4. Consider the following two logarithmic functions:

$$f(x) = \ln x \quad \text{and} \quad g(x) = \ln(x-2) + 2$$

[2 + 1 + 1 + 4 = 8 marks]

(a) State the equations of any asymptotes for these two functions.

$f(x)$ has $x=0$ ✓ ; $g(x)$ has $x=2$ ✓

(don't require them to say which is which)

(b) Determine the exact value of the x-coordinate of their point of intersection.

$\ln x = \ln(x-2)+2$ ~~✓equate~~

$x = \dfrac{2e^2}{e^2-1}$ ✓ answer

(-1) overall for decimal answers in (b) and (c).

(c) Determine the exact x-value of the root of g(x).

$0 = \ln(x-2)+2$ ✓ ~~set = 0~~

$x = 2 + \dfrac{1}{e^2}$ ✓ answer

(d) Determine, to two decimal places, the area trapped between the two curves and the x-axis.

$$A = \int_1^{2e^2/(e^2-1)} \ln x \, dx - \int_{2+1/e^2}^{2e^2/(e^2-1)} (\ln(x-2)+2)\, dx$$

$= 0.54$

✓ limits on first $\int$

✓ limits on second $\int$

✓ minus sign (difference).

✓ answer (rounding).

didn't deduct mark for extra d.p.

[END OF TEST]

8

Mathematics Methods: Units 3 & 4

Test 3: Logarithms

Calculator-Free Section

SHENTON COLLEGE

Time allowed: 20 minutes
Total marks: 35

Formula sheet provided
No notes permitted
No ClassPad (nor any other calculator) permitted

Name: MARKING KEY Teacher (circle): MARTIN SMITH MOORE

Note: You should show clear and comprehensive working out throughout to obtain part marks where these apply.

1. Evaluate the following expressions giving your answers in the simplest form.

[1 + 1 + 2 + 2 = 6 marks]

(a) $\log_6 36$

$= \log_6 6^2$

$= 2 \cdot \log_6 6$

$= 2$ ✓

(b) $\log_3 \dfrac{1}{27}$

$= \log_3 3^{-3}$

$= -3 \cdot \log_3 3$

$= -3$ ✓

(c) $\log_9 \sqrt{3}$

$= \log_9 \sqrt{\sqrt{9}}$ ✓

$= \log_9 9^{1/4}$

$= \frac{1}{4} \log_9 9$

$= \dfrac{1}{4}$ ✓

(d) $5^{3\log_5 2}$

$= (5^{\log_5 2})^3$ ✓

$= 2^3$

$= 8$ ✓

2. Express each of the following as a single logarithm. Simplify your answers where possible.

[2 + 2 + 2 = 6 marks]

(a) $3\log_{10} x + 5\log_{10} y$

$= \log_{10} x^3 + \log_{10} y^5$ ✓ indices

$= \log_{10} (x^3 y^5)$ ✓

(b) $\log_2 \dfrac{1}{x} - \log_2 \dfrac{1}{x^2} + 3$

$= \log_2 \left(\dfrac{1}{x} \cdot \dfrac{x^2}{1}\right) + 3$

$= \log_2 x + 3$ ✓ terms with x.

$= \log_2 x + \log_2 8$

$= \log_2 (8x)$ ✓

(c) $\ln(x^2 - 1) - \ln(x+1)$

$= \ln\left[\dfrac{x^2-1}{x+1}\right]$ ✓

$= \ln\left[\dfrac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}}\right]$

$= \ln(x-1)$ ✓

12

3. Find all possible values of x satisfying the following equations. Where your answers involve logarithms, express these using natural logarithms.

[2 + 2 + 3 = 7 marks]

(a) $\log_3(3x-3) = 2$

$\log_3(3x-3) = \log_3 9$ ✓

$3x-3 = 9$

$x = 4$ ✓

(b) $7^{1-x} = 6^x$

$\ln 7^{1-x} = \ln 6^x$

$(1-x)\ln 7 = x\ln 6.$ ✓ log of both sides and log law.

