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SEMESTER TWO

REVISION 1

MATHEMATICS METHODS

UNITS 3-4

2016

SOLUTIONS

SECTION ONE

1. (8 marks)

(a) $\int (2x+4)^6\,dx = \frac{(2x+4)^7}{7\times 2} + c = \frac{(2x+4)^7}{14} + c$

(b) $\int_{\pi/4}^{\pi/2} 2\sin(x) - \cos(x)\,dx$

$= \left[-\cos(x) - \sin(x)\right]_{\pi/4}^{\pi/2}$

$= -\left(\left(\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\right) - \left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right)\right)$

$= -1 + \sqrt{2}$

(c) $\int \left(x^4 + e^{2x} + \frac{2}{x}\right) dx = \frac{x^5}{5} + \frac{e^{2x}}{2} + 2\ln(x) + c$ ✓✓ -1/error

2. (16 marks)

(a) (i) $f(x) = \ln\left(\frac{x^2-3}{1+x}\right) = \ln(x^2-3) - \ln(1+x)$

$f'(x) = \frac{2x}{(x^2-3)} - \frac{1}{(1+x)}$

(ii) $g(x) = \frac{e^{\sin(x)}}{\cos(x)}$

$g'(x) = \frac{\left(e^{\sin(x)}\cos(x)\right)\cos(x) - (-\sin(x))e^{\sin(x)}}{(\cos(x))^2}$

$g'(x) = \frac{e^{\sin(x)}\left(\cos^2(x) + \sin(x)\right)}{\cos^2(x)}$

(iii) $h(x) = e^x \times \ln(x^2) = 2e^x \times \ln(x)$

$h'(x) = 2\left(e^x \times \ln(x) + \frac{e^x}{x}\right)$ ✓✓

$h'(x) = 2e^x\left(\ln(x) + \frac{1}{x}\right)$

(b) (i) Given $g(x)=\sqrt{sin(x)}$ show that $g'(x)=\frac{cos(x)}{2\sqrt{sin(x)}}$.

$$g'(x)=\frac{1}{2}(sin(x))^{\frac{-1}{2}}\times cos(x)$$

$$g'(x)=\frac{cos(x)}{2\sqrt{sin(x)}}$$

(ii)

$$\int -\frac{3cos(x)}{\sqrt{sin(x)}}dx=-3\times 2\int\frac{cos(x)}{2\sqrt{sin(x)}}dx$$

$$=-6\int\frac{cos(x)}{2\sqrt{sin(x)}}dx$$

$$=-6\sqrt{sin(x)}+c$$

(c) Given $\int_0^{10} f(x)dx=-6.4$ and $\int_4^{10} f(x)dx=2.3$.

$$2\int_0^4 \;=2\int_0^{10} \;-2\int_4^{10}$$

$$2\int_0^4(1-f(x))dx=2\left(\int_0^4 1dx-\int_0^4 f(x)dx\right)$$

$$=2\left([x]_0^4-\left(\int_0^{10} f(x)dx-\int_4^{10} f(x)dx\right)\right)$$

$$=2(4-(-6.4-2.3))$$

$$=2(4+8.7)$$

$$=2\times 12.7$$

$$=25.4$$

(d) $\frac{dr}{d\theta}=\frac{dr}{dt}\times\frac{dt}{dx}\times\frac{dx}{d\theta}$ $\qquad t=4x=4cos(\theta)$

$$\frac{dr}{d\theta}=\frac{1}{2}t^{-1/2}\times 4\times(-sin(\theta))$$

$$\frac{dr}{d\theta}=-\frac{2sin(\theta)}{\sqrt{t}}$$

$$\frac{dr}{d\theta}=-\frac{2sin(\theta)}{\sqrt{4cos(\theta)}}$$

$$\frac{dr}{d\theta}=-\frac{sin(\theta)}{\sqrt{cos(\theta)}}$$

3. (6 marks)

