Christ Church Grammar School

Semester One Examination, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNIT 3

Section One: Calculator-free

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	48	35
Section Two: Calculator-assumed	12	12	100	92	65
			Total	140	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Additional working space

Question number: __________

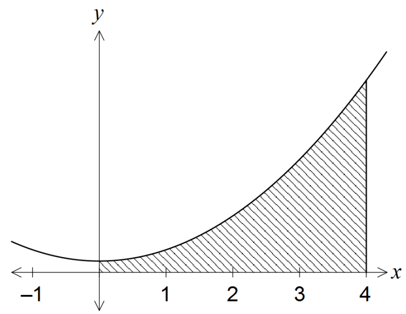
Section One: Calculator-free **35% (48 Marks)**

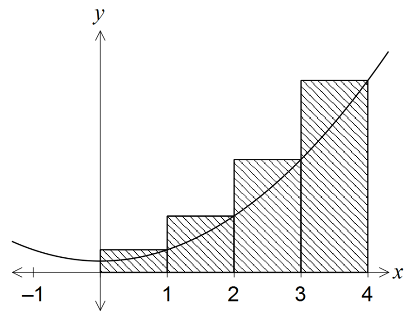
This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(5 marks)**

Part of the graph of $y = x^2 + 1$ is shown in the diagrams below.





An approximation for the area beneath the curve between $x = 0$ and $x = 4$ is made using rectangles as shown in the right-hand diagram. Determine the exact amount by which the approximate area exceeds the exact area.

Solution
$A_1 = \int_0^4 x^2 + 1dx = \left[\frac{x^3}{3} + x\right]_0^4 = \frac{64}{3} + 4 = \frac{76}{3}$ $A_2 = 2 + 5 + 10 + 17 = 34$ $A_2 - A_1 = 34 - \frac{76}{3} = \frac{102 - 76}{3} = \frac{26}{3}$ sq units
Specific behaviours
✓ antidifferentiates correctly ✓ evaluates exact area ✓ calculates rectangle heights ✓ evaluates approximate area ✓ calculates difference

See next page

Question 2 **(9 marks)**

(a) Differentiate the following with respect to *x*, simplifying your answers.

(i) $y=\int_x^1 (t-t^3)\,dt$. (2 marks)

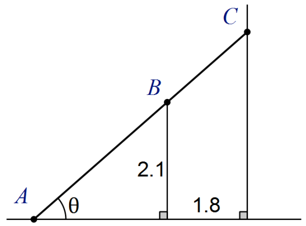
Solution
$\dfrac{d}{dx}\int_x^1 (t-t^3)\,dt=-\dfrac{d}{dx}\int_1^x (t-t^3)\,dt$ $=x^3-x$
Specific behaviours
✓ adjusts limits of integral   ✓ simplifies derivative

(ii) $y=\sin^3(2x+1)$. (3 marks)

Solution
$y=u^3 \qquad u=\sin(2x+1)$ $\dfrac{dy}{du}=3u^2 \qquad \dfrac{du}{dx}=2\cos(2x+1)$ $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$ $=3\sin^2(2x+1)\times 2\cos(2x+1)$ $=6\sin^2(2x+1)\cos(2x+1)$
Specific behaviours
✓ uses chain rule for $\sin(2x+1)$ ✓ uses chain rule for $\sin^3()$ ✓ simplifies result

Question 20 **(7 marks)**

A vertical wall, 2.1 metres tall, stands on level ground and 1.8 metres away from the wall of a house. A ladder, of negligible width, leans at an angle of θ to the ground and just touches the ground, wall and house, as shown in the diagram.



