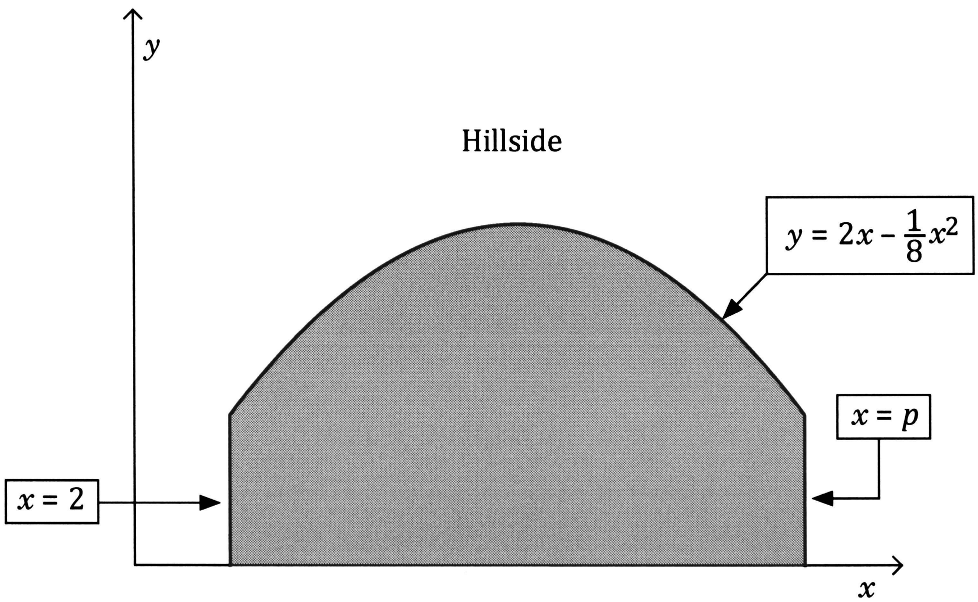
Chapter 4.

Area under a curve.

Situation. Making a tunnel.

A road tunnel is to be made in a hillside. The proposed scheme is as shown in the diagram below, with the symmetrical cross-section of the tunnel shown shaded. All units are in metres.



In the model above the tunnel is bounded by the x-axis, the lines $x = 2$, $x = p$ and the curve $y = 2x - \frac{1}{8}x^2$.

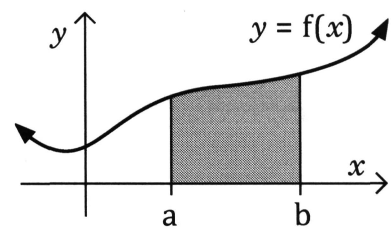
- Find the value of p.
- What will be the maximum height of the tunnel?
- Suppose that a wide rectangular load wanted to go through this tunnel. If the load was 6·4 metres wide calculate the greatest height it could be and still go through the tunnel.
- On graph paper draw an accurate version of the above diagram and use it to estimate the shaded area. Hence answer the following question: If the tunnel is to be 400 metres long what is the least volume of earth that must be removed for it to be made?

If we consider the various shapes that your mathematical studies to date allow you to find the area of, we are probably limited to:

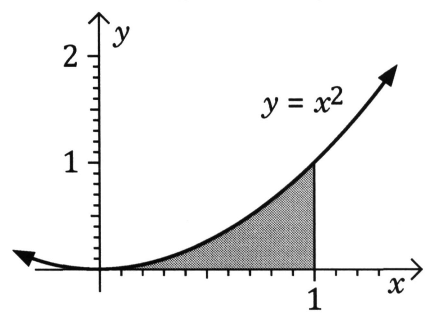
Triangles,
squares, rectangles, parallelograms, trapeziums,
circles, various parts of circles,
and any shapes that are composites of the above.

In real life many shapes that we might want to determine the area of, for example the wing of an aircraft, a canoe paddle, a ship's rudder, an insect wing, a sail, a surf board, the blade of a fan, etc, are not made up of the above shapes (as we saw in the situation on the previous page involving the cross sectional area of a tunnel). However we can obtain a reasonable approximation for such areas if we divide them up into small squares, or perhaps rectangular strips, and sum the areas of such squares or strips - the approximation becoming more accurate the more squares or strips we divide the shape into.

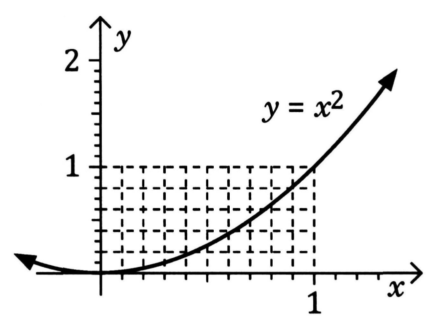
Area under a curve.



Suppose we wish to determine the area between some curve, $y = f(x)$, and the x-axis, from $x = a$ to $x = b$, as shown in the diagram on the right.
(We refer to this as the area *under* the curve from $x = a$ to $x = b$.)



Let us start by considering the area between the curve $y = x^2$ and the x-axis, from $x = 0$ to $x = 1$, as shown in the diagram.

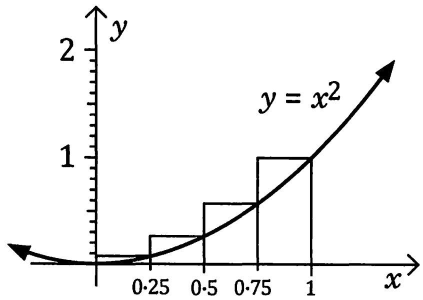


By drawing the graph accurately and by simply counting squares an approximate answer can be obtained.
In the diagram on the right one square unit consists of 50 of the little squares. Approximately 17 of these little squares lie in the required region so the area of this region is approximately:

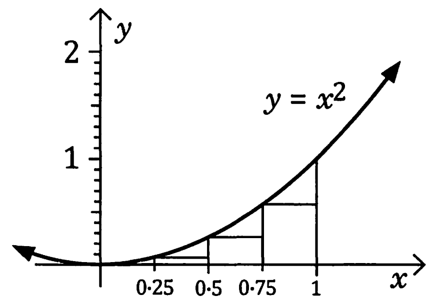
$$\frac{17}{50} = 0{\cdot}34 \text{ units}^2.$$

Alternatively we could estimate the area by dividing the region up into a number of strips, say 4, approximate each of these strips to a rectangle and sum the areas of these rectangles. We could do this in two ways, one which would overestimate the area (when the rectangles *circumscribe* the area) and one which would underestimate it (when the rectangles *inscribe* the area).

Overestimate



Underestimate



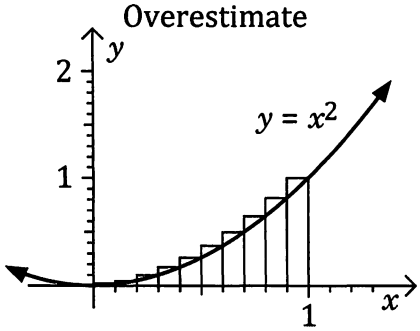
Area of 1st rect	$= 0{\cdot}25\,(0{\cdot}25)^2$	$\approx 0{\cdot}02$
Area of 2nd rect	$= 0{\cdot}25\,(0{\cdot}5)^2$	$\approx 0{\cdot}06$
Area of 3rd rect	$= 0{\cdot}25\,(0{\cdot}75)^2$	$\approx 0{\cdot}14$
Area of 4th rect	$= 0{\cdot}25\,(1)^2$	$= \underline{0{\cdot}25}$
Total area (overestimated)		$\approx 0{\cdot}47$

Area of 1st rect	$= 0{\cdot}25\,(0)^2$	$= 0$
Area of 2nd rect	$= 0{\cdot}25\,(0{\cdot}25)^2$	$\approx 0{\cdot}02$
Area of 3rd rect	$= 0{\cdot}25\,(0{\cdot}5)^2$	$\approx 0{\cdot}06$
Area of 4th rect	$= 0{\cdot}25\,(0{\cdot}75)^2$	$\approx \underline{0{\cdot}14}$
Total area (underestimated)		$\approx 0{\cdot}22$

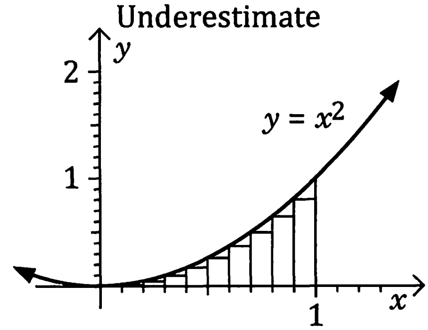
A reasonable approximation would be the mean of the two estimates.
The area is approximately $0{\cdot}34$ units^2.

