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MATHEMATICS SPECIALIST UNIT 3

Semester One

2020

SOLUTIONS

Calculator-free Solutions

1. (a) $g(f) = \dfrac{2}{f-2} = \dfrac{2}{\sqrt{4-x}-2}$ ✓

(b) $D_x = \{x \in R : x \leq 4 \wedge x \neq 0\}$ ✓✓

$R_y = \{y \in R : y > 0 \vee y \leq -1\}$ ✓✓

(c) No, because $f^{-1}(x)$ does not exist for $x < 0$. ✓✓ [7]

2. (a) $\lambda = 3$ and $\mu = 1$ given, then:

$$\begin{pmatrix} 3+3 \\ 1+3 \\ -6 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + 1 \times \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ -7 \end{pmatrix}$$ ✓✓

$$\therefore \overrightarrow{OP} = \begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix} \rightarrow |OP| = \begin{vmatrix} 6 \\ 4 \\ -6 \end{vmatrix} = 2\sqrt{22} units$$ ✓✓

(b) $$d_3 = d_1 \times d_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \\ -6 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$ ✓

answers must be parallel to $\langle 1, 3, 2 \rangle$ ✓

$$\therefore L_3 : r = \begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$ ✓

$$\therefore n = d_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \rightarrow k = n \cdot \overrightarrow{OP} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix} = 6 + 12 - 12 = 6$$

$$\therefore r \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 6 \rightarrow x + 3y + 2z = 6$$ ✓✓ [9]

3. (a) (i) $$\frac{\overline{-1+i}}{(2+i)^2} = \frac{-1-i}{3+4i} \times \frac{3-4i}{3-4i} = \frac{-7+i}{9+16} = \frac{-7}{25} + \frac{i}{25}$$ ✓✓✓

(ii) $$\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)^{2020} = \left[cis\left(\frac{-\pi}{4}\right)\right]^{2020}$$ ✓

$$¿\left[cis\left(\frac{-\pi}{4}\right)\right]^{2016} \times \left[cis\left(\frac{-\pi}{4}\right)\right]^{4}$$ ✓

$$¿1 \times \cos(-\pi) = -1 + 0i$$ ✓

(b) (i) $z = 1 + i$ is a root of $f(z)$

$$\therefore f(1+i) = (1+i)^3 - 5(1+i)^2 + 8(1+i) - 6$$ ✓

$$¿-2+2i-10i+2+8i = 0+0i$$ ✓

$\therefore (z-1-i)$ is a root of $f(z)$ as per the factor theorem.

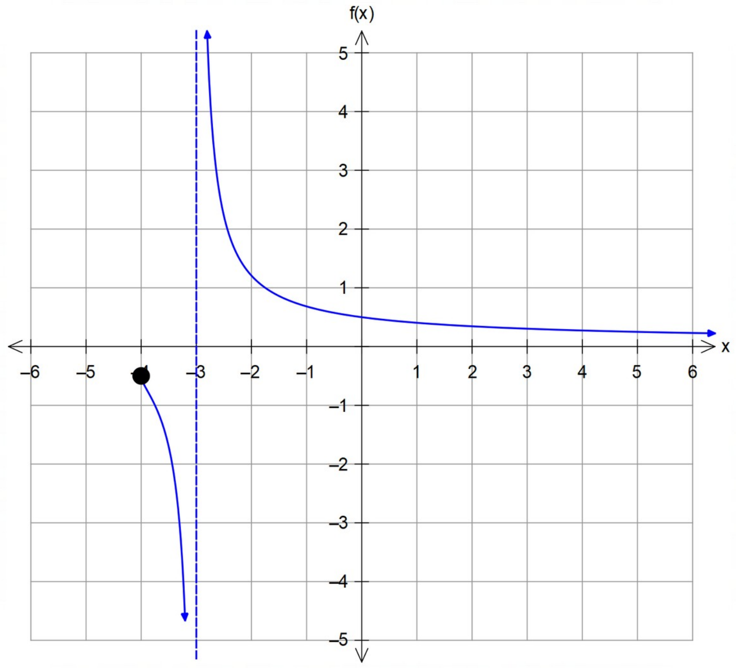
(ii) Since the coefficients of $f(z)$ are real, then $\bar{z}=1-i$ is also as root. ✓

