1

TERMINOLOGY

chain rule
composite function
Euler's number
exponential decay
exponential growth
product rule
quotient rule

FURTHER DIFFERENTIATION AND APPLICATIONS

DERIVATIVES, EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

Prior learning

EXPONENTIAL FUNCTIONS

- estimate the limit of $\frac{a^h - 1}{h}$ as $h \to 0$ using technology, for various values of $a > 0$ (ACMMM098)
- recognise that e is the unique number a for which the above limit is 1 (ACMMM099)
- establish and use the formula $\frac{d}{dx}(e^x) = e^x$ (ACMMM100)
- use exponential functions and their derivatives to solve practical problems (ACMMM101)

TRIGONOMETRIC FUNCTIONS

- establish the formulas $\frac{d}{dx}(\sin x) = \cos x$, and $\frac{d}{dx}(\cos x) = -\sin x$ by numerical estimations of the limits and informal proofs based on geometric constructions (ACMMM102)
- use trigonometric functions and their derivatives to solve practical problems (ACMMM103)

DIFFERENTIATION RULES

- understand and use the product and quotient rules (ACMMM104)
- understand the notion of composition of functions and use the chain rule for determining the derivatives of composite functions (ACMMM105)
- apply the product, quotient and chain rule to differentiate functions such as xe^x, $\tan x$, $\frac{1}{x^n}$, $x \sin x$, $e^{-x}\sin x$ and $f(ax + b)$ (ACMMM106) AC

1.01 THE PRODUCT AND QUOTIENT RULES

In Year 11 you learnt how to differentiate, using first principles and shorter rules.

IMPORTANT

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}(x^n) = nx^{n-1} \text{ for } n \in \mathbf{N}$$

$$\frac{d}{dx}(k) = 0 \text{ where } k \text{ is a constant}$$

$$\frac{d}{dx}(kx^n) = knx^{n-1} \text{ where } k \text{ is a constant}$$

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

The derivative $\frac{dy}{dx}$ is the gradient function of the curve $y = f(x)$.

A linear function through (x_1, y_1) with gradient m is given by $y - y_1 = m(x - x_1)$.

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The formula $y - y_1 = m(x - x_1)$ can be used to find the equation of a tangent at a point, as the gradient is given by $m = \frac{dy}{dx}$.

WS Basic differentiation

In this section you will learn some more differentiation rules. The derivative of a product of functions $f(x)\,g(x)$ can be found as follows.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{[f(x+h)-f(x)]g(x+h)+f(x)[g(x+h)-g(x)]}{h}\\
&= \lim_{h\to 0}\frac{[f(x+h)-f(x)]g(x+h)}{h}+\frac{f(x)[g(x+h)-g(x)]}{h}\\
&= \lim_{h\to 0}\frac{[f(x+h)-f(x)]g(x+h)}{h}+\lim_{h\to 0}\frac{f(x)[g(x+h)-g(x)]}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\times\lim_{h\to 0}g(x+h)+\lim_{h\to 0}f(x)\times\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)g(x)+f(x)g'(x)
\end{aligned}$$

It is easier to remember the **product rule** by using $u = f(x)$ and $v = g(x)$ and writing $v' = \frac{dv}{dx}$ and $u' = \frac{du}{dx}$.

IMPORTANT

The **product rule** for functions u and v can be written as $\frac{d}{dx}(uv) = u'v + uv'$.

The example below shows how you can use the product rule to avoid expanding brackets.

Example 1

a Find the derivative of $3x^2(x - 5)$.

b For the function $f(x) = (3x^4 - 2x + 7)(x^5 + 2x^2 + x - 1)$, find $f'(-2)$.

Solution

a Write the function as a product.	Let $y = 3x^2(x - 5) = uv$, where $u = 3x^2$ and $v = x - 5$
Find the derivatives.	$u' = 6x$ and $v' = 1$
Use $\frac{d}{dx}(uv) = u'v + uv'$.	$\frac{dy}{dx} = 6x \times (x-5) + 3x^2 \times 1$
Simplify.	$= 6x^2 - 30x + 3x^2$ $= 9x^2 - 30x$
Write the answer.	$\frac{d}{dx}[3x^2(x-5)] = 9x^2 - 30x$

b Write the function as a product. | Let $f(x) = uv$, where $u = 3x^4 - 2x + 7$ and $v = x^5 + 2x^2 + x - 1$

Differentiate. | $u' = 12x^3 - 2$ and $v' = 5x^4 + 4x + 1$

Write the product rule. | $f'(x) = u'v + uv'$

Substitute the functions. | $= (12x^3 - 2)(x^5 + 2x^2 + x - 1) + (3x^4 - 2x + 7)(5x^4 + 4x + 1)$

Substitute the value. | $f'(-2) = [12(-2)^3 - 2][(-2)^5 + 2(-2)^2 + (-2) - 1] + [3(-2)^4 - 2(-2) + 7][5(-2)^4 + 4(-2) + 1]$

Simplify. | $= 6953$

In part **b** above, you could simplify the derivative by multiplying out the brackets, but that wouldn't have made it any quicker to find $f'(-2)$.

WS
The product rule

You can work out the derivative of $f(x) = \dfrac{u(x)}{v(x)}$ as follows.

$$\frac{df}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\text{Now } f(x+h)-f(x) = \frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}$$

$$= \frac{u(x+h)v(x)-u(x)v(x+h)}{v(x+h)v(x)}$$

$$\text{So } \frac{df}{dx} = \lim_{h\to 0}\frac{u(x+h)v(x)-u(x)v(x+h)}{v(x+h)v(x)h}$$

$$= \lim_{h\to 0}\frac{u(x+h)v(x)-u(x)v(x+h)}{h} \times \lim_{h\to 0}\frac{1}{v(x+h)v(x)}$$

$$= \left[\lim_{h\to 0}\frac{u(x+h)v(x)-u(x)v(x)+u(x)v(x)-u(x)v(x+h)}{h}\right] \times \frac{1}{v(x)v(x)}$$

$$= \frac{1}{[v(x)]^2} \times \lim_{h\to 0}\frac{[u(x+h)-u(x)]v(x)-u(x)[v(x+h)-v(x)]}{h}$$

$$= \frac{1}{[v(x)]^2} \times \left[\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} \times v(x) - u(x) \times \lim_{h\to 0}\frac{v(x+h)-v(x)}{h}\right]$$

$$= \frac{1}{[v(x)]^2}[u'(x)v(x)-u(x)v'(x)]$$

$$= \frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}$$

This is very similar to the product rule.

IMPORTANT

The **quotient rule** for functions u and v can be written as $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{u'v - uv'}{v^2}$.

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Example 2

Find the derivative of $\dfrac{3x-5}{5x+2}$.

Solution

Write as a quotient.	Let $y=\dfrac{u}{v}$, where $u=3x-5$ and $v=5x+2$.
Find the derivatives.	$u'=3$ and $v'=5$
Write the quotient rule.	$\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{u'v-uv'}{v^2}$
Substitute the functions.	$=\dfrac{3\times(5x+2)-(3x-5)\times 5}{(5x+2)^2}$
Multiply out the brackets.	$=\dfrac{15x+6-15x+25}{(5x+2)^2}$
Simplify and write the answer.	$\dfrac{d}{dx}\left[\dfrac{3x-5}{5x+2}\right]=\dfrac{31}{(5x+2)^2}$

Sometimes you will need to differentiate functions with negative or fractional powers. Remember these definitions.

You can find the derivative of x^{-n} using the quotient rule.

IMPORTANT

$$x^{-n}=\frac{1}{x^n}$$

$$x^{\frac{1}{n}}=\sqrt[n]{x}$$

$$x^{\frac{m}{n}}=\sqrt[n]{x^m}=\left(\sqrt[n]{x}\right)^m$$

WS The quotient rule

Write as a quotient.	$x^{-n}=\dfrac{1}{x^n}=\dfrac{u}{v}$, where $u=1$ and $v=x^n$
Find the derivatives.	$u'=0$ and $v'=nx^{n-1}$
Write the quotient rule.	$\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{u'v-uv'}{v^2}$
Substitute the functions.	$\dfrac{d}{dx}(x^{-n})=\dfrac{0\times x^n-nx^{n-1}\times 1}{(x^n)^2}$
Simplify.	$=\dfrac{-nx^{n-1}}{x^{2n}}$
Express as a single power.	$=-nx^{-n-1}$

So the derivative of x^{-n} is $-nx^{-n-1}$. This means that the general rule $\dfrac{d}{dx}(x^n)=nx^{n-1}$ works when n is *any* integer (both positive and negative numbers).

This rule actually works for any real numbers, although the proof is beyond the scope of this course. You can see it for fractions like the derivative of $\sqrt{x} = x^{\frac{1}{2}}$.

It is easiest if you change $\sqrt{x+h} - \sqrt{x}$ to another form using the difference of squares.

$$\begin{aligned}
\sqrt{x+h} - \sqrt{x} &= \left(\sqrt{x+h} - \sqrt{x}\right) \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{x+h-x}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{h}{\sqrt{x+h} + \sqrt{x}}
\end{aligned}$$

Now

$$\begin{aligned}
\frac{d}{dx}\left(x^{\frac{1}{2}}\right) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \times \frac{h}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}} \\
&= \frac{1}{2} x^{-\frac{1}{2}}
\end{aligned}$$

So the derivative of $x^{\frac{1}{2}}$ is $\frac{1}{2} x^{\frac{1}{2}-1}$, which is the same rule again.

Example 3

Find the derivatives of each of the following.

a x^{-7}

b $x^{\frac{1}{5}}$.

c For $f(x) = \frac{1}{x^2}$, find $f'(5)$.

d If $g(x) = \sqrt[3]{x}$, find $g(64)$.

Solution

a Use $\frac{d}{dx}(x^n) = nx^{n-1}$.

$$\frac{d}{dx} x^{-7} = -7x^{-7-1}$$

Simplify.

$$= -7x^{-8}$$

b Use $\frac{d}{dx}(x^n) = nx^{n-1}$.

$$\frac{d}{dx}\left(x^{\frac{1}{5}}\right) = \frac{1}{5} x^{\frac{1}{5}-1} = \frac{1}{5} x^{-\frac{4}{5}}$$

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c Change $\frac{1}{x^2}$ to index form. $f(x) = \frac{1}{x^2} = x^{-2}$

Now use $\frac{d}{dx}(x^n) = nx^{n-1}$.

$$\frac{df}{dx} = -2x^{-2-1}$$
$$= -2x^{-3}$$

Substitute the value.

$$f'(5) = -2 \times 5^{-3}$$
$$= -\frac{2}{125}$$

d Change $\sqrt[3]{x}$ into index form. $g(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

Differentiate using $\frac{d}{dx}(x^n) = nx^{n-1}$.

$$g'(x) = \tfrac{1}{3}x^{\frac{1}{3}-1} = \tfrac{1}{3}x^{-\frac{2}{3}}$$

Substitute the value.

$$g'(64) = \tfrac{1}{3} \times 64^{-\frac{2}{3}}$$

Write without the fractional index.

$$= \tfrac{1}{3} \times \frac{1}{64^{\frac{2}{3}}}$$

Write in surd form.

$$= \frac{1}{3\left(\sqrt[3]{64}\right)^2}$$

Simplify.

$$= \frac{1}{48}$$

EXERCISE 1.01 The product and quotient rules

Concepts and techniques

1 Example 1 Differentiate the following.

a $x^3(2x+3)$
b $(3x-2)(2x+1)$
c $3x(5x+7)$
d $4x^4(3x^2-1)$
e $2x(3x^4-x)$

2 Find the derivative of

a $(3x-7)(2x+5)$
b $(5x+2)(x^2-1)$
c $(5x-3)(x^3+2x-1)$
d $(x^2+7)(x^2-x-1)$
e $(x^3+3)(x^4+2x^3-5x^2+x-2)$

3 Example 2 Find the derivatives of the following.

a $\frac{1}{2x-1}$
b $\frac{3x}{x+5}$
c $\frac{x^3}{x^2-4}$
d $\frac{x-3}{5x+1}$
e $\frac{x-7}{x^2}$
f $\frac{5x+4}{x+3}$
g $\frac{x}{2x^2-x}$
h $\frac{x+4}{x-2}$
i $\frac{2x+7}{4x-3}$
j $\frac{x+5}{3x+1}$

4 Example 3 Find the derivatives of the following.

a x^{-4}　b x^{-8}　c $2x^{-3}$　d $5x^{-11}$　e $\frac{x^{-9}}{5}$

f $x^{\frac{1}{2}}$　g $x^{\frac{1}{4}}$　h $3x^{\frac{1}{7}}$　i $5x^{\frac{2}{3}}$　j $2x^{-\frac{1}{2}}$

5 Differentiate

a $\frac{1}{x^5}$　b $\frac{1}{x^6}$　c $\frac{2}{x^3}$　d $\frac{1}{3x^2}$　e $\frac{4}{5x}$

f $\sqrt{x}$　g $\sqrt[6]{x}$　h $4\sqrt[3]{x}$　i $\sqrt[3]{x^2}$　j $\sqrt{x^5}$

6 Find each of the following.

a $f'(4)$ if $f(x) = \frac{1}{x}$　b $g'(-2)$ if $g(x) = \frac{x^2-5}{x+3}$

c $\left.\frac{dy}{dx}\right|_{x=3}$ if $y = (2x^2 + 3x - 5)(x^3 - x^2 + 8)$　d $p'(8)$ if $p(t) = 4t^{\frac{5}{3}} - t^{-\frac{4}{3}}$

e $h'(\frac{1}{2})$ for $h(y) = y^{-5}$

7 Find the gradient of the tangent to the curve

a $y = x^2(3x + 2)$ at the point where $x = 4$

b $y = \frac{1}{x}$ where $x = 3$

c $y = \frac{2x+1}{x-2}$ at the point $(1, -3)$

Reasoning and communication

8 For what values of x is the derivative of each of the following positive?

a $\frac{1}{x^2}$　b $x - \sqrt[3]{x}$　c $-6x^{-4}$　d $\frac{x+3}{x^2-5}$　e x^{-3}

9 The curve $y = x^2(3x - 2)$ has two tangents with gradient 5 at points M and N. Find the coordinates of these points.

10 Find any x-values on the curve $y = \frac{2x-1}{x+3}$, where the gradient of the tangent is $\frac{7}{25}$.

11 Find the equation of the tangent to the curve $y = \frac{x^2-1}{x+3}$ at the point where $x = 2$.

12 In economics, the marginal cost of production is the additional cost of a small increase in production. It is the rate of change of the cost function (i.e. cost per item) as production increases. The cost function for the production of engines is given by

$$C(x) = (x + 1)(0.04x^2 - 10x + 20)^2$$

where the cost $C(x)$ of producing x engines is given in dollars.

a Find the cost of producing 20 engines.

b Find the marginal cost if

i 20 engines are made　ii 50 engines are made.

13 a Show that $\left(x^{\frac{1}{4}} - y^{\frac{1}{4}}\right)\left(x^{\frac{3}{4}} + x^{\frac{1}{2}}y^{\frac{1}{4}} + x^{\frac{1}{4}}y^{\frac{1}{2}} + y^{\frac{3}{4}}\right) = x - y$.

b Hence show that $\frac{d}{dx}x^{\frac{1}{4}} = -\frac{1}{4}x^{-\frac{3}{4}}$.

14 Show that $(x - y)(x^2 + xy + y^2) = x^3 - y^3$ and hence that the derivative of $\sqrt[3]{x}$ is $\frac{1}{3\sqrt[3]{x^2}}$.

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1.02 THE CHAIN RULE

A **composite function** is a function of another function, like $f[g(x)]$. For example, if $f(x) = x^3$ and $g(x) = x^5$, then the composite function $f[g(x)] = f(x^5) = (x^5)^3 = x^{15}$, where $g = x^5$.

Write $h(x) = x^{15}$. Then $h(x) = f[g(x)]$ and clearly $h'(x) = 15x^{14}$.

Now $f'(x) = 3x^2$, so $f'(g) = 3(x^5)^2 = 3x^{10}$

and $g'(x) = 5x^4$

To get $h'(x) = 15x^{14}$, from $f'(g)$ and $g'(x)$ you have to multiply

$$15x^{14} = 3x^{10} \times 5x^4 = f'(g) \times g'(x)$$

This applies to *all* composite functions.

IMPORTANT

The **chain rule** states that $\frac{d}{dx}\{f[g(x)]\} = f'(g)g'(x)$, where $g = g(x)$.

You can prove this is true for all composite functions as follows.

$$\begin{aligned}
\frac{d}{dx} f[g(x)] &= \lim_{h\to 0} \frac{f[g(x+h)] - f[g(x)]}{h} \\
&= \lim_{h\to 0}\left(\frac{f[g(x+h)] - f[g(x)]}{h} \times \frac{g(x+h)-g(x)}{g(x+h)-g(x)}\right) \\
&= \lim_{h\to 0}\left(\frac{f[g(x+h)] - f[g(x)]}{g(x+h)-g(x)} \times \frac{g(x+h)-g(x)}{h}\right) \\
&= \lim_{h\to 0} \frac{f[g(x+h)] - f[g(x)]}{g(x+h)-g(x)} \times \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}
\end{aligned}$$

Now write $H = g(x + h) - g(x)$.

Then $g(x + h) = g(x) + H$.

As $h \to 0$, $g(x + h) \to g(x)$, so $g(x + h) - g(x) \to 0$, so $H \to 0$.

Substitution gives

$$\begin{aligned}
\frac{d}{dx} f[g(x)] &= \lim_{H\to 0} \frac{f(g+H) - f(g)}{H} \times \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(g) \times g'(x)
\end{aligned}$$

There are other ways to write the chain rule, and some teachers and students prefer one of these forms.

IMPORTANT

The **chain rule** can also be written as:

If $f(x) = v(u(x))$, then $f'(x) = v'(u)u(x)$, or

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example 4

a Find the derivatives of

i $(x^3 - 1)^5$ ii $y = (2x^3 + x + 7)^2$ iii $f(x) = 3(5x^2 - 4x + 2)^8$

b Find the equation of the tangent to the curve $y = (2x - 1)^6$ at the point (1, 1).

Solution

a i	Write as a function of a function.	Let $y = f[g(x)]$, where $f(x) = x^5$ and $g(x) = x^3 - 1$
	Find the derivatives.	$f'(x) = 5x^4$ and $g'(x) = 3x^2$
	Write the chain rule.	$\frac{dy}{dx} = f'(g) \times g'(x)$
	Substitute the derivatives.	$= 5g^4 \times 3x^2$
	Now substitute $g(x)$.	$= 5(x^3 - 1)^4 \times 3x^2$
	Write in the usual way.	$= 15x^2(x^3 - 1)^4$
ii	Write as a function of a function.	Let $y = y[u(x)]$, where $y(u) = u^2$ and $u(x) = 2x^3 + x + 7$
	Find the derivatives.	$\frac{dy}{du} = 2u$ and $\frac{du}{dx} = 3x^2 + 1$
	Write the rule.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
	Substitute the derivatives.	$= 2u \times (3x^2 + 1)$
	Now substitute u.	$= 2(2x^3 + x + 7)\ (6x^2 + 1)$
iii	Write as a function of a function.	Let $f(x) = v[u(x)]$, where $v(u) = 3u^8$ and $u(x) = 5x^2 - 4x + 2$
	Find the derivatives.	$v'(u) = 24u^7$ and $u(x) = 10x - 4$
	Write the chain rule.	$f'(x) = v'(u)u'(x)$
	Substitute the derivatives.	$= 24u^7 \times (10x - 4)$
	Now substitute u.	$= 24(5x^2 - 4x + 2)^7(10x - 4)$
	Take out the common factor.	$= 48(5x^2 - 4x + 2)^7(5x - 2)$

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b Write as a function of a function. Let $y = y[u(x)]$, where $y(u) = u^6$ and $u(x) = 2x - 1$

Find the derivatives. $\frac{dy}{du} = 6u^5$ and $\frac{du}{dx} = 2$

Write the chain rule. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Substitute the derivatives. $= 6u^5 \times 2$

Now substitute u. $= 12(2x - 1)^5$

Find the slope of the tangent. $m = \left.\frac{dy}{dx}\right|_{x=1} = 12 \times (2 \times 1 - 1)^5 = 12$

Check that the point is on the line. At $x = 1$, $y = (2 \times 1 - 1)^6 = 1$

Write the rule for the equation. $y - y_1 = m(x - x_1)$

Substitute values. $y - 1 = 12(x - 1)$

Simplify and write in standard form. $12x - y - 11 = 0$

State the answer. The equation of the tangent to $y = (2x - 1)^6$ at $(1, 1)$ is $12x - y - 11 = 0$.

After a while, you will be able to abbreviate the chain rule substitutions. Many problems will combine other formulas with the chain rule.

Example 5

Differentiate the following using the chain rule.

a $\frac{1}{(3x+1)^4}$ **b** $\sqrt{5x-4}$

Solution

a Write as a function of a function in index form. Let $y = \frac{1}{(3x+1)^4} = (3x + 1)^{-4} = u^{-4}$, where $u = 3x + 1$

Write the chain rule. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Substitute the derivatives. $= -4u^{-5} \times 3$

Substitute u. $= -12(3x + 1)^{-5}$

Write in the original form. $\frac{d}{dx}\left(\frac{1}{(3x+1)^4}\right) = -\frac{12}{(3x+1)^5}$

b Write as a function of a function. Let $y = \sqrt{5x-4} = u^{\frac{1}{2}}$, where $u = 5x - 4$

Write the chain rule. $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Substitute the derivatives. $= \frac{1}{2}u^{-\frac{1}{2}} \times 5$

Substitute u. $= \frac{5}{2}(5x-4)^{-\frac{1}{2}}$

Write in the original form. $\dfrac{d}{dx}\left(\sqrt{5x-4}\right) = \dfrac{5}{2\sqrt{5x-4}}$

You may have to use the chain rule and the product or quotient rules in some problems.

WS The chain rule

Example 6

Differentiate $2x^5(5x+3)^3$.

Solution

Write as a product.	Let $y = uv$, where $u = 2x^5$ and $v = (5x+3)^3$
Find the derivative of u.	$u' = 10x^4$
Write v as a function of a function.	Let $v(x) = p[q(x)]$, where $p(q) = q^3$ and $q(x) = 5x + 3$
Write the chain rule.	$v' = \dfrac{dp}{dx} = \dfrac{dp}{dq} \times \dfrac{dq}{dx}$
Substitute the derivatives.	$= 3q^2 \times 5$
Substitute q.	$= 15(5x+3)^2$
Write the product rule.	$y' = u'v + uv'$
Substitute the functions.	$= 10x^4(5x+3)^3 + 2x^5 \times 15(5x+3)^2$
Take out the common factor.	$= 10x^4(5x+3)^2[(5x+3) + 3x]$
Simplify and write the answer.	The derivative of $2x^5(5x+3)^3$ is $10x^4(5x+3)^2(8x+3)$.

Notice in Example **6** that if possible, it is usual to leave the derivative in the same form as the question. In this case it was factorised, so it has been left in factor form.

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EXERCISE 1.02 The chain rule

Concepts and techniques

WS Mixed differentiation problems

1 $m(x) = 3x + 3$, $g(x) = x^2$, $p(x) = \sqrt[3]{x}$ and $q(x) = x^2 + 4$

a What are $g(m)$, $m(x)$ and $g[m(x)]$?

b What are $m(g)$, $g(x)$ and $m[g(x)]$?

c Find $q[m(x)]$.

d Find $m[q(x)]$.

e What is $p[q(x)]$?

f What is $q[p(x)]$?

g Calculate the value of $m[p(8)]$.

h What is $p[m(8)]$?

i Find the value of $g[q(3)]$.

j Find the value of $q[g(3)]$.

2 Example 4 Differentiate the following.

a $(x+3)^4$ b $(2x-1)^3$ c $(5x^2-4)^7$

d $(8x+3)^6$ e $(1-x)^5$

3 Find the derivatives of the following.

a $3(5x+9)^9$ b $2(x-4)^2$ c $(2x^3+3x)^4$

d $(x^2+5x-1)^8$ e $(x^6-2x^2+3)^6$

4 Example 5 Differentiate the following.

a $(3x-1)^{\frac{1}{2}}$ b $(4-x)^{-2}$ c $(x^2-9)^{-3}$

d $(5x+4)^{\frac{1}{3}}$ e $(x^3-7x^2+x)^{\frac{3}{4}}$

5 Find the derivatives of the following.

a $\sqrt{3x+4}$ b $\dfrac{1}{5x-2}$ c $\dfrac{1}{(x^2+1)^4}$

d $\sqrt[3]{(7-3x)^2}$ e $\dfrac{5}{\sqrt{4+x}}$

6 Example 6 Find the derivatives of the following.

a $x^2(x+1)^3$ b $4x(3x-2)^5$ c $3x^4(4-x)^3$

d $(x+1)(2x+5)^4$ e $(x^3+5x^2-3)(x^2+1)^5$

7 Differentiate the following.

a $\dfrac{(2x-9)^3}{5x+1}$ b $\dfrac{x-1}{(7x+2)^4}$ c $\dfrac{(3x+4)^5}{(2x-5)^3}$

d $\dfrac{3x+1}{\sqrt{x+1}}$ e $\dfrac{\sqrt{x-1}}{2x-3}$

8 a If $g(x) = (3x+5)^3(2x-1)^2$, find $g'(-2)$.

b If $f(t) = \dfrac{3t^2-2t+1}{(4t-5)^3}$, find $f'(3)$.

c If $p(x) = \sqrt[3]{(4x-7)^5}$, find $p'(0.5)$.

d If $h(z) = (3z-1)^4\left(\sqrt[3]{2z-3}\right)^2$, find $h'(-1)$.

e If $m(x) = \dfrac{1}{(3x^2-2x-4)^3}$, find $m'(1)$.

Reasoning and communication

9 Find any points on the curve $y = (2x - 3)^4$ where the gradient of the tangent is -8.

10 The volume of liquid in an underground tank being gravity-filled from a tanker is given by $V = (1500t + 17t^2)^3$, where the volume V is in litres and time t is in minutes.

a What is the volume of the tank after half an hour?

b What is the rate at which the tank is being filled after 5 minutes?

c What is the flow rate in the pipe after half an hour?

(Write all answers in scientific notation correct to 1 decimal place)

Getty Images/Justin Sullivan

11 Consider the function $q(x) = \sqrt{x-4}$.

a What is the domain of $q(x)$?

b Find the gradient of the tangent at $x = 13$

c Find the equation of the tangent at $x = 13$

d Find the gradient of the tangent at the point A, where $x = a$

e Find the gradient of the line from A to the origin.

f Hence find the point A on the curve such that the tangent passes through the origin.

12 Consider the function $y = \dfrac{1}{x+2} + 2$.

a Find the gradient of the tangent at $x = -2.5$

b Find the gradient of the tangent at $x = -1.5$

c What can you say about the tangents at $x = -2.5$ and $x = -1.5$?

d Find the gradient of the tangent at $x = -2.2$

e Find the gradient of the tangent at $x = -1.8$

f What can you say about the tangents at $x = -2.2$ and $x = -1.8$?

g Find the gradient of the tangent at $x = -3$

h Find the gradient of the tangent at $x = -1$

i What can you say about the tangents at $x = -3$ and $x = -1$?

j Write a general expression for the x-coordinates of the points on this curve that have parallel tangents.

13 A normal is perpendicular to a curve. Show that the normal to $y = (x + 2)^3$ at the point $(-3, -1)$ has y-intercept -2.

14 Show that the normals to the curve $y = 4x^2$ from points the same distance on either side of the y-axis intersect on the y-axis.

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1.03 THE DERIVATIVE OF EXPONENTIAL FUNCTIONS

You can find the derivative of $f(x) = a^x$ using first principles as follows.

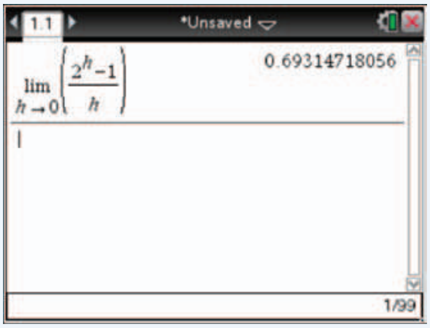
$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h\to 0} \frac{a^x \times a^h - a^x}{h} \\
&= \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} \\
&= a^x \lim_{h\to 0} \frac{a^h - 1}{h}
\end{aligned}$$

The derivative exists if $\lim\limits_{h\to 0} \frac{a^h - 1}{h}$ exists. In that case, the derivate of a^x is just a constant multiplied by the function itself!

INVESTIGATION Derivatives of a^x

1 You can estimate $\lim\limits_{h\to 0} \frac{a^h - 1}{h}$ for $a = 2$ using your CAS calculator as follows. Make sure that your calculator is set to give decimal answers (approximate calculation mode for the TI-Nspire Document settings).

TI-Nspire CAS



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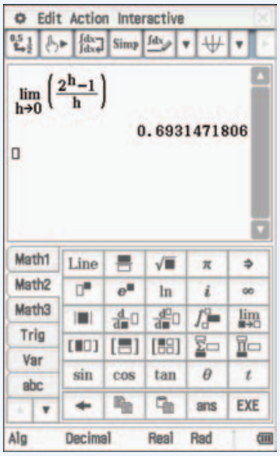
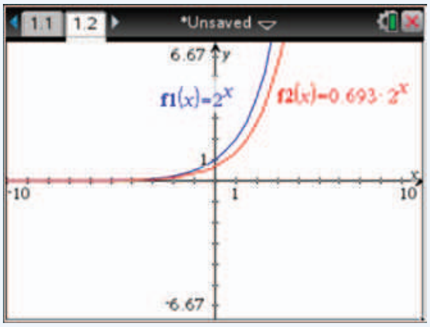


Estimate the values of $\lim_{h\to 0}\dfrac{a^h-1}{h}$ for $a = 0.5, 0.8, 1.5, 2, 2.5, 3, 3.5, 4$ and 5.

2 Use these estimates to find approximations to the derivative $\dfrac{d}{dx}(a^x) = a^x \lim_{h\to 0}\dfrac{a^h-1}{h}$ for the different values of a.

3 Using some of the results from the investigation, draw the graphs of exponential functions and their derivatives on your CAS calculator. For example, the graph of $f(x) = 2^x$ and its derivative $f'(x) \approx 0.693 \times 2^x$ looks like this.

TI-Nspire CAS



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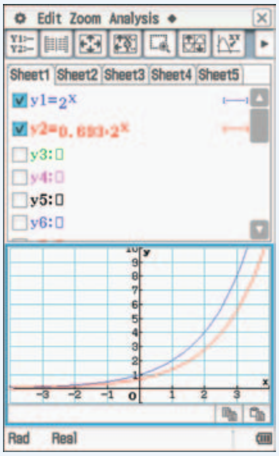

4 What happens to the value of $\lim_{h\to 0}\dfrac{a^h-1}{h}$ as a increases?

From your investigation, you should see that there must be a number close to 3 for which $\lim_{h\to 0}\dfrac{a^h-1}{h} = 1$. This number was alluded to by John Napier (1550–1617), the inventor of logarithms and was proven to be irrational by Leonard Euler (1707–1783). The number is now called e in honour of Euler and is called **Napier's number** or **Euler's number**. It is one of the most important numbers in mathematics.

IMPORTANT

The **exponential function** is the function $y = e^x$, where e is such that $\lim_{h\to 0}\dfrac{e^h-1}{h} = 1$.

$e \approx 2.718\,281\,828\,459\,045\,235$, but cannot be written exactly.

The exponential function is its own derivative, so $\dfrac{d}{dx}(e^x) = e^x$.

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You can prove the derivative property as follows.

$$\begin{aligned}\frac{d}{dx}(e^x) &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x\times e^h-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x(e^h-1)}{h}\\ &= e^x\lim_{h\to 0}\frac{(e^h-1)}{h}\\ &= e^x \qquad \text{because } \lim_{h\to 0}\frac{(e^h-1)}{h}=1\end{aligned}$$

You can find the value of e as accurately as you like using $e=\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots$

Example 7

a CAS Evaluate $3e^2+5$ to 2 decimal places.

b CAS Draw the graph of $y=e^x$.

c Find the exact equation of the tangent to the curve $y=e^x$ at the point $(2, e^2)$.

Solution

a Make sure that your calculator is set to decimal (approximate) calculation. Use e^x or e^n.

TI-Nspire CAS

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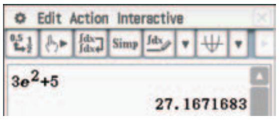
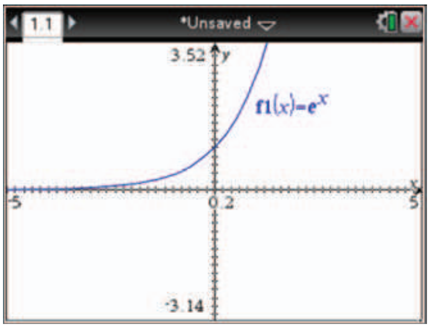


Write the answer. $3e^2+5\approx 27.17$ (correct to 2 decimal places).

b Use e^x or e^n for the graph.

TI-Nspire CAS



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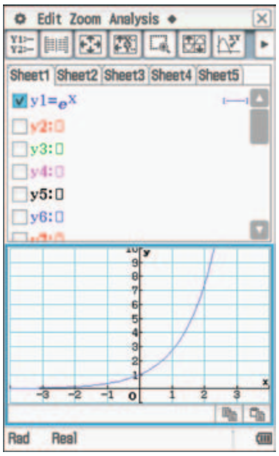

c Find the slope.	$\frac{dy}{dx} = e^x$, so $m = e^2$
Write the equation of the line.	$y - y_1 = m(x - x_1)$
Substitute values.	$y - e^2 = e^2\,(x - 2)$
Express in standard form.	The equation of the tangent is $e^2x - y - e^2 = 0$.

You can differentiate more complex functions involving the exponential function using some rules that have already been covered.

Example 8

Find the derivatives of the following.

a $4x^3 + 5e^x$

b $y = e^{x^3+x-2}$

c $f(x) = (3 + e^x)^7$

Solution

a Use the linear sum.	$\frac{d}{dx}\left(4x^3 + 5e^x\right) = 4\frac{d}{dx}\left(x^3\right) + 5\frac{d}{dx}\left(e^x\right)$
Insert the derivatives and simplify.	$= 12x^2 + 5e^x$
b Write as a function of a function.	$y = e^u$, where $u(x) = x^3 + x - 2$
Write the chain rule.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
Substitute the derivatives.	$= e^u \times (3x^2 + 1)$
Substitute u.	$= e^{x^3+x-2}\,(3x^2 + 1)$
c Write as a function of a function.	$f(x) = v[u(x)]$, where $v(u) = u^7$ and $u(x) = 3 + e^x$
Write the derivatives.	$v'(u) = 7u^6$ and $u'(x) = e^x$
Write the chain rule.	$f'(x) = v'(u)u'(x)$
Substitute the derivatives.	$= 7u^6 \times e^x$
Substitute u.	$= 7e^x(3 + e^x)^6$

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Example 9

Find the derivatives of the following.

a $x^2 e^x$ **b** $\dfrac{e^{2x}}{x}$

Solution

a Write as a product.	Let $f(x) = u(x)v(x)$, where $u(x) = x^2$ and $v(x) = e^x$
Write the derivatives.	$u'(x) = 2x$ and $v'(x) = e^x$
Write the product rule.	$f'(x) = u'v + uv'$
Substitute in the functions.	$= 2x \times e^x + x^2 \times e^x$
Factorise and write in normal order.	$= xe^x(x+2)$
Write the answer.	The derivative of $x^2 e^x$ is $xe^x(x+2)$.
b Write as a quotient.	Let $f(x) = \dfrac{u(x)}{v(x)}$, where $u(x) = e^{2x}$ and $v(x) = x$
Write $u(x)$ as a function of a function.	Let $u(x) = p[q(x)]$, where $p(q) = e^q$ and $q(x) = 2x$
Write the chain rule.	$u'(x) = p'(q) \times q'(x)$
Substitute in the derivatives.	$= e^q \times 2$
Substitute in q.	$= 2e^{2x}$
Write the derivative of v.	$v'(x) = 1$
Write the quotient rule.	$f'(x) = \dfrac{u'v - uv'}{v^2}$
Substitute in the functions.	$= \dfrac{2e^{2x} \times x - e^{2x} \times 1}{x^2}$
Factorise the numerator.	$= \dfrac{e^{2x}(2x-1)}{x^2}$
Write the answer.	The derivative of $\dfrac{e^{2x}}{x}$ is $\dfrac{e^{2x}(2x-1)}{x^2}$.

You will find after a while that you can skip many of the steps in the product, quotient and chain rules. However, it is still important to write enough so that your working is clear. If you are differentiating a function where you need to apply many rules, it is best to use 'dummy functions' like those in part **b** above to avoid errors. You will also find that you save time by doing this.

EXERCISE 1.03 The derivative of exponential functions

Concepts and techniques

1 Example 7 CAS Evaluate the following expressions, correct to 2 decimal places.

a $e^{1.5}$ b e^{-2} c $2e^{0.3}$ d $\frac{1}{e^3}$ e $-3e^{-3.1}$

2 CAS Draw these curves.

a $y = 2e^x$ b $y = e^{-x}$ c $y = -e^x$

3 a Find the exact gradient of the tangent to the curve $y = e^x$ at the point $(1, e)$.

b Find the gradient of the tangent to the curve $y = e^x$ at the point where $x = 0.58$, correct to 2 decimal places.

c Find the equation of the tangent to the curve $y = e^x$ at the point $\left(-1, \frac{1}{e}\right)$.

4 Example 8 Differentiate the following.

a $9e^x$ b $-e^x$ c $e^x + x^2$ d $2x^3 - 3x^2 + 5x - e^x$

e $(e^x + 1)^3$ f $(5 - e^x)^9$ g $(2e^x - 3)^6$ h $(e^x + x)^4$

5 Find the derivative of

a e^{3x} b e^{2x-1} c $2e^{4x}$ d e^{x^2-1} e $e^{2x^5-3x^3+x-3}$

6 Example 9 Find the derivatives of the following.

a xe^x b $(2x+3)e^x$ c $5x^3 e^x$ d $2x\,e^{3x}$ e $e^{2x}(x^2+x+2)$

7 Differentiate the following.

a $\frac{e^x}{x^2}$ b $\frac{e^{6x}}{3x}$ c $\frac{2e^{5x}}{5x^3}$ d $\frac{x-1}{e^x}$ e $\frac{e^x+1}{e^{2x}}$

8 a Find $g'(3)$ correct to 2 decimal places if $g(x) = \frac{e^x - 4}{\sqrt{e^x + 1}}$

b $y = e^{4x}(x^3 - 3x + 5)$. Find $\left.\frac{dy}{dx}\right|_{x=-1}$

c If $f(x) = \frac{xe^{3x} + 5}{x^2 + e}$, find $f'(2)$ correct to 1 decimal place.

d $h(x) = 5x^2e^{3x} + e^x$. Find $h'(2)$.

e If $y = 5e^{x^2-x-6}$, find the values of y' when $x = 1$ and $x = 3$.

Reasoning and communication

9 Find the value of x such that the rate of change of xe^{2x-1} is $5e^3$.

10 $p(x) = e^{-kx} + 3x$. For what value of k does the tangent to $p(x)$ at $x = 2$ pass through the origin?

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1.04 APPLICATIONS OF THE EXPONENTIAL FUNCTION AND ITS DERIVATIVE

The exponential function and its derivative have many applications, as there are many natural phenomena that are modelled using simple exponential functions of the form $f(x) = Ae^{kx}$, where A and k are constants. In the case where x is time, you write $f(t) = Ae^{kt}$.

What is the value of $f(t)$ when $t = 0$? What does the graph look like when $k > 0$?

IMPORTANT

For $k > 0$, $Q = Ae^{kt}$ is used to model **exponential growth** over time t, where A is the initial quantity of the variable Q.

Exponential growth is used to model growth in populations, spread of disease and advances in technology, among other things. Since the exponential function is continuous, you need to round answers that have to be whole numbers.

Example 10

The increase in the number of gum trees with pink flowers in a region of South Australia was studied. The equation $N = 1200e^{0.07t}$ was given as a model for the increase, with N as the number of gum trees over time t years.

a How many trees were there at the beginning of the study?

b How many trees were there after 10 years?

c What was the rate of increase in the number of trees after 10 years?

Photo courtesy Margaret Grove

Solution

a Substitute $t = 0$ into $N = 1200e^{0.07t}$.

$$N = 1200e^{0.07 \times 0} = 1200$$

State the result. There were 1200 trees at the beginning.

b Substitute $t = 10$ into $N = 1200e^{0.07t}$.	$N = 1200e^{0.07 \times 10}$ $= 2416.503\ldots$
Round and state the result.	There were about 2417 trees after 10 years.
c The rate of change is the derivative.	$\frac{dN}{dt} = 1200 \times 0.07e^{0.07t}$ $= 84e^{0.07t}$
Substitute $t = 10$.	Rate of change $= 84e^{0.07 \times 10}$ $= 169.155\ldots$
Round and state the result.	The rate of increase after 10 years is about 169 trees/year.

IMPORTANT

For $k < 0$, $Q = Ae^{kt}$ is used to model **exponential decay** over time t, where A is the initial quantity of the variable Q.

Examples of exponential decay include radioactive decay, cooling, leakage and decline in the populations of endangered species.

Example 11

A metal cools down according to the formula $T = T_0e^{-0.1t}$, where T is the temperature difference with the surroundings in °C and t is in minutes. The initial temperature is 228°C and the room is at 20°C.

a Evaluate T_0, the initial temperature difference.

b Find the temperature difference after
  i 5 minutes **ii** 20 minutes.

c What is the temperature after
  i 5 minutes **ii** 20 minutes?

d Find the rate at which the metal is cooling after
  i 5 minutes **ii** 20 minutes.

Solution

a Find the temperature difference with the surroundings.	$T_0 = 228°C - 20°C$ $= 208°C$
b i Substitute $t = 5$.	$T = 208e^{-0.1 \times 5}$ $= 126.158\ldots$
State the result.	The temperature difference is about 126.2°C.
ii Substitute $t = 20$.	$T = 208e^{-0.1 \times 20}$ $= 28.149\ldots$
State the result.	The temperature difference is about 28.1°C.

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c	**i** Add the room temperature.	The temperature after 5 minutes is about 146.2°C.
	ii Add the room temperature.	The temperature after 20 minutes is about 48.1°C.
d	Find the derivative.	$\frac{dT}{dt} = 208 \times (-0.1e^{-0.1t}) = -20.8e^{-0.1t}$
	i Substitute $t = 5$.	Rate of change $= -20.8e^{-0.5}$ $= -12.615\ldots$
	The negative indicates cooling.	The metal is cooling at about 12.6°C /minute.
	ii Substitute $t = 20$.	Rate of change $= -20.8e^{-2}$ $= -2.814\ldots$
	State the result.	The metal is cooling at about 2.8°C /minute.

In the examples above, the rate of change of $Q(t) = Ae^{kt}$ is given by $Q(t) = Ake^{kt} = kQ(t)$. The reverse is also true, but the proof is beyond the scope of this course.

IMPORTANT

If $\frac{dQ}{dt} = kQ$, then $Q(t) = Ae^{kt}$, where A is the quantity when $t = 0$.
This means that if the rate of change of a quantity is proportional to the quantity, then it must be a simple exponential function of time.


Shutterstock.com/Worakit Sirijinda

Example 12

A factory was producing 10 000 units per year and embarked on a productivity campaign by identifying problems in production. As a result, the production per year increased according to the equation $\frac{dP}{dt} = 0.02P$ units/year.

a Find an equation for the rate of production P.

b Find the number of units produced in the 7th year.

c Find the rate of increase after 7 years.

Solution

a Use $\frac{dP}{dt} = 0.02P$.	$\frac{dP}{dt} = 0.02P$ means $P = Ae^{0.02t}$, where A is a constant
The initial quantity is 10 000.	The equation is $P = 10\ 000e^{0.02t}$
b Substitute $t = 7$.	$P = 10\ 000e^{0.02 \times 7}$ $= 11\ 502.737\ldots$
State the result.	About 11 503 units are produced in the 7th year.
c Find the rate of change.	$\frac{dP}{dt} = 10\ 000 \times 0.02e^{0.02t}$ $= 200e^{0.02t}$
Substitute $t = 7$.	Rate of change $= 200e^{0.14}$ $= 230.054\ldots$
State the result.	Rate of increase of production after 7 years is 230 units/year.

In Example **12**, you could find the rate of increase of production after 7 years by substituting the quantity produced in the 7th year in $\frac{dP}{dt} = 0.02P = 0.02 \times 11\ 502 \approx 230$.

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EXERCISE 1.04 Applications of the exponential function and its derivative

Reasoning and communication

1 Example 10 A disease is sweeping Australia according to the formula $N = 175e^{0.062t}$, where N is the number of new cases of the disease and t is the time in weeks.

a How many new cases are there

i initially? ii after 1 week? iii after 6 weeks? iv after 6 months?

b What is the rate at which new cases are found

i after 1 week? ii after 6 weeks? iii after 6 months?

2 An advertising campaign helped increase sales of a product according to the formula $S = 180e^{0.12t}$, where S stands for the number of sales and t is the time in days.

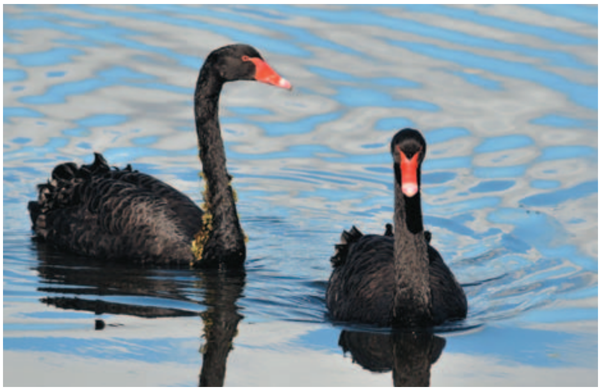
a How many sales were made at the beginning of the advertising campaign?

b How many sales were made after 2 weeks of this campaign?

c What was the rate at which sales were made after 2 weeks?

d The advertising campaign finished after 6 weeks. What was the rate of sales at this time?

3 A study of swans in an area of Western Australia showed that their numbers were gradually increasing, with the number of swans N over t months given by $N = 1100e^{0.025t}$.



iStockphoto/Bbeale

a How many swans were there at the beginning of the study?

b How many swans were there after

i 5 months?

ii a year?

iii 3 years?

c At what rate was the number of swans increasing

i initially? ii after 5 months? iii after a year?

4 Example 11 The mass of uranium decays according to the formula $M = 200e^{-0.012t}$, where M is its mass in grams and t is the time in years.

a What is the initial mass of uranium?

b What is its mass after

i 5 years? ii 20 years? iii 100 years?

c At what rate is the mass decaying after

i 5 years? ii 20 years? iii 100 years?

5 The area of rainforests is declining in a region of Queensland with the area A hectares over time t years given by $A = 120\,000e^{-0.033t}$.

a How many hectares of rainforest are there after

i 10 years? ii 25 years? iii 50 years?

b At what rate is the area of rainforest decreasing in this region after

i 2 years? ii 15 years? iii 40 years?

6 Example 12 The rate at which numbers N of bacteria are increasing over time t hours in a wound is proportional to the number of bacteria present. This can be modelled by the formula $\frac{dN}{dt} = 0.29N$.

a If there are initially 90 000 bacteria present, find a formula that models the number of bacteria present.

b How many bacteria are present after 6 hours?

c At what rate are the numbers increasing after 6 hours?

d What is the rate at which bacteria are increasing after 10 hours?

7 The average annual rainfall R is decreasing over time t years in a region of Tasmania according to the formula $\frac{dR}{dt} = -0.008R$. The first average annual rainfall measured in this region was 43 cm.

a Find a formula for the average annual rainfall in this region.

b Find the average annual rainfall after

i 10 years ii 30 years iii 100 years.

c What is the rate at which the rainfall is decreasing after

i 10 years ii 30 years iii 100 years?

8 The population P of a city after t years is given by the formula $P = P_0e^{0.024t}$.

a What is the initial population?

b By what percentage does the population increase after 6 years?

c Write a formula for the rate at which the population changes in terms of P.

9 The formula for the decay of a radioactive substance over t years is given by $Q = Q_0e^{-0.07t}$.

a What percentage of the original quantity is left after

i 2 years? ii 10 years? iii 20 years?

b The half-life is the time taken to decay to half of the original quantity. What is the approximate half-life of this substance?

10 a Find the equation of the tangent to the curve $y = e^{2x}$ at the point $M(2, e^4)$.

b Find the x-intercept N of this tangent.

c Find the exact area of triangle MNP, where P is the point on the x-axis directly below M.

11 A project is introduced to increase the number of water birds in Macquarie Marshes. The model for the population P of water birds over time t months is given by the formula $P = 100 + 2e^{0.3t}$.

a How many water birds are there at the start of the project?

b What is the population after

i 6 months? ii 2 years?

c Find a formula for the rate at which the population increases over time $\left(\frac{dP}{dt}\right)$.

d At what rate is the water bird population increasing after

i 6 months? ii 2 years?

Photo courtesy Margaret Grove

12 The number N of bacteria in a sample is given by the formula $N = N_0\, e^{1.2t}$, where t is the time in hours.

a If there are initially 30 000 bacteria found in the sample, find the value of N_0.

b Find the number of bacteria in the sample after 5 hours.

c Find the rate at which the number of bacteria is increasing after

i 5 hours ii 12 hours iii 1 day.

1.05 THE DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

You can use your CAS calculator or a computer to estimate the derivatives of sin (x) and cos (x) for different values of x. In the case of derivatives, you must work in radians, so make sure that your calculator is set to radians.

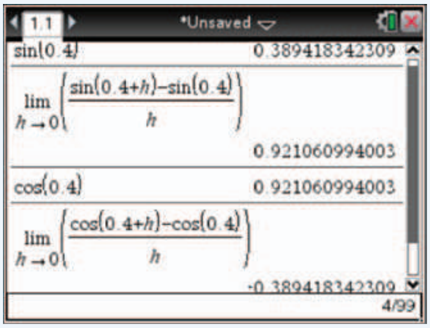
INVESTIGATION Estimating the derivatives of sin (*x*) and cos (*x*)

The derivative of sin (x) is given by $\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$,

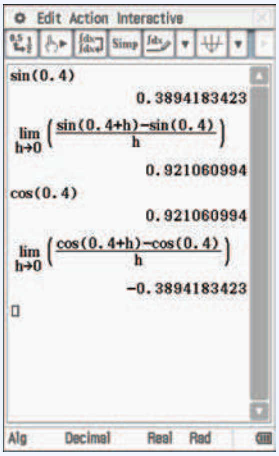
so at $x = 0.4$, the derivative is $\lim_{h \to 0} \frac{\sin(0.4+h) - \sin(0.4)}{h}$.

The derivative of cos (x) is worked out in the same way. Do this on your CAS calculator as follows, with your calculator set to approximate/decimal calculation.

TI-Nspire CAS

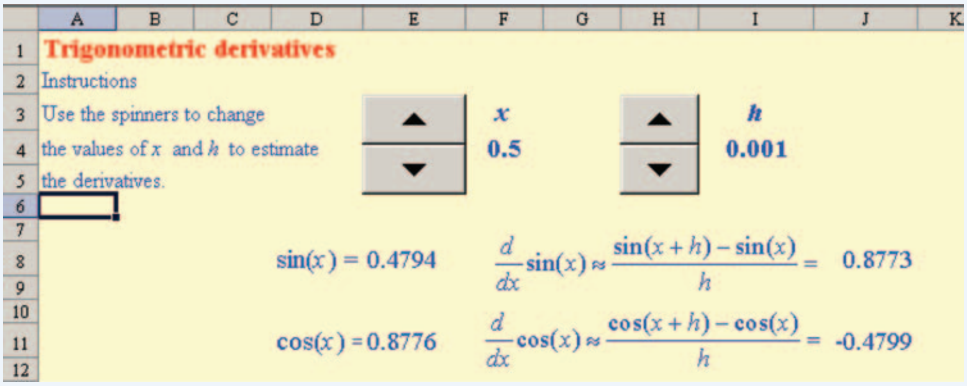


ClassPad



You can also estimate the derivatives using the Spreadsheet 'Trigonometric derivatives' from the website.

Trigonometric derivatives



What do you find about the estimates of the derivatives?

What happens as h is made smaller?

From the investigation, you can see that it appears that the derivative of sin (x) is cos (x) and that the derivative of cos (x) is −sin (x).

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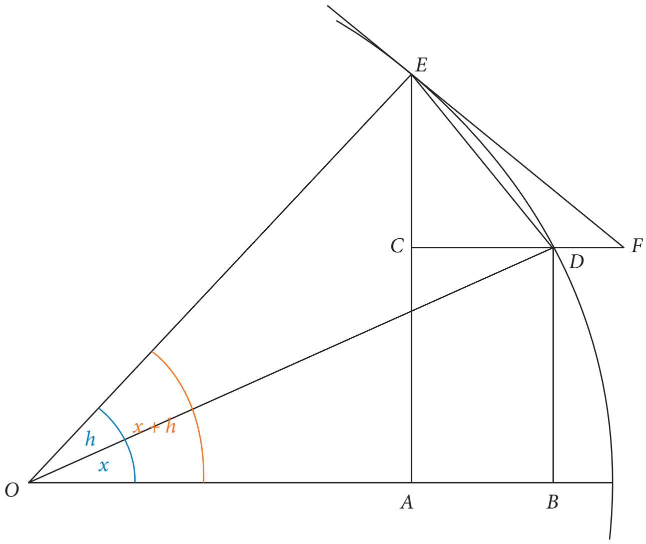
The derivative of sin (x) is given by $\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$. What does $\frac{\sin(x+h)-\sin(x)}{h}$ look like on a unit circle?

The diagram shows part of the unit circle, with angles x and $x+h$ shown.



Perpendiculars AE and BD are drawn from line OAB to radii OE and OD. Line CDF is drawn at 90° to AE to the tangent EF.

Line ED is also drawn.

From the definition of sine, sin $(x+h) = EA$ and sin $(x) = DB = CA$.

Thus, sin $(x+h)$ − sin $(x) = EA - CA = EC$.

From the definition of a radian, the length of the arc ED is h, so

$$\frac{\sin(x+h)-\sin(x)}{h}=\frac{EC}{\text{arc } ED} \qquad [1]$$

Now, in ΔECF, $\angle FEC = x+h$, so

$$\cos(x+h)=\frac{EC}{EF} \qquad [2]$$

Also, ΔOED is isosceles, so $\angle OED = \frac{\pi-h}{2}$. Now $\angle OEA = \frac{\pi}{2}-(x+h)$, so

$\angle CED = \angle OED - \angle OEA$

$$=\frac{\pi-h}{2}-\left[\frac{\pi}{2}-(x+h)\right].$$

$$=x+\frac{h}{2}$$

Thus in ΔCED,

$$\cos\left(x+\frac{h}{2}\right)=\frac{EC}{ED} \qquad [3]$$

From the diagram, it is clear that $ED <$ arc $ED < EF$, so

$$\frac{EC}{EF} < \frac{EC}{\text{arc } EF} < \frac{EC}{ED}.$$

It follows from [1], [2] and [3] that

$$\cos(x+h) < \frac{\sin(x+h)-\sin(x)}{h} < \cos\left(x+\frac{h}{2}\right).$$

In the limit, as $h \to 0$,

$$\lim_{h\to0}\cos(x+h) \le \lim_{h\to0}\frac{\sin(x+h)-\sin(x)}{h} \le \lim_{h\to0}\cos\left(x+\frac{h}{2}\right).$$

But

$$\lim_{h\to0}\cos(x+h) = \cos(x),$$

$$\lim_{h\to0}\cos\left(x+\frac{h}{2}\right) = \cos(x) \quad \text{and}$$

$$\lim_{h\to0}\frac{\sin(x+h)-\sin(x)}{h} = \frac{d}{dx}\sin(x)$$

Thus $\cos(x) \le \frac{d}{dx}\sin(x) \le \cos(x)$, so $\frac{d}{dx}\sin(x) = \cos(x)$.

The diagram is drawn for the first quartile, but it can be drawn in other quartiles and the logic of the proof would be unchanged, so the proof is valid for any value of x.

IMPORTANT

The **derivative of sin (x)** is given by $\frac{d}{dx}\sin(x) = \cos(x)$.

Example 13

a Differentiate

i $3\sin(x)$ ii $f(x) = \sin(5x)$ iii $y = x\sin(x)$ iv $g(x) = [x^3 + \sin(x)]^5$

b Find the exact gradient of the tangent to the curve $y = \sin(2x)$ at the point $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$.

Solution

a i Use the linear product rule.

$$\frac{d}{dx}[3\sin(x)] = 3\frac{d}{dx}[\sin(x)]$$
$$= 3\cos(x)$$

ii Use the chain rule with $u = 5x$.

$$f(x) = 5 \times \cos(5x)$$
$$= 5\cos(5x)$$

iii Write as a product of functions.

Let $y = uv$, where $u = x$ and $v = \sin(x)$

Write the product rule.

$$y' = u'v + uv'$$

Substitute the functions.

$$= 1 \times \sin(x) + x\cos(x)$$

Simplify.

$$= \sin(x) + x\cos(x)$$

iv Write as a function of a function. Let $g(x) = v(u(x))$, where $v(u) = u^5$ and $u(x) = x^3 + \sin(x)$

Write the chain rule. $g'(x) = v'(u)u'(x)$

Substitute the derivatives. $= 5u^4 \times [3x^2 + \cos(x)]$

Substitute u. $= 5[x^3 + \sin(x)]^4 [3x^2 + \cos(x)]$

b Find the derivative.

$$\frac{dy}{dx} = 2 \times \cos(2x)$$
$$= 2\cos(2x)$$

Substitute $x = \frac{\pi}{6}$.

$$= 2\cos\left(2 \times \frac{\pi}{6}\right)$$
$$= 2 \times 0.5$$
$$= 1$$

State the result. The gradient of the tangent is 1.

Remember that $\sin(x) = \cos\left(\frac{\pi}{2} - x\right)$ and $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$. You can use these to find the derivative of $\cos(x)$.

$$\frac{d}{dx}\cos(x) = \frac{d}{dx}\sin\left(\frac{\pi}{2} - x\right)$$
$$= -1 \times \cos\left(\frac{\pi}{2} - x\right)$$
$$= -\sin(x)$$

IMPORTANT

The **derivative of cos (x)** is given by $\frac{d}{dx}\cos(x) = -\sin(x)$

Example 14

Differentiate

a $y = 1 - 8\cos(x)$

b $\cos(\pi x)$

c $g(x) = \frac{\cos(4x)}{x}$

d $h(x) = e^{\cos(x)}$

Solution

a Use the linear product rule. $\frac{dy}{dx} = 0 - [-8\sin(x)] = 8\sin(x)$

b Use the chain rule with $u = \pi x$ $\frac{d}{dx}\cos(\pi x) = -\sin(\pi x) \times \pi = -\pi\sin(\pi x)$

c	Write as a quotient of functions.	$g(x)=\dfrac{\cos(4x)}{x}$, where $u(x)=\cos(4x)$ and $v(x)=x$
	Write the quotient rule.	$g'(x)=\dfrac{u'v-uv'}{v^2}$
	Substitute functions.	$=\dfrac{-4\sin(4x)\times x-\cos(4x)\times 1}{x^2}$
	Simplify.	$=\dfrac{-4x\sin(4x)-\cos(4x)}{x^2}$
d	Write as a function of a function.	Let $h(x)=v[u(x)]$, where $v(u)=e^u$ and $u(x)=\cos(x)$
	Write the chain rule.	$g'(x)=v'(u)u'(x)$
	Substitute the derivatives.	$=e^u\times-\sin(x)$
	Substitute u.	$=-e^{\cos(x)}\sin(x)$

You can find the derivative of $\tan(x)=\dfrac{\sin(x)}{\cos(x)}$ using the quotient rule.

Let $\tan(x)=\dfrac{u(x)}{v(x)}$.

Then $\dfrac{d}{dx}\tan(x)=\dfrac{u'v-uv'}{v^2}$

$$=\frac{\cos(x)\times\cos(x)-\sin(x)\times[-\sin(x)]}{[\cos(x)]^2}$$

$$=\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}$$

$$=\frac{1}{\cos^2(x)}, \text{ since } \sin^2(x)+\cos^2(x)=1.$$

IMPORTANT

The **derivative of tan (x)** is given by $\dfrac{d}{dx}[\tan(x)]=\dfrac{1}{\cos^2(x)}$.

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Example 15

a Differentiate

i $y = 7\tan(x)$ **ii** $g(x) = \tan(4x)$

iii $h(x) = 3x^2\tan(x)$ **iv** $\tan(1 + e^x)$

b Find the exact gradient of the tangent to the curve $y = \tan(x)$ at the point where $x = \frac{\pi}{6}$.

Solution

a i Use the linear factor rule.

$$\frac{dy}{dx} = 7 \times \frac{1}{\cos^2(x)} = \frac{7}{\cos^2(x)}$$

ii Use the chain rule with $u = 4x$

$$g'(x) = \frac{1}{\cos^2(4x)} \times 4 = \frac{4}{\cos^2(4x)}$$

iii Write as a product.

$h(x) = uv$, where $u(x) = 3x^2$ and $v(x) = \tan(x)$

Write the product rule.

$g' = u'v + uv'$

Substitute functions.

$$= 6x \times \tan(x) + 3x^2 \times \frac{1}{\cos^2(u)}$$

Simplify.

$$= 6x\tan(x) + \frac{3x^2}{\cos^2(x)}$$

iv Write as a function of a function.

Let $\tan(1 + e^x) = v[u(x)]$, where $v(u) = \tan(u)$ and $u(x) = 1 + e^x$

Write the chain rule.

$$\frac{d}{dx}\tan(1 + e^x) = v'(u)u'(x)$$

Substitute the derivatives.

$$= \frac{1}{\cos^2(u)} \times e^x$$

Substitute u.

$$= \frac{e^x}{\cos^2(1 + e^x)}$$

b Find the derivative.

$$\frac{dy}{dx} = \frac{1}{\cos^2(x)}$$

Substitute $x = \frac{\pi}{6}$.

$$= \frac{1}{\cos^2\left(\frac{\pi}{6}\right)}$$

Use $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

$$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{4}{3}$$

Write the answer.

The exact gradient is $1\frac{1}{3}$.

EXERCISE 1.05 The derivatives of trigonometric functions

Concepts and techniques

WS Differentiating trigonometric functions

1 Example 13 Find the derivatives of the following.

a $x + \sin(x)$ b $6\sin(x)$ c $\sin(6x)$

d $\sin(x^2 - 3)$ e $4\sin\left(\frac{x}{3}\right)$

2 Differentiate the following.

a $[\sin(x) + 9]^6$ b $x\sin(x)$ c $\sin(e^x)$

d $\frac{\sin(2x)}{5x^2}$ e $e^{\sin(x)}$

3 Find the gradient of the tangent to the following at the given points.

a $y = \sin(x)$ at the point where $x = \frac{\pi}{3}$

b $y = 3\sin(4x)$ at the point $\left(\frac{\pi}{16}, \frac{3}{\sqrt{2}}\right)$

c $f(x) = \sin(x)$ at the point $\left(\frac{3\pi}{2}, -1\right)$

d $y = x + \sin(x)$ at the point where $x = \frac{\pi}{6}$

e $y = \frac{\sin(x)}{x}$ at the point where $x = \frac{\pi}{2}$

4 Example 14 Differentiate the following.

a $2 + \cos(x)$ b $3\cos(x)$ c $\cos(5x)$

d $\cos(3x^2 + 1)$ e $2\cos\left(\frac{x}{2}\right)$

5 Find the derivatives of the following.

a $[4x + \cos(x)]^3$ b $x\cos(x)$ c $\cos(e^{3x})$

d $\frac{\cos(x)}{3x}$ e $\cos[\sin(x)]$

6 Example 15 Differentiate the following.

a $\tan(x)$ b $x + 6\tan(x)$ c $\tan(9x)$

d $3\tan(4x)$ e $[\tan(x) - 1]^5$

7 Differentiate the following.

a $x^2\tan(x)$ b $\frac{\tan(2x)}{x}$ c $e^{\tan(x)}$

d $\tan[\cos(x)]$ e $\frac{\tan(x)}{e^x}$

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8 Find the exact values of the derivatives of the following functions at the given value of the variable.

a $e^{2x}\sin(x)$ at $x = 0.5$

b $x^2 \tan(x)$ at $x = \frac{\pi}{4}$

c $\cos^2(x)$ at $x = \frac{\pi}{3}$

d $\frac{\cos(e^x)}{e^x}$ at $x = 1$

e $x^3 \cos(x^2)$ at $x = e$

9 CAS Find each of the following correct to three decimal places.

a The value of $p'\left(\frac{\pi}{6}\right)$, where $p(x) = x^2 \sin(x) - x\cos(x)$

b The value of $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}}$ for $y = \sqrt{\cos(x)}$

Reasoning and communication

10 Find all x-values for which the gradient of the tangent to the curve $y = \tan(x)$ is 2.

11 The equation of the displacement x cm of a particle is given by $x = 2\sin(3t)$, where the time t is in seconds. The particle is at the centre of the displacement path when $x = 0$.

a What is the particle's greatest displacement from the centre?

b At what times is the particle at the centre?

c At what rate is the particle moving after $\frac{\pi}{6}$ seconds?

d What are the first 3 times that the rate of movement is ± 3 cm/s?

12 The velocity V of a guitar string moving over time t seconds is given by $V = \sin(2t) + 3t + 1$ mm/s.

a What is its velocity after $\frac{\pi}{12}$ seconds, correct to one decimal place?

b Given that acceleration is the rate of change of velocity, find the acceleration of the string after $\frac{\pi}{4}$ seconds.

c Show that the acceleration of the string is never 0.

1.06 APPLICATIONS OF TRIGONOMETRIC FUNCTIONS AND THEIR DERIVATIVES

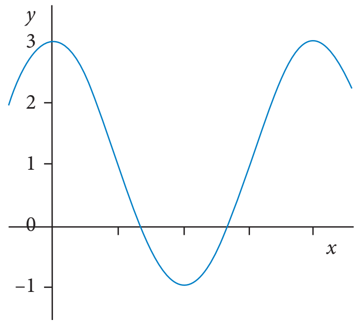
The slope m of a line is related to the angle θ it makes with the positive direction of the x-axis by $m = \tan(\theta)$; the slope of a curve at any point is given by its derivative, which is the slope of the tangent. The direction of the tangent is taken as direction of the line.

Example 16

Find the angle between the x-axis and the curve $y = 2\cos(x) + 1$ at its second point of intersection with the x-axis ($x > 0$).

Solution

Sketch the graph.



Write the equation for the intersection. $0 = 2\cos(x) + 1$

Solve to find x. $\cos(x) = -\frac{1}{2}$, so $x = \frac{2\pi}{3}$

Use your knowledge of the graph. The second intersection is at $2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$

Find the slope of the curve.

$$m = \frac{dy}{dx}$$
$$= -2\sin(x)$$

Substitute $x = \frac{4\pi}{3}$.

$$= -2\sin\left(\frac{4\pi}{3}\right)$$
$$= -2 \times \left(-\frac{\sqrt{3}}{2}\right)$$
$$= \sqrt{3}$$

Write the equation for the inclination. $m = \tan(\theta)$

Substitute the slope. $\sqrt{3} = \tan(\theta)$

Find the angle. $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

From the diagram, the angle is acute. The angle at the second intersection is $\frac{\pi}{3}$.

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Example 17

Find the equation of the tangent to the curve $y = \cos(x)$ at the point where $x = \frac{\pi}{4}$.

Solution

Find the y-coordinate.	$y = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
Differentiate $y = \cos(x)$.	$\frac{dy}{dx} = -\sin(x)$
Substitute $x = \frac{\pi}{4}$.	$m = \frac{dy}{dx} = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
Write the equation of the tangent.	$y - y_1 = m(x - x_1)$
Substitute $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and m.	$y - \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$
Simplify.	$4\sqrt{2}y - 4 = -4x + \pi$
Write the result.	The equation of the tangent is $4x + 4\sqrt{2}y - 4 - \pi = 0$.

Remember that velocity is the derivative of displacement and acceleration is the derivative of velocity.

Example 18

A spring moves so that its end is x cm from the point P at time t seconds, where $x = 2\sin(4t)$.

a Find an equation for the velocity of the spring.

b What is the initial velocity of the spring?

c When is the velocity first equal to zero?

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Solution

a Differentiate to find v.	$v = 2 \times 4\cos(4t)$ $= 8\cos(4t)$
b Substitute $t = 0$.	$= 8\cos(4 \times 0)$ $= 8$
State the result.	The initial velocity is 8 cm/s.

c Substitute $v = 0$ into the velocity equation. $0 = 8\cos(4t)$

Solve to find t. $4t = \cos^{-1}(0)$

Solve the equation. $4t = \frac{\pi}{2}$, so $t = \frac{\pi}{8} = 0.3926\ldots$

State the result. The velocity is first 0 at about 0.39 seconds.

EXERCISE 1.06 Applications of trigonometric functions and their derivatives

Reasoning and communication

1 Example 16 Find the angle between the x-axis and the curve $y = 3\sin(x) - 2$ at its first point of intersection with the x-axis ($x > 0$).

2 Find the angle between the curves $y = \sin(x)$ and $y = \cos(x)$ at the first point at which they intersect in the domain $0 \le x \le 5$.

3 Find the angle between the curves $y = \sin(3x)$ and $y = \cos(3x)$ at the first point at which they intersect in the domain $0 \le x \le 5$.

4 A cam is a non-circular shaft used to work valves and other parts in machines. A particular cam has a profile modelled by the curves $y = 3\cos\left(\frac{\pi x}{4}\right)$ and $y = -3\cos\left(\frac{\pi x}{4}\right)$ for $-2 \le x \le 2$, where x and y are both in centimetres. The 'lift' of the cam is the difference in height of the top of the cam from the centre as it rotates.

a Sketch the shape of the cam.

b What is the 'lift' of the cam?

c What is the angle at the 'points' of the profile?

5 A moving sand dune has a crescent shape such that the edges of the dune are modelled by the curves $y = 80\sin\left(\frac{\pi x}{80}\right)$ and $y = 30\sin\left(\frac{\pi x}{80}\right)$ from $x = 0$ to $x = 80$, where x and y are both in metres.

a Sketch the shape of the dune.

b What is the maximum thickness of the dune?

c Find the angle of the 'points' of the dune.

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6 Example 17 Find the equation of the tangents to the following curves at the stated points.

a $y = \sin(x)$ at the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$

b $y = -2\sin\left(\frac{x}{2}\right)$ at the point where $x = \frac{\pi}{2}$

c $y = \cos(x)$ at the point $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$

d $y = \sin(2x)$ at the point $\left(\frac{\pi}{12}, \frac{1}{2}\right)$

e $y = \tan(3x)$ at the point $\left(\frac{\pi}{12}, 1\right)$

7 Example 18 A particle moves so that its displacement is $x = 3\cos\left(\frac{t}{2}\right)$, where x is in centimetres and t is in seconds.

a Find an equation for the velocity of the particle.

b Find an equation for the acceleration of the particle.

c Find the times when the particle is at $x = 0$.

d What is the velocity and acceleration at these times?

e Find the times at which the particle has the greatest acceleration.

8 The speed of a pendulum bob is given by $v = 1.26\sin(2\pi t)$, where v is in metres/second and t is in seconds.

a Find an expression for the acceleration of the bob.

b Find the velocity and acceleration after 5 seconds.

c Find the acceleration at the times when the velocity is -1.26 m/s.

9 The tidal variation in water level in a particular bay can be modelled as $d = d_0 - 0.9\cos(0.503t)$, where d is the depth of water in metres at a particular point, t is the time in hours after low tide and d_0 is the depth of water at the same point at half-tide. Two hours after low tide the water reaches a mudflat sloped upwards at an angle of 2°. At approximately what speed does it then move horizontally across the mudflat towards the shoreline?

10 a If the displacement of a particle is given by $x = 2\sin(3t)$, show that its acceleration is given by $a = -9x$

b Given that displacement $x = a\cos(nt)$, show that its acceleration is given by $a = -n^2x$

1

CHAPTER SUMMARY
DERIVATIVES, EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

- The derivative of $f(x)$ is defined as $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$
- The derivative of a constant is zero: $\dfrac{d}{dx}(k)=0$ for $k\in \mathbf{R}$
- The derivative of a linear product is given by $\dfrac{d}{dx}[kf(x)]=kf'(x)$ for $k\in \mathbf{R}$
- The derivative of a sum of functions is given by $\dfrac{d}{dx}[f(x)+g(x)]=f'(x)+g'(x)$
- The derivative of a difference of functions is given by $\dfrac{d}{dx}[f(x)-g(x)]=f'(x)-g'(x)$
- The derivative of a linear sum is given by $\dfrac{d}{dx}[af(x)+bg(x)]=af'(x)+bg'(x)$ for $a, b\in \mathbf{R}$
- The derivative $\dfrac{dy}{dx}$ is the **gradient function** of the curve $y=f(x)$
- A linear function through (x_1, y_1) with gradient m is given by $y-y_1=m(x-x_1)$
- The **product rule** for functions u and v can be written as $\dfrac{d}{dx}(uv)=u'v+uv'$
- The **quotient rule** for functions u and v can be written as $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{u'v-uv'}{v^2}$
- Rules for negative and fractional indices: $x^{-n}=\dfrac{1}{x^n}$, $x^{\frac{1}{n}}=\sqrt[n]{x}$ and $x^{\frac{m}{n}}=\sqrt[n]{x^m}=\left(\sqrt[n]{x}\right)^m$
- The derivative of a power is given by $\dfrac{d}{dx}(x^n)=nx^{n-1}$ for $n\in \mathbf{R}$
- The **chain rule** states that $\dfrac{d}{dx}[f(g(x)]=f'(g)g'(x)$, where $g=g(x)$. It can also be written as if $f(x)=v[u(x)]$, then $f'(x)=v'(u)u(x)$, or $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$
- The **exponential function** is the function $y=e^x$, where e is such that $\lim\limits_{h\to 0}\dfrac{e^h-1}{h}=1$
- $e\approx 2.718\,281\,828\,459\,045\,235$, but cannot be written exactly.
- The exponential function is its own derivative, so $\dfrac{d}{dx}(e^x)=e^x$
- For $k>0$, $Q=Ae^{kt}$ is used to model **exponential growth** over time t, where A is the initial quantity of the variable Q
- For $k<0$, $Q=Ae^{kt}$ is used to model **exponential decay** over time t, where A is the initial quantity of the variable Q
- If the rate of change of a function is proportional to the functions, then it must be a simple exponential function.
  If $\dfrac{dQ}{dt}=kQ$, then $Q(t)=Ae^{kt}$, where A is the initial quantity
- The **derivative of sin (x)** is given by $\dfrac{d}{dx}\sin(x)=\cos(x)$
- The **derivative of cos (x)** is given by $\dfrac{d}{dx}\cos(x)=-\sin(x)$
- The **derivative of tan (x)** is given by $\dfrac{d}{dx}\tan(x)=\dfrac{1}{\cos^2(x)}$

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CHAPTER REVIEW

DERIVATIVES, EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

1

Multiple choice

1 Example 1 The derivative of $(3x^2 - 5x + 2)(x^4 + 3x^3 + x - 6)$ is:

A $24x^4 + 34x^3 - 39x^2 - 5$

B $18x^5 + 20x^4 - 52x^3 + 27x^2 - 46x + 32$

C $18x^5 - 7x^4 + 11x^3 - 21x^2 - 34x + 32$

D $18x^5 + 22x^4 - 46x^3 + 9x^2 - 14x + 8$

E $24x^4 + 34x^3 - 45x^2 + 6x - 5$

2 Example 2 The derivative of $\dfrac{2x+1}{3x-2}$ is

A $\dfrac{-1}{(3x-2)^2}$ B $\dfrac{-7}{(3x-2)^2}$ C $\dfrac{12x+7}{(3x-2)^2}$

D $\dfrac{12x-1}{(3x-2)^2}$ E $\dfrac{1}{(3x-2)^2}$

3 Example 3 The derivative of $\dfrac{1}{x^4}$ is

A $-\dfrac{1}{4x^3}$ B $-\dfrac{4}{x^3}$ C $\dfrac{4}{x^5}$ D $-\dfrac{4}{x^5}$ E $-\dfrac{1}{4x^5}$

4 Example 4 If $g(x) = (x^3 - 3x + 1)^3$, then $g'(-2) =$

A 0 B 3 C 27 D 243 E 729

5 Example 8 The gradient of the tangent to the curve $y = e^{2x}$ at the point where $x = 1.5$ is

A 7.4 B 9 C 14.8 D 20.1 E 40.2

6 Example 15 Given $y = 4\sin(e^x) + 2$, $\left.\dfrac{dy}{dx}\right|_{x=2} \approx$

A -9.3 B -7.58 C 11.3 D 13.3 E 31.3

7 Example 16 The equation of the tangent to the curve $y = \cos(3x)$ at the point where $x = \dfrac{\pi}{9}$ is

A $9x + 6y - 3\sqrt{3} - \pi = 0$

B $9\sqrt{3}x + 6y - 3 - \sqrt{3}\pi = 0$

C $9\sqrt{3}x + 18y - 9\sqrt{3} - \sqrt{3}\pi = 0$

D $9\sqrt{3}x - 6y + 3 - \sqrt{3}\pi = 0$

E $9\sqrt{3}x + 18y - 9 - \sqrt{3}\pi = 0$

Short answer

8 Example 1 Differentiate

a $x^5(3x^2 + 2x - 5)$

b $(x^2 + 1)(x^3 - 4x - 1)$

9 Example 2 Find the derivative of

a $\dfrac{5x^3}{2x+1}$

b $\dfrac{x^2 + x - 2}{4x - 3}$

10 Example 3 Find the derivative of

a x^{-6} b $x^{\frac{1}{7}}$ c $\sqrt{x^3}$ d $\frac{9}{x^2}$

11 Example 4 Differentiate

a $(2x-7)^5$ b $3(2x^3+x^2-3)^4$

12 Example 5 Differentiate

a $(x^5+1)^{-3}$ b $\sqrt[3]{x-1}$

13 Example 6 Find the derivative of

a $3x^2(x+2)^8$ b $\frac{3x^2}{(5x-1)^4}$

14 Example 7 Find

a the gradient of the tangent to the curve $y = e^x$ at the point $(3, e^3)$

b the equation of the tangent to the curve $y = 3e^x$ at the point where $x = -2$

15 Example 8 Differentiate

a $x^2 + 2e^x$ b e^{6x} c $(e^x - 3)^9$ d $2e^{4x+1}$ e $e^x + e^{-x}$

16 Example 9 Differentiate

a $(4x+3)e^{2x}$ b $\frac{e^{3x}}{x-4}$

17 Example 10 The number N of apps sold by a company over t months is given by $N = 1000e^{0.24t}$. Find

a the number of apps sold initially

b the number of apps sold after

i 6 months ii 2 years

c the rate at which apps are sold after

i 6 months ii 2 years

18 Example 11 The temperature $T°$ of a chemical that cools down over t minutes is given by $T = 75e^{-0.15t}$. Find

a the temperature after

i 2 minutes ii 5 minutes.

b the rate at which the temperature changes after

i 2 minutes ii 5 minutes.

19 Example 12 The quantity of salt increases in an evaporating rock pool according to the equation $\frac{dQ}{dt} = 0.04Q$. Initially the pool has 375 g of salt.

a Find an equation for the quantity Q of salt in the pool in terms of t, where t is in weeks.

b Find the amount of salt in the pool after 6 weeks

c Find the rate at which salt is increasing after 6 weeks.

20 Example 13 Differentiate

a $3 \sin(x) + 1$ b $\sin(5x)$ c $x \sin(2x)$

d $[3x - \sin(x)]^7$ e $\sin(x^3 + 1)$

21 Example 14 Find the derivative of

a $\cos\left(\frac{x}{3}\right)$

b $e^x \cos(x)$

c $[2+\cos(3x)]^5$

d $\cos(\pi x)$

e $\cos^2(x)$

22 Example 15 Differentiate

a $6\tan(x)$

b $\tan(3x)$

c $\tan\left(\frac{\pi x}{5}\right)$

d $x^3 \tan(2x)$

e $\frac{\tan(x)}{x}$

Application

23 Show that the derivative of $Q = Q_0 e^{kt}$ is $\frac{dQ}{dt} = kQ$

24 a Find the equation of the tangent to the curve $y = \tan(x)$ at the point $\left(\frac{\pi}{4}, 1\right)$.

b Find points A and B where this tangent meets the x- and y-axes.

c Find the area of triangle OAB, where O is the origin.

25 A pendulum moves so that its displacement x cm over time t seconds is given by $x = 6\sin(t)$.

a Find the displacement after 5 seconds.

b Find the times when the pendulum is at $x = 0$.

c Find the velocity of the pendulum after 3 seconds.

26 The profile of a skateboard 'jump' has been designed so that it follows the curve

$y = 2\cos\left(\frac{\pi x}{4}\right) + 2$ for $0 \le x \le 6$, where y and x are both in metres.

a Sketch the jump.

b What is the take-off angle at the end of the jump (in degrees)?

Practice quiz

2

TERMINOLOGY

complement
conditional probability
continuous variable
discrete variable
domain
event
expected value
failure
hypergeometric
independent
intersection
mutually exclusive
outcome
probability distribution
probability function
random
relative frequency
sample space
standard deviation
success
uniform distribution
union
variance

Prior learning

DISCRETE RANDOM VARIABLES

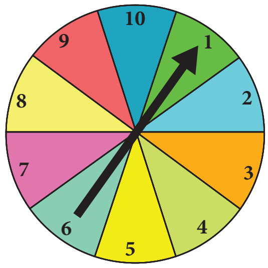
DISCRETE RANDOM VARIABLES

GENERAL DISCRETE RANDOM VARIABLES

- **understand the concepts of a discrete random variable and its associated probability function, and their use in modelling count data (ACMMM136)**
- **use relative frequencies obtained from data to obtain point estimates of probabilities associated with a discrete random variable (ACMMM137)**
- **recognise uniform discrete random variables and use them to model random phenomena with equally likely outcomes (ACMMM138)**
- **examine simple examples of non-uniform discrete random variables (ACMMM139)**
- **recognise the mean or expected value of a discrete random variable as a measurement of centre, and evaluate it in simple cases (ACMMM140)**
- **recognise the variance and standard deviation of a discrete random variable as measures of spread, and evaluate them in simple cases (ACMMM141)**
- **use discrete random variables and associated probabilities to solve practical problems (ACMMM142)** AC

2.01 DISCRETE RANDOM VARIABLES

In probability, an **experiment** has a number of **outcomes**. Each outcome is called a **sample point** and the **sample space** of an experiment is all possible outcomes. An **event** is a subset of the sample space.



Consider the spinner shown here. If event A is defined as 'an even number' and event B is defined as 'a multiple of 3', then:

$A = \{2, 4, 6, 8, 10\}$

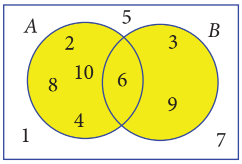
and $B = \{3, 6, 9\}$.

The **sample space** S is the set of individual possible occurrences, so in this case, $S = \{1, 2, 3, \ldots, 10\}$.

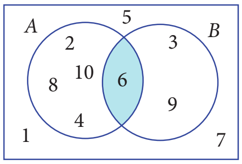
The **union** ($\cup$) of A and B is the combination of either event A or event B or both A and B occurring. So $A \cup B = \{2, 3, 4, 6, 8, 9, 10\}$.

The **intersection** ($\cap$) of A and B is the event that both A and B occur and includes the sample points that are common to A and B. So $A \cap B = \{6\}$.

Venn diagrams are used to visualise events, using circles in a rectangle to represent S.



$A \cup B$



$A \cap B$

The probability of an event is the likelihood or chance of it occurring, so $P(A) = \frac{5}{9}$ and $P(A \cup B) = \frac{7}{9}$

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IMPORTANT

The **probability** of an event A in a sample space S is written as $P(A)$ and is a real number between 0 and 1. The probability of the sample space, $P(S) = 1$ and the probability of a union of disjoint events is the sum of their probabilities.

For a finite sample space whose elements have equal probabilities,

$$P(A) = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}} = \frac{n(A)}{n(S)}$$

If $P(A) = 1$, then event A is **certain** to occur.

If $P(A) = 0$, then event A is **impossible**.

The **complement** of event A is represented as A' or $\overline{A}$. A' means 'not A', so $P(A')$ is the probability that A will not occur.

$P(A) + P(A') = 1$

$P(A') = 1 - P(A)$

You can use Venn diagrams, tree diagrams, tables and grids to help calculate the probabilities of events. You can also use the following rules.

IMPORTANT

The **addition rule** of probability states that:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$P(A \cup B)$ is also written as $P(A \text{ or } B)$.

$P(A \cap B)$ is also written as $P(A \text{ and } B)$.

Mutually exclusive events cannot occur simultaneously, so $P(A \cap B) = 0$. For mutually exclusive events: $P(A \cup B) = P(A) + P(B)$

The **conditional probability** of event A given event B is written as $P(A|B)$ or 'the probability of A given B' and defined as $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

The **multiplication rule** for any events A and B is $P(A \cap B) = P(A|B) \times P(B)$.

Events are **independent** if $P(A|B) = P(A)$. The outcome of one does not affect the probability of the other, so $P(A \cap B) = P(A) \times P(B)$.

You should remember from Year 11 that conditional probability is often used when events occur one after another, but it can be used in unordered situations as well.

Example 1

Sam knows that if she wakes up when her phone alarm goes off, she has a 90% chance of getting to school on time. If she sleeps through the alarm, she only has a 40% chance of getting to school on time. Because she sometimes sleeps heavily, she only wakes up when the alarm rings 80% of the time. Calculate these probabilities.

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a She wakes up when the alarm rings and she gets to school late.
b She gets to school on time.
c She gets to school late.
d She wakes up when the alarm rings, given that she gets to school on time.

Solution

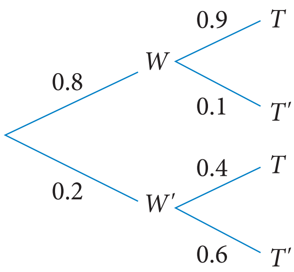
Define the events.

Let W = wakes up with alarm
Then sleeps through alarm = W'
Let T = arrives at school on time
Then arrives at school late = T'

Assign probabilities to each event.

$P(W) = 80\% = 0.8$
$P(W') = 1 - 0.8 = 0.2$
If she wakes up, $P(T|W) = 90\% = 0.9$
If she wakes up, $P(T'|W) = 1 - 0.9 = 0.1$
If she sleeps in, $P(T|W') = 40\% = 0.4$
If she sleeps in, $P(T'|W') = 1 - 0.4 = 0.6$

Draw a tree diagram and write the relevant probability on each branch.



a Write the required probability.

$P(\text{wakes up and gets to school late}) = P(W \cap T')$
$= P(T' \cap W)$

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Use the multiplication rule.

$$= P(T'|W) \times P(W)$$
$$= 0.8 \times 0.1$$
$$= 0.08$$

b Write the required probability.

$$P(\text{school on time}) = P(T)$$

Use the addition rule.

$$= P(T \cap W) + P(T \cap W')$$

Use the multiplication rule.

$$= P(T) \times P(T|W) + P(T) \times P(T|W')$$
$$= 0.8 \times 0.9 + 0.2 \times 0.4$$
$$= 0.72 + 0.08$$
$$= 0.8$$

c Getting to school late is the complement of getting to school on time.

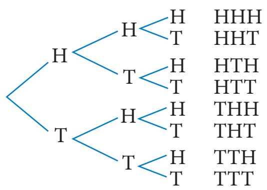
$$P(L) = P(T')$$
$$= 1 - P(T)$$
$$= 1 - 0.8$$
$$= 0.2$$

d Use the rule for conditional probability.

$$P(W|T) = \frac{P(W \cap T)}{P(T)}$$
$$= \frac{0.72}{0.8}$$
$$= 0.9$$

In Example **1**, the outcomes are occurrences in the real world. You can also have a sample space with numerical outcomes.

Consider a fair coin that is tossed three times and counting the number of tails in a row. You can use a tree diagram or list to work out the probabilities of each outcome.



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There are four different possibilities: 0, 1, 2 and 3 tails, but there are 8 equally likely elements in the sample space. The probabilities of tails in a row are shown below.

Number of tails in a row	Outcomes	Probability
0	HHH	$\frac{1}{8} = 0.125$
1	HHT, HTH, THH, THT	$\frac{1}{2} = 0.5$
2	HTT, TTH	$\frac{1}{4} = 0.25$
3	TTT	$\frac{1}{8} = 0.125$

In this case, the events are part of a sample space and have a numeric variable associated with them. The variable is the number of successive tails, T. You cannot tell what the value will be, so you say that it is **random**. It can only have the values 0, 1, 2 and 3, so it is a **discrete** variable, but you can work out the probability of each value.

IMPORTANT

A **random variable** is a variable with a numerical value determined by events from a sample space. Each value has an associated probability. A random variable is **discrete** if the possible values (the **domain**) are discrete. A **continuous random variable** has a continuous domain.

A random variable is usually denoted by a capital letter. The corresponding lower-case letter denotes specific values of the variable.

The **probability function** of a discrete random variable is the function formed by the values of the variable and their probabilities. For the random variable X, the probability function is shown as $P(X = x)$ or $p(x)$.

WS

Classifying variables 1

You can show a function as a list (set) of ordered pairs, table or rule. For the variable T above, $p(0) = 0.125$, $p(1) = 0.5$, $p(2) = 0.25$ and $p(3) = 0.125$.

Example 2

A pair of fair dice is tossed and the total of the uppermost faces is noted.

a If X = total of the uppermost faces, show that X is a discrete random variable.

b List the probability function.

Solution

a List the domain.

Domain of $X = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

State how X is determined.

The values of X are determined by the sample space from tossing fair dice.

State the result.

X is both discrete and random.

b Construct a grid to determine the outcomes for this experiment.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

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Calculate the probabilities of each value of the variable.
There are a total of 36 outcomes.

x	Number of outcomes	$p(x)$
2	1	$\frac{1}{36}$
3	2	$\frac{1}{18}$
4	3	$\frac{1}{12}$
5	4	$\frac{1}{9}$
6	5	$\frac{5}{36}$
7	6	$\frac{1}{6}$
8	5	$\frac{5}{36}$
9	4	$\frac{1}{9}$
10	3	$\frac{1}{12}$
11	2	$\frac{1}{18}$
12	1	$\frac{1}{36}$

List the probabilities.

$p(2)=\frac{1}{36}$, $p(3)=\frac{1}{18}$, $p(4)=\frac{1}{12}$, $p(5)=\frac{1}{9}$, $p(6)=\frac{5}{36}$, $p(7)=\frac{1}{6}$, $p(8)=\frac{5}{36}$, $p(9)=\frac{1}{9}$, $p(10)=\frac{1}{12}$, $p(11)=\frac{1}{18}$, $p(12)=\frac{1}{36}$

You could also show the probability function as a list of ordered pairs:

$(2,\frac{1}{36}), (3,\frac{1}{18}), (4,\frac{1}{12}), (5,\frac{1}{9}), (6,\frac{5}{36}), (7,\frac{1}{6}), (8,\frac{5}{36}), (9,\frac{1}{9}), (10,\frac{1}{12}), (11,\frac{1}{18}), (12,\frac{1}{36})$

or as the rule $p(x)=\begin{cases}\dfrac{x-1}{36} & \text{for} \quad 2\le x\le 7\\[2ex] \dfrac{13-x}{36} & \text{for} \quad 8\le x\le 12\end{cases}$.

EXERCISE 2.01 Discrete random variables

Concepts and techniques

WS
Classifying variables 2

1 Classify each of the following variables as discrete or continuous.
  a The mass of a bus
  b The number of passengers in a bus
  c The number of bus stops on a route
  d The distance between bus stops
  e The price of a washing machine
  f The capacity of a washing machine
  g The energy star rating of a washing machine
  h The volume of water used by a washing machine in a wash cycle

2 List the domain for the variable in each of the following experiments.
  a A fair die is rolled. $X =$ the number on the uppermost face.
  b A bag contains marbles numbered 1 to 10 and one marble is selected. $X =$ the number on the selected marble.
  c A fair coin is tossed three times and $X =$ the number of successive heads.
  d A card has a whole number between 1 and 10 (inclusive) written on it. A person is asked to guess the number. $X =$ the number of guesses it takes to guess the number.
  e A point inside a circle with radius 2 units is chosen at random. $X =$ the distance of the point from the centre of the circle.

3 For each experiment described in question **2**, classify the variable as either discrete or continuous and either random or non-random.

4 Which of the following random variables is not discrete?
  A The number of glass jars recycled by a family each fortnight.
  B The number of cars sold in a used car yard each month.
  C The number of runs scored by a batsman in each over of a match.
  D The height of a child measured each week from ages 5 to 10.
  E The price in cents of a kilogram of bananas at a supermarket each week for a year.

5 A driver must pass through 7 sets of traffic lights on the way to work each day in peak hour traffic. If $N =$ the number of red lights the driver stops at, which of the following best describes N?
  A continuous variable B random variable C discrete variable
  D discrete continuous variable E random continuous variable

6 For the situation described in question **5**, which of the following represents the domain of the variable?
  A $\{7\}$ B $\{0, 1, 2, \ldots, 7\}$ C $0 \le n \le 7$
  D $\{1, 2, 3, 4, 5, 6, 7\}$ E $\{1, 2, 3, \ldots, 6\}$

7 A bag contains 2 black marbles and 3 white marbles of identical size. If two marbles are drawn without replacement, what is the probability that they are different colours?
  A $\frac{6}{25}$ B $\frac{3}{10}$ C $\frac{2}{5}$ D $\frac{12}{25}$ E $\frac{3}{5}$

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8 If $P(R) = 0.2$, $P(T) = 0.6$ and $P(T|R) = 0.5$, then $P(R \cup T)$ is equal to:

A 0.1 B 0.2 C 0.3 D 0.7 E 0.8

9 If $P(M) = 0.4$, $P(Q) = 0.5$ and $P(M \cup Q) = 0.7$, which one of the following is not true?

A M and Q are independent
B $P(M \cap Q) = 0.2$
C $P(M \mid Q) = 0.4$
D M and Q are mutually exclusive
E $P(Q \mid M) = 0.5$

10 Example 1 Jamal's car starts 80% of the time. He knows that if the car starts, he has a 90% chance of getting to work on time. If the car doesn't start, his chance of getting to work on time is only 40%. What is the probability that Jamal arrives to work late?

A 0.12 B 0.2 C 0.7 D 0.72 E 0.8

11 A factory that assembles computer motherboards has three assembly lines – A, B and C. It is known that 40% of the motherboards are assembled on line A and 30% are assembled on each of lines B and C. All motherboards are tested when they come off the line and defective ones are discarded. If a motherboard is assembled on line A, the probability of it being defective is 0.01. If it comes from line B, the probability of a defective motherboard is 0.02, while the probability is 0.03 if it comes from line C.

a If a newly assembled motherboard is randomly selected for testing, what is the probability that it is not defective?

b Given that a randomly selected motherboard is defective, what is the probability that it came from line B?

12 Example 2 A pair of tetrahedral dice with faces numbered 1 through 4 is tossed and the sum of the numbers on the faces on which the dice rest is calculated.
If X = total of the faces, list the ordered pairs that make up the probability function $P(X)$.


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Reasoning and communication

13 A fair coin is tossed four times. If H = the number of successive heads, calculate:

a $P(H = 3)$ b $P(H > 3)$ c $P(H < 3)$
d $P(H = 5)$ e $P(H \geq 2)$
f List the domain of M and the probability function.

14 A pair of cubic dice are thrown. If M = the minimum of the two numbers that occur, list the domain of M and the probability function $P(M = m)$.

15 A fair coin is tossed four times and T = the number of tails.

a List the domain of T and the ordered pairs that make up the probability function.

b Calculate $P(T > 2)$.

2.02 DISCRETE PROBABILITY DISTRIBUTIONS

It is often useful to display the values of a discrete random variable with their probabilities.

You usually draw the graph of a probability distribution like a histogram, without spaces between the columns, to emphasise the connection with statistical graphs. They are called probability histograms.

You can use probability properties to get the general properties of discrete probability distributions.

IMPORTANT

A **discrete probability function (discrete probability distribution)** shows the probability of a discrete random variable as a list, table or graph.

A discrete probability function $P(X = x)$ has the following properties.

- All the values are between 0 and 1: $0 \le P(X = x) \le 1$ for all values of x
- The sum of all the probabilities is 1. This is written as $\Sigma P(X = x) = 1$.
- $p(x)$ is a function, so every value of x has only one value of $p(x)$.

The Greek letter Σ is widely used in Mathematics to mean 'the sum of'. If there are no subscripts, then it means 'the sum of all values', as in the case above.

On the other hand, $\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5$ and $\sum_{n=4}^{6} n^3 = 4^3 + 5^3 + 6^3$.

Provided that the values of X are discrete, the properties are sufficient to show that a function $p(x)$ is a discrete probability distribution. Of course, different values of x can have the same probability.

Example 3

A bag contains an equal number of identically shaped black and white discs. A disc is randomly selected, its colour noted and then it is returned to the bag. This is done three times in all.

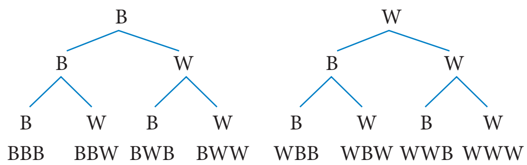
a Construct a probability distribution for the variable X, the number of black discs drawn.

b Confirm that it is a probability distribution function.

c Draw a graph of the probability distribution.

Solution

a Draw a tree diagram to calculate the outcomes for this experiment.



0, 1, 2 or 3 black discs can be drawn. List the outcomes for each event.

Number of black discs (x)	Outcomes
0	WWW
1	BWW, WBW, WWB
2	BBW, BWB, WBB
3	BBB

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Calculate the probabilities for each value of the variable.

x	$p(x)$
1	$\frac{1}{8}$
2	$\frac{3}{8}$
3	$\frac{3}{8}$
5	$\frac{1}{8}$

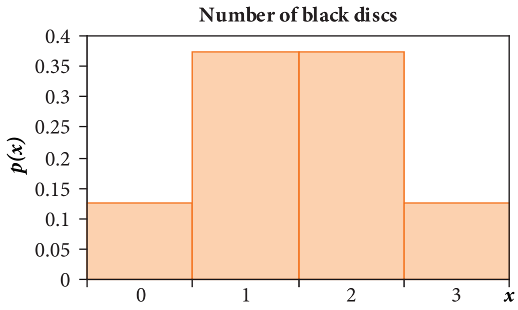
b Calculate the sum of the probabilities.

$\sum p(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

State the result.

Since $0 \leq P(X = x) \leq 1$ for all values of x, $p(x)$ is a function and the sum of the probabilities is 1, it is a probability distribution function.

c Draw the graph of the probability distribution.



You can use the properties to determine if a function could be a probability function.

Example 4

A function is given as $p(x) = \frac{x}{15}$ for $x = 1, 2, 3, 4, 5$.

a Construct a table of values for the function.

b Determine if the function is a discrete probability distribution.

Solution

a Calculate the values.

x	$p(x)$
1	$\frac{1}{15}$
2	$\frac{2}{15}$
3	$\frac{3}{15}$
4	$\frac{4}{15}$
5	$\frac{5}{15}$

b Check the values.	$0 \le p(x) \le 1$ for all values of x.
Calculate the total.	$\sum P(X=x) = \frac{1}{15} + \frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15}$ $= 1$
Check that it is a function.	Each domain value has one value of $p(x)$.
State the conclusion.	The conditions are satisfied, so $p(x)$ is a discrete probability distribution.

Example 5

A discrete probability function is defined by the following table.

a Find the value of p.

b Hence calculate $P(X > 2)$.

x	0	1	2	3	4
$P(X = x)$	0.2	0.3	$2p$	p	0.05

Solution

a The sum of the probabilities must be 1.	$0.2 + 0.3 + 2p + p + 0.05 = 1$
Simplify.	$3p + 0.55 = 1$
Solve for p.	$p = 0.15$
b Write as a sum.	$P(X > 2) = P(X = 3) + P(X = 4)$
Use the value of p.	$= 0.15 + 0.05$
Write the answer.	$P(X > 2) = 0.2$

EXERCISE 2.02 Discrete probability distributions

Concepts and techniques

1 Example 3 A pair of dice is rolled and you win \$6 if the sum is greater than 10. If you get a double you win \$2 and if you get a double with a sum greater than 10 you win \$8. Otherwise, you win nothing. If X = amount won, which of the following is the probability distribution for X?

A

x	0	2	6	8
$P(X = x)$	$\frac{7}{9}$	$\frac{5}{36}$	$\frac{1}{18}$	$\frac{1}{36}$

B

x	0	2	6	8
$P(X = x)$	$\frac{3}{4}$	$\frac{1}{6}$	$\frac{1}{18}$	$\frac{1}{36}$

C

x	0	2	6	8
$P(X = x)$	$\frac{13}{18}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{36}$

D

x	0	2	6	8
$P(X = x)$	$\frac{27}{36}$	$\frac{5}{36}$	$\frac{1}{18}$	$\frac{1}{36}$

E

x	0	2	6	8
$P(X = x)$	$\frac{7}{12}$	$\frac{5}{36}$	$\frac{1}{12}$	$\frac{1}{36}$

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2 Example 4 A function is defined as $p(x) = \dfrac{(x-2)^3}{35}$ for $x = 1, 2, 3, 4$. Choose the best of the following options.

A It is a probability function because the sum of the values is 1.

B It is not a probability function because $p(2) = 0$.

C It is a probability function because it is a function.

D It is not a probability function because $p(1) < 0$.

E Both **A** and **C** are true.

3 Example 5 If the table below represents a discrete probability distribution, calculate the value of m.

x	2	5	8	12
$P(X = x)$	$2m$	$5m$	$6m$	$3m$

A 0.01 B 0.0625 C 0.125 D 0.16 E 0.2

4 For the discrete probability distribution shown below, $P(x \le 8)$ is equal to:

x	3	5	8	9	11
$P(X = x)$	0.12	0.25	0.33	0.21	0.09

A 0.09 B 0.3 C 0.37 D 0.63 E 0.7

5 A fair coin is tossed three times. If X = number of tails in three tosses, then $p(x) = 2$ is equal to:

A $\frac{1}{8}$ B $p(x) = 1$ C $\frac{1}{4}$ D $p(x) = 3$ E $\frac{1}{2}$

6 For the discrete probability distribution shown below, calculate the value of n.

x	0	1	2	3	4
$P(X = x)$	0.15	0.25	n	0.35	0.05

A 0.15 B 0.2 C 0.25 D 0.35 E 0.4

7 State whether or not each of the following could represent a probability distribution.

a

x	$p(x)$
1	0.3
2	0.2
3	0.4

b

m	$p(m)$
−3	0.1
−1	0.2
1	0.3
3	0.4

c

t	$p(t)$
0	0.4
2	−0.1
6	0.2
8	0.5

8 Example 4 A probability distribution is defined by: $p(x) = \dfrac{x}{10}$ and $x = 1, 2, 3, 4$.

a Construct the probability distribution.

b Draw a graph of the distribution.

c Verify that it is a probability distribution.

9 Two probability distributions are defined by: $p_1 = \dfrac{x^2}{30}$ and $p_2 = \dfrac{5x^3}{12} - \dfrac{x^4}{24} - \dfrac{3}{4} - \dfrac{35x^2}{24} + \dfrac{25x}{12}$ for $x = 1, 2, 3, 4$.

a Construct the probability distributions.

b Verify that each is a probability distribution.

c Draw graphs of the distributions.

Reasoning and communication

10 A pair of tetrahedral dice with faces numbered from 1 to 4 are rolled and the sum (F) of the numbers on the faces on which the dice rest is calculated. Construct the probability distribution for F.

11 A pair of six-sided dice is rolled and the greater of the numbers on the upper faces, G, is noted.
- a Construct the probability function for G.
- b Draw a graph of the distribution.

12 A pair of six-sided dice is rolled and the sum of the numbers on the upper faces, S, is noted.
- a Construct the probability distribution for S.
- b Draw a graph of the distribution.
- c Verify that it is a probability distribution.

13 A coin is weighted so that $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$, and it is tossed three times. Let X be the random variable representing the largest number of successive heads that occur.
- a Construct the probability distribution for X.
- b Draw a graph of the distribution.

14 A player rolls a fair six-sided die. If a prime number occurs, the player wins that number of dollars. If a non-prime number occurs, the player loses that number of dollars. If X is the number of dollars the player stands to win:
- a construct the probability distribution for X
- b draw a graph of the distribution.

15 A family consists of four children. Assuming that $P(\text{girl}) = \frac{1}{2}$, draw up a probability distribution for X, the number of girls in the family.

16 A game consists of rolling three six-sided dice. If all three dice show the same number, you win \$20. If two numbers are the same you win \$5, but if all three are different you lose \$5. If X is the random variable representing the amount you win, find the probability distribution of X.

2.03 ESTIMATING PROBABILITIES

You can use data to estimate probabilities. You need to count the number of times an event occurs in the data.

IMPORTANT

The **relative frequency** of an event is an estimate of its probability. It is given by

$$\text{Relative frequency} = \frac{\text{frequency of event}}{\text{total number of observations}} = \frac{\text{frequency of event}}{\text{total frequency}}.$$

The observations are usually called **trials**.

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Example 6

1344 Australians aged 15–24 died in a 3-year period. 320 of the deaths were males killed in transport accidents. Over the same period, 389 of the females died. 1 356 900 males and 1 347 400 females were included in the study.

a Estimate the probability of an Australian man aged 15–24 dying in the 3-year period.

b What is the probability that the death of an Australian male aged 15–24 is due to a transport accident?

Solution

a Use the relative frequency. $\text{Probability} = \dfrac{1344 - 389}{1\,356\,900}$

Simplify. ≈ 0.0007

Write the answer. The probability is about 0.0007 or 0.07%.

b Use the relative frequency. $\text{Probability} = \dfrac{320}{1344}$

Simplify. ≈ 0.238

Write the answer. The probability is about 0.238 or 23.8%.

EXERCISE 2.03 Estimating probabilities

Concepts and techniques

1 Example 6 From the data in Example **6**, estimate the probability that the death of an Australian aged 15–24 is that of a female.

2 In another 3-year period, 286 of the deaths of 1309 Australians aged 15–24 were due to intentional self-harm, and 233 of these were males.

   a What is the probability that the death of an Australian male in the age group 15–24 is due to self-harm?

   b What is the probability that an Australian aged 15–24 dying from self-harm is a female?

3 From 40 people in a mall, 16 were observed to be wearing joggers. Estimate the probability that a person will be wearing joggers.

4 In 2005, 61 400 of the 85 200 babies born in Australia came from married women. Of those who were not married, 2300 chose not to name the father on the birth certificate. Use these figures to estimate the following.

   a The probability that an Australian baby will be born to an unmarried mother.

   b The probability that an Australian baby born to an unmarried mother will not have the father shown on the birth certificate.

5 In 2011, 198 300 babies were born to married women in Australia and of the 94 100 born to unmarried women, 10 100 did not name the father. Use these figures to estimate the following.

   a The probability that an Australian baby will be born to an unmarried mother.

   b The probability that an Australian baby born to an unmarried mother will not have the father shown on the birth certificate.

2.04 UNIFORM DISCRETE PROBABILITY DISTRIBUTIONS

In the previous sections the discrete probability functions were not constant. Cases where the probability function is constant occur quite frequently.

Example 7

A normal six-sided die is rolled and the uppermost number is noted.

a Draw a graph of the probability distribution.

b Calculate the probability that the number is greater than 4.

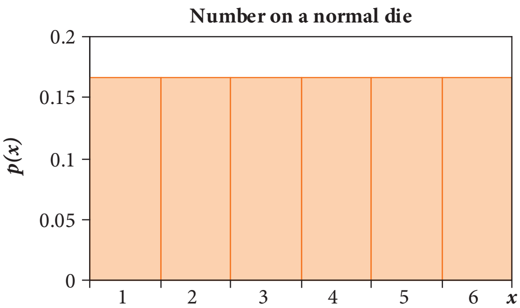
123RF/Aleksandar Kosev

Solution

a Construct the probability distribution.

x	1	2	3	4	5	6
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Draw the graph of the distribution.



b Write the probability.

$P(X > 4) = P(x = 5) + P(x = 6)$

Substitute values.

$= \frac{1}{6} + \frac{1}{6}$

Write the answer.

$= \frac{1}{3}$

The probability distribution in Example **7** is called a **uniform distribution**. The graph is a rectangle. The characteristics are as follows.

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IMPORTANT

A **uniform discrete probability distribution** has equally likely outcomes.

For a domain with n elements, the probability function is $P(X = x) = \frac{1}{n}$.

If the domain is the integers from a to b inclusive, then $n = b - a + 1$, so $P(X = x) = \frac{1}{b-a+1}$.

It follows that, for this domain, $P(c \le x \le d) = \frac{d-c+1}{b-a+1}$, where $a \le c \le d \le b$.

A uniform distribution is often called a rectangular distribution.

If the clubs in a normal pack of cards are numbered from 11 to 23, the diamonds from 24 to 36, the hearts from 37 to 49 and the spades from 50 to 62, the domain is {11, 12, 13, …, 62} and $p(x) = \frac{1}{52}$.

Example 8

A bag contains identical discs numbered 10 to 19. A disc is drawn at random and returned to the bag. If R = the number on the disc drawn, calculate:

a $P(r = 12)$ **b** $P(15 \le r \le 17)$ **c** $P(x = 7)$

Solution

a There are 10 elements. $P(r = 12) = \frac{1}{10}$

b {15, 16, 17} has 3 out of 10 elements. $P(15 \le r \le 77) = \frac{3}{10}$

c 7 is not in the domain of $p(x)$. $P(x = 7)$ is not defined.

Some people prefer to say that the answer to part **c** of Example **8** is $P(x = 7) = 0$. You can use a slightly different definition of a discrete uniform probability distribution to allow for cases like this.

IMPORTANT

A **uniform discrete probability distribution** can be defined as a distribution that has n outcomes with equal probabilities, where $p(x) = \frac{1}{n}$ for $x \in D$ and $p(x) = 0$ for $x \notin D$.

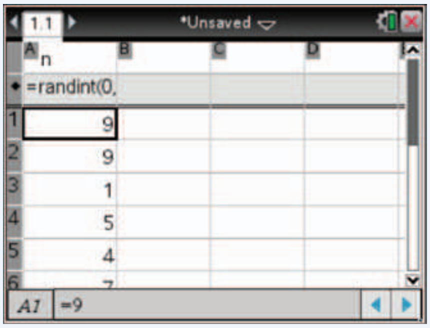
Choosing a **random number** from 10 to 50 inclusive is a use of the discrete uniform probability distribution on the integers from 10 to 50.

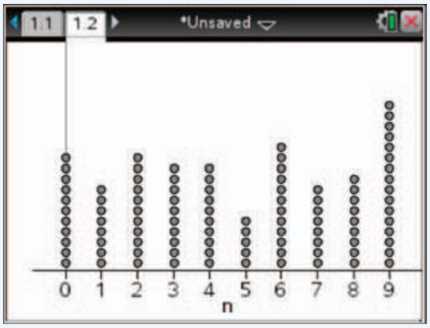
INVESTIGATION Random numbers

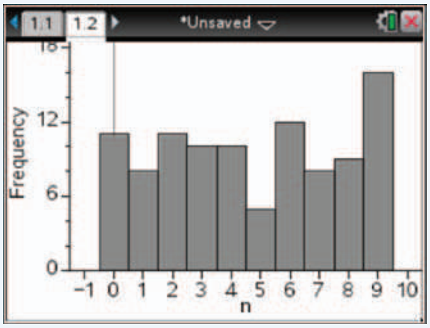
How random is your calculator?

Many calculators have the facility to generate random numbers. Use your calculator to produce 100 random integers from 0 to 9 and plot a graph.

TI-Nspire CAS

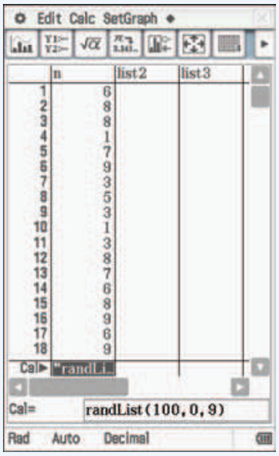






Use the Lists & Spreadsheet page ( ).
Name the first list n and type
= randint(0, 9, 100) into the second row.
This generates 100 random integers from 0 to 9 in the A column.
Then press ctrl doc▾ and insert a Data & Statistics page ( ).
Move the cursor to the bottom and click on 'add variable' or press menu, 2: Plot Properties, 5: Add X Variable and choose n to see a graph.
Then press menu, 1: Plot Type and 3: Histogram to change the display to a histogram.

ClassPad



Use the Statistics menu (Statistics).
Tap the list1 cell and type n.
Tap the cell to the right of Cal▶, and then tap the cell to the right of Cal =. Enter randList(100, 0, 9) in this cell.
Tap SetGraph and Setting and select the Type Histogram. Choose main\n for the XList.
Freq should be set to 1. Tap Set.

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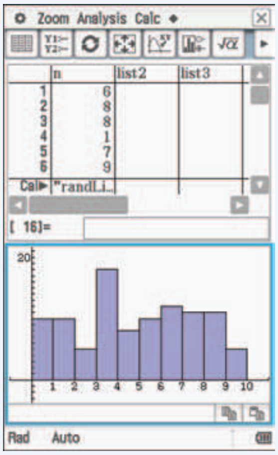
Tap SetGraph and Stat Window Auto and set to off.
Then tap to see the histogram. Set HStart to 0 and HStep to 1.
Tap to set the window to appropriate scales.



Combine your results with those of other members of your class and comment on the probability distribution for the combined data.

EXERCISE 2.04 Uniform discrete probability distributions

Concepts and techniques

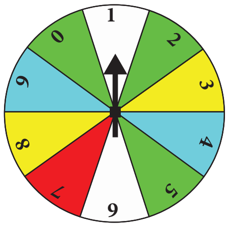
1 Example 7 A twelve-sided die is rolled and the uppermost number is noted. The probability of getting an 8 is:

A $\frac{1}{12}$ B $\frac{1}{8}$ C $\frac{1}{6}$ D $\frac{1}{4}$ E $\frac{8}{12}$

2 Example 8 A discrete random variable, X, has a uniform probability distribution. The domain of the variable is {4, 5, 6, 7, 8}. What is the probability that $x = 7$?

A $\frac{1}{8}$ B $\frac{1}{5}$ C $\frac{1}{7}$ D $\frac{4}{5}$ E 1

3 For each of the following situations describe the domain of the variable and state whether or not the variable would have a uniform probability distribution.



a A tetrahedral die is rolled and N = the number on the face on which the die rests.

b A spinner consists of 10 equal sectors, each numbered with a different digit. The spinner is spun and X = the number of the sector on which the arrow of the spinner stops.

c A circular disc consists of 10 equal sectors, each numbered with a different digit. People are asked to place their finger on the sector with their favourite digit. S = the number of the sector touched by each person.

d A bag contains 20 identical discs numbered 1 through 20. A disc is drawn at random, not replaced and then another disc is drawn. This process is repeated 10 times. D = the number on the disc.

e A fair die is rolled and M = the number on the uppermost face.

f Two fair dice are rolled and T = the total of their uppermost faces.

4 A bag contains 10 balls of identical size. The balls are numbered 1 through 10. A ball is randomly drawn from the bag. If N = the number on the ball, construct a probability distribution histogram and comment on its shape.

5 Eight identical discs, each with one of the numbers 1, 2, 3, 5, 7, 8, 10 and 11 are in a bag. A disc is drawn and then returned to the bag. If N is the number on the disc:

a construct a probability distribution histogram for N.

Use this histogram to calculate:

b $P(n \leq 4)$ c $P(n \neq 8)$ d $P(n \text{ is even})$

e $P(n \text{ is not even})$ f $P(2 \leq n \leq 8)$

6 A ten-sided die is rolled and X = the number on the uppermost face.

a Draw a graph of the probability distribution for X.

Use the graph of the probability distribution for X to calculate:

b $P(x \geq 2)$ c $P(x < 4)$ d $P(2 \leq x \leq 5)$

7 A spinner consists of 30 equal sectors numbered 1 through 30. The spinner is spun and S = the number of the sector on which the arrow of the spinner stops. Calculate the following values.

a $P(s = 17)$ b $P(s \neq 17)$ c $P(5 \leq s \leq 22)$

d $P(s > 17)$ e $P(s \leq 17)$

Reasoning and communication

8 A company sells goods online and customers ring a 1800 number to place an order. The time taken for the call centre to completely process the order is measured by a digital clock and recorded in minutes and seconds. The minimum unit of measurement is 1 second. It is known that the time taken to process and order can take as little as 150 seconds and as long as 12 minutes and that the times are randomly distributed over this range. If T = time taken to complete an order, calculate:

a $P(t > 8 \text{ min})$ b $P(t < 5 \text{ min})$

c $P(5 \text{ min} \leq t \leq 9 \text{ min})$ d $P(7 \text{ min} \leq t \leq 15 \text{ min})$

2.05 THE HYPERGEOMETRIC DISTRIBUTION

In this section, you will look at a simple non-uniform discrete probability distribution that frequently arises in quality control.

IMPORTANT

In a **hypergeometric experiment**, a sample of a particular number of items is taken from a finite population without replacement. The population is divided into two types, classified as success and failure, bad and good, or 1 and 0, with a fixed number of successes. The result of the experiment is the number of successes in the sample.

The sample size is usually written as n, the population size as N and the number of successes in the population is often written as k or M.

Example 9

People with O– blood are known as universal donors because their blood can be used in transfusions for people with any blood type without side effects. A researcher goes to a small school with 250 students. It is known from school records that 15 of the students have blood type O–. The researcher randomly selects a sample of 30 students and finds that 5 have O– blood.

a Is this an example of a hypergeometric experiment?

b List the values of the random variable X, the number of students who have O– blood type.

Solution

a	Is the population finite?	There is a finite population of 250.
	How is the sample taken?	A sample of 30 is taken without replacement.
	Are there a fixed number of successes?	The population is divided into two groups with 15 successes.
	All the conditions are met.	It is a hypergeometric experiment with $n = 30$, $N = 250$ and $k = 15$.
b	There are only 15 students with blood type O–.	$X = 0, 1, 2, 3, \ldots, 15$

As shown in Example **9**, there is a discrete random variable associated with hypergeometric experiments.

IMPORTANT

A **hypergeometric distribution** is the probability distribution of a hypergeometric experiment. The random variable is the number of successes. For a hypergeometric distribution with a sample size of n from a population of N with k successes, X has a domain from 0 to the minimum of n and k.

You can calculate the probabilities $P(X = x)$ in a hypergeometric probability distribution using the methods you learnt last year.

There are $\binom{k}{x}$ $({}^kC_x)$ ways of choosing x successes from the k successes in the population.

There are $\binom{N-k}{n-x}$ ways of choosing $n - x$ failures from the $N - k$ failures in the population.

Thus there are $\binom{k}{x}\binom{N-k}{n-x}$ ways of choosing a sample of n with k successes.

There are $\binom{N}{n}$ ways of choosing a sample of n from the population of N.

This gives the probability below.

IMPORTANT

For the hypergeometric random variable X, $P(X = x) = \dfrac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$, where $x \le n$ and $x \le k$.

Example 10

A box contains 50 electrical safety switches, of which 20% are known to be faulty. If a sample of 8 switches is selected at random and tested, calculate the probability that:

a two will be defective

b at least two will be defective.

Solution

a Identify the variable. Let X = number of faulty switches

Identify the parameters. $N = 50$, $n = 8$, $k = 0.2 \times 50 = 10$

Write the formula.

$$P(X = x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

Substitute the parameters and $x = 2$.

$$P(X = 2) = \frac{\binom{10}{2}\binom{50-10}{8-2}}{\binom{50}{8}}$$

Evaluate.

$$= \frac{\binom{10}{2}\binom{40}{6}}{\binom{50}{8}}$$

$$= 0.321\,724\ldots$$

State the result. The probability of two faulty switches is about 0.3217.

b Find the complement. $P(X < 2) = P(X = 0) + P(X = 1)$

Use the formula.

$$= \frac{\binom{10}{0}\binom{40}{8}}{\binom{50}{8}} + \frac{\binom{10}{1}\binom{40}{7}}{\binom{50}{8}}$$

Calculate the value.

$$= 0.490\,502\ldots$$

Find the complement.

$$P(X \geq 2) = 1 - 0.490\,502\ldots$$
$$= 0.509\,497\ldots$$

State the result. The probability of at least two faulty switches is about 0.5095

One area where hypergeometric probability distributions are used is known as acceptance testing. This is a process in manufacturing where a sample of the output of a full production batch is tested to determine if the entire batch is acceptable.

Example 11

A computer manufacturer orders 500 printer circuit boards from a supplier. To determine whether or not to accept the order, the manufacturer randomly selects 15 circuit boards and tests them. The manufacturer will only accept the full order if there are no defective circuit boards in those that have been randomly selected for testing. Find each of the following probabilities.

a The probability of no faulty boards if 5% of the circuit boards are defective.

b The probability of no faulty boards if 10% of the circuit boards are defective.

Solution

a Identify a success. — Success = a defective circuit board

Identify the parameters. — $N = 500$, $n = 15$, $k = 0.05 \times 500 = 25$

Find $P(X = 0)$. — $$P(X=0) = \frac{\binom{25}{0}\binom{500-25}{15-0}}{\binom{500}{15}}.$$

Evaluate. — $= 0.458\ 096\ldots$

State the result. — When 5% of all boards are defective, the probability that none of the sample boards is defective is about 46%.

b Identify the parameters. — $N = 500$, $n = 15$, $k = 0.1 \times 500 = 50$

We want to know $P(X = 0)$. — $$P(X=0) = \frac{\binom{50}{0}\binom{500-50}{15-0}}{\binom{500}{15}}$$

Evaluate. — $$= \frac{\binom{50}{0}\binom{450}{15}}{\binom{500}{15}}$$

$= 0.201\ 045\ldots$

State the result. — When 10% of all boards are defective, the probability that none of the sample boards is defective is about 20%.

In Example **11**, the manufacturer would obviously prefer to know the probability of an acceptable number of faulty circuit boards if the number in the sample was zero, but this would require much more extensive calculations.

EXERCISE 2.05 The hypergeometric distribution

Concepts and techniques

WS Hypergeometric probability experiments

1 A hypergeometric probability experiment with a population of 50 has a random variable, X. If the sample size is 5 and it is known that there are 12 successes in the population, what is the maximum value of x?

A 5 B 7 C 12 D 38 E 50

2 Example 9 Which of the following is not a characteristic of a hypergeometric probability experiment?

A The population to be sampled must be finite.

B For each trial of the experiment, there are only two possible outcomes.

C A finite sample is randomly selected from the population.

D The probability of success in each trial is constant.

E The exact number of successes in the population is known.

3 A hypergeometric probability experiment with $N = 200$ has a random variable Y. If $n = 15$ and $k = 10$, what is the maximum value of y?

A 5 B 10 C 15 D 25 E 200

Use the hypergeometric probability experiment defined by $P(X=3) = \dfrac{\binom{12}{3}\binom{38}{2}}{\binom{50}{5}}$ to answer questions **4** and **5**.

4 What is the sample size for the experiment?

A 2 B 5 C 12 D 38 E 50

5 What is the number of success in the population?

A 2 B 5 C 7 D 12 E 38

6 Example 9 For each of the following hypergeometric experiments, identify the random variable (X) and determine the values of N, n, k and the possible values of X.

a Four cards are drawn from a well shuffled standard deck of playing cards and the number of hearts selected is noted.

b An electrical appliance has 6 transistors. It is known that two of the transistors are faulty but it is not known which two. Three transistors are randomly selected and removed to test to see if they are faulty.

c It is known that 18% of a batch of 50 computer chips are defective. A computer manufacturer randomly selects a sample of 10 chips for testing before the batch is purchased to see if any are defective.

d A production run of 200 printed circuit boards contains 10% that are defective. 40 of the circuit boards are randomly selected to see if they are defective.

e A large office area has 95 workstations and 35 of these are each equipped with only a laptop computer. The remainder have only desktop computers. The laptops require a special login code in order to access the company's network. 12 staff are randomly assigned to a workstation. Each of the staff members is then asked if they require a special login code.

7 For the hypergeometric experiments described in question **6**, calculate the probability that:

a two of the four cards drawn are hearts

b one of the three transistors selected is faulty

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c four of the computer chips tested are defective

d 10 of the printed circuit boards selected are defective

e three of the staff who are asked require a special login code.

8 A hypergeometric probability experiment is conducted with the given parameters below. Find the probability correct to 4 decimal places of obtaining x successes in each case.

a $N = 120$, $n = 20$, $k = 30$, $x = 7$

b $N = 50$, $n = 15$, $k = 10$, $x = 3$

c $N = 200$, $n = 50$, $k = 40$, $x = 9$

d $N = 30$, $n = 8$, $k = 5$, $x = 3$

e $N = 150$, $n = 25$, $k = 40$, $x = 11$

Reasoning and communication

9 Six cards are drawn from a standard deck of playing cards without replacement. What is the probability of drawing two spades?

10 Example 10 A batch of 100 circuit breakers contains 15% that are defective. If a sample of 19 circuit breakers is selected at random and tested, calculate the probability that:

a three will be defective

b at least three will be defective.

11 In a group of 200 people, 12 are known to suffer from colour blindness. A researcher randomly selects a sample of 20 people from the group and tests them.

a What is the probability that exactly three are colour blind?

b What is the probability that at least one is colour blind?

12 In the game of Lotto, a player picks 6 numbers from the numbers 1, 2, 3, …, 45 and marks them on a card. This is called an entry. Six balls numbered 1 to 45 are then randomly selected by the Lotto organisers. Prizes are won by players who select 3, 4, 5 or 6 numbers that match the ones that have been randomly selected. The jackpot (major prize) is won if the player has 6 matching numbers.

a What is the probability that a single entry will win the jackpot?

b What is the probability that a single entry will win a prize?

13 Example 11 A manufacturer received an order of 250 machined components. The components are only acceptable if they are within ±1 mm of the specified dimensions. It is known that 12 of the components are not within ±1 mm of the required dimensions and are therefore unusable. The order will be rejected if there are four or more unusable components. The manufacturer decides to randomly select 20 components for testing to see if they are within the specified dimensions.

a What is the probability that no unusable components are found?

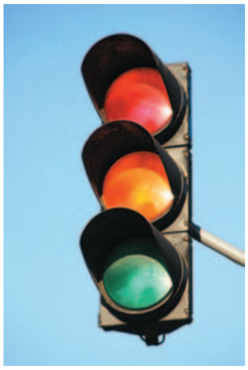
b What is the probability that three unusable components are found?

c What is the probability that the order will be rejected?

14 In certain criminal trials, a jury is required to reach a unanimous verdict in order to convict an accused person. If a unanimous verdict is not reached, the jury is said to be 'hung' and the trial is abandoned. A particular jury consists of 12 people randomly selected from a pool of 50 potential jurors, of which 3 would never be willing to convict, regardless of the evidence presented at the trial. What is the probability that the trial will result in a hung jury, regardless of the evidence presented?

2.06 EXPECTED VALUE

Probability can be an important part of the decision-making process for a broad range of activities.



123RF/alinochka5

Example 12

Susan drives through 8 sets of uncoordinated traffic lights on her way to work. She needs to stop at the following numbers of red lights on 20 successive days on her way to work.

4	5	4	5	5	1	3	4	5	8	2	4	2
6	4	4	1	5	4	6						

a Estimate the probability distribution for R, the number of red lights obtained.

b Calculate the average number of red lights she needs to stop at on a trip.

Solution

a Make a frequency table and estimate the probabilities as relative frequencies.

Red lights, r	Frequency	$P(R = r)$
0	0	0.00
1	2	0.10
2	2	0.10
3	1	0.05
4	7	0.35
5	5	0.25
6	2	0.10
7	0	0.00
8	1	0.05
Total	**20**	**1**

b Work out the average number of red lights.

$$\text{Average} = \frac{4+5+4+5+5+1+\ldots}{20}$$

Calculate the answer.

$$= 4.1$$

State the result.

There is an average number of 4.1 red lights per trip.

Expected values

In Example **9**, the average number of red lights is the number of red lights Susan would (on average) expect to get on her way to work. This value can be worked out using the estimated probabilities. Each value of $p(r)$ above has been calculated by dividing by the total frequency, so $p(r) = \dfrac{f}{\sum f}$ for each value of r.

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You already know that for the variable X, the **mean**, $\bar{x}$, is given by

$$\bar{x} = \frac{\sum fx}{\sum f}$$

You can rewrite this as $\bar{x} = \sum x \times \frac{f}{\sum f} = \sum x \cdot p(x)$

In the case of the traffic lights, this gives

$$\bar{x} = 0 \times 0.00 + 1 \times 0.10 + 2 \times 0.10 + 3 \times 0.05 + 4 \times 0.35 + 5 \times 0.25 + 6 \times 0.10 + 7 \times 0.00 + 8 \times 0.05$$

$$= 4.1.$$

You get the same result as adding individual scores or by using frequencies. Expected value is defined in this way for probability distributions in general, not just for those arising from experimental probabilities.

IMPORTANT

The **expected value**, μ or $E(X)$, of the probability distribution of the discrete random variable X is the mean value of X. For a discrete probability distribution, it can be calculated using the formula $\mu = E(X) = \sum xp(x)$.

Example 13

Find the expected value for the following distribution.

x	1	2	3	4	5
$P(X = x)$	0.2	0.1	0.3	0.1	0.3

Solution

Use the rule for expected value. $E(X) = \sum xp(x)$

Use the values from the distribution. $= 1 \times 0.2 + 2 \times 0.1 + 3 \times 0.3 + 4 \times 0.1 + 5 \times 0.3$

Evaluate and state the result. $= 3.2$

In Example **10**, the expected value for the distribution is not an actual value in the domain of the discrete random variable. This is because $E(X)$ is the theoretical mean.

A graphics calculator can be used to calculate expected value.

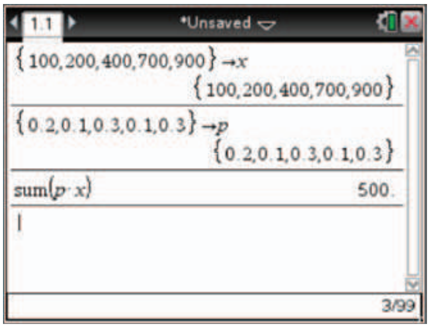
Consider the probability distribution shown here.

x	100	200	400	700	900
$P(X = x)$	0.2	0.1	0.3	0.1	0.3

You can use a graphics calculator to find the expected value of the distribution as follows.

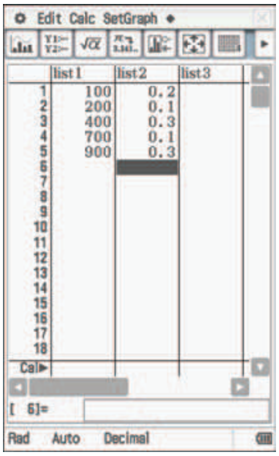
TI-Nspire CAS

Use the Calculator page.
Type in {100, 200, 400, 700, 900} and press ctrl var (sto→) x.
Type in {0.2, 0.1, 0.3, 0.1, 0.3} sto→ p.
Then add the products by typing in sum(p × x).

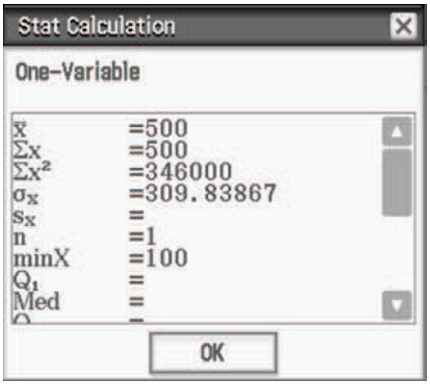


ClassPad

Use the Statistics menu and find the mean in the usual way.
Enter 100, 200, 400, 700, 900 in list 1 and the probabilities (instead of the frequencies) in list 2.
Tap Calc and choose One Variable. Set Xlist to list1 and Freq to list2.



Select the mean, $\bar{x} = 500$.



There are also other ways to use your CAS calculator to find expected values.

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EXERCISE 2.06 Expected value

Concepts and techniques

Expected values using a calculator

Using expected values

1 Example 12 The results of a test out of 40 are shown below.

22 36 12 16 35 30 25 28 26 20 18 15 32
27 21 13 8 30 17 27

Estimate the expected result for the test.

A 11.45 **B** 21.8 **C** 22.9 **D** 23.4 **E** 24.1

2 Example 13 A discrete random variable X can take the values 0, 1 and 2 with probabilities 0.2, 0.5 and 0.3 respectively. What is the expected value of X?

A 0.8 **B** 0.9 **C** 1 **D** 1.1 **E** 1.2

3 The following is the estimated probability distribution for the number (N) of laptop computers in a household in a particular town. Estimate the expected value of N.

n	0	1	2	3	4
$p(n)$	0.2401	0.4116	0.2646	0.0756	0.0081

A 1.10 **B** 1.20 **C** 1.35 **D** 1.40 **E** 1.44

4 Calculate $E(X)$ for the following probability distribution.

x	0	5	10	15
$P(X = x)$	0.5	0.1	0.2	0.2

5 Find the expected value for each of the following distributions.

a

n	2	3	11
$p(n)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

b

n	−5	−4	1	2
$p(n)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

c

n	1	3	4	5
$p(n)$	0.4	0.1	0.2	0.3

6 A random variable has the following distribution.

x	0	1	2	5
$P(X = x)$	0.1	0.2	0.3	0.4

Calculate $E(X)$.

7 The results in a subject, out of 100, are shown below.

32 45 68 90 45 63 55 17 50 60 36 67 70
85 55 65 67 57 43 80

a Find the expected value of the variable F for the test results.

The test results are rescaled to a mark M out of 30.

b Find the expected value of the variable M.

