ALL SAINTS' COLLEGE

Mathematics
Specialist
Test 4 2017

Integration Techniques & Applications of Integral Calculus

NAME: _____SOLUTIONS______________ TEACHER: Mrs Da Cruz

Resource Free Section

33 marks
35 minutes

Question 1 [3 & 3 = 6 marks]

(a) Express $\frac{2}{x^2-1}$ as partial fractions.

$$\frac{2}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1}$$

$$\therefore 2=A(x+1)+B(x-1)$$

$$\therefore \begin{cases} 2=A-B \\ 0=A+B \end{cases} \Longrightarrow A=1, B=-1$$

$$\therefore \frac{2}{x^2-1}=\frac{1}{x-1}-\frac{1}{x+1}$$

(b) Hence determine $\int_2^8 \frac{2}{x^2-1}dx$. Simplify your answer to the form $\ln\frac{p}{q}$.

$$\int_2^8 \frac{2}{x^2-1}dx=\int_2^8 \frac{1}{x-1}-\frac{1}{x+1}dx$$

¿ $\left[\ln(x-1)-\ln(x+1)\right]_2^8$

¿ $\ln 7-\ln 9-\ln 1+\ln 3$

¿ $\ln\frac{7\times 3}{9}$

¿ $\ln\frac{7}{3}$

Question 2 **[3, 2 & 2 = 7 marks]**

Determine the following indefinite integrals:

(a) $\int 1-\cos^2(5x)dx$

$$\int 1-\cos^2(5x)\,dx=\int \sin^2(5x)\,dx$$

$$¿\int \frac{1}{2}(1-\cos(10x))\,dx$$

-

$$¿\frac{1}{2}\left(x-\frac{\sin(10x)}{10}\right)+c$$

$$¿\frac{x}{2}-\frac{\sin(10x)}{20}+c$$

(b) $\int \frac{1}{2}(2x-4)(x^2-4x+1)^6\,dx$

$$\frac{1}{2}\int (2x-4)(x^2-4x+1)^6\,dx$$

$$¿\frac{1}{2}\frac{(x^2-4x+1)^7}{7}+c$$

$$¿\frac{(x^2-4x+1)^7}{14}+c$$

(c) $\int \sin^3(2x)\cos(2x)\,dx$

$$\frac{1}{2}\int [\sin(2x)]^3\cos(2x).2\,dx$$

$$¿\frac{1}{2}\frac{[\sin(2x)]^4}{4}+c$$

$$¿\frac{\sin^4(2x)}{8}+c$$

Question 3 **[5 marks]**

Use the substitution $u=4+\sqrt{x}$ to evaluate $\int \sqrt{4+\sqrt{x}}\,dx$.

Do not factorize your answer. Do not leave any fractional or negative indices.

$$\frac{du}{dx}=\frac{1}{2}x^{\frac{-1}{2}}=\frac{1}{2\sqrt{x}}$$

$$du\,2\sqrt{x}=dx$$

$$du\,2(u-4)=dx$$

$$\therefore \int \sqrt{4+\sqrt{x}}\,dx=\int \sqrt{u}\,2(u-4)\,du$$

$$\text{¿}\,2\int u^{\frac{3}{2}}-4u^{\frac{1}{2}}\,du$$

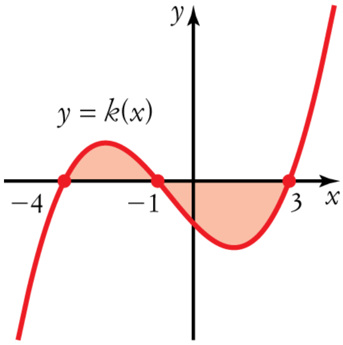
$$\text{¿}\,2\left(\frac{2u^{\frac{5}{2}}}{5}-\frac{8u^{\frac{3}{2}}}{3}\right)+c$$

$$\text{¿}\,\frac{4\sqrt{u}^5}{5}-\frac{16\sqrt{u}^3}{3}+c$$

$$\text{¿}\,\frac{4\sqrt{4+\sqrt{x}}^5}{5}-\frac{16\sqrt{4+\sqrt{x}}^3}{3}+c$$

Question 4 [1 mark]

The area under the curve $y = k(x)$ can be described by:



