PERTH MODERN SCHOOL
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Semester One Examination, 2021

Question/Answer booklet

MATHEMATICS SPECIALIST
UNIT 3

Section Two:
Calculator-assumed

Your Name

Your Teacher's Name

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Marks	Max	Question	Marks	Max
9			16		
10			17		
11			18		
12			19		
13			20		
14			21		
15					

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	49	34
Section Two: Calculator-assumed	14	14	100	96	66
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **(96 Marks)**

This section has **14** questions. Answer **all** questions. Write your answers in the spaces provided.

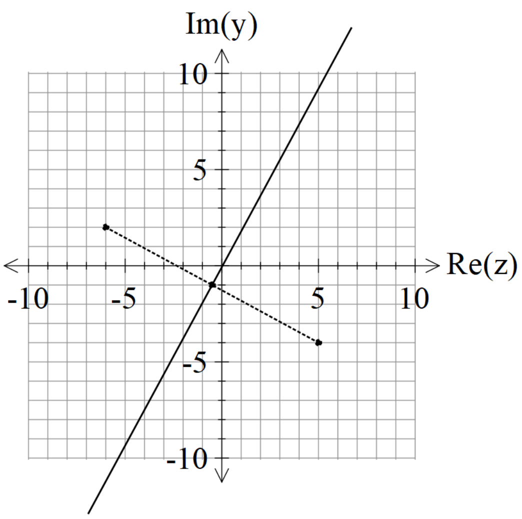
Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 100 minutes.

Question 9 **(6 marks)**

a) Sketch the locus of the equation $|z-5+4i|=|z+6-2i|$ on the axes below. (3 marks)

Solution

Specific behaviours
✓ refers to two points shown on graph ✓ uses perpendicular bisector ✓ shows locus

b) Determine the cartesian equation of this locus in terms of $x\,\&\,y$. (3 marks)

Solution

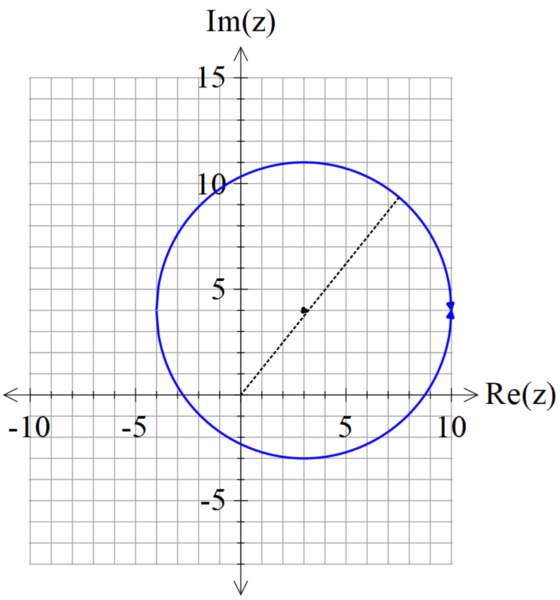
$\lvert z-5+4i\rvert=\lvert z+6-2i\rvert$ $\sqrt{(x-5)^2+(y+4)^2}=\sqrt{(x+6)^2+(y-2)^2}$ $x^2-10x+25+y^2+8y+16=x^2+12x+36+y^2-4y+4$ $1=22x-12y$
Specific behaviours
✓ subs z=x+iy ✓ squares both sides and expand real and imaginary terms ✓ states cartesian equation, no need to simplify

Question 10 (9 marks)

Consider the locus $\lvert z-3-4\rvert=7$ as graphed below.

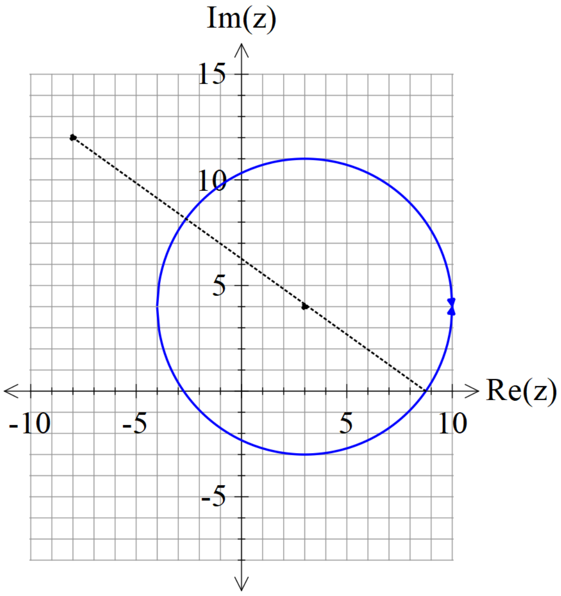
Determine the following.

a) Maximum value of $\lvert z\rvert$. (2 marks)

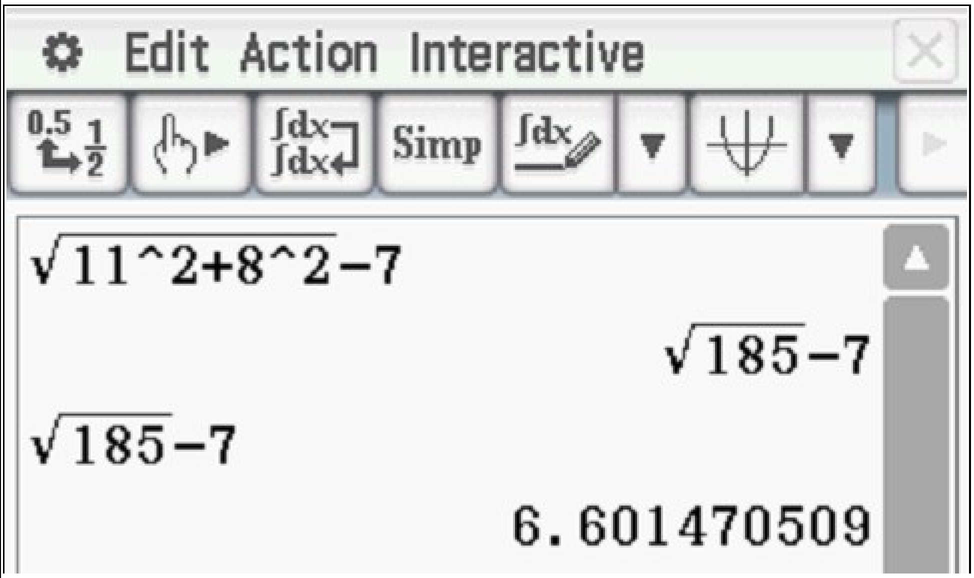
b) Solution
 $\lvert z\rvert=\sqrt{3^2+4^2}+7=12$
Specific behaviours
✓ uses modulus of centre ✓ determines maximum

c) Minimum value of $\lvert z+8-12i\rvert$ (3 marks)

Solution



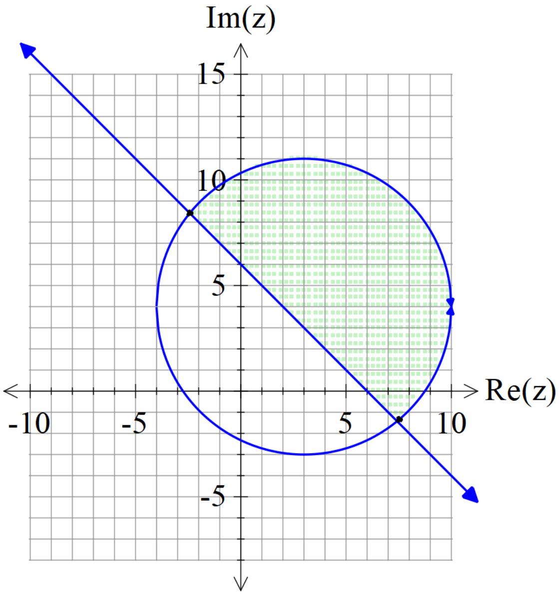
$$\sqrt{(-8-3)^2+(12-4)^2}-7$$

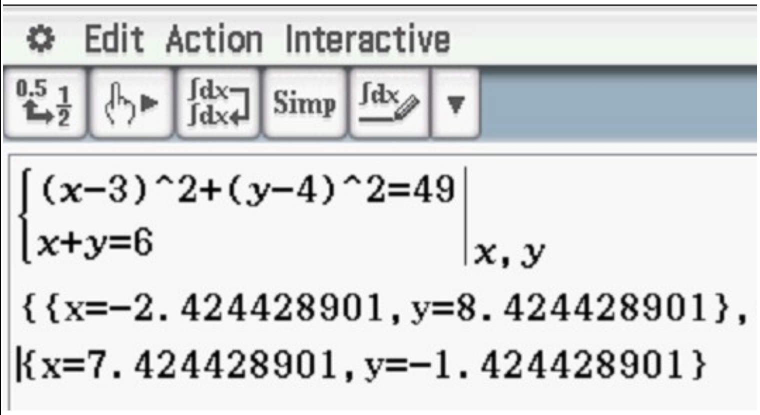


Specific behaviours
✓ determines distance from (-8,12) to centre ✓ subtracts radius ✓ states approx. distance or exact

d) Sketch the region defined by $|z-3-4i|\le 7$ and $\mathrm{Im}(z)+\mathrm{Re}(z)\ge 6$ on the axes above stating the coordinates of all boundary points. (4 marks)

