Section One: Calculator – free **(50 marks)**

This section has **six (6)** questions. Answer all questions. Write your answers in the spaces provided.

Working time: 50 minutes

Question 1 **(7 marks)**

(a) Simplify $\dfrac{3}{x^2-4} - \dfrac{5}{x+2}$ (3)

$$= \frac{3 - 5(x-2)}{(x+2)(x-2)} \checkmark \checkmark$$

$$= \frac{13-5x}{x^2-4} \checkmark$$

(b) Simplify $\dfrac{3m^2-6m-24}{m^2-5m+4} \div \dfrac{m^2-m-6}{m^2-3m}$ (4)

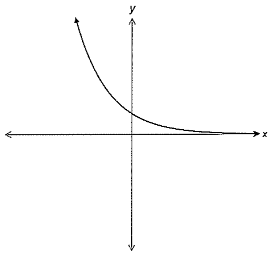
$$\frac{3\cancel{(m-4)}\cancel{(m+2)}}{\cancel{(m-4)}(m-1)} \times \frac{m\cancel{(m-3)}}{\cancel{(m-3)}\cancel{(m+2)}} \checkmark \checkmark$$

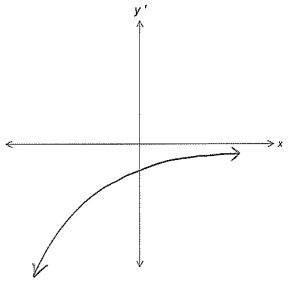
$$= \frac{3m}{m-1} \checkmark$$

$m \neq 4$
$m \neq -2$ $\checkmark$
$m \neq 3$

Question 2 **(9 marks)**

(a) Sketch the graph of the derivative function (for the function shown) on the axes provided. (2)





shape ✓
all below x-axis ✓

(b) Differentiate the following with respect to x.

(i) $f(x) = \dfrac{-x}{x^2+1}$ (express in simplest form) (3)

$$f'(x) = \frac{-1(x^2+1) - 2x(-x)}{(x^2+1)^2}$$ ✓✓ −1 per error

$$= \frac{-x^2 - 1 + 2x^2}{(x^2+1)^2}$$

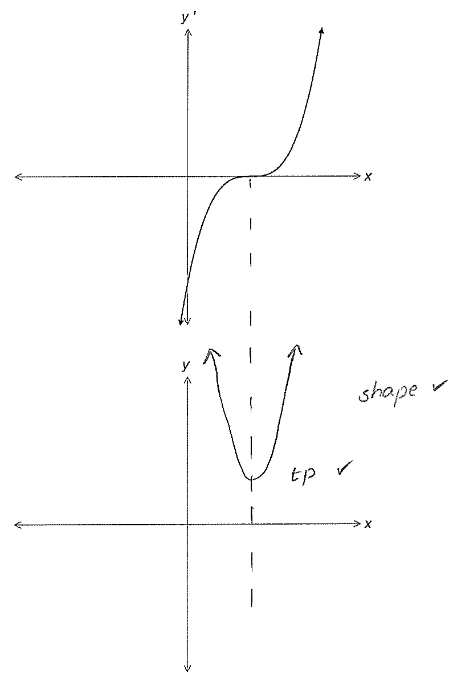
$$= \frac{x^2 - 1}{(x^2+1)^2}$$ ✓

(ii) $g(x) = (x+1)^2 e^{x^2}$ (do not simplify) (2)

$$g'(x) = 2(x+1)e^{x^2} + 2x\, e^{x^2}(x+1)^2$$
✓ ✓

Question 2 (continued)

(c) Given the derivative function, sketch the graph of the function on the axes provided. (2)



Question 3 **(12 marks)**

Consider the curve $y = x^3 - 2x^2 - 4x + 3$

(a) It is claimed that the tangent to the curve at the point where $x = 1$ passes through the point (3, 8). Is this claim valid? Justify your answer. (4)

