Practice 1 Semester Two Examination, 2016

Answers

SPECIALIST
UNITS 3 AND 4
Section One:
Calculator-free

If required by your examination administrator, please place your student identification label in this box

Student Number: In figures

In words ______________________

Your name ______________________

Time allowed for this section
Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

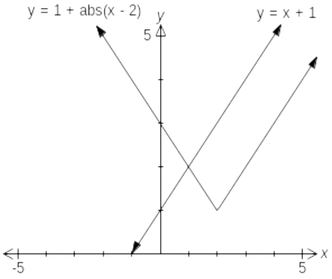
Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	12	12	100	101	65
			Total	153	100

1. [7 marks]

(a) As $|x - 2| \geq 0$, $1 + |x - 2| \geq 1$ $\Rightarrow$ No solution. ✓ [1]

(b)



From sketch: $y = 1 + |x - 2|$ and $y = x + 1$ intersect at $x = 1$ ✓

Hence, $1 + |x - 2| \leq y = x + 1$ for $x \geq 1$ ✓✓ [3]

OR

Consider $1 + |x - 2| = x + 1$ ✓

$\Rightarrow |x - 2| = x$

$\Rightarrow x - 2 = x$ or $x - 2 = -x$ ✓

But $x - 2 \neq x$, $\Rightarrow x - 2 = -x$

$x = 1$

Hence, $x \geq 1$ ✓ [3]

