



Semester Two Examination, 2019  
Question/Answer booklet

**MATHEMATICS  
METHODS  
UNITS 3 AND 4  
Section One:  
Calculator-free**

Student number:      In figures     

                                         In words      \_\_\_\_\_

                                         Your name      \_\_\_\_\_

**Time allowed for this section**  
Reading time before commencing work:      five minutes  
Working time:      fifty minutes

**Materials required/recommended for this section**  
*To be provided by the supervisor*  
This Question/Answer booklet  
Formula sheet

*To be provided by the candidate*  
Standard items:      pens (blue/black preferred), pencils (including coloured), sharpener,  
correction fluid/tape, eraser, ruler, highlighters  
Special items:      nil

**Important note to candidates**  
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

**Structure of this paper**

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
<b>Total</b>					<b>100</b>

**Instructions to candidates**

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.
2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.
3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.
4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.
5. It is recommended that you do not use pencil, except in diagrams.
6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.
7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Supplementary page

Question number: \_\_\_\_\_

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1

Determine the following:

(a)  $\int 12(2x+1)^7 dx.$

(2 marks)

(4 marks)

<b>Solution</b>
$\frac{12}{2 \times 3} (2x+1)^8 = 2(2x+1)^8 + c$
<b>Specific behaviours</b>
✓ integrates correctly ■ includes constant

(b)  $\frac{d}{dx} \cos(2x+1).$

(1 mark)

<b>Solution</b>
$-2 \sin(2x+1)$
<b>Specific behaviours</b>
✓ correct derivative

(c)  $\frac{d}{dx} \int_x^3 (2t+1) dt.$

(1 mark)

<b>Solution</b>
$2x+1$
<b>Specific behaviours</b>
✓ correct use of fundamental theorem

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Question 8

Let  $f(x) = \frac{x}{x+1}.$

(a) Determine  $f(x)$  and  $f(x+\delta x)$  when  $x=70$  and  $\delta x=5.$

(1 mark)

<b>Solution</b>
$f(70) = \frac{70}{71}, f(75) = \frac{76}{77}$
<b>Specific behaviours</b>
■ both correct fractions

(b) Use  $f(x)$  and the increments formula to estimate the difference between  $\frac{90}{92}$  and  $\frac{93}{97}.$  (5 marks)

<b>Solution</b>
$f(x) = \frac{1}{1} \times \frac{(x+1)^2}{1} = \frac{(x+1)^2}{1}$ Find $\delta y$ when $x=89$ and $\delta x=3.$ $\delta y \approx f'(x) \times \delta x \approx \frac{1}{3} \times \frac{90^2}{92} \approx \frac{1}{3} \times \frac{8100}{92} \approx \frac{2700}{92}$ Difference is approximately $\frac{1}{92}.$
<b>Specific behaviours</b>
■ use of quotient rule for $f'(x)$ ■ correct $f'(x)$ ✓ indicates values of $x$ and $\delta x$ ■ uses increments formula ■ substitutes, simplifies and states difference

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## Question 2

(7 marks)

The velocity of a small body moving in a straight line at time  $t$  seconds is given by

$$v = \frac{8}{1+t} \text{ m/s}, t \geq 0.$$

- (a) Determine the velocity of the body when its acceleration is  $-2 \text{ m/s}^2$ .

(4 marks)

Solution
$\frac{dv}{dt} = \frac{d}{dt} (8(1+t)^{-1}) = -8(1+t)^{-2}$ $-2 = \frac{-8}{(1+t)^2} \Rightarrow (1+t)^2 = 4t \Rightarrow -1 \pm 2t = 1$ $v(1) = 8 \div 2 = 4 \text{ m/s}$
Specific behaviours
<ul style="list-style-type: none"> <li>correctly differentiates</li> <li>equates to required value and simplifies</li> <li>indicates time</li> <li>correct velocity</li> </ul>

- (b) Calculate the distance travelled by the body in the first 3 seconds.

(3 marks)

Solution
$\int_0^3 \frac{8}{1+t} dt = [8 \ln(1+t)]_0^3 = 8 \ln 4 - 8 \ln 1$ $= 16 \ln 2 \text{ m}$
Specific behaviours
<ul style="list-style-type: none"> <li>writes definite integral</li> <li>correct antiderivative</li> <li>substitutes bounds and simplifies</li> </ul>

## Question 7

(7 marks)

In a class of 25 students, 20 are right-handed.