A better approximation would occur if we divided the region into more strips. Division into ten equal width strips is shown below.

Overestimate



Underestimate



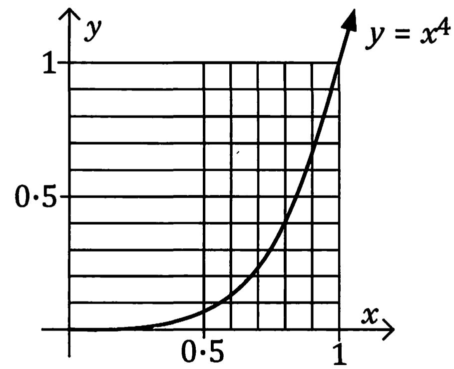
Area of 1st rect	$= 0{\cdot}1\,(0{\cdot}1)^2$	$= 0{\cdot}001$
Area of 2nd rect	$= 0{\cdot}1\,(0{\cdot}2)^2$	$= 0{\cdot}004$
Area of 3rd rect	$= 0{\cdot}1\,(0{\cdot}3)^2$	$= 0{\cdot}009$
Area of 4th rect	$= 0{\cdot}1\,(0{\cdot}4)^2$	$= 0{\cdot}016$
Area of 5th rect	$= 0{\cdot}1\,(0{\cdot}5)^2$	$= 0{\cdot}025$
Area of 6th rect	$= 0{\cdot}1\,(0{\cdot}6)^2$	$= 0{\cdot}036$
Area of 7th rect	$= 0{\cdot}1\,(0{\cdot}7)^2$	$= 0{\cdot}049$
Area of 8th rect	$= 0{\cdot}1\,(0{\cdot}8)^2$	$= 0{\cdot}064$
Area of 9th rect	$= 0{\cdot}1\,(0{\cdot}9)^2$	$= 0{\cdot}081$
Area of 10th rect	$= 0{\cdot}1\,(1)^2$	$= \underline{0{\cdot}1}$
Total area (overestimated)		$= 0{\cdot}385$

Area of 1st rect	$= 0{\cdot}1\,(0)^2$	$= 0$
Area of 2nd rect	$= 0{\cdot}1\,(0{\cdot}1)^2$	$= 0{\cdot}001$
Area of 3rd rect	$= 0{\cdot}1\,(0{\cdot}2)^2$	$= 0{\cdot}004$
Area of 4th rect	$= 0{\cdot}1\,(0{\cdot}3)^2$	$= 0{\cdot}009$
Area of 5th rect	$= 0{\cdot}1\,(0{\cdot}4)^2$	$= 0{\cdot}016$
Area of 6th rect	$= 0{\cdot}1\,(0{\cdot}5)^2$	$= 0{\cdot}025$
Area of 7th rect	$= 0{\cdot}1\,(0{\cdot}6)^2$	$= 0{\cdot}036$
Area of 8th rect	$= 0{\cdot}1\,(0{\cdot}7)^2$	$= 0{\cdot}049$
Area of 9th rect	$= 0{\cdot}1\,(0{\cdot}8)^2$	$= 0{\cdot}064$
Area of 10th rect	$= 0{\cdot}1\,(0{\cdot}9)^2$	$= \underline{0{\cdot}081}$
Total area (underestimated)		$= 0{\cdot}285$

Approximate area $= \dfrac{0{\cdot}385 + 0{\cdot}285}{2} = 0{\cdot}335$ units^2 i.e approx $\frac{1}{3}$ units^2.

Exercise 4A.

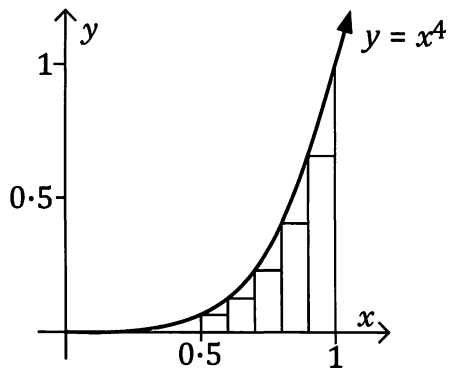
1. (a) By counting squares on the diagram on the right estimate the area under $y = x^4$ from $x = 0{\cdot}5$ to $x = 1$.

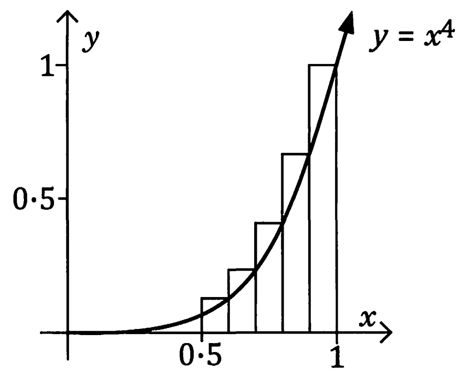


(b) Using rectangles, as shown below, find an underestimate and an overestimate for the area under

$y = x^4$ from $x = 0{\cdot}5$ to $x = 1$

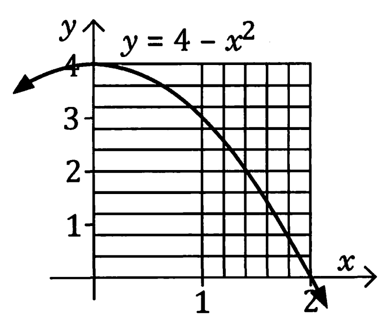
and then determine the mean of these two figures.





2. (a) By counting squares on the diagram on the right estimate the area under

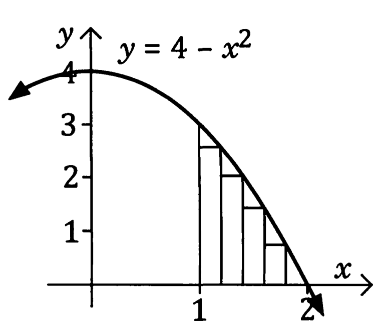
$y = 4 - x^2$ from $x = 1$ to $x = 2$.

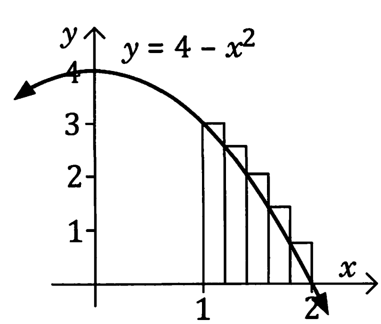


(b) Using rectangles, as shown below, find an underestimate and an overestimate for the area under

$y = 4 - x^2$ from $x = 1$ to $x = 2$

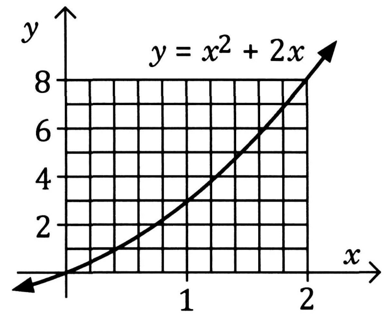
and then determine the mean of these two figures.





3. (a) By counting squares on the diagram on the right estimate the area under

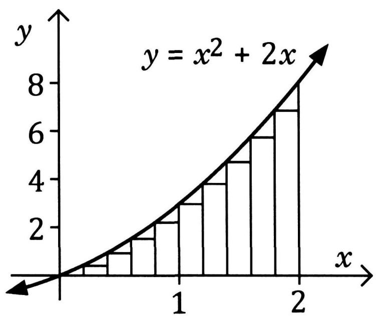
$$y = x^2 + 2x \text{ from } x = 0 \text{ to } x = 2.$$

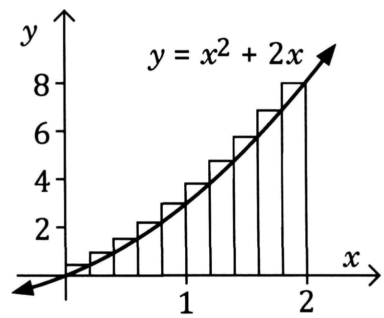


(b) Using rectangles, as shown below, find an underestimate and an overestimate for the area under

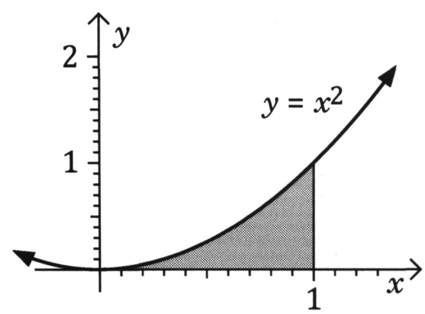
$$y = x^2 + 2x \text{ from } x = 0 \text{ to } x = 2$$

and then determine the mean of these two figures.