$\therefore (z-1-i)(z-1+i)(z-w)=0$ ✓

$(z^2-2z+2)(z-w)=0 \rightarrow w=3$

Solutions are $z=3, 1\pm i$ ✓✓ [12]

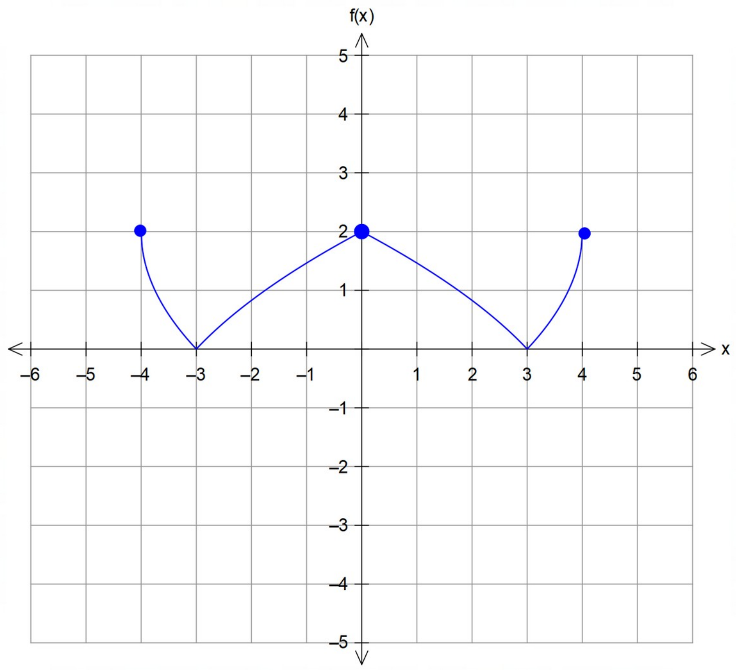
4. (a)



✓ asymptote y = 0 and a pole x=-3

✓✓ behaviour on either side of the pole.

(b)



✓ roots $x=\pm 3$ y-intercept y=2

✓✓ symmetry and curvature

(c) $k=-3$ ✓

$D_x=\{x \in R : x \geq 0\}$ ✓

$R_y = \{y \in R : y \geq -3\}$ ✓ [9]

5. (a) $x = 3, y = -1, z = 4$ ✓✓✓

(any algebraic method or matrix method accepted)

5. (b) The normal vectors to the three planes are $\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\k\end{pmatrix}$.

Solutions do not exist for $k=\frac{1}{2}$ ✓

which is when the three planes are non-parallel but do not intersect at a single point in space. ✓

Solutions exist for $k\neq\frac{1}{2}$ ✓ [6]

6. (a) $E(-2,-5,-2)$ ✓

(b) $\frac{V_I}{V_{tot}}\times 100\% = \frac{2\times 3\times 4}{4\times 8\times 6} = \frac{100}{8} = \frac{25}{2} = 12.5\%$ ✓✓

(c) centre = midpoint ¿$(0,-1,1)$ ✓

radius ¿ $\frac{1}{2}|HB| = \frac{1}{2}\begin{vmatrix}4\\-8\\-6\end{vmatrix} = \sqrt{29}$ ✓

$\therefore |r+j-k| = \sqrt{29}$ ✓ [6]

7. Solutions must satisfy:

$\left[cis\left(\frac{\pi}{3}\right)\right]^n = [cis(\pi)]^n = -1$ ✓

$\therefore cis\left(\frac{n\pi}{3}\right) = cis(n\pi) = -1$ ✓

from symmetry, the roots must be equally spaced at intervals

of $\frac{2\pi}{3},\frac{\pi}{3},\ldots etc$ therefore $n=3,9,15,\ldots,3(2k+1), k=0,1,2,3,\ldots$ ✓✓ [4]

(it must be an odd multiple of 3)

(any explanation based on symmetry is acceptable)

