SCHOOL

Trial WACE Examination, 2011

Question/Answer Booklet

MATHEMATICS SPECIALIST 3A/3B

SOLUTIONS

Section Two: Calculator-assumed

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination.

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	5	5	50	50	33
Section Two: Calculator-assumed	13	13	100	100	67
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2011*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

Section Two: Calculator-assumed **(80 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 6 **(5 marks)**

(a) Convert the polar coordinate $P\left(\frac{4\pi}{5}, \frac{3\pi}{5}\right)$ to Cartesian form, correct to 2 decimal places.

(2 marks)

$\boxed{(-0.78,\ 2.39)}$

(b) Convert the polar equation $\theta = -\frac{\pi}{4}$ into Cartesian form. (1 mark)

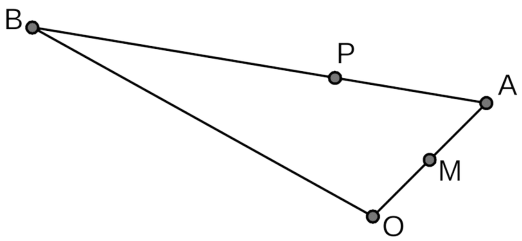
$\boxed{y = -x}$

(c) Convert the Cartesian equation $x^2 + y^2 = 4$ into polar form. (2 marks)

$\boxed{r = 2}$

Question 7 **(8 marks)**

In the triangle below, $\mathbf{a} = OA$, $\mathbf{b} = OB$, M is the midpoint of OA and P is a point on AB such that $AP : PB = 1 : 3$.



(a) Express each of the following in terms of **a** and /or **b**.

(i) BA (1 mark)

$\boxed{\mathbf{a} - \mathbf{b}}$

(ii) OP (2 marks)

$$OP = \mathbf{b} + \frac{3}{4}(\mathbf{a} - \mathbf{b})$$
$$= \frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$$

(iii) MP (2 marks)

$$MP = -\frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$$
$$= \frac{1}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$$

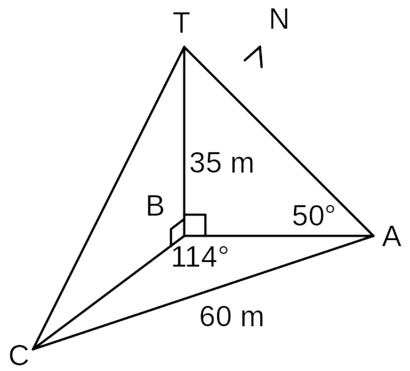
(b) If $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = -9\mathbf{i} + 4\mathbf{j}$, determine $\left| MP \right|$. (3 marks)

$$\frac{1}{4}\left| \begin{pmatrix} 1-9 \\ 2+4 \end{pmatrix} \right|$$
$$= \frac{1}{4}\left| \begin{pmatrix} -8 \\ 6 \end{pmatrix} \right|$$
$$= 2.5$$

Question 8 **(7 marks)**

The top of a vertical radio mast stands 35 m above the surrounding level ground. From point A which is on the ground and due east of the base of the mast, the angle of elevation of the top of the mast is 50˚. From another point on the ground, C, which is 60 m away from A, the bearing of the base of the mast is 024˚.

Calculate the angle of elevation of the top of the mast from C.



$$AB = 35 \div \tan 50$$
$$= 29.36849$$

$$BC = x$$
$$60^2 = x^2 + 29.36849^2 - 2x \times 29.36849 \times \cos(114)$$
$$BC = 41.72207 \quad \text{(ignore -ve soln)}$$

$$\angle BCT = \tan^{-1}\frac{35}{41.72207}$$
$$= 39.99277$$
$$\approx 40^\circ$$

