Rossmoyne Senior High School

Semester One Examination, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNIT 1

Section Two:
Calculator-assumed

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	51	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	149	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Additional working space

Question number: __________

Additional working space

Question number: __________

Section Two: Calculator-assumed **65% (99 Marks)**

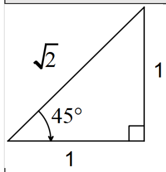
This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(6 marks)**

(a) Show how to establish that the exact value of $\cos 135^\circ$ is $-\frac{1}{\sqrt{2}}$. (3 marks)

Solution



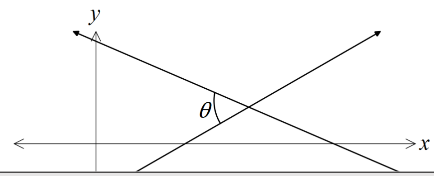
From diagram, it can be seen that $\cos 45 = \frac{1}{\sqrt{2}}$.

Using $\cos(180 - x) = -\cos x$, $\cos 135 = \cos(180 - 45) = -\cos 45 = -\frac{1}{\sqrt{2}}$.

Specific behaviours

✓ sketches isosceles triangle with angle and sides shown
✓ uses triangle to obtain cos45
✓ uses unit circle identity to obtain cos135

(b) The graphs of $x + 2y = 4$ and $2x - 3y = 3$ are shown below. Determine, to the nearest degree, the size of the angle θ. (3 marks)



Solution

$x + 2y = 4 \Rightarrow m_1 = -0.5 \Rightarrow \alpha = \tan^{-1} -0.5 = -26.6^\circ$

$2x - 3y = 3 \Rightarrow m_2 = \frac{2}{3} \Rightarrow \beta = \tan^{-1}\frac{2}{3} = 33.7^\circ$

$\theta = 26.6 + 33.7$
$= 60.3 \approx 60^\circ$ to nearest degree

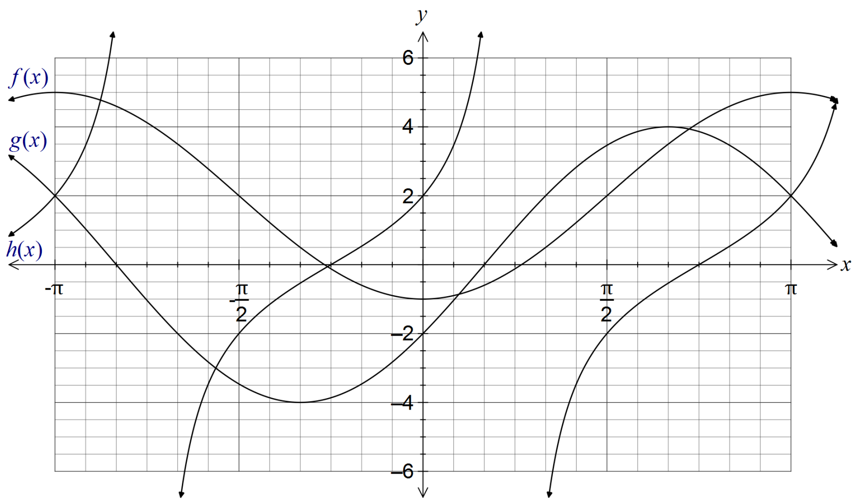
Specific behaviours

✓ determines both gradients
✓ calculates angle of lines to x-axis
✓ determines required angle

See next page

Question 9 **(7 marks)**

The graphs of the functions $f(x)=a-b\cos(x)$, $g(x)=c\sin(x-d)$ and $h(x)=m\tan(x+n)$ are shown below, where a, b, c, d, m and n are positive constants.



