MATHEMATICS METHODS

MAWA Semester 2 (Unit 3&4) Examination 2019
Calculator-assumed

Marking Key

The release date for this exam and marking scheme is

- **the end of week 1 of term 4, Fri October 18th 2019**

Section Two: Calculator-assumed (103 Marks)

Question 10 (a) (1 mark)

Solution	
$Let\ X \sim N(48.9, 3.8^2)$ $P(X>55) \approx 0.0542$ i.e. 5.42% received this invitation.	
Mathematical behaviours	Marks
• states the correct percentage	1

Question 10 (b) (1 mark)

Solution	
$P(X>k)=0.1$ $k \approx 53.77\,m$	
Mathematical behaviours	Marks
• States the correct length	1

Question 10 (c) (3 marks)

Solution	
$P(X>45)=0.8476$ $Let\ B \sim Binomial(12, 0.8476)$ P(B ≥ 7) ≈ 0.9950	
Mathematical behaviours	Marks
• Determines probability one player can kick longer than 45 m	1
• Associates this question to a Binomial Distribution	1
• Determines the correct probability	1

Question 22 (d) **(3 marks)**

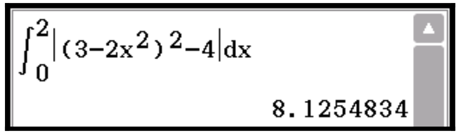
Solution
$\mu = \int_0^{120} \frac{t^2}{7200} dt ¿ 80$ $Var(T) = \int_0^{120} (t-80)^2 \frac{t}{7200} dt ¿ 800$ $Var(2T-1) = 2^2(800) ¿ 3200$

Mathematical behaviours	Marks
• determines the mean	1
• determines the variance	1
• determines the variance for $2T - 1$	1

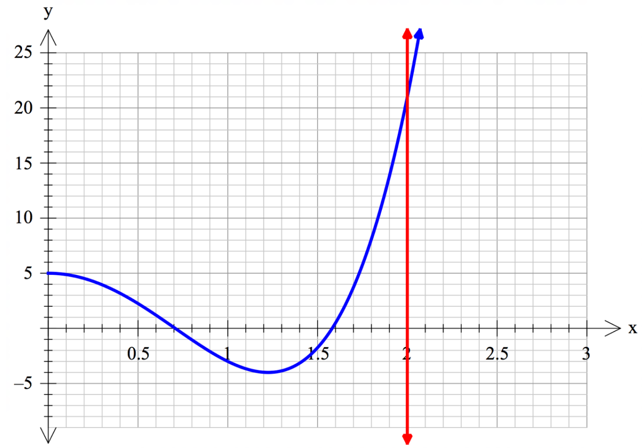
Question 11 (a) **(3 marks)**

Solution

Uing CAS,



Alternatively, sketch graph and proceed without CAS…



The x-intercepts are $\frac{1}{\sqrt{2}}, ¿ \frac{\sqrt{5}}{\sqrt{2}}$ (or 0.707, 1.581)

Let a = $\frac{1}{\sqrt{2}}$, b = $¿ \frac{\sqrt{5}}{\sqrt{2}}$

The area for the enclosed region...

= $\int_0^a (3-2x^2)^2 - 4\,dx$ + ($-\int_a^b (3-2x^2)^2 - 4\,dx$) + $\int_b^2 (3-2x^2)^2 - 4\,dx$

= $\frac{8\sqrt{2}}{5} + \frac{8\sqrt{2}}{5} + \frac{18}{5}$

= 8·125 units^2

Mathematical behaviours	Marks
• states an integral expression for the area with correct limits	2
• states correct area rounded to 3 decimal places	1
Alternatively,	
• shows integration based on three separate areas	1
• uses correct bounds on integration	1
• determines solution, correctly to three decimal places	1

Question 11 (b) (i) **(2 marks)**

Solution	
Area of P = 18 units^2 Area of Q = 7.33 units^2	define f(x)=3x done define $g(x)=\frac{x^2}{2}$ done $\int_0^6 (f(x)-g(x))dx$ 18 $\int_6^8 (f(x)-g(x))dx$ −7.333333333

