Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-assumed

Sample WACE Examination 2016

Marking Key

Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

2014/10316

Question 10 **(6 marks)**

Certain medical tests require the patient to be injected with a solution containing 0.5 micrograms (μg) of the radioactive material Technetium-99. This material decays according to the rule:

$$T = T_0 e^{-0.1155t}$$ where t is the time (in hours) from injection.

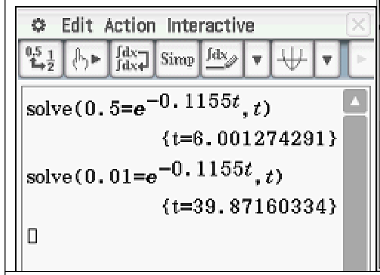
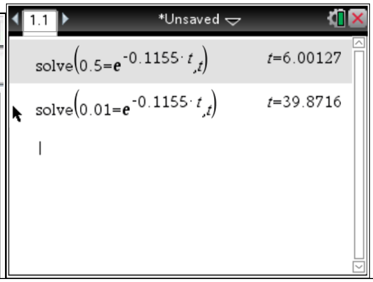
(a) What is the value of T_0? (1 mark)

Solution
0.5 μg
Specific behaviours
✓ identifies T_0 correctly

(b) What is the half-life of Technetium-99? (2 marks)

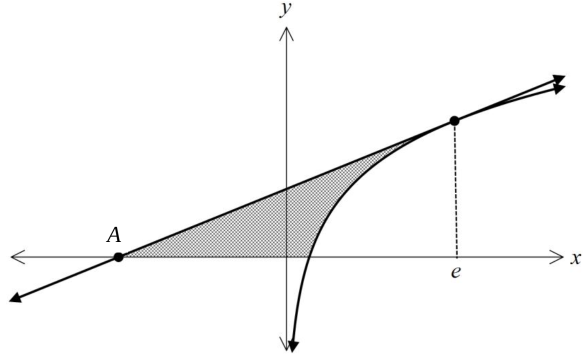
Solution
$0.5 = e^{-o.1155t}$ $t = 6$ The half-life of Technetium-99 is 6 hours.
Specific behaviours
✓ states the equation for half-life ✓ solves correctly

(c) After how long is the amount of Technetium-99 left in the patient's system less than 1% of the initial amount? Give your answer to the nearest hour. (3 marks)

Solution
$0.01T_0 = T_0 e^{-0.1155t}$ $0.01 = e^{-0.1155t}$ $t = 39.9\,\text{h}$ i.e. 40 hours 
Specific behaviours
✓ states the equation ✓ solves correctly for t ✓ rounds time to nearest hour

Question 11 **(9 marks)**

The diagram below shows the graph of the function $f(x) = \ln x + 1$ and a linear function $g(x)$, which is a tangent to $f(x)$. When $x = e$, $g(x) = f(x)$.



(a) Determine $g(x)$, the equation of the tangent. (3 marks)

Solution
When $x = e,\ f(x) = \ln e + 1 = 2$ $\therefore\ (e, 2)$ lies on the line $f(x) = \ln x + 1$ $f'(x) = \frac{1}{x}$ or Substituting $(e,\ 2)$ into $f'(x)$ $m = \frac{1}{e}$ i.e. $y = \frac{1}{e}x + c$ Substituting $(e,\ 2)$ into this equation $c = 1$ $\therefore$ Equation of tangent is $y = \frac{1}{e}x + 1$
Specific behaviours
✓ determine $f(e)$ correctly ✓ determine $f'(x)$ correctly ✓ correctly states equation of the tangent

(b) Determine the exact coordinates of A, the point where $g(x)$ intersects the x-axis. (1 mark)

Solution
$0 = \frac{1}{e}x + 1$ $x = -e$ $(-e, 0)$
Specific behaviours
✓ determines coordinates of A correctly

(c) Verify that $f(x)$ cuts the x-axis at the point $\left(\frac{1}{e}, 0\right)$. (1 mark)

