

**Papers written by
Australian Maths
Software**

SEMESTER TWO

REVISION 1

MATHEMATICS METHODS

UNITS 3-4

2016

SOLUTIONS

SECTION ONE

1. (8 marks)

$$(a) \quad \int (2x+4)^6 dx = \frac{(2x+4)^7}{7 \times 2} + c = \frac{(2x+4)^7}{14} + c$$

$$(b) \quad \int_{\pi/4}^{\pi/2} 2 \sin(x) - \cos(x) dx$$

$$= \left[-\cos(x) - \sin(x) \right]_{\pi/4}^{\pi/2}$$

$$= - \left(\left(\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) \right) - \left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) \right) \right)$$

$$= -1 + \sqrt{2}$$

$$(c) \quad \int \left(x^4 + e^{2x} + \frac{2}{x} \right) dx = \frac{x^5}{5} + \frac{e^{2x}}{2} + 2 \ln(x) + c \quad \checkmark \checkmark \quad -1/error$$

2. (16 marks)

$$(a) \quad (i) \quad f(x) = \ln\left(\frac{x^2 - 3}{1+x}\right) = \ln(x^2 - 3) - \ln(1+x)$$

$$f'(x) = \frac{2x}{(x^2 - 3)} - \frac{1}{(1+x)}$$

$$(ii) \quad g(x) = \frac{e^{\sin(x)}}{\cos(x)}$$

$$g'(x) = \frac{(e^{\sin(x)} \cos(x)) \cos(x) - (-\sin(x)) e^{\sin(x)}}{(\cos(x))^2}$$

$$g'(x) = \frac{e^{\sin(x)} (\cos^2(x) + \sin(x))}{\cos^2(x)}$$

$$(iii) \quad h(x) = e^x \times \ln(x^2) = 2e^x \times \ln(x)$$

$$h'(x) = 2 \left(e^x \times \ln(x) + \frac{e^x}{x} \right) \quad \checkmark \checkmark$$

$$h'(x) = 2e^x \left(\ln(x) + \frac{1}{x} \right)$$

(b) (i) Given $g(x) = \sqrt{\sin(x)}$ show that $g'(x) = \frac{\cos(x)}{2\sqrt{\sin(x)}}$.

$$g'(x) = \frac{1}{2} (\sin(x))^{-\frac{1}{2}} \times \cos(x)$$

$$g'(x) = \frac{\cos(x)}{2\sqrt{\sin(x)}}$$

(ii)
$$\begin{aligned} \int -\frac{3\cos(x)}{\sqrt{\sin(x)}} dx &= -3 \times 2 \int \frac{\cos(x)}{2\sqrt{\sin(x)}} dx \\ &= -6 \int \frac{\cos(x)}{2\sqrt{\sin(x)}} dx \\ &= -6\sqrt{\sin(x)} + c \end{aligned}$$

(c) Given $\int_0^4 f(x) dx = -6.4$ and $\int_4^{10} f(x) dx = 2.3$.

$$2 \int_0^4 = 2 \int_0^{10} - 2 \int_4^{10}$$

$$\begin{aligned} 2 \int_0^4 (1 - f(x)) dx &= 2 \left(\int_0^4 1 dx - \int_0^4 f(x) dx \right) \\ &= 2 \left([x]_0^4 - \left(\int_0^{10} f(x) dx - \int_4^{10} f(x) dx \right) \right) \\ &= 2(4 - (-6.4 - 2.3)) \\ &= 2(4 + 8.7) \\ &= 2 \times 12.7 \\ &= 25.4 \end{aligned}$$

(d) $\frac{dr}{d\theta} = \frac{dr}{dt} \times \frac{dt}{dx} \times \frac{dx}{d\theta}$ $t = 4x = 4\cos(\theta)$

$$\frac{dr}{d\theta} = \frac{1}{2} t^{-\frac{1}{2}} \times 4 \times (-\sin(\theta))$$

$$\frac{dr}{d\theta} = -\frac{2\sin(\theta)}{\sqrt{t}}$$

$$\frac{dr}{d\theta} = -\frac{2\sin(\theta)}{\sqrt{4\cos(\theta)}}$$

$$\frac{dr}{d\theta} = -\frac{\sin(\theta)}{\sqrt{\cos(\theta)}}$$

3. (6 marks)

