PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Test 1 Year 12

Student name: ________________ Teacher name: ________________

Task type: **Response**

Reading time for this test : 5 mins

Working time allowed for this task: 40 mins

Number of questions: **7**

Materials required: No cals allowed!!

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **42 marks**

Task weighting: **13%**

Formula sheet provided: **no**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Useful formulae

Complex numbers

Cartesian form	
$z = a + bi$	$\bar{z} = a - bi$
$\text{Mod}(z) = \lvert z\rvert = \sqrt{a^2 + b^2} = r$	$\text{Arg}(z) = \theta, \quad \tan\theta = \dfrac{b}{a}, \quad -\pi < \theta \le \pi$
$\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$	$\left\lvert \dfrac{z_1}{z_2}\right\rvert = \dfrac{\lvert z_1\rvert}{\lvert z_2\rvert}$
$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$	$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
$z\bar{z} = \lvert z\rvert^2$	$z^{-1} = \dfrac{1}{z} = \dfrac{\bar{z}}{\lvert z\rvert^2}$
$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$	$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
Polar form	
$z = a + bi = r(\cos\theta + i\sin\theta) = r\,cis\,\theta$	$\bar{z} = r\,cis(-\theta)$
$z_1 z_2 = r_1 r_2\,cis(\theta_1 + \theta_2)$	$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\,cis(\theta_1 - \theta_2)$
$cis(\theta_1 + \theta_2) = cis\,\theta_1\,cis\,\theta_2$	$cis(-\theta) = \dfrac{1}{cis\,\theta}$
De Moivre's theorem	
$z^n = \lvert z\rvert^n cis(n\theta)$	$(cis\,\theta)^n = \cos n\theta + i\sin n\theta$
$z^{\frac{1}{q}} = r^{\frac{1}{q}}\left(\cos\dfrac{\theta + 2\pi k}{q} + i\sin\dfrac{\theta + 2\pi k}{q}\right)$, for k an integer	

No cals allowed!!

Q1 (2, 2, 2 & 2 = 8 marks)

If $z=3+4i$ and $w=1-i$ determine the following exactly.

a) zw

Solution
$zw=7+i$
Specific behaviours
✓ real part ✓ imaginary part

b) z^2w

Solution
$z^2w=(3+4i)^2(1-i)=(9-16+24i)(1-i)$ $=(-7+24i)(1-i)$ $=17+31i$
Specific behaviours
✓ real part ✓ imaginary part

c) $\frac{1}{\overline{z}}$

Solution
$\frac{1}{3-4i}\frac{3+4i}{3+4i}=\frac{3+4i}{25}$
Specific behaviours
✓uses conjugate ✓ states result

d) $\frac{z}{w}$

Solution
$\frac{3+4i}{1-i}\frac{1+i}{1+i}=\frac{-1+7i}{2}$

Specific behaviours
✓uses conjugate ✓ states result

Q2 (4 marks)

Determine all possible real number pairs $a \& b$ such that $\frac{22-3i}{a+i}=5+bi$.

Solution
$22-3i=(5+bi)(a+i)=5a-b+i(ab+5)$ $22=5a-b, \quad b=5a-22$ $-3=ab+5=a(5a-22)+5=5a^2-22a+5$ $0=5a^2-22a+8=(5a-2)(a-4)$ $a=4, b=-2$ $a=\frac{2}{5}, b=-20$
Specific behaviours
✓ equates reals and imaginaries ✓ sets up an equation with only one variable ✓ solves for two a values ✓ solves for two b values

Q3 (2, 3 & 3 = 8 marks)

Consider the function $f(z)=z^3+2z^2+9z+18$.

a) Determine $f(3i)$.

Solution
$f(3i)=-27i+-18+27i+18=0$
Specific behaviours
✓ shows all 4 terms ✓ final answer of zero (Zero marks if all 4 terms not shown)

b) Hence solve $z^3+2z^2+9z+18=0$

Solution

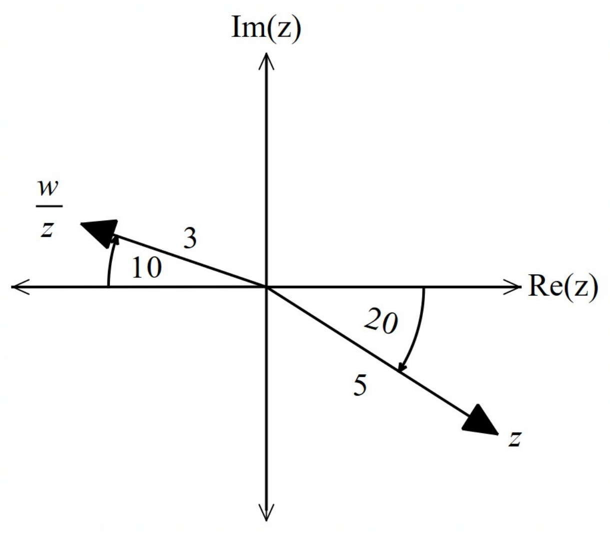
$z^3 + 2z^2 + 9z + 18 = (z - a)(z - 3i)(z + 3i) = (z - a)(z^2 + 9)$ $2 = -a, a = -2$ $z = -2, \pm 3i$
Specific behaviours
✓ uses conjugate ✓ shows full factorisation of f ✓ states all 3 roots

