Papers written by Australian Maths Software

YEAR 11

2016

REVISION 3

MATHEMATICS SPECIALIST

UNITS 1 & 2

SEMESTER TWO

SOLUTIONS

SECTION 1 – Calculator-free

Question 1 **(8 marks)**

(a) $z = 1 + 3i \quad u = 1 - 3i = \bar{z}$ ✓✓ $\quad v = -3 + i = i(1+3i)$

$v = i\,z$ ✓✓

(b) (i) $(z_2)^2 = (1+3i)^2 = 1 + 3i + 9i^2 = -8 + 3i$ ✓ ✓

(ii) $\dfrac{z_1 - z_2}{z_3} = \dfrac{2 + i - (1+3i)}{1 - 3i} = \dfrac{1-2i}{1-3i} \times \dfrac{1+3i}{1+3i} = \dfrac{7+i}{10}$ ✓ ✓

Question 2 **(9 marks)**

(a) (i) $\boldsymbol{i} + \boldsymbol{j}$, $\begin{pmatrix} -4 \\ -4 \end{pmatrix}$ ✓

(ii) $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $-4\boldsymbol{i} - 3\boldsymbol{j}$ ✓

(b) Let $\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

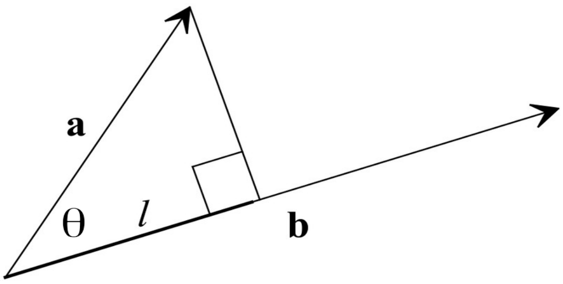
$\cos(\theta) = \dfrac{l}{|\boldsymbol{a}|} \Rightarrow l = |\boldsymbol{a}|\cos(\theta)$ ✓

$\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$

$|\boldsymbol{a}|\cos(\theta) = \dfrac{\boldsymbol{a} \bullet \boldsymbol{b}}{|\boldsymbol{b}|}$ ✓

so $l = \dfrac{\boldsymbol{a} \bullet \boldsymbol{b}}{|\boldsymbol{b}|} = \dfrac{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \end{pmatrix}}{\sqrt{9+4}}$

$l = \dfrac{5}{\sqrt{13}}$ ✓



(c) $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$

$$\begin{pmatrix} 5 \\ y \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \sqrt{25 + y^2} \times 3 \times \cos(60^o)$$

$$15 + 0 = \sqrt{25 + y^2} \times 3 \times \left(\frac{1}{2}\right) \quad ✓$$

$$10 = \sqrt{25 + y^2} \quad ✓$$

$$100 = 25 + y^2$$

$$y = \pm\sqrt{75}$$

$$y = \pm 5\sqrt{3}$$

✓ ✓

Question 3 **(7 marks)**

(a) $A + B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}$ ✓

(b) $B \times E = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & -1 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 13 & -3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$ ✓ ✓

(c) $C + D = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ cannot be determined as the sizes are not the same. ✓ ✓

(d) $E^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -1 & 1 \end{bmatrix}^{-1}$ cannot be determined as you can only get the inverse of a square matrix. ✓ ✓

Question 4 **(8 marks)**

(a) $\left(\begin{bmatrix} 0 & 3 \\ 7 & 1 \end{bmatrix} + \begin{bmatrix} -4 & -1 \\ -5 & 2 \end{bmatrix}\right) \times \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ ✓ ✓

(b) $\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} -8 \\ -5 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$x = -8, \quad y = -5$

✓ ✓

(c) $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

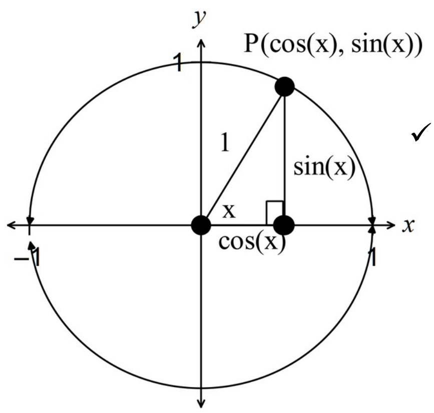
(d) $\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -2 & -1 \end{pmatrix}$ ✓

$\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}$ ✓

Therefore $\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ ✓

Question 5 **(8 marks)**

(a) (i) P(x,y) is a point on the unit circle. The height, y= sin(x) and the width, x= cos(x).
This is valid in all four quadrants.
A unit circle has a radius of 1.
Using Pythagoras theorem, $\sin^2(x) + \cos^2(x) = 1$.



