SEMESTER ONE

MATHEMATICS
METHODS
UNIT 3

2020

SOLUTIONS

Calculator-free Solutions

1. (a) $\dfrac{d}{dx}\left[\left(\sin\left(\dfrac{x}{2}\right)\right)^3\right] = 3\sin^2\left(\dfrac{x}{2}\right) \times \cos\left(\dfrac{x}{2}\right) \times \dfrac{1}{2}$ ✓✓

(b) $2t(\tan t) + \dfrac{t^2}{\cos^2 t}$ ✓✓

(c) $f(y) = \cos\left\{(\sin y)^{\frac{1}{2}}\right\}$

$\therefore \quad f\,'(y) = -\sin\sqrt{\sin y} \times \dfrac{1}{2}\left(\sin^{-\frac{1}{2}} y\right) \times \cos y$ ✓✓✓ [7]

2. (a) $v(t) = 2e^{2t} - 2e^2 t + c$ ✓

$x(t) = e^{2t} - e^2 t^2 + ct + k$ ✓

When $t = 0$, $x = 0 \rightarrow k = -1$ ✓

When $t = 1$, $x = 0 \rightarrow c = 1$ ✓

$\therefore \quad x(t) = e^{2t} - e^2 t^2 + t - 1$

(b) 0 ✓

(c) $v(0) = 3$ ✓ [6]

3. (a) $2\sin\dfrac{x}{2} - e^{\cos x} + c$ ✓✓

(b) $f'(y) = (1 - 2y)^{-\frac{1}{2}}$

$\therefore \quad f(y) = \dfrac{(1-2y)^{\frac{1}{2}}}{\left(\frac{1}{2}\right) \times (-2)} + c$ ✓✓

$\therefore \quad (-4, 3) \rightarrow 3 = -\sqrt{9} + c \rightarrow c = 6$ ✓

$\therefore \quad f(y) = -\sqrt{1 - 2y} + 6$ ✓

(c) $g(x) = -\displaystyle\int_{-1}^{2x} \frac{2}{\tan^2 t}\, dt$ ✓

$\therefore \quad g\,'(x) = -\dfrac{2}{\tan^2(2x)} \times 2 = -\dfrac{4}{\tan^2 2x}$ ✓✓ [9]

4. (a) 6 represents $\frac{1}{8}$ of sample then there were 48 responses. ✓

$\therefore \quad g = 10$ ✓

$\therefore \quad f = 12$ ✓

and $\quad h = \frac{1}{4}$ ✓

(b) $E(X) = 1 \times \frac{1}{8} + 2 \times \frac{1}{6} + 3 \times \frac{1}{4} + 4 \times \frac{1}{4} + 5 \times \frac{5}{24}$ ✓

$\therefore \quad E(X) = 3\frac{1}{4}$ ✓ [6]

5. (a) (i) $\int_{-1}^{0} f(x)\ dx - \int_{0}^{2} f(x)\ dx$ ✓✓

(ii) $\int_{u}^{p} [\ f(x) - g(x)\ ]\ dx$ ✓✓

(b) $\frac{d}{dx}(\ f(x)) = \frac{d}{dx}(x^3 - x^2 - 2x) = 3x^2 - 2x - 2$ ✓

$\therefore \quad 3x^2 - 2x - 2 = 3$ ✓

$\therefore \quad 3x^2 - 2x - 5 = 0$

$\therefore \quad (3x - 5)(x + 1) = 0$ ✓

$\therefore \quad p = \frac{5}{3}$ ✓ [8]

6. (a) (i) $\int_{-4}^{5} h(t)\ \mathrm{d}t = \int_{-4}^{0} h(t) + \int_{0}^{5} h(t)\ \mathrm{d}t$

$\therefore \quad \int_{0}^{5} h(t)\ \mathrm{d}t = 14$ ✓

(ii) $\int_{-4}^{0} [2 - h(t)]\ \mathrm{d}t = \int_{-4}^{0} 2\ \mathrm{d}t - \int_{-4}^{0} h(t)\ \mathrm{d}t$

$= \quad 8 - (-4) = 12$ ✓✓

(iii) $2 + 3\int_{-4}^{5} h(t)\ \mathrm{d}t = 2 + 30 = 32$ ✓✓

(b) We would need to know where $h(t)$ lies below the x-axis. ✓ [6]

7. $(0, 2) \rightarrow e = 2$ ✓

$y' = 4ax^3 + 3bx^2 + cx + d$ ✓

$\therefore \quad y' = 0$ when $x = 0 \rightarrow d = 0$ ✓

$y'' = 12ax^2 + 6bx + 2c$ ✓

$\therefore \quad y'' = 0$ when $x = 0 \rightarrow c = 0$ ✓

and $\quad y'' = 0$ when $x = -2 \rightarrow 48a - 12b = 0$ ✓

$(-2, 0) \rightarrow 0 = 16a - 8b + 2 \rightarrow 16a - 8b = -2$ ✓

Hence $\quad b = \dfrac{1}{2}$ and $a = \dfrac{1}{8}$ ✓ [8]

