PERTH MODERN SCHOOL
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Semester Two Examination, 2020

Question/Answer booklet

MATHEMATICS SPECIALIST
UNITs 3 & 4
Section Two:
Calculator-assumed

Your Name

Your Teacher's Name

Time allowed for this section
Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Marks	Max	Question	Marks	Max
8		8	15		7
9		7	16		5
10		16	17		6
11		8	18		10
12		10			
13		9			
14		9			

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	50	35
Section Two: Calculator-assumed	11	11	100	89	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **(89 Marks)**

This section has **11** questions. Answer **all** questions. Write your answers in the spaces provided.

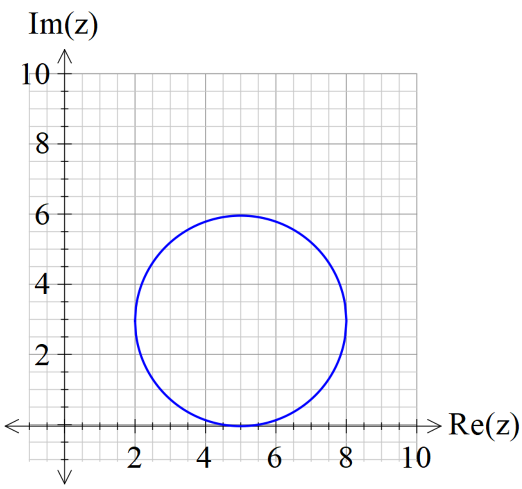
Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 100 minutes.

Question 8 **(8 marks)**

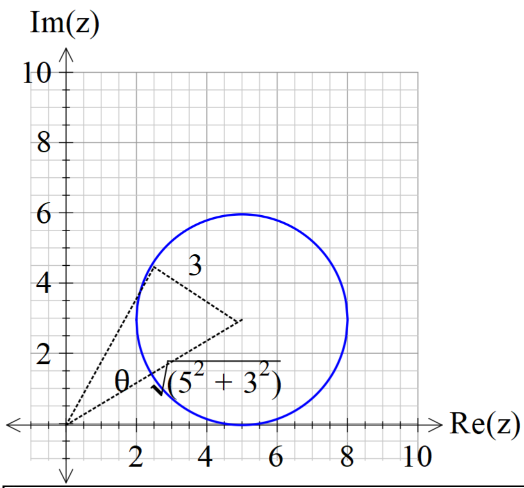
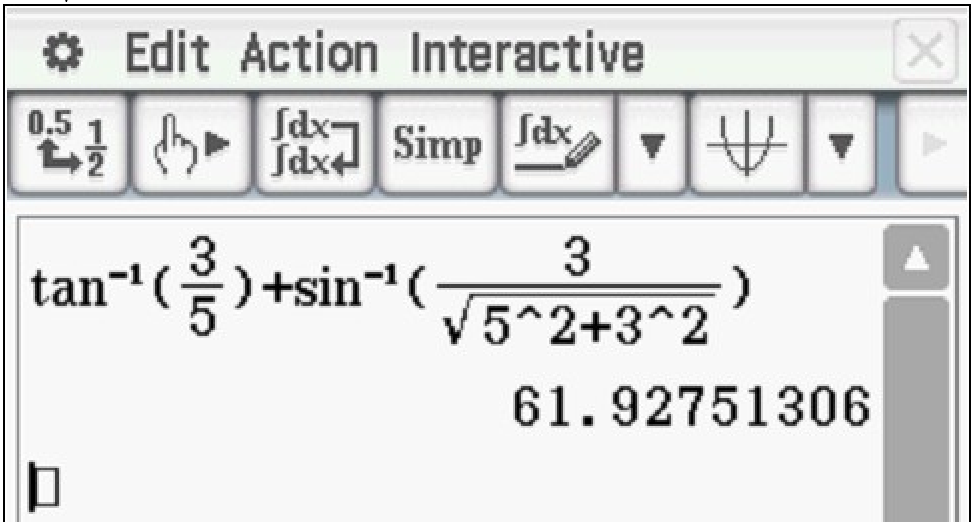
The sketch of the locus $z = x + iy$ is shown below.



a) Determine an equation for z. (3 marks)

Solution
$\lvert z - (5+3i) \rvert = 3$
Specific behaviours
✓ uses modulus ✓ uses centre ✓ uses radius

b) Determine the maximum value of $Arg(z)$ (3 marks)

Solution
 
Specific behaviours
✓ determines argument of centre ✓ uses tangent line ✓ determines max arg

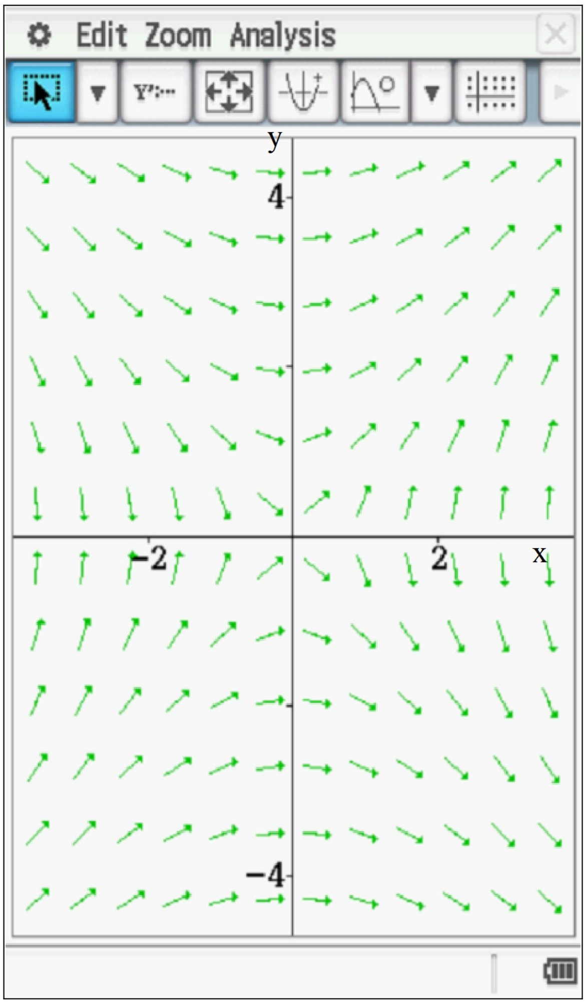
c) State the value of z from the locus above that also satisfy $Arg(z-5-3i)=\frac{\pi}{2}$