Solution





Specific behaviours
✓ sketches line ✓ shades above line and within circle to give required region ✓ states approx. coords for one point ✓ states approx. coords for two points

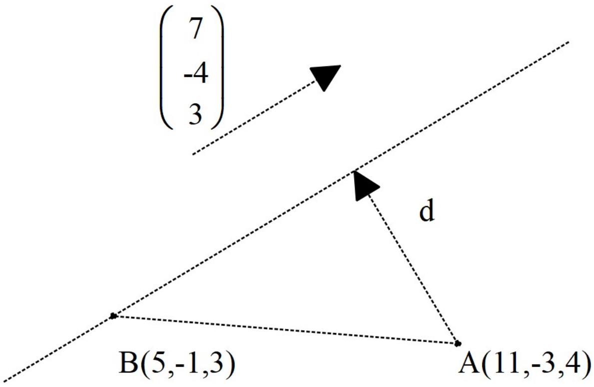
Question 11 **(6 marks)**

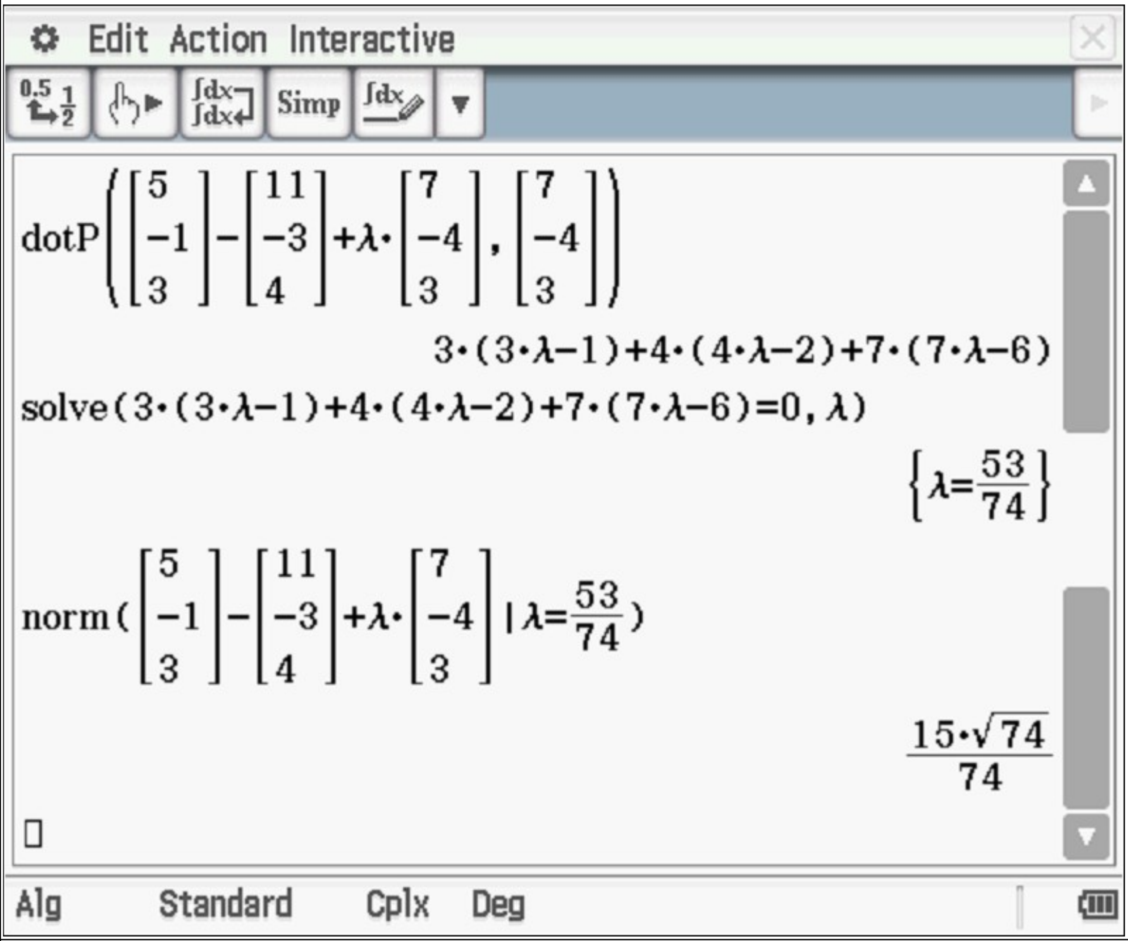
Consider the line $r = \begin{pmatrix}5\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}7\\-4\\3\end{pmatrix}$ and the point A $(11,-3,4)$.

a) Using **scalar dot** product show how to find the closest distance of point A to the line above.

(3 marks)

Solution



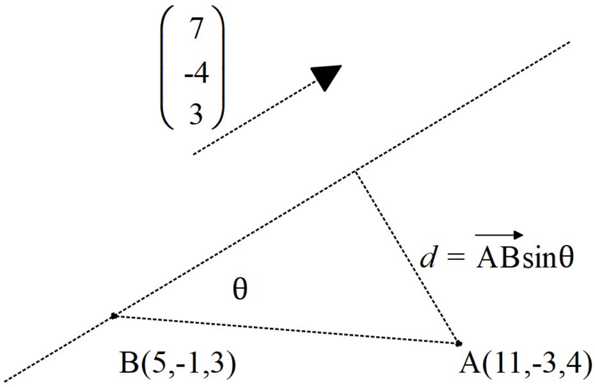


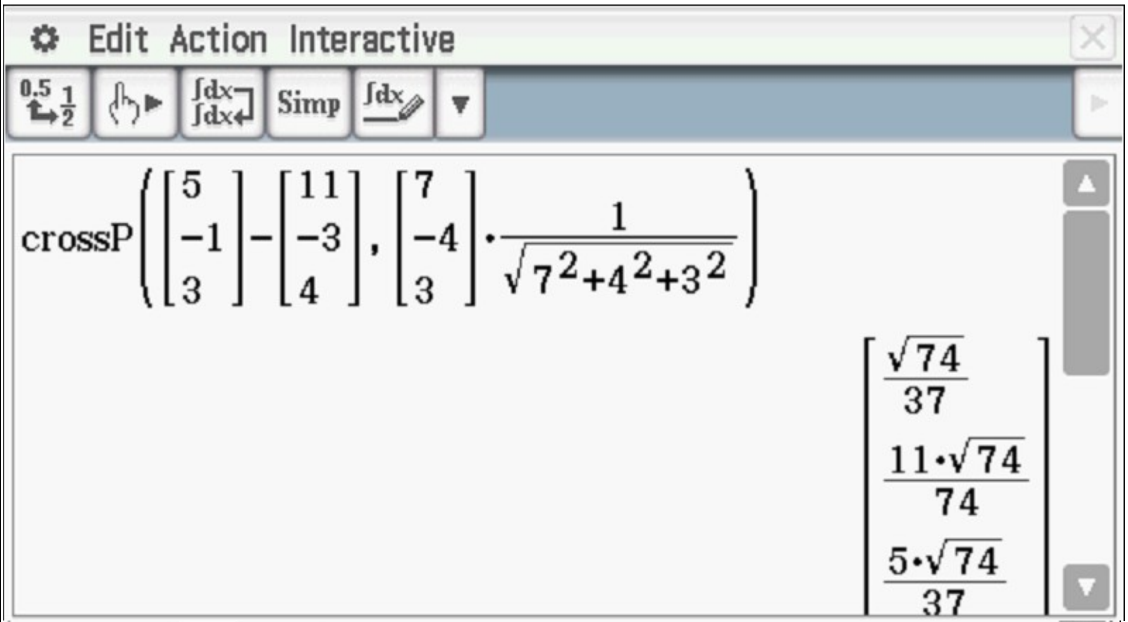
Specific behaviours
✓ obtains expression for displacement vector ✓ uses dot product and equates to zero ✓ determines magnitude, accept approx