$$(a) \quad \frac{\log_{10}(4 \times 3^2) - \log_{10}(3 \times 6) - 3\log_{10} 2}{-2\log_{10} 2}$$

$$= \frac{\log_{10}\left(\frac{4 \times 3^2}{3 \times 6 \times 8}\right)}{-2\log_{10} 2}$$

$$= \frac{\log_{10}\left(\frac{1}{4}\right)}{-2\log_{10} 2}$$

$$= \frac{-2\log_{10} 2}{-2\log_{10} 2}$$

$$= 1$$

$$(b) \quad (\log_3(x) - 1)(\ln(x) - 1) = 0$$

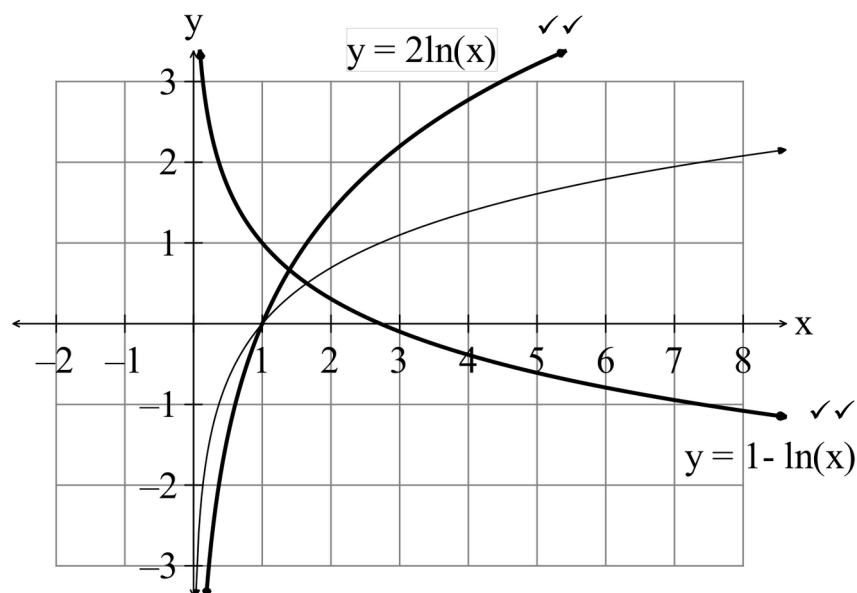
$$\log_3(x) - 1 = 0 \quad \text{or} \quad \ln(x) - 1 = 0$$

$$\log_3(x) = 1 \quad \text{or} \quad \ln(x) = 1$$

$$x = 3 \quad \text{or} \quad x = e$$

4. (6 marks)

(a) (i)



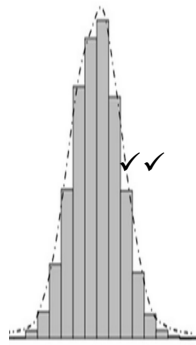
$$(ii) \quad f(x) = \ln(x) \Rightarrow f^{-1}(x) = e^x \quad \text{for } x \in \mathbb{R}$$

5. (16 marks)

(a) (i) Not a probability density functions as you cannot have negative probabilities.

(ii) Is a probability density functions as the probabilities add to one.

(b) The shape will be a tightly clustered histogram, close to a normal curve.



(c) (i) $\int_1^e \frac{1}{x} dx = [\ln(x)]_1^e = \ln(e) - \ln(1) = 1 - 0 = 1$

(ii) $P(1 \leq x \leq 2) = \int_1^2 \frac{1}{x} dx = \ln(2) - \ln(1) = \ln(2)$

(d) $F(x) = \int_0^x \frac{x}{2} dx = \frac{1}{4} [x^2]_0^x = \frac{x^2}{4}$

(e) $p(x) = \begin{cases} 0.2 & \text{for } 5 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$

$$E(x) = \int_a^b x \times p(x) dx$$

$$E(x) = \int_5^{10} x \times 0.2 dx = 0.2 \left[\frac{x^2}{2} \right]_5^{10} = 0.1(100 - 25) = 7.5$$