$x(\ln 6 + \ln 7) = \ln 7$

$x = \dfrac{\ln 7}{\ln 6 + \ln 7}$ ✓

$= \dfrac{\ln 7}{\ln 42}$ (better).

(c) $\log_{10} x + \log_{10}(x-21) = 2$

$\log_{10}[x(x-21)] = \log_{10} 100$

$x(x-21) = 100$ ✓

$x^2 - 21x - 100 = 0$

$(x-25)(x+4) = 0$ ✓ factorises

$x = 25$ ✓ positive sol^n only.

4. Determine f′(x) for each of the following functions. Simplify your answers where possible, and where your answers involve logarithms express these using natural logarithms.

[2 + 2 + 3 + 3 = 10 marks]

(a) $f(x) = 3x^2 + 2\ln x$

$f'(x) = 6x + \dfrac{2}{x}$ ✓ ✓

(b) $f(x) = \ln[(x+2)(x-5)]$

$= \ln[x^2 - 3x - 10]$ ✓ expand or log law or product rule

$f'(x) = \dfrac{2x-3}{x^2-3x-10}$ ✓ answer (single fraction)

or $= \dfrac{2x-3}{(x+2)(x-5)}$

(c) $f(x) = e^{2x}\log_2 x$

$= \dfrac{1}{\ln 2} \cdot e^{2x} \cdot \ln x$ ✓ change of base

$f'(x) = \dfrac{1}{\ln 2}\left(e^{2x} \cdot \dfrac{1}{x} + 2e^{2x} \cdot \ln x\right)$ ✓ product rule

$= \dfrac{e^{2x}}{\ln 2}\left(\dfrac{1}{x} + 2\ln x\right)$ ✓ answer ~~(factors out~~ e^{2x})

(d) $f(x) = \ln\left[\dfrac{x}{1-x}\right]$

$= \ln x - \ln(1-x)$ ✓ log law or quotient rule

$f'(x) = \dfrac{1}{x} - \dfrac{-1}{1-x}$ ✓

$= \dfrac{(1-x)+x}{x(1-x)}$

$= \dfrac{1}{x(1-x)}$ ✓ answer (single fraction)

max (-1) overall for answers as sum of two algebraic fractions.

17

3. Consider the curve defined by

$$y = \frac{\ln x}{\sqrt{x}}$$

In this question, give all of your answers using exact values.

[4 + 1 + 4 = 9 marks]

(a) Show that this curve has a local maximum and give the exact value of its coordinates.

$y' = \dfrac{2-\ln x}{2\sqrt{x^3}}$ ✓ derivative

$y' = 0 \Rightarrow 2 - \ln x = 0$

$x = e^2$ ✓ single stationary point.

$y'' = \dfrac{3\ln x - 8}{4\sqrt{x^5}}$

$y''|_{x=e^2} = \dfrac{-1}{2e^5} < 0 \Rightarrow$ local max. ✓ second derivative test or sign test.

$y|_{x=e^2} = \dfrac{2}{e}$

$(e^2, 2/e)$ ✓ coordinates

(b) Determine the equation of the tangent to this curve at the point (1,0).

$y = x - 1$ ✓

(-1) overall for decimal answers.

(c) Find the coordinates of the point of intersection of the tangent found in Part (b) and the tangent to the curve at its local maximum.

$y = x - 1$ $\quad y = 2/e.$ ✓ tangent at local max.

✓ equate lines.

$x - 1 = 2/e$

$x = 1 + 2/e$ ✓ solve.

so the point of intersection is $(1 + 2/e, 2/e)$ ✓ coordinates.

9

2. The rate at which a battery charges becomes slower the closer the battery gets to its maximum charge C_0. The time (in hours) taken for a completely flat battery to be charged to a charge C is

$$t = -k\ln\left(1 - \frac{C}{C_0}\right)$$

where k is a positive constant that depends on the battery.

