PERTH MODERN SCHOOL

UNIT 3CD MAS – 2014

TEST 1

POLAR COORDINATES, COMPLEX NUMBERS & VECTORS

NAME: ______________________________ **DATE:** **Thurs. 13th Feb.**

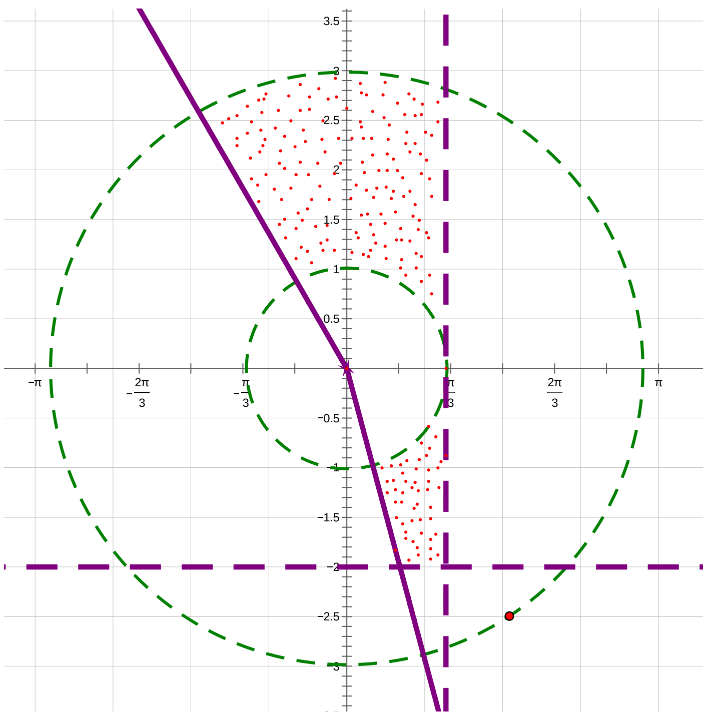
Total: **43 marks** **Time:** **45 min.**

1. Z is a complex number. Sketch the region given by [5]

$Re(Z) < 1$ and $Im(Z) > -2$ ☑

and $1 < |Z| < 3$ ☑

and $-\frac{5\pi}{12} \leq Arg Z \leq \frac{2\pi}{3}$ ☑



☑ ☑ for correct shading

2. Express $Z = -1 - \sqrt{3}\, i$ in polar form. [2]

$|z| = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2}$ $\tan\theta = \sqrt{3}$ *Quadrant* 3

$= 2$ ☑ $\therefore \theta = -\frac{2\pi}{3}$

$\therefore\ Z = 2\text{cis}\left(-\frac{2\pi}{3}\right)$ ☑

3. If $Z_1 = 5\text{cis}\frac{\pi}{6}$ and $Z_2 = 2\text{cis}\frac{\pi}{12}$, then prove $Z_1Z_2 = 5\sqrt{2}\,(1 + i)$ [4]

$Z_1Z_2 = 5\text{cis}\frac{\pi}{6} \times 2\text{cis}\frac{\pi}{12}$

$= 10\text{cis}\left(\frac{\pi}{6} + \frac{\pi}{12}\right)$

$= 10\text{cis}\frac{\pi}{4}$ ☑ ☑

$= 10\left(\cos\frac{\pi}{4} + \mathbf{i}\sin\frac{\pi}{4}\right)$

☑

$= 10\left(\frac{1}{\sqrt{2}} + \frac{\mathbf{i}}{\sqrt{2}}\right)$

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$= \frac{10}{\sqrt{2}}(1 + i)$ $\therefore Z_1Z_2 = 5\sqrt{2}\,(1 + i)$

4. Find Z if $Z\overline{Z} + 2Z = \frac{1}{4} + i$ [6]

Let $Z = a + bi$ ☑

$Z\overline{Z} + 2Z = (a + bi)$

i.e. $(\mathbf{a} + \mathbf{b}\mathbf{i})(\mathbf{a} - \mathbf{b}\mathbf{i}) + 2(\mathbf{a} + \mathbf{b}\mathbf{i}) = \frac{1}{4} + \mathbf{i}$

i.e. $\mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a} + 2\mathbf{b}\mathbf{i} = \frac{1}{4} + \mathbf{i}$ ☑