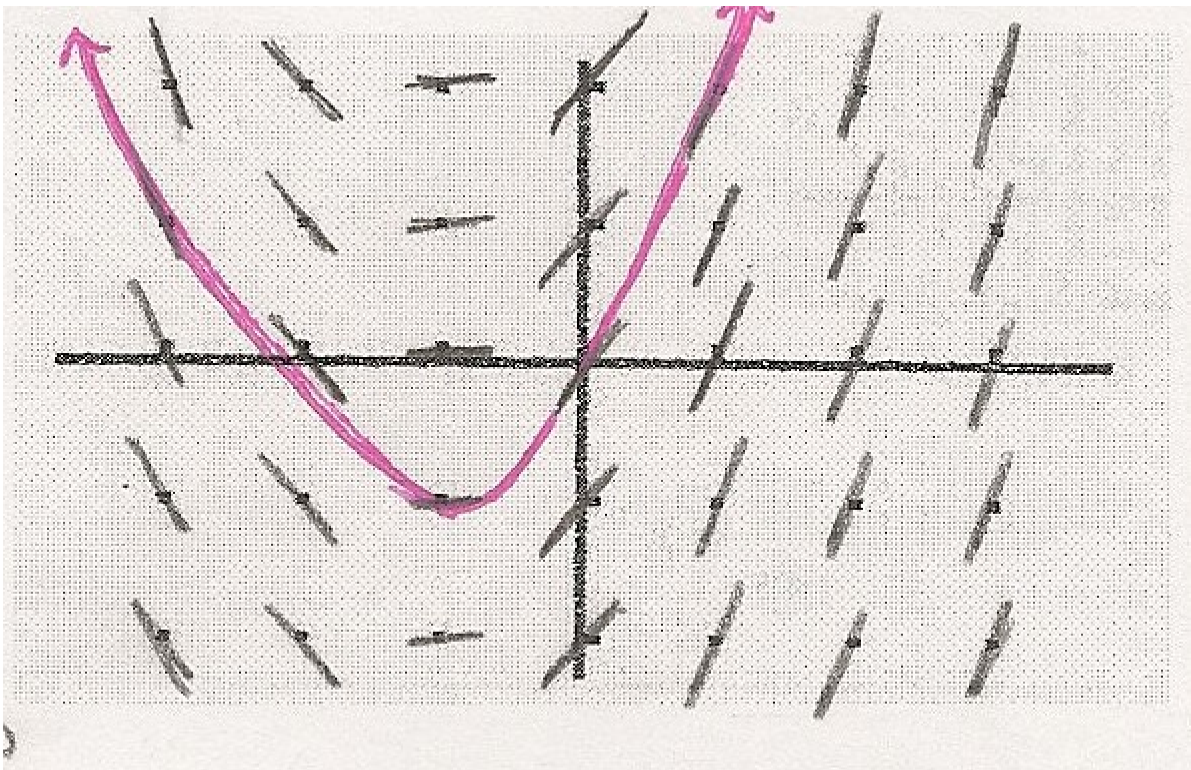
(c) $1 + |x - 2| = x - 1$

$|x - 2| = x - 2$ ✓

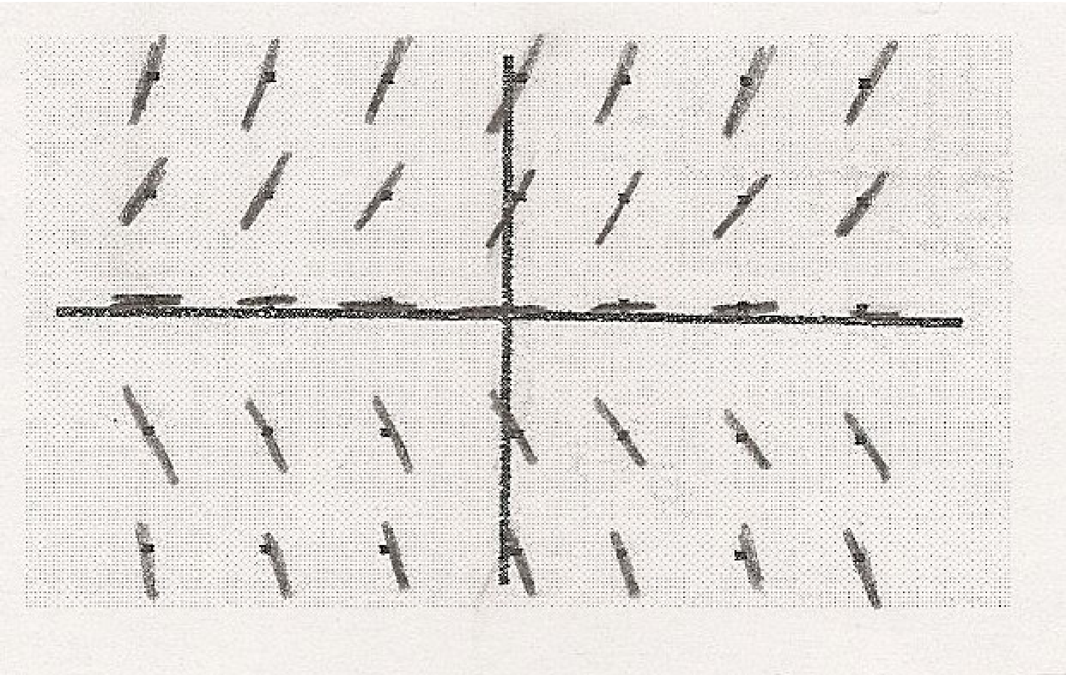
As $|x - 2| \geq 0$, $x - 2 \geq 0$ ✓ $\Rightarrow$ $x \geq 2$ ✓ [3]

2.