(2 marks)

Solution
Top of circle $z=5+3i+3i=5+6i$
Specific behaviours
✓ uses top of circle ✓ states z

Question 9 **(7 marks)**

The slope field $\frac{dy}{dx} = \frac{x}{y}$ is plotted below.

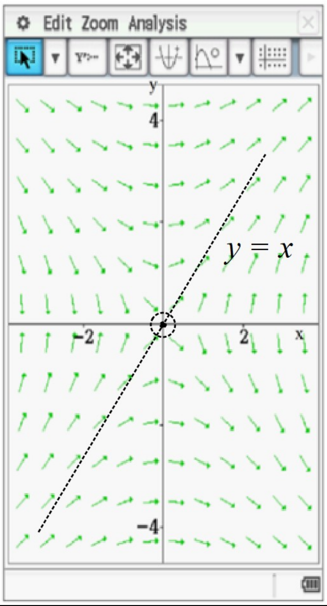


(a) Determine the value of the slope field at point A(2,2). (2 marks)

Solution
$\frac{dy}{dx} = \frac{x}{y} = \frac{2}{2} = 1$
Specific behaviours
✓ subs values ✓ states value

(b) On the diagram above, sketch the solution curve that passes through A(2,2). (2 marks)

Solution



Specific behaviours
✓ uses line y=x ✓ stops at origin or shows open circle

(c) Determine the equation of the solution curve that passes through A(2,2). (3 marks)

Solution
$\frac{dy}{dx} = \frac{x}{y}$ $\int y\,dy = \int x\,dx$ $\frac{y^2}{2} = \frac{x^2}{2} + c$ $(2,2) \quad c = 0$ $y^2 = x^2$

Specific behaviours
✓ separates variables ✓ solves for general solution ✓ solves for constant and states that the value is zero

Question 10 **(10 marks)**

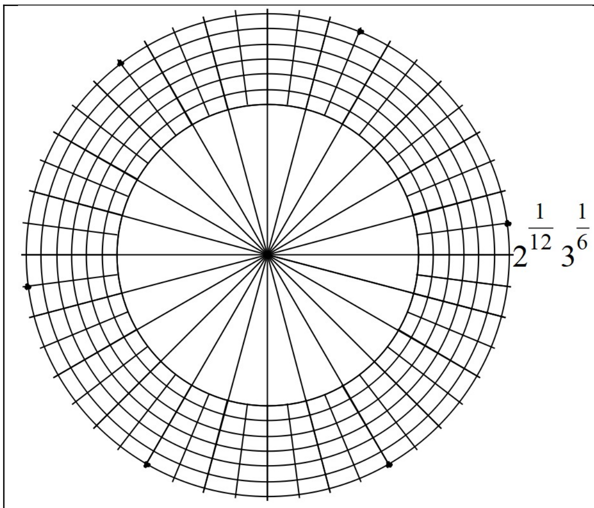
Consider $z^6 = 3 + 3i$.

a) Determine all roots to the above equation in polar form $z = r cis\theta$ with $-\pi < \theta \le \pi$.
(4 marks)

Solution
$z^6 = 3 + 3i = 3\sqrt{2} cis\left(\frac{\pi}{4} + 2n\pi\right)$ $n = 0, \pm1, \pm2...$ $z = 3^{\frac{1}{6}}\left(2^{\frac{1}{12}}\right) cis\left(\frac{\pi}{24} + \frac{8}{24}n\pi\right)$ $z_{1=} 3^{\frac{1}{6}}\left(2^{\frac{1}{12}}\right) cis\left(\frac{\pi}{24}\right)$ $z_2 = 3^{\frac{1}{6}}\left(2^{\frac{1}{12}}\right) cis\left(\frac{9\pi}{24}\right)$ $z_3 = 3^{\frac{1}{6}}\left(2^{\frac{1}{12}}\right) cis\left(-\frac{7\pi}{24}\right)$ $z_4 = 3^{\frac{1}{6}}\left(2^{\frac{1}{12}}\right) cis\left(\frac{17\pi}{24}\right)$ $z_5 = 3^{\frac{1}{6}}\left(2^{\frac{1}{12}}\right) cis\left(-\frac{15\pi}{24}\right)$ $z_6 = 3^{\frac{1}{6}}\left(2^{\frac{1}{12}}\right) cis\left(-\frac{23\pi}{24}\right)$
Specific behaviours
✓ uses De Moivres ✓ uses polar form ✓ uses correct mod for all 6 roots ✓ uses principal argument for all 6 roots

b) Plot all roots on the diagram below. (3 marks)