(a) Clearly label each of the functions f, g and h on the graph. (1 mark)

(b) Determine the values of the positive constants a, b, c, d, m and n. (6 marks)

Solution
Cos function has max value of 5, min of -1 and starts at min. $a=2,\ b=3$. Sin function has amplitude of 4 and first root at $\frac{\pi}{6}$. $c=4,\ d=\frac{\pi}{6}$. (Strictly, $d=\frac{\pi}{6}+2k\pi$) Tan function: Root at $-\frac{\pi}{4}$, and midway between root and asymptote, $h(x)=2$. $m=2,\ n=\frac{\pi}{4}$ (Strictly, $n=\frac{\pi}{4}+k\pi$)

Additional working space

Question number: __________

Additional working space

Question number: _________

Question 10 **(7 marks)**

(a) The extension, e, of a spring is directly proportional to the mass, m, hung on the end of it. When a mass of 100 g was hung on the spring, its extension was 25 mm.

(i) Write an equation that relates the variables e and m. (2 marks)

Solution
$e = km,\ 100 = k \times 25 \ \Rightarrow \ k = \frac{1}{4}$ $e = \frac{1}{4}m$
Specific behaviours
✓ writes linear equation using constant of proportionality   ✓ determines constant and writes in equation

(ii) Determine m when $e = 125$ mm. (1 mark)

Solution
$125 = \frac{1}{4}m \ \Rightarrow \ m = 500$ g
Specific behaviours
✓ determines m

(b) A full water tank can be emptied in 40 minutes using a small pump and in 10 minutes using a large pump. Assuming that the pumps do not affect each other when used together, determine the time required to empty the tank using both pumps. (4 marks)

Solution
$T = \frac{V}{R}$, where T = time, V = volume of tank, R = rate emptied $R_1 = \frac{V}{40},\ R_2 = \frac{V}{10}$ $T = \frac{V}{R_1 + R_2} = \frac{V}{\frac{V}{40} + \frac{V}{10}}$ $T = \frac{1}{\frac{1}{40} + \frac{1}{10}} = 1 \div \frac{5}{40} = \frac{40}{5} = 8$ minutes
Specific behaviours
✓ identifies inverse proportion   ✓ determines rate pumps empty in terms of V ✓ substitute rates into equation for T ✓ solves for time

NB choosing an arbitrary volume such as 40 is acceptable and simplifies working

See next page

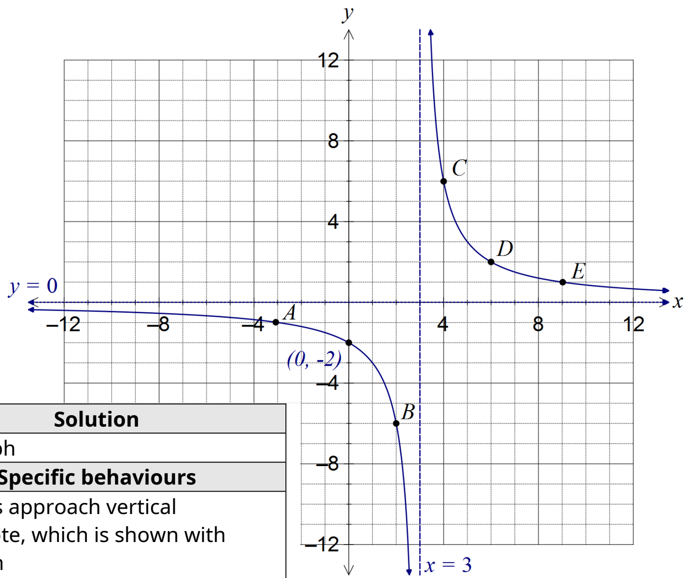
Question 11 **(7 marks)**

A function is defined by $f(x) = \dfrac{6}{x - 3}$.

(a) State the domain of this function. (1 mark)

Solution
$D_f \ \{x \in \mathbb{R},\ x \neq 3\}$
Specific behaviours
✓ states domain restriction

(b) Draw the graph of $y = f(x)$ on the axes below, clearly showing the coordinates of all axis-intercepts and equations of any asymptotes. (4 marks)



Solution
See graph
Specific behaviours
✓ curves approach vertical asymptote, which is shown with equation
✓ curves approach horizontal asymptote, without hooking upwards

(c) The graph of $y = f(x)$ is dilated vertically by a scale factor of 4 followed by a translation of three units to the right. Determine the coordinates of the y-intercept of the transformed graph. (2 marks)

