Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-free

ATAR course examination 2016

Marking Key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2016/64840

Question 1 **(5 marks)**

(a) Given that $\log_8 x = 2$ and $\log_2 y = 5$, evaluate $x - y$. (2 marks)

Solution
$8^2 - 2^5 = 32$
Specific behaviours
✓ determines x and y ✓ recognises the inverse relationship between logarithms and exponentials

(b) Express y in terms of x given that $\log_2(x+y) + 2 = \log_2(x-2y)$. (3 marks)

Solution
$\log_2(x+y)+2=\log_2(x-2y)$ $\log_2(x+y)+\log_2 4=\log_2(x-2y)$ $\log_2(4(x+y))=\log_2(x-2y)$ $4(x+y)=(x-2y)$ $4x+4y=x-2y$ $6y=-3x$ $y=\frac{-1}{2}x$
Specific behaviours
✓ expresses all terms as logarithms ✓ uses log laws to combine terms ✓ expresses y in terms of x

Question 2 **(5 marks)**

(a) Determine $\frac{d}{dx}\left(2xe^{2x}\right)$. (2 marks)

Solution
$\frac{d}{dx}\left(2xe^{2x}\right)=2x\left(2e^{2x}\right)+e^{2x}\left(2\right)$ $=2\left(2x+1\right)e^{2x}$
Specific behaviours
✓ uses product rule ✓ differentiates exponential term

(b) Use your answer in part (a) to determine $\int 4xe^{2x}dx$. (3 marks)

Solution
$\frac{d}{dx}\left(2xe^{2x}\right)=\left(4xe^{2x}\right)+e^{2x}\left(2\right)$ $\int\frac{d}{dx}\left(2xe^{2x}\right)dx=\int 4xe^{2x}dx+\int 2e^{2x}dx$ $2xe^{2x}=\int 4xe^{2x}dx+e^{2x}$ $\int 4xe^{2x}dx=\left(2x-1\right)e^{2x}+c$
Specific behaviours
✓ uses linearity of anti-differentiation ✓ uses fundamental theorem ✓ obtains an expression for required integral with a constant

See next page

Question 3 **(7 marks)**

Consider the function $f(x)=\dfrac{(x-1)^2}{e^x}$.

(a) Show that the first derivative is $f'(x)=\dfrac{-x^2+4x-3}{e^x}$. (2 marks)

Solution
$f'(x)=\dfrac{e^x 2(x-1)-e^x(x-1)^2}{e^{2x}}$ $=\dfrac{e^x(x-1)(2-x+1)}{e^{2x}}$ $=\dfrac{-(x-1)(x-3)}{e^x}$ $=\dfrac{-x^2+4x-3}{e^x}$
Specific behaviours
✓ uses quotient rule ✓ simplifies expression

(b) Use your result from part (a) to explain why there are stationary points at $x=1$ an $x=3$. (2 marks)

Solution
$f'(x)=\dfrac{-(x-1)(x-3)}{e^x}$ $f'(1)=0=f'(3)$
Specific behaviours
✓ identifies stationary points as $f'(x)=0$ ✓ shows that this is true for $x=1,3$

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(b) Using calculus, determine the value of θ that maximises the area A of the inscribed triangle. State this area in terms of r exactly. Justify your answer.
(Hint: you may need the identity $\sin^2 x = 1 - \cos^2 x$ in your working.) (5 marks)

Solution

$A = r^2 \sin\theta(1+\cos\theta)$

$\frac{dA}{d\theta} = r^2\left[\sin\theta(-\sin\theta) + (1+\cos\theta)\cos\theta\right]$

$\frac{dA}{d\theta} = r^2\left[\cos\theta + \cos^2\theta - \sin^2\theta\right]$

$\frac{dA}{d\theta} = r^2\left[\cos\theta + \cos^2\theta - (1-\cos^2\theta)\right]$

$\frac{dA}{d\theta} = r^2\left[2\cos^2\theta + \cos\theta - 1\right] = r^2(2\cos\theta - 1)(\cos\theta + 1)$

$\frac{dA}{d\theta} = 0 \quad \cos\theta = \frac{1}{2}, \theta = \frac{\pi}{3} \quad, \cos\theta \neq -1 \quad, 0 < \theta < \pi$

$A = r^2\sin\theta(1+\cos\theta) = r^2\frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) = \frac{3\sqrt{3}}{4}r^2$

Specific behaviours

✓ differentiates area with respect to θ using calculus
✓ equated derivative to zero to solve for optimal value
✓ rearranges derivative to allow solving for θ exactly
✓ solves for $0 < \theta < \pi$ allowing for one solution only
✓ states exact area for this optimal value

End of questions

It can be shown that the second derivative is $f''(x) = \frac{x^2 - 6x + 7}{e^x}$.