c) Consider $g(z) = (z^2 + bz + c)(z^2 + dz + e)$ where $b, c, d \,\&\, e$ are real constants and $g(3+i) = g(2-3i)$. Determine the values of $b, c, d \,\&\, e$.

Solution
$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$ $\alpha = 3 + i, \beta = 3 - i$ $z^2 - 6z + 10$ $\alpha = 2 - 3i, \beta = 2 + 3i$ $z^2 - 4z + 13$
Specific behaviours
✓ uses conjugates ✓ shows factorisation of each quadratic ✓ states all 4 constant values

Q4 (3 marks)
Use the diagram below to determine the complex number w in polar form with a principal argument.

(diagram not drawn to scale)

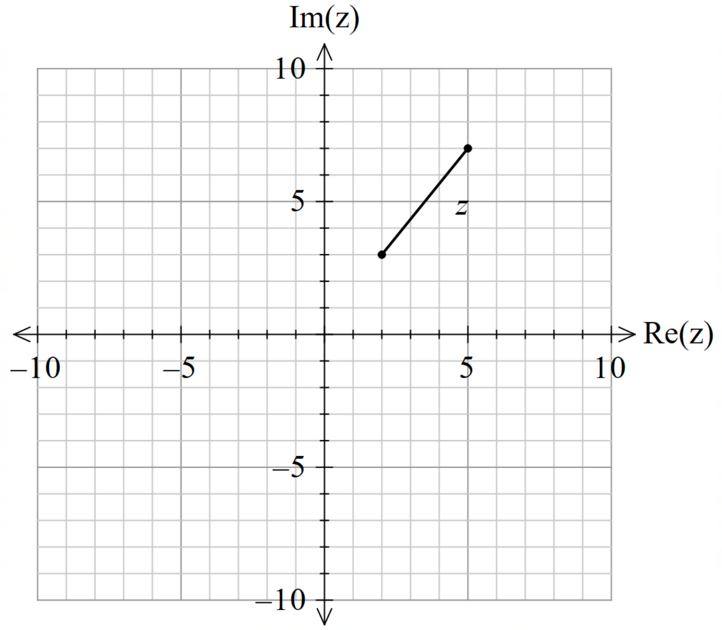


Solution
$z = 5cis20$ $w = rcis\theta$ $\frac{w}{z} = 3cis170 = \frac{r}{5}cis(\theta - 20)$ $r = 15, \theta = 150$ $w = 15cis150$

Specific behaviours
✓ determines r of w ✓ determines argument of w ✓ states w in polar form

Q5 (2 & 3 = 5 marks)
Sketch the following regions on the axes below.

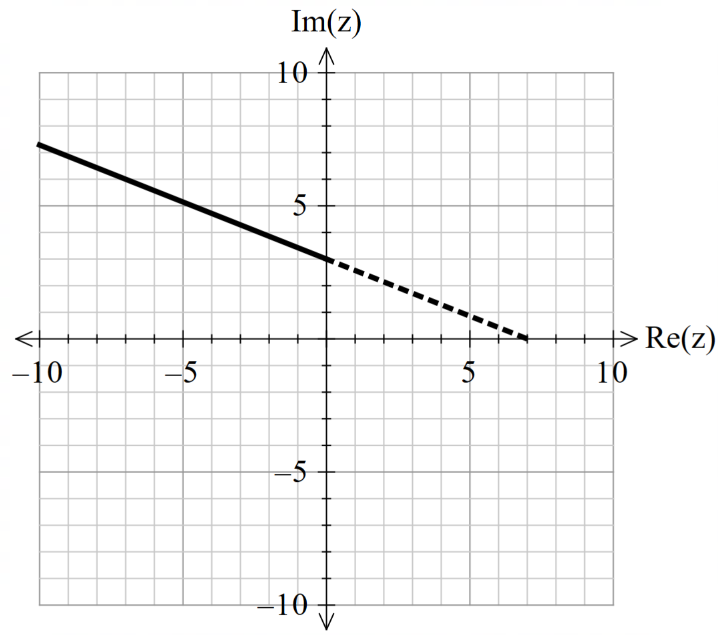
a) $|z - 2 - 3i| + |z - 5 - 7i| = 5$



Solution

Specific behaviours
✓ shows a line segment of length 5 units ✓ plots correct endpoints of closed line segment (includes endpoints)

b) $|z-7|=|z-3i|+\sqrt{58}$



Solution
Specific behaviours
✓ shows a line segment open (i.e arrow) ✓ plots y coordinate ✓ plots a dotted line segment to 7 on real axis

