2011 TRIAL WACE EXAM

CHURCHLANDS

Section Two: Calculator-assumed **(80 Marks)**

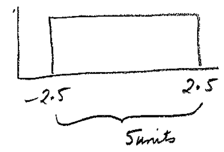
This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 **(5 marks)**

In a production facility, the lengths of metal rods are recorded to the nearest 5 mm. The rounding error, E mm, is the difference of the actual rod length minus the rounded length and is uniformly distributed between -2.5 mm and 2.5 mm.

(a) State the probability density function for E. (2 marks)



$$f(E) = \begin{cases} 0.2, & -2.5 \le E \le 2.5 \\ 0 & \text{elsewhere} \end{cases}$$ ✓ ✓

(b) Determine

(i) $P(E=1)$ $= 0$ ✓ (1 mark)

(Continuous distribution)

(ii) $P(E>-1.5 \mid E \le 2)$ $= \dfrac{P(-1.5 < E \le 2)}{P(E \le 2)}$ (1 mark)

$= \dfrac{3.5}{4.5} = \dfrac{7}{9}$ ✓

(c) What is the probability that a randomly chosen rod with a recorded length of 135 mm has a real length of a least 136 mm? (1 mark)

$P(E>1) = \dfrac{2.5-1}{5} = \dfrac{1.5}{5} = \dfrac{3}{10}.$

See next page

Question 10 **(6 marks)**

From an analysis of the median house price (M) in a city on July 1 each year from 1980 until 2010, it was observed that $\frac{dM}{dt} = 0.0772M$, where t is the time in years since July 1 1980.

(a) According to this model, how long did it take for house prices to double? (2 mark)

$M = M_0 e^{0.0772t}$

$2 = e^{0.0772t}$

$t = 8.97859$

≈ 8.98 years (2dp)

It was also observed that the median house price was $440 000 in 2008.

(b) What was the instantaneous rate of change of the median house price at this time? (1 mark)

$\frac{dM}{dt} = 0.0772 \times 440000 = \$33968$ /year.

(c) What was the median house price in 1988, to the nearest thousand dollars? (2 marks)

$\underset{2008}{440000} = \underset{1988}{M_0} e^{0.0772 \times 20}$ OR $M = 440000 e^{0.0772 \times (-20)}$

$M_0 = 93951$

$\approx \$94000$

(d) What was the average rate of change of the median house price between 1988 and 2008? (1 mark)

$\frac{440000 - 94000}{20} = \$17300$ per year.

Question 20 **(7 marks)**

A teacher introduced the following probability experiment to her class. Five cards with the letters A, B, C, D and E are thoroughly shuffled and then the letter on the top card noted. This trial is repeated a total of 20 times to complete the experiment.

Let X be the random variable 'the number of times the card with the letter A is drawn in one experiment'.

(a) Explain why X is a discrete random variable, and state the parameters of the binomial distribution which X follows. (2 marks)

Can only draw a card 0, 1, 2, 3.... times ie integer values ✓
Associated probability distribution sums to 1 ✓
∴ X is DRV

$X \sim B\left(20, \frac{1}{5}\right)$ ✓

(−1 / omission)

(b) Find $P(0 < X \le 4)$. (1 mark)

$P(1 \le X \le 4) = 0.6181$

(c) A large number of students each carry out the experiment above k times and then they share with their class the mean of their k experiments, $\bar{X}$. If approximately 90% of the means of the students' experiments are less than 4.354, use the central limit theorem to estimate k. (4 marks)

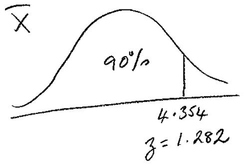
$\bar{X}$ is normal by CLT because n described as "a large #"

$np = 20 \times 0.2 = 4$

$\sigma = \sqrt{20 \times 0.2 \times 0.8} = \sqrt{\frac{16}{5}}$ or $\sqrt{3.2}$ or } population.