(a) Show that the length of the ladder, L, is given by $L=\dfrac{2.1}{\sin\theta}+\dfrac{1.8}{\cos\theta}$. (3 marks)

Solution
$\sin\theta=\dfrac{2.1}{AB}\Rightarrow AB=\dfrac{2.1}{\sin\theta},\quad \cos\theta=\dfrac{1.8}{BC}\Rightarrow BC=\dfrac{1.8}{\cos\theta}$ $L=AB+BC$ $=\dfrac{2.1}{\sin\theta}+\dfrac{1.8}{\cos\theta}$
Specific behaviours
✓ shows on diagram and writes length of AB   ✓ shows on diagram and writes length of BC   ✓ sums lengths to obtain total, using labels added to diagram

(b) Use a calculus method to determine the length of the shortest ladder that can touch the ground, wall and house at the same time. (4 marks)

Solution
$L=2.1(\sin\theta)^{-1}+1.8(\cos\theta)^{-1}$ $\dfrac{dL}{d\theta}=-2.1\cos\theta(\sin\theta)^{-2}-1.8(-\sin\theta)(\cos\theta)^{-2}$ $=\dfrac{1.8\sin^3\theta-2.1\cos^3\theta}{\sin^2\theta\cos^2\theta}$ $\dfrac{dL}{d\theta}=0\Rightarrow 1.8\sin^3\theta-2.1\cos^3\theta=0$ $\tan^3\theta=\dfrac{2.1}{1.8}\Rightarrow\theta=\tan^{-1}\sqrt[3]{\dfrac{2.1}{1.8}}\approx 0.8111$ $L(0.8111)\approx 5.51$ metres
Specific behaviours
✓ shows first derivative of L (may use CAS, but must show key results)   ✓ solves derivative equal to 0   ✓ obtains acute angle solution   ✓ substitutes into equation to obtain minimum length

Question 19 **(7 marks)**

Consider the function $f(t) = \dfrac{t-4}{2}$ and the function $A(x) = \int_0^x f(t)\,dt$.

(a) Complete the table below. (2 marks)

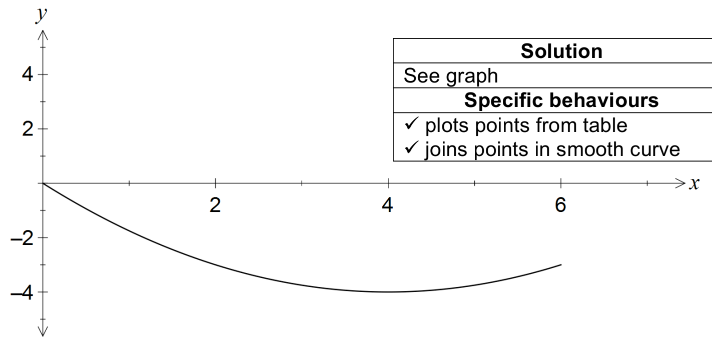
x	0	1	2	3	4	5	6
$A(x)$	0	-1.75	**-3**	**-3.75**	**-4**	**-3.75**	**-3**

Solution
See table
Specific behaviours
✓ calculates at least 3 correct values ✓ calculates all values correctly

(b) For what value(s) of x is the function $A(x)$ increasing? (1 mark)

Solution
$x > 4$
Specific behaviours
✓ correct inequality

(c) On the axes below, sketch the graph of $y = A(x)$ for $0 \le x \le 6$. (2 marks)



Solution
See graph
Specific behaviours
✓ plots points from table ✓ joins points in smooth curve

(d) Determine

(i) when $A'(x) = 0$. (1 mark)

Solution
$x = 4$
Specific behaviours
✓ states value

(ii) the function $A(x)$ in terms of x. (1 mark)

Solution
$A(x) = \int_0^x \frac{t}{2} - 2\,dt = \frac{x^2}{4} - 2x$
Specific behaviours
✓ states function in terms of x

Question 2 (continued)

(b) Determine the values of the constants a, b and c, given that $f''(x) = e^{3x}\left(ax^2 + bx + c\right)$ when $f(x) = x^2e^{3x}$. (4 marks)

Solution
$f'(x) = 2xe^{3x} + 3x^2e^{3x}$ $f''(x) = 2e^{3x} + 6xe^{3x} + 3f'(x)$ $= 2e^{3x} + 6xe^{3x} + 3\left(2xe^{3x} + 3x^2e^{3x}\right)$ $= 2e^{3x} + 12xe^{3x} + 9x^2e^{3x}$ $= e^{3x}\left(2 + 12x + 9x^2\right) \Rightarrow a = 9,\ b = 12,\ c = 2$
Specific behaviours
✓ uses product rule for first derivative ✓ uses chain rule for first derivative ✓ uses product and chain rules for second derivative ✓ simplifies and states values

Question 3 **(7 marks)**

Consider the function defined by $f(x) = \frac{x}{2} - \sqrt{x}$, $x \geq 0$.

