PERTH MODERN SCHOOL
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Semester 1 Examination 2012

Question/Answer Booklet

MATHEMATICS 3CD

Section Two:
Calculator-assumed

Name of Student: ________________Marking key____________________

Time allowed for this section
Reading time before commencing work: 10 minutes
Working time for this section: 100 minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the student
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters
Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination

Important note to students

No other items may be used in this section of the examination. It is **your** responsibility to ensure
that you do not have any unauthorised notes or other items in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One Calculator-free	6	6	50	50	
Section Two Calculator-assumed	12	12	100	100	
			Total	150	100

Instructions to students

1 Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer. If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued. i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

2 **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be

allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

3 It is recommended that you **do not use pencil**, except in diagrams.

Section Two: Calculator-assumed
(100 marks)

Question 7 **(10 marks)**

(a) The radius of a circular oil slick is increasing at a rate of 0.4 m s^{-1} . Find the rate at which the area of the oil slick is increasing when the radius is 50 m.

$$A = \pi r^2$$

$$\therefore \quad \frac{dA}{dr} = 2\pi r \quad ✓$$

By the chain rule

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} \quad ✓$$

$$\Rightarrow \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

When $r = 50$, $\frac{dr}{dt} = 0.4$, $\frac{dA}{dt} = ?$

$$\therefore \quad \frac{dA}{dt} = 2\pi \times 50 \times 0.4 \quad ✓$$

$$= 40\pi$$

Thus the rate of increase in the area of the slick is $40\pi\ m^2/s$ ✓

(4)

(b) A sink is formed by the rotation of the curve $y = \frac{x^3}{8}$, for $y > 0$, around the Y axis. If the depth of the sink is 8cm, how many cubic centimetres of water would it hold?

(6)

$8y = x^3$

When $y = 0$, $x = 0$ ✓

When $y = 8$, $x = 4$ ✓

If cylinder is rotated around the Y axis, then we have

$\delta V = \pi x^2 \delta y$

$$\therefore \quad V = \lim_{\delta y \to 0} \sum_{y=0}^{y=8} \pi x^2 \delta y \quad ✓$$

$$= \pi \int_0^8 x^2 \, dy \quad ✓$$

$$= \pi \int_0^8 \left(2y^{\frac{1}{3}}\right)^2 dy$$

$$= \frac{384\pi}{5} \quad ✓$$

$\therefore$ *A sink can hold* $\frac{384\pi}{5}$ **cm**^{3} *of water* ✓

Question 8 **(7 marks)**

(a) It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth '*y*' of fluid in the tank ,'*t*' hours after the valve is opened is given by

$y = 6\left(1 - \frac{t}{12}\right)^2$ metres.

(i) Find the rate $\frac{dy}{dt}$ m/hour at which the tank is draining at time, *t*. (2)

Solution
$\frac{dy}{dt} = -\left(1 - \frac{t}{12}\right)$
Specific behaviours
✓✓

(ii) When is the fluid in the tank draining fastest and slowest?

What are the values of $\frac{dy}{dt}$ at these times? (2)

Solution
Slowest when t = 12, $\frac{dy}{dt} = 0$

Fastest when t = 0, $\frac{dy}{dt} = -1$
Specific behaviours
✓ times ✓ values of $\frac{dy}{dt}$

Question 8 (continued)

(b) If the volume of a cylinder is given by $V = 2\pi r^3$, find the appropriate percentage change in V when r changes by $\frac{1}{2}$ % (3)

Solution
$\frac{\delta r}{r} = \frac{5}{1000}$ $\frac{dV}{dr} = 6\pi r^2$ $\delta V = \frac{dV}{dr} \times \delta r$ $\therefore \frac{\delta V}{V} = \frac{6\pi r^2}{2\pi r^3} \times \delta r$ $\frac{\delta V}{V} = 3 \times \frac{\delta r}{r}$ $\therefore \frac{\delta V}{V} = 3 \times \frac{5}{1000} \times 100\% = 1.5\%$
Specific behaviours
✓ $\frac{dV}{dr} = 6\pi r^2$ ✓ $\frac{\delta V}{V} = \frac{6\pi r^2}{2\pi r^3} \times \delta r = 3 \times \frac{\delta r}{r}$ ✓ correct answer of 1.5 %