A $\int_{-4}^{3} k(x)\,dx$

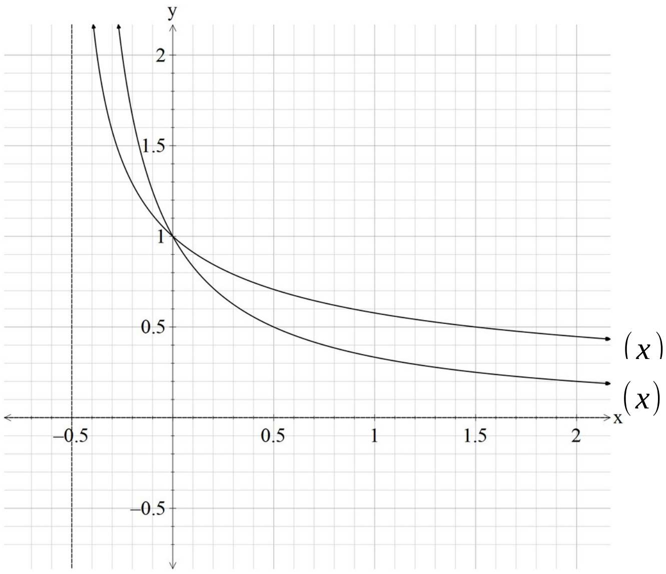
B $\int_{-4}^{-1} k(x)\,dx + \int_{-1}^{3} k(x)\,dx$

C $\left|\int_{-1}^{3} k(x)\,dx\right| + \int_{-4}^{-1} k(x)\,dx$ ✔

D $\left|\int_{1}^{3} k(x)\,dx\right|$

E $1 - \int_{1}^{3} k(x)\,dx$

Question 5 **[3 & 4 = 7 marks]**



(a)Find the area under the curve, in square units, for the function $f(x)=\frac{1}{2x+1}$ from $x=0$ to $x=1$.

$$A=\int_0^1 \frac{1}{2x+1}dx=\frac{1}{2}\int_0^1 \frac{2}{2x+1}dx$$

$$¿\frac{1}{2}¿¿$$

$$¿\frac{1}{2}(\ln 3-\ln 1)=\frac{\ln 3}{2}units^2$$

(b) Find the area enclosed by the curves, in square units, of the graphs $f(x)=\frac{1}{2x+1}$, $g(x)=\frac{1}{\sqrt{2x+1}}$ and the line $x=1$.

$$A=\int_0^1 \frac{1}{\sqrt{2x+1}}-\frac{1}{2x+1}dx$$

$$¿\left[(2x+1)^{\frac{1}{2}}-\frac{\ln(2x+1)}{2}\right]_0^1$$

$$¿\left[\sqrt{3}-\frac{\ln 3}{2}-\left(\sqrt{1}-\frac{\ln 1}{2}\right)\right]$$

$$¿\sqrt{3}-\frac{\ln 3}{2}-1\,units^2$$

Question 6 **[2 marks]**

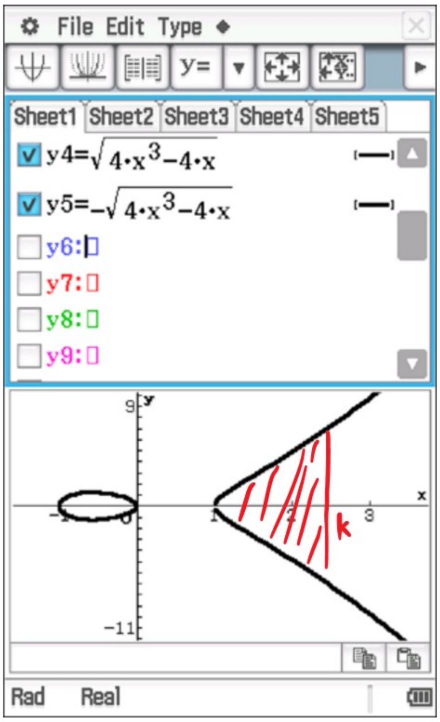
Find $\frac{d}{dx}\left(x^2 e^x\right)$ and use your answer to evaluate $\int 4xe^x(x+2)\,dx$.

$$\frac{dy}{dx}=2x.e^x+e^x.x^2=x.e^x(2+x)$$

$$\int 4x.e^x(2+x)\,dx=4\int x.e^x(2+x)\,dx=4x^2e^x+c$$

Question 7 **[5 marks]**

The region bounded by the lines $x=k$ and $x=1$ and the curve $y^2=4x^3-4x$ is rotated about the x-axis 180°. The volume formed is 9π. Determine the value of k where k is a positive integer.



Rotating the shaded area 180° is the same as rotating the top half 360°.

