Mercedes College

Semester Two Examination, 2017

Question/Answer booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section Two:
Calculator-assumed

SOLUTIONS

Student Number: In figures ☐☐ ☐☐☐ ☐☐☐

In words ______________________________________

Your name ______________________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	97	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Additional working space

Question number: _________

Additional working space

Question number: __________

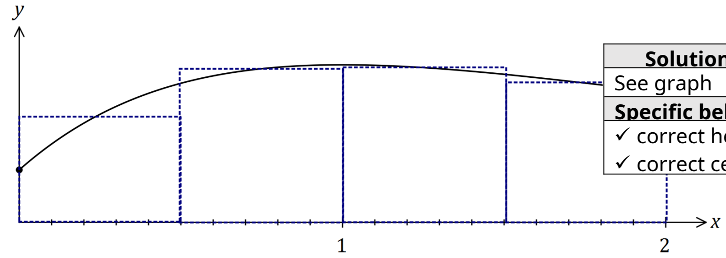
Section Two: Calculator-assumed **65% (97 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(8 marks)**

The graph of $y=f(x)$ is shown below for $0\le x\le 2$, where $f(x)=1+2xe^{1-x}$.



Solution (b)(i)
See graph
Specific behaviours
✓ correct heights ✓ correct centres

(a) Show that $f(x)$ has a stationary point at $(1,3)$. (3 marks)

Solution
$f'(x)=2(1-x)e^{1-x}$ $f'(x)=0$ when $x=1$ $f(1)=1+2e^{0}=3$ Hence stationary point at $(1,3)$
Specific behaviours
✓ shows derivative ✓ shows $f'(x)$ has a factor of $(1-x)$ ✓ indicates $f(1)$

(b) An estimate for the area bounded by the curve, the x-axis, the y-axis and the line $x=2$ is required. A suitable estimate can be calculated from the sum of the areas of four centred rectangles with heights $f(0.25), f(0.75), f(1.25)$ and $f(1.75)$, each with a width of 0.5 units.

(i) Clearly show these four rectangles on the graph above. (2 marks)

(ii) Use the rectangles to estimate the area, giving your answer correct to 2 decimal places. (3 marks)

Solution
$A=0.5(2.0585+2.9260+2.9470+2.6533)$ $A=5.2924\approx 5.29\,(2\text{dp})$
Specific behaviours
✓ indicates correct heights ✓ multiplies by width ✓ correct area

Question 10 **(7 marks)**

The capacity, X mL, of glass bottles made in a factory can be modelled by a normal distribution with mean μ and standard deviation 3.4 mL.

(a) If $\mu=784$, determine

(i) $P(X \geq 780)$. (1 mark)

Solution
$P=0.8803$
Specific behaviours
✓ states probability

(ii) $P(X<786 \mid X>780)$. (2 marks)

Solution
$P(780<X<786)=0.6021$ $P=\frac{0.6021}{0.8803}=0.6840$
Specific behaviours
✓ calculates numerator   ✓ states probability

(iii) the value of x, if $P(X \leq x)=\frac{1}{3}$. (1 mark)

Solution
$x=782.5$
Specific behaviours
✓ states value

(b) Given that $P(X>k)=0.937$,

(i) determine the value of μ in terms of k. (2 marks)

Solution
$\frac{k-\mu}{3.4}=-1.53\mu=k+5.202$
Specific behaviours
✓ equation using correct z-score   ✓ expression for μ, correct to 1 dp

(ii) determine μ if $k=503$. (1 mark)

Solution
$\mu=508.2$ mL
Specific behaviours
✓ states value

Additional working space

Question number: _________

Additional working space

Question number: __________

Question 11 **(9 marks)**

A fair die has one face numbered 1, two faces numbered 2 and three faces numbered 3.

