Topic review — Topic 3: Composite functions, transformations and inverses

Short answer

1. Consider the equations $f:R\rightarrow R$, $f(x)=x^2-4$ and $g:(2,\infty)\rightarrow R$, $g(x)=\dfrac{1}{x-2}$.

   a Prove that $f(g(x))$ is defined.

   b Find the rule for $f(g(x))$ and state the domain and range.

   c Prove that $g(f(x))$ is not defined.

   d Restrict the domain of $f(x)$ to obtain a function $f_1(x)$ such that $g(f_1(x))$ exists.

   e Find $g(f_1(x))$ and state the domain.

2. A function has the rule $y=\dfrac{x-1}{x-2}$, $x\neq 2$.

   c Sketch the graph of $y=\dfrac{x-1}{x-2}$, $x\neq 2$. State the domain and range, and give the equations of any asymptotes.

   d Find the rule for the inverse, and state its domain and range.

   e Specify whether the inverse is a function or a relation. Give reasons for your answer.

   f Sketch the graph of the inverse on the same set of axes as the original function. Include the points of intersection on your graph

3. Indicate whether each of the following functions has an inverse function. In each case, give a reason for your decision. If the inverse is a function, write the rule for the inverse function in function notation.

   a $f:R\rightarrow R$, $f(x)=\dfrac{x^3}{3}$

   b $f:R\rightarrow R$, $f(x)=2x^4$

   c $f:R\rightarrow R$, $f(x)=(3x-1)^2$

   d $f:[-5,5]\rightarrow R$, $f(x)=\sqrt{25-x^2}$

   e $f:[3,\infty)\rightarrow R$, $f(x)=\sqrt{x-3}$

6a Consider the functional equation defined by $f(x-y)=\dfrac{f(x)}{f(y)}$. Which of the following functions satisfies this equation?

i $f(x)=x^3$

ii $f(x)=e^x$

iii $f(x)=2^x$

b Consider the functional equation defined by $f(x)=f(\pi-x)$.

i Show that the function $f(x)=\sin(x)$ obeys this rule.

ii Show that the function $f(x)=-\cos(x)$ obeys this rule.

c For $g(x)=4x+2$, show that $g(x+y)$ can be written in the form $g(x)+g(y)+c$ and find the value of c.

Multiple choice

1. If $g(x)=2x-1$ and $h(x)=(x+1)^2$, then $g(h(x))$ is equal to:

A $2x^2+4x+1$

B $4x^2$

C $2x^2+4x-1$

D $(2x-1)(x+1)^2$

E $4(x+1)^2$

2. For the functions below, which of the following compositions is not defined?

$$f(x)=\sqrt{x}+1$$
$$g(x)=x^2-1$$
$$h(x)=2x+1$$

A $g(h(x))$

B $g(f(x))$

C $h(f(x))$

D $f(g(x))$

E $h(g(x))$

3. If $g(x) = \sqrt{x - 1}$, then $g(h(x))$ would exist if:

A $h: R \setminus \{0\} \to R$, $h(x) = \frac{1}{x^2} + 1$

B $h: R \to R$, $h(x) = (x - 1)^2$

C $h: [-1, \infty) \to R$, $h(x) = -(x + 1)^2$

D $h: R \setminus \{-1\} \to R$, $h(x) = \frac{1}{x + 1}$

E $h: R \to R$, $h(x) = x$

4. If $f(x) = \frac{1}{x}$, which of the following functional equations is true?

A $f(x) + f(y) = f(x + y)$

B $f(x) - f(y) = f(x - y)$

C $f(x) \times f(y) = f(xy)$

D $f(x) + f(y) = f(xy)$

E $f(x) - f(y) = f(xy)$

5. The graph of the function $f(x) = x^3$ is transformed so that its new rule is $f(x) = \frac{1}{2}(2(x - 1))^3 + 4$. The transformations that have been applied to $f(x) = x^3$ are:

A dilation by a factor of $\frac{1}{2}$ parallel to the y-axis, dilation by a factor of 2 parallel to the x-axis, a translation of 1 unit in the negative x-direction and a translation of 4 units up

B dilation by a factor of $\frac{1}{2}$ parallel to the y-axis, dilation by a factor of 2 parallel to the x-axis, a translation of 1 unit in the positive x-direction and a translation of 4 units up

C dilation by a factor of $\frac{1}{2}$ parallel to the y-axis, dilation by a factor of $\frac{1}{2}$ parallel to the x-axis, a translation of 1 unit in the negative x-direction and a translation of 4 units up

