Working out space

PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Year 12 Test One 2022

Student name: ______________ Teacher name: ______________

Task type: **Response**

Time allowed for this task: **_____40______ mins**

Number of questions: **_____8______**

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **__42____ marks**

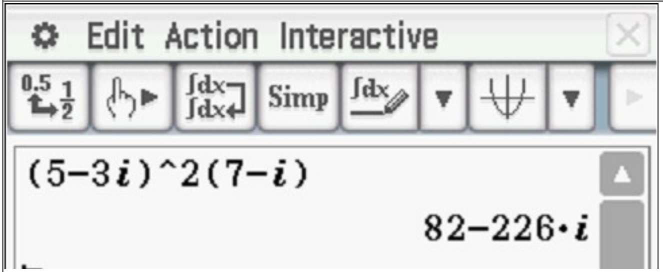
Task weighting: **_10___%**

Formula sheet provided: **Yes/No**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (2, 3 & 3 = 8 marks)
Let $z = 5 - 3i$ and $w = 7 - i$.
Simplify the following.

a) $z^2 w$

Solution

Specific behaviours
✓ real part ✓ imaginary part

b) $\dfrac{1}{w}$

Solution
$\dfrac{1}{7-i}\dfrac{7+i}{7+i} = \dfrac{7+i}{50}$
Specific behaviours
✓ shows use of conjugate ✓ numerator ✓ denominator

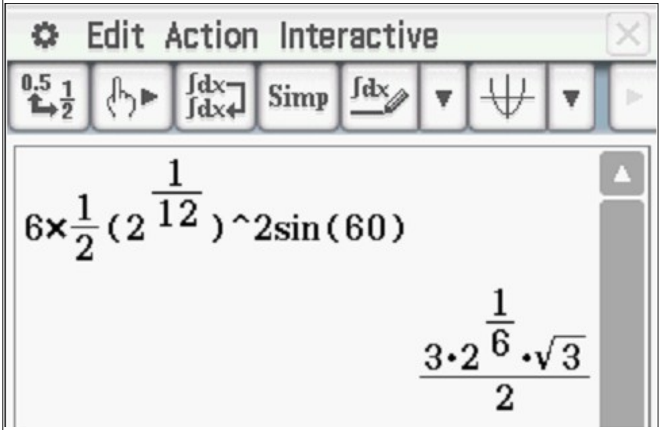
c) $\dfrac{z}{w}$

Solution
$\dfrac{5-3i}{7-i}\dfrac{7+i}{7+i} = \dfrac{35+5i-21i+3}{50} = \dfrac{38-16i}{50} = \dfrac{19-8i}{25}$
Specific behaviours

Working out space

Specific behaviours
✓ shows scale and equally distance ✓ all positions correct

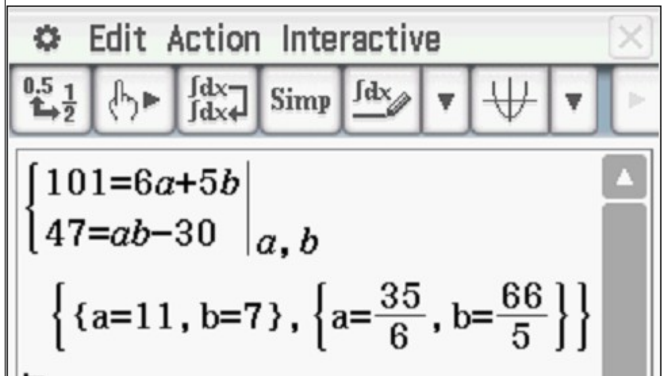
c) Adjacent points can be joined by lines to form a polygon. Determine the exact area of this polygon.

Solution

Specific behaviours
✓ identifies equilateral triangles ✓ determines side lengths ✓ shows calculation for total exact area

P shows use of conjugate or uses result from b but only if conjugate shown ✓ shows how to multiply numerators ✓ simplified expression

Q2 (3 marks)

Determine all possible real number pairs a, b such that $\frac{101+47i}{a-5i}=6+bi$

Solution
$\frac{101+47i}{a-5i}=6+bi$ $101+47i=(6+bi)(a-5i)=6a+5b+i(ab-30)$ $101=6a+5b$ $47=ab-30$ 
Specific behaviours
✓ equates real and imaginary parts of two expressions ✓ sets up two simultaneous equations P solves for two exact pairs of values

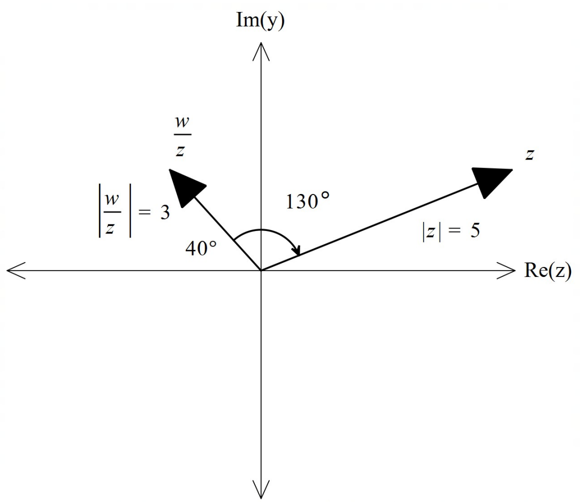
Q3 (3 marks)

Consider the polynomial $f(z) = z^3 + bz^2 + cz + d$ where $b, c \,\&\, d$ are real numbers.
Given that $f(3) = 0$ and $f(2-5i) = 0$ determine the values of $b, c \,\&\, d$.

