PERTH MODERN SCHOOL

UNIT 3C/3D MAS – 2011

TEST 1 – POLAR COORDINATES & COMPLEX NUMBERS

NAME: SOLUTIONS **DATE: 9/10 February, 2011**

[To achieve full marks and to allow assessment of particular outcomes, working and reasoning should be shown.]
[A maximum of 2 marks will be deducted for incorrect rounding, units, etc.]

This is *Resource Rich – 50 minutes for 53 marks*:

1. [1, 1, 1, 1 = 4 marks]
   Convert:
   a) (4,-6) into polar coordinates with $-180^\circ < \theta \leq 180^\circ$.

   toPol([4,-6]) ⇒ [7.21, -56.3°] ✓

   b) $(3,-\sqrt{3})$ into ***exact*** polar coordinates with $-\pi < \theta \leq \pi$.

   toPol($[3,-\sqrt{3}]$) ⇒ $[2\sqrt{3}, -\frac{\pi}{6}]$ ✓

   c) [4,35°] into Cartesian coordinates.

   toRect([4,∠35°]) ⇒ (3.28, 2.29) ✓

   d) [8,-¾π] into ***exact*** Cartesian coordinates.

   toRect([8,∠-¾π]) ⇒ $(-4\sqrt{2}, -4\sqrt{2})$ ✓

2. [3, 3 = 6 marks]

Clearly show how you obtain your answers, find:

a) the distance between [20,-210°] and [$\sqrt{5}$,-50°].

toRect([20,∠(-210°)]) ⇒ (-17.32, -10) ✓

toRect([$\sqrt{5}$,∠(-50°)]) ⇒ (1.44, -1.71) ✓

norm([-17.32, -10] – [1.44, -1.71]) ⇒ 22.11 ✓

b) the ***exact*** distance between $\left[\frac{\sqrt{5}}{3}, -\frac{2\pi}{3}\right]$ and $\left[10, -\frac{7\pi}{6}\right]$.

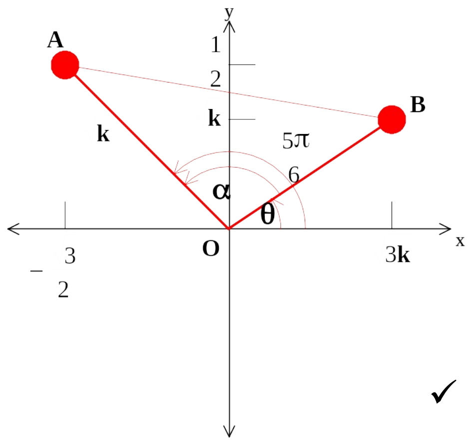
toRect([$\frac{\sqrt{5}}{3}$,∠($-\frac{2\pi}{3}$)]) ⇒ ($-\frac{\sqrt{5}}{6}$, $-\frac{\sqrt{15}}{6}$) ✓

toRect([10,∠($-\frac{7\pi}{6}$)]) ⇒ (-5 $\sqrt{3}$, 5) ✓

norm([$-\frac{\sqrt{5}}{6}$, $-\frac{\sqrt{15}}{6}$] – [-5 $\sqrt{3}$, 5]) ⇒ $\frac{\sqrt{905}}{3}$ ✓

3. [8 marks]

Find the ***exact*** distance between ($\sqrt{3k}$, $\sqrt{k}$) and $\left[\sqrt{k}, \frac{5\pi}{6}\right]$. Draw a diagram.



