MATHEMATICS SPECIALIST UNIT 3

Semester One

2019

SOLUTIONS

Calculator-free Solutions

1. $w = iz - \bar{z} = i(x + yi) - (x - yi)$

$¿ ix - y - x + yi = -(x + y) + (x + y)i$ ✓✓

$\therefore |w| = \sqrt{(x+y)^2 + (x+y)^2} = (x+y)\sqrt{2}$ units ✓

and $arg(w) = \tan^{-1}\left(\frac{-x+y}{x+y}\right) = \frac{3\pi}{4}$ only solution ✓ [4]

2. (a) (i) $f(-\sqrt{3}i) = 2(-\sqrt{3}i)^3 - (-\sqrt{3}i)^2 + 6(-\sqrt{3}i) - 3$

$¿ 2(3\sqrt{3}i) - (-3) - 6\sqrt{3}i - 3 = 0 + 0i$ ✓✓

(ii) $\bar{z} = \sqrt{3}i \rightarrow (z - \sqrt{3}i)$ is another factor ✓

(iii) $2z^3 - 2z^2 + z - 3 = 0$

$\therefore (z + \sqrt{3}i)(z - \sqrt{3}i)(az + b) = 0$ ✓

Leading term: $az^3 = 2z^3 \rightarrow a = 2$ ✓

Constant: $(\sqrt{3}i)(-\sqrt{3}i)(b) = 3b = -3 \rightarrow b = -1$ ✓

$\therefore 2z - 1 = 0 \rightarrow z = \frac{1}{2}$ ✓

Solutions $z = \pm\sqrt{3}i, \frac{1}{2}$

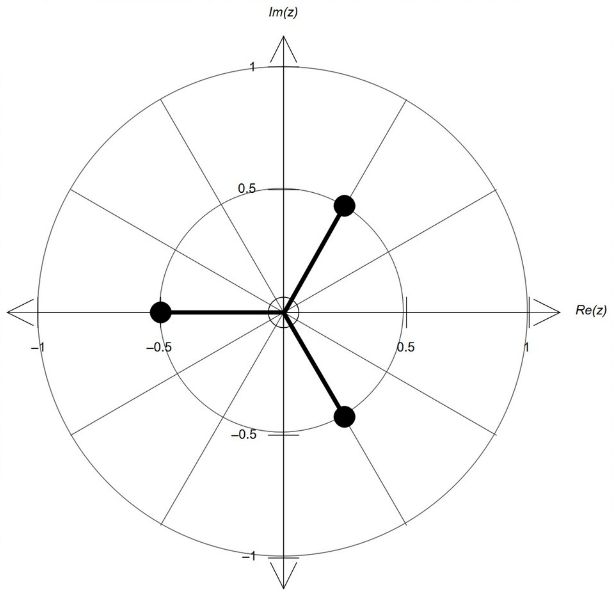
(b) $z^3 = -2^{-3} = 2^{-3} cis(\pi + 2k\pi)\, k = 0, \pm 1$

$\therefore z = 2^{-1} cis\left(\frac{\pi + 2k\pi}{3}\right) k = 0, \pm 1$ ✓

$k = 0 \rightarrow z = \frac{1}{2} cis\left(\frac{\pi}{3}\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{1}{4} + \frac{\sqrt{3}}{4}i$ ✓

$k = 1 \rightarrow z = \frac{1}{2} cis(\pi) = \frac{-1}{2}$ ✓

$k = -1 \rightarrow z = \frac{1}{2} cis\left(\frac{-\pi}{3}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{1}{4} - \frac{\sqrt{3}}{4}i$ ✓



✓ magnitude = 0.5

✓ $\frac{2\pi}{3}$ radians apart

[13]

3. (a)
$$x+y+z=1 \quad \ldots ①$$
$$2x+2y+z=2 \quad \ldots ②$$
$$x-2y-z=1 \quad \ldots ③$$

$② - ① : x+y=1 \ldots ④$

$① + ③ : 2x-y=2 \ldots ⑤$

$④ + ⑤ : 3x=3 \rightarrow x=1$ ✓

$x=1 \rightarrow ④ : y=0$ ✓

$x=1, y=0 \rightarrow ① : z=0$ ✓

(b) Entering equation into a matrix gives:

$$\begin{bmatrix} 1 & 1 & 1 & m \\ 2 & 2 & n & 2 \\ 1 & -2 & 1 & 1 \end{bmatrix}$$