Compare real, imaginary

$a^2 + b^2 + 2a = \frac{1}{4}$ ① ☑

$2b = 1$ ②

$\therefore$ $\boldsymbol{b = \frac{1}{2}}$ ☑

$sub\ b = \frac{1}{2}\ in$ ①

$a^2 + \left(\frac{1}{2}\right)^2 + 2a = \frac{1}{4}$

$a^2 + 2a = 0$

$a(a + 2) = 0$

$\therefore$ $\boldsymbol{a = 0}$ $\boldsymbol{a = -2}$ ☑

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Hence $\boldsymbol{Z = -2 + \frac{i}{2}}$ $\boldsymbol{z = \frac{i}{2}}$

5.

Consider the 3D position vectors [8]

$\overrightarrow{OA} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ *and* $\overrightarrow{OB} = 5\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

Determine:

a) the length of $\overrightarrow{AB}$.

$\overrightarrow{AB} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$ ☑

$$\left|\overrightarrow{AB}\right| = \sqrt{(3)^2 + (4)^2 + (-7)^2}$$

$$= \sqrt{74}$$ ☑

b) *∠AOB to the nearest degree.*

$$\cos \angle AOB = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{\left|\overrightarrow{OA}\right| \left|\overrightarrow{OB}\right|}$$

$$= \frac{-5}{\sqrt{29}\sqrt{35}}$$ ☑

$\therefore \quad \angle AOB = 99^\circ$ ☑

c) the vector equation of the line, in parametric form, through the points A and B.

$$r = \overrightarrow{OA} + \lambda\overrightarrow{AB}$$

$$\text{i.e. } \mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix}$$ ☑

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$$\text{i.e. } \mathbf{r} = (2 + 3\lambda)\mathbf{i} + (-3 + 4\lambda)\mathbf{j} + (4 - 7\lambda)\mathbf{k}$$

or $x = 2 + 3\lambda$ ☑

$y = -3 + 4\lambda$ ☑

6. $z = 4 - 7\lambda$ [8]

A has the rectangular coordinates $\left(-1, \sqrt{3}\right)$ *and B has polar coordinates* $\left(4, \frac{5\pi}{4}\right)$.

a) What are the exact polar coordinates of A? (1)

$(2 \;\; 120^\circ)$ ☑

or $\left(2 \;\; \frac{2\pi}{3}\right)$

b) What are the exact rectangular coordinates of B? (2)

$x = 4\cos\left(\frac{5\pi}{4}\right) \Rightarrow -2\sqrt{2}$ ☑

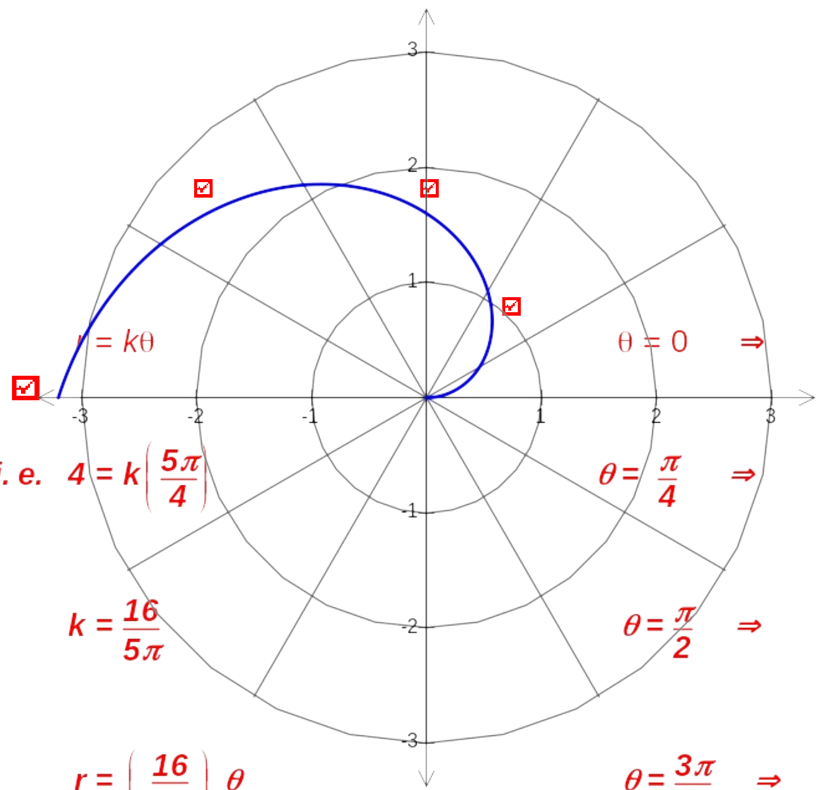
$y = 4\sin\left(\frac{5\pi}{4}\right) \Rightarrow -2\sqrt{2}$ *or* $(-2\sqrt{2} \;\; -2\sqrt{2})$ ☑

