Chapter 4

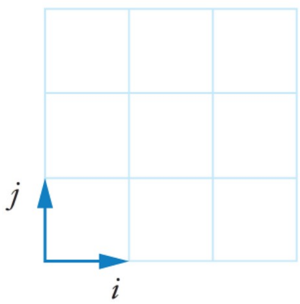
<u>Vectors in Component Form:</u>

We can use these methods where there are more than 2 vectors being applied to an object

<u>Unit vectors</u>

We use I and j to help with illustrating vectors.

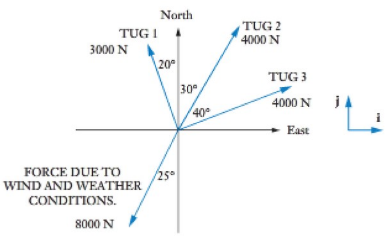
**Don’t forget the curly line underneath I and j



We can also use, sin and cos to help us to figure out the resultant for multiple vectors

e.g.

We were required to sum four vectors as shown in the diagram on the right.



Expressing each in the form $a\mathbf{i} + b\mathbf{j}$:

Remember
N = Newtons

Force from Tug 1:	$(-3000\cos 70°\,\mathbf{i} + 3000\sin 70°\,\mathbf{j})$ N
Force from Tug 2:	$(4000\cos 60°\,\mathbf{i} + 4000\sin 60°\,\mathbf{j})$ N
Force from Tug 3:	$(4000\cos 20°\,\mathbf{i} + 4000\sin 20°\,\mathbf{j})$ N
Force due to wind and water:	$(-8000\cos 65°\,\mathbf{i} - 8000\sin 65°\,\mathbf{j})$ N

Thus the resultant of these forces will be

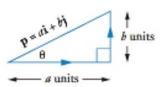
$$[(-3000\cos 70° + 4000\cos 60° + 4000\cos 20° - 8000\cos 65°)\,\mathbf{i}$$
$$+\ (3000\sin 70° + 4000\sin 60° + 4000\sin 20° - 8000\sin 65°)\,\mathbf{j}]\text{ N}$$
$$\approx (1352\mathbf{i} + 401\mathbf{j})\text{ N}$$

i.e. a vector of magnitude $\sqrt{1352^2 + 401^2} \approx 1410$ Newtons, in a direction ~073°.

Note:
- To find the magnitude and direction of the resultant from the component form we used Pythagoras and trigonometry.

  In the general case, if $\mathbf{p} = a\mathbf{i} + b\mathbf{j}$ then **magnitude**, or **modulus**, of $\mathbf{p}$ is given by

  $$|\mathbf{p}| = \sqrt{a^2 + b^2}$$

  and θ is found using $\tan\theta = \dfrac{b}{a}$.

- The vector $p\mathbf{i} + q\mathbf{j}$ is sometimes written as an ordered pair (p, q), or perhaps $<p, q>$, and sometimes as a *column matrix* $\begin{pmatrix} p \\ q \end{pmatrix}$.

  Whilst the reader needs to be aware of these alternative ways of writing the vector $p\mathbf{i} + q\mathbf{j}$, at this stage this book will tend to use the $p\mathbf{i} + q\mathbf{j}$ form most frequently.

- Some calculators have built in routines for changing a vector given in component form to its magnitude and direction. Does your calculator have this ability?

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Unit Vector:

Would be a vector in the same direction but only of one unit

- To figure it out we find out the magnitude and divide the I and k units by the

We can use this to help us do more calculations:

e.g.

If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{c} = x\mathbf{i} + \mathbf{j}$ find.

a a vector in the same direction as **a** but twice the magnitude of **a**,

b a unit vector in the same direction as **a**,

c a vector in the same direction as **a** but the same magnitude as **b**,

d the possible values of x if $|\mathbf{c}| = |\mathbf{a}|$.

Solution

a Any vector in the same direction as **a** will be a positive scalar multiple of **a**.

To be twice the magnitude the scalar multiple must be 2.

Thus the required vector is $2\mathbf{a}$, i.e. $4\mathbf{i} + 6\mathbf{j}$.

b **a** has magnitude $\sqrt{2^2+3^2}$

$= \sqrt{13}$ units

Thus the unit vector, in the same direction as **a**, would be $\frac{1}{\sqrt{13}}(2\mathbf{i} + 3\mathbf{j})$

i.e. $\frac{2}{\sqrt{13}}\mathbf{i} + \frac{3}{\sqrt{13}}\mathbf{j}$

or, with rationalised denominators

$\frac{2\sqrt{13}}{13}\mathbf{i} + \frac{3\sqrt{13}}{13}\mathbf{j}$

unitV$\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$

$\begin{bmatrix} \frac{2\cdot\sqrt{13}}{13} \\ \frac{3\cdot\sqrt{13}}{13} \end{bmatrix}$

c $|\mathbf{b}| = \sqrt{3^2+(-4)^2}$

$= 5$ units.