Earlier in this chapter, using this process of summing rectangles, we obtained an estimate for the area under $y = x^2$, from $x = 0$ to $x = 1$, of $\frac{1}{3}$ units^2.



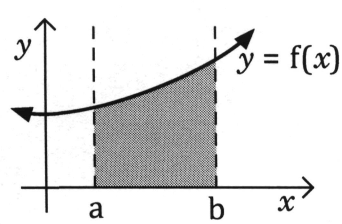
If we were to repeat this process to find estimates for the area under $y = x^2$, from $x = 0$ to $x = a$, with "a" taking the values 1, 2, 3, 4, 5 and 6 this process would give us the following table:

Area between the x-axis and $y = x^2$ from $x = 0$ to $x = a$.							
Value of a	0	1	2	3	4	5	6
Area	0	$\frac{1}{3}$	$\frac{8}{3}$	$\frac{27}{3}$	$\frac{64}{3}$	$\frac{125}{3}$	$\frac{216}{3}$

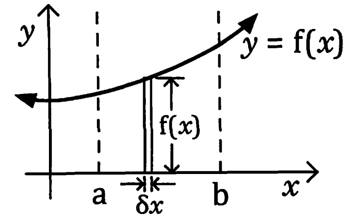
However, whilst we will reconsider these values soon, let us now move away from the particular function $y = x^2$, and consider the more general idea of the area "under"

$$y = f(x) \text{ from } x = a \text{ to } x = b,$$

shown shaded in the diagram on the right.



To determine the area "under" $y = f(x)$ from $x = a$ to $x = b$ we divide the area into elementary strips of width δx, as shown in the diagram on the right.
Each strip will approximate to a rectangle of height $f(x)$ and width δx.



Summing the areas of these rectangles will give an approximate value for the area under $y = f(x)$ from $x = a$ to $x = b$, with the approximation getting closer and closer to the exact answer as we increase the number of rectangles involved, i.e. as we allow δx to tend to zero. The value this sum "seems to be heading towards" is the *limiting* value and will equal the required area.
In mathematics we use the Greek capital letter, sigma, $\sum$, to represent a summation. Values placed below and above the summation sign indicate what the summation goes "from" and "to".
Thus the area between $y = f(x)$ and the x-axis, from $x = a$ to $x = b$, is given by:

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x.$$

We define such a **limit of a sum** as **integration** and with a **s**ummation involved we use the "stretched S" symbol that you have already encountered

$$\int$$

and write $\displaystyle\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x$ as $\displaystyle\int_a^b f(x)\,dx$.

However you do not need to be alarmed that we already use the term *integration*, and the above symbol, to mean *antidifferentiate* because it turns out that by antidifferentiating $f(x)$ we obtain the same answer as the limiting sum process would give. (As indeed you may have noticed in the table on the previous page for the area under $y = x^2$, or if you did not, glance back at that table now to confirm this.)
Hence to find the area under a curve, the limit of a sum we obtain by considering rectangles can be evaluated using antidifferentiation, an easier process than summing the areas of many rectangles (and a fact we will consider more formally in chapter 5).
Thus whilst integration is indeed "a limit of a sum", it can easily be evaluated by antidifferentiating and so we use the term integration and the "stretched S symbol" for antidifferentiation, as we have already become accustomed to.

To evaluate $\displaystyle\int_a^b f(x)\,dx$:

① Antidifferentiate $f(x)$ with respect to x (and omit the "$+c$").

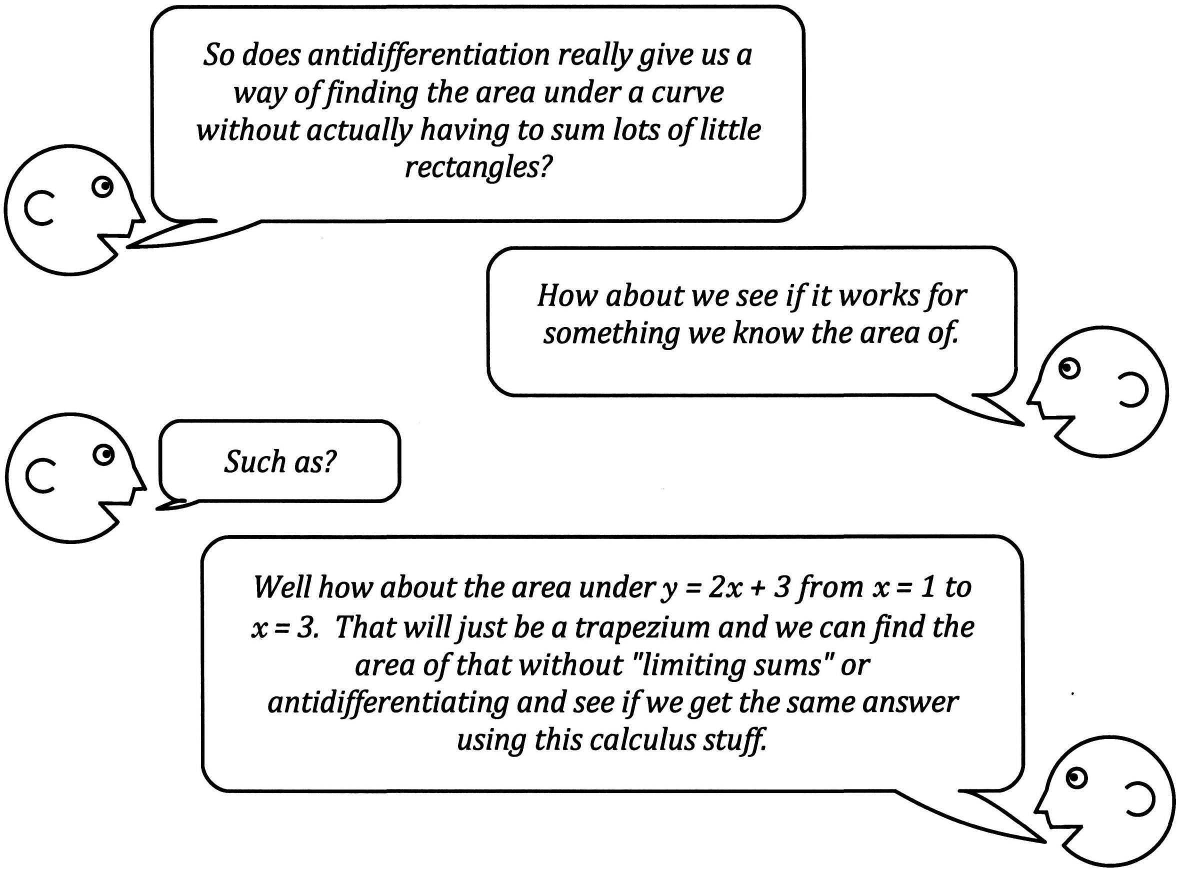
② Substitute b into your answer from ①.

③ Substitute a into your answer from ①.

④ Calculate: (Part ② answer) − (Part ③ answer).

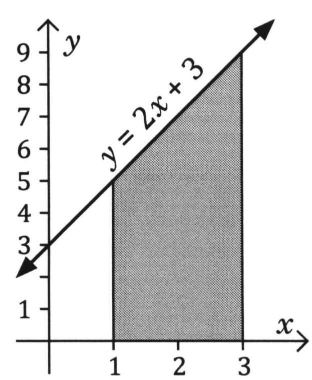
I.e.

$$\boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}$$



In the diagram on the right the shaded area is the area under $y = 2x + 3$ from $x = 1$ to $x = 3$.

The required area is a trapezium.



Area of this trapezium $= 2\left(\frac{9+5}{2}\right)$

$= 14$ units^2

Using calculus: Area $= \int_1^3 (2x+3)\,dx$

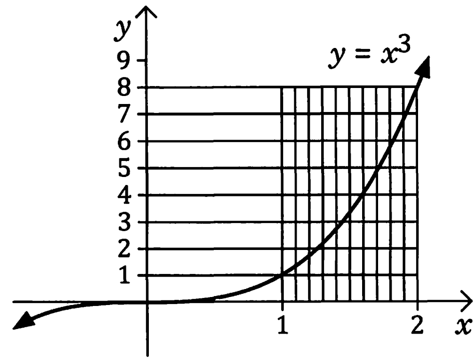
$$= \left[x^2 + 3x\right]_1^3$$

$$= (3^2 + 3(3)) - (1^2 + 3(1))$$

$$= (9 + 9) - (1 + 3)$$

$$= 14 \text{ units}^2$$

As a further check consider the area under $y = x^3$ from $x = 1$ to $x = 2$.