[3 + 2 = 5 marks]

(a) Rearrange the equation above to give an equation showing how the charge on an initially flat battery changes as a function of time. (I.e., rearrange it to the form C=)

$-t/k = \ln(1 - C/C_0)$ ✓ divide by -k

$e^{-t/k} = 1 - C/C_0$ ✓ applies inverse function.

$C/C_0 = 1 - e^{-t/k}$

$C = C_0(1 - e^{-t/k})$ ✓

(b) For a certain battery k=0.25. How long will it take for this battery to charge to 95% of its maximum charge? Give your answer to two decimal places.

$t = -0.25 \cdot \ln\left(1 - \frac{0.95\cancel{C_0}}{\cancel{C_0}}\right)$

accept sub. into exponential form.

✓ substitutes correctly

$= -0.25 \cdot \ln 0.05$

$= 0.75$ hours

✓ answer (units, d.p.)

(44.94 mins)

max. (-1) for units/rounding between Q1, Q2.

5. Evaluate the following indefinite integrals. (Assume that the domains are restricted to ensure that the denominators in any fractions are greater then zero.)

[2 + 2 + 2 = 6 marks]

(a) $\int \left(3x^2 + \frac{4}{x}\right)dx$

$= x^3 + 4\ln x + C$ ✓ ✓

{ • accept answers w/ absolute value here.
• +C missing ⇒ (-1) overall }

(b) $\int \frac{4x-10}{x^2-5x}dx$

$= 2\int \frac{2x-5}{x^2-5x}dx.$

✓ recognises derivative in numerator

$= 2\ln(x^2-5x) + C$ ✓

(c) $\int \frac{\sin 2x - \cos 2x}{\sin 2x + \cos 2x}dx$

$= -\frac{1}{2}\int \frac{+2\cos 2x - 2\sin 2x}{\sin 2x + \cos 2x}dx.$

✓ recognises derivative in numerator

$= -\frac{1}{2}\ln(\sin 2x + \cos 2x) + C.$ ✓

[END OF SECTION]

Mathematics Methods: Units 3 & 4
Test 3: Logarithms
Calculator-Assumed Section

SHENTON COLLEGE

Time allowed: 30 minutes
Total marks: 27

Formula sheet provided
1 single-sided A4 page of notes permitted
ClassPad (and/or other calculator) permitted

Name: MARKING KEY.   Teacher (circle): MARTIN   SMITH   MOORE

Note: You should show clear and comprehensive working out throughout to obtain part marks where these apply.

1. The ear is sensitive to a very wide range of sound intensities. As such, the perceived loudness of a sound is measured on a logarithmic scale in units called decibels (dB). The loudness of a sound of intensity I is given by

$$L = 10\log\frac{I}{I_0}$$

where $I_0 = 10^{-12}$ W/m² is a reference intensity defined as that of a barely audible sound.

[2 + 3 = 5 marks]

(a) Find the loudness, to the nearest decibel, of a hairdryer with a sound intensity of 1.58×10^{-5} W/m².

$L = 10\log\left(\frac{1.58\times10^{-5}}{10^{-12}}\right)$ ✓ substitutes correctly

$= 72\text{dB}$ ✓ answer (units, rounding)

(b) A normal conversation has a loudness of 50 dB. Sitting in the front row at a rock concert has a loudness of 110 dB. How many times greater is the intensity of sound at the rock concert compared with that of the normal conversation?

$L_2 - L_1 = 10\log(I_2/I_0) - 10\log(I_1/I_0)$ ✓ log law

$= 10\log(I_2/I_1)$ ✓ substitute

$110 - 50 = 10\log(I_2/I_1)$ ✓ rearrange and answer.

$\frac{I_2}{I_1} = 10^6$ i.e., it is a million times greater.

OR.
✓ $I_1 = 10^{-1}$ W/m²
✓ $I_2 = 10^{-7}$ W/m²
ratio.

5