MATHEMATICS
SPECIALIST
UNIT 1

Semester One

2018

SOLUTIONS

Calculator-free Solutions

1. (a) $b-a=\begin{pmatrix}1\\3\end{pmatrix}-\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}-1\\2\end{pmatrix}$ ✓

$\therefore a\cdot(b-a)=\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\end{pmatrix}=-2+2=0$ ✓✓

$\therefore a\perp(b-a)$

(b) (i) $\overrightarrow{OC}=\overrightarrow{AB}=b-a=-i+2j$ ✓

(ii) $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=-3i+j$ ✓

$\therefore\left|\overrightarrow{AC}\right|=\sqrt{10}$ ✓

OR

$\left|\overrightarrow{AC}\right|=\left|\overrightarrow{OB}\right|$ ✓

$\therefore\left|\overrightarrow{AC}\right|=\begin{vmatrix}1\\3\end{vmatrix}=\sqrt{10}$ ✓ [6]

2. (a) (i) $\dfrac{13!+12!}{13!-12!}=\dfrac{13\times12!+12!}{13\times12!-12!}=\dfrac{12!(13+1)}{12!(13-1)}$ ✓

$\text{¿}\dfrac{14}{12}=\dfrac{7}{6}$ ✓

(ii) $\dfrac{{}^{10}C_4}{{}^{8}C_4}=\dfrac{10!}{4!\times6!}\div\dfrac{8!}{4!\times4!}$ ✓

$\text{¿}\dfrac{10\times9\times8!\times4!\times4!}{8!\times6\times5\times4!}=\dfrac{10}{5}\times\dfrac{9}{6}\ \text{¿}2\times\dfrac{3}{2}=3$ ✓

(b) LHS $\text{¿}k\begin{pmatrix}n\\k\end{pmatrix}=k\times\dfrac{n!}{k!(n-k)!}$ ✓

$\text{¿}\dfrac{k}{k}\times\dfrac{n!}{(k-1)![n-k]!}$ ✓

$\text{¿}n\times\dfrac{(n-1)!}{(k-1)![(n-1)-(k-1)]!}$ ✓✓

$\text{¿}n\begin{pmatrix}n-1\\k-1\end{pmatrix}=\text{¿}$ RHS (Or from RHS to LHS) [8]

3. (a) $-3>-5$ but $(-3)^2=9>(-5)^2=25$ is false ✓

(b) "If the triangle is not equilateral, then the triangle does not have three equal sides." ✓
Yes is it always true since the original implication is always true by definition of equilateral triangles. ✓

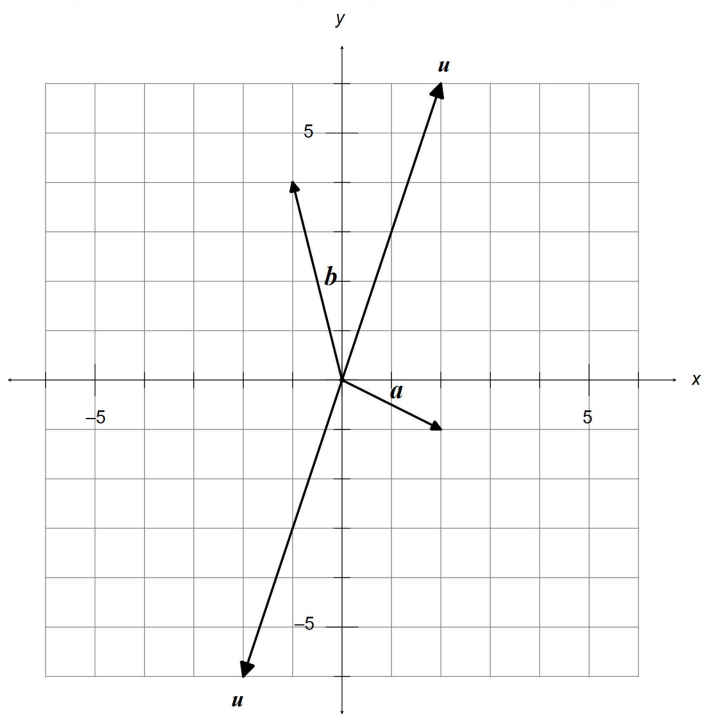
(c) "If n is divisible by 3, then n is divisible by 6". ✓
The converse is not always true, because for n to be divisible by 6 is must be divisible by both 2 and 3. ✓

