ALL SAINTS' COLLEGE

Semester One Examination, 2021

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section One: Calculator-free

SOLUTIONS

WA student number: In figures | | | | | | | | |

In words _____

_____

Your name _____

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable): | |

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

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Section One: Calculator-free **35% (52 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

(a) Determine $\frac{d}{dx}\left(\cos^4(x)\right)$. (2 marks)

Solution
$-4\sin x\cos^3 x$
Specific behaviours
✓ indicates use of chain rule
ü correct derivative

- Generally ok. Main error was forgetting to use the chain rule and the negative sign.

(b) Evaluate $f'\left(\frac{\pi}{2}\right)$ when $f(x)=\frac{x+\sin x}{\cos 2x}$. (4 marks)

Solution
$f'(x)=\frac{(1+\cos x)(\cos 2x)-(x+\sin x)(-2\sin 2x)}{\cos^2 2x}$ $f'\left(\frac{\pi}{2}\right)=\frac{(1-0)(-1)-0}{(-1)^2}¿-1$
Specific behaviours
✓ indicates use of quotient rule
ü correct u' and v'
ü correct derivative
ü substitutes and simplifies

- Finding the derivative by using the quotient rule was done well by the majority.
- Main error was incorrect exact values. Many had $\cos(\pi ¿=1$ rather than -1. Students must know their exact values for both sin and cos.

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Supplementary page

Question number: _____

The random variable $W = 10T + 2$.

(b) Determine $E(W)$ and $Var(W)$. (3 marks)

Solution
$E(W) = 10E(T) + 2 = 10\left(\frac{3}{10}\right) + 2 = 5$
$Var(T) = \frac{3}{10} \times \frac{7}{10} = \frac{21}{100}$
$Var(W) = 10^2 \times Var(T) = 10^2 \times \frac{21}{100} = 21$

Specific behaviours
✓ $E(W)$
ü indicates $Var(T)$
ü

- As a result of many not completing part(a) this was often not done.
- Those who completed part(a) correctly did this well.

Question 2 **(5 marks)**

A small body is initially at the origin. It is moving along the x-axis with velocity at time t seconds given by

$$v(t) = \left(\frac{t}{3} - 2\right)^3 \text{ cm/s}.$$

(a) Determine $x(t)$, a function for the displacement of the body at time t. (3 marks)

Solution
$x(t) = \int \left(\frac{t}{3} - 2\right)^3 dt ¿ \frac{3}{4}\left(\frac{t}{3} - 2\right)^4 + c$
$t = 0 \Rightarrow \frac{3}{4}(-2)^4 + c = 0 \Rightarrow c = -12$
$x(t) = \frac{3}{4}\left(\frac{t}{3} - 2\right)^4 - 12$

Specific behaviours
✓ reasonable attempt at using chain rule
ü correct antiderivative
ü correct displacement function

- Finding the integral was ok but not brilliant. Many still need to work on their basics.
- Main error was assuming +c = 0 rather than substituting and evaluating for c.

The small body is stationary when $t = T$.

(b) Determine the displacement of the body at $T + 3$ seconds. (2 marks)

Solution
$\frac{T}{3} - 2 = 0 \Rightarrow T = 6 \text{ s}$
$x(9) = \frac{3}{4}(1)^4 - 12 ¿ -11.25 \text{ cm}$

Specific behaviours
✓ correct value of T
ü correct displacement

- Some did not recognise that v(t)= 0 is when the body is stationary. Algebra was also lacking in solving a cubic equal to 0.
- Many substituted T+3 into the velocity equation rather than finding the value of T and then adding 3.

Question 3 **(6 marks)**

Determine the area of the finite region bounded by $y=\sqrt{2x}$ and $y=\frac{x}{2}$.

Solution

Points of intersection:

$$\sqrt{2x}=\frac{x}{2}\quad x^2-8x=0\quad x=0,x=8$$

Area:

$$A=\int_0^8 \sqrt{2x}-\frac{x}{2}\,dx = \left[\frac{(2x)^{\frac{3}{2}}}{3}-\frac{x^2}{4}\right]_0^8 = \left[\frac{(16)^{\frac{3}{2}}}{3}-\frac{8\times 8}{4}\right]-0$$

$$=\frac{64}{3}-16=\frac{16}{3}\text{ u}^2$$

Specific behaviours

✓ equates curves and squares
ü points of intersection
ü writes integral for area
ü correct antiderivative
ü substitutes
ü simplifies to obtain area

- Poorly done question. Students must be able to find the area between curves. More work required on this topic.
- Major errors include not able to solve two equations that are equal. Very poor algebra. Squaring $\frac{x}{2}=\frac{x^2}{4}$ not $\frac{x^2}{2}$. More work required on basic algebraic skills.
- Most understood the required integration and had the correct integral statement. Integrating $\sqrt{2x}$ caused many problems. Again basic integral skills need improving. Many students had $2x^{\frac{3}{2}}$ rather than $(2x)^{\frac{3}{2}}$.

