MATHEMATICS
SPECIALIST
UNIT 1

Semester One

2017

SOLUTIONS

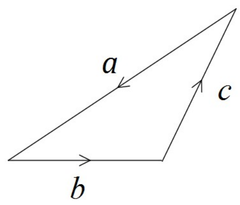
Calculator –free Solutions

1. (a) $\underset{\sim}{\mathbf{c}} = \underset{\sim}{\mathbf{a}} - \underset{\sim}{\mathbf{b}}$ ✔

$\underset{\sim}{\mathbf{c}} = \underset{\sim}{\mathbf{b}} - \underset{\sim}{\mathbf{a}}$ ✔

$\underset{\sim}{\mathbf{c}} = \underset{\sim}{\mathbf{a}} + \underset{\sim}{\mathbf{b}}$ ✔

(b) $\underset{\sim}{\mathbf{a}}$, $\underset{\sim}{\mathbf{b}}$ and $\underset{\sim}{\mathbf{c}}$ must form a closed loop, e.g.: ✔



(c) III and IV ✔✔ [6]

2. (a) $n(C) = 20$, $n(P) = 14$, $n(C \cup P) = 30 - 6 = 24$ ✔

$n(C \cup P) = n(C) + n(P) - n(C \cap P)$

$\therefore n(C \cap P) = n(C) + n(P) - n(C \cup P) = 20 + 14 - 24 = 10$ ✔✔

(b) $n(M \cup C \cup P) = 50 - 5 = 45$ ✔

$n(M) = 30$, $n(C) = 27$, $n(P) = 27$, $n(M \cap C) = 20$, $n(M \cap P) = 14$,

$n(C \cap P) = x$, and $n(M \cap C \cap P) = 10$ from (a) ✔

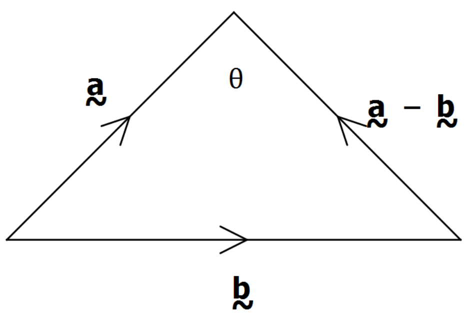
$n(M \cup C \cup P) = n(M) + n(C) + n(P) - n(M \cap C) - n(M \cap P) - n(C \cap P) + n(M \cup C \cup P)$

$\therefore n(C \cap P) = 30 + 27 + 27 - 20 - 14 + 10 - 45 = 15$ ✔

(c) 10 houses have married couples with children and pets,
therefore, 41 houses must be selected to obtain at least one
with both children and pets. ✔

The Pigeon Hole Principle. ✔ [8]

3. (a)



vector

location of

3. (b) $(\underset{\sim}{a} - \underset{\sim}{b}) \cdot (2\underset{\sim}{b} - \underset{\sim}{a}) = 2\underset{\sim}{a}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{a} - 2\underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{b}$ ✔

$= 3\underset{\sim}{a}\cdot\underset{\sim}{b} - |\underset{\sim}{a}|^2 - 2|\underset{\sim}{b}|^2$ ✔

$= 3(5) - (2)^2 - 2(3)^2 = -7$ ✔ [5]

4. (a) (i) 20 ✔

(ii) ${}^7C_3 = {}^7C_4 = 35$ ✔

(b) (i) $x = 4$; since 70 is in the 4th column of the 8th row ✔

(ii) $x = 6$; since 15 is in the 4th column of the 6th row ✔

(iii) $x = 7$; since in row 7, elements on columns 2 and 5 are equal ✔

(iv) $x = 4$; since in row 8, elements on columns 2 and 6 are equal ✔

(c) (i) ${}^8C_5 = 56$ ✔

(ii) ${}^3C_2 \times {}^5C_3 + {}^3C_3 \times {}^5C_2$ ✔✔

$= 3 \times 10 + 1 \times 10 = 40$ ✔

(d) (i) $5! = 120$ ✔

(ii) $2! \times 4! = 2 \times 24 = 48$ ✔

$\therefore 120 - 48 = 72$ ✔ [13]