D dilation by a factor of $\frac{1}{2}$ parallel to the y-axis, dilation by a factor of $\frac{1}{2}$ parallel to the x-axis, a translation of 1 unit in the positive x-direction and a translation of 4 units up

E dilation by a factor of 2 parallel to the y-axis, dilation by a factor of $\frac{1}{2}$ parallel to the x-axis, a translation of 1 unit in the negative x-direction and a translation of 4 units up

6. The following matrix equation is applied to $y = \sqrt{x}$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

This causes $y = \sqrt{x}$ to be:

A reflected in the x-axis and dilated by a factor of 2 from the y-axis

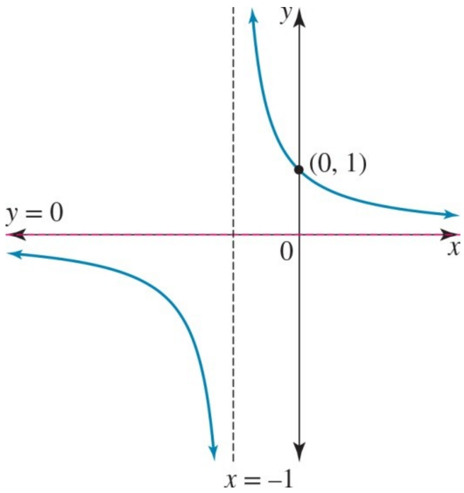
B reflected in the y-axis and dilated by a factor of 2 from the y-axis

C reflected in the y-axis and dilated by a factor of 2 from the x-axis

D reflected in the y-axis and dilated by a factor of $\frac{1}{2}$ from the x-axis

E reflected in both axes and dilated by a factor of 2 from the x-axis

7. The rule for the inverse of the graph shown would be:



A $y = \frac{1}{x} + 1$

B $y = \frac{1}{x+1}$

C $y = \frac{1}{x} - 1$

D $y = \frac{1}{x-1}$

E $y = \frac{1}{x-1} - 1$

8. For the function $f(x) = (x+1)(x-3)$ to have an inverse, its maximal domain:

A must be restricted to $[0, \infty)$

B must be restricted to $[1, \infty)$

C must be restricted to $[-4, \infty)$

D is R

E must be restricted to $(-\infty, 0]$

9. The inverse of the function defined by $f: [-1, \infty) \to R$, $f(x) = (x+1)^2$ would be:

A $f^{-1}: [-1, \infty) \to R$, $f^{-1}(x) = \sqrt{x} - 1$

B $f^{-1}: [-1, \infty) \to R$, $f^{-1}(x) = -\sqrt{x} - 1$

C $f^{-1}: [-1, \infty) \to R$, $f^{-1}(x) = (x+1)^2$

D $f^{-1}:[0,\infty)\to R$, $f^{-1}(x)=\sqrt{x}-1$

E $f^{-1}:[0,\infty)\to R$, $f^{-1}(x)=-\sqrt{x}-1$

Extended response

1. Consider the function defined by $f(x)=2(x-3)^2$.
   - **a** Sketch this graph, giving the domain and range of the function.
   - **b** Find the rule for the inverse.
   - **c** Sketch this inverse on the same set of axes that you used for $f(x)=2(x-3)^2$.
   - **d** Restrict the domain of f to the form of $[a,\infty)$ so that the inverse is also a function.
   - **e** State the rules for the restricted f and f^{-1} using function notation.
   - **f** Sketch the graphs of f and f^{-1} on one set of axes.
   - **g** Show that $f\left(f^{-1}(x)\right)=x$.
2. Consider the function defined by the rule $f:D\to R$, $f(x)=\sqrt{(3x-6)}-1$ where D is the maximal domain for f.
   - **a** Find D.
   - **b** Describe the transformations that would have been applied to $y=\sqrt{x}$ in order to achieve $y=f(x)$.
   - **c** Write a matrix equation that defines these transformations and solve the matrix equation to confirm this is correct.
   - **d** Define the rule for the inverse function f^{-1} and give its domain and range.
   - **e** Sketch the graphs of $y=f(x)$ and $y=f^{-1}(x)$ on the same set of axes.
3. **a** If $f:[3,\infty)\to R$, $f(x)=x^2+k$ and $g:[2,\infty)\to R$, $g(x)=\frac{1}{x}+k$, where k is a positive constant, find the value(s) of k such that both $f\left(g(x)\right)$ and $g\left(f(x)\right)$ are defined.

b The transformation $T: R^2 \to R^2$ is defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$, where a, b, c and d are non-zero real numbers. If the image of the curve $g(x) = -\sqrt{2x-2} + 2$ is $f(x) = \sqrt{x}$, find the values of a, b, c and d.