8 CAS Calculate the expected value of the variable in each of the following distributions.

a

x	1200	1300	1400	1500	1600
$p(x)$	0.056	0.189	0.274	0.296	0.185

b

y	300	310	320	330	340	350	360
$p(y)$	0.268	0.333	0.162	0.14	0.062	0.023	0.012

9 A random variable has the following probability distribution.

x	1	2	3	4
$p(x)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{3}$	$\frac{1}{6}$

Calculate $E(X)$

Reasoning and communication

10 A coin is tossed three times. Let X = the largest number of successive heads.

a Calculate $E(X)$ if the coin is fair.

b Calculate $E(X)$ if the coin is weighted so that $P(\text{H}) = \frac{2}{3}$.

11 Two dice are rolled and S, the equal or smaller of the two numbers on the dice, is noted.

a Draw up the probability distribution.

b Calculate the expected value.

12 Five identically-shaped balls numbered 1 to 5 are held in a bag. A ball is drawn and not replaced and then another ball is drawn. If S = the sum of the numbers drawn, calculate:

a the probability distribution for S

b $E(S)$

13 A game is played in which a pair of dice is rolled and a counter moved forward 1, 2 or 4 places according to the numbers shown on the dice. The counter is moved 1 place forward if the numbers differ by three or more, 2 places if they differ by one or two and 4 places if they are equal. How many places would you expect the counter is moved by for each roll of the dice?

14 A random variable X has the following probability distribution.

x	1	2	3	4	5
$P(X = x)$	$8k$	$5k$	$4k$	$2k$	k

a Calculate the value of k.

b Without calculating its value, explain why you expect $E(X)$ to be 3, greater than 3, or less than 3.

c Calculate $E(X)$.

15 A random variable W has the following probability distribution.

w	2	3	5	8	12
$P(W = w)$	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{1}{4}$	x	y

If $E(X) = 5\frac{2}{3}$, find the values of x and y.

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16 A fair coin is tossed until a head or 5 tails occur. If X is a random variable representing the number of tosses of the coin, find the expected value of X.

17 In a 6-cylinder car engine, 2 spark plugs are defective. Three spark plugs are removed at random and checked during a tune-up. If X is the number of defective spark plugs found, calculate the expected number of defective spark plugs found.

2.07 VARIANCE AND STANDARD DEVIATION

The expected value of a discrete random variable is a measure of its average value. In this section you will look at measuring the spread of a discrete random variable. Intuitively, the distance of a value from the mean measures how far apart that item is from the standard result, so the average distance from the mean would be a measure of spread. However, if you added the values of $x - \mu$, the ones above the mean would cancel out the ones below. Squaring the distance from the mean makes them all positive so the sum is not zero, and emphasises the values further from the mean. Hence the average of the squares of the distances from the mean gives a reasonable measure of spread. This is $E([X - \mu]^2)$. The square root of this value will give a value on the same scale as the original variable. $E([X - \mu]^2)$ can be converted to a more convenient form for calculation as follows.

$$\begin{aligned} E\left([X-\mu]^2\right) &= \sum [X-\mu]^2 p(x) \\ &= \sum X^2 p(x) - \sum 2X\mu p(x) + \sum \mu^2 p(x) \\ &= E\left(X^2\right) - 2\mu \sum Xp(x) + \mu^2 \sum p(x) \end{aligned}$$

Since $\mu = \sum Xp(x)$ and $\sum p(x) = 1$, then

$$\begin{aligned} E\left(X^2\right) - 2\mu \sum Xp(x) + \mu^2 \sum p(x) &= E\left(X^2\right) - 2\mu\mu + \mu^2 \\ &= E\left(X^2\right) - 2\mu^2 + \mu^2 \\ &= E\left(X^2\right) - \mu^2 \end{aligned}$$

Hence $E\left([X-\mu]^2\right) = E\left(X^2\right) - \mu^2$.

IMPORTANT

For the discrete random variable X with expected value $E(X) = \mu$, the **variance** is given by

$$\begin{aligned} Var(X) &= E[(X-\mu)^2] \\ &= E(X^2) - \mu^2 \end{aligned}$$

and the **standard deviation** is

$$SD(X) = \sigma = \sqrt{Var(X)}$$

Example 14

A die is rolled and X = the square of the score. Calculate the mean, variance and standard deviation of X.

Solution

Show the probability distribution for X.

x	1	4	9	16	25	36
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Calculate $E(X)$.

$$E(X) = 1 \times \frac{1}{6} + 4 \times \frac{1}{6} + 9 \times \frac{1}{6} + 16 \times \frac{1}{6} + \cdots$$

Work out the answer.

$$\mu = 15\frac{1}{6}$$

Now calculate $E(X^2)$.

$$E(X^2) = 1 \times \frac{1}{6} + 16 \times \frac{1}{6} + 81 \times \frac{1}{6} + \cdots$$

Work out the answer.

$$= 379\frac{1}{6}$$

Write the rule for variance.

$$Var(X) = E(X^2) - \mu^2$$

Substitute in the values.

$$= \frac{2275}{6} - \left(\frac{91}{6}\right)^2$$

Work out the answer.

$$= \frac{5369}{36}$$

$$\approx 149.139$$

Calculate the standard deviation.

$$\sigma = \sqrt{\frac{5369}{36}}$$

$$\approx 12.212$$

State the result.

The mean is $15\frac{1}{6}$, the variance is about 149.1 and the standard deviation is about 12.2.

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Example 15

The number of coins given in change to customers at a fast food restaurant are recorded below.

Coin	5c	10c	20c	50c	\$1	\$2
Number	143	151	215	52	141	98

a Estimate the probability distribution of X, the value of coins given in change.

b Estimate the expected value, variance and standard deviation of X.

Solution

a Use relative frequencies for the probability distribution.

x	0.05	0.1	0.2	0.5	1	2
$P(x)$	0.1788	0.1888	0.2688	0.0650	0.1763	0.1225

b Calculate $E(X)$.

$E(X) \approx 0.05 \times 0.1778 + 0.1 \times 0.1888 + \cdots$

Work out the answer.

$\mu \approx 0.5353$

Now calculate $E(X^2)$.

$E(X^2) \approx (0.05)^2 \times 0.1788 + (0.1)^2 \times 0.1888 + \cdots$

Work out the answer.

≈ 0.6956

Write the formula.

$Var(X) = E(X^2) - \mu^2$

Substitute in the values.

$= 0.6956 - (0.5353)^2$

Calculate the result.

≈ 0.4090

Find the standard deviation.

$SD(X) = \sqrt{Var(X)}$

≈ 0.6395

Write the answer.

The expected value is about 53 cents, the variance is about 0.41 and the standard deviation is about 64 cents.

In Example **15**, the values of the discrete random variable are somewhat disparate, rather than being consecutive integers.

WS

Variance and standard deviation

Example 16

CAS A probability distribution for the discrete random variable, X, is given by $P(X = x) = \dfrac{x^2 + 1}{34}$ when $x = 1, 2, 3, 4$. Work out the following.

a The probability distribution.

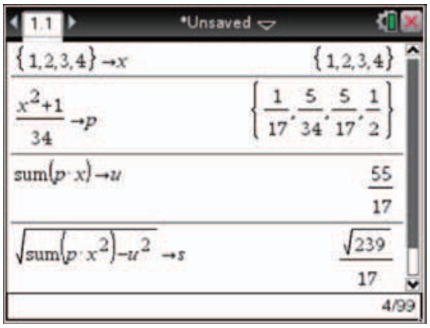
b The expected value of X, correct to 3 decimal places.

c The standard deviation of X, correct to 3 decimal places.

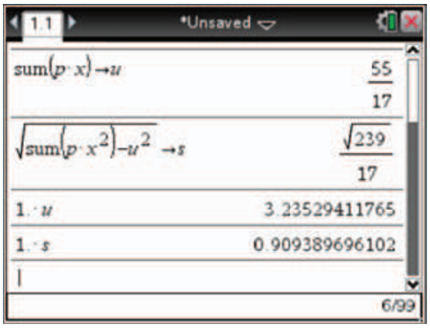
Solution

TI-Nspire CAS

a Use the Calculator page.
Store {1,2,3,4} in x.
Store the probabilities in p.



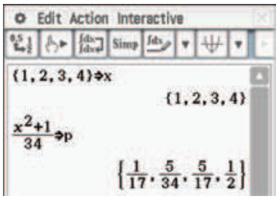
b Calculate the expected value using sum $(p \times x)$ and store in u.



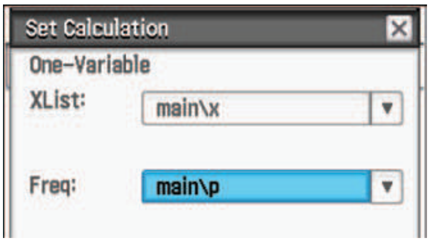
c Calculate the standard deviation using $\sqrt{\text{sum}(p \times x^2) - u^2}$ and store in s.
You switch to approximate calculation and multiply by 1 to approximate.

ClassPad

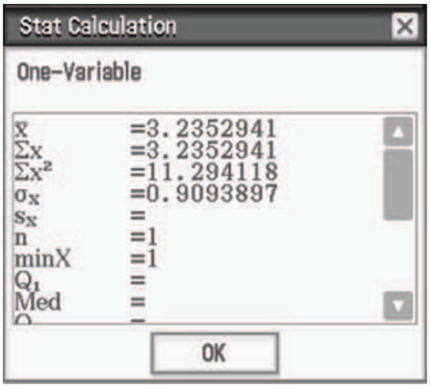
a Start with the Main menu. Set the calculator to Standard.
Store {1,2,3,4} in x.
Store the probabilities in p.



b Go to the Statistics menu.
and Rename list1 as x and list2 as p.



c The values automatically appear.
Tap Calc, One-Variable and make XList main\x and Freq main\p.
Read the mean ($\bar{x}$) and standard deviation (σ_x).
You can also use the method as shown in the TI-Nspire CAS.



Write the answers.

The probability distribution is $P(1) = \frac{1}{17}$, $P(2) = \frac{5}{34}$, $P(3) = \frac{5}{17}$, $P(4) = \frac{1}{2}$. The expected value is about 3.24 and the standard deviation is about 0.91.

The list processing shown in the last example is a powerful tool of CAS calculators.

EXERCISE 2.07 Variance and standard deviation

Concepts and techniques

1 Example 14, 15 The probability distribution for a random variable, Z, is shown below.

z	1	2	3	4
$p(z)$	0.1	0.2	0.3	0.4

The variance of Z is:

A 1.0 **B** 1.2 **C** 1.6 **D** 1.8 **E** 2.0

2 The probability distribution for a random variable, Y, is shown below.

y	100	200	300	400
$p(y)$	0.2	0.4	0.3	0.1

The standard deviation of Y is:

A 90 **B** 99 **C** 100 **D** 108 **E** 117

3 The probability distribution for a random variable, M, is shown below.

m	1	2	3	4
$p(m)$	a	$2a$	$3a$	$4a$

The standard deviation of M is:

A 1.0 **B** 1.2 **C** 1.6 **D** 1.8 **E** 2.0

4 The probability distribution of W is shown below.

w	0	2	4	6
$P(W = w)$	0.1	0.3	0.4	0.2

Find the variance and standard deviation of W.

5 For a discrete random variable X, $E(X) = 14$ and $E(X^2) = 280$. Find the variance of X.

6 For a discrete random variable Y, $E(Y) = 23$ and $E(Y^2) = 587$. Find the standard deviation of Y.

7 The values of a random variable, X, are recorded as 7, 6, 7, 3, 7, 6, 7, 8, 7, 6, 4, 7, 8, 7, 9, 8, 7, 9, 7, 8.

a Construct a probability distribution for X.

b Calculate $E(X)$.

c Calculate $Var(X)$.

8 Thirty shopkeepers in a coastal town were asked how many additional staff they intended to hire for the next holiday season. Their responses are shown below.

4, 5, 7, 9, 8, 2, 4, 2, 5, 3, 6, 7, 4, 4, 5, 6, 5, 5, 7, 8, 8, 0, 4, 6, 1, 9, 3, 3, 4, 6

If N = number of additional staff, calculate:

a the probability distribution for N

b $E(N)$

c $Var(N)$

d the standard deviation.

9 Calculate the mean (μ), variance and standard deviation (σ) for each of the following distributions.

a

n	−1	0	1	2	3
$p(N = n)$	0.1	0.3	0.1	0.2	0.3

b

r	11	12	13	14	15	16	17
$p(R = r)$	0.12	0.18	0.21	0.19	0.16	0.11	0.03

10 Example 16 CAS A probability distribution for the discrete random variable, Z, is given by $P(Z = z) = \dfrac{z^2 + 2}{22}$ when $z = 0, 1, 2, 3$. Find the following.

a The probability distribution of Z.

b The expected value of Z, correct to 3 decimal places.

c The standard deviation of Z, correct to 3 decimal places.

11 CAS A discrete random variable has the following probability distribution.

y	2	4	6	8	10
$p(y)$	0.16	0.23	0.31	0.18	0.12

Calculate $Var(Y)$

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12 CAS Calculate the mean (μ), variance and standard deviation (σ) of each of the following distributions.

a

x	1200	1300	1400	1500	1600
$p(x)$	0.056	0.189	0.274	0.296	0.185

b

y	300	310	320	330	340	350	360
$p(y)$	0.268	0.333	0.162	0.14	0.062	0.023	0.012

13 CAS The weights (to the nearest gram) of eggs laid by a number of hens are recorded as follows.

x	51	53	56	59	60	61	66	69
$p(X=x)$	0.05	0.08	0.1	0.23	0.25	0.16	0.09	0.04

where X = the weight of an egg.

Calculate:

a $E(X)$ b $Var(X)$ c σ

Reasoning and communication

14 A discrete random variable, W, has the following probability distribution.

w	1	3	k	6
$P(W=w)$	0.2	0.4	0.1	0.3

Find the value of k, a positive integer, if the variance is 3.41.

15 A discrete random variable, Y, has the following probability distribution.

y	1	k	7	11
$P(Y=y)$	0.1	0.3	0.4	0.2

Find the value of k, a positive integer, if the variance is 10.6.

16 A pair of dice is rolled and the sum, S, of the numbers on the uppermost faces is recorded. Calculate

a $E(S)$ b $Var(S)$ c σ

17 A fair six-sided die has a '2' on one face, '4' on two faces and '6' on the remaining three faces. The die is rolled twice and T = total of the two numbers rolled.

a Construct a probability distribution for T.

b Calculate the values of $E(T)$ and $Var(T)$.

18 Three cards are numbered 1, 2 and 3. The cards are placed face down, one is randomly selected, its number is noted and the card is returned face-down to the table. This procedure is then repeated. Let X = the sum of the two numbers and Y = the smaller of the two numbers.

a Construct a probability distribution for each variable.

b Calculate the values of $E(X)$ and $Var(X)$.

c Calculate the values of $E(Y)$ and $Var(Y)$.

2.08 APPLICATIONS OF DISCRETE RANDOM VARIABLES

You can use the expected value of a distribution to work out if a game of chance is fair. In a fair game of chance, your expected value would be 0. This is because you would expect to come out even in the long run.

Example 17

A dice game involves rolling a pair of dice and finding the total of the numbers on the upper faces. If the total is 11 or 12, you win \$10. If the total is 2, 3 or 4, you win \$4. For any other total, you lose \$2. Is the game fair?

Solution

First calculate the probabilities for the totals using a grid.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(11 \text{ or } 12) = \frac{3}{36} = \frac{1}{12}$

$P(2, 3 \text{ or } 4) = \frac{6}{36} = \frac{1}{6}$

$P(\text{others}) = \frac{27}{36} = \frac{3}{4}$

Choose the random variable.

Let X be the amount won on a roll.

Write out the probability distribution.

x	10	4	-2
$P(x)$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{3}{4}$

Work out the expected value.

$E(X) = 10 \times \frac{1}{12} + 4 \times \frac{1}{6} + (-2) \times \frac{3}{4} = 0$

Write the answer.

The game is fair because you win \$0 (and lose \$0) in the long run, so you 'break even'.

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Casinos, poker machines and other gambling institutions have the payments structured unfairly so the expected value for the player is negative. The proportion they expect to win is usually expressed as a percentage and is called the **house percentage**.

Example 18

A game consists of a wheel divided into twelve equal sectors numbered 1 through 12. You may place a $2 bet on any number. If your number comes up, you get $18 back. That is, you win $16 and get your $2 back.

a What is the house percentage?

b How much should you win for the game to be fair?

Solution

a Choose the random variable.

Let X be the amount won on a bet.

Write out the probability distribution. You win $16 or lose $2.

x	+16	−2
$P(x)$	$\frac{1}{12}$	$\frac{11}{12}$

Calculate the expected value.

$$E(X) = 16 \times \tfrac{1}{12} + (-2) \times \tfrac{11}{12}$$
$$= -0.5$$

Write as an amount.

You would expect to lose 50c on each bet of $2.

Express this as a percentage.

$$\text{House percentage} = \frac{0.50}{2.00}$$
$$= 25\%$$

b Choose a variable.

Let $$a$ be the amount won on a bet.

Rewrite the probability distribution, remembering you get back $$a$ + $2.

x	$+a$	−2
$P(x)$	$\frac{1}{12}$	$\frac{11}{12}$

Recalculate the expected value.

$$E(X) = a \times \tfrac{1}{12} + -2 \times \tfrac{11}{12}$$

Write the value for a fair game.

To be fair, $E(X) = 0$

Solve the equation.

$$a \times \frac{1}{12} + (-2) \times \frac{11}{12} = 0$$
$$a = 22$$

Write the answer.

For the game to be fair, you should win $22, so you should get back $24.

Insurance companies use statistical information to work out premiums. Using expected values, they set the amount you pay for insurance to cover the expected payouts, costs and profits.

Example 19

In Australia, 831 males and 387 females aged between 15 and 24 died in 2011. These worked out to relative frequencies of 0.000 513 and 0.000 253 respectively.

a If there were no costs or profits, what should it cost to insure a male aged 23 for one year with a $200 000 payout on death?

b An insurance company charges $53 for $100 000 cover for a 22-year-old non-smoking female. How much of this is to cover costs and profit?

Solution

a Choose the random variable.

Let X be the amount 'won' on death.

Let the premium be k. Write the probability distribution.

x	$200\,000 - k$	k
$P(x)$	0.000 513	0.999 487

Calculate the expected value.

$E(X) = 0.000\,513(200\,000 - k) + 0.999\,487k$

Write an equation for a fair amount.

$0.000\,513(200\,000 - k) + 0.999\,487k = 0$

Solve to find the premium.

$k \approx \$102.71$

Write the answer.

The premium would be about $102.71.

b Choose the random variable.

Let X = amount paid by client

Write the probability distribution.

x	53	–99 947
$P(x)$	0.999 747	0.000 253

Calculate the expected value.

$E(X) = 0.999\,747 \times 53 + 0.000\,253 \times (-99\,947)$

Work out the amount.

$= 27.7$

Write the answer.

$27.70 is expected for costs and profits.

INVESTIGATION Life insurance

Use information from the Australian Bureau of Statistics (Catalogue number 3303.0) to work out the probability of death for different age groups.

Use the internet or other methods to find the cost of life insurance for a year for different age groups and circumstances (gender, general health, life habits).

Compare the actual life insurance cost with the theoretical cost based on the probabilities of death. Why are there differences?

EXERCISE 2.08 Applications of discrete random variables

Reasoning and communication

1 Example 17 Three dice are rolled. It costs \$5 to play. If all 3 dice are the same, you get \$10 back. If 2 of the dice are the same, you get your \$5 back. Otherwise you lose.
   a How much can you expect to win each time you play the game?
   b Is the game fair?

2 Example 18 In a game of chance, a fair die is rolled three times. If three 4s are rolled, the player wins \$3. If the player rolls two 4s, \$2 is won and one 4 means the player wins \$1. If no 4s are rolled, then the player loses the \$1 bet. Calculate the house percentage for this game.

3 Example 19 An insurance company sells a term life insurance policy that will pay a beneficiary a certain sum of money upon the death of the policy holder. These particular policies have premiums that must be paid annually.
Suppose the company sells a \$300 000 one-year term life insurance policy to a 51-year-old male for an annual premium of \$650. According to actuarial tables, the probability that the 51-year-old male will survive the year is 0.99788. Calculate the expected value of this policy for the insurance company.

4 A local supermarket has four checkout lanes. The manager has determined that the number of lanes in use when it opens at 8:00 a.m. is a random variable having the following probability distribution.

x	0	1	2	3	4
$P(X = x)$	0.1	0.15	0.15	0.2	0.4

Calculate the expected number of checkout lanes in use at 9:00 a.m. at the supermarket.

5 New cars that come off an assembly line are fitted with five new tyres (including a spare). The probability distribution for the number of defective tyres on each car (D) is given by:

d	0	1	2	3	4	5
$P(D = d)$	0.95	0.03	0.015	0.003	0.0015	0.0005

   a Find the expected number of defective tyres per car.
   b What is the most likely number of defective tyres per car?

6 All of the households in a large regional town were surveyed to determine the number of persons living in the household. The results of the survey are shown below.

Number of persons	1	2	3	4	5	6	7
Number of households (1000s)	31.1	38.6	18.8	16.2	7.2	2.7	1.4

   a What is the probability that fewer than 4 persons are in any given household?
   b What is the expected number of persons in any given household?
   c What is the standard deviation of the number of persons in any given household?

7 An individual plays a game in which it is possible to lose $1, break even, win $3 or win $5. The probability distribution for the outcomes of this game are as follows:

Outcome ($)	−1	0	3	5
Probability	0.3	0.4	0.2	0.1

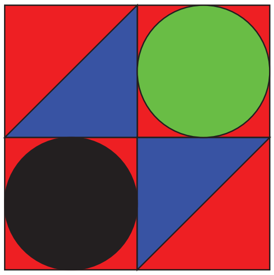
Does this game favour the player or the 'house'? Explain.

8 a The game described in the previous question is adjusted so that all payouts are reduced by $1. How does this impact the expected outcome for the player?

b The adjusted game described in part **a** is further adjusted so that all payouts are doubled. How does this impact the expected outcome for the player?

9 A lottery is run to raise funds for a school building program. The lottery offers a grand prize of $20 000, 20 prizes of $500 each and tickets cost $10. The lottery organisers know that 10 000 tickets will be sold. Determine whether purchasing a ticket is a good 'investment'.

10 Many games of chance are played in 'sideshow alley'. One game consists of throwing a dart that lands on a random spot within a square target as shown here. Each dart costs $5. If the dart lands on the blue triangles, the player gets $5 so comes out even. The black circle gets the player $8, while the green circle gets the player $10. Do you expect to make money or lose money when you play?



11 An insurance company sells term life insurance policies for a 20-year period. A 35 year-old woman takes out one of these policies and pays a premium of $2500. If the insured person dies within the 20-year period, the insurance company pays out $75 000. It is known that the probability that a 35-year-old woman will die within 20 years is 0.017. What is the amount that the insurance company can expect to gain from this policy?

12 An insurance company offers a policy that pays out $50 000 on death. The company wishes to make a profit of $225 on this type of policy. It is known that the probability of a male aged 38 dying is 0.009. What premium will the insurance company need to charge a 38-year-old male in order to achieve the desired profit?

13 Ms Hardsell, an insurance agent, offers a 1-year term life insurance policy to males in a particular age category. The cost of the insurance is $30/thousand dollars of coverage. According to actuarial tables, the probability that a male in this category will die within the next year is 0.005.

a What is the expected gain for the insurer for each thousand dollars of coverage?

b If insurance is sold only in multiples of $1000 and if the overheads for writing such a policy are $70, what is the minimum amount of insurance that Ms Hardsell should sell in a policy in order to have a positive expected gain?

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14 A worker employed to maintain quality control at an electronics plant is paid \$300 per day to test complex electronic equipment made on a production line. Only 5 units a day are produced and the quality controller is paid a bonus of \$200 for each defective unit found. The probability function for the number of defective units is given by:

x	0	1	2	3	4	5
$p(x)$	0.90	0.06	0.02	0.008	0.006	0.006

a Calculate the expected number of defective units per day.

b Find the quality controller's average daily salary.

c What is the quality controller's most likely daily salary?

15 A game involves throwing a pair of dice and adding the numbers on the top of the dice. You lose \$3 if you get a sum of 7 or 11, you win \$6 if both dice show the same number, and no money changes hands in any other case.

a Calculate your expected winnings if you play the game.

b If you had to pay an 'entrance fee' to play the game, what would you consider to be a 'fair' price?

16 In a particular business venture, a person can make a profit of \$10 000 with a probability of 0.2, a profit of \$5000 with a probability of 0.2 and a loss of \$1000 with a probability of 0.6. If X is the return that will result for the person, calculate:

a the expected return

b the most likely return.

17 Two teams, A and B, play in a tournament. The probability that A wins any game in the tournament is $\frac{1}{2}$. The first team to win two games in a row or a total of three games wins the tournament. Find the expected number of games in the tournament.

18 A class of 30 students sits for two separate maths exams. The results of the class for the first exam range from 31% to 96% while the results for the second exam range from 49% to 95%. Based on these results, what can be said about the standard deviation for the distribution of the results for the first exam compared with the standard deviation for the distribution of the results for the second exam?

19 A company is considering an investment and believes that the return in the following year will be a random amount X having the following probability distribution, where x is shown as thousands of dollars. Find the expected return.

x (\$1000s)	100	250	300	350	400
$P(X = x)$	0.15	0.35	0.25	0.15	0.1

2 CHAPTER SUMMARY
DISCRETE RANDOM VARIABLES

- A probability **experiment** has a number of **outcomes**. Each outcome is called a **sample point** and the **sample space** for an experiment consists of all possible outcomes. An **event** is a subset of the sample space.
- The **union** ($\cup$) of events A and B is the combination of either event A or event B or both A and B occurring. The **intersection** ($\cap$) of events A and B is the event that both A and B occur and includes the sample points that are common to A and B.
- The **probability** of event $A = P(A) = \dfrac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$. For any event A, $0 \le P(A) \le 1$. If $P(A) = 1$, then event A is certain to occur. If $P(A) = 0$, then event A cannot occur. The sum of the probabilities of all events for an experiment $= \sum p(x) = 1$.
- The **complement** of event A is represented as A'. The probability of A' is defined as the probability that A will not occur so $P(A) + P(A') = 1$ or $P(A') = 1 - P(A)$.
- The **addition rule** of probability states that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- **Mutually exclusive** events cannot occur simultaneously, so for mutually exclusive events $P(A \cap B) = 0$ and $P(A \cap B) = P(A) + P(B)$.
- The **conditional probability** of event A following the occurrence of event B is written as $P(A \mid B)$ or 'the probability of A given B', and defined as:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- The **multiplication rule** for any events A and B is $P(A \cap B) = P(A \mid B) \times P(B)$.
- For **independent** events $P(A \cap B) = P(A) \times P(B)$
- A variable whose value is determined by chance is said to be **random**. The values a variable can take is called the **domain** of that variable. A random variable is **discrete** if the domain is a finite or countable number of numerical values. A **continuous** random variable is one that can take any value based on measurement along a continuum.
- A **probability function**, $\boldsymbol{P(X = x)}$ or $\boldsymbol{p(x)}$, is a function of a discrete random variable that gives the probability that the outcome associated with that variable will occur. For any probability function:

$$0 \le P(X = x) \le 1 \text{ for all values of } x \text{ and } \sum P(X = x) = 1$$

- A **discrete probability distribution** shows a probability function typically as a table, list or graph.
- In a **uniform probability distribution**, all outcomes are equally likely, so:

$$P(X = x) = \frac{1}{n}, \text{ where } n = \text{the number of outcomes}$$

- For a uniform probability distribution of a discrete random variable, X, with n equally likely outcomes on the integers from a to b:

$$P(X = x) = \frac{1}{n}, \text{ where } n = b - a + 1$$

and

$$P(c \le x \le d) = \frac{d - c + 1}{b - a + 1} \text{ where } a \le c \le d \le b$$

- An integer **random number** is a number selected from a uniform discrete distribution on the integer limits of the number.

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- A **hypergeometric experiment** is the sampling of a particular number of items from a finite population without replacement. The population is divided into two types, classified as success and failure, bad and good, or 1 and 0, with a fixed number of successes. The result of the experiment is the number of successes in the sample.
- The sample size of a hypergeometric experiment is usually written as n, the population size as N, and the number of successes in the population is often written as k or M.
- A **hypergeometric distribution** is the probability distribution of a hypergeometric experiment. The random variable is the number of successes. For the hypergeometric random variable X, $P(X = x) = \dfrac{\dbinom{k}{x}\dbinom{N-k}{n-x}}{\dbinom{N}{n}}$,

  where $x \le n$ and $x \le k$.
- The **expected value**, μ or $E(X)$, of the probability distribution of the discrete random variable X is the mean value of X: $\mu = E(X) = \sum xp(x)$
- The symbol $\sum$ means the sum of all values
- For the discrete random variable X the **variance, *Var*(*X*)**, and **standard deviation**, **σ**, are measures of the 'spread' or dispersion of X.

$$Var(X) = E[(X-\mu)^2] = E(X^2) - \mu^2$$

$$\sigma = \sqrt{Var(X)}$$

- A game is fair if $E(X) = 0$, favourable to the player if $E(X) > 0$ and unfavourable to the player if $E(X) < 0$.

2 CHAPTER REVIEW
DISCRETE RANDOM VARIABLES

Multiple choice

1 Example 1 If $P(C) = 0.75$, $P(D) = 0.36$ and $P(C \cap D) = 0.29$, then $P(C \cup D)$ is equal to:

A 0.07 B 0.65 C 0.8 D 0.82 E 1.11

2 Example 1 If $P(M) = 0.56$, $P(N) = 0.24$ and $P(M \cap N) = 0.18$, find $P(M \mid N)$ correct to 2 decimal places.

A 0.11 B 0.32 C 0.43 D 0.47 E 0.75

3 Example 2 A fair coin is tossed three times and X = the number of heads. The probability function for X is:

A $P(1) = \frac{1}{8}, P(2) = \frac{3}{8}, P(3) = \frac{4}{8}$

B $P(1) = 0.2, P(2) = 0.3, P(3) = 0.5$

C $P(0) = \frac{1}{8}, P(1) = \frac{4}{8}, P(2) = \frac{2}{8}, P(3) = \frac{1}{8}$

D $P(0) = 0.1, P(1) = 0.2, P(2) = 0.3, P(3) = 0.4$

E $P(0) = \frac{1}{8}, P(1) = \frac{3}{8}, P(2) = \frac{3}{8}, P(3) = \frac{1}{8}$

4 Example 3 Which of the following is a valid probability value for a discrete random variable?

A 0.3 B 1.4 C −0.7 D $\frac{4}{3}$ E $\frac{11}{10}$

5 Example 3 A discrete variable can have:

A only integer values

B only specific values

C only rational values

D only integers or halves

E only non-negative integer values.

6 Example 4 Which one or more of the following could represent a probability distribution?

A

x	$p(x)$
0	0.4
1	0.3
2	0.2
3	0.1

B

x	$p(x)$
0	0.1
1	0.3
2	0.4
3	0.1

C

x	$p(x)$
0	−0.4
1	0.3
2	0.6
3	0.5

D

x	$p(x)$
0	0.2
1	0.2
2	0.2
3	0.2

E

x	$p(x)$
0	0.1
1	0.3
2	0.3
3	0.1

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7 Example 9 A hypergeometric probability experiment with a population of 20 has a random variable, X. If the sample size is 4 and it is known that there are 8 successes in the population, what is the maximum value of x?

A 4 B 8 C 12 D 16 E 20

8 Example 10 A hypergeometric probability experiment is defined by $P(X = 4) = \dfrac{\binom{10}{4}\binom{22}{8}}{\binom{32}{12}}$

What is the number of successes in the population?

A 4 B 8 C 10 D 22 E 32

Questions **9-10** refer to the following probability distribution.

x	1	2	3	4
$P(X = x)$	0.2	0.3	0.4	0.1

9 Example 13 $E(X)$ is equal to:

A 1.9 B 2.4 C 2.5 D 2.6 E 3.7

10 Example 14 $Var(X)$ is equal to:

A 0.78 B 0.84 C 1.04 D 1.22 E 1.46

11 Example 15 You have organised a camping weekend in the wet season. The Bureau of Meteorology says that there is a 50% chance of rain on the first day. If it rains on the first day, there is a 50% chance of rain on the second day. If it doesn't rain on the first day, there is only a 30% chance it will rain on the second day. How many rainy days can you expect on the weekend?

A 0.75 B 0.85 C 0.9 D 0.95 E 1.05

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Short answer

12 Example 1 Karla takes her car for a service every 6 months. The probability that the car will need a minor repair is 0.45 and the probability that the car will require a major repair is 0.1. The probability that the car will need both a minor and major repair is 0.04.

a What is the probability that the car will not require a minor repair but will need a major repair?

b What is the probability that the car will not need either a minor or major repair?

c Given that the car requires a minor repair, what is the probability that it will require a major repair?

13 Example 3 Classify each of the following variables as discrete or continuous.

a The number of people in a car

b The distance a car travels

c The number of trips by car in a week

d The time a car is out of the garage

e The number of cylinders in a car

f The dress sizes of women in a bus

g The weights of women in a bus

h The prices of dresses

14 Examples 2, 3 Which of the following could represent probability distributions?

a

x	$p(x)$
10	0.4
15	0.3
20	0.3

b

y	$p(y)$
−1	0.15
0	0.75
1	0.10

c

x	$p(x)$
1	0.7
2	−0.2
3	0.5

15 Example 4 Two probability distributions are defined by $p_1(x) = \frac{x}{10}$ and $p_2(x) = \frac{x^3}{6} + \frac{5x}{6} - \frac{x^4}{60} - \frac{7x^2}{12} - \frac{3}{20}$ for $x = 1, 2, 3, 4$.

a Construct the probability distributions.

b Draw graphs of the distributions.

c Verify that each is a probability distribution.

d Is either distribution uniform?

16 Example 7 A coin is tossed 4 times. Let X be the random variable representing the largest number of successive heads that occur in 4 tosses.

a Construct the probability distribution for X.

b Draw a graph of the distribution.

17 Examples 7, 8 Five marbles in a bag are numbered:

2 7 8 10 12

D is the number of the marble randomly selected from the bag.

a Draw a graph of the probability distribution for D.

b Calculate the probability that the number on the ball that is drawn is odd.

c Describe the distribution.

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18 Examples 6, 12 The numbers of tracks on some randomly selected CDs were as follows.

11	13	19	20	13	11	15	13	14	16	16	14	12
18	20	12	14	14	16	20	20	14	14	12	13	

a Use a frequency table to construct the probability distribution function for the number of tracks, T.

b What proportion of the CDs have exactly 12 tracks?

c What proportion have 15 or more tracks?

19 Example 9 For each of the following hypergeometric experiments, identify the random variable (X) and determine the values of N, n, k and the possible values of X.

a A production run of 40 components contains 7 defective ones. Five components are randomly selected and removed for testing.

b A manufacturer of car tyres reports that among a shipment of 5000 sent to a local distributor, 1000 are slightly faulty. A tyre supplier purchases 10 of these tyres at random from the manufacturer.

c A workplace has 100 employees. It is known that 12% of the employees have blood type O. A group of 15 employees is selected from the workplace for a biological case study.

20 Example 10 A hypergeometric probability experiment is conducted with the given parameters below. Find the probability of obtaining x successes in each case.

a $N = 40$, $n = 10$, $k = 12$, $x = 5$

b $N = 65$, $n = 12$, $k = 25$, $x = 4$

c $N = 250$, $n = 60$, $k = 30$, $x = 15$

21 Examples 13,14 A pair of fair dice is rolled and the difference between the upper faces noted.

a Construct the probability function for D, the random variable of the difference.

b Find the expected value of D.

22 Examples 12–15 The results of a test out of 10 are shown below.

5	8	9	1	2	4	5	5	4	3	6
7	2	3	5	5	6	8	1	10	3	7
7	5	5	8	3	5	6	8	10	2	6

Find the expected value and variance of the variable T for the test results.

23 Example 15 Calculate $E(X)$ and the variance for the following probability distribution.

x	−2	0	2	4
$p(x)$	0.25	0.125	0.5	0.125

24 Example 15 The number of mobile phones sold by a dealership each month has the following probability distribution.

y	200	300	400	500	600	800
$P(Y = y)$	0.05	0.15	0.35	0.25	0.15	0.05

Calculate:

a $E(Y)$

b $Var(Y)$

c the standard deviation for Y.

Application

25 A coin is weighted so that $P(\text{H}) = \frac{3}{4}$ and $P(\text{T}) = \frac{1}{4}$. The coin is tossed 3 times and the results are recorded. If X is a random variable representing the greatest number of successive heads that occur,

a construct the distribution for X

b find the expected value of X.

26 Two normal dice are rolled and the values on the dice are multiplied to give a number between 1 and 36. Find the expected value and variance of the random variable. Is the distribution uniform or non-uniform?

27 In one day's production at a motor vehicle factory, 80 cars are produced. Of these, 18% require paintwork touch ups before they can be sent to dealers for sale. If a sample of 15 cars from the day's production is selected at random and tested, calculate the probability that:

a four will be defective

b at least three will be defective.

28 A particular game that operates from newsagencies involves scratching the special coating away from a panel to reveal 6 amounts of money. If 3 matching amounts are revealed, the player wins that amount.

In a certain series of the game, 10 million tickets are prepared and prizes allocated as follows:

1 @ \$100 000 25 000 @ \$100
2 @ \$25 000 25 000 @ \$10
25 @ \$1000 1 200 000 @ \$2

If it costs \$1 to play the game, calculate:

a the probability of winning a prize

b the expected payout on each ticket

c the house percentage.

123RF/Scott Rothstein

29 A dice game is played according to the following rules. Two dice are thrown. If the total is less than 6 or greater than 9, the player wins \$10. Otherwise, the player loses \$5.

a How much can you expect to win each time you play the game?

b Is the game fair?

c If the game is unfair, how could it be altered to make it fair?

30 In a particular business venture, a person can make a profit of \$10 000 with a probability of 0.2, a profit of \$5000 with a probability of 0.2 and a loss of \$1000 with a probability of 0.6. If X is the profit or loss that will result for the person, calculate:

a the expected value of X

b the most likely value of X.

31 In a game of chance, a player selects (without replacement) three balls from a bag that contains 3 white and 17 red balls. If a player selects 3, 2, or 1 white ball, then the player gets back \$5, \$2 and 50c respectively. If it costs the player 50c to play, how much do you expect the player to win or lose for each play?

32 A company manufactures a mobile phone case that it sells for \$6 each. Estimates of sales demand (the number of cases that will be sold) and variable costs (in dollars) for a sales period are determined to be as follows.

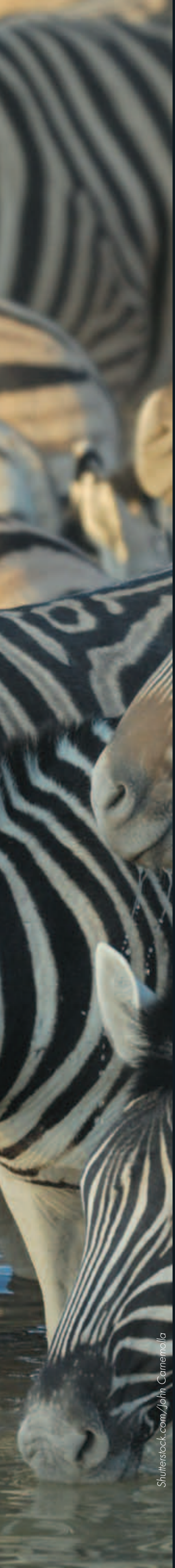
Sales demand	Probability
5000	0.3
6000	0.6
8000	0.1

Variable costs	Probability
\$3.00	0.1
\$3.50	0.3
\$4.00	0.5
\$4.50	0.1

In addition to the variable costs, there are fixed costs of \$8000 regardless of the number of units produced.

Calculate the expected profit for the sales period.

Practice quiz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3

TERMINOLOGY

acceleration
concave downwards
concave upwards
concavity
displacement
horizontal inflection
increments formula
inflection
local maximum
local minimum
optimisation
second derivative
stationary point
turning point
velocity

Prior learning

FURTHER DIFFERENTIATION AND APPLICATIONS

APPLICATIONS OF DERIVATIVES

THE SECOND DERIVATIVE AND APPLICATIONS OF DIFFERENTIATION

- **use the increments formula: $\delta y \cong \frac{dy}{dx} \times \delta x$ to estimate the change in the dependent variable y resulting from changes in the independent variable x (ACMMM107)**
- **understand the concept of the second derivative as the rate of change of the first derivative function (ACMMM108)**
- **recognise acceleration as the second derivative of position with respect to time (ACMMM109)**
- **understand the concepts of concavity and points of inflection and their relationship with the second derivative (ACMMM110)**
- **understand and use the second derivative test for finding local maxima and minima (ACMMM111)**
- **sketch the graph of a function using first and second derivatives to locate stationary points and points of inflection (ACMMM112)**
- **solve optimisation problems from a wide variety of fields using first and second derivatives. (ACMMM113)** AC

3.01 THE INCREMENTS FORMULA

Consider the function $y = f(x) = 3x^2 - 5x + 3$.

When $x = 2$, $y = 3 \times 2^2 - 5 \times 2 + 3 = 5$, so it goes through the point (2, 5).

$\frac{dy}{dx} = 6x - 5$ and the derivative at $x = 2$ is $f'(2) = 6 \times 2 - 5 = 7$.

What is the effect of a small change in the value of x, say $\delta x = 0.01$?

$f(2.01) = 3 \times (2.01)^2 - 5 \times 2.01 + 3 = 5.0703$, and $f(1.99) = 4.9303$.

The change in y, dy is about $0.07 = 7 \times 0.01$, so $\delta y \approx \frac{dy}{dx} \times \delta x$.

This is true for any *small* change in the independent variable of a function.

IMPORTANT

The change in a function is approximately equal to the product of the derivative and the change in the independent variable, provided the change is small:

This can be written as the **increments formula** $\delta y \cong \frac{dy}{dx} \times \delta x$.

You can show this as follows for a general function $y = f(x)$. Consider the small change δx and the corresponding change of the function, δy from point P to point Q. The diagram is shown on the right.

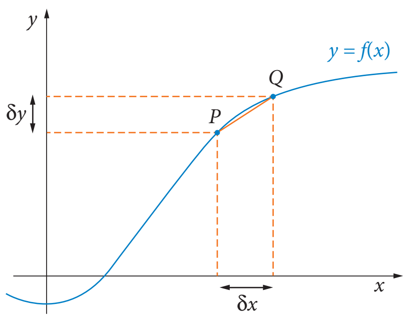
The slope of the **secant** from P to Q is given by $m = \frac{\delta y}{\delta x}$.

You can see this is almost the same as the slope of the curve between P and Q.

But the slope of a curve at any point is the value of the derivative at the point, so $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$.

Multiplying by δx gives the formula: $\delta y \cong \frac{dy}{dx} \times \delta x$.

Clearly, the smaller the value of δx, the more exact the formula will be.



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Example 1

Differentiate $\sqrt{x}$ with respect to x and use the result to find an approximate value for $\sqrt{103}$.

Solution

Write $\sqrt{x}$ as a power.	$f(x) = \sqrt{x} = x^{\frac{1}{2}}$
Differentiate.	$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$
Find a close easy value.	$f(100) = \sqrt{100} = 10$
Find the derivative as well.	$f'(100) = \frac{1}{2}(100)^{-\frac{1}{2}} = 0.05$
Apply the increments formula.	$\delta y \approx \frac{dy}{dx} \times \delta x$
Substitute $\delta x = 3$.	$= 0.05 \times 3$ $= 0.15$
Find $\sqrt{103}$.	$\sqrt{103} \approx f(100) + \frac{dy}{dx} \times 3$
Substitute and evaluate.	$= 10 + 0.15$ $= 10.15$
Write the solution.	$\sqrt{103}$ is about 10.15.

The actual $\sqrt{103} \approx 10.14889157...$ By using the increments formula you have approximated its value correct to 4 significant figures!

Example 2

The cost function to manufacture pocket radios is given by

$$C(x) = 300 + 2x + \sqrt{x} \text{ dollars}$$

iStockphoto/mgkaya

where $C(x)$ is the production cost for a week in which x radios are produced. Normal production is 10 000 radios per week. Find a linear function that approximates the production costs near 10 000 radios. Use the function to find the approximate production cost if the production increases to 10 200 radios.

Solution

Write the cost function. $C(x) = 300 + 2x + \sqrt{x}$

Calculate $C(10\,000)$. $C(10\,000) = 300 + 2 \times 10\,000 + \sqrt{10\,000}$
$= 20\,400$

Differentiate $C(x)$. $C'(x) = 2 + \frac{1}{2}x^{-\frac{1}{2}}$

Calculate $C'(10\,000)$. $C'(10\,000) = 2 + \frac{1}{2}(10000)^{-\frac{1}{2}}$
$= 2.005$

Write the increments formula. $\delta C \approx \frac{dC}{dx} \times \delta x$

Substitute values for 10 200. $= 2.005 \times 200$
$= 401$

Calculate $C(10\,200)$. $C \approx C(10\,000) + \delta C$
$= 20\,400 + 401$
$= 20\,801$

Round and write the result. 10 200 radios will cost about \$20 800 to make.

You can calculate the **percentage change** (or the percentage error) in y by expressing the change δy as a percentage, so percentage change (error) $= \frac{\delta y}{y} \times 100\%$, where $\delta y \approx \frac{dy}{dx} \times \delta x$.

Example 3

Find the percentage error made in the volume of a hot-air balloon of diameter 30 m if no allowance was made for the stretching of the fabric, resulting in a 2% error in the diameter.

123RF/Phaitoon Sutunyawatchai

Solution

Write the rule for volume of a sphere. $V = \frac{4\pi r^3}{3}$

Differentiate. $\frac{dV}{dr} = 4\pi r^2$

Write the increments formula. $\delta V \approx \frac{dV}{dr} \times \delta r$

Write as a percentage. % change of $V = \frac{\delta V}{V} \times 100\%$

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Substitute for δV and V.	$\approx \dfrac{4\pi r^2 \delta r}{\frac{4}{3}\pi r^3} \times 100\%$
Simplify.	$= 3 \times \dfrac{\delta r}{r} \times 100\%$
But $\dfrac{\delta r}{r} \times 100\%$ = % change of r.	$= 3 \times$ % change of r
But % change of r = % change of diameter.	$= 3 \times 2\%$ $= 6\%$
Write the answer.	The percentage error in volume is about 6%.

EXERCISE 3.01 The increments formula

WS Approximation with derivatives

Concepts and techniques

1 Example 1 Use derivatives to find an approximation for $\sqrt{85}$.

2 Use derivatives and the value of $\sqrt[3]{125}$ to find an approximation for $\sqrt[3]{130}$.

3 Use derivatives to find an approximate value for:

a $\sqrt{50}$ b $\sqrt[4]{85}$ c $\sqrt{2536}$ d $\sqrt[5]{250}$ e $\sqrt[6]{70}$

4 Find the value of $64^{\frac{2}{3}}$ and use derivatives to find an approximation for $67^{\frac{2}{3}}$.

5 Use the value of 10^7 to find an approximation for 10.06^7.

6 Use derivatives to find an approximation for 4.05^4. Compare your approximation with the exact value.

Reasoning and communication

7 Example 2 The cost function for a manufacturer of CD racks in dollars is given by

$$C(x) = 4000 + 2.1x + 0.01x^{\frac{3}{2}}$$

where x is the number of racks produced in a week. Use derivatives to estimate the change in production cost if production is increased from 5000 to 5100 racks per week.

123RF/Harald Biebel

8 Example 3 The radius of a sphere is measured to be 5 m ± 10 cm. Find the approximate percentage error of the calculated volume of the sphere.

9 The equation of a curve is $y = 2x^3 - 3x^2 + 4x - 1$.

a Find $\dfrac{dy}{dx}$.

b Find the value of $\dfrac{dy}{dx}$ when $x = 3$.

c Find the approximate change in y as x increases from $x = 3$ to $x = 3.02$.

10 Given that the equation of a curve is $y = 4x^3 - 3x$, find the approximate increase in y as x increases from 2 to 2.03.

11 A spherical ball of radius 12 cm is pumped with air. Find an approximation of the increase in volume of the ball when the radius increases by 0.05 cm.

12 A manufacturer of soft drink cans makes the cans 6 cm in diameter to hold 375 mL.

a What is the height of a can?

b Find an expression for the error in volume caused by a small error in the height of δx.

c Find an expression for the error in volume caused by a samll error of δy in the diameter.

d Find the error in volume caused by an error of 1 mm in the height.

e Find the error in volume caused by an error of 2 mm in the diameter.

13 The gravitational acceleration, g, can be determined by timing a pendulum. If a pendulum of length l has a period of T s, then $g = \dfrac{4\pi^2 l}{T^2}$. A 2 m pendulum is timed to take 57 s for 20 swings.

a Calculate the value of g from the data.

b Find an expression for the approximate error in g for an error of δt in the timing of 20 swings.

c Calculate the possible error in g if the timing was made 'to the nearest second'.

14 The side of a square is measured to be 1 m ± 1 mm. Use derivatives to find the approximate error of the calculated area of the square.

15 The sales manager of a car yard estimates that his staff will sell 80 cars next month. Bonuses and other incentives mean that the profit function for car sales is given by

$$P(n) = 2000n + 10n^2$$

in dollars, where n is the number of cars sold in the month. Use derivatives to find the error in the profit if the manager's estimate of sales is out by each of the following amounts.

a 5%　　b 8%　　c 10%

Alamy/Bert Klassen

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16 The edge of a cube is measured to be 17 cm. What is the approximate percentage error in the calculation of the volume of the cube if there is a 2% error in the measurement of the side?

17 The area of a circle is to be calculated using the measured length of its radius. It is necessary that the area of the circle be calculated with at most 2% error. What is the approximate maximum allowable percentage error that may be made in measuring the radius?

3.02 THE SECOND DERIVATIVE

You know that the derivative of $5x^8$ is $40x^7$. The derivative of $40x^7$ is $280x^6$. So the derivative of the derivative of $5x^8$ is $280x^6$. If you did this again, you would have the derivative of the derivative of the derivative of $5x^8$, which is $1680x^5$. You can keep differentiating derivatives many times. To avoid getting lost, we say the second derivative of $5x^8$ is $280x^6$ and the third derivative of $5x^8$ is $1680x^5$.

IMPORTANT

Derivatives can themselves be differentiated to give **higher derivatives**.

The derivative of a function $y = f(x)$, the **first derivative**, is written as $f'(x)$, $f^{(1)}(x)$ or $\frac{dy}{dx}$.

The derivative of the derivative, the **second derivative**, is written as $f''(x)$, $f^{(2)}(x)$ or $\frac{d^2y}{dx^2}$.

The derivative of $f'(x)$ is written as $f''(x)$, and so on.

The brackets around the derivative number are usually omitted as in $f^2(x)$, $f^3(x)$, …

Example 4

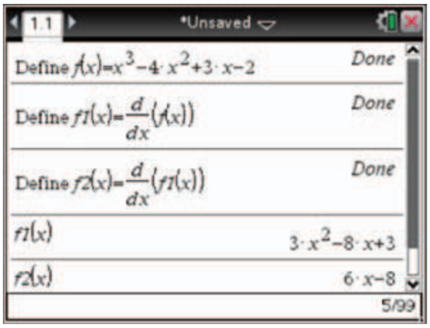
Find the first two derivatives of $f(x) = x^3 - 4x^2 + 3x - 2$ and evaluate $f''(-3)$.

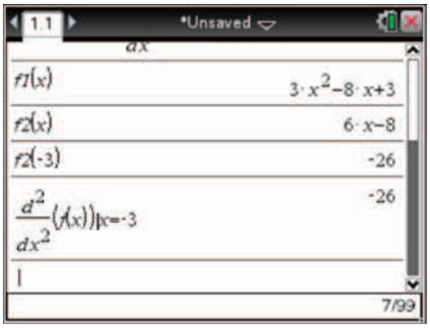
Solution

Write the function.	$f(x) = x^3 - 4x^2 + 3x - 2$
Differentiate.	$f'(x) = 3x^2 - 8x + 3$
Differentiate the first derivative.	$f''(x) = 6x - 8$
Substitute $x = -3$.	$f''(-3) = 6 \times (-3) - 8 = -26$

TI-Nspire CAS

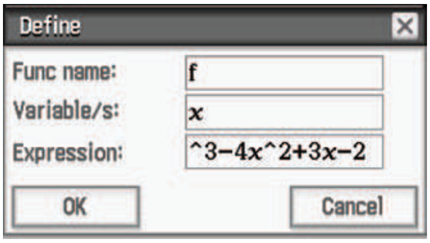
Use a Calculator page.
Use menu, 1: Actions and 1: Define or type Define to define the function.
Then define the derivative using menu, 4: Calculus and 1: Derivative to find $f'(x)$.
Do it again to find $f''(x)$.
Type $f1(x)$ and $f2(x)$ to see the results.
Type $f2(-3)$ to find $f''(-3)$.
You can also use menu, 4: Calculus and 2: Derivative at a point with the variable x, the value -3 and the 2nd derivative.

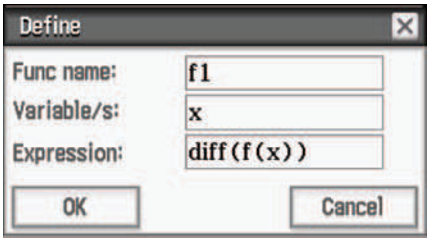




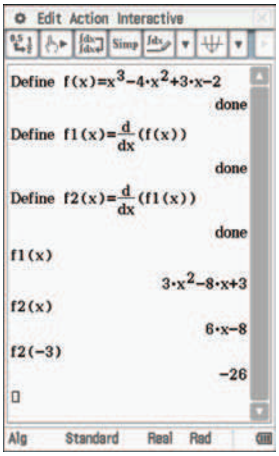
ClassPad

Use the Main menu.
Tap Interactive and Define. Fill in the box as shown.
Repeat for the second function, as shown. Use Math2 and tap , which appears as diff.
The third function, f2(x), $f''(x)$, is similarly defined as the derivative of f1(x).
Do it again to find $f''(x)$.
Type f2(–3) to find $f''(-3)$, or use diff(f(x),x,2,–3).





Type f1(x), f2(x) and f2(−3) to find the derivatives and the value of the second derivative when $x = -3$.



Write the answers. $f'(x) = 3x^2 - 8x + 3, f'(x) = 3x - 8$ and $f''(-3) = -26$

Example 5

Find the second derivative of $y = 2 \sin (3x)$.

Solution

Write the function. $y = 2 \sin (3x)$

Differentiate.
$$\frac{dy}{dx} = [2 \cos (3x)] \times 3$$
$$= 6 \cos (3x)$$

Differentiate again.
$$\frac{d^2y}{dx^2} = 6 \times [-\sin (3x)] \times 3$$
$$= -18 \sin (3x)$$

Higher derivatives

Derivatives have many practical applications. The motion of a particle is one of these applications. You should remember that **velocity** and **acceleration** are both derivatives.

IMPORTANT

The **position** of a particle relative to a fixed point (the origin) is usually shown as x.

A **displacement** is a change of position, usually shown as s.

Velocity, v, is the rate of change of displacement or position, so $v = \frac{dx}{dt} = \frac{ds}{dt}$.

Acceleration, a, is the rate of change of velocity, so $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{d^2s}{dt^2}$.

Example 6

The position of a particle is given by $x = 2t^3 - 5t$ cm, where t is time in seconds.

a Find the velocity of the particle after 3 s.

b Find the initial acceleration.

c Find the acceleration after 2 s.

d At what time will the acceleration reach 30 m/s^2?

Solution

a Write the function.	$x = 2t^3 - 5t$
Differentiate to find the velocity.	$v = \frac{dx}{dt} = 6t^2 - 5$
Substitute $t = 3$.	$v(3) = 6 \times 3^2 - 5 = 49$
Write the answer.	The velocity after 3 s is 49 cm/s.
b Differentiate again to find the acceleration.	$a = \frac{d^2x}{dt^2} = 12t$
Substitute $t = 0$ for the initial value.	$a(0) = 12 \times 0 = 0$
Write the answer.	The initial acceleration is 0 m/s^2.
c Substitute $t = 2$ into a.	$a(2) = 12 \times 2 = 24$
Write the answer.	The acceleration after 2 s is 24 cm/s^2.
d Substitute $a = 30$	$30 = 12t$
Solve to find t.	$t = 2.5$
Write the answer.	The acceleration is 30 cm/s^2 after 2.5 s.

EXERCISE 3.02 The second derivative

Concepts and techniques

1 Example 4 Find the first two derivatives of $x^7 - 2x^5 + x^4 - x - 3$.

2 If $f(x) = x^9 - 5$, find $f''(x)$.

3 Find $f'(x)$ and $f''(x)$ if $f(x) = 2x^5 - x^3 + 1$.

4 Find the second derivative of $y = x^7 - 2x^5 + 4x^4 - 7$.

5 Example 5 Find the second derivative of $y = 5 \cos (2x)$.

6 CAS Differentiate the following twice.

a $y = 2x^2 - 3x + 3$ 　　 b $y = x^{-4}$

7 Find $f'(1)$ and $f''(-2)$ given $f(x) = 3t^4 - 2t^3 + 5t - 4$.

8 CAS If $f(x) = x^4 - x^3 + 2x^2 - 5x - 1$, find $f'(-1)$ and $f''(2)$.

9 If $g(x) = \sqrt{x}$ find $g''(4)$.

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10 Given $h = 5t^3 - 2t^2 + t + 5$, find $\dfrac{d^2h}{dt^2}$ when $t = 1$.

11 Find $f'(x)$ and $f''(x)$ if $f(x) = \sqrt{2-x}$.

12 Find the first and second derivatives of $f(x) = \dfrac{x+5}{3x-1}$.

13 Find $\dfrac{d^2v}{dt^2}$ if $v = (t + 3)(2t - 1)^2$.

Reasoning and communication

14 Find the value of x for which $\dfrac{d^2y}{dx^2} = 3$, given $y = 3x^3 - 2x^2 + 5x$.

15 Find all values of x for which $f''(x) > 0$, given $f(x) = 2x^3 - x^2 + x + 9$.

16 Differentiate $y = (4 \sin(x) - 2)^5$ twice.

17 Find $f'(x)$ and $f''(x)$ if $f(x) = 2 \sin\left(\dfrac{x}{2}\right) - 3 \cos^2(x) + 1$.

18 Find the value of b in $y = bx^3 - 2x^2 + 5x + 4$ if $\dfrac{d^2y}{dx^2} = -2$ when $x = \frac{1}{2}$.

19 Find the value of b if $f(x) = 5bx^2 - 4x^3$ and $f''(-1) = -3$.

20 Example 6 The displacement of an object is given by $x = t^3 - 7t + 4$, where t is in seconds and x is in metres. Find the velocity and acceleration of the object after 3 s.

21 The position of an object is given by $x(t) = t^3 - 6t^2 + 8t + 5$, where $x(t)$ is in metres and t is in seconds.

a Find the velocity after 2 s.

b Find the velocity after 4 s.

c Find the acceleration after 2 s.

d Find the acceleration after 5 s.

22 The displacement of an object is given by $d(t) = 7t^2 - 2t^3 + 3t + 3$, where $d(t)$ is in metres and t is in seconds.

a Find the velocity after 1 s.

b Find the velocity after 3 s.

c Find the acceleration after 1 s.

d Find the acceleration after 3 s.

23 An object is travelling along a straight line. At t seconds, its displacement is given by the formula $x = t^3 + 6t^2 - 2t + 1$ m.

a Find the equations of its velocity and acceleration.

b What will its displacement be after 5 s?

c What will its velocity be after 5 s?

d Find its acceleration after 5 s.

24 The displacement of a particle is given by $s = ut + \frac{1}{2}gt^2$, where $u = 2$ m/s and $g = -10$ m/s^2.

a Find an expression for the velocity of the particle.

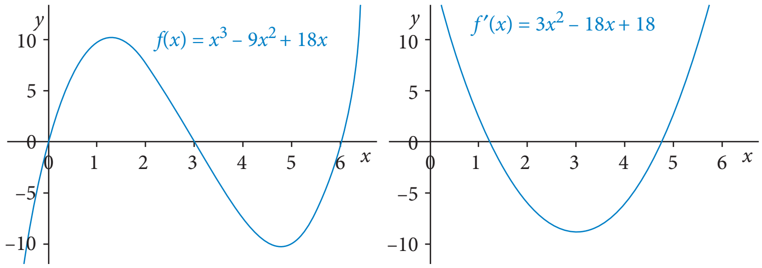
b Find the velocity after 10 s.

c Show that the acceleration is equal to g.

25 The displacement for an object is given by $s = \dfrac{2t-5}{3t+1}$, where s is in metres and t is in seconds. Find the equations for velocity and acceleration.

3.03 THE SECOND DERIVATIVE AND CONCAVITY

You can use extra information from the second derivative to help sketch the graph of a function. Look at the graphs of $f(x) = x^3 - 9x^2 + 18x$ and $f'(x) = 3x^2 - 18x + 18$.



The graph of $f(x)$ is shaped like a mound to the left of $x = 3$ and the graph $f'(x)$ is decreasing.

The graph of $f(x)$ is shaped like a bowl to the right of $x = 3$ and the graph $f'(x)$ is increasing.

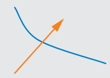
The slope of $f'(x)$ is negative below $x = 3$ and positive above $x = 3$.

This means that $f''(x) = 6x - 18$ is negative below $x = 3$ and positive above $x = 3$. It changes sign at $x = 3$, so the gradient of $f'(x)$ changes direction.

You can always use the sign of $f''(x)$ to tell if $f(x)$ is shaped like a mound or bowl.

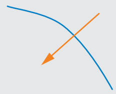
IMPORTANT

- A function is **concave upwards** (or **convex downwards**) where the second derivative is positive. An arrow drawn through the curve towards the centre of curvature will point upwards. It is sometimes called **cup** shaped.



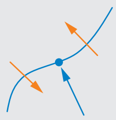
Concave upwards

- A function is **concave downwards** (or **convex upwards**) where the second derivative is negative. An arrow drawn through the curve towards the centre of curvature will point downwards. It is sometimes called **cap** shaped.



Concave downwards

- A **point of inflection** is a point where the concavity of the function changes.



Point of inflection

- A **horizontal point of inflection** is both a stationary point and a point of inflection.

Horizontal point of inflection

A point where $f''(x) = 0$ could be a minimum, a maximum or a point of inflection. It may or may not be a stationary point.

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Example 7

Find the values of x for which the curve $f(x) = 2x^3 - 7x^2 - 5x + 4$ is concave downwards.

Solution

Write the function.	$f(x) = 2x^3 - 7x^2 - 5x + 4$
Find the derivative.	$f'(x) = 6x^2 - 14x - 5$
Find the second derivative.	$f''(x) = 12x - 14$
State the condition for concavity.	$f''(x) < 0$ for concave downwards.
Substitute.	$12x - 14 < 0$
Solve the inequality.	$x < 1\frac{1}{6}$
Write the answer.	$f(x) = 2x^3 - 7x^2 - 5x + 4$ is concave downwards for $x < 1\frac{1}{6}$.

Example 8

Does $y = x^4$ have a point of inflection?

Solution

Write the function. $y = x^4$

Find the derivative. $\frac{dy}{dx} = 4x^3$

Find the second derivative. $\frac{d^2y}{dx^2} = 12x^2$

State the condition. $\frac{d^2y}{dx^2} = 0$ at points of inflection.

Find where the second derivative is zero. $12x^2 = 0$ only at $x = 0$.

State the meaning. There is a possible point of inflection at $x = 0$.

Check the signs of the second derivative before and after $x = 0$ to determine concavity.

x	−1	0	1
$\frac{d^2y}{dx^2}$	12	0	12

Make a conclusion. $\frac{d^2y}{dx^2}$ does not change sign at $x = 0$, so it is not a point of inflection.

Write the answer. $y = x^4$ does not have a point of inflection.

Example 9

Find any points of inflection on the curve $y = x^3 - 6x^2 + 5x + 9$.

Solution

Find the derivative. $\frac{dy}{dx} = 3x^2 - 12x + 5$

Find the second derivative. $\frac{d^2y}{dx^2} = 6x - 12$

State the condition. $\frac{d^2y}{dx^2} = 0$ at points of inflection.

Determine where the second derivative is zero. $6x - 12 = 0$ at $x = 2$.

State the meaning. There is a possible point of inflection at $x = 2$

Check the signs of the second derivative before and after $x = 2$ to determine concavity.

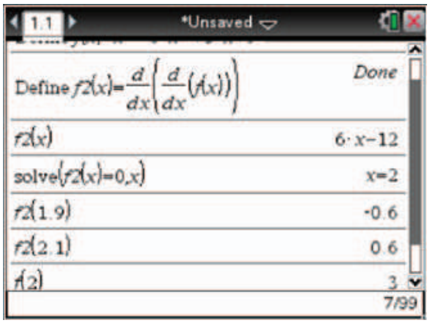
x	1	2	3
$\frac{d^2y}{dx^2}$	−6	0	6

Make a conclusion. $\frac{d^2y}{dx^2}$ changes sign at $x = 2$, so there is a point of inflection at $x = 2$.

Find the point. At $x = 2$, $y = 2^3 - 6 \times 2^2 + 5 \times 2 + 9 = 3$

TI-Nspire CAS

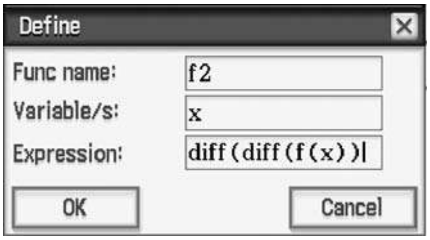
Use a calculator page.
Define the function and find the second derivative. Solve to find the values for which $f2(x) = 0$. Find the point.



ClassPad

Method 1

Use the Main menu.
Define the function and find the second derivative. If you use the interactive menu for define, you will have to take the derivative twice (see right).



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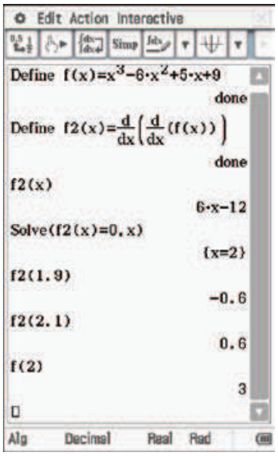
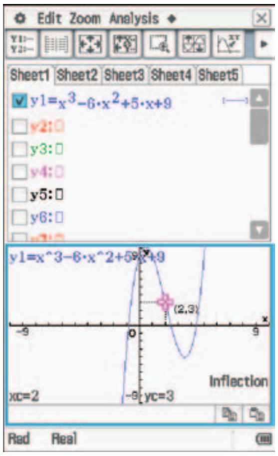
Alternatively, use Define from the Catalog, or type it in, and use }.
Solve to find the values for which $f2(x) = 0$.
Find the point.



Method 2
Use the Graph&Table menu to sketch the graph. You may need to move the graph around and use View Window to ensure that it fits.
Tap Analysis and Inflection.
The inflection is identified as (2, 3).



Answer the question.

Since the concavity changes at $x = 2$, there is a point of inflection at (2, 3).

EXERCISE 3.03 The second derivative and concavity

Concavity

Concepts and techniques

1 Example 7 For what values of x is the curve $y = x^3 + x^2 - 2x - 1$ concave upwards?

2 Find all values of x for which the curve $y = (x - 3)^3$ is concave downwards.

3 Prove that the curve $y = 8 - 6x - 4x^2$ is always concave downwards.

4 Show that the curve $y = x^2$ is always concave upwards.

5 Find the domain over which the curve $f(x) = x^3 - 7x^2 + 1$ is concave downwards.

6 Find all values of x for which the function $f(x) = x^4 + 2x^3 - 12x^2 + 12x - 1$ is concave downwards.

7 Example 8 Determine whether there are any points of inflection on these curves.

a $y = x^6$ b $y = x^7$ c $y = x^5$ d $y = x^9$ e $y = x^{12}$

8 Example 9 Find any points of inflection on the curve $g(x) = x^3 - 3x^2 + 2x + 9$.

9 Find the points of inflection on the curve $y = x^4 - 6x^2 + 12x - 24$.

10 CAS Find any points of inflection on the curve $y = x^4 - 8x^3 + 24x^2 - 4x - 9$

11 CAS For the function $f(x) = 3x^5 - 10x^3 + 7$, find any points of inflection.

Reasoning and communication

12 Sketch a curve that is always concave up.

13 Sketch a curve where $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$.

14 Show that $f(x) = \dfrac{2}{x^2}$ is concave upwards for all $x \neq 0$.

15 a Show that the curve $y = x^4 + 12x^2 - 20x + 3$ has no points of inflection.

b Describe the concavity of the curve.

16 If $y = ax^3 - 12x^2 + 3x - 5$ has a point of inflection at $x = 2$, evaluate a.

17 Evaluate p if $f(x) = x^4 - 6px^2 - 20x + 11$ has a point of inflection at $x = -2$.

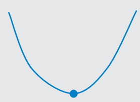
18 The curve $y = 2ax^4 + 4bx^3 - 72x^2 + 4x - 3$ has points of inflection at $x = 2$ and $x = -1$. Find the values of a and b.

3.04 THE SECOND DERIVATIVE TEST

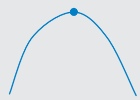
You can tell whether a **stationary point** is a minimum, maximum or point of inflection using the second derivative. You still need to find them using $f'(x) = 0$ first.

IMPORTANT

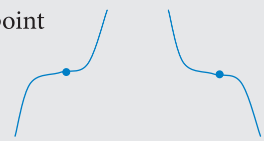
- If $f'(x) = 0$ and $f''(x) > 0$, the stationary point is concave upwards, so it is a local minimum.



- If $f'(x) = 0$ and $f''(x) < 0$, the stationary point is concave downwards, so it is a local maximum.

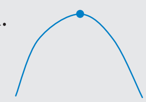


- If $f'(x) = 0$, $f''(x) = 0$, and concavity changes, there is a horizontal point of inflection.



- If $f'(x) = 0$, $f''(x) = 0$ and $f''(x) > 0$ around the value, there is a local minimum.

- If $f'(x) = 0$, $f''(x) = 0$ and $f''(x) < 0$ around the value, there is a local maximum.



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Example 10

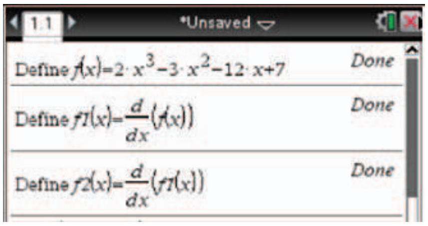
Find the stationary points on the curve $f(x) = 2x^3 - 3x^2 - 12x + 7$ and determine their nature.

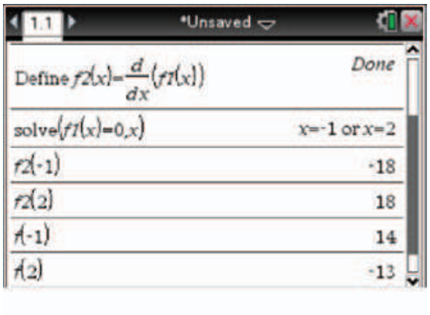
Solution

Find the derivative.	$f'(x) = 6x^2 - 6x - 12$
Find the stationary points.	$6x^2 - 6x - 12 = 0$
Factorise the LHS.	$6(x + 1)(x - 2) = 0$
Solve.	$x = -1$ or $x = 2$
Find the values.	$f(-1) = 14$ and $f(2) = -13$
State the result.	The stationary points are $(-1, 14)$ and $(2, -13)$.
Now find the second derivative.	$f''(x) = 12x - 6$
Find the value at $x = -1$.	$f''(-1) = -18$
Write the conclusion.	$f(x)$ is concave down, so $(-1, 14)$ is a local maximum.
Find the value at $x = 2$.	$f''(2) = 18$
Write the conclusion.	$f(x)$ is concave up, so $(2, -13)$ is a local minimum.

TI-Nspire CAS

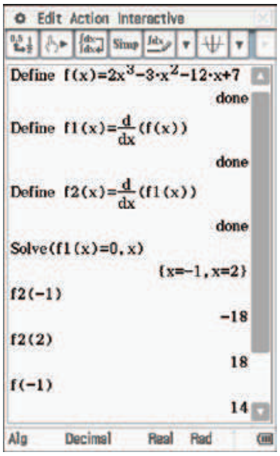
Use a calculator page.
Define the function and its derivatives. Solve $f1(x) = 0$ to find the stationary points. Find the signs of $f2(x)$ at those points.
Find the values as well.





ClassPad

Use the Main menu.
Define the function and its derivatives. Solve f1(x) = 0 to find the stationary points. Find the signs of f2(x) at those points.
Find the values as well.
Alternatively, draw the graph and search for maximum and minimum values, and points of inflection. Any points of inflection must be checked to ensure that the gradient is zero.



Example 11

Find any stationary points on the curve $y = 2x^5 - 3$ and determine their nature.

Solution

Write down the function. $y = 2x^5 - 3$

Differentiate. $\frac{dy}{dx} = 10x^4$

Find the stationary points. $10x^4 = 0$

Solve. $x = 0$ only, so there is one stationary point

Find the second derivative. $\frac{d^2y}{dx^2} = 40x^3$

Find the value at $x = 0$. $f''(0) = 0$

Write a conclusion. It could be any kind of stationary point at $x = 0$

Check concavity near $x = 0$.

x	-1	0	1
$\frac{d^2y}{dx^2}$	-40	0	40

Write a conclusion. $f(x)$ changes from concave up to concave down.

Find the point. $f(0) = 3$

State the result. There is only one stationary point, $(0, -3)$, which is a horizontal point of inflection (on a rising curve).

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EXERCISE 3.04 The second derivative test

Concepts and techniques

1 Example 10 Find any stationary points on the curve $y = x^2 - 2x + 1$ and determine their nature.

2 Find any stationary points on $y = 3x^4 + 1$ and determine what type they are.

3 Show that $y = 3x^2 - 12x + 7$ has only one stationary point and that it is a minimum.

4 Determine the stationary point on the curve $y = x - x^2$ and show that it is a maximum.

5 Example 11 Find any stationary points on the curve $f(x) = 2x^3 - 5$ and determine their nature.

6 Does the function $f(x) = 3x^5 + 8$ have a stationary point? If it does, determine its nature.

7 Find any stationary points on the curve $f(x) = 2x^3 + 15x^2 + 36x - 50$ and determine their nature.

8 Find the stationary points on the curve $y = 3x^4 - 4x^3 - 12x^2 + 1$ and determine their nature.

9 CAS Find any stationary points on the curve $y = (4x^2 - 1)^4$ and determine their nature.

10 CAS Find any stationary points on the curve $y = 2x^3 - 27x^2 + 120x$ and find their types.

11 CAS Find any stationary points on the curve $y = (x - 3)\sqrt{4 - x}$ and determine their nature.

12 CAS Find any stationary points on the curve $y = x^4 + 8x^3 + 16x^2 - 1$ and determine their nature.

Reasoning and communication

13 The curve $y = ax^2 - 4x + 1$ has a stationary point where $x = -3$.

a Find the value of a.

b Hence, or otherwise, determine the nature of the stationary point.

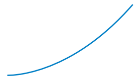
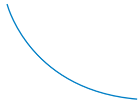
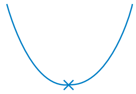
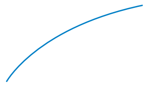
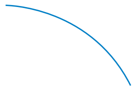
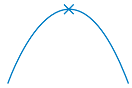
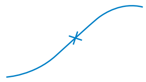
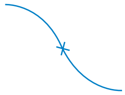
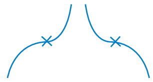
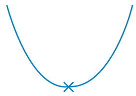
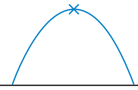
14 The curve $y = x^3 - mx^2 + 8x - 7$ has a stationary point at $x = -1$. Find the value of m.

15 The curve $y = ax^3 + bx^2 - x + 5$ has a point of inflection at $(1, -2)$. Find the values of a and b.

3.05 GRAPH SKETCHING

If you combine the information from the first and second derivatives, this will tell you about the shape of the curve. The first derivative tells you if the curve is increasing, decreasing or stationary, while at the same time, the second derivative tells you if the curve is concave upwards, downwards or stationary.

Here is a summary of the shape of a curve given the first and second derivatives.

	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$
$\frac{d^2y}{dx^2} > 0$			
$\frac{d^2y}{dx^2} < 0$			
$\frac{d^2y}{dx^2} = 0$ and concavity changes			
$\frac{d^2y}{dx^2} = 0$ and it is positive nearby			
$\frac{d^2y}{dx^2} = 0$ and it is negative nearby			

Example 12

$f(2) = -1$, $f'(2) > 0$ and $f''(2) < 0$. Sketch the graph near $x = 2$, showing its shape at this point.

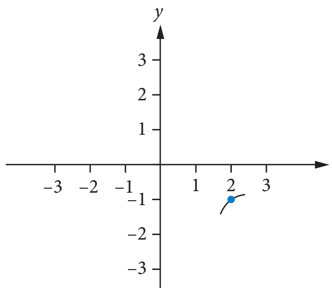
Solution

Interpret $f(2) = -1$. — $f(2) = -1$ means $(2, -1)$ is on the curve.

Interpret $f'(2) > 0$. — $f'(2) > 0$ means the curve is increasing.

Interpret $f''(2) < 0$. — $f''(2) < 0$ means the curve is concave downwards.

Sketch the graph near $(2, -1)$.



Remember that the first derivative is both the gradient and the rate of change of the function.

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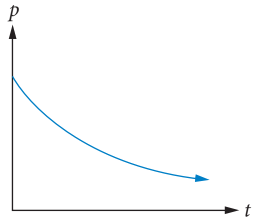
Example 13

The curve below shows the number of unemployed people P over time t months.



a State the signs of $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$.

b How is the number of unemployed people changing over time?

c Is the rate of change of unemployment increasing or decreasing?

Thinkstock/Creatas

Solution

a	The curve is decreasing.	$\frac{dP}{dt} < 0$
	The curve is concave upwards.	$\frac{d^2P}{dt^2} > 0$
b	The curve is decreasing.	The number of unemployed people is decreasing.
c	The curve is concave upwards, the gradient is increasing.	The rate of change of unemployment is increasing (becoming less negative).

You can use the information from the first and second derivatives to work out the shape of a curve at different points. This will help you to sketch the graph.

Example 14

Find any stationary points and points of inflection on $f(x) = x^3 - 3x^2 - 9x + 1$ and hence sketch the curve.

Solution

Write the function.	$f(x) = x^3 - 3x^2 - 9x + 1$
Differentiate.	$f'(x) = 3x^2 - 6x - 9$
Find any stationary points.	$3x^2 - 6x - 9 = 0$
Factorise.	$3(x - 3)(x + 1) = 0$
Solve for x.	$x = -1$ or $x = 3$
Find the values at -1 and 3.	$f(-1) = 6$ and $f(3) = -26$
State the result.	$(-1, 6)$ and $(3, -26)$ are stationary points.
Find the second derivative.	$f''(x) = 6x - 6$
Find the values at -1 and 3.	$f''(-1) = -12 < 0$ and $f''(3) = 12 > 0$
State the result.	$(-1, 6)$ is a local maximum and $(3, -26)$ is a local minimum.
Find the points of inflection.	$f''(x) = 0$ at points of inflection.
Write an equation.	$6x - 6 = 0$
Solve and write a conclusion.	$x = 1$, so there could be a point of inflection at $x = 1$
Check the concavity near $x = 1$.	(see table)

x	0	1	2
$\frac{d^2y}{dx^2}$	−6	0	6

Write a conclusion.	The concavity changes, so there is a point of inflection.
Find the value at $x = 1$.	$f(1) = -10$
State the result.	$(1, -10)$ is a point of inflection.
Find the y-intercept.	$f(0) = 1$
State the dominant term.	The dominant term is x^3.
State the result.	As $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$

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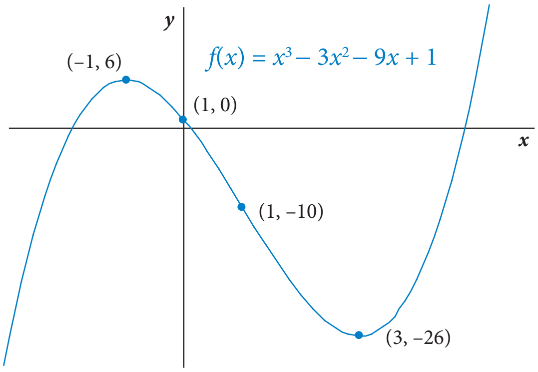
Consider the zeros.

The zeros are difficult to calculate and can be left out in this case.

Sketch the graph using all the information.



EXERCISE 3.05 Graph sketching

Curve sketching with derivatives

Concepts and techniques

1 Example 12 Draw a diagram to show the shape of each curve:

a $f'(x) < 0$ and $f''(x) < 0$

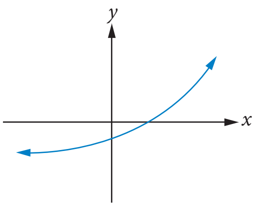
b $f'(x) > 0$ and $f''(x) < 0$

c $f'(x) < 0$ and $f''(x) > 0$

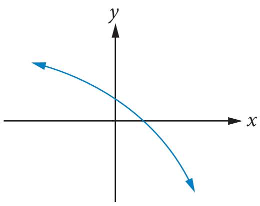
d $f'(x) > 0$ and $f''(x) > 0$

2 For each curve, describe the sign of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

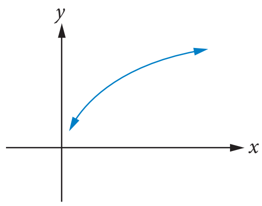
a



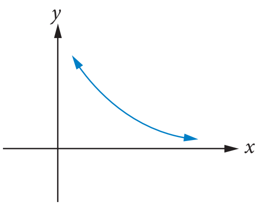
b



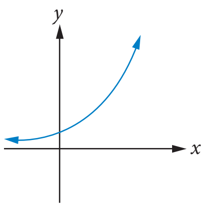
c



d



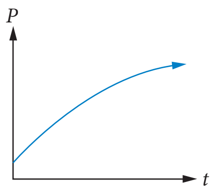
e



3 Example 13 The curve below shows the population of a colony of seals.

a Describe the sign of the first and second derivatives.

b How is the population rate changing?



4 Inflation is increasing, but the rate of increase is slowing. Draw a graph to show this trend.

Reasoning and communication

5 The size of classes at a local TAFE college is decreasing and the rate at which this is happening is decreasing. Draw a graph to show this.

6 As an iceblock melts, the rate at which it melts increases. Draw a graph to show this information.

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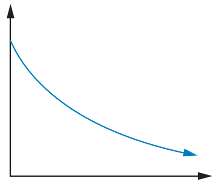
7 The graph shows the decay of a radioactive substance. State the signs of $\frac{dM}{dt}$ and $\frac{d^2M}{dt^2}$ and describe what it means in terms of the decay.



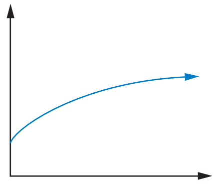
8 The population P of fish in a certain lake was studied over time, and at the start the number of fish was 2500.

a During the study, $\frac{dP}{dt} < 0$. What does this say about the number of fish during the study?

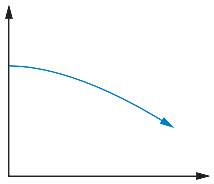
b If at the same time, $\frac{d^2P}{dt^2} > 0$, what can you say about the population rate?

c Sketch the graph of the population P against t.

9 The graph shows the level of education of youths in a certain rural area over the past 100 years.
Describe how the level of education has changed over this period of time. Include a mention about the rate of change.



10 The graph shows the number of students in a high school over several years.
Describe how the school population is changing over time, including the rate of change.



11 Example 14 Find the stationary point on the curve $f(x) = x^2 - 3x - 4$ and determine its type. Find the intercepts on the axes and sketch the curve.

12 Sketch $y = 6 - 2x - x^2$, showing the stationary point.

13 Find the stationary point on the curve $y = (x - 1)^3$ and determine its nature. Hence sketch the curve.

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14 Sketch $y = x^4 + 3$, showing any stationary points.

15 Find the stationary point on the curve $y = x^5$ and show that it is a point of inflection. Hence sketch the curve.

16 Sketch $f(x) = x^7$.

17 Identify any stationary points and points of inflection on $y = 2x^3 - 9x^2 - 24x + 30$ and sketch its graph.

18 a Determine any stationary points on the curve $y = x^3 + 6x^2 - 7$.
b Find any points of inflection on the curve.
c Sketch the curve.

19 Find any stationary points and points of inflection on the curve $y = x^3 - 6x^2 + 3$ and hence sketch the curve.

20 Find any stationary points and points of inflection on the curve $y = 2 + 9x - 3x^2 - x^3$. Hence sketch the curve.

21 Sketch the function $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$, showing all stationary points.

22 Find the stationary points on the curve $y = (x - 4)(x + 2)^2$ and hence sketch the curve.

23 Find all stationary points and inflections on the curve $y = (2x + 1)(x - 2)^4$. Sketch the curve.

3.06 OPTIMISATION

You looked at problems involving **optimisation** on an interval in Year 11 and found that, even though there is no one method you can use to solve these problems, there are general guidelines that can be helpful.

IMPORTANT

Solving optimisation problems

1 Read the problem slowly and carefully and think about the given facts.
2 Represent the unknown quantities as variables.
3 If possible, draw a sketch or diagram and label it, showing variables.
4 Make a list of known facts together with any relationships involving the variables as equations.
5 Determine the variable to be optimised (maximised or minimised) and express this variable as a function of one of the other variables. Using the relationships you have, eliminate all but one of the other variables.
6 Find the stationary points of the function obtained and test each one to see whether it is a maximum or minimum.
7 If you still can't solve the problem, don't give up. Look for another variable or relationship or redraw the diagram. Keep trying!

Example 15

The product of two positive numbers is 40. Find the numbers such that the sum of twice one number and 5 times the other is a minimum.

Solution

There are three quantities involved.	Let the numbers be a, b and m, where m is the 'sum'.
Write the relationships.	$ab = 40$, $m = 2a + 5b$, $a, b > 0$
Identify what has to be optimised/minimised.	m is to be minimised
Write b in terms of a (say).	$b = \frac{40}{a}$
Write m in terms of a.	$m = 2a + 5 \times \frac{40}{a}$
Simplify. Use index form.	$\quad = 2a + 200a^{-1}$
Differentiate.	$\frac{dm}{da} = 2 - 200a^{-2}$
Find stationary points.	$2 - 200a^{-2} = 0$
Rearrange and simplify.	$a^2 = 100$
Solve.	$a = -10$ or $a = 10$
Use $a > 0$.	Since $a > 0$, $a = 10$ is the only possibility.
Find the second derivative.	$\frac{d^2m}{da^2} = 400a^{-3}$
Find the value at $a = 10$.	$\left.\frac{d^2m}{da^2}\right\|_{a=10} = 0.4 > 0$
Identify the stationary point.	m is a minimum at $a = 10$ as it is concave down.
Find the value of b.	$b = \frac{40}{a} = \frac{40}{10} = 4$
State the result.	The required numbers are 10 and 4.

Now you will look at a more practical example.

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Example 16

A farmer is growing field tomatoes for juicing. The more plants that are crowded into a field, the more competition there is for light and nutrients, and so the smaller the mass of fruit from each plant. From previous experience, it is known that when there are n plants on a planting bed of area 100 m^2, the average mass of fruit from each plant is given by

$$m = 40 - 0.05n, \text{ where } m \text{ is in kilograms.}$$

What is the maximum total quantity of fruit, and what planting density is needed?

Shutterstock.com/Federico Rostagno

Solution

What extra variables are needed?	Let T be the mass of tomatoes from 100 m^2.
Write a relationship for T.	$T = n \times m$
Write in terms of one variable.	$= n(40 - 0.05n)$
Expand.	$= 40n - 0.05n^2$
Identify what has to be optimised.	T is to be maximised.
Find the derivative of T.	$T'(n) = 40 - 0.1n$
Find any stationary points.	$40 - 0.1n = 0$
Solve.	$n = 400$
State a conclusion.	The only possible stationary point is at $n = 400$.
Find the second derivative.	$T''(n) = -0.1 < 0$
Identify the stationary point.	There is a maximum at $n = 400$
Calculate the maximum.	$T(400) = 400(40 - 0.05 \times 400) = 8000$
State the final result.	The maximum total mass of fruit will be 8000 kg/100 m^2 from 400 plants/100 m^2.

In some questions, the equation is given as part of the question. However, you might have to work out an equation first. This is usually the hardest part.

Example 17

A north–south highway intersects an east–west highway at a point P. A vehicle crosses P at 10:00 a.m., travelling east at a constant speed of 40 km/h. At the same instant, another vehicle is 4 km north of P, travelling south at 100 km/h. Find the time at which the vehicles are closest to each other and estimate the minimum distance between them.

Solution

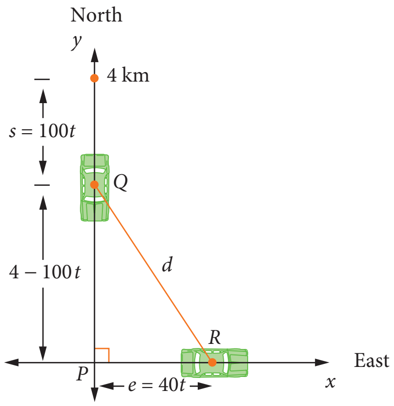
Choose some possible variables.

Let hours after 10 a.m. be t, the distances travelled be e and s respectively, and the distance between them be d.

Write the relationships.

$e = 40t$, $s = 100t$

Draw a diagram showing the positions at time t.



Use Pythagoras to find d.

$d^2 = (4 - 100t)^2 + (40t)^2$

Simplify.

$= 16 - 800t + 10\,000t^2 + 1600t^2$

Collect terms.

$= 11\,600t^2 - 800t + 16$

Identify what has to be optimised.

d, and so d^2 has to be minimised. Let $f(t) = d^2$

Write $f(t)$.

$f(t) = 11\,600t^2 - 800t + 16$

Differentiate.

$f'(t) = 23\,200t - 800$

Find any stationary points.

$23\,200t - 800 = 0$

Solve.

$t = \frac{800}{23\,200} = \frac{1}{29} \approx 0.0345$

Write a conclusion.

The only stationary point is at $t = \frac{1}{29}$

Find the second derivative.

$f''(t) = 23\,200 > 0$

Identify the stationary point.

There is a minimum at $t = \frac{1}{29}$ hr ≈ 2.1 min

Calculate the value at $t = \frac{1}{29}$.

$f\left(\frac{1}{29}\right) \approx 2.2069$

Find the square root.

$d = \sqrt{f\left(\frac{1}{29}\right)} \approx 1.49$

State the result.

The minimum distance is about 1.5 km and this is at just after 10:02 a.m.

In problems like Example **17** you should use the most accurate values on your calculator until you get to the end of the problem. It is best to use your CAS calculator for this purpose.

INVESTIGATION Heron's problem

One of the first non-trivial optimisation problems, 'The Shortest Path', was solved by Heron of Alexandria, who lived about 10–75 CE. The following is a problem which uses this theory.

One boundary of a farm is a straight river bank, and on the farm stands a house, with a shed some distance away; each is sited away from the river bank. Each morning the farmer takes a bucket from his house to the river, fills it with water, and carries the water to the shed.

Find the position on the river bank that will allow him to walk the shortest distance from house to river to shed. Further, describe how the farmer could solve the problem on the ground with the aid of a few stakes for sighting.

EXERCISE 3.06 Optimisation

Optimisation

Concepts and techniques

1 Example 15 Find two positive numbers a and b whose product is 27 and for which the value of $3a + 4b$ is the least.

2 Find two numbers whose sum is 25 and whose product is a maximum.

3 Find two numbers whose difference is 40 and whose product is the least.

4 The sum of two numbers is 32. Find the numbers if the sum of their squares is a minimum.

5 P is a movable point on the line $y = 7 - x$. Find the coordinates of P when it is closest to the origin.

6 Find the point (x, y) on the graph of $y = \sqrt{x}$ that is nearest to the point (4, 0).

Reasoning and communication

7 For the function $f(x) = \dfrac{6}{x^2 + 3}$ find the equation of the tangent with:

a the maximum slope
b the minimum slope.

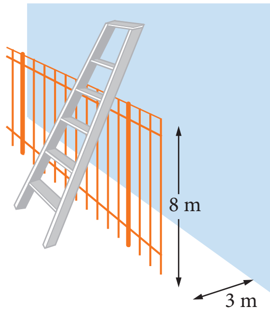
8 Example 16 A farmer is stocking a pond with fish as a supplementary form of income. The more fish there are in the pond, the more competition there will be for the available food supply, and so the more slowly the fish will gain weight. From previous experience, it is known that when there are n fish per unit area of water, the average weight gain of each fish (in grams) during the growing season is given by $w(n) = 600 - 30n$.

What value of n will lead to the maximum total production of weight in the fish?

9 Example 17 Two catamarans are 6 km apart in open water and the first is at a bearing of 045° from the second. The first begins moving directly south at 12 km/h and the second travels directly east at 18 km/h. What is the minimum distance between the catamarans?

10 A small boat moving at v km/h uses fuel at a rate that is approximated by the function $q=8+\frac{v^2}{50}$, where q is measured in litres/hour. Determine the speed of the boat at which the amount of fuel used for any given journey is the least.

11 An 8 m high fence stands 3 m from a large vertical wall. Find the length of the shortest ladder that will reach over the fence to the wall behind, as shown in the diagram.



12 An electric train uses power at the rate of 250 000 + v^3 J/h, where v is the speed of the train in kilometres/hour, and the speed it travels between stations ranges from 20–80 km/h. Find the speed at which the train should travel to minimise the use of electricity between stations.

13 An Australian test cricket player strikes the ball so that its equation of motion is given by $y = 1.4 + x - 0.004x^2$, where y is the height (m) reached by the ball and x is the horizontal distance (m) travelled by the ball. What is the greatest height reached by the ball?

14 During the course of an epidemic, the proportion of the population infected t months after the epidemic began is given by $p=\frac{t^2}{5\left(1+t^2\right)^2}$.

a Find the maximum proportion of the population that becomes infected.
b Find the time at which the proportion infected is increasing most rapidly.

15 An ironman has drifted 1 km along the beach from the finish of the swim stage. He is 400 m from shore and can swim at 4 km/h and run at 12 km/h. At what angle to the shore should he swim to reach the finish as quickly as possible?

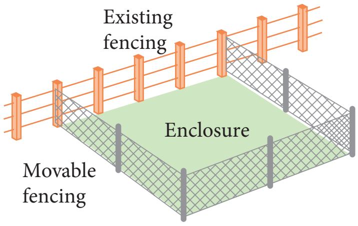
AAP Images/PR Images

3.07 OPTIMISATION IN AREA AND VOLUME

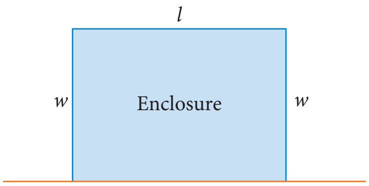
Many processes, especially manufacturing, involve the optimisation of physical quantities such as area and volume. For instance, a container manufacturer will probably want to minimise the surface area of a container designed to hold a specific quantity. This will allow the most number of containers to be produced from each unit of raw materials. When solving problems involving area and volume, you should take particular care in drawing the diagram. These problems will involve some basic two-dimensional figures and three-dimensional shapes and so you will need to know the area, surface area and volume formulas for them.

Example 18

A cattle property manager wants to construct a temporary rectangular enclosure using 200 m of movable fencing and an existing property fence as shown here. The manager wants to place the movable fencing so that the enclosure has the greatest possible area. Calculate the maximum area and state the dimensions for this area.



Solution

Choose the variables.	Let the length, width and area be l, w, and A.
Draw a diagram and label it.	
Write the relationships.	$2w + l = 200$, $A = lw$
What has to be optimised?	The area A has to be maximised.
Change to a single variable.	$l = 200 - 2w$
Write the area.	$A = (200 - 2w) \times w$
Expand.	$= 200w - 2w^2$
Differentiate to find the critical numbers.	$A'(w) = 200 - 4w$
Find any stationary points.	$200 - 4w = 0$

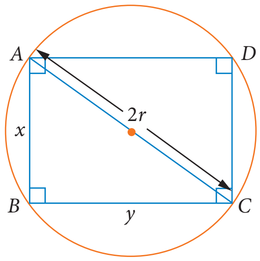
Solve.	$w = 50$
Write a conclusion.	There is only 1 stationary point, at $w = 50$.
Find the second derivative.	$A''(w) = -4 < 0$
Write a conclusion.	There is a maximum at $w = 50$.
Find the length.	$\text{Length} = 200 - 2 \times 50$ $= 100$
Find the area.	$A = 50 \times 100$ $= 5000$
State the result.	The maximum area is 5000 m^2 when the enclosure is 50 m × 100 m. The long side is parallel to the existing fence.

Example 19

Find the area of the largest rectangle that can be enclosed in a circle of radius r.

Solution

The largest rectangle must have corners that touch the circle. Draw the diagram and name the corners A, B, C and D.
The diagram shows the names chosen for the variables. The diagonal will be a diameter because the angle subtended is a right angle.



State what has to be optimised.	You want to maximise the area of the rectangle.
Write an equation for the area of the rectangle.	Let the area $= A = xy$
Write a relationship using Pythagoras' theorem.	$(2r)^2 = x^2 + y^2$
Write in terms of, say, x.	$y^2 = 4r^2 - x^2$
Consider r as fixed for any circle.	$y = \sqrt{4r^2 - x^2}$
Write the area in terms of x only.	$A = x\sqrt{4r^2 - x^2}$
Simplify by considering the square.	A is maximised when A^2 is maximised.
Write A^2 as a function.	$f(x) = A^2$ $= x^2(4r^2 - x^2)$
Expand.	$= 4r^2x^2 - x^4$
Find the derivative.	$f'(x) = 8r^2x - 4x^3$, since r is a constant.

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Find any stationary points.	$8r^2x - 4x^3 = 0$
Factorise.	$4x(2r^2 - x^2) = 0$
Use the difference of squares.	$4x\left(\sqrt{2}r - x\right)\left(\sqrt{2}r + x\right) = 0$
Solve.	$x = 0,\ x = \sqrt{2}r$ or $x = -\sqrt{2}r$
x is the length of a side.	$x = \sqrt{2}r$ is the only possibility as x is a length.
Find the second derivative.	$f''(x) = 8r^2 - 12x^2$
Check the sign at $x = \sqrt{2}r$.	$f''\left(\sqrt{2}r\right) = 8r^2 - 12\left(\sqrt{2}r\right)^2$
Simplify.	$= 8r^2 - 12 \times 2r^2$ $= -16r^2 < 0$
Identify the stationary point.	There is a maximum at $x = \sqrt{2}r$
Calculate the maximum area.	$A = \sqrt{2}r\sqrt{4r^2 - \left(\sqrt{2}r\right)^2}$
Simplify.	$= \sqrt{2}r\sqrt{4r^2 - 2r^2}$ $= \sqrt{2}r\sqrt{2}r$ $= 2r^2$
State the result.	The largest rectangle that can be enclosed in a circle of radius r has an area of $2r^2$.

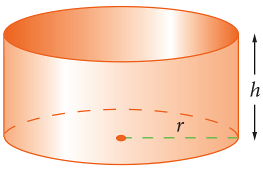
In practical situations like those in Example **19**, you can often use practical considerations to discard some of the solutions that arise from the mathematical model. Some people call these false solutions.

Example 20

A cylindrical metal can is open at the top. It must be designed to have a capacity of 1078 cm^3. The outside of the can has to be galvanised, and the manufacturer wants the surface area of the can to be minimised in order to save money. Find the dimensions of the can.

Solution

Draw a diagram and label the obvious variables, in this case the height h and the radius r of the can.



State what has to be optimised.	The external surface area A is to be minimised.
Write the relationships.	$A = 2\pi rh + \pi r^2$ and $V = 1078 = \pi r^2 h$
Use the volume to express h in terms of r.	$h = \dfrac{1078}{\pi r^2}$
Express the surface area in terms of r.	$A = 2\pi r\dfrac{1078}{\pi r^2} + \pi r^2$

Simplify.	$A(r) = \pi r^2 + \dfrac{2156}{r}$ $= \pi r^2 + 2156r^{-1}$
Differentiate.	$A'(r) = 2\pi r - 2156r^{-2}$
Find any stationary points.	$2\pi r - 2156r^{-2} = 0$
Multiply by r^2.	$2\pi r^3 - 2156 = 0$
Solve.	$r = \sqrt[3]{\dfrac{2156}{2\pi}} = \sqrt[3]{\dfrac{1078}{\pi}} = 7.000\,939\ldots$
Find the second derivative of A.	$A''(r) = 2\pi + 4312r^{-3}$
Find the value at the stationary point.	$A''\left(\sqrt[3]{\dfrac{1078}{\pi}}\right) = 2\pi + 4312\left(\sqrt[3]{\dfrac{1078}{\pi}}\right)^{-3}$
Simplify.	$= 6\pi > 0$
Identify the stationary point.	There is a minimum when $r \approx 7$.
Find the height.	$h = \dfrac{1078}{\pi}r^{-2} = \dfrac{1078}{\pi}\left(\sqrt[3]{\dfrac{1078}{\pi}}\right)^{-2} = \sqrt[3]{\dfrac{1078}{\pi}} = r$
State the result.	A 1078 cm^3 cylinder has minimal surface area when its height and radius are equal, about 7 cm.

Notice that Example **20** almost proves that an open cylinder of fixed volume has the minimal surface area when the height and radius are equal. Of course, the result for a closed cylinder would be different.

WS

Optimisation – Area and volume

EXERCISE 3.07 Optimisation in area and volume

Reasoning and communication

1 Example 18 A rectangular paddock shares one side with an existing paddock, so it requires no fence on that side. There are only 2000 m of fencing material available to fence the remaining sides. Find the maximum possible area and dimensions of the paddock.

2 Example 19 A right circular cone is machined from a solid sphere of radius 30 cm. Find the ratio of the volume of the cone to the volume of the sphere when the volume of the cone is a maximum.

3 A strip of galvanised iron is 24 cm wide. The strip of iron is to be used to form a length of guttering for a house. The guttering is going to have a rectangular cross-section and be open at the top so that water can flow from the roof of the house into the guttering. What are the dimensions of the cross-section of the guttering if it is to hold the maximum volume of water?

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4 Example 20 A manufacturer produces open-topped rectangular boxes, each with a square base and a volume of 500 cm^3. What are the dimensions of a box if the least amount of material is to be used in its construction?

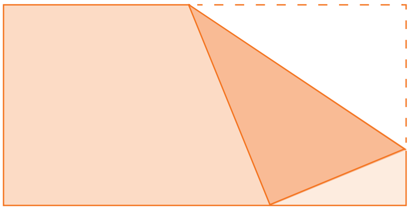
Shutterstock.com/Quang Ho

5 A metal box has square ends, rectangular sides and bottom and is open at the top. What is the least area of metal that is required to construct the box if it must have a volume of 36 m^3?

6 A right circular cone has a slant edge measuring 9 cm. What is the height of the cone that will have the greatest volume?

7 A rectangular sheet of cardboard measuring 16 cm by 10 cm is to be formed into an open rectangular box by cutting out identical squares from each corner and folding up the sides. What size must the squares be for the box to have the maximum volume?

8 A rectangular piece of paper measures 12 cm by 6 cm. One corner of the sheet of paper is folded up to just reach the opposite side as shown below. What is the minimum length of the resulting crease in the paper?



9 An opening in a wall is to be made in the shape of a semicircle over a rectangle. The distance around the edge of the opening must measure 12 m. What are the dimensions of the opening that will result in the greatest area?

10 What are the dimensions of the cone of greatest volume that can just fit inside a sphere of radius 2 m?

11 A window consists of a rectangular sheet of glass surmounted by a semicircular sheet as shown in the photograph. Calculate the maximum area obtainable if the perimeter of the window is fixed at 6 m.

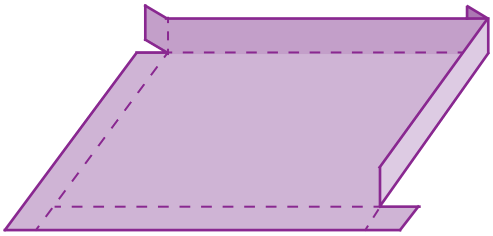
12 Find the greatest volume of a rectangular box with a square base if the sum of the height and the side of the base must not exceed 30 cm.

Thinkstock/photoncatcher

13 Postal cylinders are made according to the rule that sum of the length and circumference not be more than 150 cm. What are the dimensions of the cylinder with the largest volume?

14 A settling pond is to be constructed to hold 324 m^3 of waste. The settling pond has a square base and four vertical sides, all made of concrete, and a square top made of steel. If the steel costs twice as much per unit area as the concrete, determine the dimensions of the pond that will minimise the cost of construction.

15 Cardboard trays are made from rectangular pieces of cardboard with a length to width ratio of 3 : 2. The trays are made as shown, by cutting one side of a square corner, then bending and gluing it to make the tray. What dimensions will the trays with the largest volume have if they are made from a cardboard rectangle with length:



a 20 cm? b 30 cm? c 50 cm?

3.08 OPTIMISATION IN BUSINESS

If the management of a business knows how their profit depends on some variable, then they can choose the value that makes the profit as large as possible. You can set up revenue, cost and profit as functions for this purpose.

Example 21

A small manufacturing firm can sell everything it produces at a price of \$8 per item. It costs $C(n) = 100\,000 + 8n - 0.0156n^2 + 2 \times 10^{-6}n^3$ to produce n items per week (in dollars).

What should the weekly production target be in order to maximise profits?

Solution

Write a function for profit.	$P(n) = 8n - (100\,000 + 8n - 0.0156n^2 + 2 \times 10^{-6}n^3)$
Simplify.	$= 0.0156n^2 - 100\,000 - 2 \times 10^{-6}n^3$
Find the first derivative.	$P'(n) = 0.0312n - 6 \times 10^{-6}n^2$
Find any stationary points.	$0.0312n - 6 \times 10^{-6}n^2 = 0$
Factorise.	$n(0.0312 - 6 \times 10^{-6}n) = 0$
Solve.	$n = 0$ or $n = 5200$
Find the second derivative.	$P''(n) = 0.0312 - 1.2 \times 10^{-5}n$

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Check the relevant values. $P''(0) = 0.0312 > 0$ and $P''(5200) = -0.0312 < 0$

State the maximum and minimum points. A minimum is at $n = 0$ and a maximum $n = 5200$

Calculate the maximum profit. $P(5200) = 0.0156 \times 5200^2 - 100\,000 - 2 \times 10^{-6} \times 5200^3$

Simplify. $= 40\,608$

State the result. The maximum profit of \$40 608 is achieved with a weekly production of 5200 items.

In complicated cases, check your results using a graphics or CAS calculator by drawing a graph of the function to be optimised.

Example 22

A printer has a contract to print 20 000 copies of a one-page catalogue. It costs \$800/h to run the printing press, which will produce 120 impressions per hour. The printer can run up to 50 plates at a time, and each impression produces x copies, where x is the number of plates used. Each plate costs \$200 to set up. How many plates should be made so that the job is done the most economically?

Solution

Decide on the variable names. Let printing time = t and cost = C.

State what has to be optimised. C has to be minimised.

Write the rate of production. Copies produced = $120x$ per hour.

Write the relationships. $120x \times t = 20\,000$ and $C = 200x + 800t$

Write t in terms of x only. $t = \dfrac{20\,000}{120x} = \dfrac{500}{3x}$

Write C in terms of x only. $C(x) = 200x + 800 \times \dfrac{500}{3x}$

Simplify. $= 200x + \dfrac{400\,000}{3x}$

$= 200x + \dfrac{400\,000}{3}x^{-1}$

Differentiate. $C'(x) = 200 - \dfrac{400\,000}{3}x^{-2}$

Find any stationary points. $200 - \dfrac{400\,000}{3}x^{-2} = 0$

Multiply by $3x^2$. $600x^2 - 400\,000 = 0$

Solve. $x = \pm\sqrt{\dfrac{2000}{3}} \approx \pm 25.82$

Use common sense. Only $x = +\sqrt{\dfrac{2000}{3}}$ is valid for the number of plates.

Find the second derivative. $C''(x) = \dfrac{800\,000}{3}x^{-3}$

Find the value at $x = \sqrt{\frac{2000}{3}}$.

$C''\left(\sqrt{\frac{2000}{3}}\right) = \frac{800\,000}{3} \times \left(\sqrt{\frac{2000}{3}}\right)^{-3} > 0$

Write a conclusion.

There is a minimum at $x = \sqrt{\frac{2000}{3}} \approx 25.82$

State the possible results.

The answer has to be an integer, so it's either 25 or 26.

Check the first value.

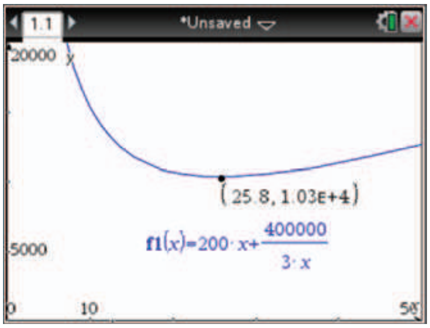
$C(25) = 200 \times 25 + \frac{400\,000}{3 \times 25} \approx 10\,333.33$

Check the cost for 26.

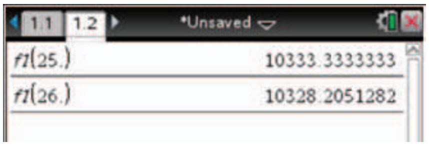
$C(26) = 200 \times 26 + \frac{400\,000}{3 \times 26} \approx 10\,328.21$

TI-Nspire CAS

Use a graph page.
Draw the graph and find the minimum using b, 6: analyse Graph and 2: Minimum.

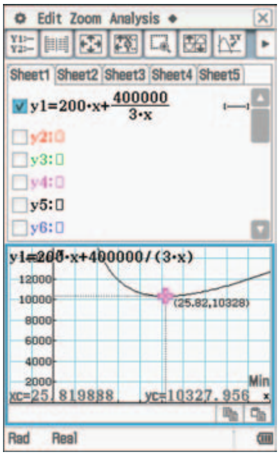


You can find the values at 25 and 26 by adding a Calculator page.

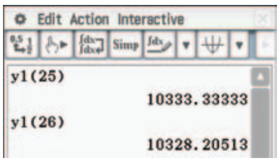


ClassPad

Use the Graph and Table menu.
Draw the graph and find the minimum by tapping Analysis, G-Solve and Min.



You can check the values at 25 and 26 by going to the Main menu.



State the result.

The job will be done most economically for \$10 328.21 when 26 plates are made.

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INVESTIGATION Fun rides

Flash owns a 'fun ride' that he sets up at country shows. Each year he brings his ride to the Moomba Festival. The ride has the capacity to take up to 90 people, but Flash knows from experience that he only gets about a third of the ride's capacity when he charges his normal price of \$10 for a ride. He also knows that each time he runs the ride it costs the same amount regardless of how many people have bought tickets. When Flash reduces the price of the ride, he attracts more customers. He wants to make the ride more profitable and has an idea that if he gets the price right, he can achieve this. He has learnt that charging too little means that he loses money.

Recently, Flash has noticed that rides costing only \$5 are usually full, and he is sure that his ride would be full every time if he dropped his price to that figure.

123RF/windu

1 Assuming that the number of customers will increase in direct proportion to the price drop, decide what price would give him the most profit.

2 How much better off would Flash be each time he runs the ride at the new fare compared with the takings at the fare of \$10?

3 Investigate other methods of approaching this problem by considering different assumptions.

EXERCISE 3.08 Optimisation in business

WS Optimisation – Cost and revenue

Reasoning and communication

1 Example 21 The Western Star Novelty Company produces x tonnes of silver per month at a total cost given by

$$C(x) = 2000 - 75x - 5x^2 + \frac{x^3}{3}$$

where C is in dollars. Find the production level for which costs are minimised.

2 Example 21 A machine shop can produce 30 items a day on one machine. The shop can use more than one machine for the job, but if it does, for technical reasons, each machine cannot then work to full capacity. In fact, production per machine is slowed by a factor equal to $\frac{1}{10}$ of the square of the number of additional machines put on the job. That is, if x additional machines are put on the job, the number of items produced by each machine is $30 - \frac{x^2}{10}$. How many machines should be used to achieve the maximum production?

3 A firm sells all the units that it produces at \$4 per unit. The firm's total cost C for producing x units is given in dollars by

$$C(x) = 50 + 1.3x + 0.001x^2$$

a Write an expression for total profit as a function of x.

b How many items should be produced so that the profit is a maximum?

4 For a particular item, the price at which x items are sold is given by the equation

$$p = 5 - 0.001x \text{ for } 0 < x < 5000$$

The cost of production is given by

$$C(x) = 2800 + x$$

a Find the value of x that maximises the revenue.

b Find the value of x that maximises the profit.

5 The cost (in dollars) of producing x plastic mounts is

$$C(x) = 4000 - 3x + 10^{-3}x^2$$

a Find the value of x that minimises production costs.

b If the plastic mounts are sold for \$4 each, find the value of x that maximises profit and calculate the maximum profit.

6 A manufacturer can sell x clock radios per week at d dollars per item, where $x = 600 - 3d$. Production costs (in dollars) are

$$C(x) = 600 + 10x + \frac{1}{2}x^2$$

How should the clock radios be priced in order to maximise profits?

7 The weekly cost (in dollars) of producing x ironing boards is given by

$$C(x) = 40\,000 - 30x + 10^{-2}x^2$$

a Find the value of x that minimises production costs.

b If the ironing boards are sold for \$40 each, find the value of x that maximises profit and calculate the maximum profit.

Alamy/Andreas von Einsiedel

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8 The total weekly cost (in dollars) of producing carpet is found to be approximated by the function

$$C(x)=100+28x-5x^2+\frac{x^3}{3}$$

where x is the number of rolls of carpet. The price (in dollars) at which carpet rolls are sold is given by $p = 5000 - 5x$ for $0 < x < 1000$.

A tax of \$222 per roll is imposed by the government. The manufacturer adds this to the production costs. The manufacturer sells to a number of different outlets and has no problem getting rid of however much is produced.

a Change the manufacturer's cost function to include the government tax.
b State the revenue function for selling x rolls of carpet.
c Write the profit function.
d Work out the number of rolls of carpet that need to be produced each week in order to maximise profit.

Dreamstime.com/Ragne Kabanova

9 By reducing the number of counter staff, a fast-food outlet can reduce its labour costs, but can also expect to lose business because of customer dissatisfaction at having to wait. Assume that the rate of pay for counter staff is \$80 per day and that the loss of profit from having only n counter staff is $\frac{5000}{n+1}$ dollars per day. Calculate the value of n that minimises the sum of the loss plus the wage cost.

10 A parcel delivery firm makes regular trips between Brisbane and Ipswich. The accountant, who also happens to be a good mathematician, has calculated that fuel and other running costs are approximated by the function

$$F(v)=2v^{\frac{3}{2}}+59 \text{ cents/trip}$$

where v = average speed in km/h. The cost of paying a driver is given by

$$D(v)=\frac{1.5\times10^5}{v}+2000 \text{ cents/trip}$$

Find the most economical speed for the trip and the cost of a trip from Brisbane to Ipswich at this speed.

3.09 GENERAL OPTIMISATION PROBLEMS

Optimisation theory is versatile and can be used to solve many complicated problems in a variety of situations. You have looked specifically at area and volume problems and the solutions to business problems. In this section you will look at the use of optimisation to solve problems in other fields. The methods you use to solve the problems remain the same.

Example 23

A tennis ball follows a path defined by $h = -t^2 + 6t + 2.5$ and $d = 14t$, where h is the height of the ball in metres, d is the horizontal distance in metres and t is the time in seconds. What is the maximum height reached by the tennis ball?

Thinkstock/nickp37

Solution

Write the relevant function.	$h = -t^2 + 6t + 2.5$
Differentiate.	$h' = -2t + 6$
Find any stationary points.	$-2t + 6 = 0$
Solve.	$t = 3$
Find the second derivative.	$h'' = -2 < 0$
Write down the nature of the stationary point.	There is a maximum at $t = 3$.
Substitute into the height.	$h(3) = -3^2 + 6 \times 3 + 2.5$ $= 11.5$
State the result.	The maximum height reached is 11.5 metres.

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Example 24

The water bug population of a lake is dependent upon the water temperature. The number of water bugs is about $N = -T^3 + 13.5T^2 + 1740T + 2575$, where T is the water temperature in °C for $20° \leq T \leq 30°$. Find the maximum water bug population.

Thinkstock/thomasmales

Solution

Write the function.	$N = -T^3 + 13.5T^2 + 1740T + 2575$
Differentiate.	$N' = -3T^2 + 27T + 1740$
Find any stationary points.	$-3T^2 + 27T + 1740 = 0$
Simplify to solve.	$T^2 - 9T - 580 = 0$
Factorise the quadratic.	$(T + 20)(T - 29) = 0$
Complete the solution.	$T = -20$ or $T = 29$
Discard the false answer.	$T = 29$ only as -20 is outside the range.
Find the second derivative.	$N'' = -6T + 27$
Find the value at $T = 29$.	$N''(29) = -6 \times 29 + 27$ $= -147 < 0$
State the nature of the stationary point.	There is a maximum at $T = 29°$
Find the maximum population.	$T(29) = -(29)^3 + 13.5 \times 29^2 + 1740 \times 29 + 2575$
Simplify.	$= 39\,999.5$
State the result.	The maximum population is about 40 000 at 29°C.

WS

Optimisation – Mixed problems

EXERCISE 3.09 General optimisation problems

Reasoning and communication

1 Example 23 A projectile is fired into the air and its height in metres is given by $h = 40t - 5t^2 + 4$, where t is in seconds. What is the maximum height that the projectile reaches?

2 The displacement in cm after time t seconds of a particle moving in a straight line is given by $x = 2 + 3t - t^2$. What is its maximum displacement?

3 A ball is rolled up a slope. Its position after time t seconds is given by $x = 15t - 3t^2$ m. How far up the slope will the ball roll before it starts to roll back down?

4 A skier decides to jump a ramp. Her height above the water, h m, t seconds after jumping, is given by $h = -4t^2 + 4t + 10$. Find the maximum height that she attains.

Getty Images/Donald Miralle

5 A toy rocket is launched from the top of an 8 metre high building so that its height, h, in metres, above the ground t seconds later is given by $h = 8 - 8t^2 + 32t$. What is the maximum height reached by the rocket?

6 A person standing on top of a 35 m high cliff throws a stone into the air. Its height after t seconds is given by $y = -5t^2 + 8t + 35$. What is the maximum height that it reaches?

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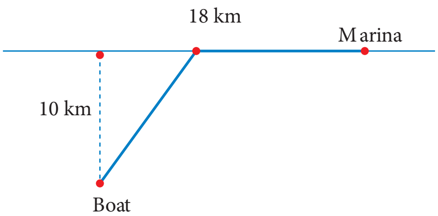
7 Example 24 A seal colony eats fish with a population dependent on the biomass of krill and other small organisms in the area. The density of these changes as the temperature of the water varies. The seal population, P, can be modelled as $P = -2\frac{2}{3}T^3 + 52T^2 - 176T + 382$, where T is the water temperature in °C for $-2° \le T \le 15°$. What will the maximum and minimum seal colony populations be, and at what temperatures will they occur? Hint: Check the population at −2°C and 15°C as well.

123RF/Ellice Lin

8 A farmer estimates that if 75 pecan nut trees are planted per hectare, the average yield per tree will be 7 kg. For every tree less that he plants on the same acreage, the average yield per tree will increase by 0.2 kg per tree. How many trees per hectare should the farmer plant to maximize the total yield?

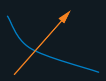
9 A man can row at 6 km/h and run at 8 km/h. He is 10 km out to sea and needs to get to a marina on the coast 18 km from the nearest point on the shore from his current position. Where should he land on the coast to get to the marina as soon as possible?



10 A 4WD driver broke down 15 km down a very winding track running away from the main road, but only 5 km from the road itself at its closest point. He needs to get to a service station on the main road, which is another 5 km down the road from the closest point. He can walk at 5 km/h on the road or track but only 3 km/h cross country. What should he do to get to the service station in the shortest possible time?

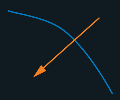
3 CHAPTER SUMMARY
APPLICATIONS OF DERIVATIVES

- For a small change δx, in the independent variable x, the corresponding change δy in the dependent variable y is given by the **increments formula** $\delta y \approx \frac{dy}{dx} \times \delta x$
- The **second derivative** can be found by differentiating a function twice. It is shown as $f''(x)$, $f^{(2)}(x)$, $f^2(x)$, y'', or $\frac{d^2y}{dx^2}$.
- The **position** of a particle relative to a fixed point (the origin) is usually shown as x.
- A **displacement** is a change of position, usually shown as s.
- **Velocity**, v, is the rate of change of displacement or position, so $v = \frac{dx}{dt} = \frac{ds}{dt}$.
- **Acceleration**, a, is the rate of change of velocity, so $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{d^2s}{dt^2}$.
- The second derivative shows the change in the first derivative and the **concavity** of a curve.
  - If $f''(x) > 0$, then $f'(x)$ is increasing: the curve is **concave upwards** (**cup** shaped).



Concave upwards

  - If $f''(x) < 0$, then $f'(x)$ is decreasing: the curve is **concave downwards** (**cap** shaped).



Concave downwards

  - If $f''(x) = 0$, then $f'(x)$ is stationary

- A **point of inflection** is a point where the concavity of the function (sign of $f''(x)$) changes.

Point of inflection

- A **horizontal point of inflection** is a both a stationary point and a point of inflection.

Horizontal point of inflection

- For stationary points, $f'(x) = 0$. The second derivative test is used to determine the kind of stationary point.
  - If $f'(x) = 0$ and $f''(x) > 0$, the curve is concave up so there is a **local minimum**
  - If $f'(x) = 0$ and $f''(x) < 0$, the curve is concave down so there is a **local maximum**
  - If $f'(x) = 0$, $f''(x) = 0$ and concavity changes, there is a **point of horizontal inflection**
  - If $f'(x) = 0$, $f''(x) = 0$, concavity does not change and $f''(x) > 0$ on either side, then there is a **local minimum**
  - If $f'(x) = 0$, $f''(x) = 0$, concavity does not change and $f''(x) < 0$ on either side, then there is a **local maximum**

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- To solve **optimisation** problems, do the following.

  1 Read the problem carefully

  2 Represent the unknown quantities as variables

  3 If possible, sketch a diagram and label it, showing variables

  4 Make a list of known facts together with any relationships involving the variables as equations

  5 Determine the variable to be optimised (maximised or minimised) and express this variable as a function of *one* of the other variables. Use the relationships you have to eliminate all but one of the other variables.

  6 Find the stationary points of the function obtained and test each one to see whether it is a maximum or minimum.

3 CHAPTER REVIEW
APPLICATIONS OF DERIVATIVES

Multiple choice

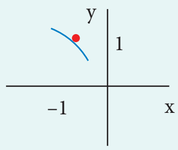
1 Example 1 Using the derivative of $\sqrt{x}$, an approximation for $\sqrt{147}$ is:

A 12 B 12.03 C 12.125 D 12.3 E 13

2 Example 5 If $y = 3\cos(4x)$, then $\dfrac{d^2y}{dx^2}$ equals

A $48\sin(4x)$ B $-48\cos(4x)$ C $-12\cos(4x)$
D $-12\sin(4x)$ E $48\cos(4x)$

3 Example 12 For a particular function, $f(-1) = 1$, $f'(-1) < 0$ and $f''(-1) < 0$. Which sketch of the curve at this point, shows its shape?

a


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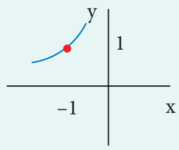

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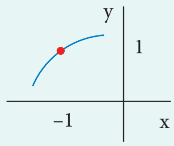
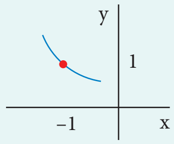
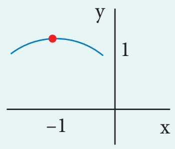


d


e


4 Example 15 Two numbers have a difference of 24. What are the two numbers if the addition of their product and their sum is a minimum?

A 13 and 37 B 11 and 35 C −13 and 11
D −11 and 13 E −13 and −37

5 Example 18 80 metres of fencing is available for enclosing a play area. What is the maximum area which can be enclosed?

A 80 m^2 B 90 m^2 C 200 m^2 D 400 m^2 E 800 m^2

Short answer

6 Example 1 Use the derivative of $x^{\frac{2}{5}}$ for $x = 32$ to find an approximation for:

a $33^{0.4}$ b $31.7^{0.4}$

7 Example 2 A spherical ball with a diameter of 12 cm is being pumped up. The volume is given by $V = \dfrac{4\pi r^3}{3}$. Find an approximation of the increase in volume when the radius expands by 0.04 cm.

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8 Example 3 A steel drum manufacturer makes 60 L drums that have a height : diameter ratio of 1.6.

a Find the dimensions of the drums.

b Find the approximate percentage error in volume, given that there is an error of x% in the height.

c Find the approximate percentage error in volume, given that there is an error of y% in the diameter.

d Find the approximate percentage error in volume if a tolerance of 3% is allowed in both the height and the diameter.

9 Example 4 Find the first two derivatives of:

a $y = 5x^6 - 3x^2 + x + 10$

b $f(t) = (2t + 9)^4$

c $f(n) = (3n - 1)^2(2n + 4)$

d $y = \dfrac{6x - 9}{3x - 1}$

10 Example 6 The position of a particle after t seconds is given by $x = t^3 - 12t^2 + 36t - 9$ cm.

a Find an expression for its velocity.

b Find an expression for its acceleration

c What is its acceleration after 2 seconds?

11 Example 7 Find all x-values for which the curve $y = 2x^3 - 7x^2 - 3x + 1$ is concave upwards.

12 Example 8 Does $f(x) = 4x^7$ have a point of inflection?

13 Example 9 Find any points of inflection on the curve $y = x^4 + 4x^3 - 48x^2 + 1$.

14 Example 10 For the following functions, find all the stationary points and determine their nature.

a $y = 3x^2 - 6x + 3$

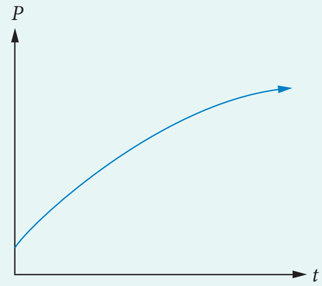
b $f(x) = 5x - x^2$

c $f(x) = 3x^4 - 4x^3 - 12x^2 + 7$

d $y = (2x - 1)^4$

15 Example 11 Find the stationary point on $y = 2x^3 - 1$ and determine its nature.

16 Example 13 The population P of possums on an island over time t years is shown on the graph.



a State the signs of $\dfrac{dP}{dt}$ and $\dfrac{d^2P}{dt^2}$.

b How is the number of possums changing over time?

c Is the rate of population growth increasing or decreasing?

17 Example 14 Sketch the graphs of the following functions, showing clearly any turning points or points of inflection.

a $y = 4x^2 - 9x + 5$

b $f(x) = 2x^3 - 6x$

c $f(x) = 2x^3 - 12x^2$

d $f(x) = x^3 - 2x^2 + x - 2$

e $f(x) = 3x^3 - x^2 - 7x - 3$

18 Example 15 Find two positive numbers, x and y, whose product is 48 such that $3x + y$ is a minimum.

Application

19 An avocado grower has 20 trees in his plot. Each tree produces 240 fruit. For each additional tree planted, the output per tree drops by 10 fruit. How many trees should be added to the existing plot in order to maximize the total output of the trees?

20 At 8 a.m. a ship going due west at 12 km/h is 10 km due north of a second ship going due north at 16 km/h.

a Find the time at which the ships are closest to each other.

b Find the minimum distance between them.

21 A block of land is bordered on one side by a straight stretch of river and on the other three sides by 640 metres of fencing. Find the dimensions of the block if its area is to be a maximum.

22 The area of a rectangle is128 cm^2. Find its dimensions so that the distance from one corner to the midpoint of a non-adjacent edge is a minimum.

23 A manufacturer wants to minimise the cost of materials used when making a closed rectangular box with a square base, which has a volume of 64 000 cm^3. Find the minimum total surface area of the box.

24 A manufacturer makes a batch of n items with the cost (in dollars) of each item being $C(n) = n^2 - 6n + 35$. The manufacturer sells the items for \$50 each. How many items should be produced in each batch to maximise profit?

25 The production cost of toy talking bears is given by $C(x) = 0.005x^2 - 2x + 250$, where x is the number of bears produced in a week. The number of bears that can be sold in a week is given by $x = 600 - 2d$, where d is the price in dollars of each bear.

a How many bears should be produced to make the greatest profit?

b What will the price of a bear then be?

c What will be the profit?

26 A skateboarder decides to do a ramp. The path of the jump can be approximated by $h = -3t^2 + 6t + 1$ and $d = 2t$, where h represents the height above the ground in metres, s is the horizontal displacement and t represents time after leaving the time ramp in seconds. Find the maximum height reached by the skateboarder.

Qz Practice quiz

27 The population of locusts in an area changes as the temperature varies. In a certain area the population, P, can be modelled as $P = -\frac{1}{3}t^3 + 16.5t^2 + 70t + 5000$, where t is the air temperature in °C for $0° \le T \le 37°$. At what temperatures will the population of locust swarm be at its maximum size?

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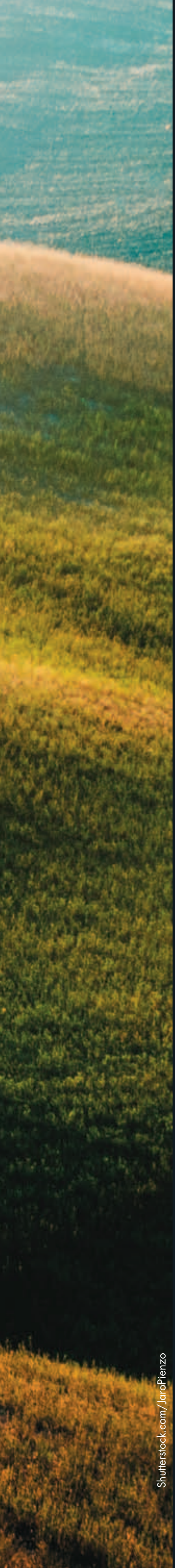

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4

TERMINOLOGY

algebraic area
antiderivative
constant of integration
definite integral
fundamental theorem of calculus
indefinite integral
integration
physical area

INTEGRALS

INTEGRATION AND AREAS

Prior learning

DEFINITE INTEGRALS

- examine the area problem, and use sums of the form $\sum_i f(x_i)\delta x_i$ to estimate the area under the curve $y = f(x)$ (ACMMM124)
- interpret the definite integral $\int_a^b f(x)dx$ as area under the curve $y = f(x)$ if $f(x) > 0$ (ACMMM125)
- recognise the definite integral $\int_a^b f(x)dx$ as a limit of sums of the form $\sum_i f(x_i)\delta x_i$ (ACMMM126)
- interpret $\int_a^b f(x)dx$ as a sum of signed areas (ACMMM127)
- recognise and use the additivity and linearity of definite integrals. (ACMMM128)

FUNDAMENTAL THEOREM

- understand the concept of the signed area function $F(x) = \int_a^x f(t)dt$ (ACMMM129)
- understand and use the theorem: $F'(x) = \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$, and illustrate its proof geometrically (ACMMM130)
- understand the formula $\int_a^b f'(x)dx = F(b) - F(a)$ and use it to calculate definite integrals. (ACMMM131)

APPLICATIONS OF INTEGRATION

- calculate the area under a curve (ACMMM132) AC

4.01 THE AREA UNDER A CURVE

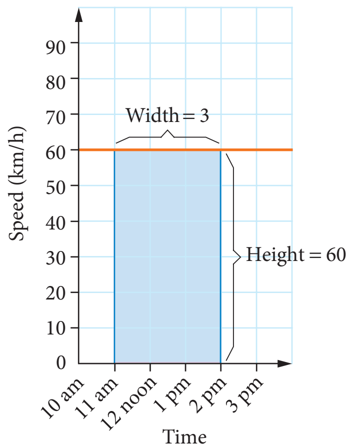
In many areas of knowledge such as surveying, physics and the social sciences, you must know how to find the area under a curve because it can give you important information. For example, the area under a speed graph shows the distance travelled.

Example 1

A car is travelling at a constant speed of 60 km/h.

a Find the distance it travels between 11 a.m. and 2 p.m. by using the graph below.

b Show that the area represents the distance that the car travels in that time.



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Solution

a Find the distance travelled at 60 km/h. — The car travels 60 km in 1 hour so it travels 180 km in 3 hours.

b Find the area of the shaded rectangle.

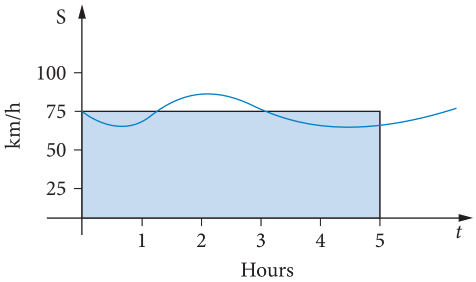
$$A = 3 \times 60$$
$$= 180$$

State the result. — The area of the rectangle shows that the distance travelled is 180 km.

In real life, however, a car would rarely travel at a constant speed, so its graph would not be a straight line.

Example 2

The graph below shows the speed S of a car over t hours. Find the approximate distance it travels in 5 hours using the rectangle shown on the graph.



Solution

Find the area of the rectangle.

$$A = 75 \times 5$$
$$= 375$$

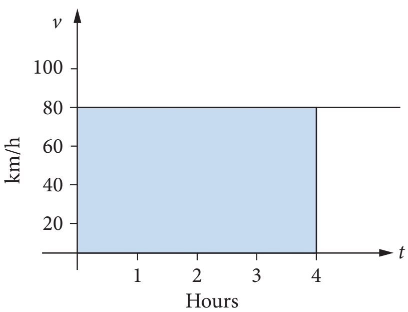
State the result. — The distance travelled in 5 hours is approximately 375 km.

