SUPPLEMENTARY PAGE

Question: _____

Question: _____

PERTH MODERN SCHOOL

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Independent Public School

Course: Methods Year: 11

Student Name: **Marking Key** Teacher Name: ________

Date: 29/07/22

Task Type: **Response**

Time Allowed: **40 minutes**

Number of Questions: **6**

Materials Required: One double-sided A4 pages of notes (to be provided by the student)

Standard Items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler and highlighters

Special Items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper (both sides)

Marks Available: **40 marks**

Task Weighting: **10 %**

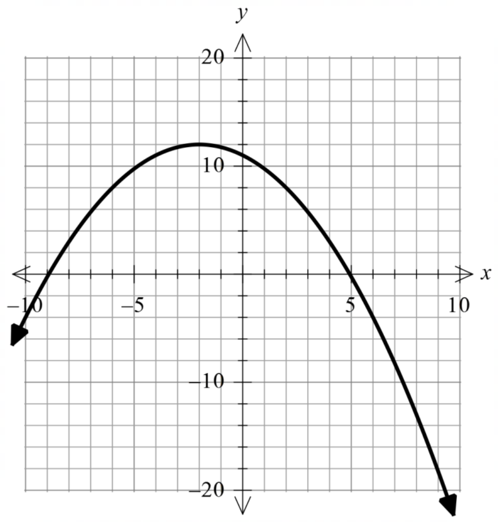
Formula Sheet Provided: Yes

Note: All questions worth more than 2 marks require working to obtain full marks.

TEST 3: DIFFERENTIAL CALCULUS

Question 1 [2 marks – 1, 1] (2.3.1-3)

Consider the function shown below. For the interval $[2, 6]$:



a) State the values of δx and δy.

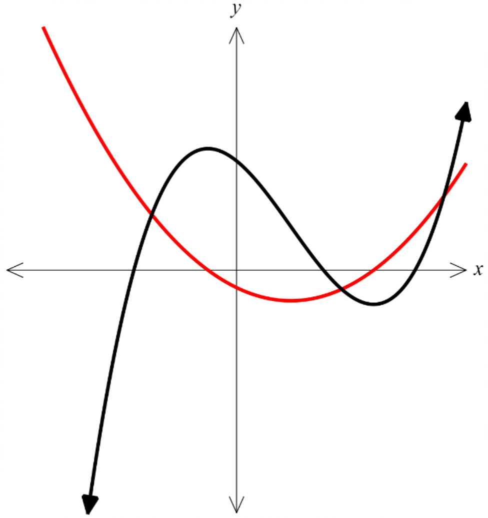
Solution
$\delta x = 4$ $\delta y = -12$
Specific behaviours
✓ States δx and δy

b) Determine the average rate of change of the function.

Solution
$\frac{\delta y}{\delta x} = -3$
Specific behaviours
✓ Calculates $\frac{\delta y}{\delta x}$

Question 2 [3 marks] (2.3.8-9, 11, 20)

Sketch a possible graph of $\frac{dy}{dx}$ for the cubic shown below, on the same axes.



Specific behaviours
✓ Roots correspond to stationary points of y
✓ Parabolic shape
✓ Concave up (correct signs)
Exact shape is not important

Question 6 (continued)

b) Given that $V(x) = 2x^3 - 33x^2 + 108x$, find the dimensions of the box that will maximise its volume, state the volume and show that it is a maximum, using calculus.

Solution

$$V'(x) = 6x^2 - 66x + 108$$

Stationary points when $V'(x) = 0$:

$$\begin{aligned} V'(x) &= 0 \\ 6x^2 - 66x + 108 &= 0 \\ x^2 - 11x + 18 &= 0 \\ (x-2)(x-9) &= 0 \\ x &= 2, \cancel{9} \quad 0 < x < \tfrac{9}{2} \text{ (from width)} \end{aligned}$$

Checking nature:

Sign Test

x	1	2	3
$V'(x)$	48	0	-36
Sign	$+$	0	$-$
Slope			

2nd Derivative Test

$$\begin{aligned} f''(x) &= 12x - 66 \\ f''(2) &= -42 \\ &= \text{negative} \end{aligned}$$

∴ Maximum at $x = 2$.

