PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Test 2 Year 12

Student name: ______________ Teacher name: ______________

Task type: **Response**

Time allowed for this task: ____40_______ **mins**

Number of questions: _____7______

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **_41_____ marks**

Task weighting: **_10___%**

Formula sheet provided: Yes

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (2, 2 & 3 = 7 marks) (3.2.1-3.2.3)

Consider the functions $f(x)=\frac{1}{x-2}$ and $g(x)=\sqrt{x}$.

a) State the natural domain and range of $f(x)$.

Solution
$d_f : x \neq 2$ $r_f : y \neq 0$
Specific behaviours
✓ states domain ✓ states range

b) Does $g \circ f(x)$ exist over the natural domain of $f(x)$? Explain.

Solution
$r_f \subseteq d_g$ To exist $y \neq 0 \not\subset y \geq 0$ therefore does not exist over natural domain
Specific behaviours
✓ states does not exist with any reason ✓ reason shows relevant domain and range

c) State the rule and natural domain and range of $f \circ g(x)$.

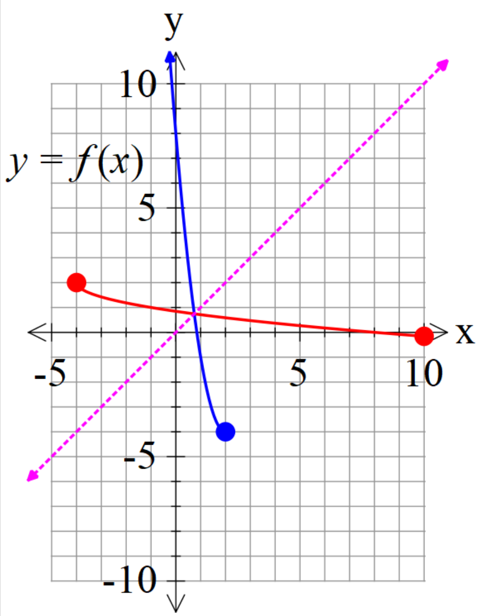
Solution
$f \circ g(x)=\frac{1}{\sqrt{x}-2}$ $d:(0 \leq x<4) \cup (x>4)$ $r: R \setminus \left\{-\frac{1}{2}<y \leq 0\right\}$
Specific behaviours
✓ states rule

✓ states domain which excludes x=4 ✓ states range with correct endpoints inequalities of excluded interval

Q2 (3, 3, 1 &2 = 9 marks) (3.2.4)

Consider the function $f(x)=3x^2-12x+8$ with domain $x\le 2$.

a) Sketch the inverse function on the axes below.

Solution

Specific behaviours
✓ appears to be reflected in y=x ✓ shows pts (-4,2) & (8,0) on iverse ✓ correct domain and range

b) Determine the inverse function $f^{-1}(x)$ stating its domain. (Show all working)

Solution

$x = 3y^2 - 13y + 8$

$3y^2 - 12y + 8 - x = 0$

$y = \dfrac{12 \pm \sqrt{144 - 4(3)(8 - x)}}{6} = \dfrac{13 \pm \sqrt{48 + 12x}}{6} = 2 \pm \dfrac{\sqrt{3(x+4)}}{3}$

$y \le 2 \therefore f^{-1}(x) = 2 - \dfrac{\sqrt{3(x+4)}}{3}$

$d : x \ge -4$

Specific behaviours

✓ shows the interchange of y & x or shows how x is made the subject of rule
✓ states inverse rule with correct sign
✓ states domain

c) Determine $f \circ f^{-1}(x)$

Solution

$f \circ f^{-1}(x) = x$

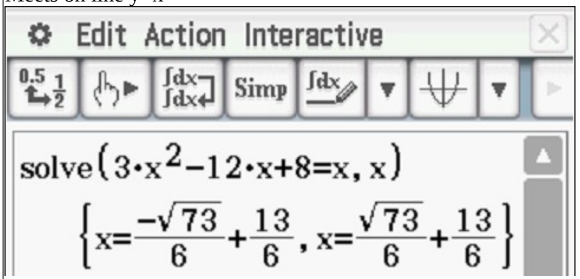
Specific behaviours

✓ states x

d) Determine when $f(x) = f^{-1}(x)$ exactly.