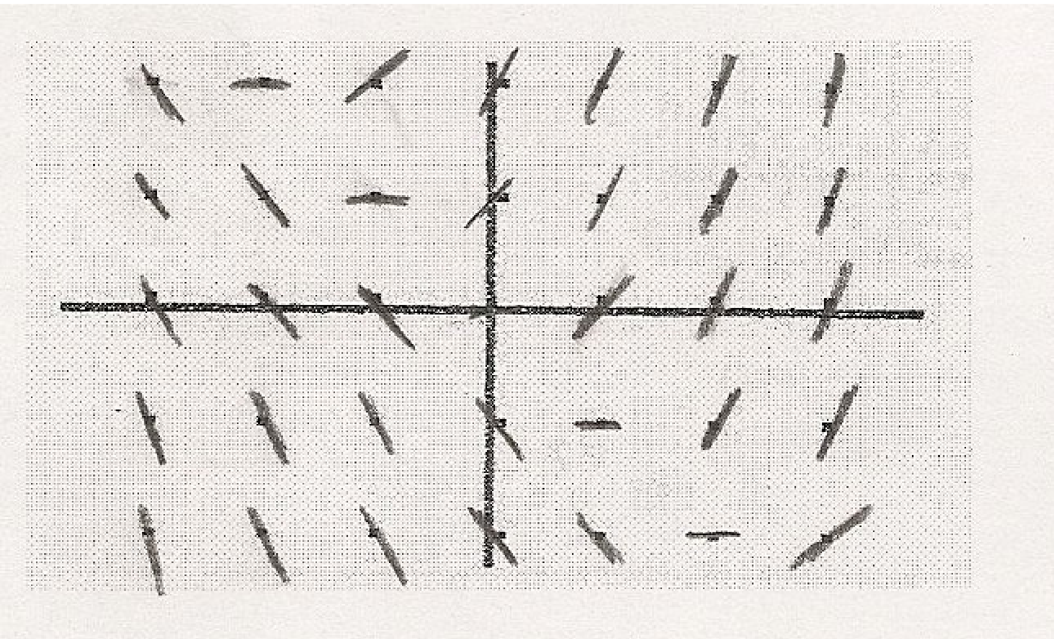
(a) $\dfrac{dy}{dx} = x + 1$



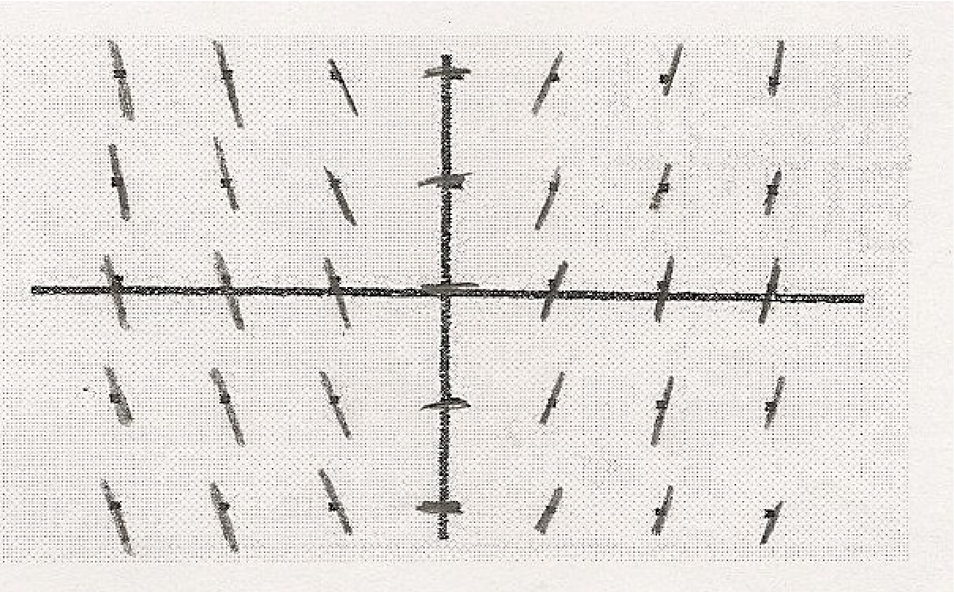
(b) $\frac{dy}{dx} = 2y$



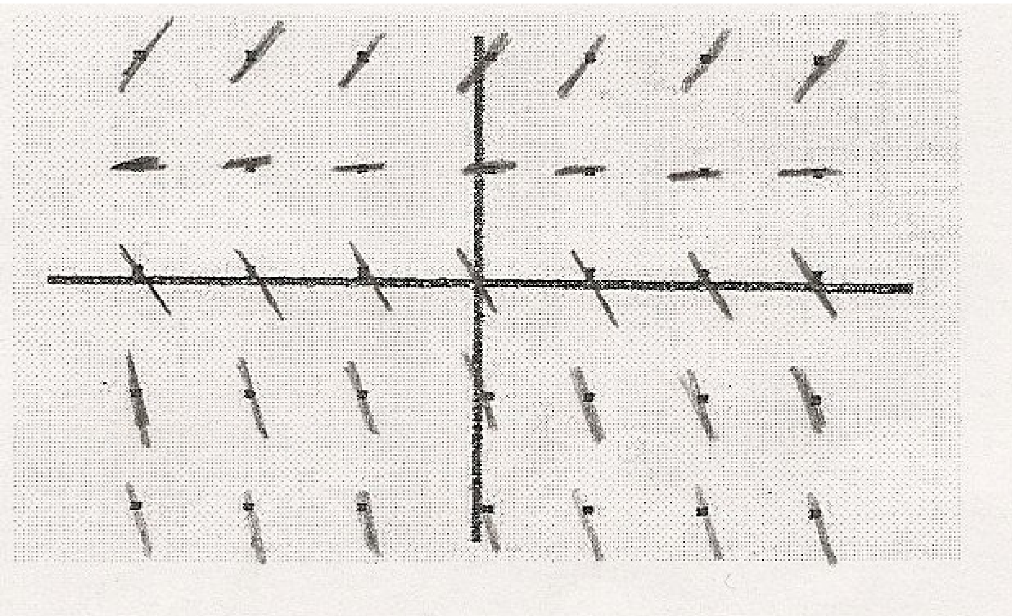
(c) $\frac{dy}{dx} = x + y$



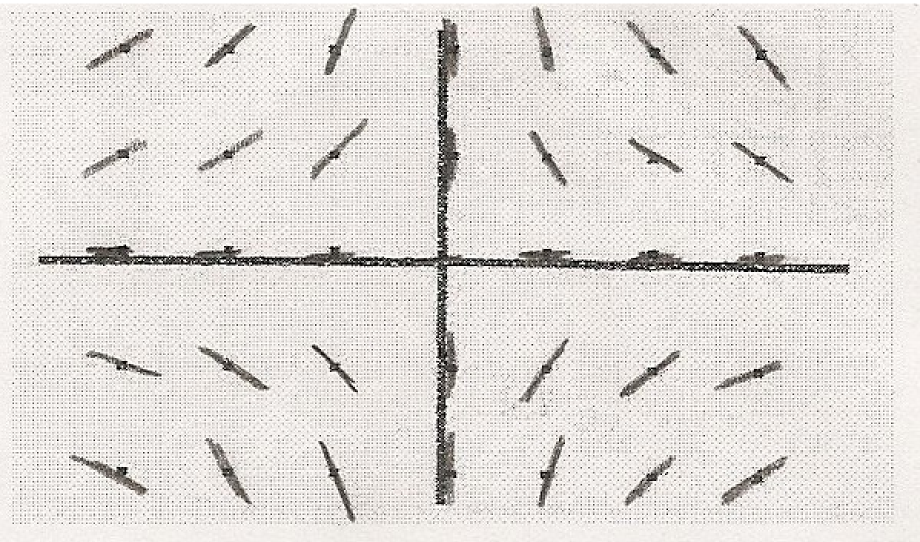
(d) $\frac{dy}{dx} = 2x$



(e) $\frac{dy}{dx} = y - 1$



(f) $\frac{dy}{dx} = \frac{-x}{y}$



3. [5 marks]

$(a + bi) \times (b + ai) = 13i$

$ab + a^2i + b^2i - ab = 13i$ ✓

Hence, $(a^2 + b^2)i = 13i$

$(a^2 + b^2) = 13$ ✓

Since a and b are real integers, $a = \pm 2$ and $b = \pm 3$ ✓✓

or $a = \pm 3$ and $b = \pm 2$ ✓ [5]

4. [6 marks]

(a) $z = 1 + \cos\theta + i\sin\theta$

