Calculator Free Section

1. [6 marks]

Points **A** and **B** have position vectors $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 15 \\ 5 \end{pmatrix}$ respectively.

Find the point **C** such that **AB** : **AC** = 3 : 5.

2. **[12 marks]**

For each of the following functions, find $\frac{dy}{dx}$.

(a) $y = 2^{x^2} \,.\, e^{\cos x}$

[4]

(b) $y = \sqrt{\cos(\sin^2 x)}$

[4]

(c) $\ln y = \dfrac{x}{x^2 + 1}$

[4]

3. **[5 marks]**

Use de Moivre's rule to determine the exact value of $(1+i)^5 - (1-i)^5$.

4. **[4 marks]**

For each of the following statements, circle either True or False.
If a statement is true, prove that it is true, showing your reasoning clearly.
If a statement is false, either explain why it is false, or provide an example to show the statement is false.

(a) The vector $\mathbf{c} = -2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ is perpendicular to both the vectors $\boldsymbol{a} = -2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j} - 7\boldsymbol{k}$.

TRUE / FALSE

[2]

(b) If the matrices MN and NM are both defined, then the size of the matrices MN and NM is the same as the size of matrix M or matrix N.

TRUE / FALSE

[2]

5. **[5 marks]**

Evaluate the following limits, showing full reasoning.

(a) $$\lim_{x \to 0} \left(\frac{\sin\left(\frac{\pi}{2} + x\right) - 1}{x} \right)$$

[2]

(b) $$\lim_{\theta \to 0} \frac{x\sqrt{12}}{\sin 3x}$$

[3]

6. [5 marks]

In triangle ABC, point D is on BC such that AD is perpendicular to BC and $|DB| = |DC|$.

Prove that triangle ABC is an isosceles triangle.

7. **[3 marks]**

There are four types of predators on the island that take chicks from the nest; cats, rats, lizards and gulls. The matrix *P* shows the proportion of chicks lost each day to each type of predator at each site.

$$P = \begin{array}{cccc} [0.015 & 0.01 & 0.005 & 0.018] \\ \text{cats} & \text{rats} & \text{lizards} & \text{gulls} \end{array}$$

The number of chicks at each nesting sites A, B and C in 2006 is given by the matrix

$$C = \begin{bmatrix} 10000 \\ 6500 \\ 9750 \end{bmatrix}$$

(a) Which of the matrix products *PC* or *CP* is defined? Explain why.

[1]

(b) (i) Form the matrix product that is defined and call it *R*.

(ii) Explain the meaning of the information that matrix *R* contains.

[2]

Calculator Assumed Section

1. [6 marks]

Determine the equation of the plane which passes through **A** <3, 2, 6>, **B** <1, –3, 10> and **C** <10, 0, 5>.

2. [6 marks]

A plane passes through the point **A** (2, –3, 4) and has a normal vector of $\begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$.

A line passes through **B** (16, –17, –8) and is parallel to the vector $\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$.

Determine where the line and plane intersect.

3. **[12 marks]**

Let $W = \begin{bmatrix} d-2 & -3 \\ -1 & d \end{bmatrix}$, $X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $Y = \begin{bmatrix} -1 & -4 \end{bmatrix}$ and $Z = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

(a) Evaluate each of the following where possible. If not possible, state this clearly and indicate the reason for your decision.

(i) WX

[2]

(ii) 2Y + Z

[2]

(b) Determine W^{-1}, stating all the necessary restrictions on d so that W^{-1} exists.

[4]

(c) Determine the matrix M which satisfies the equation:

$$\mathrm{MZ} - \frac{1}{4}\mathrm{M} = 2\mathrm{I}$$

where I is the 2×2 identity matrix.

[4]

4. [10 marks]

ABCD is a parallelogram with points E and F such that AE : EB = 1 : 2 and BF : FC = 1 : 3. G is the point where AF and ED intersect.

Let Let AB = $\boldsymbol{a}$ and AD = d.

Determine in what ratios AF and ED intersect each other.

5. **[6 marks]**

Use the method of proof by exhaustion to prove that every integer which is a perfect cube is either a multiple of 9, or 1 more, or 1 less than a multiple of 9.

6. [5 marks]

(a) Convert $z = 6 \operatorname{cis}\left(\frac{2\pi}{3}\right)$ to exact rectangular form.

[1]

(c) If w and z are both complex numbers, show that $\frac{z}{w} = \frac{z\,\overline{w}}{|w|^2}$.

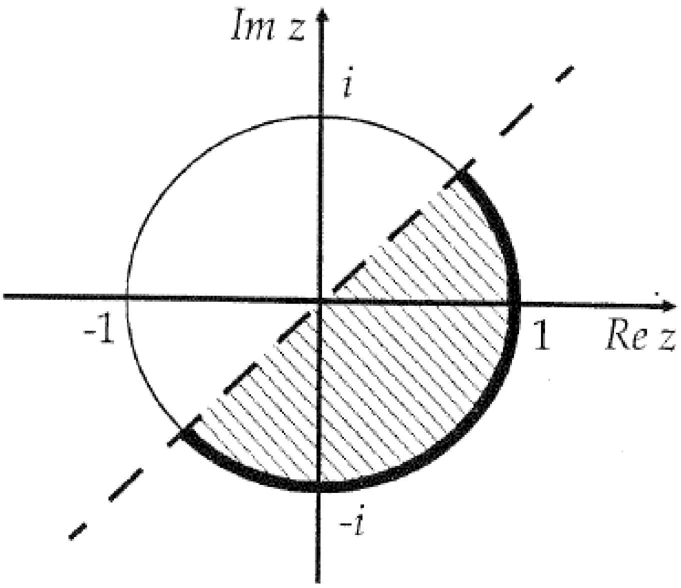
[4]

7. **[5 marks]**

Determine the equation of the tangent to the curve defined by $y^3 + 2xy = 9$ at the point P whose coordinates are (4, 1).