EXERCISE 4.01 The area under a curve

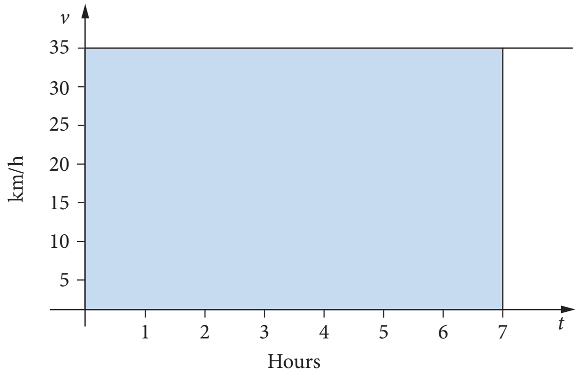
Concepts and techniques

1 Example 1 Find the distance travelled in each of the following by finding the area of the rectangle.

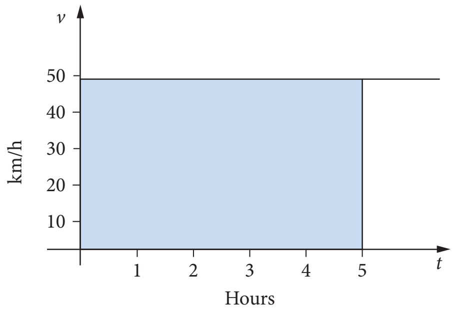
a



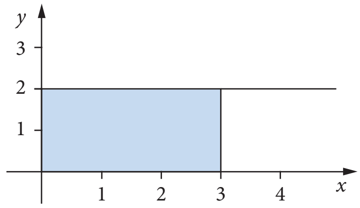
b



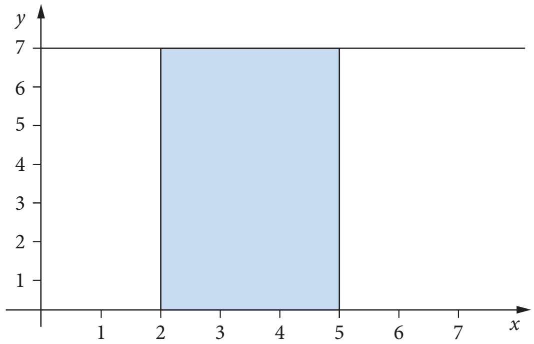
c



2 a Find the area under the graph between $x = 0$ and $x = 3$.

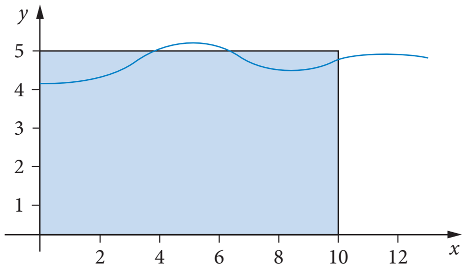


b Find the area under the graph bounded by $x = 2$ and $x = 5$.



3 Example 2 Find the approximate area under each curve by finding the shaded area of each rectangle.

a



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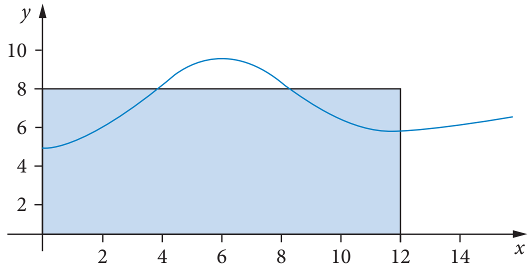
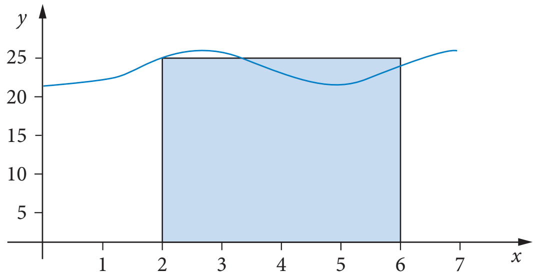
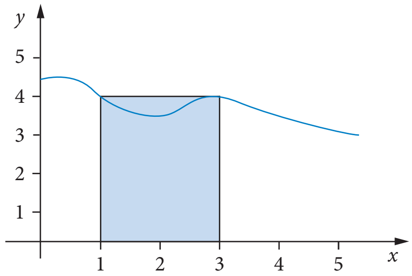
b



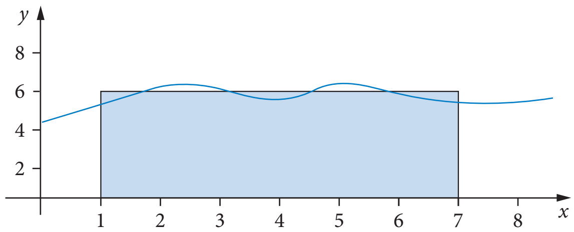
c



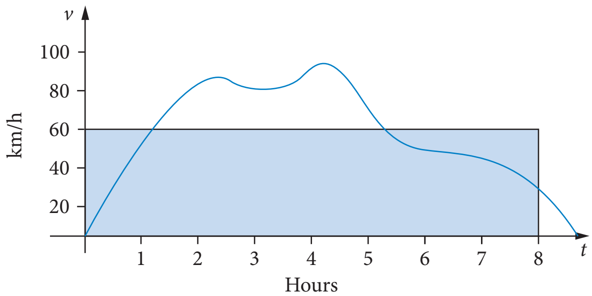
d



e

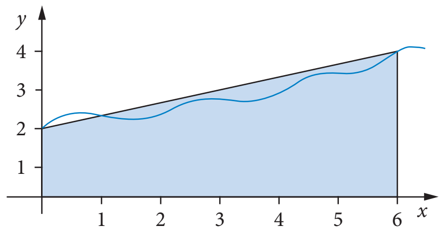


4 The travel graph shows the speed of a car as it travels along a road. Find the approximate distance travelled in 8 hours.



5 Find an approximation to each of the areas under a curve below by using the formula for the area of a trapezium.

a



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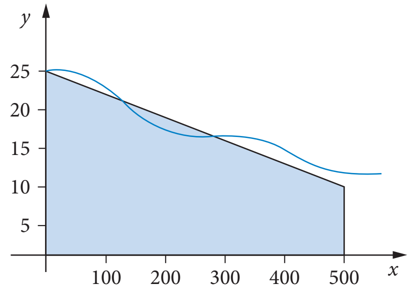

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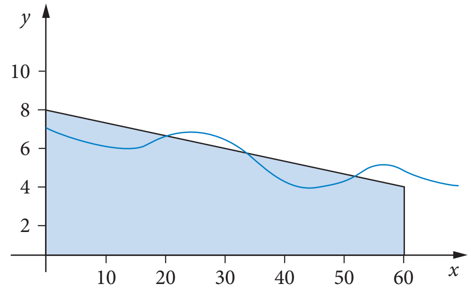
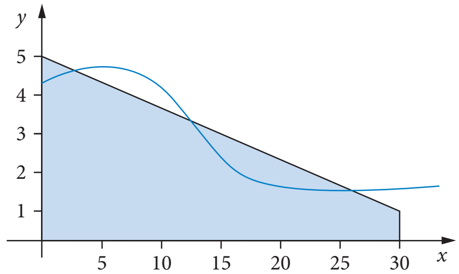
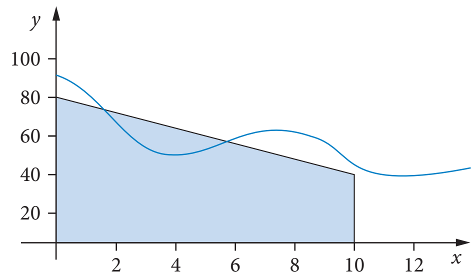


d



e



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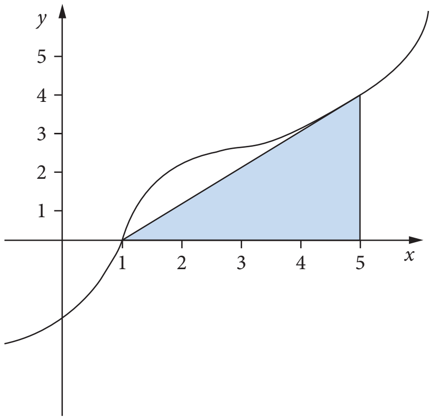
6 By finding the area of the shaded triangle, find an approximation to the area under the curve.

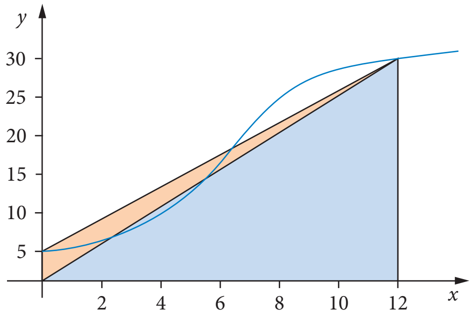


Reasoning and communication

7 Find an approximate area under the curve below by finding the area of a

a triangle

b trapezium

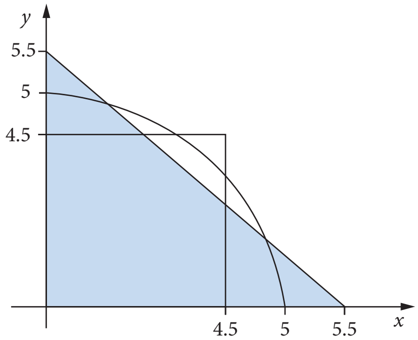


8 a Use the formula for the area of a circle to find the exact area of the quarter circle below correct to 2 decimal places.

b Find an approximation of the area under the curve by finding the area of a

i square with side 4.5 units

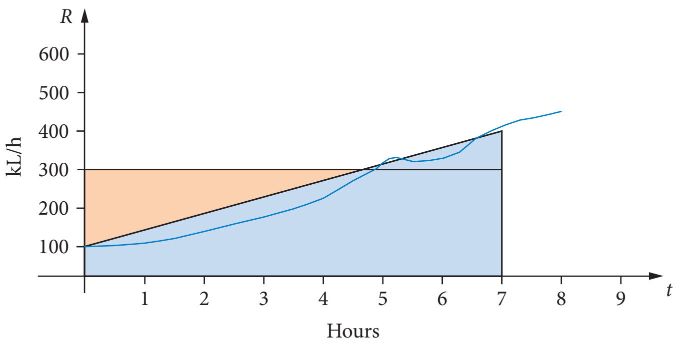
ii right-angled triangle with base and height of 5.5 units



9 The rate at which a dam is filling with water is shown in the graph below. Find the approximate volume of water in the dam after 7 hours by finding the area of a

a rectangle

b trapezium



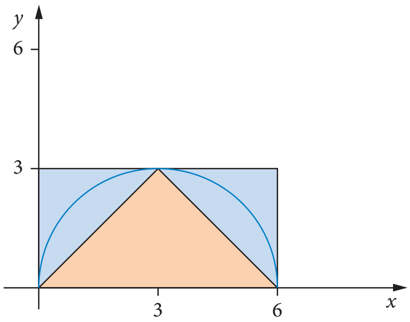
10 **a** Find the area of the semicircle below correct to 2 decimal places.

b Find the approximate area of the semicircle by finding the area of

i the shaded rectangle

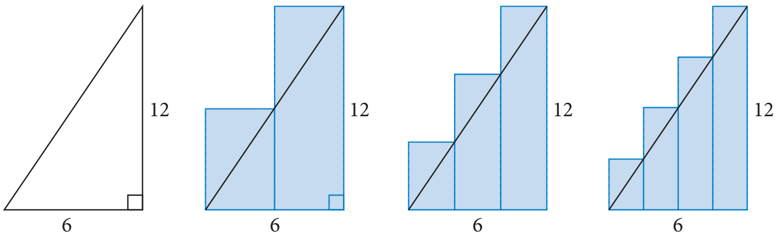
ii the shaded triangle

iii the average of these



4.02 AREA APPROXIMATIONS

You can use more than one rectangle to find the area of an irregular shape. The more rectangles you use, the more accurate the area will be. For example, the triangle below has an area of 36 units^2. On the right, it has been approximated by 2, 3 and 4 rectangles.



For the rectangle approximations you get the following.

For two rectangles, $A = 3 \times 6 + 3 \times 12 = 54$ units^2.

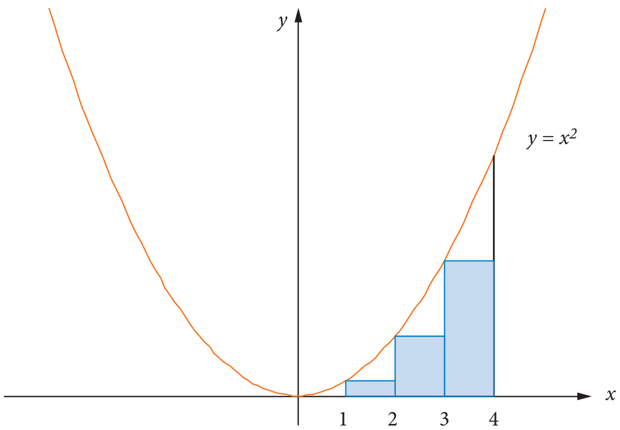
For three rectangles, $A = 2 \times 4 + 2 \times 8 + 2 \times 12 = 48$ units^2.

For four rectangles, $A = 1.5 \times 3 + 1.5 \times 6 + 1.5 \times 9 + 1.5 \times 12 = 45$ units^2.

As the number of rectangles increases, the area gets closer to the true area of the triangle.

Example 3

Find an approximation to the area under the curve $y = x^2$ between $x = 1$ and $x = 4$ using the sum of the three rectangles shown below.



Solution

Find the height of each rectangle.

$f(1) = 1^2 = 1$
$f(2) = 2^2 = 4$
$f(3) = 3^2 = 9$

Find the area of each rectangle.

$A_1 = 1 \times 1 = 1$
$A_2 = 1 \times 4 = 4$
$A_3 = 1 \times 9 = 9$

Find the sum of the rectangles.

$A = 1 + 4 + 9 = 14$

State the result.

The area under the curve is approximately 14 units^2.

In Example **3**, the rectangles have been drawn so that their tops touch the function on their left-hand sides. They are called left rectangles. You can also draw the rectangles so their tops touch the curve on the right-hand side. Obviously, they are called right rectangles.

You can use your CAS calculator to find the approximate area under a curve for large numbers of rectangles.

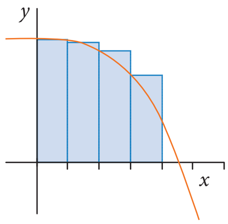
Example 4

Find an approximation to the area under the curve $y = 12 - x^3$ between $x = 0$ and $x = 2$ using

a 4 left rectangles
b 4 right rectangles
c the average of parts **a** and **b**
d CAS 50 left rectangles.

Solution

a Sketch the graph.
Draw the rectangles so that each touches the curve at the top left.



State the left value for each rectangle.

The values are at 0, 0.5, 1 and 1.5.

Find the height of each rectangle.

$f(0) = 12 - 0^3 = 12$
$f(0.5) = 12 - 0.5^3 = 11.875$
$f(1) = 12 - 1^3 = 11$
$f(1.5) = 12 - 1.5^3 = 8.625$

Find the area of each rectangle.

$A_1 = 0.5 \times 12 = 6$
$A_2 = 0.5 \times 11.875 = 5.9375$
$A_3 = 0.5 \times 11 = 5.5$
$A_4 = 0.5 \times 8.625 = 4.3125$

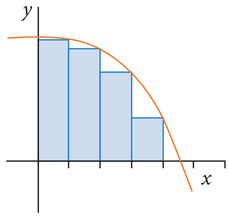
Find the sum of the rectangles.

$A = 6 + 5.9375 + 5.5 + 4.3125$
$= 21.75$

State the result.

The area is approximately 21.75 units^2.

b Sketch the graph.
Draw the rectangles so that each touches the curve at the top right.



State the right value for each rectangle.

The values are at 0.5, 1, 1.5 and 2.

Find the height of each rectangle.

$f(0.5) = 12 - 0.5^3 = 11.875$
$f(1) = 12 - 1^3 = 11$
$f(1.5) = 12 - 1.5^3 = 8.625$
$f(2) = 12 - 2^3 = 4$

Find the area of each rectangle.

$A_1 = 0.5 \times 11.875 = 5.9375$
$A_2 = 0.5 \times 11 = 5.5$
$A_3 = 0.5 \times 8.625 = 4.3125$
$A_4 = 0.5 \times 4 = 2$

Find the sum of the rectangles.

$A = 5.9375 + 5.5 + 4.3125 + 2$
$= 17.75$

State the result.

The area is approximately 17.75 units^2.

c Find the average.

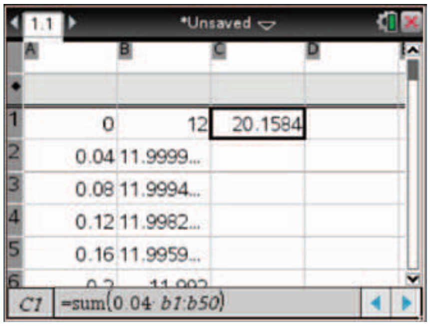
$$\text{Average area} = \frac{21.75 + 17.75}{2}$$
$$= 19.75 \text{ units}^2$$

d Find the widths of the rectangles.

Width $= 2 \div 50 = 0.04$

TI-Nspire CAS

Use the Lists and Spreadsheet page.
Enter 0 into cell A1 and “=a1+0.04” into cell A2.
Copy cell A2 using ctrl C, mark Cells A3 to A50 and copy using ctrl V.
Type “=12–a^3” into B1 and copy down to B50.
Finally type “=sum(0.04´B1:B50)” into C1.



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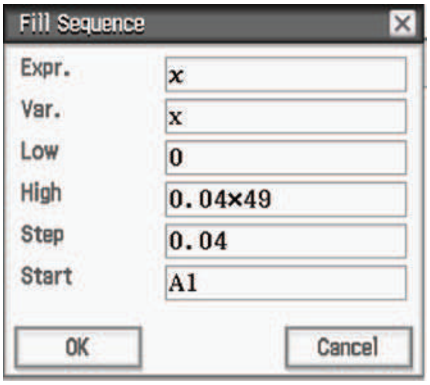
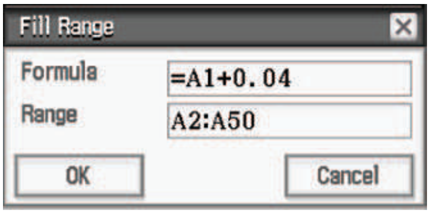
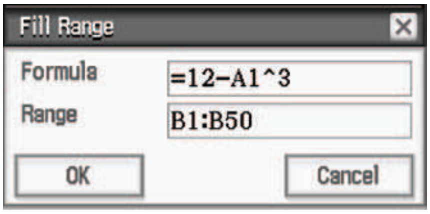
ClassPad

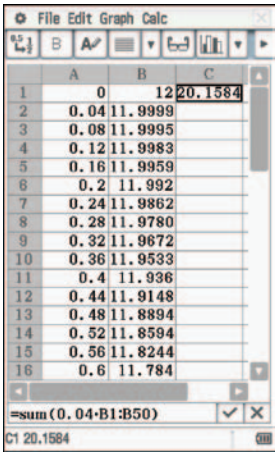
Use the Spreadsheet menu.
Tap cell A1.
Tap Edit, then Fill, then Fill Sequence.
Enter values as on the right. Most are self-explanatory. The expression is the function of x, which in this case is just the x values themselves. High will be the value in A50, which will be 49 lots of 0.04.



Alternate method
Enter 0 in A1. A2 should be selected.
Tap Edit, Fill and Fill Range.
Formula is =A1+1 and Range is A2:A50.
Tap B1, then Edit, Fill and Fill Range.
Formula is =12 – A1^3 and Range is B1:B50.
The widths are 0.04, and the heights are from B1 to B50.
Tap C1 (or any blank cell) and enter =sum(0.04´B1:B50).



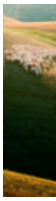




Write the answer. The area is approximately 20.158 units^2.

You can use a normal computer spreadsheet such as Excel in the same way as your CAS calculator in Example **4**.

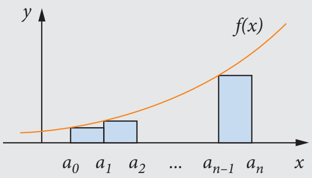
The area in Example **4** is clearly overestimated by the left rectangles and underestimated by the right rectangles. You can use the values in the centres of the rectangles to get a better estimate.



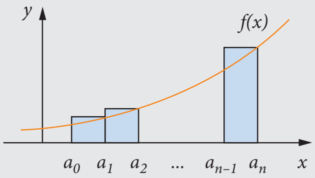
IMPORTANT

The rectangles used to find the approximate area under a curve can use function values on the left, right or in the centre of the rectangles.

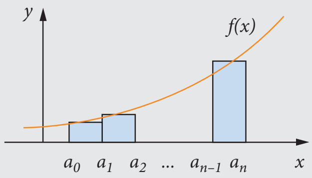
'Left' rectangles



'Right' rectangles



'Centred rectangles'



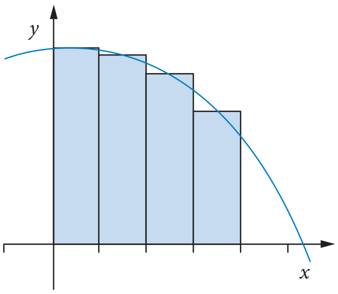
Centred rectangles can give a closer approximation to the area under the curve.

Example 5

Find an approximation to the area under the curve $y = 12 - x^3$ between $x = 0$ and $x = 2$ using 4 centred rectangles.

Solution

Sketch the graph.
Draw the rectangles so that each touches the curve at the top centre.



State the right value for each rectangle. — The values are at 0.25, 0.75, 1.25 and 1.75.

Find the height of each rectangle.

$f(0.25) = 12 - 0.25^3 \approx 11.98$
$f(0.75) = 12 - 0.75^3 \approx 11.58$
$f(1.25) = 12 - 1.25^3 \approx 10.05$
$f(1.75) = 12 - 1.75^3 = 6.64$

Find the area of each rectangle.

$A_1 \approx 0.5 \times 11.98 = 5.99$
$A_2 \approx 0.5 \times 11.58 = 5.79$
$A_3 \approx 0.5 \times 10.05 = 5.02$
$A_4 \approx 0.5 \times 6.64 = 3.32$

Find the sum of the rectangles. — $A = 5.99 + 5.79 + 5.02 + 3.32 = 20.12$

State the result. — The area is approximately 20.12 units^2.

Notice that using 4 centred rectangles gives almost the same approximation as using 50 left rectangles.

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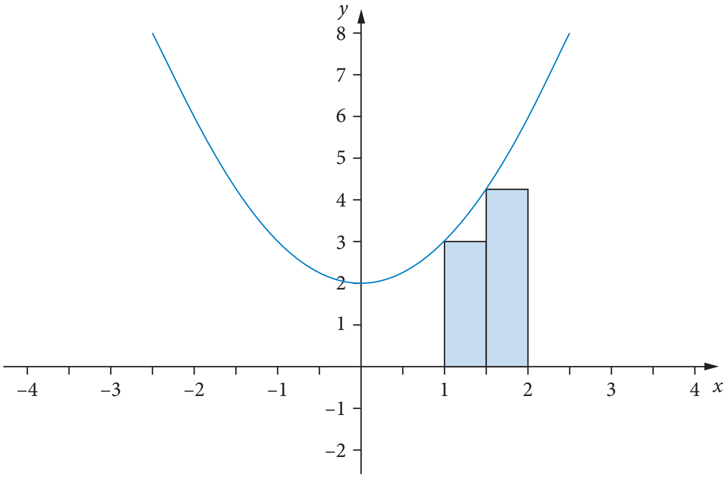
EXERCISE 4.02 Area approximations

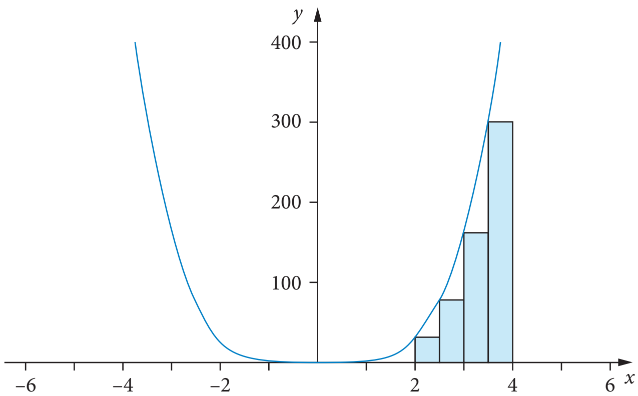
Concepts and techniques

1 Example 3 Find the approximate area of each of the following.

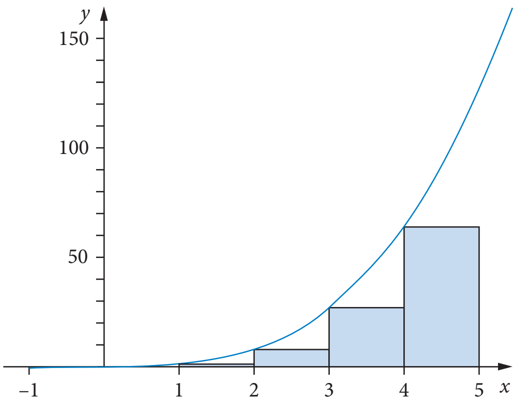
a The area under the curve $y = x^2 + 2$ between $x = 1$ and $x = 2$, using two rectangles as shown.



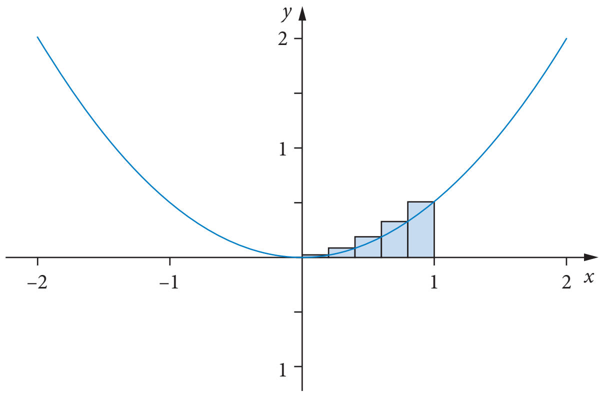
b The area under the curve $y = 2x^4$ between $x = 2$ and $x = 4$, using four rectangles as shown.



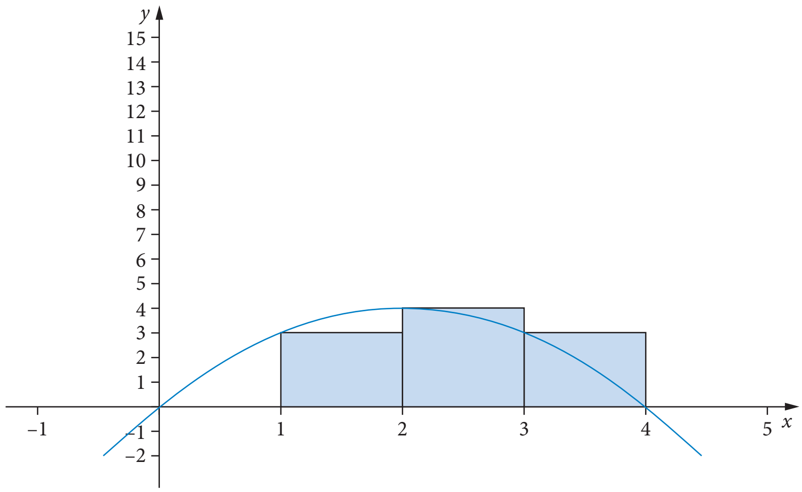
c The area under the curve $y = x^3$ between $x = 1$ and $x = 5$, using four rectangles as shown.



d The area under the curve $y = x^2$ between $x = 0$ and $x = 1$, using five right rectangles as shown. The first rectangle is very low, so doesn't show up very well on this diagram.

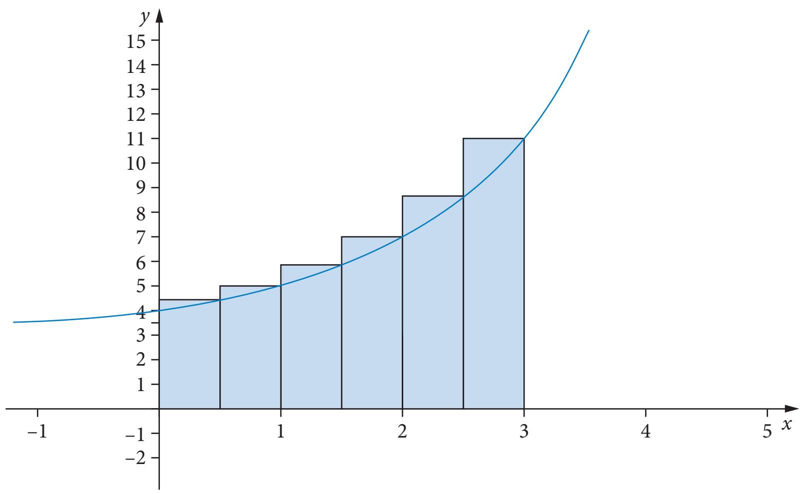


e The area under the curve $y = 4x - x^2$ between $x = 1$ and $x = 4$, using three rectangles as shown.



2 Find an approximation to each area.

a The area under the curve $y = 2^x + 3$ between $x = 0$ and $x = 3$, using six rectangles as shown.



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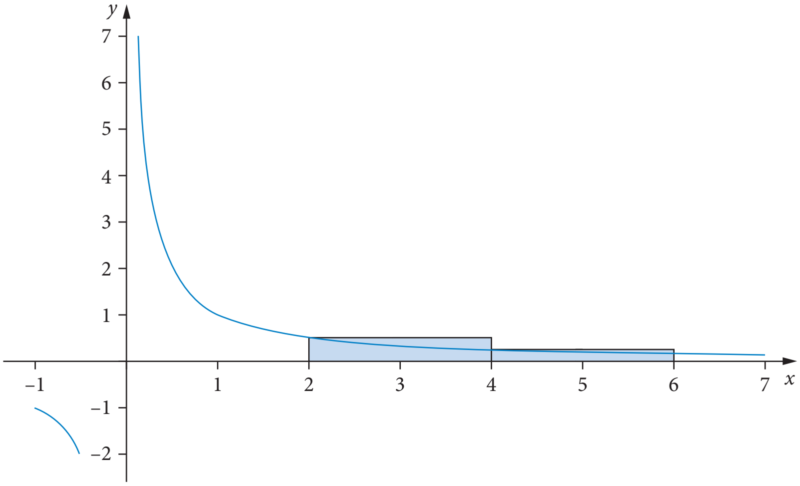
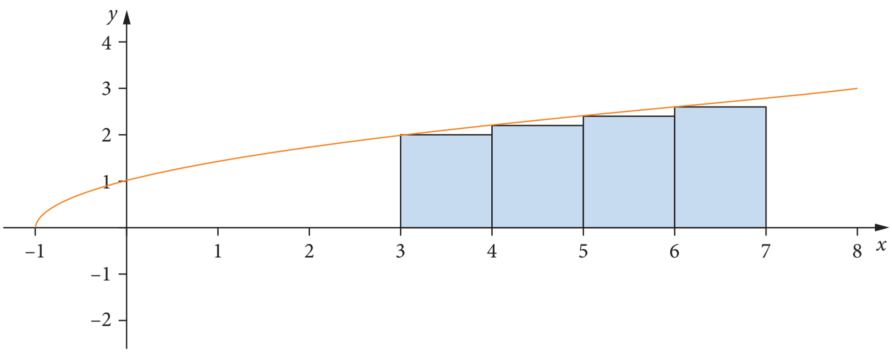
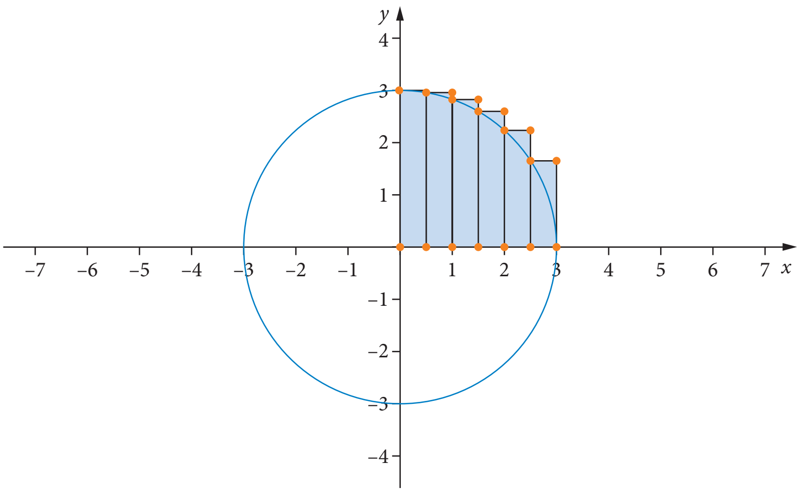
b The area under the curve $y = \frac{1}{x}$ between $x = 2$ and $x = 6$, using two rectangles as shown.



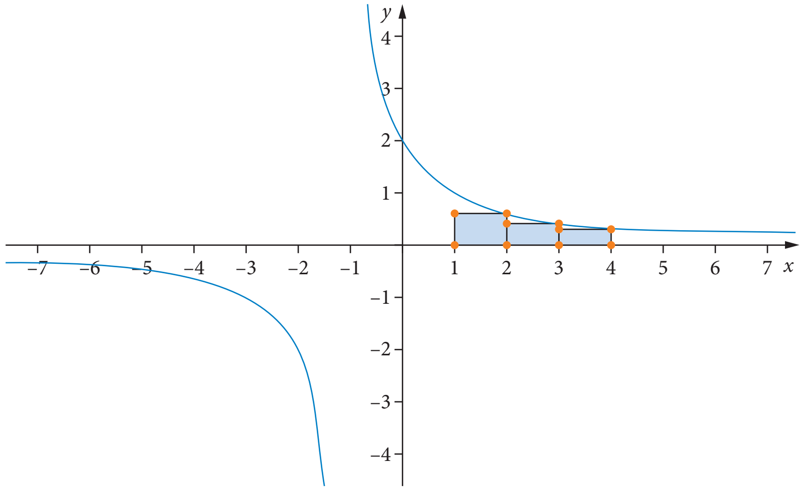
c The area under the curve $y = \sqrt{x+1}$ between $x = 3$ and $x = 7$, using four rectangles as shown.



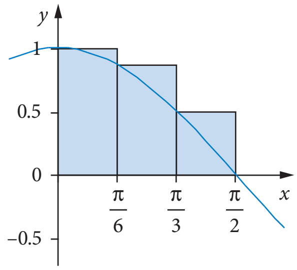
d The area under the curve $y = \sqrt{9-x^2}$ between $x = 0$ and $x = 3$, using six rectangles as shown.



e The area under the curve $y = \dfrac{2}{x+1}$ between $x = 1$ and $x = 4$, using three rectangles as shown.



3 a Find an approximation for the area under the curve $y = \cos(x)$ between $x = 0$ and $x = \frac{\pi}{2}$ using the rectangles below.



b Use a CAS calculator or computer application to find the area using 20 rectangles.

4 Example 4 For each question below,

i find an approximate area under the curve using the given number of left rectangles

ii find an approximate area under the curve using the given number of right rectangles

iii CAS find the area using 20 left rectangles.

a $y = x^2$ between $x = 1$ and $x = 2$ using two rectangles.

b $y = x^2 + 2x$ between $x = 0$ and $x = 4$ using four rectangles.

c $y = x^3 + 1$ between $x = 0$ and $x = 2$ using two rectangles.

d $y = x^2 - x - 2$ between $x = 2$ and $x = 4$ using four rectangles.

e $y = e^x$ between $x = 0$ and $x = 5$ using five rectangles.

5 Example 5 Use centred rectangles to find an approximation to each area.

a $y = x^2$ between $x = 1$ and $x = 2$ with 4 rectangles.

b $y = x^3$ between $x = 0$ and $x = 1$ with 5 rectangles.

c $y = 2x^2 + 3$ between $x = 0$ and $x = 2$ with 4 rectangles.

d $y = x^2 - 1$ between $x = 2$ and $x = 6$ with 8 rectangles.

e $y = \sin(x)$ between $x = 0$ and $x = 1$ with 10 rectangles.

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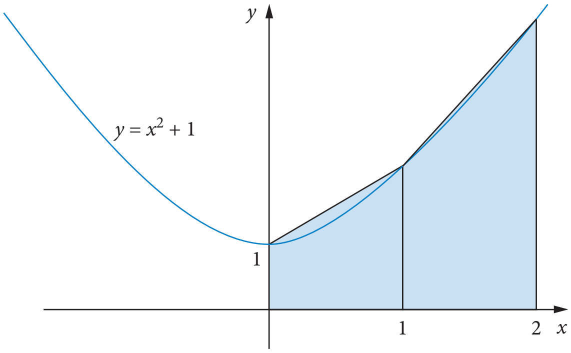
6 **a** Find the approximate area under the line $y = x - 1$ between $x = 1$ and $x = 4$ by using 3 centred rectangles.

b Find the exact area using geometry.

Reasoning and communication

7 The trapezoidal rule uses trapeziums rather than rectangles to find approximate areas under a curve. Find an approximation to the area under the curve $y = x^2 + 1$ between $x = 0$ and $x = 2$ by using the sum of each trapezium.



8 **a** Find the approximate area under the curve $y = \dfrac{1}{x+2}$ between $x = 1$ and $x = 2$ by using

i 4 left rectangles ii 4 right rectangles iii 4 centred rectangles.

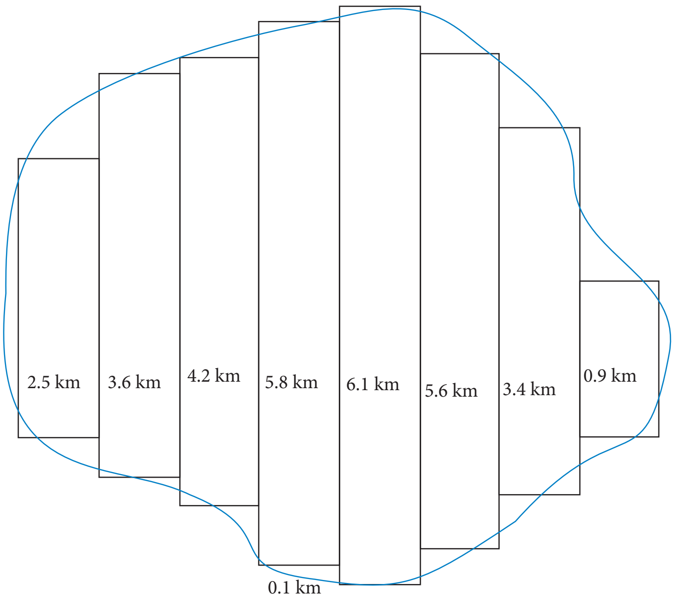
b Find the approximate area under the curve by using a trapezium with sides $f(1)$ and $f(2)$.

c Use a CAS calculator or computer application to find the area using 50 centred rectangles.

9 A lake has an irregular surface as shown below and an average depth of 850 metres.

a Find an approximation to the area of the surface of the lake using the rectangles shown with width 0.1 km.

b Find an approximation to the volume of water in the lake in km^3.

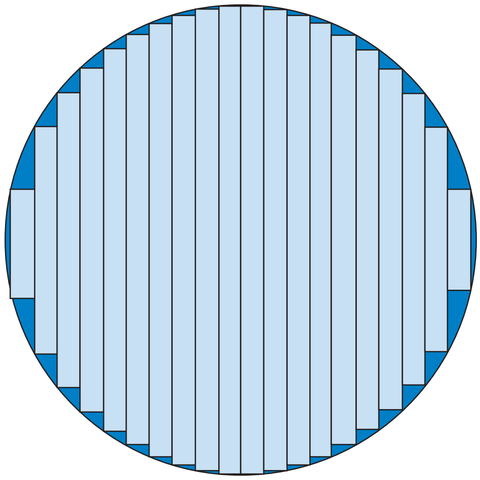


4.03 THE DEFINITE INTEGRAL

Integration is a process used to find the exact value of the area under a curve.

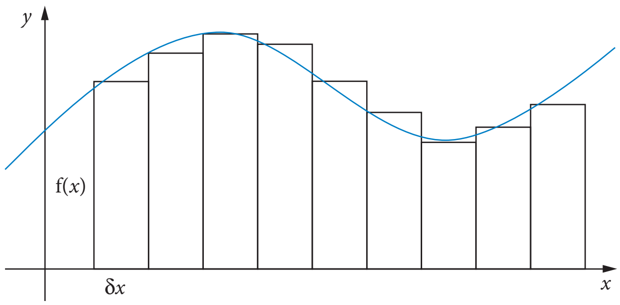
For a function $y = f(x)$, where $f(x) > 0$ between $x = a$ and $x = b$, the **definite integral** is the area under the curve.

Archimedes (287–212 BCE) found the area of a circle by cutting it into very thin layers and finding the sum of the areas of these rectangles.



You can find an approximation for the definite integral in the same way.

The area of each rectangle is given by $f(x)\delta x$, where $f(x)$ is the height of each rectangle and δx is the width.



IMPORTANT

The area under a curve can be approximated by a sum of rectangle areas. This can be written as $\sum_{i=1}^{n} f(x_i)\delta x_i$ or $\sum_{x=a}^{b} f(x_i)\delta x_i$. If the widths are all the same, this can be changed to $\delta x \sum_{x=a}^{b} f(x)$.

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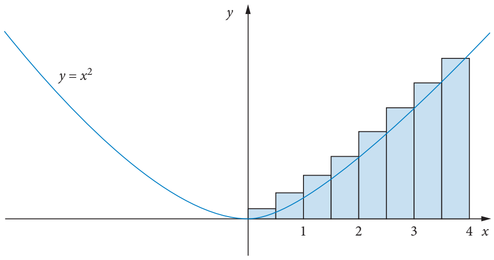
Remember that the capital Greek letter sigma means 'the sum of', so the first sum is read as 'the sum of $f(x_i)$ times delta x_i for $i = 1$ to n' and the second as 'the sum of $f(x_i)$ times delta x_i for $x = a$ to $x = b$'. The first case is for n rectangles and the second is for an unspecified number of rectangles between a and b.

Example 6

Find the approximate area under the curve $y = x^2$ between $x = 0$ to $x = 4$ using 8 right rectangles.

Solution

Draw the graph showing 8 right rectangles.



Find the value of δx.

$$\delta x = \frac{4 - 0}{8} = 0.5$$

Find the height of each rectangle.

$f(0.5) = 0.5^2 = 0.25$
$f(1) = 1^2 = 1$
$f(1.5) = 1.5^2 = 2.25$
$f(2) = 2^2 = 4$
$f(2.5) = 2.5^2 = 6.25$
$f(3) = 3^2 = 9$
$f(3.5) = 3.5^2 = 12.25$
$f(4) = 4^2 = 16$

Find the sum of the areas of the rectangles.

$$\begin{aligned} A &= 0.5 \times 0.25 + 0.5 \times 1 + 0.5 \times 2.25 + 0.5 \times 4 \\ &\quad + 0.5 \times 6.25 + 0.5 \times 9 + 0.5 \times 12.25 \\ &\quad + 0.5 \times 16 \\ &= 0.5 \times (0.25 + 1 + 2.25 + 4 + 6.25 + 9 \\ &\quad + 12.25 + 16) \\ &= 25.5 \end{aligned}$$

State the result.

The area is approximately 25.5 units^2.

You can also use a CAS calculator as shown previously in Example **4**. The more rectangles you use, the more accurate the approximation becomes. The limit of the sum as the number of rectangles increases, or as the widths decrease, will be the exact area.

IMPORTANT

The **definite integral** of the function $f(x)$ from a to b, $\int_a^b f(x)dx$, is the exact area under the curve $y = f(x)$ between $x = a$ and $x = b$. This is given by the limit of the sum of the areas of the rectangles:

$$\int_a^b f(x)dx = \lim_{\delta x \to 0} \sum f(x)\delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\delta x_i$$

For $f(x) > 0$, the definite integral is the area under the curve. However, a definite integral is usually considered to be a value, not an area. Obviously, when $f(x) < 0$, it will be negative and when $f(x)$ varies in sign, it could be either positive or negative.

Example 7

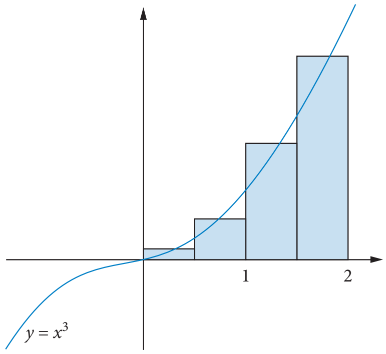
Use 4 centred rectangles to find an approximation to $\int_0^2 x^3 dx$.

Solution

Find the widths.

$\delta x = \frac{2-0}{4} = 0.5$

Draw the graph showing 4 centred rectangles.



Find the height of each rectangle.

$f(0.25) = 0.25^3 \approx 0.016$
$f(0.75) = 0.75^3 \approx 0.42$
$f(1.25) = 1.25^3 \approx 1.95$
$f(1.75) = 1.75^3 \approx 5.36$

Find the sum of the areas of the rectangles.

$$\begin{aligned} A &= 0.5 \times 0.016 + 0.5 \times 0.42 + 0.5 \times 1.95 + 0.5 \\ &\quad \times 5.36 \\ &= 3.875 \end{aligned}$$

State the result.

$\int_0^2 x^3 dx \approx 3.875$

As the number of rectangles increases, the approximate area under a curve becomes more accurate.

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Example 8

CAS Use 20 left rectangles to find an approximation to $\int_0^5 e^x\,dx$.

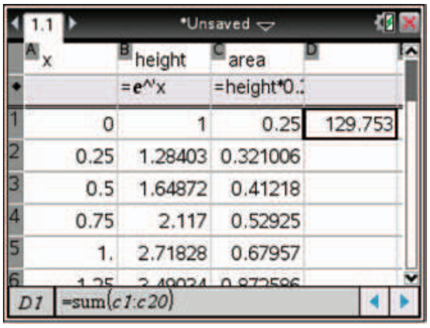
Solution

Find the widths.

$$\delta x = \frac{5-0}{20} = 0.25$$

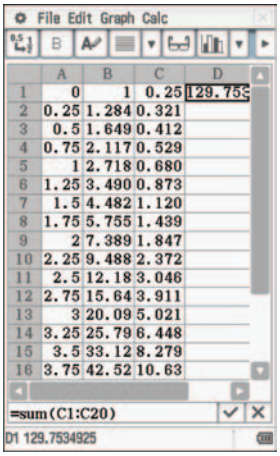
TI-Nspire CAS

Use the Lists and Spreadsheet page.
Enter 0 into cell A1 and type = a1 + 0.25 in A2 and fill down to A20.
Use the headings x, height and area for columns A, B and C
Type =e^x into the formula cell for column B and =height*0.25 into the formula cell for column C.
Finally type =sum(C1:C50)" into D1.



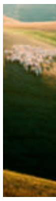
ClassPad

Use the Spreadsheet menu.
Enter 0 into cell A1 and "=a1+0.25" into cell A2 and fill down to A20 using Edit, Fill and Fill Range A2:A20.
Type =e^A1 into B1 and fill down to B20.
Type =0.25´B1 into C1 and fill down to C20.
Then type =sum(C1:C20) into D1.



State the result to reasonable accuracy.

$$\int_0^5 e^x\,dx \approx 130$$



WS

Areas using rectangles

EXERCISE 4.03 The definite integral

Concepts and techniques

1 Example 6 Find the approximate area under each of the curves below.

a $y = x^2 + x$ between $x = 0$ and $x = 3$ using 6 left rectangles.

b $y = x^3 + 1$ between $x = 0$ and $x = 5$ using 10 right rectangles.

c $y = x^2 - 1$ between $x = 1$ and $x = 3$ using 8 left rectangles.

d $y = x^4$ between $x = 0$ and $x = 6$ using 6 left rectangles.

e $y = \sin(x)$ between $x = 0$ and $x = 3$ using 6 right rectangles.

2 Example 7 Find the approximate value of each of the following using 8 centred rectangles.

a $\int_1^2 (x^2+2)dx$ b $\int_2^4 2x^4 dx$ c $\int_1^5 x^3 dx$

d $\int_3^7 \sqrt{x+1}\,dx$ e $\int_1^9 (x^2+4x)dx$

3 Find an approximation to each integral using 6 right rectangles.

a $\int_0^3 (2^x+3)dx$ b $\int_2^5 \frac{1}{x}dx$ c $\int_0^3 \sqrt{9-x^2}\,dx$

d $\int_1^7 \frac{2}{x+1}dx$ e $\int_0^6 (x^3+2)\,dx$

4 Find an approximation to $\int_0^{\frac{\pi}{2}} \cos(x)dx$ in exact form by using 2 left rectangles.

5 CAS Example 8 Use a CAS calculator or computer application with 15 left rectangles to find an approximation for each definite integral.

a $\int_0^3 2x^3 dx$ b $\int_1^4 (x^2+2)dx$

6 CAS Use 20 left rectangles to find an approximation to $\int_2^{12} \frac{x-2}{x+1}dx$.

7 Use 50 right rectangles to find an approximation to $\int_1^6 (x^2-1)\,dx$.

Reasoning and communication

8 a Find an approximation to $\int_0^2 x^3 dx$ using 8 centred rectangles.

b CAS Find $\int_{-2}^2 x^3 dx$ using 100 centred rectangles.

c Draw the graph of $y = x^3$ and explain the result in part **b**.

9 CAS Evaluate $\int_1^3 (x^2-4x)dx$ using 40 centred rectangles and explain the result with a graph.

10 The velocity of an object is given by $v = 6t - t^2$ m/s and the initial position is at $x = 0$.

a Find the approximate distance travelled in the first 4 seconds using 8 centred rectangles.

b Find the exact distance travelled in the first 4 seconds.

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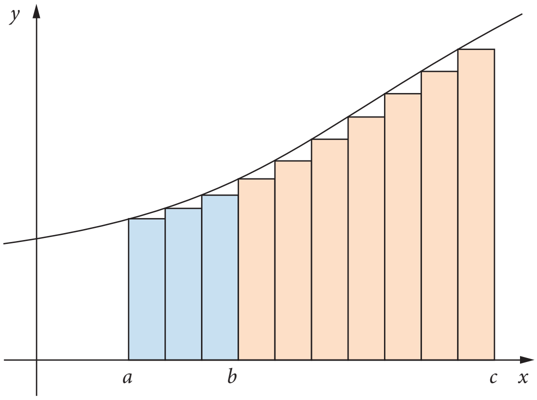
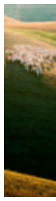
4.04 PROPERTIES OF THE DEFINITE INTEGRAL

You can work out properties of definite integrals using the definition as the limit of a sum or demonstrate them using sums of rectangle areas. The first property is shown below.

$\int_a^b f(x)dx = \lim_{\delta x \to 0} \sum f(x)\delta x$, $\int_b^c f(x)dx = \lim_{\delta x \to 0} \sum f(x)\delta x$ and $\int_a^c f(x)dx = \lim_{\delta x \to 0} \sum f(x)\delta x$ for appropriate rectangles. Consider the situation where the widths are all the same before the limit is worked out. It could look like the following for left rectangles.

> **IMPORTANT**
>
> $$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$





In this case there are, say, 3 rectangles from a to b and 7 from b to c, making 10 from a to c.

For these rectangles,

$$\begin{aligned}\int_a^b f(x)dx + \int_b^c f(x)dx &\approx \sum_{i=1}^{3} f(x_i)\delta x + \sum_{i=4}^{10} f(x_i)\delta x \\ &= \sum_{i=1}^{10} f(x_i)\delta x \\ &\approx \int_a^c f(x)dx\end{aligned}$$

When limits are taken, the approximations become exact.

You can demonstrate this using the approximations you used previously.

Example 9

a Use left rectangles of width 0.5 units to find approximations for the following.

i $\int_0^2 x^2 dx$ ii $\int_2^3 x^2 dx$ iii $\int_0^3 x^2 dx$

b Show that, for these sums, $\int_0^2 x^2 dx + \int_2^3 x^2 dx = \int_0^3 x^2 dx$

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Solution

a i Write the values of x_i. $x_1 = 0, x_2 = 0.5, x_3 = 1$ and $x_4 = 1.5$

Find the values of $f(x_i)$. $f(x_1) = 0, f(x_2) = 0.25, f(x_3) = 1$ and $f(x_4) = 2.25$

Find the sum. $$\sum_{i=1}^{4} f(x_i)\delta x = 1.75$$

State the result. $$\int_0^2 x^2 dx = 1.75$$

ii Write the values of x_i. $x_5 = 2, x_6 = 2.5$

Find the values of $f(x_i)$. $f(x_5) = 4, f(x_6) = 6.25$

Find the sum. $$\sum_{i=1}^{4} f(x_i)\delta x = 5.125$$

State the result. $$\int_2^3 x^2 dx = 5.125$$

iii Write the values of x_i. $x_1 = 0, x_2 = 0.5, x_3 = 1, x_4 = 1.5, x_5 = 2, x_6 = 2.5$

Find the values of $f(x_i)$. $f(x_1) = 0, f(x_2) = 0.25, f(x_3) = 1, f(x_4) = 2.25, f(x_5) = 4, f(x_6) = 6.25$

Find the sum. $$\sum_{i=1}^{6} f(x_i)\delta x = 6.875$$

State the result. $$\int_0^3 x^2 dx = 6.875$$

b Check the LHS.
$$\begin{aligned} \text{LHS} &= \int_0^2 x^2 dx + \int_2^3 x^2 dx \\ &= 1.75 + 5.125 \\ &= 6.875 \end{aligned}$$

Check the RHS.
$$\begin{aligned} \text{RHS} &= \int_0^3 x^2 dx \\ &= 6.875 \end{aligned}$$

State the result. $$\text{LHS} = \text{RHS, so } \int_0^2 x^2 dx + \int_2^3 x^2 dx = \int_0^3 x^2 dx$$

IMPORTANT

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

This is easily shown using the definition.

$$\begin{aligned} \int_a^b kf(x)dx &= \lim_{\delta x \to 0} \sum kf(x)\delta x \\ &= \lim_{\delta x \to 0} k\sum f(x)\delta x \\ &= k\lim_{\delta x \to 0} \sum f(x)\delta x \\ &= k\int_a^b f(x)dx \end{aligned}$$

Considering the rectangles for the sums, each rectangle for $kf(x_i)$ is k times as high as the corresponding rectangle for $f(x_i)$, so the whole sum and limit must also be k times as big.

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Example 10

CAS **a** Find approximations for the following using 100 left rectangles.

i $\int_1^3 5x^3 dx$ ii $\int_1^3 x^3 dx$

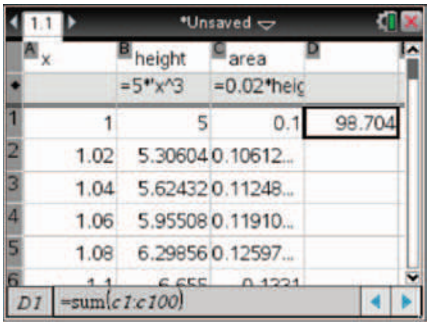
b Show that, for these sums, $\int_1^3 5x^3 dx = 5\int_1^3 x^3 dx$

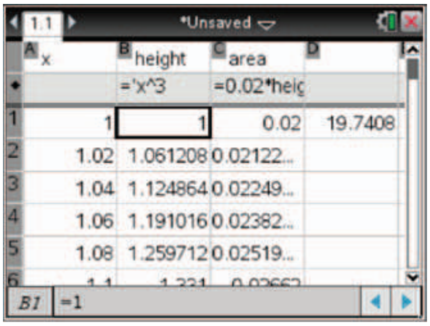
Solution

a Find the widths. $dx = \frac{3-1}{100} = 0.02$

TI-Nspire CAS

Follow the same steps as in Example **8** on page 173 for parts **i** and **ii**.





ClassPad

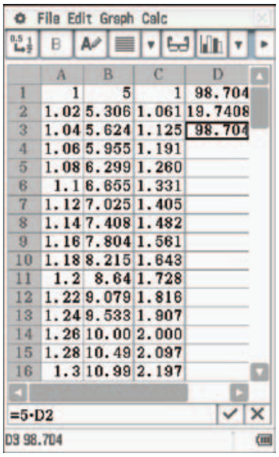
Follow the same steps as in Example **8** on page 173 for parts **i** and **ii**., but put the values for **i** in column B and the values for **ii** in column C. Add the areas by filling cells as follows.

D1: =sum(0.02×B1:B100)

D2: =sum(0.02×C1:C100)

D3: =5×D2

The columns have been narrowed to show the complete result.



Write the results.

i $\int_1^3 5x^3 dx » 98.704$

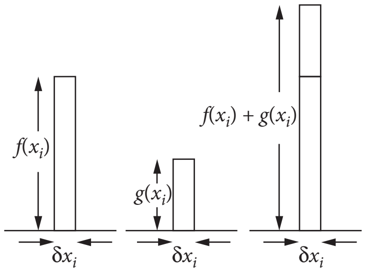
ii $\int_1^3 x^3 dx » 19.7408$

b Check the sides.

$$\begin{aligned}\text{LHS} &= \int_1^3 5x^3dx\\ &= 98.704\\ &= 5 \times 19.7408\\ &= 5\int_1^3 x^3dx\\ &= \text{RHS}\end{aligned}$$

When you work out the value of $f(x) + g(x)$, you just add the values of the two functions. This means that the height of the rectangle for $f(x_i) + g(x_i)$ is just the sum of the heights of the individual rectangles, as shown in the diagram on the right.

Since this is true for every rectangle, it is true for the sum and true for the limit.



IMPORTANT

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Example 11

a Use centred rectangles with width 1 unit to find the approximate value of the following definite integrals.

i $\int_3^8 x^2dx$ ii $\int_3^8 2x\,dx$ iii $\int_3^8 (x^2+2x)dx$

b Show that $\int_0^5 (x^2+2x)dx = \int_0^5 x^2dx + \int_0^5 2x\,dx$ for the values you have obtained.

Solution

a Write the values of x_i. $x_1 = 3.5, x_2 = 4.5, x_3 = 5.5, x_4 = 6.5, x_5 = 7.5$

i Find the values of $f(x_i)$. $f(x_1) = 12.25, f(x_2) = 20.25, f(x_3) = 30.25, f(x_4) = 42.25, f(x_5) = 56.25$

Find the sum. $\sum_{i=1}^{5} f(x_i)\delta x = 161.25$

Write the result. $\int_3^8 x^2dx \approx 161.25$

ii Find the values of $f(x_i)$. $f(x_1) = 7, f(x_2) = 9, f(x_3) = 11, f(x_4) = 13, f(x_5) = 15$

Find the sum. $\sum_{i=1}^{5} f(x_i)\delta x = 55$

Write the result. $\int_3^8 2x\,dx \approx 55$

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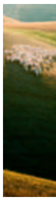
iii	Find the values of $f(x_i)$.	$f(x_1) = 19.25$, $f(x_2) = 29.25$, $f(x_3) = 41.25$, $f(x_4) = 55.25$, $f(x_5) = 71.25$
	Find the sum.	$\sum_{i=1}^{5} f(x_i)\delta x = 216.25$
	Write the result.	$\int_3^8 (x^2 + 2x)dx \approx 216.25$
b	Check the LHS.	$\text{LHS} = \int_0^5 (x^2 + 2x)dx$ $= 216.25$
	Check the RHS.	$\text{RHS} = \int_0^5 x^2 dx + \int_0^5 2x\,dx$ $= 161.25 + 55$ $= 216.25$
	State the result.	LHS = RHS, so $\int_0^5 (x^2 + 2x)dx = \int_0^5 x^2 dx + \int_0^5 2x\,dx$



You can combine the results of this section to show that the definite integral of a linear combination of functions preserves the linear sum. This is sometimes called the **linearity** property.

IMPORTANT

$\int_a^b [cf(x) + dg(x)]dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx$, for any functions and constants.

This is easily shown as follows.

$$\begin{aligned}\int_a^b [cf(x) + dg(x)]dx &= \int_a^b cf(x)dx + \int_a^b dg(x)dx \\ &= c\int_a^b f(x)dx + d\int_a^b g(x)dx\end{aligned}$$

This includes the case where $c = 1$ and $d = -1$, so it follows that

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

EXERCISE 4.04 Properties of the definite integral

Concepts and techniques

1 Example 9 **a** Use left rectangles with width 1 unit to approximate

i $\int_1^4 x^3 dx$ **ii** $\int_4^6 x^3 dx$ **iii** $\int_1^6 x^3 dx$

b Show that $\int_1^6 x^3 dx = \int_1^4 x^3 dx + \int_4^6 x^3 dx$ for these sums.

2 CAS **a** Use left rectangles of width 0.04 to find the value of

i $\int_0^2 (x^2 + 3)dx$ **ii** $\int_2^4 (x^2 + 3)dx$ **iii** $\int_0^4 (x^2 + 3)dx$

b Show that $\int_0^4 (x^2 + 3)dx = \int_0^2 (x^2 + 3)dx + \int_2^4 (x^2 + 3)dx$ for these sums.

3 Write the following as single integrals.

a $\int_0^1 x^2dx + \int_1^5 x^2dx$

b $\int_1^4 (x+1)dx + \int_4^7 (x+1)dx$

c $\int_{-2}^0 (x^3 - x - 1)dx + \int_0^2 (x^3 - x - 1)dx$

d $\int_0^2 (2x+1)dx + \int_2^3 (2x+1)dx$

e $\int_1^2 6x^3dx + \int_2^3 6x^3dx$

f $\int_{-1}^1 (3x^2 - 4x - 1)dx + \int_1^3 (3x^2 - 4x - 1)dx$

g $\int_{-2}^0 (x^2 - 2)dx + \int_0^2 (x^2 - 2)dx$

h $\int_0^3 3dx + \int_3^7 3dx$

i $\int_1^2 5x^4dx + \int_2^3 5x^4dx$

j $\int_0^4 (2x-3)dx + \int_4^6 (2x-3)dx$

4 Example 10 a Using 10 centred rectangles, find the approximate value of

i $\int_0^{10} x^2dx$ ii $\int_0^{10} 3x^2dx$

b Show that $\int_0^{10} 3x^2dx = 3\int_0^{10} x^2dx$ for these sums.

5 CAS a Use 100 centred rectangles to evaluate

i $\int_2^5 x^5dx$ ii $\int_2^5 2x^5dx$

b Show that $\int_2^5 2x^5dx = 2\int_2^5 x^5dx$ for these sums.

6 Example 11 a Use 4 left rectangles to find an approximation to

i $\int_1^2 3x\,dx$ ii $\int_1^2 2x^2dx$ iii $\int_1^2 (2x^2 + 3x)dx$

b Show that $\int_1^2 (2x^2 + 3x)dx = \int_1^2 2x^2dx + \int_1^2 3x\,dx$ for these sums.

7 Simplify each of the following.

a $\int_0^2 (3x^2 + 2)dx + \int_0^2 2x\,dx$

b $\int_1^2 x^3dx + \int_1^2 (2x^3 - 3x + 1)dx$

c $\int_{-1}^1 (2x^4 + 3)dx + \int_{-1}^1 (x^3 - x^2 - 4)dx$

d $\int_0^3 (x^2 + 4x - 3)dx + \int_0^3 (x^2 - x - 1)dx$

e $\int_1^5 2x\,dx + \int_1^5 7dx$

Reasoning and communication

8 a Use 8 left rectangles to find an approximate value of

i $\int_2^6 x^3dx$ ii $\int_2^6 x^2dx$ iii $\int_2^6 (x^3 - x^2)dx$

b Show that $\int_2^6 (x^3 - x^2)dx = \int_2^6 x^3dx - \int_2^6 x^2dx$ for these sums.

9 CAS a Use 100 right rectangles to evaluate $\int_1^3 x^3dx$.

b Use a lower boundary of 3 and upper boundary 1 to evaluate $\int_3^1 x^3dx$.

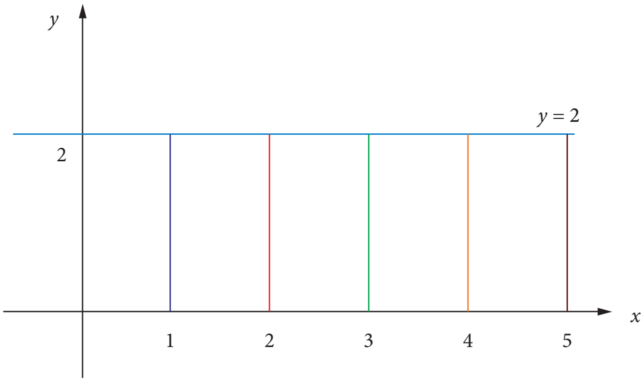
c Find a relationship between $\int_1^3 x^3dx$ and $\int_3^1 x^3dx$.

d Prove the relationship between $\int_a^b f(x)dx$ and $\int_b^a f(x)dx$.

10 The velocity of a particle is given by $v = 6t - t^2$ m/s. Find the distance that the particle travels between 1 and 5 seconds by using 8 centred rectangles.

4.05 THE FUNDAMENTAL THEOREM OF CALCULUS

You can find the area under a straight line such as $y = 2$ and draw the graph of the area function.



From the graph you can see area A under the line $y = 2$ for different values of x:

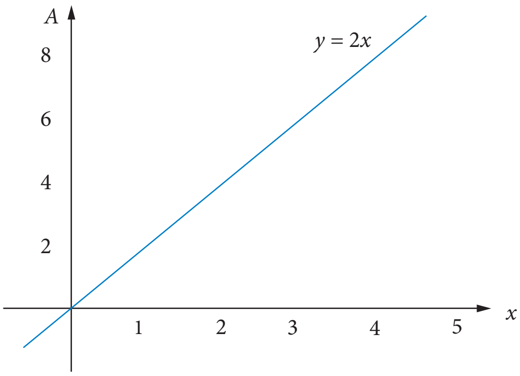
When $x = 1$, $A = 2$

When $x = 2$, $A = 4$

When $x = 3$, $A = 6$

When $x = 4$, $A = 8$ and so on.

You can draw the graph of these areas by plotting the points (1, 2), (2, 4), (3, 6) and (4, 8).



This is the graph of $y = 2x$, so the area function for the line $y = 2$ is $A(x) = 2x$.

INVESTIGATION Areas under a curve

For each of the following,

1 draw the graph of the function

2 find the area A under the curve at different values of x

3 draw the graph of the area A as a function of x

4 find the equation of the area function.

5 Can you find a relationship between the original function and its area function?

a $y = 1$　　b $y = 3$　　c $y = 4$

d $y = x$　　e $y = 2x$　　f $y = 3x$

Example 12

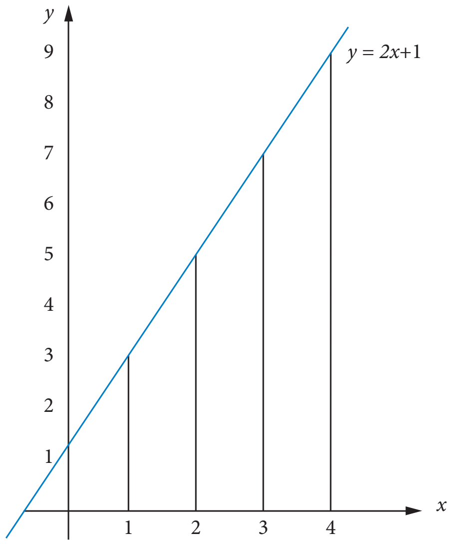
a Draw the graph of $f(x) = 2x + 1$ for $x = 0$ to 4.

b Find the area under the function from 0 to 0, 1, 2, 3 and 4 and draw the graph of the area function $A(x)$.

c CAS Find the equation of the function $y = A(x)$.

Solution

a Draw the graph of $y = 2x + 1$.



b Find the area under the curve for values of x using the area of a trapezium:

$A = \frac{1}{2}h(a+b)$.

When $x = 0$, $A = 0$

When $x = 1$: $A = \frac{1}{2} \times 1 \times (1+3) = 2$

When $x = 2$: $A = 2 + \frac{1}{2} \times 1 \times (3+5) = 6$

When $x = 3$: $A = 6 + \frac{1}{2} \times 1 \times (5+7) = 12$

When $x = 4$: $A = 12 + \frac{1}{2} \times 1 \times (7+9) = 20$

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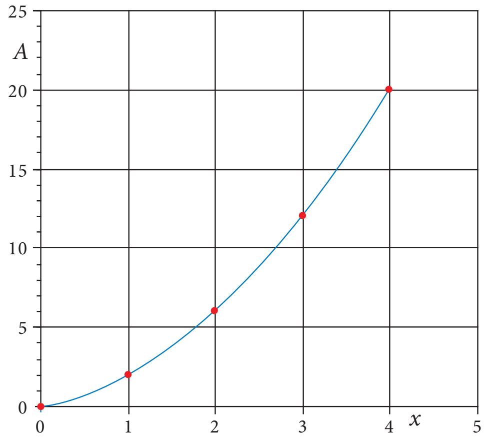
Draw the area function using points (0, 0), (1, 2), (2, 6), (3, 12) and (4, 20).

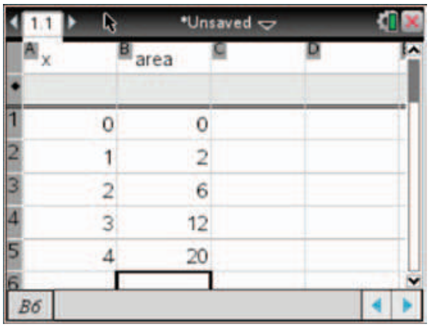


c Take a guess at the function.

The function looks like a quadratic.

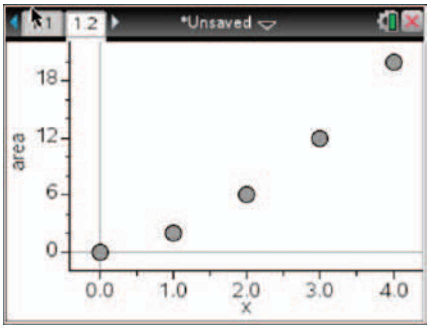
TI-Nspire CAS

Use Lists and Spreadsheet.
Put the headings x and area and type in the data.

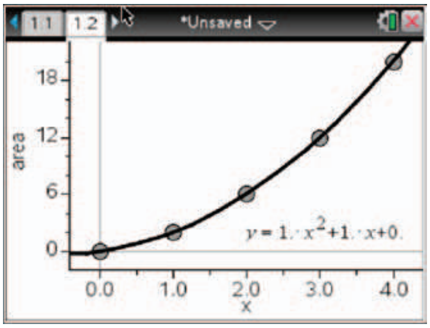


Add a Data and Statistics page and the data will be shown somewhat randomly.

Click to add variable at the bottom, and select x. Click to add variable on the left and select area. You can also press menu, 2: Plot properties and 8: Add Y Variable and choose area.

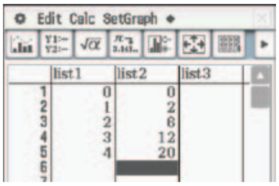


Now press menu, 4 Analyse, 6 Regression and 4 Show Quadratic to see the equation of the graph.

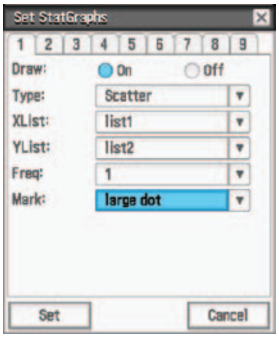


ClassPad

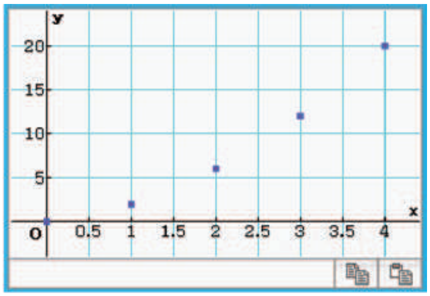
Use the Statistics menu.
Enter the x values in list1 and the corresponding areas in list2.



Tap SetGraph and make sure that only StatGraph1 is ticked. Tap Setting and define a Scatter graph as on the right. The kind of mark chosen is not important. Click Set.



Use View Window to make sure that your graph fits the values. If you want a grid, choose suitable scales for x and y.
Tap to draw the graph.



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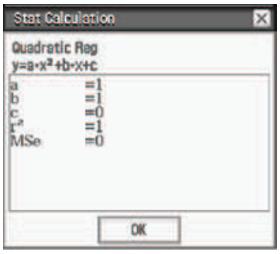
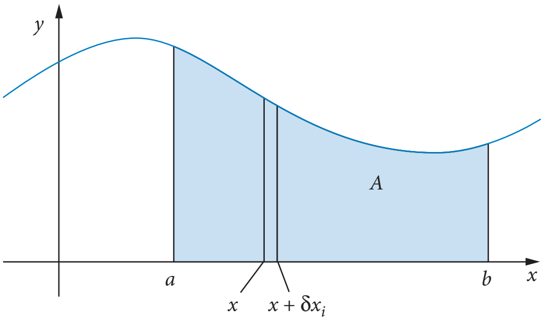
Tap anywhere in the top half of the screen, preferably on an empty cell.
Tap Calc, Regression and Quadratic Reg.
Make sure XList is list1, YList is list2 and Freq is 1.



Write the result.

The equation of the area function is $A(x) = x^2 + x$

You may notice that the area function in each of the examples and the investigation above are primitive functions (antiderivatives) of the function. You can see how this works by looking at the area from a to b for $f(x)$. Consider a rectangle $f(x_i)$ by δx_i in the sum $\sum_{i=1}^{n} f(x_i)\delta x_i$.



The value x_i could be at the left end, the right end or somewhere in between. The area of the strip is given by $\delta A_i = f(x_i)\delta x_i$. Rearranging gives $f(x_i) = \frac{\delta A_i}{\delta x_i}$. It doesn't matter which strip you pick, you will always get this result. But $\lim_{x \to 0} \frac{\delta A}{\delta x} = \frac{dA}{dx} = A'(x)$, so $f(x)$ is the derivative of the area function. From your work last year with indefinite integrals, you can write $A(x) = F(x) + c$, where $F(x)$ is a primitive of $f(x)$ and c is a constant.

The integral from a to a is zero, so $A(0) = 0$.

Thus $0 = F(a) + c$ so $c = -F(a)$.

Substituting back in $A(x) = F(x) + c$, you get $A(x) = F(x) - F(a)$, so $A(b) = F(b) - F(a)$.

Using integral notation gives $\int_a^b f(x)dx = F(b) - F(a)$.

IMPORTANT

The **fundamental theorem of calculus** states that $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is an antiderivative (primitive) of $f(x)$.

Calculation of a definite integral using an indefinite integral is normally shown as follows.

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

where $F(x)$ is a primitive, antiderivative or indefinite integral $\left(\int f(x)dx\right)$.

The 'dx' indicates what variable the integral refers to. It is *not multiplied* by the function; it is just part of the formal notation.

Example 13

Evaluate the following.

a $\int_0^5 x^2 dx$ **b** $\int_2^6 x^3 dx$

Solution

a Find an antiderivative of x^2. $F(x) = \frac{x^3}{3}$ is an antiderivative of $f(x) = x^2$

Use the antiderivative. $\int_0^5 x^2 dx = \left[\frac{x^3}{3}\right]_0^5$

Apply the theorem.

$$= \frac{5^3}{3} - \frac{0^3}{3}$$
$$= \frac{125}{3}$$
$$= 41\frac{2}{3}$$

State the result. $\int_0^5 x^2 dx = 41\frac{2}{3}$

b Find a primitive of x^3. $F(x) = \frac{x^4}{4}$ is a primitive of $f(x) = x^3$

Use the primitive. $\int_2^6 x^3 dx = \left[\frac{x^4}{4}\right]_2^6$

Apply the theorem.

$$= \frac{6^4}{4} - \frac{2^4}{4}$$
$$= 324 - 4$$
$$= 320$$

State the result. $\int_2^6 x^3 dx = 320$

EXERCISE 4.05 The fundamental theorem of calculus

Concepts and techniques

1 Example 12 **a** By drawing the graph of $y = 9$, find the area under the graph for different values of x and draw the graph of the area function $A(x)$.
  b Find the equation of the function $y = A(x)$.

2 **a** Draw the graph of $f(x) = 6x$.
  b Find the area under the function for different values of x and draw the graph of the area function $A(x)$.
  c Find the equation of the function $y = A(x)$.

3 CAS **a** Draw the graph of $f(x) = 4x + 3$.

b Find the area under the function for different values of x and draw the graph of the area function $A(x)$.

c Find the equation of the function $y = A(x)$.

4 Example 13 Evaluate each definite integral.

a $\int_0^6 x^2 dx$ **b** $\int_0^3 x^3 dx$ **c** $\int_0^2 x^5 dx$

d $\int_0^4 x^7 dx$ **e** $\int_0^5 x^4 dx$

5 Evaluate each of the following.

a $\int_1^3 x^2 dx$ **b** $\int_2^8 x\,dx$ **c** $\int_3^5 x^4 dx$

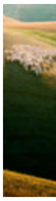
d $\int_3^4 x^3 dx$ **e** $\int_1^6 x^2 dx$

6 Find the value of each definite integral.

a $\int_2^6 x^5 dx$ **b** $\int_1^4 x^9 dx$ **c** $\int_4^6 x\,dx$ **d** $\int_1^2 x^5 dx$

e $\int_2^3 x^3 dx$ **f** $\int_1^4 x^4 dx$ **g** $\int_2^5 x\,dx$ **h** $\int_3^5 x^7 dx$

i $\int_1^2 x^9 dx$ **j** $\int_3^6 x^5 dx$



Reasoning and communication

7 The speed of a particle is given by $S = t^2$ m/s.

a Find the speed after

i 5 seconds ii 10 seconds.

b Find the distance travelled in the first 5 seconds.

c Find the distance travelled between 5 and 10 seconds.

8 **a** Draw the graph of $y = x^3$ and shade the area under the curve between $x = 2$ and $x = 4$.

b Write this area as a definite integral.

c Find this area under the curve.

9 CAS **a** Use a CAS calculator to evaluate $\int_0^5 (2t^2 + 5t + 9)dt$.

b Find the distance travelled by an object after 5 seconds if its speed is given by $v = 2t^2 + 5t + 9$ m/s.

c Find the distance that the object travels between 3 and 5 seconds.

10 The acceleration of a particle is given by $a = t^3$ m/s^2.

a Find the acceleration after 2 seconds.

b Find the speed at which the particle is moving after 2 seconds.

c Find the speed at which the particle moves in the next 2 seconds.

4.06 CALCULATION OF DEFINITE INTEGRALS

WS
Integration of power functions

You can use and combine the properties of definite integrals to help evaluate them directly.

Example 14

Evaluate each definite integral.

a $\int_1^3 5x^2 dx$

b $\int_2^6 (x^3 - x^2)dx$

c $\int_3^4 (3x^2 - 2x + 4)dx$

Solution

a Use the property $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ and integrate.

$$\int_1^3 5x^2 dx = 5\int_1^3 x^2 dx$$

$$= 5\left[\frac{x^3}{3}\right]_1^3$$

Use the fundamental theorem.

$$= 5\left[\left(\frac{3^3}{3}\right) - \left(\frac{1^3}{3}\right)\right]$$

$$= 43\frac{1}{3}$$

Write the answer.

$$\int_1^3 5x^2 dx = 43\frac{1}{3}$$

b Use the property $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$ to integrate.

$$\int_2^6 (x^3 - x^2)dx = \int_2^6 x^3 dx - \int_2^6 x^2 dx$$

$$= \left[\frac{x^4}{4}\right]_2^6 - \left[\frac{x^3}{3}\right]_2^6$$

Use the fundamental theorem.

$$= \left(\frac{6^4}{4} - \frac{2^4}{4}\right) - \left(\frac{6^3}{3} - \frac{2^3}{3}\right)$$

$$= 320 - 69\frac{1}{3}$$

$$= 250\frac{2}{3}$$

Write the result.

$$\int_2^6 (x^3 - x^2)dx = 250\frac{2}{3}$$

c Write the integral.

$$\int_3^4 (3x^2 - 2x + 4)dx$$

Use the antiderivative of $3x^2 - 2x + 4$.

$$= \left[x^3 - x^2 + 4x\right]_3^4$$

Use the fundamental theorem.

$$= (64 - 16 + 16) - (27 - 9 + 12)$$

$$= 64 - 30$$

$$= 34$$

Write the answer.

$$\int_3^4 (3x^2 - 2x + 4)dx = 34$$

In Chapter 1, you learnt that $\frac{d}{dx}(e^x) = e^x$. You can use this to find some integrals.

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Example 15

a Find the exact value of $\int_0^3 2e^x\,dx$

b Evaluate $\int_1^2 e^x\,dx + \int_2^5 e^x\,dx$

Solution

a Try the derivative of $2e^x$.

$$\frac{d}{dx}2e^x = 2\frac{d}{dx}e^x = 2e^x$$

Write the primitive.

$2e^x$ is a primitive of $2e^x$.

Use the fundamental theorem.

$$\int_0^3 2e^x\,dx = \left[2e^x\right]_0^3$$

Evaluate.

$$= 2e^3 - 2e^0 = 2e^3 - 2$$

b Simplify using $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.

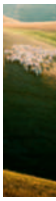
$$\int_1^2 e^x\,dx + \int_2^5 e^x\,dx = \int_1^5 e^x\,dx$$

Use that fact that $\frac{d}{dx}(e^x) = e^x$.

$$= \left[e^x\right]_1^5$$

Substitute and evaluate.

$$= e^5 - e^1 = e^5 - e = e(e^4 - 1)$$



You found these derivatives in Chapter 1: $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$

Example 16

Find the following definite integrals

a $\int_0^{\frac{\pi}{3}} \sin(x)dx$ **b** $\int_0^{\frac{\pi}{4}} 2\cos(x)dx$

Solution

a Write the integral so you can use the derivative of $\cos(x)$.

$$\int_0^{\frac{\pi}{3}} \sin(x)dx = -\int_0^{\frac{\pi}{3}} -\sin(x)dx$$

Use the fundamental theorem and $\frac{d}{dx}\cos(x) = -\sin(x)$.

$$= -\left[\cos(x)\right]_0^{\frac{\pi}{3}}$$

Substitute values.

$$= -\left[\cos\left(\frac{\pi}{3}\right) - \cos(0)\right]$$

Use exact values.

$$= -\left(\frac{1}{2} - 1\right)$$

Write the answer.

$$\int_0^{\frac{\pi}{3}} \sin(x)dx = \frac{1}{2}$$

b Use definite integral properties. $\int_0^{\frac{\pi}{4}} 2\cos(x)dx = 2\int_0^{\frac{\pi}{4}} \cos(x)dx$

Use the fundamental theorem and $\frac{d}{dx}\sin(x) = \cos(x)$. $= [\sin(x)]_0^{\frac{\pi}{4}}$

Substitute values. $= \sin\left(\frac{\pi}{4}\right) - \sin(0)$

Use exact values. $= \frac{\sqrt{2}}{2} - 0$

Write the answer. $\int_0^{\frac{\pi}{4}} 2\cos(x)dx = \frac{\sqrt{2}}{2} \approx 0.7071$

EXERCISE 4.06 Calculation of definite integrals

Concepts and techniques

1 Example 14 Find the following definite integrals.

a $\int_1^3 4x\,dx$ b $\int_0^2 7x^6 dx$ c $\int_1^2 4x^3 dx$ d $\int_2^3 (2x-1)dx$

e $\int_0^4 (x+2)dx$ f $\int_1^5 (6x-5)dx$ g $\int_0^1 (x^3-3x^2+1)dx$ h $\int_0^3 (x^2-x-2)dx$

i $\int_1^2 (8x^3-5)dx$ j $\int_0^1 (x^4-x^2+1)dx$

2 Evaluate each of the following definite integrals.

a $\int_0^2 \frac{x^2}{2}dx$ b $\int_{-1}^1 (3x^2+4x)dx$ c $\int_{-1}^2 (x^2+1)dx$

d $\int_{-2}^3 (4x^3-3)dx$ e $\int_{-1}^0 (x^2+3x+5)dx$

3 Example 15 Evaluate each of the following definite integrals.

a $\int_0^4 e^x dx$ b $\int_1^3 5e^x dx$ c $\int_0^2 (2e^x+x)dx$

d $\int_1^5 (e^x-1)dx$ e $\int_2^4 (x^3-e^x)dx$

4 Example 16 Evaluate each of the following.

a $\int_0^{\frac{\pi}{4}} \cos(x)dx$ b $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin(x)dx$ c $\int_0^{\pi} 3\sin(x)dx$

d $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cos(x)dx$ e $\int_0^{\frac{\pi}{2}} 7\sin(x)dx$

5 Find the values of the definite integrals below.

a $\int_0^{\pi} [x+\sin(x)]dx$ b $\int_0^{\frac{\pi}{4}} [\cos(x)-\sin(x)]dx$

c $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} [\cos(x)+1]dx$ d $\int_0^{\frac{\pi}{3}} [2\sin(x)+3\cos(x)]dx$

e $\int_1^3 \left[\sin(x)+3x^2\right]dx$ (answer correct to 2 decimal places)

6 Evaluate the following.

a $\int_0^{\pi} 3\cos(x)dx$ b $\int_{\pi}^{\frac{4\pi}{3}} \cos(x)dx$ c $\int_{\frac{2\pi}{3}}^{\frac{3\pi}{2}} 3\sin(x)dx$

d $\int_{\pi}^{\frac{5\pi}{4}} \sqrt{2}\cos(x)dx$ e $\int_{\pi}^{\frac{11\pi}{6}} 2\sin(x)dx$

Reasoning and communication

7 **i** Simplify and **ii** evaluate each of the definite integrals below.

a $\int_0^3 (2x-1)dx + \int_3^5 (2x-1)dx$

b $\int_0^4 e^x dx + \int_0^4 x\,dx$

c $\int_0^{\frac{\pi}{6}} \cos(x)dx - \int_0^{\frac{\pi}{6}} 2\sin(x)dx$

8 **a** What is the derivative of $\tan(x)$?

b Find $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2(x)}dx$.

9 **a** Differentiate e^{4x}

b Find $\int_0^3 4e^{4x}dx$

c Find $\int_0^3 e^{4x}dx$

10 The velocity of an object moving in a straight line is given by $\frac{dx}{dt} = 3t^2 + 2t - 5$ cm/s and its displacement is 3 cm after 2 seconds. Find

a its initial velocity

b the equation of displacement x in terms of t

c its displacement after 5 seconds

d its acceleration after 3 seconds.

4.07 AREAS UNDER CURVES

One of the main uses of definite integrals is to calculate the exact area under a curve. You have already seen that for $f(x) > 0$, the definite integral is the area under a curve. You can use the fundamental theorem to find these areas exactly.

Example 17

Find the area under $y = 6x + 2$ from $x = 3$ to 5.

Solution

Make sure that the conditions are correct.	$6x + 2 > 0$ for $x = 3$ to $x = 5$
Write the area as an integral.	Area $= \int_3^5 (6x+2)dx$
Use the fundamental theorem.	$= \left[3x^2 + 2x\right]_3^5$
Substitute values.	$= 85 - 33 = 52$ square units

EXERCISE 4.07 Areas under curves

Concepts and techniques

1 Example 17 Find the area under $y = 4x + 1$ from $x = 6$ to 9.

2 Find the area under $f(x) = x^2$ from 4 to 7.

3 Find the area under $y = x^3$ from $x = 1$ to 5.

4 Find the area under $f(x) = x^2 + 3$ from 2 to 5.

5 Find the area under $y = x^3 + 9$ from $x = -2$ to 4.

6 Find the area under $y = 7x - x^2 - 1$ from $x = 1$ to 4.

7 Find the area under $y = 6x^3 + 2x^2 + 3$ from 2 to 8.

8 Find the area under $y = x^2 - 5x + 2$ from $x = 6$ to 10.

Reasoning and communication

9 a Sketch the function $y = x^3$ from $x = -3$ to $x = 3$.

b Find $\int_{-2}^{0} x^3 dx$.

c Find $\int_{0}^{2} x^3 dx$.

d Find $\int_{-2}^{2} x^3 dx$.

e What is the area between $y = x^3$ and the x-axis from -2 and 2?

f Why are the answers for parts **d** and **e** different?

10 a What is the sign of $f(x) = x^2 - 10x + 16$ from $x = 3$ to $x = 7$?

b Find $\int_{3}^{7} (x^2 - 10x + 16) dx$.

c What is the area between $f(x) = x^2 - 10x + 16$ and the x-axis from $x = 3$ to $x = 7$?

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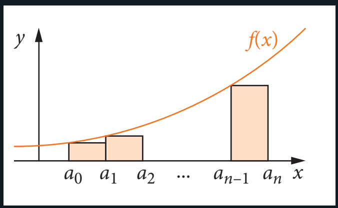
CHAPTER SUMMARY
INTEGRATION AND AREAS

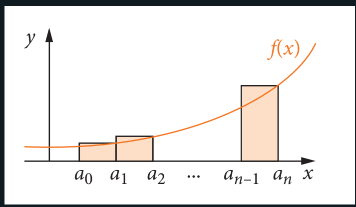
4

- Rectangles are used to find the approximate area under a curve. The rectangles can use function values on the left, right or in the centre of the rectangles, with the area written as $\sum_{i=1}^{n} f(x_i)\delta x_i$ or $\sum_{x=a}^{b} f(x_i)\delta x_i$.

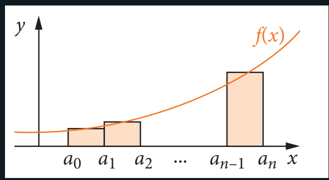
'Left' rectangles



'Right' rectangles



'Centred rectangles'



- Centred rectangles can give a closer approximation to the area under the curve.
- **Integration** is the process used to find the exact value of the area under a curve.
- The **definite integral** of the function $f(x)$ from a to b, $\int_a^b f(x)dx$, is the exact area under the curve $y = f(x)$ between $x = a$ and $x = b$. This is given by the limit of the sum of the areas of rectangles:

$$\int_a^b f(x)dx = \lim_{\delta x \to 0} \sum f(x)\delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\delta x_i$$

- The **fundamental theorem of calculus** states that

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative (primitive) of $f(x)$.

- The calculation is shown as

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

- Some of the properties of definite integrals are:

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx,$$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx \text{ and}$$

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

- The **linearity** property,

$$\int_a^b [cf(x) + dg(x)]dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx$$

preserves linear combinations of functions.

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4

CHAPTER REVIEW
INTEGRATION AND AREAS

Multiple choice

1 Example 6 Find the approximate value of $\int_1^3 x^2 dx$ using right rectangles with width 0.5 units.

A $0.5(1^2 + 1.5^2 + 2^2 + 2.5^2)$
B $0.5(1.5^2 + 2^2 + 2.5^2 + 3^2)$
C $0.5(1^2 + 1.5^2 + 2^2 + 2.5^2 + 3^2)$
D $0.5(1.5^2 + 2^2 + 2.5^2)$
E $0.5(1.25^2 + 1.75^2 + 2.25^2 + 2.75^2)$

2 Example 7 Find the approximate value of the definite integral $\int_0^5 (x^3+1)dx$ by using 5 centred rectangles.

A $0.5(0.5^3 + 1 + 1.5^3 + 1 + 2.5^3 + 1 + 3.5^3 + 1 + 4.5^3 + 1)$
B $0^3 + 1 + 1^3 + 1 + 2^3 + 1 + 3^3 + 1 + 4^3 + 1$
C $1^3 + 1 + 2^3 + 1 + 3^3 + 1 + 4^3 + 1 + 5^3 + 1$
D $0.5^3 + 1 + 1.5^3 + 1 + 2.5^3 + 1 + 3.5^3 + 1 + 4.5^3 + 1$
E $1.5^3 + 1 + 2.5^3 + 1 + 3.5^3 + 1 + 4.5^3 + 1 + 5.5^3 + 1$

3 Examples 9–11 Which of the following is true?

A $\int_2^4 (3x^3-5x^2+4x+1)dx - \int_2^4 (x^3+x^2-5x-3)dx = \int_2^4 (2x^3-6x^2-x+4)dx$
B $\int_2^4 (3x^3-5x^2+4x+1)dx - \int_2^4 (x^3+x^2-5x-3)dx = \int_2^4 (2x^3-4x^2-x-2)dx$
C $\int_2^4 (3x^3-5x^2+4x+1)dx - \int_2^4 (x^3+x^2-5x-3)dx = \int_2^4 (2x^3-6x^2+9x-2)dx$
D $\int_2^4 (3x^3-5x^2+4x+1)dx - \int_2^4 (x^3+x^2-5x-3)dx = \int_2^4 (2x^3-6x^2+9x+4)dx$
E $\int_2^4 (3x^3-5x^2+4x+1)dx - \int_2^4 (x^3+x^2-5x-3)dx = \int_2^4 (4x^3-6x^2+9x-2)dx$

4 Example 14 Find $\int_{-2}^2 (12x^2-6x+5)dx$.

A 0 B 54 C 60 D 84 E 108

5 Example 17 Find the area under the function $f(x) = x^2 - 1$ from $x = 2$ to $x = 4$.

A $7\frac{2}{3}$ B 14 C $16\frac{2}{3}$ D 56 E $56\frac{1}{3}$

Short answer

6 Example 1 a If a car is travelling at 75 km/h, find the distance it travels in 8 hours.

b By drawing the graph of the speed of the car, show that the distance it travels after 8 hours is equal to the area under the graph between 0 and 8 hours.

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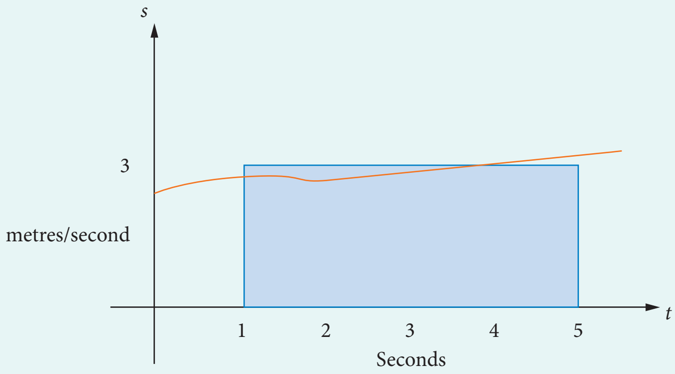
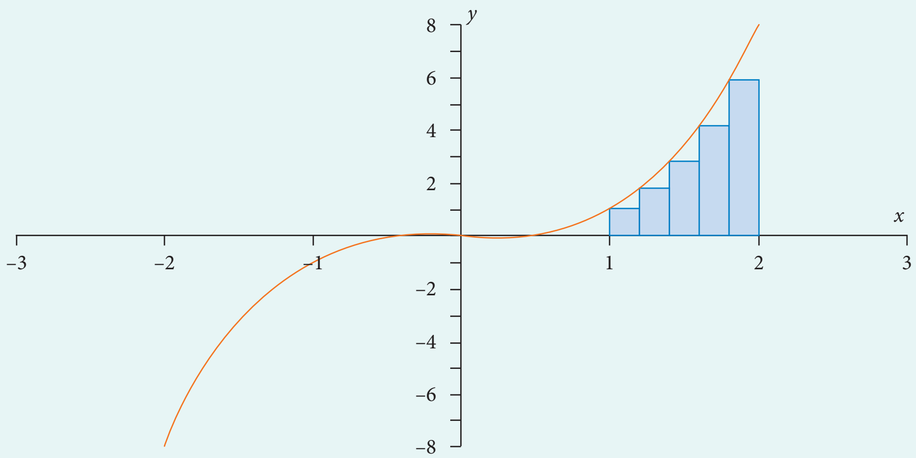
7 Example 2 Find the approximate distance travelled by a particle between 1 and 5 seconds by finding the shaded area below.



8 Example 3 Find the approximate area under the curve $y = x^3$ from $x = 1$ to $x = 2$ by finding the areas of the rectangles below.



9 Example 4 **a** Find the approximate area under the curve $y = x^2 - 2x + 1$ between $x = 1$ and $x = 3$ by using

i 4 left rectangles **ii** 4 right rectangles

b CAS 50 left rectangles.

10 Examples 5–7 Find an approximate area under the curve $y = x^2$ between $x = 0$ and $x = 2$ by using

a 8 left rectangles **b** 8 right rectangles **c** 8 centred rectangles.

11 CAS Example 8 Find the approximate area under the curve $y = x^3$ between $x = 0$ and $x = 4$ using 40 left rectangles.

12 CAS Example 8 Find an approximation to $\int_0^{10}(x^2 + 2x)dx$ by using 20 left rectangles.

13 CAS Example 9 a Sketch the graph of $y = 3x + 2$ for $x = 0$ to 5.

b Find the area under the function from 0 to 0, 1, 2, 3 and 4 and draw the graph of the area function $A(x)$.

c CAS Find the equation of the function $y = A(x)$.

14 Example 10 a Use right rectangles with width 0.5 units to find an approximation to

i $\int_1^2 (2x^2+1)dx$ ii $\int_2^4 (2x^2+1)dx$ iii $\int_1^4 (2x^2+1)dx$

b Show that $\int_1^4 (2x^2+1)dx = \int_1^2 (2x^2+1)dx + \int_2^4 (2x^2+1)dx$

15 CAS Example 11 a Use 100 left rectangles to find approximations for

i $\int_2^8 x^2 dx$ ii $\int_2^8 6x^2 dx$

b Show that $\int_2^8 6x^2 dx = 6\int_2^8 x^2 dx$ using the sums above.

16 Example 12 a Find an approximation to each of the following definite integrals using left rectangles with width 0.25 units.

i $\int_1^2 x^3 dx$ ii $\int_1^2 2x\,dx$ iii $\int_1^2 (x^3+2x)dx$

b Show that $\int_1^2 (x^3+2x)dx = \int_1^2 x^3 dx + \int_1^2 2x\,dx$ for the sums calculated.

17 Example 14 Evaluate the following definite integrals.

a $\int_0^2 x^3 dx$ b $\int_1^3 x\,dx$ c $\int_0^3 (x^2+3x-4)dx$ d $\int_1^2 (3x-2)dx$

18 Example 15 Evaluate $\int_0^7 3e^x dx$.

19 Example 16 Evaluate the following integrals.

a $\int_0^{\frac{\pi}{4}} \sin(x)dx$ b $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos(x)dx$

20 Example 17 a Find the area under the curve $f(x) = x^2 - 9$ between $x = 4$ and $x = 6$.

b Find the area under the curve $f(x) = 3x^2 - 2x - 1$ between $x = 2$ and $x = 5$.

Application

21 The velocity of an object is given by $v = 3\cos(t)$ cm/s. Find

a the velocity after 1 second

b the distance travelled in the first second

c the distance travelled between 0.6 and 0.9 seconds.

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22 a Differentiate $\frac{x}{e^x}$

b Find the definite integral $\int_0^1 \frac{1-x}{e^x} dx$.

23 a What is the sign of $f(x) = x^3 - 6x^2 + 12x - 8$ from $x = 0$ to $x = 2$?

b What is the sign of $f(x) = x^3 - 6x^2 + 12x - 8$ from $x = 2$ to $x = 4$?

c Find $\int_0^4 (x^3 - 6x^2 + 12x - 8)dx$.

d Find $\int_0^2 (x^3 - 6x^2 + 12x - 8)dx$.

e What is the area between $f(x) = x^3 - 6x^2 + 12x - 8$ and the x-axis from $x = 0$ to $x = 4$?

f Explain why the answers to **c** and **e** are different.

Qz
Practice quiz

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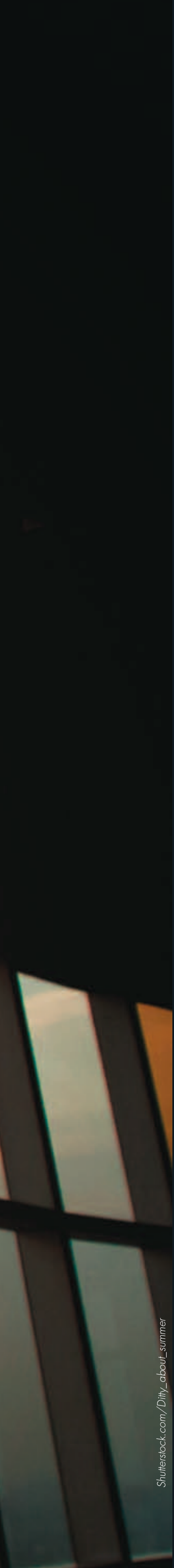

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5

TERMINOLOGY

Bernoulli distribution
Bernoulli trial
Bernoulli variable
binomial distribution
binomial variable
expected value
failure
geometric distribution
geometric variable
independent
mean
negatively skewed
positively skewed
skewed
standard deviation
success
symmetrical distribution
variance

Prior learning

DISCRETE RANDOM VARIABLES

BINOMIAL DISTRIBUTIONS

BERNOULLI DISTRIBUTIONS

- use a **Bernoulli random variable** as a model for two-outcome situations (ACMMM143)
- identify contexts suitable for modelling by **Bernoulli random variables** (ACMMM144)
- recognise the mean p and variance $p(1-p)$ of the Bernoulli distribution with parameter p (ACMMM145)
- use Bernoulli **random variables** and associated probabilities to model data and solve practical problems. (ACMMM146)

BINOMIAL DISTRIBUTIONS

- understand the concepts of **Bernoulli trials** and the concept of a binomial **random variable** as the number of 'successes' in n independent **Bernoulli trials**, with the same probability of success p in each trial (ACMMM147)
- identify contexts suitable for modelling by binomial **random variables** (ACMMM148)
- determine and use the probabilities $P(X=r)=\binom{n}{r}p^r(1-p)^{n-r}$ associated with the **binomial distribution** with parameters n and p; note the mean np and variance $np(1-p)$ of a **binomial distribution** (ACMMM149)
- use binomial distributions and associated probabilities to solve practical problems. (ACMMM150) AC

5.01 THE BERNOULLI DISTRIBUTION

There are many probability situations with only two possible outcomes, such as head/tail, boy/girl, etc. In other situations, we can classify the outcomes so that there are only two possibilities, such as drawing an ace or not drawing an ace from a pack of cards.

A distribution in which there are only two possible outcomes is known as a **Bernoulli distribution**, after Jacob Bernoulli (1654–1705).

IMPORTANT

A **Bernoulli distribution** is a distribution with two possible outcomes, called **success** and **failure**, with a fixed probability of success. The probability of success is normally written as p and the probability of failure as q, such that $p+q=1$.

The **Bernoulli random variable** X is given the values 1 for success and 0 for failure.

Example 1

Determine whether each of the following situations can be considered as Bernoulli distributions.

a Rolling a six-sided die and recording a success when a 3 occurs.

b Rolling a six-sided die and recording the number that occurs.

c Drawing a card from a deck 6 times with replacement and recording the number of hearts drawn.

d Drawing a marble from a bag containing 6 red and 4 green marbles with replacement and recording the colour of the marble drawn.

Solution

a How many outcomes are there?	There are two outcomes – rolling a 3 and rolling a number other than 3. $p=\frac{1}{6}$ and $q=\frac{5}{6}$, so $p+q=1$.
Write the answer.	This is a Bernoulli distribution.

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b	How many outcomes are there?	There are six outcomes.
	Write the answer.	This is not a Bernoulli distribution.
c	How many outcomes are there?	There are seven outcomes (0 to 6 hearts).
	Write the answer.	This is not a Bernoulli distribution.
d	How many outcomes are there?	There are two outcomes, red and green. $p = \frac{1}{2}$ and $q = \frac{1}{2}$, so $p + q = 1$.
	Write the answer.	This is a Bernoulli distribution

The trivial case of the hypergeometric distribution with a sample of size $n = 1$ is a Bernoulli distribution. Remember that a hypergeometric distribution is one where a sample of n items is taken from a population of N items, k of which are considered successes. The hypergeometric random variable is the number of successes.

Example 2

What is the probability of success if a watch is chosen from a batch of 100, of which 4 are faulty, and success is considered as getting a faulty watch?

Solution

List the hypergeometric parameters.	$x = 1, n = 1, N = 100, k = 4$
Find the probability.	$P(\text{success}) = \frac{4}{100} = \frac{1}{25}$

You could calculate the probability of success in Example **2** by using the formula for the hypergeometric distribution,

$$P(x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

with $x = 1, n = 1, N = 100$ and $k = 4$, but it would be unnecessarily complicated to do it that way.

We saw in Chapter 2 of this book that the mean (expected value) and variance of a discrete probability distribution are defined as the following.

The Bernoulli distribution

$$\mu = E(X) = \sum x \cdot p(x)$$

$$\begin{aligned} Var(X) &= E\left[(X-\mu)^2\right] \\ &= E\left(X^2\right) - \mu^2 \\ &= \sum x^2 \cdot p(x) - \left[\sum x \cdot p(x)\right]^2 \end{aligned}$$

Consider a Bernoulli distribution with probability of success p and random variable X. The calculation of the mean is easy because there are only two values of X, 0 and 1.

Then $E(X) = 0 \times q + 1 \times p = p$

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$$
\begin{aligned}
Var(X) &= E(X^2) - \mu^2 \\
&= \sum x^2 \cdot p(x) - p^2 \\
&= [0^2 \times q + 1^2 \times p] - p^2 \\
&= p - p^2 \\
&= p(1-p)
\end{aligned}
$$

Since $SD(X) = \sqrt{Var(X)}$, $SD(X) = \sqrt{p(1-p)}$

IMPORTANT

The expected value, variance and standard deviation of a Bernoulli distribution are:

$E(X) = \mu = p$, $Var(X) = p(1-p)$ and $SD(X) = \sigma = \sqrt{p(1-p)}$

Example 3

What is the standard deviation of the random variable X arising from the selection of a card from a normal pack if you want it to be hearts?

Solution

Write the values of X.	$X = 1$ for hearts and 0 for other suits.
Write the value of p.	$p = \frac{1}{4}$
Write the formula.	$\sigma = \sqrt{p(1-p)}$
Substitute in the values.	$= \sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)}$
Calculate the answer.	$= \frac{\sqrt{3}}{4}$

EXERCISE 5.01 The Bernoulli distribution

Concepts and techniques

1 Example 1 Which of the following are examples of a Bernoulli distribution?

a Selecting a marble from a bag containing red, green and blue marbles

b Selecting people at random and noting if they are over 18

c Drawing a card from a deck and recording its colour

d Tossing a coin 3 times and recording the number of tails

e Playing roulette and noting whether the ball settles on an odd number

2 Example 2 What is the probability of randomly getting an orange that you can eat from a box that contains 30 oranges, of which 3 are rotten?

A $\frac{1}{30}$ B $\frac{3}{10}$ C $\frac{1}{3}$ D $\frac{1}{2}$ E $\frac{9}{10}$

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3 Example 3 What is the standard deviation of the random variable arising from picking the multiple choice answer from this question at random?

A $\frac{\sqrt{5}}{4}$ B $\frac{1}{5}$ C $\frac{4}{5}$ D $\frac{2}{\sqrt{5}}$ E $\frac{2}{5}$

4 State whether or not each of the following are Bernoulli distributions.
   a Randomly selecting a person and recording if the person is female or male.
   b Tossing a coin 20 times and recording the number of heads.
   c Spinning a spinner with equal sized segments numbered 1 to 12 and recording the number where the spinner stops.
   d Drawing a marble from a bag containing 8 green and 5 orange marbles with replacement and recording if the marble is orange or not.
   e Drawing a card from a deck with replacement and recording if it is black or red.
   f Drawing a card from a deck without replacement and recording the number of aces drawn.
   g Rolling a six-sided die 20 times and recording the number that comes up.

5 Calculate the probability of success in each of the following Bernoulli distributions.
   a Choosing a red onion at random from a bag with an equal mixture of red and brown onions.
   b Getting a six from rolling an eight sided-die.
   c Picking a sherbet at random from a bag containing 3 sherbets, 4 jubes and 5 toffees.

6 Calculate the variances of the random variables in Bernoulli distributions with the following probabilities.

   a $p = \frac{2}{5}$ b $q = \frac{1}{3}$ c $p = 0.6$ d $q = 0.3$ e $p = \frac{3}{4}$

7 Calculate the standard deviation of the random variables in Bernoulli distributions with the following probabilities.

   a $q = 0.4$ b $p = \frac{2}{3}$ c $q = \frac{24}{25}$ d $p = 0.2$ e $p = \frac{7}{9}$

Reasoning and communication

8 It is known that 8% of all items produced on a particular assembly line are defective. They are packed in boxes of 5 items. Boxes pass quality control if none are defective. What is the probability of a box passing quality control?

9 Peter picks a marble from a bag containing two red, three green and one blue marble. He wants a red marble. What is the standard deviation of the random variable for success?

10 A card is drawn from a well-shuffled deck, checked to see whether it is an ace and then returned to the deck. What is the variance of the random variable arising from this situation?

11 On average, Simone is late for school 20% of the time. In a normal school week, what is the probability that she will be successful in getting to school every day?

12 A manufacturer will only accept a batch of 200 bags of plastic beads if a sample of 10 bags has at most 1 bag that has obvious misshaped beads, because these occasionally cause problems in the plant. What is the probability of acceptance, given that the supplier says that only 10% of the bags have misshaped beads?

5.02 THE GEOMETRIC DISTRIBUTION

The geometric distribution is a distribution that arises from multiple trials, each of which is the same Bernoulli distribution. These are called **Bernoulli trials**.

IMPORTANT

Bernoulli trials are independent trials that have only two possible outcomes, called **success** and **failure**, where all the trials have the same probability of success. The probability of success is normally written as p and the probability of failure as q, so $p + q = 1$.

In many cases involving Bernoulli trials, you may continue until you get a success. For example, if you wanted to grow an oak tree from an acorn, you would keep planting seeds until you succeeded in growing the tree.

The pattern is a sequence of failures (F) before the first success (S), symbolised as:

$$F, F, F, \ldots, S$$

Example 4

A fair eight-sided die is rolled repeatedly until a multiple of 3 (a success) is observed. What is the probability that exactly five rolls are required?

Solution

Identify the situation.	The trials are Bernoulli trials with a sequence *FFFFS*.
Calculate p.	$p = P(3 \text{ or } 6) = \frac{2}{8} = \frac{1}{4}$
Calculate q.	$q = 1 - \frac{1}{4} = \frac{3}{4}$
Use independence of trials.	$P(FFFFS) = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4}$
	$= \frac{3^4}{4^5}$
Calculate the result.	$= \frac{81}{1024}$
	$= 0.079\,101\ldots$
Write the answer.	The probability of 5 rolls is about 0.079 or 7.9%.

The pattern in Example 4 is called a **geometric distribution**. It is a discrete distribution, but unlike the ones you have previously met, it is an infinite distribution. This means that there is a finite (although very small) probability that the number of failures before you get a success is any large number that you choose.

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IMPORTANT

The **geometric distribution** is the probability distribution of the number of failures of Bernoulli trials before the first success.

The **geometric random variable** X has the values 0, 1, 2, 3, …: $X \in \boldsymbol{J}^+$.

The probability function is $P(X = x) = q^x p$.

The geometric distribution is sometimes defined as the number of trials Y needed to get a success, in which case the random variable takes the values 1, 2, … and $P(Y = y) = q^{y-1}p$.

The probability function is just a consequence of the fact that the trials are independent, so the overall probability is the product of the probabilities of x failures followed by 1 success.

The probability function is a geometric sequence.

You may have seen last year that the infinite sum of a GP with common ratio r is given by $\dfrac{a}{1-r}$, provided that $-1 \le r \le 1$. For the geometric sequence, this gives

$$\sum q^x p = \frac{p}{1-q} = \frac{1-q}{1-q} = 1$$

as expected.

You can calculate the expected value as shown below:

$$E(X) = \sum xq^x p = 0 \times q^0 \times p + 1 \times q^1 \times p + 2 \times q^2 \times p + 3 \times q^3 \times p + 4 \times q^4 \times p + \cdots$$

Thus $E(X) = 1 \times q^1 \times p + 2 \times q^2 \times p + 3 \times q^3 \times p + 4 \times q^4 \times p + 5 \times q^5 \times p + \cdots$

so $qE(X) = 1 \times q^2 \times p + 2 \times q^3 \times p + 3 \times q^4 \times p + 4 \times q^5 \times p + \cdots$

Subtracting the two expressions gives the following.

$$\begin{aligned} E(X) - qE(X) &= q^1 \times p + q^2 \times p + q^3 \times p + q^4 \times p + q^5 \times p + \cdots \\ &= \frac{pq}{1-q} \text{ (using the infinite sum of a GP again)} \\ &= \frac{(1-q)q}{1-q} = q \end{aligned}$$

Factorising the LHS, $(1 - q)E(X) = q$, so $E(X) = \dfrac{q}{1-q} = \dfrac{(1-p)}{1-(1-p)} = \dfrac{1-p}{p}$

IMPORTANT

The expected value of the geometric distribution is $E(X) = \dfrac{q}{1-q} = \dfrac{1-p}{p}$

Example 5

In a bottling plant, a problem has been discovered with a labelling machine. Over time, it has been determined that 25% of all labels are not properly fixed to bottles. If X denotes the number of labels that are fixed correctly before the first one that is not fixed properly, calculate:

a $P(X = 5)$

b $P(X \le 5)$

c $E(X)$ and explain the result.

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Solution

a Calculate p and q.

$$p = 25\% = \frac{1}{4}$$

$$q = 1 - \frac{1}{4} = \frac{3}{4}$$

Write the geometric function.

$$P(X = x) = \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)$$

Let $X = 5$.

$$P(X = 5) = \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)$$

$$= \frac{243}{4096}$$

$$\approx 0.0593$$

b Write an expression for $P(X \le 5)$.

$$P(X \le 5) = P(X = 0) + P(X = 1) + P(X = 2) + \cdots + P(X = 5)$$

Substitute.

$$= \left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^1 \left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right) + \cdots + \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)$$

State the kind of sum.

This is a GP with $a = \frac{1}{4}$, $r = \frac{3}{4}$ and $n = 6$.

Use the rule for the sum of n terms.

$$S_n = \frac{a(1 - r^6)}{1 - r}$$

Substitute for a, r and n.

$$= \frac{\frac{1}{4}\left[1 - \left(\frac{3}{4}\right)^6\right]}{1 - \frac{3}{4}}$$

Evaluate and round off.

$$= 1 - \left(\frac{3}{4}\right)^6$$

$$\approx 0.8220$$

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c Write the rule for $E(X)$.

$$E(X) = \frac{1-p}{p}$$

Substitute for p.

$$= \frac{1-\frac{1}{4}}{\frac{1}{4}}$$

Evaluate.

$$= \frac{3}{4} \times \frac{4}{1} = 3$$

State the result.

$E(X) = 3$. There will be an average of 3 bottles labelled correctly before the first incorrectly labelled bottle occurs.

Even though the average number of bottles before a dud label is found is 3, this does not mean that there couldn't be a run of 100, or 1000, or any number you like to think of. Remember that the geometric distribution is infinite.

Example 6

A basketball player knows from experience that she will make 60% of all attempted baskets from the free throw line. If Y denotes the number of free throws she will make to get her first success, find:

a $P(Y = 8)$ **b** $P(2 \le Y \le 7)$

Solution

a Identify the situation.

This is the alternate geometric distribution.

Calculate p and q.

$p = 60\% = 0.6$ and $q = 1 - 0.6 = 0.4$

Write the function.

$P(Y = y) = (0.4)^{y-1}(0.6)$

Substitute for $Y = 8$.

$$\begin{aligned} P(Y = 8) &= (0.4)^{8-1}(0.6) \\ &= (0.4)^7(0.6) \\ &\approx 0.000\,98 \end{aligned}$$

b Define $P(2 \le Y \le 7)$.

$$P(2 \le Y \le 7) = P(Y = 2) + P(Y = 3) + P(Y = 4) + \cdots + P(Y = 7)$$

Substitute.

$$= (0.4)(0.6) + (0.4)^2(0.6) + (0.4)^3(0.6) + \cdots + (0.4)^6(0.6)$$

State the kind of sum.

This is a GP with $a = (0.4)(0.6)$, $r = 0.4$ and $n = 6$

Use the rule for the sum.

$$S_n = \frac{a(1-r^6)}{1-r}$$

Substitute.

$$= \frac{(0.4)(0.6)\left[1-(0.4)^6\right]}{1-0.4}$$

Evaluate.

$$\approx 0.3984$$

EXERCISE 5.02 The geometric distribution

Concepts and techniques

WS Geometric probability distributions

1 Which of the following variables will have a geometric probability distribution?

A the number of phone calls received at a call centre in successive 10-minute periods

B the number of cards I need to deal from a well-shuffled pack of 52 cards until at least two of the cards are hearts

C the number of marbles drawn from a bag of coloured marbles until a red and a green marble have been selected

D the number of digits I read beginning at a randomly selected point in a table of random digits until I find a 9

E the number of times a coin must be flipped until 2 heads or 3 tails have been flipped

2 Example 4 A basketball player has an 80% chance of making a free throw. If this probability is the same for each free throw attempted, what is the probability that the player doesn't make a free throw in a game until the fifth attempt?

A 0.000 31 B 0.001 28 C 0.002 57 D 0.005 26 E 0.081 92

3 Example 5 X has a geometric distribution where X is number of failures before the first success. If the probability of success is 0.3. Then $P(X = 4)$ is equal to:

A 0.0057 B 0.0187 C 0.0519 D 0.07203 E 0.1029

4 Example 6 For the basketball player mentioned in question **2**, what is the probability that it takes more than 3 free throws before the player makes the first free throw?

A 0.0008 B 0.0016 C 0.0032 D 0.0064 E 0.1024

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5 For the basketball player mentioned in question **2**, what is the expected number of throws required *until the player makes the first free throw* in a game?

A 0.25 B 0.8 C 1.25 D 2 E 4

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6 Which of the following situations are examples of geometric distributions?

a If both parents carry genes for a particular condition, each child has a 25% chance of getting two genes (one from each parent) for that condition. Children inherit genes independently of each other. We wish to find the probability that the first child these parents have with this condition is their third child.

b A card is drawn from a well-shuffled deck, its suit is noted and the card is set aside before the next card is drawn. We wish to find the number of cards drawn before the first heart is selected.

c Coloured marbles are randomly selected from a bag, the colour noted and the marble returned before the next selection. The bag contains 5 red, 7 white and 2 green marbles. We wish to find the number of marbles drawn before the first green marble is selected.

d At the maternity section of a large hospital, a study is undertaken to determine the number of children born to a couple before they have a girl.

e The pool of potential jurors for a murder trial contains 500 people chosen randomly from the adult population of a large city. Each person in the pool is asked if he or she believes that judges are too lenient in sentencing people found guilty of major crimes. We are interested in the number of potential jurors interviewed before the first 'yes' response.

7 A six-sided die is rolled until the first time that a number greater than 4 is observed. What is the probability that exactly six rolls are required?

8 A baseball batter knows that he hits only one pitch out of every three. What is the probability that he misses the ball less than three times?

9 A gambler keeps a careful record of his performance and knows that, in the long run, he wins once in every five bets. If X represents the number of losses before the first win on a particular day, find:

a $P(X = 5)$ b $P(X \leq 5)$ c $E(X)$

10 In a batch of graded eggs, three-quarters of the eggs are underweight. If X represents the number of underweight eggs before an egg with the correct weight is found, calculate:

a $P(X \geq 1)$ b $P(1 \leq X \leq 5)$

11 A beginner golfer knows that, on average, he is able to hit his drive straight once in every 10 attempts. If X represents the number of bad shots before he hits a straight one, find:

a $P(4 \leq X \leq 14)$ b the expected value of X.

12 A student is sitting for a multiple choice test in which each question has 10 answer choices. The student randomly selects the answer for each question. If Y denotes the number of questions answered for the first correct answer, find:

a $P(Y = 7)$ b $P(2 \leq Y \leq 8)$

Reasoning and communication

13 If a fair six-sided die is rolled until an even number appears, what is the probability that the number of times it is rolled is:

a exactly 2? b at least 2? c no more than 2?

14 A golfer knows from experience that, if she putts within 5 m of the hole, she sinks the putt three times out of four. Find the probability that, in a particular game, she fails to sink a putt within 5 m four times before finally sinking one.

15 Police know from long experience that, on a particular stretch of road, 1 car in every 10 will exceed the speed limit. If a speed gun is used on this stretch of road, find the probability that the police will find that the first five cars will be within the speed limit and the sixth car will be speeding.

16 Assume that the Australian Tax Office (ATO) catches about 25% of all fraudulent returns each year. If you submit a fraudulent return every year:
   a what is the probability that you could get away with it five years in a row?
   b what is the expected number of fraudulent returns you could submit before being caught for the first time?

17 It is known that 75% of students hold jobs while attending a particular university. Suppose that students are selected at random from the student body. What is the probability that the first student selected who does not hold a job is the fourth student selected?

5.03 THE BINOMIAL DISTRIBUTION

You have already looked at Bernoulli distributions and Bernoulli trials. In this section you will study the most important distribution with Bernoulli trials. The geometric distribution arises from Bernoulli trials where the random variable is the number of failures before the first success. The **binomial distribution** arises from a finite number of Bernoulli trials where the random variable is the number of successes in a fixed number of trials.

IMPORTANT

The **binomial distribution** is the probability distribution of the number of successes arising from a fixed number of Bernoulli trials. A situation that produces a binomial distribution may be called a **binomial experiment**.

The properties of the binomial distribution are:

- there are a fixed number, n, of repeated trials
- each trial has only two outcomes – success and failure with probabilities p and q
- the probability of success is the same for each trial
- the trials are independent

The **binomial random variable** is the number of successes from n trials.

Example 7

Determine which of the following situations are examples of binomial experiments.

a Tossing a coin 20 times and recording the number of heads that occur.

b Rolling a six-sided die until the number 6 occurs.

c Drawing a card from a deck without replacement 10 times and recording the number of spades drawn.

d Drawing a marble from bag containing 8 black, 3 white and 5 pink marbles with replacement 12 times and recording the number of pink marbles drawn.

Solution

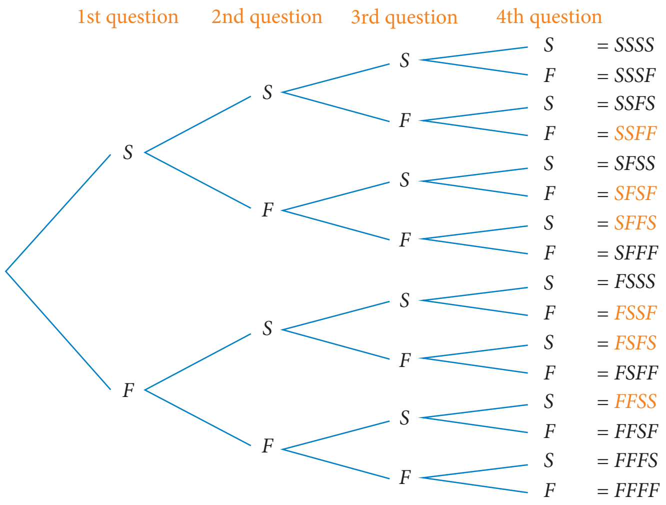
a	Are there a fixed number of trials?	There are 20 trials.
	How many outcomes are there?	There are 2 outcomes, with heads = success.
	Are the probabilities fixed?	$p = 0.5$ for every trial.
	Are the trials independent?	The trials are independent.
	What is the variable?	The variable is the number of successes.
	Write the conclusion.	This is a binomial experiment.
b	Are there a fixed number of trials?	The number of trials will vary, so it is not binomial.
c	Are there a fixed number of trials?	There are 10 trials.
	How many outcomes are there?	There are 2 outcomes, with spades = success.
	Are the probabilities fixed?	Cards are not replaced so p changes.
	Write the conclusion.	This is not binomial.
d	Are there a fixed number of trials?	There are 12 trials.
	How many outcomes are there?	There are 2 outcomes, with success = pink.
	Are the probabilities fixed?	$p = \frac{5}{16}$ for every trial
	Are the trials independent?	Replacement means the trials are independent.
	What is the variable?	The variable is the number of successes.
	Write the conclusion.	This is a binomial experiment.

Consider the experiment where a multiple choice test has four questions, each question has five alternate answers A, B, C, D and E and the answer to each is randomly selected. Only one answer is correct for any question.

This is a binomial experiment in which

- a success (S) is selecting the correct answer and $P(\text{success}) = p = \frac{1}{5}$
- a failure (F) is selecting the incorrect answer and $P(\text{failure}) = q = \frac{4}{5}$

You can construct a tree diagram to determine the various outcomes for this experiment.



This information can be summarised as follows.

Number of successes	**Possible outcomes**	**Probability**	
0	*FFFF*	$1\times\left(\frac{4}{5}\right)^4=\frac{256}{625}$	q^4
1	*SFFF, FSFF, FFSF, FFFS*	$4\times\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)=\frac{256}{625}$	$4q^3p$
2	*SSFF, SFSF, SFFS, FSSF, FSFS, FFSS*	$6\times\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^2=\frac{96}{625}$	$6q^2p^2$
3	*SSSF, SSFS, SFSS, FSSS*	$4\times\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^3=\frac{16}{625}$	$4qp^3$
4	*SSSS*	$1\times\left(\frac{1}{5}\right)^4=\frac{1}{625}$	p^4

The table above is the probability distribution for this binomial experiment. You can see that the values of the distribution are the expansion of $(p+q)^4$.

If you had to calculate the values of a binomial distribution using tree diagrams all the time, it would be extremely tedious. It is obvious from this example that you can use the expansion of $(p+q)^n$ instead. This means that you can use Pascal's triangle or combinations for the coefficients of the values.

Consider how many ways you can get 3 successes from 7 trials. You want to get 3 successes from the 7 that are possible, and the rest are failures. Since the order doesn't matter, there must be $^7C_3=\binom{7}{3}$ ways of doing this. That means that the probability of 3 successes from 7 trials must be $\binom{7}{3}p^3q^4$. This will obviously work for any value of the binomial distribution.

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IMPORTANT

Binomial distribution

If X is a binomial random variable, then the probability of x successes from n trials is given by

$$P(X=x)=\binom{n}{x}p^x q^{n-x} \quad \text{or} \quad P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$$

Example 8

a What is the probability of guessing which day of the week someone's birthday is this year?

b What is the probability of correctly guessing the day of the week of birthdays this year for 2 people from a group of 9 people?

c What is the probability of correctly guessing the day of the week of birthdays this year for 3 people from a group of 21 people?

Solution

a There are 7 possible days. — Probability of correct guess $=\frac{1}{7}$

b Write the details. — This is binomial with $n=9$, $p=\frac{1}{7}$ and $x=2$

Write the formula. — $P(X=x)=\binom{n}{x}p^x q^{n-x}$

Substitute in the values. — $P(X=2)=\binom{9}{2}\left(\frac{1}{7}\right)^2\left(\frac{6}{7}\right)^{9-2}$

Evaluate. — $=\dfrac{36\times 6^7}{7^9}$

$=0.249\ 734\ldots$

State the answer. — The probability of guessing 2 out of 9 is about 0.2497.

c Write the details. — This is binomial with $n=21$, $p=\frac{1}{7}$ and $x=3$

Write the formula. — $P(X=x)=\binom{n}{x}p^x q^{n-x}$

Substitute in the values. — $P(X=3)=\binom{21}{3}\left(\frac{1}{7}\right)^3\left(\frac{6}{7}\right)^{21-3}$

Evaluate. — $=\dfrac{1330\times 6^{18}}{7^{21}}$

$=0.241\ 832\ldots$

State the answer. — The probability of guessing 3 out of 21 is about 0.2418 or 24.18%.

Example 9

A binomial variable, X, has the probability function

$$P(X = x) = \binom{5}{x}(0.3)^x(0.7)^{5-x}$$

a What is the number of trials?

b State the probability of success in any trial.

c Show the probability distribution in a table.

Solution

a Examine the probability function. — The number of trials is $n = 5$

b Compare with the formula. — Probability of success is $p = 0.3$

c Substitute into the probability function for $x = 0$.

$$P(X = 0) = \binom{5}{0}(0.3)^0(0.7)^{5-0}$$
$$= 1 \times 1 \times 0.7^5$$
$$\approx 0.168\ 07$$

Substitute into the probability function for $x = 1$.

$$P(X = 1) = \binom{5}{1}(0.3)^1(0.7)^{5-1}$$
$$= 5 \times 0.3 \times 0.7^4$$
$$\approx 0.360\ 15$$

Continue and put the results in a table.

x	0	1	2	3	4	5
$p(x)$	0.1681	0.3602	0.3087	0.1323	0.0284	0.0024

EXERCISE 5.03 The binomial distribution

Concepts and techniques

WS Binomial probability experiments

1 Example 7 For which of the following situations would a binomial probability model be most reasonable?

A the number of phone calls made by a call centre operator in successive 10-minute periods

B the number of spades dealt in a hand of five cards from a well-shuffled deck of 52 cards

C the number of tosses of a fair coin required before two heads are observed

D the number of 9s in a randomly selected set of 10 digits from a table of random digits

E all of the above

2 There are 10 students in a school leadership group – 5 males and 5 females. The name of each student is written on a card. The cards are shuffled, one is randomly selected and the name on the card is observed. This is done a total of six times. If X is the number cards observed with the name of a female student, then which of these best describes the probability distribution for X?

A a uniform distribution with $n = 6$ and $p = 0.5$

B a binomial distribution with $n = 6$ and $p = 0.5$

C a binomial distribution with $n = 10$ and $p = 0.5$

D a geometric distribution with $p = 0.5$

E a binomial distribution with $n = 5$ and $p = \frac{6}{10}$

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3 Example 8 In a certain binomial experiment, a successful outcome occurs 5 out of every 6 trials. What is the probability that there will be exactly 3 successful outcomes in the next 4 trials?

A $\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^3$ B $6\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^3$ C $\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)^1$

D $6\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)^1$ E $4\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)^1$

4 Which of the following are examples of binomial experiments?
- a Rolling a die 15 times and recording the number that comes up.
- b Tossing a coin 10 times and recording the number of tails that occur.
- c Rolling a six-sided die 20 times and counting the number of 6s that come up.
- d Tossing a coin and counting the number of tosses required before two heads occur.
- e Drawing a card from a deck with replacement 15 times and recording the number of hearts drawn.
- f Drawing a marble from a bag containing 10 red, 4 green and 6 blue marbles without replacement 5 times and recording the number of red marbles drawn.
- g Spinning a spinner numbered 1 to 8 ten times and counting the number of odd numbers that occur.

5 Evaluate the following, correct to 4 decimal places if necessary.

a $\binom{6}{4}(0.7)^4(0.3)^2$ b $\binom{9}{3}(0.38)^3(0.62)^6$ c $\binom{5}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3$

d $\binom{8}{7}(0.25)^7(0.75)^1$ e $\binom{10}{0}(0.09)^0(0.91)^{10}$

6 Example 9 For each of the following probability distributions, identify the parameters p, q, n and x.

a $\binom{10}{2}(0.5)^2(0.5)^8$ b $\binom{20}{0}(0.85)^{20}$ c $\binom{15}{12}\left(\frac{3}{5}\right)^{12}\left(\frac{2}{5}\right)^3$

d $\binom{9}{8}(0.11)^8(0.89)$ e $\binom{7}{4}(0.25)^4(0.75)^3$

7 A binomial variable, X, has the probability function $P(X = x) = \binom{7}{x}(0.8)^x(0.2)^{7-x}$.
Find:
- a the number of trials
- b the probability of success in any trial
- c the probability distribution as a table (with probabilities correct to four decimal places).

8 A binomial variable, Z, has the probability function $P(Z = z) = \binom{7}{z}(0.15)^z(0.85)^{7-z}$.
Find:
- a the number of trials
- b the probability of success in any trial
- c the probability distribution as a table (with probabilities correct to four decimal places)

9 A fair six-sided die is rolled 7 times and the number of 4s is recorded. Find the probability of obtaining exactly three 4s.

Reasoning and communication

10 Compare the probability distributions found in questions **7** and **8** and comment on the differences you observe.

5.04 USING THE BINOMIAL DISTRIBUTION

Calculating binomial probabilities 'by hand' is very tedious. You can use your CAS calculator instead, unless you are required to give an exact answer.

Example 10

CAS Calculate the required probabilities in each of the following situations.

a A binomial experiment has 10 trials and $p = 0.35$. Find the probability that there are 3 successes obtained from the 10 trials.

b Find the binomial probability of 8 successes from 14 trials, where the probability of success is 0.75.

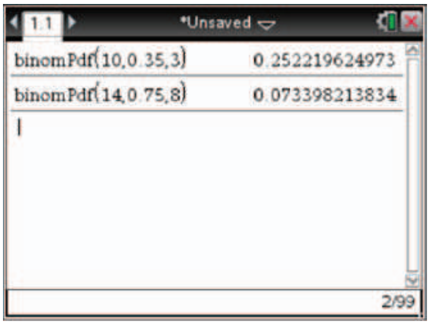
Solution

a List the parameters. $n = 10, p = 0.35, x = 3$

b List the parameters. $n = 14, p = 0.75$ and $x = 8$

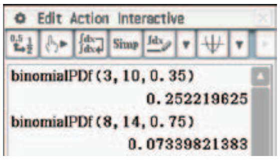
TI-Nspire CAS

Use a calculator page.
Use menu, 5: Probability, 5: Distributions and D: Binomial Pdf. Fill in the parameters.



ClassPad

Use the Main menu.
Tap Action, Distribution/Inv. Dist, Discrete and choose binomialPDf.
Complete the parameters in the order x, n, p.
You can do the same thing starting with Interactive. Fill in the table in the same order.



Write the answers.

a About 0.2522.
b About 0.0734.

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Sometimes you need to find the probability of a number of values of a binomial distribution and add them together.

Example 11

A binomial experiment has 9 trials. The probability of success in any trial is 0.7. If X = number of successes, calculate the probability that:

a X is at most 2

b X is more than 2

c X is more than 6.

