

**Papers written by
Australian Maths
Software**

SEMESTER ONE

MATHEMATICS SPECIALIST

REVISION 2

UNIT 3

2016

SOLUTIONS

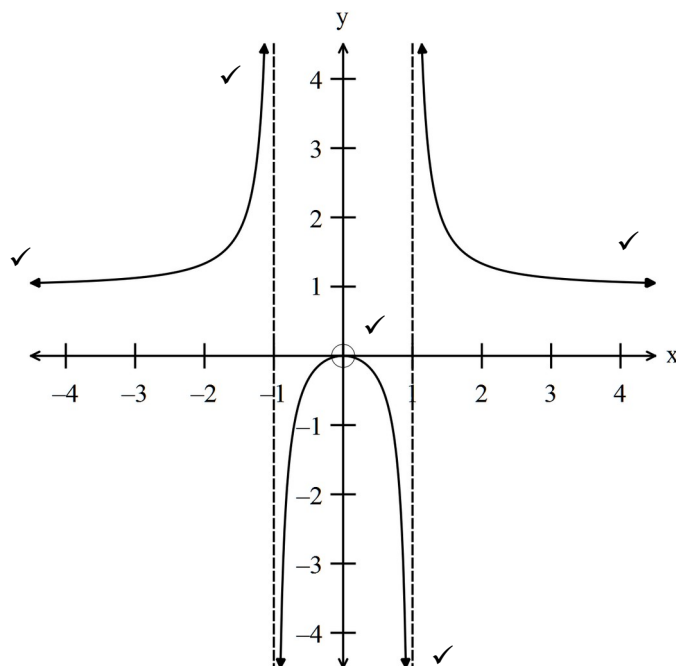
Section One

1. (6 marks)

(a) $1 + \frac{1}{x^2 - 1} = \frac{x^2 - 1 + 1}{x^2 - 1} = \frac{x^2}{x^2 - 1}$ (1)

✓

(b) (5)



2. (10 marks)

(a)

$$\begin{bmatrix} 1 & 2 & 3 & 15 \\ 1 & -1 & -1 & -3 \\ 2 & 1 & 1 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 15 \\ 0 & 3 & 4 & 18 \\ 0 & 3 & 5 & 21 \end{bmatrix}$$

$$R_1 - R_2$$

$$2R_1 - R_3$$

$$\begin{bmatrix} 1 & 2 & 3 & 15 \\ 0 & 3 & 4 & 18 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$$R_2 - R_3$$

$$-z = -3 \rightarrow z = 3$$

$$3(y) + 4(3) = 18 \rightarrow y = 2$$

$$x + 2(2) + 3(3) = 15 \rightarrow x = 2$$

The point of intersection is (2, 2, 3)

(4)

$$(b) \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 2p-7 & 2q-12 \end{bmatrix}$$

(i) Exactly one solution if $2p-7 \neq 0 \Rightarrow p \neq 3.5$ ✓✓ (2)(ii) There is no solution if $p = 3.5$ and $2q-12 \neq 0$ i.e. $q \neq 6$ ✓✓ (2)(iii) There are infinitely many solutions if $p = 3.5$ and $q = 6$ ✓✓ (2)

3. (13 marks)

$$\begin{aligned} (a) & (z - (1+2i))(z - (1-2i))(z - (3+i))(z - (3-i)) \\ &= [(z - (1+2i))(z - (1-2i))] [(z - (3+i))(z - (3-i))] \\ &= [z^2 - z(1-i+1-i) + (1+2i)(1-2i)] [z^2 - z(3-i+3-i) + (3+i)(3-i)] \\ &= (z^2 - 2z + 1 - 4i^2)(z^2 - 6z + 9 - i^2) \\ &= (z^2 - 2z + 5)(z^2 - 6z + 10) \\ &= z^4 - 8z^3 + 27z^2 - 50z + 50 \\ &\text{Therefore equation is } z^4 - 8z^3 + 27z^2 - 50z + 50 = 0 \end{aligned}$$