Solution

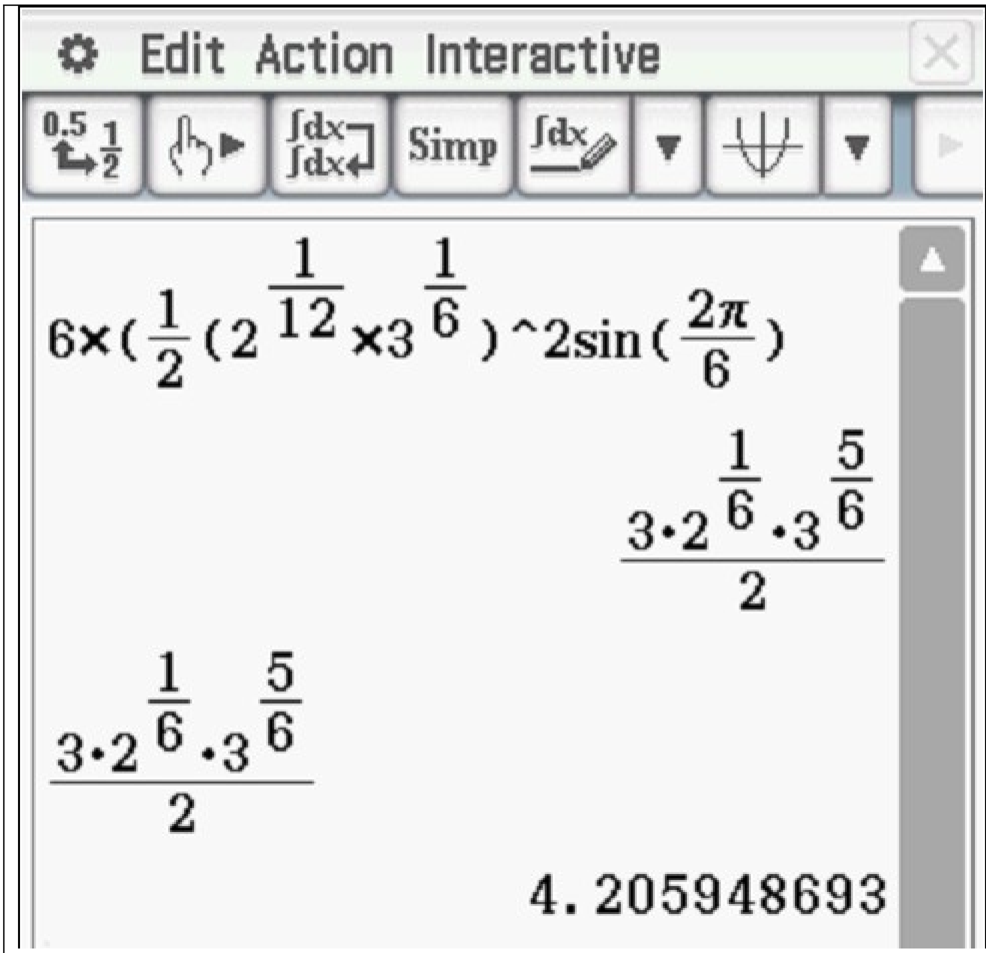


Specific behaviours
✓ states scale ✓ six dots equally spaced ✓ all at correct position

c) Joining these roots will form a polygon, determine the area of this polygon.

(3 marks)

Solution



Specific behaviours
✓ uses 6 equilateral triangles ✓ uses mod of roots for side lengths ✓ determines approx. area or exact

Question 11 (8 marks)

Consider the path of a stunt airplane that travels at a constant height according to the following position vector $r = \begin{pmatrix} 3\sin\frac{t}{2} \\ 4\cos t \end{pmatrix}$ km at t hours.

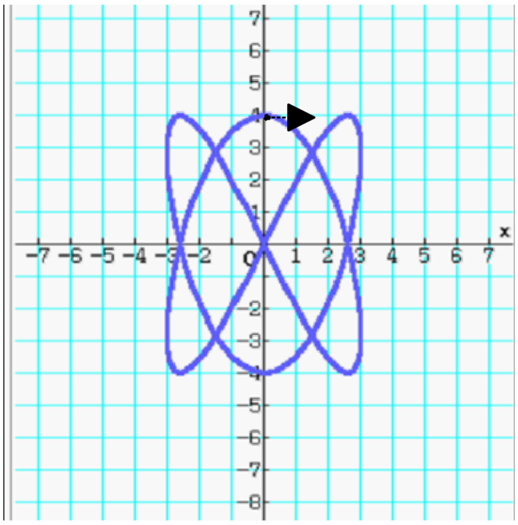
a) Plot the starting point of the plane on the above diagram showing the direction of motion. (2 marks)

Solution

$$r = \begin{pmatrix} 3\sin\frac{t}{2} \\ 4\cos t \end{pmatrix}$$

$$\dot{r} = \begin{pmatrix} \frac{3}{2}\cos\frac{t}{2} \\ -4\sin t \end{pmatrix}$$

$$\dot{r}(0) = \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$$



Specific behaviours
✓ plots (0,4) ✓ shows arrow to the right

b) Determine the initial acceleration. (3 marks)

Solution

$$r = \begin{pmatrix} 3\sin\frac{t}{2} \\ 4\cos t \end{pmatrix}$$

$$\dot{r} = \begin{pmatrix} \frac{3}{2}\cos\frac{t}{2} \\ -4\sin t \end{pmatrix}$$

$$\dot{r} = \begin{pmatrix} -\frac{3}{4}\sin\frac{t}{2} \\ -4\cos t \end{pmatrix}$$

$$\dot{r}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

Specific behaviours
✓ determines velocity function ✓ determines acceleration function ✓ determines initial t=0

c) State an expression for the distance travelled in one circuit of the motion, do not evaluate this expression. (3 marks)

Solution
Horiz motion period = 4π Vert motion period = 2π LCM= 4π = time for one circuit $\int_0^{4\pi}\left\|\sqrt{\frac{9}{4}\cos^2\frac{t}{2}+16\sin^2 t}\right\|dt$
Specific behaviours
✓ determines time for one circuit ✓ uses magnitude of velocity ✓ states correct integral