Calculator-Assumed Solutions

8. Discontinuities at $x=2,6 \rightarrow c=-2\,d=-6, \vee c=-6\,,d=-2$ ✓✓

Oblique asymptote is given by $y=ax+b=\frac{-1}{2}x+1$

$\therefore a=\frac{-1}{2}b=1$ ✓✓

$f(x)=\frac{-x}{2}+1+\frac{k}{(x-2)(x-6)}$ using the point $(4,-2)$ given:

$-2=-2+1+\frac{k}{-4} \rightarrow k=4$ ✓ [5]

9. (a) $\overrightarrow{AB}=\begin{pmatrix}-3\\3\\3\end{pmatrix}$ and $\overrightarrow{AC}=\begin{pmatrix}1\\5\\6\end{pmatrix}$ ✓✓

CAS $\rightarrow \measuredangle BAC \approx 42.84°=43°$ ✓

(b) Let $r=\overrightarrow{OA}+\lambda\overrightarrow{AB}+\mu\overrightarrow{AC}=\begin{pmatrix}1\\-2\\-1\end{pmatrix}+\lambda\begin{pmatrix}-3\\3\\3\end{pmatrix}+\mu\begin{pmatrix}1\\5\\6\end{pmatrix}$ ✓

(Other solutions exist)

(c) (i) $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}3\\21\\-18\end{pmatrix}=\beta\begin{pmatrix}m\\7\\n\end{pmatrix}$ ✓✓

$\therefore \beta=3 \rightarrow m=1\,n=-6$ as required

(ii) Normal equation of the plane needed:

$n=\begin{pmatrix}1\\7\\-6\end{pmatrix} \rightarrow k=\overrightarrow{OA}\cdot n=\begin{pmatrix}1\\-2\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\7\\-6\end{pmatrix}=-7$

$\therefore r\cdot\begin{pmatrix}1\\7\\-6\end{pmatrix}=-7$ ✓

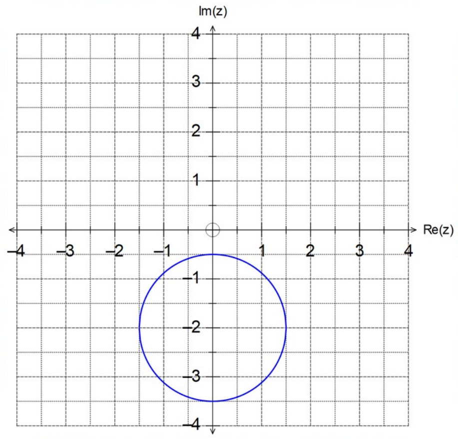
$\alpha\begin{pmatrix}1\\7\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\7\\-6\end{pmatrix}=-7 \rightarrow \alpha=\frac{-7}{86}$ ✓

$\therefore \overrightarrow{OP}=\left\langle\frac{-7}{86},-\frac{49}{86},\frac{42}{86}\right\rangle$ ✓

(d) Since L in (c)(ii) passes through the origin, then $\overrightarrow{OP}$ is the position vector of the point of shortest distance from O.

$\therefore |OP|=\frac{7}{86}\left|\begin{matrix}1\\7\\-6\end{matrix}\right|=\frac{7}{86}\sqrt{86}\approx 0.75\,units$ ✓ [10]

10. (a) (i)

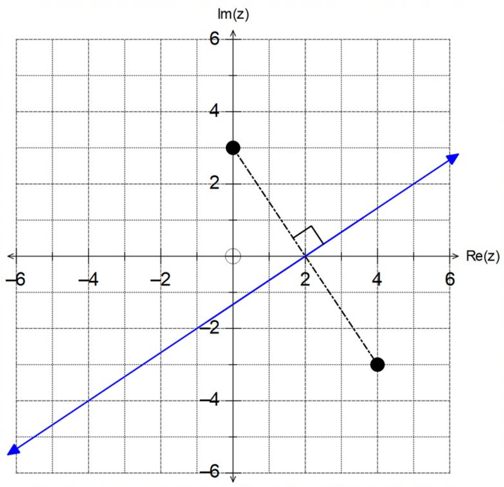


✓ circle

✓centred at (0,-2)

✓radius of 1.5 units

(ii)