4. a If $g(x) = \sqrt{3\sin(x)} - 2$, show that it obeys the functional equation defined by $g(x)^2 + 4g(x) + 4 = 3\sin(x)$.

b If $h(x) = 1 - x^2$, show that it obeys the functional equation defined by $x^2 h(x) + h(1-x) = 2x - x^4$.

Topic review — answers

Short answer

1 a For $f(g(x))$ to exist, the range of the inner function, $g(x)$, must be a subset of or equal to the domain of the outer function, $f(x)$.

$(0, \infty) \subseteq R$

$\text{ran } g \subseteq \text{dom } f$

Therefore, $f(g(x))$ is defined.

b $f(g(x)) = \dfrac{1}{(x-2)^2} - 4$

Domain = $(2, \infty)$, range = $(-4, \infty)$

c For $g(f(x))$ to exist the range of the inner function, $f(x)$ must be a subset of or equal to the domain of the outer function, $g(x)$.

$[-4, \infty) \not\subset (2, \infty)$

$\text{ran } f \not\subset \text{dom g}$

Therefore, $g(f(x))$ is not defined.

d $f_1 : \left(-\infty, -\sqrt{6}\right) \cup \left(\sqrt{6}, \infty\right) \to R$, $f(x) = x^2 - 4$

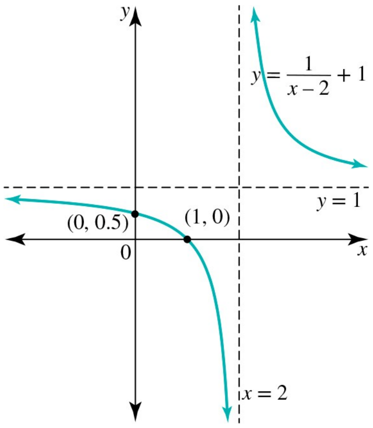
e $g(f_1(x)) = \dfrac{1}{x^2 - 6}$

Domain = $\left(-\infty, -\sqrt{6}\right) \cup \left(\sqrt{6}, \infty\right)$

2

c $y = \dfrac{1}{x - 2} + 1$; domain = $R \setminus \{2\}$ and range = $R \setminus \{1\}$

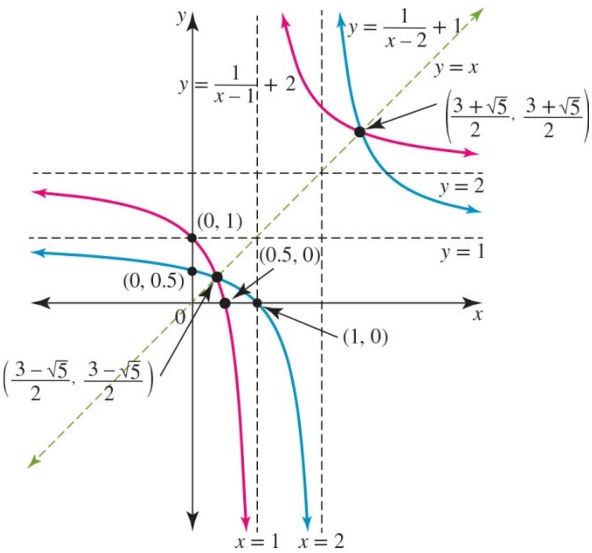
Asymptotes: $x = 2$ and $y = 1$



d $y = \dfrac{1}{x - 1} + 2$, domain = $R \setminus \{1\}$ and range = $R \setminus \{2\}$

e The inverse is a one-to-one function.

f



3 **a** One-to-one inverse function: $f^{-1}: R \to R$, $f^{-1}(x) = \sqrt[3]{3x}$

b Not a function, as it is a one-to-many mapping.

c Not a function, as it is a one-to-many mapping.

d Not a function, as it is a one-to-many mapping.

e One-to-one inverse function: $f^{-1}:[0,\infty)\to R$, $f^{-1}(x)=x^2+3$

6 a i No

ii Yes

iii Yes

b i $\text{LHS}=f(x)$

$=\sin(x)$

$\text{RHS}=f(\pi-x)$

$=\sin(\pi-x)$

Because $\sin(x)$ is positive in the 2nd quadrant,

$\sin(\pi-x)=\sin(x)$.

Therefore,

$\text{LHS}=\text{RHS}$

$f(x)=f(\pi-x)$

ii $\text{LHS}=f(x)$

$=-\cos(x)$

$\text{RHS}=f(\pi-x)$

$=\cos(\pi-x)$

Because $\cos(x)$ is negative in the 2nd quadrant,

$\cos(\pi-x)=-\cos(x)$.

