Specific behaviours
✓ subs v=0 ✓ rounds to nearest mm

PERTH MODERN SCHOOL
Exceptional schooling. Exceptional students.
Independent Public School

Course Specialist Year 12 Test Four 2022

Student name: _____ Teacher name: _____

Task type: **Response**

Time allowed for this task: **_____40_____ mins**

Number of questions: **_____6_____**

Materials required: Upto 3 Calculators with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **__40____ marks**

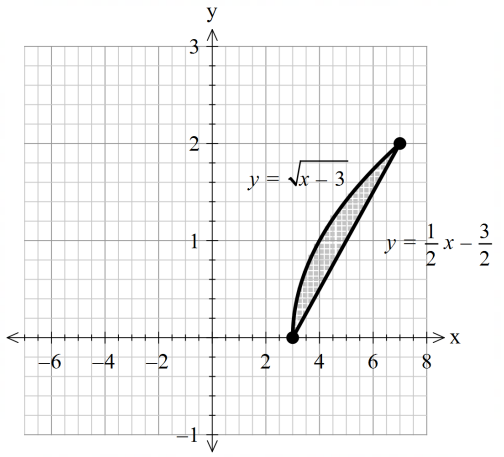
Task weighting: **_10___%**

Formula sheet provided: **Yes**

Note: All part questions worth more than 2 marks require working to obtain full marks.

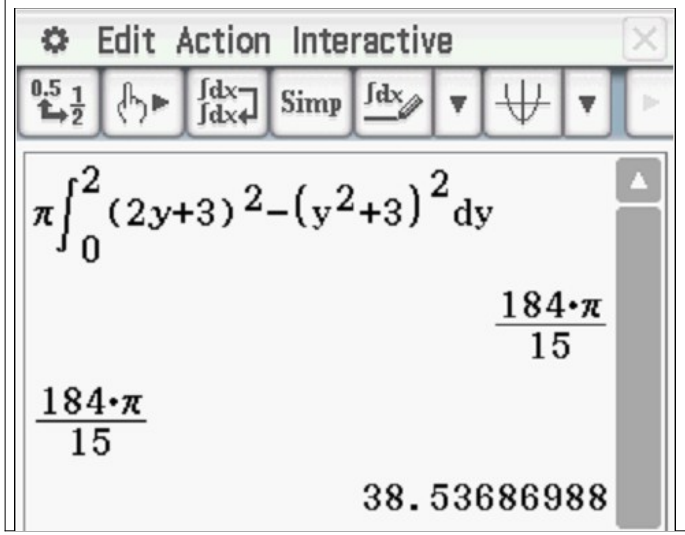
Q1 (5 marks)

Determine the volume of the solid formed by rotating the area enclosed between $y=\sqrt{x-3}$ & $y=\frac{1}{2}x-\frac{3}{2}$ about the y axis, as shown below.



Solution

$$\int_0^2 \pi(2y+3)^2\,dy - \int_0^2 \pi(y^2+3)^2\,dy$$



✓ determines a model for displacement and period ✓ solves for times at half an amplitude in one cycle/(half cycle) ✓ determines percentage of time

Q6 (4 & 2 = 6 marks)

The motion of a bullet through a wall is modelled by the equation $a=-25(v+75)^2,\quad v>0$ where $a\ m/s^2$ is its acceleration and $v\ m/s$ its velocity t seconds after impact. Initially the speed is 300 and is at the origin ($x=0$ metres)

a) Determine x in terms of v.

Solution

$$v\frac{dv}{dx}=-25(v+75)^2,\quad v>0$$

$$\int\frac{v}{(v+75)^2}dv=\int -25\,dx$$

$$let\ y=v+75$$

$$\int\frac{y-75}{y^2}dy=\int y^{-1}-75y^{-2}\,dy=\ln|y|+75y^{-1}=\ln|v+75|+\frac{75}{v+75}=-25x+c$$

$$x=0, v=300$$

$$\ln 375+\frac{75}{375}=c$$

$$x=\frac{1}{-25}\left(\ln|v+75|+\frac{75}{v+75}-\ln 375-\frac{75}{375}\right)$$

Specific behaviours

✓ uses dv & dx and separates variables v & x ✓Integrates both sides ✓ changes variable to integrate dv ✓ solves for exact constant

Q6 continued-

b) Determine how far the bullet penetrates the wall before coming to rest to the nearest mm.