Solution
$f\left(\frac{1}{e}\right) = \ln\frac{1}{e} + 1$ $= -\ln e + 1$ $= 0$ $\therefore f(x)$ cuts the x-axis at $\left(\frac{1}{e}, 0\right)$
Specific behaviours
✓ demonstrates that $f\left(\frac{1}{e}\right) = 0$

(d) Determine the area of the shaded region enclosed by $f(x)$, $g(x)$ and the x-axis. (4 marks)

Solution
$\text{Area} = \int_{-e}^{e}\left(\frac{x}{e}+1\right)dx - \int_{\frac{1}{e}}^{e}(\ln x + 1)dx$ $= \frac{1}{2}2e \times 2 - \int_{\frac{1}{e}}^{e}(\ln x + 1)dx$ $= 2.35$
Specific behaviours
✓ expresses the area as the difference between two integrals, or otherwise ✓ uses correct limits for first integral ✓ uses correct limits for second integral ✓ calculates area of shaded region correctly

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CANNINGTON WA 6107

A manufacturer of AAA batteries assumes that 99% of the batteries produced are fault-free. Ten samples of 50 packets of 50 AAA batteries are selected at random and tested. The number of faulty batteries in each of the 10 random samples is provided below.

Sample	1	2	3	4	5	6	7	8	9	10
Number of faulty batteries	34	28	22	38	28	30	22	16	28	30

(b) Using the assumption that 99% of batteries are fault free calculate the 95% confidence interval for the proportion of faulty batteries expected when sampling. (3 marks)

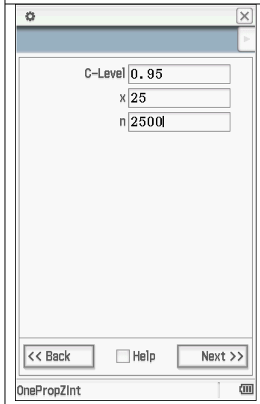
(c) Decide which of the samples, if any lie outside the 95% confidence interval. (1 mark)

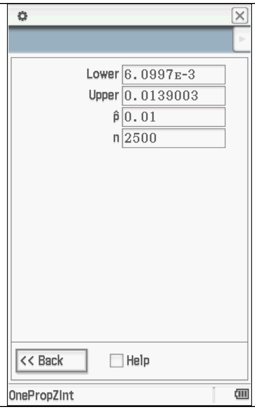
Solution

Using 25 out of 2500 batteries being faulty. This gives us $p = 0.01$ and a 95% confidence interval of:

$0.0060997 \le p \le 0.0139003 \Rightarrow 15 \le n \le 35$ where $n =$ the number of faulty batteries

Sample 4 $n=38$ lies outside the interval.





Specific behaviours

✓ identifies n = 2500 and x = 25 as the required variables to calculate the 95% confidence interval
✓ determines the 95% confidence interval
✓ calculates the interval of faulty batteries
✓ determines the sample which lies outside the 95% confidence interval

Question 12 **(12 marks)**

Rebecca sells potatoes at her organic fruit and vegetable shop that have weights normally distributed with a mean of 230 g and a standard deviation of 5 g.

(a) Determine the probability that one of Rebecca's potatoes, selected at random, will weigh between 223 g and 235 g. (1 mark)

Solution

$X \sim N(230, 5^2)$

$P(223 < x < 235) = 0.7606$

Specific behaviours

✓ calculates probability correctly

(b) Five percent of Rebecca's potatoes weigh less than w g. Determine w to the nearest gram. (2 marks)

Solution

$P(X < w) = 0.05$

$w \approx 221.8 = 222$ g to the nearest gram

Specific behaviours

✓ gives correct probability statement
✓ calculates w correctly

(c) A customer buys twelve potatoes.