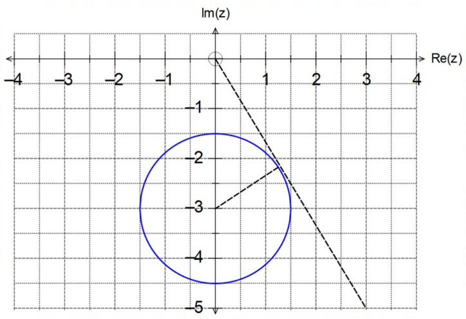


✓ line $y = \frac{2x}{3} - \frac{4}{3}$

or passes through the points (2,0) and (-1, -2)

✓✓ perpendicular bisector between points given

(b) $z - i$ is a translation of 1 unit down. ✓

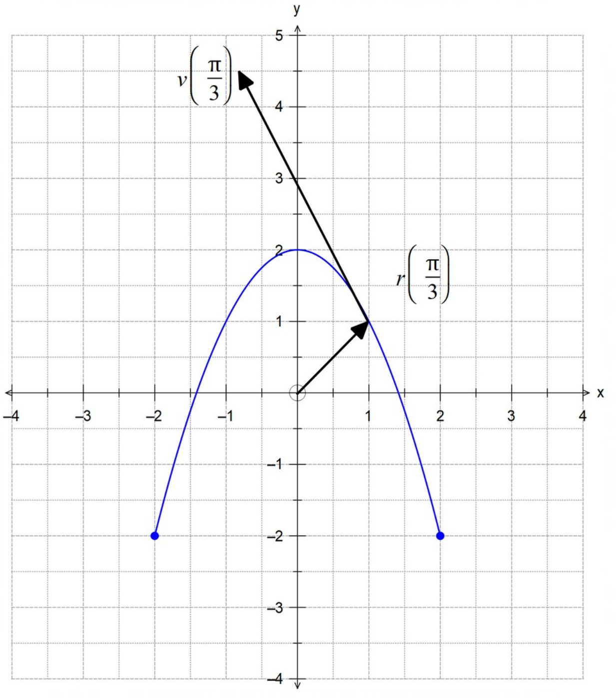


From diagram: $|z|_{max} = |-3| + \left|\frac{3}{2}\right| = 4.5\,units$ ✓

and $arg(z)_{max}$ occurs in 4th quadrant:

$arg(z)_{max} = -\pi + \sin^{-1}\left(\frac{1.5}{3}\right) = \frac{-\pi}{3}$ ✓✓ [10]

11. (a)



(a)
✓✓ parabola
$y=2-x^2$

✓ restricted domain
$-2\le x\le 2$

(c)
✓ position vector
$\langle 1,1\rangle$

✓ velocity vector
$\langle -1.7,3.5\rangle$ from (1,1)

(b) $t=2\pi\, minutes$ ✓

(c) $x=2\cos(t)=1 \rightarrow t=\frac{\pi}{3} min$ ✓

$$r\left(\frac{\pi}{3}\right)=\begin{pmatrix}2\cos\left(\frac{\pi}{3}\right)\\ -2\cos\left(\frac{2\pi}{3}\right)\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$ ✓

$$v(t)=\begin{pmatrix}-2\sin(t)\\4\sin(2t)\end{pmatrix}\rightarrow v\left(\frac{\pi}{3}\right)=\begin{pmatrix}-2\sin\left(\frac{\pi}{3}\right)\\ 4\sin\left(\frac{2\pi}{3}\right)\end{pmatrix}\approx\begin{pmatrix}-1.7\\3.5\end{pmatrix}$$ ✓✓

(d) $$a(t)=\begin{pmatrix}-2\cos(t)\\8\cos(2t)\end{pmatrix}$$

$$a\left(\frac{\pi}{6}\right)=\begin{pmatrix}-2\cos\left(\frac{\pi}{6}\right)\\ 8\cos\left(\frac{\pi}{3}\right)\end{pmatrix}=\begin{pmatrix}-\sqrt{3}\\4\end{pmatrix}$$ ✓

$$\therefore\left|\begin{matrix}-\sqrt{3}\\4\end{matrix}\right|=\sqrt{19}\approx 4.36$$ ✓

