Papers written by
Australian Maths
Software

SEMESTER TWO

MATHEMATICS SPECIALIST

REVISION 3

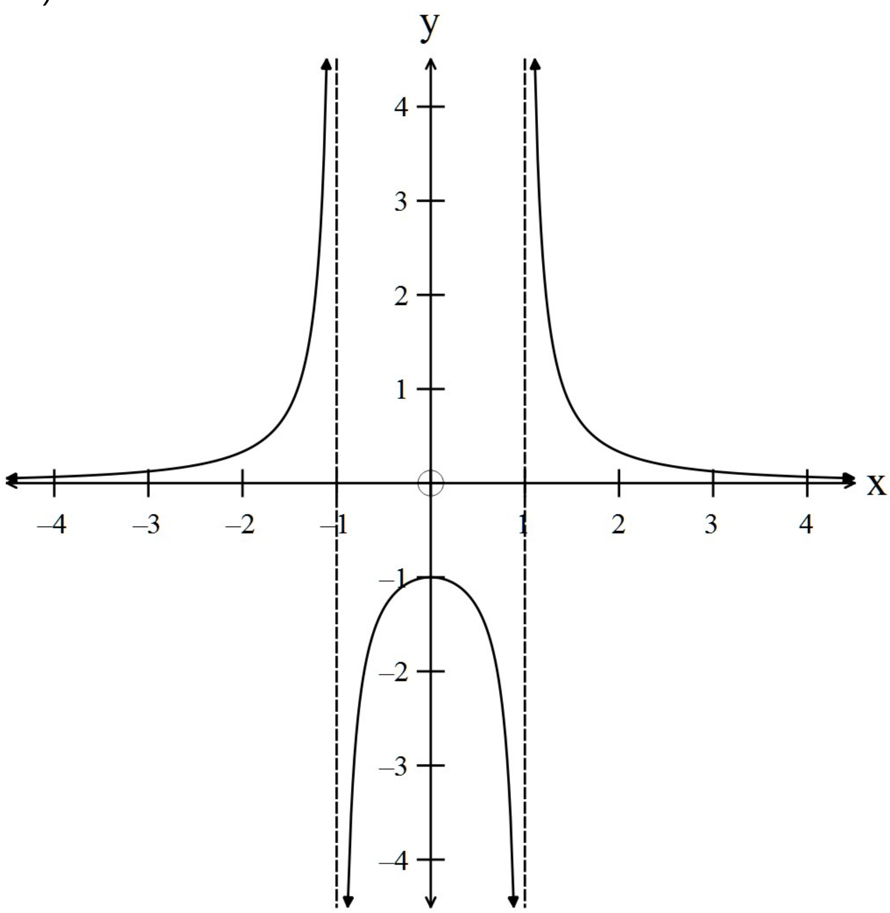
UNIT 3-4

2016

SOLUTIONS

Section One

1. (3 marks)



2. (5 marks)

(a) $x^4 + 5x^2 + 4 = 0.$

$(x^2 + 1)(x^2 + 4) = 0$

$x^2 = -1 \quad or \quad x^2 = -4$

$x = \pm i \quad or \quad x = \pm 2i$

(b) $z = 2cis\left(\frac{\pi}{4} + n\frac{\pi}{2}\right)$

$z^4 = 16\left(cis\left(\frac{\pi}{4} + n\frac{\pi}{2}\right)\right)^4$

$z^4 = 16\,cis(\pi + n2\pi)$

$\quad = 16(cos(\pi) + i\,sin(\pi))$

$z^4 = -16$

3. (7 marks)

(a) $$\frac{2x+3y}{x^2-y^2}=4$$

$$2x+3y=4x^2-4y^2 \quad for \quad x \neq \pm y$$

$$2+3\frac{dy}{dx}=8x-8y\frac{dy}{dx} \qquad ✓✓$$

$$\frac{dy}{dx}(3+8y)=8x-2$$

$$\frac{dy}{dx}=\frac{8x-2}{3+8y}$$

(b) $$If \quad x=0, \quad \frac{3y}{-y^2}=4$$

$$4y^2+3y=0$$

$$y(4y+3)=0$$

$y=0$ or $y=-\frac{3}{4}$

but $x \neq y$, so $y=-\frac{3}{4}$ only

$$\frac{dy}{dx}=\frac{8x-2}{3+8y}$$

At $\left(0,-\frac{3}{4}\right)$ $\quad \frac{dy}{dx}=\frac{-2}{3+8\left(-\frac{3}{4}\right)}$

$$\frac{dy}{dx}=\frac{2}{3}$$

4 (9 marks)