Hence, $l = 10$ cm, $w = 5$ cm and $h = 2$ cm, and $V = 10(5)(2) = 100\ \text{cm}^3$.

Specific behaviours

- ✓ Differentiates
- ✓ Equates $V'(x)$ to 0
- ✓ Solves for both values of x, then eliminates $x = 9$
- ✓ Checks nature of stationary point at $x = 2$ (must show values and signs of $V'(x)$ for sign test or $f''(x)$ for second derivative test)
- ✓ States dimensions
- ✓ States volume

End of Test

Question 5 (continued)

c) Hence, calculate the distance travelled over the given interval.

Solution

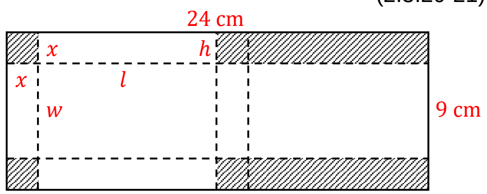
Start and end positions:
$x(0) = 0$ m
$x(5) = (5)^3 - 9(5)^2 + 24(5)$
$= 125 - 225 + 120$
$= 20$ m

Path: $0 \text{ m} \to 20 \text{ m} \to 16 \text{ m} \to 20 \text{ m}$
Distance travelled:
$d = 20 + 4 + 4$
$= 28$ m

Specific behaviours
- ✓ Finds start and end positions (both)
- ✓ Identifies path or leg lengths
- ✓ Finds distance travelled

Question 6 **[10 marks – 4, 6]** (2.3.20-21)

A rectangular sheet of metal, 9 cm by 24 cm, will be made into a closed rectangular box. Two squares of side x cm and two rectangles will be removed from the corners to form the net of the box as shown right.



a) Label the diagram with the appropriate dimensions and variables, then clearly show below that the volume of the box, V cm^3, is given by $V(x) = x(12 - x)(9 - 2x)$.

Solution

Dimensions of box:
$h = x$
$2x + 2l = 24$
$l = 12 - x$
$2x + w = 9$
$w = 9 - 2x$

Volume of box:
$V(x) = lwh$
$= x(12 - x)(9 - 2x)$

Specific behaviours
- ✓ Labels diagram (accept if labelled with $12 - x$, $9 - 2x$ and x instead of l, w and h)
- ✓ States length in terms of x (required)
- ✓ States width and height in terms of x (required)
- ✓ States volume formula and substitutes

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Question 3 **[8 marks – 1, 2, 2, 3]** (2.3.7, 12-15, 22)

a) Differentiate the following:

i) $f(x) = 4x^5 - 9x^4$

Solution

$f'(x) = 20x^4 - 36x^3$

Specific behaviours
- ✓ Differentiates

ii) $y = (2x + 3)(6x - 7)$

Solution

$y = 12x^2 + 4x - 21$
$\frac{dy}{dx} = 24x + 4$

Specific behaviours
- ✓ Expands
- ✓ Differentiates

Award full marks if product rule is used correctly

b) Anti-differentiate the following:

i) $\frac{dy}{dx} = 24x^3 + 27x^2$

Solution

$y = \int (24x^3 + 27x^2)dx$
$= 6x^4 + 9x^3 + c$

Specific behaviours
- ✓ Anti-differentiates
- ✓ Adds constant

Penalise once only for missing constant

ii) $f'(x) = \frac{12x^5 - 9x^2}{6x^2}$

Solution

$f(x) = \int \left(\frac{12x^5 - 9x^2}{6x^2}\right) dx$
$= \int \left(2x^3 - \frac{3}{2}\right) dx, \quad x \neq 0$
$= \frac{1}{2}x^4 - \frac{3}{2}x + c, \quad x \neq 0$