✓

(ii) $\sin^2(x) + \cos^2(x) = 1$ Multiply both sides by $\dfrac{1}{\cos^2(x)}$ ✓

$$\frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$\tan^2(x) + 1 = \sec^2(x)$$

$$\therefore \tan^2(x) = \sec^2(x) - 1$$ ✓

(b) (i) Show that $\tan\left(\dfrac{\pi}{4} + \theta\right) = \dfrac{1 + \tan(\theta)}{1 - \tan(\theta)}$.

$$\tan\left(\frac{\pi}{4} + \theta\right) = \frac{\tan\left(\frac{\pi}{4}\right) + \tan(\theta)}{1 + \tan\left(\frac{\pi}{4}\right)\tan(\theta)}$$ ✓ but $\tan\left(\dfrac{\pi}{4}\right) = 1$ ✓

$$\tan\left(\frac{\pi}{4} + \theta\right) = \frac{1 + \tan(\theta)}{1 - \tan(\theta)}$$

(ii) Show that $\tan\left(\frac{5\pi}{12}\right) = -2-\sqrt{3}$.

$$\tan\left(\frac{5\pi}{12}\right) = \tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$$

$$= \frac{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{6}\right)}{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)} \quad ✓$$

$$= \frac{1+\frac{1}{\sqrt{3}}}{1-1\times\frac{1}{\sqrt{3}}}$$

$$= \frac{\frac{\sqrt{3}+1}{\cancel{\sqrt{3}}}}{\frac{\sqrt{3}-1}{\cancel{\sqrt{3}}}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1} \quad ✓$$