Solution

$$x=\frac{1}{-25}\left(\ln|75|+1-\ln 375-\frac{75}{375}\right)$$

$$\approx 0.0324m$$

$$\approx 32mm$$

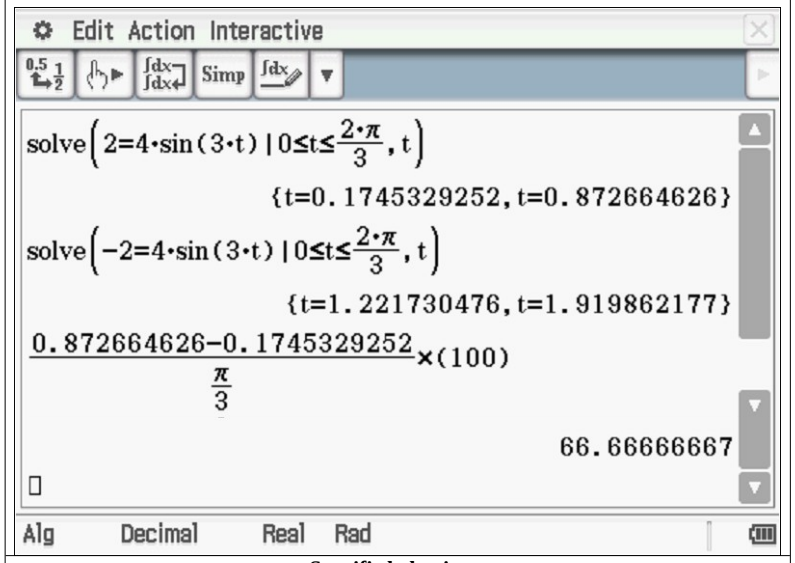
Q5 (3 & 3 = 6 marks)

Consider a particle that is moving with SHM such that $\ddot{x} = -9x$ with a maximum speed of 12 m/s.

a) Determine the exact speed when the particle is half of an amplitude from the origin.

Solution
$\ddot{x} = -9x$ $n = 3$ $12 = nA = 3A$ $A = 4$ $v^2 = n^2(A^2 - x^2) = 9(16 - 4)$ $v = \sqrt{108}$
Specific behaviours
✓ determines n & A ✓ uses correct formula ✓ states exact speed

b) Determine the percentage of the time that the particle is more than half an amplitude from the centre.

Solution
Let $x = A\sin nt = 4\sin 3t$ 
Specific behaviours

Specific behaviours
✓ uses correct integral type ✓ determines x as the subject for each graph ✓ sets up integrals for both functions with limits ✓ uses subtraction after squaring ✓states approx. volume of solid

Q2 (5, 3 & 2= 10 marks)

a) By using integration and partial fractions, show how to derive $N = \frac{a}{b + Ce^{-at}}$ from the differential equation $\frac{dN}{dt} = aN - bN^2$ ($a, b > 0$) and c is a constant

Solution
$\frac{dN}{dt} = aN - bN^2 = N(a - bN)$ $\frac{dN}{dt} \to 0, a - bN = 0 \therefore N < \frac{a}{b} \to a - bN > 0$ $\int \frac{dN}{N(a - bN)} = \int dt$ $\frac{1}{N(a - bN)} = \frac{c}{N} + \frac{d}{a - bN}$ $1 = c(a - bN) + dN$ $N = 0$ $1 = ca, c = \frac{1}{a}$ $N = \frac{a}{b}$ $1 = d\frac{a}{b}, d = \frac{b}{a}$ $\int \frac{1}{N} + \frac{b}{a - bN} dn = t + c$ $\frac{1}{a}\ln\lvert N\rvert - \frac{1}{a}\ln\lvert a - bN\rvert = t + c$ $As\ a - bN > 0$ $\ln N - \ln(a - bN) = at + c$ $\ln\frac{N}{a - bN} = at + c$ $\frac{N}{a - bN} = e^{at+c} = Ce^{at}, \frac{a - bN}{N} = Ce^{-at}$ $a - bN = NCe^{-at}$ $a = N(b + Ce^{-at}), N = \frac{a}{b + Ce^{-at}}$