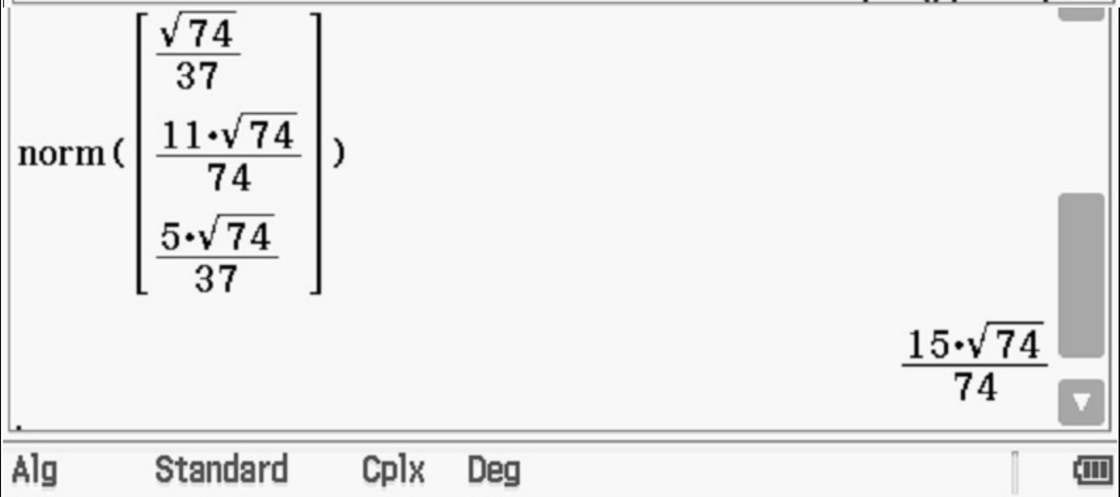
b) Using vector **cross** product show how to find the closest distance of point A to the line above.

(3 marks)

Solution







Specific behaviours

✓ shows that sine is needed
✓ uses cross product with unit vector
✓ determines magnitude of cross product

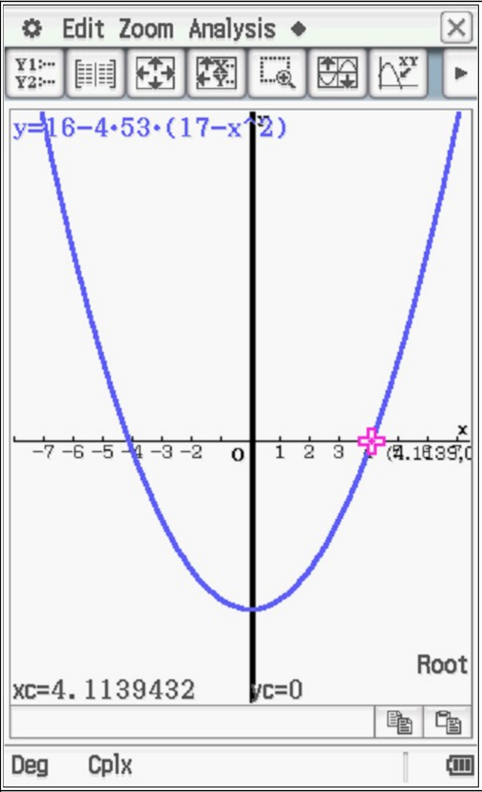
Question 12 **(9 marks)**

Consider the sphere $\left| r - \begin{pmatrix} 7 \\ -5 \\ 1 \end{pmatrix} \right| = \alpha$ with α being a positive constant and the line $r = \begin{pmatrix} 9 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -4 \\ 1 \end{pmatrix}$.

Determine all possible values of α, (2 decimal places) for the following.

i) The line does not meet the sphere at all.
ii) The line just touches the sphere at one point only.
iii) The line meets the sphere at two points.

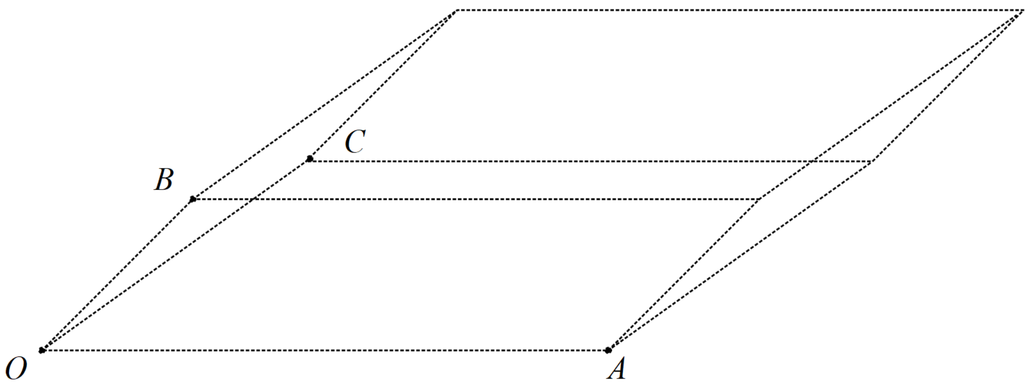
Solution
$\left\| \begin{pmatrix} 9+6\lambda \\ -2-4\lambda \\ 3+\lambda \end{pmatrix} - \begin{pmatrix} 7 \\ -5 \\ 1 \end{pmatrix} \right\| = \alpha$ $\left\| \begin{pmatrix} 2+6\lambda \\ 3-4\lambda \\ 2+\lambda \end{pmatrix} \right\| = \alpha$ $4+36\lambda^2+24\lambda+9+16\lambda^2-24\lambda+4+\lambda^2+4\lambda=\alpha^2$ $53\lambda^2+4\lambda+17-\alpha^2=0$ $\Delta = 16-4(53)(17-\alpha^2)$ $i) 0<\alpha<4.11$ $ii) \alpha = 4.11$ $iii) \alpha > 4.11$



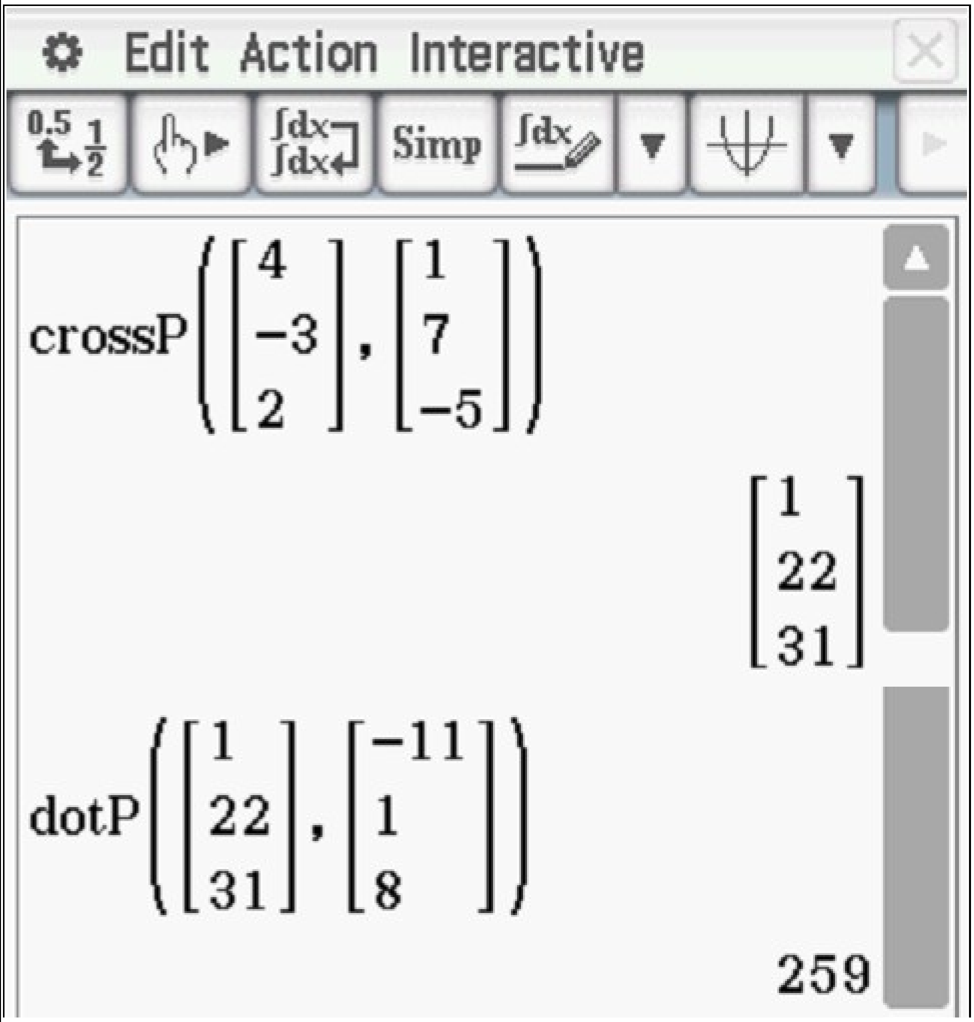
Specific behaviours
✓ subs line into sphere ✓ states a vector equation for parameter and alpha ✓ derives a quadratic equation for parameter in terms of alpha ✓ derives an expression for discriminant ✓ equates discriminant to zero ✓ writes two inequalities for discriminant ✓ states interval of values for no solution (discards negative values) and 2 dp ✓states value for touching ✓states values for meeting at two points (max -1 if not 2 dp)

