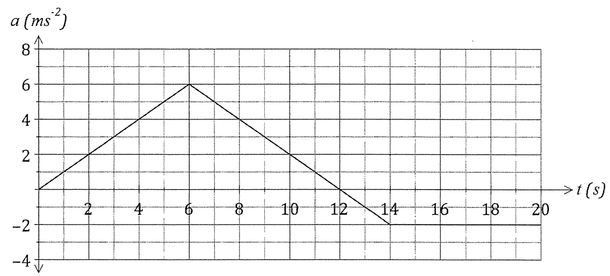
Question 20 **(8 marks)**

A particle, initially stationary and at the origin, moves subject to an acceleration, a ms^{-2}, as shown in the graph below for $0 \le t \le 20$ seconds.



(a) Determine the velocity of the object when

(i) $t = 6$. (1 mark)

$v = \frac{1}{2} \times 6 \times 6$

$= 18$ m/sec

(ii) $t = 20$. (2 marks)

$v = 18 + 18 - 2 - 12$

$= 22$ m/sec

(b) At what time is the velocity of the body a maximum, and what is the maximum velocity? (2 marks)

$t = 12$ $\quad v_{MAX} = 36$ m/sec

(c) Determine the distance of the particle from the origin after 3 seconds. (3 marks)

$a = t$

$v = \frac{t^2}{2} + c \qquad c = 0$

$x = \frac{t^3}{6} + c \qquad c = 0$

$x(3) = \frac{27}{6} \qquad (= 4.5)$

End of questions

TRINITY COLLEGE

Semester Two Examination, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section One: Calculator-free

If required by your examination administrator, please place your student identification label in this box

Student Number: In figures

In words

Your name SOLUTIONS

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Section One: Calculator-free **35% (52 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(7 marks)**

A particle leaves the origin when $t = 1$ and moves in a straight line with velocity at any time t seconds, where $t \geq 1$, given by

$$v(t) = \frac{t^2}{4} + \frac{4}{t}\,\text{ms}^{-1}$$

(a) Determine the time when the acceleration of the particle is zero. (3 marks)

$a(t) = \frac{t}{2} - \frac{4}{t^2}$

$\frac{t}{2} - \frac{4}{t^2} = 0$

$t^3 = 8$

$t = 2$

(b) Determine the exact displacement of the particle from the origin when $t = 4$. (4 marks)

$x(t) = \int \frac{t^2}{4} + \frac{4}{t}\,dt$

$= \frac{t^3}{12} + 4\ln t + c$

$t=1$
$x=0$

$0 = \frac{1}{12} + 0 + c \qquad c = -\frac{1}{12}$

$x(t) = \frac{t^3}{12} + 4\ln t - \frac{1}{12}$

$x(4) = \frac{64}{12} + 4\ln 4 - \frac{1}{12}$

$= \frac{63}{12} + 4\ln 4$

Question 19 **(7 marks)**

The moment magnitude scale M_w is used by seismologists to measure the size of earthquakes in terms of the energy released. It was developed to succeed the 1930's-era Richter magnitude scale.

The moment magnitude has no units and is defined as $M_w = \frac{2}{3}\log_{10}(M_0) - 10.7$, where M_0 is the total amount of energy that is transformed during an earthquake, measured in dyn·cm.

(a) On 28 June 2016, an estimated 2.82×10^{21} dyn·cm of energy was transformed during an earthquake near Norseman, WA. Calculate the moment magnitude for this earthquake. (1 mark)

3.6

(b) A few days later, on 8 July 2016, there was another earthquake with moment magnitude 5.2 just north of Norseman. Calculate how much energy was transformed during this earthquake. (2 marks)

$5.2 = \frac{2}{3}\log x - 10.7$

$x = 7.08 \times 10^{23}$ dyn·cm

(c) Show that an increase of 2 on the moment magnitude scale corresponds to the transformation of 1000 times more energy during an earthquake. (4 marks)

1 $\frac{2}{3}\log x - 10.7$

2 $\frac{2}{3}\log y - 10.7$

2 − 1 $\quad 2 = \frac{2}{3}\log y - 10.7 - \left(\frac{2}{3}\log x - 10.7\right)$

$2 = \frac{2}{3}(\log y - \log x)$

$3 = \log \frac{y}{x}$

$10^3 = \frac{y}{x}$

$1000 = \frac{y}{x}$

Question 18 **(7 marks)**

From a random sample of n people, it was found that 54 of them subscribe to a streaming music service. A symmetric confidence interval for the true population proportion who subscribe is $0.1842 < p < 0.2958$.