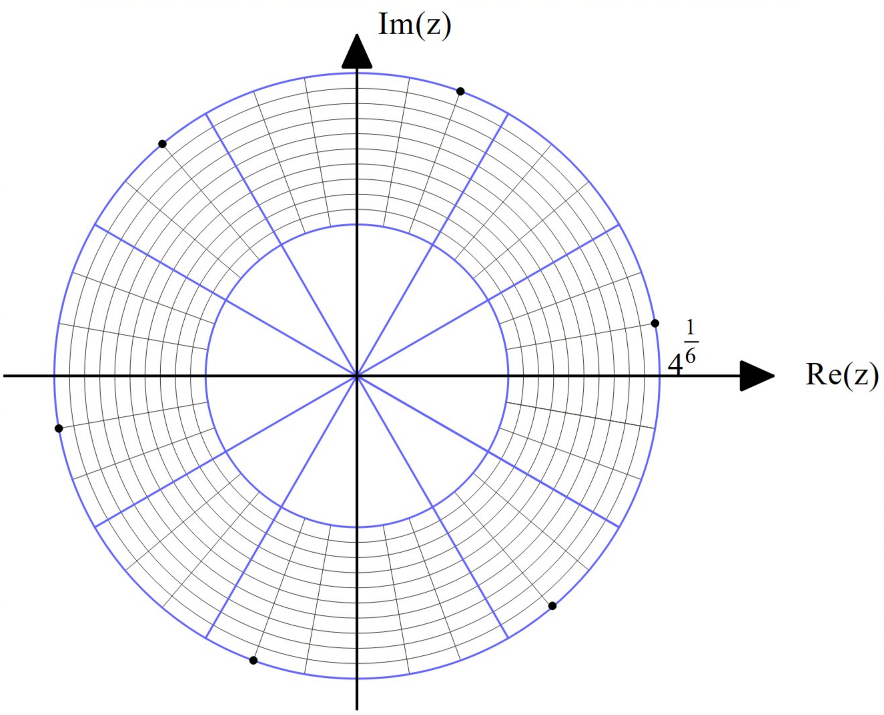
Q6 (5, 2 & 2 = 9 marks)

a) Solve $z^6=2+2\sqrt{3}i$ in polar form with principal arguments.

Solution

$z^6 = 2 + 2\sqrt{3}i = 4cis\left(\frac{\pi}{3} + 2n\pi\right) \quad n = 0, \pm 1, \pm 2, \ldots$ $z = 4^{\frac{1}{6}} cis\left(\frac{\pi}{18} + \frac{2n\pi}{6}\right) = 4^{\frac{1}{6}} cis\left(\frac{\pi}{18} + \frac{6n\pi}{18}\right)$ $z_1 = 4^{\frac{1}{6}} cis\left(\frac{\pi}{18}\right)$ $z_2 = 4^{\frac{1}{6}} cis\left(\frac{7\pi}{18}\right)$ $z_3 = 4^{\frac{1}{6}} cis\left(\frac{-5\pi}{18}\right)$ $z_4 = 4^{\frac{1}{6}} cis\left(\frac{13\pi}{18}\right)$ $z_5 = 4^{\frac{1}{6}} cis\left(\frac{-11\pi}{18}\right)$ $z_6 = 4^{\frac{1}{6}} cis\left(\frac{-17\pi}{18}\right)$
Specific behaviours
✓converts RHS to polar form ✓ shows use of De Moivres ✓ states 6 roots with same modulus ✓ all roots equally spaced ✓all arguments in Principal form

b) Plot these points on the axes below.