(d) FOR ALL natural numbers p, EXISTS a real number q, such that q is the square root of p. ✓✓ [7]

4. (a) $a+b=\begin{pmatrix}2\\-1\end{pmatrix}+\begin{pmatrix}-1\\4\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$ ✓

$\therefore \widehat{(a+b)}=\pm\frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix}$ ✓

$\therefore u=2\sqrt{10}\times\pm\frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix}=\pm\begin{pmatrix}2\\6\end{pmatrix}$ ✓



(b) (i) $c=k\,a$ ✓

$\begin{pmatrix}-4\\\alpha\end{pmatrix}=k\begin{pmatrix}2\\-1\end{pmatrix}\rightarrow k=-2\rightarrow\alpha=2$ ✓

(ii) $\left|\begin{pmatrix}-4\\ \alpha\end{pmatrix}-\begin{pmatrix}-1\\ 4\end{pmatrix}\right|=3\left|\begin{matrix}2\\ -1\end{matrix}\right|$

$\therefore \sqrt{9+(\alpha-4)^2}=3\sqrt{5}$ ✓

$9+(\alpha-4)^2=9\times5$ ✓

$(\alpha-4)^2=36$

$\alpha=4\pm6 \rightarrow \alpha=10 \vee -2$ ✓✓ [10]

5\. (a) $\overrightarrow{AC}\cdot\overrightarrow{OB}=(c-a)\cdot(a+c)=0$ ✓

$c\cdot a+c\cdot c-a\cdot a-a\cdot c=0$ ✓

$|c|^2-|a|^2=0 \rightarrow |a|=|c|$ ✓

$\therefore$ OABC is a rhombus

(b) LHS ¿ $|AC|^2+|OB|^2=|c-a|^2+|a+c|^2$ ✓

¿ $(c-a)\cdot(c-a)+(a+c)\cdot(a+c)$ ✓

¿ $c\cdot c-2a\cdot c+a\cdot a+a\cdot a+2a\cdot c+c\cdot c$ ✓

¿ $2|a|^2+2|c|^2$ ✓

¿ $|OA|^2+|AB|^2+|BC|^2+|OC|^2$ as required [7]

6\. (a) $\angle$AED = 90° ✓

Triangle in a semi-circle is always right angled. ✓

(b) $\angle$ABE = $\angle$ADE = 60° ✓

Angles within the same segment are congruent. ✓

(c) $\angle$CAE = 80° ✓

Opposite angles in a cyclic quadrilateral are supplementary ✓

(d) $\angle$TCE = $\angle$CBE = 100° ✓

The alternate segment theorem. ✓ [8]

7\. $\overrightarrow{AB}=\frac{2}{3}\overrightarrow{AC}$ ✓

$\therefore b-a=\frac{2}{3}(c-a)$

$\begin{pmatrix}1\\ 3\end{pmatrix}-\begin{pmatrix}x\\ -1\end{pmatrix}=\frac{2}{3}\begin{pmatrix}4-x\\ y+1\end{pmatrix}$ ✓

$\therefore x=-5$ and $y=5$ ✓✓ [4]

<u>Calculator-assumed Solutions</u>

8. (a) Assume all non-repeated numbers are selected from
both sets: 3, 8, 4, 6 = 4 digits ✓
Plus all remaining digits from one set: 1, 2, 5, 7 = 4 digits
Plus one more digit to make the first repetition
∴ 4 + 4 + 1 = 9 digits minimum ✓