Mathematical behaviours	Marks
• Area of P	1
• Area of Q	1

Question 11 (b) (ii) **(4 marks)**

Solution

$$Let\, f(x)=ax\, Let\, ax=\frac{x^2}{2}¿ i.e.\, x=2a\, Find\, a\, such\, that\, 2\int_0^{2a} ax-\frac{x^2}{2}dx=\int_{2a}^{8}\frac{x^2}{2}-ax\,dx$$

$$2\left[\frac{ax^2}{2}-\frac{x^3}{6}\right]_0^{2a}=\left[\frac{x^3}{6}-\frac{ax^2}{2}\right]_{2a}^{8} 4a^3-\frac{8a^3}{3}=\frac{256}{3}-32a-\frac{8a^3}{6}+2a^3 a=2.3843$$

Mathematical behaviours	Marks
• finds x value for intersection of functions f and g in terms of a.	1
• determines equation (in terms of integrals) showing region P is half the area of region Q	1
• anti-differentiates both integrals	1
• solves equation to determine the value of a	1

Question 22 (a) **(2 marks)**

Solution

$$1=\int_0^{120}\frac{a}{30}t\,dt\, 1=\left[\left(\frac{a}{30}\right)\frac{t^2}{2}\right]_0^{120} 1=\frac{14400a}{60} a=\frac{1}{240}$$

Mathematical behaviours	Marks
• states the integral equal to one	1
• determine the value of a	1

Question 22 (b) **(1 mark)**

Solution

$$P(0<T<30)=\int_0^{30}\frac{1}{7200}t\,dt ¿ \frac{1}{16}$$

Mathematical behaviours	Marks
• determines the correct probability	1

Question 22 (c) **(3 marks)**

Solution

$$P(60<T<120)=\int_{60}^{120}\frac{1}{7200}t\,dt ¿ \frac{3}{4}$$

$$P(60<T<105)=\int_{60}^{105}\frac{1}{7200}t\,dt ¿ \frac{33}{64}$$

$$P(T<105 \vee T>60)=\frac{\frac{33}{64}}{\frac{3}{4}} ¿ \frac{11}{16}$$

Mathematical behaviours	Marks
• determines the probability of arriving after 11 am	1
• determines the probability of arriving before 11.45 am and after 11 am	1
• determines the conditional probability	1

Question 21 (a) **(4 marks)**

Solution	
2019 Sample = $\frac{46}{225} \approx 0.2044$ Historically p = 0.35 Standard Deviation s = $\sqrt{\frac{(0.35)(0.65)}{225}}$ i.e. ≈ 0.0318 $p-\hat{p} \approx 0.1456$ i.e. Difference in terms of standard deviations $\approx \frac{0.1456}{0.0318} \approx 4.5775$ Given the difference between the long-term proportion and sample proportion exceeds three standard deviations, it is unlikely that this 2019 prediction is correct. Whilst it could occur, the Principal is correct to say that this is extremely unlikely.	
Mathematical behaviours	**Marks**
• determines the sample proportion for 2019.	1
• calculates the standard deviation based on n = 225.	1
• calculates the difference between the two proportions and connects this result to the standard deviation.	1
• Recognises that the Principal was justified.	1

Question 21 (b) (i) **(2 marks)**

Solution	
This method may be biased for the following reasons - only one car park was chosen (of a possible five) and the "drop off" zones were ignored. - sample was small. - the car park sample probably eliminated parents.	
Mathematical behaviours	**Marks**
• indicates a valid reason for bias.	1
• indicates a second valid reason for bias.	1

Question 21 (b) (ii) **(2 marks)**

Solution	
This method may be biased for the following reasons - only interested members are sampled. - not all community members may be on emails. eg one parent of two may be on email.	
Mathematical behaviours	**Marks**
• indicates a valid reason for bias.	1
• indicates a second valid reason for bias.	1

Question 12(a) **(1 mark)**

Solution	
P(up to 600 vehicles pass through in one hour) = $\frac{8}{24}=\frac{1}{3}$	
Mathematical behaviours	**Marks**
• states probability	1

Question 12(b) **(2 marks)**

Solution

y	0	1
$P(Y=y)$	$\frac{14}{24}=\frac{7}{12}$	$\frac{10}{24}=\frac{5}{12}$

Mathematical behaviours	Marks
• completes P(Y=0) correctly	1
• completes P(Y=1) correctly	1

Question 12(c) **(2 marks)**

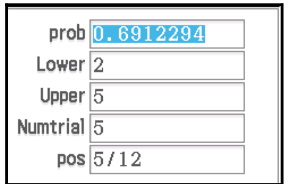
Solution	
$\sigma^2 = \frac{5}{12}\times\frac{7}{12}=\frac{35}{144} \approx 0.2431$ Bernoulli,	
Mathematical behaviours	**Marks**
• identifies the distribution as 'Bernoulli'	1
• states the variance	1