(a) Determine the coordinates of the stationary point of $f(x)$. (3 marks)

Solution
$f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{x}}$ $\frac{1}{2} - \frac{1}{2\sqrt{x}} = 0 \Rightarrow x = 1$ $f(1) = \frac{1}{2} - \sqrt{1} = -\frac{1}{2}$ ⇒ stationary point at $\left(1,\ -\frac{1}{2}\right)$
Specific behaviours
✓ differentiates function   ✓ solves $f'(x) = 0$ ✓ states coordinates of point

(b) Use the second derivative test to determine the nature of the stationary point found in (a). (3 marks)

Solution
$f''(x) = \frac{1}{4\sqrt{x^3}}$ $f''(1) = \frac{1}{4}$ $f''(1) > 0$ ⇒ local minimum
Specific behaviours
✓ determines second derivative   ✓ shows $f''(1) > 0$ ✓ states conclusion that point is local minimum

(c) State the global minimum of $f(x)$. (1 mark)

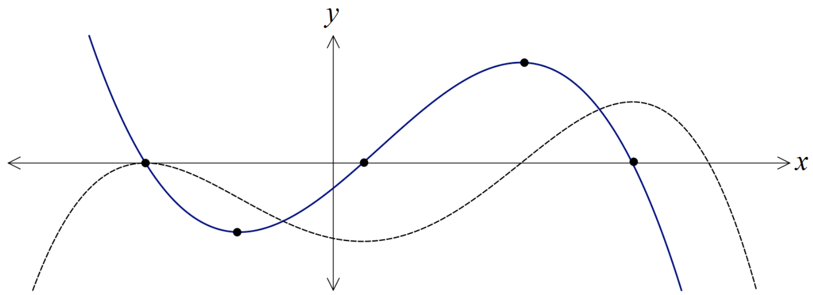
Solution
$-\frac{1}{2}$
Specific behaviours
✓ states correct value of global minimum

Question 18 (continued)

(c) Does the graph of $y = f(x)$ have a horizontal point of inflection? Justify your answer. (2 marks)

Solution
Yes, as x increases through $x = a$, the gradient of f changes from negative to zero to negative, indicating a horizontal pt of inflection.
Specific behaviours
✓ responds yes, indicating when $x = a$ ✓ explains reason

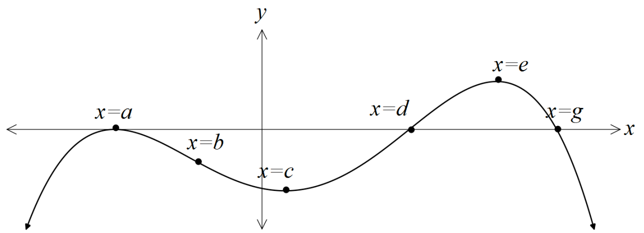
(d) On the axis below, sketch a possible graph of $y = f''(x)$. The graph of $y = f'(x)$ is shown with a broken line for your reference. (3 marks)



Solution
See graph
Specific behaviours
✓ correctly aligns three roots with turning pts   ✓ correctly aligns min and max with pts of inflection   ✓ smooth curve through five key points

Question 18 **(8 marks)**

The graph of $y = f'(x)$, the derivative of a polynomial function f, is shown below. The graph of $y = f'(x)$ has stationary points when $x = a$, $x = c$ and $x = e$, points of inflection when $x = b$ and $x = d$, and roots when $x = a$, $x = d$ and $x = g$, where $a < b < c < d < e < g$.