(a) $$\int \frac{dx}{\cos^2(3x)} = \int \sec^2(3x)\,dx = \frac{\tan(3x)}{3} + c$$
✓✓

(b) $$\int \sin^5(x)\cos^3(x)\,dx$$

$$= \int \sin^5(x)\left(1 - \sin^2(x)\right)\cos(x)\,dx$$

$$= \int \left(\sin^5(x) - \sin^7(x)\right)\cos(x)\,dx$$

put $u = \sin(x)$

$$\frac{du}{dx} = \cos(x)$$

$$du = \cos(x)\,dx$$

$$\equiv \int \left(u^5 - u^7\right)du$$

$$= \frac{u^6}{6} - \frac{u^8}{8} + c$$

$$= \frac{\sin^6(x)}{6} - \frac{\sin^8(x)}{8} + c$$

(c) $$\int_1^2 \left(\frac{1}{1+x} - e^x\right)dx = \left[\ln(1+x) - e^x\right]_1^2$$

$$= \left(\ln(3) - e^2\right) - \left(\ln(2) - e^1\right)$$

$$= \ln\left(\frac{3}{2}\right) - e + e$$

5. (5 marks)

(a) $z = 2 - 3i$ and $z = 2 + 3i$

$$\left(z - (2 - 3i)\right)\left(z - (2 + 3i)\right) = 0$$

$$z^2 - z\left(2 - \cancel{3i} + 2 + \cancel{3i}\right) + (2 - 3i)(2 + 3i) = 0$$

$$z^2 - 4z + 4 - 9i^2 = 0$$

$$z^2 - 4z + 13 = 0$$

(b) $$\left\{z : -\frac{\pi}{4} < \arg(z) < \frac{\pi}{4} \quad \cap \quad r \le 4\right\}$$ ✓✓✓

6. (10 marks)

(a)

$\mathbf{r}(t) = (10cos(2t))\mathbf{i} + 6(sin(t))\mathbf{j}$

$x = 10\,cos(2t)$ and $y = 6\,sin(t)$

$cos(2t) = 1 - 2sin^2(t)$

$\frac{x}{10} = 1 - 2\left(\frac{y}{6}\right)^2$

$\therefore x = 10\left(1 - \frac{y^2}{18}\right)$

which is parabolic

(b) At end points when $y = \pm 6$.

$y = 6\,sin(t)$

$6\,sin(t) = \pm 6 \quad when \;\; t = \pm\frac{\pi}{2}$ ✓✓

(c)

$\mathbf{r}(t) = (10cos(2t))\mathbf{i} + (6sin(t))\mathbf{j}$

$\mathbf{v}(t) = (-20sin(2t))\mathbf{i} + (6cos(t))\mathbf{j}$

$\mathbf{a}(t) = (-40cos(2t))\mathbf{i} + (-6sin(t))\mathbf{j}$

If $\mathbf{r}(t) = -\mathbf{a}(t)$

then $\begin{pmatrix} 10cos(2t) \\ 6sin(t) \end{pmatrix} = -\begin{pmatrix} -40cos(2t) \\ -6sin(t) \end{pmatrix}$

$i.e. \begin{pmatrix} 10cos(2t) \\ 6sin(t) \end{pmatrix} = \begin{pmatrix} 40cos(2t) \\ 6sin(t) \end{pmatrix}$

$i.e. \;\; cos(2t) = 0$

$2t = \pm\frac{\pi}{2} + n\pi$

$t = \pm\frac{\pi}{4} + \frac{n\pi}{2}$

$i.e. \quad t = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4} \;.....$

Where? $\mathbf{r}(t) = (10cos(2t))\mathbf{i} + (6sin(t))\mathbf{j}$

$(0, 3\sqrt{2}), (0, -3\sqrt{2})$

7. (5 marks)

Since one of the direction vectors of the plane is the same as the direction vetor of the line, the line is either parallel ti the plane or contained IN the plane. ✓
Need to determine if the point P(2, 0, 1) belongs to the plane.