$y' = 3x^2 - 4x - 4$ ✓

at $x = 1$ $y = -2$ ½
at $x = 1$ $y' = -5$ ½

$y = -5x + c$

subst $(1, -2)$
$-2 = -5(1) + c$
$c = 3$

$\therefore y = -5x + 3$ ✓

subst $x = 3$
$y = -15 + 3 = -12 \neq 8$ ✓
$\therefore$ Claim not valid

(b) Determine the value of x for which y is a maximum. (4)

max when $3x^2 - 4x - 4 = 0$
$(3x + 2)(x - 2) = 0$
$x = -\frac{2}{3}$ or $x = 2$ ✓

$y'' = 6x - 4$ ✓

if $x = -\frac{2}{3}$ $y'' < 0$ $\therefore$ max ✓ tests both points
if $x = 2$ $y'' > 0$ $\therefore$ min

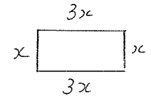
$\therefore$ max when $x = -\frac{2}{3}$ ✓

Question 16 **(10 marks)**

A piece of wire 8cm long is cut into two unequal parts. One part is used to form a rectangle that has a length three times its width. The other part of the wire is used to form a square.

(a) If the width of the rectangle is x units, show that the equation that will give the sum of the areas of the rectangle and the square in terms of x is: (5)

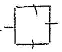
$$A = 7x^2 - 8x + 4$$



$= 8x$ ✓

$\therefore$ Square $= 8 - 8x$ ✓

$\frac{8 - 8x}{4} = 2 - 2x$ ✓



$A = 3x(x) + (2 - 2x)^2$ ✓
$= 3x^2 + 4 - 8x + 4x^2$ ✓
$= 7x^2 - 8x + 4$ as required

(b) Using Calculus, find the length of each part of the wire when the sum of the areas is a minimum. (5)

$\frac{dA}{dx} = 14x - 8$ ✓

$14x - 8 = 0$
$x = \frac{4}{7}$ ✓

$\frac{d^2A}{dx^2} = 14 > 0$ $\therefore$ min ✓

length of wire for rectangle $\frac{32}{7}$ cm ✓
square $\frac{24}{7}$ cm ✓

Question 15 (continued)

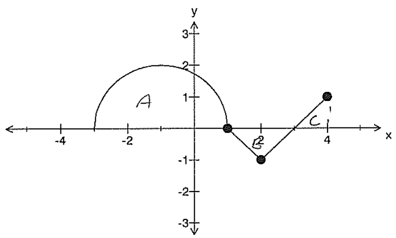
(iii) $\int_{-3}^{2} f(x)\,dx$ (1)

$= -75 - 30 + 50$

$= -55$ ✓

(b) The graph of a function $f(x)$ consists of a semi- circle and two line segments as shown.

Find the exact value of $\int_{-3}^{4} f(x)\,dx$ (3)



$A = \frac{1}{2} \times \pi \times 2^2 = 2\pi$

$B = \frac{1}{2} \times 2 \times 1 = 1$

$C = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$ ✓✓

$\int_{-3}^{4} f(x)\,dx = A + (-B) + C$

$= 2\pi - 1 + \frac{1}{2}$ ✓

$= 2\pi - \frac{1}{2}$

Question 3 (continued)

(c) Solve the following system of equations. (4)

$x - 2y + 4z = 2$ ①
$2x + y + 3z = -1$ ②
$-x - y - 2z = 0$ ③

$2 \times ① - ②$ $\quad -5y + 5z = 5$ ④ ✓
$\div 5 \Rightarrow \quad -y + z = 1$ ⑤

$① + ③$ $\quad -3y + 2z = 2$ ⑥ ✓
$2 \times ⑤$ $\quad -2y + 2z = 2$ ⑦

$⑥ - ⑦$ $\quad -y = 0$
$\therefore y = 0$ ✓

subst $y = 0$ into ⑤
$z = 1$ ½

subst $y = 0$, $z = 1$ into ③
$-x - 2 = 0$
$x = -2$ ½

Question 4 **(10 marks)**

(a) Determine c given that the graph of $f(x)=cx^2+x^{-2}$ has a point of inflection at $(1, f(1))$. (3)