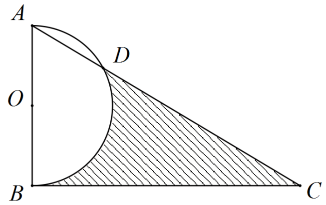
Solution
Transforms the point (-3, -1) to (-3, -4) to (0, -4)
or
$g(x) = 4f(x-3) = 4 \times \dfrac{6}{x-3-3} = \dfrac{24}{x-6} \Rightarrow g(0) = -4$ ie (0, -4)
Specific behaviours
✓ applies dilation
✓ applies translation

Question 20 **(7 marks)**

(a) Determine the exact area of a sector enclosed by an arc of length 42 cm in a circle of radius 12 cm. (2 marks)

Solution
$\theta = \dfrac{l}{r} = \dfrac{42}{12} = 3.5$
$A = \dfrac{1}{2}r^2\theta = \dfrac{1}{2} \times 12^2 \times 3.5 = 252 \text{ cm}^2$
Specific behaviours
✓ calculates angle
✓ calculates exact area

(b) In the diagram below, BC is a tangent to the circle with diameter AB and centre O. Given that $AB = 20$ cm and $BC = 30$ cm, determine the shaded area. (5 marks)



Solution
Areas: Segment $= A_{AD}$
Semi-circle $= A_{SC}$ TriangleABC $= A_{ABC}$
$\angle BAD = \tan^{-1}\dfrac{30}{20} \approx 0.9828$
$\angle AOD = \pi - 2 \times 0.9828 = 1.176$
$A_{AD} = \dfrac{1}{2}(10)^2(1.176 - \sin 1.176) \approx 12.65$
$A_{SC} = \dfrac{1}{2}\pi(10)^2 \approx 157.08$
$A_{ABC} = \dfrac{1}{2} \times 20 \times 30 = 300$
$A = 300 - (157.08 - 12.65) = 155.57 \text{ cm}^2$
Specific behaviours
✓ determines angle BAD
✓ determines angle AOD
✓ determines segment area
✓ determines semicircle and triangle area
✓ determines shaded area

Question 19 **(8 marks)**

(a) Given that $\tan\theta = -\frac{1}{3}$, where $\frac{\pi}{2} < \theta < \pi$, show how to determine the exact value of

(i) $\sin\theta$. (2 marks)

Solution
$o^2 + a^2 = h^2 \Rightarrow h = \sqrt{1^2 + 3^2} = \sqrt{10}$ $\sin\theta = \frac{o}{h} = \frac{1}{\sqrt{10}}$ (NB sin +ve in 2nd quadrant)
Specific behaviours
✓ uses right triangle to determine hypotenuse   ✓ states exact value

(ii) $\cos\theta$. (2 marks)

Solution
In 2nd quadrant, cos -ve $\cos\theta = -\frac{a}{h} = -\frac{3}{\sqrt{10}}$
Specific behaviours
✓ considers sign of cos in 2^{nd} quadrant   ✓ states exact value

(iii) $\sin 2\theta$. (2 marks)

Solution
$\sin 2\theta = 2\sin\theta\cos\theta$ $= 2 \times \frac{1}{\sqrt{10}} \times -\frac{3}{\sqrt{10}} = -\frac{3}{5}$
Specific behaviours
✓ substitutes into double angle identity   ✓ simplifies correctly

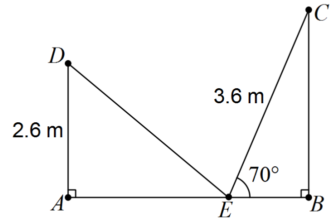
(b) Determine the two smallest solutions to the equation $6\sin\left(\frac{x}{5} - 50^\circ\right) = 3$ for $x \geq 0^\circ$. (2 marks)

Solution
Using CAS: $\theta = 400^\circ,\ 1000^\circ$, or: $\sin\left(\frac{x}{5} - 50\right) = \frac{1}{2}$ $\frac{x}{5} - 50 = 30, 150 \Rightarrow \frac{x}{5} = 80, 200 \Rightarrow x = 400^\circ,\ 1000^\circ$
Specific behaviours
✓ determines one solution   ✓ determines both solutions

See next page

Question 12 **(8 marks)**

(a) A 3.6 m long ladder first rests against a vertical wall BC, making an angle of 70° with the horizontal ground. The ladder is rotated in a vertical plane about E to rest against wall AD, reaching a point 2.6 m above the ground.