(i) Determine the probability that all twelve potatoes weigh between 223 g and 235 g. (2 marks)

Solution

$Y \sim bin(12, 0.7606)$

$P(y = 12) = 0.0375$

Specific behaviours

✓ states binomial distribution and its parameters
✓ calculates probability correctly

(ii) If the customer is selecting the twelve potatoes one at a time, determine the probability that it takes the selection of eight potatoes before six potatoes weighing between 223 g and 235 g have been selected. (3 marks)

Solution

$W \sim bin(7, 0.7606)$

$P(w = 5) \times 0.7606 = 0.3064 \times 0.7606 = 0.2330$

Specific behaviours

✓ states distribution and its parameters
✓ calculates $P(w = 5)$ correctly
✓ calculates final probability correctly

Rebecca also sells oranges. The weights of these oranges are normally distributed. It is known that 5% of the oranges weigh less than 153 g while 12% of the oranges weigh more than 210 g.

(d) Determine the mean and standard deviation of the weights of the oranges. (4 marks)

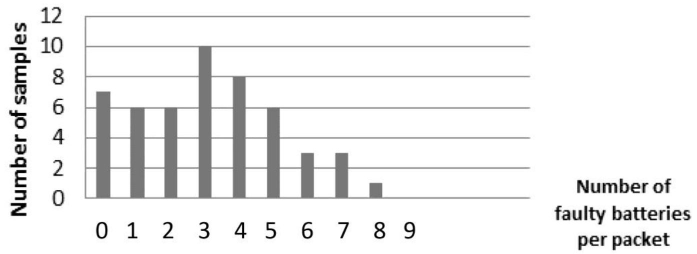
Solution
$Z \sim Norm(\mu, \sigma^2)$ $P(Z < z) = 0.05 \Rightarrow z = -1.6449$ $P(Z > z) = 0.12 \Rightarrow z = 1.1750$ $z = \dfrac{x - \mu}{\sigma}$ $-1.6449 = \dfrac{153 - \mu}{\sigma}$ *and* $1.1750 = \dfrac{210 - \mu}{\sigma}$ $\mu = 186.2 \quad \sigma = 20.2$
Specific behaviours
✓ calculates correct z-score for 0.05 ✓ calculates correct z-score for 0.12 ✓ generates simultaneous equations correctly ✓ solves correctly for μ and σ

(e) Based on your observations of the graphs in this question, make a conjecture about the defining rule for $A'(x)$. (1 mark)

Solution
$A'(x) = f(x)$
Specific behaviours
✓ states that $A'(x) = f(x)$

Question 21 **(5 marks)**

The graph below shows the number of faulty batteries per packet of 50 AAA batteries, when 50 packets are sampled at random.



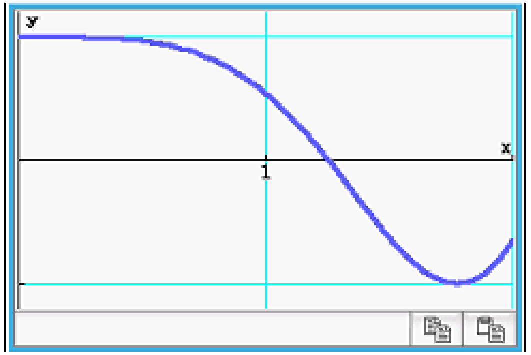
(a) Identify the type of distribution of X = the number of faulty batteries per packet of 50 AAA batteries. (1 mark)

Solution
The batteries tested are either faulty or not faulty. Each test of a battery is a Bernoulli trial. Hence the underlying distribution is binomial.
Specific behaviours
✓ identifies the distribution as binomial

(c) On the axes below, plot the values from the table in part (b), and hence sketch the graph of $A(x)$ for $0 \le x \le 2$. (2 marks)

Solution
Plot of points, jointed by curve. Should pass through the origin, (0,0) 
Specific behaviours
✓ plots points accurately ✓ joins points with curve of good shape (and labels axes)

(d) Use your graph from part (c) to sketch the graph of $A'(x)$ on the axes below. (2 marks)

Solution
Graph resembles $f(x)$ 
Specific behaviours
✓ sketches a graph which resembles the shape of $f(x)$ ✓ indicates x-intercept at approximately 1.2

Question 13 **(5 marks)**

The decibel scale for sound, measured in decibels (dB), is defined as: $D = 20\log_{10}\left(\frac{P}{P_{ref}}\right)$,

where P is the pressure of the sound being measured and P_{ref} is a fixed reference pressure.