$$\text{Var}(x) = \int_a^b p(x)(x - \mu)^2 dx$$

$$\text{Var}(x) = \int_5^{10} 0.2(x - 7.5)^2 dx \quad \checkmark \checkmark$$

END OF SECTION ONE

SECTION TWO

6. (7 marks)

$$(a) \quad A = \int_0^a e^x dx = \left[e^x \right]_0^a = e^a - e^0 = e^a - 1$$

$$(b) \quad A = e^a - 1$$

$$\frac{dA}{da} = e^a$$

$$\frac{\delta A}{\delta a} \approx \frac{dA}{da}$$

$$\delta A \approx \frac{dA}{da} \times \delta a$$

$$\text{At } a=3, \delta a=0.1$$

$$\delta A \approx e^3 \times 0.1$$

$$\delta A \approx 2.0086$$

7. (6 marks)

$$(a) \quad v = 10t - 1 \text{ ms}^{-1}.$$

$$a = 10 \text{ ms}^{-2}$$

$$x = \int (10t - 1) dt$$

$$x = 5t^2 - t + c$$

$$\text{At } t=0, x=3$$

$$x = 5t^2 - t + 3$$

(b) Changes direction when $v=0$ i.e. at $t=0.1$ (c) At $t=0$, $x=3$

$$\text{At } t=0.1, x = 0.05 - 0.1 + 3 = 2.95$$

$$\text{At } t=5, x=123$$

$$\text{Distance travelled} = 123 - 2.95 + 0.05 = 120.1 \text{ m}$$

8. (7 marks)

(a) (i) $f(t) = \sqrt{\sin(\pi t)}$

$$f'(t) = \frac{1}{2} (\sin(\pi t))^{-\frac{1}{2}} \pi \cos(\pi t) \quad \checkmark$$

$$f'(t) = \frac{\pi \cos(\pi t)}{2\sqrt{\sin(\pi t)}}$$

$$\begin{aligned}
 \text{(ii)} \quad \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\pi \cos(\pi t)}{2\sqrt{\sin(\pi t)}} dt &= \left[\sqrt{\sin(\pi t)} \right]_{\frac{1}{6}}^{\frac{1}{2}} \\
 &= \sqrt{\sin\left(\frac{\pi}{2}\right)} - \sqrt{\sin\left(\frac{\pi}{6}\right)} \\
 &= 1 - \frac{1}{\sqrt{2}}
 \end{aligned}$$

(b) $F(x) = \frac{d}{dx} \int_1^x \left(\frac{1}{t} \right) dt = \frac{1}{x}$

$$\int_1^2 F(x) dx = \int_1^2 \frac{1}{x} dx = \left[\ln(x) \right]_1^2 = \ln(2) - \ln(1) = \ln(2)$$

9. (8 marks)

(a) Turning points occur when $f'(x) = 0$.

There are no points where this occurs so there are no turning points.

Likewise, there are no points where $f''(x) = 0$, so there are no points of inflection.(b) $y = f''(x) > 0$ which suggests that the concavity is concave upwards for all x values. $\checkmark\checkmark$

(c) $f(x) = \ln(x), \quad f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}$

10. (7 marks)

(a) $0.693 \quad \checkmark\checkmark \quad (= \ln(2))$

(b) (i) True as +C can have any value

(ii) No, this is bounded and has only one solution.

$$\int_1^2 3x^2 dx = \left[x^3 \right]_1^2 = 8 - 1 = 7$$

(iii) True.

(iv) $\ln(f(x)) = \ln(3) + 2\ln(x)$

Not valid as initially, x can be negative as it is squared.

Need $2\ln|x| \quad \checkmark\checkmark$

11. (8 marks)

(a) $t=0, \quad P=23$

$$78.9 = 23e^{60(k)}$$

$$k = 0.02054478$$

(b) $t=66, \quad P=89.25$ (million)

(c) The actual population is smaller than 89.25 so the growth rate is slowing down (minimally!)

(d) 2010 $78.9 \times 10^6 \quad t=0$

2016 $87\,238\,973$

$$P = 78.9$$

$$\text{At } t=6, \quad 87\,238\,973 = 78.9 \times 10^6 e^{6k}$$

$$k = 0.01674499021$$

$$P = 78.9 \times 10^6 \times e^{0.01674499021t}$$

$$P = 100 \times 10^6, t = ?$$

$$100 = 78.9 \times e^{0.01674499021t}$$

$$t = 14.15$$

The population is expected to reach 100 million just into 2025.

