Kennedy Baptist College

Semester One Examination, 2020

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section One: Calculator-free

SOLUTIONS

WA student number: In figures | | | | | | | | |

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable): ☐

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(5 marks)**

A curve, defined for $x>0$, passes through the point $A(2,1)$ and its gradient is given by

$$\frac{dy}{dx}=3x^2-\frac{8}{x^2}-10$$

(a) Verify that A is a stationary point, determine the value of the second derivative at A and hence describe the nature of the stationary point. (3 marks)

Solution
$f'(x)=3x^2-\frac{8}{x^2}-10\Rightarrow f'(2)=12-2-10=0$ $f'(2)=0$, so A is a stationary point. $f''(x)=6x+\frac{16}{x^3}\Rightarrow f''(2)=12+2=14$ $f''(2)>0$, so A is a local minimum.
Specific behaviours
✓ simplifies $f'(2)$ to three integers that sum to zero ü correct value of second derivative

(b) Determine the equation of the curve. (2 marks)

Solution
$f(x)=x^3+\frac{8}{x}-10x+c$ $f(2)=8+4-20+c=1\Rightarrow c=9$ $y=x^3+\frac{8}{x}-10x+9$
Specific behaviours
✓ correct antiderivative ü evaluates constant and writes equation

Question 2 **(5 marks)**

Determine the area bounded by the line $y=-2x$ and the parabola $y=x^2-6x$.

Solution
Intersect when $-2x-\left(x^2-6x\right)=0 \quad 4x-x^2=0 \quad x(4-x)=0 \quad x=0,4$ Bounded area $A=\int_0^4 4x-x^2\,dx ¿ \left[2x^2-\frac{x^3}{3}\right]_0^4 ¿ \left(32-\frac{64}{3}\right)-(0) ¿ 32-21.\overline{3}$ $¿ 10.\overline{6}=10\frac{2}{3}$ square units
Specific behaviours
✓ equates functions and simplifies ü bounds of integral ü writes definite integral ü antidifferentiates ü correct area

Question 3 **(8 marks)**

Determine

(a) $f'(x)$ when $f(x)=\sqrt{4x-3}$. (2 marks)

Solution
$f'(x)=\frac{1}{2}(4)(4x-3)^{\frac{-1}{2}}¿\frac{2}{\sqrt{4x-3}}$
Specific behaviours
✓ indicates correct use of chain rule ü correct derivative (any form)

(b) $\frac{d}{d\theta}(\theta^3 e^{4\theta})$ when $\theta=2$. (3 marks)

Solution
$¿3\theta^2 e^{4\theta}+4\theta^3 e^{4\theta}\vert_{\theta=2}$ $¿12e^8+32e^8=44e^8$
Specific behaviours
ü u' or v' correct ü correct derivative in terms of θ ü correct value

(c) $f'\left(\frac{\pi}{4}\right)$ when $f(t)=\frac{1+\cos t}{\sin t}$. (3 marks)

Solution
$f'(t)=\frac{-\sin t\cdot\sin t-(1+\cos t)\cdot\cos t}{\sin^2 t}$ $¿\frac{-\cos t-\sin^2 t-\cos^2 t}{\sin^2 t}¿\frac{-1-\cos t}{\sin^2 t}$ $f'\left(\frac{\pi}{4}\right)=\left(-1-\frac{1}{\sqrt{2}}\right)\div\frac{1}{2}¿-2-\sqrt{2}$
Specific behaviours
✓ indicates correct use of quotient rule ü correct derivative ü correct value, simplified

Question 4 **(7 marks)**

A bag contains 40 counters, 15 marked with 0 and the remainder marked with 1. The random variable X is the number on a randomly selected counter from the bag.