(a) What is the decibel measure for a sound with pressure $2P_{ref}$? (1 mark)

Solution
$D = 20\log_{10}\frac{2P_{ref}}{P_{ref}} = 20\log_{10}2 = 6.0\ dB$
Specific behaviours
✓ correctly calculates the decibel measure

(b) The sound produced by a symphony orchestra measures 120 dB, while that of a rock concert measures 150 dB. How many times greater is the sound pressure of the rock concert than that of the orchestra? (4 marks)

Solution
Let P_r denote the pressure of the rock concert and P_o denote the pressure of the orchestra. Then $20\log_{10}\frac{P_r}{P_{ref}} = 150$ and $20\log_{10}\frac{P_o}{P_{ref}} = 120$. Subtracting the two gives $20\log_{10}\frac{P_r}{P_{ref}} - 20\log_{10}\frac{P_o}{P_{ref}} = 30$. Simplifying the LHS gives $20\log_{10}\frac{P_r}{P_o} = 30$ or $\log_{10}\frac{P_r}{P_o} = 1.5$. Then $\frac{P_r}{P_o} = 10^{1.5} = 31.6$. The sound pressure of the rock concert is 31.6 times greater than the symphony orchestra.
Specific behaviours
✓ states equations to concerts in logarithmic form ✓ solves the two equations ✓ simplifies LHS using log laws ✓ changes to exponential form and simplifies correctly

Question 14 **(10 marks)**

(a) The discrete random variable X has the following probability distribution:

x	1	2	3	4	5
$P(X=x)$	0.1	a	0.3	0.25	b

(i) Determine the values of a and b if the expected value, $E(X)=3.3$. (3 marks)

Solution
$0.1+2a+0.9+1+5b=3.3$ $0.1+a+0.3+0.25+b=1$ $a=0.15 \quad b=0.2$
Specific behaviours
✓ generates correct equation using $E(x)$ ✓ generates correct equation using sum of probabilities equal 1 ✓ solves simultaneously for correct values of a and b

(ii) Determine the variance, $Var(x)$. (2 marks)

Solution
$1.23^2=1.51$
Specific behaviours
✓ calculates standard deviation correctly ✓ calculates variance correctly

(iii) State the value of $Var(X+5)$. (1 mark)

Solution
$Var(X+5)=Var(X)$ $=1.51$
Specific behaviours
✓ correctly calculates new variance

(iv) State the value of $Var(2X+5)$. (1 mark)

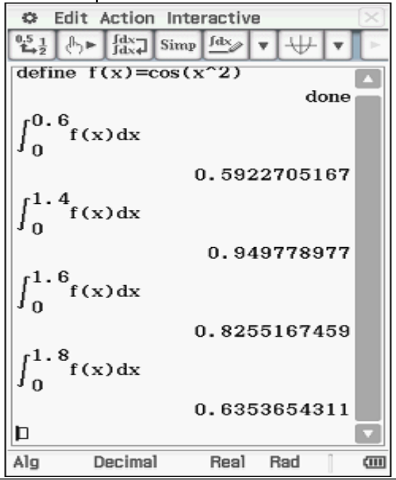
Solution
$Var(2X+5)=2^2(Var(X))$ $=2^2\times 1.51=6.04$
Specific behaviours
✓ correctly calculates new variance

(b) Complete the table below. (2 marks)

Solution

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$A(x)$	0.200	0.399	**0.592**	0.768	0.905	0.974	**0.950**	**0.826**	**0.635**	0.461

screen capture:

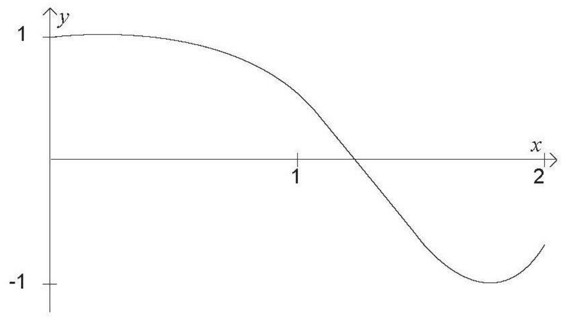


Specific behaviours

✓ demonstrates how to calculate at least one of the required numbers (by stating the integral and appropriate limits)
✓ calculates all the required numbers correctly

Question 20 **(10 marks)**

The graph of the function $f(x)=\cos x^2$ for $0 \le x \le 2$ is provided below.