Solution
$f(z) = z^3 + bz^2 + cz + d = (z-3)(z-\alpha)(z-\beta) = (z-3)\left(z^2 - (\alpha+\beta)z + \alpha\beta\right)$ $(z-3)\left(z-[2-5i]\right)\left(z-[2+5i]\right)$ $(z-3)(z^2-4z+29)$ $z^3 - 7z^2 + 41z - 87$ $b = -7, c = 41 \,\&\, d = -87$
Specific behaviours
✓ uses conjugate root ✓ solves for one constant ✓ solves for all 3 constants

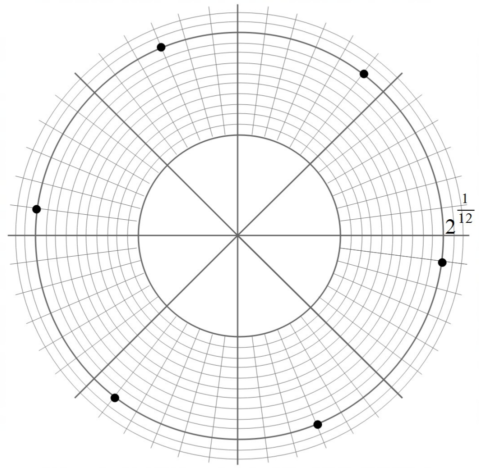
Q4 (3 marks)
Using the diagram below determine the complex number w in exact cartesian form.
(Note: Not drawn to scale)

Solution



$z^6 = 1 - i = \sqrt{2}\,cis\left(-\frac{\pi}{4} + 2n\pi\right) \quad n = 0, \pm1, \pm2, ...$ $z = 2^{\frac{1}{12}} cis\left(\frac{-\pi}{24} + \frac{2n\pi}{6}\right) = 2^{\frac{1}{12}} cis\left(\frac{-\pi}{24} + \frac{8n\pi}{24}\right) \quad n = 0, \pm1, \pm2, \pm3...$ $z_1 = 2^{\frac{1}{12}} cis(\frac{-\pi}{24})$ $z_2 = 2^{\frac{1}{12}} cis(\frac{7\pi}{24})$ $z_3 = 2^{\frac{1}{12}} cis(\frac{-9\pi}{24})$ $z_4 = 2^{\frac{1}{12}} cis(\frac{15\pi}{24})$ $z_5 = 2^{\frac{1}{12}} cis(\frac{-17\pi}{24})$ $z_6 = 2^{\frac{1}{12}} cis(\frac{23\pi}{24})$
Specific behaviours
✓ converts RHS to polar ✓ demonstrates use of De Moivre's ✓ determines modulus of all roots P determines principal arguments

b) Plot these roots on the complex plane below.



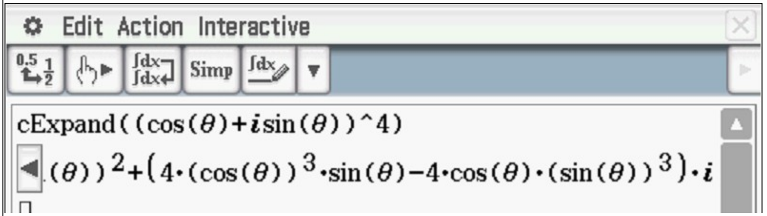
Solution

Q7 (4 marks)

Using De Moivre's Theorem, derive an expression for $\sin(4\theta)$ in terms of $\cos(\theta)\&\sin(\theta)$.

Solution

$(\cos\theta + i\sin\theta)^4 = cis(4\theta) = \cos 4\theta + i\sin 4\theta$



$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

Specific behaviours

✓ sets up equation for power 4 and uses De Moivre's
✓ states expression for power 4
✓ equates imaginary parts of both sides
P states required expression

Q8 (4, 2 & 3 = 9 marks)

a) Solve for all the roots $z^6 = 1 - i$ in polar form $z = rcis\theta$ with $-\pi < \theta \le \pi$.