(a) For what value(s) of x does the graph of $y = f(x)$ have a point of inflection? (1 mark)

Solution
$x = a,\ c,\ e$
Specific behaviours
✓ states all values

(b) Does the graph of $y = f(x)$ have a local maximum? Justify your answer. (2 marks)

Solution
Yes, as x increases through $x = g$, the gradient of f changes from positive to zero to negative, indicating a local maximum.
Specific behaviours
✓ responds yes, indicating when $x = g$ ✓ explains reason

Question 4 **(5 marks)**

The area of a segment with central angle θ in a circle of radius r is given by $A = \frac{r^2}{2}(\theta - \sin\theta)$.

Use the increments formula to find the approximate increase in area of a segment in a circle of radius 10 cm as the central angle increases from $\frac{\pi}{3}$ to $\frac{11\pi}{30}$.

Solution
$A = 50(\theta - \sin\theta)$ $\frac{dA}{d\theta} = 50(1 - \cos\theta)$ $\delta A \approx \frac{dA}{d\theta}\delta\theta$ $\delta A \approx 50\left(1 - \cos\frac{\pi}{3}\right)\times\frac{\pi}{30}$ $= 50\left(1 - \frac{1}{2}\right)\times\frac{\pi}{30}$ $= \frac{5\pi}{6}\ cm^2$
Specific behaviours
✓ put $r = 10$ and differentiates A wrt θ ✓ shows use of increments formula ✓ determines $\delta\theta = \frac{\pi}{30}$ ✓ determines $\cos\frac{\pi}{3}$ ✓ determines increase in area

Question 5 **(5 marks)**

(a) Differentiate $y = \dfrac{2x+1}{e^x}$, simplifying your answer. (3 marks)

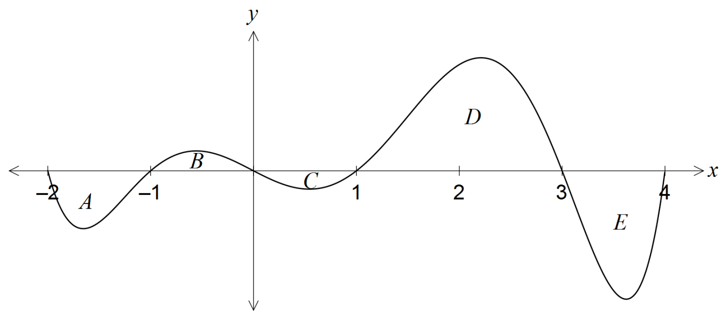
Solution
$\dfrac{dy}{dx} = \dfrac{(2)(e^x)-(2x+1)(e^x)}{(e^x)(e^x)}$ $= \dfrac{(e^x)(1-2x)}{(e^x)(e^x)}$ $= \dfrac{1-2x}{e^x}$
Specific behaviours
✓ uses quotient rule   ✓ factors out exponential term   ✓ simplifies

(b) Using the result in (a) or otherwise, evaluate $\int_1^2 \left(\dfrac{1-2x}{e^x}\right) dx$. (2 marks)

Solution
$\int_1^2 \left(\dfrac{1-2x}{e^x}\right) dx = \left[\dfrac{2x+1}{e^x}\right]_1^2$ $= \dfrac{5}{e^2} - \dfrac{3}{e}$ $= \dfrac{5-3e}{e^2}$
Specific behaviours
✓ recognises antiderivative will be function from (a)   ✓ substitutes limits

Question 17 **(8 marks)**

The graph of the function $y = f(x)$ is shown below for $-2 \le x \le 4$.



The area of regions enclosed by the *x*-axis and the curve, *A*, *B*, *C*, *D* and *E*, are 12, 7, 5, 32 and 21 square units respectively.