Sample mean distribution $\bar{X}$
has mean of 4
s.d. of $\frac{\sqrt{3.2}}{\sqrt{k}}$

OR $\bar{X} \sim N\left(4, \frac{3.2}{k}\right)$



$z = \frac{X - \mu}{\sigma}$

$1.282 = \frac{4.354 - 4}{\sqrt{\frac{3.2}{k}}}$

$k = 41.94$

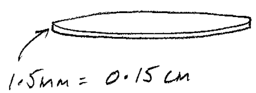
$k \approx 42$

End of questions

Question 11 **(6 marks)**

Oil is poured onto the surface of a large tank of water at a rate of 0.7 cm^3 per second. It spreads out on the surface to form a circular slick of uniform thickness 1.5 mm which can be modelled by a thin cylindrical shape.

(a) At what rate is the radius of the slick increasing one minute after pouring began? (4 marks)



$\frac{dV}{dt} = 0.7 \text{ cm}^3/\text{sec}$

Want $\frac{dr}{dt}$

$V = \pi r^2 h$

$V = 0.15\pi r^2$

$\frac{dV}{dr} = 0.3\pi r$

Find r when
$V = 0.7 \times 60$

$0.7 \times 60 = 0.15\pi r^2$

$r = 9.440697$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$0.7 = 0.3\pi r \times \frac{dr}{dt}$

$\frac{dr}{dt} = 0.078672$

≈ 0.0787 cm/second (4dp)

(b) Use the incremental formula $\delta y \approx \frac{dy}{dx} \times \delta x$ to estimate the time the slick will take to increase in radius from 55 cm to 55.5 cm. (2 marks)

$\delta r = 0.5$cm.

$\delta V = 0.3\pi(55) \times 0.5$

$\approx 25.9 \text{ cm}^3$ (1 dp)

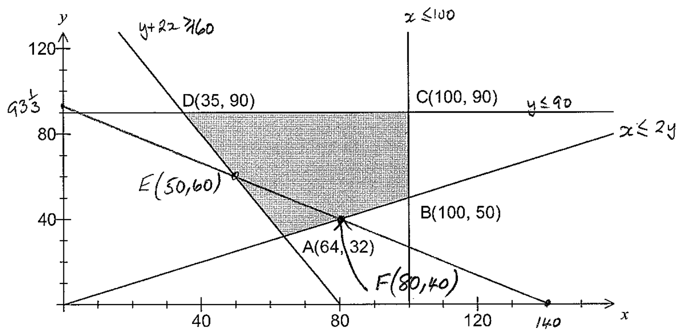
$\delta t = \frac{25.9}{0.7} \approx 37$ seconds.

(37.0259.)

Question 12 **(7 marks)**

A drink company make a fresh fruit drink every day using a combination of apples and pears. The recipe requires that the weight of apples must be no more than twice that of pears and at the same time the weight of the pears together with twice the weight of apples must be at least 160kg. Daily supplies are limited to 100kg of apples and 90kg of pears.

With x representing the weight of apples used and y the weight of pears, the feasible region for this information is shown on the graph below.



From a practical point of view, the company have another constraint such that twice the weight of the apples added to three times the weight of pears must be at least 280kg.

(a) Add this fifth constraint to the graph above and clearly label the vertices of the new feasible region. (3 marks)

$2x + 3y \geq 280$

New vertices $(50,60)$ and $(80,40)$

(b) If the price of apples is \$1.80 per kg and pears \$2.20 per kg, find the minimum daily cost of fruit whilst satisfying all the above constraints. (2 marks)

	$C = 1.8x + 2.2y$
(50, 60)	222
(80, 40)	232
(100, 50)	290
(35, 90)	261
(100, 90)	378 ← Not necessary

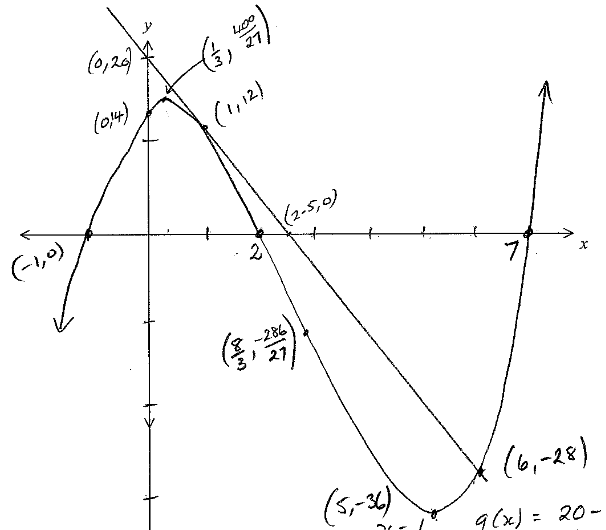
Minimum cost is \$222

Question 19 **(8 marks)**

A function $f(x)$ has derivative given by $f'(x) = 3x^2 - 16x + 5$. $f(x) = x^3 - 8x^2 + 5x + c$