Question 9 **(10 marks)**

(a) Give two reasons why the following cannot be a probability distribution. (2)

x	3	1	2	3	5	0
P(X=*x*)	0.05	0.1	0.4	0.1	0.2	0.3

Solution
- Different probabilities for same value of x i.e. P(X=3) = 0.05 and P(X=3) = 0.1 - Sum of all the probabilities is greater than 1 (1.15)
Specific behaviours
✓✓ 1 mark each

(b) The probability distribution of *x* where random variable, X is the sum of the uppermost numbers when two fair die are rolled is tabulated below.

x	2	3	4	5	6	7	8	9	10	11	12
P(X=*x*)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Find

(i) $P(X > 3)$ (2)

Solution
$\frac{33}{36} = \frac{11}{12}$
Specific behaviours
✓✓

(ii) $P(X < 10 \mid X > 3)$ (2)

Solution

$\frac{P(3<X<10)}{P(X>3)}=\frac{27/36}{11/12}=\frac{9}{11}$
Specific behaviours
✓✓

(iii) If event A is $X>3$ and event B is $X<10$, are these two events independent? Justify your answer. (4)

Solution
$P(A\cap B)=\frac{27}{36}$, $P(A)=\frac{33}{36}, P(B)=\frac{30}{36}$ Now $\frac{11}{12}\times\frac{30}{36}=\frac{55}{72}\neq\frac{27}{36}$ i.e. $P(A)\times P(B)\neq P(A\cap B)$ $\therefore$ A and B are not independent events
Specific behaviours
✓ $P(A\cap B)=\frac{27}{36}$ ✓ $P(A)=\frac{33}{36}, P(B)=\frac{30}{36}$ ✓ shows that $P(A)\times P(B)\neq P(A\cap B)$ ✓ states not independent

Question 10 **(7 marks)**

(a) The function *f(x)* is differentiable for all $x\varepsilon R$ and satisfies the conditions

f′(x) < 0 where *x* < *2*

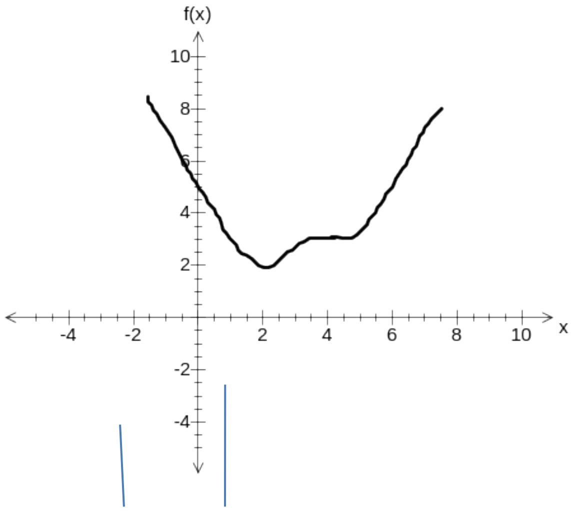
f′(x) = 0 where *x* = *2*

f′(x)= 0 where *x* = *4*

f′(x) > 0 where *2* < *x* < *4*

f′(x) > 0 where *x* > *4*

(i) Draw a sketch of this function *f(x).* (3)



Solution
✓ shape ✓ turns at x = 2 ✓ point of inflection at x = 4
Specific behaviours
As above

(ii) State whether the following statement is true or false.
"The graph *f(x)* has a stationary point of inflection where *x=4*". (1)