$f'(x) = 2cx - 2x^{-3}$ ✓

$f''(x) = 2c + 6x^{-4}$ ✓

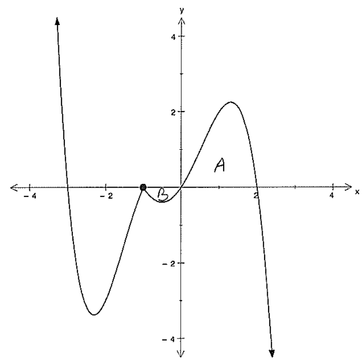
Let $2c + \frac{6}{x^4} = 0$

when $x = 1$ $\quad 2c + 6 = 0$

$c = -3$ ✓

Question 15 **(8 marks)**

(a) For the function $y = f(x)$ below



It is known that

$$\int_{-3}^{-1} f(x)\,dx = -75$$

$$\int_{-1}^{2} f(x)\,dx = 20$$

The area between the curve and the x-axis from $x=-1$ to $x=2$ is 80 square units.

Use the information above and mathematical reasoning to determine the value of each of the following.

(i) $\int_{-1}^{0} f(x)\,dx$ (3)

$B - A = 20$ ✓

$A + B = 80$

$A = 30$

$B = 50$ ✓

$= -30$ ✓

(ii) the area between the curve and the x-axis from $x=-3$ to $x=0$ (1)

$75 + 30 = 105$ ✓

Question 14 (continued)

(b) A group of anthropologists found that human tooth size is continuing to decrease, such that $\frac{dS}{dt} = kS$. In Northern Europeans, for example, tooth size reduction now has a rate of 1% per 1000 years.

(i) If ***t*** represents time in years and ***S*** represents tooth size, find the value of **k**, rounded to 8 decimal places. (2)

$S = S_0 e^{kt}$

$0.99 S_0 = S_0 e^{1000k}$

$k = -0.00001005$

(ii) In how many years will human tooth size be 90% of their present size? (2)

$0.9 = e^{-0.00001005t}$

$t = 10\,483.3$ years

($t = 10483.6$ using (i))

(iii) What will be our descendant's tooth size 20 000 years from now? (2)
(as a percentage of our present tooth size)

$S = S_0 e^{-0.00001005 \times 20\,000}$

$= 0.8179$

$= 81.79\%$

Question 4 (continued)

(b) The functions $f(x)$ and $g(x)$ are defined as follows
$f(x) = x^2 - 4$ and $g(x) = \sqrt{x-5}$

(i) Determine the simplified expressions for $f[g(x)]$ and $g[f(x)]$. (3)

$f(g(x)) = (\sqrt{x-5})^2 - 4$

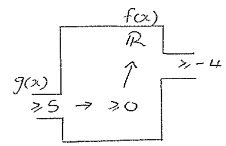
$= x - 5 - 4$

$= x - 9$

$g(f(x)) = \sqrt{x^2 - 4 - 5}$

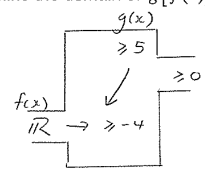
$= \sqrt{x^2 - 9}$

(ii) Determine the range of $f[g(x)]$. (2)



Range $y : y \geq -4 \quad y \in \mathbb{R}$

(iii) Determine the domain of $g[f(x)]$. (2)



$x^2 - 4 \geq 5$

$x^2 \geq 9$

domain $x : x \geq 3$ or $x \leq -3 \quad x \in \mathbb{R}$

-1 no $+c$

Question 5 **(7 marks)**

(a) Determine $\int (1+3x^2)(x-2)\,dx$ (3)

$= \int (3x^3 - 6x^2 + x - 2)\,dx$ ✓

$= \frac{3x^4}{4} - 2x^3 + \frac{x^2}{2} - 2x + c$ ✓✓

(b) Determine $\int 4x^3(3x^4-5)^7\,dx$ (2)

