-SCHOOL

Trial WACE Examination, 2012

Question/Answer Booklet

MATHEMATICS SPECIALIST 3A/3B

SOLUTIONS

Section Two: Calculator-assumed

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination.

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	50	33
Section Two: Calculator-assumed	13	13	100	100	67
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2012*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

Section Two: Calculator-assumed **(100 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(4 marks)**

A function is defined as $f(x) = e^3 - \left(1 + \frac{3}{x}\right)^x$.

(a) Find $f(1)$. (1 mark)

$$\boxed{\begin{aligned} f(1) &= 16.0855 \\ &\approx 16.1 \text{ to 3 sf} \end{aligned}}$$

(b) Calculate $f(10)$, $f(100)$ and $f(1000)$. (2 marks)

$$\boxed{\begin{aligned} f(10) &= 6.2997 \\ f(100) &= 0.8669 \\ f(1000) &= 0.0900 \end{aligned}}$$

(c) State the exact value of $\left(1 + \frac{3}{x}\right)^x$ as $x \to \infty$. (1 mark)

$$\boxed{e^3}$$

Question 9 **(10 marks)**

When a capacitor discharges through a resistor, the voltage, V in volts, across the capacitor decays according to the rule $V = 20(3)^{-0.88t}$, where t is the time, in seconds, after the discharge began.

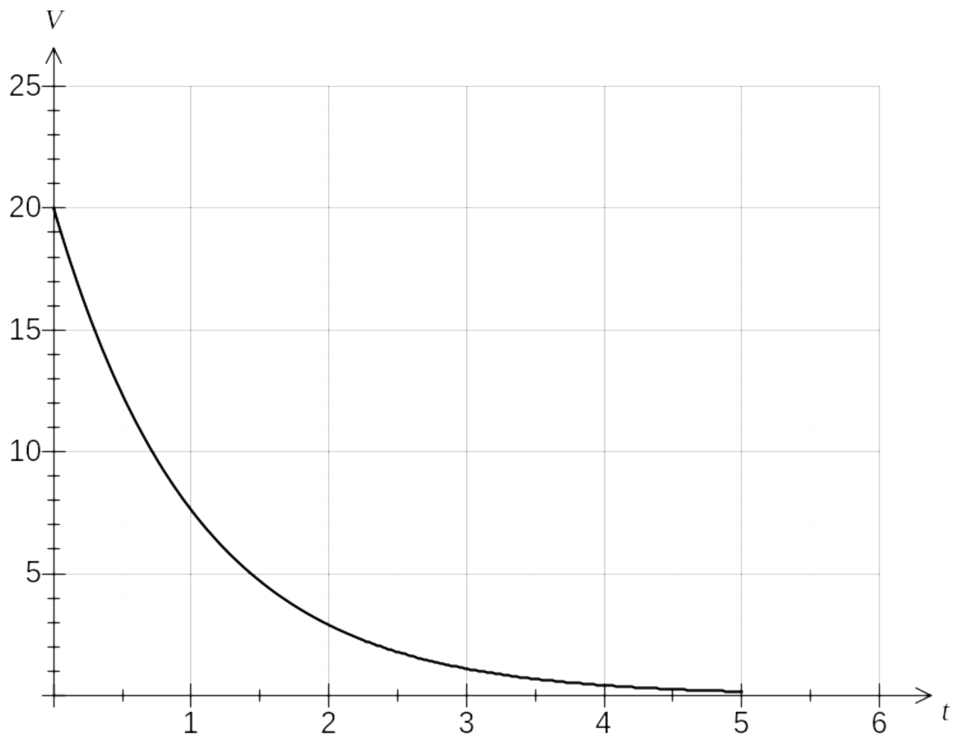
(a) What was the initial voltage across the capacitor? (1 mark)

$\boxed{V_0 = 20 \text{ volts}}$

(b) What was the voltage across the capacitor after four seconds? (1 mark)

$\boxed{V = 20(3)^{-0.88(4)} = 0.418 \text{ volts}}$

(c) Draw the graph of the voltage against time for $0 \le t \le 5$. (3 marks)



(c) How long, to the nearest millisecond, does it take for the voltage across the capacitor to halve? (3 marks)

$$10 = 20(3)^{-0.88t}$$
$$-0.88t\log 3 = \log 0.5$$
$$t = 0.716965$$
$$t = 717 \text{ milliseconds}$$

(d) How long does it take for the capacitor to become 99.9% discharged from its initial state? (2 marks)

$$0.001 = (3)^{-0.88t}$$
$$-0.88t\log 3 = \log 0.001$$
$$t = 7.145 \text{ seconds}$$

Question 10 **(10 marks)**

Three points are given by $A(1,\ 2)$, $B(4,-2)$ and $C(p,\ 4)$.