Using row-reduction in one step gives:

$$\begin{bmatrix} 1 & 1 & 1 & m \\ 0 & 3 & 0 & (m-1) \\ 0 & 0 & (2-n) & (2m-2) \end{bmatrix}$$ ✓

From the last row: $2-n \neq 0 \rightarrow \therefore n \neq 2$ ✓

and $m \in R$. ✓

(c) (i) $2x-y+z=4 \rightarrow r \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 4$

the normal to the plane is parallel to direction vector of the line,

$\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = k \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} k \in R$ ✓

hence, accept values that are multiples of k as follows:

$a=2k, b=-k, c=k \; k \in R$ ✓

(ii) Using $k=1$, the line becomes $r = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 2+2\lambda \\ -2-\lambda \\ 1+\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 4$ ✓

$\rightarrow 7+6\lambda=4 \rightarrow \lambda = \dfrac{-1}{2}$ ✓

$$\therefore r = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-3}{2} \\ \frac{1}{2} \end{pmatrix}$$

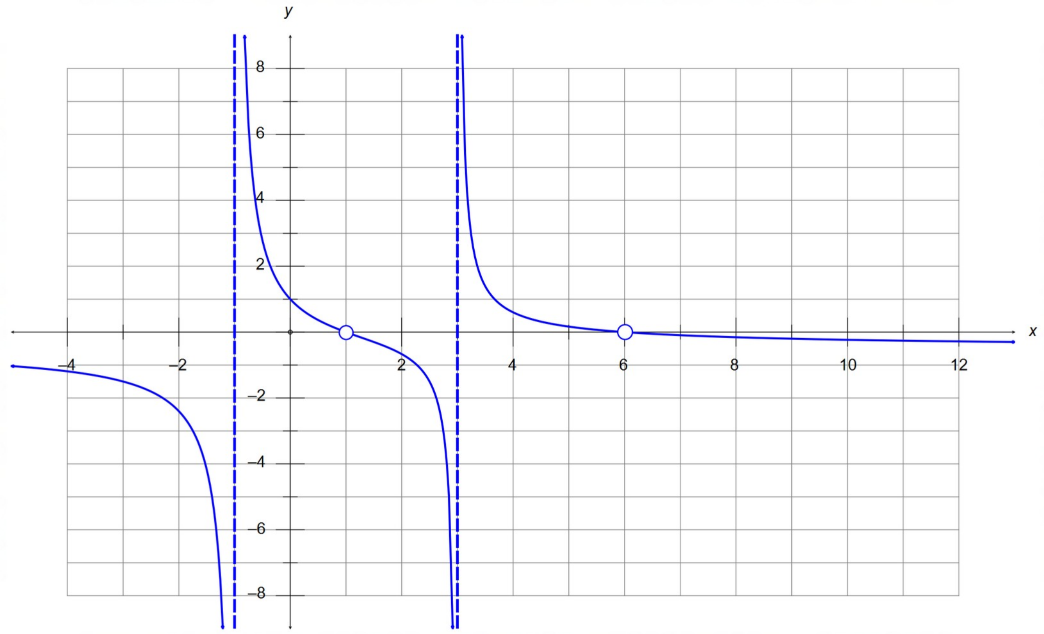
Coordinates of POI are $\left(1, -\frac{3}{2}, \frac{1}{2}\right)$ ✓ [11]

4. (a) Roots: $a=1, b=-3$ or $a=-3, b=1$ ✓✓

Poles: $c=-1, d=-6$ or $c=-6, d=-1$ ✓✓

y-intercept: $f(0)=\frac{k(1)(-3)}{(-1)(-6)}=\frac{-k}{2}=1 \rightarrow k=-2$ ✓

(b)



x < -1

(c) $x=\pm 3$ from symmetry over the y-axis ✓ [11]

5. (a) From the graphs:

$f(x)=0$ for $x=\pm 2$, hence need $g(x)=\pm 2$ ✓

but $0<g(x)<3 \rightarrow g(x)=2$ only

and $g(x)=2$ for $x=-1$, and hence $k=-1$ ✓

(b) $f(x)=4-x^2$ and $g(x)=\sqrt{x+5}$ ✓

$gf(x)=g(f)=\sqrt{f+5}=\sqrt{9-x^2}$ ✓

(c) Need $f(x)\neq -5$ and $f(x)\neq 4$ ✓

Hence, $x\neq 3$ and $x\neq 0$

∴ Domain ¿$\{-3<x<3 \wedge x\neq 0\}$ ✓✓

Range stays the same ¿$\{0<y<3\}$ ✓ [8]