Thus, using our answer for part **b**, the vector in the same direction as **a** but of magnitude 5 units will be $\frac{10}{\sqrt{13}}\mathbf{i} + \frac{15}{\sqrt{13}}\mathbf{j}$.

d $|\mathbf{c}| = \sqrt{x^2+1^2}$ and $|\mathbf{a}| = \sqrt{2^2+3^2}$.

If $|\mathbf{c}| = |\mathbf{a}|$ then $x^2 + 1 = 4 + 9$

$\therefore$ $x = \pm\sqrt{12}$.

The possible values of x are $\pm 2\sqrt{3}$.

e.g.

A body is moving with velocity $(7\mathbf{i} + 24\mathbf{j})$ m/s. How far will it travel in twenty seconds?

Solution

If the velocity $= (7\mathbf{i} + 24\mathbf{j})$ m/s

speed $= |\text{velocity}|$

$= |7\mathbf{i} + 24\mathbf{j}|$

$= 25$ m/s.

Thus in twenty seconds the body will travel 500 metres.

Equilibrium forces:

There are two approaches we could use, but the better one is

- Find the resultant of the forces
- The equal force will be the negative vector, so it will have the same magnitude
- Then by using trig we can find the direction

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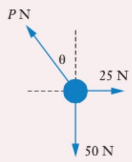
e.g.

EXAMPLE 4

The forces acting on a body are as shown in the diagram. If the body is in equilibrium find P and θ.



Solution

If the body is in equilibrium there can be no 'surplus' force in any direction because, if there was, the body would tend to move in that direction.

The horizontal forces must balance. $\therefore P\sin\theta = 25$ [1]

The vertical forces must balance. $\therefore P\cos\theta = 50$ [2]

Dividing [1] by [2] gives $\tan\theta = 0.5$

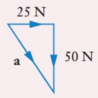
$\theta \approx 26.6°$

Substituting for θ into [1] gives $P \approx 55.9$

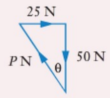
If the body is in equilibrium then $P \approx 56$ and $\theta \approx 27°$.

Alternatively, we could approach example 4 as follows:

The resultant of the 25 N force and the 50 N force is a vector **a**, see diagram on the right.



Thus the force of P N must be $-$**a** to counteract the effect of **a** and reduce the system to equilibrium.



By Pythagoras: $P = \sqrt{25^2 + 50^2}$

i.e. $P \approx 56$

By trigonometry: $\tan\theta = \dfrac{25}{50}$

i.e. $\theta \approx 27°$

If the body is in equilibrium then $P \approx 56$ and $\theta \approx 27°$, as before.

Base Vectors:

Can be used when horizontal and vertical vectors are not useful

- We will normally get given a form and then we can equate the I and j components and then solve

e.g.

Using $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - \mathbf{j}$ as base vectors, express each of the following in the form $\lambda\mathbf{a} + \mu\mathbf{b}$.

a $5\mathbf{i} + 3\mathbf{j}$ **b** $6\mathbf{i} - 4\mathbf{j}$

Solution

a Let $5\mathbf{i} + 3\mathbf{j} = \lambda\mathbf{a} + \mu\mathbf{b}$

i.e. $5\mathbf{i} + 3\mathbf{j} = \lambda(2\mathbf{i} + 3\mathbf{j}) + \mu(4\mathbf{i} - \mathbf{j})$

Equating the **i** components $5 = 2\lambda + 4\mu$ [1]

Equating the **j** components $3 = 3\lambda - \mu$ [2]

Solving [1] and [2] simultaneously: $\lambda = \dfrac{17}{14}$ and $\mu = \dfrac{9}{14}$

Thus $5\mathbf{i} + 3\mathbf{j} = \dfrac{17}{14}\mathbf{a} + \dfrac{9}{14}\mathbf{b}$

b Let $6\mathbf{i} - 4\mathbf{j} = \lambda\mathbf{a} + \mu\mathbf{b}$

i.e. $6\mathbf{i} - 4\mathbf{j} = \lambda(2\mathbf{i} + 3\mathbf{j}) + \mu(4\mathbf{i} - \mathbf{j})$

Equating the **i** components $6 = 2\lambda + 4\mu$ [1]

Equating the **j** components $-4 = 3\lambda - \mu$ [2]

Solving simultaneously $\lambda = -\dfrac{5}{7}$ and $\mu = \dfrac{13}{7}$

Thus $6\mathbf{i} - 4\mathbf{j} = -\dfrac{5}{7}\mathbf{a} + \dfrac{13}{7}\mathbf{b}$

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COME BACK TO THIS

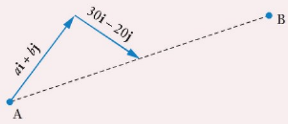
e.g.

Airports A and B are such that $\overrightarrow{AB} = (600\mathbf{i} + 200\mathbf{j})$ km. An aircraft is to be flown directly from A to B. The aircraft can maintain a steady speed of 390 km/h in still air. There is a wind blowing with velocity $(30\mathbf{i} - 20\mathbf{j})$ km/h.