Solution

$z = 5cis10$

$Arg(w) - Arg(z) = 140$

$Arg(w) = 150$

$|w| = 3|z| = 15$

$w = 15cis150 \quad = 15(-\frac{\sqrt{3}}{2} + \frac{1}{2}i)$

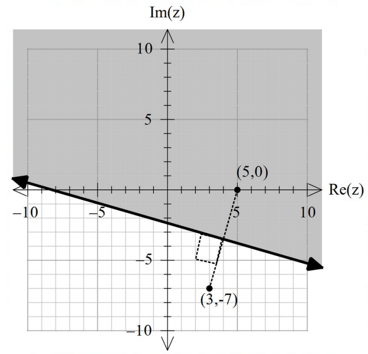
Specific behaviours

✓ determines argument of w
✓ determines modulus of w
✓ expresses in exact cartesian form

Q5 (3 & 3= 6 marks)

Sketch the locus for the following labelling important features and points.

a) $|z - 3 + 7i| \ge |z - 5|$

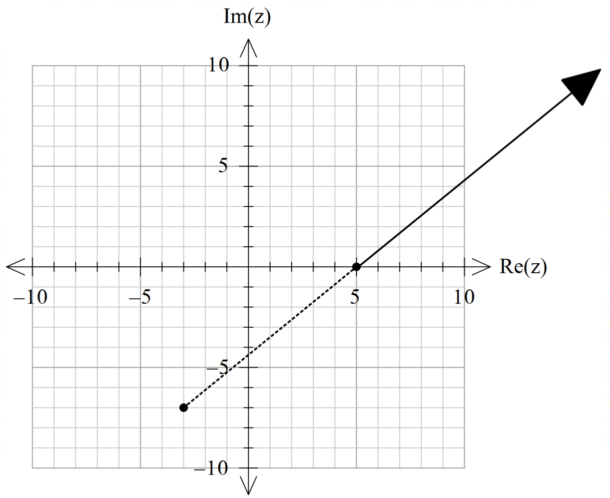


Solution

Specific behaviours

✓ plots endpoints
✓ draws perpendicular bisector & indicates right angle
✓ shades correct region

b) $|z+3+7i| = |z-5| + \sqrt{113}$

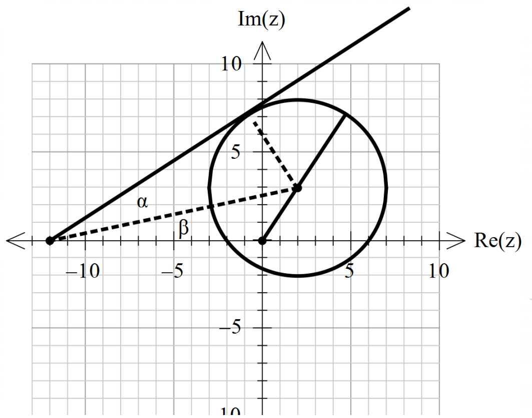


Solution
Specific behaviours
✓ plots pts (-3,-7) & (5,0) ✓ shows dotted line between ✓ plots locus line

Q6 (2 & 4 = 6 marks)

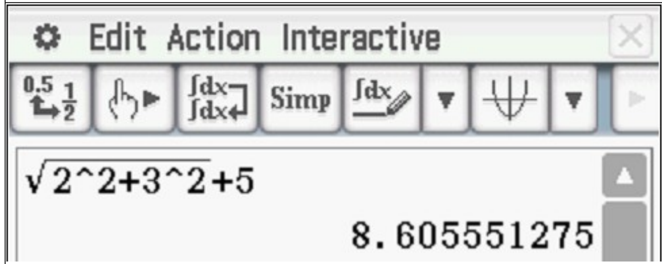
Consider the set of points z in the complex plane such that $|z-2-3i| = 5$.

a) Determine the maximum value of $|z|$



8.605551275

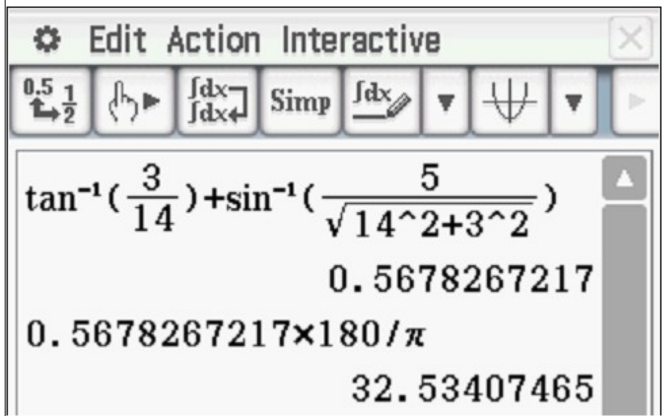
Solution



Specific behaviours
✓ determines modulus of centre ✓ adds radius (approx.)

b) Determine the maximum value of the $Arg(z+12)$.

Solution



Specific behaviours
✓ uses tangent line from (-12,0) ✓ determines alpha angle ✓ identifies right angle for beta triangle and determines two side lengths P determines sum of alpha & beta angles (see diagram)