Showing use of trigonometry, determine

(i) the angle through which the ladder was rotated. (2 marks)

Solution
$\angle AED = \sin^{-1}\frac{2.6}{3.6} \approx 46.2^\circ$ $\angle CED = 180 - 70 - 46.2 = 63.8^\circ$
Specific behaviours
✓ calculates angle in triangle   ✓ determines rotation angle

(ii) the distance AB. (2 marks)

Solution
$AB = 3.6\cos 46.2 + 3.6\cos 70$ $= 2.49 + 1.23 = 3.72$ m
Specific behaviours
✓ determines AE ✓ determines EB and adds to get AB

(iii) the distance DC. (2 marks)

Solution
$DC^2 = 3.6^2 + 3.6^2 - 2 \times 3.6 \times 3.6 \times \cos 63.8$ $DC = 3.80$ m
Specific behaviours
✓ uses cosine rule   ✓ determines length

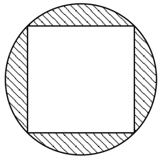
(b) A thin metal plate in the shape of an equilateral triangle has an area of 330 cm. Determine the side length of the triangle. (2 marks)

Solution
Let side length be x. Then $\frac{x^2}{2}\sin 60^\circ = 330 \Rightarrow x \approx 27.6$ cm
Specific behaviours
✓ uses area formula   ✓ determines length

See next page

Question 13 **(9 marks)**

(a) A square is inscribed in a circle of radius 16 cm, as shown below. Determine the area enclosed between the square and the circle. (3 marks)



Solution
$\theta = \frac{2\pi}{4} = \frac{\pi}{2}$
$A_s = \frac{1}{2}(16)^2\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right) = 128\left(\frac{\pi}{2} - 1\right)$
$4A_s = 4\times 128\left(\frac{\pi}{2} - 1\right) \approx 292.2$ cm^2
Specific behaviours
✓ determines segment angle ✓ determines one segment area ✓ determines area of all segments

(b) The perimeter of a sector, with central angle θ radians in a circle of radius *r*, is 12 cm.

(i) Express θ in terms of *r*. (2 marks)

Solution
$P = 2r + r\theta$
$12 - 2r = r\theta \Rightarrow \theta = \frac{12}{r} - 2$
Specific behaviours
✓ substitutes into equation for perimeter ✓ rearranges equation for θ

(ii) Show that the area of the sector is $6r - r^2$. (2 marks)

Solution
$A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{12}{r} - 2\right)$
$= 6r - r^2$
Specific behaviours
✓ substitutes *r* and θ into area formula ✓ expands and simplifies

(iii) Determine the area of the sector if $\theta = 1$. (2 marks)

Solution
$1 = \frac{12}{r} - 2 \Rightarrow r = 4$
$A = 6\times 4 - 4^2 = 8$ cm^2
Specific behaviours
✓ determines radius ✓ calculates area

Question 18 **(7 marks)**

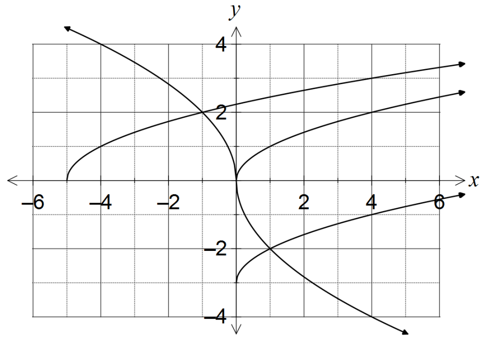
In triangle ABC, $\angle BAC = 50^\circ$, $AC = 18.4$ cm and $BC = 15$ cm.

Determine the largest possible area and smallest possible perimeter of this triangle.