(e) $|v|=\sqrt{4\sin^2 t+16\sin^2(2t)}$ ✓

¿ $\sqrt{4\sin^2 t+16(2\sin t\cos t)^2}$

¿ $\sqrt{4\sin^2 t+16\times 4\sin^2 t\cos^2 t}$ ✓

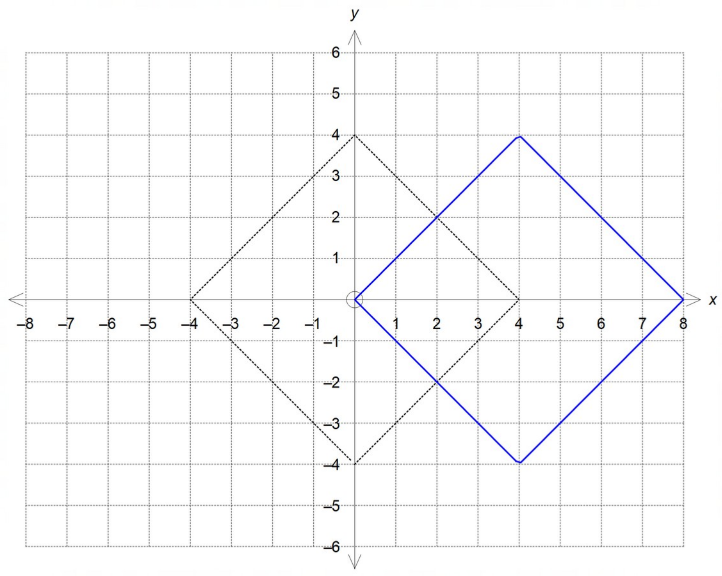
¿ $2\sin t\sqrt{1+16\cos^2 t}$ as required.

(f) $\cos t=\frac{x}{2}\rightarrow y=-2\cos(2t)=-2(\cos^2 t-\sin^2 t)$ ✓

$\therefore\ y=-2\left(\frac{x^2}{4}-\frac{4-x^2}{4}\right)=2-x^2$ ✓

with $-2\leq x\leq 2$ ✓ [17]

12. (a)



ɔe/accuracy

ıtion

(b) $x^2 + y^2 = k$ is a circle and can intersect the square four times at either its vertices, or use its sides as tangents.

$\therefore k = 16 \vee k = \left(2\sqrt{2}\right)^2 = 8$ ✓✓

(c) $n = 4 \wedge m > 1$ ✓✓ [6]

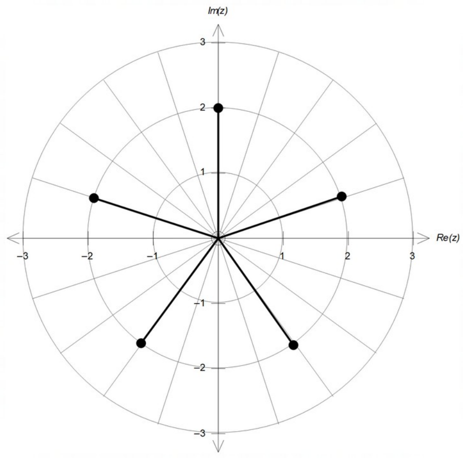
13. (a) Solution given is $z_0 = 2\,cis\left(\frac{-3\pi}{10}\right)$ ✓

$\therefore a + bi = z_0^5 = 2^5 cis\left(5 \times -\frac{3\pi}{10}\right) = 32\,cis\left(\frac{\pi}{2}\right) = 0 + 32i$

$a = 0 \wedge b = 32$ ✓✓

Other solutions can be graphed and then listed:

$z_1 = 2\,cis\left(\frac{\pi}{10}\right), z_2 = 2\,cis\left(\frac{\pi}{2}\right), z_3 = 2\,cis\left(\frac{9\pi}{10}\right), z_4 = 2\,cis\left(\frac{-7\pi}{10}\right)$ ✓✓