Plane:

$$r = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$$

$$x = 1 - s - 3t, \qquad y = 1 + s + 2t, \qquad z = 2 - s$$

$$If \ z = 1 \qquad\qquad 1 = 2 - s$$

$$s = 1$$

$\Rightarrow x = -3t$ and $y = 2 + 2t$

If $x = 2$, then $t = -\dfrac{2}{3} \Rightarrow \ y = 2 + 2\left(-\dfrac{2}{3}\right) = 2 - \dfrac{4}{3}$

$$y = \frac{2}{3}$$

But $y = 0$

Therefore the point does NOT belong to the plane.
Therefore the line is parallel to the plane.

8. (3 marks)

$$\frac{\left(cis\left(\frac{\pi}{6}\right)\right)^{-6} cis\left(\frac{3\pi}{4}\right)}{(1-i)^3(1+i)^2}$$

$$= \frac{cis\left(-\pi + \frac{3\pi}{4}\right)}{(1-i^2)^2(1-i)}$$

$$= \frac{cis\left(-\frac{\pi}{4}\right)}{4(1-i)} \times \frac{(1+i)}{(1+i)}$$

$$= \frac{cis\left(-\frac{\pi}{4}\right) \times \sqrt{2}cis\left(\frac{\pi}{4}\right)}{4(2)}$$

$$= \frac{\sqrt{2}cis(0)}{8}$$

$$= \frac{\sqrt{2}\left(cos(0) + i\cancel{sin(0)}\right)}{8}$$

$$= \frac{\sqrt{2}}{8}$$

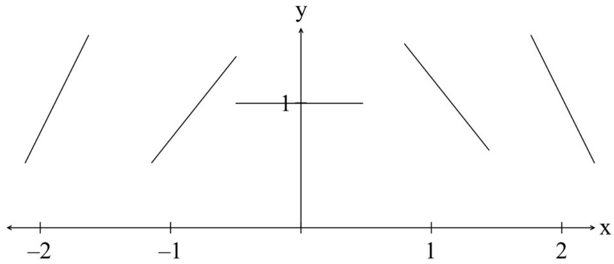
9. (6 marks)

(a)

x	-2	-1	0	1	2
y	1	1	1	1	1
$\frac{dy}{dx}$	**4**	**2**	**0**	**-2**	**-4**

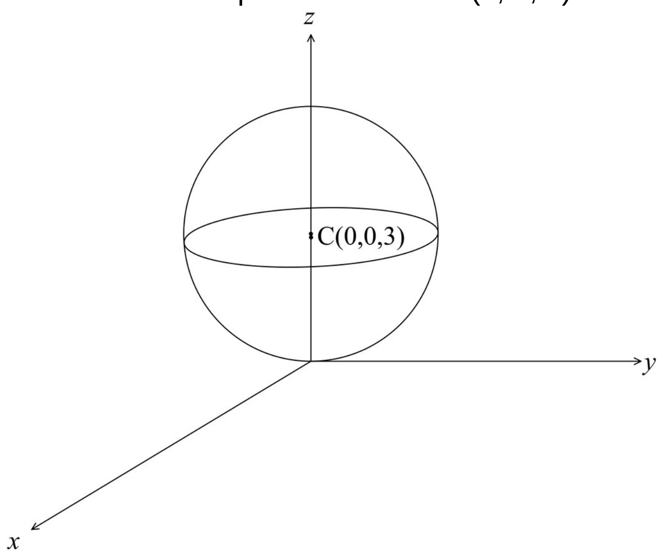
✓✓ -1/error

(b)



✓✓ -1/error

(c) $x^2 + y^2 + (z - 3)^2 = 9$ is a sphere with centre (0, 0, 3) and radius 3.