✓

$$OB = \sqrt{3\mathbf{k} + \mathbf{k}}$$

$$= 2\sqrt{\mathbf{k}} \quad ✓$$

$$\tan\theta = \frac{\sqrt{\mathbf{k}}}{\sqrt{3\mathbf{k}}} \quad ✓$$

$$= \frac{\sqrt{3}}{3}$$

$$\theta = \frac{\pi}{6} \quad ✓$$

$$\therefore \alpha = \frac{2\pi}{3} \quad ✓$$

$AB^2 = OA^2 + OB^2 – 2(OA)(OB) \cos\alpha$ ✓

$= k + 4k – 2(\sqrt{\mathbf{k}})(2\sqrt{\mathbf{k}})(-½)$

$= 7k$ ✓

$\therefore AB = \sqrt{7\mathbf{k}}$

∴ The distance between the points is $\sqrt{7\mathbf{k}}$ units. ✓

4. [3, 2 = 5 marks]

a) Find, in ***exact*** form, the modulus and principal argument of $-\sqrt{3}+i$, and hence rewrite $-\sqrt{3}+i$ in ***exact*** polar (***cis***) form.

Modulus = $\sqrt{3+1} = 2$ ✓

Principal argument = $\tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right) = \dfrac{5\pi}{6}$ ✓

$\therefore -\sqrt{3} + i = 2 \text{ cis } \dfrac{5\pi}{6}$ ✓

b) Convert $2\ cis\left(\dfrac{\pi}{4}\right)$ into ***exact*** algebraic Cartesian/rectangular form.

$$2 \text{ cis}\left(\frac{\pi}{4}\right) = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] \quad ✓$$
$$= \sqrt{2} + i\sqrt{2} \quad ✓$$

5. [3, 3 = 6 marks]

Evaluate, giving answers in ***exact*** form:

a) $4\ cis\dfrac{\pi}{3} \times 2\ cis\dfrac{3\pi}{4} = 8 \text{ cis}\left(\dfrac{\pi}{3} + \dfrac{3\pi}{4}\right)$ ✓

$$= 8 \text{ cis}\frac{13\pi}{12} \quad ✓$$

$$= 8 \text{ cis}\left(-\frac{11\pi}{12}\right) \quad ✓$$

b) $\dfrac{4\ cis\left(-\dfrac{5\pi}{6}\right)}{2\ cis\left(\dfrac{5\pi}{6}\right)} = 2 \text{ cis}\left(-\dfrac{5\pi}{6} - \dfrac{5\pi}{6}\right)$ ✓

$$= 2 \text{ cis}\left(-\frac{5\pi}{3}\right) \quad ✓$$

$$= 2 \text{ cis}\left(\frac{\pi}{3}\right) \quad ✓$$

6. [4 marks]

Given $z = 2\ cis\ \frac{\pi}{4}$, express z^{-1} and $\bar{z}$ in ***exact*** polar and rectangular form.

$$z^{-1} = \left[\ 2\ \text{cis}\ \frac{\pi}{4}\ \right]^{-1}$$

$$= \tfrac{1}{2}\ \text{cis}\left(-\frac{\pi}{4}\right) \quad \leftarrow \text{Polar form} \ \checkmark$$

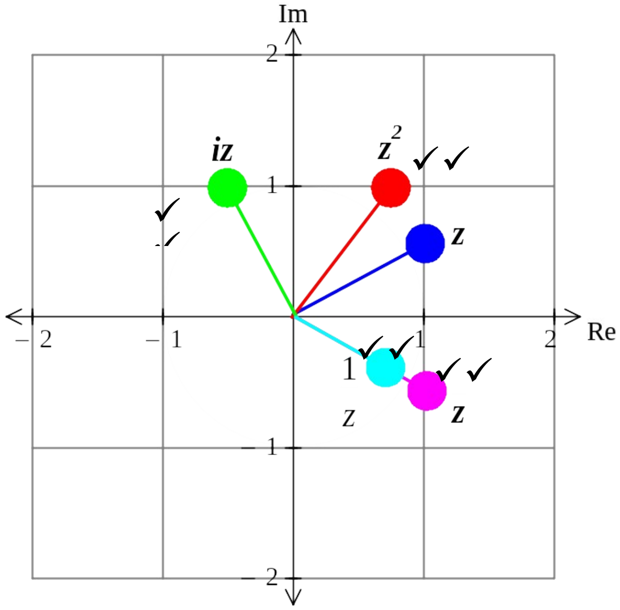
$$= \frac{\sqrt{2}}{4}(1 - \mathbf{i}) \quad \leftarrow \text{Rectangular form} \ \checkmark$$

$$\bar{z} = 2\ \text{cis}\left(-\frac{\pi}{4}\right) \quad \leftarrow \text{Polar form} \ \checkmark$$

$$= \sqrt{2}(1 - \mathbf{i}) \quad \leftarrow \text{Rectangular form} \ \checkmark$$

