Hale School

MATHEMATICS
SPECIALIST
3CD

Semester Two Examination 2011

MARKING KEY and SOLUTIONS

Section One

Calculator-Free

MARKING KEY and SOLUTIONS

Question 1 **[9 marks]**

Give exact values for the following :

(a) $e^{-\frac{i\pi}{2}}$

[1]

Solution
$e^{-\frac{i\pi}{2}}$ = cis(-π/2) = - i
Specific Behaviours
✓ Correct answer

(b) $\left(\cos\left(\frac{\pi}{5}\right) - i\sin\left(\frac{\pi}{5}\right)\right)^5$

[2]

Solution
$\left(\cos\left(\frac{\pi}{5}\right) - i\sin\left(\frac{\pi}{5}\right)\right)^5 = \left(\text{cis}\left(\frac{-\pi}{5}\right)\right)^5$ = cis(-π) = -1
Specific Behaviours
✓ Recognises cis with the correct argument ✓ Uses DeMoivre's Theorem to multiply argument correctly and give the correct answer

(c) $\lim\limits_{h\to 0}\dfrac{\cos\left(\frac{\pi}{3}+2h\right) - \cos\left(\frac{\pi}{3}\right)}{h}$

[3]

Solution
$\lim\limits_{h\to 0}\dfrac{\cos\left(\frac{\pi}{3}+2h\right) - \cos\left(\frac{\pi}{3}\right)}{h} = \lim\limits_{h\to 0}\dfrac{\cos\left(2\left(\frac{\pi}{6}+h\right)\right) - \cos\left(2\left(\frac{\pi}{6}\right)\right)}{h}$ $= f'\left(\frac{\pi}{6}\right)$ where $f(x) = \cos 2x$ $= -\sin 2\left(\frac{\pi}{6}\right).\ 2 = -\sqrt{3}$
Specific Behaviours
✓ Recognises the limit as a derivative ✓ Correct identification of the function cos 2x ✓ Determines the exact value

(d) The shaded area under the curve y = cos 2x.

[3]

Solution
$\text{Area} = 3\int_0^{\frac{\pi}{4}} \cos 2x\ dx = 3\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{3}{2}$ square units

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Specific Behaviours
✓ Correct expression for the area ✓ Anti-differentiates correctly ✓ Correct evaluation

Question 2 **[9 marks]**

Given that $z = e^{ix}$ and $w = e^{-ix}$ (where x is a real number) :

(a) express cis(3x) w in terms of z.

[2]

Solution
$\text{cis}(3x)\ w = e^{3ix} . e^{-ix} = e^{2ix} = \left(e^{ix}\right)^2 = z^2$
Specific Behaviours
✓ Expresses cis(3x) in terms of a complex exponential ✓ Correct expression in terms of z

(b) if z - w is expressed in the form a + bi determine the values of a and b.

[2]

Solution
$z - w = e^{ix} - e^{-ix} = z - \bar{z} = 2i \sin x$ Hence a = 0, b = 2 sin x
Specific Behaviours
✓ Recognises expression to give twice the imaginary part ✓ Correct values for both a and b.

(c) simplify $z^3 + w^3$

[2]

Solution
$z^3 + w^3 = e^{3ix} + e^{-3ix} = 2 \cos 3x$
Specific Behaviours
✓ Correct use of index laws with the complex exponential ✓ Simplifies correctly in terms of twice the real part

(d) solve for x given that $z^4 + 1 = 0$

[3]

Solution
$z^4 = -1 = 1\ \text{cis}(\pi + 2\pi k) \qquad k = 0, 1, 2, 3$ $z = 1^{\frac{1}{5}}\ \text{cis}\left(\dfrac{\pi + 2\pi k}{4}\right)$ $z_0 = \text{cis}\left(\dfrac{\pi}{4}\right) \quad z_1 = \text{cis}\left(\dfrac{3\pi}{4}\right) \quad z_2 = \text{cis}\left(\dfrac{5\pi}{4}\right) \quad z_3 = \text{cis}\left(\dfrac{7\pi}{4}\right)$ Hence $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, -\dfrac{3\pi}{4}, -\dfrac{\pi}{4}$
Specific Behaviours
✓ Expresses -1 in polar form with argument π ✓ Correct expression for the 4 solutions in polar form

✓ Correct values for x (using convention between $-\pi$ to π)

Question 3 **[6 marks]**

Points A, B and C have respective position vectors given by :

a = **i** + **j** - **k**
b = **i** + **j** + **k**
c = 2**i** + **j**

Determine :

(a) the value of cosine of the angle between vectors **a** and **b**.