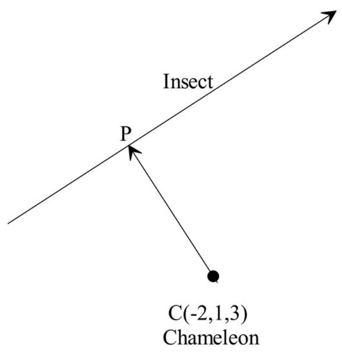
END OF SECTION ONE

Section Two

10. (5 marks)

$$\mathbf{r}(t) = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} t = \begin{pmatrix} 1 \\ -2+t \\ 2+2t \end{pmatrix}$$

(a) Path of insect:



$$\cancel{\mathbf{PC} = \begin{pmatrix} 3 \\ 2-t \\ 1-2t \end{pmatrix}}$$

At closest point $\mathbf{PC} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$

$$\cancel{\begin{pmatrix} 3 \\ 2-t \\ 1-2t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0}$$

$$0 + 2 - t + 2 - 4t = 0$$
$$\cancel{5t = 4}$$
$$\cancel{t = 0.8}$$

$$\cancel{\mathbf{r}(0.8) = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} 0.8} = \begin{pmatrix} 1 \\ -1.2 \\ 3.6 \end{pmatrix} \text{....closest point}$$

Distance between closest point and chameleon at C(-2,1,3) is

$$d = \sqrt{(-3)^2 + 2.2^2 + (-0.6)^2}$$
$$d = 3.8\,cm$$

The insect will be dinner!!!

Mrs Da Cruz's Solution:

$$r_I(t) = \begin{pmatrix} 1 \\ -2+t \\ 2+2t \end{pmatrix} cm$$

$$_I r_C(t) = \begin{pmatrix} 3 \\ -3+t \\ -1+2t \end{pmatrix} cm$$

Distance between insect and Chameleon at any time is given by:

$$|{_I r_C}| = \sqrt{9 + (-3+t)^2 + (-1+2t)^2}$$

fMin on ClassPad gives:

Minimum distance is 3.74 cm at t=1sec

The insect will be dinner.

Or Closest when $\begin{pmatrix} 3 \\ -3+t \\ -1+2t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$

-3+t-2+4t=0 $\quad \therefore t = 1 \quad \therefore |{_I r_C}| = \sqrt{14} \approx 3.74\,cm$

(b) The set of points represent a sphere of centre (1, -2, 0) and radius 3. ✓

11. (8 marks)

Given $z = cos(x) + i\,sin(x)$ and $\frac{1}{z} = cos(x) - i\,sin(x)$

(a) (i) $z + \frac{1}{z} = 2\,cos(x)$

$z - \frac{1}{z} = 2i\,sin(x)$ ✓

(ii) $z^n = (cos(x) + i\,sin(x))^n = cos(nx) + i\,sin(nx)$ ✓

$$\frac{1}{z^n} = \frac{1}{(cos(x) + i\,sin(x))^n}$$

$$= (cos(x) + i\,sin(x))^{-n}$$

$$= (cos(-nx) + i\,sin(-nx))$$

but $cos(-\theta) = cos(\theta)$ and $sin(-\theta) = -sin(\theta)$

$$\frac{1}{z^n} = (cos(nx) - i\,sin(nx))$$

$$z^n + \frac{1}{z^n} = 2\,cos(nx)$$

(iii) $z^n - \frac{1}{z^n} = 2i\,sin(nx)$ ✓

(b) Show that

$$cos^4(\theta) = \frac{1}{8}cos(4\theta) + \frac{1}{2}cos(2\theta) + \frac{3}{8}$$

$$cos^4(\theta) = \left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)^4$$

$$= \frac{1}{16}\left(z^4 + \frac{4z^3}{z} + \frac{6z^2}{z^2} + \frac{4z}{z^3} + \frac{1}{z^4}\right)$$

$$= \frac{1}{16}\left(z^4 + \frac{1}{z^4} + 4z^2 + \frac{4}{z^2} + 6\right)$$

$$= \frac{1}{16}\left(\left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6\right)$$

$$= \frac{1}{16}(2\,cos(4\theta) + 4(2\,cos(2\theta)) + 6)$$

$$cos^4(\theta) = \frac{1}{8}cos(4\theta) + \frac{1}{2}cos(2\theta) + \frac{3}{8}$$

12. (12 marks)

