ALL SAINTS' COLLEGE

Semester Two Examination, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section One: Calculator-free

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Booklet.

Additional working space

Question number: __________

Additional working space

Question number: __________

Section One: Calculator-free **35% (52 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(6 marks)**

A particle leaves the origin when $t=1$ and moves in a straight line with velocity at any time t seconds, where $t \geq 1$, given by

$$v(t)=\frac{t^2}{4}+\frac{4}{t}-\frac{7}{4}\ \text{ms}^{-1}$$

(a) Determine the time when the acceleration of the particle is zero. (2 marks)

Solution
$a(t)=\frac{dv}{dt}=\frac{t}{2}-\frac{4}{t^2}$ $\frac{t}{2}-\frac{4}{t^2}=0 \Rightarrow t^3=8 \Rightarrow t=2\text{s}$
Specific behaviours
✓ differentiates velocity   ✓ solves acceleration equal to zero

(b) Determine the exact displacement of the particle from the origin when $t=4$. (4 marks)

Solution
$x(t)=\int v(t)\,dt=\frac{t^3}{12}+4\ln t-\frac{7t}{4}+c$ $x(1)=0 \Rightarrow \frac{1}{12}+0-\frac{7}{4}+c=0 \Rightarrow c=\frac{5}{3}$ $x(4)=\frac{4^3}{12}+4\ln 4-\frac{7\times 4}{4}+\frac{5}{3}=4\ln 4\,\text{m}$
Specific behaviours
✓ integrates velocity   ✓ evaluates constant   ✓ substitutes time

Question 2 **(7 marks)**

(a) Calculate $f'(0)$ when $f(x)=e^{2x}(1+5x)^3$. (3 marks)

Solution
$f'(x)=2e^{2x}\times(1+5x)^3+e^{2x}\times 3(5)(1+5x)^2$ $f'(0)=2\times 1+1\times 15=17$
Specific behaviours
✓ uses product rule and obtains $u'v$ correctly   ✓ uses chain rule and obtains uv' correctly   ✓ substitutes to determine $f'(0)$

(b) Determine $\frac{d}{dx}\int_x^5 \sqrt{t^2+1}\,dt$. (2 marks)

Solution
$y=-\int_5^x \sqrt{t^2+1}\,dt \frac{dy}{dx}=-\sqrt{x^2+1}$
Specific behaviours
✓ swaps limits correctly   ✓ differentiates

(c) Given $f'(x)=(1-2x)^4$ and $f(1)=-1$, determine $f(x)$. (2 marks)

Solution
$f(x)=\frac{(1-2x)^5}{(-2)(5)}+c f(1)=\frac{1}{10}+c=-1\Rightarrow c=\frac{-11}{10}$ $f(x)=\frac{-(1-2x)^5}{10}-\frac{11}{10}$
Specific behaviours
✓ antidifferentiates   ✓ evaluates constant and writes complete function

Question 7 **(8 marks)**

The discrete random variable X is defined by $P(X=x)=k\log x$ for $x=2, 5$ and 10.

(a) Determine the value of k. (3 marks)

Solution
$k\log 2+k\log 5+k\log 10=1k\log(2\times 5\times 10)=1$ $k=\frac{1}{\log 100}=\frac{1}{2\log 10}=\frac{1}{2}$
Specific behaviours
✓ substitutes and sums terms to 1   ✓ uses log laws to add logs   ✓ simplifies and states k

(b) Determine $P(X=2 \mid X<10\text{¿}$. (2 marks)

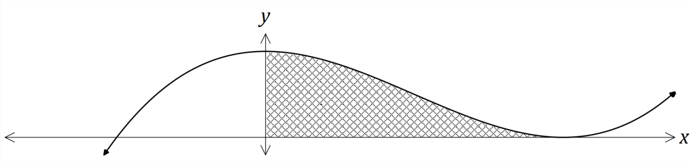
Solution
$P(X<10)=1-\frac{1}{2}\log 10=\frac{1}{2}$ $P=\frac{1}{2}\log 2\div\frac{1}{2}=\log 2$
Specific behaviours
✓ calculates $P(X<10)$ ✓ calculates conditional probability

(c) $E(X)=a(b+\log\sqrt{c})$, where the constants a, b and c are prime numbers. Determine the values of a, b and c. (3 marks)

Solution
$E(X)=2\times\frac{1}{2}\log 2+5\times\frac{1}{2}\log 5+10\times\frac{1}{2}\log 10$ $\text{¿}\log 2+\log 5+\frac{3}{2}\log 5+5\text{¿}\log 10+3\log\sqrt{5}+5$ $\text{¿}6+3\log\sqrt{5}=3(2+\log\sqrt{5})$ $a=3, b=2, c=5$
Specific behaviours
✓ expresses $E(X)$ ✓ simplifies and splits $\log 5$ term   ✓ simplifies to determine values of a, b and c

Question 6 **(8 marks)**

The diagram below shows the curve $y = x^3 - 3x^2 + k$, where k is a constant. The curve has a turning point on the x-axis.