(a) The function $A(x)$ is defined as $A(x)=\int_0^x f(t)dt$, for $0 \le x \le 2$.

Determine the value of x when $A(x)$ starts to decrease. (3 marks)

Solution
$A(x)$ starts to decrease at the point where $f(x) = 0$. i.e. where $\cos x^2 = 0$, $0 \le x \le 2$. $\Rightarrow x^2 = \frac{\pi}{2}$ $\Rightarrow x = \sqrt{\frac{\pi}{2}}$ $\Rightarrow x \approx 1.253$
Specific behaviours
✓ relates start of decrease in $A(x)$ to where $f(x) = 0$ ✓ determines this is where $x^2 = \frac{\pi}{2}$ ✓ determines correct value of x

(b) Daniel has been offered a sales position at a car yard. His weekly pay will comprise two components, a retainer of $250 and a commission of $400 for each new car sold. The following table shows the probability of his selling specific numbers of cars each week.

x	0	1	2	3	4
P(X = x)	0.3	0.4	0.25	0.04	0.01

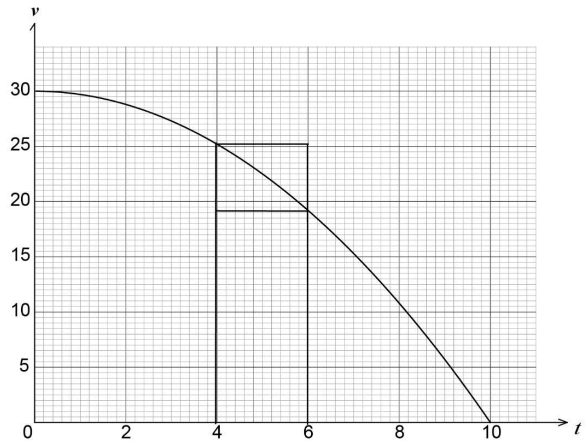
Calculate Daniel's expected weekly pay. (3 marks)

Solution
$E(N)=0+0.4+0.5+0.12+0.04=1.06$ Expected weekly pay $=250+1.06\times400=\$674$
Specific behaviours
✓ correctly calculates the expected number of cars sold per week ✓ writes an equation for the expected weekly pay in terms of the retainer and the commissions ✓ correctly calculates the expected weekly pay

Question 15 **(6 marks)**

A train is travelling at 30 metres per second when the brakes are applied. The velocity of the train is given by the equation $v = 30 - 0.3t^2$, where *t* represents the time in seconds after the brakes are applied.

Velocity-time graph



The area under a velocity-time graph gives the total distance travelled for a particular time period.

(a) Complete the tables below and estimate the distance travelled by the train during the first six seconds by calculating the mean of the areas of the circumscribed and inscribed rectangles. (The rectangles for the 4–6 seconds interval are shown on the grid above.) (5 marks)

Time *(t)*	0	2	4	6
Velocity *(v)*	**30**	28.8	**25.2**	19.2

Rectangle	0–2	2–4	4–6	Total
Circumscribed area	**60**	**57.6**	50.4	**168**
Inscribed area	**57.6**	**50.4**	38.4	**146.4**

Solution
Estimate of total distance travelled: 157.2 m
Mean = $\frac{314.4}{2} = 157.2$ m
Specific behaviours
✓ correctly completes first table, velocities
✓✓✓ calculates correctly values for areas and totals (one mark for each correct column)
✓ calculates correctly the mean distance travelled

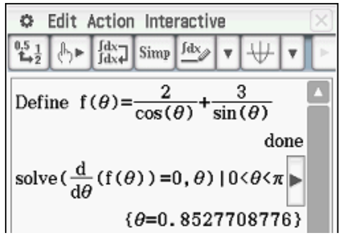
(b) The workers' ladder is 6.5 m long. Will they be able to carry their ladder along this L-shaped space? Justify your answer. (4 marks)

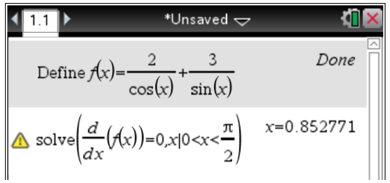
Solution

The maximum length of the ladder occurs when $\frac{dL}{d\theta} = 0$.