Therefore,

$\text{LHS}=\text{RHS}$

$f(x)=f(\pi-x)$

c $g(x+y)=4(x+y)+2$

$=4x+4y+2$

$=(4x+2)+(4y+2)-2$

$=g(x)+g(y)-2$

$\therefore c=-2$

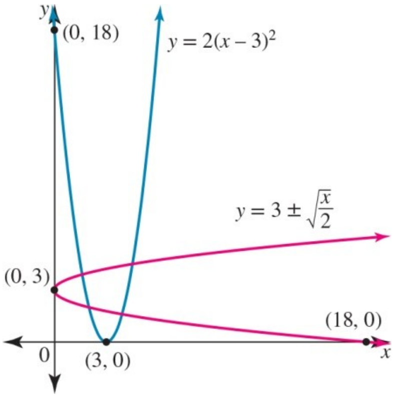
Multiple choice

1 A **2** D **3** A **4** C

6 D **7** B **8** C **9** B **10** D

Extended response

1 a, c The domain of f is R and the range of f is $[0,\infty)$.



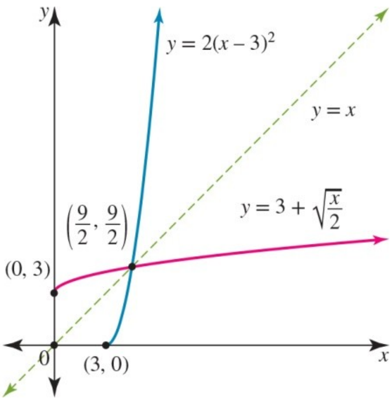
b $y=3\pm\sqrt{\dfrac{x}{2}}$; domain = $[0,\infty)$ and range = R

d The domain should be $[3,\infty)$.

e $f:[3,\infty)\to R$, $f(x)=2(x-3)^2$

$f^{-1}:[0,\infty)\to R$, $f(x)=\sqrt{\dfrac{x}{2}}+3$

f



g $f\left(f^{-1}(x)\right)=2\left(\sqrt{\dfrac{x}{2}}+3-3\right)^2$

$$=2\left(\sqrt{\frac{x}{2}}\right)^2$$

$$=2\times\frac{x}{2}$$

$$=x$$

2 a $D=[2,\infty)$

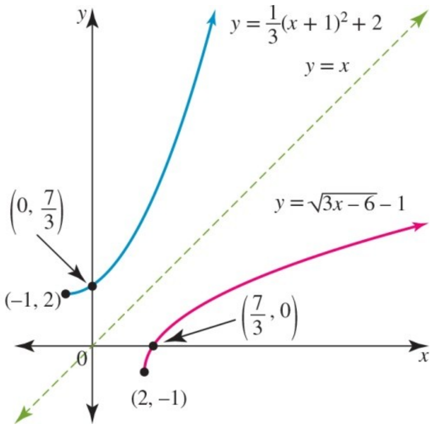
b One possible answer is:

Dilated by a factor of $\frac{1}{3}$ parallel to the x-axis or from the y-axis, translated 2 units to the right or in the positive x-direction and translated 1 unit down or in the negative y-direction

c $$\mathrm{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

d $f^{-1}:[-1,\infty)\to R,\ f^{-1}(x)=\frac{1}{3}(x+1)^2+2$ with range = $[2,\infty)$

e



3 a $k\geq 3$

b $a=2$, $b=-1$, $c=-2$ and $d=2$

4 a $\text{LHS}=g(x)^2+4g(x)+4$

$$=\left(\sqrt{3\sin(x)}-2\right)^2+4\left(\sqrt{3\sin(x)}-2\right)+4$$

$$=3\sin(x)-4\sqrt{3\sin(x)}+4+4\sqrt{3\sin(x)}-8+4$$

$$=3\sin(x)-4\sqrt{3\sin(x)}+4\sqrt{3\sin(x)}+8-8$$

$$=3\sin(x)$$

$\text{RHS}=3\sin(x)$

$\text{LHS} = \text{RHS}$:

$g(x)^2 + 4g(x) + 4 = 3\sin(x)$

b $\text{LHS} = x^2h(x) + h(1 - x)$

$= x^2(1 - x^2) + 1 - (1 - x)^2$

$= x^2 - x^4 + 1 - (1 - 2x + x^2)$

$= x^2 - x^4 + 1 - 1 + 2x + x^2$

$= 2x - x^4$

$\text{RHS} = 2x - x^4$

$\text{LHS} = \text{RHS}$:

$x^2h(x) + h(1 - x) = 2x - x^4$