- (a) One student is selected at random from the class and the random variable  $X$  is the number of right-handed students in the selection. Determine the mean and standard deviation of  $X$ .

(3 marks)

Solution
$E(X) = p = \frac{20}{25} = \frac{4}{5}$ $\text{Var}(X) = p(1-p) = \frac{4}{5} \times \frac{1}{5} = \frac{4}{25}$ $\text{Standard deviation} = \sqrt{\frac{4}{25}} = \frac{2}{5}$
Specific behaviours
<ul style="list-style-type: none"> <li>mean</li> <li>variance</li> <li>standard deviation</li> </ul>

- (b) Two students are selected at random from the class without replacement and the random variable  $Y$  is the number of right-handed students in the selection.

- (i) Complete the probability distribution table below.

(3 marks)

$y$	0	1	2
$P(Y=y)$	1/30	1/3	19/30

Solution
$P(Y=2) = \frac{20}{25} \times \frac{19}{24} = \frac{4}{5} \times \frac{19}{24} = \frac{19}{30}$ $P(Y=0) = \frac{5}{25} \times \frac{4}{24} = \frac{1}{5} \times \frac{1}{6} = \frac{1}{30}$ $P(Y=1) = 1 - \frac{19}{30} - \frac{1}{30} = \frac{10}{30}$
Specific behaviours
<ul style="list-style-type: none"> <li>each correct probability</li> </ul>

- (ii) Determine  $E(Y)$ .

(1 mark)

Solution
$E(Y) = 0 + \frac{10}{30} + \frac{2(19)}{30} = \frac{48}{30} = \frac{8}{5}$
Specific behaviours
<ul style="list-style-type: none"> <li>correct value</li> </ul>

Question 6

Let  $f(x) = [1 - x]e^{-2x}$ .

- (a) Determine the coordinates of the stationary point of the graph of  $y = f(x)$  and use the second derivative test to determine its nature. (6 marks)

Solution
$f'(x) = -e^{-2x} - 2(1 - x)e^{-2x}$
$f'(x) = 0 \Rightarrow [2x - 3]e^{-2x} = 0 \Rightarrow x = \frac{3}{2}, y = -\frac{1}{3}$
$f''(x) = 2e^{-2x} - 2[2x - 3]e^{-2x} = (8 - 4x)e^{-2x}$
$f''(\frac{3}{2}) = -2e^{-3} \Rightarrow \text{Min}$
Stationary point is at $(\frac{3}{2}, -\frac{1}{3})$ and is a minimum.
Specific behaviours
✓ correct $f'(x)$
■ equates $f'(x)$ to zero and obtains x-coordinate
■ obtains y-coordinate
■ obtains $f''(x)$
■ indicates sign of $f''(x)$ at point

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(b)

Determine the coordinates of the point of inflection of the graph of  $y = f(x)$ . (2 marks)

Solution
$(8 - 4x)e^{-2x} = 0 \Rightarrow x = 2$
$f(2) = -\frac{1}{4}e^{-\frac{1}{2}}$
Point of inflection at $(2, -\frac{1}{4}e^{-\frac{1}{2}})$
Specific behaviours
✓ solves $f''(x) = 0$

(c)

Determine  $\frac{d}{dx} \left( \log_e \left( \frac{1}{5x^2 + 1} \right) \right)$ .

Solution
$x - 2 = \ln \sqrt[3]{x} = \frac{1}{3} \ln(3) + 2$
Specific behaviours
✓ expresses using natural log
■ simplifies

- (b) Solve for x the equation  $e^{x-2} = \sqrt{3}$ .

Solution
$1 + \log_5 3 - 2 \log_5 7 = \log_5 5 + \log_5 3 - 2 \log_5 7$
$\frac{15}{49} \log_5 15 - \log_5 7^2 \log_5 49$
Specific behaviours
✓ uses $\log_a a = 1$
■ uses $x \log_a y = \log_a y^x$
■ uses $\log_a x \pm \log_a y$

- (a) Write  $1 + \log_5 3 - 2 \log_5 7$  in the form  $\log_5 k$ .

