MATHEMATICS SPECIALIST

MAWA Year 12 Examination 2017

Calculator-assumed

Marking Key

The release date for this exam and marking scheme is

- **the end of week 1 of term 4, 2017**

Question 8 (a)

Solution
$\omega = \frac{1}{2}\left(\sqrt{3} - i\right) \Rightarrow \lvert\omega\rvert = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$ Also $\arg(\omega) = -\tan^{-1}\frac{1}{\sqrt{3}} = -\frac{\pi}{6}$ $\lvert\bar{\omega}\rvert = \lvert\omega\rvert = 1$ and $\arg(\bar{\omega}) = -\arg(\omega) = +\frac{\pi}{6}$ In addition,
Specific behaviours
✓ ✓ calculates the modulus and argument of ω correctly ✓ relates the modulus and argument of $\bar{\omega}$ to those of ω

Question 8 (b)

Solution
Now $\omega^{2017} = \left[\exp\left(-\frac{i\pi}{6}\right)\right]^{2017} = \exp\left(-\frac{2017i\pi}{6}\right) = \exp\left(-336i\pi - \frac{i\pi}{6}\right)$ As $\exp(-336i\pi) = 1$ this means that $\omega^{2017} = \exp\left(-\frac{i\pi}{6}\right) = \omega = \frac{1}{2}(\sqrt{3} - i)$
Specific behaviours
✓ uses De Moivre's theorem to express answer as an exponential ✓ simplifies answer to form of an exponential of small argument ✓ obtains correct answer in Cartesian form

Question 9 (a)

Solution
Let X gm denote the average weight of a randomly chosen coffee bean
Then $Pr(X > 0.14) = Pr\left(Z > \frac{0.14 - \mu}{\sigma}\right) = 0.10$ and so $\frac{0.14 - \mu}{\sigma} = 1.282......(A)$
and $\Pr(X < 0.11) = \Pr\left(Z < \frac{0.11 - \mu}{\sigma}\right) = 0.05$ so that $\frac{0.11 - \mu}{\sigma} = -1.645........(B)$
Solving equations (A) and (B) gives $\mu \approx 0.127$ and $\sigma \approx 0.01025$
Specific behaviours
✓ obtains equation (A) ✓ obtains equation (B) ✓ solves for μ and σ correctly

Question 9 (b)

Solution
If the 10 beans in the random sample weigh more than 1.2 gm, then $\bar{X} < 0.12$ where $\bar{X}$ is the average weight of beans in the sample.
Now $\Pr(\bar{X} < 0.12) = \Pr\left(Z < \frac{0.12 - \mu}{\sigma/\sqrt{n}}\right) = \Pr\left(Z < \frac{0.12 - 0.127}{0.01025/\sqrt{10}}\right)$
i.e. $\Pr(\bar{X} < 0.12) = \Pr(Z < -2.16) \approx 0.015$
Specific behaviours
✓ ✓ uses mean and correct standard deviation for $\bar{X}$ ✓ obtains correct answer

Question 9 (c)

Solution
Now $\Pr\left(\lvert\bar{X} - \mu\rvert < 0.001\right) = \Pr\left(\lvert Z\rvert < \frac{0.001}{\sigma/\sqrt{n}}\right)$
But $\frac{0.001}{(0.01025/\sqrt{100})} = 0.976$ and $\Pr(\lvert Z\rvert < 0.976) \approx 0.670$
Hence the required probability is 0.67.
Specific behaviours
✓ obtains correct bounds for the probability in terms of Z ✓ evaluates limits correctly ✓ derives correct answer

Question 10 (a)