Using calculus: Area $= \int_1^2 x^3 \, dx$

$= \left[\frac{x^4}{4}\right]_1^2$

$= \frac{2^4}{4} - \frac{1^4}{4}$

$= 3{\cdot}75 \text{ units}^2$

The reader should confirm that this answer is consistent with the approximate answer that can be obtained by counting the little rectangles on the given diagram.

Note • The limit as δx tends to zero, of summations of the form

$$\sum_{x=a}^{x=b} f(x)\,\delta x$$

have particular importance in many mathematical applications including calculating areas, calculating volumes, locating centres of gravity, and in such fields as economics, science, engineering, psychology and others. Thus it is particularly useful that antidifferentiation provides us with a way of evaluating expressions of the form:

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x \,.$$

Definite integrals.

Example 1

Find (a) $\int 6x^2 \, dx$, (b) $\int \frac{1}{\sqrt{x}} \, dx$,

(c) $\int_1^2 (3x^2 + 4) \, dx$, (d) $\int_0^1 6(2x+1)^3 \, dx$.

(a) $\int 6x^2 \, dx = \frac{6x^3}{3} + c$

$= 2x^3 + c$

(b) $$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-0\cdot5}\,dx = \frac{x^{0\cdot5}}{0\cdot5} + c = 2\sqrt{x} + c$$

(c) $$\int_1^2 (3x^2 + 4)\,dx = \left[x^3 + 4x\right]_1^2 = (2^3 + 4 \times 2) - (1^3 + 4 \times 1) = (16) - (5) = 11$$

(d) $$\int_0^1 6(2x+1)^3\,dx = 3 \times \int_0^1 2(2x+1)^3\,dx = \left[\frac{3\,(2x+1)^4}{4}\right]_0^1 = \frac{3\,(2+1)^4}{4} - \frac{3\,(0+1)^4}{4} = \frac{243}{4} - \frac{3}{4} = 60$$

Notice that in the previous example the answers to parts (a) and (b) included the necessary constant. This constant is not necessary in part (c) because, were we to include it, it would eventually cancel itself out as shown below.

Writing the solution to (c) as follows $$\int_1^2 (3x^2 + 4)\,dx = \left[x^3 + 4x + c\right]_1^2 = (16 + c) - (5 + c) = 11 \text{ as before.}$$

Integrals of the form $\int f(x)\,dx$ will involve the constant of integration and are called **indefinite integrals**.

Integrals of the form $\int_a^b f(x)\,dx$ will not involve the constant of integration and are called **definite integrals**.

Make sure you can also use your calculator to evaluate definite integrals.

$$\int_1^2 (3x^2 + 4)\mathrm{d}x \qquad 11$$

$$\int_0^1 6(2x+1)^3\,\mathrm{d}x \qquad 60$$

Note the *linearity property* of definite integrals:

$$\int_a^b (\mathrm{f}(x) + \mathrm{g}(x))\,\mathrm{d}x = \int_a^b \mathrm{f}(x)\,\mathrm{d}x + \int_a^b \mathrm{g}(x)\,\mathrm{d}x$$

and

$$\int_a^b (\mathrm{k} \times \mathrm{f}(x))\,\mathrm{d}x = \mathrm{k} \times \int_a^b \mathrm{f}(x)\,\mathrm{d}x.$$

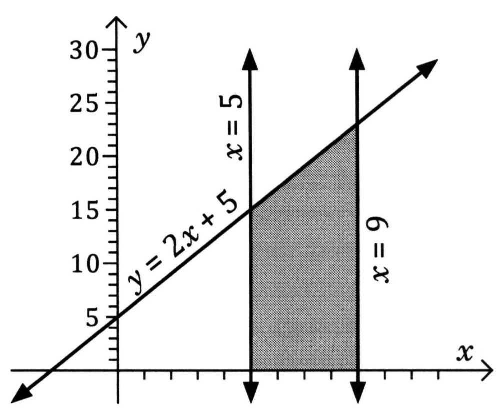
Also note that

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = -\int_b^a \mathrm{f}(x)\,\mathrm{d}x, \qquad \int_a^a \mathrm{f}(x)\,\mathrm{d}x = 0, \qquad \int_a^b \mathrm{f}(x)\,\mathrm{d}x + \int_b^c \mathrm{f}(x)\,\mathrm{d}x = \int_a^c \mathrm{f}(x)\,\mathrm{d}x.$$

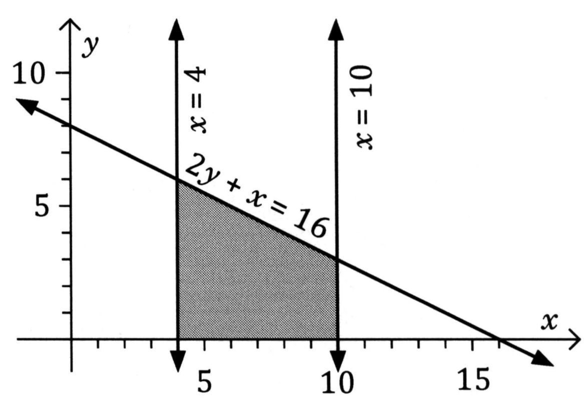
We still need to consider further the use of definite integrals to determine areas under graphs but first work through the next exercise to gain practice in evaluating definite integrals.

Exercise 4B

1. In the diagram on the right the area under $y = 2x + 5$ from $x = 5$ to $x = 9$ is shaded.
   (a) Use a method that does not use calculus to determine the shaded area.
   (b) Determine the shaded area by evaluating $\int_5^9 (2x + 5)\,\mathrm{d}x$.

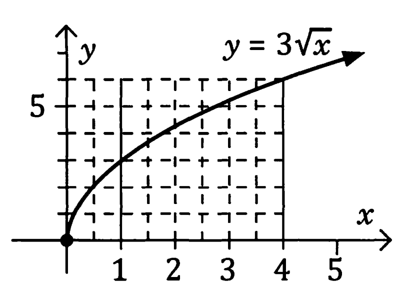


2. In the diagram on the right the area bounded by the x-axis and the lines
   $x = 4$, $x = 10$ and $2y + x = 16$
   is shown shaded.
   (a) Use a method that does not use calculus to determine the shaded area.
   (b) Determine the shaded area by evaluating a suitable definite integral.



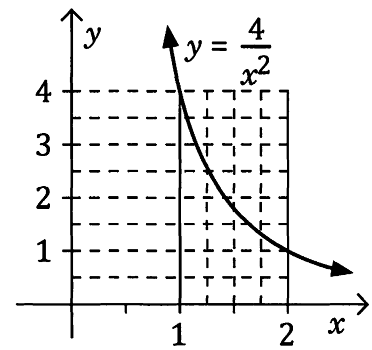
3. (a) Evaluate $\int_1^4 3\sqrt{x}\,\mathrm{d}x$.

   (b) Use the graph shown on the right to estimate the area under $y = 3\sqrt{x}$ from $x = 1$ to $x = 4$ by counting squares and check that your answer is consistent with the answer you obtained for part (a).



4. (a) Evaluate $\int_1^2 \frac{4}{x^2}\,\mathrm{d}x$.

   (b) Use the graph shown on the right to estimate the area under $y = \frac{4}{x^2}$ from $x = 1$ to $x = 2$ by counting squares and check that your answer is consistent with the answer you obtained for part (a).



Evaluate the following definite integrals "by hand" and then use your calculator to check your answers.

5. $\int_0^2 \frac{x^2}{4}\,\mathrm{d}x$

6. $\int_2^4 \frac{x^2}{4}\,\mathrm{d}x$

7. $\int_1^3 10x\,\mathrm{d}x$

8. $\int_{-1}^1 (4x + 5)\,\mathrm{d}x$

9. $\int_2^2 (4 - x^2)\,\mathrm{d}x$

10. $\int_2^3 3x^2\,\mathrm{d}x$

11. $\int_{-1}^2 (6x^2 + 7)\,\mathrm{d}x$

12. $\int_0^3 (1 + x^2)\,\mathrm{d}x$

13. $\int_3^6 x(1 + x)\,\mathrm{d}x$

14. $\int_2^3 (9 - x^2)\,\mathrm{d}x$

15. $\int_0^1 (2 + x)^4\,\mathrm{d}x$

16. $\int_0^1 (2 + 5x)^4\,\mathrm{d}x$

17. $\int_0^1 12x(1 + x^2)^2\,\mathrm{d}x$

18. $\int_3^3 (4 + x^2)^2\,\mathrm{d}x$

19. $\int_{-1}^1 (1 + x^2)^2\,\mathrm{d}x$

20. Evaluate (a) $\int_0^1 x^2\,\mathrm{d}x$, (b) $\int_1^3 x^2\,\mathrm{d}x$, (c) $\int_0^3 x^2\,\mathrm{d}x$.

21. Evaluate (a) $\int_0^4 (4x - x^2)\,\mathrm{d}x$, (b) $\int_4^5 (4x - x^2)\,\mathrm{d}x$, (c) $\int_0^5 (4x - x^2)\,\mathrm{d}x$.