(a) Explain why X is a Bernoulli random variable and determine the mean and variance of X. (3 marks)

Solution
X is a Bernoulli random variable as it can only take on two values, 0 and 1. $$E(X)=p=\frac{40-15}{40}=\frac{5}{8}$$ $$\sigma^2=\frac{5}{8}\times\frac{3}{8}=\frac{15}{64}$$
Specific behaviours
✓ states X can only take on two values   ü mean   ü variance

Each of the 32 students in a class randomly select a counter from the bag, note the number on the counter and then replace it back in the bag. The random variable Y is the number of students in the class who select a counter marked with 0.

(b) Define the distribution of Y and determine the mean and variance of Y. (3 marks)

Solution
$$Y \quad B\left(32,\frac{3}{8}\right)$$ $$E(Y)=np=32\times\frac{3}{8}=12$$ $$\sigma^2=12\times\frac{5}{8}=\frac{15}{2}=7.5$$
Specific behaviours
✓ states binomial with parameters   ü mean   ü variance

(c) Explain why it is important that the students replace their counters for the distribution of Y in part (b) to be valid. (1 mark)

Solution
If counters not replaced, the probability of a success (selecting a counter marked with 0) would not remain constant.
Specific behaviours
✓ indicates that probability of success must be constant

Question 5 **(7 marks)**

Functions f and g are such that

$$f(2)=-1, f'(x)=6(2x-7)^{-2}$$

$$g(-3)=-1, g'(x)=6(2x+7)^{-2}$$

(a) Determine $f(3)$. (3 marks)

Solution
$f(3)=f(2)+\int_2^3 6(2x-7)^{-2}dx$
$\text{¿}-1+\left[\frac{-3}{2x-7}\right]_2^3\text{¿}-1+(3-1)\text{¿}1$
Specific behaviours
✓ integrates rate of change
ü determines change
ü correct value

(b) Use the increments formula to determine an approximation for $g(-2.97)$. (3 marks)

Solution
$x=-3, \delta x=0.03$
$\delta y\approx\frac{6}{(2x+7)^2}\times\delta x\approx 6\times 0.03\approx 0.18$
$g(-2.97)\approx -1+0.18\approx -0.82$
Specific behaviours
✓ values of x and δx
ü use of increments formula
ü correct approximation

(c) Briefly discuss whether using the information given about f and the increments formula would yield a reasonable approximation for $f(3)$. (1 mark)

Solution
No, approximation wouldn't - the change $\delta x=1$ is not a small change.
(NB Yields $f(3)\approx-\frac{1}{3}$ *)*
Specific behaviours
✓ states no with reason

Question 6 **(5 marks)**

The graph of $y=f(x)$ has a stationary point at $(2,5)$ and $f'(x)=ax^2-9x+6$, where a is a constant.

Determine the interval over which $f'(x)<0$ and $f''(x)<0$.

Solution
$f'(2)=4a-18+6=0 \quad a=3$ Concave down: $f'(x)=3x^2-9x+6 \quad f''(x)=6x-9$ $f''(x)<0 \Rightarrow x<1.5$ Other stationary point: $3x^2-9x+6=0 \quad 3(x-1)(x-2)=0 \quad x=1$ Hence $f'(x)<0$ when $1<x<2$. Required interval: $1<x<1.5$.
Specific behaviours
✓ value of a ü interval where $f''(x)<0$ ü second stationary point ü interval where $f'(x)<0$ ü correct interval

Question 7 **(8 marks)**

Initially, particle P is stationary and at the origin. Particle P moves in a straight line so that at time t seconds its acceleration a cms^{-2} is given by $a=8-3\sqrt{t}$ where $t\geq 0$.