Hence, $\text{Re}(z) = 1 + \cos\theta$ ✓ and $\text{Im}(z) = \sin\theta$ ✓ [2]

(b) If $\theta = \frac{\pi}{3}$, $\text{Re}(z) = \frac{3}{2}$ ✓ and $\text{Im}(z) = \frac{\sqrt{3}}{2}$ ✓

Hence, $|z| = \sqrt{3}$ ✓ and $\arg(z) = \frac{\pi}{6}$ ✓ [4]

5. [6 marks]

Differentiate implicitly, $y' - 1 - 3e^{-y}y' = 0$ ✓✓

At (2, 0), x = 2, y = 0; $\Rightarrow$ $y' - 1 - 3y' = 0$ ✓ $\Rightarrow$ $y' = -0.5$ ✓

Hence, equation of tangent is $y = -0.5(x - 2)$ ✓✓ [6]

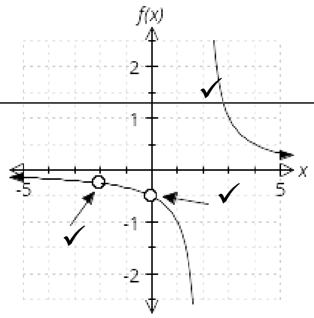
6. [8 marks]

(a) $y = 0$ ✓ [1]

(b) $x = 0$ and $x = \pm a$. ✓✓ [2]

(c) $\lim\limits_{x \to 0} \dfrac{x(x+a)}{x(x+a)(x-a)} = \lim\limits_{x \to 0} \dfrac{1}{x-a} = -\dfrac{1}{a}$ ✓✓ [2]