(a) Determine the probability that the second even number occurs on the fourth throw of the dice. (3 marks)

Solution
$P(1\text{ even in 3 throws})=\frac{1}{3}\times\left(\frac{2}{3}\right)^2\times 3=\frac{4}{9}$ $P=\frac{4}{9}\times\frac{1}{3}=\frac{4}{27}(\approx 0.148)$
Specific behaviours
✓ uses $P(\text{even})=\frac{1}{3}$ ✓ uses binomial expansion for 1 even in 3

(b) The die is thrown twice and X is the sum of the two scores.

(i) Complete the table below to show the probability distribution of X. (2 marks)

x	2	3	4	5	6
$P(X=x)$	$\frac{1}{36}$	$\frac{1}{9}$	$\frac{5}{18}$	$\frac{1}{3}$	$\frac{1}{4}$

Solution
$P(X=2)=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36},P(X=6)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$
Specific behaviours
✓ $P(X=2)$ ✓ $P(X=6)$

(ii) Determine $P(X=2|X\le 3¿$. (2 marks)

Solution
$P(X=2|X\le 3¿=\frac{1}{36}\div\left(\frac{1}{36}+\frac{1}{9}\right)=\frac{1}{5}$
Specific behaviours36
✓ $P(X\le 3)$ ✓ probability

(iii) Calculate $E(X)$. (2 marks)

Solution
$E(X)=\frac{1}{18}+\frac{6}{18}+\frac{20}{18}+\frac{30}{18}+\frac{27}{18}=\frac{84}{18}=\frac{14}{3}$
Specific behaviours
✓ uses sum of $x\times P(X=x)$ ✓ simplifies

Question 12 **(8 marks)**

From a random survey of 524 users of a free music streaming service, it was found that 386 would stop using it if they had to pay.

(a) Based on this survey, calculate the percentage of users who would stop using the service. (1 mark)

Solution
$\frac{386}{524} \times 100 = 73.7\,\%$
Specific behaviours
✓ calculates percentage

(b) Calculate the approximate margin of error for a 90% confidence interval estimate of the proportion of users who would stop using the service. (3 marks)

Solution
$z_{0.9} = 1.645$ $SE = \sqrt{\frac{0.737(1-0.737)}{524}} = 0.01924$ $E = 1.645 \times 0.01924 = 0.0316$
Specific behaviours
✓ uses correct z-score ✓ calculates standard error ✓ calculates margin of error

(c) Determine a 90% confidence interval for the proportion of users who would stop using the service. (2 marks)

Solution
$0.737 \pm 0.032 = (0.705, 0.768)$
Specific behaviours
✓ writes interval ✓ rounds to 2, 3 or 4 decimal places

(d) If 50 identical surveys were carried out and a 90% confidence interval for the proportion was calculated from each survey, determine the probability that exactly 48 of the intervals will contain the true value of the proportion. (2 marks)

Solution
$Y \quad B(50, 0.9)$ $P(Y = 48) = 0.0779$
Specific behaviours
✓ states parameters of binomial distribution

Question 21 **(6 marks)**

A popcorn container of capacity 500 mL is made from paper and has the shape of an open inverted cone of radius r and height h.

Determine the least area of paper required to make the container. (6 marks)

Solution
$A = \pi r s = \pi r\sqrt{r^2 + h^2}$ $V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{\pi r^2}$ $A = \pi r\sqrt{r^2 + \left(\frac{3(500)}{\pi r^2}\right)^2}$ $\frac{dA}{dr} = \frac{2r^6\pi^2 - 2250000}{r^2\sqrt{r^6\pi^2 + 2250000}}$ $\frac{dA}{dr} = 0 \text{ when } r = 6.963\text{ cm}$ $A_{MIN} = 263.8\text{ cm}^2$
Specific behaviours
✓ expresses A in terms of r and h ✓ expresses h in terms of r ✓ expresses A in terms of r ✓ differentiates A

Question 20 **(6 marks)**

A random sample of 510 rabbits from a nature reserve are captured, tagged and then set free. After a suitable interval, during which time it is assumed that the rabbit population does not change, another random sample of 300 rabbits is caught and 18 of these are observed to be tagged.