$= \frac{1}{3}\int 12x^3(3x^4-5)^7\,dx$ ✓

$f(x) = 3x^4 - 5$
$f'(x) = 12x^3$

$= \frac{\frac{1}{3}(3x^4-5)^8}{8} + c$

$= \frac{(3x^4-5)^8}{24} + c$ ✓

(c) Determine $\int 12xe^{x^2}\,dx$ (2)

$= 6\int 2x e^{x^2}\,dx$ ✓

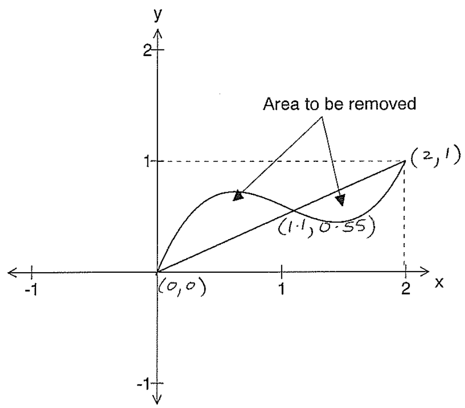
$f(x) = x^2$
$f'(x) = 2x$

$= 6e^{x^2} + c$ ✓

Question 14 **(11 marks)**

(a) A dressmaker wishes to cut a section of cloth from a piece of material measuring 2 metres by one metre. The curved edges of the piece of cloth to be removed are defined as being between the following equations:

$$y_1 = 0.5x \text{ and } y_2 = x^3 - 3.1x^2 + 2.7x$$



✓

(i) Label the 3 points of intersection with co-ordinates. (1)

(ii) Write an integral that would give the area of region bound by the two functions. (2)

$A = \int_0^{1.1} (x^3 - 3.1x^2 + 2.7x - 0.5x)\,dx + \int_{1.1}^{2} (0.5x - (x^3 - 3.1x^2 + 2.7x))\,dx$ ✓✓ -1 per error

(iii) Calculate the area of the cloth removed, correct to 2 decimal places. (2)

$A = 0.51\,m^2$ ✓✓ $-\frac{1}{2}$ units

Question 13 **(12 marks)**

Consider the letters of the word POLICE.

How many arrangements are there of these 6 letters (without repetition) if

(a) each arrangement must end with a vowel? (1)

$\underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}\ \underline{3}$ ✓ $= 360$

$5! \times 3$

(b) the vowels in each arrangement must be together? (2)

OIE , P , L , C

$4! \times 3! = 144$ ✓ ✓

(c) the vowels must be separated by the consonants (ie 2 vowels must not be together)? (2)

V C V C V C

$3! \times 3! \times 2 = 72$ ✓ ✓

Now suppose that 4 letters are chosen from this word and that the **order of selection is unimportant.**

(d) How many different 4 letter groups are possible if

(i) there is no restriction? (1)

$\binom{6}{4} = 15$ ✓

(ii) there must be at least one vowel? (2)

$\binom{3}{1}\binom{3}{3} + \binom{3}{2}\binom{3}{2} + \binom{3}{3}\binom{3}{1}$ ✓✓

$= 15$

(e) Determine the probability that in the four letter selection that is made, whenever O appears, E also appears. (4)

O E ✓✓   O E ✗✗   O E ✗✓

$$\frac{\binom{2}{2}\binom{4}{2} + \binom{2}{0}\binom{4}{4} + \binom{1}{0}\binom{1}{1}\binom{4}{3}}{\binom{6}{4}}$$ ✓✓✓ ✓

$= \frac{11}{15}$

Question 6 **(5 marks)**

$f(x)$ is defined such that $\int_{-3}^{6} f(x)\,dx = 24$ and $\int_{2}^{6} f(x)\,dx = 36$

(a) Find

(i) $\int_{-3}^{2} f(x)dx = 24 - 36$ (1)

$= -12$ ✓

(ii) $\int_{-3}^{2} (4f(x)+3)\,dx$ (3)