(3)

$$(b) \text{ Let } P(z) = z^3 - z^2 + 3z + 5$$

$$P(-1) = -1 - 1 - 3 + 5 = 0$$

$$\therefore z = -1$$

Using synthetic division with $z = -1$ You can use long division but slower

$$\begin{array}{r|rrrr} z^3 - z^2 + 3z + 5 & & & & \\ -1 & 1 & -1 & 3 & 5 \\ \hline & \downarrow & -1 & 2 & -5 \\ \hline & 1 & -2 & 5 & 0 \end{array} \quad \checkmark \text{ method}$$

$$\therefore z = -1 \text{ OR } z^2 - 2z + 5 = 0$$

$$z = \frac{2 \pm \sqrt{4 - 20}}{2}$$

$$z = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm \sqrt{16}i^2}{2}$$

$$z = \frac{2 \pm 4i}{2}$$

$$z = 1 \pm 2i \text{ or } z = -1$$

$$\therefore z = -1 \text{ so } z+1 \text{ is a factor}$$

$$\begin{array}{r} z^2 - 2z + 5 \\ z+1 \overline{) z^3 - z^2 + 3z + 5} \\ \underline{-(z^3 + z^2)} \\ -2z^2 + 3z \\ \underline{-(-2z^2 - 2z)} \\ 5z + 5 \\ \underline{-(5z + 5)} \\ 0 \end{array}$$

$$z = -1 \text{ or } z^2 - 2z + 5 = 0$$

$$(c) (i) z^4 = -16$$

$$z^4 = 16 \operatorname{cis}(\pi + n \times 2\pi) \quad n \in \mathbb{R}$$

$$z = 2 \left(\operatorname{cis}(\pi + 2n\pi) \right)^{\frac{1}{4}}$$

$$z = 2 \operatorname{cis} \left(\frac{\pi}{4} + \frac{n\pi}{2} \right)$$

$$n=0, \quad z = 2 \operatorname{cis} \left(\frac{\pi}{4} \right) = \sqrt{2} + \sqrt{2}i$$

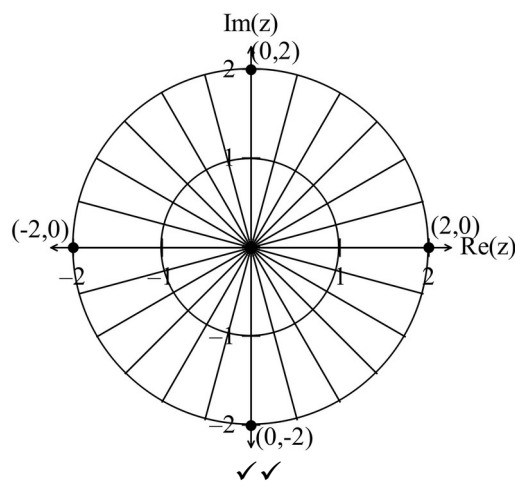
$$n=1, \quad z = 2 \operatorname{cis} \left(\frac{3\pi}{4} \right) = -\sqrt{2} + \sqrt{2}i \quad \checkmark \checkmark \quad (3)$$

$$n=2, \quad z = 2 \operatorname{cis} \left(\frac{5\pi}{4} \right)$$

$$n=-1, \quad z = 2 \operatorname{cis} \left(-\frac{\pi}{4} \right) = \sqrt{2} - \sqrt{2}i$$

$$n=-2, \quad z = 2 \operatorname{cis} \left(-\frac{3\pi}{4} \right) = -\sqrt{2} - \sqrt{2}i$$

(ii)



(2)

$$(iii) \quad z^4 = -16 \text{ is equivalent to } z = 2 \operatorname{cis} \left(\frac{\pi}{4} + \frac{n\pi}{2} \right)$$

whereas $z^4 = 16$ is equivalent to $z = 2 \operatorname{cis} \left(0 + \frac{n\pi}{2} \right)$ which means the

starting positions of the roots are different $\frac{\pi}{4}$ apart..

The roots themselves are $\frac{\pi}{2}$ apart and the roots of the two equations

start $\frac{\pi}{4}$ apart. \checkmark (1)

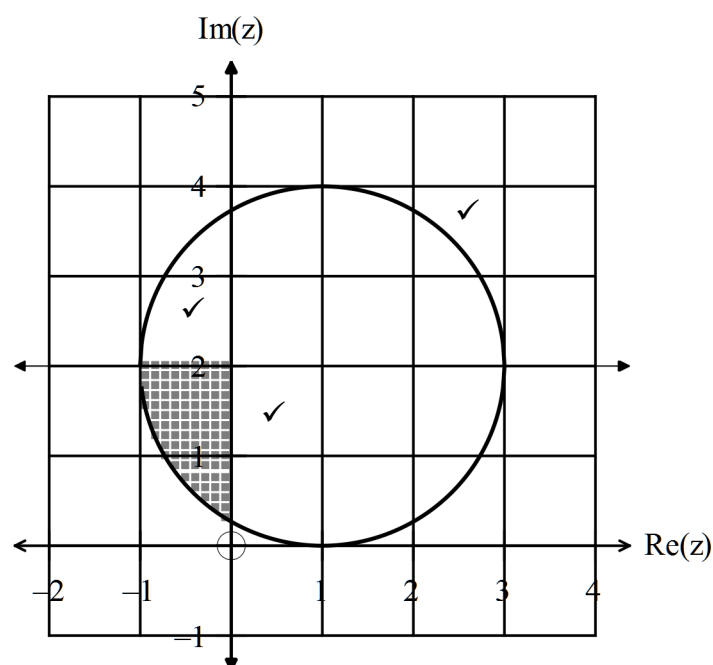
4. (13 marks)