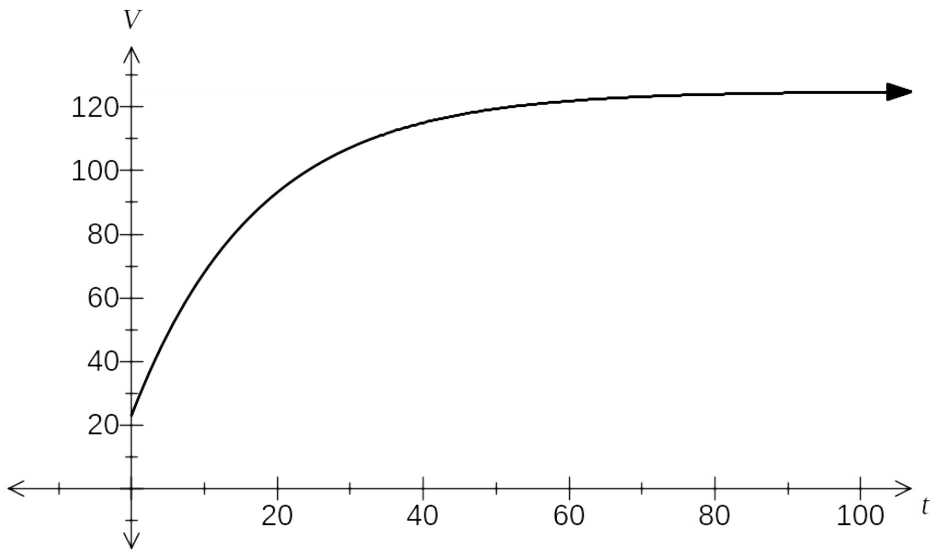
Question 9 **(8 marks)**

After a period of rain, the volume of water in a storage tank, V kL, increases according to the rule $V = 125 - 102(1.06)^{-t}$ for $t \geq 0$, where t is time in hours.

(a) What is the initial volume of water in the tank? (1 mark)

$$\boxed{\begin{aligned} V &= 125 - 102(1.06)^0 \\ &= 23 \text{ kL} \end{aligned}}$$

(b) Sketch the graph of this relationship on the axes below. (2 marks)



(c) According to the rule, what is the maximum volume of water that the tank can hold? (2 marks)

$$\boxed{\begin{aligned} V_{\max} &= 125 - 102(1.06)^{-\infty} \\ &= 125 \text{ kL} \end{aligned}}$$

(d) How long, to the nearest hour, will it take for the tank to fill to within 5% of its maximum capacity? (3 marks)

$$\boxed{\begin{aligned} &0.95 \times 125 = 118.75 \\ &118.75 = 125 - 106(1.06)^{-t} \\ &t = 47.92 \\ &\approx 48 \text{ hours} \end{aligned}}$$

Question 10 **(7 marks)**

The two functions *f* and *g* are defined by $f(x) = \frac{1}{x} + 1$ and $g(x) = \frac{1}{x+1}$.

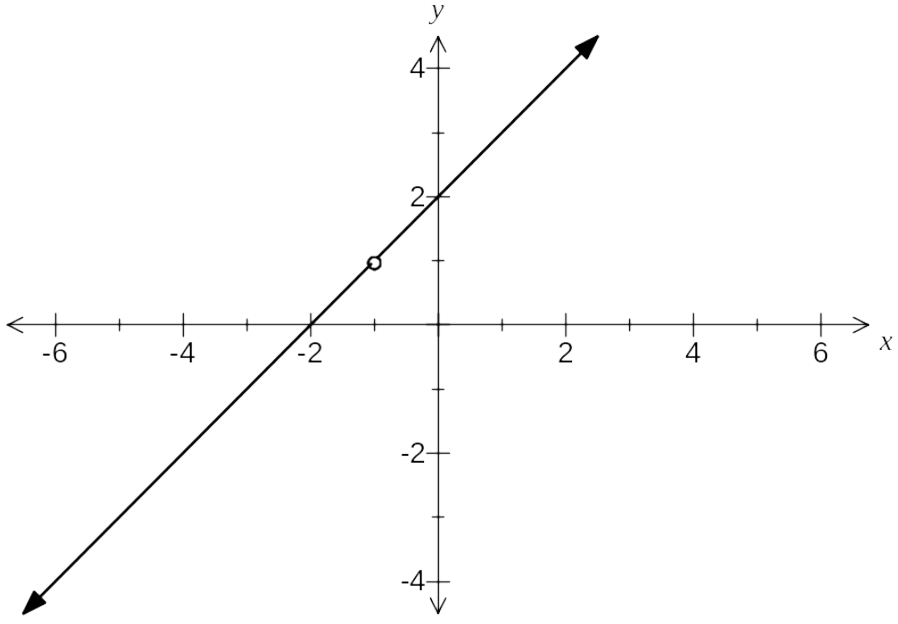
(a) Find and simplify an expression for $f \circ g(x)$. (2 marks)

$$\boxed{\begin{aligned} f \circ g(x) &= 1 \div \frac{1}{x+1} + 1 \\ &= x + 2 \end{aligned}}$$

