(\dfrac{3\pi}{4}\right)}{(1-i)^3 (1+i)^2}$$

$$= \frac{cis\left(-\pi + \dfrac{3\pi}{4}\right)}{\left(1 - i^2\right)^2 (1 - i)}$$

$$= \frac{cis\left(-\dfrac{\pi}{4}\right)}{4(1 - i)} \times \frac{(1 + i)}{(1 + i)}$$

$$= \frac{cis\left(-\dfrac{\pi}{4}\right) \times \sqrt{2} cis\left(\dfrac{\pi}{4}\right)}{4(2)}$$

$$= \frac{\sqrt{2} cis(0)}{8}$$

$$= \frac{\sqrt{2}\left(cos(0) + i\ sin(0)\right)}{8}$$

$$= \frac{\sqrt{2}}{8}$$

9.   (6 marks)

(a)

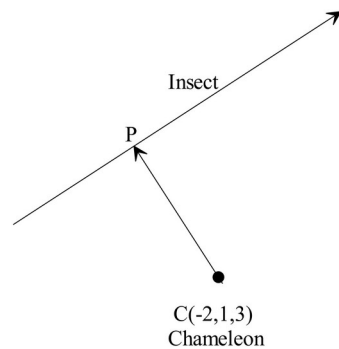| x	-2	-1	0	1	2
y	1	1	1	1	1
$\dfrac{dy}{dx}$	**4**	**2**	**0**	**-2**	**-4**

✓✓ -1/error

(b)
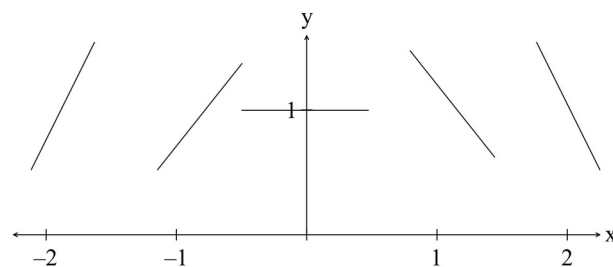
✓✓ -1/error

(c) $x^2 + y^2 + (z-3)^2 = 9$ is a sphere with centre (0, 0, 3) and radius 3.
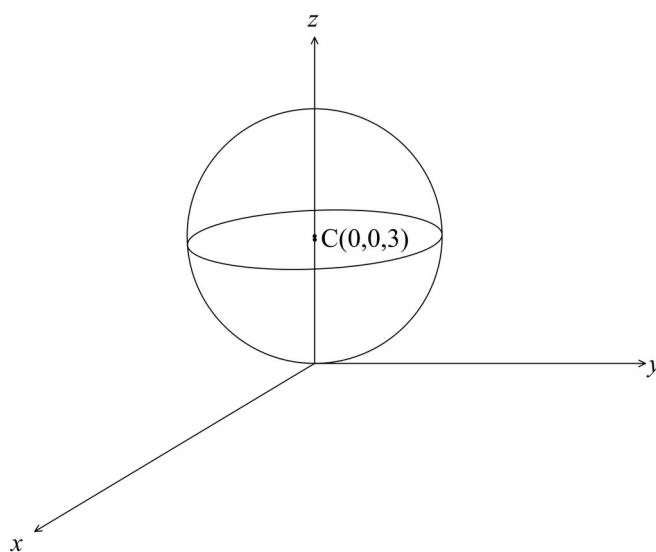
END OF SECTION ONE

Section Two

10. (5 marks)

(a) Path of insect: $r(t) = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} t = \begin{pmatrix} 1 \\ -2+t \\ 2+2t \end{pmatrix}$

P

Insect

C(-2,1,3)
Chameleon

$PC = \begin{pmatrix} -3 \\ 3-t \\ 1-2t \end{pmatrix}$

At closest point $PC \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$

$\begin{pmatrix} -3 \\ 3-t \\ 1-2t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$

$0 + 3 - t + 2 - 4t = 0$

$5t = 5$

$t = 01$

$r(1) = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times 1 = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$ ....closest point

Distance between closest point and chameleon at C(-2,1,3) is

$d = \sqrt{(-2)^2 + 2^2 + (-1)^2}$

$d = 3\,cm$

The insect will be dinner!!!