$$\begin{aligned}
 \text{(a)} \quad \left(\operatorname{cis}\left(\frac{\pi}{4}\right) \right)^5 + (1-i)^5 &= \left(\operatorname{cis}\left(\frac{\pi}{4}\right) \right)^5 + \left(\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right) \right)^5 \quad \checkmark \\
 &= \operatorname{cis}\left(\frac{5\pi}{4}\right) + (\sqrt{2})^5 \operatorname{cis}\left(-\frac{5\pi}{4}\right) \\
 &= -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + 4\sqrt{2} \left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) \quad \checkmark \\
 &= -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + 4(-1+i) \quad \checkmark
 \end{aligned}$$

$$\left(\operatorname{cis}\left(\frac{\pi}{4}\right) \right)^5 + (1-i)^5 = \left(-4 - \frac{1}{\sqrt{2}} \right) + i \left(4 - \frac{1}{\sqrt{2}} \right)$$

(3)

(b)



$$\text{(c)} \quad |z+1| = |z-i| \quad \checkmark\checkmark \quad (2)$$

$$\begin{aligned}
 \text{(d)} \quad z &= \frac{\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)} = \operatorname{cis}\left(\frac{\pi}{3} - \frac{4\pi}{3}\right) = \operatorname{cis}(-\pi) \quad \checkmark \\
 \operatorname{mod}(z) &= 1 \quad \arg(z) = \pi \quad \checkmark
 \end{aligned}$$

$$(e) \quad z = \frac{(3-2i)}{(4+3i)} \times \frac{(4-3i)}{(4-3i)} = \frac{6-17i}{25} \quad \text{Re}(z) = \frac{6}{25} \quad (2)$$

5. (8 marks)

$$(a) \quad (g(x))^2 = (1-x)^2 \quad f^{-1}(x) = x-1 \quad \checkmark$$

$$(g(x))^2 = f^{-1}(x)$$

$$(1-x)^2 = x-1$$

$$\text{i.e. } (x-1)^2 = x-1 \quad \checkmark$$

$$(x-1)^2 - (x-1) = 0$$

$$(x-1)[x-1-1] = 0$$

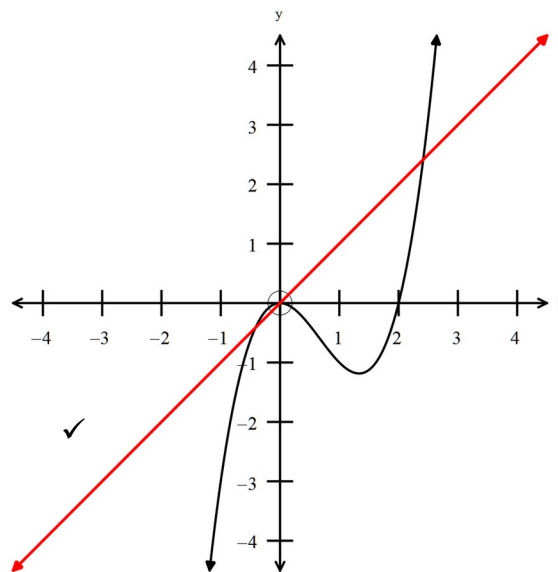
$$x=1 \text{ or } x=2 \quad \checkmark$$

$$(b) \quad (i) \quad x \leq 0 \quad \checkmark \checkmark$$

Answers will vary

(2)

$$(ii) \quad y = x \quad \checkmark$$



(2)

$$(iii) \quad f^{-1}(32) = 4 \quad \checkmark$$

(1)

END OF SECTION ONE

Section Two

6. (6 marks)

$$\begin{aligned}
 \text{(a)} \quad & \int_1^3 (2-t)\mathbf{i} + (3t^2+1)\mathbf{j} \, dt \\
 &= \left[\left(2t - \frac{t^2}{2} \right) \mathbf{i} + (t^3 + t) \mathbf{j} \right]_1^3 \quad \checkmark \\
 &= \left(\left(6 - \frac{9}{2} \right) \mathbf{i} + (27+3)\mathbf{j} \right) - \left(\left(2 - \frac{1}{2} \right) \mathbf{i} + (1+1)\mathbf{j} \right) \quad \checkmark \\
 &= 0\mathbf{i} + 28\mathbf{j} \quad \checkmark
 \end{aligned}$$