Solution
$\frac{15}{\sin 50} = \frac{18.4}{\sin B}$
$\angle B = 70.0^\circ$ or 110.0°
$\angle C = 180 - 50 - \angle B$
$= 60.0^\circ$ or 20.0°
Largest area, require $\angle C = 60.0$
Area $= \frac{1}{2}\times 18.4\times 15\times \sin 60$
$= 119.5$ cm^2
Smallest perimeter, require $\angle C = 20.0$
$AB^2 = 18.4^2 + 15^2 - 2\times 18.4\times 15\times\cos 20 \Rightarrow AB = 6.70$
Perimeter $= 18.4 + 15 + 6.7 = 40.1$ cm
Specific behaviours
✓ uses sine rule to find acute angle for B ✓ determines obtuse angle for B ✓ calculates values of angle A ✓ chooses largest value of A for maximum area ✓ calculates area ✓ chooses smallest value of A for minimum perimeter

Question 17 **(9 marks)**

(a) The diagram below shows the five graphs $y=f(x)$, $y=f(x)+a$, $y=f(x+b)$, $y=cf(x)$ and $y=f(dx)$, where a, b, c and d are constants.



(i) Determine $f(4)$. (1 mark)

Solution
$f(4)=2$
Specific behaviours
✓ determines correct value

(ii) Determine the values of the constants a, b, c and d. (4 marks)

Solution
a is vertical translation: $a=-3$ b is horizontal translation: $b=5$ c is vertical dilation and reflection in x-axis: $c=-2$ d is horizontal dilation and reflection in y-axis: $d=-4$
Specific behaviours
✓ determines a ✓ determines b ✓ determines c

(b) Describe two transformations that will transform the graph of $y=g(x)$ to:

(i) $y=g(x-1)-2$. (2 marks)

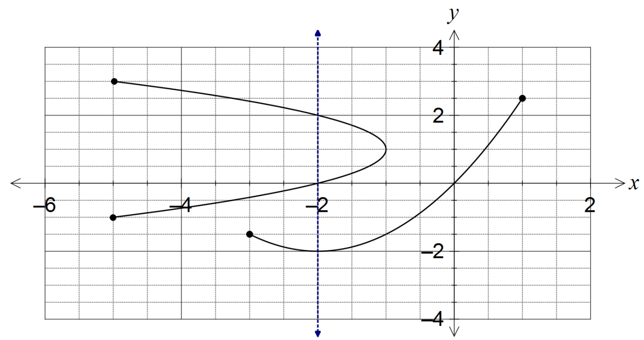
Solution
Translate original graph 1 unit right and 2 units downwards (in either order)
Specific behaviours
✓ horizontal translation ✓ vertical translation

(ii) $y=-5g(x)$. (2 marks)

Solution
Reflect in x-axis and dilate vertically by scale factor 5 (in either order)
Specific behaviours
✓ reflection ✓ dilation

Question 14 **(7 marks)**

A function and a relation have been graphed on the axes below.



(a) Draw the line $x=-2$ on the graph and explain how it can be used to identify the relation. (2 marks)

Solution
See line on graph. Vertical line cuts the relation more than once, but the function just once.
Specific behaviours
✓ draws vertical line ✓ explains vertical line test

(b) State the domain and range of the function. (2 marks)

Solution
$D_f=\{x:-3\le x\le 1\}$ $R_f=\{y:-2\le y\le 2.5\}$
Specific behaviours
✓ states domain ✓ states range

(c) The relation can be expressed in the form $y^2=ax+by-2$. Determine the values of the constants a and b. (3 marks)

Solution
When $x=-2,\ y=0,\ 2$. $0^2=-2a+0-2 \Rightarrow a=-1$ $2^2=-2(-1)+2b-2 \Rightarrow b=2$
Specific behaviours
✓ selects suitable point from relation ✓ determines a ✓ determines b