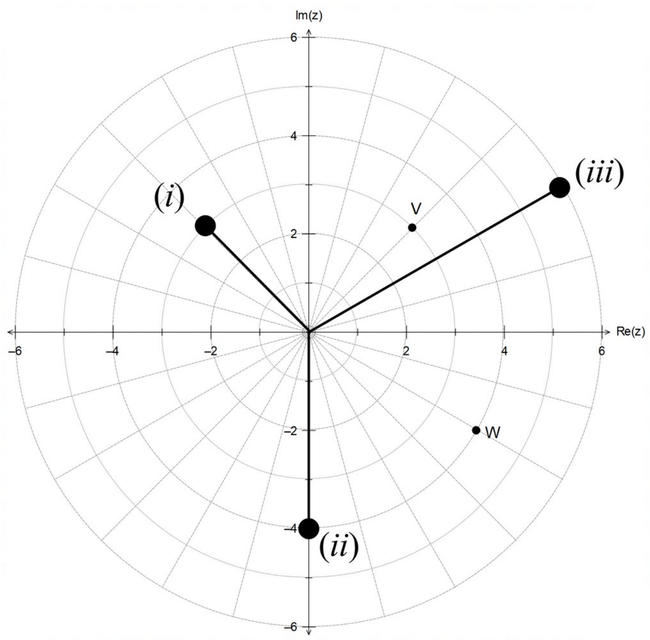
✓ magnitude of 2

✓ $\frac{2\pi}{5}$ apart (4

13. (b) (i) $\frac{-v}{i}=\frac{-1\times v}{i}=\frac{cis(\pi)\times 3\,cis\left(\frac{\pi}{4}\right)}{cis\left(\frac{\pi}{2}\right)}=3\,cis\left(\frac{3\pi}{4}\right)$

(ii) $w-\overline{w}=4\,cis\left(\frac{-\pi}{6}\right)-4\,cis\left(\frac{\pi}{6}\right)=8\,i\sin\left(\frac{-\pi}{6}\right)=-4\,i$

(iii) $v\times w^{0.5}=3\,cis\left(\frac{\pi}{4}\right)\times 2\,cis\left(\frac{-\pi}{12}\right)=6\,cis\left(\frac{\pi}{6}\right)$



✓ magnitude
✓ argument
(each)

[13]

14. (a) Location of collision:

$\overrightarrow{OA}(12)=\begin{pmatrix}3\\-2\\6\end{pmatrix}+12\begin{pmatrix}4\\3\\-2\end{pmatrix}=\begin{pmatrix}51\\34\\-18\end{pmatrix}m¿O$ ✓

$\overrightarrow{OB}(9)=\begin{pmatrix}24\\-2\\36\end{pmatrix}+9\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}51\\34\\-18\end{pmatrix}\rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\4\\-6\end{pmatrix}m/s$ ✓✓

$\therefore |v|=\begin{vmatrix}3\\4\\-6\end{vmatrix}=\sqrt{61}\approx 7.81\,m/s$ ✓

(b) Resetting timer/position for particle A;

$\overrightarrow{OA}(3)=\begin{pmatrix}3\\-2\\6\end{pmatrix}+3\begin{pmatrix}4\\3\\-2\end{pmatrix}=\begin{pmatrix}15\\7\\0\end{pmatrix}$

$$\therefore \overrightarrow{OB} = \begin{pmatrix} 24 \\ -2 \\ 36 \end{pmatrix} + t\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ 7 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = \overrightarrow{OA}$$ ✓✓

14. (c) $t\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9 \\ 9 \\ -36 \end{pmatrix} + t\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$ from (b)

$\therefore x = 4 - \frac{9}{t}, y = 3 + \frac{9}{t} \wedge z = -2 - \frac{36}{t}$ ✓

and $\sqrt{x^2 + y^2 + z^2} = 6$

$\therefore \left(4 - \frac{9}{t}\right)^2 + \left(3 + \frac{9}{t}\right)^2 + \left(2 + \frac{36}{t}\right)^2 = 36$ ✓

CAS $\rightarrow t = 9 \pm \frac{45}{\sqrt{7}} \rightarrow t \approx 26\, s ¿ t = 3$ ✓