Question 13 (4 marks)

Consider a prism where each side is a parallelogram with opposites sides congruent.
The units given are in metres.



Given that $\overrightarrow{OA} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} m, \overrightarrow{OB} = \begin{pmatrix} 1 \\ 7 \\ -5 \end{pmatrix} m, \overrightarrow{OC} = \begin{pmatrix} -11 \\ 1 \\ 8 \end{pmatrix} m$ and **using vector** methods, determine the volume of the prism.

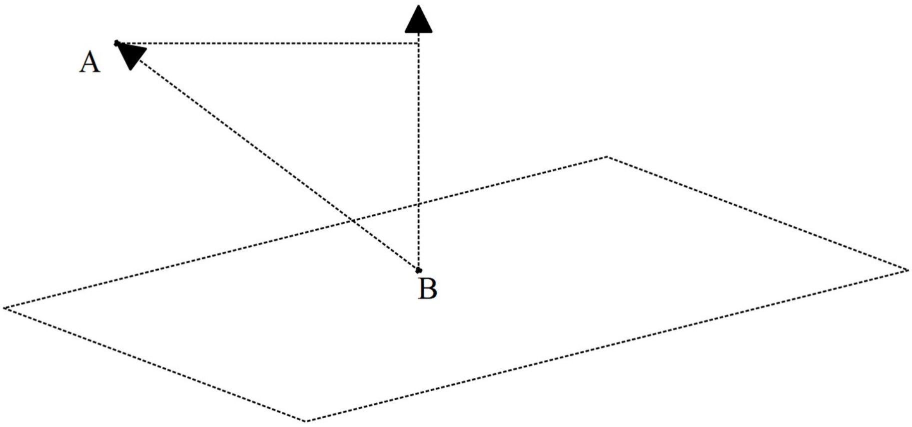
Solution
$V = OA \times OB . OC$  Volume = 259 cubic metres
Specific behaviours
✓ uses vectors in calculation ✓ uses area of face in calculation OR cross product ✓ uses perpendicular width OR dot product with normal of face ✓ determines volume with units

Question 14 (9 marks)

Consider the plane Π $5x - 2y + 6z = 9$.

a) Determine the distance of point A $(11, -3, 4)$ from the plane Π. (4 marks)

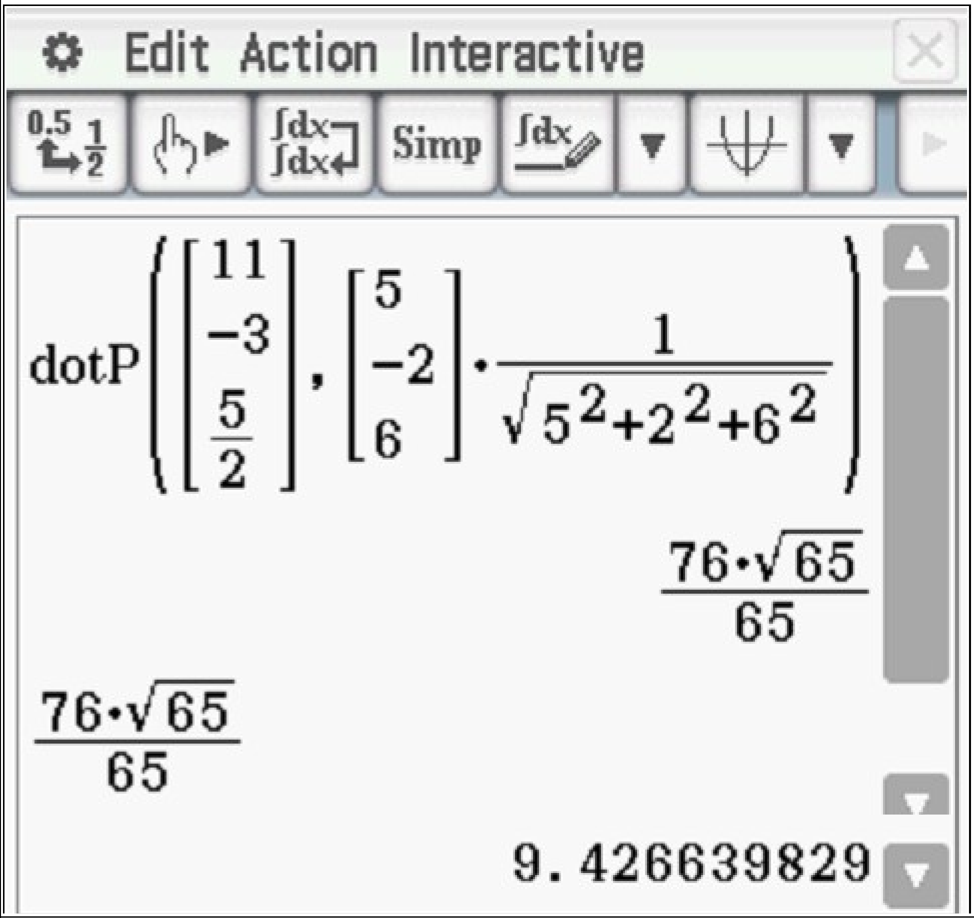
Solution



$B(0,0,\frac{9}{6})$

$$\vec{BA}=\begin{pmatrix}11\\-3\\4\end{pmatrix}-\begin{pmatrix}0\\0\\\frac{3}{2}\end{pmatrix}=\begin{pmatrix}11\\-3\\\frac{5}{2}\end{pmatrix}$$

$$\vec{d}=\left|BA.\hat{n}\right|$$



Specific behaviours
✓ determines any point on plane B OR vector equation of line ✓ uses dot product ✓ uses normal vector ✓ determines approx. distance

b) Determine an expression in terms of $x, y \& z$ for the distance of point P (x, y, z) from the plane Π. (3 marks)

Solution
$B(0,0,\frac{9}{6})$ $\overrightarrow{BA} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z - \frac{3}{2} \end{pmatrix}$ $d = \lvert \overrightarrow{BA}.\hat{n} \rvert$ $\begin{pmatrix} x \\ y \\ z - \frac{3}{2} \end{pmatrix} . \begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix} \frac{1}{\sqrt{65}} = \frac{5x - 2y + 6z - 9}{\sqrt{65}}$ $d = \left\lvert \frac{5x - 2y + 6z - 9}{\sqrt{65}} \right\rvert$
Specific behaviours
✓ uses a vector of pt P to any point on plane ✓ uses dot product with unit normal ✓ determines expression within absolute value

c) If point A $(11, -3, 4)$ is on a plane parallel to Π, determine a vector equation for this parallel plane. (2 marks)

Solution
$r . \begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix} . \begin{pmatrix} 11 \\ -3 \\ 4 \end{pmatrix} = 85$ $r . \begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix} = 85$
Specific behaviours
✓ uses normal vector and dot product ✓ states vector equation of plane

Question 15 **(7 marks)**

Consider two submarines A & B moving in deep ocean with constant velocities

$$v_A = \begin{pmatrix} 9 \\ -2 \\ 5 \end{pmatrix} km/h \quad v_b = \begin{pmatrix} 12 \\ \frac{5}{2} \\ 3 \end{pmatrix} km/h$$

.