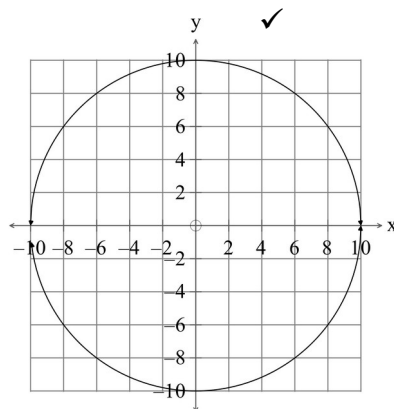
(3)

$$\begin{aligned}
 \text{(b)} \quad & \int_0^{\pi/2} (\sin(3t))\mathbf{i} + (-\cos(3t))\mathbf{j} \, dt \\
 &= - \left[\frac{\cos(3t)}{3} \mathbf{i} + \frac{\sin(3t)}{3} \mathbf{j} \right]_0^{\pi/2} \quad \checkmark \\
 &= - \frac{1}{3} \left(\left(\cos\left(\frac{3\pi}{2}\right) \mathbf{i} + \sin\left(\frac{3\pi}{2}\right) \mathbf{j} \right) - (\cos(0)\mathbf{i} + \sin(0)\mathbf{j}) \right) \quad \checkmark \\
 &= - \frac{1}{3} (-\mathbf{j} - \mathbf{i}) \\
 &= \frac{1}{3} \mathbf{i} + \frac{1}{3} \mathbf{j} \quad \checkmark
 \end{aligned}$$

(3)

7. (27 marks)

$$\begin{aligned}
 \text{(a)} \quad \text{(i)} \quad & \mathbf{r}(t) = (10\cos(t))\mathbf{i} + (10\sin(t))\mathbf{j} \\
 & x = 10\cos(t) \quad y = 10\sin(t) \quad \checkmark \\
 & \sin^2(t) + \cos^2(t) = 1 \\
 & \therefore \left(\frac{x}{10} \right)^2 + \left(\frac{y}{10} \right)^2 = 1 \\
 & x^2 + y^2 = 100 \quad \checkmark
 \end{aligned}$$



(3)

$$(ii) \quad \mathbf{r}(t) = (10 \cos(t))\mathbf{i} + (10 \sin(t))\mathbf{j}$$

$$\mathbf{v}(t) = (-10 \sin(t))\mathbf{i} + (10 \cos(t))\mathbf{j} \quad \checkmark$$

$$\mathbf{r}(t) \cdot \mathbf{v}(t) = \begin{pmatrix} 10 \cos(t) \\ 10 \sin(t) \end{pmatrix} \cdot \begin{pmatrix} -10 \sin(t) \\ 10 \cos(t) \end{pmatrix}$$

$$\mathbf{r}(t) \cdot \mathbf{v}(t) = -100 \cos(t) \sin(t) + 100 \sin(t) \cos(t) = 0 \quad \checkmark$$

$$|\mathbf{r}(t)| \neq 0, \quad |\mathbf{v}(t)| \neq 0 \quad \therefore \cos(t) = 0 \Rightarrow t = \frac{\pi}{2} \quad \checkmark$$

Therefore the position vector is always at right angles to the velocity vector.

(3)

$$(iii) \quad \mathbf{a}(t) = (-10 \cos(t))\mathbf{i} + (-10 \sin(t))\mathbf{j} \quad \checkmark$$

$$\mathbf{a}(t) = -((10 \cos(t))\mathbf{i} + (10 \sin(t))\mathbf{j}) \quad \checkmark$$

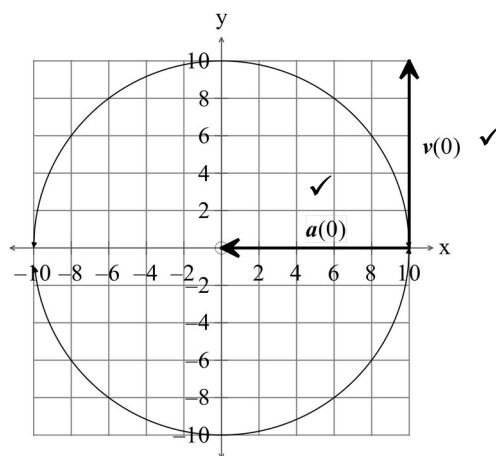
$$\mathbf{a}(t) = -\mathbf{r}(t)$$

$\mathbf{r}(t)$ is a position vector, i.e. it goes out from the origin.