Solution

a	List the parameters.	$n = 9, p = 0.7$
	'At most 2' means 2 or fewer.	$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$
	Find the values.	$= 0.3^9 + 9 \times 0.7 \times 0.3^8 + 36 \times 0.7^2 \times 0.3^7$
	Use your calculator.	$= 0.004\,290\,8\ldots$
	Write the answer.	The probability that X is at most 2 is about 0.004 291.
b	Use complements.	$P(X > 2) = 1 - P(X \le 2)$
	Substitute in the value.	$= 1 - 0.004\,290\,8\ldots$ $= 0.995\,709\,1\ldots$
	Write the answer.	The probability that X is more than 2 is about 0.9957.
c	Write the probability.	$P(X > 6) = P(X = 7) + P(X = 8) + P(X = 9)$
	Find the values.	$= 36 \times 0.7^7 \times 0.3^2 + 9 \times 0.7^8 \times 0.3 + 0.7^9$
	Use your calculator.	$= 0.462\,831\ldots$
	Write the answer.	The probability that X is more than 6 is about 0.4628.

You can also use your CAS calculator to find combined binomial probabilities using the cumulative binomial distribution.

Example 12

CAS A binomial experiment has 20 trials. The probability of success in any trial is 0.48. The random variable X is the number of successes. Calculate the probability that X is between 5 and 9 inclusive.

Solution

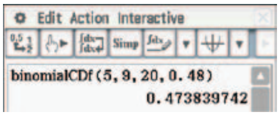
Identify the parameters. $n = 20$, $p = 0.48$, $5 \leq x \leq 9$

TI-Nspire CAS

Use a calculator page.
Use menu, 5: Probability, 5: Distributions and E: 20Binomial Cdf. Fill in the parameters.

ClassPad

Use the Main menu.
Tap Action, Distribution/Inv. Dist, Discrete and choose binomialCdf. Complete the parameters in the order *lower, upper*, n, p.
You can do the same thing if you start with Interactive and fill in the table.



Write the answer. The probability that X is between 5 and 9 inclusive is about 0.4738.

You can always type the commands into the CAS calculators instead of negotiating the menus if you prefer to do it that way.

EXERCISE 5.04 Using the binomial distribution

Concepts and techniques

WS Using the binomial probability distribution

1 Example 10 In a particular binomial experiment, a success occurs in 5 out of every 6 trials. The probability that there will be exactly 5 successes in the next 6 trials is closest to:
A 0.0231 B 0.1667 C 0.3349 D 0.4019 E 0.8333

2 Example 11 X is a random variable with a binomial probability distribution, with $n = 7$ and $p = \frac{2}{5}$. The probability that X is at least 6 is:
A 0.0139 B 0.0188 C 0.0221 D 0.9959 E 0.9992

3 In a certain binomial experiment, the probability of success is 0.4. If there are five trials, what is the probability that at most one successful outcome results?

A 0.077 76 **B** 0.2592 **C** 0.3104 **D** 0.336 96 **E** 0.663 04

4 X is a random variable with a binomial probability distribution, with $n = 8$ and $p = \frac{1}{3}$. The probability that $X \geq 1$ is closest to:

A 0.0390 **B** 0.1951 **C** 0.8049 **D** 0.9610 **E** 0.9998

5 In a certain binomial experiment, a success occurs 75% of the time. If there are 10 trials, the probability that there are fewer than 3 successful outcomes is closest to:

A 0.000 415 **B** 0.000 416 **C** 0.003 090
D 0.003 506 **E** 0.004 218

6 A binomial experiment has 7 trials. The probability of success in any trial is 0.4. If X = the number of successes, calculate the probability that:

a $X = 3$ **b** X is at least 3 **c** X is more than 5

7 Example 12 CAS In a binomial experiment, $n = 11$, $p = 0.82$ and X = the number of successes. Calculate the probability that:

a $X = 7$ **b** X is between 3 and 6 (inclusive)
c X is less than or equal to 3 **d** X is greater than 7.

8 CAS A certain binomial experiment has 15 trials. The probability of success in any trial is 0.27. The random variable X is the number of successes. Calculate the probability that:

a $X = 11$ **b** X is at least 5
c X is greater than or equal to 9 **d** X is between 2 and 9 (exclusive)

9 CAS In a binomial experiment, $n = 25$, $p = 0.725$ and X = number of successes. Calculate the probabilities of the following.

a $X = 15$ **b** X is between 5 and 15 (inclusive)
c X is greater than or equal to 14 **d** X is less than 10

10 A binomial experiment has 8 trials and $p = 0.4$. Calculate the probability that:

a 2 successes occur **b** 5 successes occur
c at least 2 successes occur.

11 CAS A binomial experiment has 17 trials and the probability of success in any trial is 0.65. Calculate the probabilities of the following.

a 5 successes occur
b between 4 and 8 successes (inclusively) occur
c at least 14 successes occur

Reasoning and communication

12 A binomial experiment has 6 trials and the probability of success in any trial is 0.5. If X is the number of successes, find the value of x if:

a $P(X = x) = 0.3125$ **b** $P(X = x) = 0.093\,75$ **c** $P(X = x) = 0.234\,375$

13 A binomial experiment has 5 trials and the probability of failure in any trial is 0.5. If X is the number of successes, find the value of x if:

a $P(X < x) = 0.1875$ **b** $P(X \geq x) = 0.1875$ **c** $P(X > x) = 0.5$

5.05 PROPERTIES OF THE BINOMIAL DISTRIBUTION

You can find the values of a binomial distribution 'by hand' or by using your calculator. It is also useful to show binomial distributions as graphs. Even though the variable is discrete, column graphs of binomial distributions are often referred to as histograms.

Example 13

Draw a histogram of the binomial distribution for $n = 14$ and $p = 0.65$. Comment on the shape of the graph.

Solution

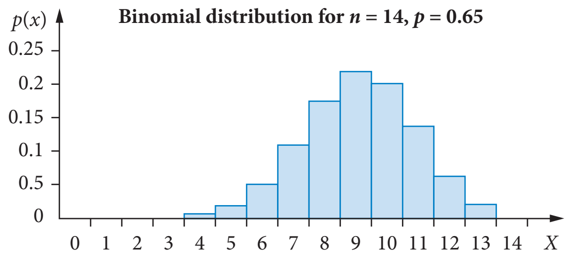
State the parameters.

$n = 14, p = 0.65, 0 \le X \le 14$

Use a your calculator or the formula to calculate the probabilities.
Many of the probabilities are too small to show on the graph.

$P(X = 0) = 0.000\ 000\ 413\ldots$
$P(X = 1) = 0.000\ 010\ldots$
$P(X = 2) = 0.000\ 129\ldots$
$P(X = 3) = 0.000\ 965\ldots$
$P(X = 4) = 0.004\ 92\ldots$
$P(X = 5) = 0.0183\ldots$
and so on.

Draw the histogram.



Comment on the shape.

The graph is skewed left.

You should remember that a **skewed** statistics or probability graph is stretched out more to one side that the other. In Example **13**, the highest point (the mode) is at 9 and the graph stretches from 0 to 9 on the left and from 9 to 14 on the right. It stretches a distance of 9 on the left but only 5 on the right, even though some of the probabilities are too small to see on the graph.

You can use your CAS calculator to investigate the shapes of binomial probability distributions.

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INVESTIGATION The effect of changing n and p on a binomial distribution histogram

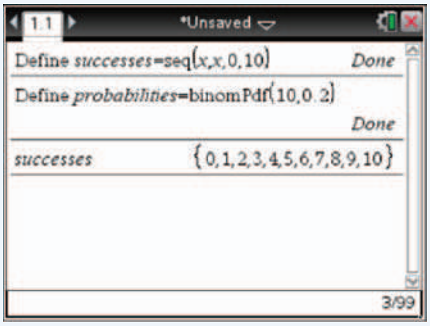
Part A

A binomial experiment has 10 trials and the probability of success in any trial is 0.2. The number of successful outcomes is observed.

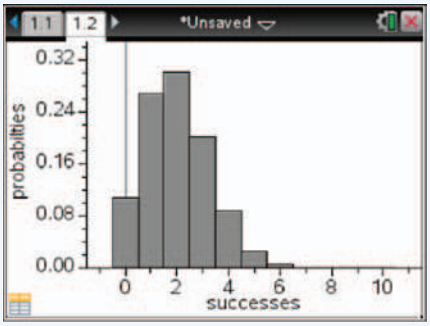
- What are the parameters for this experiment?
- Use a CAS calculator to draw a histogram of the probability distribution for this experiment.

TI-Nspire CAS

Use a Calculator page.
Type 'Define successes=seq(x,x,0,10)'.
Type 'Define probabilities=binompdf(10,0.2)
Type successes to see the list.

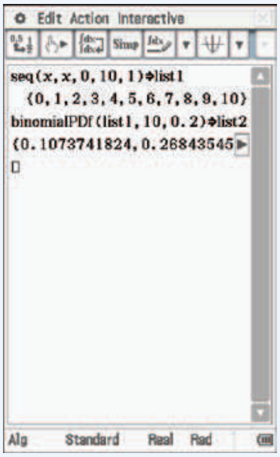


Add a Data & Statistics page.
Use menu, 2: Plot Properties and 5: Add X Variable and choose successes.
Use menu, 1: Plot Type and 3: Histogram and choose a histogram.
Finally use menu, 2: Plot Properties and 9: Add Y Summary List and choose probabilities.

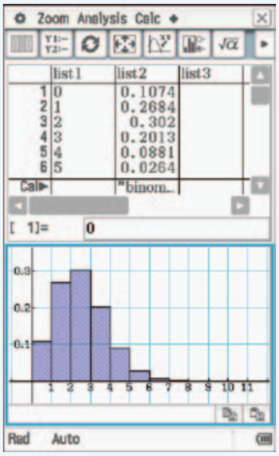


ClassPad

Tap Action, then List, Create and seq.
Enter x, x, 0, 10, 1 to create a list (sequence) of numbers from 0 to 10, going up by 1s. Name this list1.
Tap Action, Distribution/Inv. Dist, Discrete and binomialPDf.
Enter list1 for the number of successes, x, with $n = 10$ and $p = 0.2$. Name this list2.
It will create a list of probabilities for $x = 0$ to $x = 10$.



Now use the Statistics menu. The lists will already be in place.
Use the View Window ( ) to set $-1 \leq x \leq 11$, scale 1, and $-0.1 \leq y \leq 0.5$, scale 0.1.
Tap SetGraph, Setting and choose Histogram, List1 for XList and List2 for Freq.
Tap to draw the histogram.
Set Hstart to 0 and Hstep to 1.
You may have to do the View Window again.



A second binomial experiment also has 10 trials but the probability of success in any trial is 0.5. The number of successful outcomes is observed.

- What are the parameters for the second experiment?
- Use a CAS calculator to draw a histogram of the probability distribution for the second experiment.

A third binomial experiment also has 10 trials but the probability of success in any trial is 0.8. The number of successful outcomes is observed.

- What are the parameters for the third experiment?
- Use a CAS calculator to draw a histogram of the probability distribution for the third experiment.
- Compare the histograms for each of the binomial experiments and write a statement that describes the effect of changing p (the probability of success in any trial) on the shape of the histogram. What can you say about the graphs for $p = 0.2$ and $p = 0.8$?

Part B

A binomial experiment is performed in which the probability of success in any trial is 0.6. There are 6 trials and the number of successful outcomes is observed.

- What are the parameters for this experiment?
- Use a CAS calculator to draw a histogram of the probability distribution for this experiment.

This experiment is extended so that there are 20 trials and all other parameters are the same.

- Use a CAS calculator to draw a histogram of the probability distribution for the experiment.

Now the same experiment is extended to 50 trials.

- Use a CAS calculator to draw a histogram of the probability distribution for the experiment.
- Compare the three histograms for this binomial experiment. Write a statement that describes the effect of changing n (the number of trials) on the shape of the histogram.

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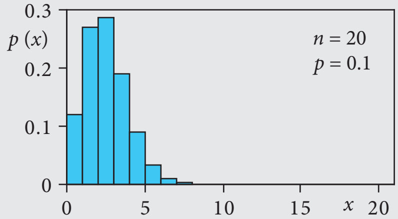
From the previous investigation you should have noticed a number of things about the shape of a probability distribution histogram.

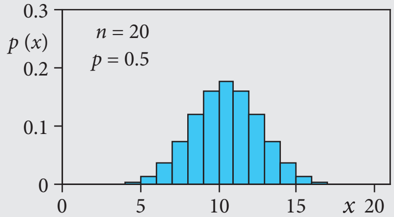
IMPORTANT

Shape of the binomial distribution

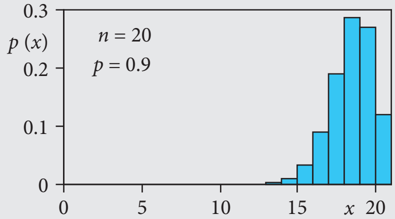
For $p < 0.5$, the distribution is **skewed to the right** (a **positive skew**).



For $p = 0.5$ the distribution is **symmetrical**.



For $p > 0.5$ the distribution is **skewed to the left** (a **negative skew**).



The most likely number of successes of a binomial distribution is the one with the highest probability. You can easily calculate the mean and variance of a binomial distribution using your CAS calculator. Remember that the expected value and variance are given by

$$\mu = E(X) = \sum x \cdot p(x) \qquad \text{and}$$

$$\begin{aligned} Var(X) &= E\left[(X-\mu)^2\right] \\ &= E\left(X^2\right) - \mu^2 \\ &= \sum x^2 \cdot p(x) - \left[\sum x \cdot p(x)\right]^2 \end{aligned}$$

Example 14

Find the expected value (mean) and variance of a binomial distribution with $n = 200$ and $p = 0.13$.

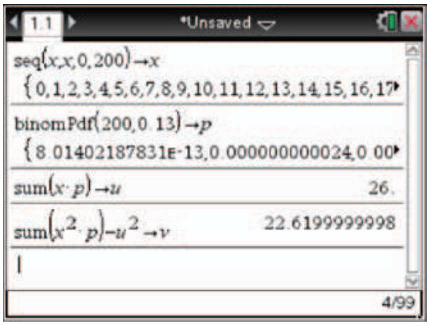
Solution

Identify the parameters.

$n = 200$, $p = 0.13$

TI-Nspire CAS

Use a calculator page.
Store 0–200 in x and store the probabilities in p.
Use sum($x \times p$) to find the expected value, u.
Use sum($x^2 \times p$) – u^2 to find the variance, v.
You could also do this as a spreadsheet, as shown in the Casio.



ClassPad

This is done in a similar way to the earlier investigation on pages 221–222.
Tap Action, then List, Create and seq.
Enter x, x, 0, 200, 1 to create a list (sequence) of numbers from 0 to 200, going up by 1s.
Name this list1.
Tap Action, Distribution/Inv. Dist, Discrete and binomialPDf.
Enter list1 for the number of successes, x, with $n = 200$ and $p = 0.13$. Name this list2.
It will create a list of probabilities from $x = 0$ to $x = 200$.
Go to the Statistics menu.
Tap Calc and One-Variable.
Make XList list1 and Freq List2.
Read the value of the mean $\bar{x}$ and the standard deviation σ_x.

The variance is the square of the standard deviation.

Variance $\approx 4.756...^2$
≈ 22.62

Write the answer.

The mean is 26 and the variance is 22.62.

Notice that there appears to be a rounding error in the calculation of the variance on the TI-Nspire in Example **14**. This does illustrate that a CAS calculator is a computer so it has limited storage. This can sometimes produce inaccurate or incorrect answers.

Notice also that the expected value for the binomial distribution with $n = 200$ and $p = 0.13$ is $200 \times 0.13 = 26$.

This follows from the fact that the binomial distribution is a fixed number of Bernoulli trials, each one of which constitutes a Bernoulli distribution. The expected value of a Bernoulli distribution is p, so the expected value of the binomial distribution will just be the sum of the expected value of each of the n trials, np.

Similarly, the variance of the Bernoulli distribution is $p(1-p)$, so the variance of the binomial distribution will be $np(1-p)$.

The standard deviation is just the square root of the variance.

IMPORTANT

The **expected value** (**mean**) of a binomial probability distribution is given by $\mu = E(X) = np$.

The **variance** of a binomial probability distribution is given by $Var(X) = np(1-p)$.

The **standard deviation** of a binomial distribution is given by $\sigma = SD(X) = \sqrt{np(1-p)}$.

You may prefer to use $q = 1 - p$ and write $\mu = np$, $Var(X) = npq$ and $\sigma = \sqrt{npq}$.

Example 15

In the general population, 15% of people are left-handed. A class has 24 students and the number who are left-handed is counted.

a Confirm this is an example of a binomial experiment.

b How many in the class would you expect to be left-handed?

c What is the probability that the number of left-handed students in the class is within one standard deviation of the mean?

Solution

a Check this situation to see if it has the characteristics of a binomial distribution.	There are 24 independent trials with 2 outcomes, p is fixed and the variable is the number of successes.
State the result.	This is a binomial distribution.
b Write the formula for expected value	$E(X) = np$
Substitute in the values and find the answer.	$= 24 \times 0.15$ $= 3.6$
State the result.	You would expect 3 or 4 left-handers.
c Write the formula for σ.	$SD(X) = \sqrt{npq}$
Substitute in the values.	$= \sqrt{24 \times 0.15 \times 0.85}$
Evaluate.	≈ 1.75
Calculate the values of X within 1 standard deviation (σ) of the mean (μ).	$\mu + \sigma \approx 3.6 + 1.75 = 5.35$ $\mu - \sigma \approx 3.6 - 1.75 = 1.85$
Identify the relevant values of X.	Only the values from 2 to 5 are *within* 1 SD.

State the required probability.	$P(\mu - \sigma \le X \le \mu + \sigma) = P(2 \le X \le 5)$
Use the CDF on your calculator.	$= 0.754\ 637\ldots$
State the result.	There is about a 75% chance that the number of left-handed students in the class is within one standard deviation of the mean.

Even if the values of $\mu - \sigma$ and $\mu + \sigma$ worked out to 1.000 000 001 and 5.999 999 999, you would use $P(2 \le X \le 5)$ in Example **15** because it had to be *within* one standard deviation. It is always very important to read a question carefully to make sure you are answering the question asked.

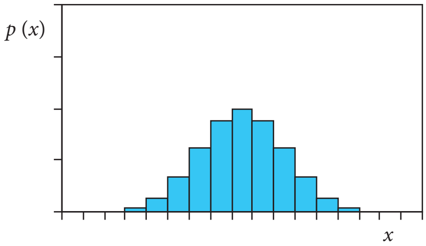
EXERCISE 5.05 Properties of the binomial distribution

Binomial probability – Mean and standard deviation

Concepts and techniques

1 Example 13 Which of the following graphs best represents the shape of a binomial probability distribution of the random variable X with 12 trials if the probability of success is 0.3?

A



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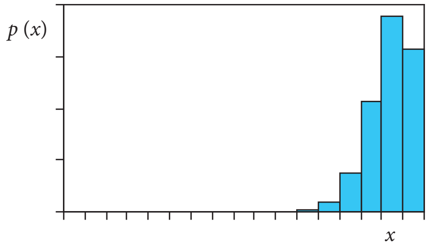

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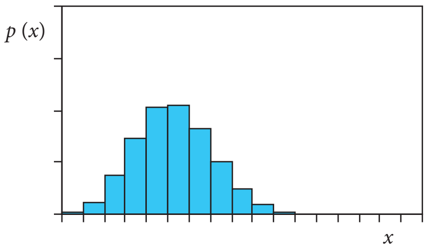

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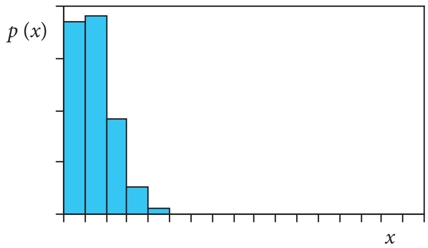
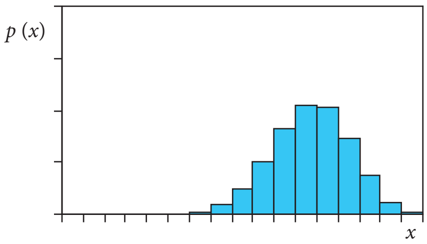


E



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2 Example 15 If a binomial random variable, X, has parameters $n = 32$ and $p = \frac{1}{4}$, then the mean and variance of X are closest to:

A $\mu = 6, \sigma^2 = 8$ B $\mu = 24, \sigma^2 = 12$ C $\mu = 16, \sigma^2 = 8$
D $\mu = 8, \sigma^2 = 6$ E $\mu = 12, \sigma^2 = 24$

3 a Draw graphs of the binomial distributions for $n = 8$ for $p = 0.2$ and $p = 0.8$.
b Comment on the shapes of the graphs.

4 a Draw graphs of the binomial distributions for $n = 10$ for $p = 0.4$ and $p = 0.6$.
b Comment on the shapes of the graphs.

5 In a binomial experiment, the probability of success in any trial is 0.8. There are 10 trials.
a Draw the histogram of the probability distribution for the number of successes.
b Use the histogram to determine the most likely number of successes.

6 Example 14 CAS For each of the following binomial distributions, calculate the mean (expected value) and standard deviation without using the formulas. Use your results to verify the formulas.
a $n = 5$ and $p = 0.5$ b $n = 8$ and $p = 0.3$
c $n = 12$ and $p = 0.8$ d $n = 9$ and $p = 0.6$

7 Use the formulas to find the mean and standard deviation of each of the following binomial distributions.
a $n = 7$ and $p = 0.1$ b $n = 7$ and $p = 0.9$
c $n = 20$ and $p = 0.65$ d $n = 30$ and $p = 0.34$

Reasoning and communication

8 A binomial experiment has a random variable, X. If there are 10 trials and $Var(X) = 0.9$, what is the probability of success?

9 A random binomial variable X has a mean of 24 and a standard deviation of 3. What is the probability of success?

10 A normal deck of playing cards is shuffled and cut. This is done 60 times. The number of aces that are cut is noted.
a Confirm this is an example of a binomial experiment.
b How many aces would you expect to get?
c What is the probability that the number of aces cut is within two standard deviations of the mean?

11 The mean of a binomial distribution is 5 and the standard deviation is 2. What is the actual probability of 5 successes?

12 About how many heads would you expect from 200 tosses of a fair coin?

13 The probability that a lettuce seed germinates is about 0.8. About how many seeds should you plant if you want 500 lettuce seedlings?

14 The probability that a car battery fails within 2 years of purchase is about 60%.
a From 420 batteries, how many would you expect to last longer than 2 years?
b What is the standard deviation of the number of failures in 2 years?

5.06 APPLICATIONS OF THE BINOMIAL DISTRIBUTION

There are many practical applications of the binomial probability distribution. The binomial probability distribution is often used in business and government to help make decisions.

Example 16

A study found that 38% of all swordfish sold in a particular seafood market contained levels of mercury that were above the Food Standards Australia (FSA) recommended maximum levels. Food inspectors took a random sample of twelve pieces of swordfish from the market for testing. Find the probability that:

Alamy/Gianni Muratore

a five of the pieces have mercury levels above the FSA maximum
b at most four pieces have mercury levels above the FSA maximum
c at least three pieces have mercury levels above the FSA maximum.

Solution

a Check this situation to see if it has the characteristics of a binomial distribution.	There are 12 independent trials with 2 outcomes (above level or not), $p = 0.38$ is fixed and the variable is the number of "successes", so it is binomial.
State the required probability.	$P(X = 5) = \binom{12}{5} p^5 q^7$
Substitute and calculate or use the PDF.	$= \binom{12}{5} (0.38)^5 (0.62)^7$
	$= 0.220\,996\ldots$
State the answer.	The probability of 5 bad fish is about 22%.
b State the required probability.	$P(\text{at most } 4) = P(0 \le X \le 4)$
Use the CDF.	$= 0.495\,719\ldots$
State the answer.	The probability of at most 4 bad fish is about 49.6%.

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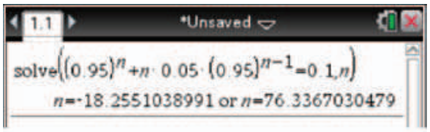
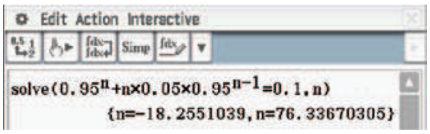
c State the required probability.	$P(\text{at least } 3) = P(3 \le X \le 12)$
Use the CDF.	$= 0.893\ 056\ldots$
State the answer.	The probability of at least 3 bad fish is about 89.3%.

FSA say consumers should eat not more than 1 serve of large predatory fish like swordfish or shark (flake) in a week, and no other fish or shellfish at all the same week.

Example 17

At a particular telemarketing company, it is known that the probability that a call to a potential client (cold calling) results in a sale is 0.05. What is the least number of calls that must be made to ensure that the probability of making at least 2 sales is 90%?

Solution

Confirm that this is an example of a binomial experiment.	There are 2 outcomes (sale or not), p is fixed and the variable is the number of "successes". The calls are independent. Determining the number of calls will make it binomial.
State the required probability.	$P(X \ge 2) > 0.9$
This is easier with the complement.	$1 - P(X < 2) > 0.9$
Simplify.	$P(X < 2) > 0.1$
Rewrite the LHS.	$P(X < 2) = P(X = 0) + P(X = 1)$
Use the binomial function with n as the number of trials.	$= \binom{n}{0}(0.05)^0(0.95)^n + \binom{n}{1}(0.05)^1(0.95)^{n-1}$
Substitute $\binom{n}{0} = 1$ and $\binom{n}{1} = n$.	$= 0.95^n + n \times 0.05(0.95)^{n-1}$
Write what needs to be solved.	$0.95^n + n \times 0.05(0.95)^{n-1} < 0.1$
Write this as an equation.	$0.95^n + n \times 0.05(0.95)^{n-1} = 0.1$
This equation cannot be solved algebraically. Use the Solve function of your CAS calculator. The syntax is the same for both calculators shown, and they both give warnings that there may be other solutions.	 
n cannot be negative.	$n = 76.336\ldots$
State the result.	At least 77 calls must be made to ensure the probability of making at least 2 sales is 90%.

You CAS calculator gives a warning because they use numerical methods to find a solution by iteration. Since it is not an algebraic method, there is no guarantee that there are no other solutions. In real situations, you would check that the solution does work.

Example 18

It is known that about 8% of all people in Australia will exceed their recommended dose of caffeine from drinking coffee or energy drinks. A company employs 200 staff at a factory and the manager wants to know how the workforce might be affected, given that they have vending machines for coffee, tea and energy drinks. Some of the effects of too much caffeine are headaches, restlessness and inattention.

Calculate the probability that the number of workers who will get too much caffeine is between two standard deviations of the mean and explain what this result means for the factory.

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Solution

Check this situation to see if it has the characteristics of a binomial distribution.	There are 200 trials with 2 outcomes (overdose or not), p is fixed and the variable is the number of "successes", so it is binomial.
Calculate the mean.	$\mu = np$
Substitute and evaluate.	$= 200 \times 0.08$ $= 16$
Calculate the standard deviation.	$\sigma = \sqrt{npq}$
Substitute and evaluate.	$= \sqrt{200 \times 0.08 \times 0.92}$ $= 3.836\,665\ldots$
Calculate the X values that are two standard deviations of either side of the mean.	$\mu + 2\sigma = 16 + 2 \times 3.836\ldots = 23.673\ldots$ $\mu - 2\sigma = 16 - 2 \times 3.836\ldots = 8.326\ldots$
State the required probability.	$P(\text{within } 2\sigma) = P(8 \le X \le 23)$
Use your CAS calculator.	$= 0.961\,419\ldots$
State the result.	There is about a 96% chance that from 8 to 23 staff will overdose with caffeine. Get rid of the vending machines and educate staff about the dangers!!

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EXERCISE 5.06 Applications of the binomial distribution

Applications of binomial probability

Concepts and techniques

In this exercise, you are expected to use your CAS calculator.

1 Example 16 A certain plant variety produces 25% red flowers and the flower colour red forms a binomial distribution. One of these plants has 3 flowers.
   a What are the values of n, p and q?
   b What is the probability that all flowers are red?
   c What is the probability that no flowers are red?
   d What is the probability that at least 2 flowers are red?

2 The probability that the Melbourne Storm win is $\frac{2}{3}$ whenever they play. Find the probabilities of winning the following numbers of games from the first 4 of the season.
   a Exactly 2 games.
   b At least 1 game.
   c More than half of the games.

3 A family has 6 children. Find the probabilities of the following family compositions.
   a 3 boys and 3 girls
   b Fewer boys than girls.

4 What proportion of families with exactly 4 children would be expected to have at least 2 girls?

5 Metal welds that are done with robotic devices have a 15% chance of containing a flaw. A pipeline contains 7 welds. Find the probability that:
   a all the welds have flaws
   b at least 1 weld has a flaw.

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6 It is a bank's experience that 5% of the cheques received in an automatic teller will 'bounce'. What is the probability that, for 8 cheques deposited
   a none will bounce?
   b exactly 2 will bounce?
   c at least 1 will bounce?

7 If, over a given period in Brisbane, rain falls at random on 4 days out of every 10, find the probability that

a the first 2 days of a given week will be wet and the remainder of the week will be fine

b rain will fall on the first 2 days of the week

c at least 2 days in the week will be wet.

8 A multiple-choice test has 7 questions, each with 4 alternatives, only 1 of which is correct. What is the probability of guessing correct answers to

a all of the 7 questions?

b none of the 7 questions?

c 3 of the 7 questions?

d at least 3 of the 7 questions?

9 A gardener plants 8 seeds. The probability that a seed will germinate is 0.85. What is the probability that at least 6 of the seeds will germinate?

10 A particular family has 3 children. It is known that the genetic makeup of the parents is such that there is a 20% chance that a child born will have curly hair.

a What are the values of n, p and q for this binomial situation?

b What is the probability that all of the children have curly hair?

c What is the probability that none of the children have curly hair?

d What is the probability that at least 1 of the children has curly hair?

Reasoning and communication

11 Example 17 The probability of Jacqui hitting a target is $\frac{1}{4}$.

a If she fires 7 times, what is the probability of her hitting the target at least twice?

b How many times must she fire so that the probability of hitting the target at least once is greater than $\frac{2}{3}$?

12 A survey revealed that 63% of all people in a particular city eat fast food at least once a week. A random sample of 6 people in the city was selected. What is the probability that:

a half of those selected ate fast food at least once a week?

b at least half of those selected ate fast food at least once a week?

13 Example 18 It is known that 68% of all school students in a certain Australian state attend government schools. A new community is planned for establishment in an area near the state's capital city. It is estimated that the community will eventually have 1500 school-aged children. Calculate the probability that the number of students who will attend a government school is between two standard deviations of the mean and explain what this result means.

14 Hospital records confirm that, of patients suffering from a particular complaint, 80% recover during their hospital stay. What is the probability that 6 randomly selected patients suffering from the complaint will all recover?

15 A judge is scheduled to hear 10 appeals for traffic violations. Each appeal has a probability of 0.4 of being approved, independently of other appeals. Find the probability that less than half of the appeals are granted.

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16 Boom Boom is the school's best tennis player. From experience, it is known that the probability that Boom Boom serves an ace is 0.15. What is the probability that Boom Boom wins a game with just 4 serves?

17 It is known that 10% of the population carry a particular gene that predisposes them to bladder cancer.

a A study group of 10 people is randomly selected for testing. Find the probability that:

i two people in the group possess the gene

ii at least one person in the group possesses the gene.

b What is the smallest number of people who should be selected for a study group to ensure that there is more than a 70% chance of having at least one person who possesses the gene?

18 After a certain drug in tablet form is stored above 30°C, it is found that an average of one-fifth of the tablets become ineffective. A person who has been on a holiday in the tropics in summer kept 8 of these tablets in a suitcase where the temperature was consistently over 30°C. What is the probability that:

a exactly 3 of the tablets are now ineffective?

b at least 1 of the tablets is now ineffective?

19 Find the probability that, of the first 6 people met in the street on a given day, at least 4 will have their birthday on a Sunday this year. Give your answer as a fraction.

20 It is known that 30% of the components manufactured in a particular assembly line will have some kind of defect.

a A random sample of 6 components is selected for testing. Find the probability that:

i none has a defect   ii at least one has a defect of some kind.

b What is the smallest number of components that should be selected to ensure that there is more that an 80% chance of having at least one defective component?

21 A manufacturer finds that, in the long run, 15% of the manufactured articles are defective. If a sample of 10 articles is randomly selected, find the probability that:

a the sample contains 2 defective articles

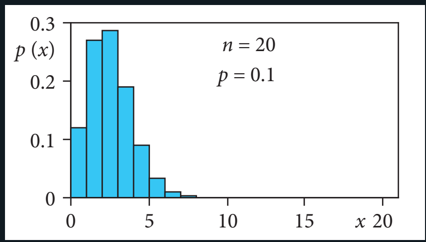
b the first 3 articles selected are defective

c the sample contains at least 3 defective articles.

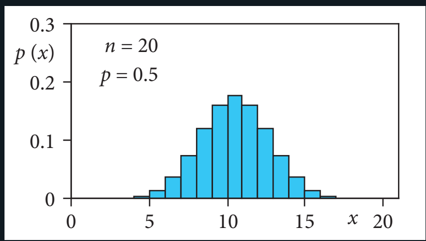
22 Wendy owns 10 holiday units in a remote location and does not want to risk leaving DVD players permanently in the units, so she hires them out to occupants as they require them. She finds that about 80% of people who occupy the units want to hire a DVD player, so she buys 7. Find the probability that, at a time when all the units are occupied, the demand for DVD players will exceed supply.

5 CHAPTER SUMMARY
BINOMIAL DISTRIBUTIONS

- A **Bernoulli distribution** is a distribution with two possible outcomes, called **success** and **failure**, with a fixed probability of success. The probability of success is normally written as p and the probability of failure as q, so that $p + q = 1$.
- The **Bernoulli random variable** X is given the values $P(X = 0) = q$, $P(X = 1) = p$.
- The expected value, variance and standard deviation of a Bernoulli distribution are given by $E(X) = \mu = p$, $Var(X) = p(1 - p)$ and $SD(X) = \sigma = \sqrt{p(1-p)}$
- **Bernoulli trials** are independent trials that have only two possible outcomes, called **success** and **failure**, where all the trials have the same probability of success. The probability of success is normally written as p and the probability of failure as q.
- The **geometric distribution** is the probability distribution of the number of failures of Bernoulli trials before the first success.
- The **geometric random variable** X has the values 0, 1, 2, 3, …: $X \in J^+$, the probability function is $P(X = x) = q^x p$ and the expected value is $E(X) = \dfrac{q}{1-q} = \dfrac{1-p}{p}$.
- The **binomial distribution** is the probability distribution of the number of successes arising from a fixed number of Bernoulli trials. A situation that produces a binomial distribution may be called a **binomial experiment**.
- A binomial distribution has the following properties:
  - each trial has only two outcomes – success and failure with probabilities p and q
  - there are a fixed number, n, of independent trials with a fixed probability of success
  - and the **binomial random variable** is the number of successes from n trials.
- If X is a binomial random variable, then the probability of x successes from n trials is given by $P(X = x) = \binom{n}{x} p^x q^{n-x}$, or $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$.
- For $p < 0.5$, the binomial distribution is **skewed to the right** (a **positive skew**).



- For $p = 0.5$, the distribution is **symmetrical**.



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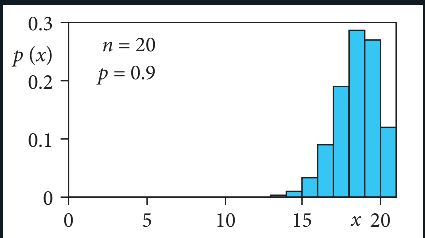
- For $p > 0.5$, the distribution is **skewed to the left** (a **negative skew**).



- The **expected value** (**mean**) of a binomial probability distribution is $\mu = E(X) = np$.
- The **variance** of a binomial probability distribution is given $Var(X) = np(1 - p)$.
- The **standard deviation** of a binomial distribution is given by $\sigma = SD(X) = \sqrt{np(1-p)}$.

5

CHAPTER REVIEW
BINOMIAL DISTRIBUTIONS

Multiple choice

1 Example 1 Which of the following can be considered as a Bernoulli distribution?

A Tossing a coin 5 times and counting the number of heads
B Rolling a die and seeing if it comes up with a 6 on top
C Cutting cards and checking the suit, with replacement
D All of the above
E None of the above

2 Example 3 A Bernoulli distribution has a probability of success of 0.2. What is the standard deviation?

A 0.04　　B 0.16　　C 0.4　　D 0.8　　E 1.6

3 Example 4 Which of the following variables will have a geometric probability distribution?

A the number of times a coin must be flipped until 2 successive heads have been flipped
B the number of red cars sold each week by a car dealership
C the number on which the ball rests when a roulette wheel is spun
D the number of cards drawn from a well-shuffled deck of playing cards before an ace occurs.
E the number of cars that are able to pass through a set of lights at an intersection while the light is green

4 Example 5 X has a geometric distribution, where X is number of failures before the first success. If the probability of success is 0.4, then $P(X = 3)$ is equal to:

A 0.0384　　B 0.0576　　C 0.0684　　D 0.0834　　E 0.0864

5 Example 7 For which of the following situations would a binomial probability model be most reasonable?

A the number of hearts that occur when 7 cards are drawn with replacement from a well-shuffled deck of 52 cards
B the number of tosses of a fair coin required before a head occurs
C the number of blue marbles that are observed when 8 marbles are drawn without replacement from a bag containing 4 blue, 6 white and 3 green marbles
D the number of spades dealt in a hand of five cards from a well-shuffled deck of 52 cards
E all of the above

6 Example 9 A card is drawn from a well-shuffled deck of 52 playing cards and its suit is noted before it is returned to the deck. This process is repeated 12 times. If X is the number of hearts observed, then which of the following best describes the probability distribution for X?

A a uniform distribution with $n = 12$ and $p = 0.25$
B a binomial distribution with $n = 52$ and $p = 0.5$
C a geometric distribution with $n = 12$ and $p = 0.75$
D a hypergeometric distribution with $k = 4$ and $n = 12$
E a binomial distribution with $n = 12$ and $p = 0.25$

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7 Example 11 In a certain binomial experiment the chances of a successful outcome are 5 in every 12 trials. If there are 7 trials, what is the probability that at most one successful outcome results?
A 0.0149 B 0.1379 C 0.1793 D 0.3842 E 0.4167

8 Example 12 A binomial experiment has 20 trials. If the probability of success in any trial is 90%, what is the probability of obtaining 18 or more successful outcomes:
A 0.190 B 0.270 C 0.285 D 0.677 E 0.900

9 Example 14 A binomial probability distribution has 20 trials and the probability of success is 0.7. The expected value is:
A 7 B 10 C 12.5 D 14 E 17

Short answer

10 Example 1 State which of the following describe a Bernoulli distribution.
- a Drawing a card from a deck with replacement and recording its suit.
- b Rolling a six-sided die 15 times and recording the number of times a prime number comes up.
- c Spinning a spinner with equal-sized segments coloured red, green, pink, blue, yellow, orange, black and white and recording the whether the spinner stops on orange.
- d Tossing a coin 15 times and recording the number of tails.
- e Drawing a card from a deck without replacement and recording if a king is drawn.
- f Drawing a marble from a bag containing 6 red and 3 blue marbles with replacement and recording the number of blue marbles drawn.
- g Observing whether a person signing a contract is left-handed.

11 Examples 2, 8 Six balls and then two more called supplementaries drawn from a tumbler with balls numbered 1 to 45 without replacement are used for a lotto game. What is the probability of successfully guessing one of the numbers drawn?

12 Example 3 When you are playing Ludo, what is the exact variance of the random variable X arising from rolling a normal die if you want to get a six to start?

13 Example 4 A fair six-sided die is rolled until the first time that a number less than 3 is observed. What is the probability that exactly five rolls are required?

14 Example 5 A putt-putt golf enthusiast keeps careful records and knows that, in the long run, she completes a hole in three putts or less 30% of the time. She regards this as a 'win'. If she takes more than three putts, she considers this to be a loss. If X represents the number of holes she loses before the first win on a particular round of putt-putt, find:
- a $P(X = 3)$
- b $P(X \leq 5)$
- c $E(X)$

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15 Example 6 An experienced rugby union player knows that he converts 68% of all kicks for goal. If Y represents the number of kicks before his first successful conversion, find:

a $P(Y = 3)$

b $P(2 \le Y \le 9)$

16 Example 7 Which of the following are examples of binomial experiments?

a Spinning a spinner numbered 1 to 12 five times and counting the number of prime numbers that occur.

b Rolling a six-sided die 20 times and recording the number of rolls required before an odd number occurs.

c Drawing a card from a deck with replacement 10 times and recording the number of spades drawn.

d Tossing a coin and counting the number of tosses required before two tails occur.

e Rolling a six-sided die 20 times and counting the number of even numbers that come up.

f Drawing a marble from bag containing 9 green, 3 red and 2 blue marbles without replacement 6 times and recording the number of green marbles drawn.

g Tossing a coin 12 times and recording the number of heads that occur.

17 Example 8 Evaluate the following, correct to 4 decimal places if necessary.

a $\binom{5}{2}(0.4)^2(0.6)^3$

b $\binom{10}{7}(0.43)^7(0.57)^3$

c $\binom{8}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^6$

18 Example 9 A binomial variable, X, has the probability function:

$$P(X = x) = \binom{8}{x}(0.45)^x(0.55)^{8-x}.$$

Find:

a the number of trials

b the probability of success in any trial

c the probability distribution as a table (with probabilities correct to four decimal places).

19 Example 9 A binomial experiment has 8 trials. The probability of success in any trial is 0.35. If X = number of successes, calculate the probability that:

a $X = 7$

b X is at least 2

c X is more than 4.

20 Example 10 A card is drawn from a well-shuffled deck, checked to see whether it is a picture card (K, Q or J) and then returned to the deck. What is the probability of obtaining 2 picture cards from 5 such draws?

21 Example 11 A certain binomial experiment has 20 trials. The probability of success in any trial is 0.68. The random variable X is the number of successes. Use a graphics calculator to find the probability that:

a $X = 12$

b X is at least 7

c X is at most 9.

22 Example 11 A binomial experiment has 9 trials and $p = 0.6$. Calculate the probability that:

a 3 successes occur

b 6 successes occur

c at least 2 successes occur.

23 Example 11 In a binomial experiment, the probability of success is 0.7. What is the probability of getting at least 2 successes if there are three trials?

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24 Examples 11, 12 CAS Find the following binomial probabilities.

a $P(x = 5)$ when $n = 15$ and $p = 0.3$

b $P(x = 7)$ when $n = 19$ and $p = 0.45$

c $P(x < 6)$ when $n = 18$ and $p = 0.75$

d $P(x \geq 4)$ when $n = 12$ and $p = 0.6$

e $P(3 \leq x \leq 9)$ when $n = 20$ and $p = 0.3$

25 Example 12 A normal coin is tossed 12 times. What is the probability of obtaining more than 3 but less than 7 heads?

26 Example 13 Draw a histogram of the binomial probability distribution for $n = 8$ and $p = 0.65$. Comment on the shape of the distribution.

27 Example 13 In a binomial experiment, the probability of success in any trial is 0.4. There are 8 trials.

a Draw the histogram of the probability distribution for the number of successes.

b Use the histogram to determine the most likely number of successes.

28 Example 14 Find the mean and standard deviation of each of the following binomial distributions.

a $n = 8$ and $p = 0.2$

b $n = 10$ and $p = 0.7$

c $n = 15$ and $p = 0.55$

29 Example 15 The probability that children will be absent from school in a flu epidemic is 15%. What is the probability that from a class of 20 students:

a 7 will be away?

b at least 7 will be away?

c at least 5 will be away?

30 Example 16 It is known that 28% of eggs placed in cartons are slightly underweight. A carton holding a dozen eggs is selected for examination. Find the probability that:

a the first 2 eggs selected are slightly underweight and the remainder are not underweight

b 2 of the eggs in the carton are slightly underweight

c at least 2 of the eggs in the carton are slightly underweight.

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Application

31 A target shooter knows from experience that she hits a target 4 times out of each 5 shots. If she is allowed 10 shots in a competition, what is the probability that she gets:

a 10 hits? b 9 hits? c at least 7 hits?

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32 A new vaccine is claimed to be 85% effective in immunising young children against a particular childhood disease. In a sample of 15 children who are vaccinated, what is the probability that:

a all become immune to the disease?

b at least 2 have no immunity to the disease?

c fewer than 4 have no immunity?

33 Five per cent of thermostats produced by a particular process are defective. The thermostats are packed 15 to a box for shipping purposes. When a box of thermostats is received by a spare parts dealer, what is the probability that:

a all are operative? b 2 are defective?

c at least 2 are defective? d no more than 2 are defective?

34 There is a 30% chance that a driver will have an accident in their first year of driving. Eighteen Year 12 students get their driver's licenses in June. What is the probability that less than a quarter will have an accident before June the next year?

35 In the general population, it is estimated that 18% of women suffer from iron deficiency. A study group of 25 women is selected and tested for iron deficiency.

a How many of the study group would you expect to be iron deficient?

b Calculate the probability that the number of women with iron deficiency in the study group is within two standard deviations of the mean.

c Explain what the result in part **b** means.

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36 The mean of a binomial distribution is 12 and the standard deviation is 3. What is the probability of 9 successes?

37 It is known that 18% of the residents in a large city object to the fluoridisation of the city's water supply.

a A random sample of 10 residents is selected. Find the probability that:

i none has an objection to water fluoridisation

ii at least two object to water fluoridisation.

b What is the smallest number of residents that should be selected to ensure that there is more that a 90% chance of having at least one who objects to water fluoridisation?

Practice quiz

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6

TERMINOLOGY

algebraic area
antiderivative
antidifferentiation
definite integral
difference function
differentiation
family of functions
indefinite integral
integration
linearity of integration
marginal rate of change
net signed area
physical area
primitive function
signed area
total change
velocity

Prior learning

INTEGRALS

APPLICATIONS OF INTEGRATION

ANTI-DIFFERENTIATION

- recognise anti-differentiation as the reverse of differentiation (ACMMM114)
- use the notation $\int f(x)dx$ for anti-derivatives or indefinite integrals (ACMMM115)
- establish and use the formula $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$ (ACMMM116)
- establish and use the formula $\int e^x\,dx = e^x + c$ (ACMMM117)
- establish and use the formulas $\int \sin(x)dx = -\cos(x)+c$ and $\int \cos(x)dx = \sin(x)+c$ (ACMMM118)
- recognise and use linearity of anti-differentiation (ACMMM119)
- determine indefinite integrals of the form $\int f(ax+b)dx$ (ACMMM120)
- identify families of curves with the same derivative function (ACMMM121)
- determine $f(x)$, given $f'(x)$ and an initial condition $f(a) = b$ (ACMMM122)
- determine displacement given velocity in linear motion problems (ACMMM123)

APPLICATIONS OF INTEGRATION

- calculate the area under a curve (ACMMM132)
- calculate total change by integrating instantaneous or marginal rate of change (ACMMM133)
- calculate the area between curves in simple cases (ACMMM134)
- determine positions given acceleration and initial values of position and velocity (ACMMM135) AC

6.01 INDEFINITE INTEGRALS

You know that the **definite integral** of a function $f(x)$ over the interval $[a, b]$, $\int_a^b f(x)dx$, is the area 'under' the graph of $f(x)$. If $f(x)$ is negative, the definite integral works out to be negative, so we say that the definite integral is the **signed area** between the curve, the x-axis, and the lines $x = a$ and $x = b$. The definite integral is given by the lmit of the sum of areas of rectangles as

$$\int_a^b f(x)dx = \lim_{\delta x \to 0} \sum f(x)\delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\delta x_i.$$

You should recall the following about the **indefinite integral**.

IMPORTANT

An **indefinite integral** $F(x)$ of $f(x)$ is a function such that $F'(x) = f(x)$.

The indefinite integral is normally written as $F(x) + c$, where c is an arbitrary constant called the **constant of integration**, because indefinite integrals differ only by a constant.

An indefinite integral is also called an **antiderivative** or **primitive** and is written as

$$F(x) = \int f(x)dx$$

The definite integral can be calculated from an indefinite integral $F(x)$ using **The Fundamental Theorem of Calculus** as

$$\int_a^b f(x)dx = F(b) - F(a)$$

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You should already be familiar with the derivatives shown below.

IMPORTANT

Basic derivatives

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \frac{d}{dx}kx^n = knx^{n-1} \qquad \frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)}$$

$$\frac{d}{dx}kf(x) = kf'(x) \qquad \frac{d}{dx}f(ax+b) = af'(ax+b)$$

You should also be able to find the derivatives of combined functions using the **product rule**, **quotient rule** and **chain rule**. You can reverse these rules to make rules for many indefinite integrals.

INVESTIGATION Basic integrals

- Find the derivative of x^{n+1}
- Find the derivative of $\frac{x^{n+1}}{n+1}$
- Hence find the indefinite integral of x^n
- What is the derivative of $\sin(x)$?
- Hence find the indefinite integral of $\cos(x)$
- Find the derivative of $-\cos(x)$
- Hence find the indefinite integral of $\sin(x)$
- What is the derivative of e^x?
- Hence find the indefinite integral of e^x
- Find the derivative of $\frac{f(ax+b)}{a}$
- Hence find the indefinite integral of $f(ax+b)$

From the investigation, we have the following rules for basic indefinite integrals.

IMPORTANT

Indefinite integrals of basic functions

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c\ (n \neq -1) \qquad \int kx^n\,dx = \frac{kx^{n+1}}{n+1} + c\ (n \neq -1)$$

$$\int \sin(x)\,dx = -\cos(x) + c \qquad \int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a} + c$$

$$\int \cos(x)\,dx = \sin(x) + c \qquad \int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + c$$

$$\int e^x\,dx = e^x + c \qquad \int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + c$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c\ (n \neq -1) \qquad \int kf'(x)\,dx = kf(x) + c$$

If you are given a value for the integral at a particular point, then you can find the constant of integration.

Example 1

Find the function $f(x)$ in each case if:

a $f'(x) = 4x^5$ and $f(1) = 5$

b $f'(x) = \dfrac{1}{x^3}$ and $f\left(\dfrac{1}{2}\right) = 3$

c $f'(x) = 5\sqrt{x}$ and $f(9) = 100$

Solution

a Write using integral notation.	$\int 4x^5\, dx = 4\int x^5\, dx$
Apply the relevant rule.	$f(x) = 4 \times \dfrac{x^6}{6} + c$
Simplify.	$= \dfrac{2x^6}{3} + c$
Substitute in the values given.	$5 = \dfrac{2 \times 1^6}{3} + c$
Find c.	$c = 4\frac{1}{3}$
Write the function.	$f(x) = \dfrac{2x^6}{3} + 4\frac{1}{3}$
b Write using integral notation.	$\int \dfrac{1}{x^3}\, dx = \int x^{-3}\, dx$
Apply the relevant rule.	$f(x) = \dfrac{x^{-2}}{-2} + c$
Write in the form given.	$f(x) = -\dfrac{1}{2x^2} + c$
Substitute in the values given.	$3 = -\dfrac{1}{2(\frac{1}{2})^2} + c$
Find c.	$c = 5$
Write the function.	$f(x) = 5 - \dfrac{1}{2x^2}$
c Write using integral notation.	$\int 5\sqrt{x}\, dx = 5\int x^{\frac{1}{2}}\, dx$
Apply the relevant rule.	$f(x) = 5 \times \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$
Substitute in the values given.	$100 = \dfrac{10 \times 9^{\frac{3}{2}}}{3} + c$
Find c.	$c = 10$
Write in the form given.	$f(x) = \dfrac{10\sqrt{x^3}}{3} + 10$ or $f(x) = \dfrac{10x\sqrt{x}}{3} + 10$

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Example 2

Find y in terms of x if:

a $\dfrac{dy}{dx} = e^{4x}$ b $\dfrac{dy}{dx} = 6\sin(3x)$ c $\dfrac{dy}{dx} = 8\cos\left(-\dfrac{x}{2}\right)$

Solution

a Write using integral notation. $y = \int e^{4x}\,dx$

Apply the relevant rule. $= \dfrac{e^{4x}}{4} + c$

b Write using integral notation. $y = \int 6\sin(3x)\,dx$

$= 6\int \sin(3x)\,dx$

Apply the relevant rule. $= 6\left[\dfrac{-\cos(3x)}{3}\right] + c$

Simplify. $= -2\cos(3x) + c$

c Write using integral notation. $y = \int 8\cos\left(-\dfrac{x}{2}\right)dx$

$= 8\int \cos\left(-\dfrac{x}{2}\right)dx$

Apply the relevant rule. $= 8\left[\dfrac{\sin\left(-\dfrac{x}{2}\right)}{-\frac{1}{2}}\right] + c$

Simplify. $= -16\sin\left(-\dfrac{x}{2}\right) + c$

$= 16\sin\left(\dfrac{x}{2}\right) + c$

Example 3

Find the antiderivative of:

a $(2x+1)^4$ **b** $(5x-1)^{-3}$

Solution

a Apply the relevant rule.

$$\int (2x+1)^4\,dx = \frac{(2x+1)^5}{2\times 5} + c$$

Simplify.

$$= \frac{1}{10}(2x+1)^5 + c \text{ or } \frac{(2x+1)^5}{10} + c$$

b Write in index form.

$$\int \frac{1}{(5x-1)^3}\,dx = \int (5x-1)^{-3}\,dx$$

Apply the relevant rule.

$$= \frac{(5x-1)^{-2}}{5\times(-2)} + c$$

Simplify.

$$= -\frac{(5x-1)^{-2}}{10} + c$$

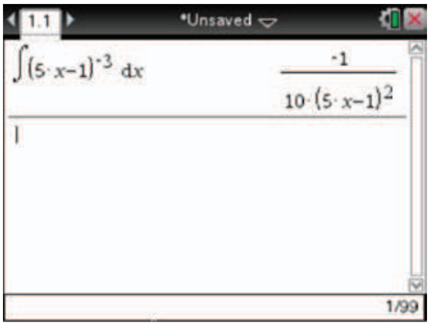
Write in the original form.

$$= -\frac{1}{10(5x-1)^2} + c$$

You can use a CAS calculator to find indefinite integrals.

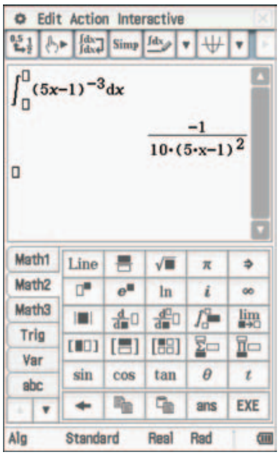
TI-Nspire CAS

In the calculation page, Press menu, 4:Calculus and 3:Integral.
To obtain the indefinite integral, leave out the bounds. Type in the expression and then x after the 'd'.



ClassPad

Use the Main menu and the Math1 soft keyboard. Set the calculator to Standard.
Tap [button]. Fill in the blanks as shown on the right and dx. To obtain the indefinite integral, leave out the bounds.



EXERCISE 6.01 Indefinite integrals

Concepts and techniques

Finding indefinite integrals 1

1 Example 1 The function $f(x)$ such that $f'(x) = 6x^2$ and $f(x) = 2$ when $x = -1$ is:

A $12x - 4$ B $2x^3 + 4$ C $2x^3 - 2$ D $3x^2 + 4$ E $2x^3 + 2$

2 Example 2 Given that $\dfrac{dy}{dx} = 4e^{-x}$ and $y = -9$ when $x = 0$, the function y is given by:

A $4e^{-x} + 5$ B $e^{-4x} + 5$ C $-e^{4x} - 4$ D $-4e^{-x} - 5$ E $-4e^{x} + 9$

3 The function $f(x)$ such that $f'(x) = -8\sin(4x)$ and $f\left(\dfrac{\pi}{4}\right) = -5$ is:

A $2\cos(4x) - 3$ B $16\cos(4x) - 9$ C $2\sin(4x) - 5$
D $-2\cos(4x) + 3$ E $-32\cos(4x) + 5$

4 Find the indefinite integral of each of the following, giving answers with positive indices.

a x b x^2 c x^6 d $2x^4$
e $5x^{-3}$ f $-3x^3$ g $-5x^{-4}$ h $\sqrt{x}$
i $5\sqrt{x}$ j $\dfrac{x^5}{7}$ k $\dfrac{x^3}{5}$ l $\dfrac{x^{-3}}{4}$
m $x^{\frac{1}{3}}$ n $3x^{\frac{2}{5}}$ o $x^{\frac{-3}{4}}$ p $\dfrac{4}{x^3}$
q $\dfrac{-5}{x^6}$ r $\dfrac{10}{\sqrt{x}}$ s $\dfrac{-6}{\sqrt[3]{x}}$ t $\dfrac{8}{x\sqrt{x}}$

5 Find the indefinite integrals of the following.

a e^{2x} b e^{4x} c e^{-x} d e^{5x}
e e^{-2x} f e^{4x+1} g $-3e^{5x}$ h e^{2t}
i $5e^{4x}$ j $-6e^{-2x}$ k $4e^{\frac{x}{2}}$ l $6e^{-\frac{x}{3}}$

6 Find the indefinite integrals of the following.

a $\cos(x)$ b $\sin(x)$ c $\sin(3x)$ d $-\sin(7x)$
e $\cos(x+1)$ f $\sin(2x-3)$ g $\cos(2x-1)$ h $4\sin\left(\dfrac{x}{2}\right)$
i $-\sin(3-x)$ j $3\cos\left(\dfrac{x}{4}\right)$ k $\sin(\pi - x)$ l $\cos(x+\pi)$
m $-2\sin\left(\dfrac{2x}{5}\right)$ n $4\cos\left(\dfrac{7x}{4}\right)$ o $2\cos\left(\dfrac{\pi x}{3}\right)$ p $-2\sin\left(\dfrac{-3x}{\pi}\right)$

7 Example 3 Find the antiderivative of each of the following.

a $(x+1)^4$ b $(5x-1)^9$ c $(3y-2)^7$ d $(4+3x)^4$
e $(7x+8)^{12}$ f $(1-x)^6$ g $\sqrt{2x-5}$ h $2(3x+1)^{-4}$
i $3(x+7)^{-2}$ j $\dfrac{1}{2(4x-5)^3}$ k $\sqrt[3]{4x+3}$ l $(2-x)^{-\frac{1}{2}}$
m $\sqrt{(t+3)^3}$ n $\sqrt{(5x+2)^5}$ o $(4-5x)^{-4}$ p $-6(3-4x)^{-5}$

Reasoning and communication

8 For the curve $y = f(x)$, $f'(x) = -3x$. If $y = 2$ when $x = 2$, find the equation of the curve.

9 The gradient at a point (x, y) on a curve is given by $3e^{2x}$. Find the equation of the curve if it passes through the point (0, 5.5).

10 Differentiate e^{x^4} and use this result to find the indefinite integral of $2x^3e^{x^4}$.

11 Differentiate $(4x^2 + 1)^3$ and hence deduce the indefinite integral of $6(4x^2 + 1)^2$.

6.02 PROPERTIES OF INDEFINITE INTEGRALS

You have seen how the indefinite integrals of certain functions can be determined using rules.

The rule for the derivative of a linear combination of functions leads directly to the property of **linearity of integration**.

IMPORTANT

Linearity of integration

Linear combination: $\int [af(x)+bg(x)]dx = a\int f(x)dx + b\int g(x)dx$

Constant multiple: $\int kf(x)dx = k\int f(x)dx$

Sum of functions: $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

Difference of functions: $\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$

Suppose $F(x)$ and $G(x)$ are primitives of $f(x)$ and $g(x)$ respectively.

Then $F'(x) = f(x)$ and $G'(x) = g(x)$.

Consider the derivative of $aF(x) + bG(x)$.

Then $\frac{d}{dx}[aF(x) + bG(x)] = aF'(x) + bG'(x)$ because a linear combination is preserved by differentiation.

Substituting $F'(x) = f(x)$ and $G'(x) = g(x)$ gives

$$\frac{d}{dx}[aF(x) + bG(x)] = af(x) + bg(x)$$

But this is just a statement that

$$\int [af(x)+bg(x)]dx = a\int f(x)dx + b\int g(x)dx$$

written in reverse form, so integration also preserves linear combinations.

Substitution of $a = k$ and $g(x)$ gives the constant multiple rule, $a = 1$ and $b = 1$ gives the addition rule and $a = 1$ and $b = -1$ gives the subtraction rule.

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Example 4

Find the indefinite integral of each of the following.

a $4x^7$ **b** $4x^3 - 7x^{-2} + 5$

Solution

a Apply the constant multiple rule. $\int 4x^7\,dx = 4\int x^7\,dx$

Apply the relevant rule of integration. $= 4 \times \frac{x^8}{8} + c$

Simplify. $= \frac{x^8}{2} + c$

b Apply the linear combination rule. $\int (4x^3 - 7x^{-2} + 5)dx = 4\int x^3\,dx - 7\int x^{-2}\,dx + \int 5dx$

Apply the relevant rule of integration.

$$= 4 \times \frac{x^4}{4} - 7 \times \frac{x^{-1}}{-1} + 5 \times x + c$$

$$= x^4 + \frac{7}{x} + 5x + c$$

As you have seen before, when additional information is provided, we can determine the value of the constant, and hence the unique antiderivative.

Example 5

Given that $f'(x) = 3x^3 - 3x^2$ and $f(2) = 7$, find $f(x)$.

Solution

Find the indefinite integral. $\int (3x^3 - 3x^2)dx = \frac{3x^4}{4} - \frac{3x^3}{3} + c$

Write the expression for $f(x)$. $f(x) = \frac{3x^4}{4} - x^3 + c$

Find $f(2)$. $f(2) = \frac{3 \times 2^4}{4} - 2^3 + c$

$$= 4 - c$$

We know that $f(2) = 7$. $7 = 4 - c$

Solve. $c = -3$

State the result. $f(x) = \frac{3x^4}{4} - x^3 - 3$

The substitution of different values for the constant of integration leads to the formation of a **family of functions**.

Example 6

CAS **a** Find the indefinite integral $\int(4x-3)(x+1)dx$.

b Substitute values of $c=-1$, $c=2$ and $c=5$ to create $f1(x)$, $f2(x)$ and $f3(x)$.

c Draw the functions on the same graph and describe this family of functions.

Solution

a Expand the product.

$$\int(4x-3)(x+1)dx=\int(4x^2+4x-3x-3)dx$$

Collect like terms.

$$=\int(4x^2+x-3)dx$$

Apply the linear combination rule.

$$=\int 4x^2\,dx+\int x\,dx-\int 3\,dx$$

$$=\frac{4x^3}{3}+\frac{x^2}{2}-3x+c$$

b Write $f1(x)$ by substituting $c=-1$.

$$f1(x)=\frac{4x^3}{3}+\frac{x^2}{2}-3x-1$$

Write $f2(x)$ by substituting $c=2$.

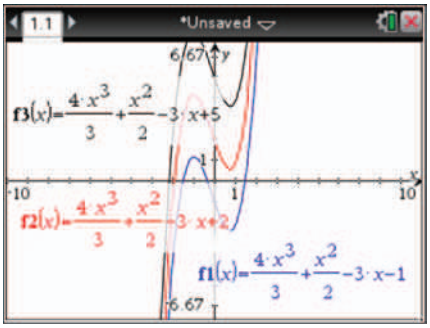
$$f2(x)=\frac{4x^3}{3}+\frac{x^2}{2}-3x+2$$

Write $f2(x)$ by substituting $c=5$.

$$f3(x)=\frac{4x^3}{3}+\frac{x^2}{2}-3x+5$$

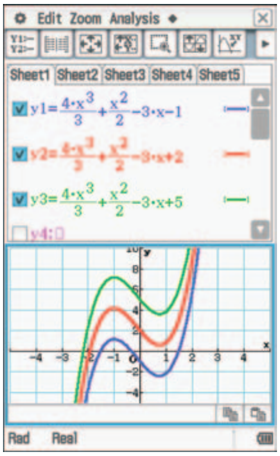
c **TI-Nspire CAS**

Start with a graph page. After entering $f1(x)$, use $\boxed{\text{menu}}$, 3: Graph Entry/Edit and 1: Function to enter $f2(x)$ and $f3(x)$.
Move the labels around to make it easier to distinguish between the graphs.



ClassPad

Use the Graph&Table menu.
You may need to reset Graph Format to default. First tap ???
Enter the functions in $y1$, $y2$ and $y3$.
Tap to see the graphs and tap to change the View Window if necessary.
Make sure that you tick the boxes to show the functions and set them to different lines.



Comment on the functions.

The functions are vertical translations of each other.

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EXERCISE 6.02 Properties of indefinite integrals

Concepts and techniques

Finding indefinite integrals 2

1 Example 4 $\int (2x^2 + x + 5)dx$ is equal to:

A $\int (2x^2 + x)dx + 5$ B $2x^2 + x + \int 5dx$ C $\int 2x^2\,dx + \int x\,dx + \int 5\,dx$

D $\int 2x^2 dx + x + 5$ E $2x^2 + \int (x + 5)dx$

2 Example 5 The function $f(x)$ such that $f'(x) = 4x + 3$ and $f(1) = 7$ is:

A $2x^3 + 3x^2 + 2x$ B $\frac{4x}{2} + 3x^2 + 1$ C $2x^2 + 3x + 2$

D $4x^2 + 3x - 2$ E $2x^2 - 3x - 2$

3 Example 6 $\int x^2(2x - 7)dx$ is equal to:

A $\int (2x^3 - 7x^2)dx$ B $\int x^2\,dx \int (2x - 7)dx$ C $(2x - 7)\int x^2\,dx$

D $x^2 \int (2x - 7)dx$ E $\int (x^2 + 2x - 7)dx$

4 Given that $\frac{dy}{dx} = \frac{ax^2}{3} - x$ and $y = 2$ when $x = 0$, an expression for y is:

A $\frac{ax^3}{3} - ax + 2$ B $ax^3 - \frac{x}{2} + 2$ C $\frac{ax^3}{3} - \frac{x}{2} + 2$

D $\frac{ax^3}{9} - \frac{x}{2} + 2$ E $ax^3 - ax + 2a$

5 Find each indefinite integral.

a $\int (m + 1)dm$ b $\int (t^2 - 7)dt$ c $\int (h^2 + 5)dh$

d $\int (y - 3)dy$ e $\int (2x + 4)dx$ f $\int (b^2 + b)db$

g $\int (a^3 - a - 1)da$ h $\int (x^2 + 2x + 5)dx$ i $\int (4x^3 - 3x^2 + 8x - 1)dx$

j $\int (6x^5 + x^4 + 2x^3)dx$ k $\int (x^7 - 3x^6 - 9)dx$ l $\int (2x^3 + x^2 - x - 2)dx$

m $\int (x^5 + x^3 + 4)dx$ n $\int (4x^2 - 5x - 8)dx$ o $\int (3x^4 - 2x^3 + x)dx$

p $\int (6x^3 + 5x^2 - 4)dx$ q $\int (3x^{-4} + x^{-3} + 2x^{-2})dx$ r $\int (7x^{\frac{3}{2}} - 4x + 6x^{-\frac{1}{3}})dx$

6 Find each indefinite integral

a $\int \frac{x^6 - 3x^5 + 2x^4}{x^3}dx$ b $\int (1 - 2x)^2 dx$ c $\int (x - 2)(x + 5)dx$

d $\int \frac{4x^3 - x^5 - 3x^2 + 7}{x^5}\,dx$ e $\int (y^2 - y^{-7} + 5)dy$ f $\int (t^2 - 4)(t - 1)dt$

g $\int \sqrt{x}\left(1 + \frac{1}{\sqrt{x}}\right)dx$ h $\int \frac{(x + 5)(x - 2)}{x^4}dx$ i $\int \frac{2x^2 - 4x + 3}{\sqrt{x}}dx$

7 Example 6 For each of the following, find y in terms of x.

a $\frac{dy}{dx} = 2x - 5$ and $y = 8$ when $x = -1$.

b $\frac{dy}{dx} = 3x^{\frac{1}{2}} - 4x$ and $y = -6$ when $x = 4$.

c $\frac{dy}{dx} = 3x^2 - x + 2$ and $y = 0$ when $x = 2$.

8 Find the equation of the curve $f(x)$ given that:

a $f'(x) = 6x - 1$ and the curve passes through the point (0, 5)

b $f'(x) = 7 - 4x$ and the curve passes through the point (−1, 1)

c $f'(x) = 3x^{-2} + 2$ and the curve passes through the point (1, 5)

d $f'(x) = \dfrac{2}{\sqrt{x}} + 3x$ and $f(1) = 3$

e $f'(x) = x^{\frac{1}{3}} + 6x^2 - 10$ and $f(1) = -7$

9 CAS a Find the indefinite integral $\int -6x\,dx$.

b Substitute in values of $c = 0$, $c = 1$ and $c = 2$ to create $f1(x)$, $f2(x)$ and $f3(x)$.

c Draw the functions on the same graph and describe this family of functions.

10 CAS a Find the indefinite integral $\int (3x^2 + 2x)dx$.

b Substitute in values of $c = -2$, $c = 1$ and $c = 3$ to create $f1(x)$, $f2(x)$ and $f3(x)$.

c Draw the functions on the same graph and describe this family of functions.

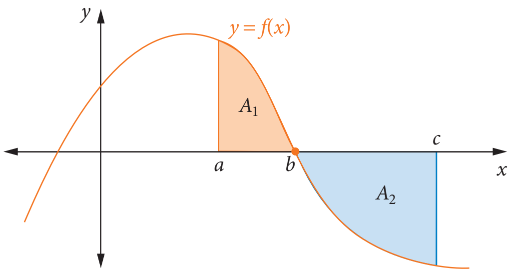
Reasoning and communication

11 The curve of the function $f(x)$ has a stationary point at (4, −2) and a gradient of $\dfrac{3x}{2} + k$, where k is a constant. Calculate $f(2)$.

12 The curve of the function $f(x)$ has a stationary point at $(\frac{1}{4}, 8)$ and a gradient of $\dfrac{16x - \sqrt{x}}{x^3}$. Find $f(x)$.

6.03 AREAS UNDER CURVES

An area must be positive. However, the definite integral is also positive for areas above the x-axis and negative for areas below the x-axis. We say that the definite integral calculates the **signed area**, so when we want the actual area, we need to change the sign for areas below the x-axis or for areas calculated from right to left. This is illustrated below for areas above and below the x-axis.



The definite integrals for the regions above are as follows.

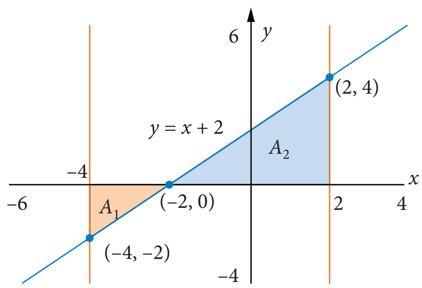
$\int_a^b f(x)dx = A_1$, $\int_b^c f(x)dx = -A_2$ and $\int_a^c f(x)dx = A_1 - A_2$

Note also that $\int_b^a f(x)dx = -A_1$, and $\int_c^b f(x)dx = A_2$.

You should think of the integral from a to c is as total signed area or **net signed area** between the curve and the x-axis.

Consider the function $f(x) = x + 2$ and the area between the graph of $f(x)$ for $-4 \le x < 2$.

The graph of $y = x + 2$ is shown here.



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Using the area of a triangle, you get:

$A_1 = \frac{1}{2} \times 2 \times 2 = 2$ square units and

$A_2 = \frac{1}{2} \times 4 \times 4 = 8$ square units.

The net signed area $= A_2 - A_1 = 8 - 2 = 6$ square units

Using integrals with the same function, you get:

$$\begin{aligned}\int_{-4}^{2}(x+2)dx &= \left[\frac{x^2}{2}+2x\right]_{-4}^{2} \\ &= \left(\frac{2^2}{2}+2\times 2\right)-\left(\frac{(-4)^2}{2}+2\times(-4)\right) \\ &= 6-0 \\ &= 6 \text{ square units}\end{aligned}$$

Also,

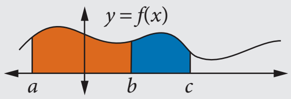
$$\int_{-4}^{-2}(x+2)dx = \left[\frac{x^2}{2}+2x\right]_{-4}^{-2} = -2$$

showing that the signed area below the axis (calculated from left to right as usual) works out to be negative.

Recall the following properties of definite integrals from Chapter 4.

IMPORTANT

Properties of definite integrals



$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^b [cf(x)+dg(x)]\,dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx$$

Example 7

Find the definite integrals of:

a $f(x) = 6x^2 - 10x + 3$ between 4 and 7

b $f(x) = \dfrac{16}{(2x+3)^3}$ between 1 and 2

Solution

a Write in symbols.

$$\int_4^7 (6x^2 - 10x + 3)\,dx$$

Find the indefinite integral and insert the limits.

$$= \left[2x^3 - 5x^2 + 3x\right]_4^7$$

Find the values at each limit.

$$= (2 \times 7^3 - 5 \times 7^2 + 3 \times 7) - (2 \times 4^3 - 5 \times 4^2 + 3 \times 4)$$

Simplify.

$$= 462 - 60$$
$$= 402 \text{ square units}$$

Write the answer.

The value of the integral of $f(x) = 6x^2 - 10x + 3$ between 4 and 7 is 402.

b Write in symbols.

$$\int_1^2 \frac{16}{(2x+3)^3}\,dx$$

Rewrite so that the expression is not a fraction.

$$= \int_1^2 16(2x+3)^{-3}\,dx$$

Find the integral using the relevant rule.

$$= \left[\frac{16(2x+3)^{-2}}{2 \times -2}\right]_1^2$$

Express with a positive power and simplify.

$$= \left[\frac{-4}{(2x+3)^2}\right]_1^2$$

Substitute in the values of the limits.

$$= \left[\frac{-4}{7^2}\right] - \left[\frac{-4}{5^2}\right]$$

Evaluate.

$$= \frac{-4}{49} + \frac{4}{25}$$

$$= \frac{96}{1225}$$

Write the answer.

The integral of $f(x) = \dfrac{16}{(2x+3)^3}$ between 1 and 2 is $\dfrac{96}{1225}$ square units.

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Example 8

Calculate $\int_{0.1}^{3} \frac{dx}{x^2}$.

Solution

Write the function in index form.

$$\int_{0.1}^{3} \frac{dx}{x^2} = \int_{0.1}^{3} x^{-2}\,dx$$

Integrate and insert the limits.

$$= \left[-x^{-1}\right]_{0.1}^{3}$$

Find the values at each limit.

$$= (-1 \times 3^{-1}) - (-1 \times 0.1^{-1})$$

Evaluate.

$$= \left(-1 \times \frac{1}{3}\right) - \left(-1 \times \frac{1}{0.1}\right)$$

$$= -\frac{1}{3} + 10$$

$$= 9\frac{2}{3}$$

Write the answer.

$$\int_{0.1}^{3} \frac{dx}{x^2} = 9\,\frac{2}{3} \text{ square units}$$

Example 9

Calculate the following.

a $\int_{0}^{\frac{\pi}{2}} \sin(2x)\,dx$ **b** $\int_{-1.5}^{1.5} 6e^{3x}\,dx$

Solution

a Integrate and insert the limits.

$$\int_{0}^{\frac{\pi}{2}} \sin(2x)\,dx = \left[-\frac{1}{2}\cos(2x)\right]_{0}^{\frac{\pi}{2}}$$

Find the values at each limit.

$$= -\frac{1}{2} \times \cos(\pi) - \left[-\frac{1}{2} \times \cos(0)\right]$$

Evaluate.

$$= \left[-\frac{1}{2} \times (-1)\right] - \left(-\frac{1}{2} \times 1\right)$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

Write the answer.

$$\int_{0}^{\frac{\pi}{2}} \sin(2x)\,dx = 1$$

b Integrate and insert limits.

$$\int_{-1.5}^{1.5} 6e^{3x}\,dx = \left[2e^{3x}\right]_{-1.5}^{1.5}$$

Find the values at each limit.

$$= 2e^{4.5} - 2e^{-4.5}$$

Evaluate.

$$= 180.0342\ldots - 0.0222\ldots$$

$$\approx 180.01$$

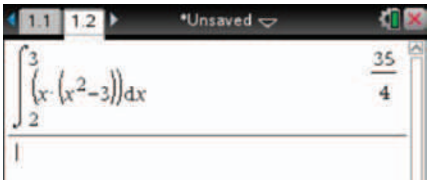
Write the answer.

$$\int_{-1.5}^{1.5} 6e^{3x}\,dx \approx 180.01 \text{ square units}$$

You can use your CAS calculator to calculate definite integrals. Instead of leaving the bounds blank like you did for indefinite integrals, put them in, as shown for $\int_2^3 x(x^2-3)dx$ below.

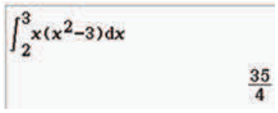
TI-Nspire CAS

Use a calculator page.
Use [menu], 4:Calculus and 3:Integral.
Complete the integral, including the bounds and dx.



ClassPad

Use the Main menu and the Math1 soft keyboard. Set the calculator to Standard. Tap [∫□]. Fill in the function including the bounds and dx.



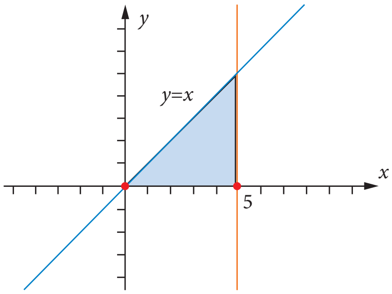
On either calculator, if you put in an integral that cannot be calculated exactly, such as $\int_0^3 e^{-x^2}\,dx$, then even if you have the calculator set to give an exact answer, the answer will be given as a decimal. However, $\int_0^3 \frac{1}{\sqrt{x}}dx$ will be given as either $2\sqrt{3}$ or 3.464…, depending whether you set the calculator for an exact or approximate answer.

EXERCISE 6.03 Areas under curves

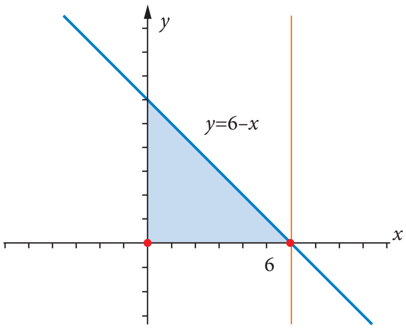
Concepts and techniques

WS
Finding definite integrals

1 Find the area of the triangle on the right:
  a geometrically
  b using integration



2 Find the area of the triangle on the right:
  a geometrically
  b using integration



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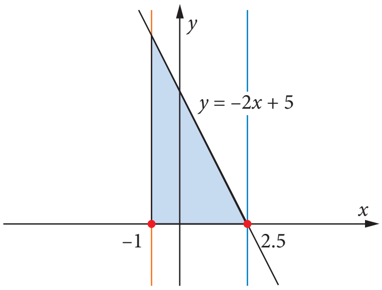
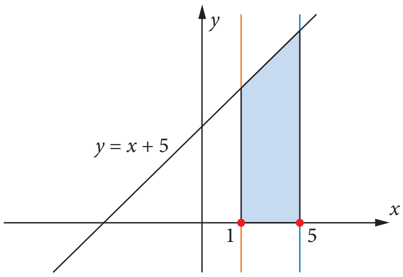
3 Express the following shaded areas as definite integrals.

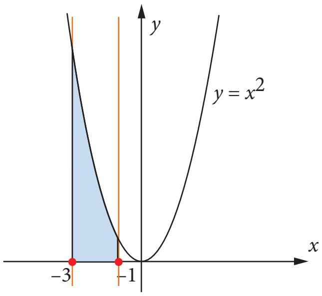
a



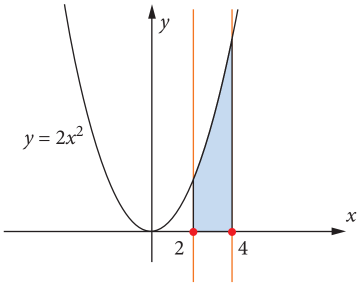
b



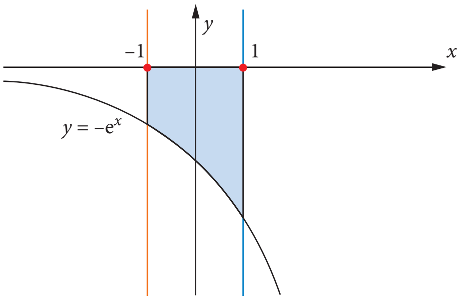
c



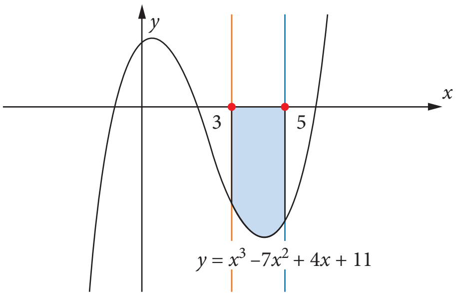
d



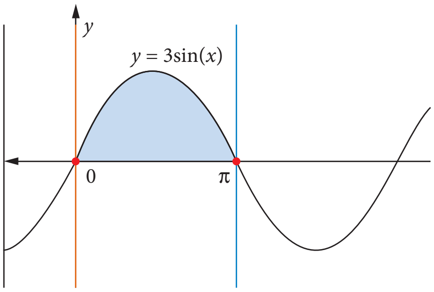
e



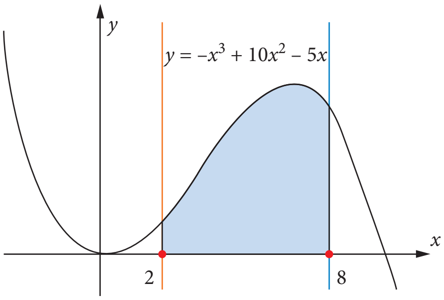
f



g



h



4 Example 7 Evaluate the following definite integrals.

a $\int_1^{10} (9x+7)\,dx$

b $\int_0^6 8\,dx$

c $\int_2^{10} 5x^3\,dx$

d $\int_{-3}^3 x^6\,dx$

e $\int_0^8 6x^3\,dx$

f $\int_{-5}^0 (2x^2 - x)\,dx$

g $\int_{-12}^{12} (20-m)\,dm$

h $\int_1^2 (4t-7)\,dt$

i $\int_{-3}^4 (2-x)^2\,dx$

j $\int_{-1}^4 (3x^2 - 2x)\,dx$

k $\int_1^3 (4x^2 + 6x - 3)\,dx$

l $\int_0^1 (x^3 - 3x^2 + 4x)\,dx$

5 Evaluate the following definite integrals.

a $\int_1^3 \frac{1}{(3x+1)^3}dx$ b $\int_0^1 \frac{1}{(2x-3)^2}dx$ c $\int_0^2 \frac{1}{(2x-5)^3}dx$

d $\int_0^1 \frac{3}{(2x+1)^4}dx$ e $\int_{-1}^0 \frac{2}{(3x+4)^4}dx$ f $\int_2^4 \frac{1}{\sqrt{2x+4}}dx$

6 Example 8 Evaluate the following definite integrals.

a $\int_1^3 \frac{1}{x^2}dx$ b $\int_1^2 \frac{1}{x^3}dx$ c $\int_5^{10} 4x^{-2}\,dx$

d $\int_{2.4}^{5.8} \frac{3}{x^4}dx$ e $\int_2^6 2x^{-3}\,dx$ f $\int_1^3 \frac{3x^2+2x}{x^4}dx$

7 Example 9 Evaluate in exact form.

a $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{3x}\,dx$ b $\int_3^8 5e^n\,dn$ c $\int_0^1 e^{5x}\,dx$

d $\int_0^2 -e^{-x}\,dx$ e $\int_1^4 2e^{3x+4}\,dx$ f $\int_2^3 (3x^2-e^{2x})dx$

g $\int_0^2 (e^{2x}+1)dx$ h $\int_1^2 (e^x-x)dx$ i $\int_0^3 (e^{2x}-e^{-x})dx$

8 Evaluate correct to 2 decimal places.

a $\int_1^4 e^{3V}\,dV$ b $\int_1^3 e^{-x}\,dx$ c $\int_0^2 2e^{3y}\,dy$

d $\int_5^6 (e^{x+5}+2x-3)dx$ e $\int_0^1 (e^{3t+4}-t)dt$ f $\int_1^2 (e^{4x}+e^{2x})dx$

9 Evaluate, giving exact answers where appropriate.

a $\int_0^{\pi} \sin(x)dx$ b $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \cos(2x)dx$ c $\int_{\frac{\pi}{2}}^{\pi} \sin\left(\frac{x}{2}\right)dx$

d $\int_0^{\frac{\pi}{2}} \cos(3x)dx$ e $\int_0^{\frac{1}{2}} \sin(\pi x)dx$ f $\int_0^{\frac{\pi}{8}} \sec^2(2x)dx$

g $\int_0^{\frac{\pi}{12}} 3\cos(2x)dx$ h $\int_0^{\frac{\pi}{10}} -\sin(5x)dx$

10 Evaluate the following.

a $\int_2^4 (5t^2+4t+5)dt$ b $\int_0^3 (v^5-4v^3+2v)dv$ c $\int_{-3}^3 (6u^5+5u^4+4)du$

d $\int_{-1}^1 \frac{72}{(4y+5)^7}dy$ e $\int_1^8 \sqrt[4]{x}\,dx$ f $\int_4^9 \frac{dt}{t^2\sqrt{t}}$

g $\int_0^2 4e^{2t-3}\,dt$ h $\int_4^6 \frac{35}{(5h-9)^2}dh$ i $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 3\sin\left(6x+\frac{\pi}{3}\right)dx$

j $\int_4^6 (e^x-x^3)dx$ k $\int_4^6 \sqrt{4x+1}\,dx$ l $\int_2^4 16(5-4v)^3\,dv$

m $\int_0^{\frac{\pi}{3}} 6\sin\left(3x-\frac{\pi}{4}\right)dx$ n $\int_{-\pi}^{\pi} [\sin(x)-\cos(x)]dx$ o $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{3}{4}\cos\left(\frac{1}{2}x+\frac{\pi}{2}\right)dx$

Reasoning and communication

11 Why can't you calculate the following?

a $\int_0^3 \frac{1}{x^2}dx$ b $\int_0^5 \frac{1}{(x-5)^2}dx$ c $\int_{-1}^3 \frac{1}{(x+1)^3}dx$

12 Explain why the value that you get for the integral $\int_0^4 \frac{1}{(x-2)^2}dx$ is not valid.

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13 What is wrong with $\int_{-2}^{2} \frac{1}{x} dx$?

14 Differentiate xe^{x^2} and hence find $\int_0^1 \left(2x^2e^{x^2} + e^{x^2}\right) dx$.

15 During a flood, the flow rate of a small creek under a road bridge increased from its normal level of 5 m^3/h to a peak after 5 hours, according to the equation $f = 5 + 30t^2$, where t is the time after the increase began. Find the amount of water that flowed under the bridge in the 5 hours before it reached its peak.

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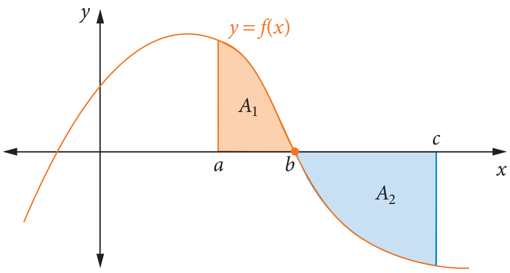
6.04 PHYSICAL AREAS

In the previous section you saw that the **signed** or **algebraic area** between a curve and the x-axis is given by the definite integral. It is the difference between the physical areas above and below the x-axis.

The **physical area** between a curve and the x-axis is always positive. Areas above and below the x-axis are calculated separately when using integrals to find a physical area.



In the diagram, the physical areas A_1 and A_2 are both positive, but the integral for A_2 is negative.

In terms of the integrals,

$$A_1 = \int_a^b f(x)dx > 0 \text{ and } A_2 = \int_b^c f(x)dx < 0$$

$$\begin{aligned}\text{So, physical area} &= A_1 - A_2\\ &= \int_a^b f(x)dx - \int_b^c f(x)dx\end{aligned}$$

When finding physical areas by integration, it is necessary to find the zeros to check where the graph crosses the x-axis and hence where the areas change sign.

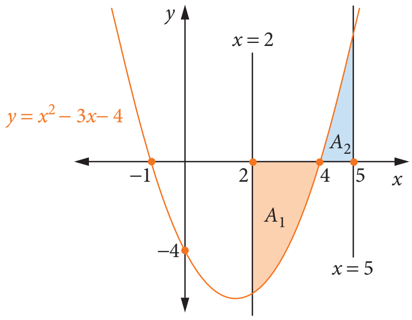
Example 10

Find the area enclosed by $y = x^2 - 3x - 4$, $x = 2$, $x = 5$ and $y = 0$.

Solution

$x = 2$ and $x = 5$ are just vertical lines, and $y = 0$ is the x-axis, so the question is actually asking for the physical area between $y = x^2 - 3x - 4$ and the x-axis from $x = 2$ to $x = 5$.

Since the question has several parts, set out your solution with headings.

Find the zeros. Let $y = 0$.	**Zeros** $x^2 - 3x - 4 = 0$
Factorise.	$(x + 1)(x - 4) = 0$
State the zeros.	The zeros are at $x = -1$ and $x = 4$
Let $x = 0$	**y-intercept** $y = -4$
The function is a quadratic with a positive coefficient of x^2, so it has a minimum. Draw in the lines $x = 2$ and $x = 5$. Shade the required areas. Shade the areas needed as A_1 and A_2.	**Sketch of graph** 
Write the integrals to find the area.	Area $= -\int_2^4 (x^2 - 3x - 4)\,dx + \int_4^5 (x^2 - 3x - 4)\,dx$
Integrate.	$= -\left[\frac{x^3}{3} - \frac{3x^2}{2} - 4x\right]_2^4 + \left[\frac{x^3}{3} - \frac{3x^2}{2} - 4x\right]_4^5$
Evaluate at the limits.	$= -[-18\frac{2}{3} - (-11\frac{1}{3})] + [-15\frac{5}{6} - (-18\frac{2}{3})]$
Finish the calculation.	$= -(-7\frac{1}{3}) + 2\frac{5}{6} = 10\frac{1}{6}$
Write the answer.	The area enclosed by $y = x^2 - 3x - 4$, $x = 2$, $x = 5$ and $y = 0$ is $10\frac{1}{6}$ square units.

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Example 11

Find the area enclosed by $f(x) = 24 - 2x - 2x^2$ and the x-axis.

Solution

The area will be between the zeros.
Let $f(x) = 0$.

Zeros

$-2(x^2 + x - 12) = 0$

Factorise and rearrange.

$-2(x - 3)(x + 4) = 0$

Write the zeros.

The zeros are at $x = -4$ and $x = 3$

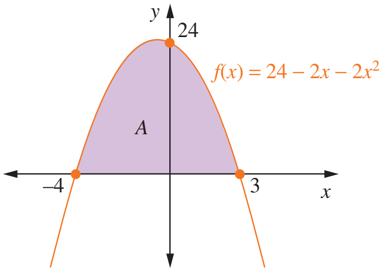
Let $x = 0$

***y*-intercept**

$y = 24$

The function is a quadratic with a negative coefficient of x^2, so it has a maximum.
Shade the required area as A.

Sketch of graph



Write the integral to find the area.

$$\text{Area } (A) = \int_{-4}^{3} (24 - 2x - 2x^2)\,dx$$

Integrate.

$$= \left[24x - x^2 - \frac{2x^3}{3} \right]_{-4}^{3}$$

Evaluate.

$$= 45 - \left(-69\tfrac{1}{3}\right)$$

$$= 114\tfrac{1}{3}$$

Write the answer.

The area enclosed by $f(x) = 24 - 2x - 2x^2$ and the x-axis is $114\frac{1}{3}$ square units.

INVESTIGATION Integrals

1 Find the following integrals.

$\int_{-1}^{1} 1\,dx$, $\int_{-1}^{1} x\,dx$, $\int_{-1}^{1} x^2\,dx$, $\int_{-1}^{1} x^3\,dx$, $\int_{-1}^{1} x^4\,dx$ and $\int_{-1}^{1} x^5\,dx$

2 Predict the result of $\int_{-1}^{1} x^n\,dx$ for n even and for n odd.

3 Use graphs to relate your integration results to the shapes of the functions.

4 Investigate further by changing the limits of integration to –5 and 5.

5 Write a report on your findings, including the integrations, graphs and predictions.

You can use your CAS calculator to calculate the area under a curve. This is especially useful when the zeros of the function are not easily found or not exact.

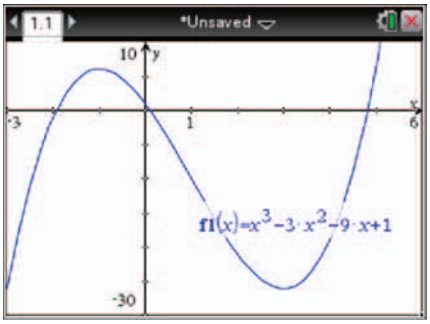
Example 12

CAS Find the physical area that is cut off the curve $f(x) = x^3 - 3x^2 - 9x + 1$ by the x-axis.

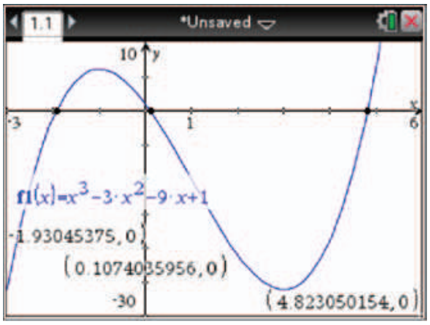
Solution

TI-Nspire CAS

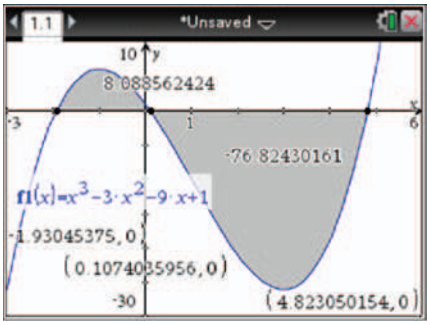
Use a Graph page.
Draw the graph and change the window settings to suitable values.
Use menu, 9: Settings to change the accuracy to, say, 10 digits.



Use menu, 6:Analyse Graph and 1:Zero to find the zeros in succession. The calculator will ask you to position the point to lower and upper bounds within which to find each zero. Grab each zero label and move them out of the way.

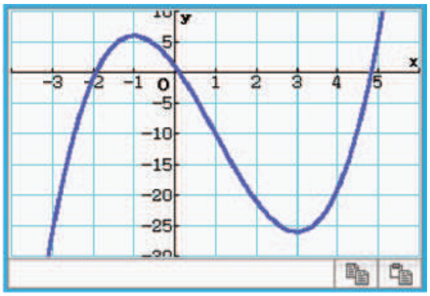


Press menu, 6:Analyse Graph and 7:Integral to find the first area. When asked for the lower bound, position the pointer at the first zero so that it says 'Intersection point' to get the exact point.
Complete with the upper bound and repeat for the second area.



ClassPad

Use the Graph&Table menu. Draw the graph and change the View Window settings to suitable values.



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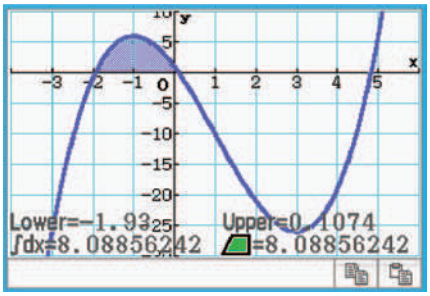
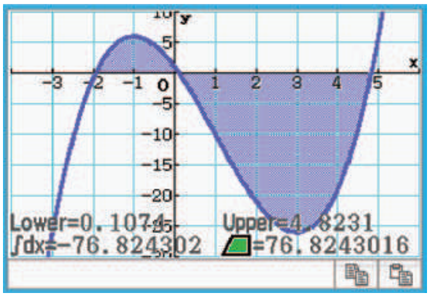
To find the first area, tap Analysis, G-solve, Integral, and ∫dx root. The smallest zero will be found to start the integral. Tap $\boxed{\text{EXE}}$ to confirm and use the right arrow to move to the next zero for the upper bound.



Repeat for the second area, but when asked for the lower bound, use the right arrow to go to the second root before confirming.



Add the physical areas.

Physical area ≈ 8.088 562… + 76.824 301…

Write the answer.

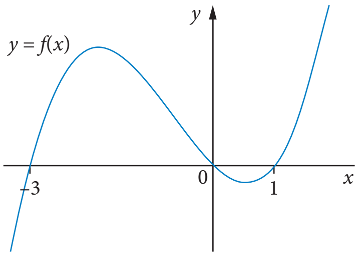
The physical area cut off the curve $f(x) = x^3 - 3x^2 - 9x + 1$ by the x-axis is about 84.91 square units.

EXERCISE 6.04 Physical areas

WS Calculating physical areas

Concepts and techniques

Use the diagram on the right to answer Questions **1** and **2**.



1 Example 10 The area between the graph of $f(x)$, the x-axis and the lines $x = -2$ and $x = -1$ is equal to:

A $\int_1^2 f(x)dx$ B $\int_{-1}^{-2} f(x)dx$

C $\int_2^1 f(x)dx$ D $\int_{-2}^{-1} f(x)dx$

E $\int_{-3}^1 f(x)dx$

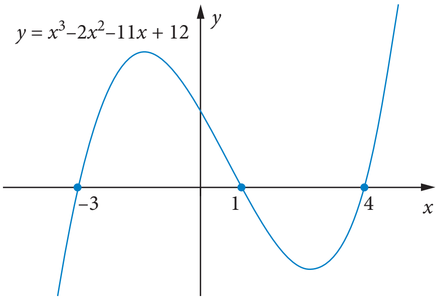
2 The area between the graph of $f(x)$, the x-axis and the lines $x = -3$ and $x = 1$ is equal to:

A $\int_0^{-3} f(x)dx + \int_0^1 f(x)dx$ B $\int_{-3}^0 f(x)dx - \int_0^1 f(x)dx$

C $\int_{-3}^1 f(x)dx$ D $\int_{-3}^0 f(x)dx + \int_0^1 f(x)dx$

E $\int_{-3}^0 f(x)dx + \int_0^1 f(x)dx$

Use the diagram on the right to answer Questions **3** to **5**.



3 The area between the graph of $y = x^3 - 2x^2 - 11x + 12$ and the x-axis from $x = -3$ to $x = 1$ is equal to:

A $15\frac{2}{3}$ B $21\frac{1}{3}$ C $23\frac{1}{3}$

D $51\frac{1}{3}$ E $53\frac{1}{3}$

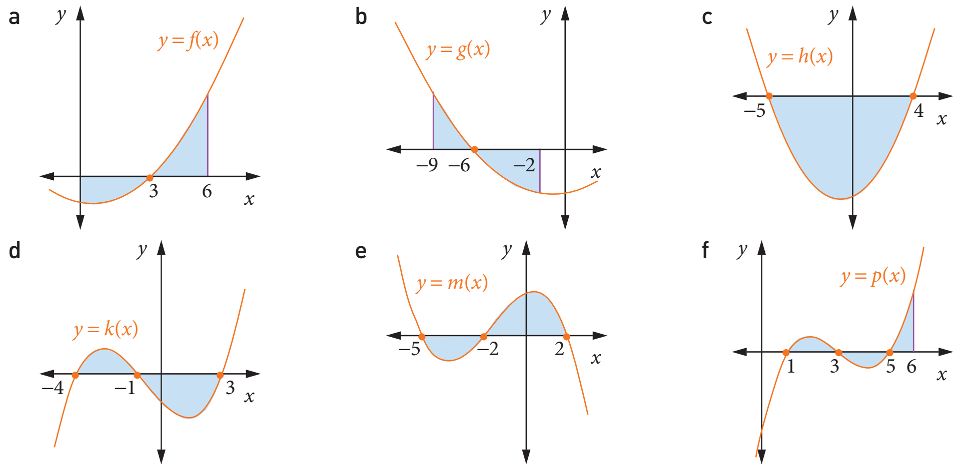
4 The area between the graph of $y = x^3 - 2x^2 - 11x + 12$ and the x-axis from $x = 1$ to $x = 4$ is equal to:

A $15\frac{1}{4}$ B $20\frac{1}{2}$ C $21\frac{1}{4}$ D $24\frac{3}{4}$ E $25\frac{1}{2}$

5 The area between the graph of $y = x^3 - 2x^2 - 11x + 12$ and the x-axis from $x = -3$ to $x = 4$ is equal to:

A $-48\frac{5}{12}$ B $-31\frac{1}{2}$ C $28\frac{7}{12}$ D $75\frac{3}{4}$ E $78\frac{1}{12}$

6 For each of the following, state how the shaded physical area can be calculated using integrals.



7 Find the area between the x-axis and each of the following curves for the domain given.

a $y = x^2 - 5x + 8$ from $x = 1$ to $x = 4$

b $f(x) = 15 + 8x - 6x^2$ between $x = -1$ and $x = 2$

c $y = 4x^3 - 3x^2 + 6x - 2$ between $x = 1$ and $x = 3$

d $f(x) = 6x^3 + 8x^2 - 2x + 8$ from $x = -1$ to $x = 2$

e $y = 7 - 6x^3$ from $x = -2$ to $x = 1$

8 Find the area enclosed by $x = 2$, $x = -2$, the x-axis and the curve $f(x) = e^x - e^{-x}$.

9 Find the area under the curve $y = (2x+1)^{-\frac{1}{3}}$ from $x = 0$ to $x = 13$.

10 Find the physical area enclosed by:

a $y = 2x^2 + 3x - 35$, $x = 2$, $x = 5$ and $y = 0$

b $y = 7x^2 + 19x - 6$, $x = -5$, $x = -2$ and the x-axis.

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11 Example 11 Find the area enclosed by:

a $f(x) = 3x^2 - 12$ and the x-axis

b $y = 3x^2 + 7x - 6$ and the x-axis

c $f(x) = 4x^2 - 16x + 15$ and the x-axis.

12 Find the physical area cut off:

a $y = (5 - x)(x + 1)(x - 3)$ by the x-axis

b $y = (x + 2)^2(x - 1)(x + 4)$ by the x-axis.

13 Find the area enclosed by $y = \sin(3x)$ and the x-axis between $x = 0$ and $x = 2\pi$.

14 Find the area enclosed by $y = e^{2x} + e^{-2x}$, $y = 0$, $x = -1.5$ and $x = 1.5$.

15 Find the area enclosed between the curve $y = \dfrac{2}{(x-3)^2}$, the x-axis and the lines $x = 0$ and $x = 1$.

16 Find the area enclosed between the curve $y = \sqrt{4 - x^2}$, the x-axis and the y-axis in the first quadrant.

17 Find the exact area enclosed between the curve $y = e^{4x-3}$, the x-axis and the lines $x = 0$ and $x = 1$.

18 Find the area enclosed between the curve $y = x + e^{-x}$, the x-axis and the lines $x = 0$ and $x = 2$ correct to 2 decimal places.

19 Find the exact area bounded by the curve $y = \cos(3x)$, the x-axis and the lines $x = 0$ and $x = \dfrac{\pi}{12}$.

20 Example 12 CAS Find an approximation for the area cut off:

a $y = 6x - 4 - x^2$ by the x-axis

b $y = 5x^2 - x^3 - 2x - 8$ by the x-axis

c $y = x^3 + 3x^2 - 10x - 3$ by the x-axis.

Reasoning and communication

21 Find the physical area enclosed by:

a $y = x^2 - x - 6$, $x = 1$, $x = 5$ and $y = 0$

b $y = 2x^2 + 7x - 30$, $x = -2$, $x = 4$ and the x-axis.

22 Find the area enclosed by each of these curves and the x-axis.

a $y = x^2 - 7x + 10$

b $y = 2x^2 + 13x + 15$

23 Find the physical area cut off:

a $y = (x + 2)(x - 2)(x - 4)$ by the x-axis

b $y = (2x + 7)(x + 1)(3 - x)$ by the x-axis

c $y = (x - 2)(x - 3)^2(x - 4)(x - 6)$ by the x-axis.

24 The marginal profit (the rate of change of profit) of a manufacturer of fencing systems is given by $M(n) = 400(1 - 4e^{-0.015n})$, where n is the number of systems sold and $M(n)$ is in dollars. Use integration to find the profit made when 500 systems are sold.

25 The force required to compress a powerful spring is given by $F = 600\,000x$ newtons, where x is the amount of compression (in metres). Use the integral $\int F dx$ to find the energy stored in the spring when it is compressed by 5 cm. Note that if the compression is in metres and the force is in newtons (N), the energy obtained is in joules (J).

6.05 AREAS BETWEEN CURVES

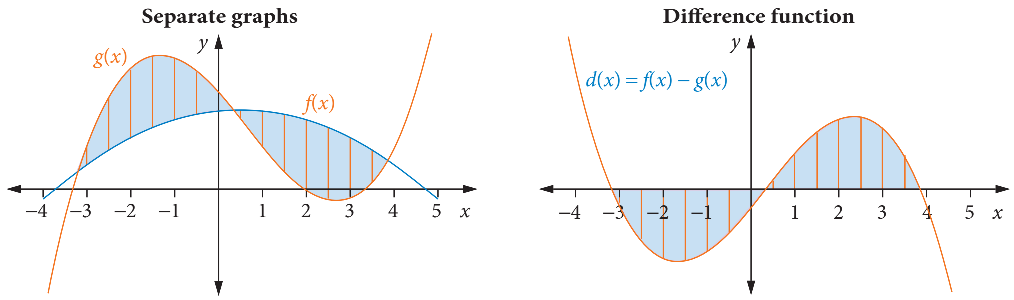
The area between a curve and the x-axis is sometimes called the *area under the curve*.

Sometimes, however, the area *between two curves* or enclosed by two curves is required. For an enclosed area, it may be necessary to find the intersection points of the curves. The area can be calculated in two ways:

- as the difference between the areas under the two functions; or
- as the area under the **difference function**, because
  $\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$

The intersections of the functions correspond to the zeros of the difference function.

The diagram below shows this geometrically using area strips.



In problems of this type it is a good idea to sketch the functions or plot them on your CAS calculator to make sure that you can see the area you are calculating. You need only do enough to find the information you need, which would not usually include stationary points.

Example 13

Find the area cut off $f(x) = x^2 - 6x + 13$ by $x - y + 3 = 0$.

Solution

$f(x) = x^2 - 6x + 13$ is a quadratic function with a minimum. The zeros are not obvious, so create a table to find some points.

$f(x) = x^2 - 6x + 13$

x	−1	1	3	5	7
$f(x)$	20	8	4	8	20

Find the x-intercept of $x - y + 3 = 0$.

At $y = 0$, $x = -3$ so the x-intercept is $(-3, 0)$.

Find the y-intercept of $x - y + 3 = 0$.

At $x = 0$, $y = 3$ so the y-intercept is $(0, 3)$.

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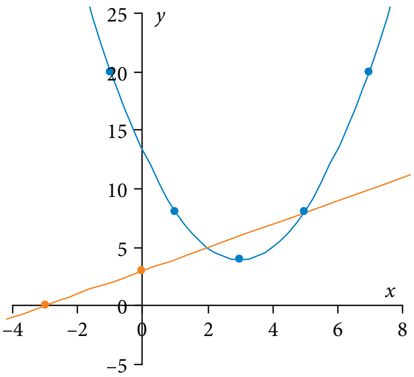
Sketch the functions.



State the result. | The lines apparently intersect.

The lines look as if they intersect at $x = 2$ and $x = 5$. Test to check. | $f(2) = 5$ and $2 - 5 = 3 = 0$ so one intersection is at (2, 5). The other is at (5, 8).

Find the area by subtracting the integrals of the lower from the higher.

$$A = \int_2^5 (x+3)\,dx - \int_2^5 (x^2 - 6x + 13)\,dx$$

Integrate.

$$= \left[\frac{x^2}{2} + 3x\right]_2^5 - \left[\frac{x^3}{3} - 3x^2 + 13x\right]_2^5$$

Evaluate.

$$= 19\frac{1}{2} - 15$$

$$= 4\frac{1}{2}$$

Write the answer. | The area cut off $f(x) = x^2 - 6x + 13$ by $x - y + 3 = 0$ is $4\frac{1}{2}$ square units.

While you could establish the intersections in Example **13** by checking your observations, this is not usually enough. It is usually quicker to use the difference function than to find intersections anyway.

Example 14

Find the area between the curves $f(x) = 12 - x^2$ and $g(x) = x^2 + 1$ and the lines $x = -1$ and $x = 2$.

Solution

Find the difference function.

$$d(x) = x^2 + 1 - (12 - x^2)$$

Simplify.

$$= 2x^2 - 11$$

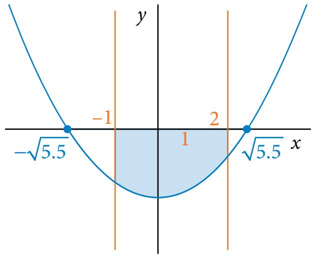
Find the zeros.

$$2x^2 - 11 = 0$$

Write the zeros.

$$x = \pm\sqrt{5.5} \approx \pm 2.35$$

Sketch the graph and lines.
Shade the area needed.



Write the necessary integral.

$$\int_{-1}^{2}(2x^2-11)dx$$

Integrate.

$$=\left[\frac{2x^3}{3}-11x\right]_{-1}^{2}$$

Evaluate.

$$=-16\tfrac{2}{3}-10\tfrac{1}{3}=-27$$

Write the answer in the form of the question.

The area between $f(x) = 12 - x^2$, $g(x) = x^2 + 1$, $x = -1$ and $x = 2$ is 27 square units.

A CAS calculator is useful when the functions are more complex.

Example 15

CAS Find the area between the curves $f(x) = (x + 1)(x - 1)(x - 2)$ and $g(x) = 4x^2 - 28$.

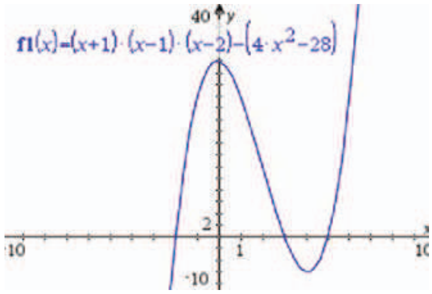
Solution

Find the difference function.

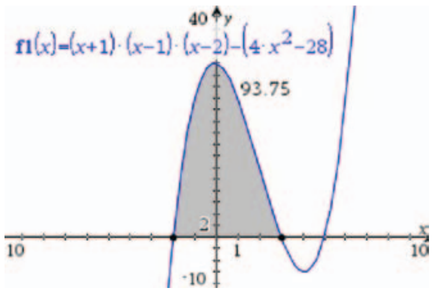
$d(x) = (x + 1)(x - 1)(x - 2) - (4x^2 - 28)$

TI-Nspire CAS

Use a Graph page.
Draw the graph of the difference function and adjust the scales. You don't need to simplify the function.



To find the first area, use menu, 6: Analyse Graph and 7: Integral. For the lower bound, move the pointer to the first intersection and press enter when the words 'intersection point' are displayed. Do the same for the upper bound.



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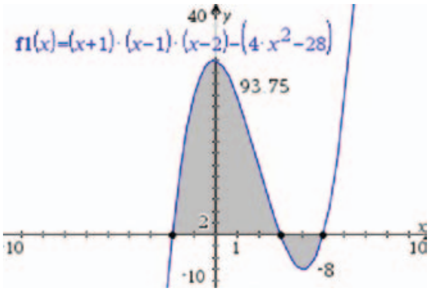
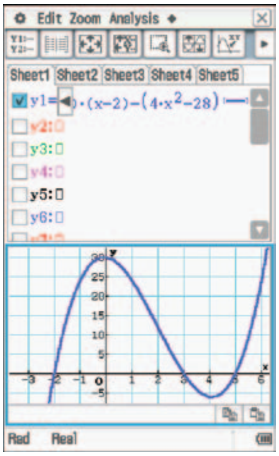
Use the same procedure for the second area.

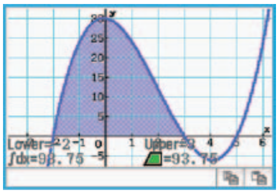


ClassPad

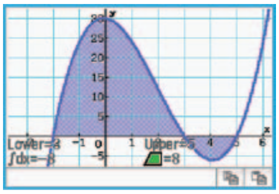
Use the Graph&Table menu.
Draw the graph of the difference function and adjust the scales using View Window.
You don't need to simplify the function.



Tap Analysis, G-Solve, Integral and ∫dx Root. Choose lower and upper bounds and find the first area.



Use the same procedure for the second area.



Calculate the total physical area.

$$\text{Area} = 93.75 + 8$$
$$= 101.75$$

Write the answer.

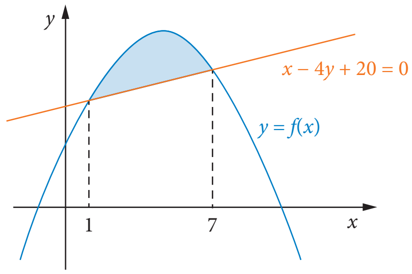
The area between the curves $f(x) = (x + 1)(x - 1)(x - 2)$ and $g(x) = 4x^2 - 28$ is $101\frac{3}{4}$ square units.

EXERCISE 6.05 Areas between curves

Concepts and techniques

WS Calculating areas between curves

1 Example 13 What expression is used to find the shaded area on the right?



A $\int_1^7 f(x)dx + \int_1^7 (4x-5)dx$

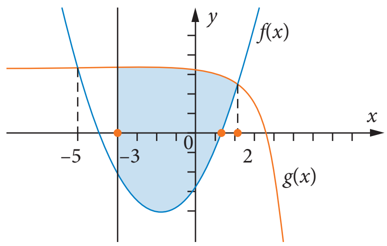
B $\int_1^7 f(x)dx - \int_1^7 (4x-5)dx$

C $\int_1^7 f(x)dx - \int_1^7 (\frac{1}{4}x+5)dx$

D $\int_1^7 f(x)dx - \int_1^7 (\frac{1}{4}x-5)dx$

E $\int_1^7 f(x)dx - \int_1^7 (x+20)dx$

2 Example 14 The area bounded by the curves $f(x)$, $g(x)$ and the line $x = -3$ is equal to:



A $\int_{-3}^2 g(x)dx + \int_{-3}^2 f(x)dx$

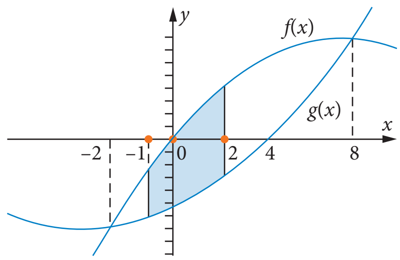
B $\int_{-3}^2 [g(x)-f(x)]dx$

C $\int_{-3}^2 [g(x)+f(x)]dx$

D $\int_{-5}^2 [g(x)-f(x)]dx$

E $\int_{-5}^2 f(x)dx - \int_{-5}^2 g(x)dx$

3 Which one of the following is equal to the shaded area shown?



A $\int_{-2}^8 f(x)dx - \int_{-2}^8 g(x)dx$

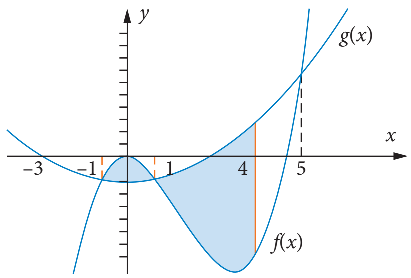
B $\int_{-1}^2 [f(x)+g(x)]dx$

C $\int_{-1}^2 f(x)dx + \int_{-1}^2 g(x)dx$

D $\int_{-1}^2 f(x)dx - \int_{-1}^2 g(x)dx$

E $\int_{-2}^8 f(x)dx + \int_{-2}^8 g(x)dx$

4 The shaded area shown here is equal to:



A $\int_{-1}^1 [f(x)-g(x)]dx + \int_1^4 [g(x)-f(x)]dx$

B $\int_{-1}^4 [g(x)-f(x)]dx$

C $\int_{-1}^1 [f(x)-g(x)]dx + \int_1^4 [f(x)-g(x)]dx$

D $\int_{-1}^5 [g(x)-f(x)]dx$

E $\int_{-1}^1 [f(x)-g(x)]dx - \int_1^4 [g(x)-f(x)]dx$

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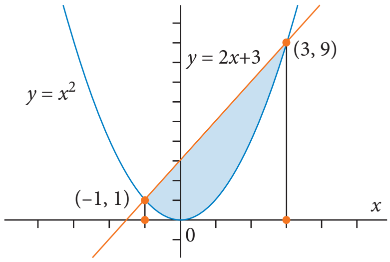
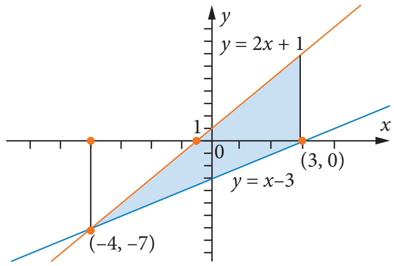
5 Find the area enclosed between the curve $y = x^2$ and the line $y = x + 6$.

6 Example 13 Calculate each of the shaded areas shown below.

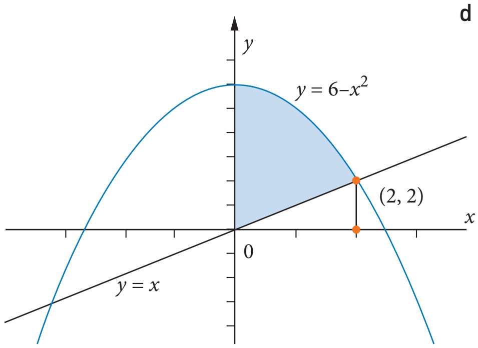
a



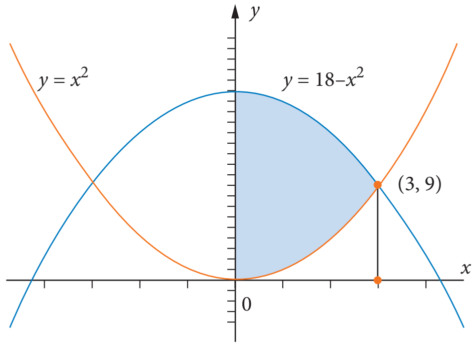
b



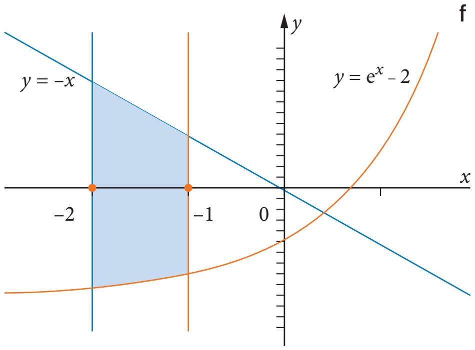
c



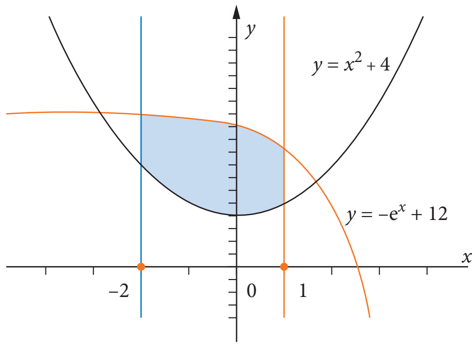
d



e



f



7 Find the area enclosed between the line $y = 2$ and the curve $y = x^2 + 1$.

8 Find the area bounded by the curve $y = 9 - x^2$ and the line $y = 5$.

9 Find the area enclosed between the curve $y = x^2$ and the line $y = -6x + 16$.

10 Find the area cut off:

a $y = 2x^2 - 12x + 20$ by $2x + y = 12$

b $f(x) = 2x + 8 - x^2$ by $y = 2x - 1$

c $f(x) = 4x^2 + 24x + 26$ by $y = 4x + 26$

d $f(x) = 4x^2 + 4x + 5$ by $y = 5 - 2x$

e $y = 4x^2 + 12x + 8$ by $y = 4x + 13$.

11 Find the area enclosed by the curves $y = x^2$ and $y = x^3$.

12 Example 14 Find the area bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$.

13 Example 15 CAS Find the area enclosed by the curves $y = (x - 1)^2$ and $y = (x + 1)^2$ and the x-axis.

14 CAS Find the area enclosed by the curve $y = x^3$, the x-axis and the line $y = -3x + 4$.

15 CAS Find the area enclosed by the curves $y = (x - 2)^2$ and $y = (x - 4)^2$, and the x-axis.

16 Find the exact area enclosed by the curve $y = \sqrt{4 - x^2}$ and the line $x - y + 2 = 0$.

17 CAS Find the areas between the following curves and lines.

a $f(x) = 3 - x^3$, $g(x) = x^2 - 9$, $x = -3$ and $x = 1$

b $f(x) = 2x^2 - 8$, $g(x) = x^2 - 2x + 6$, $x = -4$ and $x = 2$

c $f(x) = (x + 2)(x - 2)^2$ and $y = x + 2$

d $y = (x + 1)(x - 1)(x - 2)$ and $y = 2 + 5x - x^2$

e $y = 3x^2 + 2x - 21$ and $y = (x + 1)(x + 3)(x - 3)$

Reasoning and communication

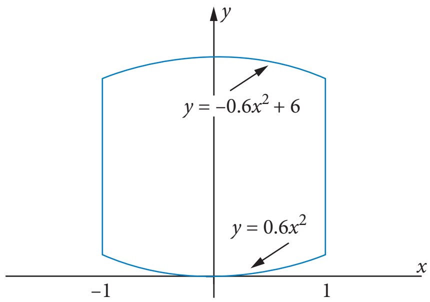
18 Find the area enclosed by the curves $y = x^2$ and $x = y^2$.

19 A cam shaft has a cross-section that is a good approximation to the area between the curves

$$y = 0.75x^2 - 3.75x + 6 \text{ and } y = 5x - 1 - x^2$$

where x is in centimetres. The shaft is 32 cm long. The shaft is made of high-tensile steel with a density of 7920 kg/m^3. Find the mass of the shaft.

20 The diagram shown on the right represents a piece of glass that has been cut from a rectangular piece of glass measuring 6 m by 2 m. All dimensions are shown in metres. What is the area of glass that was discarded from the rectangular sheet?



21 The bed of a river can be approximated by the part of the function

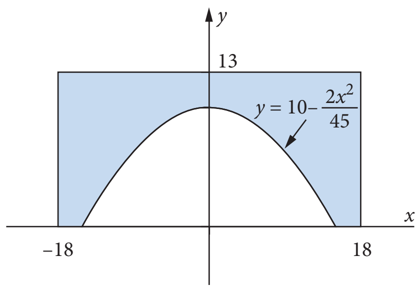
$$d = -0.007x(0.45x - 11)^2$$

cut off by the x-axis, where x is the distance in metres from one bank at water level. During normal conditions, the speed of the water is 30 cm/s. Find the amount of water in litres that flows past a point on the riverbank in half an hour.

22 The diagram on the right represents a cross-section of a concrete bridge. All dimensions are shown in metres.

a What is the area of the cross-section?

b If the bridge is 25 m wide, what is the volume of concrete it contains?



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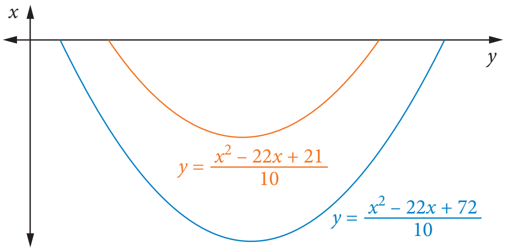
23 A sweeping 'circular' driveway actually has two parabolas as its edges to allow room for parking near the house. The x-axis is the edge of the roadway, the driveway lies between the curves $y=\dfrac{x^2-22x+21}{10}$ and $y=\dfrac{x^2-22x+72}{10}$ (in metres)



Find the area of this driveway and hence the cost of concreting to a depth of 15 cm, with concrete costing $\$350/\text{m}^3$.

6.06 TOTAL CHANGE

You have previously seen that the fundamental theorem of calculus states that:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is a primitive function of $f(x)$.

This can be restated a slightly different way using the fact that $f(x)$ is the derivative of $F(x)$.

Using, $f(x) = F'(x)$, we can state the fundamental theorem of calculus as:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

If we consider $F'(x)$ as the rate of change of a quantity, then $\int_a^b F'(x)dx$ is the total or net change in the quantity as x changes from a to b.

This can be stated as the total (or net) change theorem.

IMPORTANT

Total change theorem

The definite integral of the rate of change of a quantity, $F'(x)$, gives the total change in that quantity.

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Example 16

The rate of change of temperature in degrees Celsius per minute for a cup of coffee is given by:

$$F'(t) = 10e^{-0.25t}$$

where t is in minutes.

What is the total change in the coffee temperature between $t = 1$ and $t = 8$?

Solution

Write the integral of the rate of change function.

$$\int_1^8 F'(t)dt = \int_1^8 10e^{-0.25t}(t)dt$$

Integrate.

$$= \left[-40e^{-0.25t}\right]_1^8$$

Evaluate.

$$= -40e^{-2} - (-40e^{-0.25})$$

Factorise.

$$= 40(e^{-0.25} - e^{-2})$$

Evaluate.

$$= 40(0.7788\ldots - 0.1353\ldots)$$
$$\approx 25.7$$

State the result.

The total temperature change of the coffee between $t = 1$ and $t = 8$ is about 25.7°C.

Example 17

Following the Second World War, there was a significant increase in the birth rates among the western countries. If it is assumed that the rate of births in millions of babies per year for the post war years is approximated by:

$$B'(t) = 2t + 5 \text{ for } 0 \le t \le 15$$

a how many babies were born in the first 15 years after the war?

b how long did it take for the number of babies born after the war to reach 104 million?

Solution

a Write the integral of the birth rate function, $B'(t)$.

$$\int_0^{15} B'(t)dt = \int_0^{15} (2t+5)dt$$

Integrate.

$$= \left[t^2 + 5t\right]_0^{15}$$

Evaluate.

$$= 300$$

State the result.

300 million babies were born in the first 15 years after the war.

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b We want to know the value of t when the total number of babies born was 104 million.

Let T be the number of years after the war for 104 million babies to be born.

Write the integral.

$$\int_0^T (2t+5)dt = 104$$

Integrate.

$$\left[t^2+5t\right]_0^T = 104$$

Evaluate.

$$T^2+5T = 104$$

Rearrange.

$$T^2+5T-104 = 0$$

Factorise.

$$(T-8)(T+13) = 0$$

Solve.

$T = 8$ or $T = -13$.

Disregard any inapplicable solutions.

$T = -13$ is a false solution.

State the answer.

It took 8 years for 104 million babies to be born after the war.

In business and economics, **marginal** cost, profit, revenue, etc. is defined as the cost, profit or revenue for the last item or unit. It is approximated by the rate of change, so the derivative is taken to be the marginal quantity. This means that the total cost, profit, revenue, etc. is the integral of the marginal amount or marginal rate of change.

Example 18

A factory manufactures precision high-pressure valves. The marginal cost, $C'(x)$, and marginal revenue, $R'(x)$ are given below.

$$C'(x) = \frac{6}{5}x+5$$

$$R'(x) = 6x - \frac{3}{5}x^2 + 62$$

where the marginal cost and marginal revenue are both measured in dollars.

Calculate the total profit from the production and sale of the first 20 valves.

Solution

Find the total cost for the production of the first 20 valves.

$$\int_0^{20} C'(x)dx = \int_0^{20}\left(\frac{6}{5}x+5\right)dx$$

Integrate.

$$= \left[\frac{6x^2}{5\times 2}+5x\right]_0^{20}$$

$$= \left[\frac{3}{5}x^2+5x\right]_0^{20}$$

Evaluate.

$$= \frac{3\times 20^2}{5}+100-0$$

$$= 340$$

Find the total revenue for the sale of the first 20 valves.	$\int_0^{20} R'(x)dx = \int_0^{20}\left(6x - \frac{3}{5}x^2 + 62\right)dx$
Integrate.	$= \left[\frac{6x^2}{2} - \frac{3x^3}{5\times 3} + 62x\right]_0^{20}$
	$= \left[3x^2 - \frac{1}{5}x^3 + 62x\right]_0^{20}$
Evaluate at the limits.	$= 3\times 20^2 - \frac{20^3}{5} + 62\times 20 - 0$
	$= 840$
Use the rule for total profit.	$P(x) = R(x) - C(x)$
Evaluate.	$= 840 - 340$ $= 500$
State the answer.	The total profit from the production and sale of 20 valves is \$500.

In the previous example you could have integrated the difference function $P'(x) = R'(x) - C'(x)$ to calculate the total profit instead of calculating the total revenue and cost.

EXERCISE 6.06 Total change

WS Total change

Concepts and techniques

1 Example 16 Oil is leaking from a tank. The rate of leakage (in litres/hour) is given by:

$$\text{Rate of leakage} = 1000e^{-0.1t}$$

where t is in hours.
The total change in the volume of oil in the tank (in L) after the first 5 hours is:

A $1000e^{-0.5}$ B $\int_0^5 1000e^{-0.1t}\,dt$ C 606
D $\int_1^5 1000e^{-0.1t}\,dt$ E $1000e^{-0.5} - 1000e^0$

2 Example 17 The rate of increase in the population of lizards in a closed environment is given by:

$$P'(t) = 6 + \sqrt{10t}$$

where t is in days. The total change in the population of lizards after the first 10 days is:

A 16 B $\left[6+\sqrt{10t}\right]_0^{10}$ C $\int\left(6+\sqrt{10t}\right)dt$
D $\left[6t + \frac{2\sqrt{10}\,t^{\frac{3}{2}}}{3}\right]_0^{10}$ E $\left[6t + (10t)^{\frac{3}{2}}\right]_0^{10}$

3 If $H'(t)$ is the rate of change in the height of a conical pile of fertiliser measured in centimetres per hour, what does $\int_0^{10} H'(t)dt$ represent?

4 If $B'(t)$ represents the growth rate of the number of bacteria in a culture, where t is the time in hours, what does $\int_2^8 B'(t)dt$ represent?

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5 $P'(t)$ is the rate of growth of a population of kangaroos in an isolated area, measured in kangaroos per year. There were 150 kangaroos in the population in the year 2013 ($t = 0$). Write an integral expression that represents the population in 2017.

Shutterstock.com/Jacqui Martin

6 Fluid leaks from a storage facility at the rate:

$$F'(t) = 3500e^{-0.4t} \text{ litres/day}$$

where t is measured in days.

a How much fluid will leak in the first 5 days ($0 \le t \le 5$)?

b How much fluid will leak in the next 5 days ($5 \le t \le 10$)?

c Why does less fluid leak from the facility in the second five-day period?

7 The population of rabbits in a certain confined location is known to increase according to the function:

$$P'(t) = 4t + 1$$

where $P'(t)$ is measured in hundreds of rabbits per month and t is measured in months. The measurement of the population commences at $t = 0$.

a What is the total change in the population in the first 10 months after measuring commenced?

b How long did it take for the increase in the population of rabbits to reach 2100?

8 Example 18 The marginal cost function (in thousands of dollars) for manufacturing x components per year is given by:

$$C'(x) = 25 - \frac{1}{2}x$$

Calculate the total cost for the production of the first 50 components.

9 The marginal revenue function (in thousands of dollars) for the sale of x hundred units of a product is given by:

$$R'(x) = 12 - 3x^2 + 4x$$

Find the total revenue function and calculate the total revenue from the sale of the first 400 units.

Reasoning and communication

10 Water flows into a tank at a rate given by:

$$W'(t) = \frac{1}{75}(20t - t^2 + 600)$$

where $W'(t)$ is measured in L/hour and t is in hours. Initially there are 200 L of water in the tank. How many litres of water are in the tank after 24 hours?

11 A company manufactures toy cars. It is known that the marginal cost and revenue functions can be approximated as follows.

$$R'(x) = 10 - 0.002x$$
$$C'(x) = 2$$

There are fixed costs of \$7000 regardless of the number of toy cars produced.
Determine the total cost and total revenue functions and calculate the total profit for the first 1000 toy cars produced.

12 A manufacturer produces components. The marginal cost (\$) of producing x units is:

$$C'(x) = 5 + 16x - 3x^2$$

If the total cost of producing 5 units is \$500, find the total cost function.

13 The marginal cost, $C'(x)$ measured in dollars, of a certain product is known to be a constant multiple of the number of units (x) produced. The manufacturing process involves a fixed cost of \$5000 regardless of the number of units produced. If the cost of producing 24 units is \$5144, find the total cost function.

6.07 APPLICATION OF INTEGRATION TO MOTION

You have already studied the relationships between position, velocity and acceleration in differential calculus. You should remember that if x, v and a are the position, velocity and acceleration of an object, then $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$. You can reverse these to obtain the following.

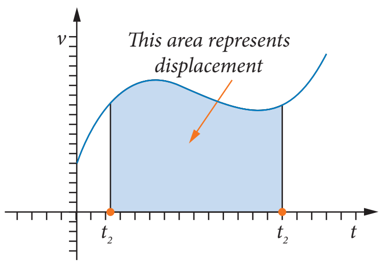
IMPORTANT

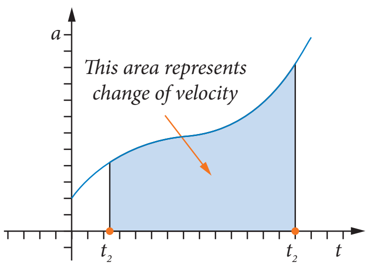
If x, v and a are the position, velocity and acceleration of an object then

$$x = \int v\,dt \qquad \text{and} \qquad v = \int a\,dt.$$

The displacement (change of position) between times T_1 and T_2 is given by $\int_{T_1}^{T_2} v\,dt$ and the change of velocity is given by $\int_{T_1}^{T_2} a\,dt$.

Using the fact that the definite integral is an area under a curve, you can interpret displacement and velocity using graphs as shown below.





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Example 19

The velocity of a vehicle increases from rest to 80 km/h in 30 seconds under constant acceleration. Find the displacement in metres of the vehicle during this time.

Solution

Express the velocity in m/s.

$$80 \text{ km/h} = \frac{80 \times 1000}{60 \times 60} \text{ m/s}$$
$$= 22\tfrac{2}{9} \text{ m/s}$$

Calculate the acceleration.

$$a = \frac{22\frac{2}{9}}{30} \text{ m/s}^2$$
$$= \frac{22}{27} \text{ m/s}^2$$

Find the velocity at time t.

$$v = \int_0^t \frac{22}{27}\,dt$$
$$= \frac{22}{27}t$$

Integrate to find the displacement.

$$\Delta x = \int_0^{30} \frac{22}{27}t\,dt$$

Integrate.

$$= \left[\frac{11}{27}t^2\right]_0^{30}$$

Evaluate.

$$= \frac{1100}{3}$$
$$= 333.333\ldots \text{ m}$$

State the result.