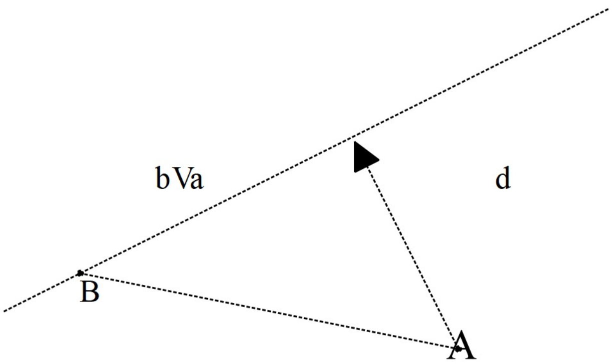
At 12:30am submarine A is at position $r_A = \begin{pmatrix} 11 \\ 8 \\ -5 \end{pmatrix} km$ and at 1am the same day submarine B is at position $r_B = \begin{pmatrix} 2 \\ -5.5 \\ 1 \end{pmatrix} km$.

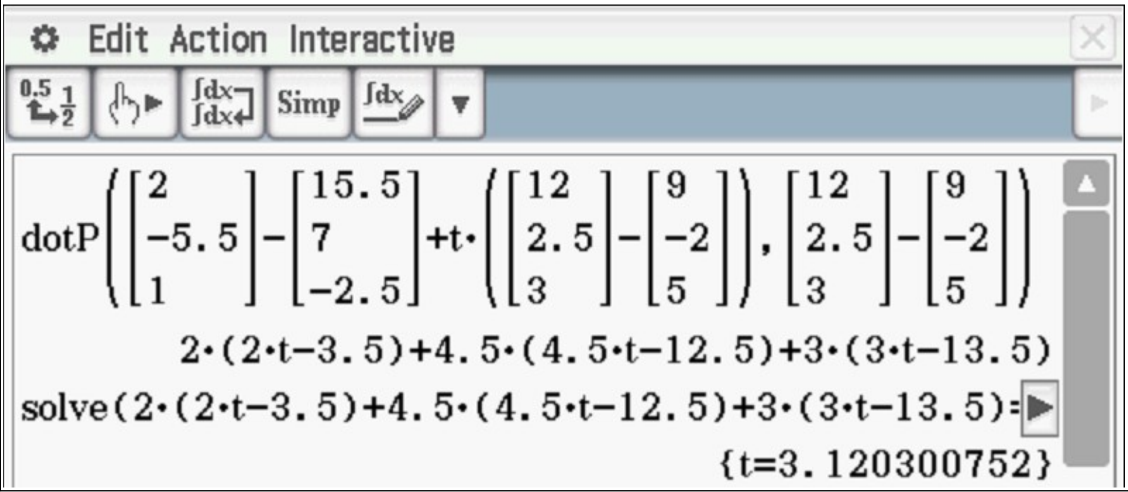
a) Determine the time, to nearest minute, that the submarines are closest to each other stating this distance to the nearest metre, (4 marks)

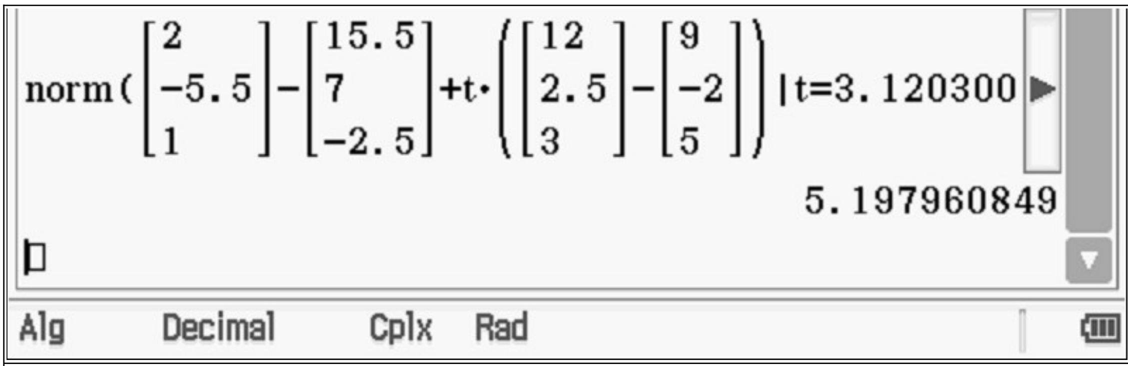
Solution

Let t=0 be at 1am

$$r_A = \begin{pmatrix} 11 \\ 8 \\ -5 \end{pmatrix} + 0.5 \begin{pmatrix} 9 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 15.5 \\ 7 \\ -2.5 \end{pmatrix} km$$







Closest approach at 4:07am at 5198 metres

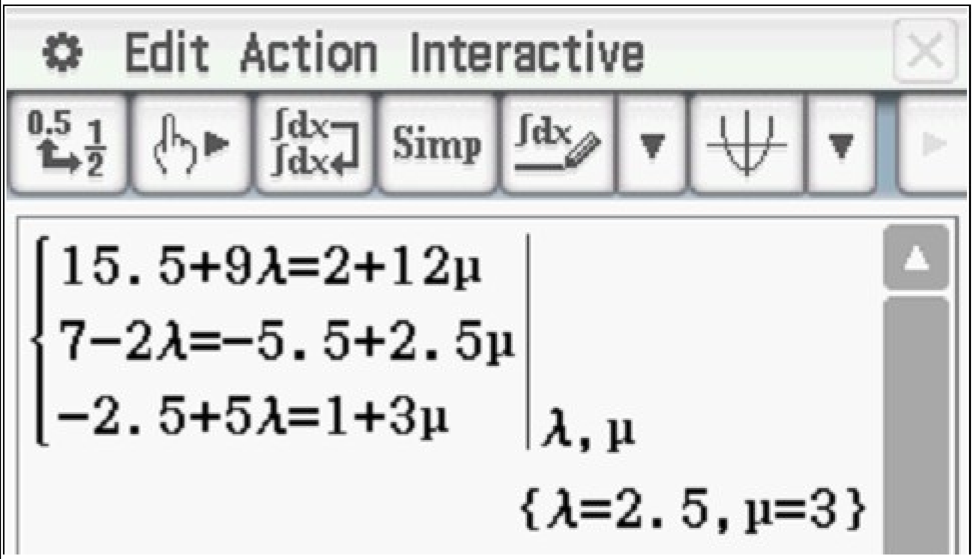
Specific behaviours

✓ determines position of both subs at the same time
✓ sets up equation to solve for time at closest approach (dot or calculus)
✓ states time at closest approach to nearest minute
✓ states distance rounded to nearest metre
(max -1 if not rounded)

b) If both submarines leave a lasting water trail of bubbles, determine if the trails cross and if they do at which position under water. (3 marks)

Solution

$$\begin{pmatrix}15.5\\7\\-2.5\end{pmatrix}+\lambda\begin{pmatrix}9\\-2\\5\end{pmatrix}=\begin{pmatrix}2\\-5.5\\1\end{pmatrix}+\mu\begin{pmatrix}12\\2.5\\3\end{pmatrix}$$



Bubble paths meet at (38,2,10)km

Specific behaviours

✓ sets up equations with different parameters
✓ shows that lines of bubbles do meet
✓ states coordinates of such point

Question 16 (5 marks)

Consider the complex numbers $s, p, w \,\&\, z$ such that:

$w = 1 + \sqrt{3}i$

$p = \sqrt{5}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i \right)$

$Arg\left(p\bar{z}\right) = \frac{7\pi}{12}$

$s = \frac{pw}{z}$

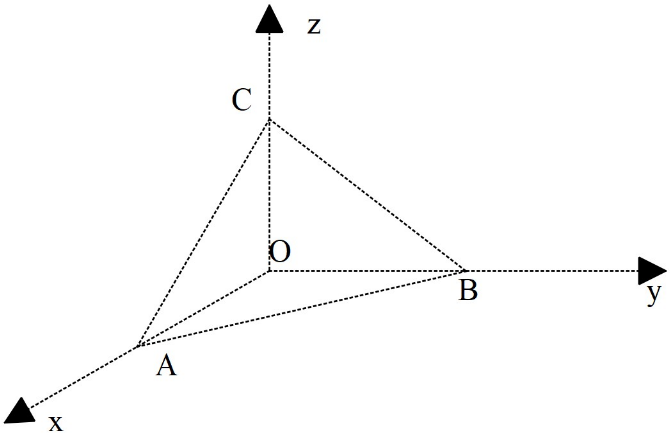
$|s| = \sqrt{10}$

Determine z in the form $z = x + iy$ where $x \,\&\, y$ are real numbers.