6. (a) Centre is midpoint between P and Q = $\begin{pmatrix}5\\0\\1\end{pmatrix}$ ✓

Radius ¿ $\frac{1}{2}|PQ|=\frac{1}{2}\begin{vmatrix}4\\2\\-2\end{vmatrix}=\begin{vmatrix}2\\1\\-1\end{vmatrix}=\sqrt{6}$ ✓

$\therefore \left|r-\begin{pmatrix}5\\0\\1\end{pmatrix}\right|=\sqrt{6}$ ✓

(b) $\overrightarrow{PQ}$ is normal to the plane, hence $n=\overrightarrow{PQ}=\begin{pmatrix}4\\2\\-2\end{pmatrix}$ ✓

$n\cdot PQ=\begin{pmatrix}4\\2\\-2\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\2\end{pmatrix}=12-2-4=6$ ✓

$\therefore r\cdot\begin{pmatrix}4\\2\\-2\end{pmatrix}=6\rightarrow 2x+y-z=3$ (simplified) ✓ [6]

Calculator-Assumed Solutions

7. Let $z=a+bi$ with $\sqrt{a^2+b^2}=4$

$|z+2i|^2+|z-2i|^2$

¿ $\sqrt{a^2+(b+2)^2}^2+\sqrt{a^2+(b-2)^2}^2$ ✓

¿ $(a^2+b^2+4b+4)+(a^2+b^2-4b+4)$ ✓

¿ $2(a^2+b^2)+8$ ✓

¿ $2\times 4^2+8=40$ ✓ [4]

8. (a) (i) $\overrightarrow{OA}+3\overrightarrow{OC}$ ¿ $\begin{pmatrix}5+3x\\ y+9\\ -1\end{pmatrix}$

since the z-coordinate is already of magnitude 1, then the x ad y coordinates must be zero. Hence,

$x=\frac{-5}{3}$ and $y=-9$ ✓✓

(ii) $\frac{3}{2}\overrightarrow{AB}=\overrightarrow{BC}$ ✓

$\therefore 3(\overrightarrow{OB}-\overrightarrow{OA})=2(\overrightarrow{OC}-\overrightarrow{OB})$

$3\overrightarrow{OB}-3\overrightarrow{OA}=2\overrightarrow{OC}-2\overrightarrow{OB}$

$5\overrightarrow{OB}=3\overrightarrow{OA}+2\overrightarrow{OC}$

$5\begin{pmatrix}9\\0\\-2\end{pmatrix}=3\begin{pmatrix}5\\y\\-4\end{pmatrix}+2\begin{pmatrix}x\\3\\1\end{pmatrix}$ ✓

$\begin{pmatrix}45\\0\\-10\end{pmatrix}=\begin{pmatrix}15+2x\\3y+6\\-10\end{pmatrix}$

$\therefore x=15$ and $y=-2$ ✓✓

(b) $|AB|=\begin{vmatrix}0\\6\\0\end{vmatrix}=6$ ✓

$|AC|=\begin{vmatrix}-4\\3\\\alpha-1\end{vmatrix}$ and $|BC|=\begin{vmatrix}-4\\-3\\\alpha-1\end{vmatrix}$ ✓

If ΔABC equilateral, then $|AB|=|AC|=|BC|=6$

$\therefore \begin{vmatrix}-4\\3\\\alpha-1\end{vmatrix}=\begin{vmatrix}-4\\-3\\\alpha-1\end{vmatrix}=6$

$4^2+3^2+(\alpha-1)^2=6^2$ ✓

$(\alpha-1)^2=11 \to \therefore \alpha=1\pm\sqrt{11}$ ✓ [10]