$L = \frac{2}{\cos\theta} + \frac{3}{\sin\theta}$

Solve$\left(\frac{d}{d\theta}(f(\theta) = 0, 1\ 0° \le \theta \le 90°)\right)$





$\theta = 48.86°\ (\theta = 0.8528)$

Hence maximum length, $L = \frac{2}{\cos 48.86°} + \frac{3}{\sin 48.86°}$

$= 7.0235$

∴ The worker will be able to carry a 6.5 m long ladder around the corner of the corridors.

Specific behaviours

✓ applies $\frac{dL}{d\theta} = 0$ to find maximum length

✓ determines correct value of θ

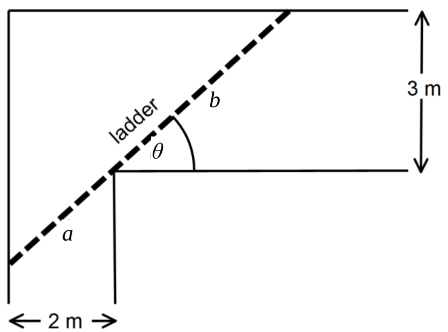
✓ substitutes to find maximum length of ladder

✓ concludes that the ladder will fit around the corner

Question 19 **(7 marks)**

Two corridors meet at right angles and are 3 m and 2 m wide respectively. The angle between the wall and the ladder is marked on the diagram as θ.

A ladder (of negligible width) is to be carried horizontally along this L-shaped space by two workers. The workers need to know the length of the longest ladder that can be carried around this corner.



(a) Show that the length of the ladder (L) is given by $L = \dfrac{2}{\cos\theta} + \dfrac{3}{\sin\theta}$. (3 marks)

Solution
Let a be the section of the ladder in the 2 m corridor and b be the section of the ladder in the 3 m corridor $\cos\theta = \dfrac{2}{a} \Rightarrow a = \dfrac{2}{\cos\theta}$ $\sin\theta = \dfrac{3}{b} \Rightarrow b = \dfrac{3}{\sin\theta}$ $L = a + b = \dfrac{2}{\cos\theta} + \dfrac{3}{\sin\theta}$
Specific behaviours
✓ writes an equation for a in terms of $\cos\theta$ ✓ writes an equation for b in terms of $\sin\theta$ ✓ expresses the length of the ladder as the sum of the section of ladder in the 2 m corridor and the section in the 3 m corridor

(b) Describe how you could better estimate the distance travelled by the train during the first six seconds than by the method used in part (a). (1 mark)

Solution
Answers could include, but not be limited to: use smaller rectangles use more rectangles
Specific behaviours
✓ describes a more accurate method to calculate the distance travelled

Question 16 **(4 marks)**

Roland spends X hours writing poetry during the day.

The probability distribution of X is given by:

$$f(x)=\begin{cases}2(1-x) & 0\le x\le 1\\ 0 & \text{elsewhere.}\end{cases}$$

(a) Evaluate $E(X)$, the expected value of X, to the nearest minute. (2 marks)

Solution
$E(X)=\int_0^1 2x(1-x)\,dx$ $=\frac{1}{3}$ h = 20 minutes
Specific behaviours
✓ writes a correct integral for $E(X)$ ✓ calculates $E(X)$ correctly to the nearest minute

(b) Determine the variance of X. (2 marks)

Solution
$Var=\int_0^1 (x-\frac{1}{3})^2 2(1-x)\,dx$ $=\frac{1}{18}$ Alternative solution $E(x^2)=2\int_0^1 (x^2-x^3)\,dx=\frac{1}{6}$ $Var=E(x^2)-(E(x))^2=\frac{1}{6}-\frac{1}{9}=\frac{1}{18}$
Specific behaviours
✓ uses variance formula correctly ✓ calculates variance correctly

The 90% confidence interval of the sample proportion p from the initial survey is

$0.649 \le p \le 0.725$.