12. (7 marks)

$$(a) \text{ Area} = \int_{0.05}^{1.47} (\ln(x) - (e^x - 4)) = 1.68 \text{ units}^2$$

$$(b) (i) P(0) = 22 \ln(3) = 24.169 \approx 24$$

$$(ii) 100 = 22 \ln(t + 3)$$

$$t = 91.203$$

$$2002 + 91 = 2093$$

The population will reach 100 in 2094 or just into 2094.

13. (7 marks)

$$(a) A = x \times y \quad y^2 = 1 - x^2$$

$$A = x\sqrt{1 - x^2}$$

$$(b) \text{ Maximum area when } \frac{dA}{dx} = 0 \text{ and } \frac{d^2A}{dx^2} < 0$$

$$\frac{dA}{dx} = -\frac{2x^2 - 1}{\sqrt{1 - x^2}}$$

$$\frac{d^2A}{dx^2} = -\frac{2x^3\sqrt{1-x^2} + 4x\sqrt{(1-x^2)^3} - x\sqrt{1-x^2}}{(x^2-1)^2}$$

$$\text{If } \frac{dA}{dx} = 0, \quad 2x^2 - 1 = 0$$

$$x^2 = 0.5$$

$$x = \frac{1}{\sqrt{2}} \quad x > 0$$

Max or min?

$$\frac{d^2A}{dx^2} = -\frac{4\left(\frac{1}{2\sqrt{2}}\right)\sqrt{0.5} + 4\left(\frac{1}{2\sqrt{2}}\right) - \frac{1}{\sqrt{2}}(\sqrt{0.5})}{2} = -\frac{\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)\sqrt{0.5} + 4\left(\frac{1}{2\sqrt{2}}\right)}{2} < 0$$

\therefore max

$$x = \frac{1}{\sqrt{2}}, \quad y = \sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}}$$

$$\text{Therefore } P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

14. (6 marks)

(a) (i)

x	0	1	2	3	4	5	≥ 6	✓✓
$P(X=x)$	0.1465	0.366	0.3945	0.068	0.015	0.0075	0.0025	

(ii) $0.366 + 0.3945 = 0.7605$

✓
✓

(iii) $E(x) = \sum x_i \times P(x_i)$
 $= 0 + 1 \times 0.366 + 2 \times 0.3945 + 3 \times 0.068 + 4 \times 0.015 + 5 \times 0.0075 + 6 \times 0.0025$
 $E(x) = 1.4715$

15. (7 marks)

(a) $P(\text{a given battery will last at least 400 hours}) = 0.006209665326 \approx 0.0062$ ✓✓

(b) $P(\text{a given battery will last between 320 and 380 hours}) = 0.8663855975 \approx 0.866$ ✓✓

(c) $P(\text{Jenny's battery will run out in a 3 hour exam if it has been used for 340 hours})$

$$= \frac{P(340 \leq x \leq 343)}{P(x \geq 340)} = \frac{0.0546318101}{0.6914624613} = 0.07900907592$$

Therefore $P(\text{Jenny's battery will run out in a 3 hour exam if it has been used for 340 hours already}) \approx 0.08$

16. (6 marks)

(a) $P(\text{there are 5 girls and 5 boys}) = 0.24609375$ ✓✓

(b) $P(\text{there are no more than 4 boys}) = 0.376953125$ ✓✓

(c) BB(8 more births) Binomial with $n = 8$

$P(\text{exactly 4 girls}) = 0.2734375$

17. (22 marks)

(a) (i) A biased sample is one in which not every sample point has equal chance of being selected. ✓✓

(ii) A sample that is chosen in such a way that each point has an equal chance of being selected. The process requires each possible selection point being numbered and then the sample is generated using random numbers. ✓✓

(iii) A skewed distribution is a set of data where a large proportion of the data is clustered at one end. ✓✓

$$p = 0.2 \quad q = 0.98 \Rightarrow np = 100 \times 0.2 = 20 > 5$$

$$nq = 100 \times 0.8 = 80 > 5 \text{ so can use normal distribution.}$$

$$\text{Mean} = p = 0.2$$

$$sd_{\hat{p}} = \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} = \sqrt{\frac{0.2 \times 0.8}{100}}$$

