

Only a formula sheet is allowed for this section. No calculator or notes allowed.

Question 1 (12 marks)

Evaluate each of the following, showing all working. Leave all answers with positive indices.

(1 mark)

(a)  $\int_{\frac{1}{2}}^4 \frac{1}{t^2} dt$

$$= \int 4t^{-2} dt$$

$$= \frac{-1}{4t^{-1}} + C$$

$$= -\frac{1}{4t} + C$$

(3 marks)

(b)  $\int 3x(x^2 - 2)^3 dx$

$$= \frac{3}{2} \int 2x(x^2 - 2)^3 dx$$

$$= \frac{3}{2} \left( \frac{(x^2 - 2)^4}{4} \right) + C$$

$$= \frac{3(x^2 - 2)^4}{8} + C$$

(3 marks)

(c)  $\int (e^{-5x} + 2\pi x - \sqrt{x}) dx$

$$= \frac{e^{-5x}}{-5} + \frac{2\pi x^2}{2} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= -\frac{5e^{-5x}}{5} + \pi x^2 - \frac{2}{3} x^{\frac{3}{2}} + C$$

(d)  $\frac{d}{dx} \left( \int_{-3}^{x^2} \frac{\sqrt{2t-3}}{t+1} dt \right)$

(2 marks)

$$= \frac{\sqrt{2x^2-3}}{x^2+1} \times 2x$$

If it is given that  $f(x)$  is continuous everywhere and that  $\int_4^{10} f(x) dx = -10$ , find:

(e)  $\int_1^3 f(3x+1) dx$

(3 marks)

$$\begin{aligned} &= \frac{1}{3} \int_3^9 f(x+1) dx \\ &= \frac{1}{3} \int_4^{10} f(x) dx \\ &= \frac{1}{3} (-10) = -\frac{10}{3} \end{aligned}$$

## Question 2

(15 marks)

Evaluate the following, showing full working.

(a)  $\int_{-1}^2 (x^2 - 1) dx$

(3 marks)

$$\begin{aligned} &= \left[ \frac{x^3}{3} - x \right]_{-1}^2 \\ &= \left( \frac{8}{3} - 2 \right) - \left( -\frac{1}{3} - (-1) \right) \\ &= \frac{2}{3} - \frac{2}{3} = 0 \end{aligned}$$

(b)  $-3 \int_{\pi}^{2\pi} \cos(3x) dx$

(3 marks)

$$\begin{aligned} &= -3 \left[ \frac{\sin 3x}{3} \right]_{\pi}^{2\pi} \\ &= - [\sin 6\pi - \sin 3\pi] \\ &= - [0 - 0] = 0 \end{aligned}$$

Question 12

(4 marks)

The area bound by the parabola  $y = 6x^2 - 6x$ , the  $x$ -axis and the lines  $x = 1$  and  $x = c$ , ( $c > 1$ ) is equal to 1 unit<sup>2</sup>. Find the value of the constant.

$x$ -intersects:

$$6x^2 - 6x = 0$$

$$6x(x-1) = 0$$

$$x = 0 \text{ or } x = 1$$



$$\int_0^1 (6x^2 - 6x) dx = 1$$

$$\left[ 2x^3 - 3x^2 \right]_0^1 = 1$$

$$(2c^3 - 3c^2) - (2 - 3) = 1$$

$$2c^3 - 3c^2 + 1 = 1$$

$$2c^3 - 3c^2 = 0$$

$$c^2(2c - 3) = 0$$

$$c \neq 0 \text{ (N.A. as } c > 1) \text{ or } c = \frac{3}{2}$$

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$$(c) \int_1^3 (-e^{-x^4} + 2) dx$$

(3 marks)

(Correct anti-diff)

$$= \left[ -\frac{1}{4}e^{-x^4} + 2x \right]_1^3$$

(Correct substitution)

$$= \left( -\frac{1}{4}e^{-81} + 6 \right) - \left( -\frac{1}{4}e^{-1} + 2 \right)$$

(Final answer)

$$= -\frac{1}{4}e^{-81} + \frac{17}{4} + 8$$

$$(d) \frac{d}{dt} \int_{x^2}^{\frac{4}{3}x^2 - 1} \frac{xp}{x^2 - 1} dt$$

(3 marks)

$$= \frac{2}{3} \frac{d}{dt} (x^2 - 1) \times 2x$$

$$= \frac{4x}{3} \frac{d}{dt} (x^2 - 1)$$

$$= \frac{4x}{3} \cdot 2x \cdot \frac{dx}{dt}$$

$$(e) \int_2^3 \frac{d}{dx} \left( \frac{xx}{x^2 + 1} \right) dx$$

(3 marks)

$$= \left[ \frac{x^2}{x^2 + 1} \right]_2^3$$

$$= \frac{9}{8} - \frac{4}{5}$$

$$= \frac{10}{40} - \frac{32}{40}$$

$$= -\frac{22}{40}$$

$$= -1.1$$

Question 3

(3 marks)