Solution
$Arg(p) = \tan^{-1}\frac{1}{-\sqrt{3}} = \frac{5\pi}{6}$ $\frac{5\pi}{6} + Arg(\bar{z}) = \frac{7\pi}{12}$ $Arg(\bar{z}) = \frac{-3\pi}{12}$ $Arg(z) = \frac{\pi}{4}$ $\lvert w\rvert = 2 \quad \lvert p\rvert = \sqrt{5}$ $\lvert z\rvert = \frac{2\sqrt{5}}{\sqrt{10}} = \sqrt{2}$ $z = \sqrt{2}cis\left(\frac{\pi}{4}\right) = 1 + i$
Specific behaviours
✓ determines Arg(p) ✓ sets up equation for Arg(z) conjugate ✓ determines modulus of p & w ✓ determines Arg(z) & modulus of z ✓ expresses z in cartesian form

Question 17 **(11 marks)**

Consider the 3D object $OABC$ as drawn below with O the origin and $A(5,0,0), B(0,4,0) \& C(0,0,3)$

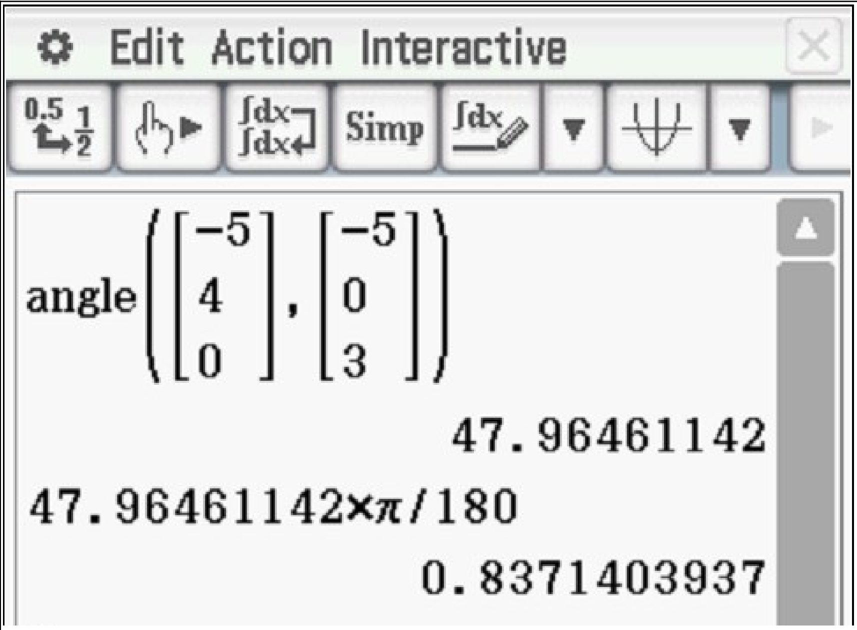


a) Determine the vectors $AB \& AC$. (2 marks)

Solution
$AB = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 0 \end{pmatrix}$ $AC = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 3 \end{pmatrix}$
Specific behaviours
✓ determines vector AB ✓ determines vector AC

b) Determine to the nearest degree the angle $\angle CAB$ (2 marks)

Solution



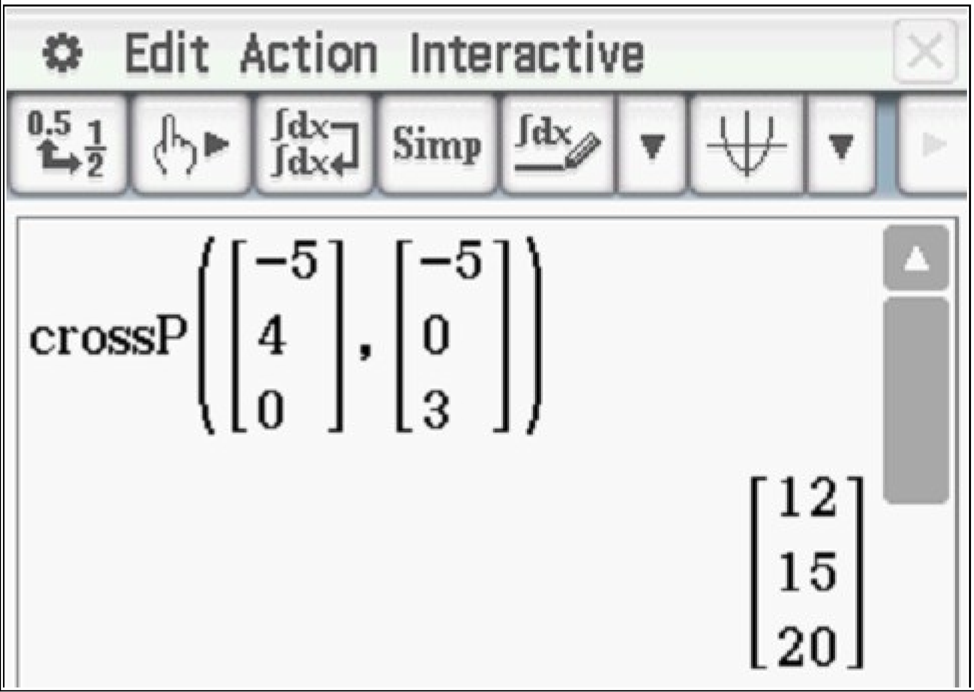
Specific behaviours

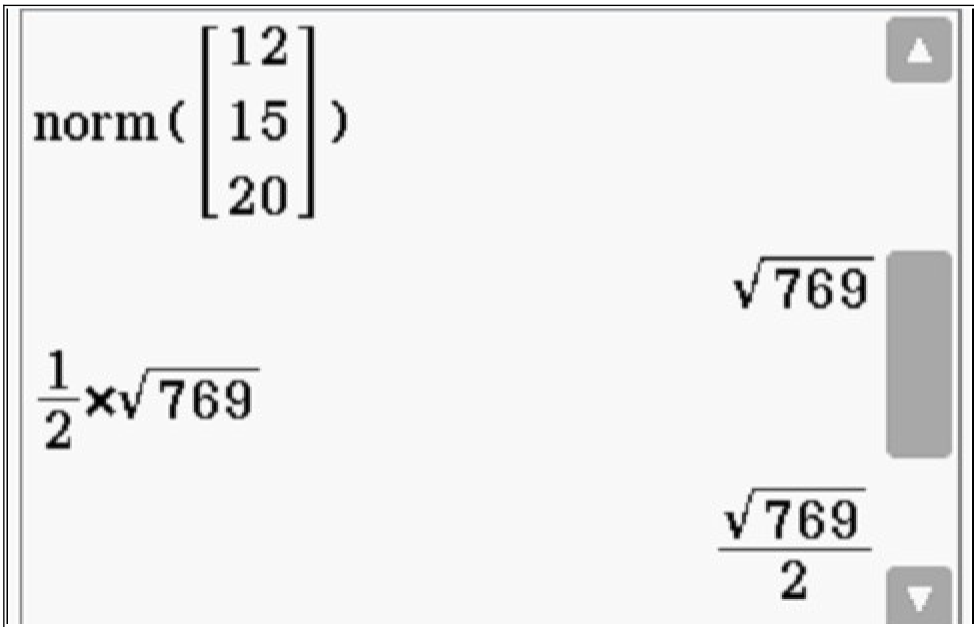
✓ uses dot product
✓ determines angle in degrees or radians(no need to round)

c) Determine the exact area of triangle ΔABC using vectors. (3 marks)