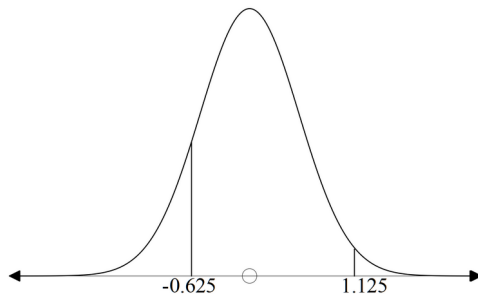
$$sd_{\hat{p}} = 0.04$$

Standardised score (using 17.5 to 24.5)

$$z = \frac{X - \mu}{\sigma}$$

$$z_{0.075} = \frac{\frac{17.5}{100} - 0.2}{0.04} = -0.625$$

$$z_{0.145} = \frac{\frac{24.5}{100} - 0.2}{0.04} = 1.125$$



$$P(-0.625 \leq z \leq 1.125) = 0.6037199538$$

The probability that between 18 and 24 of the wine tasters should not drive is 0.604.

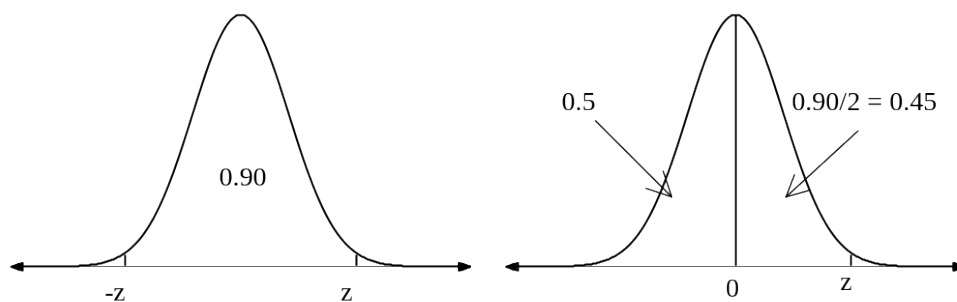
(c) (i) $\hat{p} = \frac{180}{200} = 0.9$

(ii) $\hat{p} = 0.9$

$$sd_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.9 \times 0.1}{200}}$$

$$sd_{\hat{p}} = 0.0212132$$

(iii)



$$P(X < z) = 0.95$$

$$z = 1.645$$

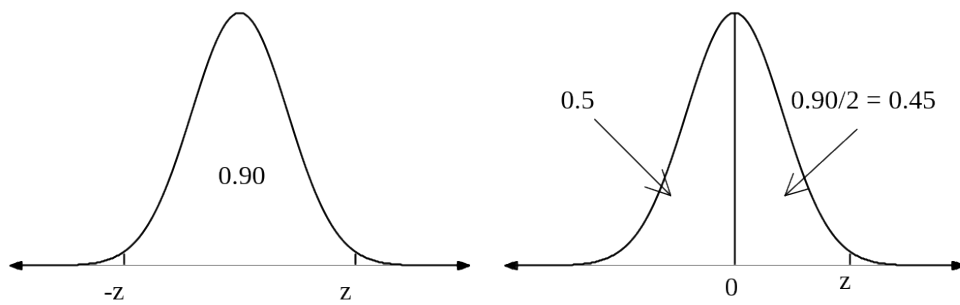
$$sd_{\hat{p}} = 0.0212132$$

$$E = z \times s = 1.645 \times 0.0212132$$

$$E \approx 0.0349$$

The 90% confidence limit are 0.9 ± 0.03 i.e. (0.90, 0.96)

(d)



$$P(X < z) = 0.95$$

$$z = 1.645$$

Use $p = 0.5$ as the maximum value as p is unknown.

$$\text{So with } p = 0.5 \quad sd = \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} = \sqrt{\frac{0.25}{n}}$$

$$E = z \times s \quad \text{but } E = 0.05$$

Therefore

$$0.05 = 1.645 \times \sqrt{\frac{0.25}{n}}$$

$$n = 270.6$$

You need to survey 271 people to have an error margin of 5% at a confidence level of 90%

END OF SECTION TWO