Calculator-Assumed Solutions

8. (a) $a + b = 26$ ✓

$$\frac{30 + 80 + 150 + 20a + 25b}{50} = 16.6$$ ✓

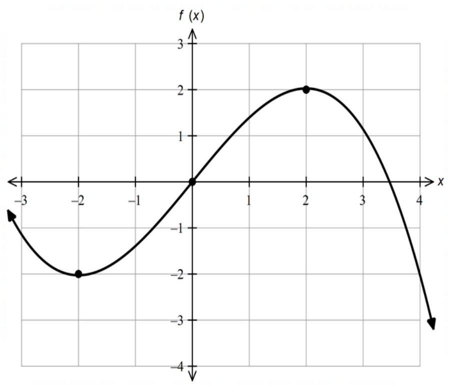
$\therefore \quad \boldsymbol{a} = 16$ and $\boldsymbol{b} = 10$ ✓✓

(b) Standard deviation = 6.44 ✓✓

(c) (i) $16.6\left(1 - \dfrac{d}{100}\right)$ ✓

(ii) $41.47\left(1 - \dfrac{d}{100}\right)^2$ ✓ ✓ [9]

9. (a)



✓✓✓✓

(b) (i) $\displaystyle\int_{-2}^{2} f'(x) = f(2) - f(-2) = 2 - (-2) = 4$ ✓✓

(ii) $\displaystyle\int_{-2}^{2} f''(x) = f'(2) - f'(-2) = 0 - 0 = 0$ ✓✓

(c) Area = $\displaystyle\int_{-2}^{2} |f'(x)|\ dx = \int_{-2}^{2} f'(x)\ dx$ since positive ✓

$= f(2) - f(-2)$

$= \ 4$ ✓ [10]

10. (a) $2.5 = 2e^{k \times 5}$ ✓

$\therefore \quad k = 0.04463$ ✓

$\therefore \quad M(t) = 2\ 000\ 000e^{0.04463t}$ ✓

(b) $M(15) = 2\ 000\ 000e^{0.04463 \times 15}$ ✓

$= 3\ 906\ 325 = 3\ 906\ 500$ ✓

(c) $M'(t) = 89\ 260e^{0.04463t}$ ✓

$\therefore \quad M'(15) = 174\ 339$ microbes/min ✓

(d) $2\ 000\ 000 = 3\ 906\ 500e^{-0.05t}$ ✓

$\therefore \quad t = 13.39$ min ✓

$\therefore$ 12:58 pm ✓ [10]

11. (a) (i) $\text{Length}(x) = \sqrt{x^2 + 16} + \sqrt{(30 - x)^2 + 100}$ ✓

$$L'(x) = \frac{1}{2}(16 + x^2)^{-\frac{1}{2}}(2x) + \frac{1}{2}(x^2 - 60x + 1000)^{-\frac{1}{2}}(2x - 60)$$ ✓

$$\therefore \quad L'(x) = \frac{x}{\sqrt{x^2 + 16}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1000}}$$ ✓

$\therefore$ Min occurs when $L'(x) = 0$ ✓

$$\therefore \quad \frac{x}{\sqrt{x^2 + 16}} = \frac{30 - x}{\sqrt{x^2 - 60x + 1000}}$$

(ii) $x = 8.57\ m$ ✓

(b) $\delta\theta = -0.01$ and $\tan\theta = \dfrac{21.43}{10} = 2.143 \rightarrow \theta = 1.134$ ✓

$\tan\theta = \dfrac{30 - x}{10}$ ✓

$\therefore \quad x = 30 - 10\tan\theta \rightarrow \dfrac{dx}{d\theta} = -\dfrac{10}{\cos^2\theta}$ ✓

$\therefore \quad \delta x = -\dfrac{10}{\cos^2\theta} \times (-0.01)$ ✓

$\therefore$ When $\theta = 1.134$, $\delta x = 0.559\ m$ ✓ [10]

12. (a) (i) $x < -2$ or $1 < x < 2$ ✓✓
(ii) $x = -2, 1, 2$ ✓✓
(iii) $x < -1.4$ or $x > 1.6$ ✓✓
(b) $x = -1.4$ or $x = 1.6$ ✓✓ [8]

13. (a) $2\sin(4\pi) = 0$ GL/h ✓

(b) $W = \int 2\sin\left(\frac{\pi t}{3}\right)\,dt = -\frac{6}{\pi}\cos\left(\frac{\pi t}{3}\right) + c$ ✓