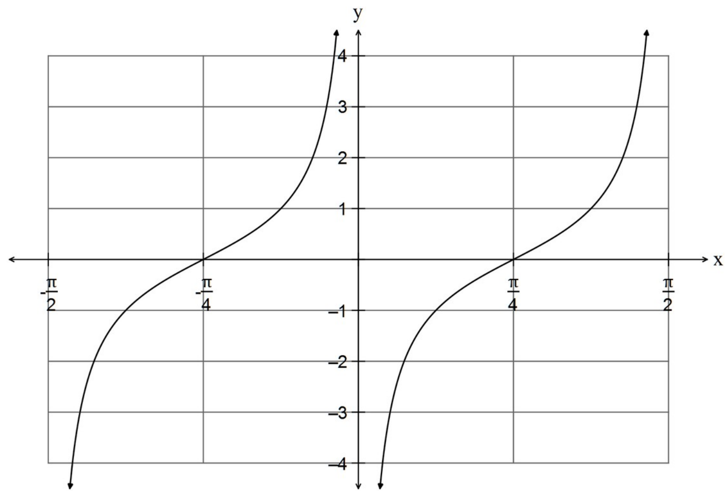
$$= \frac{3+2\sqrt{3}+1}{3-1}$$

$$= \frac{4+2\sqrt{3}}{3-1} \quad ✓$$

$$= 2+\sqrt{3}$$

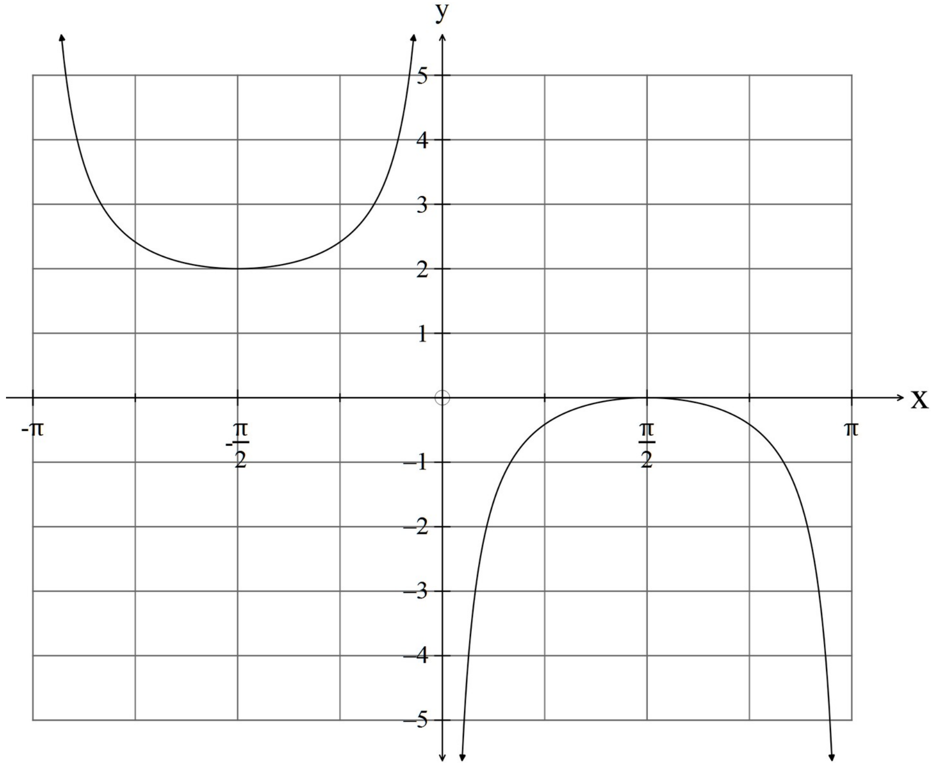
Question 6 **(4 marks)**

(a) $y = \tan\left(2x+\frac{\pi}{2}\right)$



✓✓

(b) $y = 1 - \operatorname{cosec}(x)$

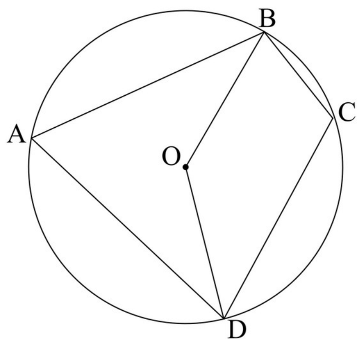


✓✓

Question 7 **(8 marks)**

(a) It could have been a rectangle. ✓

(b) Prove that "*The opposite angles of a cyclic quadrilateral are supplementary.*"



Let *ABCD* be a cyclic quadrilateral with *O* the centre of the circle.

Join the radii *OB* and *OD*. ✓

$\angle BCD = \frac{1}{2}\angle BOD$ (reflex)

$\angle BAD = \frac{1}{2}\angle BOD$ (obtuse) The angle at the circumference of a circle is half the angle at the centre subtended by the same arc. ✓

$\angle BOD$ (reflex) $+ \angle BOD$ (obtuse) $= 360^o$

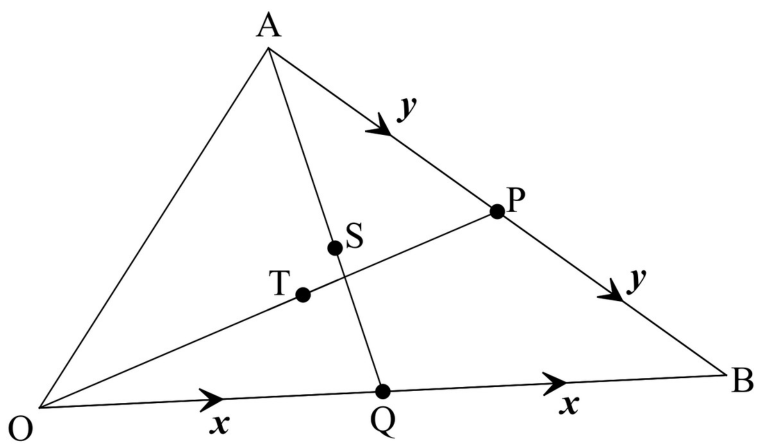
The angle around a point.

$\therefore\ \angle BCD + \angle BAD = \frac{1}{2}(360^o)$

$\angle BCD + \angle BAD = 180^o$ ✓

Therefore "*The opposite angles of a cyclic quadrilateral are supplementary.*"

(c) Prove the following using vectors
"*The three medians of a triangle are concurrent at the point of trisection*."



Let ABC be the triangle. P and Q are the midpoints of sides AB and AB respectively.

Let T and S be the points of trisection of **OP** and A**Q** respectively.

Let $\boldsymbol{OQ} = \boldsymbol{QB} = \boldsymbol{x}$

Let $\boldsymbol{AP} = \boldsymbol{PB} = \boldsymbol{y}$

$\boldsymbol{OP} = 2\boldsymbol{x} - \boldsymbol{y}$

$\boldsymbol{OT} = \frac{2}{3}\boldsymbol{OP} = \frac{2}{3}(2\boldsymbol{x} - \boldsymbol{y})$

$\boldsymbol{OT} = \frac{4}{3}\boldsymbol{x} - \frac{2}{3}\boldsymbol{y}$

$\boldsymbol{AQ} = 2\boldsymbol{y} - \boldsymbol{x}$

$\boldsymbol{AS} = \frac{2}{3}(2\boldsymbol{y} - \boldsymbol{x}) = \frac{4}{3}\boldsymbol{y} - \frac{2}{3}\boldsymbol{x}$