Another function $g(x) = 20 - 8x$ is a tangent to $f(x)$ in the first quadrant.

Sketch the curves $f(x)$ and $g(x)$, showing the **exact** coordinates of all axis-intercepts, turning points and points of inflection.



Realistically scaled, neat graph. ✓

Graph
Line
3 Roots
Pt of Intersect
C
min
max
Pt Inflect

MARKS

$3x^2 - 16x + 5 = -8$

$3x^2 - 16x + 13 = 0$

$x = 1 \quad x = 4\frac{1}{3}$ ✓

$g(1) = 12$

$g(4\frac{1}{3}) = -\frac{44}{3}$ (Not quadrant!)

$x = 1 \quad g(x) = 20 - 8 = 12$

$(1,12) \in f(x)$

$12 = 1 - 8 + 5 + c$

$c = 14$ ✓

$f(x) = x^3 - 8x^2 + 5x + 14$

Roots $-1, 2, 7$ ✓

$f''(x) = 6x - 16 = 0$

$6x = 16$

$x = \frac{16}{6} = \frac{8}{3}$

max value $\left(\frac{1}{3}, \frac{400}{27}\right)$ ✓

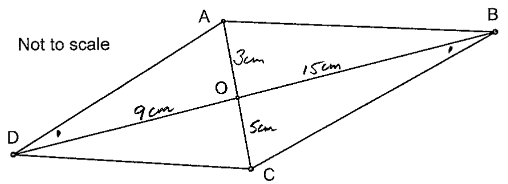
Min value $(5, -36)$ ✓

Pt of inflection $\left(\frac{8}{3}, \frac{-286}{27}\right)$ ✓

Graph of line showing intercepts $(0,20)$ $(2.5,0)$ ✓

Question 18 **(5 marks)**

The diagonals AC and BD of a quadrilateral ABCD intersect at O.



If OA = 3 cm, OB = 15 cm, AC = 8 cm and BD = 24 cm, prove that AD is parallel to BC.

Prove AD ∥ BC

Show $\triangle ADO \sim \triangle CBO$

$\angle AOD \cong \angle COB$ (Vertically opposite) ✓

$\frac{AO}{CO} = \frac{3}{5}$

$\frac{DO}{BO} = \frac{9}{15} = \frac{3}{5}$

$\therefore \frac{AO}{CO} = \frac{DO}{BO}$ (Ratios of corresponding sides equal) ✓

$\therefore \triangle ADO \sim \triangle CBO$ (SAS) ✓

$\angle ADO \cong \angle CBO$ (Corresponding angles in similar $\triangle$s) ✓

$AD \parallel BC$ (Alternate angles congruent) ✓

(c) Consider the situation where the price of apples fell to $1.70 per kg but the price of pears fell considerably more. Given that the vertex in part (b) still yielded the minimum cost, what would be the minimum price of pears on this day? (2 marks)

$C = 1.7x + ky$ E still best

$E \rightarrow D$ (Cost of apples fell)

$(50, 60) = (35, 90)$

$85 + 60k = 59.5 + 90k$

$25.5 = 30k$

$k = 0.85$

minimum price of pears 85c/kg

Question 13 **(5 marks)**

Two functions are defined by $f(x) = e^x$ and $g(x) = e^{1-2x}$.