(a) Determine a unit vector parallel to the line through AB. (2 marks)

$$\overrightarrow{AB} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{3^2 + (-4)^2} = 5$$

Unit vector is $\frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$

(b) Write down a vector equation of the line through AB. (1 mark)

$$\mathbf{r} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

(c) The lines through AB and BC are perpendicular.

(i) Write down the vector BC. (1 mark)

$$\overrightarrow{BC} = \begin{bmatrix} p \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} p-4 \\ 6 \end{bmatrix}$$

(ii) Evaluate the dot product of AB and BC. (1 mark)

$$\begin{bmatrix} 3 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} p-4 \\ 6 \end{bmatrix} = 3p - 12 - 24$$

(iii) Show that $p = 12$. (1 mark)

$$\begin{aligned} 3p - 12 - 24 &= 0 \\ p &= 12 \end{aligned}$$

(d) Find the coordinates of M, the mid point of AC. (1 mark)

$$\left(\frac{1+12}{2}, \frac{2+4}{2}\right) = (6.5,\ 3)$$

(e) Determine the vector equation of the circle through points A, B and C. (3 marks)

Since ΔABC is right angled at B, then AC must be diameter of circle and centre M:

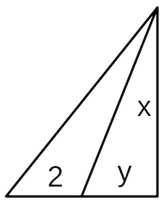
$$\vec{AC} = \begin{bmatrix} 12 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$$

$$\left|\vec{AC}\right| = \sqrt{11^2 + 2^2} = \sqrt{125} = 5\sqrt{5}$$

$$\left| \mathbf{r} - \begin{bmatrix} 6.5 \\ 3 \end{bmatrix} \right| = \frac{5\sqrt{5}}{2}$$

Question 11 **(7 marks)**

(a) A vertical pole stands on level ground. From point A on the ground, the angle of elevation of the top of the pole is 44°. Point B is 2 m nearer to the base of the pole than point A, and from point B, the angle of elevation of the top of the pole is 55°. Find the height of the pole, rounding your answer to the nearest centimetre. (3 marks)



$\tan 44 = \dfrac{x}{2+y}$

$\tan 55 = \dfrac{x}{y}$

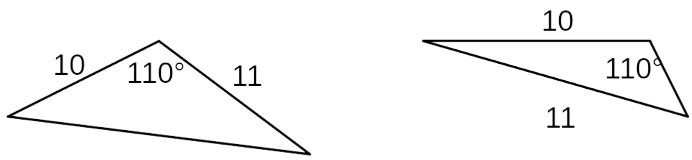
Solve simultaneously to get

$x = 5.9644$ and $y = 4.1763$

Hence, height of pole is 596 cm.

(b) A triangle has sides of length 10 cm and 11 cm and an angle of 110°.

(i) Sketch all possible triangles with these attributes. (1 mark)



(ii) Find the smallest possible area of a triangle with these attributes. (3 marks)

$\dfrac{\sin\theta}{10} = \dfrac{\sin 110}{11} \Rightarrow \theta = 58.68^\circ$

$180 - 58.68 - 110 = 11.32^\circ$

$\text{Area} = 0.5 \times 10 \times 11 \times \sin 11.32 = 10.797 \text{ cm}^2$

Question 12 **(7 marks)**

(a) Convert A(0, 5) from Cartesian to polar form. (1 mark)

$$\boxed{(5,\ \frac{\pi}{2})}$$

(b) Convert B(2, 2) from polar to Cartesian form, correct to 2 decimal places. (2 marks)

$$\boxed{(-0.8322,\ 1.8186) \Rightarrow (-0.83,\ 1.82) \text{ to 2 dp}}$$

(c) Convert the Cartesian equation $x + y = 0$, for $x \geq 0$, into polar form. (1 mark)

$$\boxed{\begin{array}{l} y = -x,\ x \geq 0 \\ \therefore\ \theta = -\dfrac{\pi}{4} \end{array}}$$

(d) Convert the polar equation $r = 3\cos\theta - 2\sin\theta$ into Cartesian form. (3 marks)

$$\boxed{\begin{array}{r} r^2 = 3r\cos\theta - 2r\sin\theta \\ x^2 + y^2 = 3x - 2y \end{array}}$$

Question 13 **(9 marks)**

The points A, B and C have position vectors $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{b} = 14\mathbf{i} - 3\mathbf{j}$ and $\mathbf{c} = -5\mathbf{i} + 2\mathbf{j}$.

