PERTH MODERN SCHOOL
Exceptional schooling. Exceptional students.
Independent Public School

# Course ________Specialist_______ Year __12______

Student name: ______________ Teacher name: ______________

Date: Fri week 5

**Task type:** **Response**

**Time allowed for this task:** **_____45______ mins**

**Number of questions:** **_____6______**

**Materials required:** No cals

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, , and up to three calculators approved for use in the WACE examinations

## NO NOTES ALLOWED

**Marks available:** **__39____ marks**

**Task weighting:** **_12___%**

**Formula sheet provided: Yes**

**Note: All part questions worth more than 2 marks require working to obtain full marks.**

**Q1 (4.1.2)** **(3, 3 & 3 = 9 marks)**

Determine the following integrals showing full working.

a) $\displaystyle\int \frac{5x}{\sqrt{7x^2-3}}\,dx \quad u=7x^2-3$

| Solution |
|---|
| $\displaystyle\int \frac{5x}{\sqrt{7x^2-3}}\,dx \quad u=7x^2-3$ <br> $\displaystyle\int \frac{5x}{\sqrt{7x^2-3}}\frac{dx}{du}\,du=\int\frac{5x}{\sqrt{u}}\frac{1}{14x}\,du=\frac{5}{14}\int u^{-\frac{1}{2}}\,du$ <br> $\displaystyle=\tfrac{5}{14}\left[2u^{\frac{1}{2}}\right]=\frac{5}{7}\left(7x^2-3\right)^{\frac{1}{2}}+c$ |
| **Specific behaviours** |
| ✓ uses change of variable and its derivative <br> ✓ derives new integral <br> ✓ obtains result with a constant |

b) $\displaystyle\int (3x+2)(5x-1)^7\,dx \quad u=5x-1$

| Solution |
|---|
| $\displaystyle\int (3x+2)(5x-1)^7\,dx \quad u=5x-1$ <br> $\displaystyle\int\left(3\left(\frac{u+1}{5}\right)+2\right)u^7\,\frac{1}{5}du$ <br> $\displaystyle=\int\left(\frac{3u+13}{25}\right)u^7\,du$ <br> $\displaystyle=\frac{1}{25}\int 3u^8+13u^7\,du$ <br> $\displaystyle=\frac{1}{25}\left[\frac{u^9}{3}+\frac{13}{8}u^8\right]+c$ <br> $\displaystyle\frac{1}{75}(5x-1)^9+\frac{13}{200}(5x-1)^8+c$ |
| **Specific behaviours** |
| ✓ uses change of variable and its derivative <br> ✓ derives new integral |





| |
|---|
| ✓ separates variables <br> ✓ sets up partial fractions for N <br> ✓ shows how to find constants for partial fractions <br> ✓ states that a-bN>0 <br> ✓derives logistical formula |

c) Consider $\dfrac{dN}{dt}=5N-3N^2$ with an initial value of $N=1$. Determine $N$ when t=50.
(No need to simplify)

| Solution |
|---|
| $\displaystyle N=\frac{5}{3+Ce^{-5t}}$ <br> $\displaystyle 1=\frac{5}{3+C} \quad C=2$ <br> $\displaystyle N=\frac{5}{3+2e^{-250}}$ |
| **Specific behaviours** |
| ✓ solves for constant <br> ✓ expresses exact value at t=50 |

| ✓ states value for N |
|---|

b) Show how to derive using integration and partial fractions that the general solution is

$N = \dfrac{a}{b + Ce^{-at}}$

| Solution |
|---|
| $\dfrac{dN}{dt} = aN - bN^2 = (a - bN)N$<br>$\displaystyle\int \dfrac{dN}{(a-bN)N} = \int dt$<br>$\dfrac{1}{(a-bN)N} = \dfrac{C}{a-bN} + \dfrac{D}{N}$<br>$1 = CN + (a-bN)D$<br>$N = 0$<br>$1 = aD \quad D = \dfrac{1}{a}$<br>$N = \dfrac{a}{b}$<br>$1 = C\dfrac{a}{b} \quad C = \dfrac{b}{a}$<br>$\displaystyle\int \dfrac{\frac{b}{a}}{a - bN} + \dfrac{\frac{1}{a}}{N} dN = t + c$<br>$-\dfrac{1}{a}\ln\lvert a - bN\rvert + \dfrac{1}{a}\ln N = t + c$<br>$\ln\dfrac{N}{a-bN} = at + c \quad Note: a - bN > 0 \quad as \quad N < \dfrac{a}{b}$<br>$Ce^{at} = \dfrac{N}{a-bN}$<br>$Ce^{-at} = \dfrac{a-bN}{N}$<br>$NCe^{-at} = a - bN$<br>$NCe^{-at} + bN = a$<br>$N = \dfrac{a}{b + Ce^{-at}}$ |
| **Specific behaviours** |