Solution

$Area = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|$

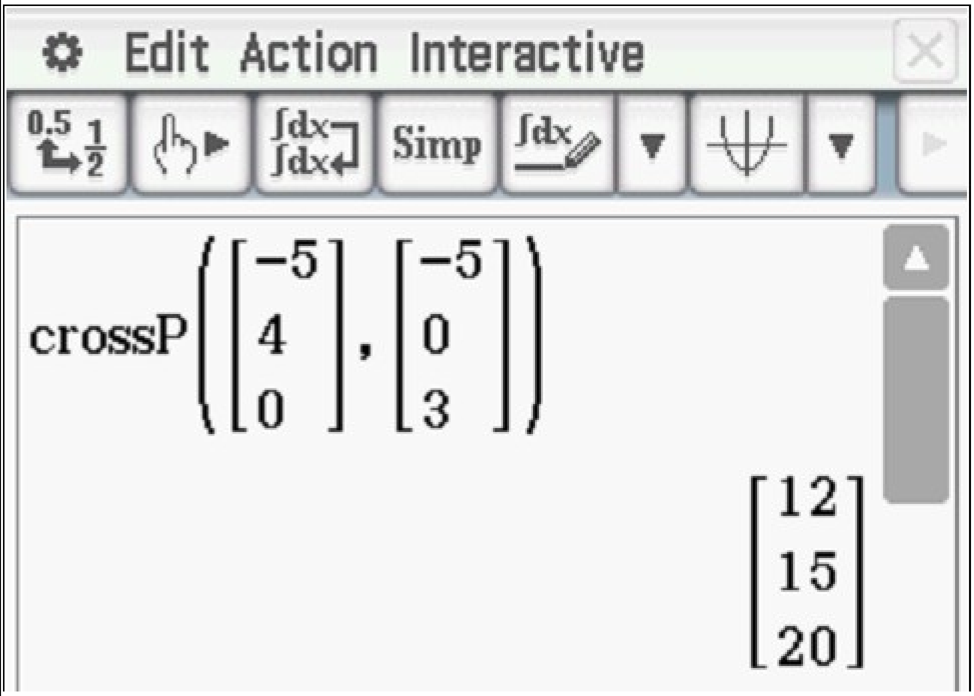




Specific behaviours
✓ uses cross product ✓ determined an expression for area ✓ determined exact area

d) Determine the cartesian equation of the plane containing triangle ΔABC. (4 marks)

Solution



$$r.\begin{pmatrix}12\\15\\20\end{pmatrix}=\begin{pmatrix}5\\0\\0\end{pmatrix}.\begin{pmatrix}12\\15\\20\end{pmatrix}=60$$

$$12x+15y+20z=60$$

Specific behaviours
✓ determines normal vector ✓ sets up equation for vector equation ✓ determines vector equation ✓ determines cartesian equation

Question 18 **(7 marks)**

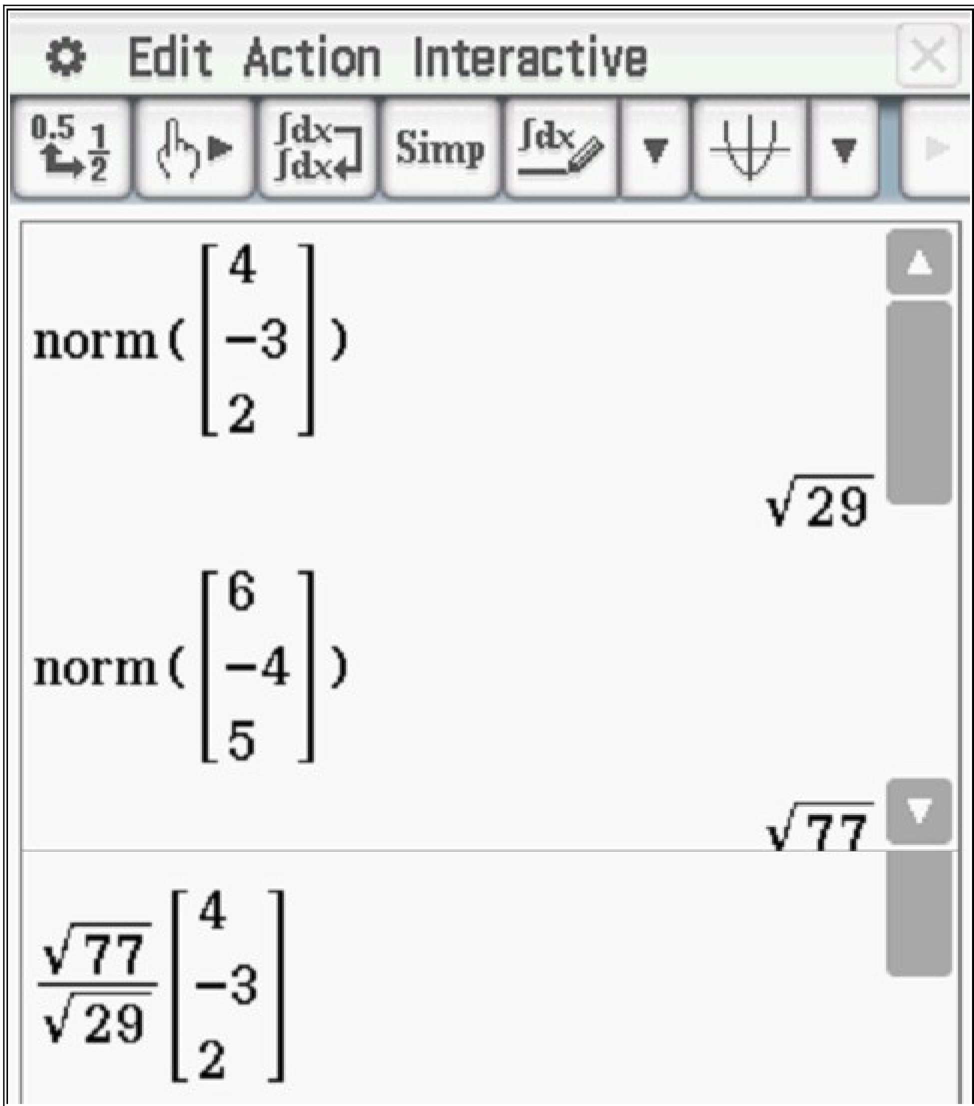
Consider the vectors $a = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}, b = \begin{pmatrix} -2 \\ \frac{3}{2} \\ q \end{pmatrix}, c = \begin{pmatrix} 7 \\ r \\ 5 \end{pmatrix} \& d = \begin{pmatrix} s \\ -11 \\ 7 \end{pmatrix}$

a) Determine $q, r \& s$ given that $a \& b$ are parallel, c is perpendicular to a and d is perpendicular to b. (4 marks)

Solution
$\begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} -2 \\ \frac{3}{2} \\ q \end{pmatrix} \quad \lambda = -2 \quad 2 = (-2)q \quad q = -1$ $c.a = 0 \quad 28 - 3r + 10 = 0 \quad r = \frac{38}{3}$ $d.b = 0 \quad -2s - \frac{33}{2} - 7 = 0 \quad s = \frac{-47}{4}$
Specific behaviours
✓ uses scalar multiple between a & b vectors ✓ uses dot product equally zero for perpendicular ✓ solves for one unknown ✓ solves for all unknowns

b) Given that $e = \begin{pmatrix} 6 \\ -4 \\ 5 \end{pmatrix}$, determine a vector parallel to a but equal in magnitude to e. (3 marks

Solution



Specific behaviours
✓ determines magnitude of a & e vectors ✓ determines unit vector parallel to a ✓ determines final vector (plus or minus)

Question 19 (7 marks)

a) Consider the cartesian equation $x^2+y^2+z^2-6x+8y-3z+20=0$. Describe what this locus of points represents and state major features and give the **vector** equation. (4 marks)

Solution

$x^2 + y^2 + z^2 - 6x + 8y - 3z + 20 = 0$ $x^2 - 6x + 9 - 9 + y^2 + 8y + 16 - 16 + z^2 - 3z + \frac{9}{4} - \frac{9}{4} = -20$ $(x-3)^2 + (y+4)^2 + \left(z - \frac{3}{2}\right)^2 = 9 + 16 + \frac{9}{4} - 20 = \frac{29}{4}$ $\left\| r - \begin{pmatrix} 3 \\ -4 \\ \frac{3}{2} \end{pmatrix} \right\| = \frac{\sqrt{29}}{2}$ Sphere with centre (3,-4,1.5) with radius root29/2
Specific behaviours
✓ completes the square for each variable ✓ states vector equation ✓ states a sphere with radius stated ✓ exact radius stated

b) Consider the equation $x^2 + y^2 + z^2 + 4x - 2y + 6z = \alpha$ where α is a constant. Determine the values of α for which the equation would be a sphere giving the centre and radius in terms of α. (3 marks)