Question 12 **(10 marks)**

The length of time in minutes that a patient spends with a dentist is normally distributed with a mean of 45 minutes and a population standard deviation of 8 minutes.
A sample of 50 patients is taken as a study of the habits at a particular dental practice.

a) State the approximate sample mean length distribution for the 50 patients. (3 marks)

Solution
$X \sim N\left(45,\left(\frac{8}{\sqrt{50}}\right)^2\right)$
Specific behaviours
✓ states Normal ✓ states mean ✓ states new standard deviation or variance

b) Determine the probability that the sample mean length will be less than 35 minutes.
(2 marks)

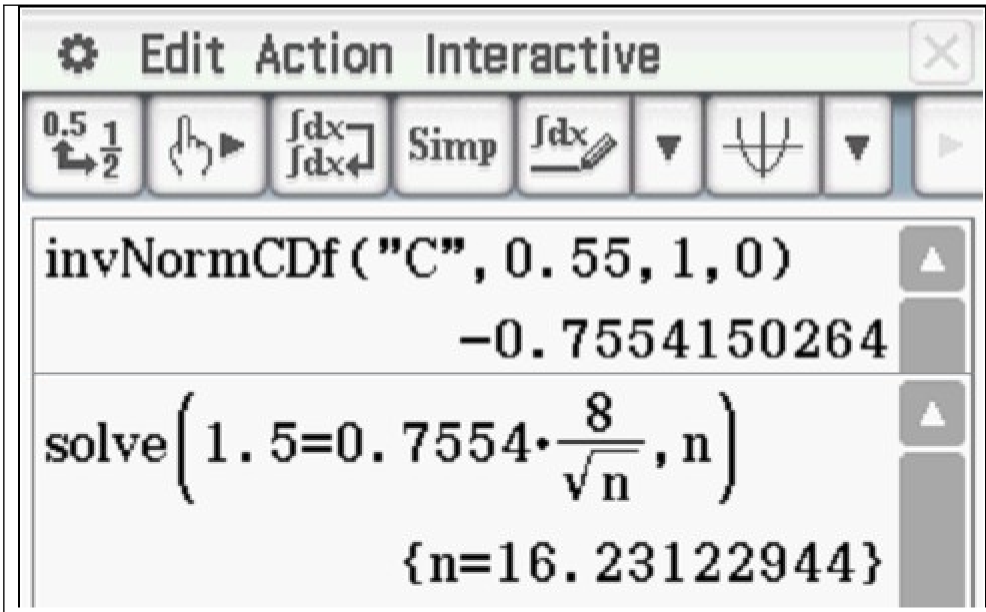
Solution
$\text{normCDf}\left(-\infty, 35, \frac{8}{\sqrt{50}}, 45\right)$ $4.836102066\text{E}-19$
Specific behaviours
✓ uses correct parameters   ✓ states prob

c) Suppose that less than 50 patients were chosen for the sample size, what would happen to the answer in (b) above, do not recalculate. Explain (2 marks)

Solution
Area to left will increase as sample mean standard deviation increases
Specific behaviours
✓ Area/Prob increases   ✓sample mean standard deviation increases(must mention mean stdev)

d) It is desired that the probability that the sample mean length time between 43.5 minutes and 46.5 minutes is at least 55%. Determine the minimum sample size for this to occur.
(3 marks)

Solution



n=17

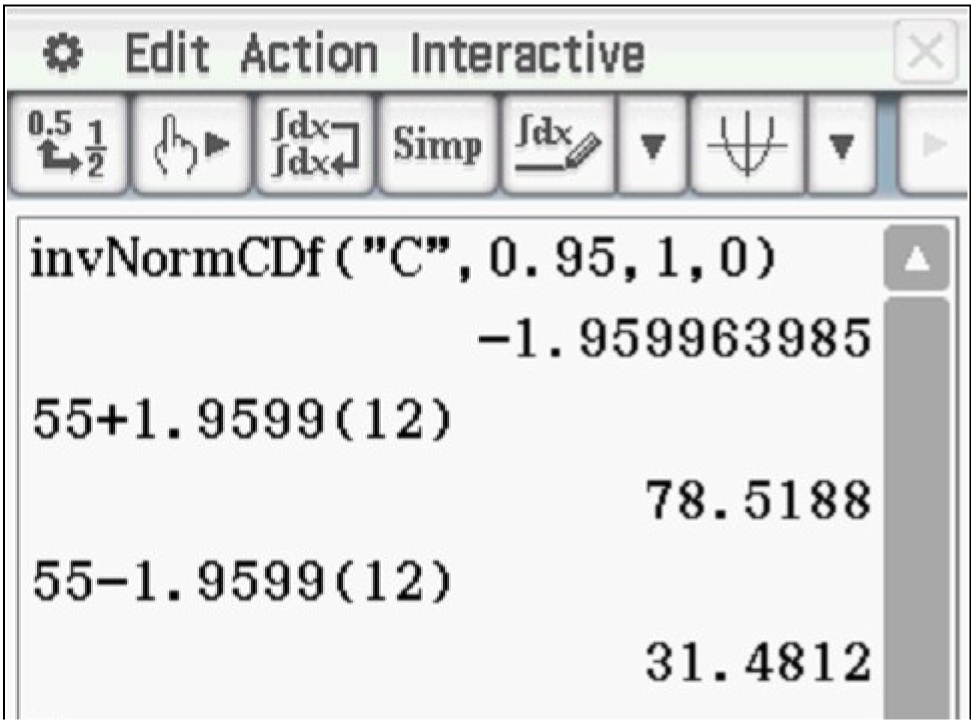
Specific behaviours
✓ determines z parameter ✓ sets up equation to solve for n ✓ rounds n up