5. (a) $(2\underset{\sim}{a} + \underset{\sim}{b}) + (\underset{\sim}{a} - \underset{\sim}{b}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ -5 \end{pmatrix}$

$3\underset{\sim}{a} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$, hence $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ ✔✔

$\therefore \underset{\sim}{b} = \underset{\sim}{a} - \begin{pmatrix} 5 \\ -5 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ ✔

Hence, $\underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ ✔

OR

$(2\underset{\sim}{a} + \underset{\sim}{b}) - \underset{\sim}{a} = \underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ ✔✔

(b) $\begin{pmatrix} 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0$, hence $\begin{pmatrix} x \\ y \end{pmatrix} = k\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $k \in R$ ✔

Unit vector $= \pm \frac{k}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ ✔ [6]

6. (a) ΔOAB isoceles, $\therefore \angle OAB = \frac{1}{2}(180° - 70°) = 55°$ ✔

$2 \times \angle ACD = \angle AOD = 55°$, $\therefore \angle ACD = 27.5°$ ✔

$\angle ABD = \angle ACD = 27.5°$ ✔

(b) $\angle QPR = \frac{1}{2} \times \angle QOR$ (angle at circumference is half angle at the centre) ✔

$= \frac{1}{2}(360° - 100° - 140°) = \frac{1}{2}(120°) = 60°$ ✔

(c) $\angle SPQ = \angle PRQ$ from the alternate-segment theorem ✔

$\therefore \angle SPQ = \frac{1}{2} \times \angle POQ = \frac{1}{2} \times 100° = 50°$ ✔ [7]

7. (a) $\overrightarrow{OF} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \underset{\sim}{\mathbf{a}} + \frac{1}{2}(\underset{\sim}{\mathbf{b}} - \underset{\sim}{\mathbf{a}}) = \frac{1}{2}(\underset{\sim}{\mathbf{b}} + \underset{\sim}{\mathbf{a}})$ ✔

$\overrightarrow{OG} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CB} = \underset{\sim}{\mathbf{c}} + \frac{1}{2}(\underset{\sim}{\mathbf{b}} - \underset{\sim}{\mathbf{c}}) = \frac{1}{2}(\underset{\sim}{\mathbf{b}} + \underset{\sim}{\mathbf{c}})$ ✔

(b) Show that opposite sides are congruent and parallel:

$\overrightarrow{EH} = \overrightarrow{OH} - \overrightarrow{OE} = \frac{1}{2}\underset{\sim}{\mathbf{c}} - \frac{1}{2}\underset{\sim}{\mathbf{a}} = \frac{1}{2}(\underset{\sim}{\mathbf{c}} - \underset{\sim}{\mathbf{a}})$ ✔

$\overrightarrow{FG} = \overrightarrow{OG} - \overrightarrow{OF} = \frac{1}{2}(\underset{\sim}{\mathbf{b}} + \underset{\sim}{\mathbf{c}}) - \frac{1}{2}(\underset{\sim}{\mathbf{b}} + \underset{\sim}{\mathbf{a}}) = \frac{1}{2}(\underset{\sim}{\mathbf{c}} - \underset{\sim}{\mathbf{a}})$

$\therefore \overrightarrow{EH} = \overrightarrow{FG}$ as required. ✔

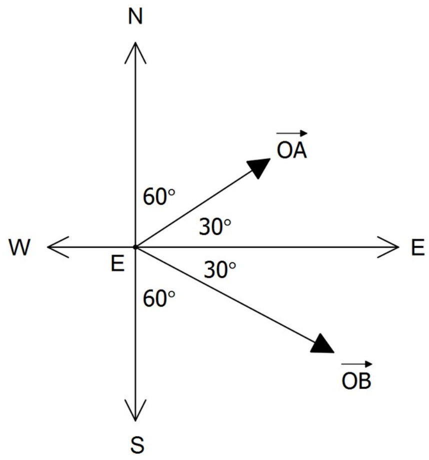
$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \frac{1}{2}(\underset{\sim}{\mathbf{b}} + \underset{\sim}{\mathbf{a}}) - \frac{1}{2}\underset{\sim}{\mathbf{a}} = \frac{1}{2}\underset{\sim}{\mathbf{b}}$