(a) Determine the angle between vectors $\mathbf{b}$ and $\mathbf{c}$, giving your answer rounded to one decimal place. (2 marks)

$$\cos^{-1}\frac{14\times(-5)+(-3)\times 2}{\sqrt{14^2+(-5)^2}\sqrt{(-3)^2+2^2}} = 170.3^\circ$$

(Or using CAS)

(b) Find the position vector of point D which divides $\overline{AC}$ internally in the ratio $5:3$.
(3 marks)

$$\begin{aligned}\mathbf{a} + \frac{5}{5+3}(\mathbf{c}-\mathbf{a}) &= \begin{bmatrix}3\\4\end{bmatrix} + \frac{5}{8}\begin{bmatrix}-5-3\\2-4\end{bmatrix}\\ &= \begin{bmatrix}3\\4\end{bmatrix} + \begin{bmatrix}-5\\-1.25\end{bmatrix}\\ &= \begin{bmatrix}-2\\2.75\end{bmatrix}\\ \mathbf{d} &= -2\mathbf{i} + 2.75\mathbf{j}\end{aligned}$$

(c) Express the vector $\mathbf{b}$ in terms of $\mathbf{a}$ and $\mathbf{c}$. (4 marks)

$\mathbf{b} = x\mathbf{a} + y\mathbf{c}$

$$\begin{bmatrix}14\\-3\end{bmatrix} = x\begin{bmatrix}3\\4\end{bmatrix} + y\begin{bmatrix}-5\\2\end{bmatrix}$$

$3x - 5y = 14$

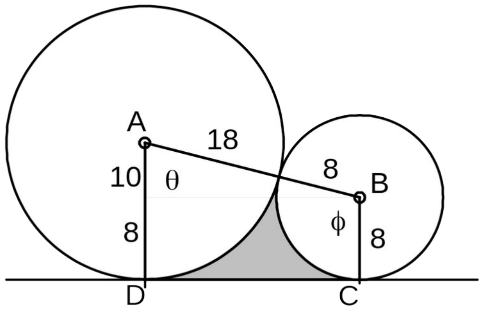
$4x + 2y = -3$

$x = 0.5,\ y = -2.5$

$\mathbf{b} = 0.5\mathbf{a} - 2.5\mathbf{c}$

Question 14 **(9 marks)**

Two circles, one of radius 8 cm and the other of radius 18 cm, with a common tangent, touch each other as shown in the diagram.



(a) Calculate the perimeter of the shaded region. (5 marks)

$CD = \sqrt{26^2 - 10^2} = 24$ cm

$\theta = \cos^{-1}\frac{10}{26} = 1.176^r$

$\phi = \pi - 1.176 = 1.966^r$

Long arc $= 18 \times 1.176 = 21.17$

Short arc $= 8 \times 1.966 = 15.72$

Perimeter $= 24 + 21.17 + 15.72 = 60.89$ cm

(b) Calculate the area of the shaded region. (4 marks)

Trapezium ABCD $= \frac{18+8}{2} \times 24 = 312$

Large sector $= \frac{1}{2} \times 18^2 \times 1.176 = 190.51$

Small sector $= \frac{1}{2} \times 8^2 \times 1.966 = 62.90$

Total area $= 312 - 190.51 - 62.90 = 58.59$ cm^2

Question 15 **(7 marks)**

A fishing boat leaves a harbour with position $2\mathbf{i}+17\mathbf{j}$ and motors with a constant velocity of $12\mathbf{i}-3\mathbf{j}$ km/h. On the shore, a radio station has position $11\mathbf{i}+2\mathbf{j}$ km, and the signal from the station can be detected up to $\sqrt{170}$ km away.