Solution
If $\dfrac{3x^2+2x+4}{(x^2+2)(x+2)}=\dfrac{Ax+B}{x^2+2}+\dfrac{C}{x+2}=\dfrac{(Ax+B)(x+2)+C(x^2+2)}{(x^2+2)(x+2)}$ $=\dfrac{(A+C)x^2+(2A+B)x+(2B+2C)}{(x^2+2)(x+2)}$ Then we have $A+C=3,\ 2A+B=2,\ 2B+2C=4$ Hence we deduce that $A=1,\ B=0,\ C=2$
Specific behaviours
✓ Combines the two given fractions correctly ✓ Expands the parts of the numerator ✓ Writes the three correct equations for the constants ✓ Solves for the constants

Question 10 (b)

Solution
$\int_0^4 \dfrac{3x^2+2x+4}{(x^2+2)(x+2)}\,dx=\int_0^4 \dfrac{x}{x^2+2}\,dx+2\int_0^4 \dfrac{dx}{x+2}$ $=\dfrac{1}{2}\left[\ln(x^2+2)\right]_0^4+2\left[\ln(x+2)\right]_0^4$ $=\dfrac{1}{2}\left[\ln 18-\ln 2\right]+2\left[\ln 6-\ln 2\right]$ $=\dfrac{1}{2}\ln 9+2\ln 3=\ln 3+\ln 9=\ln 27\Rightarrow N=27$
Specific behaviours
✓✓ integrates the two partial fractions correctly ✓ simplifies the solution to the required form

Question 11(a)

Solution
Substituting $t = 0$ into the two vector equations $\Rightarrow \mathbf{r}_A = 18\mathbf{i} + 20\mathbf{k}$ and $\mathbf{r}_B = 5\mathbf{i} + 2\mathbf{j} + 50\mathbf{k}$
Specific behaviours
✓ correct answer for $\mathbf{r}_A$ and $\mathbf{r}_B$

Question 11(b)

Solution
Differentiating with respect to t gives $\Rightarrow \mathbf{v}_A = -4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ $\mathbf{v}_B = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
Specific behaviours
✓ differentiates each position vector correctly

Question 11(c)

Solution
Since the velocity vector of each plane is independent of t, the velocity of each plane is constant, indicating motion in a straight line. The speed of plane A is $\sqrt{16+4+9} = \sqrt{29} \approx 5.4\text{km/min}$ while the speed of plane B is 3 km/min. The flight path of plane A is trending upwards while the flight path of plane B is trending downwards.
Specific behaviours
✓ remarks that the velocity of each plane is constant/linear flight path ✓ calculates the speeds of the two planes ✓ states the vertical direction of each plane.

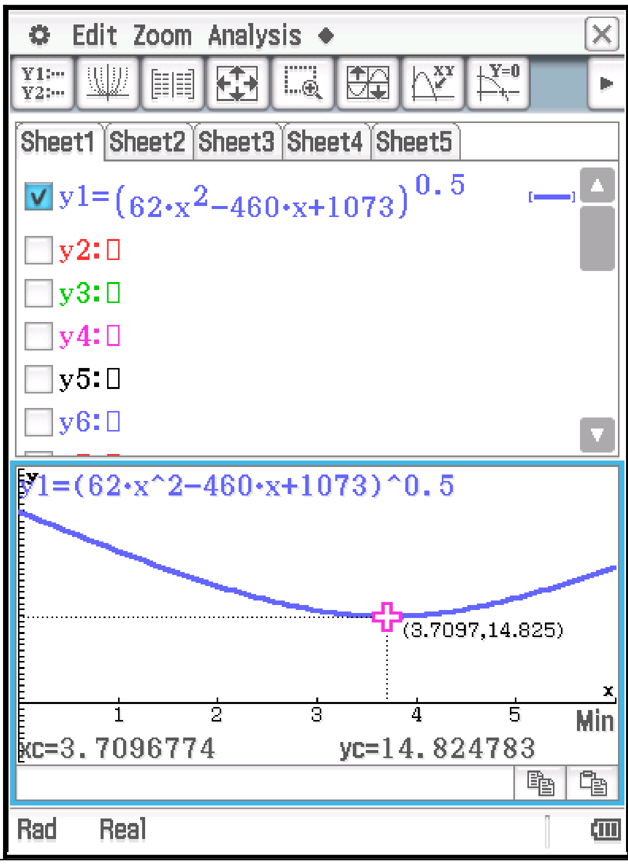
Question 11(d)

Solution

Using a CAS calculator, determine an expression for the distance $|\mathbf{r}_A - \mathbf{r}_B|$
Graph the expression using the graphing app

Use the 'Analysis' function on the CAS
To determine the minimum distance between the two planes.