(a) $$\frac{log_{10}\left(4\times3^2\right)-log_{10}(3\times6)-3log_{10}2}{-2log_{10}2}$$

$$=\frac{log_{10}\left(\frac{4\times3^2}{3\times6\times8}\right)}{-2log_{10}2}$$

$$=\frac{log_{10}\left(\frac{1}{4}\right)}{-2log_{10}2}$$

$$=\frac{-2log_{10}2}{-2log_{10}2}$$

$$=1$$

(b) $(log_3(x)-1)(ln(x)-1)=0$

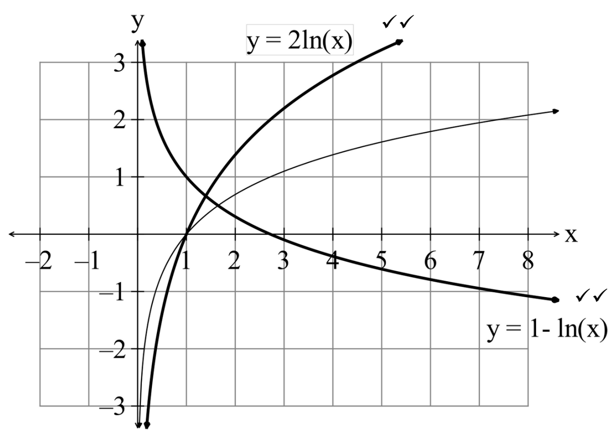
$log_3(x)-1=0 \quad or \quad ln(x)-1=0$

$log_3(x)=1 \quad or \quad ln(x)=1$

$x=3 \quad or \quad x=e$

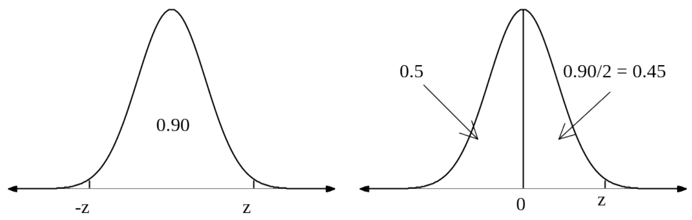
4. (6 marks)

(a) (i)



(ii) $f(x)=ln(x) \Rightarrow f^{-1}(x)=e^x \quad for \ x\in Re$

(d)



P(X< z) = 0.95
z = 1.645

Use $p=0.5$ as the maximum value as p is unknown.

So with $p=0.5$ $\quad sd=\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}=\sqrt{\frac{0.25}{n}}$

$E=z\times s$ but $E=0.05$

Therefore

$0.05=1.645\times\sqrt{\frac{0.25}{n}}$

$n=270.6$

You need to survey 271 people to have an error margin of 5% at a confidence level of 90%

END OF SECTION TWO

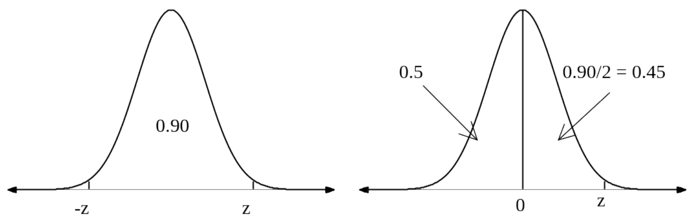
(c) (i) $\hat{p} = \frac{180}{200} = 0.9$

(ii) $\hat{p} = 0.9$

$sd_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.9 \times 0.1}{200}}$

$sd_{\hat{p}} = 0.0212132$

(iii)



$P(X < z) = 0.95$

$z = 1.645$

$sd_{\hat{p}} = 0.0212132$

$E = z \times s = 1.645 \times 0.0212132$

$E \approx 0.0349$

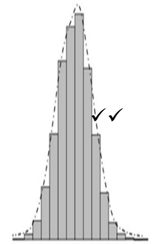
The 90% confidence limit are 0.9 ± 0.03 *i.e.* $(0.90, 0.96)$

5. (16 marks)

(a) (i) Not a probability density functions as you cannot have negative probabilities.

(ii) Is a probability density functions as the probabilities add to one.

(b) The shape will be a tightly clustered histogram, close to a normal curve.



(c) (i) $\int_1^e \frac{1}{x} dx = \left[ln(x)\right]_1^e = ln(e) - ln(1) = 1 - 0 = 1$

(ii) $P(1 \le x \le 2) = \int_1^2 \frac{1}{x} dx = ln(2) - ln(1) = ln(2)$

(d) $F(x) = \int_0^x \frac{x}{2} dx = \frac{1}{4}\left[x^2\right]_0^x = \frac{x^2}{4}$

(e) $p(x) = \begin{cases} 0.2 & \text{for } 5 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$

$E(x) = \int_a^b x \times p(x) dx$

$E(x) = \int_5^{10} x \times 0.2\, dx = 0.2\left[\frac{x^2}{2}\right]_5^{10} = 0.1(100 - 25) = 7.5$