| ✓ obtains result in terms of x |
|---|

c) $\displaystyle\int \dfrac{\sqrt{x}}{\sqrt{x}+7}\,dx$

| Solution |
|---|
| $let\ u = \sqrt{x} + 7$<br>$\dfrac{du}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}}$<br>$\displaystyle\int \dfrac{\sqrt{x}}{\sqrt{x}+7}\dfrac{dx}{du}du = \int \dfrac{2x}{u}du = \int \dfrac{2(u-7)^2}{u}du$<br>$\displaystyle = 2\int \dfrac{u^2 - 14u + 49}{u}du = 2\int u - 14 + \dfrac{49}{u}du = 2(\dfrac{u^2}{2} - 14u + 49\ln u) + c$<br>$= \left(\sqrt{x}+7\right)^2 - 28\left(\sqrt{x}+7\right)^2 + 98\ln\left(\sqrt{x}+7\right) + c$<br><br>Alternative<br>$\displaystyle\int \dfrac{\sqrt{x}}{\sqrt{x}+7}dx \quad u = \sqrt{x}$<br>$\displaystyle\int \dfrac{u}{u+7}2u\,du = \int \dfrac{2u^2}{u+7}du = \int 2u - 14 + \dfrac{98}{u+7}du = u^2 - 14u + 98\ln\lvert u+7\rvert + c$<br>$= x - 14\sqrt{x} + 98\ln\left\lvert\sqrt{x}+7\right\rvert + c$<br><br>$u + 7 \overline{)\ 2u^2}$ quotient $2u - 14$<br>$2u^2 + 14u$<br>$-14u - 98$<br>$98$ |
| **Specific behaviours** |
| ✓ uses change of variable and its derivative<br>✓ derives new integral<br>✓ obtains result in terms of x |

**Q2 (4.1.1 -4.1.3) (3, 3 & 3 = 9 marks)**

Determine the following definite integrals showing full working.

a) $\displaystyle\int_0^{\frac{\pi}{6}} \cos^2 4x\,dx$

| Solution |
|---|
| $\int_0^{\frac{\pi}{6}} \cos^2 4x\,dx = \int_0^{\frac{\pi}{6}} \frac{\cos 8x+1}{2}\,dx = \left[\frac{1}{16}\sin 8x + \frac{1}{2}x\right]_0^{\frac{\pi}{6}} = \left(\frac{-\sqrt{3}}{32} + \frac{\pi}{12}\right)$ |
| **Specific behaviours** |
| ✓ uses double angle formula for cosine<br>✓ obtains antiderivative<br>✓ obtains exact value |

b) $\int_0^{\pi} \sin^3 2x\,dx$

| Solution |
|---|
| $\int \sin^2 2x \sin 2x\,dx = \int (1-\cos^2 2x)\sin 2x\,dx$<br>$= \int_0^{\pi} \sin 2x - \cos^2 2x \sin 2x\,dx$<br>$= \left[-\frac{1}{2}\cos 2x + \frac{1}{6}\cos^3 2x\right]_0^{\pi} = \left(\frac{-1}{2}+\frac{1}{6}\right) - \left(\frac{-1}{2}+\frac{1}{6}\right) = 0$ |
| **Specific behaviours** |
| ✓ uses trig identity<br>✓ obtains antiderivative<br>✓ subs limits to show value |





**Q5 (4.2.4) (4 marks)**

Determine the solution to the following differential equation $yx^2\frac{dy}{dx} = \frac{x+x^3}{(5y^2+1)^4}$ given that (1,1) is a known point.(No need to simplify)

| Solution |
|---|
| $yx^2\frac{dy}{dx} = \frac{x+x^3}{(5y^2+1)^4}$<br>$\int y(5y^2+1)^4\,dy = \int \frac{1}{x}+x\,dx$<br>$\frac{(5y^2+1)^5}{50} = \ln x + \frac{x^2}{2} + c$<br>$\frac{6^5}{50} - \frac{1}{2} = c$<br>$\frac{(5y^2+1)^5}{50} = \ln x + \frac{x^2}{2} + \frac{6^5}{50} - \frac{1}{2}$ |
| **Specific behaviours** |
| ✓ separates variables under integration<br>✓ integrates y terms<br>✓ integrates x terms<br>✓ solves for constant (unsimplified) |