Question 13 **(9 marks)**

The time taken for a WACE marker to mark a Methods exam paper has a mean of 55 minutes. A sample of n WACE markers was obtained and the **sample mean** standard deviation was found to be 12 minutes.

a) Determine a 95% confidence interval for the population mean of marking time to the nearest 0.01 minutes. (3 marks)

Solution



31.48 to 78.52 mins

Specific behaviours

✓ uses correct z parameter
✓ determines lower level rounded to nearest 0.01 mins
✓ determines upper level

b) If a sample of $3n$ WACE markers was obtained from the same population, determine the standard deviation of this new sample mean. (2 marks)

Solution

$$\frac{\sigma}{\sqrt{n}}=12$$

$$\frac{\sigma}{\sqrt{3n}}=\frac{12}{\sqrt{3}}\approx 6.928$$

Specific behaviours

✓ uses root 3

✓ states approx. new standard deviation

c) Which of the two samples above have a greater precision for determining the true population mean μ. Explain. (2 marks)

Solution
Sample of 3n due to smaller standard deviation
Specific behaviours
✓ states 3n sample with a reason ✓ states smaller standard deviation

d) A 95% confidence interval is determined for the sample of $3n$ WACE markers. When compared to the confidence interval calculated in (a) above, which interval contains the true value of μ? Explain. (2 marks)

Solution
We do not know which interval contains the true value of population mean. A confidence interval does not always contain true value.
Specific behaviours
✓ states that we do not know ✓ explains that not all intervals contain mean

Question 14 **(9 marks)**

Consider the sphere given by the following cartesian equation
$x^2+y^2+z^2-6x+10y-2z+26=0$.

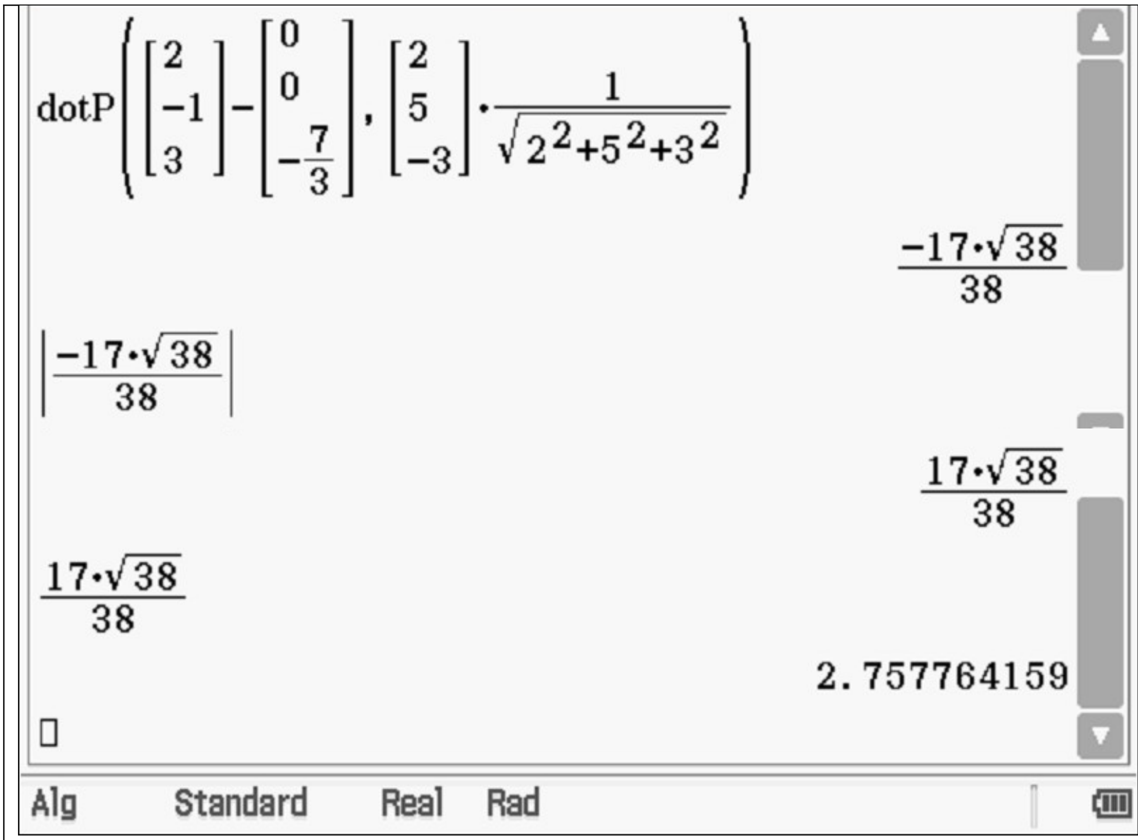
a) Determine the vector equation of this sphere. (3 marks)

Solution
$x^2+y^2+z^2-6x+10y-2z+26=0$ $x^2-6x+9-9+y^2+10y+25-25+z^2-2z+1-1+26=0$ $(x-3)^2+(y+5)^2+(z-1)^2=-26+9+25+1=9$ $\left\lvert r-\begin{pmatrix}3\\-5\\1\end{pmatrix}\right\rvert=3$
Specific behaviours
✓ completes the square for each variable ✓ determines cartesian equation with centre and radius readily seen ✓ determines vector equation

Consider a second sphere Π given by $\left\lvert r-\begin{pmatrix}2\\-1\\3\end{pmatrix}\right\rvert=4$.

b) Determine the distance of the centre of Π from the plane $2x+5y-3z=7$. (3 marks)