(b) Assume the largest numbers are chosen first: ✓
8 + 7 + 7 + 6 = 28 ✓
one more digit <u>could include</u> the number 5, making
the sum over 30.
∴ 4 digits max ✓ [54

9. (a) (i) ${}^{22}C_8 = 319770$ ✓
(ii) ${}^{12}C_4 \times {}^{10}C_4 = 103\,950$ ✓✓
(iii) ${}^{22}C_8 - {}^{20}C_6 = 281\,010$ ✓✓
(iv) ${}^{2}C_2 \times {}^{20}C_6 + {}^{2}C_0 \times {}^{20}C_8 = 38\,760 + 125\,970 = 164\,730$ ✓✓

(b) (i) $8! = 40\,320$ ✓
(ii) $3! \times 6! = 4\,320$ ✓✓
(iii) $8! - 2! \times 7! = 30\,240$ ✓✓ [12]

10. (a) II and III ✓✓
(b) ${}^{10}C_1 \times {}^{60}P_6 \times {}^{42}C_1$ OR ${}^{10}C_1 \times {}^{60}C_6 \times {}^{42}C_1 \times 6!$ ✓✓✓
(c) LHS ¿ $\dfrac{n!}{r! \times (n-r)!} + \dfrac{n!}{(r+1)! \times (n-r-1)!}$ ✓

¿ $\dfrac{n!}{r! \times (n-r-1)!} \times \left[\dfrac{1}{(n-r)} + \dfrac{1}{(r+1)}\right]$ ✓

¿ $\dfrac{n!}{r! \times (n-r-1)!} \times \left[\dfrac{r+1+n-r}{(n-r)(r+1)}\right]$

¿ $\dfrac{n! \times (n+1)}{r! \times (r+1) \times (n-r-1)! \times (n-r)}$ ✓

¿ $\dfrac{(n+1)!}{(r+1)! \times (n-r)!}$ ✓

¿ $\dfrac{(n+1)!}{(r+1)! \times [(n+1)-(r+1)]!}$ ✓

$¿^{n+1}C_{r+1}=¿$ RHS [10]

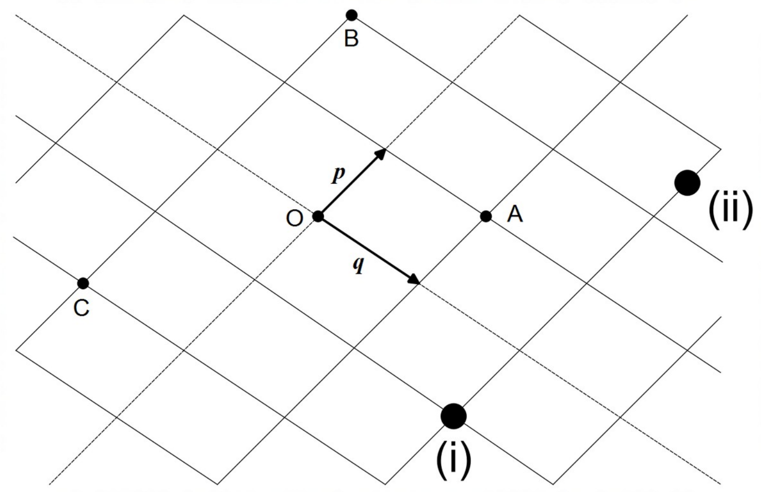
11. (a) (i) $p+q$ ✓

(ii) $2p-q$ ✓

(iii) $-(3p+2q)$ ✓

(iv) $2q-p$ ✓

(b)



[6]

12. (a) $\overrightarrow{AC}=\begin{pmatrix}2\\k\end{pmatrix}-\begin{pmatrix}-2\\0\end{pmatrix}=\begin{pmatrix}4\\k\end{pmatrix}=4i+kj$ ✓

$\overrightarrow{BC}=\begin{pmatrix}2\\k\end{pmatrix}-\begin{pmatrix}1\\-3\end{pmatrix}=\begin{pmatrix}1\\k+3\end{pmatrix}=i+(k+3)j$ ✓