(c) Use the second derivative to describe the type of stationary points at $x = 1$ and $x = 3$. (3 marks)

Solution

$f''(x) = \frac{x^2 - 6x + 7}{e^x}$

$f''(1) = \frac{2}{e}$

$f''(3) = \frac{-2}{e^3}$

when $x = 1$ $f'' > 0$ hence local minimum

when $x = 3$ $f'' < 0$ hence local maximum

Specific behaviours

✓ evaluates second derivatives for $x = 1$ and $x = 3$
✓ uses sign to determine nature
✓ states nature for each stationary point

Question 4 **(8 marks)**

The displacement x micrometres at time t seconds of a magnetic particle on a long straight superconductor is given by the rule $x = 5\sin 3t$.

(a) Determine the velocity of the particle when $t = \frac{\pi}{2}$. (3 marks)

Solution
$x = 5\sin 3t$ $v = \frac{dx}{dt} = 15\cos 3t$ $v\left(\frac{\pi}{2}\right) = 15\cos\frac{3\pi}{2} = 0$ Velocity = 0 micrometres/second
Specific behaviours
✓ differentiates to determine velocity ✓ uses chain rule ✓ evaluates velocity at $t = \frac{\pi}{2}$

(b) Determine the rate of change of the velocity when $t = \frac{\pi}{2}$. (3 marks)

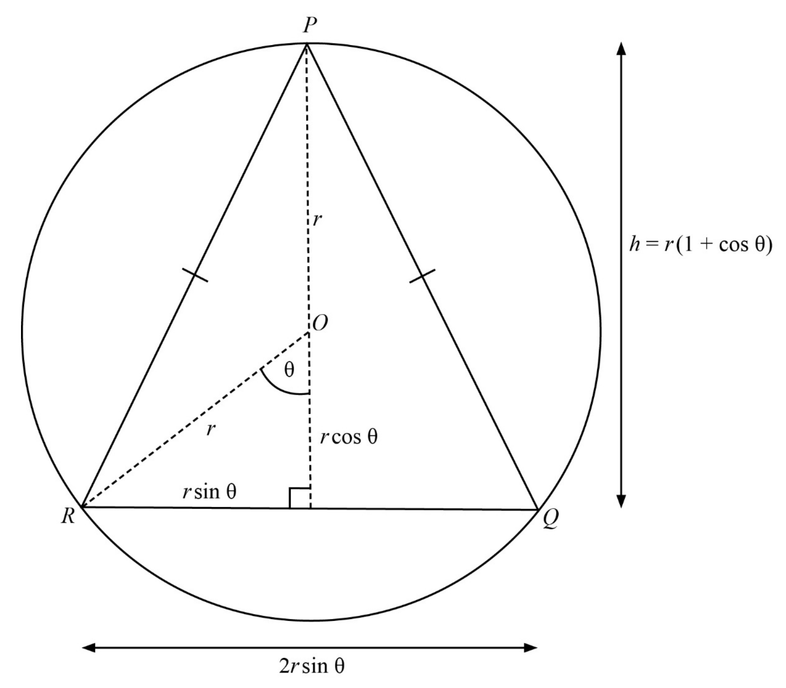
Solution
$\frac{dv}{dt} = \frac{d}{dt}(15\cos 3t)$ $= -45\sin 3t$ $\frac{dv}{dt} = 45 \quad , t = \frac{\pi}{2}$ Rate of change of velocity = 45 micrometres/second squared
Specific behaviours
✓ recognises $\frac{dv}{dt}$ as rate of change ✓ differentiates velocity ✓ evaluates rate at $t = \frac{\pi}{2}$

Question 8 **(7 marks)**

An isosceles triangle ΔPQR is inscribed inside a circle of fixed radius r and centre O.
Let θ be defined as in the diagram below.

(a) Show that the area A of the triangle ΔPQR is given by $A = r^2\sin\theta(1+\cos\theta)$. (2 marks)

Solution



Specific behaviours

✓ determines an expression of height in terms of r, θ
✓ determines an expression for base in terms of r, θ

Question 7 **(7 marks)**

Consider the graph $y = f(x)$. Both arcs have a radius of four units.