(a) Determine the value of n, by first determining the mid-point of the interval. (3 marks)

$$\frac{0.1842 + 0.2958}{2} = 0.24$$

$$p = 0.24 = \frac{54}{n}$$

$$n = 225$$

(b) Determine the confidence level of the interval. (4 marks)

$$\text{STANDARD ERROR} = \sqrt{\frac{0.24(1-0.24)}{225}}$$

$$= 0.02847$$

$$0.24 + z \times 0.02847 = 0.2958$$

$$z = 1.96$$

95% CONFIDENCE INTERVAL

Question 2 **(8 marks)**

(a) Calculate $f'(0)$ when $f(x) = e^{2x}(1+5x)^3$. (3 marks)

$$f'(x) = 2e^{2x}(1+5x)^3 + e^{2x}3(1+5x)^2 5$$

$$f'(0) = 2 \times 1 + 1 \times 3 \times 5$$

$$= 17$$

(b) Determine $\frac{d}{dx}\int_x^5 \sqrt{t^2+1}\,dt$. (2 marks)

$$y = -\int_5^x \sqrt{t^2+1}\,dt$$

$$y' = -\sqrt{x^2+1}$$

(c) Given $f'(x) = (1-2x)^4$ and $f(1) = -1$, determine $f(x)$. (3 marks)

$$f(x) = \frac{(1-2x)^5}{(-2)5} + c$$

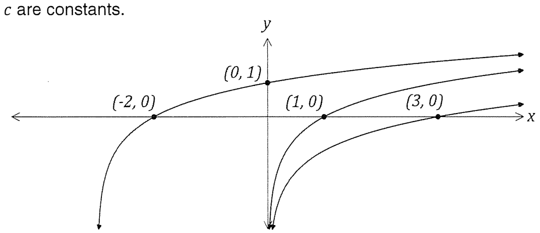
$$-1 = \frac{1}{10} + c \qquad c = -\frac{11}{10}$$

$$f(x) = -\frac{(1-2x)^5}{10} - \frac{11}{10}$$

Question 3 **(8 marks)**

(a) The function f is defined by $f(x) = \log_a x$, $x > 0$, where a is a constant, $a > 1$.

The graphs shown below have equations $y = f(x)$, $y = f(x+b)$ and $y = f(x) + c$, where b and c are constants.



Determine the values of the constants a, b and c. (4 marks)

$f(x+b)$ ONLY FUNCTION TO PASS THROUGH $(-2,0)$

$0 = f(-2+b) \quad \therefore b = 3$

USING $(0,1)$ $\quad 1 = \log_a(0+3) \quad \therefore a = 3$

$\log_3 1 = 0$ SO $f(x)$ MUST PASS THROUGH $(1,0)$

$f(x) + c$ PASSES THROUGH $(3,0)$ $\quad \therefore 0 = \log_3 3 + c$

$\therefore c = -1$

(b) Determine

(i) the equation of the asymptote of the graph of $y = ln(x-3) - 2$. (1 mark)

$x = 3$

(ii) the coordinates of the y-intercept of the graph of $y = \log_2(x+8) - 5$. (3 marks)

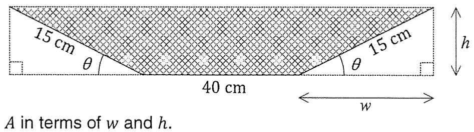
$\log_2 8 - 5 = 3\log_2 2 - 5$

$= -2$

$(0, -2)$

Question 17 **(7 marks)**

A trough for holding water is to be formed by taking a length of metal sheet 70 cm wide and folding 15 cm on either end, up through an angle of θ. The following diagram shows the cross-section of the trough with the cross-sectional area, A, shaded.