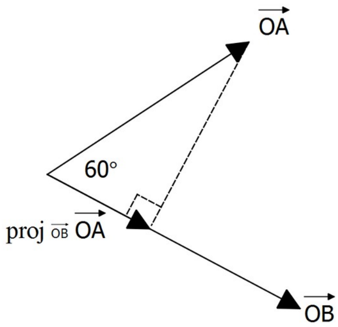
$\overrightarrow{HG} = \overrightarrow{OG} - \overrightarrow{OH} = \frac{1}{2}(\underset{\sim}{\mathbf{b}} + \underset{\sim}{\mathbf{c}}) - \frac{1}{2}\underset{\sim}{\mathbf{c}} = \frac{1}{2}\underset{\sim}{\mathbf{b}}$

$\therefore \overrightarrow{EF} = \overrightarrow{HG}$, and EFGH is a parallelogram ✔ [5]

Calculator – Assumed Solutions

8.





diagram(s)

$\overrightarrow{OA} = \begin{pmatrix} 6\cos(30°) \\ 6\sin(30°) \end{pmatrix} = \begin{pmatrix} 3\sqrt{3} \\ 3 \end{pmatrix}$ ✔

$\overrightarrow{OB} = \begin{pmatrix} 8\cos(30°) \\ -8\sin(30°) \end{pmatrix} = \begin{pmatrix} 4\sqrt{3} \\ -4 \end{pmatrix}$ ✔

$\text{proj}_{\overrightarrow{OB}}\overrightarrow{OA} = |\overrightarrow{OA}|\cos(60°) \times \dfrac{1}{|\overrightarrow{OB}|}\overrightarrow{OB}$ ✔

$= 6 \times \dfrac{1}{2} \times \dfrac{1}{8} \times \begin{pmatrix} 4\sqrt{3} \\ -4 \end{pmatrix} = \dfrac{3}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} = \dfrac{3}{2}\sqrt{3}\, i - \dfrac{3}{2} j$ ✔ [5]

9. (a) ${}^{6}\mathbf{C}_2 \times {}^{8}\mathbf{C}_2 = 420$ ✔✔

(b) (i) ${}^{5}\mathbf{C}_2 \times {}^{8}\mathbf{C}_2 = 280$ ✔✔

(ii) only choices are AB, DE, DF and EF ✔

$\therefore 4 \times {}^{8}\mathbf{C}_2 = 112$ ✔✔ [7]

10. (a) LHS $= \dfrac{(n+1)!}{(n+1-r-1)!}$ ✔

$= \dfrac{(n+1) \times n!}{(n-r)!}$ ✔

$= (n+1) \times \left[\dfrac{n!}{(n-r)!}\right] = (n+1) \times {}^{n}\mathbf{P}_r =$ RHS ✔

(b) (i) ${}^{7}\mathbf{P}_4 = 840$ ✔

(ii) $4 \times {}^{6}\mathbf{P}_3 = 480$ ✔✔

(iii) $^6\mathbf{P}_3 \times 3 = 360$ ✔✔ [8]

11. (a) $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB}$, $\therefore \overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{OC} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ ✔

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \end{pmatrix}$ ✔

(b) $\frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ ✔

$\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ -10 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ ✔✔

(4, 2) is the midpoint of both OB and AC ✔

Therefore, OB and AC bisect each other.