(a) Show that a point estimate for the size of the rabbit population is 8 500. (1 mark)

Solution
$\frac{510}{P}=\frac{18}{300}\Rightarrow P=8500$
Specific behaviours
✓ shows use of direct proportion

(b) Construct a 90% confidence interval for the proportion of rabbits in the population that are tagged. (2 marks)

Solution
$\hat{p}=\frac{18}{300}=0.06\,, E=0.02255\,, z_{0.9}=1.645$ $(0.0374\,, 0.0826)$
Specific behaviours
✓ calculates margin of error   ✓ states confidence interval

(c) Deduce an approximate 90% confidence interval for the number of rabbits in the reserve. (3 marks)

Solution
$P_L=\frac{510}{0.0826}=6174$ $P_U=\frac{510}{0.0374}=13636$ $(6200\,, 13600)$
Specific behaviours
✓ calculates lower value   ✓ calculates upper value

Question 13 **(7 marks)**

The lifetime, T hours, of an electronic component is a continuous random variable with probability density function given by

$$f(t)=0.005\,e^{-0.005t}\,, 0\le t<\infty.$$

(a) Determine the probability that a randomly chosen component has a lifetime of less than 450 hours. (2 marks)

Solution
$P(T<450)=\int_0^{450} f(t)\,dt\text{¿}0.8946$
Specific behaviours
✓ writes integral   ✓ evaluates

(b) An engineer buys 12 of the components. If they operate independently of each other, determine the probability that at least 11 of them will not last 450 hours. (2 marks)

Solution
$Y\ \ B(12\,, 0.8946)$ $P(Y\ge 11)=0.6342$
Specific behaviours
✓ indicates binomial distribution with parameters

(c) A component has already been operating for exactly 440 hours. Determine the probability that it will fail within the next 36 hours. (3 marks)

Solution
$P(T>440)=\int_{440}^{\infty} f(t)\,dt=0.11080$ $P(440<T<476)=\int_{440}^{476} f(t)\,dt=0.01825$ $P(T<476\mid T>440\text{¿}=\frac{0.01825}{0.11080}=0.1647$
Specific behaviours
✓ $P(T>440)$ ✓ $P(440<T<476)$

Question 14 **(8 marks)**

A researcher wants to estimate the proportion of Western Australian school-aged students who participate in organised sport during school holidays. The researcher plans to collect sample data by visiting schools and asking students.

(a) Discuss two different sources of bias that may occur when the researcher collects their sample data and suggest a procedure to avoid bias. (4 marks)

Solution
Undercoverage (*including volunteer or convenience sampling*) - the researcher should ensure that all students have an equal chance of being selected, rather than favouring gender, age, state, etc Nonresponse - some students may choose not to answer the question Etc, etc To avoid bias use Simple random sampling - number all students and select numbers at random Systematic sampling - number all students and select every k^{th} student Etc, etc
Specific behaviours
✓ discusses one source of bias

(b) Determine, to the nearest 10, the sample size the researcher should use to ensure that the margin of error of a 90% confidence interval is no more than 6%. (3 marks)

Solution
$n=\frac{1.645^2(0.5)(1-0.5)}{0.06^2}n=188$ Sample size of 190 students
Specific behaviours
✓ assumes $\hat{p}=0.5$ ✓ shows sample size equation ✓ calculates n

(c) Comment on how your answer to (b) would change if the researcher had a reliable estimate that the population proportion was close to 20%. (1 mark)

Solution
Size of sample would decrease (*to close to 120*)
Specific behaviours

Question 19 **(8 marks)**

The mass, X g, of wasted metal when a cast is made is a random variable with probability density function given by

$$f(x)=\begin{cases}\frac{2x}{a^2} & 0\le x\le a, \\ ¿ & ¿\end{cases}\text{elsewhere},¿$$

where a is a positive constant.