Solution
True
Specific behaviours
✓ or X

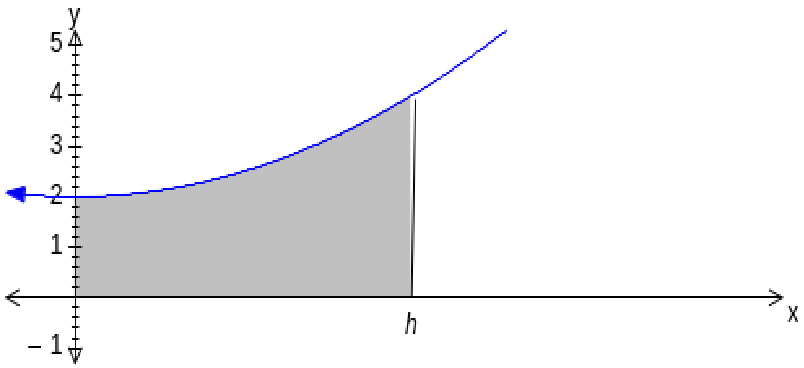
(b) If $\int_0^a f(x)\,dx = a$, find $2\int_0^{5a}\left[f\left(\frac{x}{5}\right)+3\right]\,dx$ (3)

Solution
Now if $u=\frac{x}{5}$, $5u=x$ $\int_0^{5a} f\left(\frac{x}{5}\right)dx = 5\int_0^{a} f(u)du = 5a$ $2\int_0^{5a}\left[f\left(\frac{x}{5}\right)+3\right]\,dx$

$= 2\int_0^{5a} f\left(\frac{x}{5}\right)\, dx + 2\int_0^{5a} 3\, dx$ $= 2(5a) + 2(15a) = 40a$
Specific behaviours
✓ $\int_0^{5a} f\left(\frac{x}{5}\right) dx = 5\int_0^{a} f(u) du = 5a$ ✓ $2\int_0^{5a} f\left(\frac{x}{5}\right)\, dx + 2\int_0^{5a} 3\, dx$ ✓ integrates correctly

Question 11 **(7 marks)**

A section of the function $y = 0.5x^2 + 2$ is graphed below, along with a shaded region
enclosed by the function, the axes and the line $x = h$.



(a) Show that the volume of the solid generated when the shaded region is rotated about the x-axis is given by V = $\pi[0.05h^5 + \frac{2}{3}h^3 + 4h]$. (2)

$V = \pi \int_0^h (0.5x^2 + 2)^2 \, dx$

$= \pi \int_0^h \frac{x^4}{4} + 2x^2 + 4 \, dx$ ✓

$= \pi \left[\frac{x^5}{20} + \frac{2x^3}{3} + 4x \right]_0^h$ ✓

$= \pi[0.05h^5 + \frac{2}{3}h^3 + 4h]$

(b) If h increases at the rate of 0.5 units per second, find an expression, in terms of h, for the *rate of change* of the volume of the solid generated when the shaded region is rotated about the x-axis.

(2)

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ ✓

$= \frac{\pi}{2}(0.25h^4 + 2h^2 + 4)$ ✓

(c) Use the incremental formula, $\delta V \approx \frac{dV}{dh}\delta h$,
to estimate the change in volume when h increases from 3 to 3.01.

(3)

$\delta V = \frac{dV}{dh}$ ¿ δh

$= \pi(0.25h^4 + 2h^2 + 4)$ ¿ δh ✓

When h = 3, δh = 0.01; $\Rightarrow$ δV ¿ 1.327 ✓✓

Question 12 **(9 marks)**

(a) A company produces fruit balls coated in either dark chocolate or milk chocolate. A large number of these fruit balls are placed in a box. Twenty per cent of the fruit balls in the box are coated with dark chocolate.

(i) Calculate $C_4^{10}(0.2)^4(0.8)^6$ (1)

Solution
0.08808
Specific behaviours
✓ or X

(ii) A random sample of ten fruit balls is taken from the box.