**Q6 (4.2.6) (1, 5 & 2 = 8 marks)**

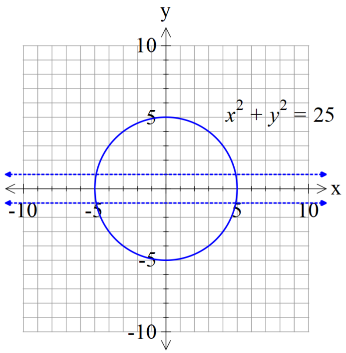
Consider the differential equation $\frac{dN}{dt} = aN - bN^2$ with $a\,\&\,b$ positive constants.

a) Determine the limiting value for $N$ as $t \to \infty$

| Solution |
|---|
| $\frac{dN}{dt} = aN - bN^2 = (a-bN)N = 0$<br>$N = \frac{a}{b}$<br>$N < \frac{a}{b}$ |
| **Specific behaviours** |

**Q4** **(4.1.5-4.1.6)** **(5 marks)**

Consider a cylindrical drill of width 2 cm that carves a cavity inside a solid sphere of radius 5 cm as shown below. Determine the volume of the sphere remaining.(Simplify)




| Solution |
| --- |
| $x^2 + 1 = 25$<br>$x = \pm\sqrt{24} = \pm 2\sqrt{6}$<br>$V = 2\pi \int_0^{2\sqrt{6}} y^2 - 1\,dx = 2\pi \int_0^{2\sqrt{6}} 24 - x^2\,dx = 2\pi \left[24x - \frac{x^3}{3}\right]_0^{2\sqrt{6}} = 2\pi\left(48\sqrt{6} - \frac{48\sqrt{6}}{3}\right)$<br>$= \frac{4\pi 48}{3}\sqrt{6}$ *accept* *or* $\frac{192\pi\sqrt{6}}{3}$<br>(Note- max 2 out of 5 if they have removed all of sphere between y=-1 and y=1) |
| **Specific behaviours** |
| ✓ solves for when y=1<br>✓ uses solid of revolutions integral<br>✓ sets up correct integral for volume remaining<br>✓ anti-differentiates<br>✓simplifies to one term/surd |





c) $\int_0^{\frac{\pi}{4}} -12\tan^2 5x\,dx$

| Solution |
| --- |
| $\int_0^{\frac{\pi}{4}} -12\tan^2 5x\,dx$<br>$= \int_0^{\frac{\pi}{4}} -12(\sec^2 5x - 1)\,dx$<br>$= \left[\frac{-12}{5}\tan 5x + 12x\right]_0^{\frac{\pi}{4}} = \left(\frac{-12}{5} + 3\pi\right)$ |
| **Specific behaviours** |
| ✓ uses trig identity<br>✓ obtains antiderivative<br>✓ subs limits to show exact value |

**Q3** **(4 marks)**

Determine the following integral showing full working.

$$\int \frac{x+7}{(x+1)(x-3)^2}\,dx$$

| Solution |
| --- |
| $\int \frac{x+7}{(x+1)(x-3)^2}\,dx$<br>$\frac{x+7}{(x+1)(x-3)^2} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$<br>$x+7 = A(x-3)^2 + B(x+1)(x-3) + C(x+1)$<br>$x = 3$<br>$10 = 4C \quad C = \frac{5}{2}$<br>$x = -1$<br>$6 = 16A \quad A = \frac{3}{8}$<br>$x = 0$<br>$7 = \frac{27}{8} - 3B + \frac{5}{2}$<br>$3B = \frac{27}{8} + \frac{20}{8} - \frac{56}{8} = -\frac{9}{8} \quad B = -\frac{3}{8}$<br>$\frac{3}{8}\ln\lvert x+1\rvert - \frac{3}{8}\ln\lvert x-3\rvert - \frac{5}{2}(x-3)^{-1} + C$<br><br>(Note: if only two terms used- max of 2 out of 4 marks) |
| **Specific behaviours** |
| ✓ uses partial fractions<br>✓ sets up equations to solve for constants<br>✓ solves all 3 constants<br>✓ anti differentiates all terms (no need to add constant) |