(a) Determine $E(X)$ in terms of a. (2 marks)

Solution
$\int_0^a \frac{2x}{a^2}\times x\,dx=\frac{2a}{3}$
Specific behaviours
✓ writes correct integral ✓ evaluates integral in terms of a

(b) The total mass of wasted metal from a random sample of 40 casts was 960 g. Estimate the value of a. (2 marks)

Solution
$\bar{x}=960\div 40=24$ $\frac{2a}{3}=24\Rightarrow a=36$
Specific behaviours
✓ calculates sample mean ✓ determines

(c) If $a=12$, determine

(i) $P(X\geq 4)$. (1 mark)

Solution
$\int_4^{12}\frac{2x}{144}dx=\frac{8}{9}$
Specific behaviours
✓ evaluates probability

(ii) $Var(X)$. (3 marks)

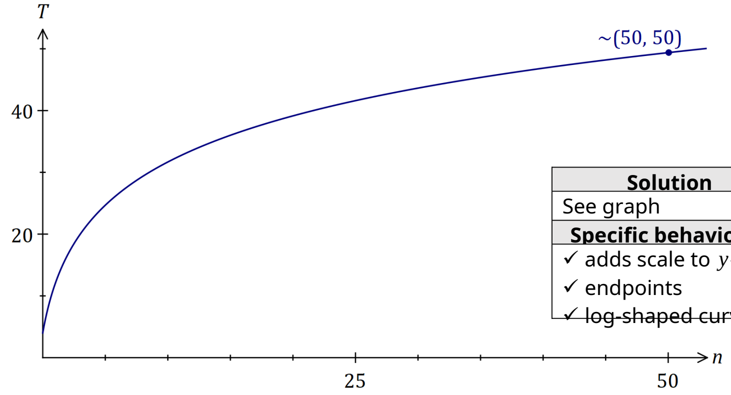
Solution
$E(X)=2\times 12\div 3=8$ $\int_0^{12}\frac{2x(x-8)^2}{144}dx=8$
Specific behaviours
✓ shows value of $E(X)$ ✓ writes correct integral ✓ evaluates variance

Question 18 **(8 marks)**

Hick's law, shown below, models the average time, T seconds, for a person to make a selection when presented with n equally probable choices.

$$T=a+b\log_2(n+1)\text{, where } a \text{ and } b \text{ are positive constants.}$$

(a) Draw the graph of T vs n on the axes below when $a=4$ and $b=8$. (3 marks)



Solution
See graph
Specific behaviours
✓ adds scale to y-axis ✓ endpoints ✓ log-shaped curve

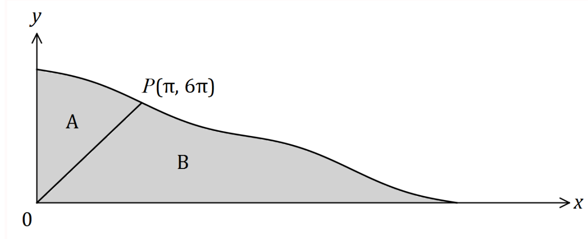
(b) When a pizzeria had 10 choices of pizza, the average time for patrons to make a choice was 40 seconds. After doubling the number of choices, the average time to make their choice increased by 25%.

Modelling the relationship with Hick's law, predict the average time to make a choice if patrons were offered a choice of 35 pizzas. (5 marks)

Solution
$40=a+b\log_2(10+1)$ $40\times 1.25=a+b\log_2(2\times 10+1)$ $a=2.917,\ b=10.719$ $T=2.917+10.719\log_2(35+1)$ $T=58.34\approx 58\text{ seconds}$
Specific behaviours
✓ writes first equation ✓ writes second equation ✓ solves for variables ✓ substitutes correctly

Question 15 **(7 marks)**

The curve $y=8\pi-2x+\sin x$ is shown below passing through $P(\pi,6\pi)$.