$$k>1, k\in\mathbb{Z}$$

$$V=\pi\int_1^k y^2\,dx$$

$$\therefore 9\pi=\pi\int_1^k 4x^3-4x\,dx$$

$$\therefore 9=\left[x^4-2x^2\right]_1^k$$

$$9=k^4-2k^2-1+2$$

$$0=k^4-2k^2-8$$

$$0=(k^2-4)(k^2+2)$$

$$k^2=4$$

$$k=2$$

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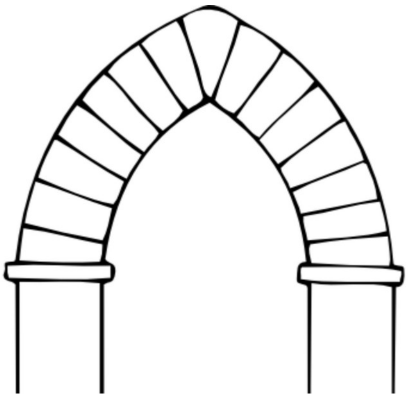
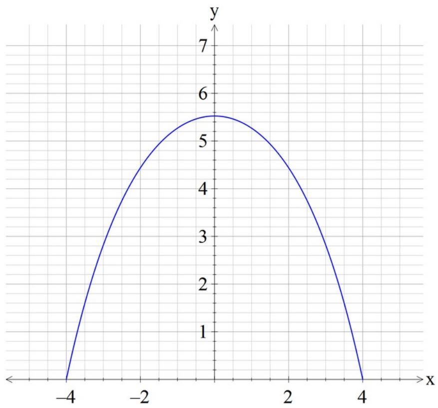
Resource Rich Section

11 marks
15 minutes

Question 8 **[1, 1 & 2 = 4 marks]**





The upper component of an archway is designed to bear the load of the wall above and around it. For this, the best shape is a catenary. A catenary is the name given to the curve formed by two simple exponential terms added together. The equation of the upper arch is $f(x) = -e^{\frac{x}{2}} - e^{\frac{-x}{2}} + c$.

The x-intercepts of the catenary are $(-4, 0)$ and $(4, 0)$.

a Use this information to determine the exact value of c.

$$c = e^2 + \frac{1}{e^2}$$

$$\text{solve}\left(-e^{\frac{4}{2}} - e^{\frac{-4}{2}} + x = 0\right)$$
$$\{x = e^2 + e^{-2}\}$$

Paint has to be applied to the area under the catenary curve.

b State a definite integral that will find the area of paint required.

$$A = \int_{-4}^{4} -e^{\frac{x}{2}} - e^{\frac{-x}{2}} + e^2 + \frac{1}{e^2}\,dx$$

c Calculate the exact area to be painted, giving your answer with positive indices.

$$A = \left[-2e^{\frac{x}{2}} + 2e^{\frac{-x}{2}} + xe^2 + \frac{x}{e^2}\right]_{-4}^{4}$$

$$A = 4e^2 + \frac{12}{e^2}\,units^2$$

$$\int_{-4}^{4} -e^{\frac{x}{2}} - e^{\frac{-x}{2}} + e^2 + e^{-2}\,dx$$
$$-4\cdot e^2 + 8\cdot(e^2 + e^{-2}) + 4\cdot e^{-2}$$

$$\text{simplify(ans)}$$
$$4\cdot(e^4 + 3)\cdot e^{-2}$$

Question 9 [2 marks]

Use a suitable definite integral to find the **exact** volume, in cubic units, that is formed by rotating about the x-axis the following curves between the limits shown.

$y = x^3$, from $x = 1$ and $x = 3$.

$$V = \pi \int_1^3 x^6 \, dx$$

$$¿ \frac{2186\pi}{7} \, units^3$$

$$\pi \int_1^3 x^6 \, dx$$

$$\frac{2186 \cdot \pi}{7}$$

Question 10 [3 marks]

Use a suitable definite integral to find the exact volume, in cubic units, that is formed by rotating about the y-axis the following curves between the limits shown.

$y = \frac{1}{5}\log_e(2x - 1)$, from $y = 0$ and $y = 1$.

$$V = \pi \int_0^1 x^2 \, dy$$

$$\therefore V = \pi \int_0^1 \left(\frac{e^{5y}}{2} + \frac{1}{2} \right)^2 dy$$

$$¿ \pi \left(\frac{e^{10}}{40} + \frac{e^5}{10} + \frac{1}{8} \right) units^3$$

$$\text{solve}\left(y = \frac{1}{5}\ln(2x-1)\right)$$

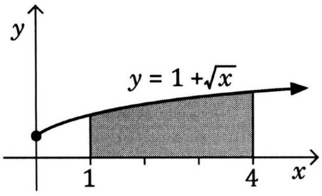
$$\left\{ x = \frac{e^{5 \cdot y}}{2} + \frac{1}{2} \right\}$$

$$\pi \int_0^1 \left(\frac{e^{5x}+1}{2} \right)^2 dx$$

$$\left(\frac{e^{10}}{40} + \frac{e^5}{10} + \frac{1}{8} \right) \cdot \pi$$

Question 10 [2 marks]

Use a suitable definite integral to find the exact volume, in cubic units, that is formed by rotating the shaded area about the y-axis.



$$V = 2\pi \int_1^4 xy\,dx$$

$$¿\,2\pi \int_1^4 x\left(1+\sqrt{x}\right)dx$$

$$¿\,\frac{199\pi}{5} units^3$$

$$2\pi\int_1^4 x(1+\sqrt{x})\,dx$$

$$\frac{199\cdot\pi}{5}$$