MATHEMATICS
METHODS
UNIT 3

Semester One

2018

SOLUTIONS

Calculator-free Solutions

1.. $f\,'(x) = 0$ when $\dfrac{1-x}{e^x} = 0$ ✓

$\therefore \quad x = 1$ ✓

$f\,''(x) = \dfrac{(e^x)(-1) - e^x(1-x)}{e^{2x}}$ ✓

Since $f\,''(1) < 0$, then maximum ✓

[4]

2. $A = \displaystyle\int_0^k e^{3-x}\,dx$ ✓

$\therefore \quad \left[-e^{3-x}\right]_0^k = \left(-e^{3-k}\right) - \left(-e^{3-0}\right)$ ✓

$= \quad -e^{3-k} + e^3$ ✓

If $A = e^3\left(1 - e^{-k}\right) = e^3$ then $e^{-k} = 0$, which is impossible. ✓

If $A = e^3\left(1 - e^{-k}\right) = e^3 - 1$ ✓

$\therefore \quad k = 3$ ✓ [6]

3. (a) Stationary point occurs when $3x^2 - kx = 0$

$\therefore \quad M\,'(6) = 108 - 6k = 0$ ✓

$\therefore \quad k = 18$ ✓

(b) $M(x) = x^3 - 9x^2 + c$ ✓

Since $M(6) = 1 \rightarrow 216 - 324 + c = 1 \rightarrow c = 109$ ✓

$\therefore \quad M(x) = x^3 - 9x^2 + 109$ ✓

$\therefore$ Vertical intercept = 109 ✓ [6]

4. (a) $e^1 = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!}$ ✓

$= 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} = 2\frac{17}{24}$ ✓

(b) $\dfrac{d}{dx}\left(1 + \dfrac{x}{1} + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \;\ldots\right)$

$= \quad 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \;\ldots$ ✓

$= \quad 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \;\ldots = e^x$ ✓ [4]

5. (a) $\Sigma f(x) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$ ✓

and $0 \leq f(x) \leq 0$ for all x ✓

$\therefore$ PDF

(b) $\Sigma f(x) = 1$ ✓

and $0 \leq f(x) \leq 0$ for all x ✓

$\therefore$ PDF

(c) Since $f(0) = -\frac{1}{3}$

$f(x)$ is not greater than or equal to 0 for all x ✓

$\therefore$ Not PDF ✓ [6]

6. (a) Since PDF $\Sigma f(x) = 0{\cdot}8 + a = 1 \rightarrow a = 0{\cdot}2$ ✓

(b) $P(X < 2 \mid X \leq 2) = \frac{P(X < 2)}{P(X \leq 2)}$ ✓

$= \frac{0{\cdot}6}{0{\cdot}7} = \frac{6}{7}$ ✓

(c) E(X) = 0 + 0.1 + 0.2 + 0.3 + 0.8 = 1.4 ✓✓

(d) Variance = (0 + 1 x 0.1 + 4 x 0.1 + 9 x 0.1 + 16 x 0.2) - 1.4^2 ✓✓

= 0.1 + 0.4 + 0.9 + 3.2 - 1.96 = 2.64 ✓ [8]

7. E(Y) = 2E(X) + 3 = 8

$\therefore$ $m = 2.5$ ✓

VAR(Y) = 4VAR(X) = 20

$\therefore$ $v = 5$ ✓✓ [3]

8. (a) $e^x \sin(e^x)$ ✓✓

(b) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(e^x \sin e^x)$ ✓

$= e^x \sin(e^x) + e^x \cos(e^x)(e^x)$ ✓✓ [5]

9. (a) $y = (\sin x)^{-1}$

$\frac{dy}{dx} = -1(\sin x)^{-2}(\cos x)$ ✓

$= \frac{-\cos x}{\sin^2 x}$ ✓

(b) $\int \frac{5\cos x}{1 - \cos^2 x}\, dx$

$= 5\int \frac{\cos x}{\sin^2 x}\, dx$ ✓

$= -5\int \frac{-\cos x}{\sin^2 x}\, dx$ ✓

$= -\frac{5}{\sin x} + c$ ✓ [5]

10. (a) $y = (ex)(e^x)$

$\therefore \dfrac{dy}{dx} = e(e^x) + (ex)(e^x)$ ✓

$\therefore \quad = e^{x+1}(x+1)$ ✓

(b) $y = \dfrac{\pi \sin x}{\cos x}$

$\therefore \dfrac{dy}{dx} = \dfrac{(\cos x)(\pi \cos x) - (\pi \sin x)(-\sin x)}{\cos^2 x}$ ✓