Solution
$x^2 + y^2 + z^2 + 4x - 2y + 6z = \alpha$ $x^2 + 4x + 4 - 4 + y^2 - 2y + 1 - 1 + z^2 + 6z + 9 - 9 = \alpha$ $(x+2)^2 + (y-1)^2 + (z+3)^2 = \alpha + 14$ $\alpha > -14$ $centre(-2, 1, -3)$ $radius = \sqrt{\alpha + 14}$
Specific behaviours
✓ completes the square and states centre ✓ all possible values of alpha (accept -14) ✓ states general rule for radius

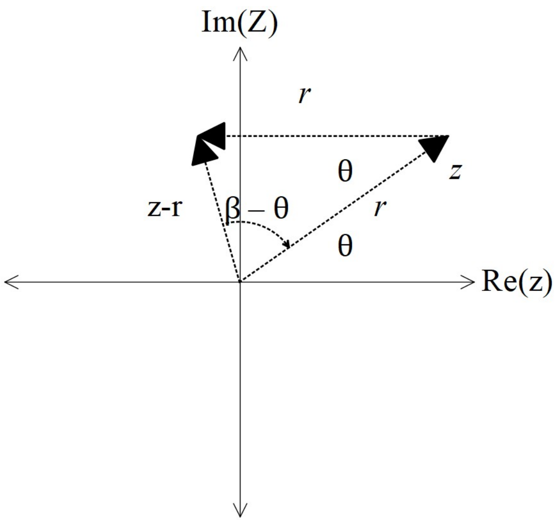
Question 20 **(6 marks)**

Let $z = r\,cis\theta$ be a complex number such that $r > 0$ and $0 < \theta < \dfrac{\pi}{2}$.

a) Express in terms of $r\ \&\ \theta$ the complex number $\dfrac{\left(\sqrt{3}+i\right)z^3}{\overline{z}\left(1-i\right)}$. (simplify) (3 marks)

Solution
$\dfrac{\left(\sqrt{3}+i\right)z^3}{\overline{z}\left(1-i\right)} = \dfrac{2cis\dfrac{\pi}{6}r^3cis\left(3\theta\right)}{rcis\left(-\theta\right)\sqrt{2}cis\dfrac{-\pi}{4}}$ $=\sqrt{2}r^2cis\left(3\theta+\dfrac{\pi}{6}+\theta+\dfrac{\pi}{4}\right)$ $=\sqrt{2}r^2cis\left(4\theta+\dfrac{5\pi}{12}\right)$
Specific behaviours
✓ converts all number in polar form   ✓ simplifies modulus of total   ✓ simplifies argument of total

b) Express $\beta = Arg\left(z - r\right)$ in terms of θ. (3 marks)

Solution

$2(\beta - \theta) + \theta = \pi$ $\beta = \dfrac{\pi + \theta}{2}$
Specific behaviours
✓ shows diagram of addition with labels   ✓ recognizes isosceles triangle

✓ derives correct expression

Question 21 **(4 marks)**

Consider the polynomial $P(z)=z^5-z^4+az^3+bz^2+cz+d$ where $a,b,c\,\&\,d$ are real constants.

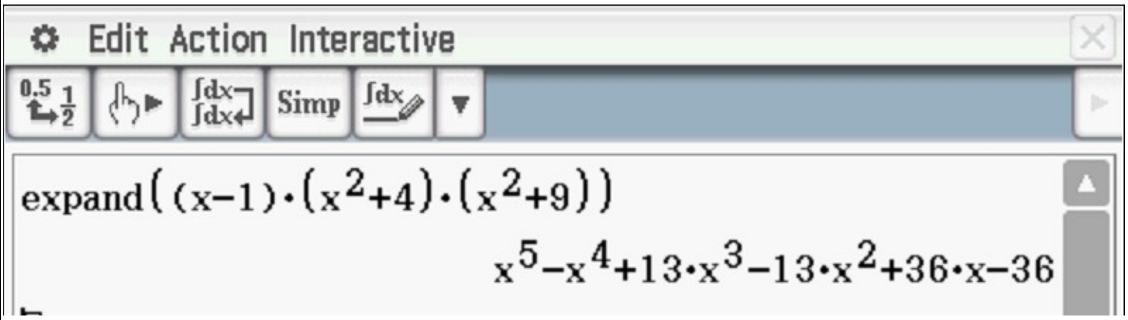
Given that $P(2i)=0=P(-3i)$ and $a+b+c+d=0$ determine the values of $a,b,c\,\&\,d$.

Solution

$P(z)=z^5-z^4+az^3+bz^2+cz+d$

$P(1)=1-1+a+b+c+d=0$

$P(z)=(z-1)(z^2+4)(z^2+9)$



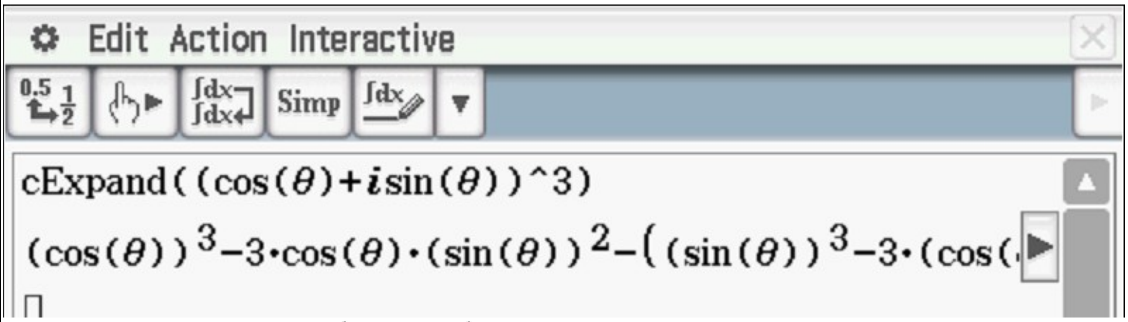
a=13, b=-13, c=36 & d=-36

Specific behaviours

✓ shows that z=1 I a root
✓ uses conjugates of complex roots in factorising
✓ expresses polynomial as a product of factors
✓ determines values of all unknowns

Question 22 **(6 marks)**

a) Using De Moivre's theorem, derive an expression for $\cos(3\theta)$ in terms of $\cos\theta$ only.
(3 marks)

Solution
$(cis\theta)^3 = cis(3\theta) = \cos(3\theta) + i\sin(3\theta)$ $real\,(\cos\theta + i\sin\theta)^3 = \cos(3\theta)$  $\cos(3\theta) = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = \cos^3\theta - 3\cos\theta + 3\cos^3\theta = 4\cos^3\theta - 3\cos\theta$
Specific behaviours
✓ shows that expression is derived from real part of De Moivre's use ✓expands expression of cubic ✓expresses all terms in cosine form

b) **Using** the result from (a) above, show how to obtain **all** solutions to $8z^3 - 6z - 1 = 0$ in the form $\cos\phi$. Express possible values of ϕ in **exact** form.
(3 marks)

Solution
$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ $let\ z = \cos\theta$ $\cos(3\theta) = 4z^3 - 3z$ $2\cos(3\theta) = 8z^3 - 6z$ $2\cos(3\theta) - 1 = 0$ $\cos(3\theta) = \frac{1}{2}$ $3\theta = \frac{\pi}{3} + 2n\pi \quad , n = 0, \pm1, \pm2 \ldots$ $\theta = \frac{\pi}{9} + \frac{6n\pi}{9}$ $\theta = \frac{\pi}{9}, \frac{7\pi}{9}, \frac{-5\pi}{9}$ $z = \cos\frac{\pi}{9}, \cos\frac{7\pi}{9}\ \&\ \cos\frac{-5\pi}{9}$

Specific behaviours
✓ expresses equation using result from cos3x ✓solves for three possible exact values of angle. NOTE- many other values possible ✓expresses all solutions in exact cosine form. NOTE- decimal values or surd expressions from classpad will not be accepted for any marks