(a) Write down a position vector for the fishing boat t hours after leaving the harbour. (1 mark)

$$\mathbf{r}=2\mathbf{i}+17\mathbf{j}+t(12\mathbf{i}-3\mathbf{j})$$

(b) Write down the vector equation for the circle representing the boundary of the radio signal from the shore station. (1 mark)

$$|\mathbf{r}-(11\mathbf{i}+2\mathbf{j})|=\sqrt{170}$$

(c) Show that the fishing boat will be within range of the radio signal for a period of 40 minutes during its journey. (5 marks)

Path of fishing boat and circle intersect when

$$\left|\begin{bmatrix}2\\17\end{bmatrix}+t\begin{bmatrix}12\\-3\end{bmatrix}-\begin{bmatrix}11\\2\end{bmatrix}\right|=\sqrt{170}$$

$$(12t-9)^2+(-3t+15)^2=170$$

$$t=\tfrac{2}{3},\ t=\tfrac{4}{3}$$

Between these times, line must be inside circle.

So within range from $\frac{4}{3}-\frac{2}{3}=\frac{2}{3}$ hour, or 40 minutes.

Question 16 **(8 marks)**

Prove that

(a) $\sin^4 x - \cos^4 x + \cos(2x) = 0$. (4 marks)

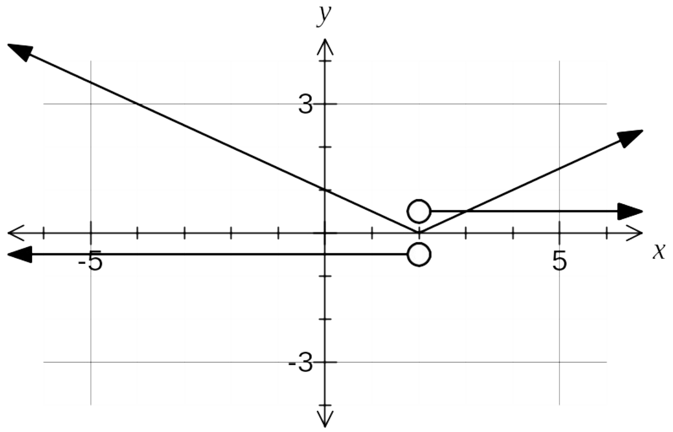
$$\begin{aligned}
LHS &= \sin^4 x - \cos^4 x + \cos(2x) \\
&= \left(\sin^2 x - \cos^2 x\right)\left(\sin^2 x + \cos^2 x\right) + \cos^2 x - \sin^2 x \\
&= \sin^2 x - \cos^2 x + \cos^2 x - \sin^2 x \\
&= 0 \\
&= RHS
\end{aligned}$$

(b) $\dfrac{1 - \cos(2x)}{\sin(2x)} = \tan x$. (4 marks)

$$\begin{aligned}
LHS &= \frac{1 - \cos(2x)}{\sin(2x)} \\
&= \frac{1 - \cos^2 x + \sin^2 x}{2\sin x \cos x} \\
&= \frac{2\sin^2 x}{2\sin x \cos x} \\
&= \frac{\sin x}{\cos x} \\
&= \tan x \\
&= RHS
\end{aligned}$$

Question 17 **(8 marks)**

The graph of $y = f(x)$ is shown, where $f(x) = |g(x)|$ and $g(x)$ is a linear function.



(a) Find a function $g(x)$. (2 marks)

$g(x) = 0.5x - 1$ or $g(x) = 1 - 0.5x$

(b) Is $f(x)$ continuous and differentiable over its natural domain? Explain your answer. (2 marks)

Continuous everywhere but not differentiable at $x = 2$, as below $x = 2$, $f'(x) = -0.5$ and above $x = 2$, $f'(x) = 0.5$.

(c) Add the graph of $y = f'(x)$ to the axes above. (2 marks)

(d) Write $f'(x)$ as a piecewise defined function. (2 marks)

$$f'(x) = \begin{cases} -0.5 & x < 2 \\ 0.5 & x > 2 \end{cases}$$

Question 18 **(7 marks)**

A helicopter, with a maximum speed through still air of 240 km/h, leaves its base at A to fly to a destination at B.

The position vector of B relative to A is $(155\mathbf{i} - 95\mathbf{j})$ km, and a steady wind of velocity $(-17\mathbf{i} - 22\mathbf{j})$ km/h is blowing over the area.