8. **[7 marks]**

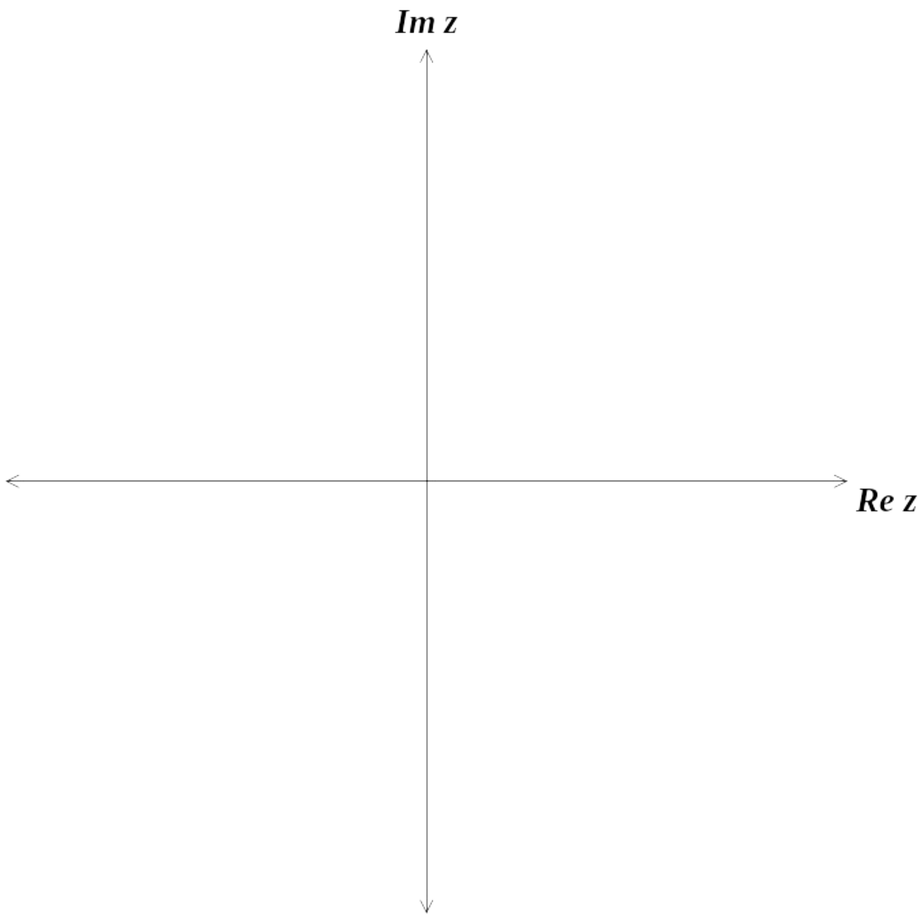
(a) Write down the inequalities on the modulus and argument of the complex number z such that together they describe the set of points shaded below.



[3]

(b) On the Argand plane provided, indicate the region defined by:

$$\{z:\ 3 \le |z - 4i| \le 4\} \cap \{z:\ -\frac{\pi}{4} \le Arg\,(z - 4i) \le \frac{\pi}{4}\}$$



[4]

9. **[10 marks]**

Two fighter jets are on a practice flight. At time t = 0 seconds, Jet A is at position $\begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$ km and jet B is at position $\begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix}$ km. Jet A is flying at a velocity of $\begin{pmatrix} 200 \\ 350 \\ 450 \end{pmatrix}$ m/s while jet B has a velocity of $\begin{pmatrix} -300 \\ -450 \\ 250 \end{pmatrix}$ m/s.

(a) At what time are the two jets closest to each other?

[6]

(b) Calculate the shortest distance between the two jets.

[4]

10. **[9 marks]**

A ladder 13 metres long rests against a vertical wall. If the bottom of the ladder slides away from the wall at the rate of 0.5 m/sec, and the bottom of the ladder is 5 feet from the wall,

(a) how fast is the top of the ladder sliding down the wall?

[5]

(b) at what rate is the angle between the ladder and the ground changing?

[4]

11. [4 marks]

The system of equations

$$2x + y + 7z = 9556$$
$$3x + y + 4z = 5899$$
$$5x + 2y + z = 3155$$

can be used to estimate the number of cats (x), rats (y), and lizards (z), on an island used as a nature reserve.

(a) Write this system of simultaneous linear equations in matrix form.

[1]

(b) Write down the inverse matrix that can be used to solve this system of simultaneous linear equations.

[1]

(c) Solve the system of simultaneous linear equations and hence estimate the number of cats, rats and lizards on the island.

[2]