(d) When $a = 2$, $f(x) = \dfrac{x(x+2)}{x(x+2)(x-2)}$



[3]

7. [10 marks]

(a) $\displaystyle\int \frac{3x-2}{\sqrt{x}}\,dx = \int 3x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}\,dx = \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{-\frac{1}{2}}}{\frac{1}{2}} + C$ ✓✓ [2]

(b) $\displaystyle\int \frac{3x}{5x^2-2}\,dx = \frac{3}{10}\int \frac{10x}{5x^2-2}\,dx = \frac{3}{10}\ln\,(5x^2-2) + C$ ✓✓ [2]

(c) Let $x = \tan\theta \;\Rightarrow\; dx = \dfrac{d\theta}{\cos^2\theta}$ ✓

When $x = 1$, $\theta = \dfrac{\pi}{4}$ and when $x = 0$, $\theta = 0$ ✓

$\displaystyle I = \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{(1+\tan^2\theta)^2} \times \frac{d\theta}{\cos^2\theta}$ ✓

$\displaystyle = \int_0^{\frac{\pi}{4}} \frac{\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}{\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)^2} \times \frac{d\theta}{\cos^2\theta} = \int_0^{\frac{\pi}{4}} \sin^2\theta\,d\theta$ ✓

$\displaystyle = \int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2}\,d\theta$ ✓

$\displaystyle = \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}$ ✓ [6]

Practice 2 Semester Two Examination, 2016

Question/Answer Booklet

SPECIALIST
UNITS 3 AND 4
Section Two:
Calculator-assumed

If required by your examination administrator, please place your student identification label in this box

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section
Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

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			Total	153	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Booklet.

8. [9 marks]

(a) $(5\ cis\left(-\frac{\pi}{4}\right)) \times (-1 - \sqrt{3}\ i) = (5\ cis\left(-\frac{\pi}{4}\right)) \times (2\ cis\left(-\frac{2\pi}{3}\right))$ ✓✓

$= 10\ cis\left(-\frac{11\pi}{12}\right)$ ✓ [3]

(b) $\overline{(-1-\sqrt{3}\,i)} = 2\ cis\left(\frac{2\pi}{3}\right)$ ✓

Hence, $\dfrac{\overline{(-1-\sqrt{3}\,i)}}{cis\left(\frac{\pi}{3}\right)} = \dfrac{2\,cis\left(\frac{2\pi}{3}\right)}{cis\left(\frac{\pi}{3}\right)} = 2\ cis\left(\frac{\pi}{3}\right)$ ✓✓ [3]

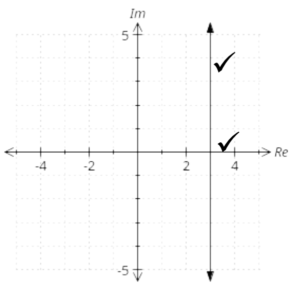
(c) $w^3 = (-4 - 4\sqrt{3}\ i) \quad \Rightarrow \quad w^3 = 8\ cis\left(-\frac{2\pi}{3}\right)$ ✓

$$w = \left[8\,cis\left(-\frac{2\pi}{3}\right)\right]^{\frac{1}{3}} = 2 \times \left[cis\left(\frac{-\frac{2\pi}{3} + 2n\pi}{3}\right)\right]$$

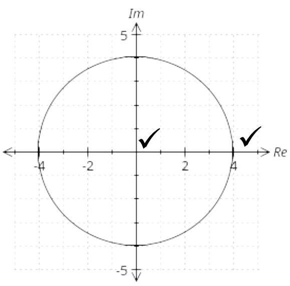
Hence, $w = 2\ cis\left(-\frac{2\pi}{9}\right),\ 2\ cis\left(\frac{4\pi}{9}\right),\ 2\ cis\left(-\frac{8\pi}{9}\right)$ ✓✓✓ [5]

✓ ✓

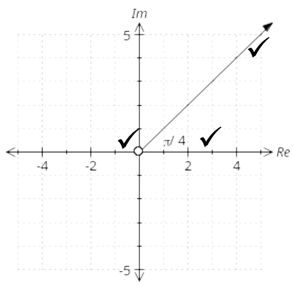
(d) (e) (f)



[2]



[2]



[3]

9. [7 marks]

Let constant acceleration be represented by a.