(a) Determine the value of $\int_{-2}^{4} f(x)\,dx$. (2 marks)

Solution
$-12+7-5+32-21=1$
Specific behaviours
✓ assigns sign to all areas   ✓ adds signed values

(b) Determine the area of the region enclosed between the graph of $y = f(x)$ and the *x*-axis from $x = 0$ to $x = 4$. (2 marks)

Solution
$5+32+21=58$ sq units
Specific behaviours
✓ chooses regions C, D and E   ✓ adds unsigned values

(c) Determine the values of

(i) $\int_0^3 f(x)+3\,dx$. (2 marks)

Solution
$\int_0^3 f(x)\,dx + \int_0^3 3\,dx = (-5+32)+(9) = 36$
Specific behaviours
✓ splits integral into two parts   ✓ evaluates and sums each part

(ii) $2\int_0^1 f'(x)dx$. (2 marks)

Solution
$2\int_0^1 f'(x)dx = 2[f(x)]_0^1$ $= 0$
Specific behaviours
✓ use FTC   ✓ evaluates integral

Question 16 **(8 marks)**

The discrete random variable Y has the probability distribution shown in the table below.

y	-2	-1	0	1	2
$P(Y=y)$	0.4	0.2	0.1	0.1	0.2

(a) Determine $P(Y \geq 0 \mid Y \leq 1)$. (2 marks)

Solution
$P(Y \geq 0 \mid Y \leq 1) = \frac{0.2}{0.8} = \frac{1}{4}$
Specific behaviours
✓ correct denominator ✓ correct numerator and simplification

(b) Calculate

(i) $E(Y)$. (2 marks)

Solution
$E(Y) = (-2)(0.4) + (-1)(0.2) + (0)(0.1) + (1)(0.1) + (2)(0.2)$ $= -0.8 - 0.2 + 0.1 + 0.4$ $= -0.5$
Specific behaviours
✓ forms products ✓ sums products to obtain expected value

(ii) $E(1-2Y)$. (1 mark)

Solution
$E(1-2Y) = 1 - 2(-0.5) = 2$
Specific behaviours
✓ applies both linear changes to obtain expected value

(c) Calculate

(i) $Var(Y)$. (2 marks)

Solution
$Var(Y) = (0.4)(-1.5)^2 + (0.2)(-0.5)^2 + (0.1)(0.5)^2 + (0.1)(1.5)^2 + (0.2)(2.5)^2$ $= \frac{49}{20} = 2.45$
Specific behaviours
✓ uses correct formula ✓ states variance

(ii) $Var(1-2Y)$. (1 mark)

Solution
$Var(1-2Y) = (-2)^2 \times Var(Y) = \frac{49}{5} = 9.8$
Specific behaviours
✓ applies square of multiplier to original variance

Question 6 **(6 marks)**

The discrete random variable X has the probability distribution shown in the table below.

x	0	1	2	3
$P(X=x)$	$\frac{2a^2}{3}$	$\frac{1-3a}{3}$	$\frac{1+2a}{3}$	$\frac{4a^2}{3}$

Determine the value of the constant *a*.

Solution
$\frac{2a^2+1-3a+1+2a+4a^2}{3} = 1$ $6a^2 - a - 1 = 0$ $(3a+1)(2a-1) = 0$ $a = -\frac{1}{3}, \frac{1}{2}$ Check $a = -\frac{1}{3}$ $\left[\frac{2}{27} \quad \frac{2}{3} \quad \frac{1}{9} \quad \frac{4}{27}\right]$ Check $a = \frac{1}{2}$ $\left[\frac{1}{6} \quad \cancel{-\frac{1}{6}} \quad \frac{2}{3} \quad \frac{1}{3}\right]$ Reject $a = \frac{1}{2}$ $a = -\frac{1}{3}$
Specific behaviours
✓ sums distribution to 1 ✓ simplifies equation ✓ solves equation for two values of *a* ✓ checks first value for valid probabilities ✓ checks second value for valid probabilities ✓ states only valid value of *a*

Question 7 **(5 marks)**

The area bounded by the curve $y = e^{2-x}$ and the lines $y = 0$, $x = 1$ and $x = k$ is exactly $e - 1$ square units. Determine the value of the constant k, given that $k > 1$.