22. Evaluate (a) $\int_1^3 (3x^2 + 2x)\,\mathrm{d}x$, (b) $\int_3^1 (3x^2 + 2x)\,\mathrm{d}x$.

23. Evaluate (a) $\int_0^3 x^2\,\mathrm{d}x$, (b) $\int_0^3 3x^2\,\mathrm{d}x$, (c) $\int_0^3 4x^2\,\mathrm{d}x$.

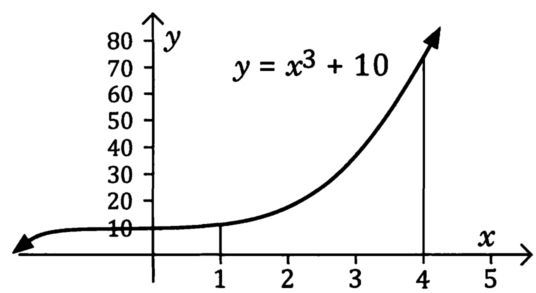
Determine each of the following giving your answers as exact values.

24. $\int_{-\pi}^{\pi} (2x + 3)\,\mathrm{d}x$

25. $\int_{\sqrt{2}}^{2} (2x + 6x^2)\,\mathrm{d}x$

Area under a curve - further examples.

Suppose we are asked to determine the area between $y = x^3 + 10$ and the x-axis from $x = 1$ to $x = 4$, see diagram.



There are a number of ways that we could proceed:

- We could (but we wouldn't want to!) approximate the area to a number of equal width rectangles, obtain an estimate for the area, increase the number of rectangles, improve our estimate, etc., etc.

- We could evaluate $\int_1^4 (x^3 + 10)\,\mathrm{d}x$, either algebraically or by calculator:

$$\int_1^4 (x^3 + 10)\,\mathrm{d}x = \left[\frac{x^4}{4} + 10x\right]_1^4$$

$$= (64 + 40) - \left(\tfrac{1}{4} + 10\right)$$

$$= 93\tfrac{3}{4}.$$

The required area is $93\frac{3}{4}$ units^2.

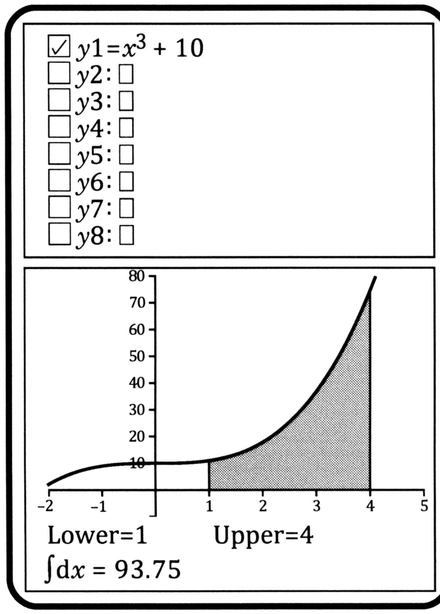
$\int_1^4 (x^3 + 10)\mathrm{d}x$

93.75

- We could use the ability of some calculators to display the graph of

$$y = x^3 + 10,$$

to show the required region shaded and to state its area.



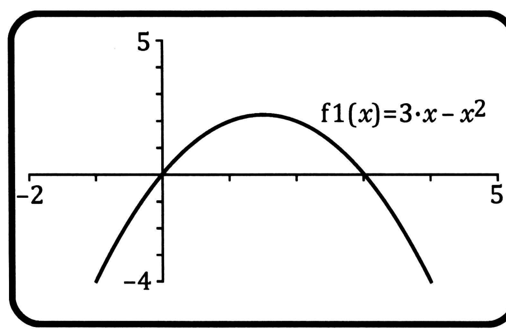
Regions that are wholly or partly below the x-axis.
Particular care needs to be taken when the area we are determining lies wholly or partly below the x-axis, as the next example shows.

Example 2

Find the area between $y = 3x - x^2$ and the x-axis from
(a) $x = 0$ to $x = 3$,
(b) $x = 3$ to $x = 4$,
(c) $x = 0$ to $x = 4$.

First view the graph of $y = 3x - x^2$ on a graphic calculator:
Or, recognising the function is quadratic, produce a sketch on paper.

(Note that the curve cuts the x-axis at (0, 0) and (3, 0).)



(a) Algebraically or by calculator:

$$\int_0^3 (3x - x^2)\,dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \left(\frac{27}{2} - \frac{27}{3}\right) - (0 - 0)$$

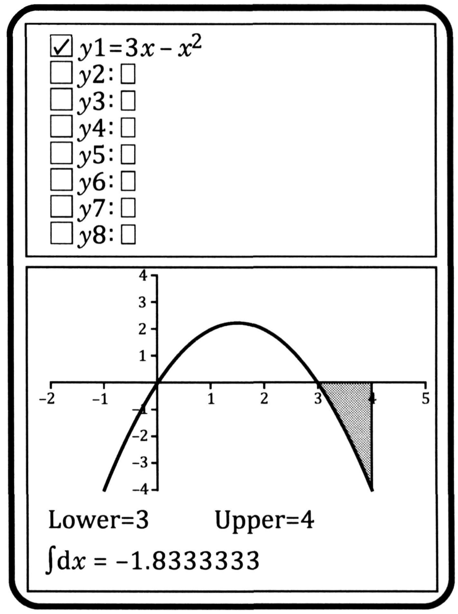
$$= 4{\cdot}5$$

The required area is 4·5 units^2.

$$\int_0^3 (3x - x^2)dx \qquad 4.5$$

(b) Similarly $\int_3^4 (3x - x^2)\,dx = -1\frac{5}{6}$

$$\int_3^4 (3x - x^2)dx \qquad -\frac{11}{6}$$



The answer is negative because the curve lies <u>below</u> the x-axis for $3 < x < 4$.

However, area cannot be negative so the required area is $1\frac{5}{6}$ units^2.

Note that while $\int_3^4 (3x - x^2)\,dx = -1\frac{5}{6}$ the required area is $1\frac{5}{6}$ units^2.

Hence: $\int_a^b f(x)\,dx$ gives the **signed** area between $f(x)$ and the x-axis from $x = a$ to $x = b$.

(c) From (a) area from $x = 0$ to $x = 3$ is $4\frac{1}{2}$ units^2.

From (b) area from $x = 3$ to $x = 4$ is $1\frac{5}{6}$ units^2.

Thus the area from $x = 0$ to $x = 4$ is $6\frac{1}{3}$ units^2.

In part (c) above had we simply evaluated

$$\int_0^4 (3x - x^2)\,dx$$

we would have obtained an answer of

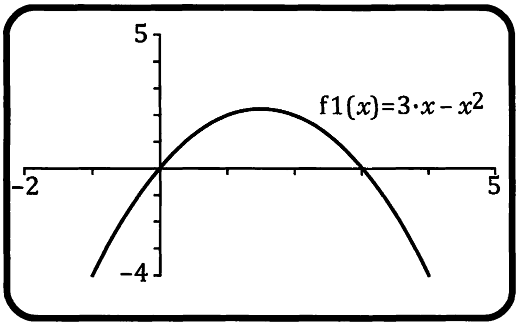
$$4\tfrac{1}{2} + \left(-1\tfrac{5}{6}\right),\text{ i.e. } 2\tfrac{2}{3}.$$

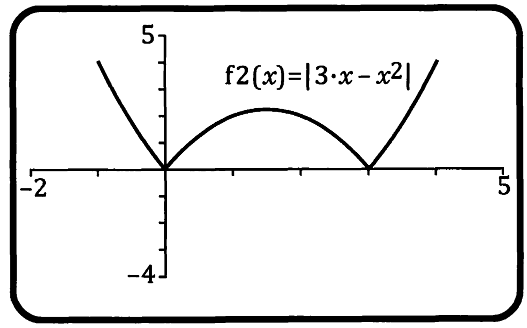
$$\int_0^4 (3x - x^2)dx \qquad \frac{8}{3}$$

This is the correct evaluation of $\int_0^4 (3x - x^2)\,dx$ but it is <u>not</u> the required area.

This shows the importance of using a graphic calculator to view the situation for these "area under a curve" questions.