9. (a) $-a-2 \le b \leftarrow a+2$ ✓✓✓

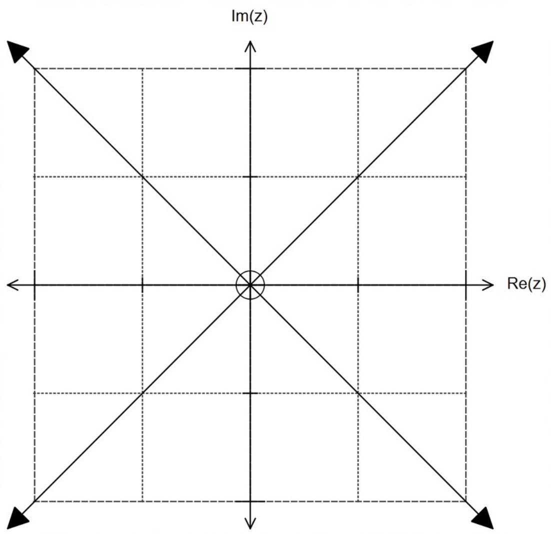
(b) $r \le \frac{3}{2}\pi \wedge -\pi < \theta \le -\frac{2\pi}{3} \wedge \frac{2\pi}{3} \le \theta \le \pi$ ✓✓✓

(c) $z^2+\bar{z}^2=(x+yi)^2+(x-yi)^2$

$¿\, x^2+2xyi-y^2+x^2-2xyi-y^2$ ✓

$¿\, 2x^2-2y^2=0$

$\therefore x^2=y^2 \rightarrow x=\pm y$ two lines through the origin. ✓



✓✓

[10]

10. (a) $v = -2\pi \sin\left(\frac{\pi t}{12}\right) i + 6\pi \cos\left(\frac{\pi t}{4}\right) j$ ✓✓

(b) $r(2) = 24\cos\left(\frac{\pi}{6}\right) i + 24\sin\left(\frac{\pi}{2}\right) j$

¿ $12\sqrt{3}\, i + 24\, j \approx \langle 20.78, 24 \rangle$ cm ✓

$v(2) = -2\pi \sin\left(\frac{\pi}{6}\right) i + 6\pi \cos\left(\frac{\pi}{2}\right) j$

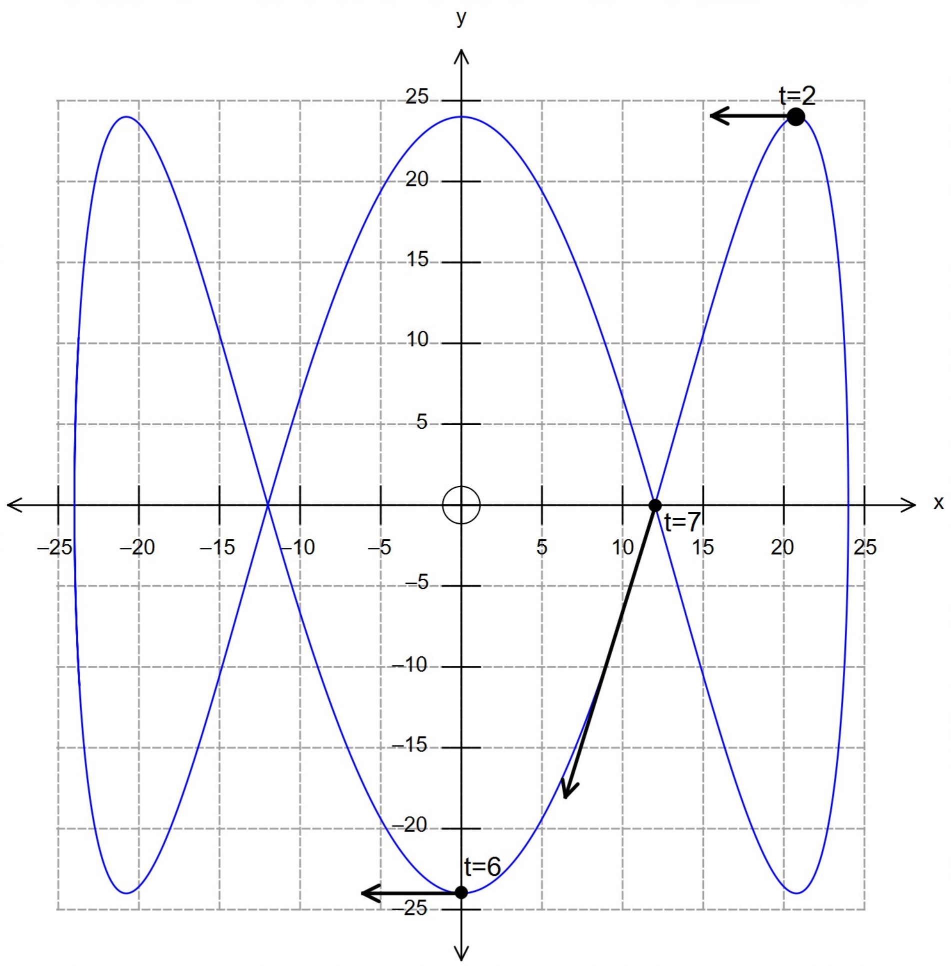
¿ $-\pi\, i + 0\, j = \langle -\pi, 0 \rangle$ cm/s ✓