Min distance = 14.825 km
Occurs when t=3.71 approx



Specific behaviours

✓ ✓ draws a graph with the appropriate shape
✓ calculates the correct min distance and gives the corresponding time

Question 12 (a)

Solution
The function $P(1-P)$ is a quadratic with a maximum at $P = 0.5$ This corresponds to half the population.
Specific behaviours
✓ recognizes the quadratic nature of the derivative function

Question 12 (b)

Solution
Separating the variables in the differential equation gives $\int \frac{dP}{P(1-P)} = \int 0.1\,dt$ and so $\int \frac{dP}{P} + \int \frac{dP}{1-P} = \int 0.1\,dt$ Therefore $\ln(P) - \ln(1-P) = 0.1t + c$ for some constant c.
Specific behaviours
✓ separates the variables correctly ✓ integrates each side correctly

Question 12 (c)

Solution
Substituting into the expression in (b) gives $\ln 0.01 - \ln 0.99 = c$ and so $c = -\ln 99$ Hence $\ln P - \ln(1-P) + \ln 99 = 0.1t \Rightarrow \ln\left(\frac{99P}{1-P}\right) = 0.1t$
Specific behaviours
✓ obtains correct value for c ✓ derives the correct equation

Question 12 (d)

Solution
Since $\ln\left(\frac{99P}{(1-P)}\right) = 0.1t$ then $\frac{99P}{(1-P)} = e^{0.1t}$ and $99P = (1-P)e^{0.1t}$ Then $P(99 + e^{0.1t}) = e^{0.1t} \Rightarrow P = \frac{e^{0.1t}}{99 + e^{0.1t}} = \frac{1}{1 + 99e^{-0.1t}}$
Specific behaviours
✓ takes exponentials correctly ✓ derives correct formula for P

Question 12 (e)

Solution
When $P = 0.95$ then $1 + 99e^{-0.1t} = 1/0.95 \Rightarrow e^{-0.1t} \approx 0.0005316 \Rightarrow t \approx 75.4$ So it takes about 75.4 days for 95% of the population to be infected.
Specific behaviours
✓ derives an equation for the required time ✓ obtains correct answer

Question 13 (a)(i) **(11 marks)**

Solution
If $y=\frac{x-2}{x+2} \Rightarrow xy+2y=x-2 \Rightarrow x(y-1)=-2(y+1) \Rightarrow x=\frac{2(y+1)}{1-y}$.   Thus $f^{-1}(x)=\frac{2(x+1)}{1-x}$
Specific behaviours
✓ Rearranges to obtain x in terms of y ✓ Interchanges x and y in formula   ✓ Deduces the form of the inverse function $f^{-1}(x)$

Question 13 (a)(ii)

Solution
Domain is $x \neq 1$; Range is $y \neq -2$
Specific behaviours
✓ States domain and range correctly

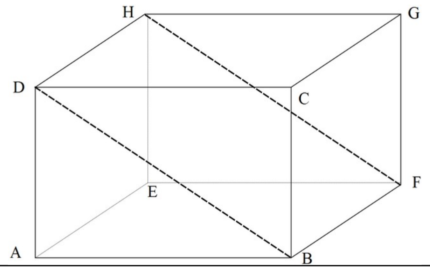
Question 13 (b)

Solution
Now $(g \circ h)(x)=g(h(x))=g(\sqrt{x-3})=(\sqrt{x-3})^2+3=x-3+3=x$ The composite is only defined where $g(x)$ is defined so that $x \geq 3$.
Specific behaviours
✓ Determines the correct composite function   ✓ Identifies the correct domain of definition.