$= \dfrac{\pi(\cos^2 x + \sin^2 x)}{\cos^2 x}$ ✓

$= \dfrac{\pi \times 1}{\cos^2 x} = \dfrac{\pi}{\cos^2 x}$ ✓ [5]

<u>Calculator-assumed Solutions</u>

11. (a) $K = \dfrac{1}{2} r^2 \sin \theta \rightarrow \dfrac{dK}{d\theta} = 8\cos \theta$ when $r = 4$ ✓

$\delta K = \dfrac{dK}{d\theta} \times \delta\theta \rightarrow \delta K = 8\cos\theta \times (0{\cdot}05\pi)$ ✓

$\therefore \delta K = 8\cos \dfrac{\pi}{4} \times 0{\cdot}05\pi$ when $\theta = \dfrac{\pi}{4}$

$\therefore \delta K = 0{\cdot}889$ ✓

(b) $\Delta K = \dfrac{1}{2} 16\left(\sin 0.3\pi - \sin \dfrac{\pi}{4}\right)$

$\therefore \Delta K = 0{\cdot}8153$ ✓

$\therefore$ Error $= \dfrac{0{\cdot}8886 - 0{\cdot}8153}{0{\cdot}8886} \times 100 = 8{\cdot}3\%$ ✓ [5]

12. $(-3, 0) \rightarrow -27a - 9b - 3c - 9 = 0$ ✓

$\dfrac{dy}{dx} = 3ax^2 - 2bx + c$ ✓

Stationary point at $x = -3 \rightarrow 0 = 27a + 6b + c$ ✓

$\dfrac{d^2y}{dx^2} = 6ax - 2b$ ✓

POI at $-\dfrac{5}{3} \rightarrow -10a - 2b = 0$ ✓

$\therefore$ Using calc $a = 1$, $b = -5$ and $c = 3$ ✓✓ [7]

13. (a) $f(x) = -4\cos\frac{x}{2} + c$ ✓

$\therefore$ $f\left(\frac{\pi}{2}\right) = -4\cos\frac{\pi}{4} + c = 4 - 2\sqrt{2} \rightarrow c = 4$ ✓

$\therefore$ $f(x) = -4\cos\frac{x}{2} + 4$ ✓

(b) $-4\cos\frac{x}{2} + 4 = 4 - 2\sqrt{2}$

$\therefore$ $\cos\frac{x}{2} = \frac{\sqrt{2}}{2} \rightarrow \frac{x}{2} = \frac{\pi}{4}, \frac{7\pi}{4}, \ldots$ ✓✓

$\therefore$ Next time is $x = \frac{7\pi}{2}$ ✓ [6]

14. (a) (i) 0 + 1 + (-5) + 4 = 0 ✓✓

(ii) -3 + 1 + (-5) + 4 = -3 ✓✓

(iii) 1 + 5 + 4 = 10 ✓✓

(b) (i) $-5 + 4 + \int_2^4 7 \; dx = -1 + (28 - 14) = 13$ ✓✓

(ii) $2\int_3^4 f(x)\,dx = 2(4) = 8$ ✓✓ [10]

15. (a) (i) $X \sim B(30, 0{\cdot}75)$ where X = the number graduated ✓

$\therefore$ $P(X = 25) = 0{\cdot}1047$ ✓✓

(ii) $P(X \geq 29 \mid X \geq 25) = \frac{0{\cdot}00196}{0{\cdot}2026}$ ✓✓

= 0.00967 ✓

(b) $Y \sim B(n, 0{\cdot}75)$ where Y is the number who graduated out of n

$\therefore$ $P(Y \geq 10) \geq 0{\cdot}99 \rightarrow n = 19$ using trial and error ✓✓✓

n	Y
18	0.981
19	0.991
20	0.996

(c) (i) $\dfrac{\binom{5}{3}\binom{6}{0} + \binom{5}{2}\binom{6}{1} + \binom{5}{1}\binom{6}{2}}{\binom{11}{3}}$ ✓✓

$= \frac{10 + 60 + 75}{165} = 0{\cdot}87879$ ✓

(ii) $P(S = n) = \begin{cases} \frac{5}{11}, & n = 1 \\ \frac{6}{11}, & n = 0 \end{cases}$ ✓✓✓ [15]

16. (a) $2\cos x \sin x + \cos x = 0$

$\therefore \quad \cos x = 0$ or $\sin x = -\dfrac{1}{2}$

$\therefore \quad x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$ ✓✓

(b) $\left(\dfrac{\pi}{2}, 0\right), \left(\dfrac{3\pi}{2}, 0\right), \left(\dfrac{7\pi}{6}, \dfrac{\sqrt{3}}{2}\right), \left(\dfrac{11\pi}{6}, -\dfrac{\sqrt{3}}{2}\right)$ ✓✓