$r(6) = 24\cos\left(\frac{\pi}{2}\right) i + 24\sin\left(\frac{3\pi}{2}\right) j$

¿ $0\, i - 24\, j = \langle 0, -24 \rangle$ cm ✓

$v(6) = -2\pi \sin\left(\frac{\pi}{2}\right) i + 6\pi \cos\left(\frac{3\pi}{2}\right) j$

¿ $-2\pi\, i + 0\, j = \langle -2\pi, 0 \rangle$ cm/s ✓



(c) $\omega_{min} = \frac{\pi}{12} \rightarrow T_{max} = \frac{2\pi}{\omega_{min}} = \frac{2\pi}{\pi/12} = 24$ sec ✓

(d) $a=\frac{-\pi^2}{6}\cos\left(\frac{\pi t}{12}\right)i-\frac{3\pi^2}{2}\sin\left(\frac{\pi t}{4}\right)j$ ✓

$\therefore a(8)=\frac{-\pi^2}{6}\cos\left(\frac{8\pi}{12}\right)i-\frac{3\pi^2}{2}\sin\left(\frac{8\pi}{4}\right)j=\frac{-\pi^2}{12}i$ ✓

10. (e) $|v|^2 = 4\pi^2 \sin^2\left(\frac{\pi t}{12}\right) + 36\pi^2 \cos^2\left(\frac{\pi t}{4}\right)$

$\therefore |v| = 2\pi\sqrt{\sin^2\left(\frac{\pi t}{12}\right) + 9\cos^2\left(\frac{\pi t}{4}\right)}$ ✓✓

(f) CAS $\rightarrow f_{max} = 19.62$ cm/s at $t = 4.0203026$ s ✓✓

$r(4.0203026) \approx \langle 11.89, -0.38 \rangle \approx \langle 12, 0 \rangle$ cm ✓ [17]

11. (a) $f(x) = |4 - |x||$ OR $f(x) = ||x| - 4|$ ✓✓

(b) $g(x) = -|x+2| + 2 \rightarrow a = b = 2$ ✓✓

(c) $0 \leq m < \frac{1}{3}$ ✓✓

$0 < n \leq \frac{20}{7}$ ✓✓ [8]

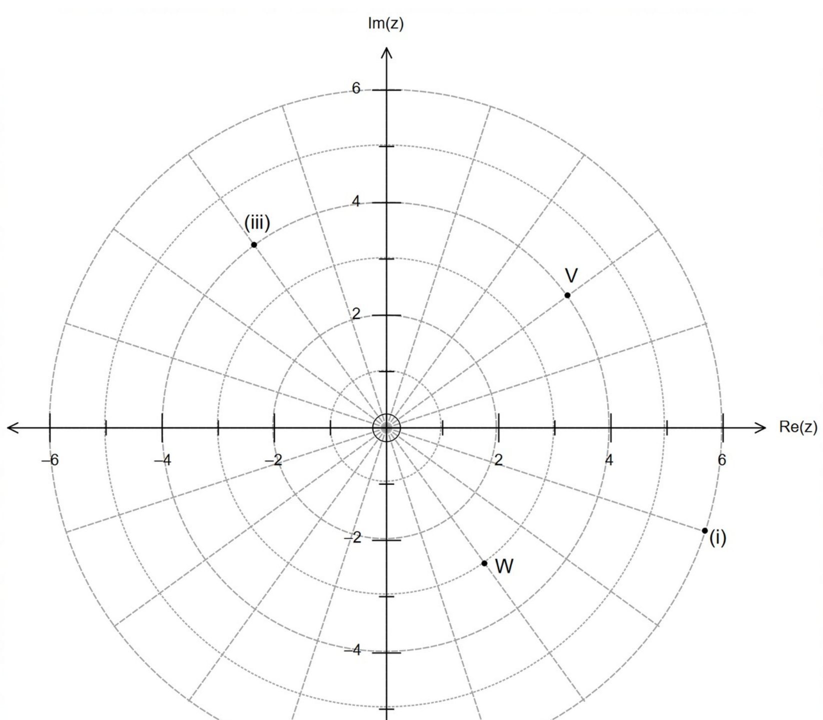
12. (a) $v = 4\,cis\left(\frac{\pi}{5}\right)$ and $w = 3\,cis\left(\frac{-3\pi}{10}\right)$