Find, in the form $a\mathbf{i} + b\mathbf{j}$, the velocity vector the pilot should set so that this velocity, together with the wind, causes the plane to travel directly from A to B.

Solution

The resultant of the planes velocity due to its engines, $(a\mathbf{i} + b\mathbf{j})$ km/h, and the wind, $(30\mathbf{i} - 20\mathbf{j})$ km/h, must be along $\overrightarrow{AB}$. i.e. the resultant must be a positive scalar multiple of $600\mathbf{i} + 200\mathbf{j}$.



Thus $\quad a\mathbf{i} + b\mathbf{j} + 30\mathbf{i} - 20\mathbf{j} = \lambda(600\mathbf{i} + 200\mathbf{j}) \qquad \lambda > 0$

Equating the $\mathbf{i}$ components $\quad a + 30 = 600\lambda \qquad [1]$

Equating the $\mathbf{j}$ components $\quad b - 20 = 200\lambda \qquad [2]$

Dividing [1] by [2] $\quad \dfrac{a+30}{b-20} = \dfrac{600}{200}$

i.e. $\quad a = 3b - 90 \qquad [3]$

We also know that $\quad |a\mathbf{i} + b\mathbf{j}| = 390 \quad$ i.e. $\quad a^2 + b^2 = 390^2 \qquad [4]$

From [3] and [4] $\quad (3b - 90)^2 + b^2 = 390^2$

Using a calculator to solve this equation gives $\quad b = 150$ or $b = -96$

But λ must be positive so, from [2], $b \neq -96$.

(Note: $b = -96$ gives a resultant in direction $\overrightarrow{BA}$, not $\overrightarrow{AB}$ as required.)

If $\quad b = 150, \quad a = 360 \quad$ i.e. $\quad$ velocity $= 360\mathbf{i} + 150\mathbf{j}$.

The required velocity is $(360\mathbf{i} + 150\mathbf{j})$ km/h.

DON'T FORGET THE DIAGRAM

<u>Position Vectors:</u>

Are vectors from the origin to that point in the I and j form

- We can use positional vectors to help us find certain vectors

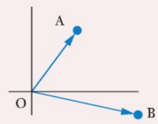
e.g.

EXAMPLE 7

Points A and B have position vectors $2\mathbf{i} + 3\mathbf{j}$ and $5\mathbf{i} - \mathbf{j}$ respectively. Find $\overrightarrow{AB}$.

Solution

Initially draw a rough sketch of the situation:



From the diagram $\quad \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\overrightarrow{OA} + \overrightarrow{OB}$

$= -(2\mathbf{i} + 3\mathbf{j}) + (5\mathbf{i} - \mathbf{j})$

$= 3\mathbf{i} - 4\mathbf{j}$

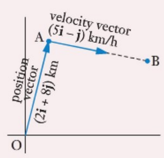
Thus $\overrightarrow{AB} = 3\mathbf{i} - 4\mathbf{j}$

EXAMPLE 8

At 1 p.m. a ship is at a location A, position vector $(2\mathbf{i} + 8\mathbf{j})$ km, and is moving with velocity $(5\mathbf{i} - \mathbf{j})$ km/h. If the ship continues with this velocity what will be its position vector at 4 p.m.?

Solution

Suppose the ship is at point B at 4 p.m. (see diagram).



Then $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= (2\mathbf{i} + 8\mathbf{j}) + 3(5\mathbf{i} - \mathbf{j})$

$= 17\mathbf{i} + 5\mathbf{j}$

By 4 p.m. the ship will be at the point with position vector $(17\mathbf{i} + 5\mathbf{j})$.

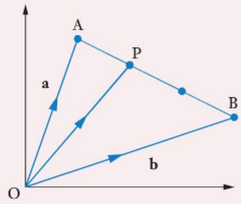
EXAMPLE 9

Points A and B have position vectors $\mathbf{i} + 7\mathbf{j}$ and $10\mathbf{i} + 4\mathbf{j}$ respectively. Find the position vector of the point that divides AB internally in the ratio 1: 2.

(Note: If a point P divides AB in the ratio 1 : 2 then AP : PB = 1 : 2.)

Solution

We require the position vector of the point P (see diagram) where



$$
\begin{aligned}
AP : PB &= 1 : 2 \\
\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP} \\
&= \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB} \\
&= \mathbf{a} + \frac{1}{3}(-\mathbf{a} + \mathbf{b}) \\
&= \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} \\
&= \frac{2}{3}(\mathbf{i} + 7\mathbf{j}) + \frac{1}{3}(10\mathbf{i} + 4\mathbf{j}) \\
&= 4\mathbf{i} + 6\mathbf{j}
\end{aligned}
$$

The point dividing AB internally in the ratio 1 : 2 has position vector $4\mathbf{i} + 6\mathbf{j}$.

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