Therefore $\mathbf{a}(t)$ is directed towards the origin. \checkmark

(3)

$$(iv) \quad \mathbf{r}(0) = \begin{pmatrix} 10 \\ 0 \end{pmatrix}, \quad \mathbf{v}(0) = \begin{pmatrix} 0 \\ 10 \end{pmatrix}, \quad \mathbf{a}(0) = \begin{pmatrix} -10 \\ 0 \end{pmatrix}$$



(4)

$$(v) \quad \text{Speed} = |\mathbf{v}(t)| \quad \checkmark$$

$$|\mathbf{v}(t)| = \sqrt{(-10 \sin(t))^2 + (10 \cos(t))^2} \quad \checkmark$$

$$= \sqrt{100(\sin^2(t) + \cos^2(t))}$$

$$= 10\sqrt{1}$$

$$= 10 \quad \checkmark$$

The speed is constant.

(3)

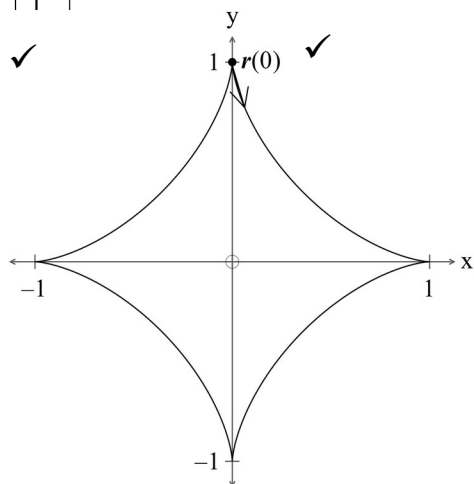
(b) (i) $\mathbf{r}(t) = (\sin^3(t))\mathbf{i} + (\cos^3(t))\mathbf{j}.$

$$\mathbf{r}(0) = (\sin^3(0))\mathbf{i} + (\cos^3(0))\mathbf{j}$$

$$\mathbf{r}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathbf{r}(0^+) = \begin{pmatrix} 0^+ \\ 1^- \end{pmatrix}$$

✓

✓



(3)

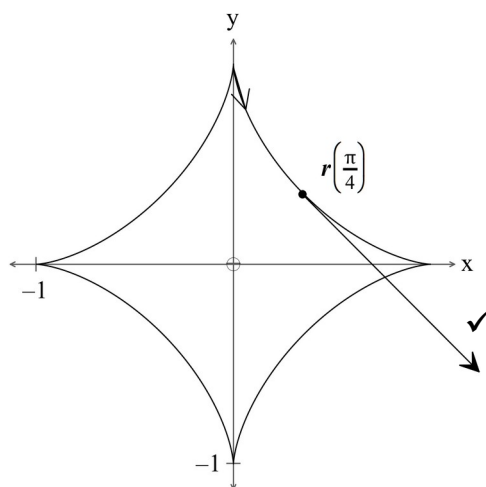
(ii) $\mathbf{v}(t) = (3\sin^2(t)\cos(t))\mathbf{i} - (3\cos^2(t)\sin(t))\mathbf{j}$ ✓✓✓

(2)

-1/error

(iii) $\mathbf{r}\left(\frac{\pi}{4}\right) = \begin{pmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{pmatrix} \approx \begin{pmatrix} 0.35 \\ 0.35 \end{pmatrix}$ ✓

$$\mathbf{v}\left(\frac{\pi}{4}\right) \approx \begin{pmatrix} 1.06 \\ -1.06 \end{pmatrix} \quad \checkmark$$



(3)

$$(iv) \quad \mathbf{v}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad t = ?$$

$$\mathbf{v}(t) = (3\sin^2(t)\cos(t))\mathbf{i} - (3\cos^2(t)\sin(t))\mathbf{j}$$

$$x = (3\sin^2(t)\cos(t)) = 1.5\sin(2t)\sin(t)$$

$$\text{If } x = 0 \quad \sin(2t) = 0 \text{ or } \sin(t) = 0$$

$$2t = 0, \pi, 2\pi \quad t = 0, \pi, 2\pi \dots$$

$$\text{If } x = 0 \quad t = 0, \frac{\pi}{2}, \pi \quad \checkmark$$

$$y = (3\cos^2(t)\sin(t)) = 1.5\sin(2t)\cos(t)$$

(3)

$$\text{If } y = 0 \quad \sin(2t) = 0 \text{ or } \cos(t) = 0$$

$$2t = 0, \pi, 2\pi \quad t = \frac{\pi}{2}, \frac{3\pi}{2} \dots$$

$$\text{If } y = 0 \quad t = 0, \frac{\pi}{2}, \pi \quad \checkmark$$

$$\text{So for } \mathbf{v}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } t > 0, \text{ first time is } t = \frac{\pi}{2} \quad \checkmark$$