Question 12(d) **(3 marks)**

Solution

Let X be the number of times that Mel faces congestion in one week

$X \sim Bin(5, 0.41670)$ $P(X \geq 2) = P(2 \leq X \leq 5) \approx 0.6912$



Mathematical behaviours	Marks
• indicates a binomial distribution	1
• states both parameters correctly	1
• determines probability	1

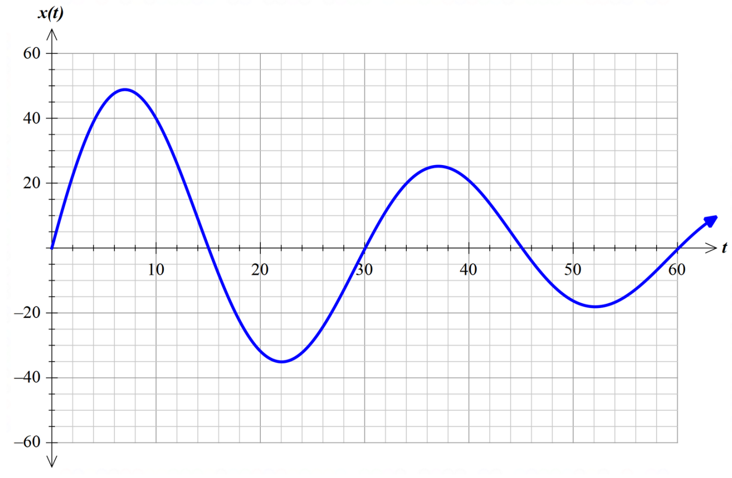
Question 12(e) **(3 marks)**

Solution
P(congestion occurs once in first 3 days) $= {}^3C_1(0.4167)^1(0.5833)^2$ P(congestion occurs on Thursday) $= \frac{7}{12}$ $\therefore {}^3C_1(0.4167)^1(0.5833)^2 \times (0.4167) = 0.1772$

Mathematical behaviours	Marks
• states expression showing that congestion occurs exactly once in the first 1st three days	1
• identifies that congestion occurs on 4th day	1
• calculates probability	1

Question 20 (b) **(2 marks)**

Solution

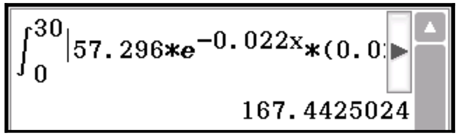


Mathematical behaviours	Marks
• shows decaying oscillatory nature	1
• shows maximum at $t=7$	1

Question 20 (c) **(3 marks)**

Solution

In the first 15 seconds the mass travels $2 \times x(7)\,cm$, i.e. $2 \times 48.84 \cong 97.7$ cm. (*)
The second turning point occurs when $t = 7 + 15 = 22$. (**)
So in the second 15 second period the mass travels $2 \times |x(22)| \cong 2 \times 35.11 \cong 70.2\,cm$. So the total distance travelled is $97.7 + 70.2 = 167.9 \cong 170\,cm$.
Alternatively, could use CAS to find absolute value of the velocity function.



Mathematical behaviours	Marks
• obtains correct distance for first 15 seconds (*)	1
• obtains second turning point (**)	1
• obtains correct answer	1
If uses CAS –	
• states the function to be integrated with correct limits	2
• states correct appropriately rounded answer.	1

Question 20 (a) **(5 marks)**

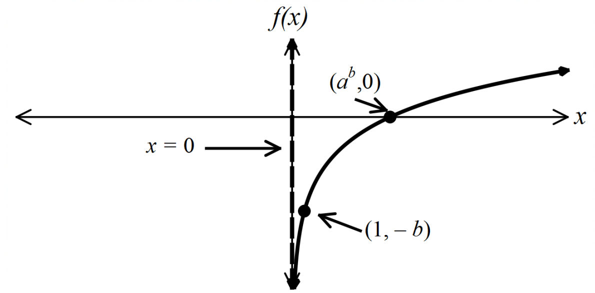
Solution	
Since $x(t)=ae^{-bt}\sin ct=0$ when $\sin ct=0$, the first zero (after $t=0$¿ occurs when $ct=\pi$ So $15c=\pi$, and so $c\cong 0.209$ Since $v(t)=\frac{d}{dt}(x(t))=-abe^{-bt}\sin ct+ace^{-bt}\cos ct$, (*) $v(0)=ac$,and so $a\times 0.209=12$, and $a\cong 57.296$ $v(t)=ae^{-bt}$¿when$t=7$, So $-b\sin 7c+c\cos 7c=0$. i.e. $b=\frac{c}{\tan 7c}\cong\frac{0.209}{\tan(7\times 0.209)}\cong 0.022$	
Mathematical behaviours	**Marks**
• obtains correct value for c • obtains correct formula for $v(t)$ (*) • obtains correct value for a • obtains correct value for b • rounds a, b and c to 3 decimal places	1 1 1 1 1

