**Semester One Examination, 2021**

**Question/Answer Booklet**

# MATHEMATICS METHODS

## ATAR Year 12

## Section Two: Calculator-assumed

Student Name: **SOLUTIONS**

Please circle your teacher's name

**Teacher: Miss Hosking Miss Rowden**

## Time allowed for this paper

| | |
|---|---|
| Reading time before commencing work: | 10 minutes |
| Working time for paper: | 100 minutes |

## Materials required/recommended for this paper

***To be provided by the supervisor***

This Question/Answer Booklet

Formula Sheet (retained from Section One)

| | |
|---|---|
| Number of additional answer booklets used (if applicable): | |

***To be provided by the candidate***

| | |
|---|---|
| Standard items: | pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters |
| Special items: | drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination |

## Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

## Structure of this paper

| Section | Number of questions available | Number of questions to be answered | Suggested working time (minutes) | Marks available | Percentage of examination |
|---|---|---|---|---|---|
| Section One: Calculator free | 8 | 8 | 50 | 51 | 35 |
| Section Two: Calculator-assumed | 13 | 13 | 100 | 97 | 65 |
| | | | | **Total** | 100 |

## Instructions to candidates



1. The rules for the conduct of the ATAR course examinations are detailed in the *Year 12 Information Handbook 2020*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Supplementary pages for the use planning/continuing your answer to a question have been provided at the end of the Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

**Section Two: Calculator-assumed** **65% (97 Marks)**

This section has thirteen (13) questions. Answer **all** questions. Write your answers in the spaces
provided.

Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

Working time: 100 minutes.

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**Question 9** **(7 marks)**

A capacitor in a circuit starts to discharge. The voltage $V$ across the capacitor after $t$ milliseconds is changing at a rate given by

$$\frac{dV}{dt}=\frac{-156}{(3t+2)^2}, t\geq 0.$$

(a) Calculate the initial rate of change of voltage. (1 mark)

| Solution |
|---|
| $V'(0)=\frac{-156}{4}=-39\text{ V/ms}$ |
| **Specific behaviours** |
| ✓ correct rate |

(b) Determine the change in voltage during the fourth millisecond. (3 marks)

| Solution |
|---|
| $\Delta V=\int_3^4 \frac{-156}{(3t+2)^2}dt¿-\frac{78}{77}\approx -1.013\text{ V}$ |
| **Specific behaviours** |
| ✓ indicates correct interval of time |
| ü writes correct integral |
| ü correct change |

(c) Given that the initial voltage across the capacitor was 25 volts, determine the time for the voltage to fall to 1 volt. (3 marks)

| Solution |
|---|
| $\Delta V=1-25=-24$<br><br>$\Delta V=\int_0^T \frac{-156}{(3t+2)^2}dt¿\frac{52}{3T+2}-26$<br>Hence require<br>$\frac{52}{3T+2}-26=-24T=8\text{ ms}$ |
| **Specific behaviours** |
| ✓ indicates required change in voltage |
| ü expression for change in $V$ |
| ü calculates time |

See next page

**Question 10** **(8 marks)**

An online employment survey on a public internet forum attracted $72$ responses from health workers, of whom $18$ said that they were employed on a casual basis.

(a) Use the survey data to construct a $95\%$ confidence interval for the population proportion of health workers employed on a casual basis. (3 marks)

| Solution |
| --- |
| $0.25 \pm 1.96\sqrt{\dfrac{0.25(1-0.25)}{72}}$ <br> $(0.150, 0.350)$ |
| **Specific behaviours** |
| ✓ indicates use of correct method <br> ü correct $z$-value <br> ü calculates interval |



(b) Assuming the survey was reliable, determine the sample size required to conduct a follow-up survey so that a $99\%$ confidence interval for the population proportion of health workers employed on a casual basis will have a margin of error close to $0.07$. (3 marks)

| Solution |
| --- |
| $n=\dfrac{2.576^2 \times 0.25 \times (1-0.25)}{0.07^2} ¿ 253.9$ <br> Hence sample size of $254$ is required. |
| **Specific behaviours** |
| ✓ indicates use of correct method <br> ü correct $z$-value <br> ü calculates integer sample size |

(c) Identify and explain a possible source of bias that may arise from this type of survey. (2 marks)

| Solution |
| --- |
| Identify: volunteer, convenience, etc., sampling. <br> Explanation: likely to be biased because it is not a true random sample drawn from the population. |
| **Specific behaviours** |
| ✓ identifies possible source of bias <br> ü explains why it is a possible source of bias |

**Question 11** **(8 marks)**

A factory makes identical plastic key fobs in four different colours. $15\%$ are red, $20\%$ are green, $25\%$ are blue and the remainder orange. The key fobs are randomly packed into boxes of $120$.