Solution
$pt(0,0,\frac{-7}{3})$



Specific behaviours

✓ uses a pt on plane
✓ uses dot product with unit normal vector
✓ determines approx. distance

c) Is the line $r = \begin{pmatrix} 11 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$ a tangent to the sphere Π? Explain. (3 marks)

Solution

$$\left| r - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right| = 4$$

$$\operatorname{norm}\left(\begin{bmatrix}11\\-2\\3\end{bmatrix}+\lambda\times\begin{bmatrix}1\\7\\-3\end{bmatrix}-\begin{bmatrix}2\\-1\\3\end{bmatrix}\right)$$

$$\sqrt{59\cdot\lambda^2+4\cdot\lambda+82}$$

$$\operatorname{solve}\left(\sqrt{59\cdot\lambda^2+4\cdot\lambda+82}=4,\ \lambda\right)$$

No Solution

Therefore line is not a tangent as does not intersect with sphere at all

Specific behaviours
✓ sets up equation for parameter ✓ shows that there is no solution for parameter ✓ states that not a tangent as does not intersect with sphere at all

Question 15 (7 marks)

a) By using integration and partial fractions, show that the solution to the differential equation $\frac{dP}{dt}=rP(k-P)$ is $P=\frac{kP_o}{P_o+(k-P_o)e^{-rkt}}$ where $r, k\ \&\ P_o$ are constants for time t hours. ($P_0=$ initial value of P)
(4 marks)

Solution

$$\frac{dP}{dt}=rP(k-P)$$ $$\int\frac{dP}{P(k-P)}dt=\int r\,dt$$ $$\frac{1}{P(k-P)}=\frac{a}{P}+\frac{b}{k-P}$$ $$1=a(k-P)+bP$$ $$P=0$$ $$1=ak \quad a=\frac{1}{k}$$ $$P=k$$ $$1=bk \quad b=\frac{1}{K}$$ $$\int\frac{\frac{1}{k}}{P}+\frac{\frac{1}{k}}{k-P}dt=\frac{1}{k}\ln P-\frac{1}{k}\ln(k-P)=rt+c$$ $$\ln\frac{P}{k-P}=rkt+c$$ $$Ce^{rkt}=\frac{P}{k-P}$$ $$\frac{k-P}{P}=Ce^{-rkt}$$ $$PCe^{-rkt}=k-P$$ $$P(1+Ce^{-rkt})=k$$ $$P=\frac{k}{(1+Ce^{-rkt})}$$ $$t=0\ P=P_o \quad P_o=\frac{k}{1+C} \quad 1+C=\frac{k}{P_o} \quad C=\frac{k-P_o}{P_o}$$ $$P=\frac{kP_o}{(P_o+CP_oe^{-rkt})}=\frac{kP_o}{(P_o+(k-P_o)e^{-rkt})}$$
Specific behaviours
✓ uses partial fractions ✓ solves for constants for partial fractions ✓ integrates to find a correct expression for P ✓ rearranges constants to required formula

a) Let P represent the number of bacteria cells present on a laboratory tray at time t hours. The initial number being 300 cells and the rate of change given by

$$\frac{dP}{dt}=\frac{1}{5}P-\frac{1}{13000}P^{2}.$$

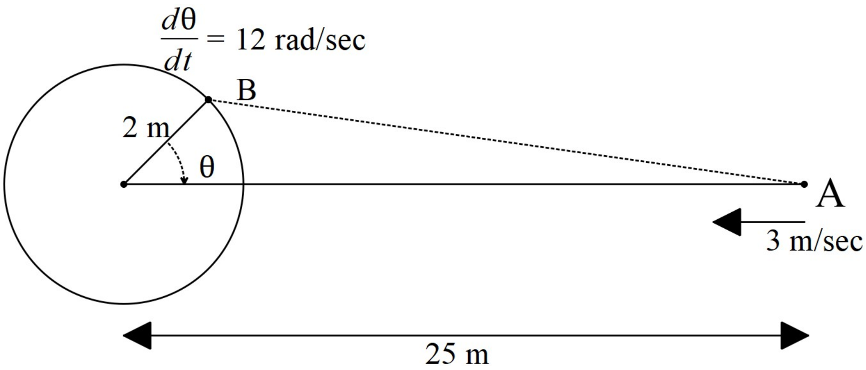
Determine the approximate number of bacteria cells after 8 hours.

(3 marks)

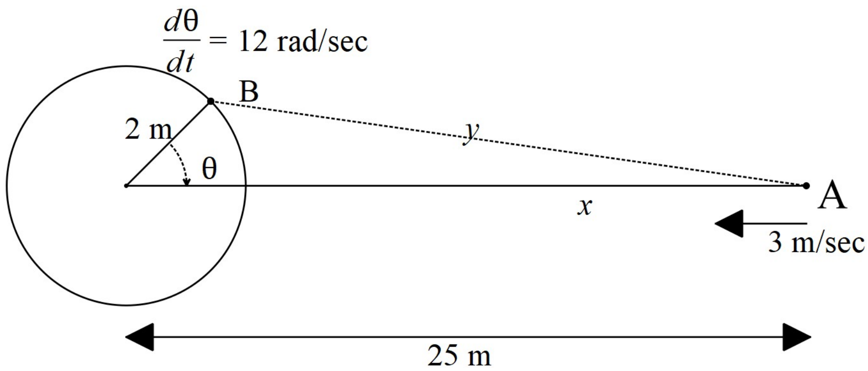
Solution
$\frac{dP}{dt}=\frac{1}{5}P-\frac{1}{13000}P^{2}=\frac{P}{13000}(2600-P)$ $r=\frac{1}{13000}$ $k=2600$ $P_{O}=300$ $P=\frac{2600(300)}{300+(2600-300)e^{-\frac{2600}{13000}t}}$ $\frac{2600\times300}{300+(2600-300)e^{-\frac{2600\times8}{13000}}}$ 1020.458904
Specific behaviours
✓ determines values of all constants ✓ uses an appropriate formula ✓ determines approx. value at 8 hours

Question 16   (5 marks)

Consider person B ridding on a merry go round at an angular speed of 12 rad/sec clockwise and person A moving towards the centre of the merry go round at a speed of 3 m/sec. Initially person A is 25 m from the centre and the angle for person B is $\frac{\pi}{3}$. Determine the initial rate of change of the distance between persons A & B.