Question 13(a) **(2 marks)**

Solution	
$f(x)=\log_a x-b$ $f(1)=-b$ x - axis intercept, $f(x)=0\Rightarrow \log_a x-b=0\Rightarrow a^b=x$	
Mathematical behaviours	**Marks**
• states $f(1)=-b$ • states x - axis intercept	1 1

Question 13(b) **(3 marks)**

Solution



Mathematical behaviours	Marks
• locates and identifies x - axis intercept • labels and shows asymptote • correct shape including containing $(1,-b)$	1 1 1

Question 13(c) **(3 mark)**

Solution

$g(x) = f(x-2) = \log_a(x-2) - b$

$g(p) = 0 \Rightarrow 0 = \log_a(p-2) - b$

$ie\ a^b = p - 2$

$ie\ p = a^b + 2$

$g(x)$ is a horizontal translation of $f(x)$, 2 units to the right.

If $g(p) = 0$ then p is the root of $f(x)$ translated 2 units to the right. Hence $p = a^b + 2$

Mathematical behaviours	Marks
• rearrange to determine $g(p)$	1
• solves algebraically for p	1
• describes that p represents the axis intercept (root) for the translated function	1

Question 19 (a) **(1 mark)**

Solution

$\frac{36}{120} \vee 0.3$

Mathematical behaviours	Marks
• determines the proportion	1

Question 19 (b) **(1 mark)**

Solution

$\sigma = \sqrt{\frac{0.3(1-0.3)}{120}}$ ¿ 0.0418

Mathematical behaviours	Marks
• determines the standard deviation	1

Question 19 (c) **(2 marks)**

Solution

Determine the relevant z - score z

$ME = z \times \sigma$

$= -1.43953 \times 0.04183$

≈ -0.06

```
invNormCDf("C",0.85,1,0)
                 -1.439531471
ans*0.0418330
                -0.06021992002
```

Mathematical behaviours	Marks
• determines the z value for 85% confidence level	1
• calculates the margin of error	1

Question 19 (d) **(2 marks)**

Solution

Graph approaches the shape of a binomial distribution.
For large sample sizes it begins to approach the shape of a normal distribution
The distribution is centred on 0.3

Mathematical behaviours	Marks
• uses one of the descriptors above.	1
• uses another one of the descriptors above.	1

Question 18 (a) **(2 marks)**

Solution	
$P(\lvert X_1-\mu_1\rvert \geq \sigma_1)=P(\lvert Z\rvert \geq 1)$ (*) $\cong 0.317$ from a calculator	
Mathematical behaviours	**Marks**
• standardises (*)	1
• obtains correct answer	1

Question 18 (b) **(2 marks)**

Solution	
$\int_0^1 \left(x-\frac{1}{2}\right)^2 dx=\left(\frac{1}{3}\left(x-\frac{1}{2}\right)^3\right)\Big\vert_0^1$ $¿\frac{1}{24}-\left(\frac{-1}{24}\right)=\frac{1}{12}$ $=0.08\dot{3}$ $\int_0^1 (x-0.5)^2 dx \quad \frac{1}{12}$	
Mathematical behaviours	**Marks**
• obtains correct indefinite integral (*)	1
• evaluates correctly	1

Question 18 (c) **(1 mark)**

Solution	
$\sigma_2^2=\int_0^1 (x-\mu_2)^2 dx=\int_0^1 \left(x-\frac{1}{2}\right)^2 dx=\frac{1}{12}$ from 12(b) So $\sigma_2=\sqrt{\frac{1}{12}}=\frac{1}{2\sqrt{3}}$	
Mathematical behaviours	**Marks**
• correct answer	1