$\boldsymbol{OS} = \boldsymbol{OA} + \boldsymbol{AS}$

$\boldsymbol{OS} = (2\boldsymbol{x} - 2\boldsymbol{y}) + \left(\frac{4}{3}\boldsymbol{y} - \frac{2}{3}\boldsymbol{x}\right)$

$\boldsymbol{OS} = \frac{4}{3}\boldsymbol{x} - \frac{2}{3}\boldsymbol{y}$

$\therefore\ \boldsymbol{OT} = \boldsymbol{OS}$

Therefore T and S are the same point.

Likewise the point of trisection of the median from B can be found to coincide with the other two points.
Therefore *"The three medians of a triangle are concurrent at the point of trisection*."

SECTION 2 – Calculator-assumed

Question 8 **(7 marks)**

(a) (i) $\begin{pmatrix} 4\cos(45^o) \\ 4\sin(45^o) \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \begin{pmatrix} 6\cos(30^o) \\ -6\sin(30^o) \end{pmatrix} = \begin{pmatrix} 8.02458 \\ 4.82843 \end{pmatrix}$ ✓

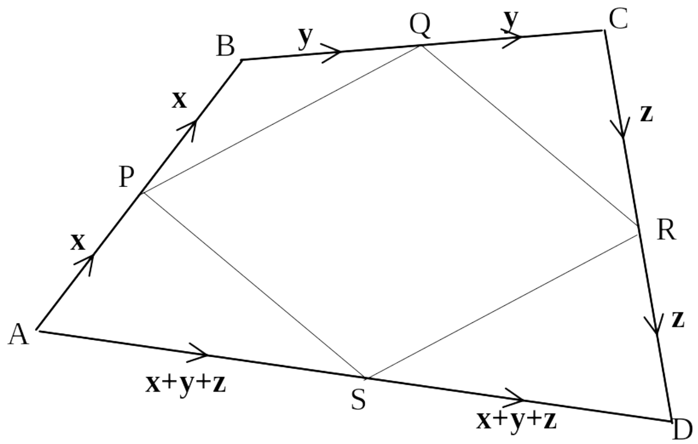
Resultant $= \sqrt{8.02458^2 + 4.82843^2}$

Resultant $\approx 9.4N$ ✓

(ii) No, the boys are not in danger form the dog. ✓

(b) Use a vector proof to show that

"*The midpoints of the sides of a quadrilateral join to from a parallelogram*."



$\mathbf{AD} = 2\mathbf{x} + 2\mathbf{y} + 2\mathbf{z} \quad \therefore \quad \mathbf{AS} = \mathbf{SD} = \mathbf{x} + \mathbf{y} + \mathbf{z}$ ✓

$\mathbf{PQ} = \mathbf{x} + \mathbf{y}$

$\mathbf{SR} = \mathbf{SD} + \mathbf{DR}$ ✓

$= \mathbf{x} + \mathbf{y} + \mathbf{z} - \mathbf{z}$

$= \mathbf{x} + \mathbf{y}$

$\mathbf{SR} = \mathbf{PQ}$

$\mathbf{PS} = \mathbf{PA} + \mathbf{AS}$ ✓

$= -\mathbf{x} + \mathbf{x} + \mathbf{y} + \mathbf{z}$

$= \mathbf{y} + \mathbf{z}$

$\mathbf{PS} = \mathbf{QR}$

$\therefore \quad \mathbf{SR} = \mathbf{PQ}$ and $\mathbf{PS} = \mathbf{QR}$

Two pairs of parallel and equal sides! ✓
Therefore PQRS is a parallelogram.

Therefore
"*If the midpoints of a quadrilateral are joined, the resulting quadrilateral is a parallelogram*."

Question 9 **(8 marks)**

(a) $$\frac{(1+2i)}{(1+i)(1-2i)} = \frac{(1+2i)}{(1-i-2i^2)} \quad \checkmark$$

$$= \frac{(1+2i)}{(3-i)} \times \frac{(3+i)}{(3+i)} \quad \checkmark$$

$$= \frac{3+7i-2}{10}$$

$$= \frac{1+7i}{10} \quad \checkmark$$

(b) (i) $\mathrm{Re}(z_1+z_2) = a+c$ ✓

$\mathrm{Im}(z_1+z_2) = b+d$ ✓

(ii) Show $\mathrm{Im}\left((z_1-z_2)^2\right) = 2(a-c)(b-d)$

$$(z_1-z_2)^2$$

$$= (a+bi-(c+di))^2$$

$$= ((a-c)+i(b-d))^2 \quad \checkmark$$

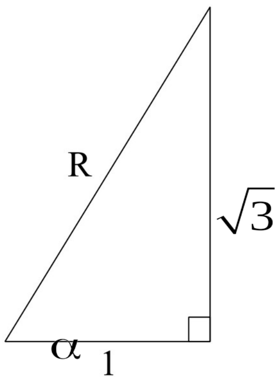
$$= (a-c)^2 + \underline{2(a-c)(b-d)i} - (b-d)^2 \quad \checkmark\checkmark$$