Explain the meaning of $C_4^{10}(0.2)^4(0.8)^6$ with respect to this sample. (2)

Solution
In a sample of 10 fruit balls, the probability of picking exactly 4 coated in dark chocolate is approximately 0.0881
Specific behaviours
✓✓

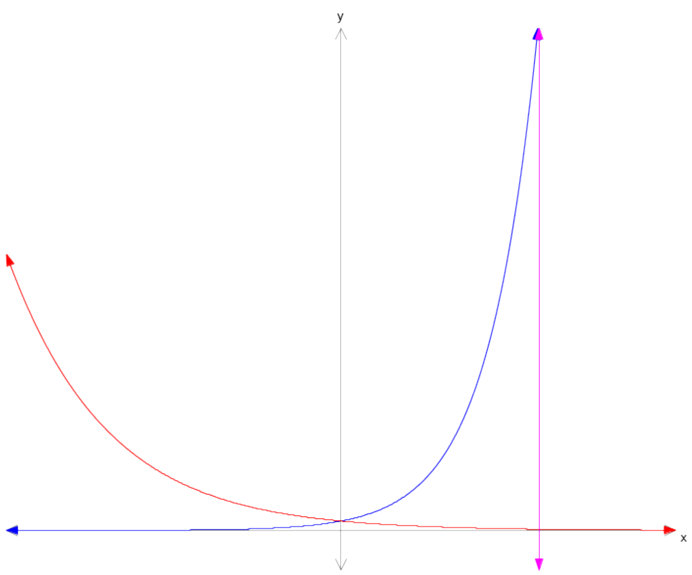
(b) (i) Find *n* given that $C_0^n(0.2)^0(0.8)^n$ =0.167 772 16 (1)

Solution
Using CAS, n = 8
Specific behaviours
✓

(ii) Explain the meaning of your answer to part (b)(i) with respect to the fruit balls. (2)

Solution
The probability of picking no dark chocolate fruit ball from 8 is 0.16777216
Specific behaviours
✓picking none ✓ from 8

(c) The curve $y = e^{2e}$ and $y = e^{-x}$ intersect at the point (0, 1) as shown in the diagram.



Find the area enclosed by the curves and the line *x=2.*
Leave your answer in terms of '*e*'.

(3)

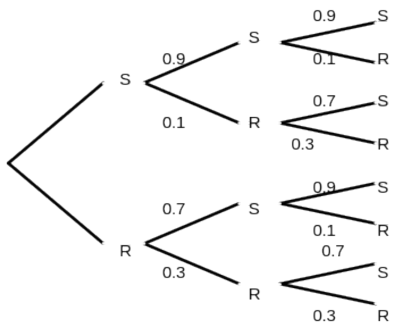
Solution
Required area = $\int_0^2 e^{2x}dx - \int_0^2 e^{-x}dx$ $= \left[\frac{e^{2x}}{2} - \left(-e^{-x}\right)\right]_0^2$ $= \frac{e^4}{2} + e^{-2} - \frac{3}{2}$
Specific behaviours
✓ $\int_0^2 e^{2x}dx - \int_0^2 e^{-x}dx$ ✓ integrates each term correctly ✓ substitutes limits of integration to get exact value of $\frac{e^4}{2} + e^{-2} - \frac{3}{2}$

Question 13 **(8 marks)**

Adam paints garden gnomes to sell. He sends the garden gnomes to his father (a qualified quality controller) in the order of completion, who classifies them as either 'Superior' or 'Regular', depending on the quality of their finish.

If the garden gnome is Superior, then the probability that the next garden gnome is superior is 0.9. If the garden gnome is Regular, then the probability that the next garden gnome is superior is 0.7.