c) The graph of the polar equation $\mathbf{r} = \mathbf{k}\theta$ passes through the point B.

If $\mathbf{k} > 0$, determine the value of **k.**

Then, on the axes below, sketch the graph of $\mathbf{r} = \mathbf{k}\theta$ *for* $0 \le \theta \le \pi$. ,

showing important features. $\frac{1}{2}$ *mark for each important feature* (5)



$r = k\theta$

$\theta = 0 \Rightarrow r = 0$

i. e. $4 = k\left(\frac{5\pi}{4}\right)$ $\qquad \theta = \frac{\pi}{4} \Rightarrow r = \frac{4}{5}$

$\therefore \quad k = \frac{16}{5\pi}$ $\qquad \theta = \frac{\pi}{2} \Rightarrow r = \frac{8}{5}$ *or* 1.6

$\therefore \quad r = \left(\frac{16}{5\pi}\right)\theta$ $\qquad \theta = \frac{3\pi}{4} \Rightarrow r = \frac{12}{5}$ *or* 2.4

and $\quad \theta = \pi \Rightarrow r = \frac{16}{5}$ *or* 3.2

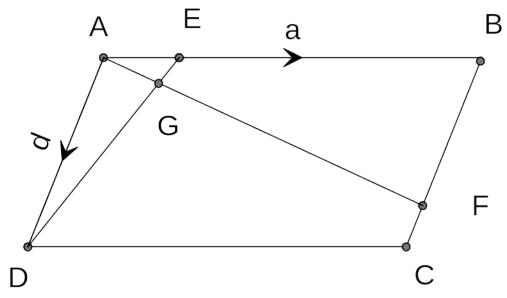
7. [10]

ABCD is a parallelogram with points E and F such that $\overrightarrow{AE} : \overrightarrow{EB} = 1 : 4$ *and* $\overrightarrow{BF} : \overrightarrow{FC} = 3 : 1$.

$\overrightarrow{ED}$ *and* $\overrightarrow{AF}$ *intersect each other at G.*

Let $\overrightarrow{AB} = \boldsymbol{a}$ ***and*** $\overrightarrow{AD} = \boldsymbol{d}$.

a) Complete the diagram below with the information given above. (2)



b) Determine the ratios in which $\overrightarrow{AF}$ *and* $\overrightarrow{ED}$ intersect each other, if the intersection point is at G. (8)

$$\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF}$$

$$= a + \frac{3}{4}\overrightarrow{BC}$$

$$= a + \frac{3}{4}d$$

$$\overrightarrow{AG} = h\overrightarrow{AF}$$

$$= h\left(a + \frac{3}{4}d\right)$$

$$\overrightarrow{ED} = \overrightarrow{EA} + \overrightarrow{AD}$$

$$= -\frac{1}{5}a + d$$

$$\overrightarrow{EG} = k\overrightarrow{ED}$$

$$= k\left(-\frac{1}{5}a + d\right)$$

$$\overrightarrow{AG} = \overrightarrow{AE} + \overrightarrow{EG}$$

$$\therefore\ ha + \frac{3}{4} = \frac{1}{5}a - \frac{1}{5}ka + kd$$

a : $h = \frac{1}{5} - \frac{1}{5}k$ ① *and* d : $k = \frac{3}{4}h$ ②

solving $h = \frac{4}{17}$, $k = \frac{3}{17}$

$\therefore$ *G divides* $\overrightarrow{AF}$ *in the ratio* 4 : 13

and G divides $\overrightarrow{ED}$ *in the ratio* 3 : 14