$Var(x) = \int_a^b p(x)(x - \mu)^2 dx$

$Var(x) = \int_5^{10} 0.2(x - 7.5)^2 dx$ ✓✓

END OF SECTION ONE

SECTION TWO

6. (7 marks)

(a) $A=\int_0^a e^x dx=\left[e^x\right]_0^a=e^a-e^0=e^a-1$

(b) $A=e^a-1$

$\frac{dA}{da}=e^a$

$\frac{\delta A}{\delta a}\approx\frac{dA}{da}$

$\delta A\approx\frac{dA}{da}\times\delta a$

$At \quad a=3,\ \delta a=0.1$

$\delta A\approx e^3\times 0.1$

$\delta A\approx 2.0086$

7. (6 marks)

(a) $v=10t-1ms^{-1}$.

$a=10\ ms^{-2}$

$x=\int(10t-1)dt$

$x=5t^2-t+c$

$At\ t=0, x=3$

$x=5t^2-t+3$

(b) Changes direction when $v=0$ i.e. at t = 0.1

(c) $At\ t=0,\ x=3$

$At\ t=0.1,\ x=0.05-\ 0.1+3=\ 2.95$

$At\ t=5,\ x=123$

Distance travelled $=123-\ 2.95+0.05=120.1\ m$

$p=0.2\ q=0.98\Rightarrow\ np=100\times 0.2=20>5$

$nq=100\times 0.8=80>5$ so can use normal distribution.

Mean $=p=0.2$

$sd_{\hat{p}}=\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}=\sqrt{\frac{0.2\times 0.8}{100}}$

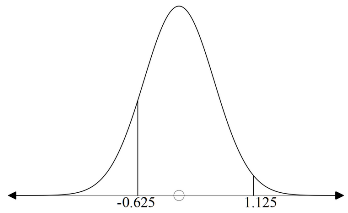
$sd_{\hat{p}}=0.04$

Standardised score (using 17.5 to 24.5)

$z=\frac{X-\mu}{\sigma}$

$z_{0.075}=\frac{\frac{17.5}{100}-0.2}{0.04}=-0.625$

$z_{0.145}=\frac{\frac{24.5}{100}-0.2}{0.04}=1.125$



$P\left(-0.625\le z\le 1.125\right)=0.6037199538$

The probability that between 18 and 24 of the wine tasters should not drive is 0.604.

14. (6 marks)

(a) (i)

x	0	1	2	3	4	5	≥ 6 ✓✓
$P(X=x)$	0.1465	0.366	0.3945	0.068	0.015	0.0075	0.0025

(ii) $0.366+0.3945 = 0.7605$ ✓ ✓

(iii) $E(x)=\sum x_i \times P(x_i)$

$=0+1\times0.366+2\times0.3945+3\times0.068+4\times0.015+5\times0.0075+6\times0.0025$

$E(x)=1.4715$

15. (7 marks)

(a) P(a given battery will last at least 400 hours) $= 0.006209665326 \approx 0.0062$ ✓ ✓

(b) P(a given battery will last between 320 and 380 hours) $= 0.8663855975 \approx 0.866$ ✓ ✓

(c) P(Jenny's battery will run out in a 3 hour exam if it has been used for 340 hours)

$$=\frac{P(340 \leq x \leq 343)}{P(x \geq 340)}=\frac{0.0546318101}{0.6914624613}=0.07900907592$$

Therefore P(Jenny's battery will run out in a 3 hour exam if it has been used for 340 hours already) ≈ 0.08

16. (6 marks)

(a) P(there are 5 girls and 5 boys) = 0.24609375 ✓ ✓

(b) P(there are no more than 4 boys) = 0.376953125 ✓ ✓

(c) BB(8 more births) Binomial with n = 8

P(exactly 4 girls) = 0.2734375

17. (22 marks)

(a) (i) A biased sample is one in which not every sample point has equal chance of being selected. ✓ ✓

(ii) A sample that is chosen in such a way that each point has an equal chance of being selected. The process requires each possible selection point being numbered and then the sample is generated using random numbers. ✓ ✓

(iii) A skewed distribution is a set of data where a large proportion of the data is clustered at one end. ✓ ✓

8. (7 marks)