(a) If the first garden gnome inspected is Superior, find the probability that the third gnome is Regular. (2)

Solution
 P(SSR) + P(SRR) = (0.9)x(0.1) + (0.1x(0.3)=0.12
Specific behaviours
✓✓

(b) If the first garden gnome inspected is Superior, find the probability that the next three gnomes are Superior. (1)

Solution
0.9x0.9x0.9 = 0.729
Specific behaviours
✓ or x

(c) A group of 3 consecutive garden gnomes is inspected and the first is a Regular. It is also found that of these three gnomes,

P(no Superior) = 0.09
P(1 Superior) = 0.28
P(2 Superior) = 0.63

Find the expected number of these gnomes that will be Superior. (2)

Solution
Expected number = 0 x 0.09 + 1 x 0.28 + 2 x 0.63 = 1.54
Specific behaviours
✓ calculation ✓ correct answer of 1.54

Question 13 (continued)

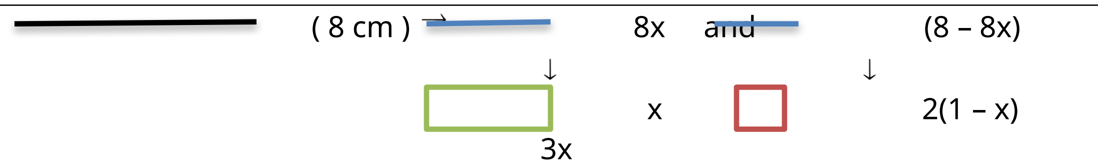
(d) Adam's little brother, Brodie joins in this business venture. The probability that any one of
Brodie's painted garden gnomes is Regular is 0.8. He wants to ensure that the probability that he paints at least two Superior is at least 0.9. Calculate the minimum number of garden
gnomes that Brodie would need to paint to achieve this aim. (3)

Solution
P(R) = 0.8, P(S) = 0.2 $P(X \geq 2) \geq 0.9$ $1 - P(X \leq 1) \geq 0.9$ $P(X \leq 1) \leq 0.1$ i.e. ${}^nC_0(0.2)^0(0.8)^n + {}^nC_1(0.2)^1(0.8)^{n-1} \leq 0.1$ Using CAS to solve, n= 17.95 ∴ minimum number is 18
Specific behaviours

Question 14 **(9 marks)**

A piece of wire 8cm long is cut into two unequal parts. One part is used to form a rectangle that has a length three times its width. The other part of the wire is used to form a square.

(i) If the width of the rectangle is *x* units, determine an equation that will give the sum of the areas of the rectangle and the square in terms of *x*. (4)

Solution

Area, A = $3x^2 + (2 - 2x)^2$ $= 7x^2 - 8x + 4$
Specific behaviours
✓✓ expressions for 8x and (8 – 8x) ✓✓ areas of rectangle and square

(ii) Using Calculus find the length of each part of the wire when the sum of the areas is a minimum. (5)

Solution
$A = 7x^2 - 8x + 4$ $\frac{dA}{dx} = 14x - 8$ $\frac{d^2A}{dx^2} = 14 \Rightarrow Min$ $14x - 8 = 0$ $x = \frac{4}{7}$ Lengths of each part of wire are $x = 4\frac{4}{7}\left(\frac{32}{7}\right) \quad and \quad 3\frac{3}{7}\left(\frac{24}{7}\right)$
Specific behaviours
✓ first derivative ✓ second derivative to confirm x value gives a minimum area ✓ x value ✓✓ the two lengths of $\left(\frac{32}{7}\right)$ and $\left(\frac{24}{7}\right)$

Question 15 **(11 marks)**

Nuts and Bolts Company manufactures 120mm bolts which are normally distributed with a mean length of 120mm and a standard deviation of 1mm. Only bolts which are between 118.6mm and 121.4mm pass inspection and are packaged as 120mm bolts.

(a) Find the probability of a randomly selected bolt being an acceptable length. (2)

Solution
$P(118.6 \le X \le 121.4) = NCDF(118.6, 121.4, 1, 120) = 0.838487$
Specific behaviours
✓✓

(b) Find the expected number of acceptable bolts in a batch of 100 000 (1)

Solution
$0.838487 \times 100000 = 83849$
Specific behaviours
✓ or X

(c) Is this a reasonable outcome for the company? Justify your answer. (2)

Solution
% of acceptable bolts = $\frac{83849}{100000} \times 100\% = 83.85\%$ % of unacceptable bolts = 16.15% which is too high – too much waste $\therefore$ outcome is not really reasonable.
Specific behaviours
✓ % of unacceptable bolts ✓ outcome not reasonable

(d) A new quality controller suggests adjusting the settings on the machines so that the

standard deviation becomes 0.85mm and that only the shortest 5% and the longest 5% of the bolts are rejected.