$$\therefore \mathrm{Im}\left((z_1-z_2)^2\right) = 2(a-c)(b-d)$$

Question 10 **(14 marks)**

(a) (i) $R\cos(\theta+\alpha) = R\underline{\cos(\theta)}\cos(\alpha) - R\underline{\underline{\sin(\theta)}}\sin(\alpha)$

$R\cos(\theta+\alpha) = \underline{\cos(\theta)} - \sqrt{3}\underline{\underline{\sin(\theta)}}$



Equating coefficients then

$R\cos(\alpha) = 1$ and $R\sin(\alpha) = \sqrt{3}$

$\cos(\alpha) = \frac{1}{R}$ and $\sin(\alpha) = \frac{\sqrt{3}}{R}$ ✓

Pythagoras $R^2 = 1^2 + \left(\sqrt{3}\right)^2 \therefore R = 2$

$$\tan(\alpha) = \sqrt{3}$$

$$\alpha = \frac{\pi}{3}$$

$$\therefore\ \cos(\theta)-\sqrt{3}\sin(\theta)=2\cos\left(\theta+\frac{\pi}{3}\right)$$
✓ ✓

(ii) $\cos(\theta)-\sqrt{3}\sin(\theta)=-1$

$$2\cos\left(\theta+\frac{\pi}{3}\right)=-1$$ ✓

$$\cos\left(\theta+\frac{\pi}{3}\right)=-\frac{1}{2}$$ ✓

$$\theta+\frac{\pi}{3}=\frac{2\pi}{3}+n(2\pi)\quad or\quad \theta+\frac{\pi}{3}=\frac{4\pi}{3}+n(2\pi)$$

$$\theta=\frac{\pi}{3}\qquad or\qquad \theta=\pi$$
✓ ✓

(b) $y=-4\sin(x)\ \ or\ \ y=4\cos\left(x+\frac{\pi}{2}\right)$ ✓✓

(c) $\boldsymbol{p}=2\boldsymbol{i}+4\boldsymbol{j}$ and $\boldsymbol{q}=3\boldsymbol{i}-5\boldsymbol{j}$

(i) $3\boldsymbol{p}-4\boldsymbol{q}=3\begin{pmatrix}2\\4\end{pmatrix}-4\begin{pmatrix}3\\-5\end{pmatrix}=\begin{pmatrix}-6\\32\end{pmatrix}=-6\boldsymbol{i}+32\boldsymbol{j}$ ✓✓