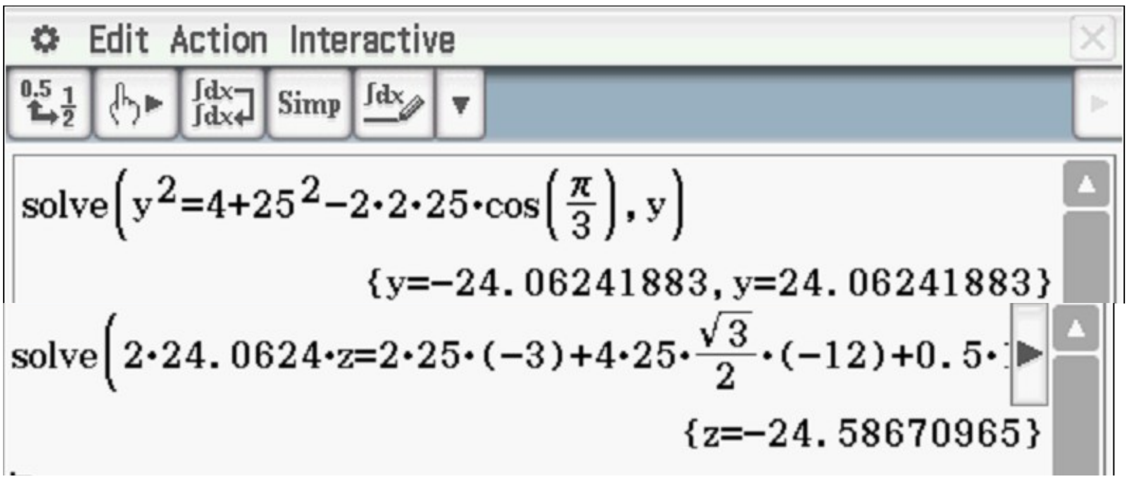


Solution



$y^2 = 4 + x^2 - 2(2)x\cos\theta$

$2y\dot{y} = 2x\dot{x} + 4x\sin\theta\dot{\theta} + \cos\theta(-4\dot{x})$



Specific behaviours
✓ uses cosine rule ✓ determines distance initially between people ✓ uses implicit did with chain and product rules ✓ sets up equation for desired rate ✓solves for initial rate

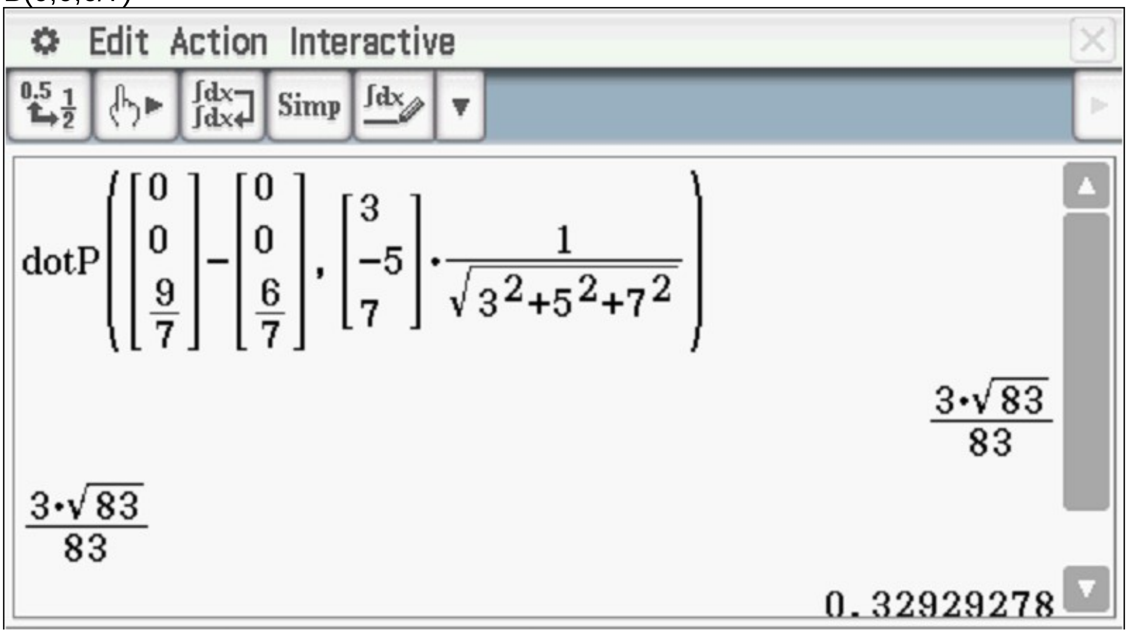
Question 17 (6 marks)

Consider the following two planes;

$\Sigma \quad 3x - 5y + 7z = 9$

$\Pi \quad r.\begin{pmatrix} -6 \\ 10 \\ -14 \end{pmatrix} = -12$

a) Determine the distance between the two planes. (4 marks)

Solution
Choose pt on each plane A(0,0,9/7) B(0,0,6/7) 