(b) $\left|\begin{matrix}4\\k\end{matrix}\right|=\left|\begin{matrix}1\\k+3\end{matrix}\right|$

$\therefore \sqrt{4^2+k^2}=\sqrt{1^2+(k+3)^2}$ ✓

$\therefore k=1$ ✓

(c) $D=\left(\dfrac{-2+1}{2},\dfrac{0-3}{2}\right)=(-0.5,-1.5)$ ✓

(d) $\overrightarrow{DC}=\begin{pmatrix}2\\1\end{pmatrix}-\begin{pmatrix}-0.5\\-1.5\end{pmatrix}=\begin{pmatrix}2.5\\2.5\end{pmatrix}$

$\overrightarrow{DB}=\begin{pmatrix}1\\-3\end{pmatrix}-\begin{pmatrix}-0.5\\-1.5\end{pmatrix}=\begin{pmatrix}1.5\\-1.5\end{pmatrix}$ ✓

$\therefore \overrightarrow{DC}\cdot\overrightarrow{DB}=\begin{pmatrix}2.5\\2.5\end{pmatrix}\cdot\begin{pmatrix}1.5\\-1.5\end{pmatrix}=2.5\times1.5-2.5\times1.5=0$ ✓

$\therefore \overrightarrow{DC}\perp\overrightarrow{DB}\rightarrow\angle$CDB is right angled [7]

13. (a) (i) True. ✓

If a number is divisible by 6, then it is also divisible by both 2 and 3. ✓

(ii) False. ✓

It must be divisible by both 2 and 3. ✓

(iii) True. ✓

The conjunction AND means that it is divisible by both 2 and 3, and therefore it is also divisible by 6. ✓

(b) Assume that n is odd AND that $3n+5$ is also odd ✓

$\therefore \exists k \in N$ such that $n=2k+1$ ✓

$\therefore 3n+5 \quad ¿3(2k+1)+5$ ✓

$¿6k+8=2(3k+4)=¿$ even ✓

Since $3n+5$ is both odd and even simultaneously, this is a contradiction, implying that n must be even. ✓

(c) $\overrightarrow{OE}+\overrightarrow{AD}+\overrightarrow{BF}$

$¿\left[b+\frac{1}{2}(a-b)\right]+\left[-a+\frac{1}{2}b\right]+\left[-b+\frac{1}{2}a\right]$ ✓✓✓

$¿\left(\frac{1}{2}a-a+\frac{1}{2}a\right)+\left(b-\frac{1}{2}b-\frac{1}{2}b\right)=0$ ✓ [15]

14. (a) $F_1\cos 30°=F_2\cos 45°$ ✓

$\therefore \frac{\sqrt{3}}{2}F_1=\frac{1}{\sqrt{2}}F_2$

$F_1\sin 30°+F_2\sin 45°=250$ ✓

$\therefore \frac{1}{2}F_1+\frac{1}{\sqrt{2}}F_2=250$

(b) $F_2=\frac{\sqrt{6}}{2}F_1$

$\therefore \frac{1}{2}F_1+\frac{\sqrt{3}}{2}F_1=250$ ✓✓

$\therefore F_1=\frac{500}{1+\sqrt{3}}=183.01\,N$

$\therefore F_2=\frac{\sqrt{6}}{2}\times\frac{500}{1+\sqrt{3}}=\frac{250\sqrt{6}}{1+\sqrt{3}}=224.14\,N$ ✓✓

(c) If $F_1=200\,N$ then $F_2=\frac{\sqrt{6}}{2}\times 200=244.95\,N>200\,N$ ✓

$\therefore$ Cable 2 exceeds its maximum load, hence Cable 1 must not reach its 200N maximum rating ✓

If $F_2 = 200\,N$ then $F_1 = \frac{2}{\sqrt{6}} \times 200 = 163.30\,N < 200\,N$ ✓✓