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Question 8 **(7 marks)**

The following table shows the probability distribution for the random variable T.

t	0	1
$P(T=t)$	$\frac{1}{5k}-\frac{3}{10}$	$\frac{1}{2}-k$

(a) Determine the value of the positive constant k and hence state $P(T=0)$. (4 marks)

Solution

$$\frac{1}{5k}-\frac{3}{10}+\frac{1}{2}-k=1\quad 2-3k+5k-10k^2=10k$$

$$10k^2+8k-2=0\quad 5k^2+4k-1=0$$

$$(5k-1)(k+1)=0\quad k=\frac{1}{5}$$

Hence

$$P(T=0)=1-\frac{3}{10}=\frac{7}{10}$$

Specific behaviours

✓ sums probabilities to 1
ü forms quadratic equal to 0
ü solves quadratic, states unique value of k
ü states probability

- The majority of students knew that the probabilities add to 1.
- Poor algebra skills in solving this algebraic fraction = 1 resulted in many not completing this question. Not knowing how to solve the equation $5k^2+4k=1$ or rewriting $\frac{1}{5k}$ as $\frac{1}{5}k$ created problems. This needs urgent attention.
- Poorly done.

End of questions

(b) Determine, with justification, the range of $f'(x)$. (4 marks)

Solution

$$f'(x)=\frac{-8x}{(x^2+12)^2}$$

As $x \to \pm\infty, f'(x) \to 0$.

Minimum and maximum of $f'(x)$ will be when its derivative $f''(x)=0$, (i.e., at points of inflection) and from part (a) this is when $x=\pm 2$.

$$f'(\pm 2)=\pm\frac{-8\times 2}{16\times 16}=\mp\frac{1}{16}$$

Hence the range is:

$$\frac{-1}{16}\le f'(x)\le\frac{1}{16}.$$

Specific behaviours

✓ expression for $f'(x)$
ü states behaviour of $f'(x)$ for $x \to \pm\infty$
ü location of minimum and maximum values of $f'(x)$

- Many blank answers. First derivative found in most cases correctly. Most had no idea of how to find the range of values.
- Poorly done and again needs more work.

Question 4 **(8 marks)**

(a) State three key characteristics of a chance experiment that make it suitable for modelling by a binomial random variable. (3 marks)

Solution

1. There are a fixed number of identical and independent trials.
2. There are only two possible outcomes for each trial ('success' and 'failure').
3. The probability of 'success' is the same in every trial.

Specific behaviours

✓ identifies one characteristic
✓ identifies second characteristic
✓ identifies third characteristic

- Quite generous with many comments. If there are 3 marks then find 3 comments – this was also stated in the question.
- Many forgot *independent trials* and *two possible outcomes for each trial.*
- In future start *Multiple Bernoulli trials* rather than just *multiple trials*.

Research has shown that 10% of dogs between the ages of 5 and 8 have some form of heart disease. A random sample of 70 dogs is selected from a large number of dogs of this age. Let X be the number of dogs in the sample with some form of heart disease.

(b) Explain why randomly selecting one dog and recording whether it has some form of heart disease is a Bernoulli trial. (1 mark)

Solution

It is a **chance experiment** (dog is selected at random) with **two possible outcomes** (dog has some form of heart disease, or it does not).

Specific behaviours

✓ mentions both bolded phrases, or their equivalent

- Generally well done.

(c) Write a numerical expression for the probability that 8 dogs in the sample have some form of heart disease. (2 marks)

Solution
$X \sim B(70, 0.1)$ $P(X=8)=\binom{70}{8}(0.1)^8(0.9)^{62}$
Specific behaviours
ü indicates binomial distribution
✓ correct expression

- Ok but not brilliant.
- State that the distribution is binomial and its associated parameters.
- Include the values in the binomial distribution formula found on the formula sheet.