Solution
$\int_1^k e^{2-x}dx = \left[-e^{2-x}\right]_1^k$ $= \left(-e^{2-k}\right) - \left(-e^1\right)$ $= e - e^{2-k}$ $e - e^{2-k} = e - 1$ $e^{2-k} = 1$ $k = 2$
Specific behaviours
✓ writes integral to determine area ✓ antidifferentiates exponential function ✓ substitutes 1 and k and simplifies ✓ equates to area ✓ solves for k

(d) After five seconds, the particle has moved a distance of k metres.

(i) Explain why $k \neq \int_0^5 v(t)\,dt$. (1 mark)

Solution
Integral will calculate change in displacement, but since particle turned around after one second, this will not be the same as distance travelled.
Specific behaviours
✓ explains change in displacement not distance travelled in this instance

(ii) Calculate k. (2 marks)

Solution
$k = \int_0^5 \left\lvert \frac{t^2 + 2t - 3}{(t+1)^2} \right\rvert dt$ $k = \frac{11}{3} \approx 3.67$
Specific behaviours
✓ use absolute function ✓ determines k

Question 15 **(9 marks)**

A particle moves in a straight line according to the function $x(t) = \dfrac{t^2+3}{t+1}$, $t \geq 0$, where t is in seconds and x is the displacement of the particle from a fixed point O, in metres.

(a) Determine the velocity function, $v(t)$, for the particle. (2 marks)

Solution
$v(t) = \dfrac{d}{dt}x(t)$ $= \dfrac{t^2+2t-3}{(t+1)^2}$
Specific behaviours
✓ relates velocity to first derivative of displacement wrt t ✓ determines the first derivative

(b) Determine the displacement of the particle at the instant it is stationary. (2 marks)

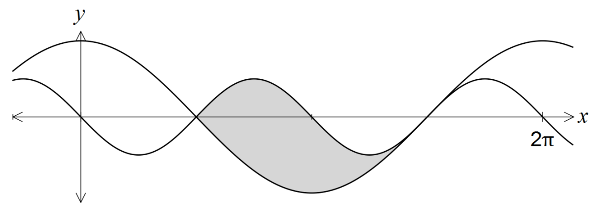
Solution
$v(t)=0 \Rightarrow t^2+2t-3=0 \Rightarrow t=\cancel{-3},\ 1$ $x(1) = 2$ m
Specific behaviours
✓ solves v=0 over domain   ✓ determines displacement

(c) Show that the acceleration of the particle is always positive. (2 marks)

Solution
$a(t) = \dfrac{dv}{dt}$ $a(t) = \dfrac{8}{(t+1)^3}$ > 0 for all t
Specific behaviours
✓ determines acceleration function   ✓ shows that acceleration always positive for $t \geq 0$

Question 8 **(6 marks)**

The shaded region on the graph below is enclosed by the curves $y = -\sin(2x)$ and $y = 2\cos x$.



(a) Given that $\sin(2x) = 2\sin x\cos x$, show that the first two roots of the equation $2\cos x = -\sin(2x)$ are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. (3 marks)

Solution
$2\cos x = -\sin(2x)$ $2\cos x = -2\sin x\cos x$ $2\cos x\,(1+\sin x) = 0$ $\cos x = 0 \ \ or \ \ \sin x = -1$ $x = \dfrac{\pi}{2} \ or \ \dfrac{3\pi}{2} \ \ or \ x = \dfrac{3\pi}{2}$ $x = \dfrac{\pi}{2} \ or \ \dfrac{3\pi}{2}$
Specific behaviours
✓ factorise terms   ✓ solve one trig. ratio   ✓ solve other trig. ratio

(b) Hence find the area of the enclosed region in the diagram above. (3 marks)

Solution
$A = \displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\sin(2x) - 2\cos x \; dx$ $A = \left[\dfrac{1}{2}\cos 2x - 2\sin x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$ $= \left(-\dfrac{1}{2}+2\right) - \left(-\dfrac{1}{2}-2\right)$ $= 4$
Specific behaviours
✓ writes integral   ✓ shows antiderivative   ✓ evaluates integral

Additional working space

Question number: __________

(c) The probability that a student misses his bus to school is 0.2, and the probability that he misses the bus on any day is independent of whether he missed it on the previous day.