Then, velocity $v = \int a \; dt$

$= at + \text{C}$ ✓

Also, displacement $x = \int at + \text{C} \; dt$

$= \dfrac{at^2}{2} + \text{C}t + \text{K}$ ✓

When $t = 0$, $x = 0 \Rightarrow \text{K} = 0$

Also, when $t = 6$, $x = 90 \Rightarrow 18a + 6\text{C} = 90$ ✓

and, when $t = 10$, $x = 180 \Rightarrow 50a + 10\text{C} = 180$ ✓

Hence, $a = 1.5$ and $\text{C} = 10.5$

Therefore, constant acceleration is 1.5 ms^{-2} . ✓ [5]

(b) When $t = 10$, $v = 1.5 \times 10 + 10.5 = 25.5 \text{ ms}^{-1}$ ✓✓ [2]

$= 91.8$ kph

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10. [11 marks]

(a) $\dfrac{d^2x}{dt^2} = -16x = -(4)^2x$. ✓

Hence, motion is simple harmonic with angular velocity $\omega = 4$. ✓

Therefore, $x = \text{A} \sin(4t + \alpha)$.

When $t = 0$, $x = 0 \Rightarrow \text{A} \sin \alpha = 0 \Rightarrow \alpha = 0$ ✓

The equation is now $x = \text{A} \sin 4t$.

Velocity $v = \dfrac{dx}{dt} = 4\text{A} \cos 4t$. ✓

When $t = 0$, $v = -1 \Rightarrow 4\text{A} = -1 \Rightarrow \text{A} = -\dfrac{1}{4}$ ✓

Hence, $x = -\dfrac{1}{4} \sin 4t$. ✓ [6]

(b) Distance travelled $= \displaystyle\int_0^{10} \left| -\sin \frac{t}{2} \right| dt$ ✓✓

$= 6.567$ ✓

Average speed $= \dfrac{6.567}{10} = 0.657 \text{ ms}^{-1}$ ✓✓

11. [8 marks]

8 .a) For line L to be parallel to π_1 the direction vector of must be perpendicular to the normal vector of π_1

$$\begin{pmatrix}1\\-5\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\4\end{pmatrix} = 1-5+4$$
$$= 0$$

$\therefore L \parallel \pi_1$

i) $\underset{\sim}{r}\cdot\underset{\sim}{n} = \underset{\sim}{a}\cdot\underset{\sim}{n}$

$$\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\4\end{pmatrix} = \begin{pmatrix}-3\\-1\\6\end{pmatrix}\begin{pmatrix}1\\1\\4\end{pmatrix}$$

$$\underline{r}\cdot\begin{pmatrix}1\\1\\4\end{pmatrix} = -3-1+24$$

$$\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\4\end{pmatrix} = 20$$

c) Line from origin that gives shortest distance is $\perp$ to Π_1 & Π_2 4

i.e. $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ 4\lambda \end{pmatrix}$

Π_1: $\begin{pmatrix} \lambda \\ \lambda \\ 4\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = 20$

$18\lambda = 20$

$\lambda = \frac{10}{9}$

distance $\frac{10}{9} \left| \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \right| = 10\frac{\sqrt{2}}{3}$ units

Π_2: $\begin{pmatrix} \lambda \\ \lambda \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = 12$

$18\lambda = 12$

$\lambda = \frac{2}{3}$

distance $= \frac{2}{3} \left| \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \right| = 2\sqrt{2}$ units

distance between Π_1 & Π_2

$= 10\frac{\sqrt{2}}{3} - 2\sqrt{2} = \frac{4\sqrt{2}}{3}$

12. [6 marks]

(a) $T = Ae^{km} + 25 \quad \Rightarrow \quad \dfrac{dT}{dm} = Ae^{km} . k$ ✓

But $Ae^{km} = T - 25 \quad \Rightarrow \quad \dfrac{dT}{dm} = k\,(T - 25)$ ✓ [2]

(b) When $m = 9$, T = 86 $\quad \Rightarrow \quad 86 = A\,e^{9k} + 25$

$A\,e^{9k} = 61$ ✓

When $m = 15$, T = 76 $\quad \Rightarrow \quad 76 = A\,e^{15k} + 25$

$A\,e^{15k} = 51$ ✓

Hence, $\quad e^{6k} = \dfrac{51}{61}$

$k = -\,0.0298414 = -0.03$ ✓

and A = 80 (nearest degree) ✓ [4]