The derivative of  $f(x)$  is given by  $f'(x) = 2e^{2x} + 3x^2$ . Given that  $f(1) = 4 + e^2$ , find an expression for  $f(x)$ .

$$f(x) = \frac{2e^{2x}}{2} + \frac{3x^3}{3} + C$$

$$4 + e^2 = e^{2(1)} + (1)^3 + C$$

$$4 + e^2 = e^2 + 1 + C$$

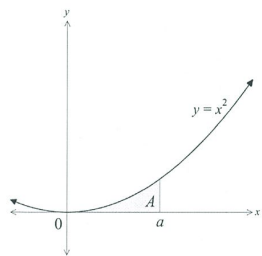
$$\leftarrow 3 = C$$

$$\therefore f(x) = e^{2x} + x^3 + 3$$

Question 4

(3 marks)

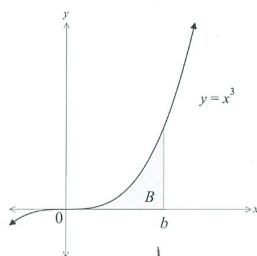
The area labelled B is two times the area labelled A. Express  $b$  in terms of  $a$ .



$$\text{area } A = \int_0^a x^2 dx$$

$$= \left[ \frac{1}{3} x^3 \right]_0^a$$

$$A = \frac{1}{3} a^3$$



$$\text{area } B = \int_0^b x^3 dx$$

$$= \left[ \frac{1}{4} x^4 \right]_0^b$$

$$B = \frac{1}{4} b^4$$

but  $B = 2A$

$$\therefore \frac{1}{4} b^4 = \frac{2}{3} a^3$$

$$\therefore b^4 = \frac{8a^3}{3}$$

$$b = \sqrt[4]{\frac{8}{3}} a^{\frac{3}{4}}$$

Question 10

(4 marks)

Show that  $\int_1^2 \left( \frac{6x+4}{\sqrt{x}} \right) dx = 16\sqrt{2} - 12$ .

(Show sufficient work out please and use exact values)

$$\int_1^2 \frac{6x+4}{\sqrt{x}} dx = \int_1^2 \left( 6\sqrt{x} + \frac{4}{\sqrt{x}} \right) dx$$

$$= \int_1^2 \left( 6x^{\frac{1}{2}} + 4x^{-\frac{1}{2}} \right) dx$$

$$= \left( \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{4(x)^{\frac{1}{2}}}{\frac{1}{2}} \right) \Big|_1^2$$

$$= \left[ 4x^{\frac{3}{2}} + 8\sqrt{x} \right]_1^2 = (4 \cdot 2^{\frac{3}{2}} + 8\sqrt{2}) - (4 + 8)$$

$$= 16\sqrt{2} - 12$$

Question 11

(3 marks)

The area under the curve  $f(x) = 4e^{kx}$  over the domain  $0 \leq x \leq 10$  is  $\frac{40}{3}(-e^{-3} + 1)$ .

Determine the value of  $k$ .

$$\therefore \int_0^{10} 4e^{kx} dx = \frac{40}{3}(-e^{-3} + 1)$$

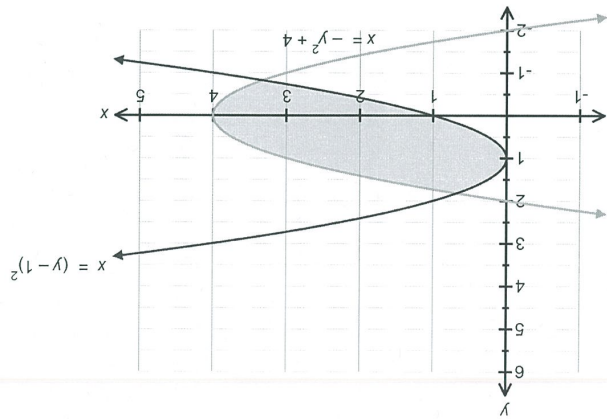
$$\left[ \frac{4e^{kx}}{k} \right]_0^{10} = \frac{40}{3}(-e^{-3} + 1)$$

$$\therefore \text{solve } \left( \frac{4e^{10k}}{k} - \frac{4}{k} \right) = \frac{40}{3}(-e^{-3} + 1), k$$

$$\therefore k = -0.3$$

Question 9

Calculate the shaded area shown below, showing all relevant working.