(b) The set of points represent a sphere of centre (1, -2, 0) and radius 3. ✓

11.   (8 marks)

Given $z = cos(x) + i\,sin(x)$ and $\dfrac{1}{z} = cos(x) - i\,sin(x)$

(a) (i) $z + \dfrac{1}{z} = 2\,cos(x)$

$z - \dfrac{1}{z} = 2i\,sin(x)$ ✓

(ii) $z^n = (cos(x) + i\,sin(x))^n = cos(nx) + i\,sin(nx)$ ✓

$\dfrac{1}{z^n} = \dfrac{1}{(cos(x) + i\,sin(x))^n}$

$= (cos(x) + i\,sin(x))^{-n}$

$= (cos(-nx) + i\,sin(-nx))$

but $cos(-\theta) = cos(\theta)$ and $sin(-\theta) = -sin(\theta)$

$\dfrac{1}{z^n} = (cos(nx) - i\,sin(nx))$

$z^n + \dfrac{1}{z^n} = 2\,cos(nx)$

(iii) $z^n - \dfrac{1}{z^n} = 2i\,sin(nx)$ ✓

(b) Show that

$cos^4(\theta) = \dfrac{1}{8}cos(4\theta) + \dfrac{1}{2}cos(2\theta) + \dfrac{3}{8}$

$cos^4(\theta) = \left(\dfrac{1}{2}\left(z + \dfrac{1}{z} \right) \right)^4$

$= \dfrac{1}{16}\left(z^4 + \dfrac{4z^3}{z} + \dfrac{6z^2}{z^2} + \dfrac{4z}{z^3} + \dfrac{1}{z^4} \right)$

$= \dfrac{1}{16}\left(z^4 + \dfrac{1}{z^4} + 4z^2 + \dfrac{4}{z^2} + 6 \right)$

$= \dfrac{1}{16}\left(\left(z^4 + \dfrac{1}{z^4} \right) + 4\left(z^2 + \dfrac{1}{z^2} \right) + 6 \right)$

$= \dfrac{1}{16}\left(2\,cos(4\theta) + 4(2\,cos(2\theta)) + 6 \right)$

$cos^4(\theta) = \dfrac{1}{8}cos(4\theta) + \dfrac{1}{2}cos(2\theta) + \dfrac{3}{8}$

12. **(12 marks)**

(a)   (i) $f(x) = \dfrac{1}{x-2} + 1 \quad for \ \ x \neq 2, y \neq 1$ ✓✓

(ii) To obtain the inverse swap x and y

$$x = \frac{1}{y-2} + 1$$

$$x - 1 = \frac{1}{y-2}$$
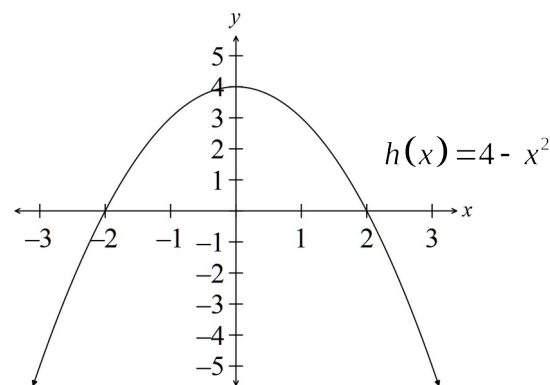
$$y - 2 = \frac{1}{x-1}$$