(d) Use the 90% confidence interval of the initial sample to compare the following samples:

(i) A random sample of 365 people at a shopping centre found that 258 had a preference for a smart phone. (2 marks)

Solution
$p=\frac{258}{365}=0.71$ and $0.668 \le p \le 0.746$ The confidence interval for this second survey overlaps, significantly, the 90% confidence interval of the initial survey so this indicates we are sampling from the same population.
Specific behaviours
✓ calculates 90% confidence interval for p correctly ✓ states the similarity of results

(ii) A random sample of 78 people at a retirement village found that 32 had a preference for a smart phone. (2 marks)

Solution
$p=\frac{32}{78}=0.41$ and $0.319 \le p \le 0.502$ The confidence interval for this sample is quite different than that of the original survey. While this could be a random outlier it is more likely to be a biased survey from inside the retirement village.
Specific behaviours
✓ calculates 90% confidence interval for p correctly ✓ states the difference of results

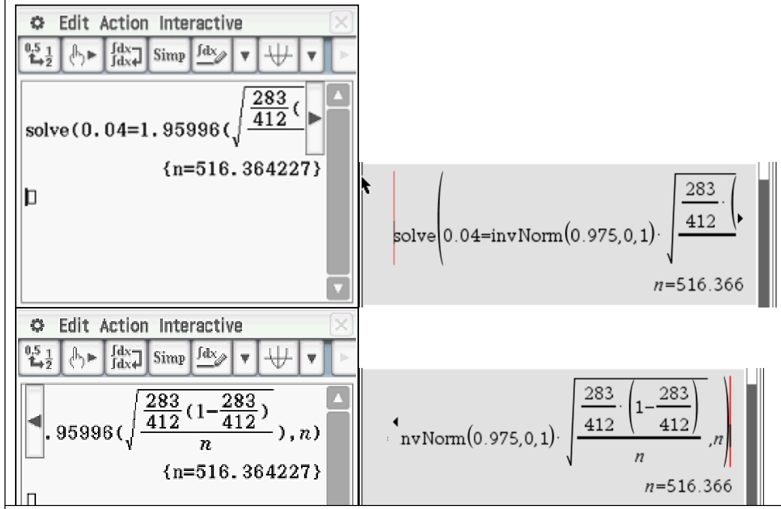
(c) A follow-up survey is to be conducted to confirm the results of the initial survey. Working with a confidence interval of 95%, estimate the sample size necessary to ensure a margin of error of at most 4%. (3 marks)

Solution

$$0.04 = 1.96\sqrt{\frac{\frac{283}{412}(1-\frac{283}{412})}{n}}$$

$n = 516.366$

$n \approx 517$



Specific behaviours

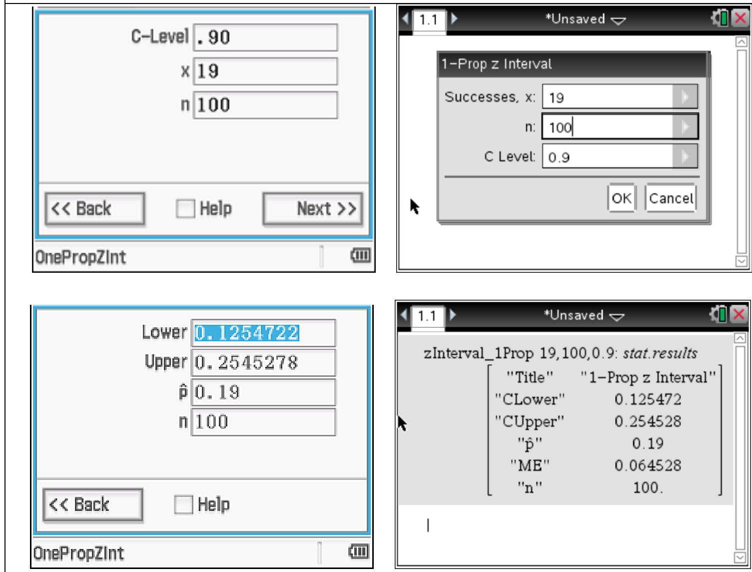
✓ writes an equation to evaluate n from the margin of error
✓ solves correctly for n
✓ rounds n up to the nearest whole number