Solution

Meets on line y=x



Discard second answer as outside domain of f

Specific behaviours

✓ sets up an equation to solve for x

Specific behaviours

✓ uses additions of $\dfrac{2\pi}{n}$
✓ sets up equation for lower boundary for n
✓ sets up equation for upper boundary for n
✓ states all allowed integer values for n

✓ subs line into sphere equation ✓ sets up equation for $\lambda \& \beta$ ✓ sets up a quadratic equation for λ in terms of β ✓ determines expression for discriminant in terms of β and solves when $\Delta = 0$ ✓ states values for one of the three scenarios with reasoning (no need for rounding) ✓states values for all three scenarios with **reasoning for each** (no need for rounding) NOTE: No follow through if mistake makes problem easier.

Q7 (4 marks) (3.1.4)

The solutions to the complex equation $z^n = k$ are plotted in the complex plane. (n is an integer & k is a complex constant). Exactly **four** of the solutions are plotted in the second quadrant, $\frac{\pi}{2} < Arg(z) < \pi$, and **no more**. Of these four solutions, the smallest argument is $\frac{7\pi}{12}$.

Determine all possible values of n.

Solution

Consecutive roots arguments separated by $\frac{2\pi}{n}$

Four arguments are: $\frac{7\pi}{12}$, $\frac{7\pi}{12}+\frac{2\pi}{n}$, $\frac{7\pi}{12}+\frac{4\pi}{n}$, $\frac{7\pi}{12}+\frac{6\pi}{n}$

$$\frac{7\pi}{12}+\frac{6\pi}{n} < \pi$$

$$\frac{6\pi}{n} < \frac{5\pi}{12} \quad 72 < 5n \quad n > 14.4$$

$5th$

$$\frac{7\pi}{12}+\frac{8\pi}{n} > \pi$$

$$\frac{8\pi}{n} > \frac{5\pi}{12} \quad 5n < 96 \quad n < 19.2$$

$values \quad 15 \le n \le 19$

✓ solves for one value of x exactly

Q3 (3 marks) (3.2.6)

Consider the inequality $\left|\frac{3}{2}x + b\right| \le 4.5$ is **only true** for $4 \le x \le 10$ with b a constant.

Determine the value of b.

Solution

$$\left|\frac{3}{2}x+b\right| \le 4.5$$

$$\frac{3}{2}\left|x+\frac{2b}{3}\right| \le \frac{9}{2}$$

$$\left|x+\frac{2b}{3}\right| \le 3$$

$$\frac{2b}{3} = -7 \quad as\ 4 \le x \le 10$$

$$b = -\frac{21}{2}$$

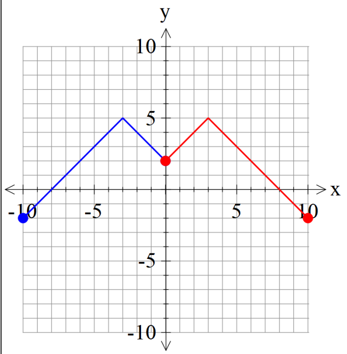
Specific behaviours

✓ uses midpoint of solution interval
✓ rearranges inequality to identify centre in terms of x
✓ states correct value of constant

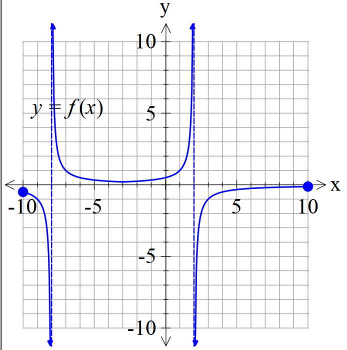
Q4 (3 & 3 = 6 marks) (3.2.7)