(ii) $|\boldsymbol{p}|=|2\boldsymbol{i}+4\boldsymbol{j}|=\sqrt{20}=2\sqrt{5}$ ✓✓

Vector required is $\boldsymbol{i}+2\boldsymbol{j}$ ✓

Question 11 **(7 marks)**

(a) $y=2\cos\left(3\left(x-\frac{\pi}{2}\right)\right)-4$ and $y=2\sin(3x)-4$

$$y=2\cos\left(3x-\frac{3\pi}{2}\right)-4$$ ✓

$$y=2\cos\left((3x-\pi)-\frac{\pi}{2}\right)-4$$

$$y=2\cos\left(\frac{\pi}{2}-(3x-\pi)\right)-4$$ ✓

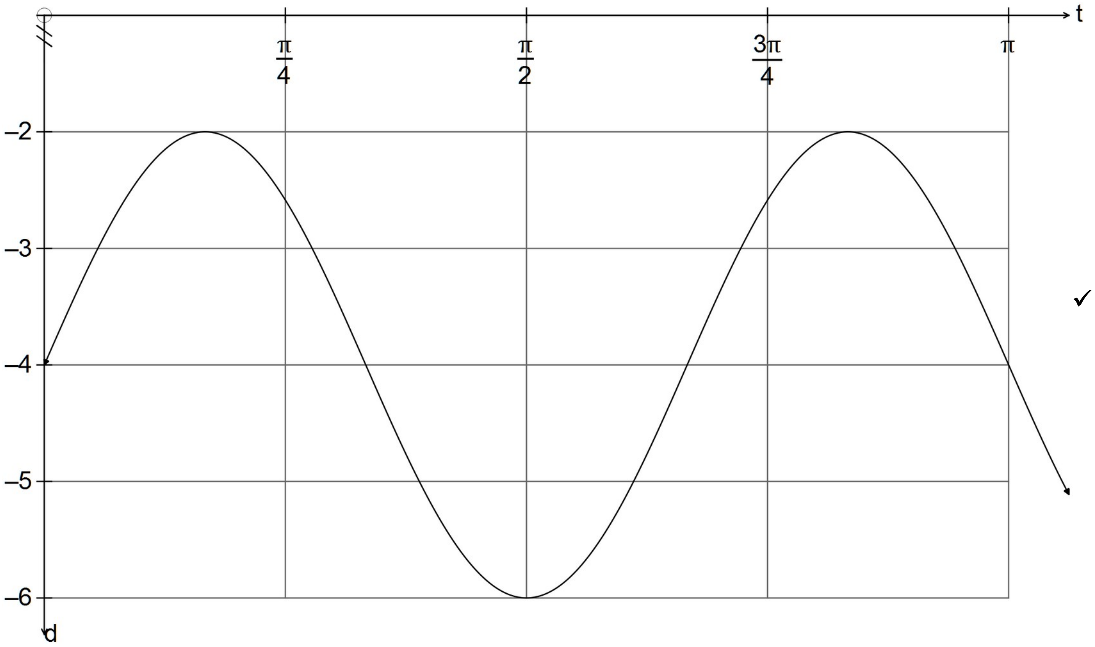
$$y=2\sin(3x-\pi)-4$$

$$y=-2\sin(\pi-3x)-4$$ ✓

$$y=-2\sin(3x)-4\neq 2\sin(3x)-4$$

so the functions are not the same

(b) $y = 2\sin(3x) - 4$



✓✓✓

Question 12 **(3 marks)**

(a) $\cos(x+y)\cos(x-y)+\sin(x+y)\sin(x-y)$

$=\cos((x+y)-(x-y))$ ✓

$=\cos(2y)$ ✓

(b) Show $\cos(x+y)\cos(x-y)+\sin(x+y)\sin(x-y)=1-2\sin^2(y)$

$\cos(x+y)\cos(x-y)+\sin(x+y)\sin(x-y)=\sin(2y)$

$=1-2\sin^2(y)$ ✓

Question 13 **(11 marks)**

(a) $z^2+4z+5=0$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$z=\frac{-4\pm\sqrt{16-20}}{2}$$ ✓

$$z=\frac{-4\pm 2i}{2} \quad where\ \sqrt{-1}=\sqrt{i^2}=i$$ ✓

$z=-2\pm i$

✓

(b) $\left(\bar{u}\right)^{-1} = 1 - 2i$

$\bar{u} = \dfrac{1}{1-2i}$ ✓

$\bar{u} = \dfrac{1}{1-2i} \times \dfrac{1+2i}{1+2i}$ ✓

$\bar{u} = \dfrac{1+2i}{1-4i^2}$

$\bar{u} = \dfrac{1+2i}{5}$

$u = \dfrac{1-2i}{5}$ ✓

(c) Real coefficients mean the roots appear in conjugate pairs.

$z = 2 - 4i$ and $z = 2 + 4i$ ✓

$\left(z - (2 - 4i)\right)\left(z - (2 + 4i)\right) = 0$

$\left((z-2) + 4i\right)\left((z-2) - 4i\right) = 0$

$(z-2)^2 + 4i\left((\cancel{z} - \cancel{2}) - (\cancel{z} - \cancel{2})\right) - 16i^2 = 0$ ✓