(i) $\frac{1}{2} v \times w = \frac{1}{2} \times 12\,cis\left(\frac{\pi}{5} - \frac{3\pi}{10}\right) = 6\,cis\left(\frac{-\pi}{10}\right)$ ✓✓

(ii) $v^{-1} \times 8w = \frac{8w}{v} = 24\,cis\left(\frac{-3\pi}{10}\right) \div 4\,cis\left(\frac{\pi}{5}\right)$

$¿ 6\,cis\left(\frac{-\pi}{2}\right)$ ✓✓

(iii) $\frac{v}{i^3} = 4\,cis\left(\frac{\pi}{5}\right) \div cis^3\left(\frac{\pi}{2}\right) = 4\,cis\left(\frac{\pi}{5} - \frac{3\pi}{2}\right) = 4\,cis\left(\frac{7\pi}{10}\right)$ ✓✓



12. (b) (i) $|z|_{max}$ and $|z|_{min}$ occur along the line connecting the centre of the circle with the origin.

$$\left|\begin{matrix}2\\1\end{matrix}\right|=\sqrt{5}$$

$$\therefore |z|_{max}=4+\sqrt{5} \text{ and } |z|_{min}=4-\sqrt{5}$$ ✓✓

(ii) $(x-2)^2+(y-1)^2=16$ and $\Re(z)=x=4$

$$\therefore 4+(y-1)^2=16 \rightarrow y=1\pm\sqrt{12}$$ ✓✓

$$\therefore arg(z)=\tan^{-1}\left(\frac{1\pm\sqrt{12}}{4}\right)=0.8402^R \vee -0.5521^R$$ ✓✓ [12]

13. (a) Speed ¿ $\left|\begin{matrix}25\\-12\\2\end{matrix}\right|=\sqrt{773}\approx 27.80$ m/s ✓

$$\begin{pmatrix}25\\-12\\2\end{pmatrix}\cdot\begin{pmatrix}25\\-12\\0\end{pmatrix}=\left|\begin{matrix}25\\-12\\2\end{matrix}\right|\times\left|\begin{matrix}25\\-12\\0\end{matrix}\right|\times\cos\theta$$ ✓

$$\therefore \cos\theta=\frac{769}{\sqrt{769\times 773}} \rightarrow \theta=4.13°$$ ✓

(b) $15\,min$ ¿ $900\,s$

$$\overrightarrow{OB}(900)=900\begin{pmatrix}25\\-12\\2\end{pmatrix}=\begin{pmatrix}22500\\-10800\\1800\end{pmatrix}$$ ✓

$$\overrightarrow{RB}=\overrightarrow{OB}-\vec{¿}=\begin{pmatrix}22500\\-10800\\1800\end{pmatrix}-\begin{pmatrix}6400\\-12500\\300\end{pmatrix}=\begin{pmatrix}16100\\1700\\1500\end{pmatrix}$$ ✓

Distance ¿ $\left|\begin{matrix}16100\\1700\\1500\end{matrix}\right|=100\left|\begin{matrix}161\\17\\15\end{matrix}\right|=100\sqrt{26435}\approx 16\,258.84$ m ✓

(c) $$\overrightarrow{OB}(960)=960\begin{pmatrix}25\\-12\\2\end{pmatrix}=\begin{pmatrix}24000\\-11520\\1920\end{pmatrix}$$ ✓

$$\vec{¿}(960)=\begin{pmatrix}6400\\-12500\\300\end{pmatrix}+960\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}24000\\-11500\\1920\end{pmatrix}$$ ✓

$$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{60}\begin{pmatrix}24000-6400\\-11520+12500\\1920-300\end{pmatrix}=\begin{pmatrix}293.\overline{3}\\16.\overline{3}\\27\end{pmatrix}$$ m/s ✓

at 1,920 m above level ground. ✓

(d) distance $= \left|\begin{matrix} 24000 \\ -11520 \\ 1920 \end{matrix}\right| \approx 26\,690.76\,m, \quad \therefore v = \frac{26690.76}{343} = 77.82\,s$ ✓✓ [12]