Question 17 **(8 marks)**

A random sample of 100 people indicated that 19% had taken a plane flight in the last year.

(a) Determine a 90% confidence interval for the proportion of the population that had taken a plane flight in the last year. (2 marks)

Solution



Hence $0.125 \le p \le 0.255$

Alternative solution

$$p - z\sqrt{\frac{p(1-p)}{n}} \le p \le p + z\sqrt{\frac{p(1-p)}{n}} \qquad n = 100, z = 1.645, p = 0.19$$

$$0.19 - 1.645\sqrt{\frac{0.19(1-0.19)}{100}} \le p \le 0.19 + 1.645\sqrt{\frac{0.19(1-0.19)}{100}}$$

$$0.125 \le p \le 0.255$$

Specific behaviours

✓ correctly calculates lower value of confidence interval
✓ correctly calculated upper value of confidence interval

Assume the 19% sample proportion applies to the whole population.

(b) A new sample of 200 people was taken and X= the number of people who had taken a plane flight in the last year was recorded. Give a range, using the 90% confidence internal, within which you would expect X to lie. (1 mark)

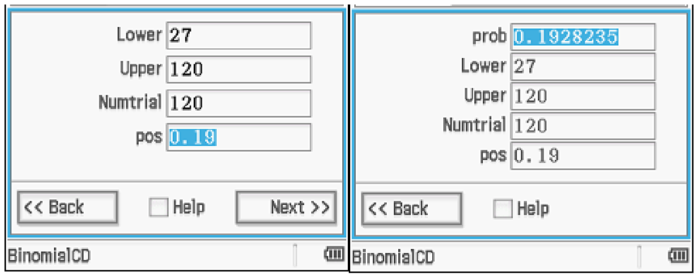
Solution
$100\times 0.125 \le X \le 100\times 0.254 \Rightarrow 13 \le X \le 25$
Specific behaviours
✓ correctly calculates upper and lower value of interval

(c) Determine the probability that in a random sample of 120 people, the number who had taken a plane flight in the last year was greater than 26. (3 marks)

Solution

The distribution is binomial with p = 0.19 and n = 120.

$P(X > 26) = P(X \ge 27)$, since n is discrete



Hence the required probability is 0.1928 (to four decimal places)

Specific behaviours

✓ identifies the distribution as binomial – bin(120,0.19)
✓ uses 27 as the lower bound in the binomial cumulative distribution
✓ states the correct probability

(d) If seven surveys were taken and for each a 95% confidence interval for p was calculated, determine the probability that at least four of the intervals included the true value of p. (2 marks)

Solution
$bin(7, 0.95) \Rightarrow P(4 \le x \le 7) = 0.9998$
Specific behaviours
✓ identifies the distribution as binomial – $bin(7,\ 0.95)$
✓ calculates the probability correctly

Question 18 **(10 marks)**

A random survey was conducted to estimate the proportion of mobile phone users who favoured smart phones over standard phones. It was found that 283 out of 412 people surveyed preferred a smart phone.

(a) Determine the proportion p of those in the survey who preferred a smart phone.(1 mark)

Solution
$p = \dfrac{283}{412} = 0.6869$
Specific behaviours
✓ calculates p correctly

(b) Use the survey results to estimate the standard deviation of p. (2 marks)

Solution
Standard deviation $= \sqrt{\dfrac{\frac{283}{412}(1-\frac{283}{412})}{412}} = 0.0228$
Specific behaviours
✓ substitutes correctly into standard deviation formula
✓ calculates standard deviation correctly