$(z-2)^2 + 16 = 0 \;\; or \;\; z^2 - 4z + 20 = 0$

✓

(d) "*If you pass your exams, you have studied hard.* ✓

(e) "*If you can't watch the news on Channel 2, then you don't have a TV.*" ✓

Question 14 **(9 marks)**

(a) $(x+iy)^2 = 5 - 12i$

$x^2 + 2ixy + i^2y^2 = 5 - 12i$

$x^2 - y^2 + 2ixy = 5 - 12i$

$Re: x^2 - y^2 = 5 \quad Im: 2xy = -12$ ✓

$y = \dfrac{-6}{x}$

$x^2 - \left(\dfrac{-6}{x}\right)^2 = 5$

$x^4 - 5x^2 - 36 = 0$ ✓

$(x^2 - 9)(x^2 + 4) = 0$

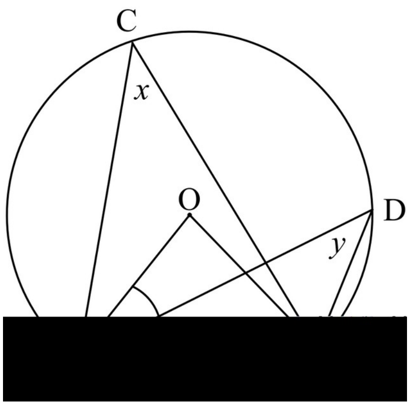
$x^2 = 9 \;\; or \;\; x^2 = -4$

but x is real

$x = \pm 3, y = \mp 2$

✓ ✓

(b) (i) Solve for x and y, giving reasons.



ΔAOB is isosceles

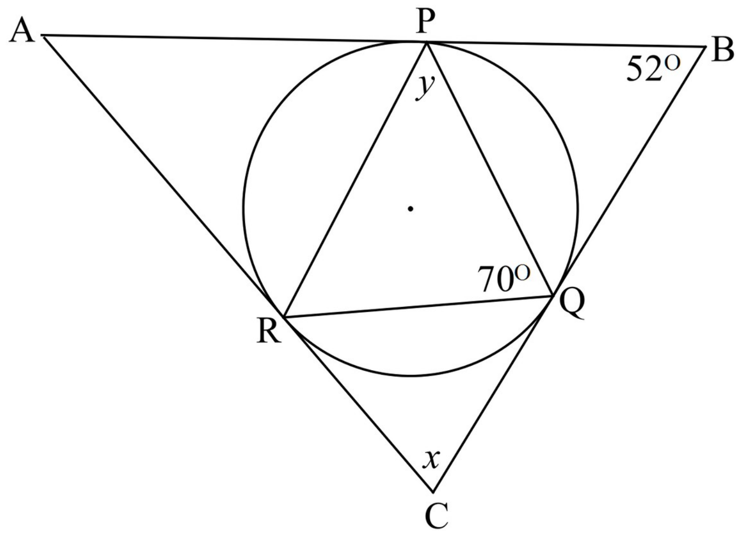
$\angle ABO = 40^o$ ΔAOB is isosceles

$\angle AOB = 100^o$ Angles in a triangle add to 180^o

✓ $x = 50^o$ An angle at the circumference of a circle is half the angle at the centre subtended by the same arc. -1 no reasons

✓ $y = 50^o$ Two angles at the circumference subtended by the same arc are equal.

(ii) Solve for x and y, giving reasons.



ΔPBQ is isosceles. Tangents from an external point are equal.

$\angle BQP = \angle BPQ = 64^o$

$\angle RQC = 46^o$ Angles on a line add to 180^o

ΔRQC is isosceles. Tangents from an external point are equal.

✓ $\therefore\ x = 88^o$ as $180^o - 2(46^o) = 88^o$

$\angle RQC = 46^o$

✓ $\therefore\ y = 46^o$ The angle between a chord and a tangent is equal to any angle in the alternate segment.

✓ Reasons

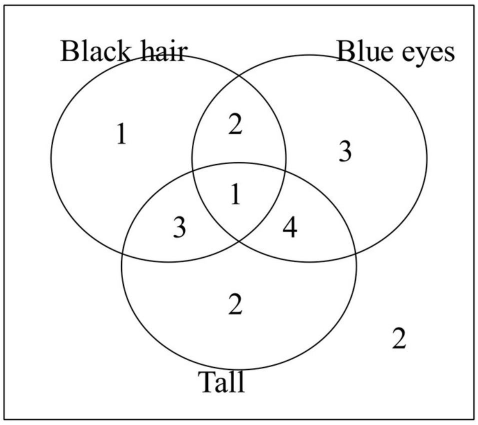
Question 15 **(10 marks)**

(a) Worst possible case: Seat then 2 at a time then leave a gap. One more person must be seated so there must be three together. ✓✓

(b) (i) 8x7x6 =336 ✓

(ii) ${}^{8}C_{3} = 56$ ✓

(c) (i)



✓

Two students did not have black hair, nor blue eyes nor were tall. ✓

(ii) 7 ✓

(d) Prove that ${}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$

$$ {}^{n}C_{r} + {}^{n}C_{r+1} = \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r-1)!\,(r+1)!} $$

$$ = \frac{n!}{(n-r-1)!\,r!}\left(\frac{1}{n-r} + \frac{1}{r+1}\right) \quad ✓ $$

$$ = \frac{n!}{(n-r-1)!\,r!}\left(\frac{\cancel{r}+1+n-\cancel{r}}{(n-r)(r+1)}\right) \quad ✓ $$

$$ = \frac{(n+1)n!}{(n-r)(n-r-1)!\,(r+1)r!} $$

$$ = \frac{(n+1)!}{(n-r)!\,(r+1)} $$

$$ = \frac{(n+1)!}{((n+1)-(r+1))!\,(r+1)} \quad ✓ $$

$$ = {}^{n+1}C_{r+1} $$

Question 16 **(10 marks)**

(a) Prove, using a proof by contradiction, that $\sqrt{2}$ is an irrational number.