Alternatively we could use the fact that the graph of $y = |3x - x^2|$ (i.e. the absolute value of $3x - x^2$) will be that of $y = 3x - x^2$ with all those parts that lie below the x-axis reflected up above the x-axis:





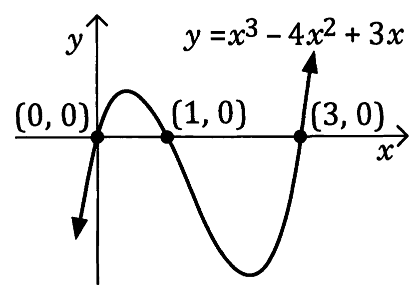
Thus part (c) of the last example could be determined using the ability of some calculators to evaluate

$$\int_0^4 |3x - x^2|\, dx$$

$$\int_0^4 \text{abs}(3x - x^2)dx \qquad \frac{19}{3}$$

Example 3

Find the area enclosed between $y = x^3 - 4x^2 + 3x$ and the x-axis.



First use your graphic calculator to view the situation. (Remember you already have a good idea what a cubic function can look like.)

Note that the function cuts the x-axis at (0, 0), (1, 0) and (3, 0).

Either algebraically or with the assistance of a calculator:

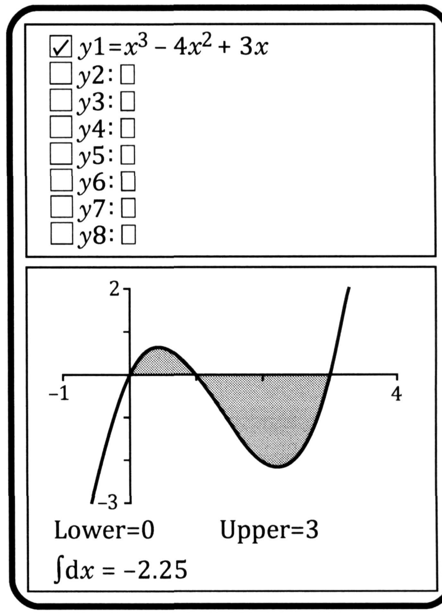
$$\int_0^1 (x^3 - 4x^2 + 3x)\, dx = \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_0^1 = \frac{5}{12}$$

$$\int_1^3 (x^3 - 4x^2 + 3x)\, dx = \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_1^3 = -\frac{8}{3}$$

The required area is $\frac{5}{12}$ units^2 + $\frac{8}{3}$ units^2 = $\frac{37}{12}$ units^2.

$$\int_0^3 |x^3 - 4x^2 + 3x|\,dx$$

$$\frac{37}{12}$$

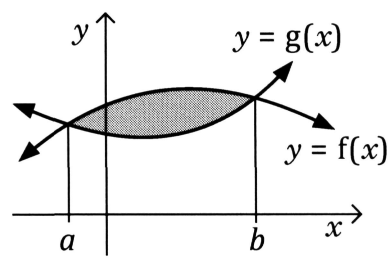


☞ Get to know the capabilities of your calculator,

☞ Practice both algebraic and calculator approaches for determining definite integrals.

Area between curves.

The diagram on the right shows two curves, $y = f(x)$ and $y = g(x)$, intersecting at $x = a$ and $x = b$.



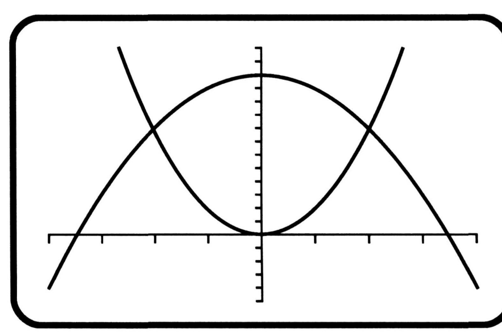
$$\text{Shaded area} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b (f(x) - g(x))\,dx$$

Example 4

Find the area enclosed between $y = 2x^2$ and $y = 12 - x^2$.

First view the situation on your calculator. Either algebraically or from the calculator we determine that the graphs intersect when $x = 2$ and again when $x = -2$.



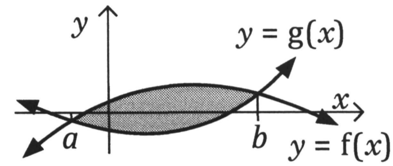
$$\text{Required area} = \int_{-2}^{2} (12 - x^2)\,dx - \int_{-2}^{2} 2x^2\,dx$$

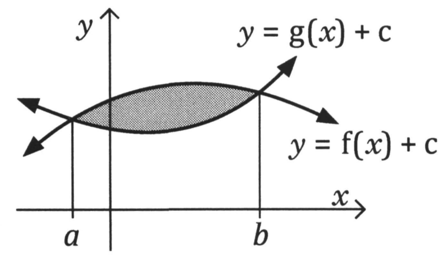
$$= \int_{-2}^{2} (12 - x^2 - 2x^2)\,dx$$

$$= \int_{-2}^{2} (12 - 3x^2)\,dx$$

$$= 32$$

The area enclosed between $y = 2x^2$ and $y = 12 - x^2$ is 32 units^2.





Suppose part of the required area between two curves lies below the x-axis? (See diagram.)

Question: Do we have to do anything different in such cases?

Answer: No. The area will still be given by

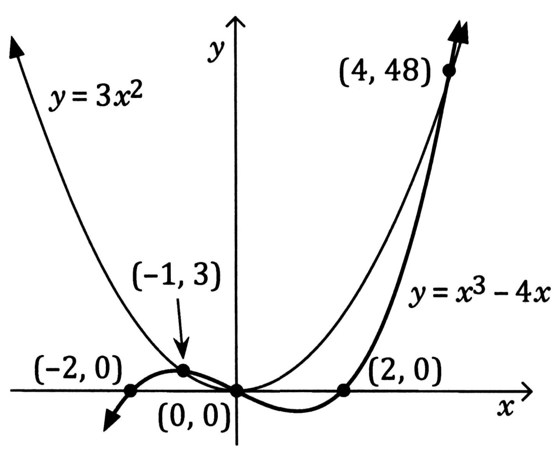
$$\int_a^b (f(x) - g(x))\, dx,$$

(provided $y = f(x)$ is the "top" function for $a < x < b$). This can be seen if a constant, c, is added to each function (as in the second diagram.) The area between the two curves is clearly the same as before and, if we use calculus:

$$\begin{aligned}\text{Shaded area} &= \int_a^b (f(x) + c)\, dx - \int_a^b (g(x) + c)\, dx \\ &= \int_a^b (f(x) + c - g(x) - c)\, dx \\ &= \int_a^b (f(x) - g(x))\, dx \quad \text{as before.}\end{aligned}$$

Example 5

Make a sketch showing the graphs of $y = x^3 - 4x$ and $y = 3x^2$ indicating clearly on your sketch the coordinates of any points where the curves intersect the axes and each other. Find the area enclosed between $y = x^3 - 4x$ and $y = 3x^2$.



Either algebraically, or from a calculator, the necessary information can be obtained for the sketch to be made.
Noting carefully which function is "above" the other in each of the two regions the required total area will be

$$\left(\int_{-1}^{0} (x^3 - 4x)\, dx - \int_{-1}^{0} 3x^2\, dx\right)$$

$$+$$

$$\left(\int_0^4 3x^2\, dx - \int_0^4 (x^3 - 4x)\, dx\right)$$

$$= \int_{-1}^{0} (x^3 - 4x - 3x^2)\, dx + \int_0^4 (3x^2 - x^3 + 4x)\, dx$$

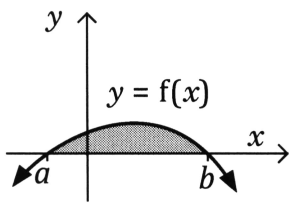
Evaluating this expression gives 32·75. The required area is 32·75 units^2.

Alternatively, in the previous example, defining $f(x)$ as $x^3 - 4x$ and $g(x)$ as $3x^2$ we could use our calculator to evaluate $\int_{-1}^{4} |\, f(x) - g(x) \,|\, dx$ and thus avoid having to decide which function is above and which below.