Over five consecutive weekdays, what is the probability that the student

(i) only misses the bus on Tuesday? (2 marks)

Solution
$0.2 \times 0.8^4 = 0.08192$
Specific behaviours
✓ uses 0.8 for catching bus ✓ determines probability

(ii) misses the bus at least twice? (2 marks)

Solution
$X \sim B(5, 0.2)$ $P(X \geq 2) = 0.26272$
Specific behaviours
✓ identifies binomial situation ✓ evaluates cumulative probability

(iii) misses the bus on Tuesday and on two other days? (3 marks)

Solution
$P = 0.2 \times P(Y = 2)$ where $Y \sim B(4, 0.2)$ $P(Y = 2) = 0.1536$ $P = 0.2 \times 0.1536$ $= 0.03072$
Specific behaviours
✓ identifies binomial situation for other two days ✓ evaluates probability of missing bus on two other days ✓ determines probability

Question 14 **(14 marks)**

(a) Determine the mean of a Bernoulli distribution with variance of 0.24. (3 marks)

Solution
$p(1-p)=0.24$ $p=0.4,\ 0.6 \Rightarrow$ mean is either 0.4 or 0.6
Specific behaviours
✓ writes variance equation ✓ solves equation ✓ states both values of p are possible means

(b) A Bernoulli trial, with probability of success *p*, is repeated *n* times. The resulting distribution of the number of successes has an expected value of 5.76 and a standard deviation of 1.92. Determine *n* and *p*. (4 marks)

Solution
$X \sim B(n,p)$ $np=5.76$ $np(1-p)=1.92^2$ $1-p=1.92^2 \div 5.76=0.64$ $p=0.36$ $n=16$
Specific behaviours
✓ identifies distribution of successes as binomial ✓ states equation for mean ✓ states equation for variance (or standard deviation) ✓ solves equations for *n* and *p*

Christ Church Grammar School

Semester One Examination, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ______________________

Your name ______________________

Time allowed for this section
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Special items: drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

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Structure of this paper

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   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Question 13 **(5 marks)**

Records of a company that has a large workforce indicate that 35 percent of employees take sick leave during any given year.

(a) If the records of five employees are selected at random from the previous year, what is the probability that fewer than three took sick leave? (2 marks)

Solution
Let X = number of employees out of 5 taking sick leave. $X \sim Bin(5, 0.35)$ $P(X \leq 2) = 0.7648$
Specific behaviours
✓ Define X and state the distribution ✓ Evaluate probability

Amongst the 20 management staff of the company, seven of them had taken sick leave during the previous year.

(b) If five management staff are selected at random, what is the probability that two or less took sick leave during the previous year? (3 marks)

Solution
Let Y = number of management staff out of 5 taking sick leave. $P(Y \leq 2) = \frac{\binom{7}{0}\binom{13}{5}}{\binom{20}{5}} + \frac{\binom{7}{1}\binom{13}{4}}{\binom{20}{5}} + \frac{\binom{7}{2}\binom{13}{3}}{\binom{20}{5}}$ = 0.7932
Specific behaviours
✓ Separate into 7 and 13 management staff ✓ Sum of three probabilities ✓ Evaluate total probability

Question 12 **(8 marks)**

The height of grain in a silo, initially 0.4 m, is increasing at a rate given by $h'(t) = 0.55t - 0.05t^2$ for $0 \le t \le 11$, where h is the height of grain in metres and t is in hours.

(a) At what time is the height of grain rising the fastest? (2 marks)

Solution
$h''(t) = 0.55 - 0.1t$ $0.55 - 0.1t = 0 \Rightarrow t = 5.5$ hours
Specific behaviours
✓ differentiates rate of change ✓ solves derivative equal to zero to obtain time.