Quality control at the factory randomly sample several boxes from the production line daily and record, amongst other things, the proportion of orange key fobs in each box.

(a) Describe the continuous probability distribution that the sample proportion of orange key fobs will approximate over time, including any parameters. (4 marks)

**Solution**

$$1-0.15-0.2-0.25=0.4$$

$$v=\frac{0.4\times(1-0.4)}{120}\approx 0.002 s=\sqrt{v}\approx 0.0447$$

The sample proportions will approximate a normal distribution with mean of $0.4$ and variance of $0.002$(or standard deviation of $0.0447$).

**Specific behaviours**

✓ indicates proportion of orange key fobs
✓ indicates normal distribution
ü correct mean
ü correct variance (or standard deviation)

(b) Calculate an approximation for the probability that the proportion of orange key fobs in a randomly chosen box is at least $35\%$. (2 marks)

**Solution**

$$X \quad N(0.4, 0.002) \quad N(0.4, 0.0447)$$

$$P(X\geq 0.35)=0.868$$

**Specific behaviours**

✓ defines sampling distribution
ü calculates probability

(c) Briefly explain why the distribution in part (a) is an approximation and state the key factor that determines the closeness of the approximation. (2 marks)

**Solution**

The true distribution of proportions is binomial.

The larger the sample size ($n$), the closer the normal distribution approximates the binomial distribution.

**Specific behaviours**

✓ states true distribution
ü states sample size as key factor

**Question 12** **(7 marks)**

A company packages salt in jars marked with a net weight of $225$ g. The weight of salt in the jars is normally distributed with a mean of $231.5$ g and a standard deviation of $3.9$ g.

(a) Determine the probability that a randomly selected jar contains less than the marked weight. (2 marks)

**Solution**

$$X \quad N(231.5, 3.9^2)$$

$$P(X<225)=0.0478$$

**Specific behaviours**

✓ states expression for probability

ü correct probability



(b) What is the probability that a randomly selected jar containing less than the marked weight contains less than $223$ g of salt? (2 marks)

**Solution**

$$P(X<223 \vee X<225)=\frac{P(X<223)}{P(X<225)}$$

$$¿\frac{0.01465}{0.0478}=0.3065$$

**Specific behaviours**

✓ states expression for conditional probability

ü calculates probability

(c) The company has decided that no more than $1$ in $300$ jars should contain less than the marked weight of salt. To achieve this, they will pack more salt in each jar and hence increase the mean of the distribution whilst maintaining the existing standard deviation. Determine the minimum increase in the mean required. (3 marks)

**Solution**

$$P(Z<z)=\frac{1}{300} \Rightarrow z=-2.713$$

$$\frac{225-\mu}{3.9}=-2.713\mu=235.6$$

$$235.6-231.5=4.1$$

Extra weight of salt is $4.1$ g.

**Specific behaviours**

✓ indicates $z$-score

ü equation for mean

ü solves for mean and states increase

**Question 13** **(7 marks)**

A small body starts from rest at point $A$ and moves in a straight line until it reaches point $B$, where it is again stationary.

The acceleration of the body $t$ seconds after leaving $A$ is $a$ m/s$^2$, where $a=0.12t-0.006t^2$.