14. (a) $\frac{1}{z}=\frac{1}{\cos\theta+i\sin\theta}\times\frac{\cos\theta-i\sin\theta}{\cos\theta-i\sin\theta}=\frac{\cos\theta-i\sin\theta}{\cos^2\theta+\sin^2\theta}=\cos\theta-i\sin\theta$ ✓✓

(b) $z^n+z^{-n}=2\cos(n\theta)\rightarrow\cos(n\theta)=\frac{z^n+z^{-n}}{z}$ ✓

$z^n-z^{-n}=2i\sin(n\theta)\rightarrow\sin(n\theta)=\frac{z^n-z^{-n}}{2i}$ ✓

(c) $\cos(2\theta)=\frac{z^2+z^{-2}}{2}$ and $\sin(4\theta)=\frac{z^4-z^{-4}}{2i}$ ✓✓

(d) $\cos(2\theta)\times\sin(4\theta)=\left(\frac{z^2+z^{-2}}{2}\right)\times\left(\frac{z^4-z^{-4}}{2i}\right)$

¿ $\frac{1}{4i}\left(z^6-z^{-2}+z^2-z^{-6}\right)$ ✓

¿ $\frac{1}{2}\left(\frac{z^6-z^{-6}}{2i}\right)+\frac{1}{2}\left(\frac{z^2-z^{-2}}{2i}\right)$ ✓

¿ $\frac{1}{2}\sin(6\theta)+\frac{1}{2}\sin(2\theta)$ ✓ [9]

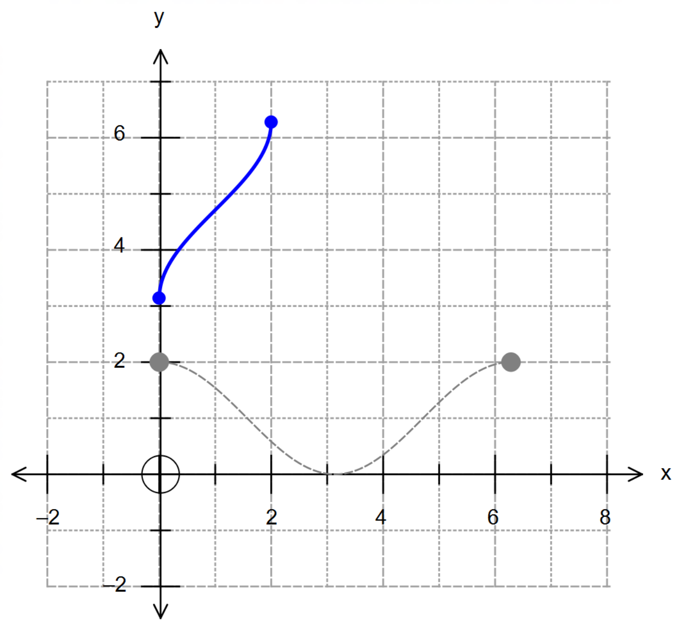
15. (a) $f^{-1}(x)$ does not exist because $f(x)$ is not a one-to-one function.✓

(b) (i) Need $\pi\leq x\leq 2\pi$, hence $k=\pi$ ✓

Then, $f^{-1}(x)=\pi+\cos^{-1}(1-x)$ ✓✓

(ii) Domain ¿ $\{0\leq x\leq 2\}$ and Range ¿ $\{\pi\leq y\leq 2\pi\}$ ✓✓

(c)



✓ sinusoidal and symmetrical over the line y=x

✓ correct location and boundaries (accuracy)

(d) $gf\left(f^{-1}\right)=g\left(x\right)$

$\therefore g\left(x\right)=gf\left(f^{-1}\right)=f^{-1}+1=\pi+\cos^{-1}\left(x-1\right)+1$ ✓✓ [10

16. $x = \cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$ ✓

$\therefore \cos^2\theta = \dfrac{x+1}{2}$ ✓

Similarly,

$x = \cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$\therefore \sin^2\theta = \dfrac{1-x}{2}$ ✓

and $y = \tan\theta$, and for domain to remain the same:

$y^2 = \tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{1-x}{2} \div \dfrac{x+1}{2}$ ✓

Hence, $y^2 = \dfrac{1-x}{x+1}$ ✓ [5]

(Other methods are possible depending on their choice of trigonometric identity. Award marks accordingly).