(b) Determine the height of grain in the silo after 11 hours. (3 marks)

Solution
$h(11) = h(0) + \int_0^{11} 0.55t - 0.05t^2 dt$ $= 0.4 + 11.09$ $= 11.49\ m$
Specific behaviours
✓ shows integral of rate of change ✓ adds initial height ✓ evaluates integral to obtain final height

(c) Calculate the time taken for the grain to reach a height of 4.45 m. (3 marks)

Solution
$h(k) = 0.4 + \int_0^k 0.55t - 0.05t^2 dt = 4.45$ That is, $\int_0^k 0.55t - 0.05t^2 dt = 4.45 - 0.4$ $k = 4.5\ hours$
Specific behaviours
✓ write integral ✓ add height to integral and equate to 4.45 m ✓ solves for k

Section Two: Calculator-assumed **65% (92 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 **(4 marks)**

A recent news report said that it took 34 months for the population of Australia to increase from 23 to 24 million people.

(a) Assuming that the rate of growth of the population can be modelled by the equation $\frac{dP}{dt} = kP$, where P is the population of Australia at time t months, determine the value of the constant k. (2 marks)

Solution
$P = P_0 e^{kt}$ $24 = 23e^{34k}$ $k = 0.001252$
Specific behaviours
✓ uses growth and decay equation with correct data ✓ solves for k

(b) Assuming the current rate of growth continues, how long will it take for the population to increase from 24 million to 25 million people? (2 marks)

Solution
$25 = 24e^{0.001252t}$ $t = 32.61\ or\ 32.7\ or\ 33\ months\ (32.6\ is\ incorrect)$
Specific behaviours
✓ sets up correct values in equation ✓ solves for t

Question 10 **(7 marks)**

A small object is moving in a straight line with acceleration $a = 6t + k$ ms^{-2}, where t is the time in seconds and k is a constant. When $t = 1$ the object was stationary and had a displacement of 4 metres relative to a fixed point O on the line. When $t = 2$ the object had a velocity of 1 ms^{-1}.

(a) Determine the value of k and hence an equation for the velocity of the object at time t. (4 marks)

Solution
$v = 3t^2 + kt + c$ $t = 1,\ 3 + k + c = 0$ $t = 2,\ 12 + 2k + c = 1$ $k = -8$ $c = 5$ $v = 3t^2 - 8t + 5$
Specific behaviours
✓ antidifferentiates acceleration, adding constant ✓ derives simultaneous equations from information ✓ solves equations for one unknown ✓ writes velocity equation

(b) Determine the displacement of the object when $t = 2$. (3 marks)

Solution
$s = t^3 - 4t^2 + 5t + c$ $t = 1,\ 4 = 1 - 4 + 5 + c$ $c = 2$ $s = t^3 - 4t^2 + 5t + 2$ $s(2) = 8 - 16 + 10 + 2$ $= 4$ m
Specific behaviours
✓ antidifferentiates velocity ✓ determines constant ✓ evaluates displacement

Question 11 **(7 marks)**

It is known that 15% of Year 12 students in a large country study advanced mathematics.

A random sample of n students is selected from all Year 12's in this country, and the random variable X is the number of those in the sample who study advanced mathematics.

(a) Describe the distribution of X. (2 marks)

Solution
$X \sim B(n,\ 0.15)$ - binomial distribution with n trials and $p = 0.15$.
Specific behaviours
✓ states binomial distribution ✓ states parameters of binomial distribution

(b) If $n = 22$, determine the probability that

(i) three of the students in the sample study advanced mathematics. (1 mark)

Solution
$P(X = 3) = 0.2370$
Specific behaviours
✓ evaluates probability

(ii) more than three of the students in the sample study advanced mathematics. (1 mark)

Solution
$P(X \geq 4) = 0.4248$
Specific behaviours
✓ evaluates probability

(iii) none of the students in the sample study advanced mathematics. (1 mark)

Solution
$P(X = 0) = 0.0280$
Specific behaviours
✓ evaluates probability

(c) If ten random samples of 22 students are selected, determine the probability that at least one of these samples has no students who study advanced mathematics. (2 marks)

Solution
$Y \sim B(10,\ 0.028)$ $P(Y \geq 1) = 0.247$
Specific behaviours
✓ states binomial distribution with parameters ✓ evaluates probability