(a) Describe, in order, the transformations which must be applied to the graph of $f(x)$ to obtain the graph of $g(x)$. (2 marks)

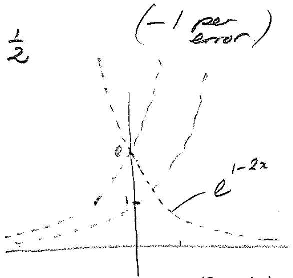
$g(x) = e^{-2x+1}$

Translate 1 unit left

Dilate // to x-axis SF $\frac{1}{2}$

Reflect over y axis

(−1 per error)



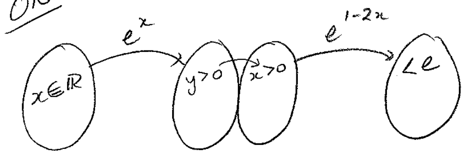
(b) Determine the domain and range of $g(f(x))$. (3 marks)

$g(f(x)) = e^{1-2e^x}$

Domain: All reals ✓

Range $x \to \infty \quad g(x) \to 0$
$x \to -\infty \quad g(x) \to e^1$ $\Big\}$ $0 < y < e$ ✓ ✓

OR



$x = 0 \quad e^{1-2x} = e^1$

$x = \infty \quad e^{1-2\infty} = 0$

$0 < y < e$

(d) What is the probability that a pallet contains at least one bottle with less than the stated contents? (2 marks)

$Z \sim B(24 \times 48, 0.005)$

$P(Z \geq 1) = 0.9969$ (4dp)

OR $1 - (0.8867)^{48} = 1 - 0.0031$
$= 0.9969$

(e) The bottling company randomly choose a pallet from the stockyard. The mean content of all the bottles from this pallet is 389.9 mL.

(i) Construct a 90% confidence interval for the mean content of all bottles. (3 marks)

$n = 24 \times 48$
$= 1152$ bottles.

$389.9 \pm 1.645\left(\frac{8.15}{\sqrt{1152}}\right)$

Sample mean

$389.5 \leq \mu \leq 390.3$

(ii) Should the interval be of concern to the bottling company? (1 mark)

The μ is supposed to be 391. This interval does not contain this value, it is significantly below 391 so the mean may not be 391 as supposed.

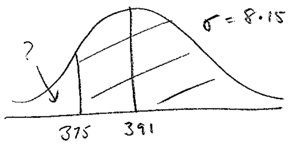
Question 17 **(11 marks)**

A bottling machine fills bottles of water. The content, X mL, of the bottles is a normally distributed random variable with a mean of 391 mL and a standard deviation of 8.15 mL.

It is known that 1 out of every 200 bottles that the machine fills has less than the stated contents on the bottle label.

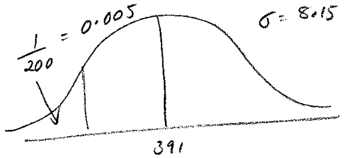
24 bottles are packed in a carton and 48 cartons are loaded onto a shipping pallet.

(a) What is the probability that a bottle contains more than 375 mL of water? (1 mark)



$P(X > 375) = 0.9752$ (4dp)

(b) What are the stated contents on the bottle label? (2 marks)



$P(X < k) = 0.005$

$k = 370 mL$

(c) What is the probability that a carton does not contain any bottles with less than the stated contents? (2 marks)

$Y \sim B(24, 0.005)$

$P(Y=0) = 0.8867$ (4dp.

Question 14 **(5 marks)**

A cubical six-sided dice is known to be biased. It is thrown 3 times and the number of sixes is noted. This experiment is then repeated 200 times in all and the results are shown in the table.

Number of sixes	0	1	2	3
Frequency	67	93	33	7

(a) What is the mean number of sixes? (1 mark)

$$\frac{0 + 93 + 66 + 21}{200} = 0.9$$

(b) What is the probability of obtaining a six when this dice is thrown? (1 mark)

Let X = "number of sixes in 3 throws of the dice."

$X \sim B(3, p)$ But $\bar{X} = 0.9 = np$

$\therefore 3p = 0.9$

$p = 0.3$

(c) Use a suitable binomial distribution to calculate the theoretical frequency distribution for the number of sixes in 200 such experiments and comment on how well your distribution models the experimental results above. (3 marks)

$X \sim B(3, 0.3)$

Graph menu Binomial$(x, 3, 0.3)$

$P(X=0)$ $0.343 \times 200 = 68.6$

$P(X=1)$ $0.441 \times 200 = 88.2$

$P(X=2)$ $0.189 \times 200 = 37.8$

$P(X=3)$ $0.027 \times 200 = 5.4$

The experimental and theoretical frequencies are similar $\therefore X \sim B(3, 0.3)$ is appropriate.