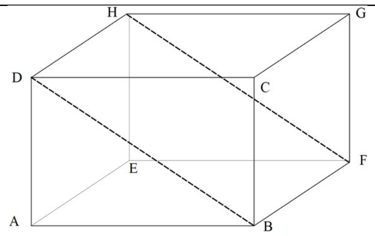
Question 13 (c)

Solution
Recall that $\lvert Z\rvert = Z$ if $Z \geq 0$ and that $\lvert Z\rvert = -Z$ for $Z < 0$. Then Case 1: $x \leq -2/3$ $\lvert 3x+2\rvert - \lvert x-3\rvert = (-2-3x)-(3-x) = -5-2x = 1 \Rightarrow x = -3$ Case 2: $-2/3 < x \leq 3$ $\lvert 3x+2\rvert - \lvert x-3\rvert = 3x+2-(3-x) = 4x-1 = 1 \Rightarrow x = \frac{1}{2}$ Case 3: $x > 3$ $\lvert 3x+2\rvert - \lvert x-3\rvert = 3x+2-(x-3) = 2x+5 = 1 \Rightarrow x = -2$. But this is a contradiction as it has been assumed that $x > 3$ Hence the solutions of the equation are $x = -3, \frac{1}{2}$.
Specific behaviours
✓ Draws correct conclusion from case 1 ✓ Solves equation in case 2 ✓ Solves equation in case 3 ✓ Recognises that the apparent solution in case 3 contradicts the assumed range ✓ Synthesises results from the three cases to deduce the correct solution of the problem

Question 14(a)

Solution
$\overrightarrow{HF} = \mathbf{b} - \mathbf{c}$ $\Rightarrow \left\|\overrightarrow{HF}\right\|^2 = \left\|\mathbf{b}-\mathbf{c}\right\|^2 = (\mathbf{b}-\mathbf{c})\bullet(\mathbf{b}-\mathbf{c})$ $= \mathbf{b}\bullet\mathbf{b} - \mathbf{b}\bullet\mathbf{c} - \mathbf{b}\bullet\mathbf{c} + \mathbf{c}\bullet\mathbf{c}$ $= \mathbf{b}\bullet\mathbf{b} + \mathbf{c}\bullet\mathbf{c} \quad \because \mathbf{b}\bullet\mathbf{c} = \mathbf{0}$ because $\mathbf{b}\perp\mathbf{c}$ $= \left\|\mathbf{b}\right\|^2 + \left\|\mathbf{c}\right\|^2$ $= \left\|\overrightarrow{EH}\right\|^2 + \left\|\overrightarrow{EF}\right\|^2$ 
Specific behaviours
✓ uses the dot product and expands $(\mathbf{b}-\mathbf{c})\bullet(\mathbf{b}-\mathbf{c})$ correctly ✓ uses $\mathbf{b}\bullet\mathbf{c} = \mathbf{0}$ and states why it is zero ✓ deduces the correct result

Question 14(b)