Specific behaviours
✓ determines apt on each plane ✓ uses a separation vector or vector line equation ✓ uses dot product with unit normal ✓ determines approx. distance

b) Consider the point (a,b,c), derive an expression for the distance of this point to the plane Π above. (2 marks)

Solution
$\begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \frac{6}{7} \end{pmatrix} = \begin{pmatrix} a \\ b \\ c - \frac{6}{7} \end{pmatrix}$ $distance = \left| \begin{pmatrix} a \\ b \\ c - \frac{6}{7} \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix} \frac{1}{\sqrt{3^2 + 5^2 + 7^2}} \right| = \frac{\lvert 3a - 5b + 7c - 6 \rvert}{\sqrt{83}}$
Specific behaviours
✓ uses separation vector   ✓ uses dot product with unit normal and absolute value

Question 18 **(10 marks)**

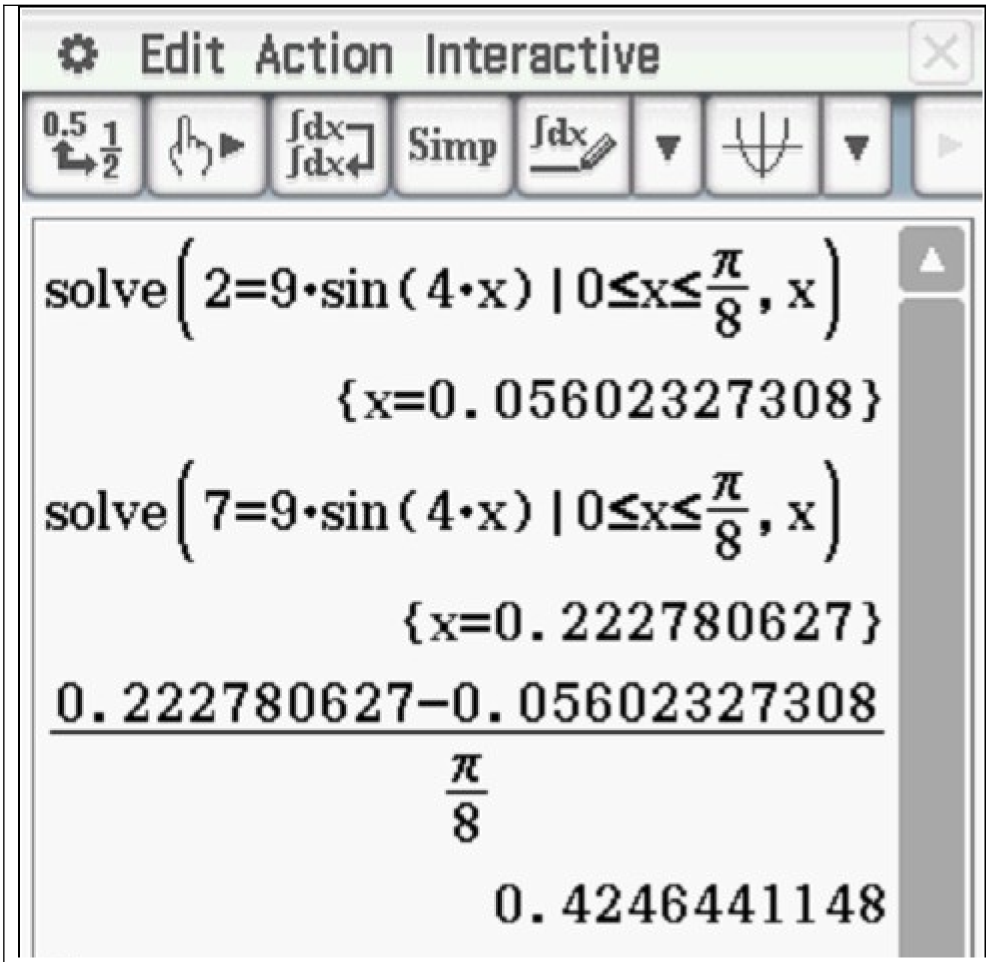
Consider an object moving with motion $\ddot{x}=-16x$ where x is the displacement from the origin in metres at time t seconds. The initial displacement is 5 metres given that the velocity is zero when $x=9$ metres.

a) Determine the displacement at any time t seconds. (2 marks)

Solution
$x=A\sin(nt+\varepsilon)$ $x=9\sin(4t+\varepsilon)$ $5=9\sin\varepsilon$ $\varepsilon\approx 0.59$ $x=9\sin(4t+0.59)$ OR $x=9\cos(4t+\varepsilon)$ $5=9\cos\varepsilon$ $\varepsilon\approx 0.98$ $x=9\cos(4t+0.98)$
Specific behaviours
✓ determines all constants ✓ uses an appropriate model for x

b) Determine the percentage of time that the distance from the origin is between 2 and 7 metres. (4 marks)

Solution



Percentage is 42.5%

Specific behaviours
✓ solves for a time dist=2 ✓ solves for a time dist=7 ✓ uses a known interval in aperiodic cycle ✓ determines approx. percentage

c) Show by **using integration** that for $\ddot{x} = -n^2 x$ the following can be derived $v^2 = n^2\left(A^2 - x^2\right)$ where A equals the amplitude. (4 marks)

Solution
$v\dfrac{dv}{dx} = -n^2 x$ $\int v\,dv = \int -n^2 x\,dx$ $\dfrac{v^2}{2} = \dfrac{-n^2x^2}{2} + c$ $x = A, v = 0$ $c = \dfrac{n^2A^2}{2}$ $v^2 = n^2\left(A^2 - x^2\right)$

Specific behaviours
✓ uses separation of variables ✓integrates both sides ✓solves for constant in terms of amplitude ✓states required formula

Additional working space

Question number: ________

Acknowledgements