The displacement is about 333 m.

Many situations involve motion subject to gravity. In these cases, acceleration due to gravity must be taken into account. For practical purposes, acceleration due to gravity (g) is taken to be 9.8 m/s^2 acting towards the centre of the Earth. This means that if you consider upwards as positive, then, $g = -9.8$ m/s^2, but if downwards is positive, then $g = 9.8$ m/s^2.

Getty Images

Example 20

An object is propelled vertically upwards from 2 m above the ground at 25 m/s.

a Find the velocity after 2 seconds.

b When does the object first come to rest?

c Find the position of the object after 3 s.

Solution

a Find the change of velocity, with up as the positive direction.

$$\text{Change of velocity} = \int_0^2 -9.8\,dt = -19.6$$

Find the velocity.

$$v = 25 + -19.6 = 5.4 \text{ m/s}$$

State the result.

After 2 seconds, the object has a velocity of 5.4 m/s upwards.

b Write an expression for v at time t.

$$v(t) = 25 + \int_0^t -9.8\,dt = 25 - 9.8t$$

Find when the velocity is 0.

$$25 - 9.8t = 0$$

Solve for t.

$$t = \frac{25}{9.8} \approx 2.55 \text{ s}$$

State the answer.

The object comes to rest after about 2.55 s.

c Find the displacement.

$$\text{Displacement} = \int_0^3 (25 - 9.8t)\,dt$$

Find the integral.

$$= \left[25t - 4.9t^2\right]_0^3$$

Evaluate.

$$= 30.9 \text{ m}$$

Find the position.

$$\text{Position} = 2 + 30.9 = 32.9$$

State the answer.

After 3 seconds the object is 32.9 m above the ground.

When air resistance is taken into account, the acceleration of objects falling under gravity is not constant. In fact, they eventually reach a terminal velocity where the forces of gravity and air friction are balanced. For a human body, the terminal velocity is about 200 km/h (56 m/s). Parachutes are designed to reduce the terminal velocity to about 8 m/s. The equations for velocity and displacement under these conditions are exponential functions.

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Example 21

The velocity of a particle moving in a straight line is given by:

$$\frac{dx}{dt} = 20 - 8e^{-0.4t}$$

Calculate the total distance travelled by the particle in the first 3 seconds.

Solution

State the equation for velocity. $\frac{dx}{dt} = 20 - 8e^{-0.4}$

Integrate to find displacement. $x = \int_0^3 (20 - 8e^{-0.4t})\,dt$

Find the integral. $= \left[20t - \frac{8e^{-0.4t}}{-0.4}\right]_0^3$

Simplify. $= \left[20t + 20e^{-0.4t}\right]_0^3$

Evaluate. $= 60 + 20e^{-1.2}$

≈ 66

State the answer. The displacement of the particle after 3 s is about 66 m.

EXERCISE 6.07 Application of integration to motion

WS

Displacement, velocity and acceleration

Concepts and techniques

1 Example 19 If $v(t)$ is the velocity of a particle moving along the x-axis, measured in metres per second, what does $\int_2^{10} v(t)\,dt$ represent?

2 An object is initially at rest. It is then subject to a constant acceleration of 6 m/s^2. Calculate its displacement after 4.1 s.

3 The velocity (cm/s) of a particle moving horizontally to the right of the origin is given by $v = 3t^2 + 2t + 1$. If the particle is initially 2 cm to the left of the origin, find the displacement of the particle after 5 s and its position relative to its initial position.

4 The acceleration of a particle is given by $a = -9 \sin(3t)$ cm/s^2. If the initial velocity is 5 cm/s and the particle is 3 cm to the left of the origin, find its exact position after π s.

5 The velocity of an object is given by $v = 4\cos(2t)$ m/s. If the object is 3 m to the right of the origin after π s, find the exact:

a position after $\frac{\pi}{6}$ s

b acceleration after $\frac{\pi}{6}$ s

6 Example 20 An object is propelled vertically upwards with an initial velocity of 20 m/s from the top of a cliff that is 300 m high. Consider the height relative to the bottom of the cliff.

a Find its height after 5 seconds.
b What is the velocity of the object after 5 seconds?
c When does the object reach its greatest height?

7 A ball is thrown vertically upwards from the ground at 14 m/s.

a How high does it go?
b What is the time taken for the ball to hit the ground?
c What is the velocity of the ball when it hits the ground?

8 A stone is dropped from a bridge over a river. The splash from the stone hitting the river is heard 2.5 seconds after it is dropped. How high is the bridge, correct to two decimal places?

9 A rocket is launched with an initial velocity of 2.1×10^3 m/s upwards. How high does it rise before returning to the ground?

10 Example 21 A particle accelerates according to the equation $a = -e^{2t}$ (cm/s^2). If the particle is initially at rest, find its displacement after 4 seconds, correct to the nearest centimetre.

11 The acceleration (m/s^2) of an object is given by $a = e^{3t}$. The particle is initially at the origin with velocity –2 m/s. Find its displacement after 3 seconds, correct to 3 significant figures.

12 The acceleration of a particle is given by $a = 25e^{5t}$ m/s^2 and its velocity is 5 m/s initially. If its initial position is 1 m to the right of the origin:

a find its velocity after 9 seconds
b find its position after 6 seconds.

Reasoning and communication

13 A manilla folder falls from an aircraft and has a velocity given by

$$v(t) = \frac{g}{2}(1 - e^{-2t})$$

How far does it fall in the first 100 s?

14 A person can jump 2 m high on Earth.

a What is the initial velocity of the jumper?
b How high could the same person jump on the moon if the acceleration due to gravity on the moon is 1.6m/s^2, assuming they started with the same velocity?

15 The velocity of an object is given by $v = t^2(t^3 + 1)$ cm/s and the object is initially 2 cm to the right of the origin.

a Find its acceleration after 1 second.
b Find the position of the object after 2 seconds.

16 A weight at the end of a spring accelerates according to the equation:

$$a = \cos^2\left(t + \frac{\pi}{4}\right) - \sin^2\left(t + \frac{\pi}{4}\right)$$

Initially the weight is at rest at the origin.

a Find the velocity of the weight after $\frac{\pi}{2}$ s.
b Find its displacement after $\frac{\pi}{4}$ s.
c Find the times at which the weight has a velocity of $-\frac{1}{2}$ cm/s.

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CHAPTER SUMMARY
APPLICATIONS OF INTEGRATION

6

- The **indefinite integral** of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$ and is written as $\int f(x)dx$. It is usually written as $\int f(x)dx = F(x) + c$, where c is an arbitrary constant called the **constant of integration**, because indefinite integrals differ only by a constant.
- An indefinite integral is also called an **antiderivative** or a **primitive**.
- The indefinite integrals of basic functions are:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int kx^n\,dx = \frac{kx^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int \sin(x)dx = -\cos(x) + c$$

$$\int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a} + c$$

$$\int \cos(x)\,dx = \sin(x) + c \qquad \int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + c$$

$$\int e^x\,dx = e^x + c \qquad \int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + c$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad (n \neq -1)$$

$$\int kf'(x)dx = kf(x) + c$$

- The **linearity of integration** refers to the properties below.

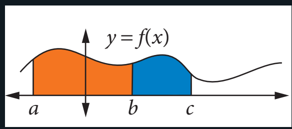
Linear combination: $\int [af(x)+bg(x)]dx = a\int f(x)dx + b\int g(x)dx$

Constant multiple: $\int kf(x)dx = k\int f(x)dx$

Sum of functions: $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

Difference of functions: $\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$

- The substitution of different values for the constant of integration leads to the formation of a **family of functions** that are vertical translations of each other.
- The properties of definite integrals include:



$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

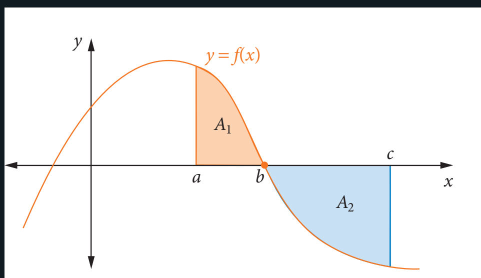
$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

- The **signed** or **algebraic area** between a curve and the x-axis is given by the integral. It is the difference between the areas above and below the x-axis.
- The **physical area** between a curve and the x-axis is always positive. Areas above and below the x-axis must be calculated separately and negative areas are subtracted.



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- The area can be calculated in two ways:
  - as the difference between the areas under the two functions:
    $$\int_a^b f(x)dx - \int_a^b g(x)dx$$
  - as the area under the **difference function**:
    $$\int_a^b [f(x)-g(x)]dx$$
- The **total change theorem** states that the definite integral of the rate of change of a quantity, $F'(x)$, gives the total change in that quantity.
  $$\int_a^b F'(x)dx = F(b)-F(a)$$
- The **marginal** value of a quantity such as cost, revenue or profit is the cost, profit or revenue for the last item or unit. It is normally taken as being the rate of change of the quantity.
- If x, v and a are the position, velocity and acceleration of an object, then $x = \int v\,dt$ and $v = \int a\,dt$.
- The displacement (change of position) between times T_1 and T_2 is given by $\int_{T_1}^{T_2} v\,dt$ and the change of velocity is $\int_{T_1}^{T_2} a\,dt$
  $$\int_{T_1}^{T_2} v(t)dt = [x(t)]_{T_1}^{T_2} = x(T_2) - x(T_1)$$

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CHAPTER REVIEW
APPLICATIONS OF INTEGRATION

6

Multiple choice

1 Example 1 The function $f(x)$ such that $f'(x) = 15x^2 - 3$ and $f(x) = 6$ when $x = 1$ is:

A $5x^3 - 3x$ B $5x^3 - 3x - 1$ C $30x$

D $\frac{15x^3}{2} - 3x^2 + 6$ E $5x^3 - 3x + 4$

2 Example 2 $\int 12x\sqrt{x}\,dx =$

A $6x\sqrt{x} + c$ B $4x\sqrt{x} + c$ C $2.4x\sqrt{x} + c$

D $8x^2\sqrt{x} + c$ E $4.8x^2\sqrt{x} + c$

3 Example 4 $\int (3x^3 - 5x + 2)dx$ is equal to:

A $\int 3x^3\,dx - \int 5x\,dx + \int 2\,dx$ B $3x^3 - 5x + \int 2\,dx$ C $\int (3x^3 - 5x)dx + 2$

D $\int 3x^3 dx - 5x + 2$ E $3x^3 - \int (5x + 2)dx$

4 Example 5 Given that $\frac{dy}{dx} = \frac{mx^3}{2} + 3x$ and $y = 4$ when $x = 0$, an expression for y is:

A $\frac{mx^4}{4} + 3x^2 + 1$ B $2mx^4 + 3x^2 + 1$ C $\frac{mx^4}{8} + \frac{3x^2}{2} + 4$

D $2mx^4 + \frac{3x^2}{2} - x - 2$ E $ax^3 - ax + 2a$

5 Example 6 $\int x^3(4x + 3)dx$ is equal to:

A $\int (4x^4 + 3x^3)dx$ B $\int x^3\,dx \int (4x + 3)dx$ C $(4x + 3)\int x^3\,dx$

D $x^3 \int (4x + 3)dx$ E $\int (x^3 + 4x + 3)dx$

6 Example 9 $\int_0^{\frac{\pi}{6}} \cos(3x)dx =$

A -1 B $-\frac{1}{3}$ C 0 D $\frac{1}{3}$ E 1

7 Example 10 The algebraic area (signed area) between the graph of $f(x)$, the x-axis and the lines $x = -3$ and $x = 2$ is equal to:

A $\int_2^{-3} f(x)dx$ B $\int_{-2}^{-3} f(x)dx$ C $\int_{-3}^{0} f(x)dx - \int_0^2 f(x)dx$

D $\int_{-3}^{2} f(x)dx$ E $-\int_{-3}^{0} f(x)dx + \int_0^2 f(x)dx$

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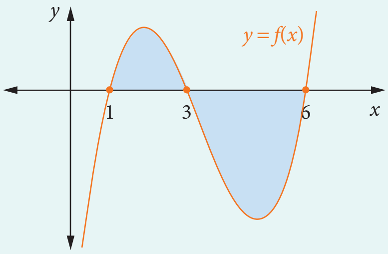
8 Example 11 The shaded area shown here is equal to:



A $\int_1^6 f(x)dx$

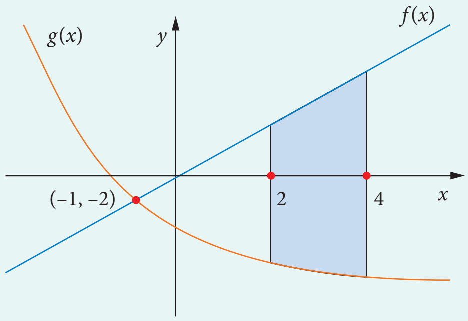
B $-\int_1^6 f(x)dx$

C $\int_1^3 f(x)dx - \int_3^6 f(x)dx$

D $\int_1^3 f(x)dx + \int_3^6 f(x)dx$

E $\int_1^6 f(x)dx - \int_3^6 f(x)dx$

9 Example 13 Which one of the following is equal to the shaded area shown?



A $\int_{-1}^4 f(x)dx - \int_{-1}^4 g(x)dx$

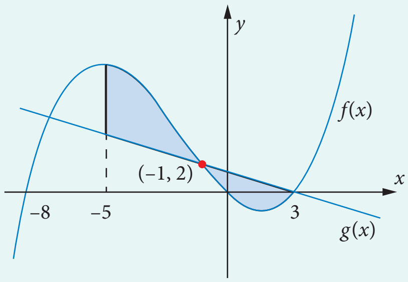
B $\int_2^4 [f(x) - g(x)]dx$

C $\int_2^4 [f(x) + g(x)]dx$

D $\int_2^4 f(x)dx + \int_2^4 g(x)dx$

E $\int_2^4 [g(x) - f(x)]dx$

10 Example 13 The shaded area shown here is equal to:



A $\int_{-5}^{-1} [g(x) - f(x)]dx + \int_{-1}^3 [f(x) - g(x)]dx$

B $\int_{-5}^3 [f(x) - g(x)]dx$

C $\int_{-5}^{-1} [f(x) - g(x)]dx + \int_{-1}^3 [g(x) - f(x)]dx$

D $\int_{-8}^3 [g(x) - f(x)]dx$

E $\int_{-5}^{-1} [f(x) - g(x)]dx - \int_{-1}^3 [g(x) - f(x)]dx$

11 Example 16 Water is flowing out of a tank. The flow rate (in litres/hour) is given by:

$$R'(t) = 100e^{-0.2t}$$

where t is in hours.

The total change in the volume of water in the tank (in L) after the first 3 hours is:

A $\int_0^3 100e^{-0.2t}\,dt$ B $500e^{-0.6} - 500e^0$ C 274

D $500e^{-0.6}$ E $\int_1^3 100e^{-0.2t}\,dt$

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Short answer

12 Example 1–3 Find the indefinite integral of each of the following, giving answers with positive indices.

a $\int (y^3 - 3y^2 + 4y + 1)\,dy$ b $\int -n^{-2}\,dn$ c $\int \frac{-2}{x^2}\,dx$

d $\int (9x^2 - 2)(3x^3 - 2x + 4)\,dx$ e $\int \sin(x)\,dx$ f $\int -3\cos(6x)\,dx$

g $\int -5\sin(10x)\,dx$ h $\int e^{3t}\,dt$ i $\int \frac{3}{e^{2x}}\,dx$

j $4(x-5)^{-3}$ k $\frac{1}{3(2x+7)^4}$ l $\sqrt{4x+7}$

13 Example 4 Find each indefinite integral.

a $\int (x^4 + 7)\,dx$ b $\int (5x^4 - 2x^3 + 4x)\,dx$ c $\int (6x^3 - 8x^2 - 3)\,dx$

14 Example 5 Find y in terms of x if $\frac{dy}{dx} = 6x - 4$ and $y = 22$ when $x = -2$.

15 Example 5 Find the equation of the curve $f(x)$ given that $f'(x) = \frac{3}{2\sqrt{x}}$ and the curve passes through the point $(1, 5)$.

16 Example 6 Find each indefinite integral.

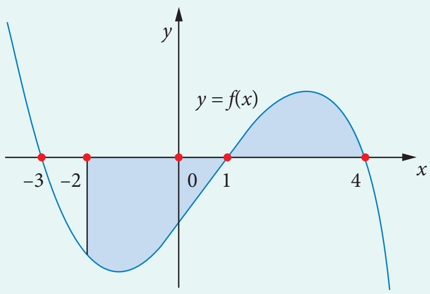
a $\int \frac{x^5 - 3x^3 + 7x}{x}\,dx$ b $\int (2-3x)^2\,dx$ c $\int \frac{3x^2 - 5x + 2}{\sqrt{x}}\,dx$

17 Examples 7–9 Evaluate the following.

a $\int_{-1}^{2} (12x^2 - 6x + 1)\,dx$ b $\int_{1}^{9} x^{\frac{1}{2}}\,dx$ c $\int_{5}^{10} \frac{dx}{(x-4)^2}$

d $\int_{0}^{\frac{\pi}{3}} \sin(3x)\,dx$ e $\int_{0}^{\frac{\pi}{4}} \sin\left(2x + \frac{\pi}{3}\right)dx$ f $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 12\cos(3x)\,dx$

18 Example 10 Express the shaded area in terms of definite integrals.



19 Example 10 Find the area under each of the following curves for the domain given.

a $y = x^2 - 7x - 8$ from $x = -2$ to $x = 6$

b $f(x) = e^x + e^{-x}$ from $x = -1$ and $x = 2$

c $y = (1-2x)^{\frac{1}{3}}$ from $x = -1$ to $x = 3$.

20 Example 11 Use your graphics calculator to find an approximation for the area cut off $y = x^3 - 3x^2 - 8x + 9$ by the x-axis.

21 Example 11 Find the area enclosed by $f(x) = 3(x-2)^2$, $x = -1$ and $x = 1$.

22 Example 12 CAS Find the area under the curve $y = 8x^3 - 3x^2 + 6x$ between $x = 3$ and $x = 7$.

23 Example 12 CAS Find the physical area cut off $y = (x+3)(x-1)(x-2)$ by the x-axis.

24 Example 13 Find the area enclosed between the curve $y = x^2 - 2x - 2$ and the line $y = x + 2$.

25 Example 14 Find the area enclosed between the curves $y = 3x^2 - 8x - 3$ and $y = 2x^2 - 5x + 7$.

26 Example 15 CAS Find the area between the curves $f(x) = e^x$, $g(x) = 16 - x^2$ between $x = -1$ and $x = 1$.

27 Example 16 If $C'(t)$ is the rate of temperature change of a hot liquid measured in °C per minute, what does $\int_0^5 C'(t)\,dt$ represent?

28 Example 16 Fluid leaks from a vessel at the rate:

$$V'(t) = 150e^{-0.2t} \text{ litres/hour}$$

where t is measured in hours.

a How much fluid will leak in the first 3 hours ($0 \leq t \leq 3$)?

b How much fluid will leak in the next 3 hours ($3 \leq t \leq 6$)?

29 Example 17 The population of mice in a closed habitat is known to increase according to the function:

$$P'(t) = \frac{t}{3} + 6$$

where $P'(t)$ is measured in hundreds of mice per month and t is measured in months. The measurement of the population commences at $t = 0$.

a What is the total change in the population in the first 3 months after measuring commenced?

b How long will it take for the increase in the population of mice to reach 4200?

30 Example 18 The marginal revenue function (in dollars) for the sale of x units of a product is given by:

$$R'(x) = 1500 - 3x^2 - 4x$$

Find the total revenue function and calculate the total revenue from the sale of the first 30 units.

31 Example 19 The acceleration (m/s^2) of a particle is given by $a = 6t - 12$. If the particle is initially at rest 2 m to the left of the origin, find the displacement of the particle after 5 s and its position relative to its initial position.

32 Example 20 An object is propelled vertically upwards with an initial velocity of 30 m/s.

a Find its height after 4 seconds.

b What is the velocity of the object after 5 seconds?

c When does the object reach its greatest height?

33 Example 21 A particle accelerates according to the equation $a = -20(1+2t)^2$ (cm/s^2), where t is in seconds. If the particle has an initial velocity of 30 cm/s, find an expression for the velocity as a function of time.

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Application

34 The diagram to the right shows the graph of the function $y = -e^{\frac{x}{4}}$. PQ is the perpendicular to the curve at the point P. Find:

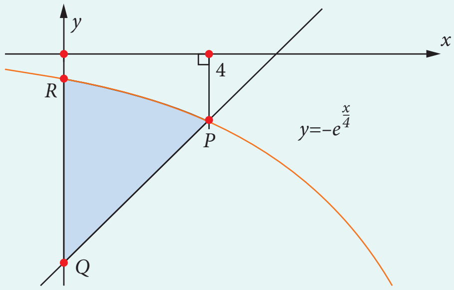
a the coordinates of P

b the equation of PQ

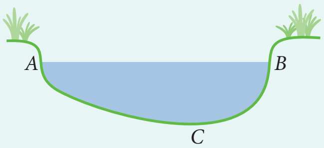
c the coordinates of Q

d the coordinates of R.

e the physical area bound by the curve and the normal and the y-axis.



35 The cross-section of a river bed is shown at right. The deepest part of the river is towards one side, at C, 6 m from bank B and 12 m from bank A. The water level is the line AB. If the origin of a coordinate system is placed at A, then the curve from A to C can be written as



$$y = \frac{x^2}{24} - x$$

and the curve from C to B can be written as

$$y = \frac{x^2}{6} - 4x + 18$$

where both x and y are in metres.

a Find the maximum depth of the river.

b Find the cross-sectional area of the river.

c If the water flows at an average rate of 1.4 m/s, how much water flows in 1 s?

d Find the amount of water that flows in a day.

Vertical levee banks that allow for a further height of water of 1.5 m are placed at A and B. During floods, the flow rate increases to 2.5 m/s and the water rises up the levees.

e Find the ratio of the flow of water in a flood to the flow at the height AB, if the flood reaches the top of the levee banks.

36 The shape of a *shade sail* is formed by three intersecting parabolas. The parabolas have equations

$$y = x^2 - 8x + 16,\ y = x^2 + 8x + 16 \text{ and } y = 2 - \frac{x^2}{8}$$

Find the area of the sail if the dimensions are in metres.

37 The acceleration of a particle is given by $a = 3t + 1$ m/s^2, where t is in seconds. If the particle is initially at the origin and moving with a velocity of 15 m/s:

a find its velocity after 3 s, correct to 1 decimal place

b show that the particle is never at rest.

38 Find the exact area enclosed between the curve $y = x\sqrt{x^2 - 1}$, the x-axis and the lines $x = 1$ and $x = 2$.

39 A driveway sweeps from a narrow entrance to a double garage with its doors perpendicular to the road (the x-axis). The driveway can be modelled as the area between the positive axes and the curves

$$y=\frac{12-x^2}{4} \text{ and } y=\frac{36-x^2}{4} \text{ (in metres)}$$

Find the area of this driveway and the cost of concreting to a depth of 10 cm with concrete costing \$325/m^3.

40 A canvas sheet has two opposite edges sewn to allow steel poles to be run through the edges. The canvas is then held up flat by poles and guy ropes to make a shelter. Unfortunately, when it rains water pools on top of the shelter and stretches the canvas so that it sags in the middle. While the curve is actually a type of 3D catenary, the side-to-side cross-section can be approximated by the quadratic

$$y=\frac{x(4-x)h}{400}$$

where h is the amount of sag in the centre, and all are in metres. The canvas will rip if there is more than 5 kg of water on top. The shelter is 4 m wide and 6 m long and water weighs 1000 kg/m^3. Find the maximum sag that can be allowed before the canvas rips.

41 A water trough is a prism with a parabolic cross-section. The trough is 3 m long and the cross-section corresponds to the part of

$$y=\frac{5x^2-400x+350}{90}$$

that is below the x-axis, where the units are in centimetres. Find the area of the cross-section and the volume of water that can be contained in the trough.

Qz Practice quiz

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7

TERMINOLOGY

asymptote
base
common logarithm
decibel
exponent
exponential
index (indices)
logarithm
Naperian logarithm
natural logarithm
pH scale
power
Richter scale

Prior learning

THE LOGARITHMIC FUNCTION

LOGARITHMIC FUNCTIONS

LOGARITHMIC FUNCTIONS

- define logarithms as indices: $a^x = b$ is equivalent to $x = \log_a(b)$, i.e. $a^{\log_a(b)} = b$ (ACMMM151)
- establish and use the algebraic properties of logarithms (ACMMM152)
- recognise the inverse relationship between logarithms and exponentials: $y = a^x$ is equivalent to $x = \log_a(y)$ (ACMMM153)
- interpret and use logarithmic scales such as decibels in acoustics, Richter Scale for earthquake magnitude, octaves in music, pH in chemistry (ACMMM154)
- solve equations involving indices using logarithms (ACMMM155)
- recognise the qualitative features of the graph of $y = \log_a(x)$ $(a > 1)$ including asymptotes, and of its translations $y = \log_a(x) + b$ and $y = \log_a(x + c)$ (ACMMM156)
- solve simple equations involving logarithmic functions algebraically and graphically (ACMMM157)
- identify contexts suitable for modelling by logarithmic functions and use them to solve practical problems (ACMMM158)

CALCULUS OF LOGARITHMIC FUNCTIONS

- define the natural logarithm $\ln(x) = \log_e(x)$ (ACMMM159)
- recognise and use the inverse relationship of the functions $y = e^x$ and $y = \ln(x)$ (ACMMM160)
- establish and use the formula $\frac{d}{dx}\ln(x) = \frac{1}{x}$ (ACMMM161)
- establish and use the formula $\int \frac{1}{x}\,dx = \ln(x) + c$ for $x > 0$ (ACMMM162)
- use logarithmic functions and their derivatives to solve practical problems (ACMMM163)

7.01 INDICES AND LOGARITHMS

You already know that indices are useful for large numbers; $3^{16} = 43\,046\,721$. This means that when you multiply 3 by itself 16 times, you get 43 046 721.

What if you wanted to write it the other way around? This is done using a **logarithm**. You would write $\log_3(43\,046\,721)$, which means the power of 3 needed to get 42 046 721, so $\log_3(43\,046\,721) = 16$.

What is $\log_5(625)$?

This means the power of 5 needed to get 625.

Since $5^4 = 625$, $\log_5(625) = 4$.

The logarithm is actually just the index, not the whole power, so you should ask 'How many factors of 5 do I need to multiply?'

$5 \times 5 \times 5 \times 5 = 625$ so $5^4 = 625$ so $\log_5(625) = 4$.

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IMPORTANT

The **logarithm** of b to the **base** a, where a, $b > 0$ and $a \neq 1$ is the **exponent** x such that $a^x = b$.

In symbols: $a^x = b \Leftrightarrow \log_a(b) = x$, where a, $b > 0$ and $a \neq 1$.

x is also called the **index** of a^x and the expression is called the xth **power** of a.

The index statement $a^x = b$ means the same as the logarithm statement $\log_a(b) = x$. You say they are *equivalent*.

Example 1

Write the following in index form.

a $\log_2(64) = 6$

b $\log_7\left(\frac{1}{7}\right) = -1$

Solution

a Write the equivalence backwards.	$\log_a(b) = x \Leftrightarrow a^x = b$
Substitute $a = 2$, $b = 64$ and $x = 6$.	$\log_2(64) = 6 \Leftrightarrow 2^6 = 64$
Write the answer.	The index form of $\log_2(64) = 6$ is $2^6 = 64$
b Write the equivalence backwards.	$\log_a(b) = x \Leftrightarrow a^x = b$
Substitute $a = 7$, $b = \frac{1}{7}$ and $x = -1$.	$\log_7\left(\frac{1}{7}\right) = -1 \Leftrightarrow 7^{-1} = \frac{1}{7}$
Write the answer.	The index form of $\log_7\left(\frac{1}{7}\right) = -1$ is $7^{-1} = \frac{1}{7}$

Example 2

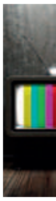
Write the following in logarithmic form.

a $5^2 = 25$

b $2^{-3} = \frac{1}{8}$

Solution

a Write the equivalence.	$a^x = b \Leftrightarrow \log_a(b) = x$
Substitute $a = 5$, $x = 2$ and $b = 25$.	$5^2 = 25 \Leftrightarrow \log_5(25) = 2$
Write the answer.	The logarithmic form of $5^2 = 25$ is $\log_5(25) = 2$
b Write the equivalence.	$a^x = b \Leftrightarrow \log_a(b) = x$
Substitute $a = 2$, $x = -3$ and $b = \frac{1}{8}$.	$2^{-3} = \frac{1}{8} \Leftrightarrow \log_2\left(\frac{1}{8}\right) = -3$
Write the answer.	The logarithmic form of $2^{-3} = \frac{1}{8}$ is $\log_2\left(\frac{1}{8}\right) = -3$



When you have to find the logarithm of a number to a particular base, you need to think 'What power of this will give the number?' or 'How many times do I need to multiply the base to get the number?'.

Example 3

Evaluate each of the following.

a $\log_4(64)$ **b** $\log_3(9)$ **c** $\log_{\frac{1}{2}}\left(\frac{1}{32}\right)$

d $\log_6\left(\frac{1}{216}\right)$ **e** $\log_{\frac{1}{5}}(25)$ **f** $\log_{96}(1)$

Solution

a	Write the problem.	$\log_4(64)$
	Think $4^{???} = 64$, $4 \times 4 \times 4 = 64$.	$4^3 = 64$, so $\log_4(64) = 3$
b	Write the problem.	$\log_3(9)$
	Think $3^{???} = 9$, $3 \times 3 = 9$.	$3^2 = 9$, so $\log_3(9) = 2$
c	Write the problem.	$\log_{\frac{1}{2}}\left(\frac{1}{32}\right)$
	Think $\left(\frac{1}{2}\right)^{???} = \frac{1}{32}, \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{32}$.	$\left(\frac{1}{2}\right)^5 = \frac{1}{32}$ so $\log_{\frac{1}{2}}\left(\frac{1}{32}\right) = 5$
d	Write the problem.	$\log_6\left(\frac{1}{216}\right)$
	Think $6^{???} = \frac{1}{216}$.	It must be negative.
	Think $6^{???} = 216$, $6 \times 6 \times 6 = 216$.	$6^{-3} = \frac{1}{216}$, so $\log_6\left(\frac{1}{216}\right) = -3$
e	Write the problem.	$\log_{\frac{1}{5}}(25)$
	Think $\left(\frac{1}{5}\right)^{???} = 25$	It must be negative.
	Think $5^{???} = 25$, $5 \times 5 = 25$.	$\left(\frac{1}{5}\right)^{-2} = 25$, so $\log_{\frac{1}{5}}(25) = -2$
f	Write the problem.	$\log_{96}(1)$
	Think $96^{???} = 1$.	$96^0 = 1$, so $\log_{96}(1) = 0$

You can see from Example **3** that if $a > 1$ and $b < 1$ or $a < 1$ and $b > 1$, then $\log_a(b)$ is negative. If both a and b are less than 1, then $\log_a(b)$ is positive. Since the zeroth power of any number is 1, the log of 1 is 0, no matter what the base is.

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EXERCISE 7.01 Indices and logarithms

Logarithms

Concepts and techniques

1 Example 1 Write the following in index form.

a $\log_5(25) = 2$ b $\log_4(16) = 2$ c $\log_5(125) = 3$ d $\log_2(16) = 4$
e $\log_3(3) = 1$ f $\log_7(49) = 2$ g $\log_2(128) = 7$ h $\log_5(1) = 0$

2 Write the following in index form.

a $\log_8(2) = \frac{1}{3}$ b $\log_4\left(\frac{1}{2}\right) = -\frac{1}{2}$ c $\log_4\left(\sqrt[4]{7}\right) = \frac{1}{4}$ d $\log_3\left(\frac{1}{\sqrt[3]{3}}\right) = -\frac{1}{3}$

e $\log_2\left(\frac{\sqrt{2}}{4}\right) = -\frac{3}{2}$ f $\log_a(b) = c$ g $\log_c\left(\sqrt{a}\right) = 3m$

3 Example 2 Write the following in logarithmic form.

a $7^2 = 49$ b $3^3 = 27$ c $2^4 = 16$
d $5^3 = 125$ e $11^0 = 1$ f $(2)^0 = 1$

4 Write the following in logarithmic form.

a $5^{-2} = \frac{1}{25}$ b $4^{-2} = \frac{1}{16}$ c $10^{-3} = \frac{1}{1000}$ d $\left(\frac{1}{3}\right)^4 = \frac{1}{81}$

e $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$ f $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ g $6^{\frac{2}{3}} = \sqrt[3]{36}$ h $7^{\frac{3}{5}} = \sqrt[5]{343}$

i $a^k = m$ j $b^3 = d$

5 Example 3 Evaluate the following.

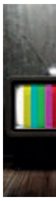
a $\log_2(64)$ b $\log_9(81)$ c $\log_3(81)$ d $\log_7(343)$
e $\log_6(216)$ f $\log_5(1)$ g $\log_3(3)$ h $\log_{10}(100\,000)$
i $\log_3(243)$ j $\log_4(1024)$

6 Evaluate the following.

a $\log_{\frac{1}{2}}\left(\frac{1}{16}\right)$ b $\log_5\left(\frac{1}{125}\right)$ c $\log_{\frac{1}{4}}(16)$ d $\log_{\frac{1}{4}}\left(\frac{1}{256}\right)$

e $\log_2\left(\frac{1}{128}\right)$ f $\log_{\frac{1}{2}}(512)$ g $\log_{\frac{1}{3}}\left(\frac{1}{81}\right)$ h $\log_7\left(\frac{1}{7}\right)$

i $\log_{\frac{1}{3}}(27)$ j $\log_{\frac{1}{10}}(0.001)$



Reasoning and communication

7 Evaluate each of the following.

a $\log_2\left(\sqrt{2}\right)$ b $\log_9\left(9\sqrt{9}\right)$ c $\log_4\left(\sqrt{64}\right)$ d $\log_7\left(\sqrt{343}\right)$ e $\log_6\left(\sqrt[3]{36}\right)$

8 Evaluate each of the following.

a $\log_4(8)$ b $\log_9(27)$ c $\log_8(16)$ d $\log_{64}(32)$ e $\log_{27}(243)$

9 If $x = \sqrt[m]{a^p}$ show that $\log_a\left(x^m\right) = p$

10 If $a = \frac{1}{\sqrt[y]{b^p}}$ show that $\log_b\left(a^{-y}\right) = p$

11 If $y = \sqrt[3a]{p^m}$ show that $\log_p\left(y^{3a}\right) = m$

7.02 PROPERTIES OF LOGARITHMS

In the last section, you found that $\log_5(1) = 0$, $\log_3(3) = 1$ and $\log_7(\frac{1}{7}) = -1$. In fact, it should be clear that will be true for any base, and this is easily proven from the definition of a logarithm.

IMPORTANT

The logarithm of 1 with any base is zero: $\log_a(1) = 0$ for $a > 0$ and $a \neq 1$.

The logarithm of any number to its own base is 1: $\log_a(a) = 1$ for $a > 0$ and $a \neq 1$.

The logarithm of the reciprocal of any number to its own base is –1: $\log_a\left(\frac{1}{a}\right) = -1$ for $a > 0$ and $a \neq 1$.

Example 4

Evaluate the following.

a $\log_3(1)$ b $\log_7(7)$ c $\log_5(0)$

d $\log_2(-3)$ e $\log_1(5)$ f $\log_{10}(0.1)$

g $\log_{0.25}(4)$

Solution

a The log of 1 to any allowable base is 0. $\log_3(1) = 0$

b The log of any number to its own base is 1. $\log_7(7) = 1$

c A number has to be positive to have a log ($b > 0$). $\log_5(0)$ is undefined

d A number has to be positive to have a log ($b > 0$). $\log_2(-3)$ is undefined.

e The base of a log cannot be 1 ($a \neq 0$). $\log_1(5)$ is undefined.

f The logarithm of the reciprocal of any number to its own base is –1, and $0.1 = \frac{1}{10}$. $\log_{10}(0.1) = -1$

g The logarithm of the reciprocal of any number to its own base is –1, and $4 = \frac{1}{0.25}$. $\log_{0.25}(4) = -1$

Consider the values of $\log_3(9)$, $\log_3(81)$ and $\log_3(729)$. Since $3^2 = 9$, $3^4 = 81$ and $3^6 = 729$, $\log_3(9) = 2$, $\log_3(81) = 4$ and $\log_3(729) = 6$.

Notice that $729 = 9 \times 81$ and $6 = 2 + 4$ so $\log_3(9 \times 81) = \log_3(9) + \log_3(81)$.

This is a consequence of the index law $\mathbf{a}^m \times \mathbf{a}^n = \mathbf{a}^{m+n}$, so it will work in all cases.

You can write the first index law in words as follows.

The product of two powers with the same base number is the same base number whose index is the sum of the other two indices.

Translating this to a log statement, you get the following.

IMPORTANT

First law of logarithms: logarithm of a product

The logarithm of a product is equal to the sum of the logarithms to the same base:

$$\log_a(xy) = \log_a(x) + \log_a(y) \text{ for } a, x, y > 0 \text{ and } a \neq 1.$$

This is proven as follows.

Let $m = \log_a(x)$ and $n = \log_a(y)$.

Then $a^m = x$ and $a^n = y$.

Now $xy = a^m \times a^n = a^{m+n}$.

In logarithmic form, this is $\log_a(xy) = m + n$.

But $m = \log_a(x)$ and $n = \log_a(y)$, so $\log_a(xy) = \log_a(x) + \log_a(y)$. QED

The index laws $\dfrac{a^m}{a^n} = a^{m-n}$ and $a^{nm} = (a^n)^m$ lead to the other log laws.

IMPORTANT

Second law of logarithms: logarithm of a quotient

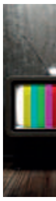
The logarithm of a quotient is equal to the difference of the logarithms to the same base:

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y) \text{ for } a, x, y > 0 \text{ and } a \neq 1.$$

Third law of logarithms: logarithm of a power

The logarithm of a power is equal to the product of the power and the logarithm:

$$\log_a(x^p) = p\log_a(x) \text{ for } a, x > 0 \text{ and } a \neq 1.$$



You can use the definition, laws and other properties of logarithms to simplify expressions involving logarithms.

Example 5

Simplify the following without using a calculator.

a $\log_4(\sqrt{4})$

b $\log_6(18) + \log_6(12) - 2$

c $\dfrac{\log_2(27)}{\log_2(81)}$

Solution

a Write $\sqrt{4}$ as a power. $\log_4(\sqrt{4}) = \log_4\left(4^{\frac{1}{2}}\right)$

Use the definition. $= \frac{1}{2}$

b Write the expression. $\log_6(18) + \log_6(12) - 2$

Use the product law in reverse. $= \log_6(18 \times 12) - 2$

Write as a product with 6s. $= \log_6(6 \times 3 \times 6 \times 2) - 2$

Write as a power. $= \log_6(6^3) - 2$

Use the definition. $= 3 - 2$

$= 1$

c Write the expression. $\dfrac{\log_2(27)}{\log_2(81)}$

Write in terms of powers. $= \dfrac{\log_2(3^3)}{\log_2(3^4)}$

Use the power law. $= \dfrac{3\log_2(3)}{4\log_2(3)}$

Cancel $\log_2(3)$. $= \frac{3}{4}$

Example 6

Write the expressions below as single logarithms.

a $3\log_2(x) - 4\log_2(x+3) + \log_2(y)$

b $2\log_5\left(\sqrt{x}\right) - \log_5(5x)$

Solution

a Write the expression. $3\log_2(x) - 4\log_2(x+3) + \log_2(y)$

Use the power law in reverse. $= \log_2(x)^3 - \log_2(x+3)^4 + \log_2(y)$

Use the product and quotient laws in reverse. $= \log_2\left(\dfrac{x^3y}{(x+3)^4}\right)$

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b	Write the expression.	$2\log_5\left(\sqrt{x}\right) - \log_5(5x)$
	Use the product and power laws.	$= \log_5\left[\left(\sqrt{x}\right)^2\right] - [\log_5(5) + \log_5(x)]$
	Simplify and use $\log_a(a) = 1$.	$= \log_5(x) - 1 - \log_5(x)$
	Simplify	$= -1$

EXERCISE 7.02 Properties of logarithms

WS Logarithm laws

Concepts and techniques

1 Example 4 Evaluate the following.

a $\log_3(1)$ **b** $\log_2(1)$ **c** $2\log_5(1)$ **d** $\log_x(1), x > 0$
e $[\log_3(1)]^2$ **f** $\log_3(3)$ **g** $\log_2(2)$ **h** $2\log_5(5)$
i $\log_x(x), x > 0$ **j** $5\log_a(a), a > 0$

2 Evaluate the following.

a $\log_7(0)$ **b** $\log_2(0)$ **c** $2\log_5(0)$ **d** $\log_y(0), y > 0$
e $\log_7(-1)$ **f** $\log_6(-7)$ **g** $\log_4(-x), x > 0$

3 Example 5 Simplify the following without using a calculator.

a $\log_2(64)$ **b** $\log_4(64)$ **c** $\log_3\left(\sqrt{3}\right)$ **d** $\log_5\left(5\sqrt{5}\right)$
e $\log_7\left(\frac{1}{\sqrt{7}}\right)$ **f** $4\log_a\left(\sqrt{a}\right)$ **g** $2\log_a(a^3)$

4 Example 6 Write as single logarithms.

a $\log_4(10) + \log_4(2) - \log_4(5)$ **b** $\log_5(25) + \log_5(125) - \log_5(625)$
c $\log_{27}\left(\frac{1}{9}\right) + \log_8(4)$ **d** $\log_2(16) + \log_2(4) + \log_2(8)$
e $\log_4(40) - \log_4(10) - \log_4(4)$ **f** $\log_5(8) - \log_5(4) + 2$
g $\log_8(2) - \log_8\left(\frac{1}{4}\right)$ **h** $\log_6(125) - \log_6(32) - \log_6\left(\frac{2}{5}\right)$

5 Evaluate, without the use of a calculator.

a $\dfrac{\log_6(16)}{\log_6(2)}$ **b** $\dfrac{\log_2(81)}{\log_2(27)}$ **c** $\dfrac{\log_3(81)}{\log_3(\frac{1}{3})}$ **d** $\dfrac{\log_7(2)}{\log_7(0.25)}$

6 Write as single logarithms.

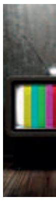
a $5\log_4(x) + \log_4\left(x^2\right) - \log_4\left(x^3\right)$ **b** $3\log_7(x) - 5\log_7(x) + 4\log_7(x)$
c $4\log_6(x) - \log_6\left(x^2\right) - \log_6\left(x^3\right)$ **d** $\log_2(x+2) + \log_2(x+2)^2$
e $\log_4[(x-1)^3] - \log_4[(x-1)^2]$ **f** $\log_3(x-3) + \log_3(x+3) - \log_3(x^2-9)$

7 Expand using the logarithmic laws.

a $\log\left(\frac{12a}{10}\right)$ **b** $\log_6\left[\left(\frac{a}{b}\right)^5\right]$ **c** $\log_3\left(\sqrt[5]{10x^3}\right)$ **d** $\log_4\left(\frac{\sqrt[3]{x^2a}}{y^2}\right)$

8 If $\log_6(3) = 0.613$, then correct to the nearest thousandth, find the approximate value of each of the following.

a $\log_6\left(\sqrt[4]{3}\right)$ **b** $\log_6(2)$ **c** $\log_6(108)$



Reasoning and communication

9 Given that $\log_p(7) + \log_p(k) = 0$, find k.

10 If $\log_a(x) = 4$ and $\log_a(y) = 5$, find the exact values of

a $\log_a(x^2y)$

b $\log_a(axy)$

c $\log_a\left(\frac{\sqrt{x}}{y}\right)$

11 Show that $\log_3\left(\sqrt[4]{\frac{x^2}{y^8z^6}}\right)$ can be expressed as $\frac{1}{2}\log_3(x) - 2\log_3(y) - \frac{3}{2}\log_3(z)$.

12 Use the definition of a logarithm to prove each of the following.

a $\log_a(1) = 0$ for $a > 0$ and $a \neq 1$

b $\log_a(a) = 1$ for $a > 0$ and $a \neq 1$.

13 Prove the quotient law of logarithms.

7.03 COMMON LOGARITHMS AND THE CHANGE OF BASE THEOREM

Logarithms were invented simultaneously by the Scotsman John Napier and Joost Burgi from Switzerland about the same time that decimal fractions were invented. Napier published his version in 1614 and then collaborated with the Englishman Henry Briggs to change them to base 10. The use of logarithms was taught in schools up until the 1970s for calculations, particularly for trigonometry.

IMPORTANT

A **common logarithm** (or **Briggsian logarithm**) is in base 10 and is normally written without the base, so $\log(16) = \log_{10}(16)$

Briggs published tables of common logarithms that were used for nearly 400 years for calculations until the invention of electronic calculators. Your calculator has two kinds of logarithms on it, common logarithms shown by log and **natural logarithms** shown by ln. If you want to find a logarithm to another base you need to use the **change of base** theorem.

IMPORTANT

Change of base theorem

The logarithm of a number to a given base is the quotient of the logarithms of the number and base to any other base: $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for $a, b, x > 0$ and $a, b \neq 0$.

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The change of base theorem is proven from the definition.

Let $y = \log_a(x)$.

Then $x = a^y$.

By taking logarithms of both sides, we get $\log_b(x) = \log_b(a^y)$.

So from the power rule of logarithms, $\log_b(x) = y\log_b(a)$

But $y = \log_a(x)$, so $\log_b(x) = \log_a(x) \times \log_b(a)$.

Divide by $\log_b(a)$ to get $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ QED

Example 7

CAS Find the value of $\log_5(27)$.

Solution

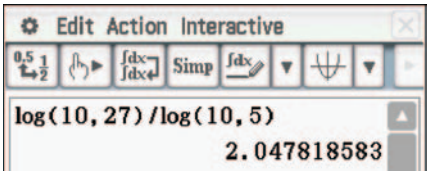
Use the change of base theorem. $\log_5(27) = \dfrac{\log(27)}{\log(5)}$

TI-Nspire CAS

Use a calculator page and set the calculation mode to approximate.
Press [ctrl] [10^x] to get log. Leave the base blank and press [enter].

ClassPad

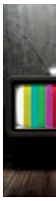
Use the Main menu and the Math1 menu. It is easiest to tap [$\log_{10}(\blacksquare)$], but note how the calculator writes base 10 logarithms.
$\log_{10}(27)$ is written as log(10, 27).



Write the answer. $\log_5(27) \approx 2.047$

You can use the change of base theorem to determine the value of an exponent when given two bases.

For example, you can solve simple equations involving indices using logarithms.



Example 8

CAS Solve the equation $(0.993)^x = 0.5$ correct to 4 significant figures.

Solution

Write the problem.	$(0.993)^x = 0.5$
Change to logarithms.	$x = \log_{0.993}(0.5)$
Use the change of base theorem.	$= \dfrac{\log(0.5)}{\log(0.993)}$
Use your calculator.	$= 98.674\ldots$
Write the answer.	$x \approx 98.67$

EXERCISE 7.03 Common logarithms and the change of base theorem

Concepts and techniques

1 Example 7 Express in terms of common logarithms.

a $\log_4(9)$ b $\log_8(6)$ c $\log_2(20)$
d $\log_7(200)$ e $\log_9(0.2)$

2 Find the values of each of the following, correct to 4 significant features.

a $\log_3(6)$ b $\log_{12}(2)$ c $\log_5(15)$
d $\log_{25}(4)$ e $\log_8(1.3)$

3 Example 8 Solve the following equations, correct to 4 significant figures.

a $2^x = 100$ b $4^x = 9$ c $5^x = 70$
d $(0.75)^x = 0.01$ e $(1.045)^x = 2$

4 Solve for x in the following equations, correct to 3 decimal places.

a $3^x = 5$ b $7^x = 14.3$ c $3^x = 15$
d $5^x = 100$ e $6^x = 4$

5 Solve the following equations, correct to 4 significant figures.

a $3^{x+1} = 85.7$ b $9^{4x+1} = 64$ c $5^{2x+1} = 32$
d $3^{7x-2} = 13$ e $6^{5-3x} = 17$

Reasoning and communication

6 If $\log_3(a) = b$ and $\log_a(2) = c$, find $\log_a(48)$.

7 Prove that $\log_a(b) = \dfrac{1}{\log_b(a)}$

8 Given that $3^x = 4^y = 12^z$, show that $z = \dfrac{xy}{x+y}$

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7.04 SOLVING EQUATIONS WITH LOGARITHMS

You may have to solve equations that are more involved than $2^x = 7$. You saw in the last section that you can do this by changing to logs and using the change of base theorem.

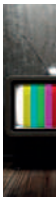
In most cases, you will need to use your general equation solving skills as well. You can rearrange some equations to find the value of a power such as 2^x. If you have to solve a quadratic as part of the process, you should check the solutions in the original equation.

Example 9

Solve $4^x = 2^{x+1} + 3$.

Solution

Write the equation.	$4^x = 2^{x+1} + 3$
Rewrite so that 2 is the only base.	$(2^2)^x = 2^{x+1} + 3$
Simplify.	$2^{2x} = 2^{x+1} + 3$
Write in terms of 2^x.	$(2^x)^2 = 2^x \times 2^1 + 3$
You may find it easier to substitute $a = 2^x$.	$a^2 = 2a + 3$
Write in standard form.	$a^2 - 2a - 3 = 0$
Factorise.	$(a - 3)(a + 1) = 0$
Solve for a.	$a = 3$ or $a = -1$
Substitute back 2^x.	$2^x = 3$ or $2^x = -1$
Eliminate the false solution.	But $2^x > 0$, so $2^x = 3$ only.
Write as logarithms.	$x = \log_2(3)$
Write the answer as a decimal.	≈ 1.584



In Example **9**, the same power was hidden in the equation by the fact that one base was a power of the other. Some equations involving several bases that are not powers of each other can still be solved by changing to only one power.

Example 10

Solve $3^{x+1} = 5^{x-4}$ and then evaluate, correct to four significant figures.

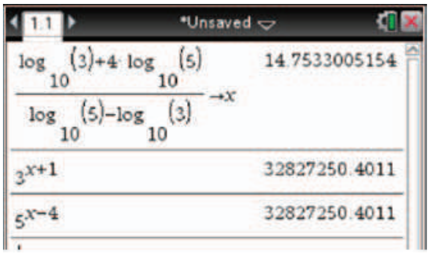
Solution

Write the equation.	$3^{x+1} = 5^{x-4}$
Write in terms of simple powers.	$3^x \times 3^1 = \dfrac{5^x}{5^4}$
Isolate the variables.	$\dfrac{5^x}{3^x} = 3 \times 5^4$
Express the variable as a single index.	$\left(\dfrac{5}{3}\right)^x = 3 \times 5^4$
Write as logarithms.	$x = \log_{\frac{5}{3}}(3 \times 5^4)$
Use the change of base theorem.	$= \dfrac{\log(3 \times 5^4)}{\log(\frac{5}{3})}$
Change to simpler logs.	$= \dfrac{\log(3) + 4\log(5)}{\log(5) - \log(3)}$
Evaluate.	$= 14.753\ldots$
Write the answer to the required accuracy.	$\dfrac{\log(3) + 4\log(5)}{\log(5) - \log(3)} \approx 14.75$

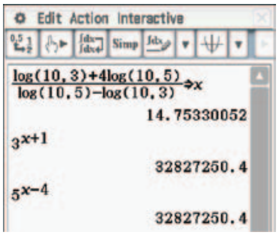
You could evaluate the expression $\dfrac{\log(3 \times 5^4)}{\log(\frac{5}{3})}$ without expressing it in the more elegant form $\dfrac{\log(3) + 4\log(5)}{\log(5) - \log(3)}$, but the latter is certainly preferred. You could also do this problem by taking common logs of both sides of the original equation or the equation $\dfrac{5^x}{3^x} = 3 \times 5^4$, and solving for x. Do you get the same answer?

You should use your CAS calculator to check the answers to problems like those in Examples **9** and **10** by substituting the answer into the original equation. Store the answer before you do the checking.

TI-Nspire CAS



ClassPad



You can also have equations that involve a logarithm of the variable. In this case you need to find a single logarithm and then use the definition. You may have to solve an equation to find a single logarithm or solve an equation after converting it to powers.

Example 11

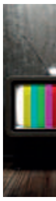
Solve the following, correct to 4 significant figures if necessary.

a $\log_3(x+4) - \log_3(x-2) = 2$

b $[\log(x)]^2 + \log(x) - 3 = 0$

Solution

a Write the equation.	$\log_3(x+4) - \log_3(x-2) = 2$
Write a single logarithm.	$\log_3\left(\dfrac{x+4}{x-2}\right) = 2$
Change to a power.	$\dfrac{x+4}{x-2} = 3^2 = 9$
Simplify.	$x + 4 = 9x - 18$
Solve.	$x = 2\frac{3}{4}$
b The equation is a quadratic in $\log(x)$.	$[\log(x)]^2 + \log(x) - 3 = 0$
Substitute $y = \log(x)$ to make it easier.	$y^2 + y - 3 = 0$
Write the quadratic formula.	$y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Substitute $a = 1$, $b = 1$ and $c = -3$	$= \dfrac{-1 \pm \sqrt{1^2 - 4\times1\times-3}}{2\times1}$
Use your calculator.	$y = -2.302\ldots$ or $y = 1.302\ldots$
Substitute $y = \log(x)$ back in.	$\log(x) = -2.302\ldots$ or $\log(x) = 1.302\ldots$
Use the definition.	$x = 10^{-2.302\ldots}$ or $x = 10^{1.302\ldots}$
Use your calculator again.	$x = 0.004\,979\ldots$ or $x = 20.080\ldots$
Use your calculator to check the solutions.	Both solutions work.
Write the answer.	$x \approx 0.0050$ or $x \approx 20.08$



Logarithms – Solving equations 1

Logarithms – Solving equations 2

EXERCISE 7.04 Solving equations with logarithms

Concepts and techniques

1 Example 9 Solve the following.

a $9^x + 3^x = 12$ b $5^{2x+1} + 5^x - 4 = 0$ c $1 + 6^{1-x} = 6^x$

d $2^{2x+1} + 20 = 3 \times 2^x$ e $11 \times 8^x - 30 = 8^{2x}$

2 Example 10 Find exact solutions to each of the following.

a $9^x = 5^{x+3}$ b $8^x = 49^{x-3}$ c $4^{x+5} = 350^{x-5}$

d $2^{3x} = 15^{x-1}$ e $7^{2x-1} = 17^{x+2}$

3 Solve the following, correct to 4 significant figures.

a $4^x = 7^{x-2}$ b $58^x = 4^{x+4}$ c $5^{x+2} = 46^{x-2}$

d $6^{2x} = 5^{x+3}$ e $28^{x+1} = 9^{2x-4}$

4 Example 11 Solve each of the following.

a $\log_3 (x-2) = 4$ b $\log (2x - 10) = 2$

c $\log_2 (2x + 12) - \log_2 (x) = \log_2 (4)$ d $\log_2 (2x + 1) - \log_2 (x - 1) = \log_2 (4x - 4) + 2$

e $\log (3x + 6) - \log (x + 2) = \log (x - 2)$ f $\log_3 (2x - 4) - \log_3 (x - 1) = \log_3 (x - 2)$

5 Solve each of the following.

a $[\log (x)]^2 - 2 \log (x) - 3 = 0$ b $[\log_2(x)]^2 - 2 \log_2 (x) = 8$

c $[\log_2 (x)]^2 + \log_2 (x) - 2 = 0$ d $[\log_5 (x)]^2 = \log_5 (x) + 2$

e $[\log_3 (x)]^2 - \log_3 (x^4) + 3 = 0$ f $[\log_5 (x)]^2 - \log_5 (x^5) - 24 = 0$

6 Solve the following, correct to 3 significant figures.

a $3 [\log(x)]^2 + 5 \log (x) - 4 = 0$ b $[\log_2 (x)]^2 = 5 \log_2 (x) - 3$

c $4 [\log_3 (x)]^2 = 6 - \log_3 (x)$ d $2 \log (x) - 4 = 3 [\log (x)]^2$

e $[\log_5 (x)]^2 + 5 \log_5 (x) - 3 = 0$

Reasoning and communication

7 The compound interest formula is $A = P\left(1+\dfrac{i}{k}\right)^{kn}$, where A is the amount after n years, i is the interest rate as a decimal, P is the principal (starting amount) and k is the number of times that interest is calculated each year (the 'rest').
What interest rate would you need to turn \$1000 into \$5000 in 20 years if it was compounded half-yearly?

8 A sum of money doubles itself at compound interest in 15 years. In how many years will it become eight times as big?

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7.05 LOGARITHMIC GRAPHS

You can use the graphs of logarithmic functions to solve logarithm equations instead of doing them algebraically.

Example 12

Plot a graph to solve each of the following.

a $\log_3(x) = 1.6$

b $\log_{0.5}(x) = -1.5$

c CAS $\log_4(x) = 1.3$

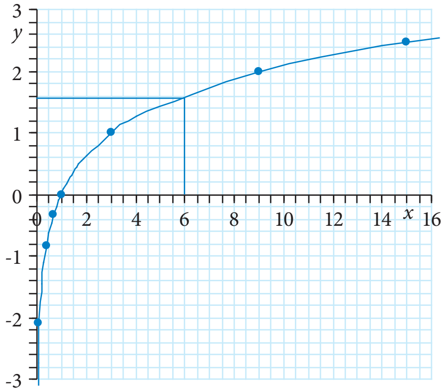
Solution

a Make a table of values.
Remember $x > 0$.

$y = \log_3(x)$

x	0.1	0.4	0.7	1	3	9	15
y	−2.1	−0.83	0.32	0	1	2	2.46

Plot the points and join with a smooth curve.



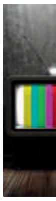
Use the graph.

$\log_3(6) \approx 1.6$, so $x \approx 6$

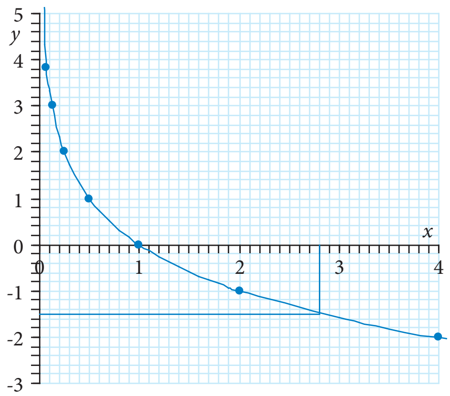
b Make a table of values.
Remember $x > 0$

$y = \log_{0.5}(x)$

x	0.07	0.125	0.25	0.5	1	2	4
y	3.84	3	2	1	0	−1	−2



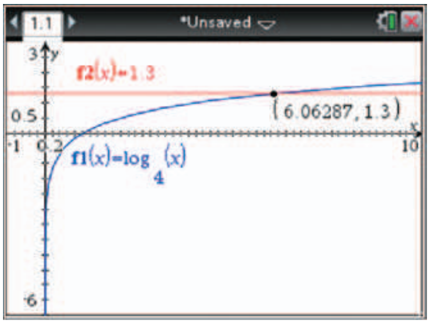
Plot the points and join with a smooth curve.



Use the graph. $\log_3(2.8) \approx -1.5$, so $x \approx 2.8$

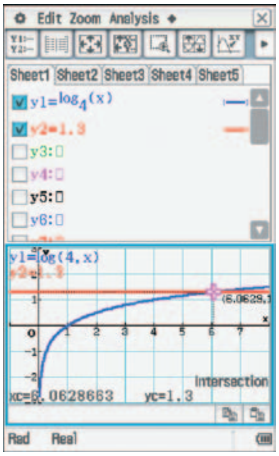
TI-Nspire CAS

Use a graph page.
Complete $f1(x) = \log_4(x)$ and change the Window Settings to appropriate values.
Use menu, 3: Graph Entry/Edit and 1: Function to put in $f2(x) = 2.3$.
Then use menu, 6: Analyse Graph and 4: Intersection to find the point needed.



ClassPad

Use the Graph&Table menu.
Enter the graphs in y1 and y2.
Tap $\log_{\blacksquare}\square$ in the Math1 menu.
You may have to set the View Window appropriately.
Tap Analysis, G-Solve and Intersection.



Write the answer. $x \approx 6.063$

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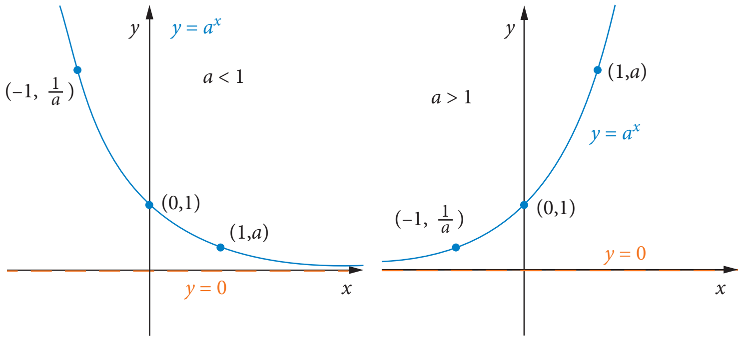
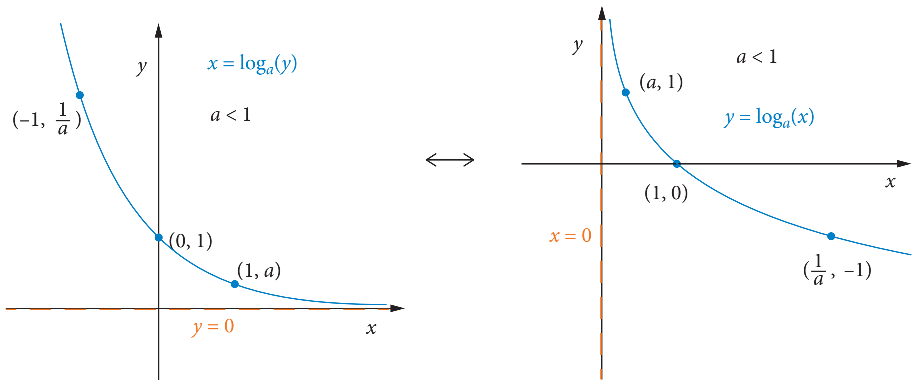
You already know how to sketch exponential functions from your work in Year 11. The domain of an exponential function such as $y = a^x$ has domain $\boldsymbol{R}$ and range $\boldsymbol{R}^+$, with y-intercept 1 and horizontal asymptote $y = 0$ (the x-axis). For $a > 1$ the graph is always increasing and for $a < 1$ it is always decreasing. The graph passes through the points $(1, a)$ and $(-1, -a)$. The graphs look like those below.

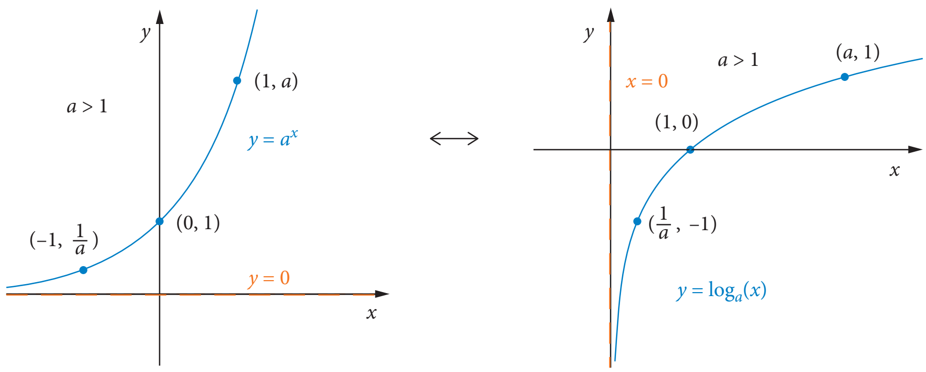


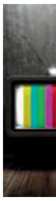
The definition of a logarithm means that $y = a^x$ and $x = \log_a(y)$ are equivalent, so the graphs above are also graphs of $x = \log_a(y)$.

To change them to graphs of $y = \log_a(x)$, you only need to swap the x and y coordinates. This means that $\left(-1, \frac{1}{a}\right) \leftrightarrow \left(\frac{1}{a}, -1\right)$, $(0, 1) \leftrightarrow (1, 0)$ and $(1, a) \leftrightarrow (a, 1)$. The shape stays the same and the horizontal asymptote becomes a vertical asymptote.

The swapped over graphs are shown below.

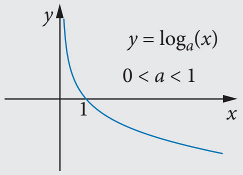




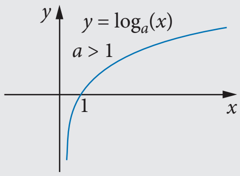


IMPORTANT

Graphs of logarithmic functions of the form $y = \log_a(x)$



Increasing logarithmic function



Decreasing logarithmic function

The graphs of exponential functions of the form $y = \log_a(x)$ have a zero of 1.

For increasing functions, as $x \to 0$, $y \to -\infty$.

For decreasing functions, as $x \to 0$, $y \to \infty$.

The y-axis is a vertical asymptote.

What shapes are logarithmic functions of the form $y = \log_a(x) + b$ or $y = \log_a(x + c)$?

INVESTIGATION Transforming logarithms

a Consider the function $f(x) = \log_2(x)$

- **i** State the domain and range.
- **ii** What is the zero for this function?
- **iii** What is the asymptote for this function?
- **iv** Find $f(2)$ and explain why this value is so important for this function.

b Now consider the function $f(x) = \log_2(x + 3)$.

Use your CAS calculator or computer software to draw the graph.

- **i** State the domain and range.
- **ii** What is the zero for this function?
- **iii** What is the asymptote for this function?
- **iv** Find $f(-1)$ and explain why this value is so important for this function.

c Now consider the function $f(x) = \log_2(x) + 4$.

Use your CAS calculator or computer software to draw the graph.

- **i** State the domain and range.
- **ii** What is the zero for this function?
- **iii** What is the asymptote for this function?
- **iv** Find $f(2)$ and explain why this value is so important for this function.

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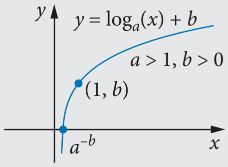
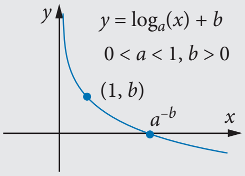
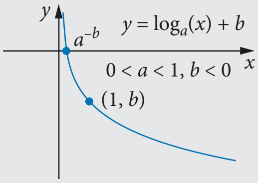
From your investigation you should be able to see the following.

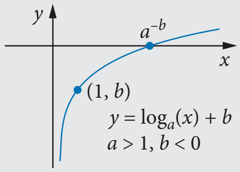
IMPORTANT

The graph of $y = \log_a(x) + b$ is a vertical translation of $y = \log_a(x)$ by b. For b positive, the translation is upwards, and for b negative it is down. The asymptote is still the vertical axis, but the zero becomes $(a^{-b}, 0)$.

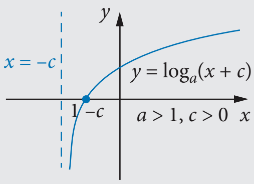


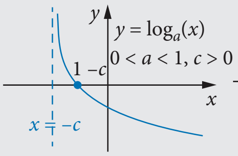


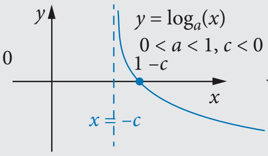


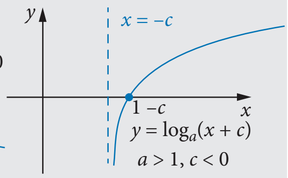


The graph of $y = \log_a(x + c)$ is a horizontal translation of $y = \log_a(x)$ by c. For c positive, the translation is to the left, and for b negative it is to the right. The zero becomes $(1 - c, 0)$ and the vertical asymptote becomes $x = -c$.









Example 13

Sketch the graph of $f(x) = \log_2(x) - 3$, labelling important features.

Solution

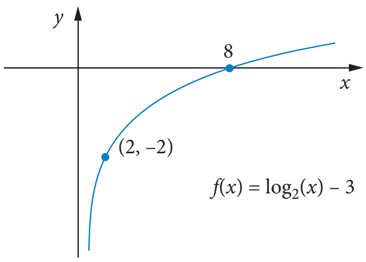
State the translation. $\log_2(x)$ is translated 3 units down.

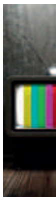
State the zero. The zero is $(2^3, 0) = (8, 0)$.

State the asymptote. The asymptote is at $x = 0$.

State another point. The point $(2, -2)$ is on the graph.

Sketch the graph.





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Example 14

Sketch the graph of $f(x) = \log_{\frac{1}{4}}(x-1)$, labelling important features.

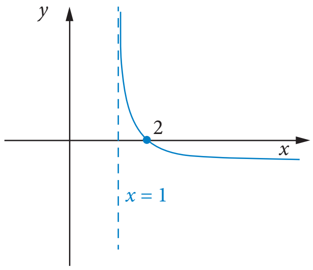
Solution

State the translation. $\log_{\frac{1}{4}}(x)$ is translated 1 unit to the right.

State the zero. The zero is (2, 0).

State the asymptote. The asymptote is at $x = 1$.

Sketch the graph.



EXERCISE 7.05 Logarithmic graphs

Concepts and techniques

WS Plotting log functions

1 Example 12 Plot a graph to solve each of the following.

a $\log_2(x) = 2.3$ b $\log_{\frac{1}{6}}(y) = 1.4$ c $\log_5(z) = -0.8$ d $\log_{0.6}(k) = -1.7$

2 CAS Use a graph to solve each of the following.

a $\log_4(x) = 3.1$ b $\log_{0.3}(y) = 2.5$ c $\log_7(z) = -1.6$ d $\log_{0.2}(k) = -0.4$

3 Example 13 Sketch the graphs of the following, labelling important features.

a $f(x) = \log_2(x) - 2$ b $f(x) = \log_{0.5}(x) - 2$
c $f(x) = \log_4(x) + 1$ d $f(x) = \log_{0.6}(x) + 3$
e $f(x) = \log_3(x) - 2$ f $f(x) = \log_{0.2}(x) + 2$

4 Example 14 Sketch the graphs of the following, labelling important features.

a $f(x) = \log_{\frac{1}{6}}(x-2)$ b $f(x) = \log_4(x+3)$
c $f(x) = \log_{0.8}(x-3)$ d $f(x) = \log_3(x+2)$
e $f(x) = \log_{\frac{1}{6}}(x-1)$ f $f(x) = \log_2(x+1)$

5 Find the equations of the new functions produced by the following.

a Translation of $\log_7(x)$ by 3 units up.
b Translation of $\log_{0.5}(x)$ by 2 units down.
c Translation of $\log_{\frac{1}{6}}(x)$ by 1 unit up.
d Translation of $\log_3(x)$ by 4 units down.

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6 Find the equations of the new functions produced by the following.
   a Translation of $\log_5(x)$ by 4 units to the left.
   b Translation of $\log_{0.3}(x)$ by 2 units to the right.
   c Translation of $\log_4(x)$ by 3 units to the left.
   d Translation of $\log_{0.8}(x)$ by 5 units to the right.

Reasoning and communication

7 Find the equations of the new functions produced by the following.
   a Translation of $y = \log_2(x)$ by 4 units to the left and 3 units up.
   b Translation of $y = \log_{0.1}(x)$ by 2 units to the right and 1 unit up.
   c Translation of $y = \log_4(x)$ by 3 units to the left and 4 units down.
   d Translation of $y = \log_{0.6}(x)$ by 1 unit to the right and 2 units down.

8 Sketch the graphs of the following, labelling important features.
   a $y = \log_2(x+3)+1$
   b $y = \log_{0.5}(x-2)-2$
   c $y = \log_3(x+1)-2$
   d $y = \log_4(x-2)+1$

7.06 APPLICATIONS OF LOGARITHMS

You can use logarithmic models in a number of physical applications, such as determining the magnitude of earthquakes, intensity of sound, and acidity of a solution. A logarithmic model initially changes rapidly but the growth slows as it continues.

The intensity of earthquakes were originally measured by a scale of physical experiences such as 'like vibrations from heavy traffic' or 'chimney fall'. The Richter scale is a logarithmic scale based on measurements by a seismograph to measure the magnitude of an earthquake. The intensity refers to the local effect of an earthquake, but the magnitude measures the energy.

One of the Richter formulas used to measure the magnitude of an earthquake is $M(x) = \log\left(\frac{x}{x_0}\right)$, where x_0 is defined as the magnitude of an earthquake (x_0) with a seismographic reading of 0.001 millimetre at a distance of 100 kilometres from the epicenter. Modern seismographs are calibrated to this scale

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Example 15

The San Francisco Earthquake of 1906 would have given a seismographic reading of 7.943 metres 100 kilometres from the centre. What was its magnitude?

Solution

Write the formula.	$M(x) = \log\left(\frac{x}{x_0}\right)$
Substitute in $x = 7943$, $x_0 = 0.001$	$= \log\left(\frac{7943}{0.001}\right)$
Evaluate.	$= 6.9$
Write the answer.	The magnitude was 6.9.

Example 16

The pH scale measures the acidity of a solution using the concentration of hydrogen ions. In pure water, the concentration is 10^{-7} moles/litre and the pH is defined as $-\log([H^+])$, where $[H^+]$ is the concentration in moles/litre.

a What is the pH of pure water?

b In 0.1 M hydrochloric acid solution, all the molecules are disassociated so the concentration of hydrogen ions is 0.1 moles/litre. What is the pH of this solution?

Solution

a Write the formula.	$pH = -\log([H^+])$
Substitute $[H^+] = 10^{-7}$.	$= -\log(10^{-7})$
Use the definition of $\log(x)$.	$= 7$
Write the answer.	Pure water has a pH of 7.
b Find the concentration of H^+ ions.	$[H^+] = 10^{-7} + 0.1\ (\approx 0.1)$
Substitute $[H^+] = 0.1$.	$= -\log(0.1)$
Use the definition of $\log(x)$.	$= 1$
Write the answer.	0.1 molar hydrochloric acid has a pH of 1.

EXERCISE 7.06 Applications of logarithms

Reasoning and communication

1 Examples 15, 16 The wind speed s (in km/h) near the center of a tornado is related to the distance d (in thousands of km) the tornado has travelled over a warm ocean surface. The tornado travels according to the equation $s = 930\log(d) + 65$. A tornado whose wind speed is about 280 kilometres per hour struck the coast. How far had this tornado travelled over warm ocean water?

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2 The pH of a solution is given by pH = –log ([H^+]), where [H^+] is the molar concentration of hydrogen ions in moles/litre. The equation for the dissociation of water molecules is [H^+] [OH^-] = 10^{-14}.

a What is the pH of a sodium hydroxide solution with [OH^-] = 10^{-1}?

b When ingested, dishwasher detergent dissolves in the moisture in the mouth, oesophagus and stomach to give a pH of 15. What is the concentration of OH^- ions?

3 Pure sulfuric acid is carried in steel tankers because it needs water to produce hydrogen ions. A tanker carrying sulfuric acid spills some on a wet road and the pH of the resulting solution was measured to be –2 as it burned through the road. What was the concentration of hydrogen ions?

4 Biologists use the logarithmic model $n = k\log(A)$ to estimate the number of species (n) that live in a region of area (A km^2). In the model, k represents a constant. If 2800 species live in a rain forest of 500 square kilometres, then how many species will be left when half of this rainforest is destroyed by logging?

5 The intensity of a sound wave is interpreted by our ears as its loudness. The weakest sound wave that a human ear can hear has an intensity of 1×10^{-12} watts/m^2 and is called the threshold of human hearing, I_0. To compare relative sound intensities, S, we use a scale called decibels, dB, which is calculated with the formula $S = 10\log\left(\frac{I}{I_0}\right)$. In this formula, the threshold of human hearing has a decibel reading of 0 dB. What is the intensity in watt/m^2 of a sound wave that has a sound level reading of 125 dB, the loudness of an average fire alarm?

6 A telescope is limited in its usefulness by the brightness of the star that it is aimed at and by the diameter of its lens. One measure of a star's brightness is its magnitude – the dimmer the star, the larger its magnitude. A formula for the limiting magnitude L of a telescope, that is, the magnitude of the dimmest star that it can be used to view, is given by $L = 9 + 5.1\log(d)$, where d is the diameter (in cm) of the lens.

a What is the limiting magnitude of a telescope 6 cm in diameter?

b What diameter is required to view a star of magnitude 10?

7 The Richter scale can be given by $R = 0.67\log(0.37E) + 1.46$, where E is the energy (in kilowatt-hours) released by the earthquake.

a An earthquake releases 15 500 000 000 kilowatt-hours of energy. What is the earthquake's magnitude?

b How many kilowatt-hours of energy would an earthquake have to release in order to be a 8.5 on the Richter scale?

8 Desalination is the process of producing fresh water from salt water using the formula $y = a + b\log_2(t)$, where y is the amount of fresh water produced (in litres) in time, t (hours). How much fresh water can be produced after 10 hours from a desalination process, given that after 1 hour, 18.27 litres of fresh water can be produced and after 2 hours, 25.41 litres of fresh water can be produced?

9 In the modern scale of musical notes the note names repeat every octave, and each note is double the frequency of the note of the same name in the octave below. The A note below middle C has a standard frequency of 440 Hertz. There are actually 12 different notes, including sharps and flats, in an octave. This is called the chromatic scale and the ratio of the frequency of one note to the previous note in the chromatic scale is a constant.

a What is the ratio of a musical note to the previous note in the chromatic scale?

b What is the frequency of middle C?

7.07 THE NATURAL LOGARITHM AND ITS DERIVATIVE

The exponential function is given by e^x, where $e = 2.728\,281\ldots$ is the number such that $\frac{d}{dx}(e^x) = e^x$. This number is also the most important base for logarithms.

IMPORTANT

The **natural logarithm** is given by $\ln(x) = \log_e(x)$.

The natural logarithm is the inverse of the exponential function.

Remember that inverse functions must be 1-to-1 and they are such that if $y = f(x)$, then $x = f^{-1}(y)$, where f^{-1} is the inverse of f.

Since $e > 1$, $y = e^x$ is equivalent to $x = \ln(y)$ by the definition of logarithms, so the functions must be inverses of each other. Calculators normally have a ln button for calculating the natural logarithm of a number.

The exponential function and the natural logarithm are extremely important in calculus.

IMPORTANT

The derivative of $\ln(x)$ is given by $\frac{d}{dx}\ln(x) = \frac{1}{x}$.

This can be proven as follows.

Let $y = \ln(x)$.

Then $x = e^y$.

Differentiate both sides to get $\frac{d}{dx}(x) = \frac{d}{dx}(e^y)$.

But $\frac{d}{dx}(x) = 1$, so $1 = \frac{d}{dx}(e^y)$.

Using the chain rule, $\frac{d}{dx}(e^y) = \frac{d}{dy}(e^y) \times \frac{dy}{dx}$.

But $\frac{d}{dy}(e^y) = e^y$, so $1 = e^y \times \frac{dy}{dx}$.

But $x = e^y$, so $1 = x \times \frac{dy}{dx}$ and this gives $\frac{dy}{dx} = \frac{1}{x}$. QED

All the normal derivative rules apply to the derivative of $\ln(x)$.

Therefore, if $f(x) = \ln(kx)$, then $f'(x) = \frac{1}{x}$, $x > 0$

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Example 17

Differentiate the following.

a $f(x) = \ln(5x)$ **b** $\log_7(x)$

Solution

a Write as a function of a function. $\quad$ Write $f(x) = \ln(u)$, where $u(x) = 5x$

Find the derivatives. $\quad f'(u) = \dfrac{1}{u}$ and $u'(x) = 5$

Use the chain rule. $\quad f'(x) = f'(u) \times u'(x)$

Substitute the derivatives. $\quad = \dfrac{1}{u} \times 5$

Substitute $u = 5x$. $\quad = \dfrac{1}{5x} \times 5$

Cancel the 5s. $\quad = \dfrac{1}{x}$

b Write the problem. $\quad \dfrac{d}{dx}\log_7(x)$

Use the change of base theorem. $\quad = \dfrac{d}{dx}\left[\dfrac{\log_e(x)}{\log_e(7)}\right]$

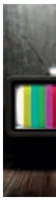
Rewrite to emphasise that $\dfrac{1}{\log_e(7)}$ is a constant. $\quad = \dfrac{d}{dx}\left[\dfrac{1}{\log_e(7)} \times \log_e(x)\right]$

Use the rule $\dfrac{d}{dx}[kf(x)] = k\dfrac{d}{dx}[f(x)]$ $\quad = \dfrac{1}{\log_e(7)} \cdot \dfrac{d}{dx}[\log_e(x)]$

Use the derivative of $\ln(x)$. $\quad = \dfrac{1}{\ln(7)} \times \dfrac{1}{x}$

Simplify. $\quad = \dfrac{1}{x\ln(7)}$

Form part **a**, you should be able to see that it doesn't matter what the constant is in the derivative of $\ln(kx)$, the derivative will still be the same, even though it seems a bit odd at first. That is also true for part **b**.



IMPORTANT

The derivative of $\ln(kx)$ is given by $\dfrac{d}{dx}\ln(kx) = \dfrac{1}{x}$.

The derivative of $\log_a(x)$ is given by $\dfrac{d}{dx}\log_a(x) = \dfrac{1}{x\ln(a)}$ for $a > 0$.

Example 18

Differentiate $y = \ln(2x - 5)^2$

Solution

Write the problem.

$$\frac{d}{dx}\ln(2x-5)^2$$

Use the chain rule.

$$= \frac{d}{du}\ln(u) \times \frac{d}{dx}(2x-5)^2 \text{ where } u = (2x-5)^2$$

Find the first derivative and use the chain rule again.

$$= \frac{1}{u} \times \frac{d}{dx}(v^2) \times \frac{d}{dx}(2x-5) \text{ where } v = 2x-5$$

Find the last two derivatives.

$$= \frac{1}{u} \times 2v \times 2$$

Substitute u and v.

$$= \frac{1}{(2x-5)^2} \times 2(2x-5) \times 2$$

Simplify.

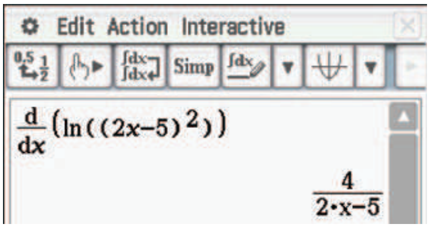
$$= \frac{4}{2x-5}$$

TI-Nspire CAS

Use a calculator page.
Use [menu], 4: Calculus and 1: Derivative.

ClassPad

Use the Main menu and Math2. You may have to clear all variables (Edit menu) first.



You are expected to become competent in finding the derivatives of a wide variety of functions. However, you should get into the habit of checking complex derivatives with your CAS calculator if it is available.

It is worth noting the following rule, which is easily proven using the chain rule.

IMPORTANT

The derivative of $\ln[f(x)]$ is given by $\dfrac{d}{dx}\ln[f(x)] = \dfrac{f'(x)}{f(x)}$

Example 19

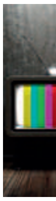
Find the derivative of $f(x) = x^2 \ln(x)$.

Solution

Write as a product.	Let $f(x) = u(x)v(x)$ where $u(x) = x^2$ and $v(x) = \ln(x)$
Write the product rule.	$f'(x) = u'(x)v(x) + u(x)v'(x)$
Substitute.	$= 2x\ln(x) + x^2 \times \dfrac{1}{x}$
Simplify.	$= x + 2x\ln(x)$

EXERCISE 7.07 The natural logarithm and its derivative

WS Differentiating exponential and logarithmic functions



Concepts and techniques

1 Example 17 Differentiate each of the following.

a $y = \ln(x)$ b $y = \ln(10x)$ c $y = 3\ln(2x)$

d $y = \ln(0.3x)$ e $y = 6\ln(9x)$ f $y = \ln\left(\dfrac{x}{2}\right)$

g $y = 4\ln\left(\dfrac{x}{3}\right)$ h $y = 2\ln\left(-\dfrac{2x}{3}\right)$

2 Find the derivatives of the following.

a $\log_4(x)$ b $\log(x)$ c $\log_9(x)$ d $\log_2(x)$ e $\log_{0.2}(x)$

3 Example 18 Find the derivatives of the following.

a $y = \ln(3x - 1)$ b $y = \ln(2x + 7)$ c $y = 2\ln(4x - 3)$ d $y = 5\ln(6x + 7)$

e $y = \ln(2x + 1)$ f $y = 3\ln(5x - 1)$ g $y = 6\ln(3 - 4x)$ h $y = 12\ln(5 - 8x)$

4 Find the derivatives of the following.

a $y = \ln(3x^5)$ b $y = \ln(4x^3)$ c $y = \ln(2x^2 + 1)$

d $y = 2\ln(5 - 8x^2)$ e $y = \ln(x^3 - 2x^2 + 3x - 4)$ f $y = 3\ln(2x^4 - 7x^5 + x)$

5 Differentiate the following.

a $y = \ln\left(\sqrt{3x+1}\right)$ b $y = \ln\left(\sqrt{5-7x}\right)$ c $y = \ln\left(\sqrt[3]{4x+9}\right)$

d $y = \ln\left(\sqrt[5]{8-x}\right)$ e $y = \ln(3x - 7)^4$ f $y = \ln(5x - 2)^3$

g $y = \ln\left(\dfrac{1}{x+2}\right)$ h $y = \ln\left(\dfrac{2}{5-3x}\right)$

i $y = \ln\left(\dfrac{3}{6x+1}\right)^{-2}$ j $y = \ln\left(\dfrac{7}{4-x}\right)^{-5}$

6 Find the derivatives of the following.

a $y = \ln (x^2 + 2)^2$

b $y = \ln (3 - x^2)^2$

c $y = \ln (x^3 - 2x + 3)^3$

d $y = \ln (2x^3 - 3x^2 + 4x - 1)^3$

7 Example 19 Find the derivatives of the following.

a $(x^2 - 2x + 1) \ln (x)$

b $(x^3 + 3x^2 + 5) \ln (x^3 + 3x^2 + 5)$

c $x \ln (x)$

d $e^x \ln (x)$

e $\ln (x) \sin (x)$

f $\ln (x) \cos (x) + \dfrac{\sin(x)}{x}$

Reasoning and communication

8 Given $f(x) = 6 \ln (3 - 4x)$, find:

a $f'(x)$

b $f'(2)$

c x such that $f'(x) = 2$

9 Given $f(x) = 6 \ln \left(\sqrt{x^2 - 1}\right)$, find:

a $f'(x)$

b $f'(2)$

c x such that $f'(x) = 6$

10 Given $f(x) = 4x^2 + 3 \ln (x^2 + 2x)$, find:

a $f'(x)$

b $f'(2)$

c x such that $f'(x) = 2$

11 If $g(x) = \ln\left[f(x)\right], f(1) = 3$ and $f'(1) = 6$ find the derivative of $g(x)$ when $x = 1$.

12 Prove that $\dfrac{d}{dx} \ln[f(x)] = \dfrac{f'(x)}{f(x)}$.

7.08 THE INTEGRAL OF $\dfrac{1}{x}$

The integral of $\dfrac{1}{x}$ is just the reverse of the derivative of $\ln (x)$.

IMPORTANT

$\displaystyle\int \frac{1}{x}\, dx = \ln (x) + c$ for $x > 0$.

$\displaystyle\int \frac{1}{x}\, dx$ is sometimes written as $\displaystyle\int \frac{dx}{x}$.

Since you already know how to find the integral of x^n for $n \neq -1$, knowing the integral of $\dfrac{1}{x} = x^{-1}$ means that you can integrate x^n for any value of n.

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Example 20

Find $\int \frac{3}{5x} dx$

Solution

Write the problem. $\int \frac{3}{5x} dx$

Use the rule for $\int kf(x)dx$. $= \frac{3}{5}\int \frac{1}{x} dx$

Integrate. $= \frac{3}{5} \ln (x) + c$ for $x > 0$

If you have to integrate any function involving a denominator, then you should test the derivative of the natural logarithm of the denominator.

Example 21

Find the integral of $\frac{2x-3}{x^2-3x+5}$

Solution

Examine the denominator and numerator. $\frac{d}{dx}(x^2 - 3x + 5) = 2x - 3$

Test the log of the denominator. $\frac{d}{dx}\ln (x^2 - 3x + 5)$

Write as a function of a function with $u = (x^2 - 3x + 5)$ $= \frac{d}{du}\ln(u) \times \frac{d}{dx}(x^2 - 3x + 5)$

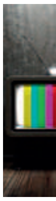
Find the derivatives. $= \frac{1}{u} \times (2x - 3)$

Substitute u back in. $= \frac{1}{x^2-3x+5} \times (2x - 3)$

Simplify. $= \frac{2x-3}{x^2-3x+5}$

Use the meaning of an indefinite integral. $\int \frac{2x-3}{x^2-3x+5} dx = \ln (x^2 - 3x + 5) + c$

Example **21** uses the method of trying to find the function whose derivative is the function you want to integrate. You can use this method for many functions that have a denominator that is itself a function. You should also remember that the integral of x^{-1} exists only for $x > 0$. In the case of Example **21**, the integral exists for all values of x because $x^2 - 3x + 5 > 0$ for $x \in \boldsymbol{R}$. You should always state any restrictions on the domain.



Example 22

Find the integral of $\dfrac{x^2 - 2x + 5}{x - 3}$

Solution

Divide to simplify the problem.

$$\begin{array}{r@{}l} & \quad\;\; x+1 \\ x-3 \big) & \overline{x^2 - 2x - 5} \\ & \underline{x^2 - 3x} \\ & \qquad\; x+5 \\ & \qquad\; \underline{x-3} \\ & \qquad\qquad 8 \end{array}$$

Write the simplification.

$$\frac{x^2 - 2x + 5}{x - 3} = x + 1 + \frac{8}{x - 3}$$

Test the derivative of the denominator.

$$\frac{d}{dx}\ln(x-3) = \frac{1}{x-3} \times 1$$
$$= \frac{1}{x-3}$$

Now do the integral.

$$\int \frac{x^2 - 2x + 5}{x - 3}\,dx$$

Use the simplification.

$$= \int \left(x + 1 + \frac{8}{x-3} \right) dx$$

Write as a sum and take out the 8.

$$= \int (x+1)\,dx + 8\int \frac{1}{x-3}\,dx$$

Complete the integral.

$$= \frac{1}{2}x^2 + x + 8\ln(x-3) + c$$

Work out the restriction of the domain.

For $x - 3 > 0$, $x > 3$

State the final answer.

$$\int \frac{x^2 - 2x + 5}{x - 3}\,dx$$
$$= \frac{1}{2}x^2 + x + 8\ln(x-3) + c \text{ for } x > 3$$

EXERCISE 7.08 The integral of $\frac{1}{x}$

WS Integration of reciprocals

Concepts and techniques

1 Example 20 Find the integrals of the following for $x > 0$.

a $\frac{2}{x}$ b $\frac{7}{x}$ c $\frac{6}{5x}$

d $\frac{4}{7x}$ e $-\frac{8}{11x}$ f $-\frac{9}{4x}$

2 Example 21 Find the integrals of the following with an appropriate restriction of the domain.

a $\frac{1}{x+4}$ b $\frac{1}{x-2}$ c $\frac{1}{3x+1}$ d $\frac{1}{5x-9}$

e $\frac{11}{7x-9}$ f $\frac{13}{4x-1}$ g $\frac{6}{5-2x}$ h $\frac{7}{3-x}$

Reasoning and communication

3 Example 22 Find the following, stating any restrictions of the domain.

a $\int \frac{x^3+x^2}{x^3}dx$ b $\int \frac{4x^4-3x^2+x}{x^3}dx$

c $\int \frac{5x+2x^3-1}{x^2}dx$ d $\int \frac{4x^2+8x^5-2x}{2x^3}dx$

e $\int \frac{3x^{10}-2x^4+15x^2}{x^3}dx$

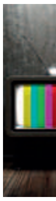
4 Find the equation of the curve $f(x)$ given that $f'(x) = \frac{1}{x-2}$ and the curve passes through (3, 6).

5 Find the equation of the curve $f(x)$ given that $f'(x) = \frac{7}{5-3x}$ and $f(2) = 7$.

6 Find $\frac{d}{dx}[\ln(x^2+2)]$. Hence find $\int \frac{4x}{x^2+2}dx$.

7 Find $\frac{d}{dx}[\ln(x^2-5)]$. Hence find $\int \frac{x}{x^2-5}dx$.

8 Show that $\frac{4}{x^2-4} = \frac{1}{x-2} - \frac{1}{x+2}$ and hence find $\int \frac{4}{x^2-4}dx$.



7.09 APPLICATIONS OF NATURAL LOGARITHMS

Logarithmic growth models apply to phenomena where the rate of increase decreases over time. For example, training to improve a skill generally follows a logarithmic model because you make a lot of progress to start with, but as you improve, it gets harder to go further. If you start from zero on day 0 of a program, the function must be of the type $a\ln(t+1)$. If you start with a level of, c, then it becomes $a\ln(t+1)+c$. More complex models may be modelled as $a\ln[b(t+1)]+c$.

Example 23

Paula is learning to speak Spanish before going to South America for 12 months. After 3 days of a crash course she has a vocabulary of 150 words and can use them to communicate in Spanish. 2 days later she has learned another 120. She needs a very basic vocabulary of 600 words before the trip, which will be in 5 months time.

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a Make a simple logarithmic model for her vocabulary after t days.

b How long will it take her to learn the very basic vocabulary?

c Find an expression for her rate of learning.

d How long would it take before her rate of learning dropped below 1 word per day?

Solution

a Write the basic model to start at 3 days.	Let $w(3) = a \ln(t-2) + 150$
Substitute $w(5) = 150 + 120 = 270$	$270 = a \ln(5-2) + 150$
Solve for a.	$a = \dfrac{120}{\ln(3)} = 109.2\ldots$
b Write the equation of the model.	$w(t) = 109.2\ldots\ln(t-2) + 150,\ t \geq 3$
Substitute $w(t) = 600$.	$600 = 109.2\ldots \ln(t-2) + 150$
Solve to find t.	$t = 63.54\ldots$
Write the answer.	At this rate, it would take her a bit over 2 months to acquire enough Spanish.
c Find the derivative.	$w'(t) = \dfrac{109.2\ldots}{t-2},\ t \geq 3$
d Substitute $w'(t) = 1$.	$1 = \dfrac{109.2\ldots}{t-2}$
Solve for t.	$t = 111.2\ldots$
Write the answer.	She would be learning less than 1 word a day after 111 days (nearly 16 weeks).

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EXERCISE 7.09 Applications of natural logarithms

Reasoning and communication

1 Example 23 A psychologist uses the function $L(t) = 100 \ln (kt)$ to measure the amount learnt, $L(t)$ at time t minutes where k is a constant. The psychologist determines that a student learnt 129 words after 20 minutes.

  a Determine the exact value of k
  b How many words will the student have learnt after 10 minutes?
  c How many words will the student have learnt after 60 minutes?
  d How long does it take for the student to learn 180 words?
  e At what rate is the student learning after 45 minutes?

2 A small colony of black peppered moths live on a small isolated island. In summer the population begins to increase. If t is the number of days after 12 midnight on 1 January, the equation that best models the number of moths in the colony at any given time is

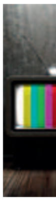
$$N(t) = 500 \ln (21t + 3),\ t \in [0, 40]$$

  a What is the population of the species on 1 January?
  b What is the population of moths after 30 days?
  c On which day is the population first greater than 2000?

  A related species, the white peppered moth, shares the same habitat with the black peppered moth. It reproduces in a similar pattern to the black peppered moth, with its population modelled by

$$P(t) = P \ln (Qt + 3),\ t \in [0, 40]$$

  d The initial population of white peppered moths is 769 and the population when $t = 15$ is 2750. Find the value of P, correct to the nearest whole number and Q, correct to 3 decimal places.
  e Sketch the graphs of $P(t)$ and $N(t)$ on the same set of axes, labelling all special features.
  f Using your graph, find an approximate time when the populations of the black and white peppered moths will be the same.
  g What are the population growth rates at this time?



3 A swimmer is training for a 100 m freestyle race and wants to get under 50 seconds. With ordinary training his best result is 1 minute. After 12 days of intensive training he is taking 55 seconds to swim 100 metres.

  a Construct a model using the function $T(t) = 60 - a \log (t - b)$ for his time.
  b How long will it take to get under 50 seconds?
  c What is his rate of reduction of his 100 m time at this point?
  d How long would it take him to be an Olympic champion contender (under 46 s), assuming his body could stand the training regime?

4 Weight loss on strict diets generally follows a logarithmic pattern such as $W(t) = W_0 - a \ln (t + 1)$, where W_0 is the mass at time $t = 0$. After 30 days on a very strict diet, a man who began the diet with a weight of 185 kg claims that although he lost a massive amount of weight in the first week, he is now losing only 0.2 kg per day. How long would it take him to get down to 100 kg?

5 David can currently make about 5 porcelain figures in a day. He starts to improve his productivity by focussing more clearly on the task and after 2 weeks has increased his productivity by 2 figures per day.

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a Construct a simple productivity model for his productivity t weeks after starting.
b How long will it take him to get his productivity up to 10 per day?
c What will be his rate of improvement after 4 weeks?
d What will be his rate of improvement after 10 weeks?

7

CHAPTER SUMMARY
LOGARITHMIC FUNCTIONS

- The **logarithm** of b to the **base** a, where $a, b > 0$ and $a \neq 1$ is the **exponent** x such that $a^x = b$. In symbols: $a^x = b \Leftrightarrow \log_a(b) = x$, where $a, b > 0$ and $a \neq 1$. x is also called the **index** of a^x and the expression is called the xth **power** of a.
- The logarithm of 1 with any base is zero: $\log_a(1) = 0$ for $a > 0$ and $a \neq 1$.
- The logarithm of any number to its own base is 1: $\log_a(a) = 1$ for $a > 0$ and $a \neq 1$.
- The logarithm of the reciprocal of any number to its own base is -1:

  $$\log_a\left(\frac{1}{a}\right) = -1 \text{ for } a > 0 \text{ and } a \neq 1.$$

- **First law of logarithms: logarithm of a product**

  The logarithm of a product is the sum of the logarithms to the same base:

  $\log_a(xy) = \log_a(x) + \log_a(y)$ for $a, x, y > 0$ and $a \neq 1$.

- **Second law of logarithms: logarithm of a quotient**

  The logarithm of a quotient is the difference of the logarithms to the same base:

  $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$ for $a, x, y > 0$ and $a \neq 1$.

- **Third law of logarithms: logarithm of a power**

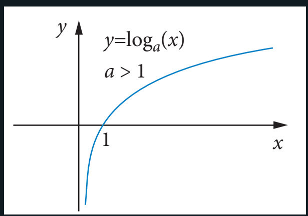
  The logarithm of a power is the the product of the power and the logarithm:

  $\log_a(x^p) = p\log_a(x)$ for $a, x > 0$ and $a \neq 1$.

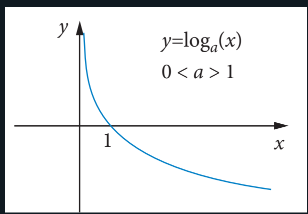
- A **common logarithm** (or **Briggsian logarithm**) is in base 10 and is normally written without the base, so $\log(16) = \log_{10}(16)$
- **Change of base theorem**

  The logarithm of a number to a given base is the quotient of the logarithms of the number and base to any other base: $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for $a, b, x > 0$ and $a, b \neq 0$.

- **Graphs of logarithmic functions of the form $y = \log_a(x)$**



Increasing logarithmic function



Decreasing logarithmic function

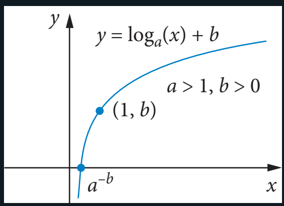
The graphs of exponential functions of the form $y = \log_a(x)$ have a zero of 1.

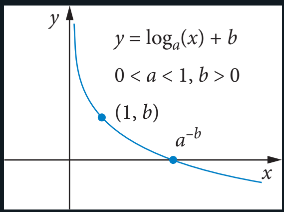
For increasing functions, as $x \to 0, y \to -\infty$.

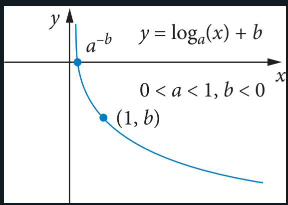
For decreasing functions, as $x \to 0, y \to \infty$.

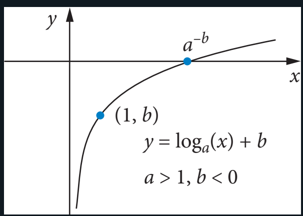
The y-axis is a vertical asymptote.

- The graph of $y = \log_a(x) + b$ is a vertical translation of $y = \log_a(x)$ by b. For b positive, the translation is upwards, and for b negative, it is down. The asymptote is still the vertical axis, but the zero becomes $(a^{-b}, 0)$.









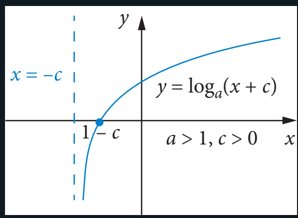
- The **natural logarithm** is given by $\ln(x) = \log_e(x)$. The natural logarithm is the inverse of the exponential function.

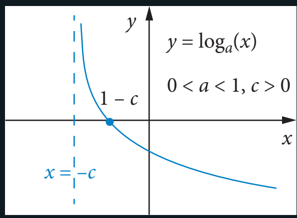
- The derivative of $\ln(x)$ is given by $\frac{d}{dx}\ln(x) = \frac{1}{x}$.

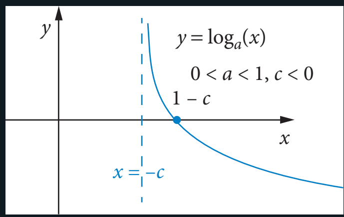
- The derivative of $\ln(kx)$ is given by $\frac{d}{dx}\ln(kx) = \frac{1}{x}$.

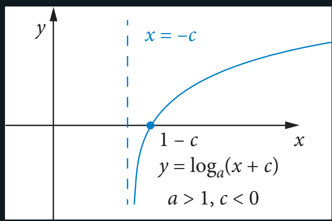
- The derivative of $\log_a(x)$ is given by $\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$ for $a > 0$.

- The graph of $y = \log_a(x + c)$ is a horizontal translation of $y = \log_a(x)$ by c. For c positive the translation is to the left, and for b negative it is to the right. The zero becomes $(1 - c, 0)$ and the vertical asymptote becomes $x = -c$.









- The derivative of $\ln[f(x)]$ is given by $\frac{d}{dx}\ln[f(x)] = \frac{f'(x)}{f(x)}$

- $\int \frac{1}{x}dx = \ln(x) + c$ for $x > 0$. $\int \frac{1}{x}dx$ is sometimes written as $\int \frac{dx}{x}$.

- A simple logarithmic model is of the form $a \ln(t + 1) + c$, where c is the initial value and a is a constant determined by the situation.

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CHAPTER REVIEW
LOGARITHMIC FUNCTIONS 7

Multiple choice

1 Example 3 What is the value of $\log_2(8)$?

A $-\frac{1}{4}$ B $\frac{1}{4}$ C 3 D 4 E 6

2 Example 4 What is the value of $\log_3(-2)$?

A undefined B $\frac{1}{9}$ C $-\frac{1}{9}$ D $\frac{1}{8}$ E -9

3 Example 4 State the domain for $y = \log_2(x - 5)$

A R B $x < -5$ C $x > 2$ D $x > 5$ E $(2, 5)$

4 Example 5 $\log_4(2) + \log_4(8) + \log_4\left(\frac{1}{4}\right) =$

A 0 B 1 C 2 D 3 E 4

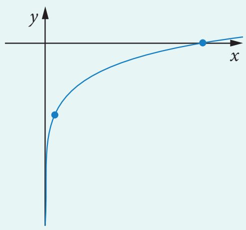
5 Example 6 Write $5 \log(x) + 6 \log(x + 6)$ as a single logarithm.

A $\log[x(x + 6)]$ B $30 \log[x(x + 6)]$ C $\log[x^5(x + 6)^6]$

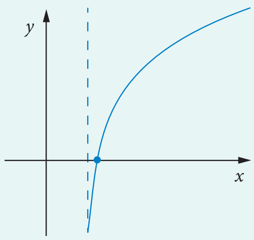
D $11 \log(2x + 6)$ E None of these

6 Examples 13, 14 The graph of $y = \log_2(x) + 4$ is most like:

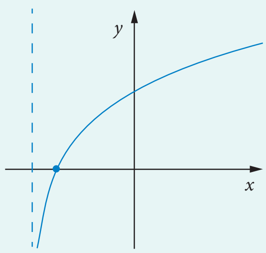
A



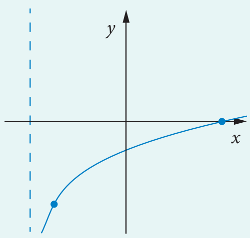
B



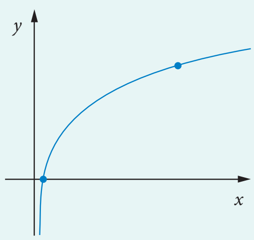
C



D



E



7 Example 17 What is the derivative of $\ln(2x-3)$?

A $\dfrac{2}{2x-3}$ B $\dfrac{-3}{2x-3}$ C $\dfrac{1}{x}$ D $\dfrac{1}{2x-3}$ E $\dfrac{1}{2x}$

8 Example 20 What is the integral of $\dfrac{1}{5x}$?

A $\ln(x)$ B $\log_5(x)$ C $\frac{1}{5}\ln(x)$ D $\ln(5x)$ E $\ln(\frac{1}{5}x)$

Short answer

9 Example 1 Write $\log_3(81) = 4$ in index form.

10 Example 2 Write $5^{-2} = 0.04$ in logarithmic form.

11 Example 7 CAS What is the value of $\log_6(15)$?

12 Example 8 CAS Solve $4^x = 23$ correct to 3 decimal places.

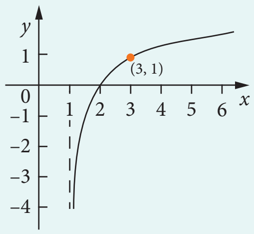
13 Example 9 Solve $3^{2x+1} + 3^x + 4 = 3^{x+2}$.

14 Example 10 Solve $4^{3x+2} = 6^{2x-1}$ and then evaluate correct to 4 significant figures.

15 Example 11 Solve $2\log_3(x) + \log_3(2x-1) - \log_3(x) = 1$

16 Example 12 Plot a graph to solve $\log_3(x) = -0.7$

17 Examples 13, 14 The function f has domain $(1, \infty)$ and its graph is as shown below. Given that $y = \log_a(x+b)$, find the equation of this function.



18 Examples 18, 19 Differentiate:

a $\log_6(x)$

b $\log_e[(3x^2+8)]$

c $(3x^4 - x^3 + 5)\ln(7x+1)$

d $\dfrac{\ln(x)}{\ln(3x-5)}$

19 Example 21 Find the integral of $\dfrac{3x^2-4}{x^3-4x+1}$.

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Application

20 Sketch the graph of $f(x) = \log_3 (x + 2) + 2$, labelling important features.

21 The *Shade number* of welding glass is given by $-\frac{7}{3}\log\left(\frac{T}{I}\right) + 1$, where T measures the intensity of light transmitted through the glass when light of intensity I is incident on the glass. What percentage of light is transmitted through welding glass rated at shade factor 10?

22 Find the integral of $y = \dfrac{x^3 - 3x^2 + 2x - 4}{x + 1}$.

23 A patient is given a dose of a drug. At a later stage, the patient is given a *second* dose of the drug, and the amount x units of a metaboloid in the patient's bloodstream t minutes from administering this second dose is modelled by $x = \log_e (2t + e^2)$, $t \geq 0$.

a How much of the metaboloid is present in the patient's bloodstream:

i at the instant the second dose is given?

ii 5 hours later (to the nearest tenth of a unit)?

b Find the rate of change of the amount of the metaboloid in the patient's bloodstream after 3 hours, correct to 3 decimal places.

Practice quiz

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8

CONTINUOUS RANDOM SAMPLES AND THE NORMAL DISTRIBUTION

CONTINUOUS RANDOM SAMPLES AND THE NORMAL DISTRIBUTION

TERMINOLOGY

closed interval
continuous
cumulative distribution function (cdf)
discrete
expected value
linear transformation
mean
median
normal distribution
open interval
probability density function (pdf)
quantile
quartile
random variable
range
rectangular distribution
semi-closed interval
standard deviation
standard normal distribution
standard normal score
triangular distribution
uniform distribution
variance
Z-score

Prior learning

GENERAL CONTINUOUS RANDOM VARIABLES

- **use relative frequencies and histograms obtained from data to estimate probabilities associated with a continuous random variable (ACMMM164)**
- **understand the concepts of a probability density function, cumulative distribution function, and probabilities associated with a continuous random variable given by integrals; examine simple examples of continuous random variables and use them in appropriate contexts (ACMMM165)**
- **recognise the expected value, variance and standard deviation of a continuous random variable and evaluate them in simple cases (ACMMM166)**
- **understand the effects of linear changes of scale and origin on the mean and the standard deviation (ACMMM167)**

NORMAL DISTRIBUTIONS

- **identify contexts such as naturally occurring variation that are suitable for modelling by normal random variables (ACMMM168)**
- **recognise features of the probability density function of the normal distribution with mean μ and standard deviation σ and the use of the standard normal distribution (ACMMM169)**
- **calculate probabilities and quantiles associated with a given normal distribution using technology, and use these to solve practical problems (ACMMM170)** AC

8.01 CONTINUOUS RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

You have already studied discrete random variables and distributions. You have also examined the most important of the discrete distributions, the binomial distribution.

In this chapter you will study continuous random variables and the **normal distribution**, which is the most important continuous distribution.

You should already know the difference between a **discrete variable** and a **continuous variable** from your work in earlier years. You have also used **random variables** in Chapters 2 and 5.

IMPORTANT

A **random variable** is a variable with a numerical value that depends on the outcome of a chance experiment. A random variable is **discrete** if the values of the variable (**range**) are either a finite set, the counting numbers or an equivalent set. A random variable is **continuous** if it can take any real value, or any real value from an interval.

A random variable is usually denoted by a capital letter. The corresponding lower-case letter denotes specific values of the variable.

The heights of people selected at random, the time you wait for a bus, the mass of a pebble selected at random from some gravel and length of a randomly selected city street are all examples of continuous random variables whose values are numbers from an interval.

You can use data and relative frequencies to examine the distributions of continuous random variables and estimate associated probabilities. When estimating probabilities across groups, you need to take account of the proportion of each group that is included.

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Example 1

Julian timed the duration of his bus trips to and from school each day for a fortnight. the nearest 30 seconds) it took to get from his stop to school and vice versa, including the time at stops along the way. The times in minutes he obtained were as follows:

Getty Images

$17\frac{1}{2}$, $18\frac{1}{2}$, $19\frac{1}{2}$, $17\frac{1}{2}$, 18, 15, $17\frac{1}{2}$, 15, 15, $15\frac{1}{2}$, 19, $16\frac{1}{2}$, 21, $13\frac{1}{2}$, 17, 17, 15, 19, 15, $14\frac{1}{2}$, 16, $14\frac{1}{2}$, $16\frac{1}{2}$, $17\frac{1}{2}$, 16, 14, 16, $19\frac{1}{2}$

a Draw a frequency table and histogram of the times in 1 minute intervals.

b Use the histogram to estimate the probability of Julian's bus trip taking 14–16 minutes, to the nearest minute.

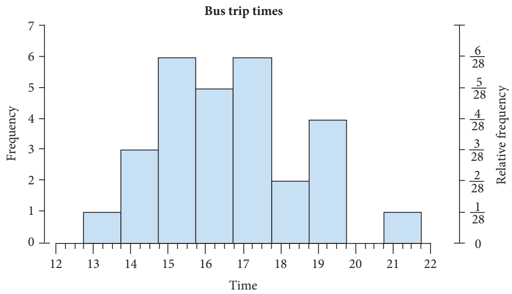
Solution

a The times are from 13.5 minutes to 21 minutes. Draw the table for 13–13.5, 14–14.5, … minutes. It is also a good idea to add a relative frequency column to the table.

Time (minutes)	Frequency	Relative frequency
13–13.5	1	$\frac{1}{28}$
14–14.5	3	$\frac{3}{28}$
15–15.5	6	$\frac{3}{14}$
16–16.5	5	$\frac{5}{28}$
17–17.5	6	$\frac{3}{14}$
18–18.5	2	$\frac{1}{14}$
19–19.5	4	$\frac{1}{7}$
20–20.5	0	0
21–21.5	1	$\frac{1}{28}$
Total	28	1

Draw the histogram, with true class intervals from 11.75–12.75, 12.75–13.75, …

You can put a relative frequency on the RHS to make part **b** easier.



b Write down what parts of the class intervals need to be used for 13.5–16.5 minutes.

13.5–13.75 is 1 of 12.75–13.75, all of 13.75–14.75 and 14.75–15.75 are included and 15.75–16.5 is $\frac{3}{4}$ of 15.75–16.75.

Find the probability for 13.5–16.5 minutes.

$$P(14\text{–}16) = \frac{1}{4}\times\frac{1}{28} + 1\times\frac{3}{28} + 1\times\frac{6}{28} + \frac{3}{4}\times\frac{5}{28}$$
$$= \frac{13}{28} \approx 0.46$$

Write the answer.

The estimated probability of the trip taking between 14–16 minutes, to the nearest minute, is about 46%.

In fact, since the universe itself is finite, there cannot be a real frequency distribution that is on the whole set of real numbers, but for practical purposes there are many that are effectively on the whole set of real numbers. Example **1** showed a practical example of a continuous random variable on a small interval. Although the data was collected discretely, it is still actually continuous. This is an important distinction, because it affects the mathematical treatment of the information.

WS

Discrete and continuous random variables

Some continuous random variables are on intervals that include negative numbers, others start at 0, but many are similar to Example **1**, having intervals between positive bounds. The class interval 5–10 usually means $5 \le x < 10$, but 5–10 inclusive means $5 \le x \le 10$. Strictly between 5 and 10 means $5 < x < 10$.

Example 2

The thickness of a layer of oil on top of some water depends on the amount of oil that is used, the viscosity of the oil, the temperature and the time it is allowed to spread. The thickness can be measured by observing the way angled light is reflected from the oil/water. A researcher made the following observations of thicknesses for different oils at temperatures like those at coastal locations. The thicknesses are in millimetres.

1.62, 0.71, 0.97, 0.04, 1.87, 1.22, 1.39, 0.34, 2.14, 1.27, 2.09, 0.49, 0.31, 2.47, 1.71, 0.91, 2.57, 0.51, 0.67, 0.61, 0.68, 0.74, 0.63, 0.04, 0.46, 2.48, 2.31, 1.69, 0.43, 2.29, 1.88, 2.53, 1.35, 0.43, 1.86

a Draw a histogram of the thicknesses with class widths of 0.25.

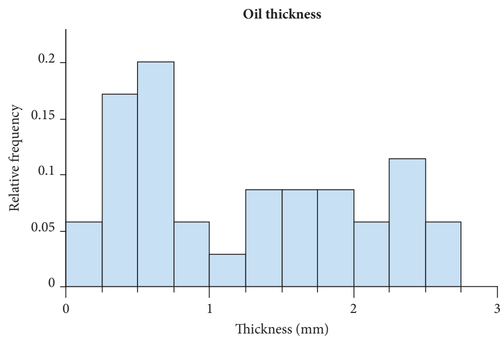
b Find the probability that the thickness of a randomly selected sample is strictly between 0.9 and 1.4 mm.

Solution

a Draw a table first. The thickness cannot be less than 0.

Thickness	Frequency	R. F.
0–0.25	2	0.057
0.25–0.5	6	0.171
0.5–0.75	7	0.2
0.75–1	2	0.057
1–1.25	1	0.029
1.25–1.5	3	0.086
1.5–1.75	3	0.086
1.75–2	3	0.086
2–2.25	2	0.057
2.25–2.5	4	0.114
2.5–2.75	2	0.057
Totals	35	1.000

Now draw the histogram. In this case, the true class intervals are almost the same as the stated class intervals. Use the relative frequency as the vertical scale.



b Write down what parts of the class intervals need to be used.

$0.9–1$ is $\frac{2}{5}$ of $0.75–1$, all of $1–1.25$ is included and $1.25–1.4$ is $\frac{3}{5}$ of $1.25–1.5$.

Find the probability for 0.9–1.4 mm.

$$P(0.9–1.4) = \frac{2}{5} \times 0.057 + 1 \times 0.029 + \frac{3}{5} \times 0.086$$
$$= \frac{18}{56} \approx 0.103$$

Write the answer.

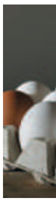
The estimated probability of the oil thickness being strictly between 0.9 and 1.4 mm is about 0.103.

INVESTIGATION Belly button heights

The belly button heights of some Australian Year 12 students who did the Census in schools survey are as follows, showing gender and height in cm.

M, 120; M, 100; M, 106; F, 109; F, 100; M, 110; F, 92; F, 110; M, 95; F, 108; M, 112; F, 119; M, 103; M, 105; F, 94; F, 91; M, 120; F, 100; F, 102; F, 105; F, 100; M, 113; M, 95; F, 97; F, 105; F, 104; F, 100; M, 115; M, 82; M, 108; F, 94; M, 100; M, 100; F, 106; M, 100; M, 110; F, 100; F, 112; F, 80; F, 100; F, 105; F, 100; M, 110; F, 97; F, 108; F, 100; F, 101; F, 100; M, 109; M, 100; F, 94; F, 97; F, 93; M, 106; M, 105

- Construct a histogram of the data.
- Find the probability of a randomly selected Australian student having a BB height within 3 cm of your own.
- Find the mean and standard deviation of the data.
- Collect the data for your class.
- Compare the data for your class with the data for Australian students in general.
- What do you notice about the patterns of the data?



EXERCISE 8.01 Continuous random variables and probability distributions

Concepts and techniques

1 Examples 1, 2 The data below shows the palm widths, to the nearest 0.1 cm, of a group of Year 12 students.

7.2, 8.2, 10.2, 6.4, 6.6, 5.7, 6.4, 5.7, 9.1, 8.8, 11.1, 6.8, 8.8, 9.1, 7.5, 6.5, 9.4, 8.8, 9.3, 7.1, 9.9, 8.9, 5.8, 10, 9.2, 8.3, 11.2, 9, 7.8

a Draw up a frequency table with classes 5.5–5.9, 6.0–6.4, 6.5–6.9, …
b Construct a relative frequency histogram.
c Use the histogram to estimate the probability of a palm width of strictly between 6 and 7 cm?
d Use the histogram to estimate the probability of a palm width of 8 cm, measured to the nearest cm?

2 The data below shows the heights of a group of 17-year-olds to the nearest cm.

162, 161, 158, 169, 161, 180, 166, 159, 171, 163, 166, 163, 181, 170, 178, 175, 164, 169, 174, 160

a Make a relative frequency table with class widths of 5 cm.
b Draw a relative frequency histogram
c Use the histogram to estimate the probability of a randomly selected 17-year-old having a height of 161–164 cm.
d Use the histogram to estimate the probability of having a height over 168 cm?
e Use the histogram to estimate the probability of having a height strictly between 165 and 175 cm?

3 Sarah has kept track of the amount of time she waits to get served when she goes to the post office with the afternoon mail. The times to the nearest minute are shown below.

6, 7, 4, 9, 0, 6, 4, 9, 1, 6, 1, 7, 2, 7, 2, 10, 2, 9, 1, 9, 0, 6, 6, 9, 6, 8, 4, 12

a Make a frequency table with class widths of 2 minutes.
b Draw a relative frequency histogram.
c Estimate the probability of waiting less than 5 minutes.
d Estimate the probability of waiting more than 10 minutes?
e Estimate the probability of waiting strictly between 5 and 10 minutes?

Reasoning and communication

4 The following are the IQs of a group of Year 12 students. IQ is a continuous variable.

90, 111, 89, 123, 137, 97, 120, 101, 101, 101, 110, 108, 101, 85, 103, 121, 92, 130, 109, 112, 121, 139, 100, 83, 141, 82, 94, 94, 119, 104, 95, 87, 108, 104, 101, 115, 106, 93, 90, 128, 107, 110, 98, 138

a Draw up a frequency table with class widths of 10 and construct a histogram.
b What is the probability of an IQ of 100–110?
c What is the probability of an IQ strictly between 100 and 110?
d What is the mean IQ?
e IQ tests are standardised to have a mean of 100. Why isn't the mean of this group 100?

5 Said waits for a tram to go to work in the city each morning. He walks across the King's domain to St Kilda Rd but often stops for a chat on the way. The times in minutes (to the nearest 30 seconds) that he takes to get to the tram stop are given below.

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Alamy/William Caram

2, 7.5, 5.5, 3.5, 1, 4.5, 2.5, 12.5, 11.5, 1.5, 3.5, 0, 3.5, 7, 3.5, 10.5, 8.5, 7.5, 7.5, 4.5, 9.5, 10.5, 12, 15.5, 0.5, 10.5, 0, 1, 2.5, 4, 8, 4.5, 2.5, 7.5, 1.5

a Draw up a frequency table with classes of width 2 minutes and construct a histogram.

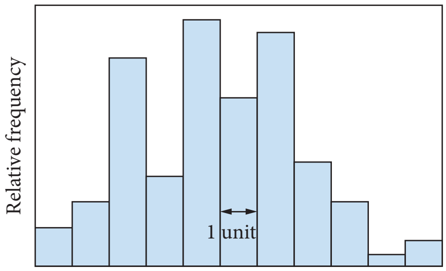
b Estimate the probability of waiting for strictly less than 5 minutes?

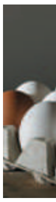
c Can you make any conclusion about the trams going along St Kilda Rd in the morning?

8.02 PROBABILITY DENSITY AND CUMULATIVE DISTRIBUTION FUNCTIONS

The probability of you getting *an exact value* of a continuous random variable is theoretically zero because it is one value from an infinite number of values. In practice, continuous quantities are measurements, and they can never be exact because anything you use for measurement has limited accuracy. As in the previous section, you will want to know the probability of a continuous random variable being between certain limits. Even if you ask 'what is the probability of someone weighing 70 kg?', you really mean a value from 69.5 kg to 70.5 kg, unless you state greater or less accuracy.

Consider the following histogram of relative frequencies worked out from data for class widths of 1 unit.





Each column must be of area equal to the relative frequency. Since the relative frequencies add up to 1, this means that the area of the histogram must also be 1. You work out the probability of getting between particular scores by finding the area of that part of the histogram.

For very large collections of data, you will have more columns and the graph will start to look like a line graph, but the area under the line will still be 1. The probability of getting between particular scores will still be the area under the curve.

IMPORTANT

The **probability density function** for a continuous random variable X is defined as a function $p(x)$ such that $P(c \le X \le d) = \int_c^d p(x)dx$ for all intervals (c, d) on which X is defined.

The probability density function is often referred to by its initials **pdf**.

You should be able to see that a probability density function cannot have negative values because this would give a negative probability. Because the area under the function is a probability, the total area under the function must be 1.

IMPORTANT

A probability density function $p(x)$ for a continuous random variable X defined on the interval $[a, b]$ must satisfy the conditions $p(x) \ge 0$ for $x \in [a, b]$ and $\int_a^b p(x)dx = 1$.

The **closed interval** $a \le x \le b$ includes the end values a and b and is written as $[a, b]$.

The **open interval** $a < x < b$ excludes the end values a and b and is written as (a, b).

The semi-closed interval $a \le x < b$ includes a and excludes b and is written as $[a, b)$.

The real numbers can be written as $(-\infty, \infty)$. It is conventional to exclude 'infinity' because the real numbers are unbounded in either direction, so you cannot reach the end.

Example 3

For each of the following functions, determine whether or not it could be a probability density function on the given interval.

a $f(x) = \frac{3}{32}(4 - x^2)$ for the interval $[-2, 2]$

b $f(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{for } -\infty < x < 0 \text{ or } x > 1 \end{cases}$

c $f(x) = x$ for the interval $[0, 3]$

d $f(x) = \dfrac{1}{2(|x|+1)^2}$ for the real numbers

Solution

a Check that $f(x) \ge 0$.

$|x| \le 2$ on $[-2, 2]$, so $4 - x^2 \ge 0$ and $f(x) \ge 0$

Find $\int_{-2}^2 f(x)dx$.

$$\begin{aligned}\int_{-2}^2 \frac{3}{32}(4 - x^2)dx &= \frac{3}{32}\left[4x - \frac{x^3}{3}\right]_{-2}^2 \\ &= \frac{3}{32}\left[\left(8 - \frac{8}{3}\right) - \left(-8 - \frac{-8}{3}\right)\right] \\ &= \frac{3}{32}\left[\frac{48 - 16}{3}\right] \\ &= 1\end{aligned}$$

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Write the conclusion.

Since $f(x) \geq 0$ and $\int_{-2}^{2} f(x)dx = 1$, $f(x)$ is a pdf on $[-2, 2]$.

b Check that $f(x) \geq 0$.

$3x^2 \geq 0$ so $f(x) \geq 0$

Find $\int_0^1 f(x)dx$.

$$\int_0^1 3x^2 dx = \left[x^3\right]_0^1 = 1$$

Write the conclusion.

Since $f(x) \geq 0$ and $\int_0^1 f(x)dx = 1$,

$$f(x) = \begin{cases} 3x^2 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } -\infty < x < 0 \text{ or } x > 1 \end{cases}$$

is a pdf

c Check that $f(x) \geq 0$.

$x \geq 0$ for $x \in [0, 3]$ so $f(x) \geq 0$

Find $\int_0^3 f(x)dx$.

$$\begin{aligned} \int_0^3 x\,dx &= \left[\frac{x^2}{2}\right]_0^3 \\ &= \frac{9}{2} - 0 \\ &= 4\frac{1}{2} \end{aligned}$$

Write the conclusion.

Since $\int_0^3 f(x)dx \neq 1$, $f(x)$ is not a pdf on $[0, 3]$.

d Check that $f(x) \geq 0$.

$\dfrac{1}{2(|x|+1)^2} \geq 0$ for all $x \in \boldsymbol{R}$ so $f(x) \geq 0$.

Find $\int_{-\infty}^{\infty} f(x)dx$.

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{2(|x|+1)^2} &= \int_{-\infty}^{0} \frac{dx}{2(|x|+1)^2} + \int_{0}^{\infty} \frac{dx}{2(|x|+1)^2} \\ &= \left[-\frac{-1}{2(1-x)}\right]_{-\infty}^{0} + \left[\frac{-1}{2(x+1)}\right]_{0}^{\infty} \\ &= \left(\frac{1}{2} - 0\right) + \left(0 - \frac{-1}{2}\right) \\ &= 1 \end{aligned}$$

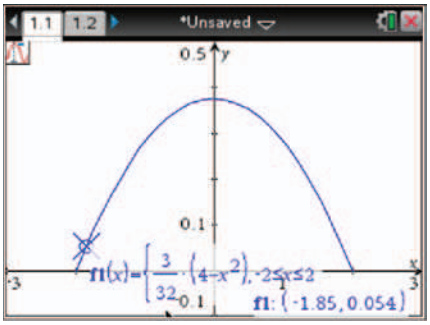
Write the conclusion.

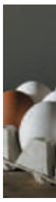
Since $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$,

$f(x) = \dfrac{1}{2(|x|+1)^2}$ is a pdf on $\boldsymbol{R}$.

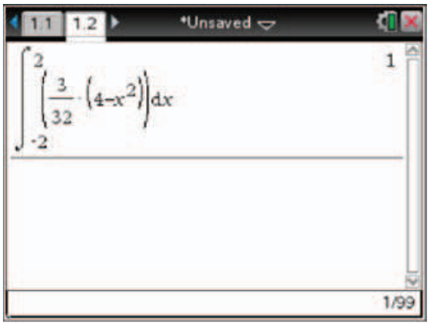
TI-Nspire CAS

a Use the graph page to draw the function for the interval $-2 \leq x \leq 2$ to check the sign. Use ctrl = to get the signs and use Trace and Trace step to check the whole graph.



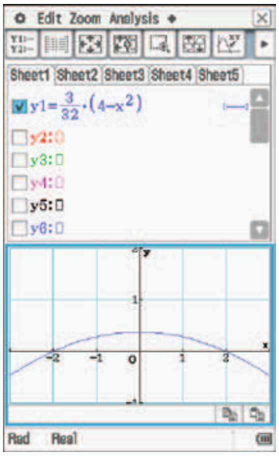


Then use the calculator page to find the integral.

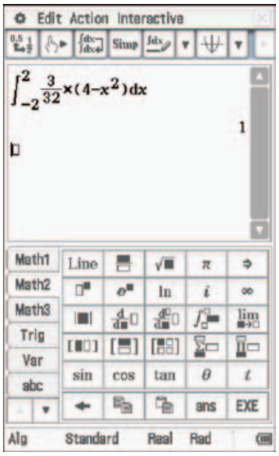


ClassPad

a Use the Graph & Table menu to draw the function for the interval $-3 \le x \le 3$ to check the sign. Use View Window to set $-2 \le x \le 2$ and $-1 \le y \le 2$.



Use the Main menu and (Keyboard) Math2 to calculate the integral. Integrals are found in the Math2 menu. Make sure the calculator is set to Standard.



The area under a curve for the interval (a, b) is the same as the area under the curve for the closed interval $[a, b]$. This means that for a continuous random variable the probabilities $P(a \le X \le b)$, $P(a < X \le b)$, $P(a \le X < b)$ and $P(a < X < b)$ are all the same.

You can make a probability density function from any continuous function on an interval where it is positive by dividing the function by its integral over the interval. It will only be useful when it corresponds to a probability situation.

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Example 4

Make a probability density function using $x^2 - x + 2$ on the interval [−3, 4].

Solution

Check that the function is positive on the interval.

$$x^2 - x + 2 = \left(x - \frac{1}{2}\right)^2 + 1\frac{3}{4} > 0 \text{ for all } x \in \boldsymbol{R}$$

Find $\int_{-3}^{4}(x^2 - x + 2)dx$.

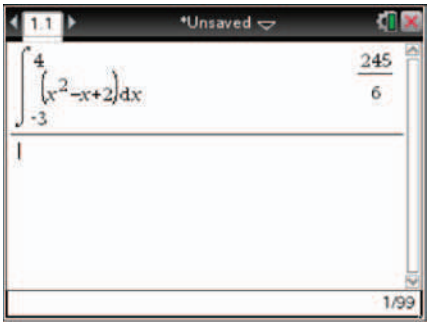
$$\int_{-3}^{4}(x^2 - x + 2)dx = \left[\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-3}^{4}$$
$$= \left(\frac{64}{3} - \frac{16}{2} + 8\right) - \left(\frac{-27}{3} - \frac{9}{2} - 6\right)$$
$$= \frac{245}{6}$$

Make a function such that $\int_{-3}^{4}(x^2 - x + 2)dx = 1$.

$f(x) = \dfrac{6(x^2 - x + 2)}{245}$ is a pdf on [−3, 4].

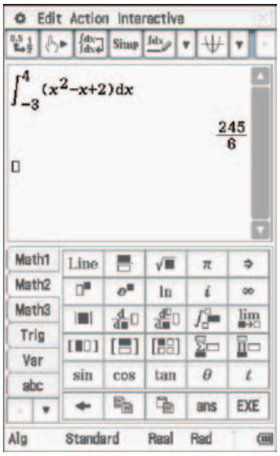
TI-Nspire CAS

You can find the integral using your CAS calculator.



ClassPad

Make sure that the calculator is set to Standard. Use the Main menu and the Math2 menu.



You can use probability density functions to work out probabilities associated with continuous random variables by integrating the function, but it is more useful to use a function that gives the probability directly.

IMPORTANT

The **cumulative distribution function** for the probability density function $f(x)$ of a continuous random variable X defined on the interval (a, b) is given by $F(x) = \int_a^x f(x)dx$.

The cumulative distribution function is often referred to by its initials **cdf**.

It follows immediately that $P(c \le x \le d) = F(d) - F(c)$.

Example 5

a Find the cumulative distribution function for the probability density function $f(X = x) = x^2$ defined on the interval $\left[0, \sqrt[3]{3}\right]$.

b Use the cdf to find the probability that $0.7 < X < 1.3$.

Solution

a Check that f is a pdf.

$$\int_0^{\sqrt[3]{3}} x^2 dx = \left[\frac{x^3}{3}\right]_0^{\sqrt[3]{3}} = 1$$

State the result.

Since $x^2 \ge 0$ as well, it is a pdf.

Find the integral.

$$F(X) = \int_0^x x^2 dx$$
$$= \left[\frac{x^3}{3}\right]_0^x$$
$$= \frac{x^3}{3}$$

b Write $P(a < X < b)$ as the difference of the values of $F(x)$.

$P(0.7 < X < 1.3) = F(1.3) - F(0.7)$

Calculate the value.

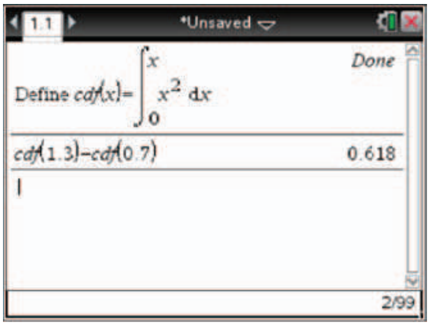
$$= \frac{(1.3)^3}{3} - \frac{(0.7)^3}{3}$$
$$= 0.618$$

Write the answer.

The probability is 0.618.

TI-Nspire CAS

You can define the cdf using the integral on your CAS calculator.