Solution
 $\overrightarrow{DB} = \overrightarrow{HF} = \mathbf{b} - \mathbf{c}$ $\Rightarrow \left\|\overrightarrow{DB}\right\|^2 = \left\|\overrightarrow{HF}\right\|^2 = \left\|\mathbf{b}\right\|^2 + \left\|\mathbf{c}\right\|^2$ (from part (a)) ....(1) $\left\|\overrightarrow{DF}\right\|^2 = \left\|\overrightarrow{DB}\right\|^2 + \left\|\overrightarrow{BF}\right\|^2 = \left\|\mathbf{b}\right\|^2 + \left\|\mathbf{c}\right\|^2 + \left\|\mathbf{a}\right\|^2$ using Pythagoras, $\because$ DHFB is a rectangle. $\left\|\overrightarrow{HB}\right\|^2 = \left\|\overrightarrow{HF}\right\|^2 + \left\|\overrightarrow{BF}\right\|^2 = \left\|\mathbf{b}\right\|^2 + \left\|\mathbf{c}\right\|^2 + \left\|\mathbf{a}\right\|^2$ $\therefore \left\|\overrightarrow{DF}\right\|^2 + \left\|\overrightarrow{HB}\right\|^2 = 2(\left\|\mathbf{b}\right\|^2 + \left\|\mathbf{c}\right\|^2) + 2\left\|\mathbf{a}\right\|^2$ $= 2\left\|\overrightarrow{DB}\right\|^2 + 2\left\|\mathbf{a}\right\|^2$ .... from (1) and $\mathbf{a} = \overrightarrow{EA} = \overrightarrow{FB}$ $= 2\left\|\overrightarrow{DB}\right\|^2 + 2\left\|\overrightarrow{BF}\right\|^2$ ....$\because \left\|\overrightarrow{BF}\right\|^2 = \left\|\overrightarrow{EA}\right\|^2 = \left\|\mathbf{a}\right\|^2$
Specific behaviours
✓ determines $\left\|DB\right\|^2$ in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ using part (a) ✓ determines $\left\|DF\right\|^2$ and $\left\|HB\right\|^2$ in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ ✓ sums to find the result = $2(\left\|\mathbf{b}\right\|^2 + \left\|\mathbf{c}\right\|^2) + 2\left\|\mathbf{a}\right\|^2$ ✓show this equals $2\left\|DB\right\|^2 + 2\left\|BF\right\|^2$

Question 14(c)

Solution
The sum of the squares of the diagonals of a rectangle are equal to twice the sum of the squares of any two adjacent sides of the rectangle.
Specific behaviours
✓ states a correct interpretation of the result

Question 15 (a)

Solution
We want $E = z\frac{\sigma}{\sqrt{n}} < 1000$ i.e. $\sqrt{n} > 1.96 \times \frac{\sigma}{1000} = 1.96 \times 3.5 = 6.86$ Therefore $n > (6.86)^2 \approx 47.06$ and so we need to test at least 48 tyres.
Specific behaviours
✓ obtains correct inequality ✓ solves for n correctly

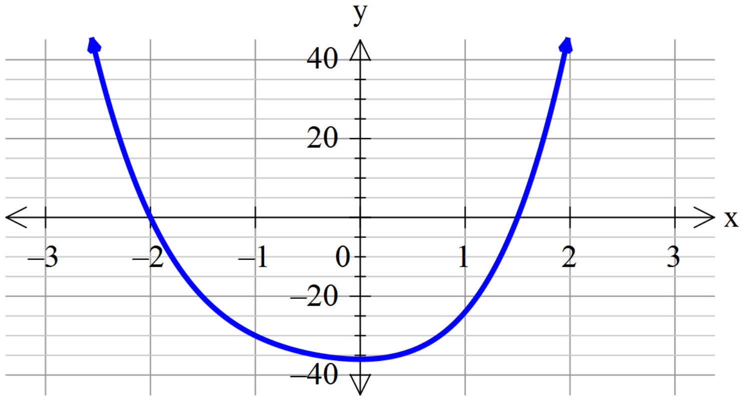
Question 15 (b)

Solution
The confidence interval is $\bar{X} - E \le \mu \le \bar{X} + E$ where $E = z\frac{\sigma}{\sqrt{n}} = 1.96 \times \frac{3500}{\sqrt{50}} \approx 970.15$ Hence the confidence interval is $39103 - 970 < \mu < 39103 + 970$ i.e. $38133 < \mu < 40073$
Specific behaviours
✓ uses correct form for the confidence interval ✓ obtains correct value for E the margin of error ✓ obtains correct limits for the confidence interval

Question 15 (c)

Solution
The sample does provide some evidence to dispute the manfacturer's claim because the sample mean is less than 40000. However, the confidence interval $38133 < \mu < 40073$ contains some numbers greater than 40000, and so the evidence for rejecting the claim is not compelling.
Specific behaviours
✓ notes that there is some evidence for disputing the claim ✓ recognises that the evidence for rejecting is not overwhelming.