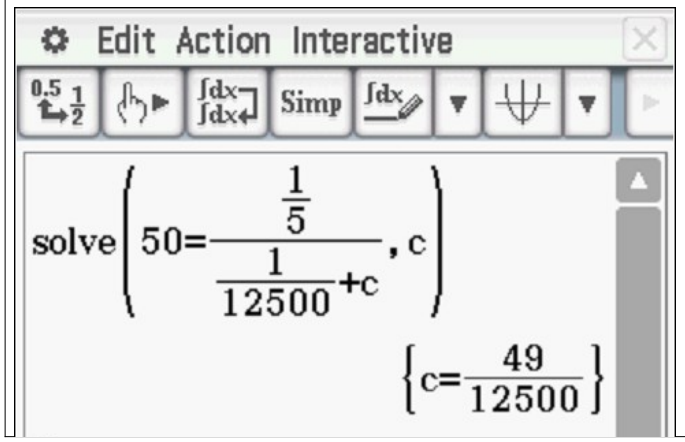
Specific behaviours
✓ explains limit of N and sign of a-bN
✓ separates dN & dt and integrates
✓ uses partial fractions
✓ uses logs and obtains expression of N in terms of t
✓shows derivation of final rule

Q2 continued

b) Let N equal the number of kangaroos living in a habitat after t years and $\frac{dN}{dt} = \frac{1}{5}N - \frac{1}{12500}N^2$ .If initially there are 50 kangaroos, determine the number in 10 years time.

Solution

$$N = \frac{\frac{1}{5}}{\frac{1}{12500} + ce^{-0.2t}}$$



Q4 (5 marks)

Determine expressions in terms of $x \& y$ only for $\frac{dy}{dx} \& \frac{d^2y}{dx^2}$ in terms of $x, y \& y'$ using the following equation $x^3y^2 = 5 - xy$

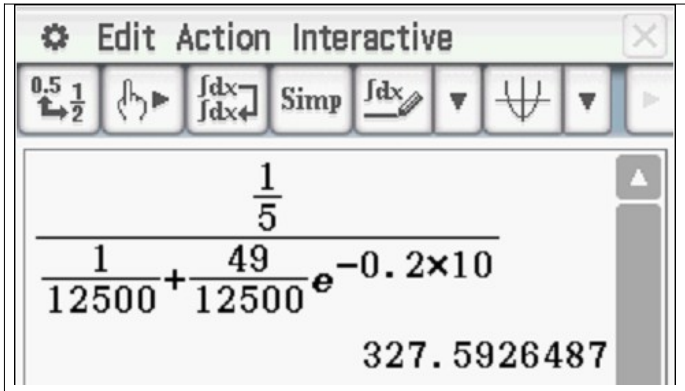
Solution

$$x^3y^2 = 5 - xy$$
$$x^3(2yy') + y^2 3x^2 = -xy' - y$$
$$y'(2x^3y + x) = -y(1 + 3x^2y)$$
$$y' = \frac{-y(1 + 3x^2y)}{(2x^3y + x)}$$
$$y'(2x^3y' + 6x^2y + 1) + (2x^3y + x)y'' = -y(3x^2y' + 6xy) - y'(1 + 3x^2y)$$
$$y'' = \frac{-y(3x^2y' + 6xy) - y'(1 + 3x^2y) - y'(2x^3y' + 6x^2y + 1)}{(2x^3y + x)}$$

Specific behaviours
✓ uses product rule on both sides for first derivative
✓ uses implicit diff in terms of x
✓ uses product/quotient rule for second drivative
✓ obtains an expression with second derivative
✓makes second derivative subject in terms of x,y and first derivative

Hint – use logarithmic differentiation. Show all working.

Solution
$y = 3^x$ $\ln y = \ln 3^x = x \ln 3$ $\frac{1}{y} y' = \ln 3$ $y' = y \ln 3 = 3^x \ln 3$ $\int 3^x dx = \frac{1}{\ln 3} 3^x + c$ $\frac{dy}{dx} = 3^x$ $y = \frac{1}{\ln 3} 3^x + c$ $2 = \frac{9}{ln3} + c$ $c = 2 - \frac{9}{\ln 3}$ $y = \frac{1}{\ln 3} 3^x + 2 - \frac{9}{\ln 3}$
Specific behaviours
✓ uses log diff to diff exponential ✓ uses integration ✓ solves for exact constant **Note max 1 out of 3 if log diff not shown**



Specific behaviours
✓ solves for constant ✓ subs t=10 into correct expression ✓ states population (accept decimal)