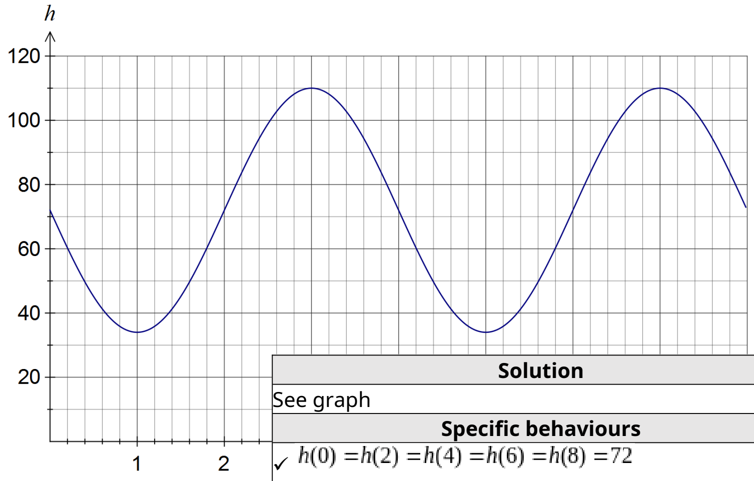
Question 15 **(9 marks)**

A sensor was fitted to the tip of a blade on a wind turbine to measure the height, h metres, of the blade above the ground. The height was observed to vary according to the function $h(t)=72-38\sin\left(\frac{\pi t}{2}\right)$, where t is the time in seconds since measurements began.

(a) Determine the height of the blade tip above the ground when $t=3$. (1 mark)

Solution
$h(3)=110$ m
Specific behaviours
✓ calculates height

(b) Sketch the graph of $h(t)$ on the axes below for $0\le t\le 8$. (4 marks)



Solution
See graph
Specific behaviours
✓ $h(0)=h(2)=h(4)=h(6)=h(8)=72$
✓ minimums at (1, 34) and (5, 34)
✓ maximums at (3,110) and (7, 110)
✓ smooth curve through above points, no abrupt

(c) How long does the blade take to rotate once? (1 mark)

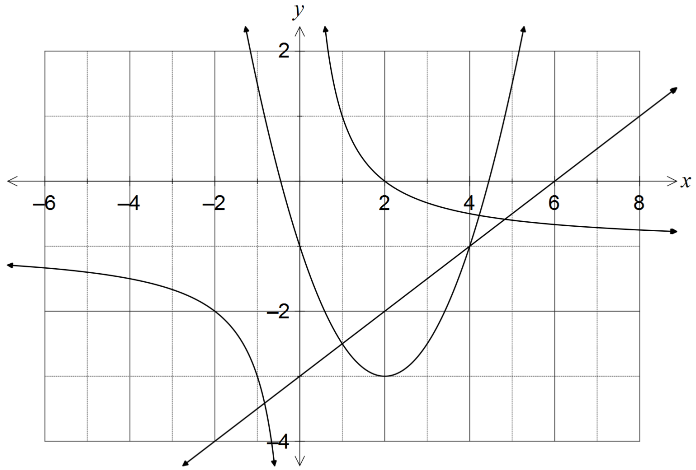
Solution
4 seconds
Specific behaviours
✓ states time

(d) Assuming the blade continues to rotate in this manner, determine the percentage of time during which the blade tip is at least 90 m above the ground. (3 marks)

Solution
$h(t)=90 \Rightarrow t=2.3142,\ 3.6858,\ \ldots$
$3.6858-2.3142=1.3716$
$\frac{1.3716}{4}\times 100\approx 34.3\%$
Specific behaviours
✓ solves for height of 90 m
✓ determines interval above 90
✓ determines percentage

Question 16 **(7 marks)**

The graphs of $ax+by=6$, $y=\frac{c}{x}+d$ and $y=n(x-p)^2+q$ are shown below. Determine the values of the constants a, b, c, d, n, p and q.



Solution
Linear function:
$a(6)+b(0)=6 \Rightarrow a=1$
$a(0)+b(-3)=6 \Rightarrow b=-2$
Quadratic:
Turning point $\Rightarrow p=2,\ q=-3$
$y=n(x-2)^2-3$
$-1=n(0-2)^2-3 \Rightarrow n=\frac{1}{2}$
Hyperbolic:
Asymptote $\Rightarrow d=-1$
$0=\frac{c}{2}-1 \Rightarrow c=2$
Specific behaviours
✓ each correct value