(d) State the mean and variance of X. (2 marks)

Solution
$E(X)=70\times0.1=7$ $Var(X)=7\times0.9=6.3$
Specific behaviours
✓ correct mean
ü correct variance

- Ok but not brilliant. Many did not use n = 70 and used only 0.1 and 0.9.More work needed on expected value and variance.

Question 7 **(8 marks)**

The function f is defined by $f(x)=\frac{4}{x^2+12}$, so that $f''(x)=\frac{24(x^2-4)}{(x^2+12)^3}$.

(a) Describe the concavity of the graph of $y=f(x)$. (4 marks)

Solution
$f''(x)=0\Rightarrow x^2-4=0\Rightarrow x=\pm2$ $x<-2, f''(x)>0 \quad -2<x<2, f''(x)<0 \quad x>2, f''(x)>0$ f is concave up when $x<-2$ and $x>2$. f is concave down when $-2<x<2$.
Specific behaviours
✓ solves $f''(x)=0$
ü indicates sign of $f''(x)$ in three intervals
ü states domains for concave up, down
ü uses correct inequalities in domains
(penalise ambiguous language such as between -2 and 2, etc.)

- Badly done question. Most had no idea of concavity and its relationship with f. Many played around with f'.
- Most commented on concavity related to $f'(x)>0 \vee ¿0$rather than on the values of x.
- This needs much more work.

Question 5 **(7 marks)**

A four-sided die has faces marked with the numbers $1, 1, 2$ and 3. All faces have an equal chance of landing face down after the die is rolled. A game, that costs $\$2$ to play, involves throwing the die twice and adding the two numbers that land face down. If the total score is 6, the player wins $\$30$, and otherwise they win nothing.

Let X be the total score obtained in one play of the game.

(a) Construct a probability distribution table for X. (3 marks)

Solution

x	2	3	4	5	6
$P(X=x)$	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{5}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

Specific behaviours

✓ table with label x and correct x values

ü label $P(X=x)$ and at least two correct probabilities

ü wholly correct pd table

- Generally well done.
- Major errors were not adding the two values together and some incorrect probabilities.

(b) Determine $E(X)$. (1 mark)

Solution

$$E(X)=\frac{8+12+20+10+6}{16}=3.5$$

Specific behaviours

ü correct $E(X)$

- Well done.

Let Y be the net monetary loss, in dollars, of a player in **two** plays of the game.

(c) Determine $E(Y)$. (3 marks)

Solution

t	2	-28
$P(T=t)$	$\frac{15}{16}$	$\frac{1}{16}$

Let T be monetary loss in one game, then $E(T)=\frac{30-28}{16}=\frac{1}{8}$

.

Hence $E(Y)=2\times E(T)=\frac{2}{8}=\$0.25$.

Specific behaviours

✓ indicates possible losses with probabilities in one game

- Not well done by the majority.
- Most did not read the question – two plays and Y = monetary loss.
- Incorrect values for t and P(T = t).

Question 6 **(5 marks)**

(a) Determine $\frac{d}{dx}\left(3x\cdot\sqrt[3]{e^x}\right)$. (2 marks)

Solution

$$\frac{d}{dx}\left(3x\cdot e^{\frac{x}{3}}\right)=3e^{\frac{x}{3}}+xe^{\frac{x}{3}}$$

Specific behaviours

✓ uses product rule

ü obtains correct result

- Generally well done.

(b) Hence, or otherwise, determine $\int\left(3x\cdot\sqrt[3]{e^x}\right)dx$. (3 marks)

Solution

$$\int\frac{d}{dx}\left(3x\cdot e^{\frac{x}{3}}\right)dx=\int 3e^{\frac{x}{3}}dx+\int xe^{\frac{x}{3}}dx$$

$$3xe^{\frac{x}{3}}=9e^{\frac{x}{3}}+\int xe^{\frac{x}{3}}dx$$

$$3\int xe^{\frac{x}{3}}dx=\int\left(3x\cdot\sqrt[3]{e^x}\right)dx=9xe^{\frac{x}{3}}-27e^{\frac{x}{3}}+c$$

Specific behaviours

✓ integrates all terms of result from (a)

ü uses fundamental theorem to simplify LHS

ü obtains required result, with constant

- Poorly done by most students. More work needed on this topic as this is a common question.
- First statement was often done well. Poor setting out resulted in incorrect working. On LHS - Integral of derivative resulted in $3xe^{\frac{x}{3}}$. Many then took the integral of this rather than using the RHS integral of $xe^{\frac{x}{3}}$.