Question 3

(7 marks)

(3 marks)

Solution
$\log_e \left( \frac{1}{5x^2 + 1} \right) = -\ln(5x^2 + 1)$
$\frac{d}{dx} (-\ln(5x^2 + 1)) = \frac{-10x}{5x^2 + 1}$
Specific behaviours
✓ uses log law
■ correct derivative

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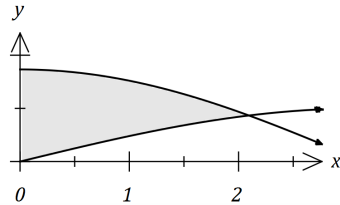
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## Question 4

(6 marks)

Let  $f(x) = \sqrt{3} \cos\left(\frac{x}{2}\right)$  and  $g(x) = \sin\left(\frac{x}{2}\right)$ .

The shaded region on the graph below is enclosed by  $x=0$ ,  $y=f(x)$  and  $y=g(x)$ .



(a) Show that  $f\left(\frac{2\pi}{3}\right) = g\left(\frac{2\pi}{3}\right)$ .

## Solution

$$f\left(\frac{2\pi}{3}\right) = \sqrt{3} \cos\left(\frac{\pi}{3}\right) = \sqrt{3} \times \frac{1}{2} = \frac{\sqrt{3}}{2}$$

$$g\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\text{Hence } f\left(\frac{2\pi}{3}\right) = g\left(\frac{2\pi}{3}\right).$$

## Specific behaviours

✓ evaluates  $f(x)$

✓ evaluates  $g(x)$ , stating same as

(b) Determine the area of the shaded region.

## Solution

$$\int_0^{\frac{2\pi}{3}} \sqrt{3} \cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) dx$$

$$= \left[ 2\sqrt{3} \sin\left(\frac{x}{2}\right) + 2 \cos\left(\frac{x}{2}\right) \right]_0^{\frac{2\pi}{3}}$$

$$= \left[ 2\sqrt{3} \sin\left(\frac{\pi}{3}\right) + 2 \cos\left(\frac{\pi}{3}\right) \right] - \left[ 2\sqrt{3} \sin(0) + 2 \cos(0) \right]$$

$$= \left( 2\sqrt{3} \times \frac{\sqrt{3}}{2} + 2 \times \frac{1}{2} \right) - (0 + 2) = 3 + 1 - 2 = 2 \text{ sq units}$$

## Specific behaviours

✓ writes correct integral

✓ integrates correctly

✓ substitutes correctly

✓ correct area

## Question 5

(7 marks)

The random variable  $X$  has probability density function  $f(x)$  shown below, where  $k$  is a positive constant.

$$f(x) = \begin{cases} kx + \frac{1}{20} & 0 \leq x \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Deduce that  $k = \frac{1}{10}$ .

(3 marks)

## Solution

$$\int_0^4 kx + \frac{1}{20} dx = \left[ \frac{kx^2}{2} + \frac{x}{20} \right]_0^4 = 8k + \frac{1}{5}$$

$$8k + \frac{1}{5} = 1 \Rightarrow 8k = \frac{4}{5} \Rightarrow k = \frac{1}{10}$$

## Specific behaviours

✓ integrates  $f(x)$

✓ evaluates definite integral

✓ equates to 1 and shows steps to solve for  $k$

(b) Determine the value of  $a$  if  $P(1 < X < a) = \frac{1}{5}$ .

(4 marks)

## Solution

$$\int_1^a \left( \frac{x}{10} + \frac{1}{20} \right) dx = \left[ \frac{x^2}{20} + \frac{x}{20} \right]_1^a = \frac{1}{20} (a^2 + a) - \frac{2}{20}$$

$$\frac{1}{20} (a^2 + a - 2) = \frac{1}{5} \Rightarrow a^2 + a - 6 = 0 \Rightarrow (a+3)(a-2) = 0 \Rightarrow a = 2$$

## Specific behaviours

✓ integrates  $f(x)$

✓ evaluates definite integral

✓ equates to probability and simplifies quadratic

✓ factorises and states the only valid value of  $a$