Using the graph of $y = f(x)$, $x \geq 0$, evaluate exactly the following integrals.

(a) $\int_0^{12} f(x)\,dx$ (3 marks)

Solution
$36+\frac{\pi 4^2}{4}+4\times 2+\frac{1}{2}2^2=46+4\pi$
Specific behaviours
✓ determines areas of two rectangles ✓ determines area of triangle and sector ✓ adds areas together

(b) $\int_0^{18} f(x)\,dx$ (2 marks)

Solution
$46+4\pi-\left[\frac{1}{2}2^2+\left(4\times 6-\frac{\pi 4^2}{4}\right)\right]=20+8\pi$
Specific behaviours
✓ determines area under axis ✓ uses signed areas to find net result

(c) Determine the value of the constant α such that $\int_0^{\alpha} f(x)\,dx = 0$. There is no need to simplify your answer. (2 marks)

Solution
$6(\alpha-18)+26-4\pi=46+4\pi$ $6(\alpha-18)=(20+8\pi)$ $\alpha=\frac{(20+8\pi)}{6}+18$
Specific behaviours
✓ determines a value so that signed areas balance ✓ derives an expression for α

Let v = velocity of the particle at t seconds.

(c) Determine $\int_0^{\frac{\pi}{2}} \frac{dv}{dt}\,dt$. (2 marks)

Solution
$\int_0^{\frac{\pi}{2}} \frac{dv}{dt}\,dt = v\left(\frac{\pi}{2}\right)-v(0)$ $=0-15$ $=-15$ Integral = –15 micrometres/second
Specific behaviours
✓ uses fundamental theorem ✓ subtracts velocities at the two limits

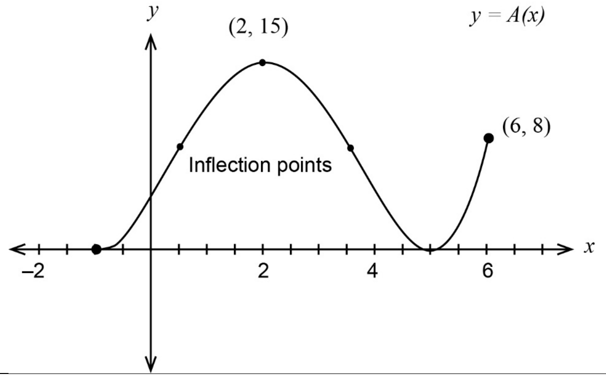
Question 5 **(6 marks)**

Consider the graph of $y = f(x)$ below.

Let $A(x)$ be defined by the integral $A(x) = \int_{-1}^{x} f(t)\,dt$ for $-1 \le x \le 6$.

It is known that $A(2) = 15$, $A(5) = 0$ and $A(6) = 8$.

Sketch on the axes below the function $A(x)$ for $-1 \le x \le 6$ labelling clearly key features such as x intercepts, turning points and inflection points if any.

Solution


Specific behaviours
✓ sketched only for $-1 \le x \le 6$ ✓ both x intercepts given ✓ local maximum shown at (2,15) ✓ endpoint labelled with A value ✓ at least one inflection point marked near a turning point of $y = f(x)$ ✓ both inflection points marked near both turning points of $y = f(x)$

Question 6 **(4 marks)**

The graphs $y = 6 - 2e^{x-4}$ and $y = -\frac{1}{4}x + 5$ intersect at $x = 4$ for $x \ge 0$.

Determine the exact area between $y = 6 - 2e^{x-4}$, $y = -\frac{1}{4}x + 5$ and the y axis for $x \ge 0$.

Solution
$A = \int_0^4 \left(6 - 2e^{x-4} - \left[-\frac{1}{4}x + 5\right]\right) dx$ $= \int_0^4 \left(-2e^{x-4} + \frac{1}{4}x + 1\right) dx$ $= \left[-2e^{x-4} + \frac{x^2}{8} + x\right]_0^4$ $= (-2 + 2 + 4) - (-2e^{-4})$ $= 2\left(2 + \frac{1}{e^4}\right)$

Specific behaviours
✓ sets up an appropriate integral for area ✓ uses correct limits ✓ anti-differentiates correctly ✓ calculates area