(a) (i) $f(x)=\frac{1}{x-2}+1 \quad for \ \ x\neq 2, y\neq 1$ ✓✓

(ii) To obtain the inverse swap x and y

$$x=\frac{1}{y-2}+1$$

$$x-1=\frac{1}{y-2}$$

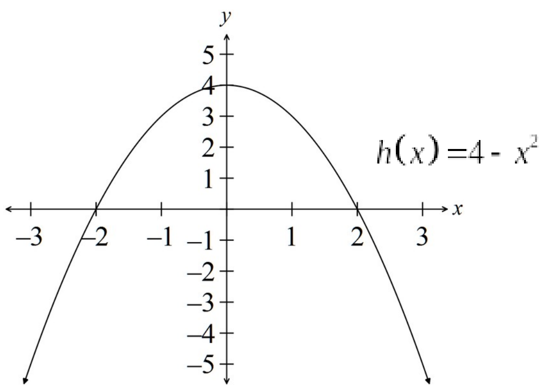
$$y-2=\frac{1}{x-1}$$

$$y=\frac{1}{x-1}+2$$

$$\therefore \ f^{-1}(x)=\frac{1}{x-1}+2 \qquad for \quad x\neq 1, \ y\neq 2$$

✓ ✓ ✓

(b) (i) $f(h(x))=f(4-x^2)=ln(4-x^2)$ ✓



The domain is $-2<x<2$ and the range is $y\leq ln4$

✓ ✓

(ii) $j(x)=x^2-2$

$$g(j(x))=5-x^2$$

$$=-(x^2-2)+3$$

$$\therefore \ g(x)=-x+3$$

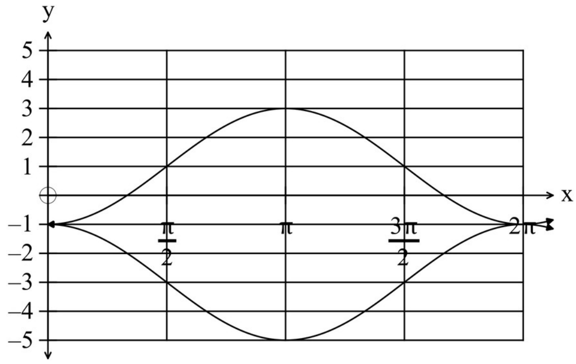
(iii) $g(f(x))=g(ln(x))=3-ln(x)$

$ln(x)$ is a one to one function so $g(f(x))$ is also a one to one function.

13. (6 marks)

(a) $x = 0,\ x = 2\pi$ ✓✓✓

(b) Area = $\int_0^{2\pi} \left| 1 - 2sin\left(x + \frac{\pi}{2}\right) - \left(2cos(x) - 3\right) \right| dx = 25.13\text{units}^2$ ✓✓✓



14. (7 marks)

(a)
$$\boldsymbol{v}_0 = \begin{pmatrix} 30cos30^o \\ 30sin30^o \end{pmatrix} = \begin{pmatrix} 15\sqrt{3} \\ 15 \end{pmatrix}$$
$$\boldsymbol{a} = -9.8\,\boldsymbol{j}$$
$$\boldsymbol{v} = \int -9.8\,\boldsymbol{j}\ dt$$
$$\boldsymbol{v} = -9.8t\,\boldsymbol{j} + \boldsymbol{c}_1$$
$$At\ t = 0 \quad \begin{pmatrix} 15\sqrt{3} \\ 15 \end{pmatrix} = \boldsymbol{c}_1$$
$$\boldsymbol{v} = 15\sqrt{3}\,\boldsymbol{i} + (15 - 9.8t)\,\boldsymbol{j}$$
$$\boldsymbol{x} = \int 15\sqrt{3}\,\boldsymbol{i} + (15 - 9.8t)\,\boldsymbol{j} \quad dt$$
$$\boldsymbol{x} = 15\sqrt{3}t\,\boldsymbol{i} + \left(15t - 4.9t^2\right)\boldsymbol{j} + \boldsymbol{c}_2$$
$$\boldsymbol{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{c}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ \boldsymbol{x} = 15\sqrt{3}t\,\boldsymbol{i} + \left(15t - 4.9t^2\right)\boldsymbol{j}$$
$$At\ y = 2, x = ?$$
$y = 2 = 15t - 4.9t^2 \Rightarrow t = 2.92151503\ or\ \ t = 0.1397$ (still going up)
$$At\ t = 2.92151503,\ x = 15\sqrt{3}t = 15\sqrt{3} \times 2.92151503 \approx 75m$$

Yes, the arrow goes over the back fence.