Question 15 **(8 marks)**

(a) A team of 3 students is chosen at random from a group of 4 girls and 5 boys for a TV game show. What is the probability that the team chosen consists of more boys than girls? (2 marks)

P(2B ∩ 1G) OR P(3 Boys)

$$\frac{\binom{5}{2}\binom{4}{1}}{\binom{9}{3}} + \frac{\binom{5}{3}\binom{4}{0}}{\binom{9}{3}} = \frac{25}{42}$$

(b) In one of the games, the team choose one of four closed doors. The doors then open to reveal a prize placed at random behind just one of them. The team keep the prize if they are correct. How many rounds of this game must the team play so that the probability of them obtaining at least one prize is greater than 0.95? (3 marks)

$B \sim (n, 0.25)$

P(at Least 1 prize ≥ 1) > 0.95

∴ P(No prizes) < 0.05

$P(x=0) < 0.05$

$\binom{n}{0} p^0 (1-p)^n < 0.5$

$\left(\frac{3}{4}\right)^n < 0.05$

OR P(At least 1 prize)

$= 1 -$ P(no prizes)

$1 - \left(\frac{3}{4}\right)^n \geq 0.95$

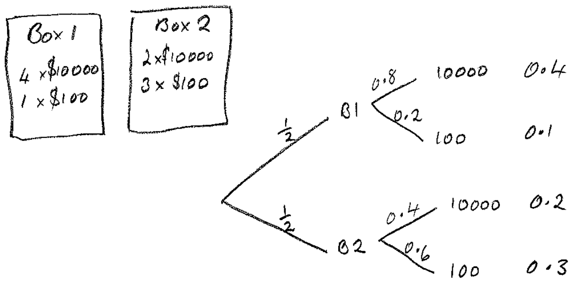
$n \geq 10.4$

∴ at least 11 rounds

(c) At the close of the show, the team can select one of two boxes to keep as another prize. Inside each of the boxes are five sealed envelopes, each containing a voucher. In one of the boxes, four of the vouchers are worth \$10 000 and the fifth \$100, whilst in the other box two of the vouchers are worth \$10 000 and the other three, \$100 each.

The team is allowed to choose an envelope from one of the boxes and open it. They must then decide whether to keep that box or choose the other one. The team plan to keep the box that the envelope they opened came from if it contains a \$10 000 voucher. Otherwise they will take the other box.

What is the probability that the team wins more than \$30 000? (3 marks)



Keep & get >30 000

← Will take other box ie get >30000

∴ P(>30 000) = 0.7

Question 16 **(7 marks)**

The velocity $v(t)$ ms^{-1} of a body moving along a straight track after t seconds, is given by

$$v(t) = \frac{t^2 + 2t + 3}{(t+1)^2}, \ t \geq 0.$$

(a) Find the acceleration of the body after 4 seconds. (1 mark)

$v'(4) = \frac{-4}{125}$ or -0.032 m/s^2

$a = \left((t+1)^2(2t+2) - (t^2+2t+3)\,2(t+1)\right) \div (t+1)^4$ Too much work for 1 mark!

(b) Explain why the body is never stationary over the given domain. (1 mark)

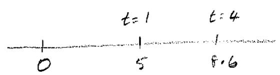
Stationary if $v = 0$

$\frac{t^2+2t+3}{(t+1)^2} = 0$

$t^2 + 2t + 3 = 0$ No real roots ∴ velocity cannot be zero.

(c) If $x(t)$ m is the displacement of the body from a fixed point on the track and $x(1) = 5$ determine $x(4)$. (2 marks)

$x(4) = x(1) + \int_1^4 v(t)\,dt = 5 + 3.6$
$= 8.6$



(d) The average speed of the body over the first T seconds is 1.2 ms^{-1}. Determine the value of T. (3 marks)

$$\frac{\int_0^T v(t)\,dt}{T} = 1.2$$

$T - \frac{2}{T+1} + 2 = 1.2T$

Solve $T = 9$