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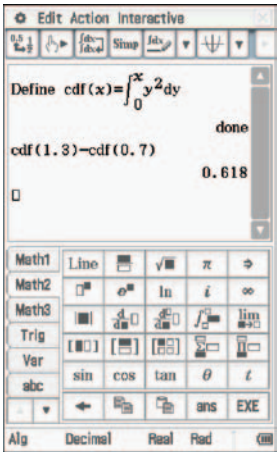
ClassPad

Note that $\int_0^x x^2dx$ does not really make sense, as it literally means the integral from $x = 0$ to $x = x$. The TI-Nspire seems to be able to cope with this, but you must use two variables for the ClassPad.



EXERCISE 8.02 Probability density and cumulative distribution functions

Concepts and techniques

WS
Probability density functions

1 Example 3 For each of the following functions, determine whether or not it could be a probability density function on the given interval.

a $f(x) = 0.5x$ for the interval $[0, 2]$

b $f(x) = \dfrac{1}{(x+1)^2}$ for the interval $[0, \infty)$

c $f(x) = \frac{1}{3}(3 - x)(x + 1)$ for the interval $[0, 3]$

d CAS $f(x) = 4x^3 - 4x$ for the interval $\left[-1, \sqrt{2}\right]$

e CAS $f(x) = \dfrac{e}{7x}$ for the interval $[1, 7]$

2 Example 4 Make probability density functions from each of the following functions on the given intervals.

a $f(x) = \dfrac{1}{(x-1)^2}$ for the interval $[5, \infty)$

b $f(x) = x^3$ for the interval $[0, 4]$

c CAS $f(x) = (3 - x)(x + 3)$ for the interval $[-1, 2]$

d CAS $f(x) = x$ for the interval $[0, 20]$

e CAS $f(x) = e^{2x} - 1$ for the interval $[0, \ln(2)]$

3 Example 5 a Find the cumulative distribution function for the probability density function $f(x) = x^{-2}$ defined on the interval $[1, \infty)$.

b Use the cdf to find the probability that $1 < X < 2$.

c Use the cdf to find the probability that $2 < X < 3$.

d Use the cdf to find the probability that $2 < X < 4$.

e Use the answers to **c** and **d** to find the probability that $3 < X < 4$.

4 CAS a Find the cumulative distribution function for the probability density function $f(x) = \frac{1}{70}(x+2)$ defined on the interval [0, 10].

b Use the cdf to find the probability that $0.5 < X < 1$.

c Use the cdf to find the probability that $2 < X < 4$.

d Use the cdf to find the probability that $5 < X < 10$.

e Use the cdf to find the probability that $4 < X < 6$.

5 CAS a Find the cumulative distribution function for the probability density function $f(x) = e^{-x}$ defined on the interval $[-\ln(1.2), \ln(5)]$.

b Use the cdf to find the probability that $2 < X < 3$.

c Use the cdf to find the probability that $-0.1 < X < 1$.

d Use the cdf to find the probability that $0 < X < 4$.

e Use the cdf to find the probability that $-0.15 < X < 0.15$.

Reasoning and communication

6 The cumulative distribution function for the continuous random variable Q defined for the domain [0, 5] is given by $F(x) = 0.4x - 0.04x^2$. Find the probability density function $f(x)$, describe the shape of $f(x)$ and prove that it is actually a probability density function.

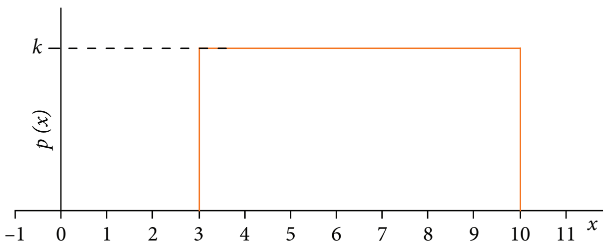
7 A cumulative distribution function is given by $F(t) = 1 - e^{-4t}$ on the domain $[0, \infty)$. Find the probability density function $f(t)$, describe its shape and prove that it is actually a probability density function.

8 CAS A cumulative distribution function is given by $F(m) = 0.1m$ for the domain [0, 10]. Find the probability density function, describe its shape and prove that it is actually a probability density function.

8.03 SIMPLE CONTINUOUS RANDOM VARIABLES

The simplest functions are constants and straight lines. These are also the simplest probability density functions that have practical applications.

Consider a constant probability density function $p(x) = k$ on the interval [3, 10]. Can you work out the value of the constant k?



Since $p(x)$ is a probability density function, the total area under the line from 3 to 10 is 1.

It is a rectangle of length $10 - 3 = 7$. You should be able to see that the area is $7k$, so $7k = 1$ and $k = \frac{1}{7}$. The same method applies for all constant probability density functions.

IMPORTANT

A **uniform continuous probability variable** is one whose probability density function has a constant value on the domain of X.

If X is defined for the domain $[a, b]$, then $p(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise (i.e., } x \notin [a,b]) \end{cases}$

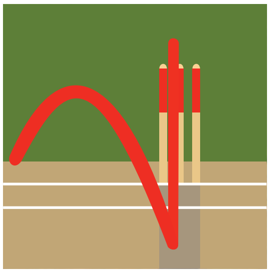
Example 6

A fast bowler aiming for the left stump has equal chances of the ball landing and going anywhere across a 2-foot wide cross-section centred on the outside of the stump. The stumps themselves occupy 9 inches of the cross-section and 1 foot = 12 inches.

a Find the probability density function for X, the distance from the left-hand end of the cross-section that the ball crosses.

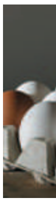
b What is the cumulative distribution function for X?

c What is the probability of the ball being in-line with the stumps?



Solution

a The cross-section is 24 inches wide.	X has domain [0, 24].
Write the probability density function.	$p(x) = \frac{1}{24}$
b Integrate to find the cumulative distribution function.	$P(x) = \int_0^x \frac{1}{24}\,dx = \frac{x}{24}$
c Where are the stumps?	The stumps occupy the interval [12, 21].
Find the probability that they are in-line.	$P(\text{Stumps in-line}) = P(21) - P(12)$
Calculate the value.	$= \frac{21}{24} - \frac{12}{24}$ $= \frac{9}{24}$ $= \frac{3}{8}$
Write the answer.	The probability that the ball is in-line with the stumps is $\frac{3}{8}$.



A uniform probability distribution is sometimes called a **rectangular distribution**. Could there be a useful distribution with a triangular shape?

IMPORTANT

A **triangular continuous random variable** is one whose probability density function $p(x)$ has a graph in the shape of a triangle.

Example 7

A measurement is reported as being 15 cm ± 0.2 cm.

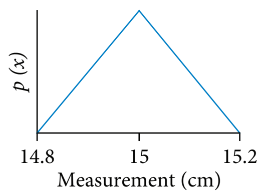
a Draw a triangular graph to show the probability density function.

b Write an expression for the probability density function.

c Find the probability that the measurement is actually 15 cm ± 0.07 cm.

Solution

a The most likely measurement is 15 cm, and the stated error means that the probability of errors diminishes to 0 at 14.8 and 15.2 cm. Draw the graph as a triangle.



b The area $\frac{1}{2}bh$ has to be 1, where $h = p(15)$.

$\frac{1}{2} \times 0.4 \times h = 1$

Solve for the maximum value of $p(x)$.

$h = 5$

Find the slope of the first line.

$m = \frac{5}{0.2} = 25$

Use (14.8, 0) in $y - y_1 = m(x - x_1)$.

$y - 0 = 25(x - 14.8)$

Find the slope of the second line.

$m = \frac{-5}{0.2} = -25$

Use (15.2, 0) in $y - y_1 = m(x - x_1)$.

$y - 0 = -25(x - 15.2)$

Write the probability density function.

$$p(x) = \begin{cases} 25(x - 14.8) & \text{for } 14.8 \le x < 15 \\ -25(x - 15.2) & \text{for } 15 \le x \le 15.2 \end{cases}$$

c There are two parts to integrate.

$P(14.93 \le x < 15) = \int_{14.93}^{15} (25x - 370)\,dx$

Simplify the first integral.

$= 0.288\,75.$

Find the second integral.

$P(15 \le x \le 15.07) = \int_{15}^{15.07} (-25x + 380)\,dx$

Simplify.

$= 0.288\,75$

Write the answer.

The probability that the measurement is15 cm ± 0.07 cm is $2 \times 0.288\,75 = 0.5775$

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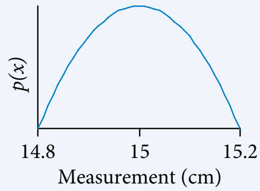
In Example **7** the continuous random variable is defined for $[0, \infty)$, but in practical terms it is really $[14.8, 15.2]$. In cases like this where it is obvious that the domain is restricted to a particular interval, the '0 elsewhere' is usually omitted. Notice that the either one or both parts of the domain could include $x = 15$.

INVESTIGATION Continuous probability functions for measurement errors

The main criticism of using a triangular continuous random variable for measurement errors, as shown in Example **8**, is that it has discontinuities at the ends and in the centre. It also gives equal weight to the possible error being anywhere within the limits.



Some researchers prefer to use a quadratic continuous random variable for measurement errors to give greater 'trust' to the central measurement, so their functions look like the diagram given here.

- Assume that the value of the quadratic probability density function at 15 is h
- Write an expression for the function above using the zeros
- Use $(15, h)$ to write $p(x)$ in terms of h
- Find an expression for the area under the curve in terms of h
- Use the fact that the area = 1 to find the function
- Compare the probability that the measurement is actually 15 cm ± 0.07 cm for a quadratic function with that for a triangular function
- Find other probability density functions used for measurement errors

In some cases, triangular probability density functions have the highest probability at a non-central value.

Example 8

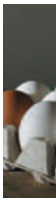
A produce merchant sells chicken feed by the kilogram. Over time, they find that their sales range from 2 kg to 70 kg, with most sales being approximately 20 kg.

a Draw a triangular graph to show the probability density function.

b Write an expression for the probability density function.

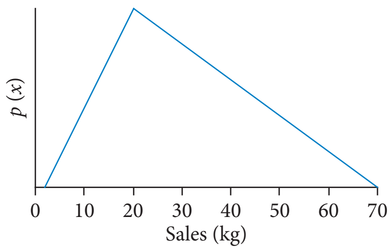
c Find the probability that the next sale is between 30 and 40 kg.

Shutterstock.com/Lisa S.



Solution

a The most likely measurement is 20 kg, and the graph has zeros at 2 kg and 70 kg. Draw the graph as a triangle.



b The area $\frac{1}{2}bh$ has to be 1, and $h = p(20)$.

$\frac{1}{2} \times 68 \times h = 1$

Solve for h.

$h = \frac{1}{34}$

Find the slope of the first line.

$m = \frac{\frac{1}{34}}{18} = \frac{1}{612}$.

Use (2, 0) in $y - y_1 = m(x - x_1)$.

$y - 0 = \frac{1}{612}(x - 2)$

Find the slope of the second line.

$m = \frac{-\frac{1}{34}}{50} = -\frac{1}{1700}$

Use (70, 0) in $y - y_1 = m(x - x_1)$.

$y - 0 = -\frac{1}{1700}(x - 70)$

Write the probability density function.

$$p(x) = \begin{cases} \frac{1}{612}(x-2) & \text{for } 2 \le x < 20 \\ -\frac{1}{1700}(x-70) & \text{for } 20 \le x \le 70 \end{cases}$$

c Find the integral.

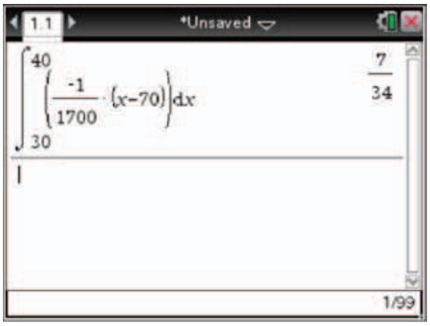
$P(30 \le x < 40) = \int_{30}^{40} -\frac{1}{1700}(x-70)\,dx$

Calculate the answer.

$= 0.205\ 88$

TI-Nspire CAS

You can use your CAS calculator.



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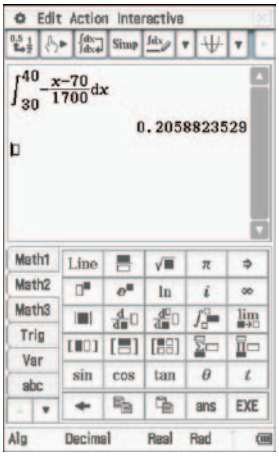
ClassPad

Use CALC from the 2D menu and set the calculator to Decimal.



Write the answer.

The probability that the next sale is between 30 and 40 kg is about 21%.

EXERCISE 8.03 Simple continuous random variables

WS Uniform and triangular probability distribution functions

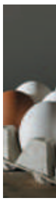
Concepts and techniques

1 Example 6 Find the uniform probability density functions for continuous random variables defined on the following domains.
   a [0, 20] b [0, 18] c [10, 20] d [5, 15] e [6, 36]

2 Example 7 Find the triangular probability density functions with maximum values in the centres for continuous random variables defined on the following domains.
   a [0, 12] b [0, 16] c [12, 22] d [4, 24] e [2, 34]

3 Example 8 A triangular probability density function is defined on the domain [4, 10] with a maximum value at $x = 6$.
   a What is the maximum value of $p(x)$?
   b What is the slope of the line on the left of 6?
   c What is the slope of the line on the right of 6?
   d Write an expression for the probability density function.

4 Find triangular probability density functions with maximum values as stated for continuous random variables defined on the following domains.
   a [5, 15] with a maximum value at 7
   b [4, 10] with a maximum value at 8
   c [20, 30] with a maximum value at 23
   d [0, 20] with a maximum value at 15
   e [20, 90] with a maximum value at 60



Reasoning and communication

5 Johannes has found that the time it takes him to drive to work is between 15 and 20 minutes, with equal chances of being any of the times in between.
   a Construct a probability density function $p(t)$ for the time it takes him to get to work.
   b What is the probability of being late if he leaves 16 minutes before he is due at work?
   c What is the probability of being late if he leaves 18 minutes before he is due at work?

6 Buses go past Carol's stop into the city every 10 minutes, so she doesn't bother looking at the timetable before catching the bus.
   a What are the maximum and minimum times she would wait for the next bus?
   b What is the probability of catching a bus between 0 and 5 minutes after getting to the stop?
   c Construct a probability density function $b(t)$ for the probability of waiting t minutes for the next bus when she reaches the stop.
   d What is the probability of Carol getting a bus within 3 minutes of arriving at the stop?

7 The record long jump at a particular high school is 6.84 m, accurate to ±2 cm. Use a triangular probability density function to find the probability that the jump was less than 6.85 m.

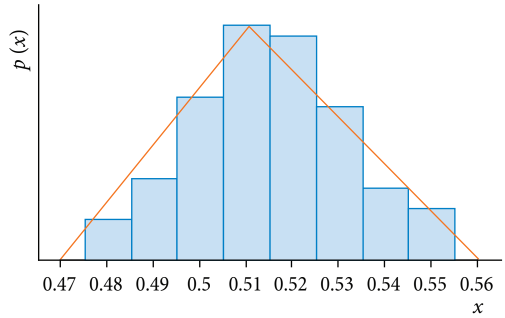
8 Bathroom scales are accurate to about ±2% after they have been zeroed. Use a triangular probability density function to find the probability that someone whose bathroom scales show them as weighing 82 kg is between 81 and 83 kg.

9 In the game of darts, the triple 20 can be considered to be almost a rectangle 8 mm high and 17 mm wide. A world-class player can hit within 8 mm laterally from his target point and 12 mm vertically. With his first dart, the player aims for the centre of the triple 20.
   a Draw a probability density function for the vertical distance from the centre.
   b Construct a triangular probability density function for being within d mm vertically from the centre.
   c What is the probability of getting a triple 20 on the first dart?
   d The shaft of the darts is about 8 mm in diameter. If the player does get a triple 20, how will this affect his target area?

10 Simone has kept note of the time it takes her to get to work. The most common time is 22 minutes, with a minimum of 17 minutes and a maximum of 31 minutes.
   a Construct a triangular probability density function for the time it takes her to get to work.
   b Use your function to find the probability of her taking between strictly between 18 and 20 minutes to get to work.
   c What is the probability of Simone taking 25 minutes to get to work (correct to the nearest minute)?
   d What is the probability of her taking under 25 minutes to get to work?
   e How much time should she allow to get to work?

8.04 EXPECTED VALUE

Front-end loaders are used to load trucks and trailers at a landscape supply business. The bucket on the loader has a nominal size of 0.5 m^3. The actual amount in a bucket was checked 100 times by weight with dry sand, to the nearest 0.002 m^3. The smallest amount was 0.48 m^3, the most common amount was 0.51 m^3 and the maximum was 0.55 m^3. The amounts were rounded to the nearest 0.01 m^3 for a probability histogram drawn with a triangular probability density function on the same graph.



The triangular graph and the histogram are very alike and would both have areas equal to 1.

If the values were discrete, the expected value would be $\sum x \times p(x)$. This is the same as the mean of the values. If the values were rounded to the nearest 0.004 m^3, the graph would have narrower columns and be more like a curve. Since the values are actually continuous, we should use the limit of $\sum x \times p(x)$ as the widths of the columns is decreased. This corresponds to the integral.

IMPORTANT

The **expected value** of a continuous random variable X defined on the interval $[a, b]$ is calculated from the probability density function $p(x)$ as $\mu = E(X) = \int_a^b x \cdot p(x)dx$.

The expected value is the term used for the theoretical average. The actual average for any set of observations is the **mean** calculated from the histogram. The theoretical average corresponds to an infinite number of observations and is the theoretical **mean**.

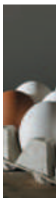
Example 9

Calculate the expected value for a uniform probability density distribution defined on the interval [5, 25].

Solution

Write the pdf.	$p(x) = 0.05$
Write the formula for the expected value.	$E(X) = \int_a^b x \cdot p(x)dx$
Substitute for $[a, b]$ and $p(x)$.	$= \int_5^{25} x \times 0.05\,dx$
Calculate the answer.	$= 15$
Write the answer.	The expected value is 15.

As you would expect, the expected value for the uniform continuous random variable in Example **9** is the centre of the distribution.



IMPORTANT

The expected value of a uniform continuous random variable defined on the interval $[a, b]$ is given by $E(X) = \frac{1}{2}(a+b)$.

You might expect the same result for a triangular distribution. However, this is only true for a symmetrical distribution.

Example 10

Find the expected value of a continuous random variable with a triangular probability density function defined on the interval [0.48, 0.55] with a maximum value at 0.51.

Solution

Find the maximum value, at $p(0.51)$.

$\frac{1}{2} \times 0.07 \times h = 1$, so $p(0.51) = h = \frac{200}{7}$

Find the slopes of the lines.

$m_L = \frac{\frac{200}{7}}{0.03} = \frac{20\,000}{21}$, $m_R = \frac{-\frac{200}{7}}{0.04} = -\frac{5000}{7}$

Write $p(x)$.

$$p(x) = \begin{cases} \frac{20\,000}{21}(x - 0.48) & \text{for } 0.48 \le x < 0.51 \\ -\frac{5000}{7}(x - 0.55) & \text{for } 0.51 \le x \le 0.55 \end{cases}$$

Find the expected value.

$E(X) = \int_a^b x \cdot p(x)dx$

The integral has two parts.

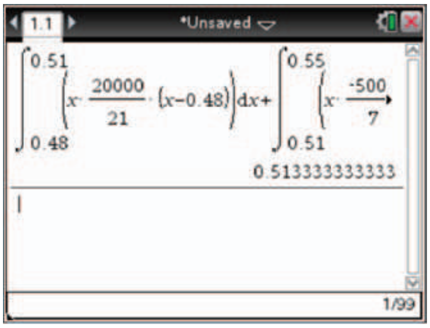
$$= \int_{0.48}^{0.51} x \times \frac{20\,000}{21}(x - 0.48)dx + \int_{0.51}^{0.55} x \times \left[-\frac{5000}{7}(x - 0.55)\right]dx$$

Work out the answer.

≈ 0.513

You can do this on your CAS calculator.

TI-Nspire CAS



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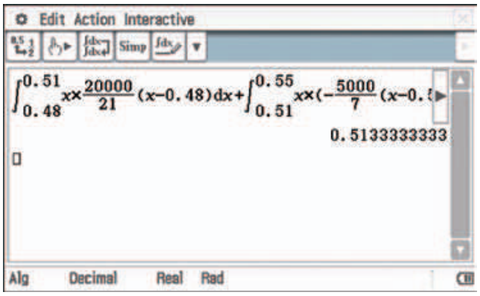

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The expected value in Example **10** is a little higher than the point with the maximum value. This corresponds to the mean being higher than the mode in a distribution that is skewed to the right.

EXERCISE 8.04 Expected value

Concepts and techniques

1 Example 9 **a** What is the probability density function for a uniform continuous random variable S on the interval $[3, 27]$?
b Use integration to find the expected value of S.

2 What is the expected value for a uniform continuous random variable S on the interval $[20, 90]$?

3 What is the expected value for a uniform continuous random variable S on the interval $[8, 36]$?

4 Example 10 Find the expected value of a continuous random variable with a triangular probability density function defined on the interval $[6, 30]$ with a maximum value at 24.

5 CAS Find the expected value of a continuous random variable with a triangular probability density function defined on the interval $[0, 45]$ with a maximum value at 10.

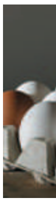
6 CAS Find the expected value of a continuous random variable with a triangular probability density function defined on the interval $[20, 60]$ with a maximum value at 40.

Reasoning and communication

7 Prove that a continuous random variable on $[a, b]$ with a triangular probability density function and maximum in the centre has an expected value of $\frac{a+b}{2}$.

8 CAS The sales of houses in a particular area over a year have a range in prices between \$300 000 and \$750 000, with a mode of about \$450 000.
a Use a triangular distribution function to find the average sale price.
b What is the value of $\int_a^m p(x)dx$ for the median m of a distribution?
c What would you expect the median sale price of houses in this area to be?

8.05 VARIANCE AND STANDARD DEVIATION



In Chapter 5, you saw that the **variance** of a discrete random variable is given by $\sum p(x)(x-\mu)^2$, where μ is the expected value. As with the expected value, the variance for a continuous random variable is worked out using the integral over the interval for which it is defined instead of the sum.

IMPORTANT

The **variance** of a continuous random variable X is calculated using

$$Var(X) = \int_a^b p(x)(x-\mu)^2\,dx$$

where X is defined on $[a, b]$, $p(x)$ is the probability density function and μ is the expected value.

Example 11

Calculate the variance of a uniform continuous random variable defined on the interval [20, 50].

Solution

Find the expected value. $\mu = E(X) = \frac{1}{2}(a+b) = 35$

Find $p(x)$. $p(x) = \frac{1}{50-20} = \frac{1}{30}$

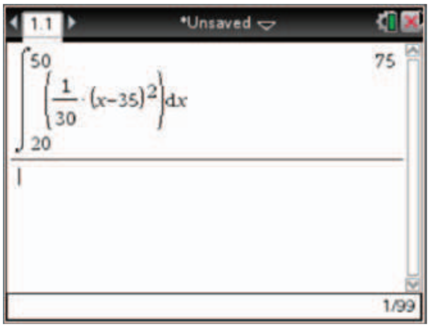
Write the formula. $Var(X) = \int_a^b p(x)(x-\mu)^2\,dx$

Substitute the values. $= \int_{20}^{50} \frac{1}{30}(x-35)^2\,dx$

Calculate the value. $= 75$

You can do the integral on your CAS calculator.

TI-Nspire CAS



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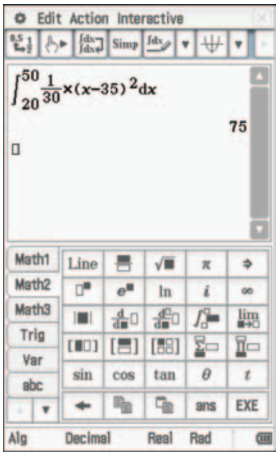


You can find a formula for the variance of a uniform probability distribution. The **standard deviation** is the square root of the variance.

IMPORTANT

The **standard deviation** of a continuous random variable X is given by $\sigma = \sqrt{Var(X)}$. The standard deviation is also written as $SD(X)$ to emphasise the random variable.

The variance of a uniform continuous random variable X defined on the interval $[a, b]$ is given by $Var(X) = \dfrac{(b-a)^2}{12}$, so the standard deviation is $\dfrac{(b-a)}{2\sqrt{3}}$

Example 12

Find the variance and standard deviation of a continuous random variable X defined on the interval [20, 80] with a symmetrical triangular probability density function.

Solution

The maximum value is at $p(50)$.

$\frac{1}{2} \times 60 \times h = 1$, so $p(50) = h = \frac{1}{30}$

Find the slopes of the lines.

$m_L = \dfrac{\frac{1}{30}}{30} = \dfrac{1}{900}$, $m_R = \dfrac{-\frac{1}{30}}{30} = -\dfrac{1}{900}$

Write $p(x)$.

$$p(x) = \begin{cases} \dfrac{1}{900}(x-20) & \text{for } 20 \le x < 50 \\ -\dfrac{1}{900}(x-80) & \text{for } 50 \le x \le 80 \end{cases}$$

Find the expected value.

By symmetry, $E(X) = 50$

Write the formula.

$$Var(X) = \int_a^b p(x)(x-\mu)^2\,dx$$

Split into the two sections.

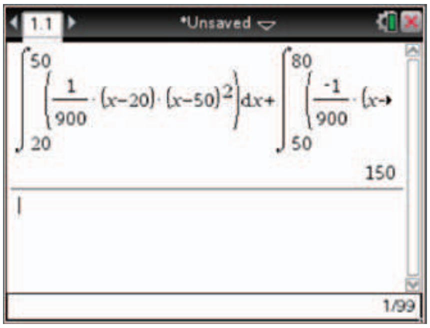
$$= \int_{20}^{50} \frac{1}{900}(x-20)(x-50)^2\,dx + \int_{50}^{80} -\frac{1}{900}(x-80)\,(x-50)^2\,dx$$

Multiply out to evaluate or use your CAS calculator.

$$= \int_{20}^{50} \frac{1}{900}(x^3 - 120x^2 + 4500x - 50000)dx + \int_{50}^{80} -\frac{1}{900}(x^3 - 180x^2 + 10500x - 200000)dx$$

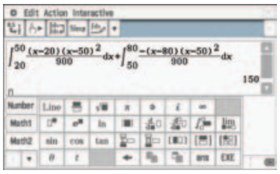
TI-Nspire CAS

Use the integration function on your CAS calculator for each part.



ClassPad

Use the rotate option Rotate so the screen spreads out lengthways.
Enter the two integrals as a sum.
The calculator should be set to decimal mode.



Write the variance. $Var(X) = 150$

Find the standard deviation. $\sigma = \sqrt{150} \approx 12.25$

You can find a formula for the variance of a continuous random variable with a symmetrical triangular probability density function.

IMPORTANT

The variance of a continuous random variable X defined on the interval $[a, b]$ with a symmetrical triangular probability density function is given by $Var(X) = \dfrac{(b-a)^2}{24}$, so the standard deviation is $\sigma = \dfrac{(b-a)}{2\sqrt{6}}$.

EXERCISE 8.05 Variance and standard deviation

Concepts and techniques

1 Example 11 A uniform continuous random variable X is defined on the interval $[4, 16]$.
  a What is the probability density function?
  b What is the expected value?
  c Use calculus to find the variance and standard deviation.

2 Use the formula to find the variance and standard deviation for uniform continuous random variables X defined on the following intervals.
  a $[5, 25]$ b $[0, 50]$ c $[0, 20]$ d $[80, 120]$ e $[0.6, 2.1]$

3 Examples 12 CAS A continuous random variable X is defined on the interval $[12, 20]$ and has a symmetrical triangular probability density function.
  a Find $p(x)$ b Find $E(X)$ c Find $Var(X)$ and σ.

4 CAS A continuous random variable X is defined on the interval $[0, 50]$ and has a symmetrical triangular probability density function.
  a Find $p(x)$ b Find $E(X)$ c Find $Var(X)$ and σ.

5 Use the formula to find the variance and standard deviation for the continuous random variables with symmetrical triangular probability density functions defined on the following intervals.
  a $[5, 15]$ b $[0, 54]$ c $[6, 60]$

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6 CAS A continuous random variable X is defined on the interval [0, 10]. It has a triangular probability density function with its maximum value at $x = 0$.

a Find $p(x)$ b Find $E(X)$ c Find $Var(X)$ and σ.

Reasoning and communication

7 Prove that the variance of a continuous random variable X defined on the interval $[a, b]$ with a symmetrical triangular probability density function is given by $Var(X) = \dfrac{(b-a)^2}{24}$.

8 Find an expression for the variance of a continuous random variable X defined on the interval $[0, a]$ with a symmetrical triangular probability density function that has a maximum at $x = 0$.

8.06 LINEAR CHANGES OF SCALE AND ORIGIN

A linear function is of the form $y = ax + b$.

IMPORTANT

For a random variable X, a **linear change of scale and origin** produces a new random variable $Y = aX + b$, where a and b are constants.

You can use the definitions of expected value, variance and standard deviation to find the effects of a linear change of scale and origin.

Example 13

A uniform continuous random variable X is defined on the interval [24, 144]. It is transformed into the random variable Y according to the equation $Y = 3X - 4$.

a What are the expected value, variance and standard deviation of X?

b What is the interval on which Y is defined?

c What are the expected value, variance and standard deviation of Y?

d What is the relationship between $E(X)$ and $E(Y)$?

e What is the relationship between $Var(X)$ and $Var(Y)$?

Solution

a Use the formula for $E(X)$. $E(X) = \dfrac{a+b}{2} = \dfrac{24+144}{2} = 84$

Use the formula for $Var(X)$. $Var(X) = \dfrac{(b-a)^2}{12} = \dfrac{(144-24)^2}{12} = 1200$

Now find $SD(X)$. $SD(X) = \sqrt{1200} \approx 34.64$

b Transform 24. For $X = 24$, $Y = 3 \times 24 - 4 = 68$

Transform 144. For $X = 144$, $Y = 3 \times 144 - 4 = 428$

Write the answer. Y is defined on the interval [68, 428].

c Use the formula for $E(Y)$. $E(Y) = \frac{a+b}{2} = \frac{68+428}{2} = 248$

Use the formula for $Var(Y)$. $Var(Y) = \frac{(b-a)^2}{12} = \frac{(428-68)^2}{12} = 10\,800$

Now find $SD(Y)$. $SD(Y) = \sqrt{10\,800} \approx 103.92$

d Compare $E(X)$ and $E(Y)$. $3 \times E(X) = 3 \times 84 = 252$

$E(Y) = 3 \times E(X) - 4$

e Compare $Var(X)$ and $Var(Y)$. $9 \times Var(X) = 9 \times 1200 = 10\,800$

$Var(Y) = 3^2 \times Var(X)$

Notice that $SD(Y) = 3 \times SD(X)$ in Example **13**.

Example 14

A continuous random variable X is defined on the interval [30, 90] and has a symmetrical triangular probability density function. It is transformed to the random variable Y according to the equation $Y = 0.5X + 3$.

a What are the expected value, variance and standard deviation of X?

b What is the interval on which Y is defined?

c What are the expected value, variance and standard deviation of Y?

d What is the relationship between $E(X)$ and $E(Y)$?

e What is the relationship between $Var(X)$ and $Var(Y)$?

Solution

a Use the formula for $E(X)$. $E(X) = \frac{a+b}{2} = \frac{30+90}{2} = 60$

Use the formula for $Var(X)$. $Var(X) = \frac{(b-a)^2}{24} = \frac{(90-30)^2}{24} = 150$

Now find $SD(X)$. $SD(X) = \sqrt{150} \approx 12.25$

b Transform 30. For $X = 30$, $Y = 0.5 \times 30 + 3 = 18$

Transform 90. For $X = 90$, $Y = 0.5 \times 90 + 3 = 48$

Write the answer. Y is defined on the interval [18, 48].

c Use the formula for $E(Y)$. $E(Y) = \frac{a+b}{2} = \frac{18+48}{2} = 33$

Use the formula for $Var(Y)$. $Var(Y) = \frac{(b-a)^2}{24} = \frac{(48-18)^2}{24} = 37.5$

Now find $SD(Y)$. $SD(Y) = \sqrt{37.5} \approx 6.12$

d Compare $E(X)$ and $E(Y)$. $0.5 \times E(X) + 3 = 0.5 \times 60 + 3 = 33 = E(Y)$

e Compare $Var(X)$ and $Var(Y)$. $0.25 \times Var(X) = 0.25 \times 150 = 37.5 = Var(Y)$

$Var(Y) = (0.5)^2 \times Var(X)$

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Notice that $SD(Y) = 0.5 \times SD(X)$ in Example **14**.

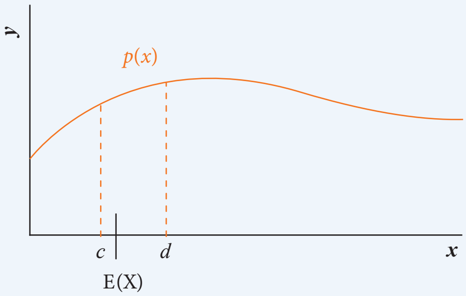
You can apply the methods of Examples **13** and **14** to any uniform or triangular continuous random variables, so the same results will hold for linear transformations of any continuous random variables with uniform or triangular distributions.

INVESTIGATION Linear transformations of continuous random variables

Consider the linear transformation $Y = aX + b$ of some continuous random variable X with probability density function $p(x)$.

Obviously, the graph of probability density function $q(y)$ of the variable Y must be the same basic shape as $p(x)$, but will be altered by the factor a and the shift a along the x-axis.

Consider the part of $p(x)$ from c to d that contains the expected value $E(X)$. It could be shaped as shown below.



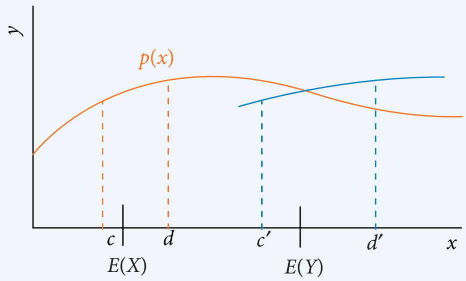
What happens to the interval $[c, d]$ under the linear transformation?

What is c', the value of y corresponding to c under the transformation?

What is d', the value of y corresponding to d under the transformation?

The transformed part of the probability density function could look like the graph below.

The graph is shifted along by the factor b and stretched by the factor a.



Obviously, the probabilities of the two sections must be the same, and the value of $E(Y)$ must be between c' and d'.

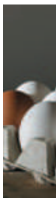
$$P(c \leq x \leq d) = P(c' \leq y \leq d')$$

For this to be true, what must happen to the vertical scale of the blue shifted version of $p(x)$ so that the areas are the same?

What is the value of x corresponding to a value of y?

Write an expression for the probability density function $q(y)$ of Y in terms of $p(x)$.

You can use the expression for the probability density function of Y from the investigation to find the mean, variance and standard deviation of the transformed random variable.



IMPORTANT

For the linear change of scale and origin of a random variable X with probability density function $p(x)$ given by $Y = aX + b$, where a and b are constants, the probability density function of Y is

$$q(y) = \frac{1}{a} p\left(\frac{y-b}{a}\right),$$
$$E(Y) = aE(X) + b,$$
$$Var(Y) = a^2 Var(X) \text{ and}$$
$$SD(Y) = SD(X)$$

EXERCISE 8.06 Linear changes of scale and origin

Concepts and techniques

1 Example 13 A uniform continuous random variable X is defined on the interval [0, 100]. It is transformed to the random variable Y according to the equation $Y = 2X + 5$.
 a What are the expected value, variance and standard deviation of X?
 b What is the interval on which Y is defined?
 c What are the expected value, variance and standard deviation of Y?
 d What are the relationships between $E(X)$, $Var(X)$, $SD(X)$ and $E(Y)$, $Var(Y)$ and $SD(Y)$?

2 A uniform continuous random variable X is defined on the interval [30, 50]. It is transformed to the random variable Y according to the equation $Y = 5X - 2$.
 a What are the expected value, variance and standard deviation of X?
 b What is the interval on which Y is defined?
 c What are the expected value, variance and standard deviation of Y?
 d What are the relationships between $E(X)$, $Var(X)$, $SD(X)$ and $E(Y)$, $Var(Y)$ and $SD(Y)$?

3 A uniform continuous random variable X is defined on the interval [20, 60]. It is transformed to the random variable Y according to the equation $Y = 0.2X + 4$.
 a What are the expected value, variance and standard deviation of X?
 b What is the interval on which Y is defined?
 c What are the expected value, variance and standard deviation of Y?
 d What are the relationships between $E(X)$, $Var(X)$, $SD(X)$ and $E(Y)$, $Var(Y)$ and $SD(Y)$?

4 Example 14 A continuous random variable X is defined on the interval [7, 8] and has a symmetrical triangular probability density function. It is transformed to the random variable Y according to the equation $Y = 20X - 30$.
 a What are the expected value, variance and standard deviation of X?
 b What is the interval on which Y is defined?
 c What are the expected value, variance and standard deviation of Y?
 d What are the relationships between $E(X)$, $Var(X)$, $SD(X)$ and $E(Y)$, $Var(Y)$ and $SD(Y)$?

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5 A continuous random variable X is defined on the interval [0, 50] and has a symmetrical triangular probability density function. It is transformed to the random variable Y according to the equation $Y = 3X - 5$.

a What are the expected value, variance and standard deviation of X?

b What is the interval on which Y is defined?

c What are the expected value, variance and standard deviation of Y?

d What are the relationships between $E(X)$, $Var(X)$, $SD(X)$ and $E(Y)$, $Var(Y)$ and $SD(Y)$?

6 A continuous random variable X is defined on the interval [70, 140] and has a symmetrical triangular probability density function. It is transformed to the random variable Y according to the equation $Y = 0.1X + 2.5$.

a What are the expected value, variance and standard deviation of X?

b What is the interval on which Y is defined?

c What are the expected value, variance and standard deviation of Y?

d What are the relationships between $E(X)$, $Var(X)$, $SD(X)$ and $E(Y)$, $Var(Y)$ and $SD(Y)$?

7 A continuous random variable X is defined on the interval [40, 90] and has an average value of 55 with a standard deviation of 5. It is transformed to the random variable Y according to the equation $Y = 3X + 8$. What are the values of $E(Y)$, $Var(Y)$ and $SD(Y)$?

8 A continuous random variable X is defined on the interval [4, 19]. $E(X) = 14$ and $Var(X) = 8$. X is transformed to the random variable Y according to the equation $Y = 4X - 10$. What are the mean and standard deviation of Y?

Reasoning and communication

9 CAS A continuous random variable X is defined on the interval [0, 25]. It has a triangular probability density function with its maximum value at $x = 0$. It is transformed to the random variable Y by the equation $Y = 8X + 200$.

a Find $p(x)$.

b Find $E(X)$, $Var(X)$ and $SD(X)$.

c What is the interval on which Y is defined?

d Find the probability density function of Y.

e Find $E(Y)$, $Var(Y)$ and $SD(Y)$.

f What are the relationships between $E(X)$, $Var(X)$, $SD(X)$ and $E(Y)$, $Var(Y)$ and $SD(Y)$?

10 CAS A continuous random variable X is defined on the interval [40, 100]. It has a triangular probability density function with its maximum value at $x = 90$. It is transformed to the random variable Y by the equation $Y = 2X - 15$.

a Find $p(x)$.

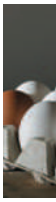
b Find $E(X)$, $VarX)$ and $SD(X)$.

c What is the interval on which Y is defined?

d Find the probability density function of Y.

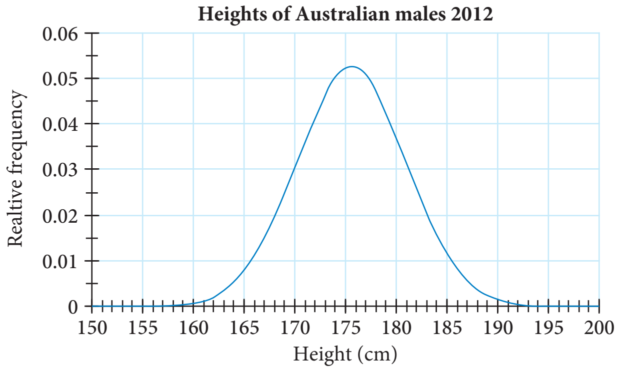
e Find $E(Y)$, $Var(Y)$ and $SD(Y)$.

f What are the relationships between $E(X)$, $Var(X)$, $SD(X)$ and $E(Y)$, $Var(Y)$ and $SD(Y)$?



8.07 THE NORMAL DISTRIBUTION AND STANDARD NORMAL DISTRIBUTION

The tallest man ever recorded was 272 cm and the shortest man in the world is only 53.5 cm. However, most people are between 120 cm and 210 cm tall. There are healthy people with heights outside this range, as some basketball players are 230 cm, but it is extremely rare. The average height of Australian men in 2012 was found to be 175.6 cm. A relative frequency graph of male height in Australia is shown below.

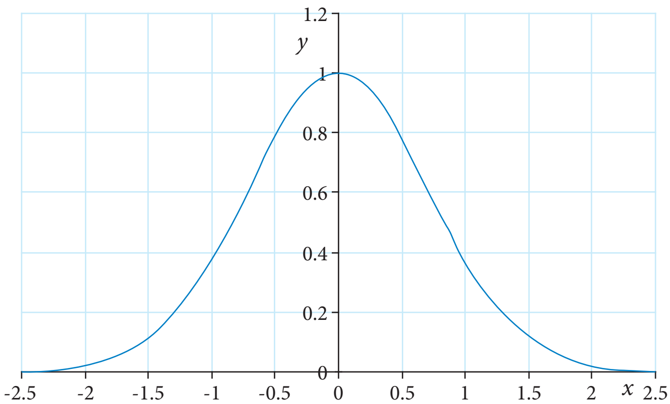


Source: ABS

Although there are adult men in normal health outside the height range shown, the relative frequencies (less than 0.000 03) are too small to be shown on the graph.

A graph like this is often referred to as a 'bell-shaped curve'. It is the most common shape found for continuous random variables that have a central mean and diminishing values to each side, such as human height, IQ, size of raindrops, time taken to complete a task and so on.

The graph of the function $y = e^{-x^2}$ is shown below.



As you can see, the shape of $y = e^{-x^2}$ is identical to the shape of the graph of human height.

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It can be shown that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ so the function needs modification to become a probability density function. Writing $p(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ gives a probability density function that has $E(Z) = 0$ and $Var(Z) = 1$, so $SD(Z) = 1$.

The linear transformation $X = \sigma Z + \mu$ changes the distribution to one with mean $E(X) = \mu$, $D(X) = \sigma$, $Var(X) = \sigma^2$ and probability density function $q(x) = \frac{1}{\sigma} p\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, as shown in the last section.

IMPORTANT

The **standard normal distribution** has the probability density function $p(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$.

The mean and standard deviation of the standard normal variable Z are $\mu = 0$ and $\sigma = 1$.

A **normal distribution** is a linear transformation of the standard normal distribution with a probability density function of the form $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$,

where μ and σ are the mean and standard deviation of the random variable X.

The standard normal distribution and hence any normal distribution assume that the sample space is the set of real numbers, so it is both *continuous* and *infinite*. Notice that the standard normal variable is usually shown as Z.

Example 15

What is the probability density function of a normal distribution with $\mu = 100$ and $\sigma = 15$?

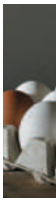
Solution

Write the formula. $\quad p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$,

Substitute $\mu = 100$ and $\sigma = 15$. $\quad = \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2\times 15^2}}$

Simplify. $\quad = 0.0266e^{-0.002\,222(x-100)^2}$

Since the universe is finite, there are no measurements from an infinite sample space. This means that application of a normal distribution function to practical measurements are necessarily approximations, but they are *very good* ones. Example **15** is the probability density function used for IQ, which is standardised to an average of 100 with a standard deviation of 15, even though scores on the tests used have been rising steadily by the equivalent of about 3 IQ points per ten years.



Example 16

In 2012, the ABS found that the average height of adult Australian females was 161.8 cm with a standard deviation of 6.69 cm.

a Use the normal distribution to model the probability density function of Australian women.

b Use the integral on your CAS calculator to find the probability of an Australian woman being under 152 cm (about 5 feet) tall.

Solution

a Substitute $\mu = 161.8$ and $\sigma = 6.69$ in the formula.

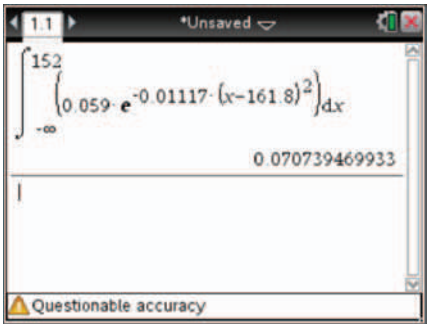
$$p(x) = \frac{1}{6.69\sqrt{2\pi}} e^{-\frac{(x-161.8)^2}{2\times 6.69^2}}$$

Simplify.

$$= 0.0596e^{-0.01117(x-161.8)^2}$$

b Find $\int_{-\infty}^{152} 0.0596e^{-0.01117(x-161.8)^2}\,dx$.

TI-Nspire CAS



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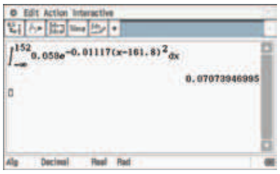

Write the answer.

The probability of an Australian woman being under 152 cm tall is about 7%.

The TI-Nspire gives a warning of 'Questionable accuracy' because the integral cannot be evaluated exactly. The calculator evaluates this integral numerically. In fact, the integral cannot be evaluated algebraically.

As with many complex calculations, tables of values for the standard normal distribution were used until quite recently for calculations normal distributions. The tables showed the areas under the standard normal curve for areas from 0 to Z, where $0 \le Z \le 3$. Since areas outside $-3 \le Z \le 3$ are very small and the curve is symmetrical, this was enough for all but the rarest situations. Modern calculators include normal distribution calculations, so it is unnecessary to use tables.

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Example 17

Use your CAS calculator to find the following.

a The area of the standard normal distribution for $-2 < Z \leq 2$.

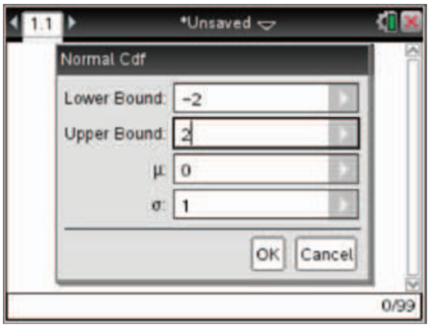
b $P(40 \leq X < 70)$ for the normal variable X with mean 58 and standard deviation 13.

c $P(X > 160)$ for the normal variable X with mean 175 and standard deviation 11.

Solution

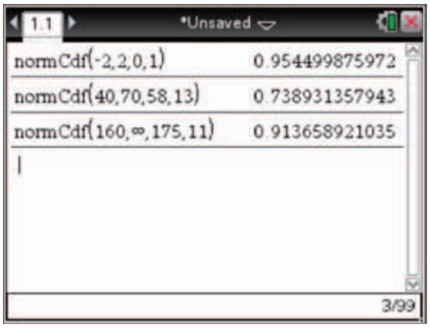
TI-Nspire CAS

a In the **Calculator** ( ).mode, press menu, 6: Statistics ▶, 5: Distributions ▶ and 2: Normal Cdf. Choose –2 for the Lower bound, 2 for the Upper bound, and leave 0 for μ and 1 for σ.



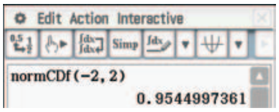
b Choose 40 for the Lower bound, 70 for the Upper bound, and put 58 for μ and 13 for σ.

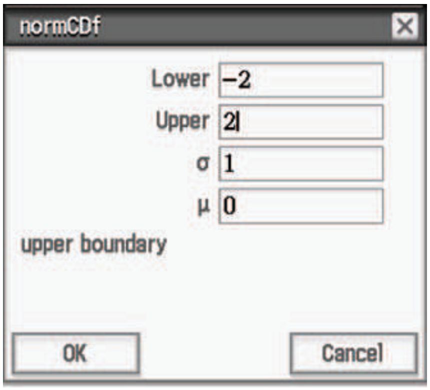
c Use ∞ for the Upper bound.

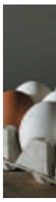


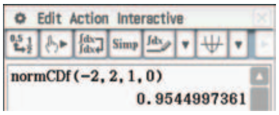
ClassPad

a Tap Action, Distribution/Inv. Dist, Continuous and normCDf. Enter the lower and upper bounds. There is no need to enter a standard deviation or mean as the calculator will assume a standard normal distribution if they are not mentioned. Alternatively tap Interactive, Distribution/ Inv. Dist, Continuous and normCDf. Fill in the table as on the right.



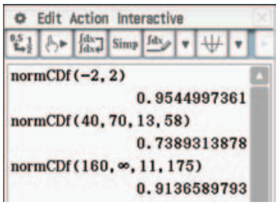






b As for **a**, but add the standard deviation of 13 first, then the mean (58).

c As for **b**, but use ∞ for the upper bound.



Write the answers.

a The area is about 0.9545.
b The probability is about 0.7389.
c The probability is about 0.9137.

What does Example **17** part **a** mean?

For normal distributions, the probability of being within 1 standard deviation of the mean is about 68%, 2 standard deviations is about 95% and 3 standard deviations is about 99.7%. You can check this on your calculator.

WS
The standard normal curve

EXERCISE 8.07 The normal distribution and standard normal distribution

Concepts and techniques

1 Example 15 For each of the following, what is the probability density function for the normal distribution?

a $\mu = 28.5$ and $\sigma = 3.2$
b $\mu = 28.5$ and $\sigma = 5.7$
c $\mu = 48.6$ and $\sigma = 5.7$
d $\mu = 246$ and $\sigma = 78$
e $\mu = 0.07$ and $\sigma = 0.0024$

2 Example 16 CAS For each of the following, state the normal probability density function and find the integral for the stated interval.

a $\mu = 163$ and $\sigma = 8.5$, 150 to 160
b $\mu = 678$ and $\sigma = 147$, 500 to 700
c $\mu = 4240$ and $\sigma = 355$, 4000 to 5000
d $\mu = 6.5$ and $\sigma = 1.8$, 4.9 to 5.8
e $\mu = 74.9$ and $\sigma = 22.6$, 50 to 85

3 Example 17 CAS Find the area of the standard normal distribution table for each of the following intervals.

a $0 \le Z \le 2.3$
b $0 \le Z \le 2.34$
c $0 \le Z \le 1.687$
d $0 \le Z \le 0.058$
e $0 \le Z \le 1.242$

4 CAS Find the following for the standard normal variable Z.

a $P(Z < -1.305)$
b $P(Z > 0.623)$
c $P(Z > 0.596)$
d $P(-1.307 < Z < 2.6)$
e $P(Z < -1.646 \text{ or } Z > 0.831)$

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5 CAS Find the following for the standard normal variable Z.

a $P(-1.356 < Z < 1.873)$ b $P(Z < 0.823)$ c $P(Z > -2.306)$

d $P(Z < 1.6)$ e $P(Z < -0.628 \text{ or } Z > 1.636)$

6 CAS For each random normal variable X with the specified mean and standard deviation given below, find the probability shown.

a $\mu = 8463, \sigma = 2976, P(6766.68 < X < 16766.04)$

b $\mu = 5192, \sigma = 2200, P(2288 < X < 9196)$

c $\mu = 7.15, \sigma = 3.3, P(4.807 < X < 14.773)$

d $\mu = 16, \sigma = 6, P(27.88 < X < 33.04)$

e $\mu = 2.66, \sigma = 0.7, P(4.095 < X < 5.495)$

7 CAS For each random normal variable X with the specified mean and standard deviation given below, use your CAS calculator to find the probability shown.

a $\mu = 18.33, \sigma = 4.68, P(20.9508 < X < 28.4388)$

b $\mu = 2175, \sigma = 625, P(837.5 < X < 3437.5)$

c $\mu = 167.5, \sigma = 82.5, P(21.475 < X < 491.725)$

d $\mu = 26.98, \sigma = 4.94, P(30.7344 < X < 45.6532)$

e $\mu = 25.62, \sigma = 13.44, P(56.6664 < X < 60.1608)$

Reasoning and communication

8 Car tyres have an average life of 50 000 km with a standard deviation of 6150 km. Assuming that the life is normally distributed, find the following.

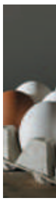
a The probability of a tyre lasting more than 60 000 km

b The probability of a tyre lasting between 45 000 and 55 000 km

c The percentage of tyres that can be expected to last less than 42 000 km

9 A manufacturer of tyres for 4WDs decides to make tyres that last an average of 90 000 km, even though the performance of long-lasting tyres is relatively poor. The standard deviation of the life of these tyres is 8300 km. What percentage of such tyres can be expected to last more than 100 000 km?

10 Over a six-month period, the average exchange rate for the Australian dollar was US\$1.03, with a standard deviation of US\$0.027. The exchange rate is a 'random walk', which means that it is not possible to predict the specific exchange rate at any time and that it is normally distributed. Assume that the pattern is maintained to find the probability that the Australian dollar is worth less than one US dollar.



11 Red wrigglers are an average of 4 cm long, with a standard deviation of 1.2 cm. What is the probability that one of these worms is longer than 5 cm?

12 The average queue time to speak to someone at a bank is 6 minutes, with a standard deviation of 1.5 minutes. What proportion of callers are answered within 3 minutes of calling?

8.08 STANDARDISATION AND QUANTILES

You only need the mean and standard deviation to define a normal distribution. Changes of scale and origin preserve the basic shape and characteristics of a distribution. This means that you can change any normal distribution to any other by using suitable changes to the scale and origin. It is common practice to compare results from different distributions by standardising a score of x to the **standard normal score** z using the linear transformation $Z = \frac{X-\mu}{\sigma}$, where μ and σ are the mean and standard deviation of the normal variable X. The standard normal score is often called the ***Z*-score**.

Example 18

Antoine had a score of 65% for English but only 57% for Physics. The class mean and standard deviation for English were 62% and 8% respectively; for Physics they were 52% and 10%. In which subject did Antoine appear to do better?

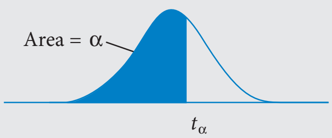
Solution

Calculate the standard normal score for English.	$Z(\text{English}) = \frac{65-62}{8}$ $= 0.375$
Calculate the standard normal score for Physics.	$Z(\text{Physics}) = \frac{57-52}{10}$ $= 0.5$
Write the answer.	Assuming that the classes are of equal ability, Antoine appeared to do better in Physics.

The median divides statistical distributions in half. Quartiles divide distributions into quarters. Percentiles divide distributions into 100 equal groups. **Quantiles** are extensions that divide distributions into proportions of any size.

IMPORTANT

A **quantile** t_α for a continuous random variable X is the value of x such that $P(X < t_\alpha) = \alpha$, where $0 < \alpha < 1$. It is the value that divides the distribution in the ratio $\alpha : 1 - \alpha$.



The **median** is the value m such that $\alpha = 0.5$: $P(X < m) = 0.5$.

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You will already be familiar with quartiles. For a continuous random variable, the **first quartile** is the quantile for $\alpha = 0.25$, so a quarter of the distribution lies below it. To find quartiles of normal distributions, you need to use the *inverse* of the cumulative normal distribution function. This function calculates the quantile t_α.

Example 19

V is a random normal variable with mean \$420 000 and standard deviation \$28 000.

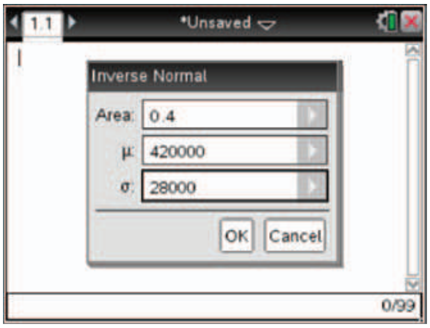
a Find the 40% quantile, the value which 40% of the distribution is below.

b Find the value a such that $P(v > a) = 0.27$.

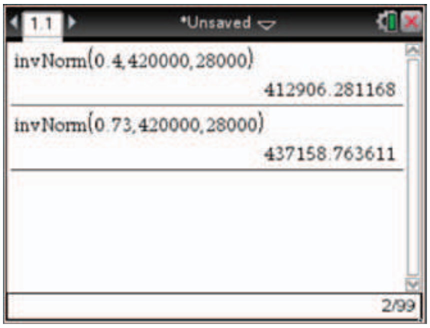
Solution

TI-Nspire CAS

a You need to find $t_{0.4}$.
In the **Calculator** ( ).mode, press menu, 6: Statistics ▶, 5: Distributions ▶ and 3: Inverse Normal. Put 0.4, 420 000 and 28 000 for the Area, μ and σ respectively.

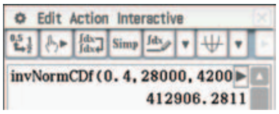


b You want to find the quantile that divides the distribution in the ratio 0.73 : 0.27, so you want $t_{0.73}$.

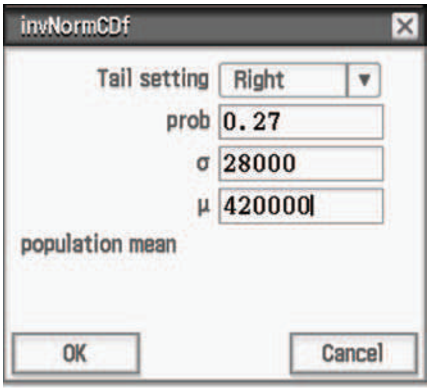


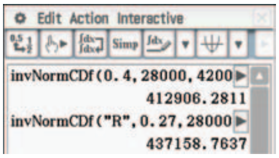
ClassPad

a You need to find $t_{0.4}$.
Tap Action, Distribution/Inv. Dist, Inverse and invNormCDf.
Enter the area, followed by the standard deviation and the mean.
Alternatively, tap Interactive and the same menus and use the table, with the tail setting on Left.



b Tap Interactive, Distribution/Inv. Dist, Inverse and invNormCDf.
Set the Tail to Right, the area is the probability, 0.27, and standard deviation and mean are the same as for part **a**. You can also use the Action menu, but insert the "R" to show that it's a right tail.
Note: Tail setting can also be Center. In this case it assumes the middle area, with both tails cut off symmetrically. In this case use "C", and only the left bound is given.





Write the answers.

a The 40% quantile is about \$413 000.

b The amount which values have a probability of 0.27 of being above is about \$437 000.

EXERCISE 8.08 Standardisation and quantiles

Concepts and techniques

1 Example 19 CAS A random normal variable X has a mean of 10.6 and a standard deviation of 3.4. Find each of the following quantiles, correct to 4 significant figures.

a $t_{0.31}$ b $t_{0.85}$ c $t_{0.224}$ d $t_{0.753}$ e $t_{0.1314}$

2 CAS Find the specified values for each of the following normal variables.

a $\mu = 320, \sigma = 245, P(x < a) = 0.324$
b $\mu = 846, \sigma = 29.7, P(x > a) = 0.592$
c $\mu = 27.8, \sigma = 4.6, P(x \le a) = 0.54$
d $\mu = 39.4, \sigma = 12.6, P(x > a) = 0.82$
e $\mu = 104.5, \sigma = 14.92, P(x > a) = 0.415$

3 A random normal variable X has a mean of 74 and a standard deviation of 8.2.

a What is $P(-\infty \le X \le 64)$?
b Find the value a such that $P(64 \le X < a) = 0.6$.

4 CAS For each of the following random normal variables below, find the specified quantile or value.

a $\mu = 3.63, \sigma = 4.95, P(-\infty < X < a) = 0.254\,627$
b $\mu = 862.4, \sigma = 362.6, P(a < X < 2262.036) = 0.015\,721$
c $\mu = 442, \sigma = 272, P(a < X < \infty) = 0.977\,784$
d $\mu = 720, \sigma = 900, P(-\infty < X < a) = 0.571\,424$
e $\mu = 29.28, \sigma = 12.2, P(13.664 < X < a) = 0.857\,011$

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f $\mu = 686, \sigma = 156.8, P(618.576 < X < a) = 0.666\ 384$
g $\mu = 325.5, \sigma = 148.8, P(a < X < 859.692) = 0.882\ 811$
h $\mu = 4260, \sigma = 1740, P(a < X < \infty) = 0.059\ 38$

5 CAS For each of the following random normal variables below, find the specified quantile or value using your CAS calculator.
a $\mu = 418, \sigma = 199.5, P(-\infty < X < a) = 0.053\ 699$
b $\mu = 530.4, \sigma = 102, P(a < X < 913.92) = 0.408\ 961$
c $\mu = 168, \sigma = 196, P(a < X < \infty) = 0.070\ 781$
d $\mu = 326.8, \sigma = 114, P(-\infty < X < a) = 0.492\ 022$
e $\mu = 25.5, \sigma = 37.4, P(46.818 < X < a) = 0.283\ 855$
f $\mu = 86.49, \sigma = 34.41, P(160.1274 < X < a) = 0.010\ 944$
g $\mu = 73.5, \sigma = 20.3, P(a < X < 164.85) = 0.178\ 783$
h $\mu = 1988, \sigma = 1420, P(a < X < \infty) = 0.397\ 432$

Reasoning and communication

6 Example 18 Callum got 18 out of 25 for English and 15 out of 20 for Maths Methods. The means were 15 and 13 and the standard deviations were 8 and 5 for English and Maths Methods respectively. Use Z-scores to determine which subject he did better in.

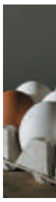
7 Deirdre, who is 17, has an IQ of 110 and a height of 175 cm. The average IQ is 100 and the average height for 17-year-old females is 171 cm with a standard deviation of 12 cm. The standard deviation for IQ is 15. Which is more unusual: Deidre's height as a 17-year-old, or her IQ?

8 Two basketball players have average game totals of 25 points and 29 points respectively. The first has a standard deviation of 9 points and the second a standard deviation of 5 points. Is the first or second player more likely to score more than 37 points in a particular game?

9 Ten-year-old girls can do an average of 11.2 push-ups with a standard deviation of 5.4 push-ups. Which is more unusual for 10-year-old girls: being able to do only 2 push-ups or being able to do 17?

10 The mean and standard deviation of a spelling test were 45 and 13.7. What should the pass mark be set at to ensure that 75% of the people who sat the test passed?

8.09 USING THE NORMAL DISTRIBUTION



The normal distribution is a very good approximation to many situations, particularly for large populations. Real measurements that cluster symmetrically about a mean, with the number of measurements diminishing as the distance from the mean increases, are often approximately normal distributions. The graphs of such measurements are often 'bell-shaped'. Examples include heights and weights of people, other measurements of living things, capacities of manufactured containers and masses of products packed by machine. Most errors are normally distributed, so continuous measurements are usually assumed to be normally distributed.

Example 20

A study of digestion rates found that normal meals were fully digested in an average time of 9.4 hours, with a standard deviation of 2.1 hours. Use this information to find the percentage of people who will digest a normal meal in 8 hours or less.

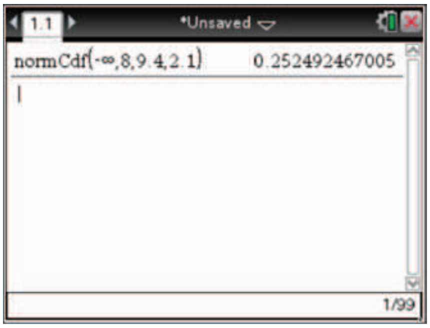
Shutterstock.com/Monkey Business Images

Solution

Write the required proportion for the normal cumulative distribution function.

$P(-\infty < X \leq 8)$

TI-Nspire CAS



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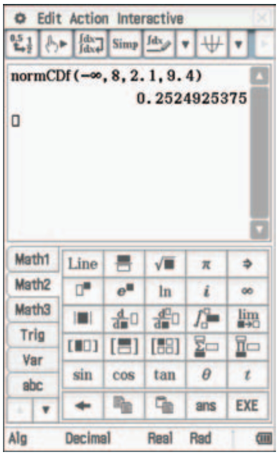

Write the answer.

About 25% of people will have fully digested the meal in 8 hours or less.

When someone says they are 172 cm tall, there is an implied assumption that the measurement is taken to the nearest centimetre. This means that their height is actually between 171.5 cm and 172.5 cm. In the same way, the question ‘What is the probability that a Year 12 boy will weigh 76 kg?’ is actually asking for the probability that his weight is between 75.5 kg and 76.5 kg.

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Example 21

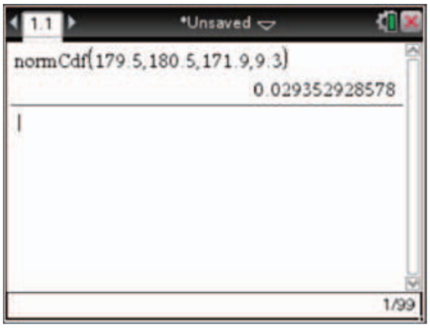
In Australia, Year 12 girls have an average height of 171.9 cm, with a standard deviation of 9.3 cm. What is the probability that a Year 12 girl selected at random will have a height of 180 cm?

Solution

Write the required probability.

$P(179.5 \le X < 170.5)$

TI-Nspire CAS



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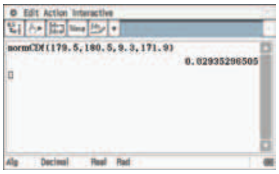
Use the Rotate option to view the screen lengthways.



Write the answer.

The probability that a Year 12 girl has a height of 180 cm is about 0.0294.

Example 22

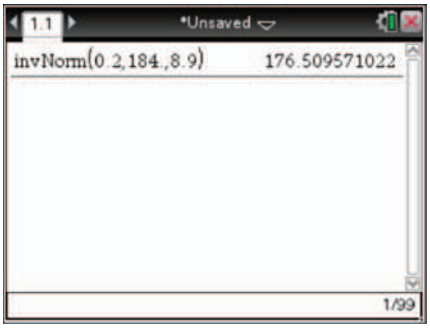
The average height of Year 12 boys is 184.0 cm with a standard deviation of 8.9 cm. Estimate the height of a boy shorter than 80% of all Year 12 boys.

Solution

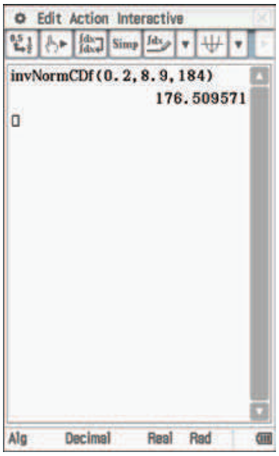
Change to a quantile.

$P(-\infty < X < a) = 0.2$

TI-Nspire CAS

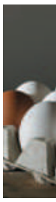


ClassPad



Write the answer.

A Year 12 boy who is 176.5 cm tall will be shorter than 80% of all Year 12 boys.



EXERCISE 8.09 Using the normal distribution

WS Normal distribution – Worded problems 1

WS Normal distribution – Worded problems 2

Reasoning and communication

1 Example 20 The 'life' of young women's fashions measures how long a store can expect a fashion to sell in sufficient quantities to be worth keeping in stock. For some types of fashions it is about 2 months (60 days) but with a wide variation, reflected in the standard deviation of 3 weeks (actually 19 days). What percentage of these fashions would be expected to:

a have a life of over 3 months (90 days)?

b have a life of less than a month?

c be a '9-day-wonder' (less than 10 days)?

2 The average rainfall in a northern Queensland town in January is 268.5 mm, with a standard deviation of 84.5. What is the probability that the rainfall for a particular January is:

a above 220 mm? b less than 190 mm?

3 The average earnings of plumbers are \$36/h, with a standard deviation of \$7.

a From a group of 20 plumbers, how many would you expect to be earning over \$43/h?

b From 50 plumbers, how many would you expect to be earning less than \$38/h?

4 Example 21 Terri is an office worker who starts work at 9:00 a.m. She has found that it takes her an average of 40 minutes to get to work, with a standard deviation of 3 minutes. If she leaves home at 8:15 to allow some extra time, what is the probability that she gets to work at:

a 9:00 a.m. b 8:59 a.m. c 9:01 a.m. d 8:58 a.m. e 9:02 a.m.

f What is the probability that she arrives early?

g What is the probability that she is late?

5 A large factory has 3000 fluorescent tubes. The tubes have an average life of 4500 hours with a standard deviation of 400 hours. What is the probability that a particular tube has a life of:

a 4500 hours b 4000 hours c 4800 hours

d 5000 hours e 4100 hours?

6 Avalon Airport has an average maximum temperature in August of 15.5° with a standard deviation of 2.6°. What is the probability that 15 August this year will have a maximum of 15.5°? What is the probability that the maximum will be about 15°C?

7 Example 22 A car battery manufacturer found that their batteries have an average life of 26 months, with a standard deviation of 2.5 months. For what period could they guarantee the batteries if they want no more than 3% to fail within the guarantee period?

8 40 playing cards are dealt at random from a well-shuffled 'shoe' containing 4 ordinary packs in an ESP test. The people being tested have to guess the correct suit for each card as it is dealt. The psychologist doing the test has calculated that they will get an average of 10 correct, with a standard deviation of 2.7. Even if someone does have 'second-sight', she does not expect them to get the suit right every time. She has decided to use a 5% probability level as her threshold to find people who might have 'second-sight'.

a What number would someone have to get right in order to have less than a 5% chance of doing it by chance?

b When she did the test on 100 people, she found 7 who exceeded the threshold. Is this sufficient evidence to say that some people do have 'second-sight'?

9 The length of bolts produced by a machine is supposed to be 40 mm. The bolts produced actually have an average length of 40 mm with a standard deviation of 0.6 mm. The manager does not want to reject more than 2% as being too long or short. What is the range of length for this level of acceptability?

10 The towns of Weldon and Betterdon have average annual rainfalls of 1048 mm and 839 mm respectively. The standard deviation for Weldon is 255 mm and for Betterdon it is 122 mm. In which town is the annual rainfall more likely to fall below 500 mm?

11 CAS IQ is normally distributed with a mean of 100 and a standard deviation of 15. Students entering university from school are thought to have an IQ of at least 105.

a What is the probability of a person having an IQ over 120?

b Assuming university students do have an IQ over 105, what is the probability that a university student has an IQ of 120 or more? Hint: Use conditional probability.

12 CAS Potatoes are harvested when the tops brown off. Potatoes between 3 cm and 5 cm in diameter are packed as 'chat' potatoes, those smaller than 3 cm are discarded and others are packed as normal potatoes. The average diameter of harvested potatoes is 7 cm with a standard deviation of 2 cm.

a What is the probability that a harvested potato will be a 'chat' potato?

b What is the probability that a packed potato will be a normal potato?

Alamy/Arco Images/Huetter, C.

8 CHAPTER SUMMARY

CONTINUOUS RANDOM SAMPLES AND THE NORMAL DISTRIBUTION

- The **closed interval** $a \le x \le b$ includes the end values a and b and is written as $[a, b]$.
- The **open interval** $a < x < b$ excludes the end values a and b and is written as (a, b).
- The **semi-closed interval** $a \le x < b$ includes a and excludes b and is written as $[a, b)$.
- A **random variable** has a numerical value that depends on the outcome of a chance experiment. It is usually denoted by a capital letter. The corresponding lower-case letter denotes specific values of the variable.
- The **range** (values) of a **discrete** random variable is a finite set, the counting numbers or an equivalent set.
- The range of a **continuous** random variable is an interval of real numbers. The interval can be $(-\infty, \infty)$, the entire set of real numbers.
- The **probability density function** (**pdf**) of a continuous random variable X defined on the interval (a, b) is a function $p(x)$ such that $p(x) \ge 0$ for $x \in (a, b)$ and $\int_a^b p(x)dx = 1$.

  $P(c \le X \le d) = \int_c^d p(x)dx$ for all intervals (c, d) on which X is defined.
- $P(c \le X \le d)$, $P(c < X \le d)$, $P(c \le X < d)$ and $P(c < X < d)$ are all the same because the area of the interval is the same for open and closed intervals.
- The **cumulative distribution function** (**cdf**) for the probability density function $f(x)$ of a continuous random variable X defined on the interval (a, b) is given by $F(x) = \int_a^x p(x)dx$.
- The **expected value** of a continuous random variable X defined on the interval $[a, b]$ is calculated from the probability density function $p(x)$ as $\mu = E(X) = \int_a^b x \cdot p(x)dx$.
- The variance of a continuous random variable X is calculated using $Var(X) = \int_a^b p(x)(x-\mu)^2 dx$, where X is defined on $[a, b]$, $p(x)$ is the probability density function and $\mu = E(X)$.
- The **standard deviation** of a continuous random variable X is given by $SD(X) = \sigma = \sqrt{Var(X)}$.
- A **uniform continuous probability variable** (**rectangular distribution**) is one whose probability density function has a constant value on the domain of X. If X is defined for the domain $[a, b]$, then

  $$p(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise (i.e., } x \notin [a,b]) \end{cases}$$

  The expected value is given by $E(X) = \frac{1}{2}(a+b)$, the variance is $Var(X) = \dfrac{(b-a)^2}{12}$ and the standard deviation is $\sigma = \dfrac{(b-a)}{2\sqrt{3}}$
- A **triangular continuous random variable** is one whose probability density function $p(x)$ has a graph in the shape of a triangle. A **symmetrical triangular distribution** has expected value $E(X) = \frac{1}{2}(a+b)$, variance $Var(X) = \dfrac{(b-a)^2}{24}$ and standard deviation $\sigma = \dfrac{(b-a)}{2\sqrt{6}}$.

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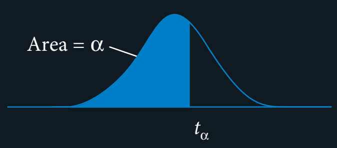
- A **linear change of scale and origin** of a random variable X gives a new random variable $Y = aX + b$, where a and b are constants. The statistics are related by: $E(Y) = aE(X) + b$, $Var(Y) = a^2 Var(X)$ and $SD(Y) = aSD(X)$.
- The **standard normal distribution** has the probability density function $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.
  The mean and standard deviation of the standard normal distribution are $\mu = 0$ and $\sigma = 1$.
- A **normal distribution** is a linear transformation of the standard normal distribution with a probability density function of the form
  $$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$
  where μ and σ are the mean and standard deviation of the random variable.
- The standard normal score z **of a random normal value x is obtained by** $Z = \frac{x-\mu}{\sigma}$, where μ and σ are respectively the mean and standard deviation of X. The standard normal score is called the Z-score.
- The **standard normal score** may be used to compare values in different normal distributions.
- A **quantile** t_α for a continuous random variable X is the value of x such that $P(X < t_\alpha) = \alpha$, where $0 < \alpha < 1$. It is the value that divides the distribution in the ratio $\alpha : 1 - \alpha$.



- The **median** is the value such that $\alpha = 0.5$: $P(X < m) = 0.5$.
- The **first quartile** and **third quartile** of a continuous random variable are the quantiles for $\alpha = 0.25$ and $\alpha = 0.75$ respectively.

CHAPTER REVIEW

CONTINUOUS RANDOM SAMPLES AND THE NORMAL DISTRIBUTION

1 Example 3 Which of the following could be probability density functions on the interval [1, 5]?

I $f(x) = 0.4x$

II $f(x) = \dfrac{1}{x\ln(5)}$

III $f(x) = 0.56x - 0.08x^2 - 0.47$

A I B II C III D II and III E I and III

2 Example 5 The cumulative distribution function of the continuous random variable X on the interval [−1, 1] is $\text{cdf}(x) = 0.5x^3$. The probability density function is:

A $1.5x^2$ B $\dfrac{x^4}{8}$ C $0.15x^2$ D $\dfrac{x^3}{6}$ E $\dfrac{x^2}{6}$

3 Example 9 The expected value of a uniform probability density function on the interval [4, 44] is

A $0.0208\overline{3}$ B $0.0\overline{227}$ C 0.025 D 22 E 24

4 Example 12 The variance of a continuous random variable defined on the interval [16, 40] with symmetrical triangular distribution is:

A $2\sqrt{6}$ B 27 C 48 D $4\sqrt{3}$ E 24

5 Example 17 What is the area under the standard normal distribution for $0 \le Z \le 2.14$?

A 0.4780 B 0.4793 C 0.4823 D 0.4838 E 0.4840

6 Example 18 What is the standard normal score of a score of $x = 28$ for a random normal variable with mean 24 and standard deviation 5?

A 0.45 B 0.8 C 1.17 D 1.25 E 1.4

7 Example 17 X is a random normal variable with a mean of 21 and a standard deviation of 5.3. Find the probability that a random value of X lies between 19 and 22.

A 0.0448 B 0.2219 C 0.3773 D 0.5660 E 0.7547

8 Example 19 A random normal variable G has a mean of 50 and a standard deviation of 12. What is the value of g such that 30% of the scores are below g?

a 32.8 b 38.2 c 39.9 d 43.7 e 48.6

Short answer

9 Example 2 The distances driven by some Year 12 students with learner's licences during the last fortnight were as follows.

217, 152, 127, 101, 110, 330, 301, 308, 127, 161, 136, 199, 136, 138, 158, 106, 250, 198, 75, 102, 320, 58, 111, 133, 127, 113, 147, 373, 108, 368, 207, 144, 176, 132, 150, 117, 237, 125, 224, 262, 119, 217, 283, 113, 159, 175, 145, 244, 253, 172, 124, 109, 290, 242, 144, 94, 125, 188, 225, 165, 357, 131, 289, 293, 229, 194, 137, 99, 179, 147, 189, 116, 144, 138, 273, 166, 150, 216, 119, 187, 136, 225, 74, 108, 296, 144, 121, 173, 221, 147

Draw a histogram of the distances with 50 km class widths and use your graph to find the probability that a randomly selected Year 12 learner driver will drive between 180 and 220 km inclusive in a fortnight.

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10 Example 4 Make a probability density function using $x^3 - 2x^2 + 2$ on the interval [2, 5].

11 Examples 6, 7 An unreinforced concrete path can crack anywhere along its length. An unreinforced clothesline path is 6 m long. Construct a probability density function for the distance of the first crack from the beginning of the path and hence find the probability that the first crack is 2.3 m from the beginning of the path.

12 Example 10 Find the expected value of a continuous random variable with a triangular probability density function defined on the interval [3, 27] with a maximum value at 18.

13 Examples 13, 14 A continuous random variable X is transformed to the random variable Y according to the equation $Y = 2X + 3$. The mean and standard deviation of X are 27.8 and 5.6 respectively. What are the mean and standard deviation of Y?

14 Example 15 What is the probability density function of a normal distribution with $\mu = 76$ and $\sigma = 5.2$?

15 Example 18 Danielle's class got an average of 18.8 on an English test with a standard deviation of 5.4. The same group scored an average of 22.3 on a Maths Methods test with a standard deviation of 3.6. Danielle scored 27 on both tests. In which test did she do better?

16 Example 17 M is a standard normal variable. Calculate each of the following.

a $P(M > -0.7)$

b $P(0.2 \le M \le 2.4)$

17 Example 19 The mean odometer reading of cars pulled up for safety checks was found to be 124 000 km, with a standard deviation of 38 000 km. Assuming the odometer readings were normally distributed, what was the first quartile?

Application

18 Customers in a material shop need to wait until an assistant is available to measure and cut cloth from a roll. The waiting time can be anything from 0 to 15 minutes, with equal chances of any time in between. What is the probability of waiting between 5 and 8 minutes?

19 200 students were asked to estimate the temperature outside. The estimates varied from 23° to 35°. Use a symmetrical triangular distribution to find the probability that a randomly selected student made an estimate of 30°.

20 Students learning to use a pottery wheel take an average time of 25 minutes to make a simple pot. 30% of such students complete their pots within 20 minutes. Assuming that the times are normally distributed, what is the probability of a student taking longer than 28 minutes?

Practice quiz

9

TERMINOLOGY

bias
central limit theorem
cluster sampling
completion bias
convenience sampling
interviewer bias
judgement sampling
non-response bias
parameter
population
pseudo-random number
quota sampling
random number
recall/reporting bias
sample
sample proportion
sampling distribution
selection bias
self-selection bias
simple random sample
statistic
stratified random sampling
systematic sampling

Prior learning

INTERVAL ESTIMATES FOR PROPORTIONS

RANDOM SAMPLES AND PROPORTIONS

RANDOM SAMPLING

- understand the concept of a random sample (ACMMM171)
- discuss sources of bias in samples and procedures to ensure randomness (ACMMM172)
- use graphical displays of simulated data to investigate the variability of random samples from various types of distributions, including uniform, normal and Bernoulli (ACMMM173)

SAMPLE PROPORTIONS

- understand the concept of the sample proportion $\hat{p}$ as a random variable whose value varies between samples, and the formulas for the mean p and standard deviation $\sqrt{\frac{p(1-p)}{n}}$ of the sample proportion $\hat{p}$ (ACMMM174)
- examine the approximate normality of the distribution of $\hat{p}$ for large samples (ACMMM175)
- simulate repeated random sampling, for a variety of values of p and a range of sample sizes, to illustrate the distribution of $\hat{p}$ and the approximate standard normality of $\frac{\hat{p}-p}{\sqrt{\hat{p}(1-\hat{p})/n}}$ where the closeness of the approximation depends on both n and p (ACMMM176)

9.01 RANDOM SAMPLES AND BIAS

Many variables involve the collection of data from very large groups. It may not be practical to use the whole group. It may be more practical to collect information from only part of the group.

IMPORTANT

For any variable or group of variables, the **population** (or **population of interest**) is the whole group from which data could be collected. It is the universal set for the data.

A **sample** is a part of the population.

In a **census**, data is collected from the whole population.

A **parameter** is a characteristic value of a particular population, such as the mean.

A **statistic** is an estimate of a parameter obtained using a sample.

A **survey** obtains the same information from each member of the sample or population. For a survey of people you would ask each person the same questions.

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Example 1

A sample of twenty people waiting in an ATM queue at 7:30 a.m. were asked how much they intended to withdraw. The smallest amount was \$20, the average amount was \$78, and the greatest was \$500. Identify the population, some parameters and statistics.

Getty Images/Bloomberg

Solution

The population is the whole group that could be asked about the amount they withdraw from an ATM.

The population is all the people who use ATMs.

Parameters are clearly defined values from the whole population. You don't need to know the value to define it clearly.

There are 3 parameters: the minimum withdrawal, the average amount withdrawn and the maximum withdrawal.

Statistics are the values you get from the sample. The number of people (20) is not a statistic because it is not an estimate of a parameter.

There are 3 statistics: the minimum withdrawal (\$20), the average withdrawal (\$78) and the maximum withdrawal (\$500).

The sample size is not a parameter because it is not a population property. The size of the population is a parameter.

When you use a sample to find a statistic, you want the statistic to be as close as possible to the population parameter. You need to choose the sample so that it is representative of the population.

Using a very small sample will not usually give you an accurate representation of the population. If you used the extreme case of a sample of size 1, it is obvious that this will not give good results.

You cannot guarantee that a sample will be representative. Suppose that you wanted to find the average income of people in a particular area. Unless you checked everyone, you might miss the one person who was a millionaire. This would obviously have a big effect on the statistic from your sample.

IMPORTANT

A **fair sample** is one that is representative of the population.

A **biased sample** is not representative: it favours some section of the population.

A **random sample** ensures that every member of the population has an equal chance of being chosen.

The statistics from a fair sample are likely to be close to the parameters of the population. Those from a biased sample are unlikely to be close to those of the population.

A random sample is more likely to be fair than one chosen by other means, so statisticians prefer random samples. Unfortunately, random samples of large groups are generally difficult and expensive to obtain.

There are many sources of bias in statistical studies. Investigations involving opinion or feelings are more likely to involve bias than those where you are making measurements.

- **Selection bias** arises from the choice of the sample. This is best avoided by using a random sample.
- **Design flaw bias** arises from faults in the design of the study. Use objective measures wherever possible. If opinion or other subjective measures are required, focus clearly on the question. For example, rather than asking 'Do you support the Liberal party?', ask 'If an election were held this Saturday, which party would you be most likely to vote for?'.

Bias can arise during data collection from differences in the way that data is collected.

- **Interviewer bias** can occur through differences in the way that different interviewers seek information. This can be minimised using standard questions and question options.

In medical trials, special procedures are used to reduce bias. For example, in a double-blind trial some patients are given a drug and others are given a placebo, but the doctors and patients are not told which. This reduces bias caused by the doctor or patient knowledge of treatment.

- **Recall/reporting bias** arises when knowledge of the outcome of one answer affects recall or reporting of the answer. For example, a question about voting intention could affect reporting of past voting practice. Even asking about something that happened the previous week could give false results due to inaccurate recollection.
- **Completion bias** occurs when surveys are incomplete. This can mean that later questions are biased because the questionnaire or interview is abandoned. Surveys should be as short as practical. Longitudinal studies use similar surveys of the same group over an extended period. Completion bias is a big problem, as lost subjects could have a systematic affect on outcomes. A longitudinal study of rural employment in remote areas would be affected by the loss of people who moved away.

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Some people will refuse to answer sensitive questions. This is minimised by avoiding questions of potential embarrassment. For example, rather than asking 'how much do you earn in a week?' put it as 'tick the box that shows your income category'.

- **Non-response bias** occurs when some subjects do not take part in the survey. You should be sure that this does not systematically affect results. The most extreme example of non-response bias is a **self-selected sample**. An example of a such a sample is the group of people who respond to a media poll, such as a TV program that asks people to respond 'Yes' or 'No' to a question by ringing up, texting or logging on to a site.

Even if a survey is framed and conducted with minimum bias, bias can still be introduced by inappropriate analysis or reporting of results.

Example 2

In each of the following cases, state whether or not the sampling method is fair, and if it is biased, state the kind(s) of bias.

a An interviewer outside a supermarket on Saturday morning asked people going in: 'Do you prefer *Razzle* dishwasher detergent or an inferior brand?'

b 2000 mobile phone numbers were telephoned at random and people answering were asked: 'What kind of dishwasher detergent do you use?'

c 200 people are chosen at random from the electoral roll of a 'litmus-test' electorate and interviewed at home on a Sunday morning. They are asked 'If an election were held tomorrow, who would you vote for?' Anyone who wasn't home was contacted later at a follow-up that evening. Altogether, 190 people were successfully contacted and only 10 refused to answer. They were put into the 'don't know' category.

Solution

a Those interviewed were available at a particular place and time only. They were asked a leading question.

The survey is biased, with both selection bias and design flaw bias.

b Mobile phones are still not common among elderly people. The question seems fair, but some people would just hang up.

While the design of the question is good, there is some selection bias and there is likely to be some non-response bias.

c Within the electorate, the method and follow-up gives as close to a random selection as practical. However, just because this electorate has gone with the government in the past doesn't mean it will in the future.

The method is fair within the electorate, but if the intention is to predict the result in terms of government, there is selection bias.

Example **2** shows that it is virtually impossible to avoid some bias in a survey, and that this may even be true for a census because of non-response bias. The Australian Bureau of Statistics (ABS) has the legal power to demand answers for its surveys, but this does not guarantee that people will give genuine answers. For important surveys, they use advanced statistical methods to ensure that the final analysis is as free of bias as possible.

INVESTIGATION Political polls

Polls of voting intentions generally have samples of 1000–3000 voters from a large part of the state or nation. Examine some recent polls to find how they have selected the people they use, and how they ensure as much randomness as possible.

Such polls often have the collected data analysed to produce a 'two-party preferred' result. In this kind of analysis, what do they do about people who did not answer or who gave an answer of 'don't know'? Is the method reasonable?

What do they do about people who said they would vote for one of the candidates or parties not selected as one of the two parties? Is this a reasonable method?

Historically, one of the most famous political polls was the 'Readers Digest' poll of 1936. It predicted a 60–40 massive loss for the US presidential candidate who actually won a short time later in a 60–40 landslide. This poll was sent out to 10 million voters. What went wrong with this survey?

Shutterstock.com/PRANAV VORA

It is not always easy to collect data for statistics to estimate a population parameter. Sometimes it is better to use a sample to find a *related* statistic that can be used in combination with other statistics to obtain the desired information.

Example 3

Which statistic is easier (and cheaper) to collect?

A The proportion of full-time workers who are women.

B The proportion of women who are full-time workers.

Solution

To find an unbiased sample of full-time workers might prove difficult, as people work in so many different industries and have such different work hours. However, it is relatively easy (and cheap) to obtain an unbiased sample of women by using the electoral rolls.

Write the answer. **B** is the easier (and thus cheaper) statistic to collect.

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EXERCISE 9.01 Random samples and bias

Concepts and techniques

1 Example 1 Identify the population, some parameters and some statistics for each of the following.

a The weights of 5 meat pies produced at a pie factory were 105 g, 110 g, 98 g, 101 g and 102 g. The quality control officer also found that all pies were of even colour.

b A feedback sheet left in guest rooms at a hotel had 2 questions about room service (each on a 5-point scale of Very good to Very poor). The responses of 10 guests were as follows:
Food quality: Very good 3, Good 4, Average 1, Poor 1, Very poor 1
Service speed: Very good 3, Good 3, Average 1, Poor 2, Very poor 1
The manager told kitchen staff that they had scored a rating of 'only 3.7' but this was 'better than the service staff'.

c The numbers of passengers on 10 successive 55-seat buses at Holland Park bus station were 15, 30, 45, 20, 20, 25, 46, 48, 16 and 32.

d Thirty people arriving at Cairns airport to catch flights were asked how long it had taken them to reach the airport. Five said less than 10 minutes, 17 said between 10 and 20 minutes, 6 said between 20 and 30 minutes and the other 2 said between 30 and 40 minutes.

e A large supermarket kept records of checkout operators. From 15 shift records selected at random, there were 4 operators with more than 5 errors, 3 operators with incorrect till totals and 2 operators who had processed less than $10 000 in their shift.

Alamy/Blend Images

2 Example 3 Which statistic is easier to collect?

A The proportion of male drivers who have serious car accidents.

B The proportion of drivers in serious car accidents who are male.

3 Which statistic is easier to collect?

A The proportion of timber workers who make Work Cover claims.

B The proportion of Work Cover claims made by timber workers.

4 Which statistic is easier to collect?

A The proportion of serious injuries to children incurred on school playground equipment.

B The proportion of children using school playground equipment who are seriously injured on the equipment.

Reasoning and communication

5 Example 2 In each of the following cases, state whether or not the sampling method is fair, and if it isn't, state the kind(s) of bias.

a A hardware store wants to know if it would be worthwhile staying open for longer hours. A survey placed on the counter asks customers to tick the times they are likely to shop in the store from a list.

b A student in Year 12 goes to Year 8 classes to investigate the amount of pocket money that they receive from their parents. He asks students in the classes to tell him how much pocket money they got last week and writes down the responses from each.

c A reality TV show eliminates one contestant each week by having people SMS their choice of who gets eliminated to a particular number each week. They have the system set up so that only one vote is accepted from each mobile number.

d A council needs to establish a new landfill site for rubbish as the old one is almost full. Some councillors urge the council to use 'people power' to decide the new site by asking residents close to each of the possible sites for their opinion of the best site. It is proposed that people attending public meetings near each site will be asked to vote on the suitability of the nearby site.

e A food critic goes to different restaurants and tries a selection of dishes on the menu before rating the restaurants with 1–10 scales on ambience, presentation, quality of cooking and variety of the menu. The critic is very well-known in the area and his opinion is valued highly. Reviews of three restaurants are published the following week.

6 A school surveys its students at the beginning of the year to determine the amount of part-time work they do.

a What is the population for this survey?

b What are the parameters?

c What problems may this survey have in determining the parameters?

7 A high school hall is set up for Year 12 examinations, with students seated alphabetically in rows of 25 from front to back with 11 desks in each row. Some desks are vacant due to student illness. Students are selected to provide feedback on the conduct of the exam. Every 15th student is chosen, starting with the 3rd desk in the front row and working across to the right, then back to the left in the next row, and so on until 10 students are selected. Comment on the method of sampling.

8 Here are some ways a student proposes to collect a sample of students in a school for interview. State the bias that may be present for each method, and select the one that you think is the fairest.

a Ask everyone in your class.

b Ask the first 80 students who walk into the resource centre.

c Ask all students in your year level.

d In school assembly, announce that there will be a poll and ask the first 70 students who volunteer to do the survey.

e Obtain an alphabetical list of all the students at the school and ask every 20th student on the list.

f Leave a 'nomination sheet' in the resource centre and ask only those people who write their names on it.

g Ask for 5 volunteers from every form class in the school.

h Ask 1 in every 15 students from each year level in the school.

i Call a meeting of all interested students and ask all the people who attend the meeting.

j Wait at the entrance of the school and ask the first 100 students who arrive after 7:30 a.m.

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9.02 SELECTION OF SAMPLES

However well a study or survey is designed, administered, analysed and reported, if a poor sample is chosen, the results are likely to be poor. The statistics will probably be bad estimates of the parameters.

The simplest way to obtain a random sample is to number the population in some way and then use **random numbers** to select the sample. You can choose to use a numbering system that is already present in the population. For example, you could use serial numbers of manufactured goods or identification numbers of the target population, like student numbers for school students. For very large populations, such as 'all Australians', it could be difficult to assign numbers.

Even if you do have a numbering system, you then need random numbers. Tables of random numbers produced from random physical phenomena are available and are commonly used when truly random numbers are required. Some hardware random number generators are also available.

The short table of 2-digit random numbers on page 468 shows what you might expect, although books of random numbers generally have at least 4 digits. If you need fewer digits, it is usual to use the first ones in the number. If you want more digits, two or more numbers are taken together.

Example 4

a Use the two-digit random number table on page 468 to randomly select 6 numbers from 87 to 524.

b CAS Select 6 random numbers from 87 to 524.

Solution

a Start at, say, the 16th column in the 11th row. — The first two numbers are 24 70.

04 92 03 87 51	08 13 11 48 36	98 73 32 94 11	01 78 95 19 70	13 84 91 57 67
05 04 13 40 88	75 68 99 63 19	56 69 99 33 68	24 70 05 25 64	42 41 85 04 88
30 64 49 26 22	93 66 84 39 90	57 91 05 63 53	86 05 39 32 61	67 10 68 26 73
16 02 93 88 42	32 97 19 48 39	27 00 17 29 98	95 33 02 15 35	84 54 88 77 88
90 72 79 41 71	30 19 99 89 25	18 77 55 49 03	75 26 66 89 31	45 75 85 95 16
99 31 34 95 97	50 56 14 09 36	63 23 12 58 28	64 16 96 92 62	73 96 99 48 21
55 38 06 44 27	29 38 61 58 15	66 43 42 97 45	51 03 81 16 99	06 55 69 43 88

Use the first 3 digits. — The number is 247.

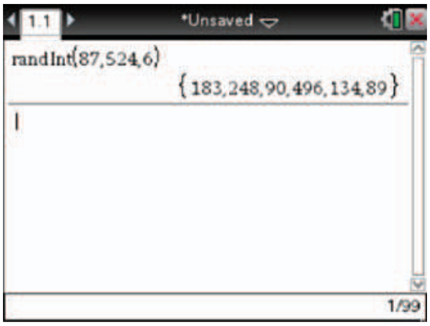
Get the next number. — Next is 05 25, but 052 is less than 87, so discard it.

Get the next number. — Next is 64 42, so 644.

Keep going in the same way. — 6 random numbers are 247, 644, 418, 306, 492 and 229.

TI-Nspire CAS

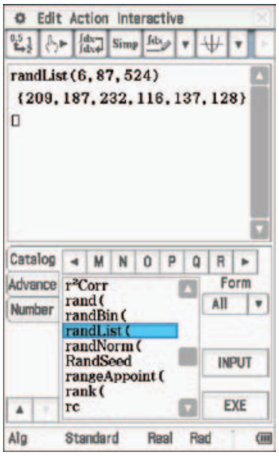
Press menu 5: Probability, 4: Random and 2: Integer and put in randInt(87, 524,6).
A different set of numbers is chosen each time.



b Write the answer. 6 random numbers are 183, 248, 90, 496, 134, 89.

ClassPad

Use the Main menu. Press Keyboard. Tap ▾ to get to the Catalog of functions.
Select randList(and fill in the order as follows: the number of random numbers required, the lowest integer in the range, the highest integer in the range.
A different set of numbers is chosen each time.
You may find it useful to use the screen rotation if there are more random numbers, or simply choose fewer at the start.



b Write the answer. 6 random numbers are 209, 187, 232, 116, 137, 128.

Once you have started using a table of random numbers, you usually continue any subsequent use from the place you last finished. This avoids the possibility of repeating the same set.

Using truly random numbers is very time-consuming, so it is more common to use **pseudo-random** numbers generated by a rule that produces numbers that are difficult to distinguish from random numbers. This is the method used by most modern calculators and computers, such as your CAS calculator.

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INVESTIGATION Pseudo-random numbers

A very simple pseudo-random number rule is

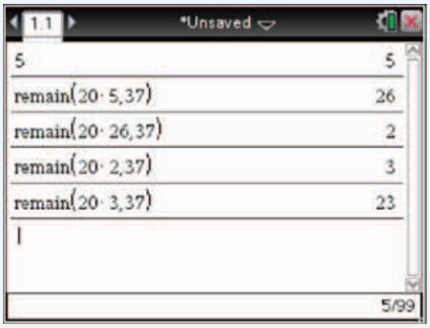
$x_n = 20x_{n-1} \bmod (37)$, where $a \bmod (b)$ is the remainder when a is divided by b.

Start with 5 (the seed) and generate random numbers with this rule. What do you get?

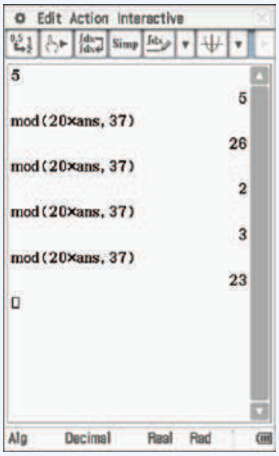
How long does it take for the sequence to repeat itself?

You can use your CAS calculator by entering 5 and using the rule on the answer. Note that in the TI-Nspire, the mod function is called remain and 'ans' is replaced by the number. Pressing enter will repeat the calculation.

TI-Nspire CAS



ClassPad



How long does it take before this sequence of pseudo-random numbers repeats?

Try the rule $x_n = 12x_{n-1} \bmod 19$, starting from 7. How long does this take to repeat?

Random numbers are used extensively in encryption of computer messages, particularly payments over the internet. Obviously, the pseudo-random number used must be non-repeating over a very long sequence.

The Blum Blum Shub generator, invented in 1986, is given by $x_n = (x_{n-1})^2 \bmod (pq)$, where p and q are large prime numbers. Try this rule for the prime numbers 6047 and 4723, starting from $n = 3$.

Investigate how random numbers are used for internet encryption.

There are a number of variations of random sampling that are used to avoid numbering the whole population. You can choose representatives of different groups, you can choose different groups or you can choose individuals from a group.

IMPORTANT

Simple random sampling is choosing your sample at random from the whole population.

Stratified random sampling is choosing representatives at random from identifiable groups of the population in proportion to the size of each group.

Cluster sampling is choosing **group(s)** at random from the population as the sample.

Systematic sampling is choosing representatives from the population by taking every nth to make the desired sample size.

Example 5

A distributor has 34 office staff, 23 store employees and 43 delivery drivers. How many of each should be selected to make a stratified random sample of 10?

Solution

Find the proportion of each.	The proportions are $\frac{34}{100}:\frac{23}{100}:\frac{43}{100}$, office : store : delivery.
Find the number of office staff.	Number of office staff $=\frac{34}{100}\times 10=3.4\approx 3$
Find the number of store staff.	Number of store staff $=\frac{23}{100}\times 10=2.3\approx 2$
Find the number of delivery staff.	Number of delivery staff $=\frac{43}{100}\times 10=4.3\approx 4$
Write the answer.	For the balanced sample closest to 10, there should be 2 store staff, 4 drivers and 3 from the office, selected at random from each group.

If you had to have a sample of exactly 10 in Example **5**, you would choose an extra person from the office. It is closest to an extra half more, so would upset the balance by the smallest amount. If you were one over and needed the exact number, you would choose the one closest to needing a half less.

When you use a systematic sample, you should start at a random place in your list. If you go past the end, you just go back to the start and keep counting.

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Example 6

The library ID numbers of Year 8 students in a school run from 247 to 395 inclusive, since Year 8 library cards were issued after the Year 7 cards. Use the ID numbers for a systematic sample of 20 Year 8 students.

Solution

Find the number of Year 8 students.	Number of Year 8s = 395 − 247 + 1 = 149
Find the repeat for selection.	$149 \div 20 \approx 7$
Pick a place to start at random.	Start with student number 328.
Choose every 7th student. When you get to the end of the list, still count in 7s into the start of the list.	The students chosen should be those with library IDs 328, 335, 342, 349, 356, 363, 370, 377, 384, 391, 249, 256, 263, 270, 277, 284, 291, 298, 305 and 312.

While the best samples are generally considered to be random samples, they can be difficult and expensive to use. For a random sample to be valid, you need to make sure that the whole selected sample is used. When people are involved, this can be difficult, as some may not be available when you try to obtain responses. You cannot ignore people who are absent as you could end up excluding a particular group when you only do surveys at particular times, and this could cause significant bias. In practice, cost dictates the number of times you try to include responses from absent selections.

Random sampling of people is difficult and expensive, so surveys may be conducted using non-random samples. You might just choose a convenient group, such as people who live close by; you might try to include people you think would be representative of the population; you might decide to include people who attend a particular event; or you might know that 6% of the adult population are unemployed, 28% do not work (retired, disabled, etc.) and the rest are working so you choose quotas of 3 unemployed people, 14 non-workers and 33 workers from a crowd of people.

IMPORTANT

Convenience sampling is choosing from a convenient group of the population.

Judgment sampling is the use of judgement to determine a representative sample.

Purposive sampling is choosing representatives that meet particular conditions.

Quota sampling is choosing the first convenient representatives for a sample according to the proportions of particular divisions of the population, such as male and female.

If you were choosing a sample from the company staff in Example **5** and you were in the office, then just using office staff would be a convenient sample. If you decided on people you thought would make the best sample, it would be judgement sampling. If you decided to choose people who had been employed for more than 5 years at the company, it would be purposive sampling and if you just used the first 4 office staff, the first 4 drivers and the first 2 store staff you came across, it would be quota sampling.

INVESTIGATION Random and non-random samples

WS Sampling methods

Use newspapers, magazines, TV reports, advertisements and internet reports of surveys.

Classify the surveys as random or non-random and then further classify them into types.

Order the surveys so that they show how the samples used are representative of the population involved. Write a report of your findings.

EXERCISE 9.02 Selection of samples

Concepts and techniques

1 Example 4 Start at row 8, column 17 of the 2-digit random number table on page 468 and select 8 different numbers that are:
   a 2-digit numbers between 20 and 99
   b 6-digit numbers between 100 000 and 400 000
   c 1-digit numbers
   d 2-digit numbers between 30 and 84.

2 Start at row 9, column 25 of the 2-digit random number table on page 468 and select:
   a 10 different numbers from 1 to 50
   b 6 different numbers from 450 to 700
   c 8 different numbers from 1500 to 2500
   d 12 different 3-digit numbers.

3 CAS Select 8 random integers from 28 to 2198.

4 Example 5 How many of each group should be selected to make stratified random samples from each of the following?
   a A sample of 10 swimsuit-wearers from 15 men in board shorts, 10 men in briefs, 25 women in bikinis and 7 women in one-pieces.
   b A sample of 20 chocolates from 180 soft-centred, 140 hard-centred, 85 liquid-centred and 108 nutty-centred chocolates.
   c A sample of 16 from 5 fifteen-year-olds, 35 sixteen-year-olds, 10 seventeen-year-olds and 3 eighteen-year-olds.
   d A sample of 15 staff from a manufacturing firm that employs 42 assembly workers, 10 office staff and 3 supervisors.

5 Example 6 Starting from the given number, state which numbers to use for systematic samples for each of the following.
   a A sample of 10, starting at 28 from a group numbered from 5 to 146.
   b A sample of 8, starting at 216 from a group numbered from 105 to 327.
   c A sample of 15, starting at 64 from a group numbered from 1 to 427.
   d A sample of 9, starting at 1472 from a group numbered from 1257 to 2832.

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6 State the kind of sampling used in each of the following non-random samples of students at a school.

a Peter asked the first 10 girls and 10 boys who came into the library
b Vera asked 10 of her friends
c David asked students he thought would be good representatives
d Sally interviewed the students who played netball
e Ami interviewed the students on her bus on the way to school
f Michael asked students at lunchtime until he had 3 from each of Years 7 to 12
g Celia thought people on the school council would be best, so chose 6 of them in Years 11 and 12
h Corey chose people who had been at the school since Grade 7

7 Use systematic sampling to select 10 students using the enrolment numbers at a school. Start from enrolment number 5130 of the current enrolments, which run from 4928 to 5672 inclusive.

Reasoning and communication

8 Biotech Industries Pty Ltd wishes to form a staff social committee consisting of 18 members. The firm decides to use the method of stratified random sampling for selecting the committee members. The employment details of the firm are given in the table.

	Administrative staff	Factory workers
Males	11	73
Females	24	52

a How many from each group should be selected to represent all groups fairly?
b How many from each group should be selected if no distinction is made between males and females?

9 The table below shows the Australian population in 2012. Use stratified random sampling to determine how many should be chosen from each state to make a sample of 500:

a if persons are selected regardless of sex
b if males and females are selected in proportion.

Australian population, September 2012

State	Males	Females	Persons
New South Wales	3 628 553	3 685 546	7 314 099
Victoria	2 793 330	2 855 722	5 649 052
Queensland	2 285 309	2 299 280	4 584 589
South Australia	820 740	837 408	1 658 148
Western Australia	1 236 308	1 215 137	2 451 445
Tasmania	255 169	257 006	512 175
Northern Territory	124 080	112 269	236 349
Australian Capital Territory	187 329	189 131	376 460
Australia	**11 332 884**	**11 452 580**	22 785 464

Source: ABS (3101.0 Table 4)

10 O'Hea Street in Coburg has numbers from 1 to 386. Use systematic sampling to choose 20 houses for an employment survey, starting from 205 O'Hea St.

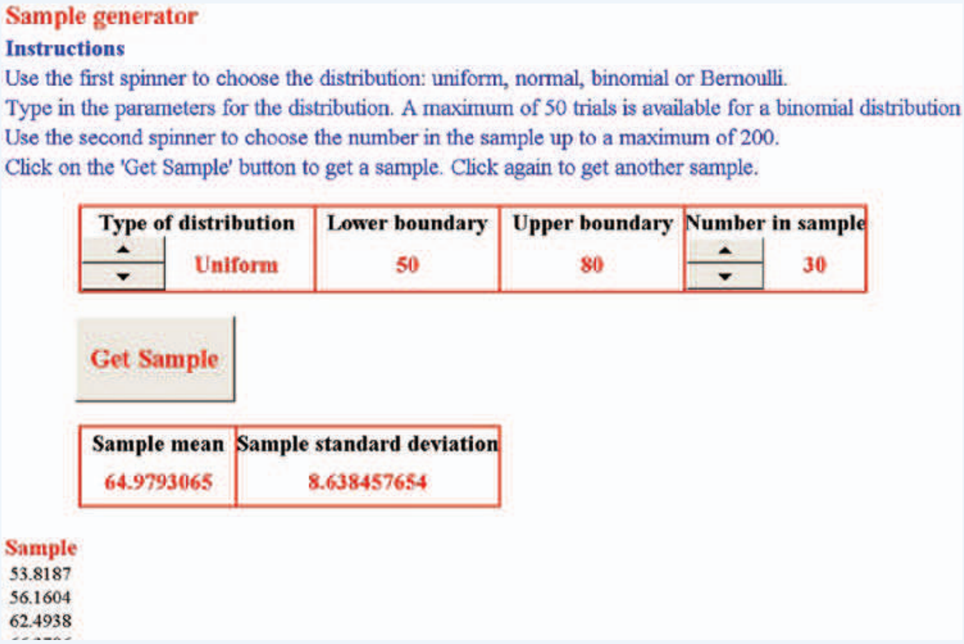
9.03 VARIABILITY OF RANDOM SAMPLES

Unless you use the whole population, random samples will vary. For a small population like {3, 5, 7, 8, 11} you could take samples of 2 items. Some of those samples would be {3, 5}, {3, 11} and {7, 8}. These are clearly very different and it is obvious that samples as small as this might give you a very misleading idea about the population distribution.

Sample generator

INVESTIGATION Sample variation

You can use the Excel spreadsheet *Sample generator* on the website to generate samples from uniform, normal, binomial or Bernoulli distributions to compare them. The spreadsheet also calculates the mean and standard deviation of the sample.


Sample generator

Instructions

Use the first spinner to choose the distribution: uniform, normal, binomial or Bernoulli.
Type in the parameters for the distribution. A maximum of 50 trials is available for a binomial distribution
Use the second spinner to choose the number in the sample up to a maximum of 200.
Click on the 'Get Sample' button to get a sample. Click again to get another sample.

Type of distribution	Lower boundary	Upper boundary	Number in sample
Uniform	50	80	30

Get Sample

Sample mean	Sample standard deviation
64.9793065	8.638457654

Sample
53.8187
56.1604
62.4938

- Use the spreadsheet to generate a sample of 20 numbers from a uniform distribution on the interval 50–80.
- Draw a graph of the sample.
- Now get another sample and draw a graph of this sample.
- How do the graphs compare to each other?
- Compare box-and-whisker plots of the samples.
- Get another two samples and draw box-and-whisker plots of the samples.

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- Now get 4 samples of 20 values from the binomial distribution with $n = 30$ trials and probability of success $p = 0.4$.
- Compare box-and-whisker plots of your binomial samples.
- Get 4 samples of 20 values from the normal distribution with mean $\mu = 40$ and standard deviation $\sigma = 8$.
- Compare box-and-whisker plots of your normal samples.

You will have seen from the investigation that each sample is likely to be different. Their means and standard deviations are also different, but the larger the samples, the more similar they are likely to be.

Example 7

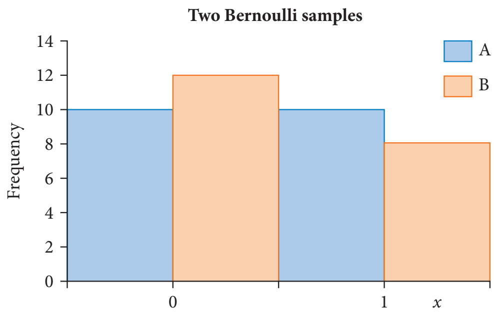
CAS Two random samples were taken from a Bernoulli distribution with probability $p = 0.42$. Each sample has 20 values. The samples are shown below.

A: 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1
B: 1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0

a Show the samples as a side-by-side column graph.
b Calculate the means and standard deviations.
c Compare the samples.

Solution

a Draw the graphs with different colours for each sample.



b Calculate the mean and standard deviation of each sample.

$\mu_A = 0.5, \sigma_A = 0.5$
$\mu_B = 0.4, \sigma_B \approx 0.49$

Compare the samples.

Sample A has a higher mean than sample B, and this is evident on the graph. However, their spreads are similar, as shown on the graph, and by their standard deviations.

Your CAS calculator may be used to generate a list of random numbers between 0 and 1 and you can manipulate this list to produce a random sample from a uniform distribution, for a random number $0 < x < 1$, $a < a + (b - a)x < b$.

Example 8

CAS **a** Generate two samples of 15 items from a uniform distribution on the interval [30, 50], placing each sample in a different column of a spreadsheet.

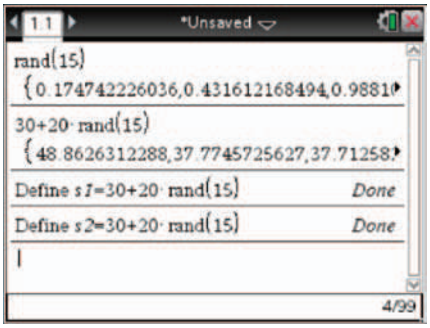
b Draw a graph of the first distribution

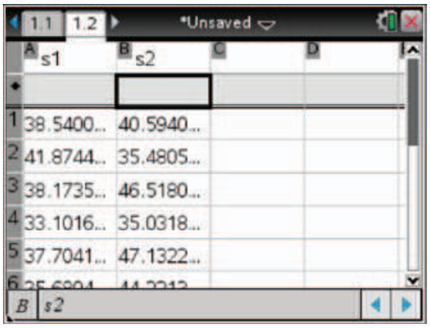
c Compare to the second distribution and comment on the appearance.

Solution

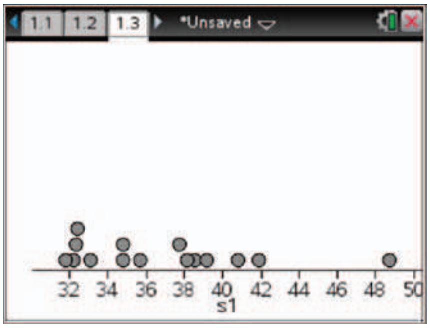
TI-Nspire CAS

a rand(15) will produce 15 random numbers, so $30 + 20 \times \text{rand}(15)$ produces a sample of 15 items from the required distribution.
Define two such samples as *s1* and *s2*.
You can easily examine the lists by inserting a Lists & Spreadsheet page. Type the variables *s1* and *s2* into the column headings and they will be copied into the columns.

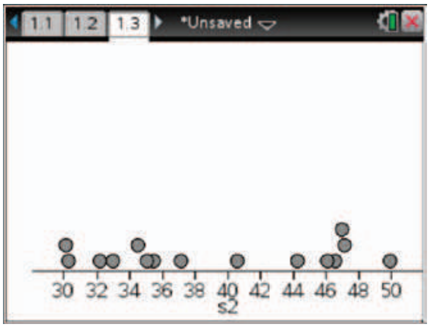




b Insert a Data & Statistics page and 'Click to add variable' at the bottom. Choose *s1*. A dot graph of the sample will be shown.



c Click on the variable *s1* at the bottom of the graph and choose *s2* instead. The display will change to a dot plot of *s2*.



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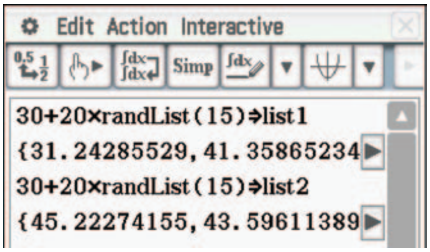
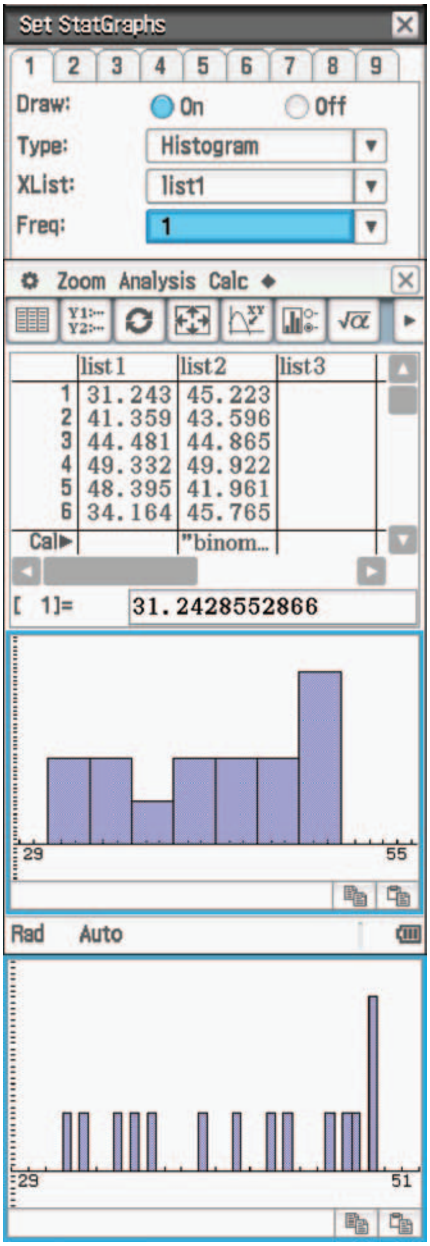
ClassPad

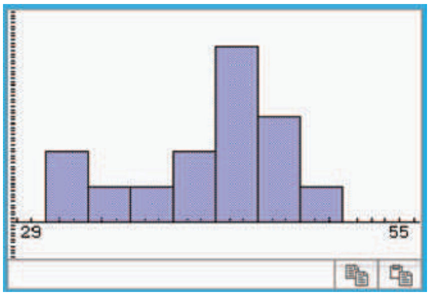
a Make sure the calculator is set to decimal. randList(15) will produce 15 decimal numbers between 0 and 1, so randList(15) $\boxed{\times}$ 20 $\boxed{+}$ 30 will produce 15 decimal numbers between 30 and 50. Name this list1. Repeat for list2. A different set of random numbers will be produced.
You can see both distributions in the Statistics menu.

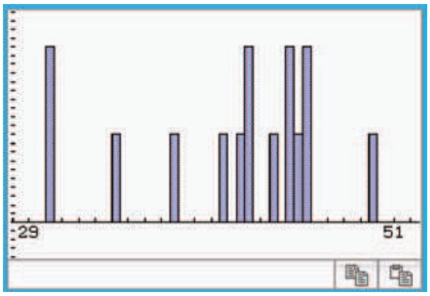


b Go to the Statistics menu.
Tap SetGraph and Setting. Choose as shown on the right.
Tap to draw the graph. Make HStart 30 and HStep 3.
Tap the top half of the graph, and repeat using HStart 30 and HStep 0.5.
You will get a different type of graph.



c Repeat for the second distribution. The only change will be that XList will now be list2 and not list1.
Only the graphs are shown on the right.





Compare the sample graphs. — The samples appear to be different.

In Example **8**, each sample is different. If you follow the instructions for Example **8**, your samples will almost certainly be different to those shown, but the general conclusions will be similar.

You can use randNorm(μ, σ, #*s*) on the TI-Nspire CAS or randNorm(σ, μ, #*s*) on the ClassPad to get a list of #*s* random values from a normal distribution with mean m and standard deviation s. Similarly, randBin(n, p, #*s*) gives a list of #*s* random values from a binomial distribution with n trials and probability p on both calculators. A Bernoulli distribution is a binomial distribution with one trial, so randBin(1, p, #*s*) gives a list of #*s* random values of 0 or 1 where success (1) has a probability p on both calculators. You can also find the mean and standard deviation of a list using 1-variable statistics.

Example 9

CAS **a** Generate two random samples of 20 items from a normal distribution with mean $\mu = 25$ and standard deviation $\sigma = 6$.

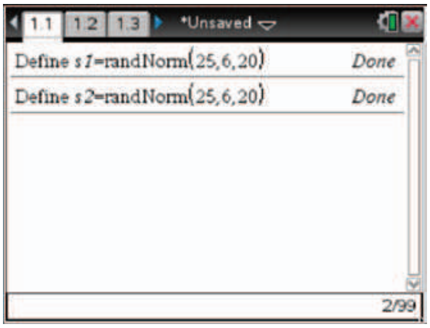
b Find the mean and standard deviation of each sample.

c Compare the samples.

Solution

TI-Nspire CAS

a Define the random samples as *s1* and *s2* using the randNorm function.
You might want to look at the samples as columns in a spreadsheet of dot plots as shown in Example 7.



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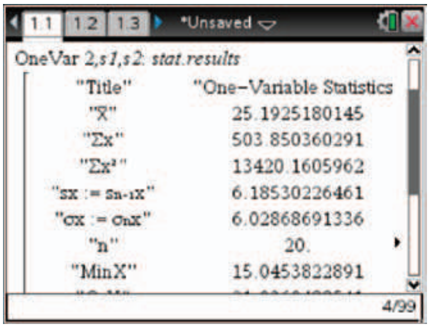
b Press menu, 6: Statistics, 1: Stat Calculations and 1: One-Variable Statistics. Choose 2 lists and type *s1* and *s2* as the X1 List and X2 List.
Scroll the lists up and down and to the right to find the mean and standard deviations of the lists, shown as $\overline{x}$ and $\sigma x := \sigma nx$.
Write the results.



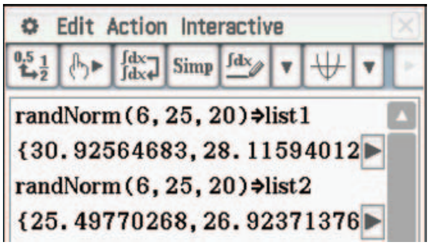
$\mu_{s1} \approx 25.19$, $\sigma_{s1} \approx 6.03$, $\mu_{s2} \approx 27.25$, $\sigma_{s2} \approx 7.09$

c Compare the samples.

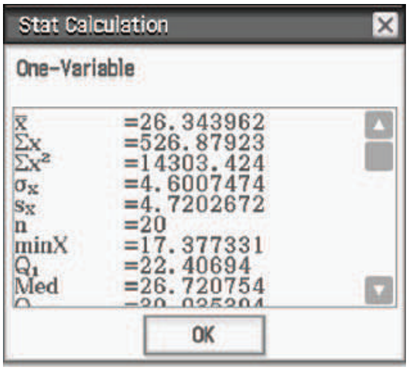
The samples have different means, standard deviations and appearances.

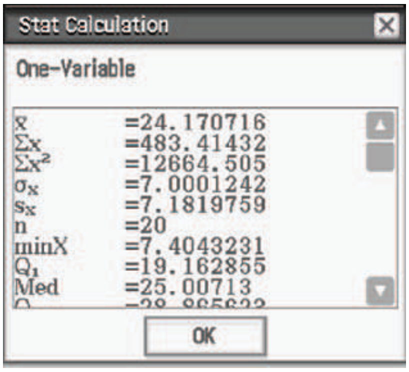
ClassPad

a Use the Catalog to select randNorm. Enter in the order standard deviation, mean, and number of random samples. Store the data in list1 and list2.



b Go to the Statistics menu.
Tap Calc, One-Variable and set XList to list1 and Freq to 1.
Read the mean, $\overline{x} \approx 26.34$ and standard deviation $\sigma_x \approx 4.60$.
Repeat for list2.
Read the mean, $\overline{x} \approx 24.17$ and standard deviation $\sigma_x \approx 7.00$.





$\mu_{s1} \approx 26.34$, $\sigma_{s1} \approx 4.60$, $\mu_{s2} \approx 24.17$, $\sigma_{s2} \approx 7.00$

c Compare the samples.

The samples have different means, standard deviations and appearances.

EXERCISE 9.03 Variability of random samples

Concepts and techniques

1 Example 7 Two random samples were taken from a Bernoulli distribution with $p = 0.6$. Each sample has 20 values. The samples are shown below.

A: 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1, 0, 1, 1, 1

B: 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1

a Represent the samples using a side-by-side column graph.

b Compare the samples.

2 Two random samples were taken from a uniform distribution on the interval [10, 25]. Each sample has 18 values. The samples, rounded correct to 1 decimal place, are shown below.

A: 14.1, 17.1, 19.5, 10.3, 16.8, 22.7, 23.8, 21.7, 22.2, 24.9, 14.2, 18.2, 16.4, 17.9, 17.8, 13.2, 21.7, 18.6

B: 12.6, 16.5, 12.2, 10.8, 21.6, 12.9, 21.3, 23.8, 11.9, 20.5, 21.8, 24.7, 23.1, 14.5, 20.7, 17.3, 13.7, 18.1

a Represent the samples using side-by-side histograms with class widths of 2.

b Compare the samples.

3 Two random samples were taken from a normal distribution with a mean of 50 and a standard deviation of 8. Each sample has 25 values. The samples, rounded correct to 1 decimal place, are shown below.

A: 50.4, 62.3, 49.0, 45.3, 59.1, 45.1, 50.2, 49.7, 40.0, 41.6, 38.5, 46.6, 47.5, 47.1, 63.2, 55.4, 43.0, 52.1, 53.3, 54.9, 58.4, 34.2, 54.4, 37.1, 56.9

B: 37.3, 49.3, 35.9, 45.7, 53.5, 40.2, 46.7, 44.8, 52.8, 41.7, 59.4, 48.7, 50.2, 39.4, 50.3, 41.6, 43.4, 41.9, 46.1, 47.8, 45.1, 53.3, 44.9, 44.5, 50.6

a Represent the samples using side-by-side histograms with class widths of 5.

b Compare the samples.

4 Examples 8, 9 CAS a Generate two random samples of 25 items from a binomial distribution with $n = 30$ trials and probability $p = 0.7$ and draw dot plots of the samples.

b Find the mean and standard deviations of the samples.

c Compare the samples.

5 a Generate two random samples of 20 items from a Bernoulli distribution with probability $p = 0.25$.

b Find the mean and standard deviations of the samples.

c Compare the samples.

6 a Generate two random samples of 30 items from a uniform distribution on the interval [5, 25] and draw dot plots of the samples.

b Find the mean and standard deviations of the samples.

c Compare the samples.

7 a Generate two random samples of 35 items from a normal distribution with mean $\mu = 50$ and standard deviation $\sigma = 10$ and draw dot plots of the samples.

b Find the mean and standard deviations of the samples.

c Compare the samples.

Reasoning and communication

8 a Generate two random samples of 9 items from a binomial distribution with $n = 15$ trials and probability $p = 0.6$ and find the mean and standard deviations of the samples.

b Generate two random samples of 64 items from a binomial distribution with $n = 15$ trials and probability $p = 0.6$ and find the mean and standard deviations of the samples.

c Comment on the variation of the means and standard deviations for samples of 9 and samples of 64 items.

9 a Generate two random samples of 16 items from a uniform distribution on the interval [35, 45] and find the mean and standard deviations of the samples.

b Generate two random samples of 64 items from a uniform distribution on the interval [35, 45] and find the mean and standard deviations of the samples.

c Comment on the variation of the means and standard deviations for samples of 16 and samples of 64 items.

10 a Generate two random samples of 9 items from a normal distribution with mean $\mu = 40$ and standard deviation $\sigma = 8$ and find the mean and standard deviations of the samples.

b Generate two random samples of 81 items from a normal distribution with mean $\mu = 40$ and standard deviation $\sigma = 8$ and find the mean and standard deviations of the samples.

c Comment on the variation of the means and standard deviations for samples of 9 and samples of 81 items.

9.04 SAMPLE PROPORTIONS

If you were checking whether Vitamin C supplements decrease the chances of catching a cold, you might ask one sample of people to take no vitamin C, another to take one capsule per day and a third sample to take 2 capsules per day. For each sample, the variable would be a random Bernoulli variable with success (if you can call it that) being 'catching a cold'. You would then look at the frequencies of people in each group who caught colds. The actual frequencies would not be as important as the ratio of the numbers who caught colds to the numbers in the samples. Each ratio is an example of a **sample proportion**.

IMPORTANT

A **sample proportion** is the ratio of the number of times a property (characteristic) occurs in a sample, divided by the number in the sample. The sample proportion is denoted by $\hat{p}$ (read as p hat).

The occurrence of the property is normally called a **success**, so for x successes in a sample of n, the sample proportion is given by $\hat{p} = \frac{x}{n}$.

Example 10

From 20 people who took no vitamin C, 8 got colds during one winter. What is the sample proportion of colds?

Solution

Success is getting a cold. Use the formula.

$$\hat{p} = \frac{x}{n}$$

Substitute $x = 8$ and $n = 20$ and calculate the answer.

$$= \frac{8}{20} = 0.4$$

Write the solution.

The sample proportion for those who got colds is 0.4.

You would expect the probability of success in a population to be close to the sample proportion. From the work you did in the last section, you would expect the variability of the sample proportion to decrease as the sample size increased. In the limiting case, when the sample is the whole population, it must be equal to the probability of success.

IMPORTANT

The sample proportion $\hat{p}$ for the occurrence of a property (characteristic) in a population is an **estimator** of the probability p of the occurrence in the population.

Example 11

Police at a roadside checkpoint stopped 55 cars to check their roadworthiness. 7 of the drivers were issued with notices to have faults fixed within a week and 2 cars had such severe problems that they were immediately stopped from driving any further.

a Use the information to estimate the probability that a randomly selected car has a fault.

b State any problems with treating this as a reliable estimate.

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Solution

a 9 out of 55 cars had faults affecting roadworthiness.

Sample proportion $= \frac{9}{55} \approx 0.16$

State the answer.

The estimated probability of faults is about 0.16.

b Experienced police would be likely to stop cars they thought would be likely to have faults.

Since the police are unlikely to choose obviously new cars, the estimate may be higher than the true probability.

EXERCISE 9.04 Sample proportions

WS Sample proportions

Concepts and techniques

1 Example 10 From a sample of 200 Irish High School students, 9 had red hair. What is the sample proportion of red hair for Irish High School students?

2 A normal die was tested by throwing it 120 times. It landed with 6 uppermost a total of 18 times. What is the sample proportion of '6' for this die?

3 A stove manufacturer checked stoves leaving the factory for faults. From 125 checked in one day, 8 were found to have faults in the paintwork that would make them 'factory seconds'. What was the sample proportion for factory seconds?

4 Example 11 From a sample of 40 Australian Year 12 students, 9 were found to have heights of 180 cm or greater. Only one of the 9 was female.

a Estimate the probability of an Australian Year 12 student being 180 cm tall or greater.

b 19 of the 40 students were male. Estimate the probability of a male Australian Year 12 student being 180+ cm.

c Estimate the probability of a female Australian Year 12 student being 180+ cm.

5 From a sample of 14 male South African Year 12 students, 3 had heights of 180 cm or more. Estimate the probability of a male South African Year 12 student being 180+ cm.

6 A commercial art gallery had an exhibition of paintings with prices ranging from \$180 to \$7900, with a total of 74 paintings on show. 5 of the paintings were under \$400. Estimate the probability of paintings being priced under \$400.

Reasoning and communication

7 Are there any problems with the estimate in question **6**?

8 The ages of 70 people towing caravans or driving motor homes on the Bruce Highway in Queensland in July were checked and 24 were found to be over 60 years old.

a Estimate the probability of caravan or motor home drivers being over 60.

b Are there likely to be any problems with this estimate?

9.05 PARAMETERS OF SAMPLE PROPORTIONS

How are sample proportions related to the probability of the property occurring in the population? The sample proportion you get for, say, blue eyes from a sample of 20 Australian Year 12 students, will not be the same for each sample. However, you would expect the mean of the sample proportions to be representative of the probability.

The probability of an Australian Year 12 student having blue eyes is 0.32. Since there are a very large number of Australian Year 12 students, the probability for a sample of 20 students will be the same for each student chosen for the sample. Each student chosen constitutes a Bernoulli trial because there are only two outcomes and the probability of blue eyes is the same for each. This means that for a sample of 20, the probability of success is 0.32 for each trial so the number of successes is a binomial distribution with $p = 0.32$ and $n = 20$. The number of blue-eyed students in the sample is a random variable, say B. You know from your work in Chapter 5 that

$$E(B) = np = 0.32 \times 20 = 6.4 \text{ and } \mathrm{Var}(B) = npq = 20 \times 0.32 \times 0.68 = 4.352$$

For a linear transformation $Y = aX + b$ of a random variable X, you also know that

$$E(Y) = a \times E(X) + b \text{ and } \mathrm{Var}(Y) = a^2 \times \mathrm{Var}(X).$$

Consider the random variable $H = \frac{1}{20} \times B$.

Then $E(H) = \frac{1}{20} \times E(B) = \frac{1}{20} \times 6.4 = 0.32$ and $\mathrm{Var}(H) = \left(\frac{1}{20}\right)^2 \times \mathrm{Var}(B) = 0.01088$

But $\hat{p} = \frac{x}{n} = \frac{b}{20}$, so H is the random variable for the sample proportion $\hat{p}$ of blue eyes in a sample of 20 Australian Year 12 students.

This implies that the mean of the sample proportion is 0.32 and the variance is 0.010 88.

IMPORTANT

For samples that are small compared to the population, the sample proportion is effectively a random binomial variable.

If the probability of a particular property is p, then for samples of n items

$E(\hat{p}) = p$, $\mathrm{Var}(\hat{p}) = \frac{pq}{n}$ and $\mathrm{SD}(\hat{p}) = \sqrt{\frac{pq}{n}} = \sqrt{\frac{p(1-p)}{n}}$.

You can prove that the parameters of sample proportions have the values above by using the same method as shown for the example of blue eyes in Year 12 Australian students.

Since $E(\hat{p}) = p$, $\hat{p}$ is the best estimator of p.

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Example 12

The probability of a normal car tyre lasting more than 60 000 km is about 0.34. What is the variance and standard deviation of the proportion of samples of 30 such tyres lasting more than 60 000 km?

Solution

Write down the parameters for $\hat{p}$.

$p = 0.34,\ q = 0.66,\ n = 30$

Write the formula for Var($\hat{p}$).

$\text{Var}(\hat{p}) = \dfrac{pq}{n}$

Substitute values and calculate the answer.

$= \dfrac{0.34 \times 0.66}{30}$

$= 0.00748$

Find SD($\hat{p}$).

$\text{SD}(\hat{p}) = \sqrt{\text{Var}(\hat{p})}$

≈ 0.0865

Write the answer.

The variance is 0.007 48 and the standard deviation is about 0.0865.

Example 13

A coin is tossed 20 times and the number of heads is noted. This experiment is repeated many times. What is the expected value and standard deviation of the sample proportion of heads?

Solution

Write down the parameters for $\hat{p}$.

$p = 0.5,\ q = 0.5,\ n = 20$

Write the formulas for $E(\hat{p})$ and SD($\hat{p}$).

$E(\hat{p}) = p$ and $\text{SD}(\hat{p}) = \sqrt{\dfrac{pq}{n}}$

Substitute values.

$E(\hat{p}) = 0.5$ and $\text{SD}(\hat{p}) = \sqrt{\dfrac{0.5 \times 0.5}{20}}$

Simplify SD($\hat{p}$).

≈ 0.11

Write the answer.

The mean proportion of heads would be 0.5 with a standard deviation of 0.11.

Example 14

A class of Year 7 students investigated the results of dealing a card from a well-shuffled pack and checking its suit. Each student in the class dealt a card 50 times, replacing and shuffling the cards before dealing the next one. They each counted the number of times the card was a heart and recorded the proportion of times out of 50 as a decimal. What would be the mean and standard deviation of these results?