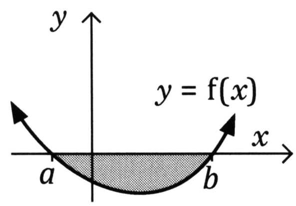
Exercise 4C

1. For which of the graphs drawn below is the shaded area equal to $\int_a^b f(x)\, dx$?

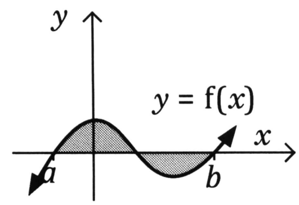
(a)



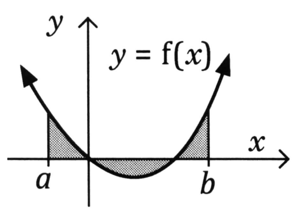
(b)



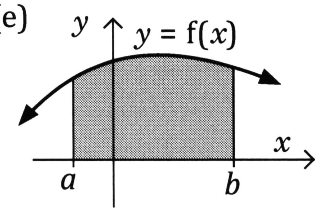
(c)



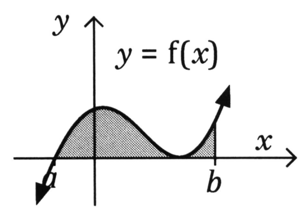
(d)



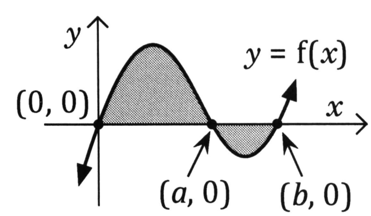
(e)



(f)



2. For each of the following state whether evaluating the given expression would give the total area shown shaded on the right.



(a) $\int_0^a f(x)\, dx + \int_a^b f(x)\, dx$

(b) $\int_0^b f(x)\, dx$

(c) $\int_0^a f(x)\, dx - \int_a^b f(x)\, dx$

(d) $\left| \int_0^b f(x)\, dx \right|$

(e) $\int_0^b |f(x)|\, dx$

(f) $\int_0^a |f(x)|\, dx + \int_a^b |f(x)|\, dx$

(g) $\left| \int_0^a f(x)\, dx \right| + \left| \int_a^b f(x)\, dx \right|$

3. Use calculus to determine the area between $y = 2x + 1$ and the x-axis from $x = 0$ to $x = 2$. Check your answer using area formulae.

4. Find the area between $y = \frac{x^4}{4}$ and the x-axis from $x = 2$ to $x = 4$.

5. Find the area between $y = (x - 2)^2 + 3$ and the x-axis from $x = 1$ to $x = 3$.

6. Find the area bounded by $y = 8 - 2x^2$ and the x-axis.

7. Find the area between $y = 1 - x^3$ and the x-axis from $x = 0$ to $x = 1$.

8. Find the area between $y = (x + 1)^3 + 1$ and the x-axis from $x = -2$ to $x = 0$.

9. Find the area between $y = x^2 - 1$ and the x-axis from $x = 0$ to $x = 2$.

10. Find the area between $y = 1 - x^3$ and the x-axis from $x = 0$ to $x = 2$.

11. Find the area between $y = (x + 1)^3$ and the x-axis from $x = -2$ to $x = 0$.

12. Find the area between $y = 12x(1 - x^2)^3$ and the x-axis from $x = -1$ to $x = 1$.

13. Find the exact area enclosed between $y = 2 - x^2$ and the x-axis.

14. Make a sketch showing the graphs of $y = x^2$ and $y = 3 - 2x$ indicating clearly on your sketch the coordinates of any points where the functions intersect the axes and each other.

    Find the area enclosed between $y = x^2$ and $y = 3 - 2x$.

15. Make a sketch showing the graphs of $y = (x - 3)^2$ and $y = x - 1$ indicating clearly on your sketch the coordinates any stationary points and of any points where the functions intersect the axes and each other.

    Find the area enclosed between $y = (x - 3)^2$ and $y = x - 1$.

16. Make a sketch showing the graphs of $y = x^2 - 2x + 3$ and $y = 2x^2$ indicating clearly on your sketch the coordinates of any stationary points and of any points where the functions intersect the axes and each other.

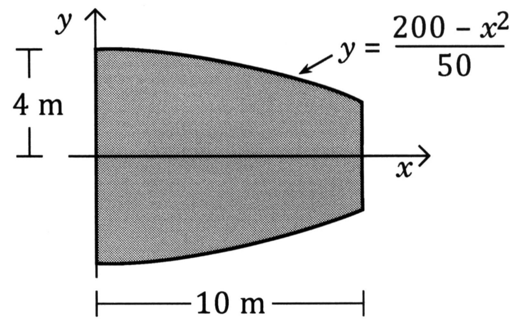
    Find the area enclosed between $y = x^2 - 2x + 3$ and $y = 2x^2$.

17. Make a sketch showing the graphs of $y = x$ and $y = x^3$ indicating clearly on your sketch the coordinates of any stationary points and of any points where the functions intersect the axes and each other.

    Find the area enclosed between $y = x$ and $y = x^3$.

18. Make a sketch showing the graphs of $y = 2x^3 - 3x$ and $y = 7x$ indicating clearly on your sketch the <u>exact</u> coordinates of any points where the functions intersect the axes and each other.
Showing full algebraic reasoning determine the area enclosed between the curve with equation $y = 2x^3 - 3x$ and the line with equation $y = 7x$.

19. Part of a flat deck of a ship is shaped as shown shaded in the diagram on the right. The area is symmetrical with the x-axis the line of symmetry. (1 metre = 1 unit on each axis).
This part of the deck is to be given a special fire resistant coating.
The firm carrying out this process quotes a price of \$45 per square metre.
How much will the job cost?



Using definite integrals to find total change from rate of change.

Suppose that some quantity, Q, changes with respect to some other variable, r, such that

$$\frac{dQ}{dr} = 3r^2 + 2r + 3.$$

Integration gives $\qquad Q = r^3 + r^2 + 3r + c.$

Without further information we cannot determine c, the constant of integration. However, if we require the change in Q when r changes, say from 5 to 10, this can be determined without knowing c.

$$\begin{aligned} Q(10) &= 10^3 + 10^2 + 3(10) + c \\ &= 1130 + c \\ \\ Q(5) &= 5^3 + 5^2 + 3(5) + c \\ &= 165 + c \\ \\ Q(10) - Q(5) &= (1130 + c) - (165 + c) \\ &= 965 \end{aligned}$$

When r changes from 5 to 10, Q increases by 965.

What we have found here is the definite integral:

$$\int_5^{10} (3r^2 + 2r + 3)\, dr.$$

Thus we could set our method out as follows:

Change in Q as r changes from 5 to 10 $= \displaystyle\int_5^{10} (3r^2 + 2r + 3)\ \mathrm{d}r$

$$= \left[r^3 + r^2 + 3r\right]_5^{10}$$

$$= 965 \quad \text{as before.}$$

$\int_5^{10}(3r^2+2r+3)\mathrm{d}r$

965

Thus $\displaystyle\int_a^b \frac{\mathrm{d}Q}{\mathrm{d}r}\ \mathrm{d}r$ gives the total change in Q when r changes from a to b.

Example 6

An oil tank is drained of oil such that if V kL of oil are in the tank t minutes after draining commences then $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 2t - 20$.

The initially full tank is emptied in 5 minutes.

(a) Write an expression which, if evaluated, would give the number of kilolitres of oil drained from the tank in the first minute.

(b) Determine the number of kilolitres of oil drained from the tank in the fourth minute.

(a) Total change in V in the first minute is given by: $\displaystyle\int_0^1 (2t - 20)\ \mathrm{d}t$

V is decreasing so this expression will give a negative answer.

Thus the number of kilolitres drained in the 1st minute is $\displaystyle -\int_0^1 (2t - 20)\ \mathrm{d}t$

(b) Algebraically or by calculator.

$$\int_3^4 (2t - 20)\ \mathrm{d}t = -13$$

Thus 13 kL are drained from the tank in the fourth minute.

$\int_3^4(2t-20)\mathrm{d}t$

−13

Exercise 4D

1. \$$C$ is the cost of producing x tonnes of a certain product where C is such that:

$$\frac{dC}{dx} = 3x^2 - 60x + 500.$$

Find the extra cost of producing (a) 20 tonnes rather than 10 tonnes,

(b) 50 tonnes rather than 40 tonnes.

2. \$C is the cost of producing x units of a certain product where C is such that:

$$\frac{dC}{dx} = \frac{250}{\sqrt{x}}.$$

Find the extra cost incurred by producing 100 units rather than 25.

3. \$C is the cost of producing x units of a certain product where C is such that:

$$C'(x) = \frac{400}{x+1}.$$

Using your calculator to evaluate the appropriate definite integral, find, to the nearest dollar, the extra cost incurred by producing

(a) 20 units rather than 10, (b) 40 units rather than 20.

4. To test the safety of various designs of hot air balloons, models of each design are constructed in the laboratory and then each is punctured and their deflation is monitored. For one such model the rate of change in the volume of air in the balloon with respect to time, from the time of puncture ($t = 0$) to total deflation, is approximately given by the rule:

$$\frac{dV}{dt} = 40\,(t - 25)\text{ cm}^3\text{/sec}$$

(a) Write a definite integral which, if evaluated, would give the number of cubic centimetres of air escaping from the balloon from $t = 5$ to $t = 8$. (You may assume that the balloon takes longer than 8 seconds to deflate.)