Question 18 (d) **(3 marks)**

Solution	
For the Bernoulli random variable X_3, $\mu_3=p=0.5$ and $\sigma_3=\sqrt{p(1-p)}=0.5$ So $P(\lvert X_3-\mu_3\rvert \geq \sigma_3)=P(\lvert X_3-0.5\rvert \geq 0.5)=P(X_3=0)+P(X_3=1)$ (*) $¿0.5+0.5=1$	
Mathematical behaviours	**Marks**
• uses correct values for μ_3 and σ_3	1
• obtains (*)	1
• obtains correct answer	1

Question 14 (a) **(1 mark)**

Solution	
Confidence interval is $(\hat{p}-E, \hat{p}+E)=(0.39, 0.53)$ So $\hat{p}=\frac{0.39+0.53}{2}=0.46$	
Mathematical behaviours	**Marks**
• answers correctly	1

Question 14 (b) **(1 mark)**

Solution	
Confidence interval is $(\hat{p}-E, \hat{p}+E)=(0.35, 0.49)$ So $E=\frac{0.53-0.39}{2}=0.07$	
Mathematical behaviours	**Marks**
• answers correctly	1

Question 14 (c) **(2 marks)**

Solution	
$E=z_\alpha\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ i.e. $0.07=1.96\sqrt{\frac{0.46\times 0.54}{n}}$ Solving for n gives $n\cong 194.7$ and so the sample size was 195 (approximately)	
Mathematical behaviours	**Marks**
• uses $z_\alpha=1.96$	1
• solves for n and rounds	1

Question 14 (d) **(3 marks)**

Solution	
For this interval $E=0.04$ $E=z_\alpha\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, and so $0.04=z_\alpha\times\sqrt{\frac{0.46\times 0.54}{195}}$ (*) and so $z_\alpha=1.12$ So $\alpha\cong 0.746$ and hence the confidence level is approximately 75%	
Mathematical behaviours	**Marks**
• obtains equation for z_α (*)	1
• solves for z_α	1
• obtains correct answer	1

Question 14 (e) **(2 marks)**

Solution	
The sample provides strong evidence that a majority opposes the plan, but it is hardly compelling because the 95% confidence interval extends into the region $p>0.5$	
Mathematical behaviours	**Marks**
• gives a sensible answer	1
• provides a good reason	1

• obtains correct answer to the required level of accuracy	1

• correct shape	1

Question 17 (a) **(3 marks)**

Area of rectangle is xy (m^2) Area of triangle ¿ half base × height ¿ $\frac{y}{2} \times \frac{\sqrt{3}y}{2}$ $(m^2$¿ So total area is $xy + \frac{\sqrt{3}y^2}{4}$	
Mathematical behaviours	Marks
• gives correct area of rectangle	1
• gives correct area of triangle and sums to give total area	1+1

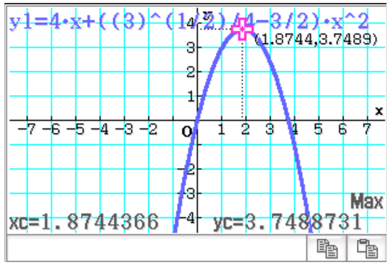
Question 17 (b) **(2 marks)**

Solution	
$P = 2x + 3y = 8$ So $x = 4 - 3y/2$ and so $A = \left(4 - \frac{3y}{2}\right)y + \frac{\sqrt{3}y^2}{4} = 4y + \left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)y^2$ (*)	
Mathematical behaviours	Marks
• correct expression for perimeter P	1
• correct (one-variable) expression for area A (*)	1

Question 17 (c) **(3 marks)**

Solution

$A = 4y + \left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)y^2 \cong 4y - 1.067y^2$



So $\frac{dA}{dy} = 4 - 2.134y$ and

$\frac{dA}{dy} = 0$ when $y \cong 1.874$

Since $\frac{d^2A}{dy^2} = -2.134 < 0$,

A has a maximum when $y \cong 1.874$.