Determine

(a) the time taken for the body to travel from $A$ to $B$. (3 marks)

| Solution |
|---|
| $v(t)=\int a\,dt=0.06t^2-0.002t^3+c$ <br> $v(0)=0\Rightarrow c=0$ <br><br> $v(t)=0\Rightarrow t=0,30$ <br><br> Hence body took 30 seconds to travel from $A$ to $B$. |
| **Specific behaviours** |
| ✓ indicates use of integration to obtain velocity <br> ü expression for $v(t)$ <br> ü solves $v(t)=0$ and states travel time |

(b) the distance from $A$ to $B$. (2 marks)

| Solution |
|---|
| Since $v(t)\neq 0$ within interval, then: <br><br> $d=\int_0^{30} v(t)\,dt=135\,\text{m}$ |
| **Specific behaviours** |
| ✓ writes integral <br> ü correct distance |

(c) the maximum velocity of the body between $A$ and $B$. (2 marks)

| Solution |
|---|
| Maximum velocity when $a=0$: <br> $a(t)=0\Rightarrow t=0,20$ <br><br> $v(20)=8\,\text{m/s}$ |
| **Specific behaviours** |
| ✓ indicates time <br> ü correct velocity |

**Question 14** **(8 marks)**

The level of atmospheric carbon dioxide $C$ in parts per million was measured by scientists at an Arctic base and was observed to increase from $322.9$ ppm on $1$ January $1967$, to $335.4$ ppm by
$1$ January $1976$.

The level can be modelled by equation $C = C_0 e^{kt}$, where $t$ is the number of years from the start of the year $1960$.

(a) Determine an expression for the constant $k$ in the form $a \ln(b)$ and hence show that its value is approximately $0.00422$. (3 marks)

| Solution |
|---|
| $1976 - 1967 = 9$ years |
| $e^{9k} = \frac{335.4}{322.9} k = \frac{1}{9} \ln\left(\frac{3354}{3229}\right) ¿ 0.00422$ |
| **Specific behaviours** |
| ✓ indicates correct time interval<br>ü uses ratio of values<br>ü correct expression for $k$ and evaluates |



(b) Determine the value of the constant $C_0$. (2 marks)

| Solution |
|---|
| $322.9 = C_0 e^{0.00422(7)} C_0 = 313.5$ |
| **Specific behaviours** |
| ✓ substitutes into equation<br>ü solves for $C_0$ |

(c) Calculate the level of atmospheric carbon dioxide at the start of the year $1995$. (1 mark)

| Solution |
|---|
| $C(35) = 363.4$ ppm |
| **Specific behaviours** |
| ✓ correct value |

(d) Determine the rate at which the level of atmospheric carbon dioxide was increasing at the start of the year 1995. (2 marks)

| Solution |
|---|
| $\Delta C = 363.4 \times 0.00422$<br>$¿ 1.53$ ppm/yr |
| **Specific behaviours** |
| ✓ indicates method<br>✓ calculates correct rate |

**Question 15** **(9 marks)**

A person drives to work $n$ times each month and on any one journey, the probability that they arrive late for work is $p$.

(a) When $n=16$ and $p=0.14$ determine the probability that

(i) they are late for work exactly twice in a month. (2 marks)

| Solution |
|---|
| $X \quad B(16, 0.14)$<br>$P(X=2)=0.2847$ |
| **Specific behaviours** |
| ✓ indicates binomial distribution<br>ü correct probability |

(ii) they are late for work at least once in a month. (1 mark)

| Solution |
|---|
| $P(X \geq 1)=0.9105$ |
| **Specific behaviours** |
| ✓ correct probability |

(iii) they are never late for work in at least one of three consecutive months. (3 marks)

| Solution |
|---|
| $1-0.91047=0.08953, Y \quad B(3, 0.08953)$<br>$P(Y \geq 1)=0.2453$ |
| **Specific behaviours** |
| ✓ indicates probability for never late<br>✓ indicates appropriate distribution<br>ü correct probability |