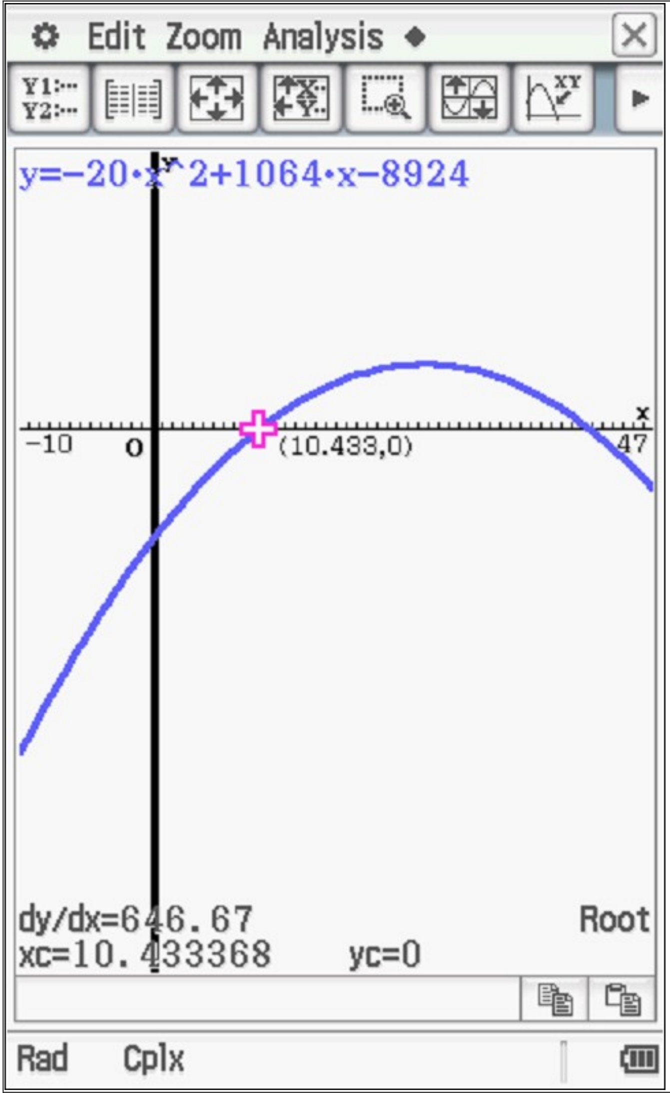
Consider the following function $f(x)$.

a) Sketch $y = f(-|x|)$ on the axes below.

Solution

Specific behaviours
✓ appears to reflect left side ✓ correct x intercepts ✓ y intercept and drawn with correct domain

b) Sketch $y = \dfrac{1}{f(x)}$ on the axes below.

Solution




Tangent $\beta = 10.4, 42.8$

Two places $10.4 < \beta < 42.8$

No intersection $\beta < 10.4, \quad \beta > 42.8$

Specific behaviours

$$\left|\begin{pmatrix} 3+\lambda \\ 7\lambda \\ 1-2\lambda \end{pmatrix} - \begin{pmatrix} 6 \\ \beta \\ -7 \end{pmatrix}\right| = 5$$

$$\sqrt{(\lambda-3)^2 + (7\lambda-\beta)^2 + (8-2\lambda)^2} = 5$$

$$\lambda^2 - 6\lambda + 9 + 49\lambda^2 - 14\beta\lambda + \beta^2 + 64 - 32\lambda + 4\lambda^2 = 25$$

$$54\lambda^2 - (38+14\beta)\lambda + 48 + \beta^2 = 0$$

$$\Delta = (38+14\beta)^2 - 4(54)(48+\beta^2)$$

Edit Action Interactive

$\text{expand}\left((38+14\cdot x)^2 - 4\cdot 54\cdot(48+x^2)\right)$

$-20\cdot x^2 + 1064\cdot x - 8924$

Specific behaviours
✓ both asymptotes shown ✓ y intercept correct with y endpoints ✓ shape in all 3 section