A straight line joins the origin to P, dividing the shaded area into two regions, A and B.

(a) Show that when $x=4\pi$, $y=0$. (1 mark)

Solution
$y=8\pi-2(4\pi)+\sin 2\pi=0$
Specific behaviours
✓ substitutes

(b) Determine the value of $\int_0^{\pi}(8\pi-2x+\sin x)\,dx$. (2 marks)

Solution
$I=7\pi^2+2\ (\approx 71.1)$
Specific behaviours
✓ evaluates integral ✓ states exact value

(c) Determine the ratio of the area of region A to the area of region B in the form $1:k$. (4 marks)

Solution
$\int_0^{\pi}(6x)\,dx=3\pi^2$ $\int_{\pi}^{4\pi}(8\pi-2x+\sin x)\,dx=9\pi^2-2$ $A=7\pi^2+2-3\pi^2=4\pi^2+2$ $B=9\pi^2-2+3\pi^2=12\pi^2-2$ Ratio A:B is $1:2.807$
Specific behaviours
✓ evaluates area of triangle ✓ evaluates area A ✓ evaluates area B

Question 16 **(8 marks)**

160 black and 840 white spherical beads, identical except for their colour, are placed in a container and thoroughly mixed.

In experiment A, a bead is randomly selected, its colour noted and then replaced until a total of 20 beads have been selected.

(a) The random variable X is the number of black beads selected in experiment A. Determine $P(X>5)$. (2 marks)

Solution
$X \quad B(20, 0.16)$ $P(X \geq 6) = 0.0870$
Specific behaviours
✓ indicates binomial RV, with parameters ✓ states P

(b) Experiment A is repeated 10 times. Determine the probability that at least one black bead is selected in each of these experiments. (2 marks)

Solution
$P(X \geq 1) = 0.9694$ $0.9694^{10} = 0.7329$
Specific behaviours
✓ calculates P(at least one black) in one experiment

In experiment B, a bead is randomly selected, its colour noted and then replaced until a total of 65 beads have been selected.

Experiments A and B are repeated a large number of times, with the proportions of black beads in each experiment, $\hat{p}_A$ and $\hat{p}_B$ respectively, recorded.

(c) The distribution of which proportion, $\hat{p}_A$ or $\hat{p}_B$, is most likely to approximate normality? Explain your answer and state the mean and standard deviation of the normal distribution for the proportion you have chosen. (4 marks)

Solution
$\hat{p}_B$ most likely, as it is based on much larger sample size (*65 rather than 20*). Parameters: Mean: 0.16 Variance: $\frac{0.16(1-0.16)}{65} = 0.00206$, $s_x = 0.045$
Specific behaviours
✓ chooses $\hat{p}_B$ ✓ explains $\hat{p}_B$ is based on larger sample size ✓ states mean

Question 17 **(7 marks)**

A polynomial function $f(x)$ is such that $\int_2^6 4f(x)\,dx = 12$.

(a) Show that $\int_6^2 f(x)\,dx = -3$. (2 marks)

Solution
$4\int_6^2 f(x)\,dx = -12 \int_6^2 f(x)\,dx = -3$
Specific behaviours
✓ reverses limits and changes sign ✓ factors and divides

(b) Determine the value of $\int_2^3 (f(x) + 3x^2)\,dx + \int_3^6 (1 + f(x))\,dx$. (5 marks)

Solution
$¿ \int_2^3 (f(x))\,dx + \int_2^3 (3x^2)\,dx + \int_3^6 (f(x))\,dx + \int_3^6 (1)\,dx$ $¿ \int_2^6 (f(x))\,dx + \int_2^3 (3x^2)\,dx + \int_3^6 (1)\,dx ¿ 3 + [x^3]_2^3 + 3 ¿ 3 + 19 + 3 ¿ 25$
Specific behaviours
✓ uses linearity to split ✓ uses interval addition with f ✓ integrates ✓ evaluates ✓ correct sum