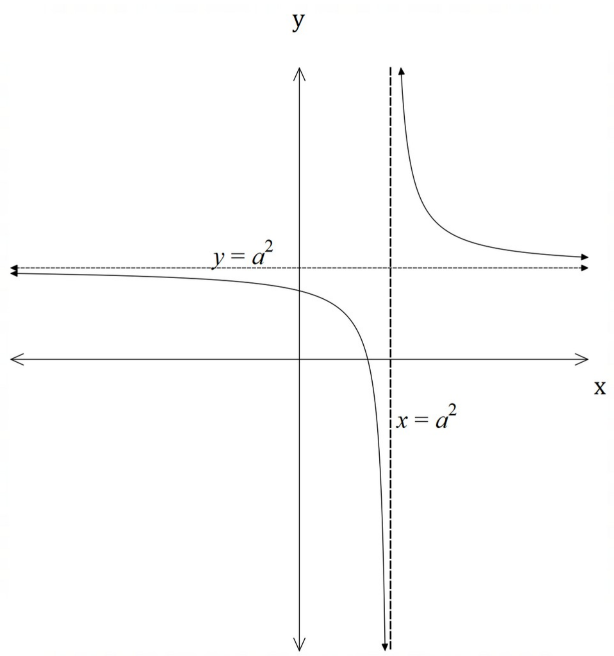
$\therefore t = 29\, s ¿ t = 0$ ✓

$\overrightarrow{OA}(29) \approx \begin{pmatrix} 119 \\ 85 \\ -52 \end{pmatrix} m ¿ O$ ✓ [11]

15. (a) $¿(x) = a^2 + \frac{1}{\left(a^2 + \frac{1}{x - a^2}\right) - a^2} = a^2 + \frac{1}{\left(\frac{1}{x - a^2}\right)} = a^2 + x - a^2 = x$ ✓

$D_x = \{x \in R : x \neq a^2\}\ R_y = \{y \in R : y \neq a^2\}$ ✓✓

(b)



✓ asymptote $y = a^2$

✓ pole $x = a^2$

✓ positive rectangular hyperbola

(c) (i) $h(x) = 2 + \frac{1}{x - 2} = \frac{2x - 3}{x - 2} \rightarrow k = -3$ ✓✓

(ii) $h(x)$ is an involution and hence it is its own inverse.

The graphs of $h(x)$ and $h^{-1}(x)$ are symmetrical over the line $y = x$.

They have the same set for both its domain and range: $x, y \neq 2$.

They have the same asymptote $x = 2$.

They have the same pole $y = 2$ ✓✓✓✓ [12]

(any four of these)

16. (a) $\sin(6\theta) = \Im\left[\left(\cos(2\theta) + i\sin(2\theta)\right)\right]^3$ from De Moivre's theorem ✓

CAS $\rightarrow \sin(6\theta) = 3\cos^2(2\theta)\sin(2\theta) - \sin^3(2\theta)$ ✓

¿ $3\sin(2\theta)\left[1 - \cos^2(2\theta)\right] - \sin^3(2\theta)$ ✓

¿ $3\sin(2\theta) - 4\sin^3(2\theta)$ ✓

(b) Let $x = \sin(2\theta)$

$\therefore 3x - 4x^3 = 3\sin(2\theta) - 4\sin^3(2\theta) = \sin(6\theta) = 1$ from (a) ✓

$\therefore 6\theta = \frac{-3\pi}{2}, \frac{\pi}{2} \rightarrow \theta = \frac{-\pi}{4}, \frac{\pi}{12}$ ✓

$\therefore x = \sin\left(2 \times \frac{-\pi}{4}\right) = \sin\left(\frac{-\pi}{2}\right) = -1$

$x = \sin\left(2 \times \frac{\pi}{12}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ ✓ [7]

17. $(f \circ g)(x) = 2g - 1 = 2\left(\frac{x}{2} - 4\right) - 2 = x - 9$ ✓

$\therefore (f \circ g)^{-1}(x) = x + 9$ ✓

$f^{-1}(x) = \frac{x+1}{2}$ and $g^{-1}(x) = 2x + 8$ ✓

$\therefore \left(f^{-1} \circ g^{-1}\right)(x) = \frac{1}{2}g^{-1} + \frac{1}{2} = \frac{1}{2}(2x + 8) + \frac{1}{2} = x + \frac{9}{2}$ ✓

$\left(g^{-1} \circ f^{-1}\right)(x) = 2f^{-1} + 8 = 2\left(\frac{x+1}{2}\right) + 8 = x + 9$ ✓

$\therefore (f \circ g)^{-1}(x) = \left(g^{-1} \circ f^{-1}\right)(x)$ ✓ [6]