(b) State the domain of $f \circ g(x)$. (2 marks)

$$\boxed{\text{Domain of } f \circ g(x) = \{\, x \in R : x \neq -1 \,\}}$$

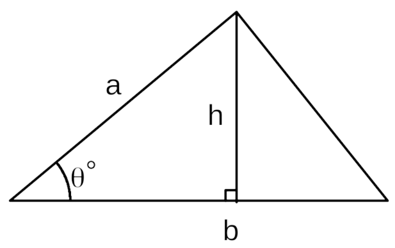
(c) Sketch the graph of $f \circ g(x)$. (3 marks)



Question 11 **(10 marks)**

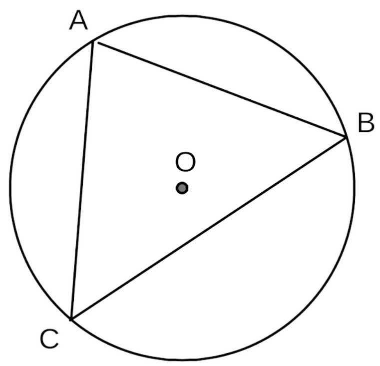
(a) Using the formula $\text{Area} = \frac{1}{2} \times \text{Base} \times \text{Perpendicular Height}$, show that the area of a triangle with two sides of lengths a and b, and included angle of θ is given by $\text{area} = \frac{ab\sin\theta}{2}$.

(2 marks)



$$h = a \sin \theta$$
$$\text{area} = \frac{1}{2} \times b \times h$$
$$= \frac{1}{2} \times b \times a \sin \theta$$
$$= \frac{ab \sin \theta}{2}$$

(b) A triangle is inscribed in a circle, centre O, with minor arcs AB, BC and CA having lengths 5π, 8π and 5π cm respectively.



(i) Show that the radius of the circle is 9 cm. (2 marks)

$$C = 2\pi r$$
$$18\pi = 2\pi r$$
$$r = 9 \text{ cm}$$

(ii) Show that $\angle CAB = 80^\circ$. (3 marks)

$$\angle AOB = \frac{5}{18} \times 360^\circ$$
$$= 100^\circ$$

$$\angle OAB = \frac{180^\circ - 100^\circ}{2} \text{ (isosceles triangle)}$$
$$= 40^\circ$$

$$\angle BAC = 2 \times 40^\circ$$
$$= 80^\circ$$

(iii) Find the area of triangle ABC. (3 marks)

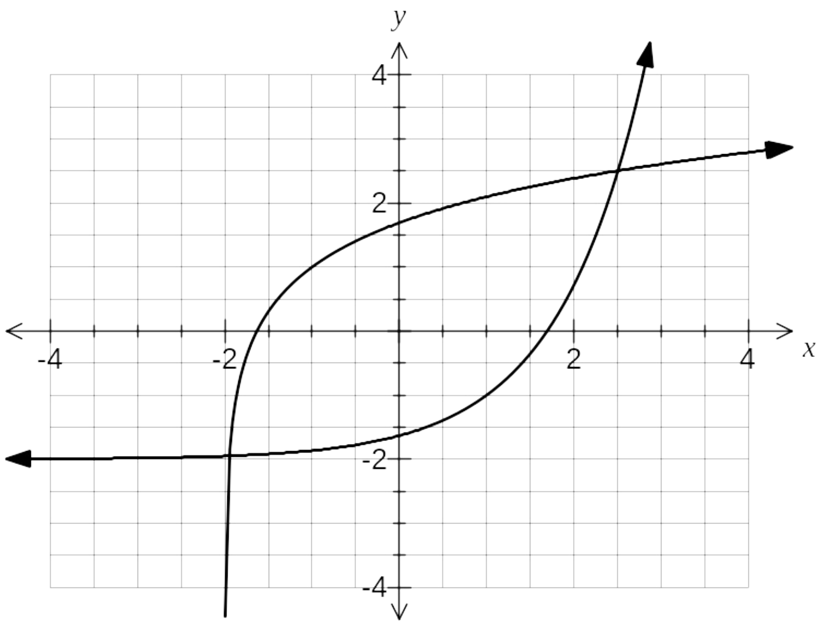
$$AB^2 = 9^2 + 9^2 - 2 \times 9 \times 9 \times \cos 100^\circ$$
$$AB = 13.789$$

$$Area = \frac{1}{2} \times 13.789 \times 13.789 \times \sin 80^\circ$$
$$= 93.624$$
$$\approx 93.6 \text{ cm}^2$$

Question 12 **(7 marks)**

The function f is defined by $f(x) = e^{x-1} - 2$.