(c) A = $\displaystyle\int_{\frac{\pi}{2}}^{\frac{7\pi}{6}} (-\cos x - \sin 2x)\ dx$ ✓

= 2.25 units^2 ✓ [6]

17. (a) $\dfrac{x+1}{x-1} = 0 \rightarrow x = -1$ ✓

(b) (i) $\dfrac{dy}{dx} = \dfrac{(x-1)-(x+1)}{(x-1)^2} = \dfrac{-2}{(x-1)^2}$ ✓

(ii) True since $\dfrac{dy}{dx} \neq 0$ for any value of x. ✓

(c) (i) $\dfrac{d^2y}{dx^2} = \dfrac{4}{(x-1)^3}$ ✓

(ii) True since $\dfrac{d^2y}{dx^2} \neq 0$ for any value of x. ✓ [5]

18. (a) $v(0) = 3$ ✓

(b) Stationary when $2t^2 - 5t + 3 = 0$ ✓

$\therefore \quad t = 1$ and 1.5 ✓✓

(c) (i) $a = 4t - 5$ ✓

(ii) v is a minimum when a = 0 $\rightarrow a = 1{\cdot}25$ ✓✓

(d) $x = \dfrac{2}{3}t^3 - \dfrac{5}{2}t^2 + 3t + c$ ✓

Since $x(0) = 0 \rightarrow c = 0$

$\therefore \quad x(t) = \dfrac{2}{3}t^3 - \dfrac{5}{2}t^2 + 3t$ ✓

$\therefore \quad x(3) = 4{\cdot}5$ ✓

(e) Distance = $\displaystyle\int_0^3 |v(t)|\ dt$ ✓

= 4.58 ✓

(f) Distance travelled ≠ Displacement ✓ [13]

19. (a) $A = A_0\, e^{0\cdot 05t}$ ✓

$\therefore \quad A = 0\cdot 6e^{0\cdot 05(10)} = 0\cdot 99$ Ha ✓✓

(b) $5 = 0\cdot 6e^{0\cdot 05t}$ ✓

$\therefore \quad t = 42.4$ hours ✓✓ [6]

20. (a) (i) $f'(x) = -\sin x + \cos x$ ✓

(ii) $f''(x) = -\cos x - \sin x$ ✓

(b) Maximum value occurs when $f\,'(x) = 0$

$\therefore \quad -\sin x + \cos x = 0 \rightarrow f(x) = \sqrt{2}$ when $x = \dfrac{\pi}{4}$ ✓✓

Since $f''\left(\dfrac{\pi}{4}\right) < 0$ then maximum ✓✓

(c) POI occurs when $f\,''(x) = 0$

$\therefore \quad -\cos x - \sin x = 0 \rightarrow \tan x = -1$ ✓

$\therefore \quad x = \dfrac{3\pi}{4}$ or $\dfrac{7\pi}{4}$ ✓✓

$\therefore \quad \left(\dfrac{3\pi}{4}, 0\right)$ and $\left(\dfrac{7\pi}{4}, 0\right)$ ✓ [10]

21. (a) 10 L/sec ✓

(b) $10 - \dfrac{t}{20} = 0 \rightarrow t = 200$ secs ✓

(c) $F = \displaystyle\int_0^{200} \left(10 - \dfrac{t}{20}\right)$ dt ✓

$=$ 1000 L ✓ [4]

22. (a) $r' = e^{-t}$ ✓

$\therefore \quad r\,'(4) = e^{-4} = 0\cdot 018$ ✓

and $r\,'(5) = e^{-5} = 0\cdot 0067$ ✓

(b) $r''(t) = -e^{-t}$ ✓

Since $-e^{-t} \neq 0$, then the fire is spreading at its greatest rate at the start of the fire,
Or the graph has greatest slope at t = 0. ✓

(c) $A = \pi r^2 \rightarrow \dfrac{dA}{dt} = 2\pi r$ ✓

$\therefore \quad \dfrac{dA}{dt} = \dfrac{dA}{dr} \times \dfrac{dr}{dt} = 2\pi r \times e^{-t}$ ✓

When $t = 4$ $\dfrac{dA}{dt} = 2\pi(-e^{-4} + 4)(0\cdot 018) = 0\cdot 458\ \text{cm}^2/\text{sec}$ ✓✓ [9]