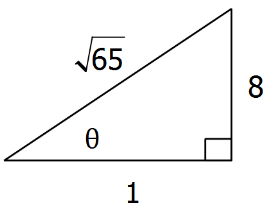
(c) $\theta = \angle AOC$ since OABC is a parallelogram

$\therefore \overrightarrow{OA} \cdot \overrightarrow{OC} = |\overrightarrow{OA}||\overrightarrow{OC}|\cos\theta$

$\begin{pmatrix} 4 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{vmatrix} 4 \\ 7 \end{vmatrix} \begin{vmatrix} 4 \\ -3 \end{vmatrix} \cos\theta$ ✔

$\therefore \cos\theta = -\frac{1}{\sqrt{65}}$ ✔

(d) From (c):



$\therefore \sin\theta = \frac{8}{\sqrt{65}}$ ✔

(e) Area OABC = 2 × Area ΔOAC

$= 2 \times \frac{1}{2} \times |\overrightarrow{OA}| \times |\overrightarrow{OC}| \times \sin\theta$

$= \begin{vmatrix} 4 \\ 7 \end{vmatrix} \times \begin{vmatrix} 4 \\ -3 \end{vmatrix} \times \frac{8}{\sqrt{65}}$ ✔

$= \sqrt{65} \times 5 \times \frac{8}{\sqrt{65}} = 40$ units^2 ✔ [11]

12. (a) (i) <u>For all</u> natural/counting numbers represented by n ✔

<u>there exists</u> another natural number m ✔

such that n is the square of m

OR such that m is a whole number root of n. ✔

12. (a) (ii) 5 is natural and there is no natural number that when squared gives 5.
(any acceptable answer) ✔

(b) A rhombus has two pairs of parallel sides, therefore B $\Rightarrow$ A is a valid statement. ✔

Not all parallelograms are rhombi, e.g. rectangles, therefore A $\Rightarrow$ B is not a valid statement. ✔

Hence, A is not equivalent to B, i.e. A $\Leftrightarrow$ B is invalid. ✔

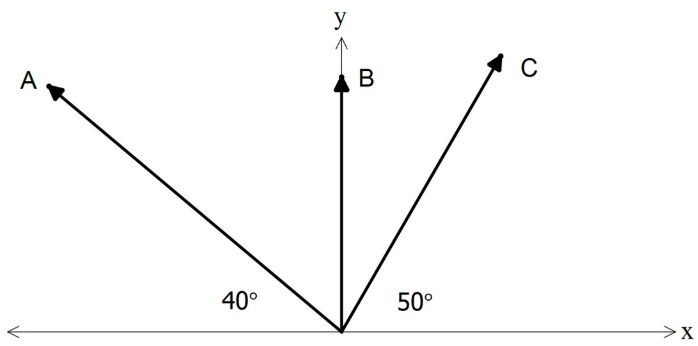
(c) (i) If a triangle inscribed in a circle is right angled, then the triangle has the diameter as one of its sides. ✔

(ii) Yes, because ALL right-angled triangles inscribed in a circle will have the diameter as the hypotenuse. ✔

(iii) If a triangle inscribed in a circle does not have the diameter as one of its sides, then the triangle is not right angled. ✔

(iv) Yes, because if a statement is true then so is the contrapositive of that statement ✔ [11]

13. (a)



Combined force vector: $\underset{\sim}{r} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$

$$= \begin{pmatrix} -1200\cos(40°) \\ 1200\sin(40°) \end{pmatrix} + \begin{pmatrix} 0 \\ 800 \end{pmatrix} + \begin{pmatrix} 1000\cos(50°) \\ 1000\sin(50°) \end{pmatrix}$$ ✔✔✔

$$\therefore \underset{\sim}{r} = \begin{pmatrix} -276.47 \\ 2337.39 \end{pmatrix}$$ ✔

and $|\underset{\sim}{r}| = 2353.68\ N$ ✔

bearing = $270° + \tan^{-1}\left(\dfrac{2337.39}{276.47}\right) = 353.25°T$ ✔

13. (b) $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = 3000\boldsymbol{j}$ ✔

$\therefore \underset{\sim}{b} = \begin{pmatrix} 0 \\ 3000 \end{pmatrix} - \begin{pmatrix} -1200\cos(40°) \\ 1200\sin(40°) \end{pmatrix} - \begin{pmatrix} 1000\cos(50°) \\ 1000\sin(50°) \end{pmatrix}$