8. (3 marks)

$$(a) \quad \mathbf{AG} = \mathbf{AO} + \mathbf{OG} = -\mathbf{OA} + \mathbf{OG} \quad \checkmark$$

$$= -\mathbf{a} + \mathbf{g} \quad \checkmark \quad (2)$$

$$(b) \quad \mathbf{OM} = \mathbf{OA} + \frac{1}{2}\mathbf{AG} = \mathbf{a} + \frac{1}{2}(-\mathbf{a} + \mathbf{g}) = \frac{1}{2}(\mathbf{a} + \mathbf{g}) \quad \checkmark \quad (1)$$

9. (13 marks)

$$(a) \quad (i) \quad C(-1, 4, 0) \quad \checkmark \quad r^2 = (1 - (-1))^2 + (2 - 4)^2 + (4 - 0)^2 = 4 + 4 + 16 = 24 \quad \checkmark$$

$$(x+1)^2 + (y-4)^2 + z^2 = 24 \quad \checkmark \quad (3)$$

$$(ii) \quad \mathbf{PQ} = \begin{pmatrix} -4 \\ 4 \\ -8 \end{pmatrix}, \quad \mathbf{PR} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} \quad \checkmark$$

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -4 \\ 4 \\ -8 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} \quad \checkmark \checkmark$$

Other solutions are possible (3)

$$(iii) \quad \mathbf{PQ} = \begin{pmatrix} -4 \\ 4 \\ -8 \end{pmatrix}, \quad \mathbf{PR} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$$

$$\mathbf{PQ} \times \mathbf{PR} = \begin{pmatrix} -20 \\ -12 \\ 4 \end{pmatrix} \quad \checkmark \quad (1)$$

$$(b) \quad (i) \quad \mathbf{r}_{bird}(t) = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + t \begin{pmatrix} 2.5 \\ -1 \\ -3 \end{pmatrix}$$

$$x = 4 + 2.5t, \quad y = 5 - t, \quad z = 6 - 3t$$

$$\text{If } z = 0, t = 2 \quad \text{check } x = 4 + 5 = 9$$

$$z = 6 - 6 = 0$$

$$\text{So at } (9, 0, 3) \quad t = 2 \quad \checkmark \quad (1)$$

$$(ii) \quad (9, 3, 0) \text{ to } (9, 4, 0) \quad \text{Mouse takes 1 second to get to its hole. } \checkmark \quad (1)$$

$$(iii) \quad \left| \begin{pmatrix} 2.5 \\ -1 \\ -3 \end{pmatrix} \right| = 4.03 \text{ m/s} \quad \checkmark \quad \left| \begin{pmatrix} 2.5 \\ 0 \\ -3 \end{pmatrix} \right| = 3.91 \text{ m/s} \quad \checkmark$$

$$\text{Change in speed is } 0.12 \text{ m/s} \quad \checkmark \quad (2)$$

$$(iv) \quad \text{At } t=1 \text{ the bird is at } P(6.5, 4, 3)$$

$$\mathbf{r}_{bird}(t) = \begin{pmatrix} 6.5 \\ 4 \\ 3 \end{pmatrix} + t \begin{pmatrix} 2.5 \\ 0 \\ -3 \end{pmatrix}$$

After one second (when the mouse gets to its hole) \checkmark

$$\mathbf{r}_{bird}(1) = \begin{pmatrix} 6.5 \\ 4 \\ 3 \end{pmatrix} + 1 \begin{pmatrix} 2.5 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \\ 0 \end{pmatrix} \quad \text{the bird arrives at the nest,}$$

so they both arrive at the hole together. \checkmark

Let's hope the mouse does not have a long tail!!! (2)



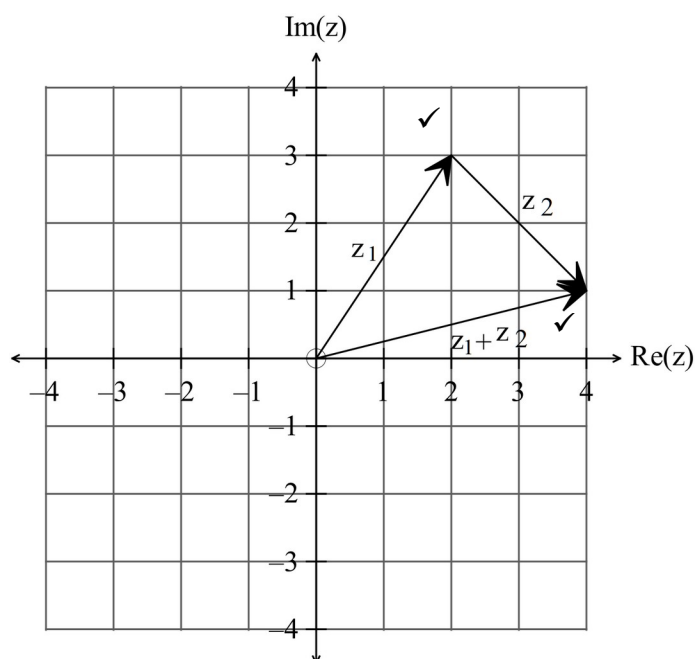
10. (12 marks)