(a) Determine the speed of P after 1 second. (3 marks)

Solution
$v=\int 8-3t^{0.5}\,dt\text{¿}8t-2t^{1.5}+c$ $v(0)=0\Rightarrow c=0$ $v=8t-2t^{1.5}$ $v(1)=8(1)-2(1)^{1.5}=6$ Hence speed is 6 cm/s.
Specific behaviours
ü indicates v is integral of a ✓ expression for velocity v ü correct speed

(b) Determine the speed of P when it returns to the origin. (5 marks)

Solution
Require 0 change in displacement for $0\leq t\leq T$ $\Delta x=\int_0^T 8t-2t^{1.5}\,dt0=\left[4t^2-\frac{4}{5}t^{2.5}\right]_0^T 4T^2-\frac{4}{5}T^{2.5}=0$ $4T^2\left(1-\frac{1}{5}\sqrt{T}\right)=0\sqrt{T}=5T=25$ $v(25)=8(25)-2(25)^{\frac{3}{2}}\text{¿}200-2(5)^3\text{¿}200-250=-50$ Hence speed is 50 cm/s.
Specific behaviours
ü obtains expression for Δx in terms of T ü equates $\Delta x=0$ ü solves for T ü obtains velocity   ü correct speed, with units

Question 8 **(7 marks)**

(a) Determine an expression for $\frac{d}{dt}\left(8t\sin\left(\frac{\pi t}{8}\right)\right)$. (2 marks)

Solution

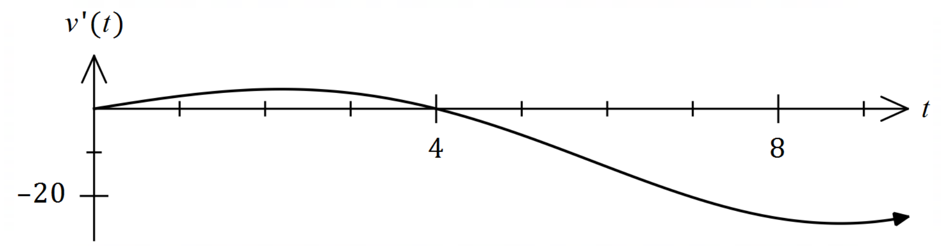
$$\frac{d}{dt}\left(8t\sin\left(\frac{\pi t}{8}\right)\right)=8\sin\left(\frac{\pi t}{8}\right)+\pi t\cos\left(\frac{\pi t}{8}\right)$$

Specific behaviours

✓ correct use of product rule

ü correct derivative

The volume of water in a tank, v litres, is changing at a rate given by $v'(t)=\pi t\cos\left(\frac{\pi t}{8}\right)$, where t is the time in hours. The rate of change is shown in the graph below.



(b) Using the result from part (a) or otherwise, determine the change in volume of water in the tank between $t=0$ and $t=8$ hours. (5 marks)

Solution

$$\Delta v=\int_0^8 v'(t)\,dt¿\int_0^8 \pi t\cos\left(\frac{\pi t}{8}\right)dt$$

1. Using (a):

$$\int \frac{d}{dt}\left(8t\sin\left(\frac{\pi t}{8}\right)\right)dt=\int 8\sin\left(\frac{\pi t}{8}\right)dt+\int \pi t\cos\left(\frac{\pi t}{8}\right)dt$$

2. And so:

$$\int \pi t\cos\left(\frac{\pi t}{8}\right)dt=8t\sin\left(\frac{\pi t}{8}\right)-\int 8\sin\left(\frac{\pi t}{8}\right)dt$$

3. Hence:

$$\int_0^8 \pi t\cos\left(\frac{\pi t}{8}\right)dt=\left[8t\sin\left(\frac{\pi t}{8}\right)\right]_0^8+\left[\frac{64}{\pi}\cos\left(\frac{\pi t}{8}\right)\right]_0^8 ¿\,[0-0]+\left[\frac{-64}{\pi}-\frac{64}{\pi}\right]$$

$$\Delta v=\frac{-128}{\pi}\text{ L}$$

Specific behaviours

✓ indicates required definite integral

ü line 1 - uses part (a)

ü line 2 - expression to evaluate integral

ü line 3 - antidifferentiates ready for substitution

ü correct change in volume, with units

End of questions

Supplementary page

Question number: _________