$= \begin{pmatrix} 276.47 \\ 1462.61 \end{pmatrix}$ ✔

maximum force = $|\underset{\sim}{b}| = 1488.51\ N$ ✔

bearing = $\tan^{-1}\left(\dfrac{276.47}{1462.61}\right) = 10.70°$

$\therefore$ 79.30°T ✔ [10]

14. (a) Assume that n is even and n^2 is odd. ✔

Then $\exists\ k \in N$ such that $n = 2k$ ✔

Thus, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$

which implies that n^2 must be even. ✔

Since n^2 cannot be both even and odd, this is a contradiction ✔

and therefore n must be even.

(b) $\angle DOB = 2\alpha$ and reflex $\angle DOB = 2\beta$ ✔

$\therefore 2\alpha + 2\beta = 360°$ ✔

and hence $\alpha + \beta = 180°$ as required ✔

$\alpha = \beta$ if O,B,D are collinear OR DB=diameter ✔ [8]

15. (a) $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} k \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} k-1 \\ -5 \end{pmatrix}$ ✔

$\overrightarrow{DB} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ ✔

(b) In a rhombus the diagonals are perpendicular.

$\therefore \overrightarrow{AC} \cdot \overrightarrow{BD} = 0$ ✔

$\begin{pmatrix} k-1 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \end{pmatrix} = 0$ hence $k = 2$ ✔

(c) $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $\overrightarrow{AD} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ ✔

$2\,|\overrightarrow{AB}|^2 \; + \; 2\,|\overrightarrow{AD}|^2 = 2(\sqrt{13})^2 + 2(\sqrt{13})^2 = 52$ ✔

$|\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2 = (\sqrt{26})^2 + (\sqrt{26})^2 = 52$ ✔

$\therefore 2|\overrightarrow{AB}|^2 + 2|\overrightarrow{AD}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2$ as required. [7]

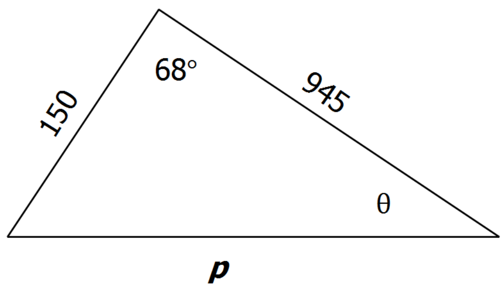
16. (a)

wind vector

plane vector

flight direction and resultant vector using parallelogram method

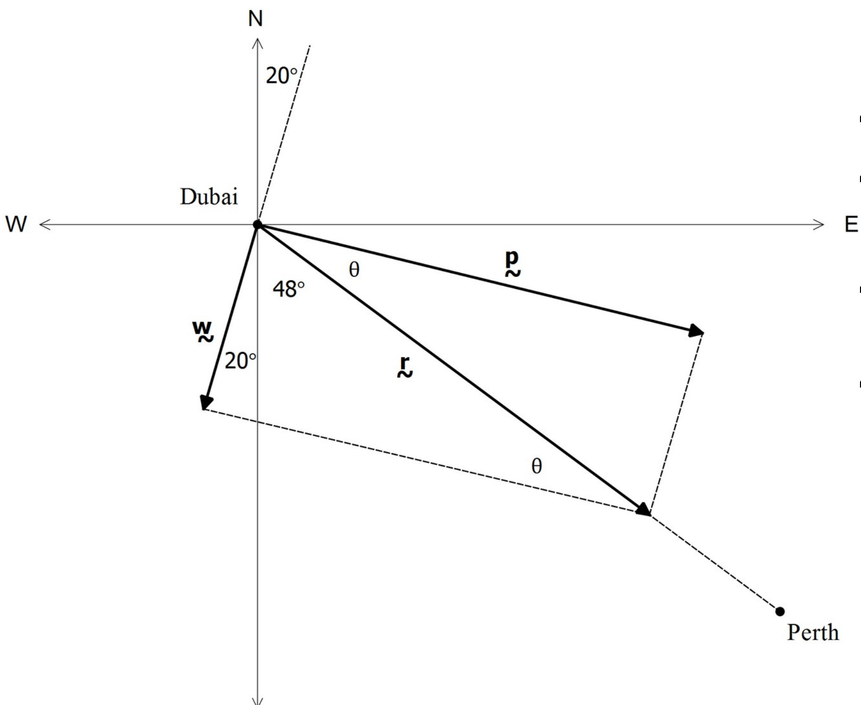
(b)



