

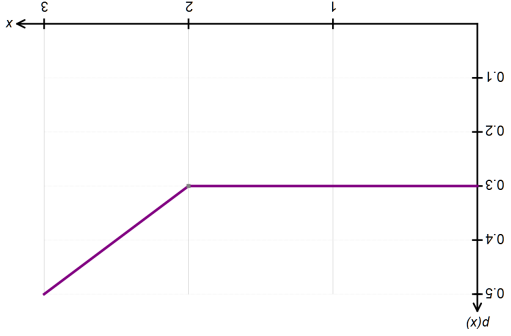


**Calculator Free**  
**General Continuous Random Variables and**  
**the Normal Distribution**  
Time: 45 minutes  
Total Marks: 45  
Your Score: / 45

CF

**Question One: [3, 5 = 8 marks]**

Consider the probability density function drawn below:



(a) Confirm, with appropriate calculations, that this above graph represents a probability density function.

(b) State the piecewise function that defines this continuous random variable.

**Question Two: [4, 4, 4 = 12 marks]****CF**

Determine the value(s) of  $k$  which make each of the following functions a probability density function.

(a) 
$$h(x) = \begin{cases} kx + 2 & ; 1 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

(b) 
$$f(x) = \begin{cases} k(1 - x^2) & ; -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) 
$$h(x) = \begin{cases} k\sqrt{x} & ; 0 < x \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

**Question Five: [2, 2, 2, 2 = 10 marks]****CF**

The heights of fairy penguins in a particular geographic location are normally distributed with a mean height of 32 cm and a standard deviation of 1.5 cm.

Use the 68%, 95% and 99.7% rule to calculate each of the following.

- (a) Determine the probability that a randomly selected fairy penguin is taller than 30.5 cm.

$$P(X > 30.5) = 0.34 + 0.5 = 0.84$$

✓ ✓

- (b) Determine the probability of a randomly selected fairy penguin being shorter than 30.5 cm if it is known that they are in the 0.5 quantile.

$$P(X < 30.5 | X < 32) = \frac{0.16}{0.5} = \frac{16}{50}$$

✓ ✓

- (c) In a sample of 2000 penguins, how many would you expect to be taller than 35cm?

$$P(X > 35) = 1 - 0.5 - 47.5 = 0.025$$

$$0.025 \times 2000 = 50$$

✓ ✓

- (d) What is the maximum height of the shortest 2.5% of penguins in this location?

$$P(X < k) = 0.025$$

$$k = 29\text{cm}$$

✓ ✓

- (e) In a different geographic location the mean height of the fairy penguins found there is 33 cm. If 97.5% of the penguins are shorter than 35cm, and their heights are also normally distributed, what is the standard deviation for this population?

$$2 = \frac{35 - 33}{\sigma}$$

$$\sigma = \frac{2}{2} = 1\text{cm}$$

✓ ✓

✓ ✓

- (e) The expected value of a uniform distribution is calculated by  $E(X) = \frac{b+a}{2}$ , where  $a$  and  $b$  are the endpoints over which the distribution is defined. Calculate  $E(X)$ .

$$E(X) = \frac{360+0}{2} = 180$$

- (f) The variance of a uniform distribution is calculated by  $V(X) = \frac{(b-a)^2}{12}$ . Calculate  $V(X)$ .

$$V(X) = \frac{(360-0)^2}{12} = \frac{360 \times 360}{12} = 10800$$

**Question Four:** [4 marks] **CF**

On a recent test, Tom scored 70% and his standard score was 1. Jerry sat the same test and his standard score was -0.5 when he scored 55%.

Calculate the mean and standard deviation for these test results.

$$\begin{aligned} 1 &= \frac{70 - \mu}{\sigma} \\ -0.5 &= \frac{55 - \mu}{\sigma} \\ \mu &= 70 - \sigma \\ \mu &= 55 + 0.5\sigma \\ 0 &= 15 - 1.5\sigma \\ \sigma &= 10\% \\ \mu &= 70 - 10 = 60\% \end{aligned}$$

**Question Three:** [1, 2, 2, 2, 2, 2 = 11 marks] **CF**

Columbus is playing with a broken compass and he spins the compass needle around. Determine the probability the compass needle lands between the North and East.



- (b) Determine the probability the compass needle lands between the South and North West.

- (c) Use your answers to parts (a) and (b) to sketch the uniform probability density function that models each spin of the compass needle. Let  $X$  be the true bearing.

- (d) Hence define the probability density function,  $p(x)$ .

- (e) The expected value of a uniform distribution is calculated by  $E(X) = \frac{b+a}{2}$ , where  $a$  and  $b$  are the endpoints over which the distribution is defined. Calculate  $E(X)$ .