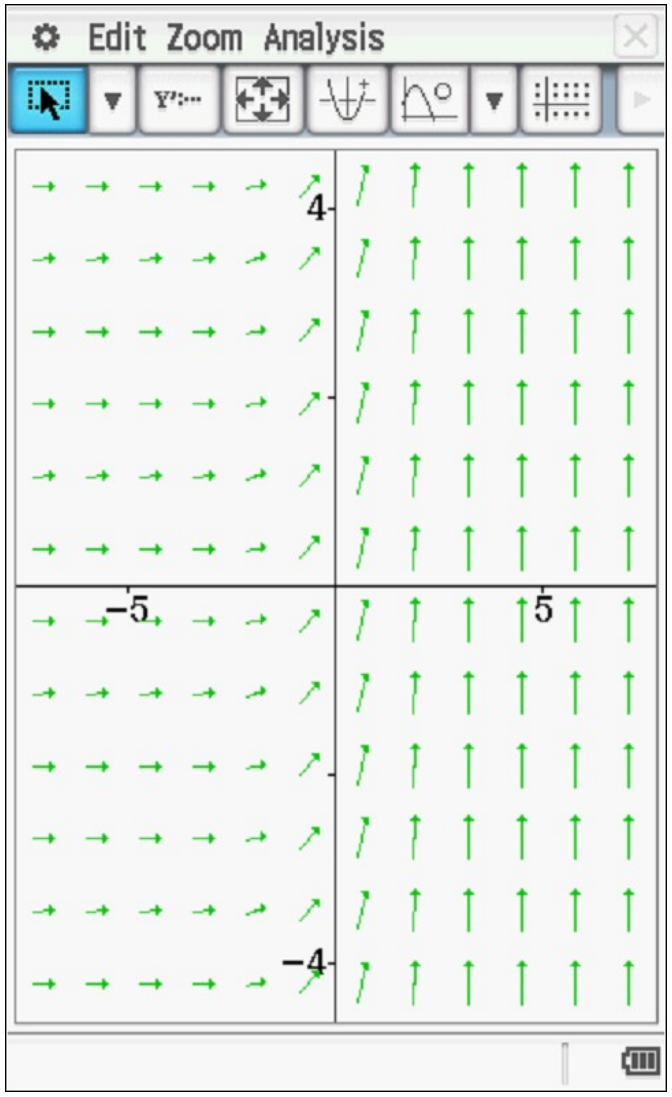
c) Determine the size of the population at the maximum growth rate.

Solution
$N = \frac{1}{2}\frac{a}{b}$ $\frac{1}{2}\frac{\frac{1}{5}}{\frac{1}{12500}}$ 1250
Specific behaviours
✓ uses half of maximum ✓ States population

Q3 (3, 2 & 3 = 8 marks)

a) Sketch the slope field on the axes below for $\frac{dy}{dx} = 3^x$

Solution

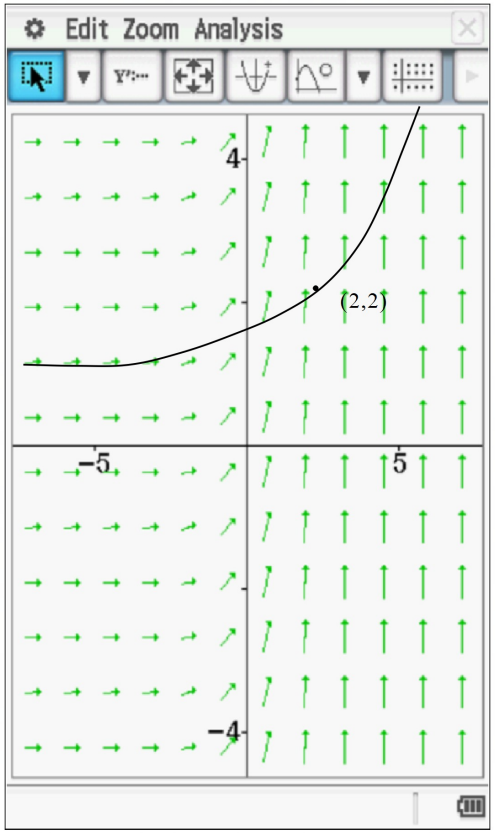


Specific behaviours

- ✓ left side near zero gradients
- ✓ 45 degrees on y axis i.e 1
- ✓ right side approaches vertical lines, i.e infinite

b) Show the solution curve on the axes above that passes through point (2,2).

Solution



Specific behaviours

- ✓ follows contours
- ✓ passes through (2,2)

c) Determine in cartesian form the solution curve for b above **without using a classpad**.