Question 16 (a)

Solution
 The graph suggests zeroes at $x=-2$ and $x=1.5$. To check this we calculate $p(-2)=2(-2)^4+3(-2)^3+7(-2)^2-36=32-24+28-36=0$ and $p(1.5)=2(1.5)^4+3(1.5)^3+7(1.5)^2-36$ $=10.125+10.125+15.75-36$ $=0$
Specific behaviours
✓ estimates zeroes at -2 and 1.5 from a graph ✓ checks these are correct by direct substitution

Question 16 (b)

Solution
Since roots of $p(z)=0$ are -2 and 1.5 this suggests that $(z+2)(2z-3)=2z^2+z-6$ is a factor of $p(z)$. By long division we observe that $\dfrac{2z^4+3z^3+7z^2-36}{2z^2+z-6}=z^2+z+6$ If $z^2+z+6=0 \Rightarrow z=\dfrac{-1+\sqrt{-23}}{2}=\dfrac{1}{2}\pm i\dfrac{\sqrt{23}}{2}$ Hence the solutions of $p(z)=0$ are $z=-2, 1.5$ and $-\dfrac{1}{2}\pm i\dfrac{\sqrt{23}}{2}$.
Specific behaviours
✓ ✓ identifies the two linear factors that follow from (a) ✓ ✓ conducts the long division correctly ✓ writes down the solution of the quadratic and hence finds correct complex zeros

Question 17 (a)

Solution
If $y(t) = Ce^{-t}\sin 2t$ then $$\frac{dy}{dt} = Ce^{-t}(2\cos 2t - \sin 2t)$$ and $$\frac{d^2y}{dt^2} = Ce^{-t}(-2\cos 2t + \sin 2t - 4\sin 2t - 2\cos 2t)$$ $$= Ce^{-t}(-3\sin 2t - 4\cos 2t)$$ Hence $$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = Ce^{-t}(-3\sin 2t - 4\cos 2t + 4\cos 2t - 2\sin 2t + 5\sin 2t)$$ $$= 0$$ as required
Specific behaviours
✓ obtains correct expression for $\frac{dy}{dt}$ ✓ obtains correct expression for $\frac{d^2y}{dt^2}$ ✓ completes proof

Question 17 (b)

Solution
Since $V(t) = \frac{dy}{dt} = Ce^{-t}(2\cos 2t - \sin 2t)$ so $V(0) = 2C = 14 \quad \Rightarrow \quad C = 7$
Specific behaviours
✓ obtains correct expression for the velocity ✓ deduces the correct value for the constant

Question 17 (c)

Solution
The spring is stationary when $V(t) = 0$ i.e. when $2\cos 2t - \sin 2t = 0$ i.e. when $\tan(2t) = 2$ The least positive solution of this equation is $2t = \arctan(2) \approx 1.1071$ So the spring is first stationary at time 0.5536 seconds approximately

Specific behaviours
✓ obtains equation for stationary times ✓ calculates the correct answer

Question 17 (d)

Solution
From part (c) the spring is stationary when $\tan(2t)=2$ The solutions are then $t=\frac{1}{2}\arctan(2)+\frac{n\pi}{2}$ where n is any non-negative integer. These form an arithmetic sequence with common difference is $\frac{\pi}{2}$.
Specific behaviours
✓ obtains general solution for the stationary times ✓ states the correct common difference between these times

Question 17 (e)