(6 marks)

for intersection:

$$\text{solve } (y-1)^2 = -y^2 + 4, y$$

$$y = 1.82288 \text{ or } y = -0.822876$$

$$x = 3.32$$

$$\therefore y \approx 1.82$$

$$x = 0.68$$

$$\text{Area} = \int_{-0.83}^{1.83} [(-y^2 + 4) - (y-1)^2] dy$$

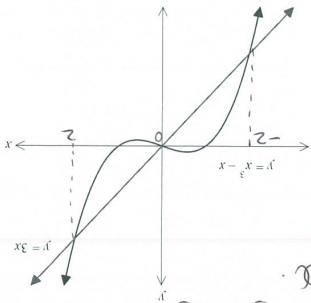
$$= \left[ -\frac{y^3}{3} + 4y - (y-1)^3 \right]_{-0.83}^{1.83}$$

$$= 6.17326$$

$$\approx 6.173 \text{ units}^2$$

Question 5

Find the exact area bound by the two curves shown below. (symmetry)



$$\text{Area} = 2 \int_{-2}^2 [3x - (x^3 - x)] dx$$

$$= 2 \int_{-2}^2 (4x - x^3) dx$$

$$= 2 \left[ 2x^2 - \frac{x^4}{4} \right]_{-2}^2$$

$$= 2(8 - 4) = 8 \text{ units}^2$$

(3 marks)

Question 6

Determine the function  $y$  given that  $\frac{d^2y}{dx^2} = 3e^x + 2$  and  $\frac{dy}{dx} = 5$  when  $x = 0$  and  $y = 3e^2 + 15$  when  $x = 2$ .

(4 marks)

$$\frac{d^2y}{dx^2} = 3e^x + 2$$

$$\frac{dy}{dx} = 3e^x + 2x + C$$

$$y = 3e^x + x^2 + 2x + d$$

$$3e^2 + 15 = 3e^2 + 2^2 + 2(2) + d$$

$$d = 7$$

$$\therefore y = 3e^x + x^2 + 2x + 7$$

when  $x = 2$ .

## Question 7

(6 marks)

The gradient function of  $f(x)$  is given by  $f'(x) = ax^2 + b$ . Determine the values of  $a$  and  $b$  if  $f'(-2) = 28$ ,  $f(0) = 1$  and  $f(1) = 7$ .

$$f'(-2) = 4a + b$$

$$\therefore 28 = 4a + b$$

$$f(x) = \frac{ax^3}{3} + bx + C$$

$$1 = C$$

$$f(1) = 7$$

$$7 = \frac{a}{3} + b + 1$$

$$\therefore b = \frac{a}{3} + 6$$

$$\text{and } 28 = 4a + b$$

$$22 = \frac{11a}{3}$$

$$\frac{66}{11} = a$$

$$\therefore a = 6$$

$$28 = 4a + b$$

$$28 = 24 + b$$

$$b = 4$$

METHODS YEAR 12 Test 3 2017  
Anti-DifferentiationName: Sol Key

Resource Assumed

Time: 25 minutes

Marks: / 25

CAS calculator + A4 page 1 side of notes

## Question 8

(8 marks)

Sam has invested \$A in a fund which compounds her investment continuously at a rate of  $k\%$  per annum.

The rate of change of her investment is given by  $\frac{dV}{dt} = k(Ae^{kt})$  where  $V$  is the value of her investment in dollars and  $t$  is the time in years.

The net change in the value of her investment in the first 10 years is \$12 331.78.

The net change in the value of her investment in the next 10 years is \$22 469.97.

(a) Determine the values of  $A$  and  $k$ .

(6 marks)

$$\int_0^{10} k A e^{kt} dt = 12\,331.78$$

$$\int_{10}^{20} k A e^{kt} dt = 22\,469.97$$

$$\therefore [A e^{kt}]_0^{10} = 12\,331.78 \quad \text{and} \quad [A e^{kt}]_{10}^{20} = 22\,469.97$$

$$\textcircled{1} \quad A e^{10k} - A = 12\,331.78$$

$$\textcircled{2} \quad A e^{20k} - A e^{10k} = 22\,469.97$$

Solve ① and ②

$$\therefore k = 0.06$$

and

$$A = 15\,000$$

(b) Hence determine the function that defines the value of her investment.

(2 marks)

$$V(t) = 15\,000 e^{0.06t}$$