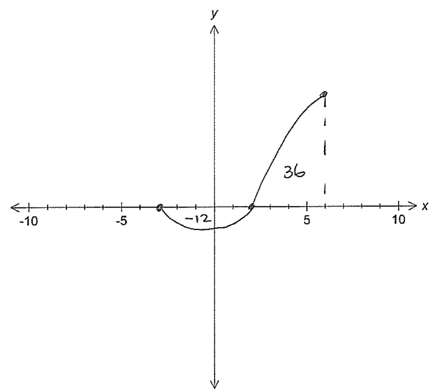
$= \int_{-3}^{2} 4f(x)\,dx + \int_{-3}^{2} (3)\,dx$

$= 4(-12) + [3x]_{-3}^{2}$ ✓ ✓

$= -48 + (6 - -9)$

$= -33$ ✓

(b) Sketch a possible graph of $y=f(x)$ for $-3 \le x \le 6$. Your graph should display the relative areas of important regions but you do not need to draw this graph to scale. (1)



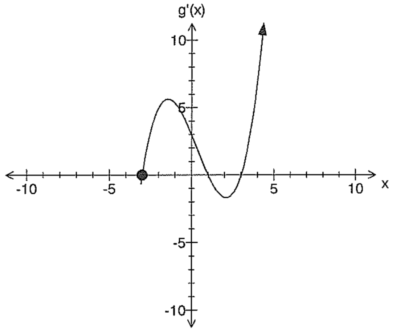
Section Two: Calculator – assumed **(100 marks)**

This section has **ten (10)** questions. Answer all questions. Write your answers in the spaces provided.

Working time: 100 minutes

Question 7 **(7 marks)**

The graph of g′(x) is given below.



(a) What can be said about the gradient of the function $g(x)$ between x = –3 to x = 1? (1)

positive ✓

(b) When does the function, $g(x)$ have a negative gradient? (2)

$1 < x < 3$ ✓✓ $(\leq)$ ✓

(c) State an equation for the tangent to the graph of $g(x)$ at x = 3. (1)

$y = k$, k a constant ✓

(d) Find the value of x at which $g(x)$ has a relative maximum for $-3 \leq x \leq 4$ (1)

$x = 1$ ✓

(e) Find the x-coordinate of each point of inflection of the graph of $g(x)$ for $-3 \leq x \leq 4$ (2)

$x = -1.5$ ✓
$x = 2$ ✓

Question 12 **(9 marks)**

Research has been conducted to determine the benefits of a flu vaccine before winter for adults over 65. The following information has been obtained:

60% of the target population (i.e. adults over 65) had the flu vaccine and of these 22% actually developed the flu, 3% developed a chest infection and the remainder had no flu-like symptoms over the winter.

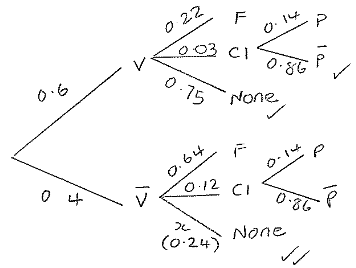
Of those who did not have the flu vaccine 12% developed a chest infection.

The proportion of those studied who did not have the vaccine and had no flu-like symptoms over the winter was 0.096.

14% of all those who developed a chest infection also got pneumonia.

(Note that in this same sample no one developed both the flu and a chest infection)

(a) Draw a tree diagram to represent the above information. (4)



$0.4x = 0.096$
$x = 0.24$

(b) For a randomly chosen person from this study determine the probability that:

(i) the person developed the flu if they did not have the flu vaccine. (1)

0.64 ✓

(ii) the person had the flu vaccine and developed pneumonia. (1)

$0.6 \times 0.03 \times 0.14 = 0.00252$ ✓

(iii) the person had the vaccine if they developed pneumonia. (3)

$$\frac{0.00252 \checkmark}{0.00252 + 0.4 \times 0.12 \times 0.14 \checkmark\checkmark} = 0.2727$$

Question 11 (continued)

(c) State the resulting equation when the graph of $y = e^x$ undergoes the following transformations **in succession**: (3)

- horizontal dilation of factor $\frac{1}{3}$
- reflection about the y-axis
- vertical translation 5 units in the direction of the negative y-axis
- horizontal translation 3 units in the direction of the positive x-axis
- vertical dilation of factor 2