(a) Draw the graph of $y = f(x)$ on the axes below. (2 marks)



(b) State the domain and range of $f(x)$. (2 marks)

Domain of $f(x) = \{x \in R\}$

Range of $f(x) = \{y \in R : y > -2\}$

(c) Add the graph of $y = f^{-1}(x)$, the inverse of $f(x)$, to the axes above. (2 marks)

(d) State the domain and range of $f^{-1}(x)$. (1 mark)

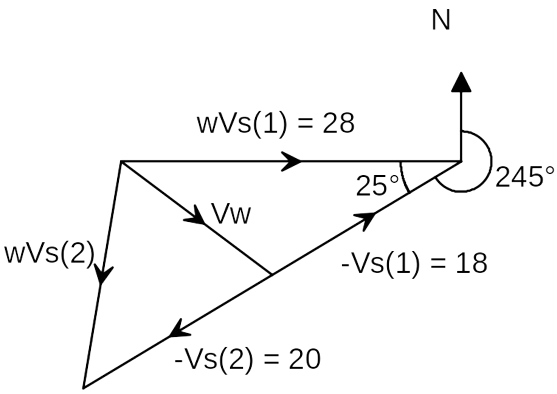
Domain of $f^{-1}(x) = \{x > -2\}$

Range of $f(x) = \{y \in R\}$

Question 13 **(7 marks)**

A boat is motoring at 18 km/h on a bearing of 245°. To an observer on board the boat, the wind appears to be blowing at 28 km/h from due west.

If the boat turned through 180° and increased its speed to 20 km/h, find the new apparent wind speed and direction.



$$wVs(2) = \sqrt{38^2 + 28^2 - 2 \times 38 \times 28 \times \cos 25^\circ}$$
$$= 17.3 \text{ km/h}$$

$$\frac{\sin\theta}{38} = \frac{\sin 25}{17.3}$$
$$\theta = 112^\circ$$

$$\text{Bearing} = 112 + 90$$
$$= 202^\circ$$

Question 14 **(11 marks)**

Two robots can be programmed to travel in a straight line with constant velocity.

Relative to an origin O, robot A leaves position $-13\mathbf{i}+22\mathbf{j}$ m and travels with velocity $3\mathbf{i}-2\mathbf{j}$ m/s.

One second later, robot B starts from position $5\mathbf{i}+15\mathbf{j}$ m and travels with velocity $-4\mathbf{i}-\mathbf{j}$ m/s.

(a) Calculate the position and velocity of robot A relative to robot B at the instant robot B starts and hence explain why the robots will not collide. (5 marks)

When B starts A is at $\begin{bmatrix}-13+3\\22-2\end{bmatrix}=\begin{bmatrix}-10\\20\end{bmatrix}$

$${}_A\mathbf{r}_B=\begin{bmatrix}-10\\20\end{bmatrix}-\begin{bmatrix}5\\15\end{bmatrix}=\begin{bmatrix}-15\\5\end{bmatrix}$$

$${}_A\mathbf{v}_B=\begin{bmatrix}3\\-2\end{bmatrix}-\begin{bmatrix}-4\\-1\end{bmatrix}=\begin{bmatrix}7\\-1\end{bmatrix}$$

No collision because ${}_A\mathbf{v}_B$ is clearly not a multiple of ${}_A\mathbf{r}_B$.

(b) Determine the minimum distance between the two robots after they both start moving. (6 marks)

Minimum distance will occur when ${}_A\mathbf{v}_B \bullet (t\,{}_A\mathbf{v}_B + {}_A\mathbf{r}_B) = 0$.

$$\begin{bmatrix} 7 \\ -1 \end{bmatrix} \bullet \left(t \begin{bmatrix} 7 \\ -1 \end{bmatrix} + \begin{bmatrix} -15 \\ 5 \end{bmatrix} \right) = 0$$

$7(7t - 15) + t - 5 = 0$

$t = 2.2$

$$2.2 \times \begin{bmatrix} 7 \\ -1 \end{bmatrix} + \begin{bmatrix} -15 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 2.8 \end{bmatrix}$$

$\sqrt{0.4^2 + 2.8^2} = 2\sqrt{2}$

≈ 2.83 m

Question 15 **(5 marks)**

$v,\ u$ and t are related by the equations $v = 20\log_e u$ and $t = 40u$, for $u > 0$.