7. [2, 2, 2, 2 = 8 marks]
The Argand diagram below shows the point representing the complex number z where $|\,z\,| > 1$. Plot on the same diagram, the points representing the complex numbers:

a) $\bar{z}$ b) iz c) z^2 d) $\frac{1}{z}$



If $z = 1 + \tfrac{1}{2}i$

$\Rightarrow \bar{z} = 1 - \tfrac{1}{2}i$

$iz = -\tfrac{1}{2} + i$

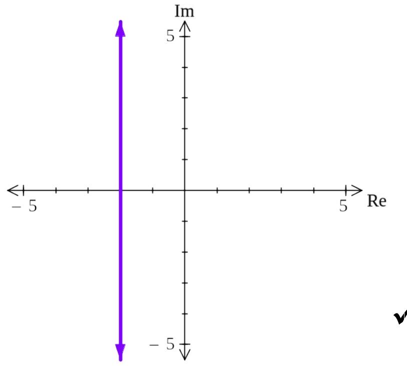
$z^2 = \tfrac{3}{4} + i$

$\frac{1}{z} = \frac{4}{5}\left(1 - \tfrac{1}{2}i\right)$

8. [3, 3, 3, 3 = 12 marks]

Sketch on an Argand diagram, the locus of the point $z = x + iy$, satisfying each of the following conditions. In each case, give the Cartesian equation or inequality of the locus.

a) $\mathrm{Re}(z) = -2 \Rightarrow x = -2$ ✓



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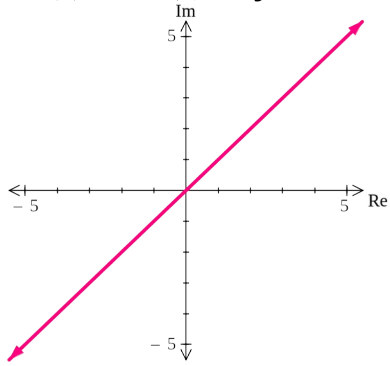
b) $\mathrm{Im}(z) = \mathrm{Re}(z) \Rightarrow y = x$ ✓



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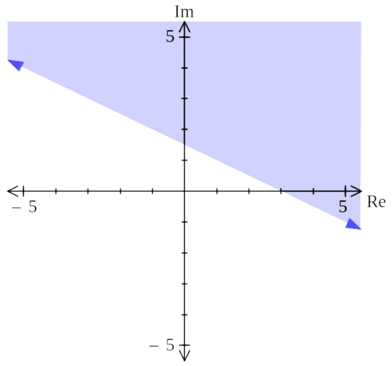
c) $\mathrm{Re}(z) + 2\mathrm{Im}(z) > 3 \Rightarrow x + 2y > 3$ ✓



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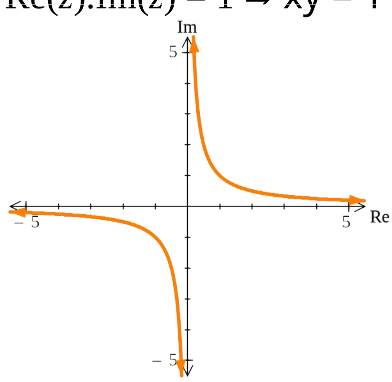
d) $\mathrm{Re}(z).\mathrm{Im}(z) = 1 \Rightarrow xy = 1$ ✓



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