(a) Determine A in terms of w and h. (1 mark)

$A = 40h + wh$

(b) Show that $A = 600 \sin\theta + 225 \sin\theta \cos\theta$. (2 marks)

$w = 15\cos\theta \qquad h = 15\sin\theta$

$A = 40 \times 15\sin\theta + 15\cos\theta \times 15\sin\theta$

$= 600\sin\theta + 225\sin\theta\cos\theta$

(c) Use calculus to determine the maximum possible cross-sectional area. (4 marks)

$\frac{dA}{d\theta} = 600\cos\theta + 225(\cos^2\theta - \sin^2\theta)$

$\frac{dA}{d\theta} = 0$ when $\theta = 1.26$

$A(1.26) = 636.8$

$\approx 637 \text{ cm}^2$

(b) The stationery company that supplies pens to the conference centre claim that no more than 3 in 50 pens fail to write. Use your previous working to comment on the validity of this claim. (2 marks)

$\frac{3}{50} = 0.06$

INSIDE INTERVAL IN (a)

CLAIM VALID

(c) Comment on how the margin of error would change in (a) (ii) if

(i) the quality of the pens had been better. (1 mark)

DECREASE

(ii) the required level of confidence decreased. (1 mark)

DECREASE

Question 4 **(7 marks)**

A curve has equation $y = 2x^5 - 5x^4 + 10$.

(a) Point A lies on the curve at $(-1, 3)$. Use the increments formula $\delta y \approx \frac{dy}{dx} \times \delta x$ to estimate the y-coordinate of point B that has an x-coordinate of -0.99. (3 marks)

$\delta y \approx \frac{dy}{dx} \times \delta x$

$\approx (10x^4 - 20x^3) \times (0.01)$

$(x = -1)$ ≈ 0.3

y COORD $= 3 + 0.3$

$= 3.3$

(b) Point C also lies on the curve, at $(2, -6)$. Determine whether C is either a minimum or maximum point of the curve. Justify your answer. (4 marks)

$\frac{dy}{dx} = 10x^4 - 20x^3$

$x = 2$ $\frac{dy}{dx} = 0$ $\therefore$ T.P.

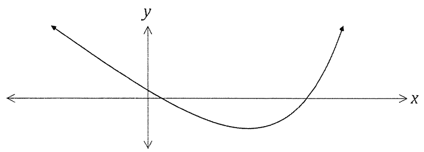
$\frac{d^2y}{dx^2} = 40x^3 - 60x^2$

$x = 2$ $\frac{d^2y}{dx^2} = +ve$

$\therefore$ MIN

Question 5 **(5 marks)**

The graph of $y = e^{2x-1} - 4x$ has a single stationary point, as shown on the graph below.



Determine the exact coordinates of the stationary point. (5 marks)

$$\frac{dy}{dx} = 2e^{2x-1} - 4$$

$$2e^{2x-1} = 4$$

$$e^{2x-1} = 2$$

$$2x-1 = \ln 2$$

$$x = \frac{\ln 2}{2} + \frac{1}{2}$$

$$y = e^{2\left(\frac{\ln 2}{2}+\frac{1}{2}\right)-1} - 4\left(\frac{\ln 2}{2} + \frac{1}{2}\right)$$

$$= e^{\ln 2 + 1 - 1} - 2\ln 2 - 2$$

$$= 2 - 2\ln 2 - 2$$

$$= -2\ln 2$$

$$\left(\frac{\ln 2}{2} + \frac{1}{2},\ -2\ln 2\right)$$

Question 16 **(9 marks)**

The management at a conference centre was concerned about the quality of the free pens that it provided in its meeting rooms. A staff member tested a random sample of 150 pens and found that 18 of them fail to write.

(a) If p is the true proportion of pens that fail to write and $\hat{p}$ is the corresponding sample proportion, use the above sample to determine

(i) $\hat{p}$. (1 mark)

$$\frac{18}{150} \quad (= 0.12)$$

(ii) the approximate margin of error for a 98% confidence interval for p. (3 marks)

98% ie $z = 2.326$

$$\text{MARGIN OF ERROR} = \sqrt{\frac{0.12(1-0.12)}{150}} \times 2.326$$

$$= 0.0617$$

(iii) an approximate 98% confidence interval for p. (1 mark)

$$0.12 \pm 0.0617$$

$$0.0583 < p < 0.1817$$

Question 15 **(8 marks)**

An analysis of the number of dogs registered by each household within a suburb resulted in the following information:

Number of dogs registered	0	1	2	3 or more
Percentage of households	21	44	27	8

(a) A council worker selects households at random to visit. What is the probability that the first five households visited all have at least one dog registered? (2 marks)

$p = 1 - 0.21$
$= 0.79$

$0.79^2 = 0.3077$

(b) A random sample of 40 households within the suburb is selected.