(a) (i) $f(t)=\sqrt{sin(\pi t)}$

$f'(t)=\frac{1}{2}(sin(\pi t))^{\frac{-1}{2}}\pi cos(\pi t)$ ✓

$f'(t)=\frac{\pi cos(\pi t)}{2\sqrt{sin(\pi t)}}$

(ii) $\int_{1/6}^{1/2}\frac{\pi cos(\pi t)}{2\sqrt{sin(\pi t)}}dt=\left[\sqrt{sin(\pi t)}\right]_{1/6}^{1/2}$

$=\sqrt{sin\left(\frac{\pi}{2}\right)}-\sqrt{sin\left(\frac{\pi}{6}\right)}$

$=1-\frac{1}{\sqrt{2}}$

(b) $F(x)=\frac{d}{dx}\int_1^x\left(\frac{1}{t}\right)dt=\frac{1}{x}$

$\int_1^2 F(x)dx=\int_1^2\frac{1}{x}dx=\left[ln(x)\right]_1^2=ln(2)-ln(1)=ln(2)$

9. (8 marks)

(a) Turning points occur when $f'(x)=0$.

There are no points where this occurs so there are no turning points.

Likewise, there are no points where $f''(x)=0$, so there are no points of inflection.

(b) $y=f''(x)>0$ which suggests that the concavity is concave upwards for all x values. ✓✓

(c) $f(x)=ln(x)$, $f'(x)=\frac{1}{x}$, $f''(x)=\frac{-1}{x^2}$

10. (7 marks)

(a) 0.693 ✓✓ (= ln(2))

(b) (i) True as +C can have any value

(ii) No, this is bounded and has only one solution.

$\int_1^2 3x^2 dx = \left[x^3\right]_1^2 = 8 - 1 = 7$

(iii) True.

(iv) $ln(f(x)) = ln(3) + 2\,ln(x)$

Not valid as initially, x can be negative as it is squared.

Need $2\,ln|(x)|$ ✓✓

11. (8 marks)

(a) $t = 0, \quad P = 23$

$78.9 = 23e^{60(k)}$

$k = 0.02054478$

(b) $t = 66, \quad P = 89.25$ (million)

(c) Te actual population is smaller than 89.25 so the growth rate is slowing down (minimally!)

(d) $2010 \qquad 78.9 \times 10^6 \quad t = 0$

$2016 \qquad 87\,238\,973$

$P = 78.9$

$At \quad t = 6, \qquad 87\,238\,973 = 78.9 \times 10^6 e^{6k}$

$k = 0.01674499021$

$P = 78.9 \times 10^6 \times e^{0.01674499021 \times t}$

$P = 100 \times 10^6, t = ?$

$100 = 78.9 \times e^{0.01674499021 \times t}$

$t = 14.15$

The population is expected to reach 100 million just into 2025.

Mathematics Methods Units 3-4, 2016, Semester Two Solutions

12. (7 marks)

(a) Area $= \int_{0.05}^{1.47} \left(ln(x) - (e^x - 4)\right) = 1.68$ units^2

(b) (i) $P(0) = 22\,ln(3) = 24.169 \approx 24$

(ii) $100 = 22\,ln(t + 3)$

$t = 91.203$

$2002 + 91 = 2093$

The population will reach 100 in 2094 or just into 2094.

13. (7 marks)

(a) $A = x \times y \qquad y^2 = 1 - x^2$

$A = x\sqrt{1 - x^2}$

(b) Maximum area when $\frac{dA}{dx} = 0$ and $\frac{d^2A}{dx^2} < 0$

$$\frac{dA}{dx} = -\frac{2x^2 - 1}{\sqrt{1 - x^2}}$$

$$\frac{d^2A}{dx^2} = -\frac{2x^3\sqrt{1 - x^2} + 4x\sqrt{(1 - x^2)^3} - x\sqrt{1 - x^2}}{(x^2 - 1)^2}$$

$If \ \frac{dA}{dx} = 0, \quad 2x^2 - 1 = 0$

$x^2 = 0.5$

$x = \frac{1}{\sqrt{2}} \quad x > 0$

Max or min?

$$\frac{d^2A}{dx^2} = -\frac{4\left(\frac{1}{2\sqrt{2}}\right)\sqrt{0.5} + 4\left(\frac{1}{2\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\left(\sqrt{0.5}\right)}{2} = -\frac{\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)\sqrt{0.5} + 4\left(\frac{1}{2\sqrt{2}}\right)}{2} < 0$$

$\therefore$ max

$x = \frac{1}{\sqrt{2}}, \qquad y = \sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}}$

Therefore $P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$