(a) Find expressions for $\frac{dv}{du}$ and $\frac{du}{dt}$ and hence use the chain rule to find $\frac{dv}{dt}$ in terms of t.

(3 marks)

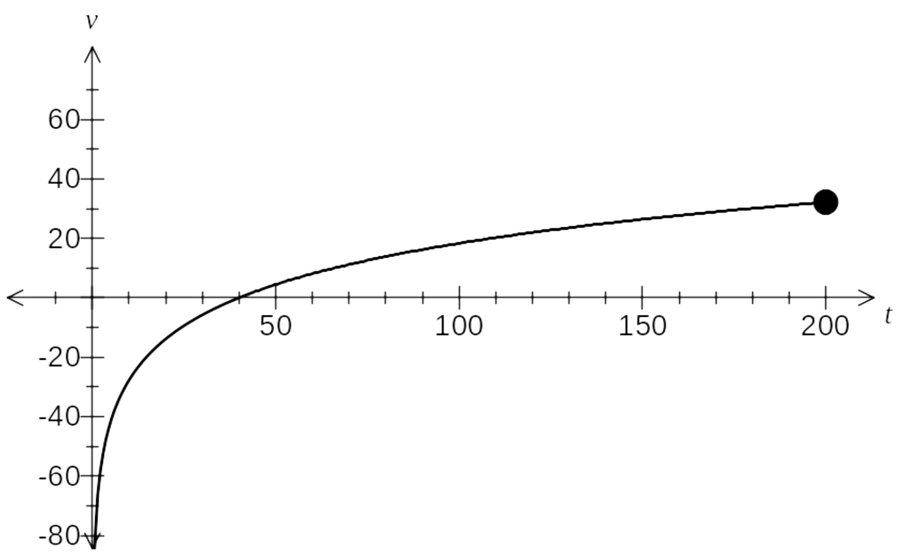
$$\frac{dv}{du} = \frac{20}{u}$$

$$u = \frac{t}{40}$$

$$\frac{du}{dt} = \frac{1}{40}$$

$$\begin{aligned}\frac{dv}{dt} &= \frac{20}{u} \times \frac{1}{40} \\ &= \frac{1}{2u} \\ &= \frac{1}{2} \times \frac{40}{t} \\ &= \frac{20}{t}\end{aligned}$$

(b) Sketch the graph of v against t for $0 \le t \le 200$. (2 marks)



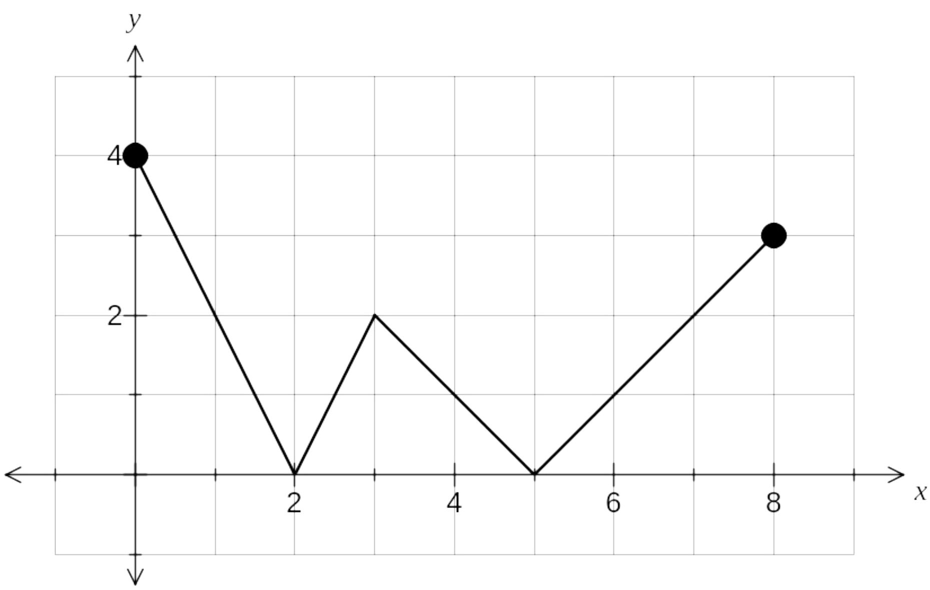
Question 16 **(9 marks)**

The function f is continuous on $[0, 8]$ and is defined by $f(x) = \begin{cases} |2x - 4| & \text{for } 0 \le x \le k \\ |x - 5| & \text{for } k < x \le 8 \end{cases}$