Determine the probability that the sample contains:

(i) exactly 6 households with no dogs registered. (2 marks)

$X \sim bin(40, 0.21)$

$P(x=6) = 0.1088$

(ii) no more than 15 households with at least two dogs registered. (2 marks)

$0.28 + 0.08 = 0.35$

$X \sim bin(40, 0.35)$

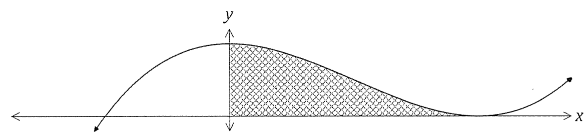
$P(x \le 15) = 0.6946$

(c) A random sample of 25 households within the city is to be selected. If X is the number of households in the sample that have exactly one dog registered, determine the mean of X. (2 marks)

$\bar{x} = np$
$= 25 \times 0.44$
$= 11$

Question 6 **(8 marks)**

The diagram below shows the curve $y = x^3 - 3x^2 + k$, where k is a constant. The curve has a turning point on the x-axis.



(a) Determine the value of k. (3 marks)

$\frac{dy}{dx} = 3x^2 - 6x$
$3x^2 - 6x = 0$
$x = 0, 2$

$(2,0)$ $0 = 8 - 12 + k$ $k = 4$

(b) Determine the set of values of x for which $\frac{dy}{dx}$ is increasing. (2 marks)

$\frac{d^2y}{dx^2} = 6x - 6$
$6x - 6 = 0$
$x = 1$

INCREASING FOR $x > 1$

(c) Calculate the area of the shaded region. (3 marks)

$A = \int_0^2 x^3 - 3x^2 + 4 \, dx$

$= \left[\frac{x^4}{4} - x^3 + 4x \right]_0^2$

$= 4$

Question 7 **(9 marks)**

The discrete random variable X is defined by $P(X = x) = k \log x$ for $x = 2, 5$ and 10.

(a) Determine the value of k. (3 marks)

$$k \log 2 + k \log 5 + k \log 10 = 1$$
$$k \log(2 \times 5 \times 10) = 1$$
$$k = \frac{1}{\log 100}$$
$$= \frac{1}{2}$$

(b) Determine $P(X = 2 \mid X < 10)$. (2 marks)

$$P(X < 10) = 1 - \tfrac{1}{2}\log 10$$
$$= \tfrac{1}{2}$$

$$P(X = 2 / X < 10) = \frac{\frac{1}{2}\log 2}{\frac{1}{2}}$$
$$= \log 2$$

(c) $E(X) = a(b + \log\sqrt{c})$, where the constants a, b and c are prime numbers. Determine the values of a, b and c. (4 marks)

$$E(X) = 2 \times \tfrac{1}{2}\log 2 + 5 \times \tfrac{1}{2}\log 5 + 10 \times \tfrac{1}{2}\log 10$$
$$= \log 2 + \log 5 + \tfrac{3}{2}\log 5 + 5$$
$$= \log 10 + 3\log\sqrt{5} + 5$$
$$= 6 + 3\log\sqrt{5}$$
$$= 3(2 + \log\sqrt{5})$$

$a = 3 \quad b = 2 \quad c = 5$

End of questions

Question 14 **(8 marks)**

The random variable X denotes the time, in hours, that a business telephone line is in use per nine hour working day.

The probability density function of X is given by $f(x) = \begin{cases} \frac{(x-a)^2+b}{k} & 0 \le x \le 9 \\ 0 & \text{otherwise} \end{cases}$,

where a, b and k are constants.

(a) If $a = 15$ and $b = 3$, determine the value of k. (2 marks)

$$\int_6^9 \frac{(x-15)^2 + 3}{k}\,dx = 1$$

$$k = 1080$$

(b) Let $a = 16$, $b = 1$ and $k = 1260$.

(i) The business is open for work for 308 days per year. On how many of these days can the business expect the phone line to be in use for more than eight hours? (2 marks)

$$\int_0^9 \frac{(x-16)^2+1}{1260}\,dx = 0.0455$$

$$0.0455 \times 308 = 14 \text{ days}$$

(ii) Determine the mean and variance of X. (4 marks)

$$E(X) = \int_0^9 x \times \frac{(x-16)^2+1}{1260}\,dx$$
$$= 3.39$$

$$Var(X) = \int_0^9 (x - 3.39)^2 \times \frac{(x-16)^2+1}{1260}\,dx$$
$$= 5.78$$

Question 13 **(7 marks)**

A hardware store sells stakes, of nominal length 1.8 metres, to be used for supporting newly planted trees. The length, X metres, of the stakes can be modelled by a normal distribution with mean 1.85 and standard deviation σ.