Solution
At a local maximum of $y(t)$ we have $V(t)=0$ This occurs when $t=\theta+\frac{n\pi}{2}$ where $2\theta=\arctan(2)$ Now $y\left(\theta+\frac{n\pi}{2}\right)=7e^{-\theta-\frac{1}{2}n\pi}\sin(2\theta+n\pi)$ We know that $\sin(2\theta+n\pi)$ will be positive if n is even and negative if n is odd Hence the local maxima of $y(t)$ correspond to n even. So the local maxima of $y(t)$ are given by $7e^{-\theta-m\pi}\sin 2\theta$ where m is any non-negative integer. These form a geometric sequence in which the common ratio is $e^{-\pi}$
Specific behaviours
✓ evaluates $y(t)$ at the stationary point ✓ shows that every second stationary point is a local maximum ✓ realises the maxima correspond to even values of n ✓ observes the maxima values constitute a GP and states the common ratio

Question 18 (a)

Solution
For the region $0 \le x \le 2$ the parabola is above the x-axis so the required area is $A = \int_0^2 x(2-x)\,dx = \int_0^2 (2x - x^2)\,dx$ $= \left[x^2 - \frac{1}{3}x^3 \right]_0^2$ $= 4 - \frac{8}{3} = \frac{4}{3}$
Specific behaviours
✓ states correct limits of integration ✓ writes a correct expression for the area ✓ integrates all the terms correctly ✓ substitutes in values to determine the area

Question 18 (b)

Solution

If rotating about x-axis have that

$$V_1 = \pi\int_0^2 y^2\,dx = \pi\int_0^2 x^2(2-x)^2dx = \pi\int_0^2 \left(4x^2 - 4x^3 + x^4\right)dx$$

$$= \pi\left[\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5\right]_0^2$$

$$= \pi\left[\frac{32}{3} - 16 + \frac{32}{5}\right] = 32\pi\left[\frac{8}{15} - \frac{1}{2}\right] = \frac{16}{15}\pi$$

To compute volume around y-axis need to evaluate $\pi\int x^2\,dy$

The second volume is determined by calculating the volume V_R generated by rotating the right-hand branch of the parabola about the axis and subtracting the volume V_L formed using the left-hand branch of the parabola.

The parabola is $(x-1)^2 = 1 - y \Rightarrow x = 1 \pm\sqrt{1-y}$. Hence $x^2 = 2 - y \pm 2\sqrt{1-y}$

Right hand branch is with +sign and is defined by $0 \le y \le 1$. Then

$$V_R = \pi\int_0^1\left[2 - y + 2\sqrt{1-y}\right]dy = \pi\left[2y - \frac{1}{2}y^2 - \frac{4}{3}(1-y)^{3/2}\right]_0^1 = \pi\left[2 - \frac{1}{2} + \frac{4}{3}\right] = \frac{17}{6}\pi$$

Left hand branch is with – sign and is also defined by $0 \le y \le 1$. Then

$$V_L = \pi\int_0^1\left[2 - y - 2\sqrt{1-y}\right]dy = \pi\left[2y - \frac{1}{2}y^2 + \frac{4}{3}(1-y)^{3/2}\right]_0^1 = \pi\left[2 - \frac{1}{2} - \frac{4}{3}\right] = \frac{1}{6}\pi$$

Thus volume generated by rotating about the y axis is $V_2 = V_R - V_L = \frac{8}{3}\pi$

Hence we see that $V_1 = \frac{16}{15}\times\frac{3}{8}V_2 = \frac{2}{5}V_2$

Specific behaviours

✓ writes down correct form of integral for first volume

✓✓ integrates correctly and hence evaluates to determine the volume V_1 (candidates are expected to use calculator for the manipulation)

✓ realises that for rotation is about y-axis the required integral is of form $\pi\int x^2\ dy$

✓ gives clear statement of strategy adopted to determine the volume V_2

✓ determines the appropriate formulae for the two branches of the parabola

✓✓ for volume V_R forms correct integral with limits. Evaluates correctly (calc allowed)
✓✓ for volume V_L forms correct integral with limits. Evaluates correctly
✓ deduces the correct relationship between V_1 and V_2