$y = e^{3x}$ ✓½

$y = e^{-3x}$ ✓½

$y = e^{-3x} - 5$ ✓½

$y = e^{-3(x-3)} - 5$ ✓

$y = 2(e^{-3(x-3)} - 5)$ ✓½

$y = 2e^{-3(x-3)} - 10$

(d) The point $(3, 0.5e^4)$ lies on the curve of $y = 0.5e^{x+1}$. Identify the subsequent location of this point if the transformations listed below are applied **in succession**. (2)

- reflection about the x-axis
- horizontal translation 7 units in the direction of the negative x-axis
- vertical translation 3 units in the direction of the positive y-axis
- reflection about the y-axis

$(3, -0.5e^4)$

$(-4, -0.5e^4)$

$(-4, -0.5e^4 + 3)$

$(4, -0.5e^4 + 3)$

✓ ✓

Question 8 **(12 marks)**

(a) Events A and B are such $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\overline{A \cup B}) = \frac{1}{4}$

Show that events A and B are not mutually exclusive. (3)

$P(A \cup B) = \frac{3}{4}$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= \frac{1}{2} + \frac{7}{12} - \frac{3}{4}$ ✓

$= \frac{1}{3}$ ✓

$\neq 0$ ∴ not mutually exclusive ✓

(b) A toy robot has 3 main components (X, Y and Z) which are manufactured separately and then assembled together. Previous random testing of components has shown that:

P (X defective) = 0.002, P (Y defective) = 0.015, P (Z defective) = 0.003

If a toy robot is selected at random, what is the probability that:

(i) only component Y is defective, (2)

$0.998 \times 0.015 \times 0.997$ ✓✓

$= 0.01492509$

(ii) at least one of its components are defective. (2)

1 − none defective

$= 1 - 0.998 \times 0.985 \times 0.997$ ✓✓

$= 0.0199$ (4dp)

Question 8 (continued)

(c) If X and Y are independent events and P (X) = 0.75 and P(X ∪ Y) = 0.875, find

(i) P (Y) (3)

if independent then

$P(X) \cdot P(Y) = P(X \cap Y)$

$0 \cdot 75\, P(Y) = P(X \cap Y)$ ✓

$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$0 \cdot 875 = 0 \cdot 75 + P(Y) - 0 \cdot 75\, P(Y)$ ✓

$0 \cdot 125 = 0 \cdot 25\, P(Y)$

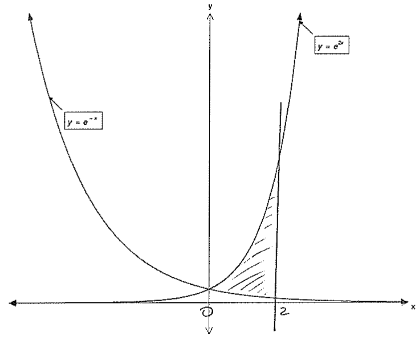
$P(Y) = 0 \cdot 5$ ✓

(ii) P (Y | X) $= 0 \cdot 5$ ✓ (1)

(iii) P (X | Y′) $= 0 \cdot 75$ ✓ (1)

Question 11 (continued)

(b) The curve $y = e^{2x}$ and $y = e^{-x}$ intersect at the point (0, 1) as shown in the diagram.



Find the area enclosed by the curves and the line *x=2*.
Leave your answer in terms of *e*. (4)

$\int_0^2 (e^{2x} - e^{-x})\,dx$ ✓

$= \left[\frac{e^{2x}}{2} - (-e^{-x})\right]_0^2$ ✓ ✓

$= \frac{e^4}{2} + e^{-2} - \left(\frac{1}{2} + 1\right)$

$= \frac{e^4}{2} + \frac{1}{e^2} - \frac{3}{2}$ ✓

Question 11 **(12 marks)**

(a) The function $f(x)$ is differentiable for all $x \in R$ and satisfies the conditions