Solution

Write down the parameters for $\hat{p}$.	$p = 0.25$, $q = 0.75$, $n = 50$
Write the formulas for $E(\hat{p})$ and $SD(\hat{p})$.	$E(\hat{p}) = p$ and $SD(\hat{p}) = \sqrt{\dfrac{pq}{n}}$
Substitute in the values.	$E(\hat{p}) = 0.25$ and $SD(\hat{p}) = \sqrt{\dfrac{0.25\times0.75}{50}}$
Simplify $SD(\hat{p})$.	≈ 0.061
Write the answer.	The mean and standard deviation of the results would be about 0.25 and 0.061 respectively.

EXERCISE 9.05 Parameters of sample proportions

WS Sample proportion calculations

Concepts and techniques

1 Example 12 Find the mean, variance and standard deviations of sample proportions for samples with the following probabilities and number in each sample.

a $p = 0.2$ and $n = 50$
b $p = 0.7$ and $n = 25$
c $p = 0.81$ and $n = 120$
d $p = 0.22$ and $n = 80$

2 Example 13 A normal die is tossed and the number it lands on is noted. Samples of 45 tosses are taken and in each case the proportion of times that the number is less than 3 is calculated. What is the mean and standard deviation of the sample proportion when this experiment is repeated multiple times?

3 45% of Canadian high school students catch a bus to get to school. Samples of 200 students from high schools across Canada are surveyed to determine the proportion travelling by bus to school. What is the expected proportion and standard deviation of the sample proportion?

Reasoning and communication

4 Example 14 A normal pair of dice is thrown and the total is noted. Samples of 30 such throws are performed and the proportion of times the total is more than 9 is calculated for each sample. What is the mean and standard deviation of the sample proportions for totals more than 9?

5 A card is cut from a well shuffled deck. This is done 100 times and the number of times that a picture card appears is noted in each case. When this is repeated many times, what is the mean and standard deviation of the sample proportion of picture cards?

6 A weighted coin was tossed 25 times and the proportion of heads was noted. This was done a total of 40 times and the mean number of heads was found to be 15.02 and the standard deviation of the sample proportion of heads was 0.1. Estimate the probability of heads for this coin.

7 Show that for a property with a probability of p, the mean sample proportion of samples of size n is $E(\hat{p}) = p$.

8 Show that for a characteristic with a probability of p, the variance of sample proportions of samples of size n is $\text{Var}(\hat{p}) = \frac{pq}{n}$.

9.06 THE CENTRAL LIMIT THEOREM

You can calculate the mean and standard deviation of random variables from different samples. As you have seen already, these vary between samples, but as the sample size increases, the variation decreases and they become closer to the population mean and standard deviation. What happens with sample proportions?

INVESTIGATION Cabbage moths

Colin has a patch of cabbages, but the patch has attracted cabbage moths. He has used a large net to catch all the cabbage moths, so it is possible to work out the average size of the cabbage moths. Work in groups of about five people, so there are 5 groups in the class. Your teacher will give you some paper models of the cabbage moths to check and measure.

1 Work out the mean and standard deviation of the wing span of a sample of 6 cabbage moths.

2 How many of your sample are blue? What proportion are blue?

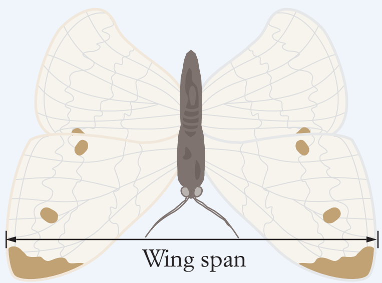
3 Now use samples of 12 cabbage moths.

4 Find the mean, standard deviation and proportion for samples of 18 and 25.

5 Compare your results with those of other groups.

- What happens to the mean as the sample size is increased?
- What happens to the standard deviation as the sample size is increased?
- What happens to the proportion as the sample size is increased?

6 Work out the class averages for samples of 6, 12, 18 and 25. What happens to the class average as the sample size is increased?



To understand what happens to a statistic as sample size increases, it is useful to examine the *distribution* of the statistic for multiple samples. The distribution of a statistic for many samples is called a **sampling distribution**.

Collection of real data for multiple samples is very time-consuming, so you will use a simulation. Consider samples of Australian high school students and the property of having blue eyes. For Australian students, the probability of having blue eyes is 0.32. The occurrence of blue eyes is a Bernoulli random variable with a probability of success of 0.32. A sample of 20 students corresponds to 20 trials. The number of successes is a value x of the random binomial variable X and the sample proportion is $\frac{x}{20}$.

Example 15

CAS **a** Simulate the sample proportion for blue eyes ($p = 0.32$) for 100 samples of 10 Australian students in the first column of a spreadsheet.

b Create a dot plot of the distribution.

c What shape does it appear to be?

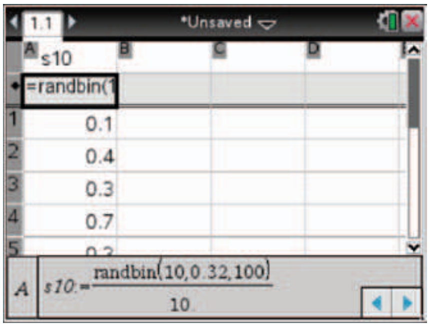
d Repeat the simulation for samples of 50 students in the next column and do a dot plot.

e What happens to the shape of the distribution?

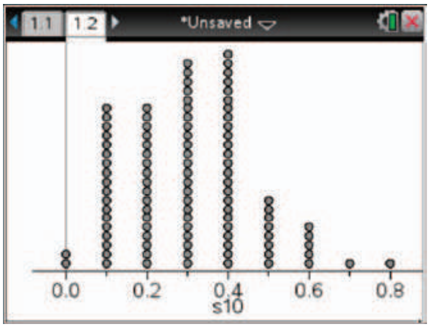
Solution

TI-Nspire CAS

a Use column A of a Lists & Spreadsheet page.
Type the variable name *s10*, say, into the top cell. Then type randBin(10, 0.32, 100) ÷ 10. into the next (formula) cell at the top. The simulated sample proportions for blue eyes for 100 samples of 20 students each will appear in cells A1–A100. The decimal point forces approximate calculation.

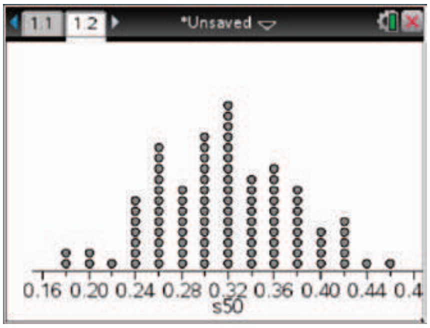


b Add a Data & Statistics page and click on the variable *s10* at the bottom of the page.



c Comment on the shape. The distribution is not symmetrical.

d Repeat in column B but use the variable name *s50* and the randBin(50, 0.32, 30) ÷ 50.
Click on *s50* in the dot plot.

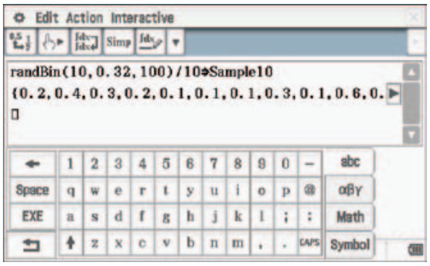


e Comment on the change of distribution.

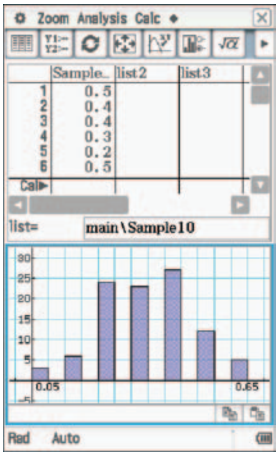
The distribution looks like a normal distribution.

ClassPad

a Start at the Main menu. The randBin function generates a list of numbers and thus can be used to generate a list we shall call Sample10. Make sure the calculator is set to decimal.
It is not necessary to see all the numbers in the list, but you can by tapping the arrow on the right of the numbers.



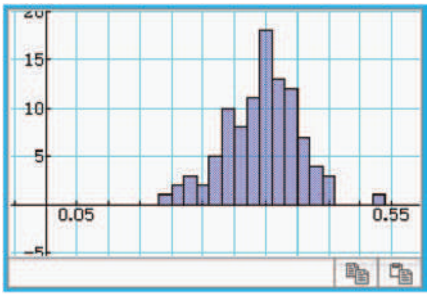
b Go to the Statistics menu. Tap list1 and enter Sample10. If the numbers appear as fractions, tap the heading and then the decimal icon.
From SetGraph, tap Setting and choose Histogram, Main\Sample10 and 1.
Use View Window to set the scale for x to 0.05 and the scale for y to 5.
Tap the graph icon, and set HStart to 0 and HStep to 0.05.
For a graph without gaps, set HStep to 1.



c Comment on the shape.

The distribution is not symmetrical.

d Repeat but use the name Sample50 and randBin(50, 0.32, 100) ÷ 100. Use the second column and since the numbers change by a minimum of $\frac{1}{50} = 0.02$, make HStep 0.02.
The value falls on the left edge of each column.



e Comment on the change of distribution.

The distribution looks more like a normal distribution with the mean a little less than 0.35. The graph roughly falls between 2 and 4.4, giving a mean of $\frac{2+4.4}{2} = 3.2$.

When the number in the sample is large, the distribution looks approximately normal. Remember that each time you get a sample, it is different, so your samples may appear a little different than those in Example **15**, but the general pattern should be the same.

What happens to the sampling distribution for sample proportions as the value of p changes?

Example 16

CAS **a** Simulate 100 samples of sample proportions with $p = 0.005$ and $n = 40$ and create a dot plot of the sampling distribution.

b Simulate 100 samples of sample proportions with $p = 0.9$ and $n = 40$ and do a dot plot of the sampling distribution.

c Simulate 100 samples of sample proportions with $p = 0.5$ and $n = 40$ and do a dot plot of the sampling distribution.

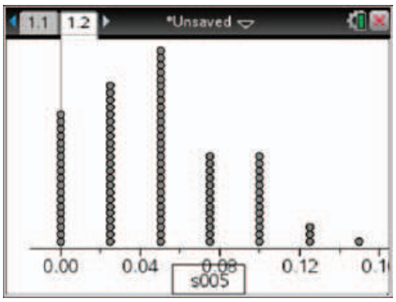
d Compare the sampling distributions.

Solution

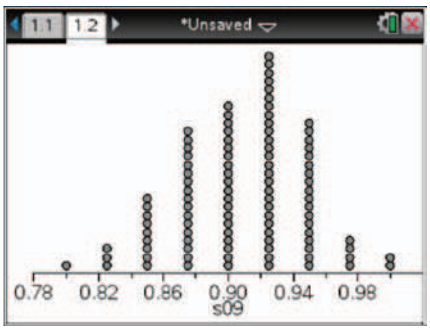
TI-Nspire CAS

Use randBin(40, 0.005, 100) ÷ 40 labelled as s005, randBin(40, 0.9, 100) ÷ 40 labelled as s09 and randBin(40, 0.5, 100) ÷ 40 labelled as s05.

a



b



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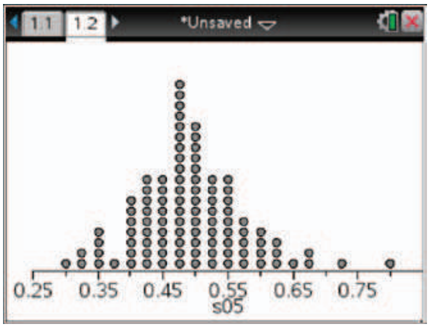
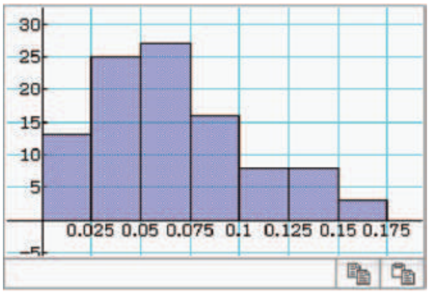
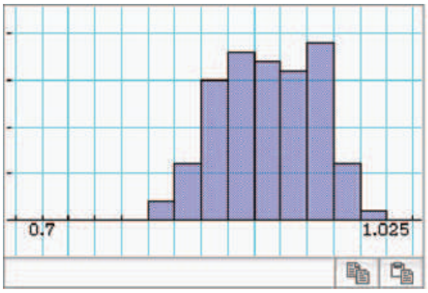
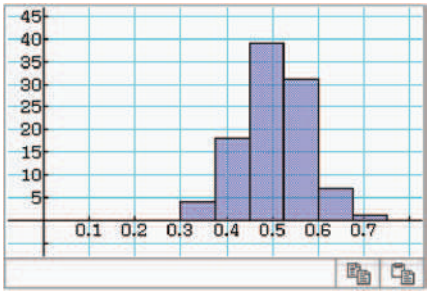
c 

ClassPad

a 

b 

c 

d Write a comment about the change in the shape of the distribution.

The distribution becomes more symmetrical and similar to a normal distribution as $p \to 0.5$.

When the probability of the property gets close to 0.5, the sampling distribution of the sample proportion approximates a normal distribution.

Both the number in a sample and the probability of the property for which the statistic is calculated affect the nature of the sampling distribution of a sample proportion. Statisticians use a guide to determine when the distribution of a sample proportion may be approximated by a normal distribution. This is important in applications because calculations involving the normal distribution are much simpler and quicker than those for a binomial distribution.

IMPORTANT

For $np > 5$ and $nq > 5$, the distribution of the sample proportion $\hat{p}$ for a random Bernoulli variable is approximately normal with mean p and standard deviation $\sqrt{\dfrac{pq}{n}}$.

In fact, any binomial distribution for which $np > 5$ and $nq > 5$ may be approximated by a normal distribution with the same mean (np) and standard deviation ($\sqrt{npq}$).

What about other sampling distributions? The sample proportion is rather like the mean, which is the most common statistic in general distributions, along with the standard deviation.

Example 17

CAS **a** Simulate 100 samples of 9 items from a uniform distribution on the interval [30, 50] and draw a dot plot of the means of the samples.

b Simulate 100 samples of 64 items from a uniform distribution on the interval [30, 45] and draw a dot plot of the means of the samples.

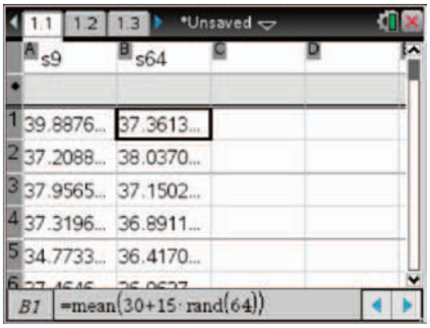
c Compare the shapes of the sampling distributions.

d Compare the means and standard deviations of the sampling distributions.

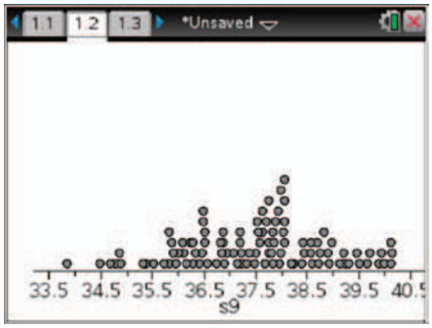
Solution

TI-Nspire CAS

You can use 30 + 15 × rand(9) to produce a sample of 9 items from the distribution (see Example **8**). You can find the mean using mean(), so put the formulas mean(10+15×rand(9)) and mean(10+15×rand(64)) into cells A1 and B1 of a Lists & Spreadsheet page. Call the columns *s9* and *s64* and copy cells A1 and B1 down to cells A100 and B100. Add a Data & Statistics page and examine the dot plots.



a



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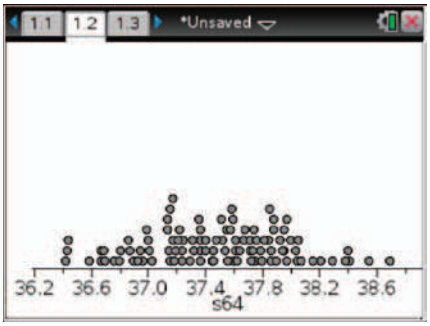
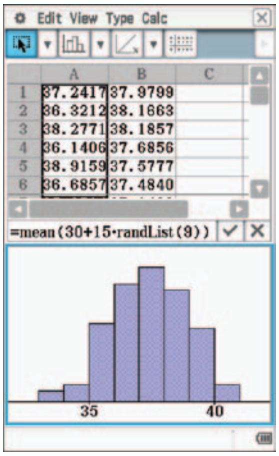
b



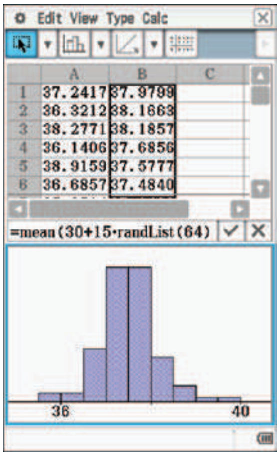
ClassPad

Use the spreadsheet menu.
Tap Edit, Fill and Fill Range.
Enter the Formula =mean(30+15×rand(9)) for the Range A1:A100. Note the symbols at the top of the calculator.
Repeat for column B, using the formula =mean(30+15×randList(64)), and Range B1:B100.
Tap Edit, then Select, then Select Range, and enter A1:A100.
Tap Graph and then Histogram to draw a histogram. The ClassPad does not draw dot plots.
Repeat for a histogram of column B.

a



b

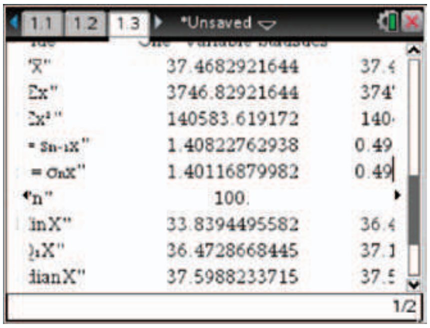


c Comment on the shapes of the distributions.

The sampling distributions do not look like uniform distributions. They are more like normal distributions, with the larger sample even more like a normal distribution.

TI-Nspire CAS

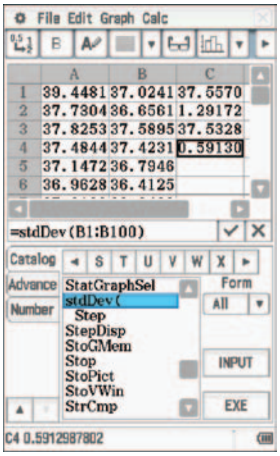
d Insert a Calculator page and find the 1-variable statistics for *s9* and *s64* as in Example **9**.



Write the means and standard deviations. $\overline{x}_9 \approx 37.468$, $s_9 \approx 1.40$, $\overline{x}_{64} \approx 37.475$, $s_{64} \approx 0.490$

ClassPad

Use the catalog to enter the following.
C1 =mean(A1:A100)
C2 =stdDev(A1:A100)
C3 =mean(B1:B100)
C4 =stdDev(B1:B100)



Write the means and standard deviations. $\overline{x}_9 \approx 37.557$, $s_9 \approx 1.29$, $\overline{x}_{64} \approx 37.533$, $s_{64} \approx 0.591$

Write a comment. The means are close to the mean of the original distribution (37.5), but the standard deviations are smaller than the original (about 4.3).

The uniform distribution and the normal distribution are very different from each other. However, the distribution of the sampling distribution of the means of a uniform distribution is similar to a normal distribution. This is true for all sampling distributions of means, no matter what the original distribution. As the number in a random sample increases, the similarity to a normal distribution increases. This is called the **central limit theorem**.

IMPORTANT

The **central limit theorem** states that for 'relatively large' random samples of a random variable X from a distribution with a finite mean μ and a finite standard deviation σ, the sampling distribution of the means is approximately normal. The approximation is also better for larger samples.

It can also be shown that $\overline{X} = \mu$ and $\sigma_{\overline{X}} = \dfrac{\sigma}{\sqrt{n}}$, where n is the number in the sample.

A random sample means that the values of X are *independent*.

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EXERCISE 9.06 The central limit theorem

Concepts and techniques

1 Example 16 CAS **a** Simulate 100 samples of sample proportions with $p = 0.1$ and $n = 20$ and create a dot plot of the sampling distribution.

b Simulate 100 samples of sample proportions with $p = 0.1$ and $n = 50$ and create a dot plot of the sampling distribution.

c Simulate 100 samples of sample proportions with $p = 0.1$ and $n = 200$ and create a dot plot of the sampling distribution.

d Compare the sampling distributions.

2 CAS **a** Simulate 100 samples of sample proportions with $p = 0.8$ and $n = 20$ and create a dot plot of the sampling distribution.

b Simulate 100 samples of sample proportions with $p = 0.8$ and $n = 50$ and create a dot plot of the sampling distribution.

c Simulate 100 samples of sample proportions with $p = 0.8$ and $n = 200$ and create a dot plot of the sampling distribution.

d Compare the sampling distributions.

3 CAS **a** Simulate 100 samples of sample proportions with $p = 0.1$ and $n = 30$ and create a dot plot of the sampling distribution.

b Simulate 100 samples of sample proportions with $p = 0.3$ and $n = 30$ and create a dot plot of the sampling distribution.

c Simulate 100 samples of sample proportions with $p = 0.5$ and $n = 30$ and create a dot plot of the sampling distribution.

d Compare the sampling distributions.

4 CAS **a** Simulate 100 samples of sample proportions with $p = 0.08$ and $n = 20$ and create a dot plot of the sampling distribution.

b Simulate 100 samples of sample proportions with $p = 0.25$ and $n = 50$ and create a dot plot of the sampling distribution.

c Simulate 100 samples of sample proportions with $p = 0.5$ and $n = 90$ and create a dot plot of the sampling distribution.

d Compare the sampling distributions.

5 Example 17 CAS **a** Simulate 100 samples of 12 items from a uniform distribution on the interval [10, 40] and draw a dot plot of the means of the samples.

b Simulate 100 samples of 80 items from a uniform distribution on the interval [10, 40] and draw a dot plot of the means of the samples.

c Simulate 100 samples of 200 items from a uniform distribution on the interval [10, 40] and draw a dot plot of the means of the samples.

d Compare the shapes, means and standard deviations of the sampling distributions with the original distribution.

6 CAS a Simulate 100 samples from a binomial distribution with $p = 0.2$ and $n = 10$ and draw a dot plot of the means of the samples.

b Simulate 100 samples from a binomial distribution with $p = 0.2$ and $n = 30$ and draw a dot plot of the means of the samples.

c Simulate 100 samples from a binomial distribution with $p = 0.2$ and $n = 100$ and draw a dot plot of the means of the samples.

d Compare the shapes, means and standard deviations of the sampling distributions and the original distribution.

7 CAS a Simulate 100 samples of 6 items from a normal distribution with $\mu = 55$ and $\sigma = 12$ and draw a dot plot of the means of the samples.

b Simulate 100 samples of 25 items from a normal distribution with $\mu = 55$ and $\sigma = 12$ and draw a dot plot of the means of the samples.

c Simulate 100 samples of 120 items from a normal distribution with $\mu = 55$ and $\sigma = 12$ and draw a dot plot of the means of the samples.

d Compare the shapes, means and standard deviations of the sampling distributions with the original distribution.

Reasoning and communication

8 Example 15 CAS The Bureau of Meteorology issues long-range forecasts of rain based on ocean temperatures. One June, they say that there is a 65% chance of above median rainfall in Victoria for the 3-month period July–September.

a Simulate the sample proportion for above median rainfall for 100 samples of 12 Victorian locations in the first column of a spreadsheet.

b Draw a dot plot of the distribution.

c What shape does it appear to be?

d Repeat the simulation for 100 samples of 80 Victorian locations in the next column and draw a dot plot.

e What happens to the shape of the distribution?

9 CAS In Australia, about 24% of high school students can speak more than one language.

a Simulate the sample proportion of high school students who can speak more than one language for 100 samples of 12 students in the first column of a spreadsheet.

b Draw a dot plot of the distribution.

c What shape does it appear to be?

d Repeat the simulation for 100 samples of 80 students in the next column and draw a dot plot.

e What happens to the shape of the distribution?

10 CAS The average height of Year 12 boys is reported to be 184 cm with a standard deviation of 8.9 cm.

a Simulate 100 random samples for 15 heights of Year 12 Australian boys and find the means of each sample.

b Draw a dot plot of the sample distribution of the means.

c What shape does it appear to be?

d Repeat the simulation for 100 samples of 50 in the next column and draw a dot plot.

e What happens to the shape and standard deviation of the distribution?

9.07 SAMPLE PROPORTIONS AND THE STANDARD NORMAL DISTRIBUTION

You have seen that the distribution of sample proportions gets closer to the normal distribution as $n \to \infty$ and $p \to 0.5$. What happens to the distribution of $\dfrac{\hat{p} - p}{\sigma_{\hat{p}}}$?

INVESTIGATION Normalised sample proportions

- Simulate the sample proportion of 100 samples of 12 items with $p = 0.6$ in the first column of a spreadsheet.
- Find the mean and standard deviation of the sampling distribution.
- In the second column of the spreadsheet, subtract the mean and divide by the standard deviation. This gives the normalised sample proportions (like z-scores).
- Draw a graph of the new distribution.
- Draw the standard normal distribution on the same graph.
- Repeat the whole process for 100 samples of 40 items with $p = 0.6$.
- Do it again for 100 samples with of 200 items and $p = 0.6$.
- What happens to the normalised graph as n is increased?

Whether or not a particular property occurs in a sample from a population is a Bernoulli variable, say X, which has the possible values 0 or 1. For x occurrences with a probability p from samples of n, the sample proportion $\hat{p} = \dfrac{x}{n}$ is the mean value $\bar{X}$. The mean value of a binomial variable is np and the standard deviation is $\sqrt{npq}$. In this case, $n = 1$, so $\mu = p$ and $\sigma = \sqrt{pq} = \sqrt{p(1-p)}$.

According to the central limit theorem, the distribution of $\bar{X}$ is approximately normal with $\bar{X} = \mu$ and $\sigma_{\bar{X}} = \dfrac{\sigma}{\sqrt{n}}$, so $E(\hat{p}) = E(\bar{X}) = \mu = p$ and

$$\mathrm{SD}(\hat{p}) = \mathrm{SD}(\bar{X}) = \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{pq}}{\sqrt{n}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{p(1-p)/n}.$$

This is exactly the same result as you saw in Section 9.05 for the parameters of $\hat{p}$.

Alamy/Rtimagse

In Chapter 8, you saw that any normal distribution can be transformed into a standard normal distribution using the linear transformation $Z = aX + b$, where $a = \frac{1}{\sigma}$ and $b = -\frac{\mu}{\sigma}$, so $Z = \frac{X - \mu}{\sigma}$. For example, IQ has a mean of 100 and a standard deviation of 15. For an IQ of 130, $Z = (130 - 100)/15 = 2$. It is 2 standard deviations above the mean. Applying this transformation to $\hat{p}$, the distribution approximates a standard normal distribution with $Z = \frac{\hat{p} - p}{\sqrt{p(1-p)/n}}$.

IMPORTANT

For a Bernoulli variable X with parameter p, as $n \to \infty$, the distribution of $\frac{\hat{p} - p}{\sqrt{p(1-p)/n}}$ approaches the standard normal distribution.

From the work you did in the last section, it should be clear that the closeness of the approximation to the standard normal distribution depends on the values of n and p. The larger the value of n and the closer p is to 0.5, the better the approximation.

Example 18

CAS What percentage of values of $\hat{p}$ would lie between 0.45 and 0.55 for samples with $n = 80$ and $p = 0.4$?

Solution

Check the values of np and nq.

$np = 80 \times 0.4 = 32$
$nq = 80 \times 0.6 = 48$

Make a conclusion.

$np > 5$ and $nq > 5$, so the normal distribution can be used.

Find the mean and standard deviation.

$\mu = p = 0.4$

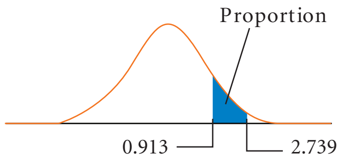
$\text{SD}(\hat{p}) = \sqrt{\frac{pq}{n}} = \sqrt{\frac{0.4 \times 0.6}{80}} \approx 0.055$

Find the standard values of $\hat{p}$.

Standard value of $0.45 = \frac{0.45 - 0.4}{0.055} \approx 0.913$

Standard value of $0.55 = \frac{0.55 - 0.4}{0.055} \approx 2.739$

Find the proportion of the standard normal distribution.



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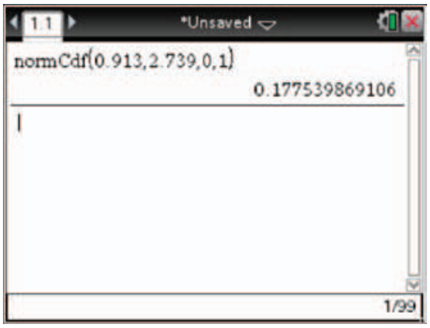

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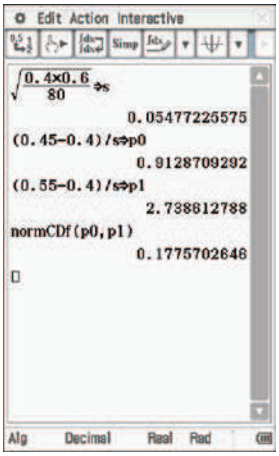


Note that if a mean and standard deviation aren't given, normCDf treats it as a standard normal distribution.

Write the answer.

About 17.8% of the values would be between 0.45 and 0.55.

You can apply sample proportions to many situations involving samples.

Example 19

CAS Given that about 15% of Australians are left-handed, what is the probability that in a sample of 200 Australians, from 20 to 30 of them are left-handed?

Solution

Check np and nq.

$np = 200 \times 0.15 = 30$
$nq = 200 \times 0.85 = 170$

Write the conclusion.

$np > 5$ and $nq > 5$, so the normal distribution can be used.

Find the mean and standard deviation.

$\mu = p = 0.15$

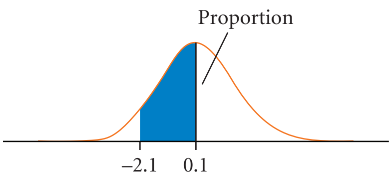
$$\text{SD}(\hat{p}) = \sqrt{\frac{pq}{n}} = \sqrt{\frac{0.15\times0.85}{200}} \approx 0.0252$$

Find the standard values of $\hat{p}$, using 19.5 to 30.5 to find the probability of the integers from 20 to 30 inclusive.

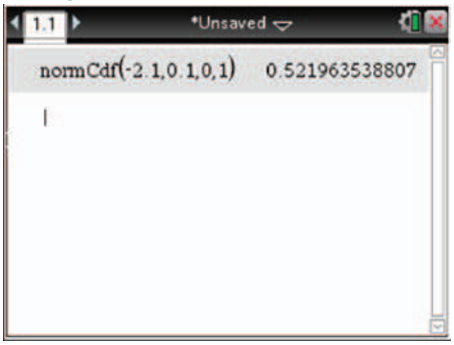
Standard value of $\frac{19.5}{200} = \frac{0.0975-0.15}{0.025} \approx -2.1$

Standard value of $\frac{30.5}{200} = \frac{0.1525-0.15}{0.025} \approx 0.1$

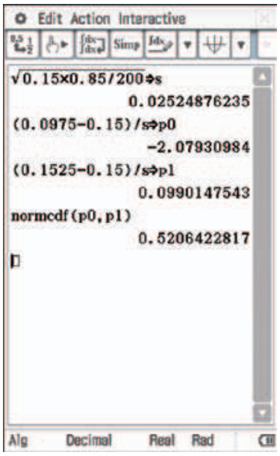
Find the proportion of the standard normal distribution.



TI-Nspire CAS



ClassPad



Write the answer.

The probability that 20 to 30 Australians from a sample of 200 are left-handed is about 0.52.

EXERCISE 9.07 Sample proportions and the standard normal distribution

WS
Sample proportion probabilities

Concepts and techniques

1 Example 18 CAS What percentage of values of $\hat{p}$ would lie between 0.5 and 0.6 for samples with $n = 30$ and $p = 0.7$?

2 CAS What percentage of values of $\hat{p}$ would lie between 0.15 and 0.2 for samples with $n = 50$ and $p = 0.2$?

3 CAS What percentage of values of $\hat{p}$ would lie between 0.75 and 0.85 for samples with $n = 100$ and $p = 0.9$?

4 CAS What proportion of values of $\hat{p}$ would lie between 0.6 and 0.7 for samples with $n = 300$ and $p = 0.72$?

5 CAS What proportion of values of $\hat{p}$ would lie between 0.48 and 0.52 for samples with $n = 45$ and $p = 0.25$?

Reasoning and communication

6 Example 19 CAS The probability of an adult having fair hair is 0.21. What is the probability that a sample of 400 adults has from 70 to 75 people with fair hair?

7 CAS About 25% of Australians are Victorians and of these, 76% live in Melbourne.

a From 300 Australians, what is the probability that from 50 to 60 are Victorian?

b From 300 Victorians, what is the probability that from 200 to 250 live in Melbourne?

c What is the probability that of the 300 Victorians, from 200 to 210 live in Melbourne?

8 CAS About 6.5% of Melbourne students travel to school by tram. What is the probability that from 50 to 60 Melbourne students from a random sample of 1000 travel to school by tram?

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CHAPTER SUMMARY
RANDOM SAMPLES AND PROPORTIONS

9

- For any variable or group of variables, the **population** is the whole group from which data could be collected.
- A **sample** is a part of the population.
- Data that is collected from the whole population is called a **census**.
- Data that is collected using a sample is called a **survey**.
- A **parameter** is a characteristic value of a particular population, such as the mean.
- A **statistic** is an estimate of a parameter obtained using a sample.
- A **fair sample** is one that is representative of the population.
- A **biased sample** is not representative: it favours some section of the population.
- A **random sample** is one chosen by a method that ensures that every member of the population has an equal chance of being chosen.
- **Selection bias** arises from the way the sample is chosen.
- **Design flaw bias** arises from faults in the design of a survey or census.
- **Interviewer bias** arises from differences in the way that interviewers seek information.
- **Recall/reporting bias** occurs when knowledge of the outcome of one answer affects recall or reporting of the answer of another.
- **Completion bias** occurs when surveys are incomplete.
- **Non-response bias** occurs when some subjects do not respond to the survey. **Self-selection bias** is an extreme form where subjects choose whether or not to take part.
- A **simple random sample** is one in which every member of the population has an equal chance of being selected.
- **Random numbers** are often used to choose a random sample after numbering a population.
- **Pseudo-random** numbers are generated by a rule that gives numbers that seem random.
- **Stratified random sampling** is a random choice of representatives from identifiable groups of the population, usually in proportion to the size of each group.
- **Cluster sampling** is a random choice of group(s) from the population as the sample.
- **Systematic sampling** is a choice of representatives from one group of the population by taking every nth one to make the desired sample size.
- **Convenience sampling** is a choice from a convenient group of the population.
- **Judgment sampling** involves the use of judgement to decide on a representative sample.
- **Purposive sampling** is a choice of representatives that meet particular conditions.
- **Quota sampling** is a choice of the first convenient representatives for a sample according to the proportions of particular divisions of the population, such as male and female.

- A **sample proportion** is the ratio $\hat{p} = \frac{x}{n}$ of the number of times x a property (characteristic) occurs in a sample, divided by the number n in the sample. The occurrence of the property is normally called a **success**. For samples that are small compared to the population, sample proportion is a random binomial variable.
- The sample proportion $\hat{p}$ for the occurrence of a property in a population is an **estimator** of the probability p of the occurrence in the population.
- If the probability of a particular property is p, then for samples of n items

$$E(\hat{p}) = p,\ \text{Var}(\hat{p}) = \frac{pq}{n} \text{ and}$$

$$\text{SD}(\hat{p}) = \sqrt{\frac{pq}{n}} = \sqrt{\frac{p(1-p)}{n}}.$$

- The distribution of a statistic for many samples is called a **sampling distribution**.
- For $np > 5$ and $nq > 5$, the distribution of the sample proportion $\hat{p}$ for a random Bernoulli variable is approximately normal with mean p and standard deviation $\sqrt{\frac{pq}{n}}$.
- The **central limit theorem** states that for 'relatively large' random samples of a random variable X from a distribution with a finite mean m and a finite standard deviation s, the sampling distribution of the means is approximately normal. The approximation is also better for larger samples.
- It can also be shown that $\bar{X} = \mu$ and $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$, where n is the number in the sample.
- A random sample means that the values of X are *independent*.
- For a Bernoulli variable X with parameter p, as $n \to \infty$, the distribution of $\frac{\hat{p} - p}{\sqrt{p(1-p)/n}}$ approaches the standard normal distribution.

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CHAPTER REVIEW
RANDOM SAMPLES AND PROPORTIONS

9

Multiple choice

1 Example 2 A Government department wanted to work out the proportions of students in Victoria who intended to seek employment in various categories, undertake trade apprenticeships, go to university or have other plans at the end of Year 12. They did a survey of a school in each of Toorak, Ballarat, St Kilda and Portland, choosing 40 Year 12 students at random from each school. Each student was asked to write down their three choices in order of what they intended to do at the end of Year 12. Which is the most accurate statement about this survey?

A It is fair
B It has selection bias
C It has non-response bias
D It has recall/reporting bias
E It has interviewer bias

2 Example 5 A survey of members of a veteran and vintage car-owners club was conducted. The club had 21 members with pre-1925 cars, 42 with cars made between 1925 and 1934, 17 with cars from 1935 to 1944, 62 with cars from 1945 to 1954 and 160 with cars from 1955 on. For a survey of 30 members, how many should be chosen who own cars from 1925 to 1934?

A 4 **B** 5 **C** 6 **D** 7 **E** 9

3 Example 6 A survey was conducted by interviewing people in Little Collins Street about Chinese restaurants. To ensure a reasonable balance of views, the interviewer continued to approach people until she had obtained responses from 10 women under 30 years old, 10 women over 30, 10 men under 30 and 10 men over thirty. What kind of sampling was involved?

A Cluster
B Stratified
C Quota
D Judgement
E Systematic

4 Example 12 A die is loaded so that the probability of getting a 6 is $\frac{1}{4}$, the probability of getting a 1 is $\frac{1}{12}$ and the probability for the other numbers is $\frac{1}{6}$ each. The die is thrown 36 times and the number of 6s is noted. This experiment is repeated many times. The mean and standard deviation of the proportion of sixes will be close to:

A 0.25 and 0.18
B 6 and 2.2
C 0.25 and 0.072
D 9 and 2.6
E 0.17 and 0.028

5 Example 18 What percentage of values of $\hat{p}$ would lie between 0.56 and 0.63 for samples with $n = 70$ and $p = 0.62$?

A About 15%
B About 18%
C About 33%
D About 42%
E About 71%

6 Example 19 The probability of an Australian having fair hair is about 0.21. What is the probability that there are between 10 and 20 fair-haired people in a random sample of 80 Australians?

A 0.085 **B** 0.45 **C** 0.60 **D** 0.78 **E** 0.82

Short answer

7 Example 1 Identify the population, some parameters and some statistics for each of the following.

a 40 people on a tram were asked which football team they most disliked. 10 said Collingwood, 4 said Hawthorn, 3 said Brisbane and the rest of the teams were chosen by at most 2 people.

b From the odd-numbered houses on one side of a street, 2 have one bedroom, 12 have 2 bedrooms, 46 have 3 bedrooms, 9 have 4 bedrooms and 3 have 5 bedrooms.

8 Example 3 Which statistic is easier to collect?

A The proportion of serious accidents that occur in DIY home renovation.

B The proportion of DIY home renovators who have serious accidents.

9 Example 4 CAS Select 7 random integers between 38 and 240 inclusive.

10 Example 5 A workshop has 28 welders, 14 boilermakers, 16 sheet metal workers and 40 labourers. How many of each should be selected for a stratified sample of 10 workers?

11 Example 6 The account numbers for the customers of a business run from 10 000 to 21 314. Use systematic sampling to obtain a sample of 8 customer account numbers, starting from customer number 17 113.

12 Example 7 Two random samples of 20 items each were taken from a binomial distribution with $n = 24$ and $p = 0.45$. The samples were as follows.

A: 14, 12, 16, 14, 13, 12, 14, 8, 17, 12, 11, 9, 12, 11, 8, 11, 13, 14, 13, 14

B: 14, 8, 12, 7, 9, 12, 5, 9, 9, 13, 12, 9, 12, 9, 10, 9, 11, 13, 9, 10

a Draw a side-by-side column graph of the samples.

b Compare the samples.

13 Example 10 From 24 students in a Maths Methods class, 18 achieved a pass, C or better. What is the sample proportion of students at least passing?

14 Example 11 A fashion shop in a large shopping centre had jeans on clearance at a sale. Of the 8 brands on sale, 3 brands were \$75–\$124, 4 were \$125–\$165 and one brand was \$210.

a Estimate the probability of jeans at the shopping centre being priced under \$125.

b Are there any problems with this estimate?

15 Examples 12–14 The probability of a weighted coin landing with 'heads' up is 0.59. The coin is tossed 30 times. What is the expected proportion of heads and the standard deviation of this proportion?

16 Examples 15–17 a What happens to the distribution of sample proportions for a particular value of p as n gets larger?

b What happens to the distribution of sample proportions for a particular value of n as p gets further from 0.5?

17 Examples 18,19 CAS What percentage of sample proportions would be expected to lie between 0.4 and 0.6 for samples with $n = 40$ and $p = 0.55$?

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Application

18 In each of the following cases, state the kind of sampling employed and whether or not it is fair. If it is biased, state the kind(s) of bias.

- a A Year 11 student asks each of the people in his class what kind of mobile they have and how many SMSs they send each week to determine mobile phone use among students.
- b A market research company rings 100 phone numbers taken at random from the residential phone directory to ask whether they vacationed in Victoria, interstate or overseas last year as part of a study for the tourism industry in Victoria. They were also asked to give the reasons for their choices.

19 CAS a Generate two samples of 20 items from a uniform distribution on the interval [15, 25].

- b Find the mean and standard deviations of each sample.
- c Compare the sample results with each other and the theoretical mean and standard deviation.

20 A case of oranges with cardboard 'dimple spacers' between each layer contains 63 oranges. The probability that an orange packed in this way is rotten is 0.04. A fruit shop bought 20 boxes of oranges. Assuming that the probability of a distribution of 20 boxes having any values outside 3σ from the mean is negligible, what is the maximum number of rotten oranges the fruit shop can expect to find in a box?

Practice quiz

21 46% of Australian students travel to school by car, but only about 23% of students in Britain travel to school by car. What is the probability that between 20 and 30 students from 50 students travel by car in each country?

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10

TERMINOLOGY

central limit theorem
confidence interval
confidence level
estimator
interval estimate
margin of error
parameter
point estimate
population
probability
proportion
quantile
sample
sampling distribution
sample proportion
simulation
standard deviation
standard normal distribution
standard normal variable
statistic
variance
z-score

Prior learning

INTERVAL ESTIMATES FOR PROPORTIONS

CONFIDENCE INTERVALS

CONFIDENCE INTERVALS FOR PROPORTIONS

- the concept of an interval estimate for a parameter associated with a random variable (ACMMM177)
- the approximate confidence interval $\left(\hat{p}-z\sqrt{\hat{p}(1-\hat{p})/n}, \hat{p}+z\sqrt{\hat{p}(1-\hat{p})/n}\right)$, as an interval estimate for p, where z is the appropriate quantile for the standard normal distribution (ACMMM178)
- define the approximate margin of error $E = z\sqrt{\hat{p}(1-\hat{p})/n}$ and understand the trade-off between margin of error and level of confidence (ACMMM179)
- use simulation to illustrate variations in confidence intervals between samples and to show that most but not all confidence intervals contain p. (ACMMM180) AC

10.01 INTERVAL ESTIMATES FOR SAMPLE PROPORTIONS

You cannot predict the value of a random variable on any particular occasion. However, you do expect the typical value of a random variable to be close to the mean value. You can use a value from a sample as an estimate of the value for the population concerned.

Remember that a value obtained from a population is called a **parameter**, while a value obtained from a sample is called a **statistic**. You use statistics as estimates of parameters.

IMPORTANT

A **point estimate** of a parameter is a single value obtained from a sample. The statistic used is called an **estimator** of the parameter.

You have already seen that the expected value of a sample proportion is the probability of a property (characteristic) in a population; $E(\hat{p}) = p$. This means that sample proportion is the best estimator of that probability. You also expect that as the sample size is increased, the accuracy of the estimate will improve. In the extreme case where the whole population is used as the sample, the sample proportion must actually be the probability.

You can estimate the probability of Australians having blue eyes using the sample proportion, but if you said you thought it was between 20% and 40%, this would be an **interval estimate**.

IMPORTANT

An **interval estimate** of a parameter is an interval that is likely to include the value of the parameter.

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Example 1

State whether each of the following is a point or interval estimate.

a The average height of Year 12 Australian boys is thought to be about 181 cm.

b Average adult human height is between 165 cm and 175 cm.

Solution

a 181 cm is a single value. This is a point estimate.

b This is a range. This is an interval estimate.

INVESTIGATION Point and interval estimates in the media

The use of estimates in the media is very common, both in reporting and in advertising. In many cases, the estimates are based on samples, but it may not be clear whether a point estimate or an interval estimate has been used. For example, a headline might read '60% of Australians support a republic'. This is apparently a point estimate of the probability of an Australian supporting such a change. However, you might find that details of the survey contain a phrase such as '60% with an error of 4%', which indicates an interval estimate of 0.56 £ p £ 0.64. Before you could decide whether such an estimate is likely to be worthwhile you should probably ask 'how big was the sample?'.

- Examine media reports to determine whether statistical estimates are point estimates or interval estimates.
- What proportion of media reports use interval estimates?
- What proportion of reports state the sample size?
- What proportion of reports state the nature of the sample?

The variance and standard deviation of sample proportions are given by $Var(\hat{p}) = \frac{pq}{n}$ and $SD(\hat{p}) = \sqrt{\frac{pq}{n}} = \sqrt{\frac{p(1-p)}{n}}$, where p is the probability of the characteristic and n is the number in the sample.

$\hat{p}$ is the best estimator of p, so you can *estimate* the variance and standard deviation of the sample proportion by replacing p by $\hat{p}$.

IMPORTANT

The **estimated variance** and **estimated standard deviation** of a sample proportion $\hat{p}$ with sample size n are $Var(\hat{p}) \approx \frac{\hat{p}\hat{q}}{n} = \frac{\hat{p}(1-\hat{p})}{n}$ and $SD(\hat{p}) \approx \sqrt{\frac{\hat{p}\hat{q}}{n}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

This means that you can estimate both mean and the standard deviation of the **sampling distribution** of proportions from a single sample. Remember that a sampling distribution is the distribution of a statistic obtained from multiple samples (of the same size) from the same population.

Example 2

From a sample of 60 Year 12 students, 45 said they like chocolate. Estimate the probability of Year 12 students liking chocolate and the variance of the sampling distribution.

Solution

Use the sample proportion to estimate p.	$p \approx \hat{p} = \frac{45}{60} = 0.75$
Write the formula for the estimated variance.	$\text{Var}(\hat{p}) \approx \frac{\hat{p}(1-\hat{p})}{n}$
Substitute $\hat{p}$ and n.	$= \frac{0.75 \times (1-0.75)}{60}$ $= 0.003\,125$
Write the answer.	The probability of Year 12 students liking chocolate is about 0.75 and the variance of the sampling distribution is about 0.0031.

It doesn't make sense to show more than 2 significant figures in the estimates because the figures used in the calculation were only accurate to 2 figures.

WS
Calculations with sample proportions

You can use the standard deviation of the sampling distribution to work out an interval estimate of probabilities obtained from a sample proportion.

Example 3

In 2010, a survey of Victorians found 268 out of every 327 people aged 20–24 had completed Year 12. Use this to estimate the probability that a Victorian aged 20–24 has completed Year 12, within one standard deviation.

Solution

Estimate the probability.	$\hat{p} \approx \frac{268}{327} = 0.8195\ldots$
Write the formula for the standard deviation.	$SD(\hat{p}) \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
Substitute $\hat{p}$ and n.	$= \sqrt{\frac{0.8195\ldots(1-0.8195\ldots)}{327}}$ $= 0.0212\ldots$
Find the interval.	$\hat{p} \approx 0.8195\ldots \pm 0.0212\ldots$ $= 0.7983\ldots$ to $0.8408\ldots$
Write the answer.	The probability that a Victorian aged 20–24 has completed Year 12 is between 0.798 and 0.841, within one standard deviation.

Is it reasonable to give the estimate in the previous example to 3 significant figures, or should it be given to greater or less accuracy?

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EXERCISE 10.01 Interval estimates for sample proportions

WS Simple interval estimates

Concepts and techniques

1 Example 1 State whether each of the following is a point or interval estimate.

a The mean income is about \$47 000.

b The median house price in suburbs is between \$350 000 and \$720 000.

c The variance is between 800 and 1500.

d The probability of dark hair is about 0.6.

2 Example 2 From 80 people interviewed at Spencer St Station, 57 said they were competent in using chopsticks. Estimate the probability that Melbourne residents are competent in using chopsticks and the variance and standard deviation of the sampling distribution.

3 From 40 tosses, a coin lands heads up 18 times. Estimate the probability of this coin landing with heads up and the variance and standard deviation of the sampling distribution.

4 From a sample of 87 Australian men over the age of 30, 52 were overweight or obese. Estimate the probability of Australian men over 30 being overweight or obese and the variance and standard deviation of the sampling distribution.

5 The protractors of 50 students were checked for accuracy, and 16 were found to be inaccurate in measuring an angle of 60° (more than 1° out). Estimate the probability of students' protractors being inaccurate and the variance and standard deviation of the sampling distribution.

6 Example 3 From a group of 50 computer professionals, 32 said they used the Linux operating system on their home computers. Estimate the probability of a computer professional using Linux on their home computer to within one standard deviation.

7 From a class of 28 Year 12 students, only 19 had completed all of their homework the previous night. Use this information to estimate the probability of a Year 12 student completing all of their homework to within 1.5 standard deviations.

8 A group of 32 parents waiting for interviews at a high school Parent-teacher evening were discussing the habits of teenagers and 20 of them agreed that 'teenagers take too long in the bathroom'. What is the probability that parents of high school students think they take too long in the bathroom, to within 1.6 standard deviations?

Reasoning and communication

9 A poll of 500 voters found that 44% supported a particular political party. What is the standard deviation of the sampling distribution?

10 A sample showed 30% support for a proposal and the estimated variance of the sampling distribution was 0.005. How many people were surveyed?

10.02 CONFIDENCE LEVELS AND MARGIN OF ERROR

The **Central limit theorem** guarantees that for sufficiently large samples, the distribution of the sample means is approximately normal. This means that you can use the normal distribution to work out the size an interval must be to contain any particular proportion of the sample means.

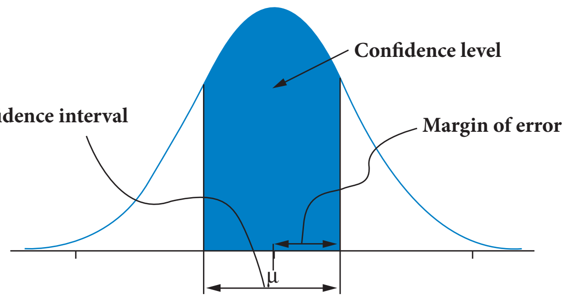
If you choose an interval of a normal distribution from 2 standard deviations below the mean to 2 standard deviations above the mean, then about 95% of the values will be in the interval. The standard normal distribution has the mean $\mu = 0$ and standard deviation $\sigma = 1$. This means that for the standard normal variable Z, $P(-2 \le z \le 2) \approx 0.95$. You say that the 95% **confidence level** for the standard normal variable has a **margin of error** of ± 2.

IMPORTANT

For a **confidence interval** symmetric around the mean in a statistical distribution:

- the **confidence level** is the proportion of values that lie within the interval
- the **margin of error** is the distance of the ends of the interval from the mean.

For a normal distribution, you can picture this as shown below.



The confidence level is the area under the curve, which is both the proportion of values and the probability of lying within the interval.

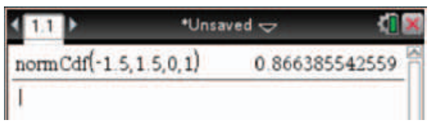
Example 4

CAS What confidence level is produced by a margin of error of 1.5 in a standard normal variable?

Solution

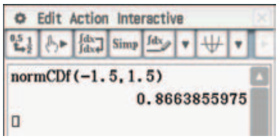
TI-Nspire CAS

Use a calculator page.
Press menu, 6: Statistics, 5: Distributions and 2: Normal Cdf. Choose boundaries of –1.5 and 1.5; or type normCdf(–1.5, 1.5, 0, 1).



ClassPad

Use the Main menu.
Tap Action, Distribution/Inv. Dist, Continuous and choose normCDf. Choose boundaries of −1.5 and 1.5; or type normCDf(−1.5, 1.5, 1, 0).

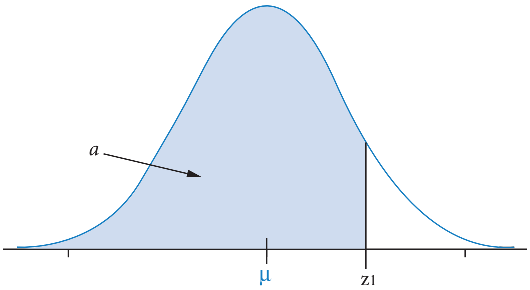


Round and write the answer.

The confidence level is about 87%.

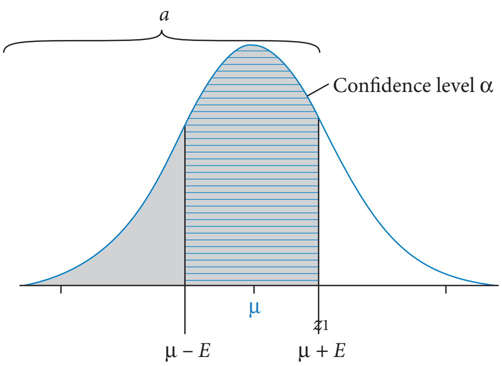
For both CAS calculators, you can just use normCdf(−1.5, 1.5) and the standard normal distribution with $\mu = 0$ and $\sigma = 1$ will be assumed.

You need to use the inverse normal distribution function to work out a margin of error for a given confidence level. The inverse function finds the value z_1 so the given proportion of values lies below z_1. In symbols, they find the value z_1 such that $P(-\infty \le z \le z_1) = a$, where a is the given area. You can see this on the diagram below.



For the confidence level a, you want the margin of error E such that $P(\mu - E \le \mu + E) = \alpha$.

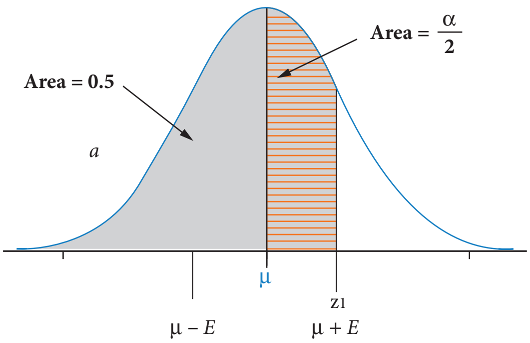
If you choose $z_1 = \mu + E$, then you get the following diagram.



You can divide the area differently to give the diagram below.

Half of a normal distribution is below the mean, so $a = 0.5 + \dfrac{\alpha}{2}$.

In Chapter 8, you saw that the value t_a for which $P(-\infty < t_a) = a$ is a **quantile**. For the proportion a of the standard normal variable Z, the quantile is written as z_a.



Example 5

CAS Find the margin of error needed to give a confidence level of 90% for a standard normal variable.

Solution

Find the area above the mean.

$$\text{Area above the mean} = \frac{0.9}{2} = 0.45$$

Find the area required for the inverse function.

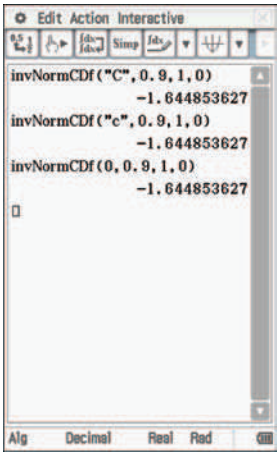
$$\text{Inverse function area } a = 0.5 + 0.45 = 0.95$$

TI-Nspire CAS

Use a calculator page.
Press menu, 6: Statistics, 5: Distributions and 3: Inverse Normal to find the quantile $z_{0.95}$; or type invNormCdf(0.95, 0, 1).

ClassPad

Use the Main menu.
Tap Action, Distribution/Inv. Dist, Inverse and invNormCDf.
Enter 0 or "C" or "c" (for central), 0.9 (confidence limit), 1 (s.d.) and 0 (mean).
Only the lower boundary is given. The confidence level of 0.9 is approximately contained in $-1.64 \le Z \le 1.64$.



Write the answer.

The margin of error is about 1.64.

WS
Margins of error for standard normal variables

In the TI, you can omit the mean and standard deviation and $\mu = 0$ and $\sigma = 1$ will be assumed.

Example 6

Find the confidence interval needed to give a confidence level of 80% for a standard normal variable.

Solution

Find the area above the mean.

$$\text{Area above the mean} = \frac{0.8}{2} = 0.4$$

Find the area required for the inverse function.

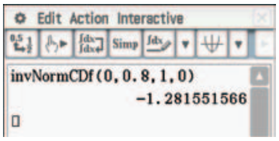
$$\text{Inverse function area } a = 0.5 + 0.4 = 0.9$$

TI-Nspire CAS

Use a calculator page.
Press menu, 6: Statistics, 5: Distributions and 3: Inverse Normal to find the quantile $z_{0.9}$; or type invNorm(0.9).

ClassPad

Use the Main menu.
Tap Action, Distribution/Inv. Dist, Inverse and invNormCDf.
Enter 0 (central), 0.8, 1 and 0.



Write the margin of error. $E \approx 1.28$

Write the answer. The confidence interval is $-1.28 < z < 1.28$.

It doesn't matter whether you write the confidence interval as $-1.28 < z < 1.28$ or as $-1.28 \le z \le 1.28$, because the probability of getting the exact boundary is 0 for a continuous distribution.

EXERCISE 10.02 Confidence levels and margin of error

Concepts and techniques

1 Example 4 CAS Find the confidence levels corresponding to the following margins of error in standard normal variables.
 a $E = 2.5$ b $E = 3$ c a margin error of 1 d $E = 2.7$

2 Example 5 CAS Find the margins of error needed in standard normal variables to give the following confidence levels.
 a $\alpha = 0.98$ b 85% confidence c 99% d $\alpha = 0.75$

3 Example 6 CAS Find the confidence intervals in standard normal variables needed to give the following confidence levels.
 a $\alpha = 0.9$ b $\alpha = 0.995$ c 95% confidence d 99% confidence

4 CAS Find the confidence levels corresponding to the following confidence intervals in standard normal variables.
 a $-2.5 \le z \le 2.5$ b $-1.5 \le z \le 1.5$ c $-1 \le z \le 1$ d $-3 \le z \le 3$

Reasoning and communication

5 Can you have a small margin of error and a high confidence level for a standard normal variable? Explain your answer.

10.03 CONFIDENCE INTERVALS AND CONFIDENCE LEVELS FOR SAMPLE PROPORTIONS

When you use a sample to work out the probability of something, how do you know whether the results are reasonable? Intuitively, a large sample would give better results than a small sample. You can use a confidence interval to give a reasonable range for the answer.

Example 7

CAS 12 students from a sample of 30 Year 12 students said they liked the 'Pirates of the Caribbean' movies.

a Estimate the probability of Year 12 students liking these movies.
b Estimate the standard deviation of the sampling distribution.
c Estimate the 90% confidence interval for the probability of liking these movies.
d What does the confidence interval mean in this case?

Solution

a Use the sample proportion. $p \approx \hat{p} = \frac{12}{30} = 0.4$

Write the answer. The probability is about 0.4.

b Write the formula for the standard deviation. $SD(\hat{p}) \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Substitute $\hat{p}$ and n. $= \sqrt{\frac{0.4(1-0.4)}{30}}$

$= 0.089\,44\ldots$

Write the answer. The standard deviation is about 0.089.

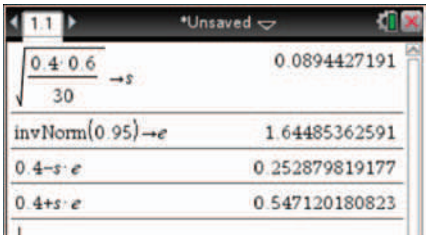
c Find the area above the mean. Area above the mean $= \frac{0.9}{2}$

$= 0.45$

Find the area required for the inverse function. Inverse function area $a = 0.5 + 0.45 = 0.95$

TI-Nspire CAS

Use a calculator page to find the quantile $z_{0.95}$ for the standard normal distribution.
Type invNorm(0.95)



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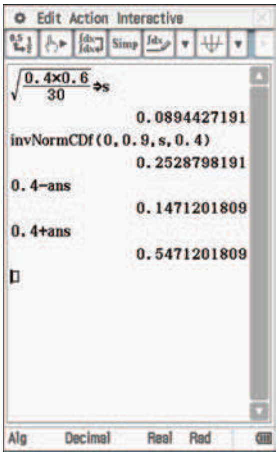
ClassPad

Find the lower bound for 'central' values.
invNormCDf(0, 0.9, s, 0.4) gives about 0.25.
Find the difference from the mean.
0.4 – ans (0.25) ≈ 0.15
Add this value (ans) to the mean.
The upper bound ≈ 0.4 + 0.15 = 0.55



Work out the margin of error for $\sigma \approx 0.089$. | $E = 0.089\,44\ldots \times 1.64\ldots$
$= 0.1471\ldots$

Find the lower boundary of the interval. | Lower boundary $= 0.4 - 0.1471\ldots$
≈ 0.25

Find the upper boundary. | Upper boundary $= 0.4 + 0.1471\ldots$
≈ 0.55

Write the answer. | The confidence interval is about (0.25, 0.55).

d 90% of the values of the sampling distribution lie in the 90% confidence interval . | The probability of a Year 12 student liking these movies has a 90% chance of lying in the interval from 0.25 to 0.55.

You can use the same method as shown in Example **7** to find any confidence interval for a sample proportion.

For the confidence level α, you need the quantile z_a for $a = 0.5 + \frac{\alpha}{2}$.

The estimated standard deviation for the sample proportion $SD(\hat{p}) \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, so the margin of error is about $z_a \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$. You can use this to find the confidence level.

IMPORTANT

The area for the **standard normal quantile** z_a of the z-score for the confidence level α is given by $a = 0.5 + \frac{\alpha}{2}$.

The corresponding margin of error for the sample proportion $\hat{p}$ is given by $E \approx z_a\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

and the confidence interval is approximately $\left(\hat{p} - z_a\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z_a\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$.

The area a is often omitted in the formulas for the margin of error and confidence interval.

Example **8** shows the use of the formula.

Example 8

CAS A sample of 80 has a sample proportion of 0.6. Find each of the following.

a The margin of error for a confidence level of 80% for the corresponding probability.

b The 80% confidence interval for the probability.

Solution

a Find the area of the required quantile.

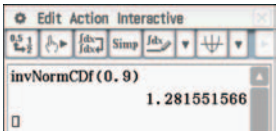
$a = 0.5 + 0.8 \div 2$
$\quad = 0.9$

TI-Nspire CAS

Use the calculator page to find the quantile.

ClassPad

Find the main menu to find the quantile.



Write the margin of error formula.

$E \approx z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

Substitute in the values.

$= 1.28\ldots \times \sqrt{\dfrac{0.6(1-0.6)}{80}}$

Calculate the value.

$= 0.0701\ldots$

Write the answer.

The margin of error is about 0.07.

b Find the confidence interval.

Interval = (0.6 – 0.07, 0.6 + 0.07)

Write the answer.

The 80% confidence interval is (0.53, 0.67).

It doesn't matter whether you write the confidence interval as the open interval (0.53, 0.67) or the closed interval [0.53, 0.67] because it is continuous.

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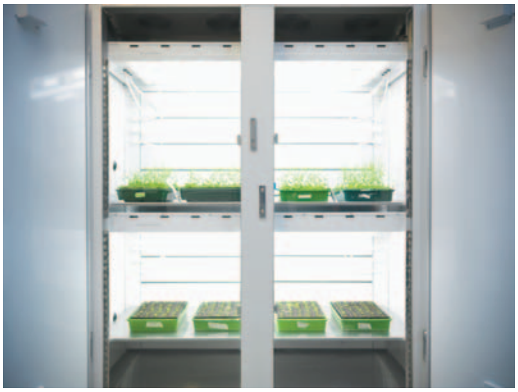
Example 9

CAS As a test of the germination rate, 300 carrot seeds were moistened and placed in an incubator. When they were checked 5 days later, 250 were found to have germinated. Estimate the 95% confidence interval for the germination rate.



Science Photo Library/Monty Rakusen/Cultura

Solution

Estimate the germination rate.

$$p \approx \hat{p} = \frac{250}{300} = 0.833\ldots$$

Find the area of the required quantile.

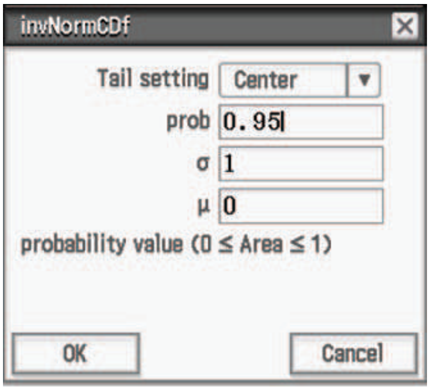
$$\begin{aligned} a &= 0.5 + 0.95 \div 2 \\ &= 0.975 \end{aligned}$$

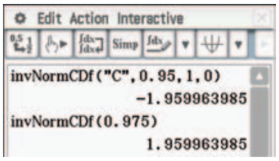
TI-Nspire CAS

Use the calculator page to find the quantile.

ClassPad

Use the main menu to find the quantile. Tap Interactive, Distribution/Inv. Dist, Inverse and invNormCDf. Tail setting is Center and prob (area) is 0.95.
The answer is the lower bound, −1.95...
Since the mean is 0, the upper bound is 1.95...
Alternatively, use a left bound with an area of 0.975.





Write the margin of error formula.	$E \approx z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
Substitute in the values.	$= 1.95\ldots \times \sqrt{\frac{0.833\ldots(1-0.833\ldots)}{300}}$
Calculate the value.	≈ 0.0422
Work out the lower boundary.	0.8333 – 0.0422 = 0.7911
Work out the upper boundary.	0.8333 + 0.0422 = 0.8744
Write the answer.	The 95% confidence interval for the germination rate is about (0.791, 0.874).

You could work out the error margin directly on your CAS calculator by including the standard deviation of the sample proportion in the inverse normal function, but it is probably safer to do it as shown in the examples.

WS

Sample proportion confidence intervals

EXERCISE 10.03 Confidence intervals and confidence levels for sample proportions

Concepts and techniques

1 Example 7 CAS 35 out of 50 people crossing the intersection of Collins and Swanston Streets in July were wearing dark-coloured clothing.

a Estimate the probability of people wearing dark clothing in the city in winter.

b Estimate the standard deviation of the sampling distribution.

c Estimate the 95% confidence interval for the probability of wearing dark clothing in the city.

2 Example 8 CAS Find the approximate margin of error for each of the following confidence levels for the stated values of $\hat{p}$ and n.

a $\hat{p} = 0.8, n = 40, \alpha = 90$

b $\hat{p} = 0.5, n = 60, \alpha = 95$

c $\hat{p} = 0.45, n = 140, \alpha = 99$

d $\hat{p} = 0.65, n = 100, \alpha = 80$

3 CAS Find the confidence intervals for each of the stated values of $\hat{p}$, n and a.

a $\hat{p} = 0.1, n = 60, \alpha = 85$

b $\hat{p} = 0.75, n = 40, \alpha = 95$

c $\hat{p} = 0.32, n = 55, \alpha = 98$

d $\hat{p} = 0.61, n = 70, \alpha = 90$

Reasoning and communication

4 Example 9 CAS The songs on popular radio stations in 'drive-time' were checked over several weeks, and from a total of 140 songs, 125 were of less than 3 minutes in duration. Estimate the 90% confidence interval for songs on 'drive-time' radio being less than 3 minutes long.

5 CAS A survey of 178 Victorian students found that 87 travelled to school by car. Estimate the 95% confidence interval for the probability of Victorian students travelling to school by car.

6 CAS A flour mill produced 2 kg packets of flour. From 100 packets checked at the end of the run, 6 were found to be underweight. What is the 95% confidence interval for the proportion of underweight packets?

7 CAS A survey of 1200 voters commissioned by a political party found that 48.5% on a two-party preferred basis intended to vote for the party. What is the 90% margin of error for this survey?

8 CAS Estimate the 90% error margins for $n = 100$ for $\hat{p} = 0.1, 0.3, 0.5, 0.7$ and 0.9 and comment on the differences.

10.04 VARIATION OF CONFIDENCE INTERVALS

You can use a CAS calculator simulation to check whether or not the population probability for a sample proportion is within a confidence interval.

Example 10

CAS A property has a probability of occurrence of $p = 0.6$.

a Simulate a sample of 40 items and estimate the 80% error margin from the sample proportion.

b Repeat the simulation and calculation another 5 times.

c Comment on the position of p in the confidence intervals.

Solution

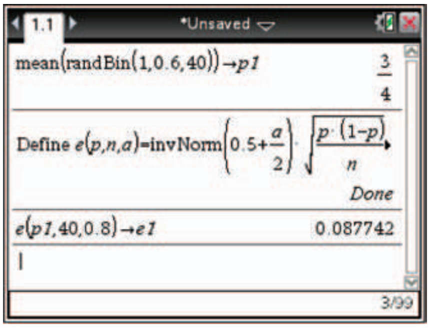
TI-Nspire CAS

a Use the calculator page.
Generate 40 Bernoulli trials with $p = 0.6$ using randBin(1,0.6,40) and find the sample proportion $p1$ using mean(randBin(1,0.6,40)).

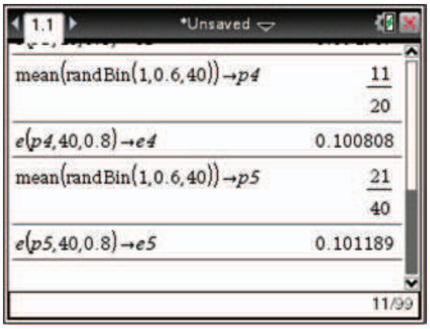
Define $e(p,n,a) =$

$$\text{invNorm}\left(0.5+\frac{a}{2}\right)\times\sqrt{\frac{p\times(1-p)}{n}}.$$

Find the 80% sample error $e1$ using $e(p1,40,0.8)$.



b Generate 40 new Bernoulli trials and find the sample proportion $p2$. Find the new 80% sample error $e2$.
Do this another 3 times.



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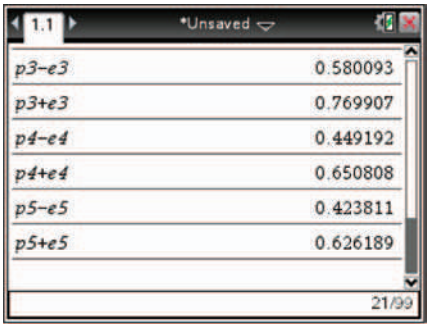
c Find the confidence intervals using ($p1-e1$, $p1+e1$) and so on for your calculator.



Write the results.

The confidence intervals were about (0.66, 0.84), (0.55, 0.75), (0.58, 0.77), (0.45, 0.65) and (0.42, 0.63).

Comment on the positions of p.

$p = 0.6$ was in all except one of the confidence intervals.

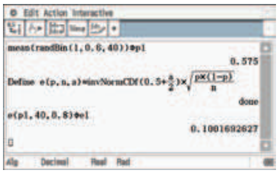
ClassPad

Use the main menu.

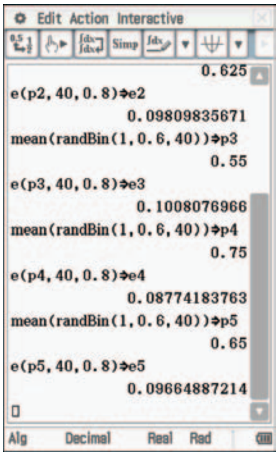
a You can generate 40 Bernoulli trials with $p = 0.6$ using randBin(1,0.6,40) and find the sample proportion $p1$ using mean(randBin(1,0.6,40)).
Define $e(p,n,a) =$

$$\text{invNormCDf}\,(0.5 + \frac{a}{2}) \times \sqrt{\frac{p\times(1-p)}{n}}$$

The multiplication signs are essential!
Find the 80% sample error $e1$using $e(p1,40,0.8)$.



b Generate 40 new Bernoulli trials and find the sample proportion $p2$. Find the new 80% sample error $e2$.
Do this another 3 times.



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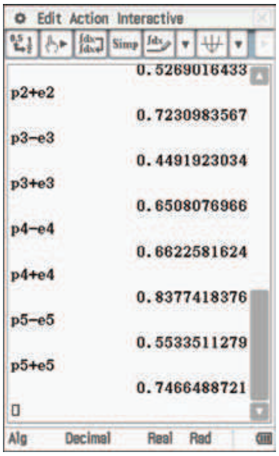
c Find the confidence intervals using $p1 - e1$, $p1 + e1$, and so on for your calculator.



Write the results.

The confidence intervals were about (0.58, 0.78), (0.53, 0.72), (0.45, 0.65), (0.66, 0.84) and (0.55, 0.74)

Comment on the positions of p.

$p = 0.6$ was in all of the confidence intervals.

Once you understand what you are doing, you can use the spreadsheet facility to do this. The spreadsheet method is much quicker, but you need to be sure you know how and why it works. This can be done using CAS calculators, or a computer spreadsheet such as Excel.

TI-Nspire CAS

First calculate $a = 0.5 + (0.8 \div 2) = 0.9$

Enter the following formulas for $p1$ and $e1$ into A1 and B1, using $p = 0.6$, $a = 0.9$ and sample size $n = 40$.

A1: =mean(randBin(1, 0.6, 40))

B1: =invNorm(0.9)×(A1×(1−A1)/40)^0.5

Since A1 is storing $p1$ and B1 is storing $e1$, make C1 and D1 the confidence limits by entering

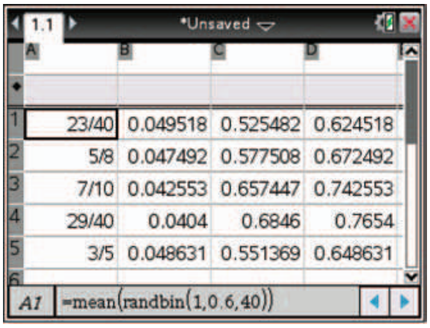
C1: = A1−B1 $(p1 - e1)$

D1: =A1+B1 $(p1 + e1)$

In this case, the confidence limits are (0.61, 0.79).

Fill each column down (Edit, Fill, Fill Range) and you can generate as many sets of results as you like.

A new set can be generated by filling column A down to row 6. The other columns will adjust automatically, but row 1 will be unchanged. Take rows 2 to 6 as the next set of five results. This can be repeated as often as required. Alternatively, all columns can be filled down to row 10 for the second set of results.



Write the answer.

The confidence limits are (0.53, 0.62), (0.58, 0.67), (0.66, 0.74), (0.68, 0.76) and (0.55, 0.65).

Comment on the positions of p.

$p = 0.6$ was in all except two of the confidence intervals.

ClassPad

First calculate $a = 0.5 + (0.8 \div 2) = 0.9$

Enter the following formulas for $p1$ and $e1$ into A1 and B1, using $p = 0.6$, $a = 0.9$ and sample size $n = 40$.

A1: =mean(randBin(1, 0.6, 40))

B1: =invNormCDf(0.9)×(A1×(1–A1)/40)^0.5

Since A1 is storing $p1$ and B1 is storing $e1$, make C1 and D1 the confidence limits by entering

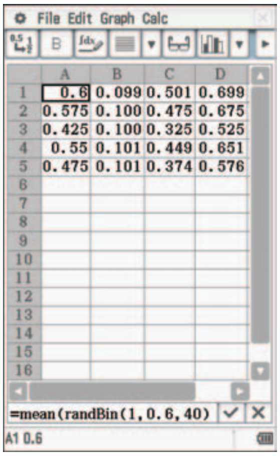
C1: = A1–B1 $(p1 - e1)$

D1: =A1+B1 $(p1 + e1)$

In this case, the confidence limits are (0.61, 0.79).

Fill each column down (Edit, Fill, Fill Range) and you can generate as many sets of results as you like.

A new set can be generated by filling column A down again. New data will appear in A1 to A5 and the other columns will adjust automatically. Alternatively, all columns can be filled down to row 10 for the second set of results.



Note that the columns have been narrowed to display all the data on one screen. This is not necessary.

Write the answer.

The confidence limits are (0.61, 0.79), (0.53, 0.72), (0.61, 0.79), (0.40, 0.60) and (0.37, 0.58).

Comment on the positions of p.

$p = 0.6$ was in all except one of the confidence intervals.

Your teacher will probably want everybody to work through Example **10** separately and compare results. You might find that all of your confidence intervals include the probability of 0.6, or you could find that 2 or 3 of the 5 do not include the probability. You are dealing with random variables, so should expect your results to vary. Across the whole class, you are likely to find that on average, about 20% of the class confidence intervals do not include the probability of 0.6.

For sample proportions, a margin error of 90% for a confidence interval means that if you take multiple samples, then you can expect to find the population probability within the approximate confidence interval calculated from the sample proportions about 90% of the time.

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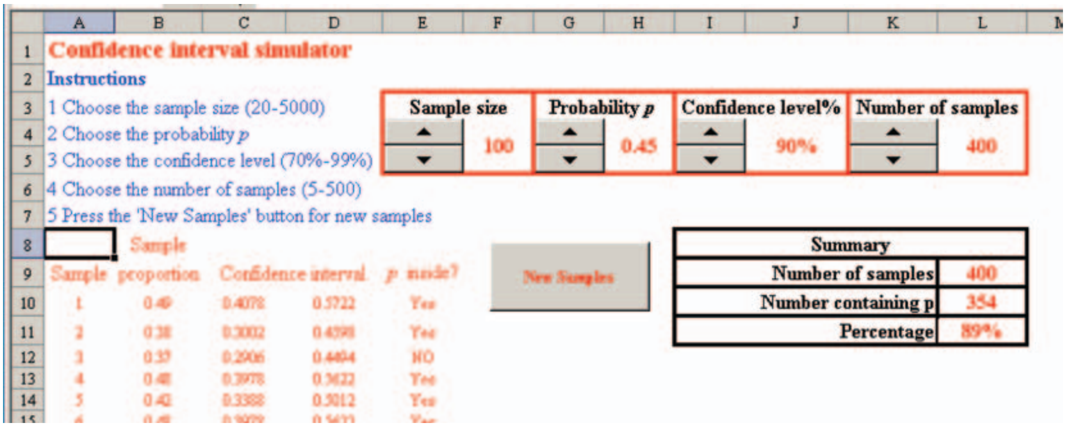
INVESTIGATION Dice simulation of confidence intervals

In this investigation you will pool results from the whole class to simulate sample proportions and confidence intervals. The probability of the event for the simulation is $p = \frac{1}{3}$ and $n = 20$.

- Roll a normal die and note when it lands with 5 or 6 uppermost
- Repeat the roll another 19 times and find the sample proportion of high numbers (5 or 6)
- Calculate the approximate 90% confidence interval for your 20 rolls
- Do another 20 rolls and find the new sample proportion and 90% confidence interval
- Do this whole procedure another 3 times
- How many of your 5 simulations of the confidence interval contained $p = \frac{1}{3}$?
- Put the results of the whole class together and compare the total number of simulations and the number of times that the confidence intervals contained the probability
- Is this what you would have expected?

The Excel spreadsheet program 'confidence interval simulator' allows you to simulate many samples and calculations of $\hat{p}$ and confidence intervals for various values of p to find the proportion of values of p that lie in different confidence intervals.

You teacher will tell you what simulations you should perform.



Confidence interval simulator

EXERCISE 10.04 Variation of confidence intervals

Reasoning and communication

Your teacher will tell you which questions each person in the class should do and probably want you to pool results with groups who have done the same questions. You may have to give a report to the class as a whole.

1 Example 10 CAS A property has a probability of occurrence of $p = 0.3$.
   a Simulate a sample of 20 items and estimate the 75% error margin from the sample proportion.
   b Repeat the simulation and calculation another 9 times.
   c Comment on the position of p in the confidence intervals

2 Example 10 CAS A characteristic has a probability of occurrence of $p = 0.8$.
   a Simulate a sample of 20 items and estimate the 90% error margin from the sample proportion.
   b Repeat the simulation and calculation another 9 times.
   c Comment on the position of p in the confidence intervals.

3 Example 10 CAS A property has a probability of occurrence of $p = 0.4$.
   a Simulate a sample of 20 items and estimate the 95% error margin from the sample proportion.
   b Repeat the simulation and calculation another 9 times.
   c Comment on the position of p in the confidence intervals.

4 Example 10 CAS A characteristic has a probability of occurrence of $p = 0.7$.
   a Simulate a sample of 20 items and estimate the 85% error margin from the sample proportion.
   b Repeat the simulation and calculation another 9 times.
   c Comment on the position of p in the confidence intervals.

5 Example 10 CAS A property has a probability of occurrence of $p = 0.2$.
   a Simulate a sample of 20 items and estimate the 80% error margin from the sample proportion.
   b Repeat the simulation and calculation another 9 times.
   c Comment on the position of p in the confidence intervals

10.05 APPLICATIONS OF CONFIDENCE INTERVALS

The only way to be *certain* about a parameter is by using a census. Any sample has a level of confidence, but it is never 100%. When you design a survey, you need to decide what level of confidence is acceptable. Since larger surveys cost more to conduct, the level of confidence will be traded off with the cost of the survey. If you have some idea of the likely value of the parameter, then you can calculate the size of the survey needed for any particular level of confidence.

IMPORTANT

The table shows the most commonly used levels of confidence and corresponding z-scores for the standard normal distribution.

Level of confidence α	90%	95%	98%	99%
***z*-score**	1.645	1.960	2.326	2.576

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Example 11

The probability of an Australian Year 12 student driving to school is thought to be about 6%. How large a sample of students should be used to establish the probability to a 95% confidence level within 0.5% of the true probability?

Solution

Write the formula for E for a sample proportion.	$E \approx z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$
Write the known values.	$E = 0.005,\ z = 1.96,\ \hat{p} \approx 0.06$
Substitute in the formula.	$0.005 = 1.96 \times \sqrt{\dfrac{0.06(1-0.06)}{n}}$
Square both sides.	$(0.005)^2 = (1.96)^2 \times \dfrac{0.06 \times 0.94}{n}$
Solve for n.	$n = \dfrac{1.96^2 \times 0.06 \times 0.94}{0.005^2} \approx 8667$
Write the answer.	About 9000 Year 12 students would need to be asked for a margin error of 0.5% to a level of confidence of 95%.

It would clearly be impractical to perform a survey to the level of accuracy requested in Example **11**. The example shows that the most critical influence on the sample size is the margin of error. To decrease the size of the sample, a larger margin of error must be accepted.

Example 12

It is thought that about 60% of Year 12 students obtain their driver's licence before they complete Year 12. How large a sample would be needed to establish this to within a margin of error of 5% at the 90% confidence level?

Solution

Write the formula for E for a sample proportion.	$E \approx z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$
Write the known values.	$E = 0.05,\ z = 1.645,\ \hat{p} \approx 0.6$
Substitute in the formula.	$0.05 = 1.645 \times \sqrt{\dfrac{0.6(1-0.6)}{n}}$
Square both sides.	$(0.05)^2 = (1.645)^2 \times \dfrac{0.6 \times 0.4}{n}$
Solve for n.	$n = \dfrac{1.645^2 \times 0.6 \times 0.4}{0.05^2} \approx 260$
Write the answer.	About 260 Year 12 students at the end of the year would need to be asked to get this accuracy.

What should you do if you have no real idea of the likely value of $\hat{p}$? In this case, you use the value that produces the largest margin of error to make sure that whatever value found, the margin of error will be within the accuracy required.

WS

Sample sizes

This is easily shown using calculus as demonstrated below.

If everything else is the same, then the size of the error margin is determined by $\sqrt{\hat{p}(1-\hat{p})}$, as this is the only part of the equation for E that changes. The largest value of the square root is produced by the largest value of $f(\hat{p}) = \hat{p}(1 - \hat{p}) = \hat{p} - \hat{p}^2$.

IMPORTANT

For the same sample size and z-score, the largest margin of error is produced by $\hat{p} = 0.5$.

Now $f'(\hat{p}) = 1 - 2\hat{p}$ and $f''(\hat{p}) = -2$, so there is a maximum at $\left(\frac{1}{2}, \frac{1}{4}\right)$.

You may find it easier in problems of this kind to transform the formula before substituting values.

Example 13

The transport department wants to determine the proportion of cars on the road that have faulty brakes to within 10% of the correct value to a level of confidence of 98% to work out whether they should request a police crackdown. How many vehicles should have their brakes tested at a random spot check?

Solution

What value of $\hat{p}$ should be used?	Use $\hat{p} = 0.5$ as the worst case.
Write the formula for E for a sample proportion.	$E = z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$
Square both sides.	$E^2 = z^2 \times \dfrac{\hat{p}(1-\hat{p})}{n}$
Transform to get n.	$n = \dfrac{z^2\hat{p}(1-\hat{p})}{E^2}$
Write the values.	$z = 2.326$, $\hat{p} = 0.5$, $E = 0.1$
Substitute in the values.	$n = \dfrac{2.326^2 \times 0.5 \times 0.5}{0.1^2}$ ≈ 135
Write the answer.	Testing the brakes of 135 vehicles at random should be enough to determine the proportion of cars with faulty brakes to within 10% at a level of confidence of 98%.

Example **13** shows that you can obtain a small size by accepting a low margin of error. In that particular case, it is unlikely that the transport department would accept such a low margin of error.

If you want to use levels of confidence other than the common ones listed in the table, you will need to use the inverse normal function to find the value of z needed.

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EXERCISE 10.05 Applications of confidence intervals

Concepts and techniques

1 Examples 11, 12 What sample size would be needed to establish an error margin of 0.03 at a 98% confidence level for the probability of a characteristic occurring in a proportion of about 0.3 in a population?

2 Example 13 The probability of a particular property in a population is unknown. What sample size should be used to establish the probability with an error margin of 5% at a confidence level of 90%?

Reasoning and communication

3 It is generally assumed that the chances of a baby being a girl or boy are 50-50. How many births would need to be checked to establish the true proportion in Australia to within 0.1% at a confidence level of 95%?

4 Suppose you were organising the venue for a Year 12 formal. How many students should you survey to establish the preferred venue from a list of 3 at an error margin of 10% and confidence level of 90%?

5 The tourist information centre counter staff in Federation square said that about 2 in every 5 of the people coming in asked about local attractions. How many enquiries would need to be noted to establish the proportion to within 2% at a level of confidence of 99%?

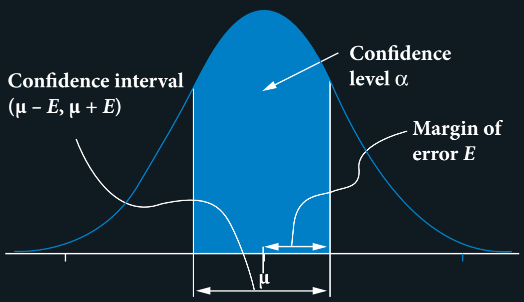
6 An advertising company wanted to conduct a small survey of consumers to establish a baseline for the proportion who were aware of a particular brand of ice-cream before a marketing campaign. How many consumers should they target to get a result accurate to within 3% at a confidence level of 95%?

7 CAS A survey of 50 people claimed that 71% ± 5% of the population could roll their tongues. What is the confidence level of this result?

8 CAS A poll of voting intentions among 300 voters claimed that the vote for a new political party would be 15% ± 3%. What level of confidence has been used in the survey?

10 CHAPTER SUMMARY
CONFIDENCE INTERVALS

- A value obtained from a population is called a **parameter**, while a value obtained from a sample is called a **statistic**.
- A **point estimate** of a parameter is a single value obtained from a sample. The statistic used is called an **estimator** of the parameter.
- An **interval estimate** of a parameter is an interval that is likely to include the value of the parameter.
- The **estimated variance** and **estimated standard deviation** of a sample proportion $\hat{p}$ with sample size n are $Var(\hat{p}) \approx \dfrac{\hat{p}\hat{q}}{n} = \dfrac{\hat{p}(1-\hat{p})}{n}$ and $SD(\hat{p}) \approx \sqrt{\dfrac{\hat{p}\hat{q}}{n}} = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$.
- For a **confidence interval** symmetric around the mean in a statistical distribution, the **confidence level** α is the proportion of values that lie within the interval and the **margin of error** E is the distance of the ends of the interval from the mean.



- For the sample proportion $\hat{p}$, the margin of error is given by $E \approx z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ and the confidence interval is approximately $\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$, where z is the **standard normal quantile** for the value $0.5 + \dfrac{\alpha}{2}$, so the proportion of the standard normal distribution below z is $P(-\infty < x < z) = 0.5 + \dfrac{\alpha}{2}$.
- The most commonly used levels of confidence and z values are:

Level of confidence α	90%	95%	98%	99%
z-score	1.645	1.960	2.326	2.576

- The **largest margin of error** for a sample proportion occurs when $\hat{p} = 0.5$, assuming that the sample size n and level of confidence are set.

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CHAPTER REVIEW
CONFIDENCE INTERVALS

10

Multiple choice

1 Example 1 The following estimations were contained in a document about human development.

P 2-3 Months: Can lift head, follows objects with eyes, opens and closes hands

Q 6 Months: Enjoys 'peekaboo', rolls over, remains sitting without support, recognises others

R 10 ± 2 months: Moves around by crawling or scooting, babbles, first words, eats finger food

S 18 months: Walks, talks in simple words or phrases, shy with strangers, becoming independent

Which of **P**, **Q**, **R** and **S** are interval estimates?

A P only　　B P and Q　　C P and R
D Q and S　　E All of them

2 Example 2 From 40 Maths Methods students, 16 were also studying Chemistry. The estimated probability and variance of the sampling distribution of students who do Maths Methods and Chemistry are:

A 0.4 and 0.06　　B 0.6 and 0.04　　C 0.4 and about 0.24
D 0.16 and about 0.024　　E 0.4 and 0.006

3 Example 4 A margin of error of 2 in a standard normal variable corresponds to a confidence level of about:

A 85%　　B 90%　　C 95%
D 98%　　E 99%

4 Examples 5, 6 What is the confidence interval of a standard normal variable for a level of confidence of 85%?

A (−1.44, 1.44)　　B (−1.04, 1.04)　　C (−1.96, 1.96)
D (−0.45, 0.45)　　E (−1.645, 1.645)

5 Examples 8, 9 30% of a group of 80 Year 12 students were strong swimmers. Estimate the 90% margin of error for the probability of a Year 12 student being a strong swimmer.

A 0.0026　　B 0.029　　C 0.051
D 0.084　　E 0.17

6 Examples 11, 12 In Australia, the probability of someone voting number 1 for one of the major parties is generally about 0.4. How large a sample would be needed to determine the level of support for a major party to within 3% at a 99% confidence level?

A About 50　　B About 260　　C About 1100
D About 1800　　E About 2100

7 Example 13 What sample size should be used to determine the proportion of students who swim at least once a week to an accuracy of 10% at a 95% confidence level?

A About 50　　B About 100　　C About 200
D About 400　　E About 800

Short answer

8 Examples 2, 3 A survey of 90 Australians found that 62 owned their own homes, with some having mortgages. Estimate the probability of Australians owning their own home to within 1.5 standard deviations.

9 Example 4 CAS Find the confidence level that corresponds to an error margin of 2.5 standard deviations of a standard normal variable.

10 Examples 5, 6 CAS What margin of error corresponds to a confidence level of 92% for a standard normal variable?

11 Example 7 CAS 11 out of 40 Year 12 students said that they had part-time jobs.

- a What is the sample proportion for Year 12 students having part-time jobs?
- b Estimate the standard deviation of the sampling distribution.
- c Estimate the 95% confidence interval for the probability of Year 12 students having part-time jobs.
- d What does the 95% confidence interval mean in this case?

12 Examples 8, 9 CAS A sample of 120 items has a sample proportion of 0.75.

- a Estimate the 90% margin of error for the corresponding probability.
- b What is the 90% confidence interval for this probability?

13 Example 10 CAS a Simulate a sample of 30 for a property that has a probability of occurrence of 0.67.

- b Find the 95% confidence interval for the sample proportion.
- c Comment on the position of the probability within the confidence interval.

14 Examples 11, 12 CAS What sample size should be used to establish a 98% confidence interval with a margin of error of 5% for a characteristic that occurs about 70% of the time?

15 Example 13 CAS A survey is to be conducted to determine the proportion of a population having a certain property.

- a What should be assumed about the occurrence to find the sample size required?
- b What sample size is needed to establish the occurrence within 8% at the 90% confidence level?

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Application

16 The estimated probability and 95% margin of error from a survey were given as 0.31 and 0.058. What was the sample size?

17 A survey of 500 Australians renting houses in outer suburbs found that 32% of their income was paid in rent.

- **a** Estimate the standard deviation of the sampling distribution.
- **b** What z-score should be used for an 85% confidence interval?
- **c** Estimate the 85% confidence interval for the proportion of income used for rent in outer suburbs in Australia.

18 A surfer who regularly surfs at Bell's beach says that you get the best rides from about 1 in 5 waves. Another surfer standing at the lookout watching the waves sees that they are coming about every 11 seconds.

- **a** Assuming that the first surfer is close to being correct, how many waves would you have to watch to determine the proportion of good waves to an accuracy of 10% at a 95% level of confidence?
- **b** How long would you need to watch?

Practice quiz

ANSWERS

1.01

1 a $8x^3+9x^2$ b $12x-1$ c $30x+21$
  d $72x^5-16x^3$ e $30x^4-4x$

2 a $12x+1$ b $15x^2+4x-5$
  c $5(x^3+2x-1)+(5x-3)(3x^2+2)$
  d $2x(x^2-x-1)+(2x-1)(x^2+7)$
  e $3x^2(x^4+2x^3-5x^2+x-2)+(x^3+3)(4x^3+6x^2-10x+1)$

3 a $\frac{-2}{(2x-1)^2}$ b $\frac{15}{(x+5)^2}$
  c $\frac{x^4-12x^2}{\left(x^2-4\right)^2}=\frac{x^2\left(x^2-12\right)}{\left(x^2-4\right)^2}$
  d $\frac{16}{(5x+1)^2}$ e $\frac{-x^2+14x}{x^4}=\frac{-x+14}{x^3}$
  f $\frac{11}{(x+3)^2}$ g $\frac{-2x^2}{\left(2x^2-x\right)^2}$
  h $\frac{-6}{(x-2)^2}$ i $\frac{-34}{(4x-3)^2}$
  j $\frac{-14}{(3x+1)^2}$

4 a $-4x^{-5}$ b $-8x^{-9}$ c $-6x^{-4}$
  d $-55x^{-12}$ e $-\frac{9x^{-10}}{5}$ f $\frac{1}{2}x^{-\frac{1}{2}}$ g $\frac{1}{4}x^{-\frac{3}{4}}$
  h $\frac{3}{7}x^{-\frac{6}{7}}$ i $\frac{10}{3}x^{-\frac{1}{3}}$ j $-x^{-\frac{3}{2}}$

5 a $-\frac{5}{x^6}$ b $-\frac{6}{x^7}$ c $-\frac{6}{x^4}$ d $-\frac{2}{3x^3}$
  e $-\frac{4}{5x^2}$ f $\frac{1}{2\sqrt{x}}$ g $\frac{1}{6\sqrt[6]{x^5}}$ h $\frac{4}{3\sqrt[3]{x^2}}$
  i $\frac{2}{3\sqrt[3]{x}}$ j $\frac{5\sqrt{x^3}}{2}$

6 a $-\frac{1}{16}$ b -3 c 852
  d $26\frac{65}{96}$ e -320

7 a 160 b $-\frac{1}{9}$ c -5

8 a $x<0$ b $x>\frac{1}{\sqrt{27}}$ or $x<-\frac{1}{\sqrt{27}}$ c $x>0$
  d $-5<x<-1$ e None

9 $(1, 1)$ and $\left(-\frac{5}{9},-1\frac{32}{243}\right)$

10 $x=-8, 2$

11 $17x-25y-19=0$

12 a \$564 816 b i \$84 755.20 ii \$376 960

13, 14 Proofs

1.02

1 a $m^2, 3x+3, (3x+3)^2$
  b $3g+3$, x^2 and $3x^2+3$
  c $(3x+3)^2+4=9x^2+18x+13$
  d $3(x^2+4)+3=3x^2+15$
  e $\sqrt[3]{x^2+4}$ f $\sqrt[3]{x^2}+4$
  g 9 h 3 i 169 j 85

2 a $4(x+3)^3$ b $6(2x-1)^2$
  c $70x(5x^2-4)^6$ d $48(8x+3)^5$
  e $-5(1-x)^4$

3 a $135(5x+9)^8$
  b $4(x-4)$
  c $4(6x^2+3)(2x^3+3x)^3$
  d $8(2x+5)(x^2+5x-1)^7$
  e $12(3x^5-2x)(x^6-2x^2+3)^5$

4 a $\frac{3}{2}(3x-1)^{-\frac{1}{2}}$ b $2(4-x)^{-3}$
  c $-6x(x^2-9)^{-4}$ d $\frac{5}{3}(5x+4)^{-\frac{2}{3}}$
  e $\frac{3}{4}(3x^2-14x+1)(x^3-7x^2+x)^{-\frac{1}{4}}$

5 a $\frac{3}{2\sqrt{3x+4}}$ b $-\frac{5}{(5x-2)^2}$
  c $-\frac{8x}{(x^2+1)^5}$ d $-\frac{2}{\sqrt[3]{7-3x}}$
  e $-\frac{5}{2\sqrt{(4+x)^3}}$

6 a $x(x+1)^2(5x+2)$ b $8(9x-1)(3x-2)^4$
  c $3x^3(16-7x)(4-x)^2$ d $(10x+13)(2x+5)^3$
  e $10x(x^3+5x^2-3)(x^2+1)^4+(3x^2+10x)(x^2+1)^5$

7 a $\frac{6(5x+1)(2x-9)^2-5(2x-9)^3}{(5x+1)^2}$
  $=\frac{(2x-9)^2(20x+51)}{(5x+1)^2}$
  b $\frac{(7x+2)^4-28(x-1)(7x+2)^3}{(7x+2)^8}=\frac{-21x+30}{(7x+2)^5}$
  c $\frac{15(2x-5)^3(3x+4)^4-6(3x+4)^5(2x-5)^2}{(2x-5)^6}$
  $=\frac{3(3x+4)^4(4x-33)}{(2x-5)^4}$

d $\dfrac{3\sqrt{x+1}-\dfrac{3x+1}{2\sqrt{x+1}}}{x+1}=\dfrac{3x+5}{2\sqrt{(x+1)^3}}$

e $\dfrac{\dfrac{2x-3}{2\sqrt{x-1}}-2\sqrt{x-1}}{(2x-3)^2}=\dfrac{-2x+1}{2\sqrt{x-1}(2x-3)^2}$

8 a 245 b $-\dfrac{152}{2401}$ c $\dfrac{20\sqrt[3]{25}}{3}$

d $-\dfrac{12\,544\sqrt[3]{25}}{15}$ e $-\dfrac{4}{27}$

9 (1, 1)

10 a 2.2×10^{14} L b 3.1×10^{11} L/min

c 2.7×10^{13} L/min

11 a $x\geq4$ b $\dfrac{1}{6}$ c $y=\dfrac{1}{6}x+\dfrac{5}{6}$

d $\dfrac{1}{2\sqrt{a-4}}$ e $\dfrac{\sqrt{a-4}}{a}$ f (8, 2)

12 a −4 b −4 c Parallel d −25

e −25 f Parallel g −1 h −1

i Parallel j $x=-2\pm k$

13, 14 Proofs

INVESTIGATION: DERIVATIVES OF a^x

1

x	0.5	0.8	1.5	2	2.5
$\lim\limits_{h\to0}\dfrac{a^h-1}{h}$	−0.6931	−0.2231	0.4055	0.6931	0.9163
x	3	3.5	4	5	
$\lim\limits_{h\to0}\dfrac{a^h-1}{h}$	1.0986	1.2528	1.3863	1.6094	

2 $\dfrac{d}{dx}(a^x)\approx-0.6931a^x$ for $x=0.5$

$\dfrac{d}{dx}(a^x)\approx-0.2231a^x$ for $x=0.8$

$\dfrac{d}{dx}(a^x)\approx0.4055a^x$ for $x=1.5$

$\dfrac{d}{dx}(a^x)\approx0.6931a^x$ for $x=2$

$\dfrac{d}{dx}(a^x)\approx0.9163a^x$ for $x=2.5$

$\dfrac{d}{dx}(a^x)\approx1.0986a^x$ for $x=3.5$

$\dfrac{d}{dx}(a^x)\approx1.2528a^x$ for $x=3.5$

$\dfrac{d}{dx}(a^x)\approx1.3863a^x$ for $x=4$

$\dfrac{d}{dx}(a^x)\approx1.6094a^x$ for $x=5$

3 The graph for $a=0.5$ is reflected in the y-axis compared to $a=2$. The graphs the derivatives for $a=0.8$ and $a=0.5$ are below those of the functions for $x<0$.

The graphs for $a=1.5$ and 2 look similar, with the graphs of the derivatives below those of the functions for $x>0$.

The graphs for $a=3$, 3.5, 4 and 5 look similar, with the graphs of the derivatives above those of the functions for $x>0$.

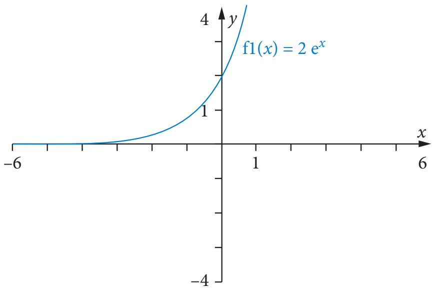
4 As a increases, the graph of the derivative moves from below that of the function to above and further away.

1.03

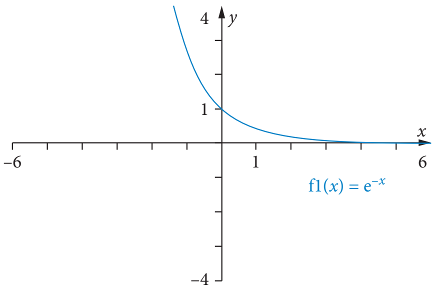
1 a 4.48 b 0.14 c 2.70

d 0.05 e − 0.14

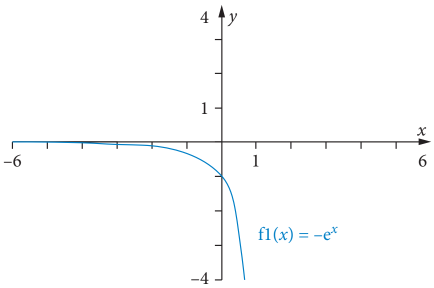
2 a



b



c



3 a e b 1.79 c $x-ey+2=0$

4 a $9e^x$ b $-e^x$

c e^x+2x d $6x^2-6x+5-e^x$

e $3e^x(e^x+1)^2$ f $-9e^x(5-e^x)^8$

g $12e^x(2e^x-3)^5$ h $4(e^x+1)(e^x+x)^3$

5 a $3e^{3x}$ b $2e^{2x-1}$ c $8e^{4x}$

d $2x\,e^{x^2-1}$ e $(10x^4-9x^2+1)\,e^{2x^5-3x^3+x-3}$

6 a $e^x+xe^x=e^x(x+1)$

b $2e^x+e^x(2x+3)=e^x(2x+5)$

c $15x^2e^x+5x^3e^x=5x^2e^x(x+3)$

d $2e^{3x}+6xe^{3x}=2e^{3x}(3x+1)$

e $2e^{2x}(x^2+x+2)+e^{2x}(2x+1)=e^{2x}(2x^2+4x+5)$

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7 a $\dfrac{xe^x - 2e^x}{x^3} = \dfrac{e^x(x-2)}{x^3}$

b $\dfrac{6xe^{6x} - e^{6x}}{3x^2} = \dfrac{e^{6x}(6x-1)}{3x^2}$

c $\dfrac{10xe^{5x} - 6e^{5x}}{5x^4} = \dfrac{2e^{5x}(5x-3)}{5x^4}$

d $\dfrac{e^x - (x-1)e^x}{e^{2x}} = \dfrac{2-x}{e^x}$

e $\dfrac{e^{3x} - 2e^{2x}(e^x+1)}{e^{4x}} = \dfrac{-(e^x+2)}{e^{2x}}$

8 a 2.71 b $28e^{-4} \approx 0.5128$ c 348.4

d $80e^6 + e^2$ e $5e^{-6}$, 25

9 2

10 $k = -\dfrac{1}{2}$

1.04

1 a i 175 cases ii 186 cases
iii 254 cases iv 877 cases

b i 11.5 cases/week ii 15.7 cases/week
iii 54.4 cases/week

2 a 180 sales b 966 sales
c 116 sales/day d 3337 sales/day

3 a 1100 swans
b i 1246 swans ii 1485 swans
iii 2706 swans
c i 28 swans/month ii 31 swans/month
iii 37 swans/month

4 a 200 g
b i 188.4 g ii 157.3 g iii 60.2 g
c i 2.26 g/year ii 1.89 g/year iii 0.72 g/year

5 a i 86 270.8 ha ii 52 588.2 ha iii 23 046 ha
b i 3707.1 ha/year ii 2413.9 ha/year
iii 1057.9 ha/year

6 a $N = 90\,000e^{0.29t}$ b 512 761 bacteria
c 148 701 bacteria/hour d 474 345 bacteria/hour

7 a $R = 43e^{-0.008t}$
b i 39.7 cm ii 33.8 cm iii 19.3 cm
c i 0.32 cm/year ii 0.27 cm/year iii 0.15 cm/year

8 a P_0 b 15.5% c $\dfrac{dP}{dt} = 0.024P$

9 a i 87% ii 49.7% iii 24.7%
b 10 years

10 a $2e^4x - y - 3e^4 = 0$ b $(1\frac{1}{2}, 0)$
c $\dfrac{e^4}{4}$ units^2

11 a 102
b i 112 ii 2779
c $\dfrac{dP}{dt} = 0.6e^{0.3t}$
d i 4 birds/month ii 804 birds/month

12 a $N_0 = 30\,000$ b 12 102 864
c i 14 523 437 bacteria/hour
ii 6.5×10^{10} iii 1.2×10^{17}

INVESTIGATION: ESTIMATION OF DERIVATIVES OF sin (x) AND cos (x)

The derivative of sin (x) is close to cos (x).
The derivative of cos (x) is close to −sin (x).
As h is made smaller, the approximations become more exact.

1.05

1 a $1 + \cos(x)$ b $6\cos(x)$
c $6\cos(6x)$ d $2x\cos(x^2 - 3)$
e $\dfrac{4}{3}\cos\left(\dfrac{x}{3}\right)$

2 a $6\cos(x)(\sin x + 9)^5$ b $\sin(x) + x\cos(x)$
c $e^x\cos(e^x)$
d $\dfrac{2[x\cos(2x) - \sin(2x)]}{5x^3}$ e $\cos(x)\,e^{\sin(x)}$

3 a $\dfrac{1}{2}$ b $\dfrac{12}{\sqrt{2}} = 6\sqrt{2}$ c 0
d $1 + \dfrac{\sqrt{3}}{2} = \dfrac{2+\sqrt{3}}{2}$ e $-\dfrac{4}{\pi^2}$

4 a $-\sin(x)$ b $-3\sin(x)$
c $-5\sin(5x)$ d $-6x\sin(3x^2 + 1)$
e $-\sin\left(\dfrac{x}{2}\right)$

5 a $3[4 - \sin(x)][4x + \cos(x)]^2$
b $\cos(x) - x\sin(x)$ c $-3e^{3x}\sin(e^{3x})$
d $\dfrac{-x\sin(x) - \cos(x)}{3x^2}$ e $-\cos(x)\sin[\sin(x)]$

6 a $\dfrac{1}{\cos^2(x)}$ b $1 + \dfrac{6}{\cos^2(x)}$
c $\dfrac{9}{\cos^2(9x)}$ d $\dfrac{12}{\cos^2(4x)}$
e $\dfrac{5}{\cos^2(x)}[\tan(x) - 1]^4$

7 a $2x\tan(x) + \dfrac{x^2}{\cos^2(x)}$
b $\dfrac{2x - \tan(2x)\cos^2(2x)}{x^2\cos^2(2x)}$ c $\dfrac{e^{\tan(x)}}{\cos^2(x)}$
d $\dfrac{-\sin(x)}{\cos^2[\cos(x)]}$ e $\dfrac{1 - \tan(x)\cos^2(x)}{e^x\cos^2(x)}$

8 a $e(2\sin(0.5) + \cos(0.5))$ b $\dfrac{\pi^2 + 4\pi}{8}$
c $-\dfrac{\sqrt{3}}{2}$ d $-\sin(e) - \dfrac{\cos(e)}{e}$
e $3e^2\cos(e^2) - 2e^4\sin(e^2)$

9 a 0.157 b −0.269

10 $x = \pm\dfrac{\pi}{4}, \pm\dfrac{3\pi}{4}, \pm\dfrac{5\pi}{4}, \pm\dfrac{7\pi}{4}, \ldots$

11 a 2 cm b $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \ldots$ s

c 0 cm/s d $\frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}$ s

12 a 2.3 mm/s b 3 mm/s^2

c Proof

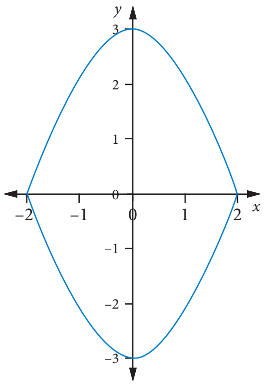
1.06

1 About 1.15^c or $65.9°$.

2 About 1.23^c or $70.5°$.

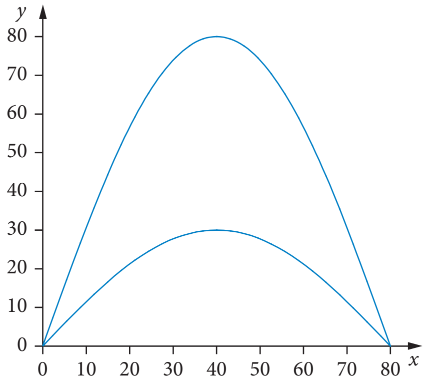
3 About 0.881^c or $50.5°$.

4 a



b 1 cm c About 2.34^c or $134.0°$

5 a



b 50 m c About 0.396^c or $22.7°$.

6 a $6\sqrt{3}x - 12y + 6 - \sqrt{3}\pi = 0$

b $2x + 2\sqrt{2}y + 4 - \pi = 0$

c $6x + 12y - 6\sqrt{3} - \pi = 0$

d $12\sqrt{3}x - 24y + 12 - \sqrt{3}\pi = 0$

e $12x - 2y + 2 - \pi = 0$

7 a $v = -\frac{3}{2}\sin\left(\frac{t}{2}\right)$ b $a = -\frac{3}{4}\cos\left(\frac{t}{2}\right)$

c $\pi, 3\pi, 5\pi, \ldots$ s d $\pm\frac{3}{2}$ cm/s; 0 cm/s^2

e 0.75 at $2\pi, 6\pi, \ldots$ s

8 a $a = 2.52\pi\cos(2\pi t)$ b 0 m/s; 7.9 m/s^2

c $\frac{3}{4}, \frac{7}{4}, \frac{11}{4}, \ldots$ s

9 Approximately 18 cm/min.

10 a $x = 2\sin(3t)$

$$v = \frac{dx}{dt} = 6\cos(3t)$$
$$a = \frac{dv}{dt} = -18\sin(3t)$$
$$= -9[2\sin(3t)]$$
$$= -9x$$

b $x = a\cos(nt)$

$$v = \frac{dx}{dt} = -an\sin(nt)$$
$$a = \frac{dv}{dt} = -an^2\cos(nt)$$
$$= -n^2[a\cos(nt)]$$
$$= -n^2x$$

CHAPTER 1 REVIEW

1 B

2 B

3 D

4 C

5 E

6 D

7 B

8 a $5x^4(3x^2 + 2x - 5) + 2x^5(3x + 1)$

b $2x(x^3 - 4x - 1) + (x^2 + 1)(3x^2 - 4)$

9 a $\frac{5x^2(4x+3)}{(2x+1)^2}$ b $\frac{4x^2 - 6x + 5}{(4x-3)^2}$

10 a $-6x^{-7}$ b $\frac{1}{7}x^{-\frac{6}{7}}$ c $\frac{3\sqrt{x}}{2}$ d $-\frac{18}{x^3}$

11 a $10(2x-7)^4$

b $24x(3x+1)(2x^3 + x^2 - 3)^3$

12 a $-15x^4(x^5+1)^{-4}$ b $\frac{1}{3\sqrt[3]{(x-1)^2}}$

13 a $6x(x+2)^8 + 24x^2(x+2)^7 = 6x(x+2)^7(5x+2)$

b $\frac{-6x(5x+1)}{(5x-1)^5}$

14 a e^3 b $3x - e^2y + 9 = 0$

15 a $2x + 2e^x$ b $6e^{6x}$ c $9e^x(e^x - 3)^8$

d $8e^{4x+1}$ e $e^x - e^{-x}$

16 a $4e^{2x} + 2e^{2x}(4x+3) = 2e^{2x}(4x+5)$

b $\frac{e^{3x}(3x-13)}{(x-4)^2}$

17 a 1000 apps

b i 4221 apps ii 317 348 apps

c i 1013 apps/month ii 76 164 apps/month

18 a i $55.6°$ ii $35.4°$

b i $8.3°$/min ii $5.3°$/min

19 a $Q = 375e^{0.04t}$ b 476.7 g

c 19.1 g/week

20 a $3\cos(x)$ b $5\cos(5x)$

c $\sin(2x) + 2x\cos(2x)$

d $7[3 - \cos(x)][3x - \sin(x)]^6$

e $3x^2\cos(x^3 + 1)$

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21 a $-\frac{1}{3}\sin\left(\frac{x}{3}\right)$

b $e^x[\cos(x) - \sin(x)]$

c $-15\sin(3x)\,[2 + \cos(3x)]^4$

d $-\pi\sin(\pi x)$

e $-2\cos(x)\sin(x)$

22 a $\frac{6}{\cos^2(x)}$ b $\frac{3}{\cos^2(3x)}$ c $\frac{\pi}{5\cos^2\left(\frac{\pi x}{5}\right)}$

d $3x^2\tan(2x) + \frac{2x^3}{\cos^2(2x)}$ e $\frac{x - \tan(x)\cos^2(x)}{x^2\cos^2(x)}$

23 Proof

24 a $4x - 2y - \pi + 2 = 0$

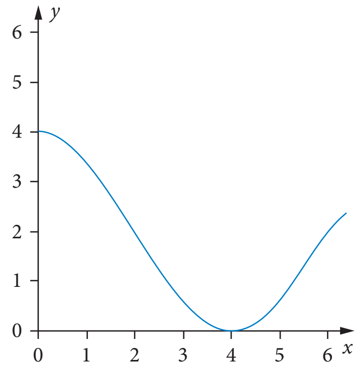
b $\left(\frac{\pi - 2}{4}, 0\right); \left(0, \frac{2-\pi}{2}\right)$

c $\frac{(\pi - 2)^2}{16}$ units^2

25 a -5.8 cm b $0, \pi, 2\pi, \ldots$ s

c -6 cm/s

26 a



b About 57.5°.

2.01

1 a Continuous b Discrete

c Discrete d Continuous

e Discrete f Continuous

g Discrete h Continuous

2 a $X = \{1, 2, 3, 4, 5, 6\}$

b $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

c $X = \{0, 1, 2, 3\}$

d $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

e $X =$ any real number between 0 and 2.

3 a Discrete, random

b Discrete, random

c Discrete, random

d Discrete, non-random (the number chosen by a person could be influenced by their personal preferences and is therefore not random).

e Continuous, random

4 D

5 C

6 B

7 E

8 D

9 D

10 B

11 a 0.981 b $\frac{6}{19}$

12 $(2, \frac{1}{16})$, $(3, \frac{1}{8})$, $(4, \frac{3}{16})$, $(5, \frac{1}{4})$, $(6, \frac{3}{16})$, $(7, \frac{1}{8})$, $(8, \frac{1}{16})$

13 a 0.125 b 0.0625 c 0.8125

d 0 e 0.5

f $\{0, 1, 2, 3, 4\}$, $p(0) = 0.0625$, $p(1) = 0.4375$, $p(2) = 0.3125$, $p(3) = 0.125$, $p(4) = 0.0625$

14 $M = 1, 2, 3, 4, 5, 6$; $P(M = m) =$ $(1, \frac{11}{36})$, $(2, \frac{1}{4})$, $(3, \frac{7}{36})$, $(4, \frac{5}{36})$, $(5, \frac{1}{12})$, $(6, \frac{1}{36})$

15 a $T = 0, 1, 2, 3, 4$;

$P(T = t) = (0, \frac{1}{16})$, $(1, \frac{1}{4})$, $(2, \frac{3}{8})$, $(3, \frac{1}{4})$, $(4, \frac{1}{16})$

b $\frac{5}{16}$

2.02

1 A

2 D

3 B

4 E

5 B

6 B

7 a No, as $p(x)$ does not add to 1.

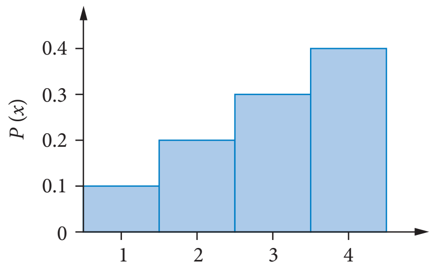
b Yes

c No, as $p(t)$ has a negative value.

8 a

x	$p(x)$
1	0.1
2	0.2
3	0.3
4	0.4

b



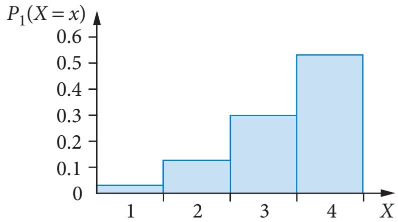
c $0 \le P(X = x) \le 1$ for all values of x; $\sum P(X = x) = 1$ and every value of x has a unique value of $p(x)$.

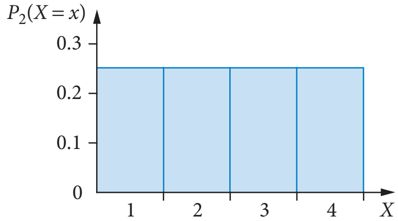
9 a

x	1	2	3	4
$P_1(X = x)$	$\frac{1}{30}$	$\frac{4}{30}$	$\frac{9}{30}$	$\frac{16}{30}$
$P_2(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

b For both distributions, $0 \le p(x) \le 1$ and $\sum p(x) = 1$.

c





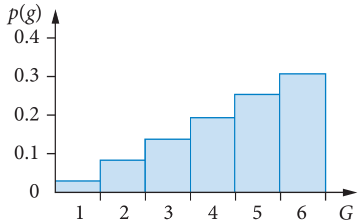
10

f	2	3	4	5	6	7	8
$P(F = f)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

11 a

g	1	2	3	4	5	6
$P(G = g)$	$\frac{1}{36}$	$\frac{1}{12}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{1}{4}$	$\frac{11}{36}$

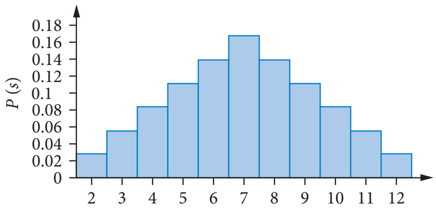
b



12 a

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

b

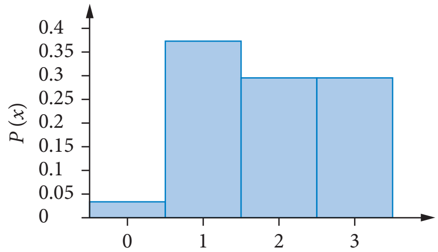


c $0 \le P(X = x) \le 1$ for all values of x; $\sum P(X = x) = 1$ and every value of x has a unique value of $p(x)$.

13 a

x	0	1	2	3
$p(x)$	$\frac{1}{27}$	$\frac{10}{27}$	$\frac{8}{27}$	$\frac{8}{27}$

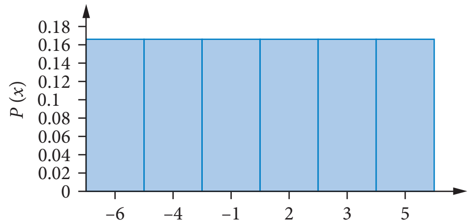
b



14 a

x	−6	−4	−1	2	3	5
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b



15

x	0	1	2	3	4
$p(x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

16

x	−5	5	20
$p(x)$	$\frac{5}{9}$	$\frac{5}{12}$	$\frac{1}{36}$

2.03

1 0.289

2 a 0.178 b 0.185

3 0.4

4 a 0.279 b 0.097

5 a 0.322 b 0.107

INVESTIGATION: RANDOM NUMBERS

The more results that are combined, the closer the graph comes to a uniform distribution.

2.04

1 A

2 B

3 a $N = 1, 2, 3, 4$; uniform

b $X = 0, 1, 3, \ldots, 9$; uniform

c $S = 0, 1, 3, \ldots, 9$; uniform

d $D = 1, 2, 3, \ldots, 20$; non-uniform.

e $M = 1, 2, 3, 4, 5, 6$; uniform

f $T = 2, 3, 4, \ldots, 12$; non-uniform

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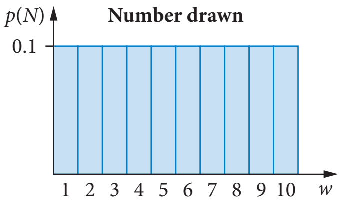
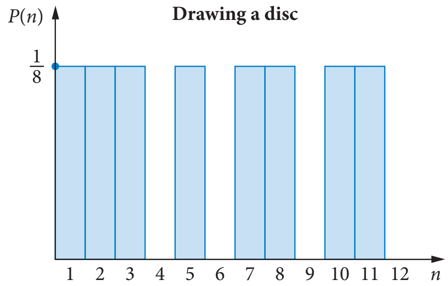
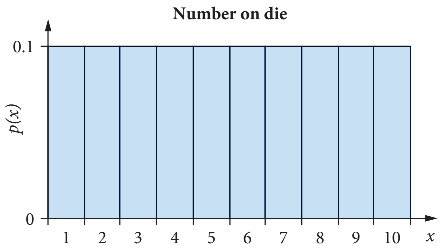
4 

The probability distribution for N is uniform.

5 a 

b $\frac{3}{8}$ c $\frac{7}{8}$ d $\frac{3}{8}$

e $\frac{5}{8}$ f $\frac{5}{8}$

6 a 

b 0.9 c 0.3 d 0.4

7 a $\frac{1}{30}$ b $\frac{29}{30}$ c $\frac{3}{5}$

d $\frac{13}{30}$ e $\frac{17}{30}$

8 a $\frac{240}{571} \approx 0.4203$ b $\frac{150}{571} \approx 0.2627$

c $\frac{241}{571} \approx 0.4221$ d $\frac{301}{571} \approx 0.5271$

2.05

1 A
2 D
3 B
4 B
5 D
6 a X = number of hearts selected; $N = 52$; $n = 4$; $k = 13$; $X = 0, 1, 2, 3, 4$
b X = number of faulty transistors selected; $N = 6$; $n = 3$; $k = 2$; $X = 0, 1, 2$
c X = number of defective chips selected; $N = 50$; $n = 10$; $k = 9$; $X = 0, 1, 2, \ldots, 9$
d X = number of defective boards selected; $N = 200$; $n = 40$; $k = 20$; $X = 0, 1, 2, \ldots, 20$
e X = number of workstations requiring a special login code selected; $N = 95$; $n = 12$; $k = 35$; $X = 0, 1, 2, \ldots, 12$
7 a 0.2135 b 0.6 c 0.0552
d 0.0012 e 0.1770
8 a 0.1136 b 0.2979 c 0.1523
d 0.0908 e 0.0217
9 0.3151
10 a 0.2705 b 0.5777
11 a 0.0833 b 0.7282
12 a 0.000 000 1228 b 0.0238
13 a 0.3590 b 0.0507 c 0.0098
14 0.5696

2.06

1 C
2 D
3 B
4 5.5
5 a 4 b −1 c 3
6 2.8
7 a $E(F) = 57.5$ b $E(M) = 17.25$
8 a About 1437 b About 315
9 $2\frac{7}{12}$
10 a 1.375 b 1.852
11 a

s	1	2	3	4	5	6
$p(s)$	$\frac{11}{36}$	$\frac{1}{4}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{1}{12}$	$\frac{1}{36}$

b 2.528
12 a

s	3	4	5	6	7	8	9
$P(S = s)$	0.1	0.1	0.2	0.2	0.2	0.1	0.1

b $E(S) = 6$
13 2
14 a $k = 0.05$
b The value of X is more likely to be 1 or 2 than it is to be 4 or 5. This means that $E(X)$ is likely to be less than 3.
c 2.15
15 $x = \frac{1}{12}$ and $y = \frac{5}{24}$.
16 1.9
17 1

2.07

1 A
2 A
3 A
4 $Var(W) \approx 3.24$, $SD(W) \approx 1.8$
5 84

6 7.62

7 a

x	3	4	6	7	8	9
$p(x)$	0.05	0.05	0.15	0.45	0.2	0.1

b 6.9 c About 1.99.

8 a

n	0	1	2	3	4	5	6	7	8	9
$p(n)$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{2}{15}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{15}$

b 5 c About 5.07.

d About 2.25.

9 a $\mu = 1.3$; $Var(N) \approx 2.01$; $\sigma \approx 1.42$

b $\mu = 13.54$; $Var(R) \approx 2.67$; $\sigma \approx 1.63$

10 a

Z	0	1	2	3
$P(Z = z)$	$\frac{1}{11}$	$\frac{3}{22}$	$\frac{3}{11}$	$\frac{1}{2}$

b 2.182 c 0.983

11 About 6.05.

12 a $\mu \approx 1437$; $Var(X) \approx 13\,158$; $\sigma \approx 114.7$

b $\mu \approx 315$; $Var(Y) \approx 195$; $\sigma \approx 14$

13 a 59.42 b 16.1 c 4.01

14 $k = 5$

15 $k = 3$

16 a 7 b $5\frac{5}{6}$ c About 2.42.

17 a

t	4	6	8	10	12
$p(t)$	$\frac{1}{36}$	$\frac{1}{9}$	$\frac{5}{18}$	$\frac{1}{3}$	$\frac{1}{4}$

b $E(T) = 9\frac{1}{3}$ and $Var(T) = 4\frac{4}{9}$

18 a

x	2	3	4	5	6
$p(x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{9}$

y	1	2	3
$p(y)$	$\frac{5}{9}$	$\frac{1}{3}$	$\frac{1}{9}$

b 4, 1.33 c 1.56, 0.469

INVESTIGATION: LIFE INSURANCE

The actual life insurance cost is higher than the theoretical cost based on the probability of death. The differences are because of the costs of administration and the profits made by the companies.

2.08

1 a −\$2.64 b No

2 8%

3 \$14

4 2.65

5 a 0.0775 b 0

6 a About 0.763 b About 2.513

c About 1.4

7 The player can expect to win \$0.80 playing this game, so it favours the player.

8 a The player can now expect to lose \$0.20 playing this game, so it favours the house.

b The player can now expect to lose \$0.40 playing this game, so it is even more favourable to the house.

9 No, you are likely to lose about \$7 on average for each ticket purchased.

10 You can expect to lose about 22c on each play.

11 \$1225

12 \$675

13 a \$25 b \$3000

14 a 0.178 b \$335.60 c \$300

15 a \$0.33 b 33c or \$1 for 3 games

16 a \$2400 b −\$1000

17 2.875

18 You would expect the standard deviation for the first exam to be more than in the second.

19 \$270 000

CHAPTER 2 REVIEW

1 D

2 E

3 E

4 A

5 B

6 A

7 A

8 C

9 B

10 B

11 C

12 a 0.06 b 0.49 c About 0.089

13 a Discrete b Continuous

c Discrete d Continuous

e Discrete f Discrete

g Continuous

h Discrete (must be to 5 cents)

14 a Yes b Yes

c No, as $p(x)$ has a negative value.

15 a

x	1	2	3	4
$P_1(X = x)$	0.1	0.2	0.3	0.4
$P_2(X = x)$	0.25	0.25	0.25	0.25

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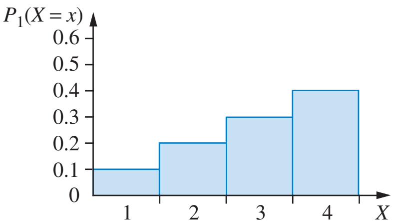
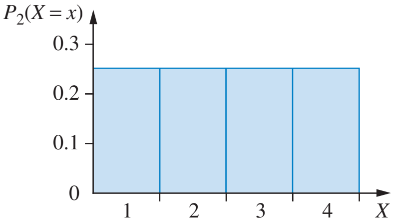
b 

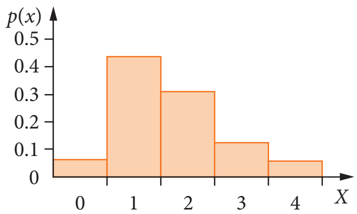


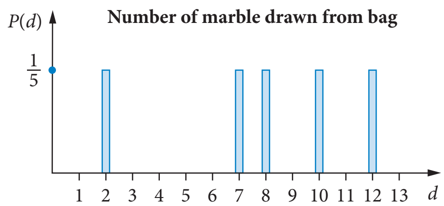
c $\sum p(x) = 1$ and $0 \le p(x) \le 1$, for each distribution.

d The second distribution is uniform.

16 a

x	0	1	2	3	4
$P(X=x)$	$\frac{1}{16}$	$\frac{7}{16}$	$\frac{5}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

b 

17 a 

b $P(d \text{ is odd}) = \frac{1}{5}$

c The distribution is uniform.

18 a

t	11	12	13	14	15	16	17
f	2	3	4	6	1	3	0
$P(T=t)$	0.08	0.12	0.16	0.24	0.04	0.12	0

t	18	19	20
f	1	1	4
$P(T=t)$	0.04	0.04	0.16

b 0.12 c 0.4

19 a $N = 40$, $n = 5$, $k = 7$, $X = 0, 1, 2, \ldots, 5$

b $N = 5000$, $n = 10$, $k = 1000$, $X = 0, 1, 2, \ldots, 10$

c $N = 100$, $n = 15$, $k = 12$, $X = 0, 1, 2, \ldots, 12$

20 a 0.0918 b 0.2415 c 0.000 656

21 a

d	0	1	2	3	4	5
$P(D=d)$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

b $E(D) = 1\frac{17}{18} \approx 1.94$

22 $E(T) \approx 5.27$; $Var(T) \approx 5.83$

23 $E(X) = 1$; $Var(X) = 4$

24 a 450 b 17 500

c About 132.3

25 a

x	0	1	2	3
$P(X=x)$	$\frac{1}{64}$	$\frac{9}{32}$	$\frac{9}{32}$	$\frac{27}{64}$

b $\frac{135}{64} \approx 2.1$

26 $E(X) = 12.25$; $Var(X) \approx 80$; the distribution is non-uniform.

27 a 0.1620 b 0.5158

28 a 0.125 0028 b \$0.53 c 47%

29 a \$1.67

b No, you should win in the long run.

c Change the sums that are included so winning and losing were equal. Eg. less than 6 or an 8, 11 or 12; more than 7 or a 4.

30 a \$2400 b −\$1000

31 The expected value is about −23c, so the player loses 23c.

32 \$4980

3.01

1 $9\frac{2}{9} = 9.\overline{2}$

2 $5\frac{1}{15} = 5.0\overline{6}$

3 a $7\frac{1}{14} \approx 7.071$ b $3\frac{1}{27} \approx 3.037$

c $50\frac{9}{25} = 50.36$ d $3\frac{7}{405} \approx 3.017$

e $2\frac{1}{32} \approx 2.031$

4 16, 16.5

5 10 420 000

6 268.8 compared with 269.042 006 25 (exact)

7 \$316

8 6%

9 a $6x^2 - 6x + 4$ b 40 c 0.8

10 1.35

11 28.8π cm^3 ≈ 90.48 cm^3

12 a 13.26 cm b $\delta V = 9\pi\delta x$ mL

c $\delta V = 125\delta y$ mL d About 2.8 mL.

e About 25 mL.