$(0, 3) \rightarrow c = \frac{6}{\pi} + 3 \rightarrow W(t) = -\frac{6}{\pi}\cos\left(\frac{\pi t}{3}\right) + 3 + \frac{6}{\pi}$ ✓

$\therefore$ At midday $W(12) = 3\ GL$ ✓

(c) Amount = $\int_0^5 2\sin\left(\frac{\pi t}{3}\right)\,dt = 0.955\ GL$ ✓✓

(d) $W(t) = -\frac{6}{\pi}\cos\left(\frac{\pi t}{3}\right) + 3 + \frac{6}{\pi} = 4 \rightarrow t = 1.03$ ✓

$\therefore$ 1:02 am ✓ [8]

14. (a) DRV and involves success or failure ✓

(b) (i) $(0.65)^5 = 0.1160$ ✓✓

(ii) $(0.65)^3(0.35)^2 = 0.0336$ ✓✓

(iii) $X \sim B(5, 0.65)$

$\therefore$ $P(X \geq 2) = 0.9460$ ✓✓

(iv) $(1 - h)^5$ ✓✓

(c) 1 - P(0) = 0.99757
$\therefore$ P(0) = 0.00243

$\therefore$ $(1 - h)^5 = 0.00243$ ✓
$\therefore$ $h = 0.7$ ✓
(d) VAR = 0.65 x 0.35 = 0.2275 ✓
$\therefore$ St dev = 0.4770 ✓ [13]

15. (a) $f'(x) = \frac{\sqrt{3}}{4} + \frac{1}{2}\cos\left(\frac{x}{2}\right)$ ✓

TP occurs when $f'(x) = 0$

$\therefore \quad \frac{1}{2}\cos\left(\frac{x}{2}\right) = -\frac{\sqrt{3}}{4} \rightarrow \cos\left(\frac{x}{2}\right) = -\frac{\sqrt{3}}{2}$ ✓

$\therefore \quad \frac{x}{2} = \frac{5\pi}{6} \rightarrow x = \frac{5\pi}{3}$ km ✓✓

and height = 2.767 m ✓

(b) Max gradient occurs when $f''(x) = 0$

$\therefore \quad -\frac{1}{4}\sin\left(\frac{x}{2}\right) = 0$ ✓

$\therefore \quad x = 0, 6.28, 12.57, 18.85$ ✓

$\therefore$ Max gradient = 0.93 ✓ [8]

16. (a) $f'(x) = \sin x + x\cos x$ ✓✓

(b) (i) $\int f'(x)\,dx = \int \sin x\,dx + \int x\cos x\,dx$ ✓

$\therefore \quad f(x) = -\cos x + \int x\cos x\,dx$ ✓

$\therefore \quad \int x\cos x\,dx = f(x) + \cos x + c$ ✓

$\therefore \quad \int x\cos x\,dx = x\sin x + \cos x + c$

(ii) $\int_0^{\pi} x\cos x\,dx = \left[x\sin x + \cos x\right]_0^{\pi}$ ✓

$= (0 - 1) - (0 + 1) = -2$ ✓

(c) $\int_0^{\pi} |x\cos x|\,dx = 3.14$ ✓ ✓ [9]

17. (a) Uniform ✓

(b) (i) $\frac{1}{5}$ ✓

(ii) $\frac{\frac{2}{5}}{\frac{3}{5}} = \frac{2}{3}$ ✓

(iii) $\frac{1}{2}(2 + 6) = 4$ ✓

(iv) $VAR = \frac{1}{12}(6 - 2 + 1)^2 = 2$ ✓

∴ Std Dev = $\sqrt{2}$ ✓

(c) (i) E(Y) = 3 - 2(4) = -5 ✓

(ii) Std Dev(Y) = 2(Std Dev(X))

∴ = $2\sqrt{2}$ ✓

∴ VAR(Y) = 8 ✓ [9]

18. (a) $x(t) = \int e^{\sin 2t} \cos 2t \, dt$

Since $\frac{d}{dx}\left(e^{\sin 2t}\right) = 2e^{\sin 2t}\cos 2t$

∴ $x(t) = \int e^{\sin 2t} \cos 2t \, dt = \frac{1}{2} e^{\sin 2t} + c$ ✓✓

Since x(0) = 0 $\rightarrow c = -\frac{1}{2}$

∴ $x\left(\frac{\pi}{4}\right) = \frac{1}{2} e - \frac{1}{2}$ ✓

(b) $\frac{d}{dt}(v(t)) = 0$ when $t = 0.3331$ ✓

$v(t) = 0$ when $t = 0.7854$

∴ Total distance = $\int_0^{0.3331} e^{\sin 2t} \cos 2t \, dt = 0.428$ ✓✓ [6]