(i) Find the new minimum and maximum acceptable lengths correct to the nearest 0.1mm. (3)

Solution
$\bar{x} = 120, \quad \sigma = 0.85$ $P(a < X < b) = 0.9$ INV NormCDF(‘c’,0.9,0.85,120) results in a = 118.6 and b = 121.4
Specific behaviours
✓ uses inverse NormCDF ✓✓ “a” and “b” values

(ii) Do the packages now contain bolts that are more consistent in length?

(1)

Solution

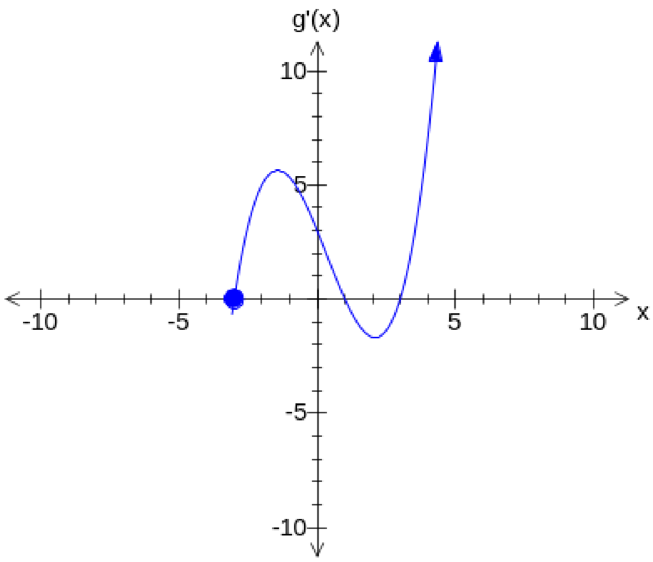
Range of size of bolts is the same at 118.6 mm to 121.4 mm ∴ in terms of consistency the bolts have the same range.
Specific behaviours
✓ valid reason

(iii) Is the manufacturer better off? Justify. (2)

Solution
Yes, as wastage is reduced from 16.15% to 10%. i.e. 6150 more bolts will be accepted
Specific behaviours
✓ yes ✓ justification

Question 16 **(7 marks)**

The graph of g′(x) is given below.



(a) What can be said about the gradient of the function g(x) between x =-3 to x= 1? (1)

Solution
Gradient is positive
Specific behaviours
✓ or X

(b) When does the function, g(x) have a negative gradient? (2)

Solution
1 < x < 3
Specific behaviours
✓✓ correct interval

(c) State an equation for the tangent to the graph of g(x) at x = 3. (1)

Solution
Horizontal line y = k where k is a constant
Specific behaviours
✓ or X

(d) Find the value of *x* at which $g(x)$ has a relative maximum for $-3 \le x \le 4$ (1)

Solution
x = 1
Specific behaviours
✓ or X

(e) Find the *x*-coordinate of each point of inflection of the graph of $g(x)$ for $-3 \le x \le 4$

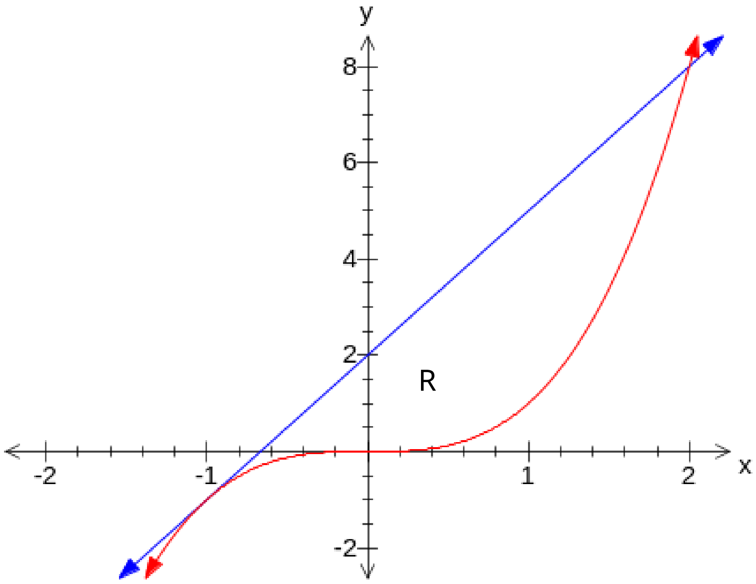
(2)