Q5 (3 & 3 = 6 marks) (3.3.3-3.3.6)

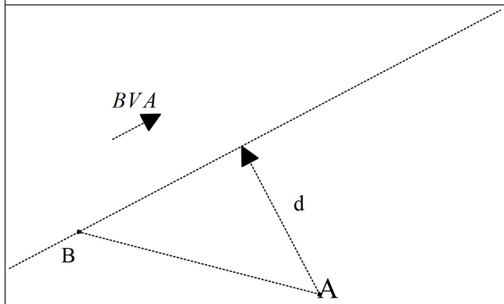
Consider two rockets A&B, moving with constant velocities such that at time $t=0$ hours their positions and velocities are as follows:

$$r_A = \begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix} km \quad r_B = \begin{pmatrix} 6 \\ -2 \\ -1 \end{pmatrix} km$$

$$v_A = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} km/h \quad v_B = \begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix} km/h$$

a) Determine the time and distance of their closest approach.

Solution



Edit Action Interactive

$\text{dotP}\left(\begin{bmatrix} 6 \\ -2 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \\ 7 \end{bmatrix} + t\cdot\left(\begin{bmatrix} -2 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}\right), \begin{bmatrix} -2 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}\right)$

$6\cdot(3\cdot t-8)$

$\text{solve}(6\cdot(3\cdot t-8)=0, t)$

$\left\{t=\frac{8}{3}\right\}$

$$\text{norm}\left(\begin{bmatrix}6\\-2\\-1\end{bmatrix}-\begin{bmatrix}-2\\3\\7\end{bmatrix}+t\cdot\left(\begin{bmatrix}-2\\5\\2\end{bmatrix}-\begin{bmatrix}1\\5\\-1\end{bmatrix}\right)\middle| t=\frac{8}{3}\right)$$

$$5$$

Alg Standard Cplx Rad

Distance=5 mins

Specific behaviours
✓ uses displacement vector with dot product OR calculus with separation vector ✓ states time of closest approach (no need for units) ✓ states distance approx. or exact (no need for units)

b) Given that the rockets leave smoke trails that stays in the air for a long period of time, determine if the smoke trails cross at all and if they do, its position in space. Justify.

Solution

$$r_A = \begin{pmatrix}-2\\3\\7\end{pmatrix} + \lambda\begin{pmatrix}1\\5\\-1\end{pmatrix}$$

$$r_B = \begin{pmatrix}6\\-2\\-1\end{pmatrix} + \mu\begin{pmatrix}-2\\5\\2\end{pmatrix}$$

Edit Action Interactive

$$\begin{cases}-2+\lambda=6-2\mu\\3+5\lambda=-2+5\mu\\7-\lambda=-1+2\mu\end{cases}\Bigg|_{\lambda,\mu}$$

$$\{\lambda=2, \mu=3\}$$

$$\begin{bmatrix}0\\13\\5\end{bmatrix}$$

Specific behaviours
✓ uses lines with **different** parameters ✓ shows solution to **stated** simultaneous equations(all 3) with values of parameters ✓ states point of intersection of smoke trails

Q6 (6 marks) (3.3.4, 3.3.6)

Consider the line $r = \begin{pmatrix}3\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\7\\-2\end{pmatrix}$ and the sphere $\left| r - \begin{pmatrix}6\\\beta\\-7\end{pmatrix}\right| = 5$ with β a constant.

Determine the value(s) of β, to one decimal place, such that:

a) The line is a tangent to sphere.
b) The line meets the sphere in two places.
c) The line misses the sphere completely.

Solution