Assume $\sqrt{2}$ is rational.

i.e. $\sqrt{2}=\frac{p}{q}$ where p and q are relatively prime. ✓

$$\sqrt{2}=\frac{p}{q}$$

$$\Rightarrow\ 2=\frac{p^2}{q^2}$$

$$\Rightarrow\ 2q^2=p^2 \quad ✓$$

Since p^2 is a square number then p is a multiple of 2.

If p is a multiple of 2, then p = 2n where n is an integer. ✓

$$p^2=2q^2$$

$$\Rightarrow\ (2n)^2=2q^2 \quad ✓$$

$$\Rightarrow\ 4n^2=2q^2$$

$$\Rightarrow\ q^2=2n^2$$

$\Rightarrow$ q is a multiple of 2. ✓

This is a contradiction to p and q being relatively prime. ✓

Therefore our original assumption is false.
Therefore $\sqrt{2}$ is irrational.

(b) Prove using mathematical induction, that $1+3+5+....+(2n-1)=n^2$ for any integer $n\geq 1$.

Test for n = 1 1 = 1^2 so valid for n = 1. ✓

Assume valid for n = k i.e. $1+3+5+....+(2k-1)=k^2$

Test for n = k+1 ✓

$$1+3+5+....+(2k-1)+(2(k+1)-1)=k^2+(2(k+1)-1)$$

$$=k^2+2k+1$$

$$=(k+1)^2 \quad ✓$$

So works for n = k+1
Valid for n = 1, so valid for n = 2 etc ✓

Therefore $1+3+5+....+(2n-1)=n^2$

Question 17 **(12 marks)**

(a) (i) $\begin{pmatrix} -4 & 0 \\ 0 & -1 \end{pmatrix}$ ✓

(ii) $\begin{pmatrix} \cos(90^o) & -\sin(90^o) \\ \sin(90^o) & \cos(90^o) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ✓

(iii) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$ ✓

$A'(0,0), B'(0,-2), C'(-4,-2), D'(-4,0)$

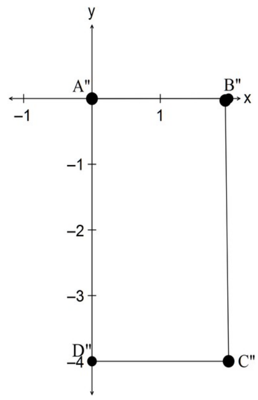
$\therefore A''(0,0), B''(2,0), C''(2,-4), D''(0,-4)$ ✓

(iv) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} -4 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$ ✓✓

(v) $\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$ ✓✓

(b) (i) Area ABCD x $\left|\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\right|$ = Area A''B''C''D'' ✓✓

(ii) Area ABCD = 2



$\left|\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\right| = 4$

Area A''B''C''D'' = 8 ✓

Area ABCD x $\left|\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\right|$ = 2 x 4 = 8 ✓

Yes, the method works.

Question 18 **(8 marks)**

(a) $p = \dfrac{1-2^2}{1+2^2} \qquad q = \dfrac{2(2)}{1+2^2}$

$p = -\dfrac{3}{5} \qquad q = \dfrac{4}{5}$ ✓

$M = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ ✓

$\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \times \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ ✓

The first reflected point is P'(1, 7)

(b) $p = \dfrac{1-\left(-\frac{1}{2}\right)^2}{1+\left(-\frac{1}{2}\right)^2} \qquad q = \dfrac{2\left(-\frac{1}{2}\right)}{1+\left(-\frac{1}{2}\right)^2}$

$p = \dfrac{3}{5} \qquad q = -\dfrac{4}{5}$ ✓

$M = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix}$ ✓

$\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix} \times \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \end{pmatrix}$ ✓

The second reflected point is P''(-1, -7)

(c) $P(5, 5)$ is exactly $\sqrt{50}$ from the origin. ✓

$P''(-1, -7)$ is also exactly $\sqrt{50}$ from the origin. ✓

Therefore the second reflected point is a rotation of P(5, 5) about the origin.

End of solutions