[2]

Solution
a . b $= 1 + 1 - 1 = \sqrt{3}.\sqrt{3}.\cos\theta$ $1 = 3\cos\theta$ Hence $\cos\theta = \frac{1}{3}$
Specific Behaviours
✓ Correct use of dot product and the magnitudes of each vector   ✓ Correct value for $\cos\theta$

(b) the vector equation of the line containing points A and B.

[2]

Solution
Direction vector **d** = **b** - **a** = (1 ¿)(1 ¿)¿¿¿ ¿   Vector equation line **r** = (1 ¿)(1 ¿)¿¿¿ ¿
Specific Behaviours
✓ Finds an appropriate direction vector   ✓ Expresses in correct point-direction vector form (does not have to express as a single vector)

(c) the vector equation of the plane containing vectors **a** and **b** and also containing the point C.

[2]

Solution
Vector equation plane **r** = (2 ¿)(1 ¿)¿¿¿ ¿

Alternative answer **r .** $\left(\quad 1 ¿\right)\left(-1 ¿\right) ¿ ¿ ¿$ ¿
Specific Behaviours
✓ Uses 2 parameters with the direction vectors a and b ✓ Expresses in correct vector form (does not have to express as a single vector) NO MARKS if students uses vectors a and b as points in the plane

Question 4 **[3 marks]**

Evaluate the definite integral $\int_0^{\pi} \sin^2 x \, dx$ exactly :

Solution
$\int_0^{\pi} \sin^2 x \ dx = \int_0^{\pi} \frac{1}{2}(1 - \cos 2x) \ dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{\pi}{2}$
Specific Behaviours
✓ Correct use of the cosine DOUBLE angle identity ✓ Anti-differentiates correctly ✓ Correct evaluation NO marks for use of $\sin^3 x$ as the anti-derivative

Question 5 **[4 marks]**

(a) Determine matrix T = $[2 \ \ 0 ¿] ¿ ¿ ¿$ ¿

[1]

Solution
T = $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$
Specific Behaviours
✓ ALL Matrix elements are correct

(b) Hence if matrix T represents a transformation matrix, describe the actions of matrix T.

[3]

Solution
T = A B i.e. B then A i. Reflect about the line y = -x and then ii. Dilate horizontally about x = 0 with factor 2

Specific Behaviours
✓ Description of the reflection matrix ✓ Description of the dilation matrix ✓ Correct order i.e. reflect then dilate

Question 6 **[5 marks]**

The natural logarithm function can be defined as $\ln(x) = \int_1^x \frac{dt}{t}$ where $x > 0$.

(a) Given that $a, b > 0$, using the substitution $u = \frac{t}{a}$ find an expression for the definite integral $\int_a^{ab} \frac{dt}{t}$.

[3]

Solution
$\int_a^{ab} \frac{dt}{t} = \int_1^b \frac{a \cdot du}{au} = \int_1^b \frac{du}{u} = \ln b$
Specific Behaviours
✓ Changes limits correctly ✓ Expresses integrand correctly ✓ Recognises answer using the natural logarithm definition

(b) By considering $\int_1^{ab} \frac{dt}{t} = \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t}$ and using the result from part (a) make a deduction about the natural logarithm function.

[2]

Solution
$\int_1^{ab} \frac{dt}{t} = \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t}$ $\ln(ab) = \ln a + \ln b$
Specific Behaviours
✓ Uses the result from part (a) ✓ Deduces that the log(Product) = sum of logarithms

Question 7 **[4 marks]**

Prove, by any method, that the cube of any number that is 2 more than a multiple of 3 is always 1 less than a multiple of 9.

Solution
Let n be any counting number. Hence $3n + 2$ is two more than a multiple of 3 (the particular number) Consider $(3n+2)^3 = (3n)^3 + 3(3n)^2(2) + 3(3n)(2^2) + 2^3$ $= 27n^3 + 54n^2 + 36n + 8$ $= 27n^3 + 54n^2 + 36n + 9 - 1$ $= 9(3n^3 + 6n^2 + 4n + 1) - 1$ Hence $(3n+2)^3$ is always of the form $9k - 1$
Specific Behaviours
✓ Express the cube of the particular number ✓ Expand correctly the cube of the binomial ✓ Simplify each term correctly ✓ Express in the form $9k - 1$