(b)
$$At\ y = 0,\ t = 3.06122445$$
$$At\ t = 3.06122445,\ x = 79.53\ m$$

15. (10 marks)

(a) $V = \int_0^4 \left(\frac{y+2}{1.5} \right) dy - \int_2^4 (y-2)\, dy$

✓ ✓ ✓ ✓

(b) (i) The period is 2 cm. ✓ $k = \pi$ ✓

(ii) $V = 20 \int_0^1 \pi \sin^2(\pi x)\, dx = 31.415\, cc$

✓ ✓ ✓ ✓

16. (14 marks)

(a) $x = e^{-2t}$ cm

$v = \frac{dx}{dt} = -2e^{-2t}$

$a = \frac{dv}{dt} = 4e^{-2t}$

$At\ t = 1 \quad v = -2e^{-2}$

$a = 4e^{-2}$

(b) (i) $x = -3\sin\left(2t + \frac{\pi}{6}\right)$

SHM if $a = -n^2 x$

$v = -6\cos\left(2t + \frac{\pi}{6}\right)$

$a = 12\sin\left(2t + \frac{\pi}{6}\right)$

$a = -4\left(-3\sin\left(2t + \frac{\pi}{6}\right)\right)$

$a = -4x$

$\therefore SHM$

(ii) $v_0 = -6\cos\left(\frac{\pi}{6}\right) = -3\sqrt{3}$ $\qquad a_0 = 12\sin\left(\frac{\pi}{6}\right) = 6$

(iii) $If\ v = 0,\ \cos\left(2t + \frac{\pi}{6}\right) = 0 \Rightarrow \sin\left(2t + \frac{\pi}{6}\right) = \pm 1 \Rightarrow x = \pm 3$ ✓

(c) $A = \pi r^2$

$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi r \times \frac{1}{r}$

$\frac{dA}{dt} = 2\pi \quad m^2/min$

which is constant

(d) $2xy\frac{dy}{dx} = 1 - x.$

$\int 2y\,dy = \int \frac{1-x}{x}\,dx$

$y^2 = \int \left(\frac{1}{x} - 1\right) dx$

$y^2 = ln(x) - x + c$

$(1,4) \quad 16 = \cancel{ln(1)} - 1 + c \quad \rightarrow \quad c = 17$

$y^2 = ln(x) - x + 17$

$y = \pm\sqrt{ln(x) - x + 17}$

But, if $x = 1, y = +4$

$\therefore\ y = \sqrt{ln(x) - x + 17}$

17. (8 marks)

(a) Let equation of the plane be $ax + by + cz = d$

$A(12,0,0) so\ 12a = d$

$B(0,4,0) so\ 4b = d$

$C(0,0,4) so\ 4c = d$

$\therefore\ \frac{dx}{12} + \frac{dy}{4} + \frac{dz}{4} = d$

$\Rightarrow \frac{x}{12} + \frac{y}{4} + \frac{z}{4} = 1$

$\therefore\ x + 3y + 3z = 12$

(b) $P(6,0,0),\ Q(0,2,0),\ R(0,0,2)$

Likewise $x + 3y + 3z = 6$

The planes are parallel as the coefficients of x, y, and z are all identical. ✓

NB (a) and (b) can be done using vectors.

(c) $$\boldsymbol{AC}\times\boldsymbol{AB}=\begin{pmatrix}-12\\0\\4\end{pmatrix}\times\begin{pmatrix}-12\\4\\0\end{pmatrix}=0\,\boldsymbol{i}-48\,\boldsymbol{j}-48\,\boldsymbol{k}$$ ✓✓

(d) $$|\boldsymbol{AC}\times\boldsymbol{AB}|=|\boldsymbol{AC}||\boldsymbol{AB}|\sin(\theta)$$

so $$Area_{\Delta}=\frac{1}{2}|\boldsymbol{AC}\times\boldsymbol{AB}|$$
$$=\frac{1}{2}\sqrt{(-48)^2+(-48)^2}$$
$$=\frac{48\sqrt{2}}{2}$$
$$Area\,\triangle=24\sqrt{2}\ \ units^2$$

18. (8 marks)

