Mathematics: Specialist
Formula sheet Units 3C and 3D

Vectors

$\left|(a_1, a_2, a_3)\right| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ $\qquad$ $|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3$

Vector equation of a line in space:

one point and the slope: $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$

two points A and B: $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

Cartesian equations of a line in space: $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3}$

Parametric form of vector equation of a line in space:

$x = a_1 + \lambda b_1$ ....(1)

$y = a_2 + \lambda b_2$ ....(2)

$z = a_3 + \lambda b_3$ ....(3)

Vector equation of a plane in space: $\mathbf{r} \bullet \mathbf{n} = c$ or $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$

Trigonometry

In any triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$a^2 = b^2 + c^2 - 2bc\cos A$ $\qquad$ $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$$A = \frac{1}{2}ab\sin C$$

In a circle of radius r, for an arc subtending angle θ (radians) at the centre:

Length of arc $= r\theta$ $\qquad$ Area of sector $= \dfrac{1}{2}r^2\theta$

Area of segment $= \dfrac{1}{2}r^2(\theta - \sin\theta)$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$ $\qquad$ $\cos^2\theta + \sin^2\theta = 1$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\tan(\theta \pm \phi) = \dfrac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$ $\qquad$ $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

$\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$ $\qquad$ $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Simple Harmonic Motion: If $\frac{d^2x}{dt^2} = -k^2x$ then $x = A\sin(kt + \alpha)$ or $x = A\cos(kt + \beta)$

Exponentials and logarithms

For $a, b > 0$ and m, n real,

$a^m a^n = a^{m+n}$ $\qquad$ $a^{m-n} = \dfrac{a^m}{a^n}$

$a^0 = 1$ $\qquad$ $a^{-n} = \dfrac{1}{a^n}$

$\left(a^m\right)^n = a^{mn}$ $\qquad$ $a^m b^m = (ab)^m$

For m an integer and n a positive integer:

$a^{\frac{1}{n}} = \sqrt[n]{a}$ $\qquad$ $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

For $a, y > 0$ then

$y = a^x \Leftrightarrow \log_a y = x$

$1 = a^0 \Leftrightarrow \log_a 1 = 0$ $\qquad$ $a = a^1 \Leftrightarrow \log_a a = 1$

$\log_a(mn) = \log_a(m) + \log_a(n)$ $\qquad$ $\log_a\left(m^n\right) = n\log_a(m)$

$\log_a(m) = \dfrac{\log_b(m)}{\log_b(a)}$ (change of base) $\qquad$ If $\dfrac{dp}{dt} = kP$ then $P = P_0 e^{kt}$

Functions

If $f(x) = y$, then $f'(x) = \dfrac{dy}{dx}$ $\qquad$ If $f(x) = x^n$, then $f'(x) = n\,x^{n-1}$

If $f(x) = e^x$, then $f'(x) = e^x$ $\qquad$ If $f(x) = \ln x$, then $f'(x) = \dfrac{1}{x}$

If $\mathrm{f}(x) = \sin x$ then $\mathrm{f}'(x) = \cos x$ $\qquad$ If $\mathrm{f}(x) = \cos x$ then $\mathrm{f}'(x) = -\sin x$

If $\mathrm{f}(x) = \tan x$ then $\mathrm{f}'(x) = \sec^2 x = \dfrac{1}{\cos^2 x}$

Product rule:

If $y = f(x)\,g(x)$

then $y' = f'(x)\,g(x) + f(x)\,g'(x)$

or

If $y = uv$

then $\dfrac{dy}{dx} = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$

Quotient rule:

If $y = \dfrac{f(x)}{g(x)}$

then $y' = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$

or

If $y = \dfrac{u}{v}$

then $\dfrac{dy}{dx} = \dfrac{\dfrac{du}{dx}v - u\dfrac{dv}{dx}}{v^2}$

Incremental formula: $\delta y \simeq \dfrac{dy}{dx}\delta x$ $\qquad$ or $\qquad$ $f(x+h) - f(x) \simeq f'(x)h$

Chain rule:

If $y = f(g(x))$ or If $y = f(u)$ and $u = g(x)$

then $y' = f'(g(x))\, g'(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Powers: $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + c, \; n \neq -1$

Exponentials: $\int e^x dx = e^x + c$

Logarithms: $\int \dfrac{1}{x} dx = \ln x + c$

Trigonometric:

$\int \sin x \; dx = -\cos x + c$

$\int \cos x \; dx = \sin x + c$

$\int \dfrac{1}{\cos^2 x} \; dx = \tan x + c$

Fundamental Theorem of Calculus:

$\dfrac{d}{dx} \int_a^x f(t)dt = f(x)$ and $\int_a^b f'(x)dx = f(b) - f(a)$

Complex numbers

For $z = a + ib$, where $i^2 = -1$

Argument: $\text{Arg } z = \theta$, where $\tan \theta = \dfrac{b}{a}$ and $-\pi < \theta \leq \pi$

Modulus: $\text{mod } z = |z| = |a + ib| = \sqrt{a^2 + b^2} = r$

Product: $|z_1 z_2| = |z_1||z_2|$ $\quad \arg z_1 z_2 = \arg z_1 + \arg z_2$

Quotient: $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ $\quad \arg \dfrac{z_1}{z_2} = \arg z_1 - \arg z_2$

Polar form:

For $z = r cis\theta$ where $r = |z|$ and $\theta = \arg z$

$cis\theta = \cos\theta + i \sin\theta$

$cis(\theta + \phi) = cis\theta cis\phi$

$cis(0) = 1$

$cis(-\theta) = \dfrac{1}{cis\theta}$

$z_1 z_2 = r_1 r_2 \, cis(\theta + \phi)$

$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} cis(\theta - \phi)$

Exponential form:

$z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg z$

For complex conjugates:

$z = a + bi$ $\qquad$ $\bar{z} = a - bi$

$z = r\,cis\,\theta$ $\qquad$ $\bar{z} = r\,cis(-\theta)$

$z = re^{i\theta}$ $\qquad$ $\bar{z} = re^{-i\theta}$

$z\bar{z} = |z|^2$ $\qquad$ $z^{-1} = \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$

$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ $\qquad$ $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

Mathematical reasoning

DeMoivre's theorem:

$(cis\theta)^n = (\cos\theta + i\sin\theta)^n$ $\qquad$ $(cis\theta)^n = \cos n\theta + i\sin n\theta$

$z^n = |z|^n\,cis(n\theta)$

$z^{\frac{1}{q}} = |z|^{\frac{1}{q}}\left(\cos\left(\dfrac{\theta + 2\pi k}{q}\right) + i\sin\left(\dfrac{\theta + 2\pi k}{q}\right)\right)$ for $k \in \{\text{Integers}\}$

Matrices

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A| = \det A = ad - bc$ $\qquad$ $A^{-1} = \dfrac{1}{(ad - bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Dilation = $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ $\qquad$ Shear = $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

Rotation = $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\qquad$ Reflection = $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

Measurement

Circle: $C = 2\pi r = \pi D$, where C is the circumference, r is the radius and D is the diameter

$A = \pi r^2$, where A is the area

Triangle: $A = \frac{1}{2} b h$, where b is the base and h is the perpendicular height

Parallelogram: $A = b h$

Trapezium: $A = \frac{1}{2}(a + b)h$ where a and b are the lengths of the parallel sides and h is the perpendicular height

Prism: $V = Ah$, where V is the volume, A is the area of the base and h is the perpendicular height

Pyramid: $V = \frac{1}{3}\mathrm{A}h$

Cylinder: $S = 2\pi r h + 2\pi r^2$, where S is the total surface area

$V = \pi r^2 h$

Cone: $S = \pi r s + \pi r^2$ where s is the slant height

$V = \frac{1}{3}\pi r^2 h$

Sphere: $S = 4\pi r^2$

$V = \frac{4}{3}\pi r^3$

Note: Any additional formulas identified by the examination panel as necessary will be included in the body of the particular question.