Specific behaviours
- ✓ Rearranges (no need to state $x \neq 0$)
- ✓ Anti-differentiates
- ✓ Adds constant

Question 4 [7 marks – 3, 4] (2.3.4, 6, 9, 17)

Consider points $A(3, 18)$ and $B\big(3+h, f(3+h)\big)$ on the curve $f(x) = 2x^2$.

a) Determine the expression for the gradient of chord AB, using the difference quotient formula $\frac{\delta y}{\delta x} = \frac{f(x+h) - f(x)}{h}$.

Solution
$$\begin{aligned} m_{AB} &= \frac{f(3+h) - f(3)}{h} \\ &= \frac{\big(2(3+h)^2\big) - 18}{h} \\ &= \frac{2\big(9 + 6h + h^2\big) - 18}{h} \\ &= \frac{18 + 12h + 2h^2 - 18}{h} \\ &= \frac{12h + 2h^2}{h} \\ &= \frac{\cancel{h}(12 + 2h)}{\cancel{h}} \\ &= 12 + 2h \end{aligned}$$
Specific behaviours
✓ Substitutes into difference quotient formula ✓ Fully expands expression ✓ Fully simplifies expression

b) Hence, by applying first principles to your answer above, determine the gradient and equation of the tangent to point A.

Solution
$$\begin{aligned} m_A &= \lim_{h \to 0}(12 + 2h) \\ &= 12 \end{aligned}$$ $$\begin{aligned} y - 18 &= 12(x - 3) \\ y - 18 &= 12x - 36 \\ y &= 12x - 18 \end{aligned}$$
Specific behaviours
✓ Applies first principles (must show $\lim_{h \to 0}$) ✓ Finds gradient ✓ Substitutes gradient and point A into any linear relationship formula ✓ States equation of tangent

Question 5 [10 marks – 3, 4, 3] (2.3.16, 18-20)

An object moves such that its position x metres from point O after t seconds is given by $x(t) = t^3 + at^2 + 24t$ for $0 \le t \le 5$. After 1 second, it has a velocity of $9\ \text{m/s}$.

a) Show that $a = -9$.

Solution
$$\begin{aligned} v(t) &= 3t^2 + 2at + 24 \\ v(1) &= 9 \\ 3(1)^2 + 2a(1) + 24 &= 9 \\ 3 + 2a + 24 &= 9 \\ 2a + 27 &= 9 \\ 2a &= -18 \\ a &= -9 \end{aligned}$$
Specific behaviours
✓ Differentiates ✓ Substitutes in given information ✓ Calculates a with at least one prior line of working

b) Determine when the object is stationary and its positions at those times.
You do not need to prove the nature of these stationary points.

Solution
Since $a = -9$, $$x(t) = t^3 - 9t^2 + 24t$$ $$v(t) = 3t^2 - 18t + 24$$ Stationary when $v(t) = 0$: $$\begin{aligned} 3t^2 - 18t + 24 &= 0 \\ t^2 - 6t + 8 &= 0 \\ (t-2)(t-4) &= 0 \\ t &= 2\text{ s and }4\text{ s} \end{aligned}$$ Positions: $$\begin{aligned} x(2) &= (2)^3 - 9(2)^2 + 24(2) \\ &= 8 - 36 + 48 \\ &= 20\text{ m after 2 seconds} \end{aligned}$$ $$\begin{aligned} x(4) &= (4)^3 - 9(4)^2 + 24(4) \\ &= 64 - 144 + 96 \\ &= 16\text{ m after 4 seconds} \end{aligned}$$
Specific behaviours
✓ Substitutes in $a = -9$ for position and velocity (okay if implicit) ✓ Equates velocity to 0 ✓ Solves for both times ✓ Finds both positions (okay if corresponding times are missing) *Award 1 mark if one time and position are found*

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