(a)

$$\begin{aligned}
 \operatorname{Re} \left(\frac{(1+i)^6 \operatorname{cis} \left(\frac{\pi}{2} \right)}{(1-i)^2} \right) &= \operatorname{Re} \left(\frac{\left(\sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} \right) \right)^6 \operatorname{cis} \left(\frac{\pi}{2} \right)}{\left(\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right) \right)^2} \right) \\
 &= \frac{8}{2} \operatorname{Re} \left(\operatorname{cis} \left(\frac{6\pi}{4} + \frac{\pi}{2} + \frac{2\pi}{4} \right) \right) \quad \checkmark \\
 &= 4 \operatorname{Re} \left(\operatorname{cis} \left(\frac{5\pi}{2} \right) \right) \\
 &= 4 \operatorname{Re} \left(\operatorname{cis} \left(\frac{\pi}{2} \right) \right) \quad \checkmark \\
 &= 4 \operatorname{Re} \left(\cos \left(\frac{\pi}{2} \right) + i \sin \left(\frac{\pi}{2} \right) \right) \\
 &= 4 \operatorname{Re} (0 + i) \\
 &= 0 \quad \checkmark
 \end{aligned}$$

(3)

(b)



(2)

$$\begin{aligned}
 \text{(c) (i)} \quad x + yi &= \frac{2+3i}{1+i} - \frac{1+5i}{3-i} \\
 \frac{2+3i}{1+i} - \frac{1+5i}{3-i} &= \frac{2+3i}{1+i} \times \frac{1-i}{1-i} - \frac{1+5i}{3-i} \times \frac{3+i}{3+i} \quad \checkmark \quad \checkmark \\
 &= \frac{2+3i-2i-3i^2}{1-i^2} - \frac{3+15i+i+5i^2}{9-i^2} \quad \checkmark \quad \checkmark \\
 &= \frac{5+i}{2} - \left(\frac{-2+16i}{10} \right) \\
 &= \frac{5}{2} + \frac{1}{5} + i \left(\frac{1}{2} - \frac{8}{5} \right) \\
 &= \frac{27}{10} - \frac{11i}{10} \\
 x &= 2.7 \text{ and } y = -1.1 \quad \checkmark \checkmark
 \end{aligned}$$

(6)

$$\text{(ii)} \quad x + yi = \sqrt{4+3i} \quad x = 2.12, y = 0.71 \quad \checkmark$$

(1)

11. (6 marks)

$$\text{(a)} \quad \cos(\theta) = \frac{\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}}{\left\| \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right\| \left\| \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \right\|} \quad \checkmark$$

$$\begin{aligned}
 \cos(\theta) &= \frac{6+6+4}{\sqrt{4+9+1}\sqrt{9+4+16}} \\
 &= \frac{16}{\sqrt{14}\sqrt{29}} \\
 \theta &= 37.43^\circ \quad \checkmark
 \end{aligned}$$

(2)

$$\text{(b)} \quad \text{The projection of } \mathbf{a} \text{ on } \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{16}{\sqrt{29}} \quad \checkmark \quad \checkmark \quad (2)$$

$$\text{(c)} \quad \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} = 0 \quad \mathbf{p} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \quad \checkmark \checkmark \quad \text{Answers will vary} \quad (2)$$

12. (17 marks)

(a) $p(q(x)) = (x+1)(x+3)$ and $p(x) = x^2 - 1$ find $y = q(x)$.

$$p(q(x)) = (q(x))^2 - 1 \quad \checkmark$$

$$p(q(x)) = (x+1)(x+3)$$

$$= x^2 + 4x + 3$$

$$= x^2 + 4x + 4 - 1 \quad \checkmark$$

$$p(q(x)) = (x+2)^2 - 1 \quad \checkmark$$

$$\therefore q(x) = x+2 \quad \checkmark$$

(4)

$$(b) \quad (i) \quad f(x) = |x(x-1)(x+1)| \quad \checkmark$$

(1)

$$(ii) \quad f(x) = |x|(|x|-1)(|x|+1) \quad \checkmark \checkmark$$

(2)

$$(iii) \quad f(x) = \frac{1}{x(x-1)(x+1)} \quad \checkmark \checkmark \checkmark$$