(a) Find the velocity vector the helicopter pilot should set in order to fly directly from A to B in the shortest time. (5 marks)

Let velocity vector be $a\mathbf{i} + b\mathbf{j}$. Then

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} -17 \\ -22 \end{bmatrix} = \lambda \begin{bmatrix} 155 \\ -95 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 155\lambda + 17 \\ -95\lambda + 22 \end{bmatrix}$$

But $a^2 + b^2 = 240^2$

$$(155\lambda + 17)^2 + (-95\lambda + 22)^2 = 240^2$$

$\lambda = 1.2949$ (ignore $\lambda = -1.3279$ as $\lambda > 0$)

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 217.7 \\ -101.0 \end{bmatrix}$$

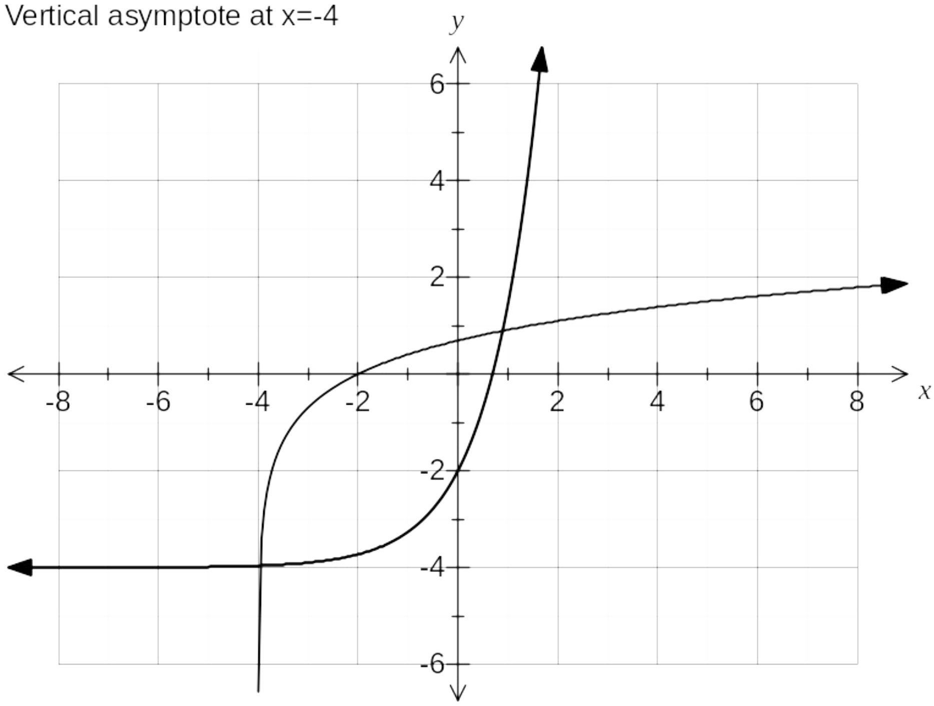
(b) What is the shortest journey time, to the nearest minute? (2 marks)

$$t = \frac{1}{1.2949} \times 60$$
$$= 46.34$$
$$= 46 \text{ minutes}$$

Question 19 **(9 marks)**

A function is given by $f(x) = \log_e\left(2 + \frac{x}{2}\right)$.

(a) Sketch the graph of $y = f(x)$ on the axes below, clearly showing any asymptotes. (3 marks)



(b) Add the graph of $y = f^{-1}(x)$, the inverse of $f(x)$, to the axes above. (2 marks)

(c) State

(i) the domain of $f(x)$. (1 mark)

$\boxed{x : x > -4}$

(ii) the range of $f^{-1}(x)$. (1 mark)

$\boxed{y : y > -4}$

(d) Describe how to obtain the graph of $y = f(x)$ from the graph of $y = \log_e x$. (2 marks)

Translate $y = \log_e x$ 2 units to the left and then dilate horizontally by a scale factor of 2.

Question 20 **(5 marks)**

(a) If $f(x) = \dfrac{5}{x}$, write down an expression for $f'(x)$. (1 mark)

$$\boxed{f'(x) = -\frac{5}{x^2}}$$

(b) Differentiate $f(x) = \dfrac{5}{x}$ from first principles, using the formula $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

(4 marks)

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\frac{5}{x+h} - \frac{5}{x}}{h} \\
&= \lim_{h \to 0} \frac{\frac{5x}{x(x+h)} - \frac{5(x+h)}{x(x+h)}}{h} \\
&= \lim_{h \to 0} \frac{5x - 5(x+h)}{x(x+h)h} \\
&= \lim_{h \to 0} \frac{-5h}{x(x+h)h} \\
&= \lim_{h \to 0} \frac{-5}{x(x+h)} \\
&= -\frac{5}{x^2}
\end{aligned}
$$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

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