So $A_{max} \cong 4 \times 1.874 - 1.067 \times 1.874^2 \cong 3.75$

So the maximum total area is 3.75 (m^2)

Mathematical behaviours	Marks
• draws a sketch of $A(y)$ as found in part (b) – or states that uses a calculator sketch of $A(y)$	1
• indicates the maximum area as the y-value of the TP	1
• provides this value correctly rounded	1
Alternatively,	
• differentiates correctly	1
• obtains critical point	1

Question 15(a) **(2 marks)**

Solution	
$pH = -\log H^+$ $= -\log 1 \times 10^{-7}$ $= \log\left(10^{-7}\right)^{-1} = \log 10^7 = 7$ Hence distilled water is neutral	
Mathematical behaviours	Marks
• demonstrates use of $\log$ law, $a\log = \log b^a$	1
• evaluates $pH = 7$ and draws conclusion	1

Question 15(b) **(1 mark)**

Solution	
$11 = -\log H^+$ $10^{11} = \frac{1}{H^A} \Rightarrow H^+ = 10^{-11}$ Hence the concentration of hydrogen ions is 10^{-11} moles per litre.	
Mathematical behaviours	Marks
• states solution with unit	1

Question 15(c) **(1 mark)**

Solution	
$\log\frac{H_A}{H_B} = \log H_A - \log H_B$ $= -pH_A + pH_B$	
Mathematical behaviours	Marks
• states correct expression	1

Question 15(d) **(2 marks)**

Solution

From part (c)

$\log\frac{H_{BC}}{H_{LJ}} = -pH_{BC} + pH_{LJ}$

$= -5 + 2$

$= -3$

$\frac{H_{BC}}{H_{LJ}} = 10^{-3}$

$H_{BC} = 10^{-3} H_{LJ}$

ie The number of hydrogen ions in Black Coffee is 10^{-3} times the number of hydrogen ions in Lemon Juice.

Mathematical behaviours	Marks
• substitutes into formula	1
• rewrites logarithmic equation as an exponential and states ratio.	1

Question 16(a) **(4 marks)**

Solution

$$f(x)=x^2\ln 2x$$

$$f'(x)=2x\ln 2x+x^2\times\frac{1}{2x}\times 2=2x\ln 2x+x$$

$$f'(x)=0\Rightarrow x(2\ln 2x+1)=0\Rightarrow 2\ln 2x+1=0,\ x>0$$

$$2\ln 2x=-1\Rightarrow e^{-\frac{1}{2}}=2x\Rightarrow x=\frac{1}{2\sqrt{e}}$$

$$f(\frac{1}{2\sqrt{e}})=\left(\frac{1}{2\sqrt{e}}\right)^2\ln\left(2\times\frac{1}{2\sqrt{e}}\right)=\frac{1}{4e}\left(\frac{-1}{2}\right)=\frac{-1}{8e}$$

$$f''(x)=2\ln 2x+3$$

$$f''(\frac{1}{2\sqrt{e}})=2\ln\left(2\frac{1}{2\sqrt{e}}\right)+3=2>0\Rightarrow \text{min at}\left(\frac{1}{2\sqrt{e}},\frac{-1}{8e}\right)$$

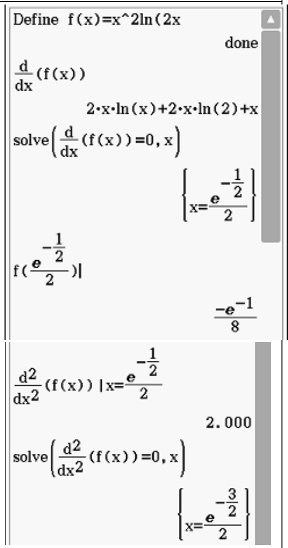
POI at $f''(x)=0$

$ie\ 2\ln 2x+3=0$

$ie\ 2\ln x=-3$

$ie\ x=e^{\frac{-3}{2}}$

ie Point of inflection at $\left(\frac{1}{2\sqrt{e^3}},\frac{-3}{8e^3}\right)$

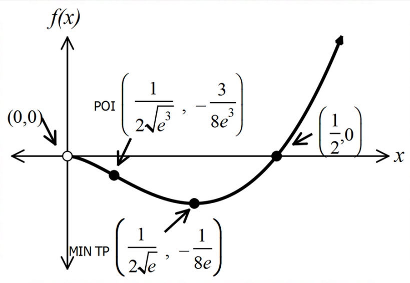


Mathematical behaviours	Marks
• determines first derivative	1
• equates $f'(x)=0$ and solves, rejecting $x=0$	1
• evaluates y coordinate and justifies minimum	1
• equates 2nd derivative to 0 and locates point of inflection	1

Question 16(b) **(3 marks)**

Solution

$f(x)=0\Rightarrow \ln 2x=0\Rightarrow x=\frac{1}{2}\Rightarrow (\frac{1}{2},0)$



Mathematical behaviours	Marks
• function undefined for $x\le 0$	1
• point of inflection, minimum and x axis intercept identified (labelled)	1