(a) Find the value of k. (3 marks)

Over defined interval, two pieces intersect once between 2 and 5.
Hence for continuity
$2k - 4 = -(k - 5)$
$3k = 9$
$k = 3$

(b) Sketch the graph of $y = f(x)$. (3 marks)



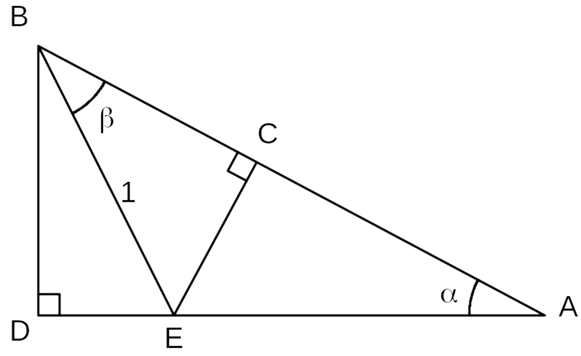
(c) Discuss the differentiability of f on $[0, 8]$. (3 marks)

f is differentiable everywhere except when $x = 2$, $x = 3$ and $x = 5$.

At all these values of x, there is an obvious cusp, indicating that $\lim_{x \to a^-} f'(x) \neq \lim_{x \to a^+} f'(x)$.

Question 17 **(11 marks)**

(a)



(i) Explain why $\angle \text{BED} = \alpha + \beta$ and hence show that $\text{BD} = \sin(\alpha + \beta)$. (3 marks)

$$\angle CEA = 90 - \alpha$$
$$\angle CEB = 90 - \beta$$
$$\angle BED = 180 - (90 - \alpha) - (90 - \beta)$$
$$= \alpha + \beta$$

$$BD = 1 \times \sin(\alpha + \beta)$$
$$= \sin(\alpha + \beta)$$

(ii) Given that $\text{BC} = \cos\beta$ and $\text{CE} = \sin\beta$, use triangle ACE to show that $\text{AB} = \dfrac{\cos\alpha \sin\beta}{\sin\alpha} + \cos\beta$. (2 marks)

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\sin\beta}{\text{AC}}$$
$$\text{AC} = \frac{\cos\alpha \sin\beta}{\sin\alpha}$$

$$\text{AB} = \text{AC} + \text{CB}$$
$$= \frac{\cos\alpha \sin\beta}{\sin\alpha} + \cos\beta$$

(iii) Use triangle ABD to show that $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$. (3 marks)

$$\sin\alpha=\frac{\sin(\alpha+\beta)}{\dfrac{\cos\alpha\sin\beta}{\sin\alpha}+\cos\beta}$$

$$\sin\alpha\left(\frac{\cos\alpha\sin\beta}{\sin\alpha}+\cos\beta\right)=\sin(\alpha+\beta)$$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

(b) Show use of the above result to find an exact value for $\sin 105^\circ$. (3 marks)

$$\sin(105^\circ)=\sin(45^\circ+60^\circ)$$

$$=\sin 45^\circ\cos 60^\circ+\cos 45^\circ\sin 60^\circ$$

$$=\frac{\sqrt{2}}{2}\frac{1}{2}+\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}$$

$$=\frac{\sqrt{2}\left(1+\sqrt{3}\right)}{4}$$

Question 18 **(5 marks)**

(a) Use your calculator to evaluate $\lim_{h \to 0} \frac{e^h - 1}{h}$. (1 mark)

$$\boxed{1}$$

(b) Using your result from part (a), show that the derivative of e^{3x} from first principles is $3e^{3x}$. (4 marks)

$$\begin{aligned}
\frac{d}{dx}e^{3x} &= \lim_{x \to 0} \frac{e^{3(x+h)} - e^{3x}}{h} \\
&= \lim_{x \to 0} \frac{e^{3x} \times e^{3h} - e^{3x}}{h} \\
&= \lim_{x \to 0} \frac{e^{3x}\left(e^{3h} - 1\right)}{h} \\
&= 3e^{3x} \times \lim_{x \to 0} \frac{\left(e^{3h} - 1\right)}{3h} \\
&= 3e^{3x} \times 1 \\
&= 3e^{3x}
\end{aligned}$$

End of questions

Additional working space

Question number: _________

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