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13. [10 marks]

(a) Area = $\displaystyle\int_0^2 e^{-(x-2)^2}\,dx$ ✓✓ = 0.8821 ✓ [3]

(b) Area between x =2 and x = 50 = $\displaystyle\int_2^{50} e^{-(x-2)^2}\,dx = 0.8862$

Area between x =2 and x = 100 = $\displaystyle\int_2^{100} e^{-(x-2)^2}\,dx = 0.8862$

Area between x =2 and x = 200 = $\displaystyle\int_2^{100} e^{-(x-2)^2}\,dx = 0.8862$ ✓✓✓

Hence, the area to the right of x = 2 approaches 0.8862 (limiting area) ✓ [4]

(c) Volume = $\displaystyle\pi\int_0^2 \left[e^{-(x-2)^2}\right]^2 dx$ ✓✓

$= 0.6266\pi = 1.9686$ ✓ [3]

14. [5 marks]

Rewrite $P = \frac{2\pi}{\sqrt{g}} x^{\frac{1}{2}}$. $\Rightarrow$ $\frac{dP}{dx} = \frac{2\pi}{\sqrt{g}} \times \frac{1}{2} x^{-\frac{1}{2}}$ ✓

Hence, $\delta P \approx \frac{dP}{dx} \times \delta x$

$= \frac{2\pi}{\sqrt{g}} \times \frac{1}{2} x^{-\frac{1}{2}} \times \delta x = \frac{\pi x^{-\frac{1}{2}}}{\sqrt{g}} \times \delta x$ ✓

If x increases by 1%, then, $\delta x = 0.01 \times x$;

$\delta P \approx \frac{\pi x^{-\frac{1}{2}}}{\sqrt{g}} \times 0.01x$ ✓

$= 0.01 \times \frac{\pi x^{\frac{1}{2}}}{\sqrt{g}}$ ✓

$= 0.01 \times \frac{P}{2} = 0.005P$ ✓ [5]

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15. [9 marks]

(a) $\frac{dV}{ds} = \frac{25}{\sqrt{s}}$. ✓

Hence, when $s = 36$, $\frac{dV}{ds} = \frac{25}{6}$ ✓ [2]

(b) $\frac{dV}{dt} = \frac{25}{\sqrt{s}} \times -\frac{3}{4} t^{\frac{1}{2}}$ ✓

When $s = 36$, $100 - 0.5t^{\frac{3}{2}} = 36$ $\Rightarrow$ $t = 25.3984$. ✓✓

Hence, when $s = 36$, $\frac{dV}{dt} = -15.75$ km^3 min^{-1} ✓ [4]

(c) When it hits the earth, $s = 0$.

Speed = $\left|\frac{dS}{dt}\right| = \left|-\frac{3}{4}t^{\frac{1}{2}}\right| = \frac{3}{4}t^{\frac{1}{2}}$ ✓

When $s = 0$, $100 - 0.5t^{\frac{3}{2}} = 0$ $\Rightarrow$ $t = 34.1995$ ✓

Hence, when $s = 0$, speed = 4.39 km min^{-1} ✓ [3]

16. [12 marks]

	Domain	Range
$f(x)$	$x \neq 1$ ✓	$y \neq 1$ ✓
$f^{-1}(x)$	$x \neq 1$ ✓	$y \neq 1$ ✓
$g(x)$	$x \neq 1$ ✓	$y \geq 0$ ✓✓
$h(x)$	$x \neq \pm 1$ ✓	$y \leq -1$ and $y > 1$ ✓✓
$f(f(x))$	$x \neq 1$ ✓	$y \neq 1$ ✓

17. [10 marks]

(a) $\boldsymbol{a} = -9.8\,\mathbf{j} \quad \Rightarrow \quad \boldsymbol{v} = \int -9.8\mathbf{j}\,dt = -9.8t\,\mathbf{j} + \boldsymbol{c}$ ✓

Since $\boldsymbol{v}(0) = 100\cos 30\,\mathbf{i} + 100\sin 30\,\mathbf{j} = 50\sqrt{3}\,\mathbf{i} + 50\,\mathbf{j}$,