$f'(x) < 0$ where $x < 2$

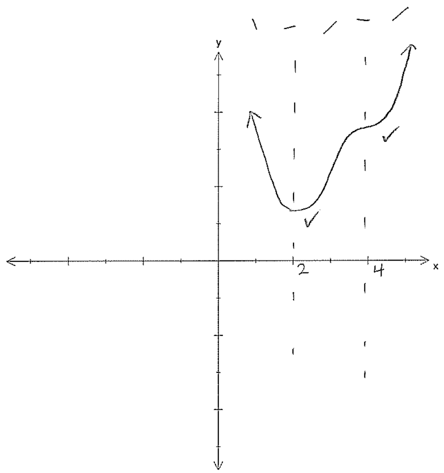
$f'(x) = 0$ where $x = 2$

$f'(x) = 0$ where $x = 4$

$f'(x) > 0$ where $2 < x < 4$

$f'(x) > 0$ where $x > 4$

(i) Draw a sketch of this function $f(x)$. (3)



shape ✓

Question 9 **(12 marks)**

(a) It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth, y, of fluid in the tank t hours after the valve is opened is given by

$y = 6\left(1 - \frac{t}{12}\right)^2$ metres.

(i) Find the rate $\frac{dy}{dt}$ m/hour at which the tank is draining at time, t. (2)

$$\frac{dy}{dt} = 12\left(1 - \frac{t}{12}\right)^1 \cdot -\frac{1}{12} \checkmark$$
$$= -\left(1 - \frac{t}{12}\right) \checkmark$$
$$= \frac{t}{12} - 1$$

(ii) When is the fluid in the tank falling fastest and slowest?
What are the values of $\frac{dy}{dt}$ at these times? (3)

$0 \le t \le 12$

$\frac{t}{12} - 1 = 0$

$t = 12$ ✓ minimum, slowest $\frac{dy}{dt} = 0$ ✓

fastest

at $t = 0$ $\frac{dy}{dx} = -1$ ✓

Question 9 (continued)

(c) If $y=kx^3$ for some constant k, use the incremental formula to establish the percentage increase in x required to yield a 15% increase in y. (3)

$\delta y = \frac{dy}{dx}\delta x$

$0.15y = 3kx^2\delta x$ ✓

$0.15kx^3 = 3kx^2\delta x$ ✓

$\delta x = 0.05x$

5% change in x required ✓

(d) A company sells goods such that its revenue, in dollars, from selling x items is given by the equation,

$$R(x) = 5x(20x - x^2)$$
$$= 100x^2 - 5x^3$$

(i) Determine the marginal revenue when $x = 10$. (2)

$R'(x) = 200x - 15x^2$ ✓

$R'(10) = 500$

∴ marginal revenue is \$500 ✓

(ii) What does marginal revenue represent? (2)

The revenue for the 11^{th} item ✓ is approximately \$500 ✓

Question 10 **(7 marks)**

The Australian Kayak team must select 4 elite rowers from 14 possible contenders to be the new 'Awesome Foursome'.

(a) How many different selections are possible? (1)

$\binom{14}{4}$ ✓ $= 1001$

Mike is the singles kayak champion and Geoff is the runner up champion.

(b) What is the probability that of the 4 rowers chosen at random:

(i) Mike is included? (1)

$\frac{\binom{1}{1}\binom{13}{3}}{\binom{14}{4}}$ ✓ $= \frac{286}{1001} = 0.2857$

(ii) Mike and Geoff are included? (1)

$\frac{\binom{2}{2}\binom{12}{2}}{\binom{14}{4}}$ ✓ $= \frac{66}{1001} = 0.0659$

(iii) Mike or Geoff is selected? (2)

$\frac{286}{1001} + \frac{286}{1001} - \frac{66}{1001} = \frac{506}{1001} = 0.5055$ ✓✓

OR

$1 - \frac{\binom{2}{0}\binom{12}{4}}{\binom{14}{4}} = 0.5055$

(c) If Mike is selected for the Kayak team, what is the probability that Geoff is also selected? (2)

$\frac{\frac{66}{1001}}{\frac{278}{1001}}$ ✓ $= \frac{66}{286}$ ✓ $= 0.2307$.