Solution
x = -1.5 and x = 2
Specific behaviours
✓✓ 1 mark each

Question 17 **(9 marks)**

(a) Shade the region, R bounded by the curves $y = x^3$, $y = 3x + 2$, and $x = 0$ in the diagram.

Find the area of the region R, showing all working steps. (4)



Solution
Area of region R = $\int_{-1}^{0} 3x + 2 - x^3 \, dx$ $= \left[\frac{3x^2}{2} + 2x - \frac{x^4}{4}\right]_{-1}^{0}$ $= \frac{3}{4}$ units^2
Specific behaviours
✓ correct shading of R ✓ $\int_{-1}^{0} 3x + 2 - x^3 \, dx$ ✓ integrates each term correctly ✓ correct answer

Question 17 (continued)

(b) A group of anthropologists found that human tooth size is continuing to decrease, such that

$$\frac{dS}{dt} = kS$$

In Northern Europeans, for example, tooth size reduction now has a rate of 1% per 1000 years.

(i) If ***t*** represents time in years and ***S*** represents tooth size, find the value of **k**.

(2)

Solution
$S = S_0 e^{kt}$ $0.99S_0 = S_0 e^{1000k}$ i.e. $0.99 = e^{1000k}$ k = -0.000 01
Specific behaviours
✓ $0.99S_0 = S_0 e^{1000k}$ ✓ solves correctly for k

(ii) In how many years will human tooth size be 90% of their present size?

(2)

Solution
$0.9 = e^{-0.00001t}$ t = 10 536 years
Specific behaviours
✓ $0.9 = e^{-0.00001t}$ ✓ solves correctly for t

(iii) What will be our descendant's tooth size 20 000 years from now? (1)

(as a percentage of our present tooth size)

Solution
$S = S_0 e^{-0.00001x20000}$ $S = 0.8187S_0$ $\therefore \sim 82\%$

Specific behaviours
✓ or X

Question 18 **(7 marks)**

A particle is moving in rectilinear motion with acceleration a at any time t, in m s^{-2}, given as

$$a = 6t - 1$$

Initially the particle is at the origin with a velocity of -2 m/s.

Determine:

(a) the velocity of the particle at any time t. [1]

$v = 3t^2 - t + c$
$v(0) = -2 \rightarrow c = -2$
$\therefore \quad v(t) = 3t^2 - t - 2$ ✓

(b) when the particle is again at the origin. [2]

$x = t^3 - \frac{1}{2}t^2 - 2t + k$ ✓

$x(0) = 0 \rightarrow k = 0$

$\therefore \quad x(t) = t^3 - \frac{1}{2}t^2 - 2t$

$\therefore \quad x = 0$ when $t(t^2 - 0.5t - 2) = 0$

$\therefore \quad t = 1.69$ s ✓

(c) the minimum velocity of the particle. [2]

$a = 0$ when $t = \frac{1}{6}$ ✓

$\therefore \quad v\left(\frac{1}{6}\right) = -2{\cdot}08$ m/s ✓

(d) the total distance travelled by the particle in the first three seconds. [2]

$$\text{Total distance} = \int_0^3 |\,3t^2 - t - 2\,|\,dt \quad ✓$$

$$= 1.5 + 16.5 + 1.5$$

$$= 19.5 \text{ m} \quad ✓$$