(a) (i) $\mu_{\bar{x}}=130$ grams

$$\sigma_{\bar{x}}=\frac{\sigma}{\sqrt{n}}=\frac{15}{\sqrt{6}}$$
$$\sigma_{\bar{x}}\approx 6.12$$

(ii) When a sample is taken, the range is usually smaller as it does not usually contain any outliers (which are rare scores).
Most apples would be close to the mean weight, and it is unlikely to select an unusually big or small apple in a small sample, so a sample of 6 would have a small standard deviation.
✓✓

(b) The average time they would wait is is 20 minutes.

$$\sigma_{\bar{x}}=\frac{\sigma}{\sqrt{n}}=\frac{6}{\sqrt{3}}$$
$$\sigma_{\bar{x}}\approx 3.46$$

19. (7 marks)

(a) The expected mean is 160 cm.

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$
$$1.5 \times \sqrt{20} = \sigma$$
$$\sigma \approx 6.7 \text{ cm}$$

(b) 95% confidence limits $\Rightarrow z = 1.96$

$$160 \pm 1.96 \times 1.5 \times \sqrt{20}$$
$$= 160 \pm 13.15$$
$$146.85 \le \mu \le 173.15$$

(c) To be 90% sure the confidence limits are correct, you can have a smaller ranger as there is a 10% error margin.

To be surer, i.e. a 95% confidence limit then you have a wider range so there is a bigger chance that the mean will be included. ✓✓

20. (13 marks)

(a) (i) $t = 34 \quad N \approx 165$ ✓

(ii) $t = 85 \quad N \approx 459$ ✓

(b)
$$\lim_{x \to \infty}\left(\frac{540}{1 + 12.5e^{-0.0502t}}\right)$$
$$as\ t \to \infty\ \ e^{-0.0502t} \to 0$$
$$= \frac{540}{1}$$
$$= 540$$

(c)
$$N = \frac{540}{1 + 12.5e^{-0.0502t}}$$
$$\frac{dN}{dt} = \frac{-540 \times 12.5 \times (-0.0502) \times e^{-0.0502t}}{\left(1 + 12.5e^{-0.0502t}\right)^2}$$
$$\frac{dN}{dt} = \frac{338.85 \times e^{-0.0502t}}{\left(1 + 12.5e^{-0.0502t}\right)^2}$$

$In\ 1990,\ t = 60,\ \frac{dN}{dt} = 6.392$, i.e. about 6 cases per year.

$At\ \ t = 60, P = 334$ which is more than half way up to the maximum (of 540).

so the rate of increase will be decreasing then.

(d) $\frac{dN}{dt} = kN\left(1 - \frac{N}{K}\right) = kN\left(\frac{K - N}{K}\right)$

$\therefore \quad kdt = \frac{K}{N(K - N)}dN$

$let \quad \frac{1}{N(K-N)} = \frac{a}{N} + \frac{b}{(K-N)}$

$= \frac{a(K-N)+bN}{N(K-N)}$

$\frac{0N+1}{N(K-N)} = \frac{N(-a+b)+aK}{N(K-N)}$

Equating coefficients $0 = -a + b \quad 1 = aK$

$a = b \qquad b = a = \frac{1}{K}$

$\int kdt = \int \frac{K}{N(K-N)}dN = \int K\left(\frac{a}{N} + \frac{b}{(K-N)}\right) = \frac{K}{K}\int\left(\frac{1}{N} + \frac{1}{(K-N)}\right)dN$

$\int kdt = \int\left(\frac{1}{N} + \frac{1}{(K-N)}\right)dN$

$kt = \ln N - \ln(K-N) + c$

$kt - c = \ln\frac{N}{K-N}$

$\frac{N}{K-N} = e^{kt-c}$ ....$let \ e^{-c} = M$

$N = (K-N)Me^{kt}$

$KMe^{kt} = N\left(1 + Me^{kt}\right)$

$N = \frac{KMe^{kt}}{1+Me^{kt}} \times \frac{1/e^{kt}}{1/e^{kt}}$

$N = \frac{KM}{e^{-kt}+M} \times \frac{1/M}{1/M}$

$N = \frac{K}{1 + \frac{1}{M}e^{-kt}}$

END OF SECTION TWO