Max Force ¿ $F_1 \sin 30° + F_2 \sin 45°$ ✓

¿ $\frac{400}{\sqrt{6}} \times \frac{1}{2} + 200 \times \frac{1}{\sqrt{2}} = 223.07\,N$ ✓ [12]

15. (a) $n(D \cup C) = n(D) + n(C) - n(D \cap C)$ ✓

$810 = 400 + 500 - n(D \cap C)$ ✓

$\therefore n(D \cap C) = 90$ ✓

(b) $n(D \cup C \cup B) = n(D) + n(C) + n(B)$

$-n(D \cap C) - n(D \cap B) - n(C \cap B)$

$+n(D \cap C \cap B)$ ✓

$900 = 400 + 500 + 210 - 90 - 60 - 110 + n(D \cap C \cap B)$ ✓

$\therefore n(D \cap C \cap B) = 50$ ✓ [6]

16. (a) $({}^3C_1)^2 = {}^3C_2 + {}^4C_2$ ✓

$({}^5C_1)^2 = {}^5C_2 + {}^6C_2$ ✓

(b) $a = 2$ ✓

$b = n + 1$ ✓

(c) $({}^6C_1)^2 = {}^8C_3 - {}^6C_3$ ✓✓

(d) $({}^nC_1)^2 = {}^{n+2}C_3 - {}^nC_3$ ✓✓ [8]

17. (a) $\overrightarrow{OE} = a + \frac{1}{2}(b - a) = \frac{1}{2}(a + b)$ ✓

$\overrightarrow{OF} = c + \frac{1}{2}(b - c) = \frac{1}{2}(b + c)$ ✓

(b) $\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = \frac{1}{2}(a + b) - \frac{1}{2}a = \frac{1}{2}b$ ✓

$\overrightarrow{GF} = \overrightarrow{OF} - \overrightarrow{OG} = \frac{1}{2}(b + c) - \frac{1}{2}c = \frac{1}{2}b = \overrightarrow{DE}$ ✓

$\overrightarrow{DG} = \overrightarrow{OG} - \overrightarrow{OD} = \frac{1}{2}c - \frac{1}{2}a = \frac{1}{2}(c - a)$ ✓

$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \frac{1}{2}(b + c) - \frac{1}{2}(a + b) = \frac{1}{2}(c - a) = \overrightarrow{DG}$ ✓

$\therefore \overrightarrow{DE} = \overrightarrow{GF} \wedge \overrightarrow{DG} = \overrightarrow{EF} \Rightarrow$ DEFG is a parallelogram ✓ [7]

18. $BF \times FD = CF \times FA$ ✓

$\therefore x(2y) = 2x(6) \rightarrow y = 6\,cm$ ✓

$MD \times MB = MT^2$ ✓

$\therefore 4 \times (4 + 12 + x) = \sqrt{76}^2 \rightarrow x = 3\,cm$ ✓

$NT^2 = NC \times (NC + FC + FA)$ ✓

$\therefore z^2 = 6 \times (6+6+6) \rightarrow z = 6\sqrt{3}\,cm$ ✓ [6]

19. Let $\hat{n}$ be a unit vector perpendicular to b

then, $a = u + |u| \tan 60° \hat{n}$ ✓

$|u| = \left|\begin{matrix} 3 \\ 1.5 \end{matrix}\right| = \frac{3}{2}\sqrt{5}$ ✓

$n \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 0 \rightarrow$ let $n = \pm\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ ✓

$\therefore \hat{n} = \pm\frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ ✓

$\therefore a = \begin{pmatrix} 3 \\ 1.5 \end{pmatrix} + \frac{3}{2}\sqrt{5} \times \sqrt{3} \times \pm\frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1.5 \end{pmatrix} \pm \frac{3}{2}\sqrt{3}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\therefore x = 3 \pm \frac{3}{2}\sqrt{3}$ and $y = \frac{3}{2} \mp 3\sqrt{3}$ ✓✓ [6]