(b) Evaluate your expression from (a).

5. In a particular chemical reaction the temperature of the reagents, T°C, increases such that $\frac{dT}{dt} = \frac{t^{0\cdot1}}{2}$ where the t is the time in seconds from the start of the reaction.

(a) Write a definite integral which, if evaluated, would give the number of °C by which the temperature of the reagents rises from $t = 5$ to $t = 10$.

(b) Evaluate your expression from (a) giving your answer correct to 1 decimal place.

6. The population of a particular country (P million people) was changing such that t years after records were first kept

$$\frac{dP}{dt} \approx 5{\cdot}1 + 0{\cdot}04t$$

If it is now 20 years since records were first kept use the above rule to determine an approximate value, to the nearest half million, for the increase in the population in the next eight years.

7. An oil tank is drained of oil such that if V kL of oil are in the tank t minutes after draining commences then $\dfrac{dV}{dt} = 0{\cdot}15t^2 - 20.$

The initially full tank is emptied in 10 minutes.

For each of the following (i) write a definite integral which, if evaluated, would determine the required answer in kL,

and (ii) determine the answer (to the nearest kL).

(a) How much oil was in the full tank?

(b) How much oil was drained from the tank in the first minute?

(c) How much oil was drained from the tank in the tenth minute?

8. The total sales, N, of a new product, t weeks after its "launch" is such that:

$$\frac{dN}{dt} = 600 + \frac{600}{(t+1)^2}$$

Find the number of sales in (a) the first 4 weeks,
(b) the fifth week.

9. A motor car manufacturer introduces a new sports model. The total number, N, that the company has produced t weeks after production commenced is such that:

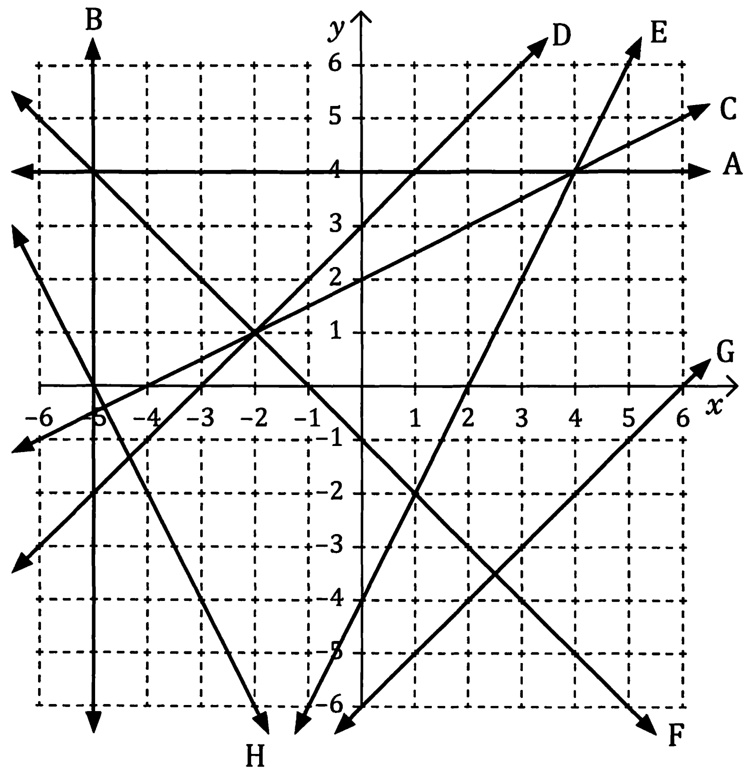
$$\frac{dN}{dt} = 150 - \frac{600}{(t+2)^2}.$$

Find the number produced in (a) the first 4 weeks,
(b) the second 4 weeks,
(c) the second week,
(d) the fourth week.

Miscellaneous Exercise Four.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Determine the equation of each of the straight lines A to H shown below.



2. If $f(x) = 2x - 3x^3$ find (a) $f'(x)$ (b) $f'(5)$ (c) $f''(x)$ (d) $f''(-5)$

3. Evaluate the following definite integrals.

   (a) $\int_1^3 2x\,dx$ (b) $\int_1^4 \sqrt{x}\,dx$

Use the product rule to determine $\frac{dy}{dx}$ for each of the following.

4. $y = (x + 5)(x - 3)$

5. $y = (x + 5)(3 - x)$

6. $y = (2x + 1)(x + 5)$

7. $y = (5 - 2x)(2x + 1)$

8. $y = (x + 1)^2(2x + 7)$

9. $y = (2x + 5)^3(5x + 6)$

10. Find the equation of the tangent to each of the following at the point indicated.

    (a) $y = (3x + 1)^2$ at $(-1, 4)$.
    (b) $y = (4x)^{-1}$ at $(0{\cdot}25, 1)$.
    (c) $y = (3x - 5)^4$ at $(2, 1)$.
    (d) $y = \dfrac{2x - 1}{x - 3}$ at $(4, 7)$.

11. Determine the coordinates of any points on the curve
$$y = (2x - 3)(x^2 - 1)$$
where the gradient is equal to -2.

12. The curve $y = ax^3 + bx^2 + cx + d$, for constant a, b, c and d, cuts the y-axis at $(0, 30)$ and the gradient of the curve at this point is -1.
The gradient at the point on this curve where $x = 1$, is -17, and at this point the second derivative with respect to x is -10.
With the assistance of a calculator if you wish, find the coordinates of all those points where $y = ax^3 + bx^2 + cx + d$ cuts the x-axis.

13. Find the following indefinite integrals.

    (a) $\displaystyle\int 20x^3\, dx$
    (b) $\displaystyle\int 6\sqrt{x}\, dx$
    (c) $\displaystyle\int (x + 3)^4\, dx$
    (d) $\displaystyle\int (2x + 3)^4\, dx$
    (e) $\displaystyle\int 60x^2 (1 + x^3)^4\, dx$
    (f) $\displaystyle\int (1 + x^2)^2\, dx$

14. A firm produces and sells x units of a particular item with the cost and revenue functions given in dollars by $\quad C(x) = 6\,000 + 18x,$
and $\quad R(x) = 25{\cdot}5x.$

    Find (a) an expression for $P(x)$, the profit function in dollars,
    (b) how many units must be produced and sold for the firm to break even with this product,
    (c) the marginal cost, marginal revenue and marginal profit.

15. A particle travels along a straight line with its acceleration at time t seconds equal to
$$6(t + 1)\ \text{m/s}^2.$$
When $t = 1$ the displacement is 3 metres. When $t = 2$ the displacement is 19 metres.
Find the displacement and velocity when $t = 3$.

16. Find A in terms of p in each of the following cases:

    (a) $\dfrac{dA}{dp} = (2p - 1)^3$ and $A = 0{\cdot}5$ when $p = 0$.
    (b) $\dfrac{dA}{dp} = 8p(p^2 - 1)^3$ and $A = 45$ when $p = 2$.

17. (a) Find the area between $y = -3x^2$ and the x-axis from $x = 0$ to $x = 2$.
    (b) Find the area between $y = 3 - 3x^2$ and the x-axis from $x = 0$ to $x = 2$.

18. Without the assistance of a calculator (a) find $\displaystyle\int \frac{3x+1}{\sqrt{x}}\,dx$,

    and show that (b) $\displaystyle\int_4^5 \frac{3x+1}{\sqrt{x}}\,dx = 12\sqrt{5} - 20$.

19. Without the assistance of your calculator:
    (a) Find the coordinates of any points where $y = x^3 - 5x^2 - 6x$
    and $y = x^2 - 9x - 10$
    cut the axes.
    (b) Show algebraically that the curve $y = x^3 - 5x^2 - 6x$
    cuts the curve $y = x^2 - 9x - 10$
    at the points (−1, a), (2, b) and (5, c) and determine a, b and c.
    (c) Show the points from (a) and (b) on a sketch of the two curves. (Coordinates of turning points need not be given on the sketch.)

    With the assistance of your calculator:
    (d) Find the area enclosed between $y = x^3 - 5x^2 - 6x$
    and $y = x^2 - 9x - 10$.

20. (a) The cost, \$C, for the production of x units of a certain product is given by

    $$C = ax^3 - bx^2 + cx$$

    for a, b and c positive constants.

    Show that for the value of x that makes the average cost per unit a minimum then: average cost per unit = marginal cost.

    (b) The cost, \$C, for the production of x units of a certain product is given by: $C = f(x)$.

    Show that the task of finding the value of x for which the average cost per unit is minimised, if such a minimum exists, involves solving the equation:

    $$x\,f'(x) = f(x), \qquad x \neq 0.$$

    Show that this same equation results from attempting to find the value of x for which: average cost per unit = marginal cost.