(a) If $\sigma = 0.035$, determine

(i) the probability that a randomly chosen stake is shorter than 1.8 metres. (1 mark)

$P(X < 1.8) = 0.0766$

(ii) the probability that a randomly chosen stake is longer than 1.79 m given that it is shorter than 1.8 metres. (2 marks)

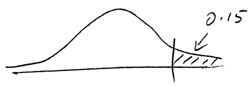
$$\frac{P(1.79 < X < 1.8)}{P(X < 1.8)}$$

$$= \frac{0.0333}{0.0766}$$

$$= 0.435$$

(iii) the value of k, if the longest 15% of stakes exceed k metres in length. (1 mark)

$k = 1.886$

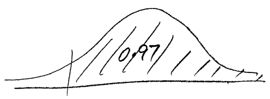


(b) A large number of stakes were measured and it was found that 97% of them were longer than their nominal length. Show how to use this information to deduce that the value of σ is 0.027 (to three decimal places). (3 marks)

$$z = \frac{x - \mu}{\sigma}$$

$$-1.881 = \frac{1.8 - 1.85}{\sigma}$$

$$\sigma = 0.027$$



$z = -1.881$

TRINITY COLLEGE

Semester Two Examination, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section Two: Calculator-assumed

If required by your examination administrator, please place your student identification label in this box

Student Number: In figures

In words ______________

Your name ______________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

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Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(5 marks)**

Zebra mussels are an invasive species of shellfish recently discovered in some North American waterways. The mussel density, D, in shellfish per square metre, observed in a power station water supply pipe t days after a colony began, was modelled by the following equation, where k is a positive constant:

$$D = 200e^{kt}$$

(a) What was the mussel density in the colony when observations began? (1 mark)

200

The mussel density was observed to double every eight days.

(b) Determine the value of k, rounded to four decimal places. (2 marks)

$e^{8k} = 2$

$k = 0.0866$

(c) The water supply pipe was seriously compromised when the mussel density reached 85 thousand shellfish per square metre. After how many days from the commencement of observations did this happen? (2 marks)

$85000 = 200e^{0.0866t}$

$t = 69.9$ days

≈ 70

Question 12 **(9 marks)**

A box contains a large number of packets of buttons. The number of buttons in a packet may be modelled by the random variable X, with the probability distribution shown below. It is also known that $E(X) = 6.25$.

x	3 or fewer	4	5	6	7	8	9 or more
$P(X = x)$	0	0.05	a	b	0.25	0.15	0

(a) Two packets are randomly chosen from the box. Determine the probability that there are at least 15 buttons altogether in the two packets. (3 marks)

(7, 8) (8, 7) (8, 8)

$P = 0.25 \times 0.15 + 0.15 \times 0.25 + 0.15 \times 0.15$

$= 0.0975$

(b) Determine the values of a and b. (3 marks)

$a + b + 0.45 = 1$

$5a + 6b + 3.15 = 6.25$

$a = 0.2$

$b = 0.35$

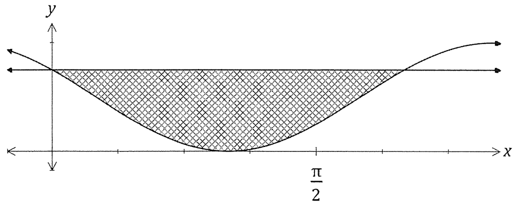
(c) Calculate $\mathrm{Var}(X)$. 1.1875 (1 mark)

(d) As part of a fundraiser, patrons pay 75 cents to select a packet at random and then win back 10 cents for each button in the packet. If the random variable W represents the net gain per game for a patron in cents, determine $E(W)$. (2 marks)

$E(W) = 10 \times E(X) - 75$

$= -12.5$

(c) The graphs of $y = \cos^2\left(x + \frac{\pi}{6}\right)$ and $y = \frac{3}{4}$ are shown below. Determine the exact area of the shaded region they enclose. (4 marks)



$\cos^2\left(x + \frac{\pi}{6}\right) = \frac{3}{4}$

$x = 0, \frac{2\pi}{3}$

$A = \int_0^{\frac{2\pi}{3}} \frac{3}{4} - \cos^2\left(x + \frac{\pi}{6}\right)\, dx$

$= \frac{\pi}{6} + \frac{\sqrt{3}}{4}$ sq units

Question 9 **(7 marks)**

The speeds of 250 vehicles, on a section of freeway undergoing road works with a speed limit of 60 kmh^{-1}, had a mean and standard deviation of 56.9 kmh^{-1} and 3.6 kmh^{-1} respectively. A summary of the data is shown in the table below.