$$y = \frac{1}{x-1} + 2$$

$$\therefore f^{-1}(x) = \frac{1}{x-1} + 2 \quad for \ \ x \neq 1, \ y \neq 2$$

   ✓   ✓   ✓

(b)   (i) $f(h(x)) = f(4 - x^2) = ln(4 - x^2)$ ✓

$h(x) = 4 - x^2$

The domain is $-2 < x < 2$ and the range is $y \leq ln \, 4$

   ✓   ✓

(ii) $j(x) = x^2 - 2$

$g(j(x)) = 5 - x^2$

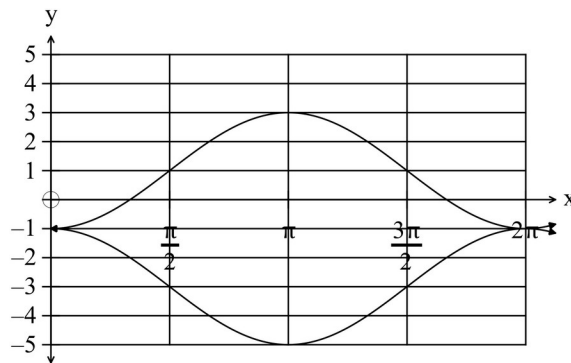$$\qquad = -(x^2 - 2) + 3$$

$\therefore \ g(x) = -x + 3$

(iii) $g(f(x)) = g(ln(x)) = 3 - ln(x)$

$ln(x)$ is a one to one function so $g(f(x))$ is also a one to one function.

13.   (6 marks)

(a) $x = 0, \; x = 2\pi$ ✓✓✓

(b)   Area = $\int_0^{2\pi} \left| 1 - 2\sin\left(x + \dfrac{\pi}{2}\right) - \left(2\cos(x) - 3\right) \right| dx = 25.13 \text{ units}^2$ ✓✓✓

14.   (7 marks)

(a) $\mathbf{v}_0 = \begin{pmatrix} 30\cos 30^o \\ 30\sin 30^o \end{pmatrix} = \begin{pmatrix} 15\sqrt{3} \\ 15 \end{pmatrix}$

$\mathbf{a} = -9.8\,\mathbf{j}$

$\mathbf{v} = \int -9.8\,\mathbf{j}\; dt$

$\mathbf{v} = -9.8t\,\mathbf{j} + \mathbf{c}_1$

$At\; t = 0 \quad \begin{pmatrix} 15\sqrt{3} \\ 15 \end{pmatrix} = \mathbf{c}_1$

$\mathbf{v} = 15\sqrt{3}\,\mathbf{i} + (15 - 9.8t)\,\mathbf{j}$

$\mathbf{x} = \int 15\sqrt{3}\,\mathbf{i} + (15 - 9.8t)\,\mathbf{j} \;\; dt$

$\mathbf{x} = 15\sqrt{3}t\,\mathbf{i} + (15t - 4.9t^2)\,\mathbf{j} + \mathbf{c}_2$

$\mathbf{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{c}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore \; \mathbf{x} = 15\sqrt{3}t\,\mathbf{i} + (15t - 4.9t^2)\,\mathbf{j}$

$At\; y = 2, x = ?$

$y = 2 = 15t - 4.9t^2 \; \Rightarrow \; t = 2.92151503 \;\; or \;\; t = 0.1397 \text{ (still going up)}$

$At\; t = 2.92151503, \; x = 15\sqrt{3}t = 15\sqrt{3} \times 2.92151503 \approx 75m$

Yes, the arrow goes over the back fence.