- (f) The variance of a uniform distribution is calculated by  $V(X) = \frac{(b-a)^2}{12}$ . Calculate  $V(X)$ .

**Question Four: [4 marks] CF**

On a recent test, Tom scored 70% and his standard score was 1. Jerry sat the same test and his standard score was -0.5 when he scored 55%.

Calculate the mean and standard deviation for these test results.

**Question Three: [1, 2, 2, 2, 2 =11 marks] CF**

Columbus is playing with a broken compass and he spins the compass needle around.

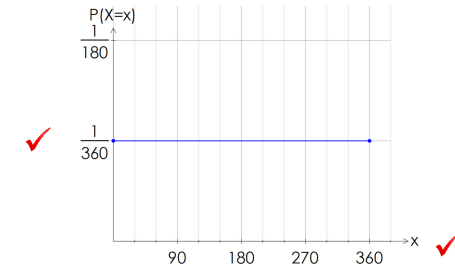
- (a) Determine the probability the compass needle lands between the North and East.

0.25 ✓

- (b) Determine the probability the compass needle lands between the South and NorthWest.

$\frac{135}{360}$  ✓✓

- (c) Use your answers to parts (a) and (b) to sketch the uniform probability density function that models each spin of the compass needle. Let  $X$  be the true bearing.



- (d) Hence define the probability density function,  $p(x)$ .

$$p(x) = \begin{cases} \frac{1}{360} & 0 \leq x \leq 360 \\ 0 & \text{otherwise} \end{cases}$$



$$(c) \quad h(x) = \begin{cases} k\sqrt{x} & ; 0 < x \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^9 k\sqrt{x} \, dx = 1$$

$$k \left[ \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^9 = 1$$

$$k [18 - 0] = 1$$

$$k = \frac{1}{18}$$

**Question Five: [2, 2, 2, 2 = 10 marks]**

**CP**

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(b) Determine the probability of a randomly selected fairy penguin being shorter than 30.5 cm if it is known that they are in the 0.5 quantile.

(c) In a sample of 2000 penguins, how many would you expect to be taller than 35cm?

(d) What is the maximum height of the shortest 2.5% of penguins in this location?

(e) In a different geographic location the mean height of the fairy penguins found there is 33 cm. If 97.5% of the penguins are shorter than 35cm, and their heights are also normally distributed, what is the standard deviation for this population?



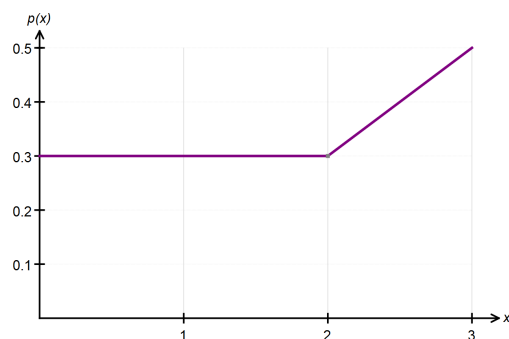
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**Question One: [3, 5 = 8 marks]**

**CF**

Consider the probability density function drawn below:



- (a) Confirm, with appropriate calculations, that this above graph represents a probability density function.

$$\text{Area} = 3 \times 0.3 + 0.5 \times 1 \times 0.2 = 1$$

- (b) State the piecewise function that defines this continuous random variable.

$$p(x) = \begin{cases} 0.3; & 0 \leq x < 2 \\ 0.2x - 0.1; & 2 \leq x \leq 3 \end{cases}$$

**Question Two: [4, 4, 4 = 12 marks]**

**CF**

Determine the value(s) of  $k$  which make each of the following functions a probability density function.

(a) 
$$h(x) = \begin{cases} kx + 2; & 1 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \int_1^4 kx + 2 \, dx &= 1 \\ \left[ \frac{kx^2}{2} + 2x \right]_1^4 &= 1 \\ (8k + 8) - (0.5k + 2) &= 1 \\ 7.5k + 6 &= 1 \\ 7.5k &= -5 \\ k &= \frac{-10}{15} = \frac{-2}{3} \end{aligned}$$

(b) 
$$f(x) = \begin{cases} k(1 - x^2); & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \int_{-1}^1 k(1 - x^2) \, dx &= 1 \\ k \left[ x - \frac{x^3}{3} \right]_{-1}^1 &= 1 \\ k \left[ \left( 1 - \frac{1}{3} \right) - \left( -1 + \frac{1}{3} \right) \right] &= 1 \\ \frac{4k}{3} &= 1 \\ k &= \frac{3}{4} \end{aligned}$$