68° angle between wind and plane speeds

(c) $|\boldsymbol{p}|^2 = 150^2 + 945^2 - 2(150)(945)\cos 68°$ ✔

$\therefore |\boldsymbol{p}| = 899.62$ km/h ✔



✓
✓
✓
✓

[11]

17. $\overrightarrow{AB} = \begin{pmatrix}1\\y\end{pmatrix} - \begin{pmatrix}-2\\-2\end{pmatrix} = \begin{pmatrix}3\\y+2\end{pmatrix}$ ✔

$\overrightarrow{BC} = \begin{pmatrix}x\\1\end{pmatrix} - \begin{pmatrix}1\\y\end{pmatrix} = \begin{pmatrix}x-1\\1-y\end{pmatrix}$ ✔

since A,B,C collinear then $\overrightarrow{AB} = \alpha\,\overrightarrow{BC}$ ✔

Given AB:BC = 1:2 then $2\,\overrightarrow{AB} = \overrightarrow{BC}$ ✔

$\therefore 2\begin{pmatrix}3\\y+2\end{pmatrix} = \begin{pmatrix}x-1\\1-y\end{pmatrix}$

$\therefore x = 7$ and $y = -1$ ✔✔ [6]

18. (a) $\dfrac{(x+2)!}{(x-2)!(x+2-x+2)!} = 210$ ✔

$\dfrac{(x+2)(x+1)(x)(x-1)(x-2)!}{(x-2)!} = 210 \times 4!$ ✔

$(x+2)(x+1)(x)(x-1) = 10 \times 9 \times 8 \times 7$

$\therefore x = 8$ ✔

(b) LHS $= \dfrac{n!}{r!(n-r)!} \times \dfrac{(n-r)!}{2!(n-r-2)!}$ ✔

$= \dfrac{n!}{2!r!(n-r-2)!}$ ✔

$= \dfrac{n!}{2!} \times \dfrac{1}{r![(n-2)-r]!} \times \dfrac{(n-2)!}{(n-2)!}$ ✔

$= \dfrac{n!}{2!(n-2)!} \times \dfrac{(n-2)!}{r![(n-2)-r)!}$ ✔

$= \begin{pmatrix}n\\2\end{pmatrix} \times \begin{pmatrix}n-2\\r\end{pmatrix}$ = RHS [7]

19. (a) Let $PB = x$ and $AP = 2x$

$FE^2 = AF \times BF$

$\therefore (\sqrt{10})^2 = (2x + x + 1) \times 1$ ✔

$\therefore x = 3$ cm ✔

and hence $|AB| = 2(3) + 3 = 9$ cm ✔

(b) $AP \times PB = CP \times PD$

$\therefore 6 \times 3 = CP \times 5$ ✔

$\therefore y = \frac{18}{5} = 3.6$ cm ✔

and hence $|CD| = 5 + 3.6 = 8.6$ cm ✔

(c) ΔPBD is right angled since ΔABD is right angled (triangle in semicircle)

$\therefore PB^2 + BD^2 = PD^2$

$3^2 + BD^2 = 5^2$

$\therefore |BD| = 4$ cm ✔

(d) ΔABD is right angled in semicircle

$\therefore AB^2 + BD^2 = AD^2$ ✔

$9^2 + 4^2 = AD^2$

$\therefore |AD| = \sqrt{97}$ cm

hence, radius is $\frac{\sqrt{97}}{2}$ cm ✔ [9]