$\boldsymbol{c} = 50\sqrt{3}\,\mathbf{i} + 50\,\mathbf{j}$

Hence, $\boldsymbol{v} = 50\sqrt{3}\,\mathbf{i} + (50 - 9.8t)\,\mathbf{j}$ ✓

$\boldsymbol{r} = \int 50\sqrt{3}\,\mathbf{i} + (50 - 9.8t)\,\mathbf{j}\,dt = 50\sqrt{3}t\,\mathbf{i} + (50t - 4.9t^2)\,\mathbf{j} + \boldsymbol{k}$ ✓

Since $\boldsymbol{r}(0) = \mathbf{0}$, $\boldsymbol{k} = \mathbf{0} \Rightarrow \boldsymbol{r} = 50\sqrt{3}t\,\mathbf{i} + (50t - 4.9t^2)\,\mathbf{j}$ [3]

(b) Rocket lands when $\boldsymbol{r}_y = 5\,\mathbf{j}$.

Hence, $(50t - 4.9t^2) = 5$ ✓

$\Rightarrow \quad t \approx 10.1031$ ✓ [2]

(c) $\boldsymbol{r} = 100\cos\theta\, t\,\mathbf{i} + (100\sin\theta\, t - 4.9t^2)\,\mathbf{j}$ ✓

Rocket lands at the point $700\,\mathbf{i} + 5\,\mathbf{j}$. ✓

Hence, $100\cos\theta\, t = 700 \quad \Rightarrow \quad t = \dfrac{7}{\cos\theta}$ ✓

But $100\sin\theta\, t - 4.9t^2 = 5$, $\Rightarrow \; 100\sin\theta \times \dfrac{7}{\cos\theta} - 4.9\left(\dfrac{7}{\cos\theta}\right)^2 = 5$ ✓

$\theta = 22.10^{\circ}$ or 68.27° ✓ [5]

18.

$$\frac{dp}{dq} = 2pq(p+3)$$

Separate variables, $\frac{dp}{p(p+3)} = 2qdq$

Thus $\int \frac{dp}{p(p+3)} = \int 2qdq$

Expressing as partial fractions $\frac{1}{p(p+3)} = \frac{A}{p} + \frac{B}{p+3}$

Thus $1 = A(p+3) + Bp$

This means $1 = 3A$ and $A = \frac{1}{3}$.

$0 = A + B$. Hence $B = \frac{-1}{3}$.

Thus $\frac{1}{3}\int\left(\frac{1}{p} - \frac{1}{p+3}\right)dp = \int 2qdq$

Hence $\frac{1}{3}\ln\frac{p}{p+3} = q^2 + c$

Which rearranges to $\frac{p}{p+3} = Ae^{3q^2}$, where $A = e^{3c}$.

This also can be written as $p = Ape^{3q^2} + 3Ae^{q^2}$.

Making p the subject we have $p(1 - Ae^{3q^2}) = 3A^{3q^2}$

Hence $p = \frac{3Ae^{3q^2}}{1 - Ae^{3q^2}} = \frac{3}{Be^{-3q^2} - 1}$, where $B = \frac{1}{A}$

Since $p = 10$, when $q = 3$, by substitution we obtain $10 = \frac{3}{Be^{-27} - 1}$.

Thus $B = 6.916 \times 10^{11}$

19.

4a) $x = \sin 2t$

$$x^2 = (\sin 2t)^2$$

$$x^2 = (2\sin t\cos t)^2$$

$$x^2 = 4\sin^2 t\cos^2 t$$

$$x^2 = 4(1-\cos^2 t)\cos^2 t$$

$$x^2 = 4\cos^2 t - 4\cos^4 t$$

$$x^2 = (2\cos t)^2 - \frac{1}{4}(2\cos t)^4$$

$$x^2 = y^2 - \frac{1}{4}y^4$$

$$4x^2 + y^4 = 4y^2$$

b) $4x^2 + y^4 = 4y^2$

$$8x + 4y^3 \frac{dy}{dx} = 8y \frac{dy}{dx}$$

$$8(1) + 4(\sqrt{2})^3 \frac{dy}{dx} = 8\sqrt{2} \frac{dy}{dx}$$

$$8 = \frac{dy}{dx}\left(8\sqrt{2} - 8\sqrt{2}\right)$$

$$\frac{dy}{dx} = \frac{8}{0}$$

$$\frac{dx}{dy} = \frac{0}{8}$$

$$\frac{dx}{dy} = 0$$

$\therefore$ $\underline{x = 1}$