(a) Determine the value of k. (3 marks)

Solution

$\frac{dy}{dx} = 3x^2 - 6x3x(x-2) = 0 \Rightarrow x = 0, x = 2$

$(2,0) \Rightarrow 8 - 12 + k = 0 \Rightarrow k = 4$

Specific behaviours

✓ differentiates
✓ solves derivative equal to zero
✓ determines k

(b) Determine the set of values of x for which $\frac{dy}{dx}$ is increasing. (2 marks)

Solution

$\frac{d^2y}{dx^2} = 6x - 6$

$6x - 6 = 0 \Rightarrow x = 1 \Rightarrow \frac{dy}{dx}$ is increasing for $x > 1$

Specific behaviours

✓ determines where 2nd derivative is zero
✓ states inequality, not including 1

(c) Calculate the area of the shaded region. (3 marks)

Solution

$A = \int_0^2 x^3 - 3x^2 + 4\,dx$

$¿\left[\frac{x^4}{4} - x^3 + 4x\right]_0^2 ¿ 4$ sq units

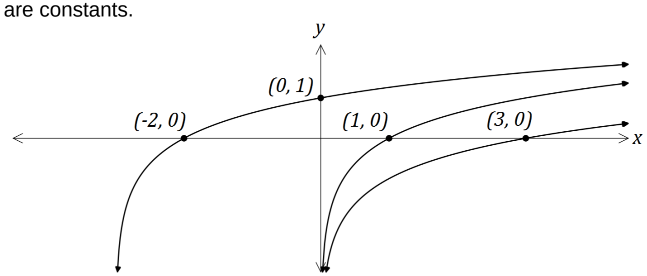
Specific behaviours

✓ writes integral
✓ antidifferentiates
✓ evaluates

Question 3 **(7 marks)**

(a) The function f is defined by $f(x) = \log_a x$, $x > 0$, where a is a constant, $a > 1$.

The graphs shown below have equations $y = f(x)$, $y = f(x+b)$ and $y = f(x) + c$, where b and c are constants.



Determine the values of the constants a, b and c. (4 marks)

Solution

$f(x+b)$ is only function that could pass through $(-2, 0)$.

Hence $0 = f(-2+b)$ and so $b = 3$.

Using $(0,1)$, $1 = \log_a(0+3) \Rightarrow a = 3$

$\log_3 1 = 0$ and so $f(x)$ must pass through $(1, 0)$

$f(x) + c$ passes through $(3,0) \Rightarrow 0 = \log_3 3 + c = 0$ and so $c = -1$

Specific behaviours

✓ starts by using $f(x+b)$ and $(-2, 0)$
✓ determines b
✓ determines

(b) Determine

(i) the equation of the asymptote of the graph of $y = \log_e(x-3) - 2$. (1 mark)

Solution

$x = 3$

Specific behaviours

✓ writes asymptote as

(ii) the coordinates of the y-intercept of the graph of $y = \log_2(x+8) - 5$. (2 marks)

Solution

$\log_2(8) - 5 = 3\log_2 2 - 5 = -2$

At $(0, -2)$

Specific behaviours

✓ substitutes and simplifies
✓ writes using coordinates

Question 4 **(8 marks)**

A curve has equation $y=2x^5-5x^4+10$.

(a) Point A lies on the curve at $(-1,3$¿. Use the increments formula $\delta y \approx \frac{dy}{dx} \times \delta x$ to estimate the y-coordinate of point B that has an x-coordinate of -0.99.

(4 marks)

Solution
$\frac{dy}{dx}=10x^4-20x^3 x=-1 \Rightarrow \frac{dy}{dx}=10+20=30$ $\delta y \approx 30 \times 0.01 \approx 0.3$ Estimate for y-coord is $3+0.3=3.3$
Specific behaviours
✓ differentiates   ✓ substitutes to get gradient   ✓ finds change in y using increments   ✓ states new y-coordinate

(b) Point C also lies on the curve, at $(2,-6)$. Verify that C is either a minimum or maximum point of the curve.

(4 marks)

Solution
$x=2 \Rightarrow \frac{dy}{dx}=160-160=0$ Hence C is a stationary point as $\frac{dy}{dx}=0$ $\frac{d^2y}{dx^2}=40x^3-60x^2$ $x=2 \Rightarrow \frac{d^2y}{dx^2}=320-240=80$ Hence C is a minimum, as $\frac{d^2y}{dx^2}>0$
Specific behaviours
✓ substitutes into first derivative   ✓ concludes that C is a stationary point

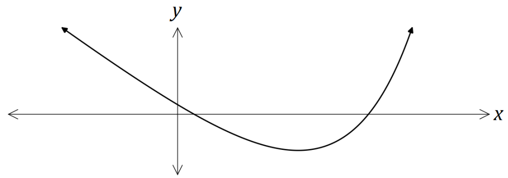
✓ obtains second derivative

Question 5 **(8 marks)**

(a) Determine the coordinates of the root of the graph of $y=\log_3(2x+1)-2$. (3 marks)

Solution
$0=\log_3(2x+1)-2\log_3(2x+1)=2 2x+1=3^2$ $x=4$ At $(4,0)$
Specific behaviours
✓ substitutes and simplifies   ✓ writes as exponential equation   ✓ evaluates x and writes as coordinates

(b) The graph of $y=e^{2x-1}-4x$ has a single stationary point, as shown on the graph below.



Determine the exact coordinates of the stationary point. (5 marks)

Solution
$\frac{dy}{dx}=2e^{2x-1}-4 \frac{dy}{dx}=0 \Rightarrow e^{2x-1}=2 2x-1=\ln 2$ $x=\frac{1}{2}+\frac{1}{2}\ln 2 y=e^{\ln 2}-4\left(\frac{1}{2}+\frac{1}{2}\ln 2\right)=2-2-2\ln 2$ Stationary point at $\left(\frac{1}{2}+\frac{1}{2}\ln 2,-2\ln 2\right)$
Specific behaviours
✓ obtains first derivative   ✓ equates to 0 and simplifies   ✓ takes logs of both sides   ✓ solves for x ✓ substitutes to find y, simplifying