13 a 9.72 m/s^2 b $\delta g \approx -0.34\delta t$

c 0.17 m/s^2

14 2000 mm^2

15 a \$14 400 b \$23 040 c \$28 800

16 6%

17 1%

3.02

1 $7x^6 - 10x^4 + 4x^3 - 1$; $42x^5 - 40x^3 + 12x^2$

2 $72x^7$

3 $f'(x) = 10x^4 - 3x^2, f''(x) = 40x^3 - 6x$
4 $42x^5 - 40x^3 + 48x^2$
5 $-20\cos(2x)$
6 a $\dfrac{dy}{dx} = 4x - 3, \dfrac{d^2y}{dx^2} = 4$ b $-4x^{-5}, 20x^{-6}$
7 $f'(1) = 11, f''(-2) = 168$
8 $f'(-1) = -16, f''(2) = 40$
9 $g''(4) = -\dfrac{1}{32}$
10 $\dfrac{d^2h}{dt^2} = 26$ when $t = 1$
11 $f'(x) = -\dfrac{1}{2\sqrt{2-x}}; f''(x) = -\dfrac{1}{4\sqrt{(2-x)^3}}$
12 $f'(x) = -\dfrac{16}{(3x-1)^2}; f''(x) = \dfrac{96}{(3x-1)^3}$
13 $\dfrac{d^2v}{dt^2} = 24t + 16$
14 $x = \dfrac{7}{18}$
15 $x > \dfrac{1}{6}$
16 $\dfrac{dy}{dx} = 20\cos(x)[4\sin(x) - 2]^4;$

$$\frac{d^2y}{dx^2} = 320\cos^2(x)[4\sin(x)-2]^3 - 20\sin(x)[4\sin(x)-2]^4$$

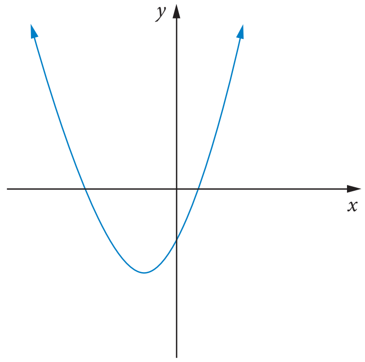
17 $f'(x) = \cos\left(\dfrac{x}{2}\right) + 3\sin(2x);$

$$f''(x) = -\frac{1}{2}\sin\left(\frac{x}{2}\right) + 6\cos(2x)$$

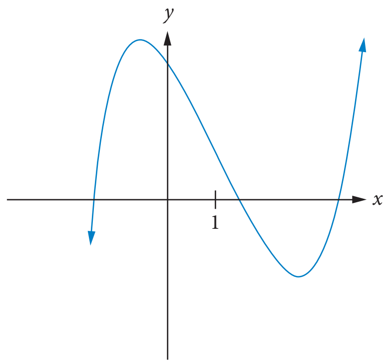
18 $b = \dfrac{2}{3}$
19 $b = -2.7$
20 velocity = 20 m/s; acceleration = 18 m/s^2
21 a -4 m/s b 8 m/s
c 0 m/s^2 d 18 m/s^2
22 a 11 m/s b -9 m/s
c 2 m/s^2 d -22 m/s^2
23 a $v = 3t^2 + 12t - 2;\ a = 6t + 12$
b 266 m c 133 m/s d 42 m/s^2
24 a $v = 2 - 10t$ b -98 m/s
c $a = -10 = g$
25 $v = \dfrac{17}{(3t+1)^2}; a = -\dfrac{102}{(3t+1)^3}$

3.03

1 $x > -\dfrac{1}{3}$
2 $x < 3$
3 $\dfrac{d^2y}{dx^2} = -8 < 0$
4 $\dfrac{d^2y}{dx^2} = 2 > 0$
5 $x < 2\dfrac{1}{3}$
6 $-2 < x < 1$
7 a No
b Yes—inflection at (0, 0).
c Yes—inflection at (0, 0).
d Yes—inflection at (0, 0).
e No
8 (1, 9)
9 (1, −17) and (−1, −41)
10 None: (2, 31) is not an inflection since concavity does not change.
11 (0, 7), (1, 0) and (−1, 14)
12 Other answers are possible.



13 Other answers are possible.



14 $f''(x) = \dfrac{12}{x^4}$
$x^4 > 0$ for all $x \neq 0$
So $\dfrac{12}{x^4} > 0$ for all $x \neq 0$
So the function is concave upward for all $x \neq 0$.
15 a $\dfrac{d^2y}{dx^2} = 12x^2 + 24$
$x^2 \geq 0$ for all x
So $12x^2 \geq 0$ for all x
$12x^2 + 24 \geq 24$
So $12x^2 + 24 \neq 0$ and there are no points of inflection.
b $12x^2 + 24 \geq 24$ The curve is always concave upwards.
16 $a = 2$

17 $p=4$

18 $a=3, b=-3$

3.04

1 (1, 0) minimum

2 (0, 1) minimum

3 $(2, -5), \dfrac{d^2y}{dx^2}=6>0$

4 $(0.5, 0.25), \dfrac{d^2y}{dx^2}<0$, so a maximum

5 (0, –5), horizontal point of inflection

6 Yes—horizontal point of inflection at (0, 8).

7 (–2, –78) minimum, (–3, –77) maximum

8 (0, 1) maximum, (–1, –4) minimum, (2, –31) minimum

9 (0, 1) maximum, (0.5, 0) minimum, (–0.5, 0) minimum

10 (4, 176) maximum, (5, 175) minimum

11 (3.67, 0.38) maximum

12 (0, –1) minimum, (–2, 15) maximum, (–4, –1) minimum

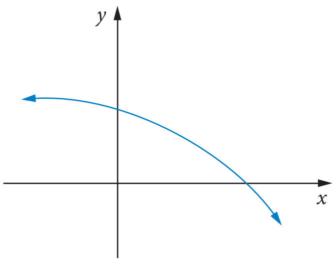
13 a $a=-\dfrac{2}{3}$ b maximum, as $\dfrac{d^2y}{dx^2}<0$

14 $m=-5\dfrac{1}{2}$

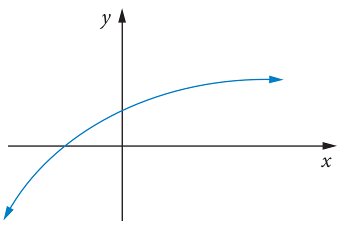
15 $a=3, b=-9$

3.05

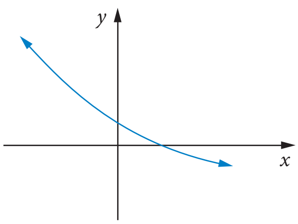
1 a



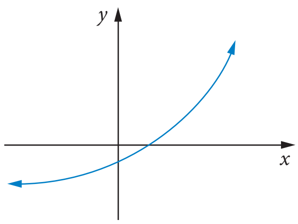
b



c



d



2 a $\dfrac{dy}{dx}>0, \dfrac{d^2y}{dx^2}>0$

b $\dfrac{dy}{dx}<0, \dfrac{d^2y}{dx^2}<0$

c $\dfrac{dy}{dx}>0, \dfrac{d^2y}{dx^2}<0$

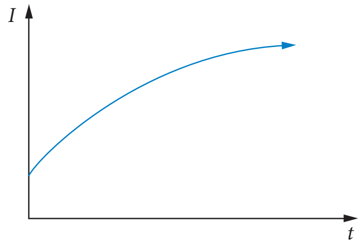
d $\dfrac{dy}{dx}<0, \dfrac{d^2y}{dx^2}>0$

e $\dfrac{dy}{dx}>0, \dfrac{d^2y}{dx^2}>0$

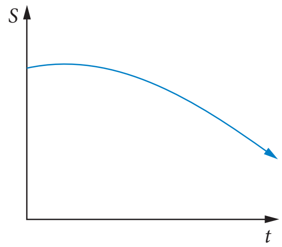
3 a $\dfrac{dP}{dt}>0, \dfrac{d^2P}{dt^2}<0$

b The rate is decreasing.

4



5



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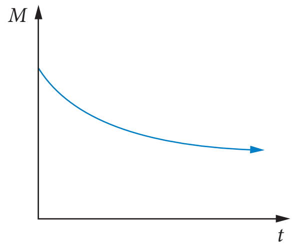

7 $\dfrac{dM}{dt}<0, \dfrac{d^2M}{dt^2}>0$. The mass is decreasing but at a decreasing rate.

8 a The number of fish is decreasing.

b The population rate is increasing.

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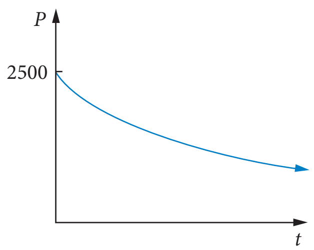
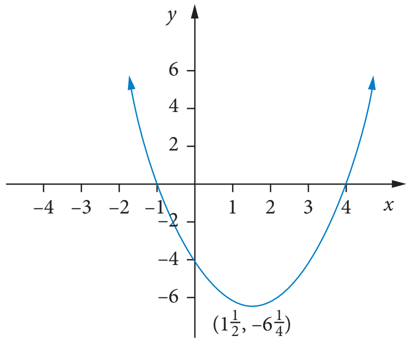
c



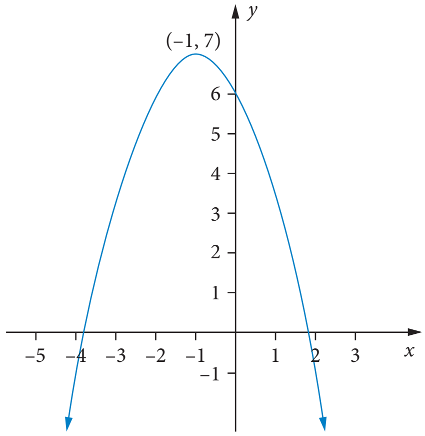
9 The level of education is increasing, but the rate is slowing down.

10 The population is decreasing, and the population rate is decreasing.

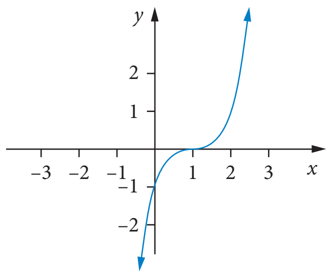
11



12



13



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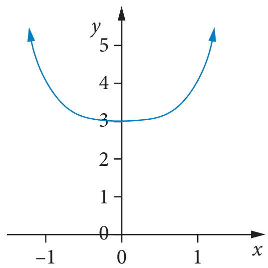

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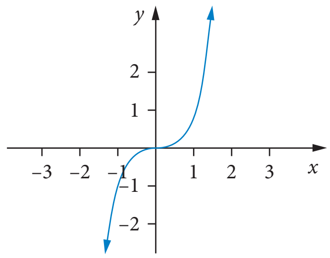

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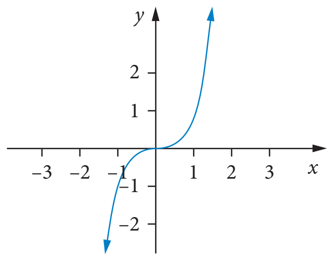

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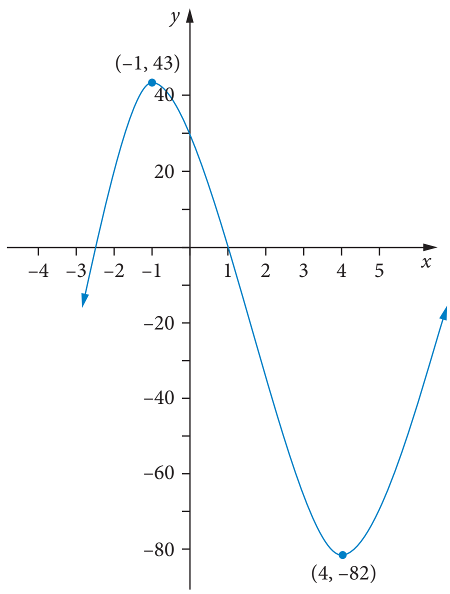

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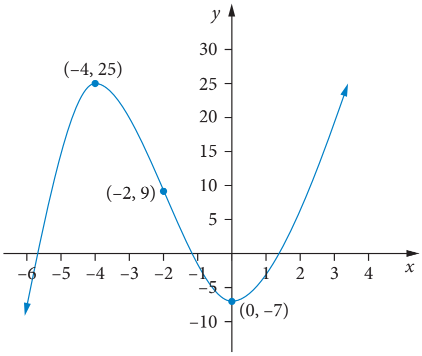
18 a (0, –7) minimum, (–4, 25) maximum

b (–2, 9)

c



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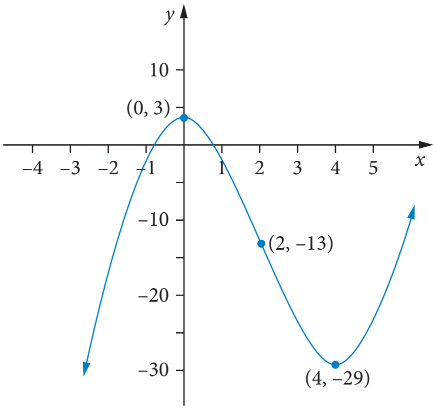

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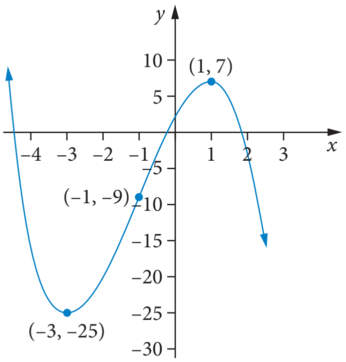

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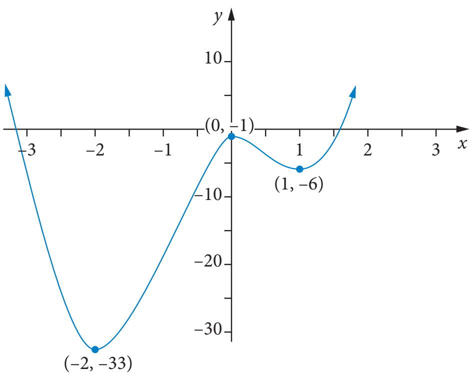

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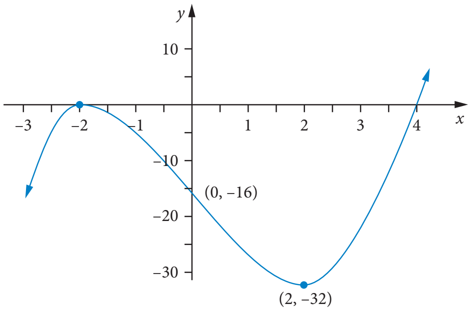

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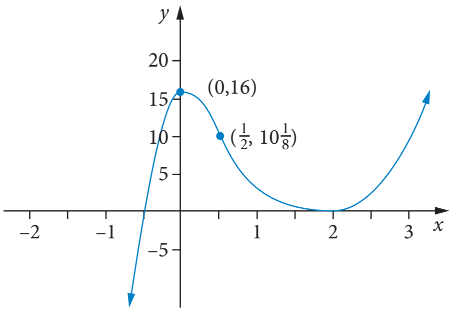
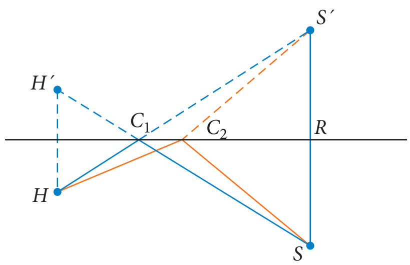


INVESTIGATION: HERON'S PROBLEM

Consider the case where the house is closer to the bank than the shed. If it's the other way round, just rename *H* and *S* the other way round.



Put a sticks at *S′* and *H′* on the other side of the river, directly opposite the shed and house and the same distance from the riverbank. Set off towards the stick. When you get to the riverbank, get the water and go directly to the shed.

The path HC_1S' is the same length as HC_1S as $\triangle C_1SR$ is congruent to $\triangle C_1S'R$.

The same applies to any other path like as HC_2S' and as HC_1S.

The shortest path between *H* and *S′* is a straight line, so the path described is the shortest.

3.06

1 $a = 6$, $b = 4.5$

2 Both 12.5.

3 20 and –20

4 Both 16.

5 (3.5, 3.5)

6 $\left(3\frac{1}{2}, \sqrt{\frac{7}{2}}\right)$

7 a $y=\frac{3}{4}x+2\frac{1}{4}$ b $y=-\frac{3}{4}x+2\frac{1}{4}$

8 10

9 About 1.18 km

10 20 km/h

11 About 15 m

12 50 km/h

13 About 63.9 m

14 a $\frac{1}{20}$ b 0.36 months (10 or 11 days)

15 About 71° (70.528…), taking about 10.7 min to finish.

3.07

1 500 000 m^2

2 $1:2\sqrt{2} \approx 1:2.83$

3 Width ≈ 12 cm, depth ≈ 6 cm

4 10 cm × 10 cm × 5 cm

5 54 m^2

6 5.2 cm

7 2 cm × 2 cm

8 About 7.79 cm

9 Height ≈ 1.68 m, width ≈ 3.36 m

10 Radius ≈ 1.89 m $\left(\frac{4\sqrt{2}}{3}\text{ m}\right)$, height ≈ 2.67 m $\left(2\frac{2}{3}\text{ m}\right)$

11 $\frac{18}{4+\pi}\text{ m}^2$

12 4000 cm^3

13 Height ≈ 50 cm, diameter ≈ 31 .8 cm

14 6 m × 6 m × 9 m

15 a About 2.62 cm × 8.10 cm × 14.77 cm.
b About 3.92 cm × 12.15 cm × 22.15 cm.
c About 6.54 cm × 20.25 cm × 36.92 cm.

INVESTIGATION: FUN RIDES

1 He should drop the price by \$3.75 to \$6.25.

2 The amount extra is \$262.50.

3 Changing parameters (the charges, proportion full, capacity of the fun ride) will change the point of highest return.

3.08

1 15 t per month

2 11

3 a $P(x) = 2.7x - 0.001x^2 - 50$ b 1350

4 a 2500 b 2000

5 a 1500 b 3500, \$8250

6 \$162

7 a 1500 b 3500, \$82 500

8 a $C(x)=100+250x-5x^2+\frac{x^3}{3}$
b $R(x) = x(5000 - 5x)$
c $P(x)=-100+4750x-\frac{x^3}{3}$
d 69

9 7 (integer value)

10 75.8 km/h, \$53.58

3.09

1 84 m

2 4.25 cm

3 18.75 m

4 11 m

5 40 m

6 38.2 m

7 Minimum of about 217 at 2°C, maximum of about 1189 at 11°C.

8 55 trees/ha

9 About 11.34 km along the coast (i.e 1.66 km from the marina).

10 He should walk through the bush to a point 3.75 km down the main road from the nearest point (i.e. 1.25 km from the service station) and then walk down the road to get there in 2 h 20 m.

CHAPTER 3 REVIEW

1 C

2 B

3 A

4 C

5 D

6 a 4.05 b 3.985

7 about 18.1 cm^3

8 a diameter ≈ 36.3 cm, height ≈ 58.0 cm
b $x\%$ c $2y\%$ d 9%

9 a $30x^5 - 6x + 1,\ 150x^4 - 6$
b $8(2t + 9)^3,\ 48(2t + 9)^2$
c $54n^2 + 48n - 22,\ 108n + 48$
d $\frac{21}{(3x-1)^2}, -\frac{126}{(3x-1)^3}$

10 a $\frac{dx}{dt}=3t^2-24t+36$ b $\frac{d^2x}{dt^2}=6t-24$
c -12 cm/s^2

11 $x>1\frac{1}{6}$

12 Yes, point of inflection at (0, 0).

13 (–4, –767) and (–2, –143)

14 a Minimum at (1, 0).
b Maximum at (2.5, 6.25).
c Maximum at (0, 7) and minimums at (–1, 2) and (2, –25).
d Minimum at (0.5, 0).

15 Point of horizontal inflection at (0, –1).

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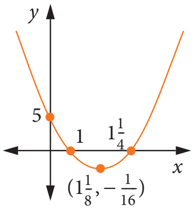
16 a $\dfrac{dP}{dt} > 0$ and $\dfrac{d^2P}{dt^2} < 0$

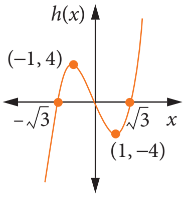
b The number of possums is increasing.

c The rate of growth of the population is decreasing.

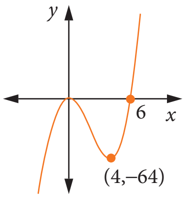
17 $y = 4x^2 - 9x + 5$



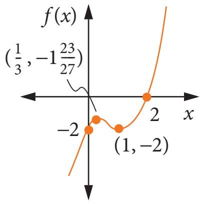
$h(x) = 2x^3 - 6x$



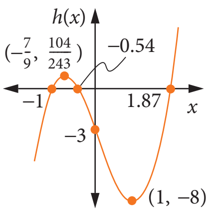
$y = 2x^3 - 12x^2$



$f(x) = x^3 - 2x^2 + x - 2$



$h(x) = 3x^3 - x^2 - 7x - 3$



18 $x = 4, y = 12$

19 2 trees

20 a 8:24 a.m.

b 6 km.

21 160 m by 320 m

22 16 cm by 8 cm

23 9 600 cm^2

24 5

25 a 299 b \$150.50 c \$44 900.50

26 4 m

27 35°C

MIXED REVISION 1

Multiple choice

1 E

2 D

3 E

4 C

5 C

6 C

7 C

8 B

9 B

Short answer

1 a 175 b 7

2 a $(2x+5)(8x^7-3)+2(x^8-3x+4)$

b $7(2x+3)(x^2+3x+4)^6$

c $\dfrac{31}{(3x+7)^2}$ d $2e^{2x} + \dfrac{1}{\cos^2(x)}$

e $\dfrac{1}{4}\cos\left(\dfrac{x}{4}\right)$

3 26 m

4 a

X	2	3	4	5
$P(X=x)$	$\frac{5}{58}$	$\frac{5}{29}$	$\frac{17}{58}$	$\frac{13}{29}$

b About 4.103. c About 0.955.

d About 91.4%. 5 60 cm by 60 cm by 15 cm.

6 a $12x - 3y + 3\sqrt{3} - 4\pi = 0$

b $x - 4y + 5 = 0$

Application

1 a

R	0	1	2	3
$P(R=r)$	0.064	0.288	0.432	0.216

b $\dfrac{6}{13}$

2 a $32e^{20}$ cm/s^2

b $x = 2e^{4t}$

$v = 8e^{4t}$

$a = 32e^{4t}$

$= 16(2e^{4t})$

$= 16x$

3 \$2 327.09

4 No, the game is not fair as you stand to lose 10c every time you play.

5 50

6 a 241 650 b 16 915 people/year

c 12 165 people/year

4.01

1 a 320 km b 245 km c 250 km
2 a 6 units^2 b 21 units^2
3 a 50 units^2 b 96 units^2 c 100 units^2
d 8 units^2 e 36 units^2
4 480 km
5 a 18 units^2 b 8750 units^2 c 360 units^2
d 90 units^2 e 600 units^2
6 8 units^2
7 a 180 units^2 b 210 units^2
8 a $6.25\pi \approx 19.63$ units^2
b i 20.25 units^2 ii 15.125 units^2
9 a 2100 kL b 1750 kL
10 a 14.14 units^2
b i 18 units^2 ii 9 units^2 iii 13.5 units^2

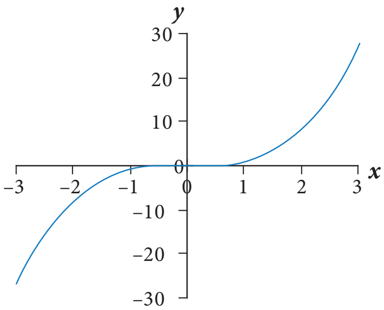
4.02

1 a 3.625 units^2 b 286.125 units^2
c 100 units^2 d 0.44 units^2
e 10 units^2
2 a 20.95 units^2 b 1.5 units^2
c 9.33 units^2 d 7.64 units^2
e 1.6 units^2
3 a 1.24 units^2 b 1.039 units^2
4 a i 1.625 units^2 ii 3.125 units^2
iii 2.259 units^2
b i 26 units^2 ii 50 units^2
iii 34.96 units^2
c i 3 units^2 ii 11 units^2 iii 5.61 units^2
d i 6.25 units^2 ii 11.25 units^2 iii 8.17 units^2
e i 85.79 units^2 ii 233.2 units^2 iii 129.75 units^2
5 a 2.33 units^2 b 0.245 units^2
c 11.25 units^2 d 65.25 units^2
e 0.46 units^2
6 a 4.5 units^2 b 4.5 units^2
7 5 units^2
8 a i 0.298 units^2 ii 0.278 units^2
iii 0.289 units^2
b 0.292 units^2 c 0.288 units^2
9 a 3.21 km^2 b 2.7285 km^3

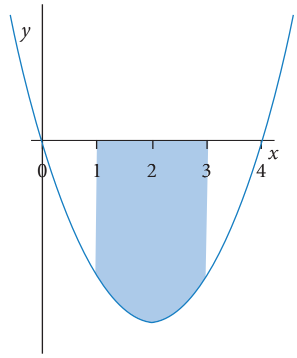
4.03

1 a 10.625 units^2 b 194.06 units^2
c 5.69 units^2 d 979 units^2
e 1.98 units^2
2 a 4.33 b 395.63 c 155.25
d 9.75 e 402
3 a 20.95 b 0.85 c 6.14
d 2.44 e 453
4 $\dfrac{\pi(2+\sqrt{2})}{8}$
5 a 35.28 b 25.52
6 5.402
7 68.425

8 a 3.968 b 0
c



The two areas cancel each other out.
9 −7.33; negative since it is below the x-axis as the graph shows.



10 a 26.75 m b $26\frac{2}{3}$ m

INVESTIGATION: AREAS UNDER A CURVE

1 $y = x$
2 $y = 3x$
3 $y = 4x$
4 $y = 0.5x^2$
5 $y = x^2$
6 $y = 1.5x^2$
The original function is the derivative of the area function.

4.04

1 a i 36 ii 189 iii 225
b $\int_1^4 x^3dx + \int_4^6 x^3dx = 36 + 189 = 225 = \int_1^6 x^3dx$
2 a i 8.58 ii 24.43 iii 33.01
b $\int_0^2 (x^2+3)dx + \int_2^4 (x^2+3)dx = \int_0^4 (x^2+3)dx$
3 a $\int_0^5 x^2dx$ b $\int_1^7 (x+1)dx$
c $\int_{-2}^2 (x^3-x-1)dx$ d $\int_0^3 (2x+1)dx$
e $\int_1^3 6x^3dx$ f $\int_{-1}^3 (3x^2-4x-1)dx$

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g $\int_{-2}^{2}(x^2-2)dx$ h $\int_{0}^{7}3dx$

i $\int_{1}^{3}5x^4dx$ j $\int_{0}^{6}(2x-3)dx$

4 a i 332.5 ii 997.5

b $3\int_{0}^{10}x^2dx = 3\times 166.25 = 498.75 = \int_{0}^{10}3x^2dx$

5 a i 2593.5 ii 5187

b $2\int_{2}^{5}x^5dx = 2\times 2593.5 = 5187 = \int_{2}^{5}2x^5dx$

6 a i 4.125 ii 3.9375 iii 8.0625

b $\int_{1}^{2}2x^2dx + \int_{1}^{2}3xdx = 4.125 + 3.9375 = 8.0625$

$= \int_{1}^{2}(2x^2+3x)dx$

7 a $\int_{0}^{2}(3x^2+2x+2)dx$ b $\int_{1}^{2}(3x^3-3x+1)dx$

c $\int_{-1}^{1}(2x^4+x^3-x^2-1)dx$

d $\int_{0}^{3}(2x^2+3x-4)dx$ e $\int_{1}^{5}(2x+7)dx$

8 a i 270 ii 61.5 iii 208.5

b $\int_{2}^{6}x^3dx - \int_{2}^{6}x^2dx = 270 - 61.5 = 208.5 =$

$\int_{2}^{6}(x^3-x^2)dx$

9 a 20.26 b −20.26 c $\int_{3}^{1}x^3dx = -\int_{1}^{3}x^3dx$

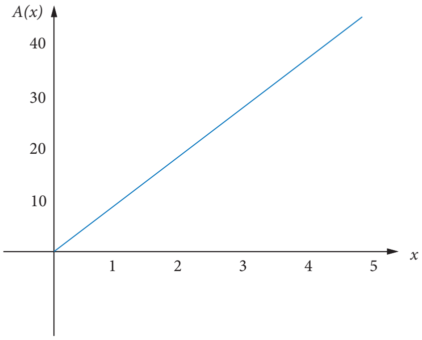
d $\int_{a}^{b}f(x)dx + \int_{b}^{a}f(x)dx = \int_{a}^{a}f(x)dx = 0,$

so $\int_{a}^{b}f(x)dx = -\int_{b}^{a}f(x)dx$

10 30.75 m

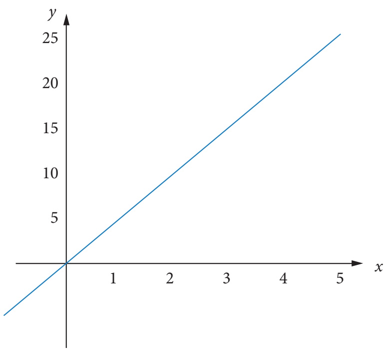
4.05

1 a

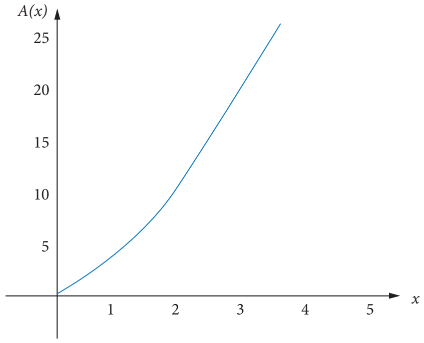


b $A(x) = 9x$

2 a

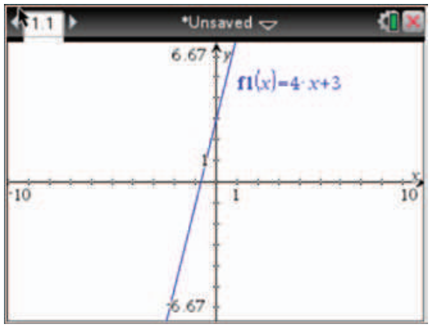


b

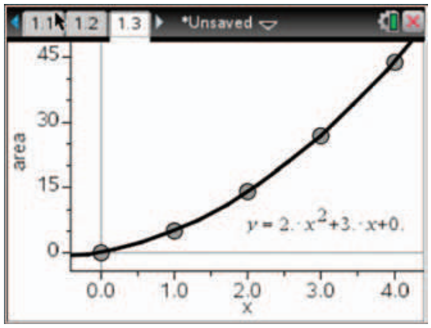


c $A(x) = 3x^2$

3 a



b



c $A(x) = 2x^2 + 3x$

4 a 72 b 20.25 c 10.67

d 8192 e 625

5 a 8.67 b 30 c 576.4

d 43.75 e 71.67

6 a 7765.33 b 104 857.5 c 10 d 10.5

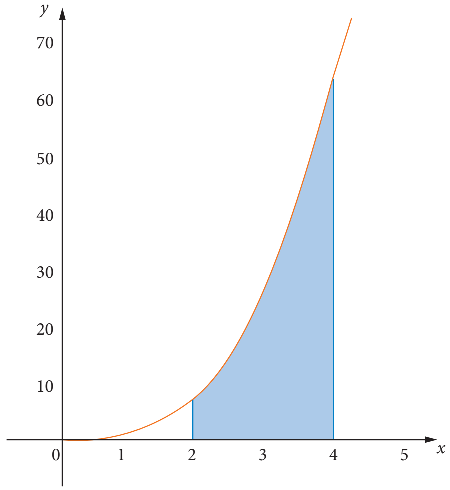
e 16.25 f 204.6 g 10.5 h 48 008

i 102.3 j 7654.5

7 a i 25 m/s ii 100 m/s

b 41.67 m c 291.67 m

8 a



b $\int_2^4 x^3 dx$ c 60 units^2

9 a 190.833 b 190.833 m

c 123.33 m

10 a 8 m/s^2 b 4 m/s c 60 m/s

4.06

1 a 16 b 128 c 15 d 4

e 16 f 52 g $\frac{1}{4}$ h $-1\frac{1}{2}$

i 25 j $\frac{13}{15}$

2 a $1\frac{1}{3}$ b 2 c 6

d 50 e $3\frac{5}{6}$

3 a $e^4 - 1$ b $5e(e^2 - 1)$ c $2e^2$

d $e^5 - e - 4$ e $60 - e^4 + e^2$

4 a $\frac{1}{\sqrt{2}}$ b $\frac{\sqrt{3}-1}{2}$ c 6

d $\frac{2(\sqrt{2}-1)}{\sqrt{2}} = 2 - \sqrt{2}$ e 7

5 a $\frac{\pi^2 + 4}{2}$ b $\sqrt{2} - 1$

c $\frac{3\sqrt{3} + \pi - 3}{6}$ d $\frac{3\sqrt{3} + 2}{2}$

e 27.53

6 a 0 b $-\frac{\sqrt{3}}{2}$ c $-1\frac{1}{2}$

d -1 e $-\sqrt{3} - 2$

7 a i $\int_0^5 (2x - 1)dx$ ii 20

b i $\int_0^4 (e^x + x)dx$ ii $e^4 + 7$

c i $\int_0^{\frac{\pi}{6}} [\cos(x) - 2\sin(x)]dx$ ii $\frac{2\sqrt{3} - 3}{2}$

8 a $\frac{1}{\cos^2(x)}$ b $\sqrt{3} - 1$

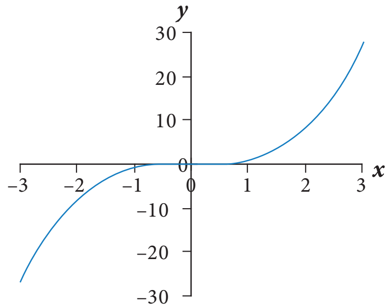
9 a $4e^{4x}$ b $e^{12} - 1$ c $\frac{e^{12} - 1}{4}$

10 a -5 cm/s b $x = t^3 + t^2 - 5t + 1$

c 126 cm d 20 cm/s^2

4.07

1 93
2 93
3 156
4 48
5 114
6 $28\frac{1}{2}$
7 6474
8 $109\frac{1}{3}$
9 a



b -4 c 4 d 0 e 8

f Because one half is below the axis and the other half is above, they cancel out in the integral.

10 a Negative

b $-30\frac{2}{3}$

c $30\frac{2}{3}$, since an area must be positive.

CHAPTER 4 REVIEW

1 B
2 D
3 D
4 D
5 C
6 a 600 km b Area = 75 × 8 = 600 km
7 12 m
8 3.08 units^2
9 a i 1.75 units^2 ii 3.75 units^2

b 2.5872 units^2

10 a 2.1875 units^2 b 3.1875 units^2

c 2.65625 units^2

11 60.84 units^2
12 403.75

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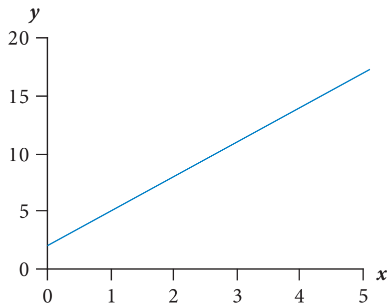
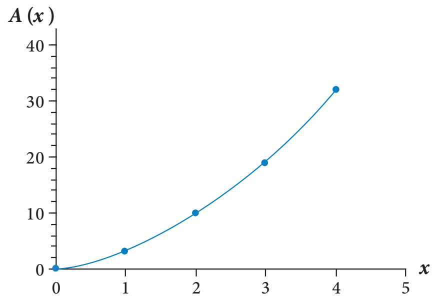
13 a



b



c $y = \frac{3x^2}{2} + 2x$

14 a i 7.25 ii 45.5 iii 52.75

b $\int_1^2 (2x^2 + 1)dx + \int_2^4 (2x^2 + 1)dx = 7.25 + 45.5$

$= 52.75 = \int_1^4 (2x^2 + 1)dx$

15 a i 166.204 ii 997.222

b $\int_2^8 6x^2 dx = 1020.2616 = 6 \times 170.0436 = 6\int_2^8 x^2 dx$

16 a i 2.92 ii 2.75 iii 5.67

b $\int_1^2 x^3 dx + \int_1^2 2x\,dx = 2.92 + 2.75 = 5.67$

$= \int_1^2 (x^3 + 2x)dx$

17 a 4 b 4 c 10.5 d 2.5

18 $3(e^7 - 1)$

19 a $\frac{2-\sqrt{2}}{2}$ b $\frac{\sqrt{3}-1}{2}$

20 a $32\frac{2}{3}$ units^2 b 93 units^2

21 a 1.62 cm/s b 2.52 cm c 0.656 cm

22 a $\frac{1-x}{e^x}$ b $\frac{1}{e}$

23 a Negative b Positive c 0

d −4 e 8

f Because the part from 0 to 2 is below the axis and the part from 2 to 4 is above the axis, but anti-symmetrical.

5.01

1 b, c and e

2 E

3 E

4 a Yes b No c No d Yes

e Yes f No g No

5 a $\frac{1}{2}$ b $\frac{1}{8}$ c $\frac{1}{4}$

6 a $\frac{6}{25}$ b $\frac{2}{9}$ c 0.24

d 0.21 e $\frac{3}{16}$

7 a About 0.4899 b $\frac{\sqrt{2}}{3}$

c $\frac{2\sqrt{6}}{25} \approx 0.1960$ d 0.4

e $\frac{\sqrt{14}}{9}$

8 $\frac{6\,436\,343}{9\,765\,625} \approx 0.659\,082$

9 $\frac{\sqrt{2}}{3}$

10 $\frac{12}{169}$

11 $\frac{1024}{3125} = 0.327\,68$

12 About 74% (0.7371…)

5.02

1 D

2 B

3 D

4 B

5 A

6 **a**, **c**, **d** and **e** are geometric distributions.

7 0.0439

8 0.7037

9 a 0.0655 b 0.738 c 4

10 a 0.75 b 0.5720

11 a 0.4502 b 9

12 a 0.0531 b 0.4695

13 a 0.25 b 0.25 c 0.875

14 0.00293

15 0.059

16 a 0.2373

b 2 (you can expect that your third fraudulent return will be detected)

17 About 0.0117

5.03

1 D

2 B

3 E

4 **b**, **c**, **e** and **g**

5 a 0.3241 b 0.2618 c 0.2048

d 0.0004 e 0.3894

6 a $p = 0.5, q = 0.5, n = 10, x = 2$

b $p = 0.15, q = 0.85, n = 20, x = 0$

c $p = \frac{3}{5}, q = \frac{2}{5}, n = 15, x = 12$

d $p = 0.11, q = 0.89, n = 9, x = 8$

e $p = 0.25, q = 0.75, n = 7, x = 4$

7 a 7 b 0.8

c

x	0	1	2	3
$P(X=x)$	0.0000	0.0004	0.0043	0.0287

x	4	5	6	7
$P(X=x)$	0.1147	0.2753	0.3670	0.2097

8 a 7 b 0.15

c

z	0	1	2	3
$P(Z=z)$	0.3206	0.3960	0.2097	0.0617

z	4	5	6	7
$P(Z=z)$	0.0109	0.0012	0.0001	0.0000

9 0.0781

10 The value of p in question **7** is much higher than the value of p in question **8** and the probabilities of more successes in question **7** are much higher than the corresponding probabilities in question **8**. Similarly, the probabilities of more failures in question **7** are much lower than the corresponding probabilities in question **8**.

5.04

1 D
2 B
3 D
4 D
5 B
6 a 0.2903 b 0.5801 c 0.0188
7 a 0.0864 b 0.0334 (0.033 378...)
c 0.0001 (0.000 107…) d 0.8803
8 a 0.0002 b 0.3810 c 0.0073 d 0.8064
9 a 0.0650 b 0.1216
c 0.9770 d 0.0001 (0.000 149…)
10 a 0.2090 b 0.1239 c 0.8936
11 a 0.0024 b 0.0993 c 0.1028
12 a 3 b 1, 5 c 2, 4
13 a 2 b 4 c 2

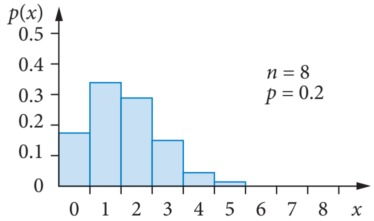
INVESTIGATION: THE EFFECT OF CHANGING *n* AND *p* ON A BINOMIAL DISTRIBUTION HISTOGRAM

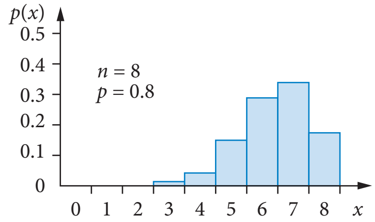
A As p increases from 0.2 to 0.5 to 0.8 the histograms change from being skewed right to symmetrical to skewed left.

B As n increases, the graph becomes 'smoother' and the amount at the ends decreases.

5.05

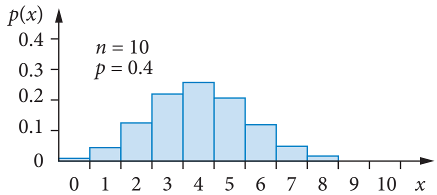
1 C
2 D
3 a

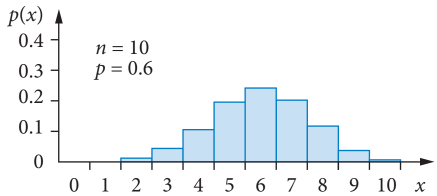




b The graphs are mirror images of each other, skewed positively and negatively respectively.

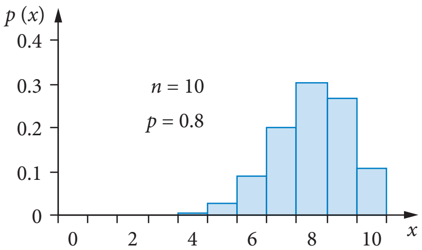
4 a





b The graphs are mirror images of each other, skewed positively and negatively respectively.

5 a



b 8

6 a

x	0	1	2	3	4	5
$P(X=x)$	0.0313	0.1563	0.3125	0.3125	0.1563	0.0313

$E(X) = 2.5$, $SD(X) \approx 1.118$

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b

x	0	1	2	3
$P(X=x)$	0.0576	0.1977	0.2965	0.2541

x	4	5	6	7	8
$P(X=x)$	0.1361	0.0467	0.01	0.0012	0.0001

$E(X) = 2.4$, $SD(X) \approx 1.296$

c

x	0	1	2	3
$P(X=x)$	0.0 …	0.0 …	0.0 …	0.0 …
x	4	5	6	7
$P(X=x)$	0.000	0.0033	0.0155	0.0532
x	8	9	10	11
$P(X=x)$	0.1329	0.2362	0.2835	0.2062
x	12			
$P(X=x)$	0.0687			

$E(X) = 9.6$, $SD(X) = 1.386$

d

x	0	1	2	3
$P(X=x)$	0.0003	0.0035	0.0212	0.0743
x	4	5	6	7
$P(X=x)$	0.1672	0.2508	0.2508	0.1612
x	8	9		
$P(X=x)$	0.0605	0.0101		

$E(X) = 5.4$, $SD(X) \approx 1.470$

7 a $E(X) = 0.7$, $SD(X) \approx 0.794$
b $E(X) = 6.3$, $SD(X) \approx 0.794$
c $E(X) = 13$, $SD(X) \approx 2.133$
d $E(X) = 10.2$, $SD(X) \approx 2.595$

8 0.1 or 0.9

9 $\frac{5}{8}$

10 a The experiment is binomial.
b About 5 (4.6). c About 95%.

11 $n = 25$ and $p = 0.2$, so $p(5) \approx 0.196$

12 About 100.

13 625

14 a 168 b About 10.04.

5.06

1 a $n = 3, p = \frac{1}{4}, q = \frac{3}{4}$ b 0.016
c 0.422 d 0.156

2 a 0.296 b 0.988 c 0.593

3 a $\frac{5}{16} = 0.3125$ b $\frac{11}{32} = 0.343\ 75$

4 $\frac{11}{16} = 0.6875$

5 a $1.708 \times 10^{-6} \approx 0$ b 0.679

6 a 0.663 b 0.051 c 0.337

7 a 0.012 b 0.16 c 0.841

8 a 0.000 061 b 0.133 c 0.173 d 0.244

9 0.895

10 a $n = 3, p = 0.2, q = 0.8$ b 0.008
c 0.512 d 0.488

11 a 0.555 b 4 times

12 a 0.2533 b 0.8596

13 There is a 95% chance that from 984 to 1056 students from the new community will attend government schools.

14 0.262

15 0.633

16 0.0005

17 a i 0.1937 ii 0.6513 b 12

18 a 0.147 b 0.832

19 $\frac{577}{117649}$

20 a i 0.1176 ii 0.8824 b 5

21 a 0.276 b 0.003 c 0.180

22 0.678

CHAPTER 5 REVIEW

1 B
2 C
3 D
4 E
5 A
6 E
7 B
8 D
9 D
10 **c**, **e** and **g**

11 $\frac{8}{45}$

12 $\frac{5}{36}$

13 0.0658

14 a 0.1209 b 0.8824 c 2.333...

15 a 0.0223 b 0.1024

16 **a**, **c**, **e**

17 a 0.3456 b 0.0604 c 0.0171

18 a 8 b 0.45
c

x	0	1	2	3
$P(X=x)$	0.0084	0.0548	0.1569	0.2568

x	4	5	6	7	8
$P(X=x)$	0.2627	0.1719	0.0703	0.0164	0.0017

19 a 0.003 346 b 0.8309 0.1061

20 0.2424

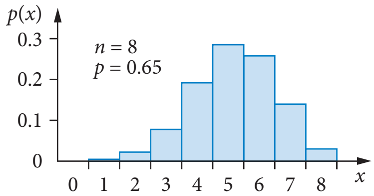
21 a 0.1354 b 0.9994 c 0.0279

22 a 0.0743 b 0.2508 c 0.9962

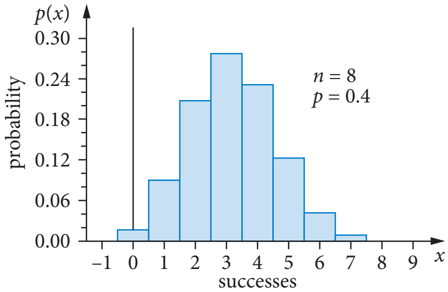
23 0.784

24 a 0.206 b 0.144 c 0.000 03
d 0.985 e 0.9166

25 0.5398

26 

The graph is negatively skewed.

27 **a** 

b Most likely number of successes = 3.

28 **a** $\mu = 1.6, \sigma = 1.13$ **b** $\mu = 7, \sigma = 1.45$

c $\mu = 8.25, \sigma = 1.9268$

29 **a** 0.016 **b** 0.022 **c** 0.170

30 **a** 0.0029 **b** 0.1937 **c** 0.8900

31 **a** 0.107 **b** 0.268 **c** 0.879

32 **a** 0.087 **b** 0.681 **c** 0.823

33 **a** 0.463 **b** 0.135 **c** 0.171 **d** 0.964

34 0.3327

35 **a** About 5. **b** 0.9678

c There is about a 97% chance that from 1 to 8 women in the study group will suffer from iron deficiency.

36 0.0858

37 **a** **i** 0.1374 **ii** 0.5608 **b** 12

INVESTIGATION: BASIC INTEGRALS

$(n+1)x^n$

x^n

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\cos(x)$

$\int \cos(x)dx = \sin(x) + c$

$\sin(x)$

$\int \sin(x)dx = -\cos(x) + c$

e^x

$\int e^x dx = e^x + c$

$f'(ax+b)$

$\int \frac{f(ax+b)}{a}dx = F(ax+b) + c$ where $F(x)$ is a primitive of $f(x)$

6.01

1 B

2 D

3 A

4 **a** $\frac{x^2}{2}+c$ **b** $\frac{x^3}{3}+c$ **c** $\frac{x^7}{7}+c$ **d** $\frac{2x^5}{5}+c$

e $-\frac{5}{2x^2}+c$ **f** $-\frac{3x^4}{4}+c$ **g** $\frac{5}{3x^3}+c$ **h** $\frac{2x^{\frac{3}{2}}}{3}+c$

i $\frac{10x^{\frac{3}{2}}}{3}+c$ **j** $\frac{x^6}{42}+c$ **k** $\frac{x^4}{20}+c$

l $-\frac{1}{8x^2}+c$ **m** $\frac{3x^{\frac{4}{3}}}{4}+c$ **n** $\frac{15x^{\frac{7}{5}}}{7}+c$

o $4x^{\frac{1}{4}}+c$ **p** $-\frac{2}{x^2}+c$ **q** $\frac{1}{x^5}+c$

r $20\sqrt{x}+c$ **s** $-9x^{\frac{2}{3}}+c$ **t** $-\frac{16}{\sqrt{x}}+c$

5 **a** $\frac{1}{2}e^{2x}+c$ **b** $\frac{1}{4}e^{4x}+c$ **c** $-e^{-x}+c$

d $\frac{1}{5}e^{5x}+c$ **e** $-\frac{1}{2}e^{-2x}+c$ **f** $\frac{1}{4}e^{4x+1}+c$

g $-\frac{3}{5}e^{5x}+c$ **h** $\frac{1}{2}e^{2t}+c$ **i** $\frac{5}{4}e^{4x}+c$

j $3e^{-2x}+c$ **k** $8e^{\frac{x}{2}}+c$ **l** $-18e^{-\frac{x}{3}}+c$

6 **a** $\sin(x)+c$ **b** $-\cos(x)+c$

c $-\frac{1}{3}\cos(3x)+c$ **d** $\frac{1}{7}\cos(7x)+c$

e $\sin(x+1)+c$ **f** $-\frac{1}{2}\cos(2x-3)+c$

g $\frac{1}{2}\sin(2x-1)+c$ **h** $-8\cos\left(\frac{x}{2}\right)+c$

i $-\cos(3-x)+c$ **j** $12\sin\left(\frac{x}{4}\right)+c$

k $\cos(\pi-x)+c$ **l** $\sin(x+\pi)+c$

m $5\cos\left(\frac{2x}{5}\right)+c$ **n** $\frac{16}{7}\sin\left(\frac{7x}{4}\right)+c$

o $\frac{6}{\pi}\sin\left(\frac{\pi x}{3}\right)+c$ **p** $-\frac{2\pi}{3}\cos\left(\frac{-3x}{\pi}\right)+c$

7 **a** $\frac{(x+1)^5}{5}+c$ **b** $\frac{(5x-1)^{10}}{50}+c$

c $\frac{(3y-2)^8}{24}+c$ **d** $\frac{(4+3x)^5}{15}+c$

e $\frac{(7x+8)^{13}}{91}+c$ **f** $-\frac{(1-x)^7}{7}+c$

g $\frac{\sqrt{(2x-5)^3}}{3}+c$ **h** $-\frac{2(3x+1)^{-3}}{9}+c$

i $-3(x+7)^{-1}+c$ **j** $-\frac{1}{16(4x-5)^2}+c$

k $\frac{3\sqrt[3]{(4x+3)^4}}{16}+c$ **l** $-2(2-x)^{\frac{1}{2}}+c$

m $\frac{2\sqrt{(t+3)^5}}{5}+c$ **n** $\frac{2\sqrt{(5x+2)^7}}{35}+c$

o $-\frac{1}{15(5x-4)^3}+c$ **p** $-\frac{3}{8(4x-3)^4}+c$

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8 $y = -\frac{3}{2}x^2 + 8$

9 $f(x) = \frac{3}{2}e^{2x} + 4$

10 $\frac{1}{2}e^{x^4} + c$

11 $\frac{1}{4}(4x^2+1)^3 + c$

6.02

1 C

2 C

3 A

4 D

5 a $\frac{m^2}{2} + m + c$ b $\frac{t^3}{3} - 7t + c$

c $\frac{h^3}{3} + 5h + c$ d $\frac{y^2}{2} - 3y + c$

e $x^2 + 4x + c$ f $\frac{b^3}{3} + \frac{b^2}{2} + c$

g $\frac{a^4}{4} - \frac{a^2}{2} - a + c$ h $\frac{x^3}{3} + x^2 + 5x + c$

i $x^4 - x^3 + 4x^2 - x + c$ j $x^6 + \frac{x^5}{5} + \frac{x^4}{2} + c$

k $\frac{x^8}{8} - \frac{3x^7}{7} - 9x + c$ l $\frac{x^4}{2} + \frac{x^3}{3} - \frac{x^2}{2} - 2x + c$

m $\frac{x^6}{6} + \frac{x^4}{4} + 4x + c$ n $\frac{4x^3}{3} - \frac{5x^2}{2} - 8x + c$

o $\frac{3x^5}{5} - \frac{x^4}{2} + \frac{x^2}{2} + c$ p $\frac{3x^4}{2} + \frac{5x^3}{3} - 4x + c$

q $-x^{-3} - \frac{x^{-2}}{2} - 2x^{-1} + c$ r $\frac{14x^{\frac{5}{2}}}{5} - 2x^2 + 9x^{\frac{2}{3}} + c$

6 a $\frac{x^4}{4} - x^3 + x^2 + c$ b $x - 2x^2 + \frac{4x^3}{3} + c$

c $\frac{x^3}{3} + \frac{3x^2}{2} - 10x + c$

d $-\frac{4}{x} - x + \frac{3}{2x^2} - \frac{7}{4x^4} + c$ e $\frac{y^3}{3} + \frac{y^{-6}}{6} + 5y + c$

f $\frac{t^4}{4} - \frac{t^3}{3} - 2t^2 + 4t + c$ g $\frac{2\sqrt{x^3}}{3} + x + c$

h $\frac{-1}{x} - \frac{3}{2x^2} + \frac{10}{3x^3} + c$ i $\frac{4x^{\frac{5}{2}}}{5} - \frac{8x^{\frac{3}{2}}}{3} + 6\sqrt{x} + c$

7 a $y = x^2 - 5x + 2$ b $y = 2x^{\frac{3}{2}} - 2x^2 + 10$

c $y = x^3 - \frac{x^2}{2} + 2x - 10$

8 a $y = 3x^2 - x + 5$ b $y = -2x^2 + 7x + 10$

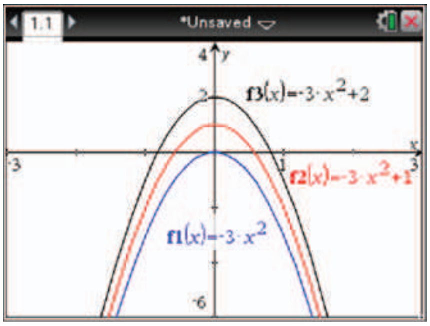
c $y = \frac{-3}{x} + 2x + 6$ d $y = 4\sqrt{x} + \frac{3x^2}{2} - \frac{5}{2}$

e $y = \frac{3x^{\frac{4}{3}}}{4} + 2x^3 - 10x + \frac{1}{4}$

9 a $-3x^2$

b $f1(x) = -3x^2, f2(x) = -3x^2 + 1, f3(x) = -3x^2 + 2$

c

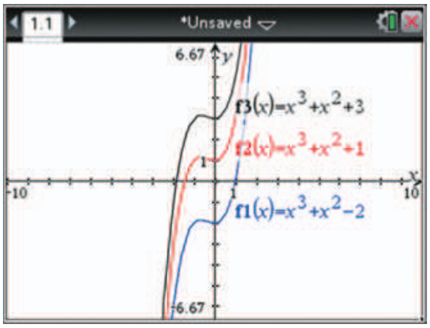


The functions are vertical translations of each other.

10 a $x^3 + x^2$

b $f1(x) = x^3 + x^2 - 2, f2(x) = x^3 + x^2 + 1,$
$f3(x) = x^3 + x^2 + 3$

c



The functions are vertical translations of each other.

11 $f(2) = 1$

12 $f(x) = \frac{2}{3x^{\frac{3}{2}}} - \frac{16}{x} + 66\frac{2}{3}$

6.03

All areas are in square units.

1 12.5

2 18

3 a $\int_{-1}^{2.5}(-2x+5)dx$ b $\int_{1}^{5}(x+5)dx$

c $\int_{-3}^{-1}(x^2)dx$ d $\int_{2}^{4}(2x^2)dx$

e $-\int_{-1}^{1}(-e^x)dx$ f $-\int_{3}^{5}(x^3 - 7x^2 + 4x + 11)dx$

g $\int_{0}^{\pi}[3\sin(x)]dx$ h $\int_{2}^{8}(-x^3 + 10x^2 - 5x)dx$

4 a $508\frac{1}{2}$ b 48 c 12 480 d $624\frac{6}{7}$

e 6144 f $95\frac{5}{6}$ g 480 h −1

i $44\frac{1}{3}$ j 50 k $52\frac{2}{3}$ l $1\frac{1}{4}$

5 a $\frac{7}{800}$ b $\frac{1}{3}$ c $\frac{-6}{25}$

d $\frac{13}{27}$ e $\frac{7}{32}$ f $2(\sqrt{3} - \sqrt{2}) \approx 0.6357$

6 a $\frac{2}{3}$ b $\frac{3}{8}$ c $\frac{2}{5}$

d About 0.0672 e $\frac{2}{9}$ f $\frac{26}{9}$

7 a $\frac{1}{3}\left(e^{\frac{3\pi}{4}} - e^{-\frac{3\pi}{4}}\right)$ b $5e^3(e^5 - 1)$

c $\frac{1}{5}(e^5 - 1)$ d $e^{-2} - 1 = \frac{1}{e^2} - 1$

e $\frac{2}{3}e^7(e^9 - 1)$ f $19 - \frac{1}{2}e^4(e^2 - 1)$

g $\frac{1}{2}e^4 + 1\frac{1}{2}$ h $e^2 - e - 1\frac{1}{2}$

i $\frac{1}{2}e^6 + e^{-3} - 1\frac{1}{2}$

8 a 54 244.90 b 0.32 c 268.29

d 37 855.68 e 346.85 f 755.19

9 a 2 b $\frac{1}{\sqrt{2}}$ c $\frac{2}{\sqrt{2}} = \sqrt{2}$ d $-\frac{1}{3}$

e $\frac{1}{\pi}$ f $\frac{1}{2}$ g $\frac{3}{4}$ h $-\frac{1}{5}$

10 a $127\frac{1}{3}$ b $49\frac{1}{2}$ c 510

d $3(1 - 9^{-6}) \approx 3.00$ e $\frac{4}{5}\left(8\sqrt[4]{8} - 1\right) \approx 9.96$

f $\frac{19}{324} \approx 0.06$ g $2(e - e^{-3}) \approx 5.34$

h $\frac{10}{33} \approx 0.30$ i 0

j $e^6 - e^4 - 260 \approx 88.83$

k $\frac{1}{6}\left(125 - 17\sqrt{17}\right) \approx 9.15$ l $-14\ 560$

m $2\sqrt{2} \approx 2.83$ n 0 o 0

11 a The function and indefinite integral are not defined at the limit $x = 0$.

b The function and indefinite integral are not defined at the limit $x = 5$.

c The function and indefinite integral are not defined at the limit $x = -1$.

12 The function and indefinite integral are not defined at the value $x = 2$, which is included in the limits of integration.

13 The function and indefinite integral are not defined at 0, which is included in the limits of integration.

14 $\frac{d}{dx}xe^{x^2} = e^{x^2} + 2x^2e^{x^2}$, so $\int_0^1 (2x^2e^{x^2} + e^{x^2})dx = e$.

15 1275 m^3

INVESTIGATION: SIMPLE POWERS

1 2, 0, $\frac{2}{3}$, 0, $\frac{2}{5}$, 0

2 For n even, $\int_{-1}^{1} x^n dx = \frac{2}{n+1}$ and for n odd $\int_{-1}^{1} x^n dx = 0$.

3 The even powers have the same area either side of 0, but the odd powers have opposite areas.

4 The results are similar as the odd powers are all 0.

6.04

All areas are in square units.

1 D

2 B

3 E

4 D

5 E

6 a $-\int_0^3 f(x)dx + \int_3^6 f(x)dx$

b $\int_{-9}^{-6} g(x)dx - \int_{-6}^{-2} g(x)dx$

c $-\int_{-5}^{4} h(x)dx$

d $\int_{-4}^{-1} k(x)dx - \int_{-1}^{3} k(x)dx$

e $-\int_{-5}^{-2} m(x)dx + \int_{-2}^{2} m(x)dx$

f $\int_1^3 p(x)dx - \int_3^5 p(x)dx + \int_5^6 p(x)dx$

7 a $7\frac{1}{2}$ b 39 c 74

d $67\frac{1}{2}$ e $43\frac{1}{2}$

8 $2(e^2 + e^{-2} - 2) \approx 11.05$

9 6

10 a $38\frac{1}{4}$ b $73\frac{5}{6}$

11 a 32 b $24\frac{35}{54}$ c $\frac{2}{3}$

12 a $49\frac{1}{3}$ b $31\frac{1}{4}$

13 4

14 $e^3 - e^{-3} \approx 20.04$

15 $\frac{1}{3}$

16 $\pi \approx 3.14$

17 $\frac{1}{4}(e - e^{-3})$

18 $3 - \frac{1}{e^2} \approx 2.86$

19 $\frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}$

20 a 14.91 b 21.08 c 94.03

21 a 20 b $132\frac{3}{4}$

22 a $4\frac{1}{2}$ b $14\frac{7}{24}$

23 a $49\frac{1}{3}$ b $123\frac{11}{32}$ c $18\frac{2}{3}$

24 \$93 392

25 750 J

6.05

All areas are in square units.

1 C
2 B
3 D
4 A
5 $20\frac{5}{6}$
6 a $10\frac{2}{3}$ b $24\frac{1}{2}$ c $7\frac{1}{3}$
d 36 e 3.267... f 18.42
7 $1\frac{1}{3}$
8 $10\frac{2}{3}$
9 $166\frac{2}{3}$
10 a 9 b 36 c $83\frac{1}{3}$
d $2\frac{1}{4}$ e 18
11 $\frac{1}{12}$
12 $2\frac{2}{3}$
13 $\frac{2}{3}$
14 $\frac{5}{12}$
15 $\frac{2}{3}$
16 $\pi - 2$
17 a $58\frac{2}{3}$ b 72 c $21\frac{1}{12}$
d $21\frac{1}{12}$ e $78\frac{1}{12}$
18 $\frac{1}{3}$
19 $1.996 \approx 2$ kg
20 0.8 m^2
21 About 22 775 000 L ≈ 23 ML.
22 a 268 m^2 b 6700 m^3
23 87.6 m^2, \$4599

6.06

1 B
2 D
3 The change in the height (in cm) of the pile from time $t = 0$ to time $t = 10$ (the first 10 hours).
4 The increase in the number of bacteria in the culture from time $t = 2$ to time $t = 8$ (from the start of the second hour to the end of the eighth hour).
5 $150 + \int_0^4 P'(t)dt$
6 a About 7566 L.
b About 1024 L.
c The rate of leakage decreases as time goes on.
7 a 21 000 b 3 months
8 \$625 000
9 $R(x) = 12x - x^3 + 2x^2$; \$16 000
10 407.36 L
11 $R(x) = x(10 - 0.001x)$; $C(x) = 7000 + 2x$;
Total profit = \$0, break-even point
12 $C(x) = 8x^2 - x^3 + 5x + 400$
13 $C(x) = \frac{1}{4}x^2 + 5000$

6.07

1 The displacement of the particle from time = 2 s to time = 10 s.
2 50.4 m
3 Displacement = 155 cm, position = 153 cm to the right of the origin.
4 3 cm to the left of the origin.
5 a $\left(\sqrt{3}+3\right)$ m to the right of the origin
b $-4\sqrt{3}$ m/s^2
6 a 277.5 m b −29 m/s c 2.04 s
7 a 10 m b $2\frac{6}{7}$ s c −14 m/s
8 30.63 m
9 225 km
10 About −743 cm
11 893 m
12 a $5e^{45}$ m/s b e^{30} m
13 490 m
14 a 6.3 m/s b 12.25 m
15 a 7 cm/s^2 b $15\frac{1}{3}$ cm to the right of the origin.
16 a −1 cm/s b $\frac{2-\pi}{8}$ cm c $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$ s

CHAPTER 6 REVIEW

All areas are in square units.

1 E
2 E
3 A
4 C
5 A
6 D
7 D
8 C
9 B
10 C
11 A
12 a $\frac{y^4}{4} - y^3 + 2y^2 + y + c$ b $\frac{1}{n} + c$ c $\frac{2}{x} + c$
d $\frac{1}{2}(3x^3 - 2x + 4)^2 + c$ e $-\cos(x) + c$
f $-\frac{1}{2}\sin(6x) + c$ g $\frac{1}{2}\cos(10x) + c$
h $\frac{1}{3}e^{3t} + c$ i $-\frac{3}{2e^{2x}} + c$

j $-\dfrac{2}{(x-5)^2}+c$ k $-\dfrac{1}{18(2x+7)^3}+c$

l $\dfrac{\sqrt{(4x+7)^3}}{6}+c$

13 a $\dfrac{x^5}{5}+7x+c$ b $x^5-\dfrac{x^4}{2}+2x^2+c$

c $\dfrac{3}{2}x^4-\dfrac{8}{3}x^3-3x+c$

14 $y=3x^2-4x+2$

15 $f(x)=3\sqrt{x}+2$

16 a $\dfrac{1}{5}x^5-x^3+7x+c$ b $-\dfrac{(2-3x)^3}{9}+c$

c $\dfrac{6}{5}x^{\frac{5}{2}}-\dfrac{10}{3}x^{\frac{3}{2}}+4\sqrt{x}+c$

17 a 30 b $17\frac{1}{3}$ c $\dfrac{5}{6}$

d $\dfrac{2}{3}$ e $\dfrac{\sqrt{3}+1}{4}\approx 0.68$ f 0

18 $\int_1^4 f(x)dx-\int_{-2}^1 f(x)dx$

19 a 111 b $e^2+e-e^{-2}-e^{-1}\approx 9.60$

c 4.83

20 About 60.64.

21 26

22 4444

23 $32\frac{3}{4}$

24 $20\frac{5}{6}$

25 $57\frac{1}{6}$

26 28.98

27 The total change (°C) in temperature of the liquid in the first 5 minutes.

28 a 338.39 L b 185.71 L

29 a 1950 b 6 months

30 $R(x)=1500x-2x^2-x^3$; \$16 200

31 Displacement = 25 m; position = 27 m to the left of the origin.

32 a 41.6 m b −19 m/s c 3.06 s

33 $v=\dfrac{100-10(1+2t)^3}{3}$ cm/s

34 a $(4,-e)$ b $y=(4e^{-1})x-e-16e^{-1}$

c $(0,-e-16e^{-1})$ d $(0,-1)$

e $|-32e^{-1}-4|\approx 15.77$

35 a 6 m b 72 m^2

c 100.8 m^3 d 8 709 120 m^3

e 247.5 : 100.8 ≈ 2.46 : 1

36 32.26 m^2

37 a 31.5 m/s

b For $v=0$, $3t^2+2t+30=0$ has no real solutions, so $v\neq 0$.

38 $\sqrt{3}$

39 29.07 m^2, \$944.83

40 3.125 cm

41 4433 cm^2, 13 300 L

MIXED REVISION 2

Multiple choice

1 B
2 C
3 D
4 D
5 D
6 E
7 B
8 A
9 A

Short answer

1 a x^3-4x^2+5x+c b $-\dfrac{e^{-2x}}{2}+c$

c $\dfrac{\sin(2x)}{2}+c$ d $\dfrac{(2x-3)^{10}}{20}+c$

e $-5\cos\left(\dfrac{x}{5}\right)+c$

2 0.0988

3 $y=3x^2-5x-7$

4 a 6 b 0.4

c

x	0	1	2	3
$p(x)$	0.0467	0.1866	0.3110	0.2765

x	4	5	6
$p(x)$	0.1382	0.0369	0.0041

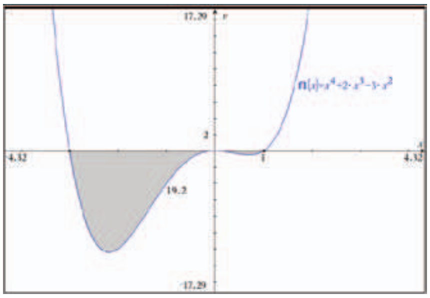
5 a $\dfrac{e^{18}-1}{3}$ b 14 c $\dfrac{3(\sqrt{2}+1)}{\sqrt{2}}$

6 21.76

Application

1 a $x=-3, 0, 1$

b



c 19.2 $units^2$

2 128 m^2

3 a About 0.0388 b About 0.3585

4 a $10\,482\times 3.71\times 10^7\approx 3.89\times 10^{11}$

b Number of atoms decayed.

c 693 s ≈ $11\frac{1}{2}$ min

5 a $3e^{30}$ cm/s b $x=e^{3t}$ c $a=9e^{3t}=9x$

6 11

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7.01

1 a $5^2=25$ b $4^2=16$ c $5^3=125$ d $2^4=16$
e $3^1=3$ f $7^2=49$ g $2^7=128$ h $5^0=1$

2 a $8^{\frac{1}{3}}=2$ b $4^{-\frac{1}{2}}=\frac{1}{2}$ c $7^{\frac{1}{4}}=\sqrt[4]{7}$
d $3^{-\frac{1}{3}}=\frac{1}{\sqrt[3]{3}}$ e $2^{-\frac{3}{2}}=\frac{\sqrt{2}}{4}$ f $a^c=b$
g $c^{3m}=\sqrt{a}$

3 a $\log_7(49)=2$ b $\log_3(27)=3$
c $\log_2(16)=4$ d $\log_5(125)=3$
e $\log_{11}(1)=0$ f $\log_2(1)=0$

4 a $\log_5\left(\frac{1}{25}\right)=-2$ b $\log_4\left(\frac{1}{16}\right)=-2$
c $\log_{10}\left(\frac{1}{1000}\right)=-3$
d $\log_{\frac{1}{3}}\left(\frac{1}{81}\right)=4, \log_3\left(\frac{1}{81}\right)=-4$
e $\log_{\frac{1}{4}}\left(\frac{1}{64}\right)=3, \log_4\left(\frac{1}{64}\right)=-3$
f $\log_{\frac{1}{2}}\left(\frac{1}{8}\right)=3, \log_2\left(\frac{1}{8}\right)=-3$
g $\log_6\left(\sqrt[3]{36}\right)=\frac{2}{3}$ h $\log_7\left(\sqrt[5]{343}\right)=\frac{3}{5}$
i $\log_a(m)=k$ j $\log_b(d)=3$

5 a 6 b 2 c 4 d 3
e 3 f 0 g 1 h 5
i 5 j 5

6 a 4 b −3 c −2 d 4
e −7 f −9 g 4 h −1
i −3 j 3

7 a $\frac{1}{2}$ b $\frac{3}{2}$ c $\frac{3}{2}$
d $\frac{3}{2}$ e $\frac{2}{3}$

8 a $\frac{3}{2}$ b $\frac{3}{2}$ c $\frac{4}{3}$
d $\frac{5}{6}$ e $\frac{5}{3}$

9–11 Proofs

7.02

1 a 0 b 0 c 0 d 0
e 0 f 1 g 1 h 2
i 1 j 5

2 a undefined b undefined
c undefined d undefined
e undefined f undefined
g undefined

3 a 6 b 3 c $\frac{1}{2}$ d $\frac{3}{2}$
e $-\frac{1}{2}$ f 2 g 6

4 a 1 b 1 c 0
d 9 e 0 f $\log_5(50)$
g 1 h $\log_6\left(\frac{625}{64}\right)$

5 a 4 b $\frac{4}{3}$ c −4 d $-\frac{1}{2}$

6 a $\log_4(x^4)$ b $\log_7(x^2)$
c $\log_6\left(\frac{1}{x}\right)$ d $\log_2(x+2)^3$
e $\log_4(x-1)$ f 0

7 a $\log_{10}(12)+\log_{10}(a)-1$
b $5\log_6(a)-5\log_6(b)$
c $\frac{1}{5}\log_3(10)+\frac{3}{5}\log_3(x)$
d $\frac{2}{3}\log_4(x)+\frac{1}{3}\log_4(a)-2\log_4(y)$

8 a 0.153 b 0.387 c 2.613

9 $\frac{1}{7}$

10 a 13 b 10 c −3

11–13 Proofs

7.03

1 a $\frac{\log(9)}{\log(4)}$ b $\frac{\log(6)}{\log(8)}$
c $\frac{\log(20)}{\log(2)}=1+\frac{1}{\log(2)}$ d $\frac{\log(200)}{\log(7)}$
e $\frac{\log(0.2)}{\log(9)}$

2 a 1.631 b 0.2789 c 1.683
d 0.4307 e 0.1262

3 a 6.644 b 1.585 c 2.640
d 16.01 e 15.75

4 a $x=1.465$ b $x=1.367$ c $x=2.465$
d 2.861 e 0.774

5 a 3.051 b 0.2232 c 0.5767
d 0.6192 e 1.140

6 $4c+\frac{1}{b}$

7–8 Proofs

7.04

1 a $x=1$
b $x=\log_5(0.8)\approx$ -0.1386
c $x=\log_6(3)\approx 0.6131$
d $x=2$
e $x=\log_8(5)\approx 0.7740$ or $x=\log_8(6)\approx 0.8617$

2 a $x=\frac{3\log(5)}{\log(9)-\log(5)}$
b $x=\frac{3\log(49)}{\log(49)-\log(8)}$
c $x=\frac{5[\log(350)+\log(4)]}{\log(350)-\log(4)}$

d $x = \dfrac{\log(15)}{\log(15) - 3\log(2)}$

e $x = \dfrac{2\log(17) + \log(7)}{2\log(7) - \log(17)}$

3 a $x = \dfrac{2\log(7)}{\log(7) - \log(4)} \approx 6.954$

b $x = \dfrac{4\log(4)}{\log(58) - \log(4)} \approx 2.074$

c $x = \dfrac{2[\log(5) + \log(46)]}{\log(46) - \log(5)} \approx 4.901$

d $x = \dfrac{3\log(5)}{2\log(6) - \log(5)} \approx 2.446$

e $x = \dfrac{\log(28) + 4\log(9)}{2\log(9) - \log(28)} \approx 11.41$

4 a $x = 83$ b $x = 55$ c $x = 6$

d $x = \dfrac{5}{8}$ or $x = 1\dfrac{1}{2}$ e $x = 5$ f $x = 3$

5 a $x = \dfrac{1}{10}, 1000$ b $x = \dfrac{1}{4}, 16$

c $x = \dfrac{1}{4}, 2$ d $x = \dfrac{1}{5}, 25$

e $x = 3, 27$ f $x = 390\,625, \dfrac{1}{125}$

6 a $x \approx 0.005\,53$ or $x \approx 3.90$

b $x \approx 1.62$ or $x \approx 19.7$

c $x \approx 0.225$ or $x \approx 3.37$

d No real solutions.

e $x \approx 0.001\,34$ or $x \approx 2.39$

7 0.0821 (8.21%)

8 45 years

INVESTIGATION: TRANSFORMING LOGARITHMS

a i $x > 0, y \in R$

ii $x = 1$

iii $x = 0$

iv $f(2) = 1$. It is the value of the function at the base of the function

b i $x > -3, y \in R$

ii $x = -2$

iii $x = -3$

iv $f(-1) = 1$. It corresponds to the base of the function

c i $x > -3, y \in R$

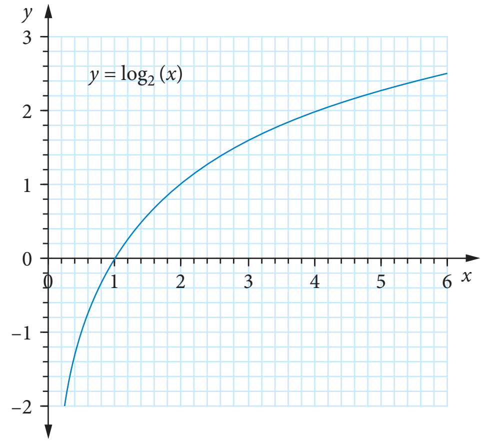
ii $x = 0$

iii $x = 2^{-4}$

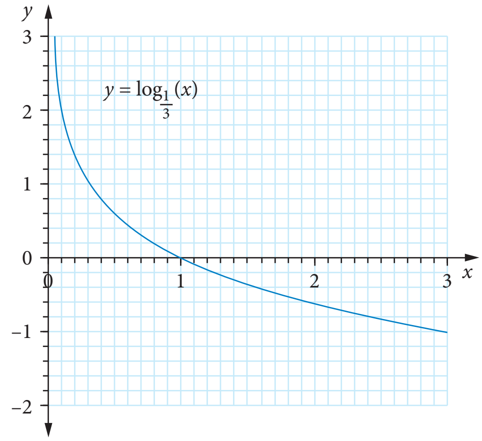
iv $f(2) = 5$. It corresponds to the base of the function.

7.05

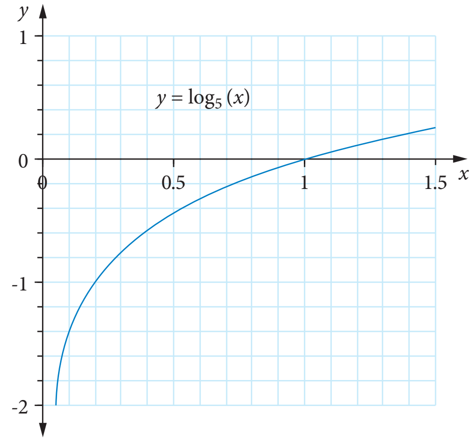
1 a $x \approx 4.9$



b $y \approx 0.2$



c $z \approx 0.27$



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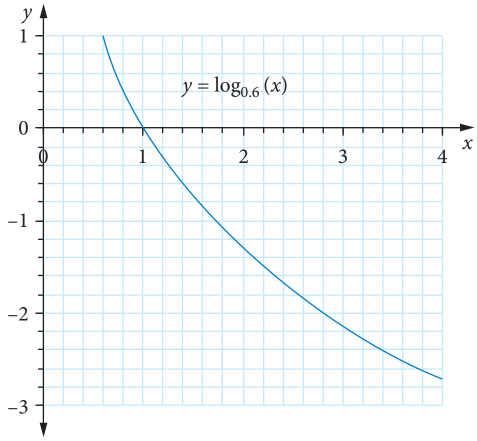
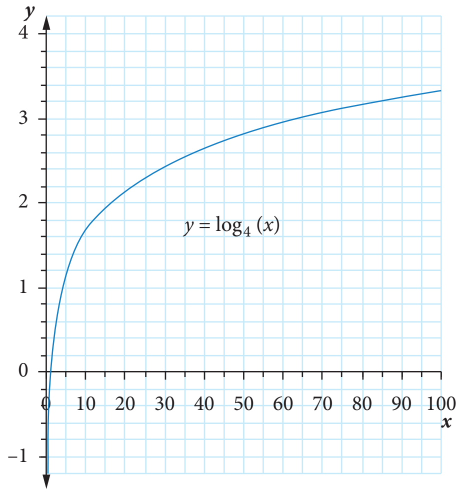
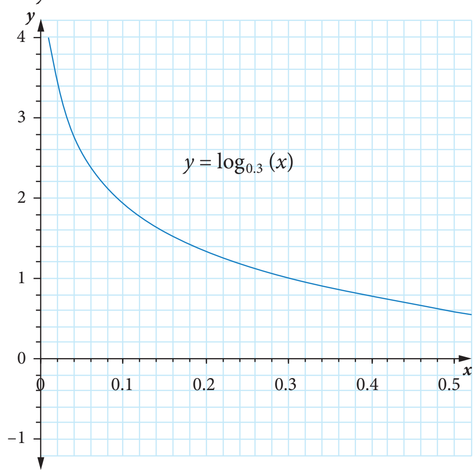
d $k \approx 2.4$



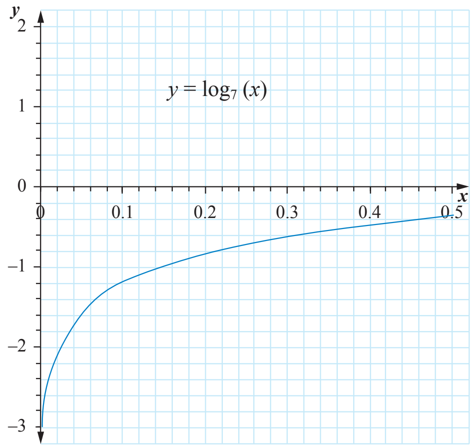
2 a $x \approx 75$



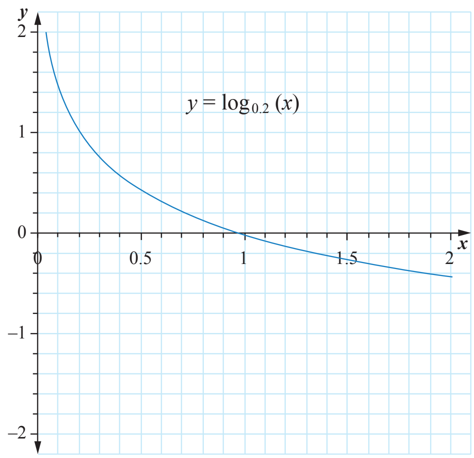
b $y \approx 0.05$



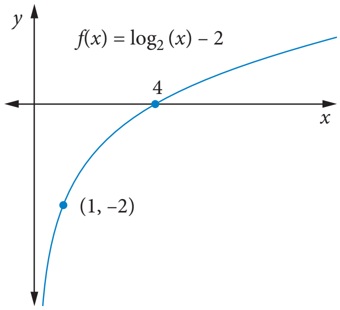
c $z \approx 0.04$



d $k \approx 1.9$



3 a



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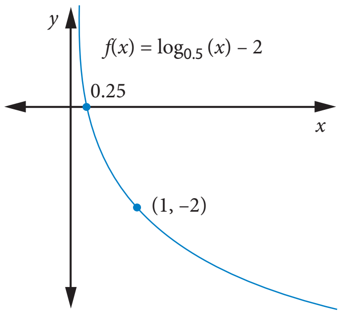

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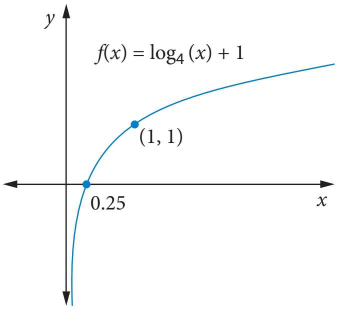

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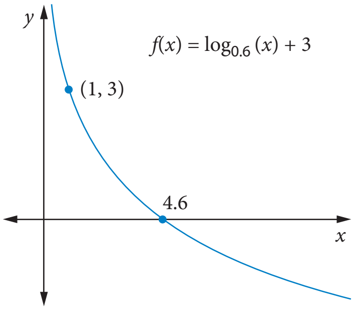

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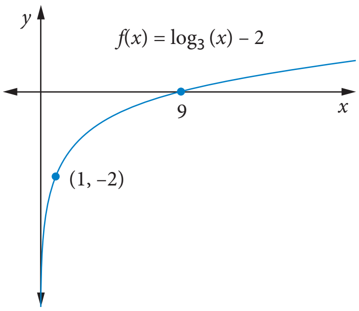

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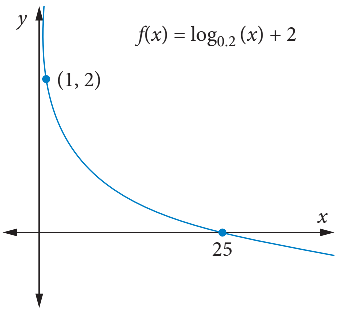
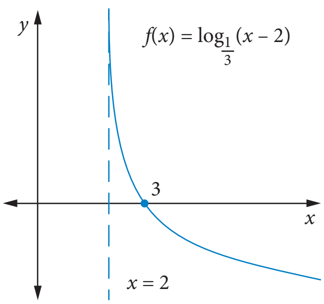


4 a



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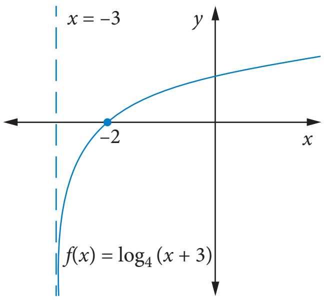

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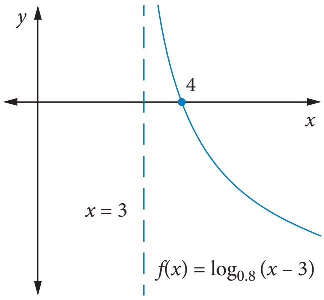

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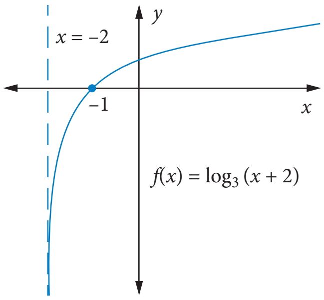

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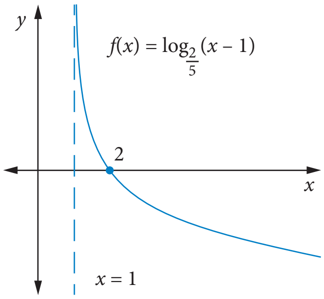

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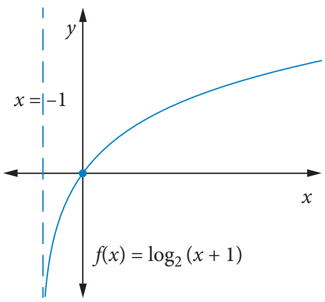

5 a $f(x) = \log_7(x) + 3$ b $f(x) = \log_{0.5}(x) - 2$
c $f(x) = \log_{\frac{1}{6}}(x) + 1$ d $f(x) = \log_3(x) - 4$

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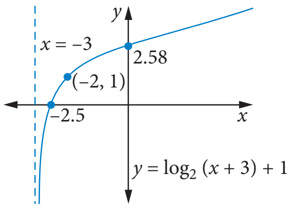
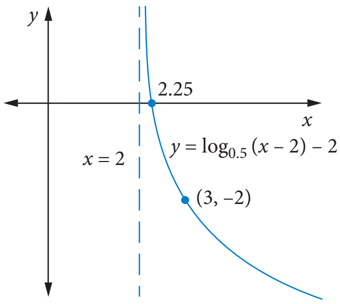
6 a $f(x) = \log_5(x+4)$ b $f(x) = \log_{0.3}(x-2)$
c $f(x) = \log_4(x+3)$ d $f(x) = \log_{0.8}(x-5)$

7 a $f(x) = \log_2(x+4)+3$ b $f(x) = \log_{0.1}(x-2)+1$
c $f(x) = \log_4(x+3)-4$ d $f(x) = \log_{0.6}(x-1)-2$

8 a



b



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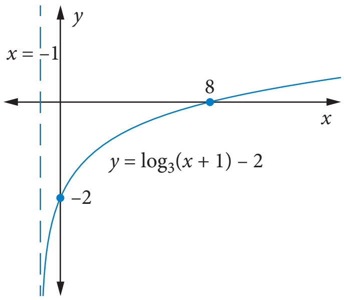

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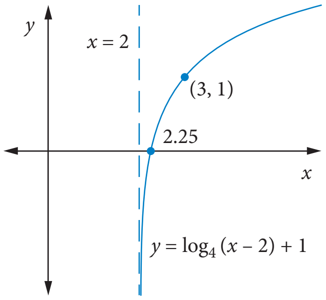


7.06

1 About 1700 km.

2 a 13 b 10 moles/litre

3 100 moles/litre

4 2488

5 3.16 watt/m^2

6 a 12.9686 b 1.570 65 cm

7 a 8 b 8.69×10^{10} kilowatt-hours

8 41.9886 litres

9 a $\sqrt[12]{2}$ b $440\sqrt[4]{2}$ Hz

7.07

1 a $\frac{dy}{dx} = \frac{1}{x}$ b $\frac{dy}{dx} = \frac{1}{x}$ c $\frac{dy}{dx} = \frac{3}{x}$
d $\frac{dy}{dx} = \frac{1}{x}$ e $\frac{dy}{dx} = \frac{6}{x}$ f $\frac{dy}{dx} = \frac{1}{x}$
g $\frac{dy}{dx} = \frac{4}{x}$ h $\frac{dy}{dx} = -\frac{2}{x}$

2 a $\frac{1}{x\ln(4)}$ b $\frac{1}{x\ln(10)}$ c $\frac{1}{x\ln(9)}$
d $\frac{1}{x\ln(2)}$ e $\frac{1}{x\ln(0.2)} = -\frac{1}{x\ln(5)}$

3 a $\frac{dy}{dx} = \frac{3}{3x-1}$ b $\frac{dy}{dx} = \frac{2}{2x+7}$
c $\frac{dy}{dx} = \frac{8}{4x-3}$ d $\frac{dy}{dx} = \frac{30}{6x+7}$
e $\frac{dy}{dx} = \frac{2}{2x+1}$ f $\frac{dy}{dx} = \frac{15}{5x-1}$
g $\frac{dy}{dx} = \frac{24}{4x-3}$ h $\frac{dy}{dx} = \frac{96}{8x-5}$

4 a $\frac{dy}{dx} = \frac{5}{x}$ b $\frac{dy}{dx} = \frac{3}{x}$
c $\frac{dy}{dx} = \frac{4x}{2x^2+1}$ d $\frac{dy}{dx} = \frac{32x}{8x^2-5}$
e $\frac{dy}{dx} = \frac{3x^2-4x+3}{x^3-2x^2+3x-4}$
f $\frac{dy}{dx} = \frac{24x^3-105x^4+3}{2x^4-7x^5+x}$

5 a $\frac{dy}{dx} = \frac{3}{2(3x+1)}$ b $\frac{dy}{dx} = \frac{7}{2(7x-5)}$
c $\frac{dy}{dx} = \frac{4}{3(4x+9)}$ d $\frac{dy}{dx} = \frac{1}{5(x-8)}$
e $\frac{dy}{dx} = \frac{12x}{3x-7}$
f $\frac{dy}{dx} = \frac{15}{5x-2}$
g $\frac{dy}{dx} = -\frac{1}{x+2}$
h $\frac{dy}{dx} = -\frac{3}{3x-5}$
i $\frac{dy}{dx} = \frac{12}{6x+1}$
j $\frac{dy}{dx} = \frac{5}{x-4}$

6 a $\frac{dy}{dx} = \frac{4x}{x^2+2}$
b $\frac{dy}{dx} = \frac{4x}{x^2-3}$
c $\frac{dy}{dx} = \frac{3(3x^2-2)}{x^3-2x+3}$
d $\frac{dy}{dx} = \frac{6\left(3x^2-3x+2\right)}{2x^3-3x^2+4x-1}$

7 a $2(x-2)\ln(x)+\dfrac{x^2-2x+1}{x}$

b $(3x^2+6x)\ln(x^3+3x^2+5)+3x(x+2)$

c $\ln(x)+1$

d $e^x\ln(x)+\dfrac{e^x}{x}$

e $\ln(x)\cos(x)+\dfrac{1}{x}\sin(x)$

f $\dfrac{2}{x}\cos(x)-\ln(x)\sin(x)-\dfrac{1}{x^2}\sin(x)$

8 a $f'(x)=\dfrac{24}{4x-3}$ b $f'(2)=\dfrac{24}{5}$ c $x=\dfrac{15}{4}$

9 a $f'(x)=\dfrac{6x}{x^2-1}$ b $f'(2)=4$

c $x=\dfrac{1+\sqrt{5}}{2}$

10 a $f'(x)=8x+\dfrac{6(x+1)}{x(x+2)}$ b $f'(2)=18.25$

c $x=-1.8363$

11 $g'(1)=2$

12 Proof

7.08

1 a $2\ln(x)+c$ b $7\ln(x)+c$

c $\dfrac{6}{5}\ln(x)+c$ d $\dfrac{4}{7}\ln(x)+c$

e $-\dfrac{8}{11}\ln(x)+c$ f $-\dfrac{9}{4}\ln(x)+c$

2 a $\ln(x+4)+c, x>-4$

b $\ln(x-2)+c, x>2$

c $\dfrac{1}{3}\ln(3x+1)+c, x>-\dfrac{1}{3}$

d $\dfrac{1}{5}\ln(5x-9)+c, x>1\dfrac{4}{5}$

e $\dfrac{11}{7}\ln(7x-9)+c, x>1\dfrac{2}{7}$

f $\dfrac{13}{4}\ln(4x-1)+c, x>\dfrac{1}{4}$

g $-3\ln(2x-5)+c, x>\dfrac{5}{2}$

h $-7\ln(x-3)+c, x>3$

3 a $x+\ln(x)+c, x>0$

b $2x^2-3\ln(x)-\dfrac{1}{x}+c, x>0$

c $5\ln(x)+x^2+\dfrac{1}{x}+c, x>0$

d $2\ln(x)+\dfrac{4x^3}{3}+\dfrac{1}{x}+c, x>0$

e $\dfrac{3x^8}{8}-x^2+15\ln(x)+c, x>0$

4 $f(x)=\ln(x-2)+6, x>2$

5 $f(x)=-\dfrac{7}{3}\ln(3x-5)+7, x>\dfrac{5}{3}$

6 $y=2\ln(x^2+2)+c$

7 $y=\dfrac{1}{2}\ln(x^2-5)+c, x^2>5$

8 $\ln(x-2)-\ln(x+2)+c, x>2$

7.09

1 a $k=\dfrac{1}{20}e^{1.29}$ b 59 words

c 238 words d 34 (33.3) minutes

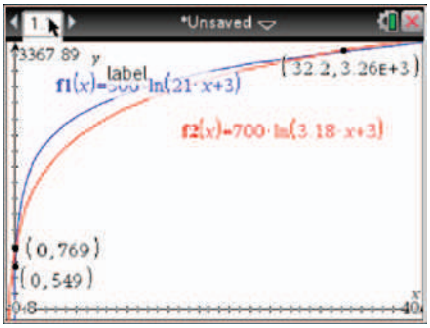
e About 2.2 words/minute.

2 a 549 black peppered moths

b 3225 black peppered moths

c $t=2.46$ days on 3 January

d $P=700$, $Q=3.189$

e 

f 3 February

g About 15 and 21 moths/day respectively.

3 a $T(t)=60-4.488\ldots\log(t+1)$

b 168 days

c About 0.027 s/day.

d Over $3\dfrac{1}{2}$ years (1315 days). It is very unlikely his body could stand that kind of continuous training for that length of time.

4 Longer than his lifetime (nearly 900 000 days).

5 a $P(t)=1.820\ldots\ln(t+1)+5$ per day.

b About 15 weeks (14.6).

c About 0.36 per week.

d About 0.17 per week.

CHAPTER 7 REVIEW

1 C
2 A
3 D
4 B
5 C
6 E
7 A
8 C
9 $3^4=81$
10 $\log_5(0.04)=-2$
11 About 1.511.
12 2.262
13 $x\approx-0.369$ or $x\approx0.631$
14 $x=\dfrac{-\log(6)-2\log(4)}{3\log(4)-2\log(6)}\approx-7.933$

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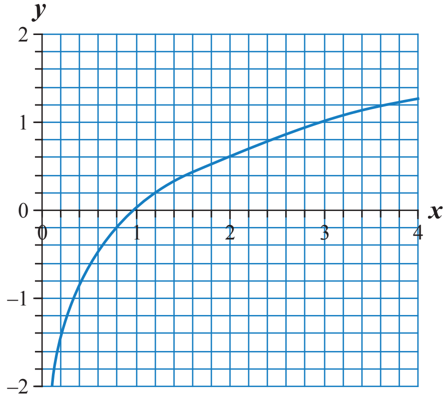
15 1.5

16 $x \approx 0.46$



17 $y = \log_2(x - 1)$

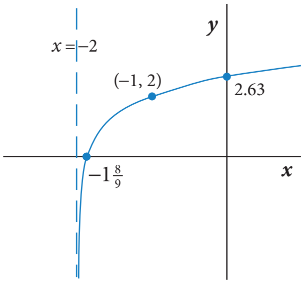
18 a $\dfrac{1}{x\ln(6)}$ b $\dfrac{6x}{3x^2+8}$

c $\dfrac{7(3x^4 - x^3 + 5)}{7x+1} + (12x^3 - 3x^2)\ln(7x+1)$

d $\dfrac{(3x-5)\ln(3x-5) - 3x\ln(x)}{x(3x-5)[\ln(3x-5]^2}$

19 $\ln(x^3 - 4x + 1) + c$

20



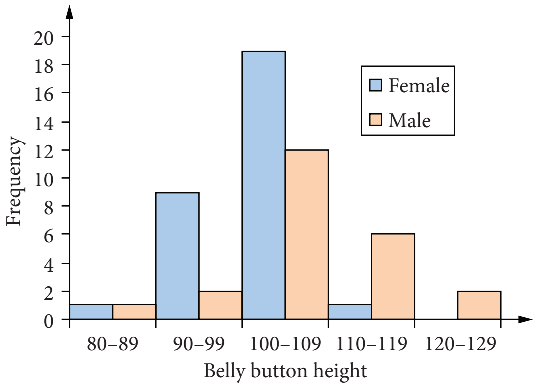
21 About 0.014%.

22 $\frac{1}{3}x^3 - 2x^2 + 6x - 10\ln(x+1) + c$

23 a i 2 units ii 6.4 units

b About 0.005 units/minute

INVESTIGATION: BELLY BUTTON HEIGHTS



Mean ≈ 103 cm, SD ≈ 8.1 cm

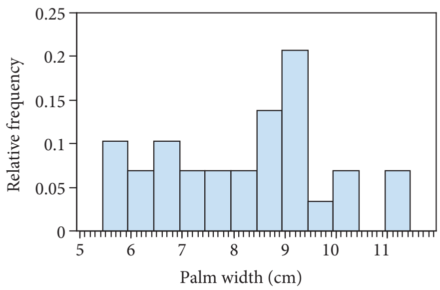
Other answers will vary.

8.01

1 a

Width (cm)	F	R.F.
5.5–5.9	3	0.103
6–6.4	2	0.069
6.5–6.9	3	0.103
7–7.4	2	0.069
7.5–7.9	2	0.069
8–8.4	2	0.069
8.5–8.9	4	0.138
9–9.4	6	0.207
9.5–9.9	1	0.034
10–10.4	2	0.069
10.5–10.9	0	0
11–11.4	2	0.069
Total	29	0.999

b

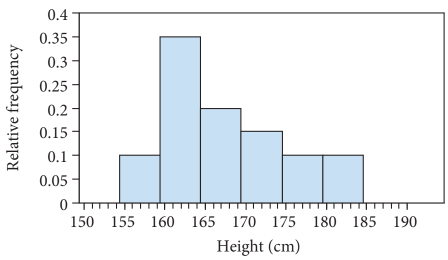


c 0.172 d 0.145

2 a

Height (cm)	Frequency	R.F.
155–159	2	0.1
160–164	7	0.35
165–169	4	0.2
170–174	3	0.15
175–179	2	0.1
180–184	2	0.1

b

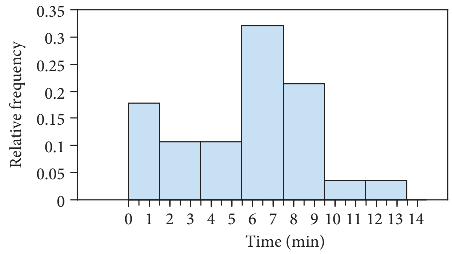


c 0.28 d 0.39 e 0.31

3 a

Time (min)	F	R.F.
0–1	5	0.178 571 4
2–3	3	0.107 142 9
4–5	3	0.107 142 9
6–7	9	0.321 428 6
8–9	6	0.214 285 7
10–11	1	0.035 714 3
12–13	1	0.035 714 3

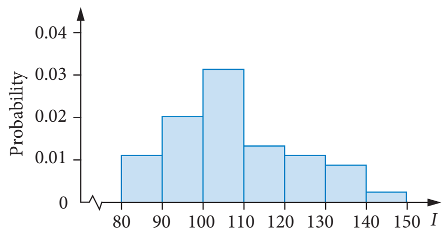
b



c 0.339   d 0.054   e 0.536

4 a

IQ	F	R.F.
80–89	5	0.113
90–99	9	0.205
100–109	14	0.318
110–119	6	0.136
120–129	5	0.114
130–139	4	0.091
140–149	1	0.023
Total	44	1.000

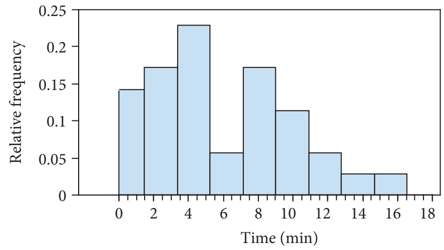


b 0.332   c 0.286   d 107.5

e It doesn't include students of very low IQ. They go to special schools.

5 a

Time (min)	F	R.F.
0–1	5	0.142 857 1
2–3	6	0.171 428 6
4–5	8	0.228 571 4
6–7	2	0.057 142 9
8–9	6	0.171 428 6
10–11	4	0.114 285 7
12–13	2	0.057 142 9
14–15	1	0.028 571 4
16–17	1	0.028 571 4



b 0.429

c They come at intervals of less than 16 minutes.

8.02

1 a Yes   b Yes   c No, area is not 1

d No, it is negative for (0, 1).

e No, area is not 1

2 a $p(x) = \dfrac{4}{(x-1)^2}$ b $p(x) = \dfrac{x^3}{64}$

c $p(x) = \dfrac{(3-x)(x+3)}{24}$ d $p(x) = 0.005x$

e $p(x) = \dfrac{2(e^{2x}-1)}{3-2\ln(2)}$

3 a $cdf(x) = 1 - \dfrac{1}{x}$ b $\dfrac{1}{2}$ c $\dfrac{1}{6}$

d $\dfrac{1}{4}$ e $\dfrac{1}{12}$

4 a $cdf(x) = \dfrac{x(x+4)}{140}$ b $\dfrac{1}{56}$ c $\dfrac{1}{7}$

d $\dfrac{19}{28}$ e $\dfrac{1}{5}$

5 a $cdf(x) = 1.2 - e^{-x}$ b 0

c About 0.737   d 0.8

e About 0.301

6 $f(x) = 0.4 - 0.02x$, it is a straight line with slope -0.02. $f(x) \geq 0$ for [0, 5] and $\int_0^5 f(x)dx = 1$

7 $f(t) = 4e^{-4t}$, it is a decreasing exponential curve. $f(t) \geq 0$ for [0, 5] and $\int_0^\infty f(t)dt = 1$

8 $f(m) = 0.1$, it is a constant function. $f(m) \geq 0$ for [0, 5] and $\int_0^{10} f(m)dm = 1$

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INVESTIGATION: CONTINUOUS PROBABILITY FUNCTIONS FOR MEASUREMENT ERRORS

$p(x) = k(x - 14.8)(x - 15.2)$

$p(x) = -25h(x - 14.8)(x - 15.2)$

$A = \frac{4h}{15}$

$p(x) = -\frac{375}{4}(x - 14.8)(x - 15.2)$

$\int_{14.93}^{15.07} p(x)dx = 0.503\ 562$ is a bit less than the triangular function.

Various functions including the normal distribution

8.03

1 a $p(x) = 0.05$ b $p(x) = \frac{1}{18}$

c $p(x) = 0.1$ d $p(x) = 0.1$

e $p(x) = \frac{1}{30}$

2 a $p(x) = \begin{cases} \frac{1}{36}x & \text{for } 0 \le x < 6 \\ -\frac{1}{36}(x-12) & \text{for } 6 \le x \le 12 \end{cases}$

b $p(x) = \begin{cases} \frac{1}{64}x & \text{for } 0 \le x < 8 \\ -\frac{1}{64}(x-16) & \text{for } 8 \le x \le 16 \end{cases}$

c $p(x) = \begin{cases} \frac{1}{25}(x-12) & \text{for } 12 \le x < 17 \\ -\frac{1}{25}(x-22) & \text{for } 17 \le x \le 22 \end{cases}$

d $p(x) = \begin{cases} \frac{1}{100}(x-4) & \text{for } 4 \le x < 14 \\ -\frac{1}{100}(x-24) & \text{for } 14 \le x \le 24 \end{cases}$

e $p(x) = \begin{cases} \frac{1}{256}(x-2) & \text{for } 2 \le x < 18 \\ -\frac{1}{256}(x-34) & \text{for } 18 \le x \le 34 \end{cases}$

3 a $\frac{1}{3}$ b $\frac{1}{6}$ c $-\frac{1}{12}$

d $p(x) = \begin{cases} \frac{1}{6}(x-4) & \text{for } 4 \le x < 6 \\ -\frac{1}{12}(x-10) & \text{for } 6 \le x \le 10 \end{cases}$

4 a $p(x) = \begin{cases} \frac{1}{10}(x-5) & \text{for } 5 \le x < 7 \\ -\frac{1}{40}(x-15) & \text{for } 7 \le x \le 15 \end{cases}$

b $p(x) = \begin{cases} \frac{1}{12}(x-4) & \text{for } 4 \le x < 8 \\ -\frac{1}{6}(x-10) & \text{for } 8 \le x \le 10 \end{cases}$

c $p(x) = \begin{cases} \frac{1}{15}(x-20) & \text{for } 20 \le x < 23 \\ -\frac{1}{35}(x-30) & \text{for } 23 \le x \le 30 \end{cases}$

d $p(x) = \begin{cases} \frac{1}{150}x & \text{for } 0 \le x < 15 \\ -\frac{1}{50}(x-20) & \text{for } 15 \le x \le 20 \end{cases}$

e $p(x) = \begin{cases} \frac{1}{1400}(x-20) & \text{for } 20 \le x < 60 \\ -\frac{1}{1050}(x-90) & \text{for } 60 \le x \le 90 \end{cases}$

5 a $p(t) = \frac{1}{5}$ for $15 \le t < 20$

b 0.8 c 0.4

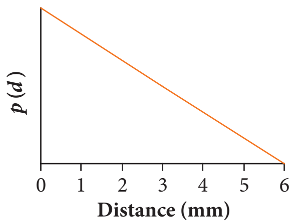
6 a 0 and 10 minutes b 0.5

c $b(t) = 0.1$ d 0.3

7 0.875

8 About 0.848.

9 a



b $p(x) = \frac{1}{18}(6-d)$ c $\frac{8}{9}$

d It reduces the size of the target area but he can still have the same probability as with the first dart.

10 a $p(t) = \begin{cases} \frac{1}{35}(t-17) & \text{for } 17 \le t < 22 \\ -\frac{1}{63}(t-31) & \text{for } 22 \le t \le 31 \end{cases}$

b $\frac{4}{35}$ c About 0.095. d $\frac{5}{7}$

e It depends how often she can afford to be late.

8.04

1 a $p(x) = \frac{1}{24}$ b 15

2 55

3 22

4 20

5 $18\frac{1}{3}$

6 40

7 Proof

8 a \$500 000 b 0.5

c About \$490 000 (\$490 192.38).

8.05

1 a $p(x)=\frac{1}{12}$ b $E(X)=10$

c $Var(X)=12, \sigma=2\sqrt{3}\approx 3.46$

2 a $Var(X)=33\frac{1}{3}, \sigma=\frac{10\sqrt{3}}{3}\approx 5.77$

b $Var(X)=208\frac{1}{3}, \sigma=\frac{25\sqrt{3}}{3}\approx 14.43$

c $Var(X)=33\frac{1}{3}, \sigma=\frac{10\sqrt{3}}{3}\approx 5.77$

d $Var(X)=133\frac{1}{3}, \sigma=\frac{20\sqrt{3}}{3}\approx 11.55$

e $Var(X)=\frac{3}{16}, \sigma=\frac{\sqrt{3}}{4}\approx 0.433$

3 a $p(x)=\begin{cases}\frac{1}{16}(x-12) & \text{for } 12\le x<16\\ -\frac{1}{16}(x-20) & \text{for } 16\le x\le 20\end{cases}$

b $E(X)=16$

c $Var(X)=2\frac{2}{3}, \sigma=\frac{2\sqrt{6}}{3}\approx 1.63$

4 a $p(x)=\begin{cases}\frac{1}{625}x & \text{for } 0\le x<25\\ -\frac{1}{625}(x-50) & \text{for } 25\le x\le 50\end{cases}$

b $E(X)=25$

c $Var(X)=104\frac{1}{6}, \sigma=\frac{25\sqrt{6}}{6}\approx 10.21$

5 a $Var(X)=4\frac{1}{6}, \sigma=\frac{5\sqrt{6}}{6}\approx 2.04$

b $Var(X)=121.5, \sigma=\frac{9\sqrt{6}}{2}\approx 11.02$

c $Var(X)=121.5, \sigma=\frac{9\sqrt{6}}{2}\approx 11.02$

6 a $p(x)=-0.02(x-10)$ for $0\le x\le 10$

b $E(x)=3\frac{1}{3}$

c $Var(X)=5\frac{5}{9}, \sigma=\frac{5\sqrt{2}}{3}\approx 2.36$

7 Proof

8 $Var(X)=\frac{a^2}{18}$

INVESTIGATION: LINEAR TRANSFORMATIONS OF CONTINUOUS RANDOM VARIABLES

$c'=ac+b$ and $d'=ad+b$

The vertical scale has to be multiplied by $\frac{1}{a}$.

$y=ax+b$ so $x=\frac{y-b}{a}$.

$q(y)=\frac{1}{a}p\left(\frac{y-b}{a}\right)$.

8.06

1 a $50, 833\frac{1}{3}$ and $\frac{50\sqrt{3}}{3}\approx 28.87$

b [5, 205]

c $105, 3333\frac{1}{3}$ and $\frac{100\sqrt{3}}{3}\approx 57.74$

d $E(Y)=2E(X)+5$, $Var(Y)=4\ Var(X)$ and $SD(Y)=2SD(X)$

2 a $40, 33\frac{1}{3}$ and $\frac{10\sqrt{3}}{3}\approx 5.77$

b [148, 248]

c $198, 833\frac{1}{3}$ and $\frac{50\sqrt{3}}{3}\approx 28.87$

d $E(Y)=5E(X)-2$, $Var(Y)=25Var(X)$ and $SD(Y)=5SD(X)$

3 a $40, 133\frac{1}{3}$ and $\frac{20\sqrt{3}}{3}\approx 11.55$

b [8, 16]

c $12, 5\frac{1}{3}$ and $\frac{4\sqrt{3}}{3}\approx 2.31$

d $E(Y)=0.2E(X)+4$, $Var(Y)=0.04\ Var(X)$ and $SD(Y)=0.2SD(X)$

4 a $7.5, \frac{1}{24}$ and $\frac{\sqrt{6}}{12}\approx 0.2041$

b [110, 130]

c $120, 16\frac{2}{3}$ and $\frac{5\sqrt{6}}{3}\approx 4.082$

d $E(Y)=20E(X)-30$, $Var(Y)=400\ Var(X)$ and $SD(Y)=20SD(X)$

5 a $25, 104\frac{1}{6}$ and $\frac{25\sqrt{6}}{6}\approx 10.21$

b [–5, 145]

c $70, 937\frac{1}{2}$ and $\frac{25\sqrt{6}}{2}\approx 30.62$

d $E(Y)=3E(X)-5$, $Var(Y)=9\ Var(X)$ and $SD(Y)=3SD(X)$

6 a $105, 204\frac{1}{6}$ and $\frac{35\sqrt{6}}{6}\approx 14.29$

b [9.5, 16.5]

c $13, 2\frac{1}{24}$ and $\frac{7\sqrt{6}}{12}\approx 1.429$

d $E(Y)=0.1E(X)+2.5$, $Var(Y)=0.01\ Var(X)$ and $SD(Y)=0.1SD(X)$

7 $E(Y)=173$, $Var(Y)=225$, $SD(Y)=15$

8 $\mu=46, \sigma=8\sqrt{2}\approx 11.32$

9 a $p(x)=0.08-0.0032x$

b $8\frac{1}{3}, 34\frac{13}{18}$ and $\frac{25\sqrt{2}}{6}\approx 5.89$

c [200, 400]

d $p(y)=-0.000\,05(y-400)$

e $266\frac{2}{3}, 2222\frac{2}{9}$ and $\frac{100\sqrt{2}}{3}\approx 47.14$

f $E(Y)=8E(X)+200$, $Var(Y)=64\ Var(X)$ and $SD(Y)=8SD(X)$

10 a $p(x)=\begin{cases}\frac{1}{1500}(x-40) & \text{for } 40\le x<90\\ -\frac{1}{300}(x-100) & \text{for } 90\le x\le 100\end{cases}$

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b $76\frac{2}{3}$, $172\frac{2}{9}$ and $\frac{5\sqrt{62}}{3} \approx 13.12$

c [65, 185]

d $p(y) = \begin{cases} -\frac{1}{6000}(y-65) & \text{for} \quad 65 \le y < 165 \\ -\frac{1}{1200}(y-185) & \text{for} \quad 165 \le y \le 185 \end{cases}$

e $138\frac{1}{3}$, $688\frac{8}{9}$ and $\frac{10\sqrt{62}}{3} \approx 26.25$

f $E(Y) = 2E(X) - 15$, $\text{Var}(Y) = 4\,Var(X)$ and $SD(Y) = 2SD(X)$

8.07

1 a $p(x) = 0.125e^{-0.0488(x-28.5)^2}$
b $p(x) = 0.07e^{-0.0154(x-28.5)^2}$
c $p(x) = 0.07e^{-0.0154(x-48.6)^2}$
d $p(x) = 0.005\,11e^{-0.000\,0822(x-246)^2}$
e $p(x) = 166e^{-86\,806(x-0.07)^2}$

2 a $p(x) = 0.0469e^{-0.006\,92(x-163)^2}$, 0.299
b $p(x) = 2.714e^{-0.000\,0231(x-678)^2}$, 0.447
c $p(x) = 0.001\,124e^{-0.000\,003\,97(x-4240)^2}$, 0.734
d $p(x) = 0.221e^{-0.1543(x-6.5)^2}$, 0.162
e $p(x) = 0.0177e^{-0.000\,979(x-74.9)^2}$, 0.537

3 a 0.4893 b 0.4904 c 0.4542
d 0.0231 e 0.3929

4 a 0.0959 b 0.2666 c 0.2756
d 0.8997 e 0.2529

5 a 0.8819 b 0.7947 c 0.9894
d 0.9452 e 0.3159

6 a 0.713 026 b 0.872 203 c 0.750 704
d 0.021 596 e 0.020 157

7 a 0.272 353 b 0.962 131 c 0.961 594
d 0.223 549 e 0.005 359

8 a 0.051 973 b 0.5838 c 9.67%

9 11.41%

10 0.1333

11 0.2023

12 0.022 75

8.08

1 a 8.914 b 14.12 c 8.020
d 12.93 e 6.793

2 a 208.2 b 839.1 c 28.26
d 27.87 e 107.7

3 a 0.1113 b 78.57

4 a 0.363 b 1641.99 c −104.72
d 882 e 50.264 f 1333.584
g 148.428 h 6974.4

5 a 96.805 b 553.86 c 456.12
d 324.52 e 148.92 f 174.5796
g 92.176 h 2357.2

6 $Z_E \approx 0.375$ and $Z_M \approx 0.4$, so he did better in Maths Methods.

7 $Z_{IQ} \approx 0.67$ and $Z_H \approx 0.33$, so her IQ is more unusual than her height.

8 First player $P(X > 37) = 0.091 > P(X > 37) = 0.055$ for second player

9 Only 2, using standard normal scores.

10 35

8.09

Note that all answers in this exercise are approximate.

1 a 5.72% b 5.72% c 0.42%

2 a 0.717 b 0.1764

3 a 3 b 31

4 a 0.0334 b 0.0549 c 0.0182
d 0.0807 e 0.008 92 f 0.952
g 0.048

5 a 0.0995 b 0.0457 c 0.0752
d 0.0457 e 0.0605

6 0.0153, 0.1497

7 21 months

8 a 15
b No, because there is a 12.8% of this result by chance.

9 38.6–41.4 mm

10 Weldon (2%) is more likely than Betterdon (0.3%).

11 a 0.0912 b 0.2469

12 a 0.1359 b 0.8609

CHAPTER 8 REVIEW

1 B

2 A

3 E

4 E

5 D

6 B

7 B

8 D

9 0.137

10 $p(x) = \frac{4(x^3 - 2x^2 + 2)}{321}$

11 $p(x) = \frac{1}{6}$, $P(2.25 < x < 2.35) = \frac{1}{60}$

12 16

13 58.6, 11.2

14 $p(x) = 0.0767e^{-0.0185(x-76)^2}$

15 English, as $Z_E = 1.52$ and $Z_M = 1.31$

16 a 0.7580 b 0.4125

17 About 98 000 km

18 0.2

19 $\frac{5}{36}$

20 0.3765

MIXED REVISION 3

Multiple choice

1 B
2 C
3 C
4 A
5 E
6 A

Short answer

1 $2\frac{2}{3}$
2 $x < \frac{2}{3}$
3 The constant C translates the graph horizontally left ($C > 0$) or right ($C < 0$).
4 About 79.0 kg.

Application

1 41, 57 and 68, to the nearest mark.
2 Proof
3 $y = 3\log_e(x + 1) + 2$
4 a $E = \log_{10}\left(\frac{C_2}{C_1}\right)$
  b $E = 0.42$ kilocalories per gram molecule
  c $E = 1.09$ kilocalories per gram molecule

INVESTIGATION: POLITICAL POLLS

They discard those who did not answer or who answered 'don't know'. It is generally reasonable, but may introduce some non-response bias.
They allocate their preference partly to each of the two parties based on historical preferences of people who voted that way in the past. It is reasonable because it produces reasonable results.
The Reader's Digest poll went to a selected group: Reader's Digest readers, and these were not typical of the whole population. There was also considerable non-response bias and the poll was conducted some time before the election.

9.01

1 a Pies produced; mass and colour; mean = 103.2 g, range = 12 g, all even colour
  b Guests at hotel; room service speed, room service food quality; speed 3.5, food quality 3.7, on averaged scale.
  c Brisbane passenger buses; number of passengers; mean ≈ 29.7 passengers, range = 33 passengers.
  d People catching planes from Cairns; time to reach airport; mean ≈ 16.7 minutes, range = 40 minutes.
  e Checkout operators; number of errors, correctness of till totals, value processed; 27% made 5 errors, 20% incorrect till totals, 13% processed $10 000
2 B
3 B
4 A
5 a Reasonably fair; Selection bias, only current opening hours customers polled; Non-response bias, people who don't care may not respond.
  b Biased; Selection bias; only students from this school; Recall/reporting bias, non-secret system so students may not report low amounts.
  c Biased; Self-selection bias, only people who are very interested will respond and it could be open to manipulation by friends of the contestants.
  d Biased; Non-response bias, people who have positive views or don't care are unlikely to attend the meetings; Selection bias, local meetings.
  e Biased; Selection bias, the reviewer decides on restaurants; Reporting bias, restaurant's knowledge of the critic may change its response to the critic.
6 a Students at the school who are legally able to work.
  b Proportion with part-time jobs, hours worked.
  c Population will change as more students reach working age during the year; Reporting bias, students may under-report hours to the school.
7 Selection bias, students later in the alphabet (the last 125) are not chosen, students from the same family cannot be selected; Non-response bias, absent students may be those who find conduct of the test poor.
8 a Selection bias, only one year group.
  b Selection bias, only students who use the resource centre.
  c Selection bias, only one year group.
  d Self-selection bias.
  e Possible selection bias as students in the same family cannot be surveyed.
  f Self-selection bias.
  g Possible selection bias as classes may not be the same size. Self-selection bias.
  h Possible selection bias as year groups may not be the same size.
  i Self-selection bias.
  j Selection bias, as only early-birds will be selected.

The best methods are likely to be (in order) **e**, **g** and **h.**

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INVESTIGATION: PSEUDO-RANDOM NUMBERS

$x_n = 20x_{n-1}$ mod 37, starting with 5, repeats after 37 numbers

$x_n = 12x_{n-1}$ mod 19, starting from 7, repeats after 6 numbers

The repeat cyle for the Blum Blum Shub generator is very long.

One method uses the Blum Blum Shub generator where the receiver generates the encoding generator number and sends it to the sender. The sender then encodes using this generator number, but cannot decode. The receiver can decode the message.

Breaking the code takes a lot of computer power.

INVESTIGATION: RANDOM AND NON-RANDOM SAMPLES

Results will vary, but there is usually some attempt at randomisation.

9.02

1 a 50, 42, 76, 44, 43, 79, 47, 40
  b 374 995, 380 868, 139 324, 347 249, 105 078, 110 178, 138 491, 336 824
  c 7, 0, 2, 6, 4, 8, 3, 9
  d 66, 39, 57, 63, 53, 32, 61, 67

2 a 4, 3, 8, 13, 11, 48, 36, 32, 1, 19
  b 576, 689, 631, 566, 682, 644
  c 1068, 1602, 1729, 1535, 1696, 2155, 1566, 2079
  d 344, 866, 759, 507, 466, 928, 462, 672, 257, 916, 855, 553

3 Answers will vary.

4 a 3 men in board shorts, 2 men in briefs, 4 women in bikinis and 1 woman in a one-piece.
  b 7 soft-centred, 5 hard-centred, 3 liquid-centred and 4 nutty-centred chocolates (= 19). An extra hard-centred would be chosen if exactly 20 were needed.
  c 2 fifteen-year-olds, 11 sixteen-year-olds, 3 seventeen-year-olds and 1 eighteen-year-old (= 17). If exactly 16 were required, you would choose only one 15-year-old.
  d 11 assembly workers, 3 office staff and 1 supervisor.

5 a 28, 42, 56, 70, 84, 98, 112, 126, 140, 12
  b 216, 244, 272, 300, 105, 133, 161, 189
  c 64, 92, 120, 148, 176, 204, 232, 260, 288, 316, 344, 372, 400, 1, 29
  d 1472, 1647, 1822, 1997, 2172, 2347, 2522, 2697, 1296

6 a Quota b Convenience
  c Judgement d Purposive
  e Convenience f Quota
  g Judgement h Purposive

7 5130, 5205, 5280, 5355, 5430, 5505, 5580, 5655, 4985, 5060

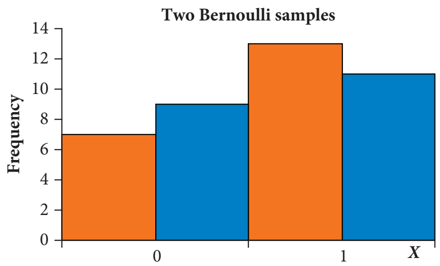
8 a Administration: males 1, females 3; Factory: males 8, females 6
  b Administration 4, factory 14

9 a NSW 160 or 161 Vic. 124, Qld 101, SA 36, WA 54, Tas. 11, NT 5, ACT 8
  b Men: NSW 80, Vic. 61, Qld 50, SA 18, WA 27, Tas. 6, NT 3, ACT 4
  Women: NSW 81, Vic. 63, Qld 50, SA 18, WA 27, Tas. 6, NT 2, ACT 4

10 Numbers 205, 224, 243, 262, 281, 300, 319, 338, 357, 376, 9, 28, 47, 66, 85, 104, 123, 142, 161, 180

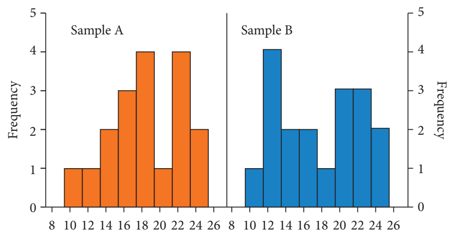
9.03

1 a



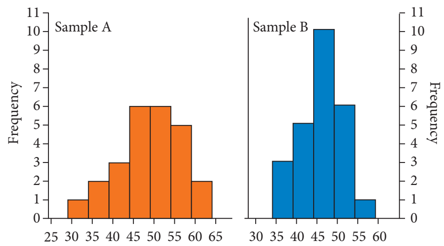
  b $\mu_A = 0.65$, $\sigma_A \approx 0.477$, $\mu_B = 0.55$, $\sigma_B \approx 0.497$; Sample A has a higher mean and less spread than sample B.

2 a



  b $\mu_A \approx 18.4$, $\sigma_A \approx 3.84$, $\mu_B \approx 17.7$, $\sigma_B \approx 4.51$; Sample A has a higher mean and less spread than sample B.

3 a



  b $\mu_A \approx 49.4$, $\sigma_A \approx 7.61$, $\mu_B \approx 46.2$, $\sigma_B \approx 5.42$; Sample A has a higher mean and more spread than sample B.

Specific answers for questions **4–10** will vary, but the general conclusions will be the same.

4 a The dot plots will be similar in overall shape.
b The means will be close to 17.5 and the standard deviations will be close to 2.5.
c The samples will probably have different frequencies but they will have similar patterns.
5 a The dot plots will be different.
b The means will be close to 0.2 and the standard deviations will be close to 0.4.
c The samples will have different appearances.
6 a The dot plots will be different.
b The means will be close to 15 and the standard deviations will be close to 6.
c The samples will have different appearances.
7 a The dot plots will be different.
b The means will be close to 50 and the standard deviations will be close to 10.
c The samples will have different appearances.
8 a The means will be close to 9 and the standard deviations will be close to 2.
b The means will be close to 9 and the standard deviations will be close to 2.
c The difference between the means and the standard deviations for the samples of 64 items will be less than the difference for the samples of 9 items.
9 a The means will be close to 40 and the standard deviations will be close to 3.
b The means will be close to 40 and the standard deviations will be close to 3.
c The difference between the means and the standard deviations for the samples of 64 items will be less than the difference for the samples of 16 items.
10 a The means will be close to 40 and the standard deviations will be close to 8.
b The means will be close to 40 and the standard deviations will be close to 8.
c The difference between the means and the standard deviations for the samples of 81 items will be less than the difference for the samples of 9 items.

9.04

1 0.045
2 0.15
3 0.064
4 a 0.225 b 0.421 c 0.048
5 0.214
6 0.068
7 The cheaper paintings may be under-represented in a commercial gallery.
8 a 0.343
b Older caravan and motor home drivers may be over-represented in Queensland in winter as 'grey nomads' tend to travel north in winter.

9.05

1 a $E(\hat{p}) = 0.2$, $Var(\hat{p}) = 0.0032$, $SD(\hat{p}) \approx 0.057$
b $E(\hat{p}) = 0.7$, $Var(\hat{p}) = 0.0084$, $SD(\hat{p}) \approx 0.092$
c $E(\hat{p}) = 0.81$, $Var(\hat{p}) \approx 0.0013$, $SD(\hat{p}) \approx 0.036$
d $E(\hat{p}) = 0.22$, $Var(\hat{p}) = 0.0022$, $SD(\hat{p}) \approx 0.046$
2 $E(\hat{p}) = 0.\overline{3}$, $SD(\hat{p}) \approx 0.0703$
3 $E(\hat{p}) = 0.45$, $SD(\hat{p}) \approx 0.035$
4 $E(\hat{p}) = \frac{1}{6}$, $SD(\hat{p}) \approx 0.068$
5 $E(\hat{p}) = \frac{3}{13}$, $SD(\hat{p}) \approx 0.042$
6 About 0.6.
7, 8 Proofs

INVESTIGATION: CABBAGE MOTHS

As the sample size is increased, the sample means, standard deviations and proportions get closer to the class average, which becomes more stable.

9.06

1 a–c Results will vary.
d As n increases, the distribution more closely approximates a normal distribution.
2 a–c Results will vary.
d As n increases, the distribution more closely approximates a normal distribution.
3 a–c Results will vary.
d As p gets closer to 0.5, the distribution more closely approximates a normal distribution.
4 a–c Results will vary.
d As p gets closer to 0.5 and n increases, the distribution more closely approximates a normal distribution.
5 a–c Results will vary.
d As n increases, the distribution more closely approximates a normal distribution; the mean gets closer to 25 and the standard deviation decreases from about 2 to 0.8 to 0.5 compared to the original 7.2.
6 a–c Results will vary.
d As n increases, the distribution more closely approximates a normal distribution;the mean gets closer to the original distribution mean and the ratio of the standard deviation of the sampling distribution to that of the original distribution decreases.
7 a–c Results will vary.
d As n increases the distribution more closely approximates a normal distribution, the mean gets closer to 55 and the standard deviation decreases from about 5 to about 2.4 to about 1 compared to the original 12

8 a, b, c, d Results will vary.
 e The distribution becomes more like a normal distribution.
9 a, b, c, d Results will vary.
 e The distribution becomes more like a normal distribution.
10 a, b, c, d Results will vary.
 e The mean gets closer to 184 cm and the standard deviation decreases from about 2.3 to about 1.3.

INVESTIGATION: NORMALISED SAMPLE PROPORTIONS

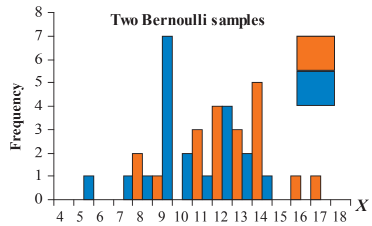
The normalised graph becomes more and more like a normal distribution as the sample size increases, but the width decreases.

9.07

1 10.8%
2 31.2%
3 4.78%
4 0.220
5 0.000169
6 0.1108
7 a 0.026 26 b 0.9988 c 0.008 939
8 0.2585

CHAPTER 9 REVIEW

1 B
2 A
3 C
4 C
5 D
6 E
7 a People who follow AFL, proportion most disliking each team, proportion disliking Collingwood = 0.25.
 b Houses in the area, number of bedrooms, mean ≈ 3, range 1 to 5.
8 A
9 Answers will vary.
10 3 welders, 1 boilermaker, 2 sheet metal workers and 4 labourers.
11 17 113, 18 527, 19 942, 10 042, 11 456, 12 870, 14 285, 15 699
12 a



 b $\mu_A = 12.4 > \mu_B = 10.1$, but $\sigma_A \approx 2.29 \approx \sigma_B \approx 2.19$; this is also shown by the graphs.
13 0.75
14 a 0.375
 b This is only one shop, and since it is a sale, the estimate may be low.
15 $\hat{p} = 0.59$ and $\sigma \approx 0.0898$
16 a The distribution becomes more like a normal distribution.
 b The distribution gets less like a normal distribution.
17 About 71%.
18 a Convenience sampling, selection bias, recall/reporting bias.
 b Convenience sampling, as not everyone has a landline, some selection bias; probably some completion bias (reasons) and probably non-response bias.
19 a Answers will vary.
 b Answers will vary but they should both be between about 18 and 22. The standard deviation is likely to be between about 2 and 4.
 c The theoretical mean is $\dfrac{15+25}{2} = 20$ and the theoretical SD $= \dfrac{25-15}{2\sqrt{3}} \approx 2.89$. The mean is close to the theoretical mean but the standard deviation could vary considerably.
20 10 (10.8)
21 Australia: about 0.82; Britain: about 0.0036.

INVESTIGATION: POINT AND INTERVAL ESTIMATES IN THE MEDIA

Results will vary.

10.01

1 a Point b Interval c Interval d Point
2 $p \approx 0.71$, $\sigma^2 \approx 0.0026$, $\sigma \approx 0.051$
3 $p \approx 0.45$, $\sigma^2 \approx 0.0062$, $\sigma \approx 0.079$
4 $p \approx 0.60$, $\sigma^2 \approx 0.0028$, $\sigma \approx 0.053$
5 $p \approx 0.32$, $\sigma^2 \approx 0.0044$, $\sigma \approx 0.066$
6 0.57 to 0.71
7 0.55 to 0.81
8 0.49 to 0.76
9 0.0222
10 42

10.02

1 a About 98.8%. b $\alpha \approx 0.997$
 c About 0.683. d $\alpha \approx 0.993$
2 a $E \approx 2.33$ b $E \approx 1.44$
 c $E \approx 2.58$ d $E \approx 1.15$

3 a About $-1.64 \le z < 1.64$
   b About $-2.81 \le z < 2.81$
   c About $-1.96 < z < 1.96$
   d About $-2.58 \le z < 2.58$
4 a About 0.988 b $\alpha \approx 0.866$
   c $\alpha \approx 0.683$ d About 99.7%.
5 The confidence level is the area under the curve. To get a larger area under the curve, you need a wider interval. The margin of error is half the interval length, so you cannot have a high confidence level and a low margin of error.

10.03

1 a 0.70 b 0.065 c (0.57, 0.83)
2 a $E \approx 0.104$ b $E \approx 0.127$
   c $E \approx 0.108$ d $E \approx 0.061$
3 a (0.044, 0.156) b (0.616, 0.884)
   c (0.174, 0.466) d (0.514, 0.706)
4 (0.850, 0.936)
5 (0.415, 0.562)
6 (0.013, 0.107)
7 About 2.4%.
8 0.049, 0.075, 0.082, 0.075 and 0.049 so the error margin increases as the sample proportion gets closer to 0.5 for the same size sample and confidence level.

INVESTIGATION: DICE SIMULATION OF CONFIDENCE INTERVALS

For 20 rolls the proportion of high rolls is likely to be between 0.06 and 0.60 and in most cases will be close to the centre (0.33).
You are likely to have between 0 and 2 confidence intervals that do not contain 0.33.
The class average is likely to be between 5% and 15% not containing 0.33.

10.04

1–5 Answers will vary, but pooled results should approach the proportion given by the confidence level in each case.

10.05

1 About 1300 (1262).
2 About 300 (271).
3 About a million (960 400).
4 About 70 (68).
5 About 4000 (3981).
6 About 1000 (1067).
7 About 56%.
8 About 85%.

CHAPTER 10 REVIEW

1 C
2 E
3 C
4 A
5 D
6 D
7 B
8 0.64–0.74
9 98.8% (0.98758…)
10 1.75
11 a 0.275 b 0.071 c (0.14, 0.41)
   d The proportion of Year 12 students having a part-time job has a probability of 95% of being in the interval (0.14, 0.41).
12 a About 0.065. b (0.685, 0.815)
13 a Answers will vary, but should be of the form {0,1,1,1,0,0,1,1,1,1,1,1,1,1,1,1,0,1,1,1,0,0,1,1,1,0, 1,0,0,1}.
   b In this case, it is (0.54, 0.86); they should be about the same width.
   c Mention that in most cases (95%), 0.67 will be inside the interval, whether it is or not.
14 About 500 (454).
15 a That the sample proportion is 0.5.
   b About 110 (106).
16 244
17 a 0.0209 b 1.440 c (0.29, 0.35)
18 a About 60 (61). b About 11 minutes.

MIXED REVISION 4

Multiple choice

1 B
2 D
3 B
4 C
5 E
6 D

Short answer

1 6 Year 7s, 7 Year 8s, 7 Year 9s, 8 Year 10s, 6 Year 11s and 5 Year 12s making 39 altogether, rather than 40 because of rounding errors. To get 40, an extra Year 8 student would be chosen.
2 (0.24, 0.36)
3 a $\frac{1}{12}, \frac{1}{6}$ and $\frac{1}{4}$ respectively.
   b About 0.001 060, 0.001 93 and 0.002 60 respectively.
   c About 0.0326, 0.0439 and 0.0510 respectively.
4 a 0.0162 b 2.326 c (91.2%, 98.8%)

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Application

1 99%
2 About 660 (664).
3 a Simple random sampling of voters, possible interviewer bias if they were not given the same script.
  b Convenience/cluster sampling, possible selection bias.
  c Quota sampling, selection bias and some design bias as the question is one-sided.
  d Simple random sampling of those with mobile phones, non-response bias.
4 About 0.88.