Speed (x kmh^{-1})	$45 \le x < 50$	$50 \le x < 55$	$55 \le x < 60$	$60 \le x < 65$	$65 \le x < 70$
Relative frequency	0.024	0.272	0.504	0.188	0.012

(a) Use the table of relative frequencies to estimate the probability that the next vehicle to pass the road works

(i) was not exceeding the speed limit. (1 mark)

$0.024 + 0.272 + 0.504$

$= 0.8$

(ii) had a speed of less than 65 kmh^{-1}, given they were exceeding the speed limit. (2 mark)

$\frac{0.188}{1-0.8} = 0.94$

(b) Subsequent tests on the measuring equipment discovered that it had been wrongly calibrated. The correct speed of each vehicle, v, could be calculated from the measured speed, x, by increasing x by 6% and then adding 1.7.

(i) Calculate the adjusted mean and standard deviation of the vehicle speeds. (2 marks)

$\bar{v} = 56.9 \times 1.06 + 1.7$

$= 62$ km/hr

$SD_v = 3.6 \times 1.06$

$= 3.82$ km/hr

(ii) Determine the correct proportion of vehicles that were speeding. (2 marks)

$60 = x \times 1.06 + 1.7$

$55 = x$

$\therefore 0.504 + 0.188 + 0.012$

$= 0.704$

Question 10 **(6 marks)**

A student planned to investigate what proportion of the 1260 students at their school had access to more than one computer at home.

(a) The student thought of the following three ways to select a sample from the population. Briefly discuss the main source of bias in each method.

(i) Wait at the bus-bay after school and ask the first 50 students who show up. (1 mark)

BIASED TOWARDS STUDENTS WHO CATCH A BUS

(ii) Advertise the survey in a whole school assembly and ask the first 50 students who volunteer to stay behind. (1 mark)

NON – RESPONSE BIAS

(iii) Select and ask every 100th student from the school roll. (1 mark)

SMALL SAMPLE BIAS (JUST 13)

(b) Assuming that 80% of students had access to more than one computer at home, the student carried out 100 simulations in which a sample proportion was calculated from a random sample of 64 students.

(i) Explain why it is reasonable to expect that the distribution of the sample proportions would approximate normality. (1 marks)

SAMPLE SIZE (64) LARGE ENOUGH

(ii) Determine the mean and standard deviation of the normal distribution that the sample proportions would approximate. (2 marks)

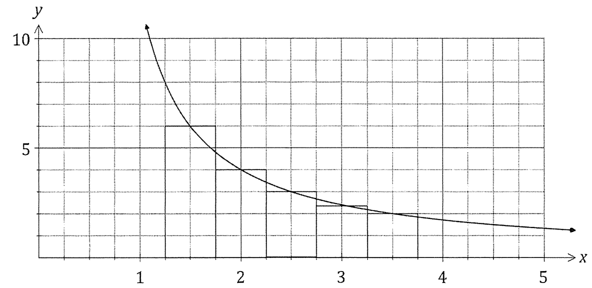
MEAN 0.8

$SD = \sqrt{\frac{0.8(1-0.8)}{64}}$

$= 0.05$

Question 11 **(10 marks)**

(a) The graph below shows the curve $y = f(x)$, where $f(x) = \frac{12}{2x-1}$.



Use the five centred rectangles shown to estimate the shaded area under the curve from $x = 1.25$ to $x = 3.75$. (3 marks)

$A \approx \frac{1}{2}\times 6 + \frac{1}{2}\times 4 + \frac{1}{2}\times 3 + \frac{1}{2}\times 2.4 + \frac{1}{2}\times 2$

$= 8.7$ sq units

(b) Given $\int_a^b h(x)\,dx = k$ and $h(x)$ is a polynomial, determine the following in terms of the constants a, b and k:

(i) $\int_a^b 3\,h(x)\,dx$. $3k$ (1 mark)

(ii) $\int_a^b 2 - h(x)\,dx$. $= \int_a^b 2 - \int_a^b h(x)\,dx$ (2 marks)

$= 2b - 2a - k$