(3)

$$(c) \quad (i) \quad 2f(1) = 2 \times (-3) = -6 \quad \checkmark$$

(1)

$$(ii) \quad f(|-1|) = f(1) = -3 \quad \checkmark$$

(1)

$$(iii) \quad |f^{-1}(3)| = |-1| = 1 \quad \checkmark$$

(1)

$$(iv) \quad \text{True} \quad \checkmark$$

$$(\text{as } f(-1) = 3 \text{ and } f(1) = -3$$

$$\text{and } f^{-1} \text{ is monotonically decreasing})$$

(1)

$$(v) \quad \text{True} \quad \checkmark$$

(1)

$$(d) \quad (i) \quad f(x) = e^{2x} \Rightarrow y = e^{2x}$$

$$\text{To get inverse } x = e^{2y}$$

$$2y = \ln(x)$$

$$y = f^{-1}(x) = \frac{\ln(x)}{2}$$

 \checkmark

(1)

$$(ii) \quad f(f^{-1}(f^{-1}(1))) = f^{-1}(1) = \frac{\ln(1)}{2} = 0 \quad \checkmark$$

(1)

13. (4 marks)

$$(a) \quad f(g(x)) = f(x^2) = \sqrt{1-x^2} \quad -1 \leq x \leq 1 \quad 0 \leq f(g(x)) \leq 1$$

✓ ✓

(2)

$$(b) \quad (i) \quad h(x) = 1 + e^x$$

To get inverse:

$$x = 1 + e^y$$

$$x - 1 = e^y$$

$$\ln(x - 1) = y$$

$$y = h^{-1}(x) = \ln(x - 1)$$

✓

(1)

$$(ii) \quad h^{-1}(2) = \ln(2 - 1) = 0$$

✓

(1)

14. (4 marks)

$$(a) \quad y = -2|x| + 2 = \begin{cases} -2x + 2 & \text{for } x \geq 0 \\ 2x + 2 & \text{for } x < 0 \end{cases}$$
(1)

$$(b) \quad y = |1 - x| = \begin{cases} 1 - x & \text{for } x \leq 1 \\ x - 1 & \text{for } x > 1 \end{cases} \quad \checkmark$$
(1)

(c)

$$\begin{aligned} \text{For } x > 1 \quad -2x + 2 &= x - 1 \\ 3 &= 3x \\ 1 &= x \end{aligned}$$

$$\begin{aligned} \text{For } 0 < x < 1 \quad -2x + 2 &= 1 - x \\ 1 &= x \end{aligned}$$

$$\begin{aligned} \text{For } x < 0 \quad 2x + 2 &= 1 - x \\ 3x &= -1 \\ x &= -\frac{1}{3} \end{aligned}$$

(2)

15. (3 marks)

$$a = -1, \quad b = -2, \quad c = 0$$

✓

✓

✓

(2)

16. (5 marks)

Prove that $\cos(5\theta) = 16\cos^5(\theta) - 20\cos^3(\theta) + 5\cos(\theta)$

$$\cos(5\theta) = \operatorname{Re}(\operatorname{cis}(5\theta))$$

$$= \operatorname{Re}(\cos(\theta) + i\sin(\theta))^5$$

$$= \operatorname{Re}(\cos^5(\theta) + 5\cos^4(\theta)(i\sin(\theta)) + 10\cos^3(\theta)(i\sin(\theta))^2 \\ + 10\cos^2(\theta)(i\sin(\theta))^3 + 5\cos(\theta)(i\sin(\theta))^4 + (i\sin(\theta))^5)$$

$$= \cos^5(\theta) - 10\cos^3(\theta)\sin^2(\theta) + 5\cos(\theta)\sin^4(\theta)$$

$$\text{BUT } \sin^2(\theta) = 1 - \cos^2(\theta)$$

$$\begin{aligned} \cos(5\theta) &= \cos^5(\theta) - 10\cos^3(\theta)[1 - \cos^2(\theta)] + 5\cos(\theta)[1 - \cos^2(\theta)]^2 \\ &= \cos^5(\theta) - 10\cos^3(\theta) + 10\cos^5(\theta) + 5\cos(\theta)[1 - 2\cos^2(\theta) + \cos^4(\theta)] \\ &= 11\cos^5(\theta) - 10\cos^3(\theta) + 5\cos(\theta) - 10\cos^3(\theta) + 5\cos^5(\theta) \\ \cos(5\theta) &= 16\cos^5(\theta) - 20\cos^3(\theta) + 5\cos(\theta) \end{aligned}$$

(5)

END OF SECTION TWO