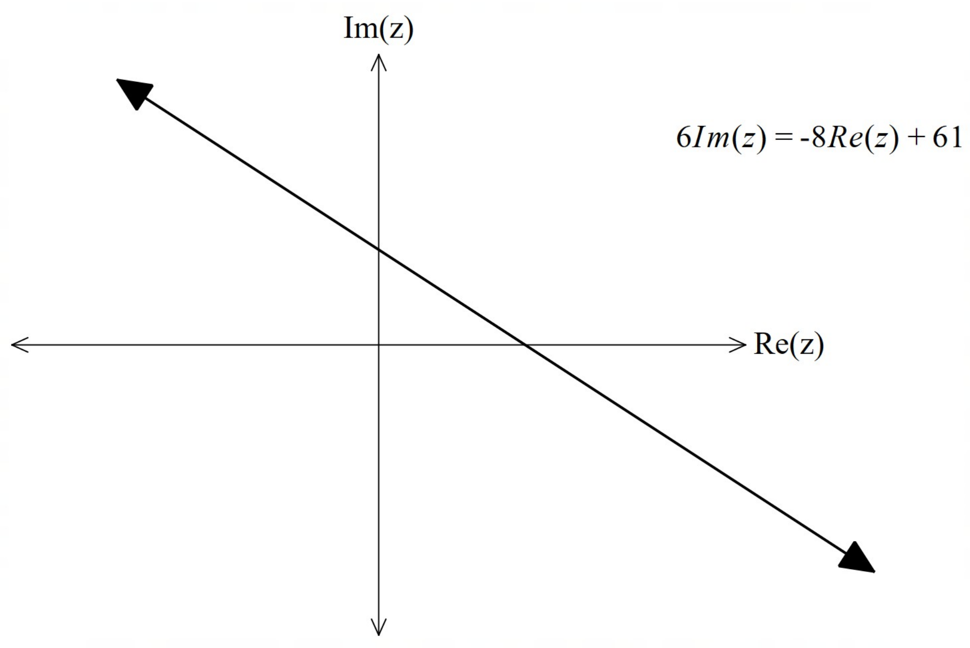
Solution
Specific behaviours
✓ scale shown with 6 roots equally spaced ✓ correct positions for all roots

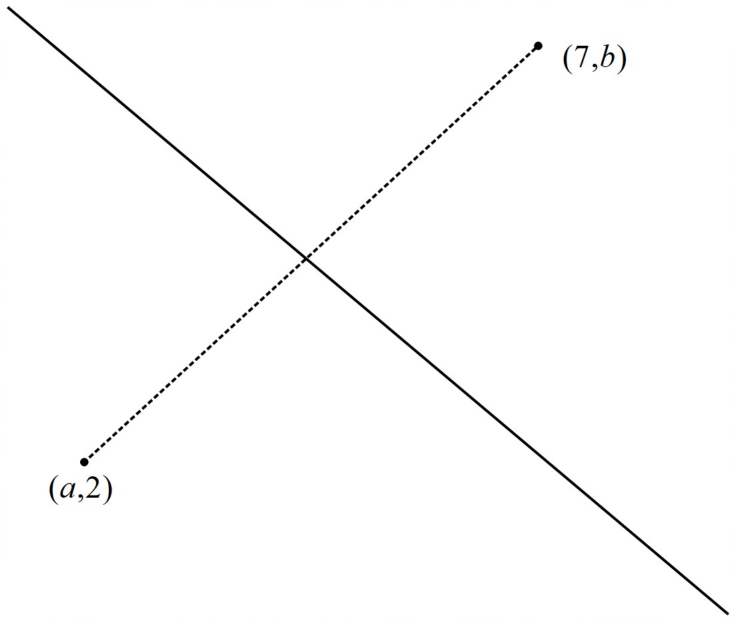
c) Determine the area of the polygon formed by joining the points in (b) above.

Solution
$6\times\frac{1}{2}\left(4^{\frac{1}{6}}\right)^{2}\sin\frac{\pi}{3}=3\frac{\sqrt{3}}{2}\left(4^{\frac{1}{3}}\right)$
Specific behaviours
✓ uses 6 equilateral triangles ✓ states exact value in surd form

Q7 (5 marks)

The locus of $|z - a - 2i| = |z - 7 - bi|$ where $a \,\&\, b$ are real constants is plotted below and can also be defined as $6\,\mathrm{Im}(z) = -8\,\mathrm{Re}(z) + 61$. Determine the values of $a \,\&\, b$ showing full reasoning.



Solution
 Midpoint $\left(\frac{a+7}{2},\frac{2+b}{2}\right)$ Gradient of dotted line $\frac{b-2}{7-a}$ perpendicular to locus line $\frac{b-2}{7-a}=\frac{3}{4}$ $4b-8=21-3a$ $b=\frac{29-3a}{4}=\frac{87-9a}{12}$ $6\left(\frac{b+2}{2}\right)=-8\left(\frac{7+a}{2}\right)+61$ $3b+6=-28-4a+61$ $b=\frac{27-4a}{3}=\frac{108-16a}{12}$ $87-9a=108-16a$ $7a=21$ $a=3$ $b=5$

Specific behaviours
✓ identifies two major points in terms of a&b OR subs $z=x+iy$ into both sides ✓ uses midpoint and subs into line equation OR squares both sides and eliminates squared terms. ✓ uses perpendicular gradient and major points OR subs eqn of line ✓ sets up two simultaneous eqns for a&b ✓solves for both a & b