(b) $At\; y = 0, \; t = 3.06122445$

$At\; t = 3.06122445, \; x = 79.53 \; m$

15. (10 marks)

(a) $V = \int_0^4 \left(\frac{y+2}{1.5} \right) dy - \int_2^4 (y-2) dy$

✓   ✓   ✓   ✓

(b) (i)   The period is 2 cm. ✓ $k = \pi$ ✓

(ii) $V = 20 \int_0^1 \pi \sin^2 (\pi x) dx = 31.415 \, cc$

✓ ✓   ✓   ✓

16. (14 marks)

(a) $x = e^{-2t}$ cm

$v = \dfrac{dx}{dt} = -2e^{-2t}$

$a = \dfrac{dv}{dt} = 4e^{-2t}$

$At \ t = 1 \ \ v = -2e^{-2}$

$a = 4e^{-2}$

(b) (i) $x = -3\sin\left(2t + \dfrac{\pi}{6} \right)$

SHM if $a = -n^2 x$

$v = -6\cos\left(2t + \dfrac{\pi}{6} \right)$

$a = 12\sin\left(2t + \dfrac{\pi}{6} \right)$

$a = -4\left(-3\sin\left(2t + \dfrac{\pi}{6} \right) \right)$

$a = -4x$

$\therefore SHM$

(ii) $v_0 = -6\cos\left(\dfrac{\pi}{6} \right) = -3\sqrt{3}$ $a_0 = 12\sin\left(\dfrac{\pi}{6} \right) = 6$

(iii) $If \ v = 0, \ \cos\left(2t + \dfrac{\pi}{6} \right) = 0 \ \Rightarrow \ \sin\left(2t + \dfrac{\pi}{6} \right) = \pm 1 \ \Rightarrow \ x = \pm 3$ ✓

(c) $A = \pi r^2$

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r \times \frac{1}{r}$$

$$\frac{dA}{dt} = 2\pi \quad m^2/min$$

which is constant

(d) $2xy \frac{dy}{dx} = 1 - x.$

$$\int 2y \, dy = \int \frac{1-x}{x} dx$$

$$y^2 = \int \left(\frac{1}{x} - 1 \right) dx$$

$$y^2 = \ln(x) - x + c$$

$(1,4) \quad 16 = \ln(1) - 1 + c \quad \rightarrow \quad c = 17$

$$y^2 = \ln(x) - x + 17$$

$$y = \pm\sqrt{\ln(x) - x + 17}$$

But, if $x = 1, y = +4$

$$\therefore y = \sqrt{\ln(x) - x + 17}$$

17.   (8 marks)

(a)   Let equation of the plane be $ax + by + cz = d$

$A(12,0,0) \, so \, 12a = d$

$B(0,4,0) \, so \, 4b = d$

$C(0,0,4) \, so \, 4c = d$

$$\therefore \frac{dx}{12} + \frac{dy}{4} + \frac{dz}{4} = d$$

$$\Rightarrow \frac{x}{12} + \frac{y}{4} + \frac{z}{4} = 1$$

$$\therefore x + 3y + 3z = 12$$

(b) $P(6,0,0), \, Q(0,2,0), \, R(0,0,2)$

Likewise $x + 3y + 3z = 6$

The planes are parallel as the coefficients of x, y, and z are all identical.   ✓

NB (a) and (b) can be done using vectors.

(c) $\quad AC \times AB = \begin{pmatrix} -12 \\ 0 \\ 4 \end{pmatrix} \times \begin{pmatrix} -12 \\ 4 \\ 0 \end{pmatrix} = -16\boldsymbol{i} - 48\boldsymbol{j} - 48\boldsymbol{k} \quad$ ✓✓

(d) $\quad |AC \times AB| = |AC||AB| \sin(\theta)$

so $Area_\triangle = \dfrac{1}{2}|AC \times AB|$

$$= \dfrac{1}{2}\sqrt{(-16)^2 + (-48)^2 + (-48)^2} = \dfrac{\sqrt{4864}}{2}$$

$$= \dfrac{16\sqrt{19}}{2}$$

$Area \triangle = 8\sqrt{19} \ units^2$

18. (8 marks)

(a) (i) $\mu_{\bar{x}} = 130$ grams

$\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{15}{\sqrt{6}}$

$\sigma_{\bar{x}} \approx 6.12$

(ii) When a sample is taken, the range is usually smaller as it does not
usually contain any outliers (which are rare scores).
Most apples would be close to the mean weight, and it is unlikely to select
an unusually big or small apple in a small sample, so a sample
of 6 would have a small standard deviation.

$\qquad\qquad\qquad\qquad$ ✓✓

(b) The average time they would wait is is 20 minutes.

$\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{6}{\sqrt{3}}$

$\sigma_{\bar{x}} \approx 3.46$

19.   (7 marks)

(a)   The expected mean is 160 cm.

$sd \approx \sigma$

$\sigma \approx 1.5$ cm

(b)   95% confidence limits $\Rightarrow z = 1.96$

$160 \pm 1.96 \times 1.5 \times \sqrt{20}$

$= 160 \pm 13.15$

$146.85 \leq \mu \leq 173.15$

(c)   To be 90% sure the confidence limits are correct, you can have a smaller ranger as there is a 10% error margin.

To be surer, i.e. a 95% confidence limit then you have a wider range so there is a bigger chance that the mean will be included.   ✓✓

20.   (13 marks)

(a)   (i) $t = 34 \quad N \approx 165$ ✓

(ii) $t = 85 \quad N \approx 459$ ✓

(b) $\lim\limits_{x \to \infty} \left(\dfrac{540}{1 + 12.5e^{-0.0502t}} \right)$

$as\ t \to \infty\ e^{-0.0502t} \to 0$

$= \dfrac{540}{1}$

$= 540$

(c) $N = \dfrac{540}{1 + 12.5e^{-0.0502t}}$

$\dfrac{dN}{dt} = \dfrac{-540 \times 12.5 \times (-0.0502) \times e^{-0.0502t}}{\left(1 + 12.5e^{-0.0502t}\right)^2}$

$\dfrac{dN}{dt} = \dfrac{338.85 \times e^{-0.0502t}}{\left(1 + 12.5e^{-0.0502t}\right)^2}$

$In\ 1990, t = 60,\ \dfrac{dN}{dt} = 6.392,$i.e. about 6 cases per year.

$At\ t = 60, P = 334$ which is more than half way up to the maximum (of 540), so the rate of increase will be decreasing then.

(d) $\quad \dfrac{dN}{dt} = kN\left(1 - \dfrac{N}{K}\right) = kN\left(\dfrac{K - N}{K}\right)$

$\therefore\ kdt = \dfrac{K}{N(K - N)}dN$

$let\ \dfrac{1}{N(K - N)} = \dfrac{a}{N} + \dfrac{b}{(K - N)}$

$\qquad\qquad\qquad = \dfrac{a(K - N) + bN}{N(K - N)}$

$\dfrac{0N + 1}{N(K - N)} = \dfrac{N(-a + b) + aK}{N(K - N)}$

Equating   coefficients $0 = -a + b \quad 1 = aK$

$\qquad\qquad\qquad a = b \qquad b = a = \dfrac{1}{K}$

$\int kdt = \int \dfrac{K}{N(K - N)}dN = \int K\left(\dfrac{a}{N} + \dfrac{b}{(K - N)}\right) = \dfrac{K}{K}\int\left(\dfrac{1}{N} + \dfrac{1}{(K - N)}\right)dN$

$\int kdt = \int\left(\dfrac{1}{N} + \dfrac{1}{(K - N)}\right)dN$

$kt = ln\,N - ln(K - N) + c$

$kt - c = ln\dfrac{N}{K - N}$

$\dfrac{N}{K - N} = e^{kt-c}.............let\ e^{-c} = M$

$N = (K - N)Me^{kt}$

$KMe^{kt} = N(1 + Me^{kt})$

$N = \dfrac{KMe^{kt}}{1 + Me^{kt}} \times \dfrac{1/e^{kt}}{1/e^{kt}}$

$N = \dfrac{KM}{e^{-kt} + M} \times \dfrac{1/M}{1/M}$

$N = \dfrac{K}{1 + \dfrac{1}{M}e^{-kt}}$

END OF SECTION TWO