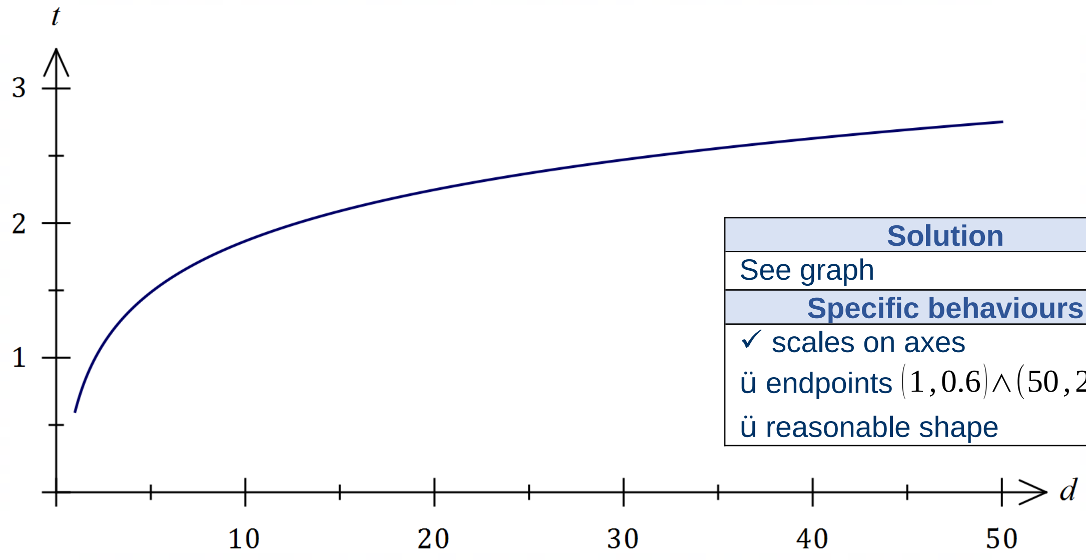
(b) Determine $n$ and $p$ when the mean and variance of the number of times the person is late for work each month is 3.2 and 2.688 respectively. (3 marks)

| Solution |
|---|
| $np=3.2, np(1-p)=2.688$<br>Solve simultaneously:<br>$n=20, p=0.16$ |
| **Specific behaviours** |
| ✓ equation for mean<br>ü equation for sd or variance<br>ü correct values |

**Question 16** **(7 marks)**

The time, $t$ seconds, for a trained rat to pick a bead out of a container and drop it into a small hole when the distance of the bead container from the hole was $d$ cm can be modelled by the relationship $t=0.6+0.55\ln(d)$ for $d\geq 1$.

(a) Sketch the graph of $t$ as a function of $d$ for $1\leq d\leq 50$ cm. (3 marks)






| Solution |
|---|
| See graph |
| **Specific behaviours** |
| ✓ scales on axes |
| ü endpoints $(1,0.6)\wedge(50,2.8)$ |
| ü reasonable shape |

(b) Determine the extra time taken by the rat to move a bead when the distance of the bead container from the hole increases from 20 cm to 60 cm. (1 mark)

| Solution |
|---|
| $t(60)-t(20)=2.852-2.248\text{¿}0.604\text{ s}$ |
| **Specific behaviours** |
| ✓ calculates change |

(c) Use the relationship to show that if the distance of the bead container from the hole increases from $x$ cm to $3x$ cm, the change in time is constant. (3 marks)

| Solution |
|---|
| $t(x)=0.6+0.55\ln(x)$ <br> $t(3x)=0.6+0.55\ln(3x)$ <br> $\text{¿}0.6+0.55(\ln 3+\ln x)$ <br> $\text{¿}0.6+0.55\ln 3+0.55\ln x$ <br> $\Delta t=t(3x)-t(x)\text{¿}0.55\ln 3$ <br> Hence change in time is a constant. |
| **Specific behaviours** |
| ✓ expressions for $t(x)$ and $t(3x)$ |
| ü isolates bolded term from $t(3x)$ |
| ü calculates change and deduces constant |

| Alternative Solution |
|---|
| $\Delta t=\int_x^{3x}\frac{dt}{dd}\,dd\text{¿}\int_x^{3x}\frac{0.55}{d}\,dd\text{¿}\frac{11\ln 3}{20}$ <br> Hence change in time is a constant. |
| **Specific behaviours** |
| ✓ integral for total change from rate of change |
| ü expression for rate of change |
| ü calculates change and deduces constant |

**Question 17** **(8 marks)**

The local newspaper in a large city claimed that over $75\%$ of the city's population trusted them. To check this claim, a research group took a random sample of $625$ people in the city and found that $450$ of them trusted the newspaper.

(a) Construct a $99\%$ confidence interval for the proportion of all people in the city who trust the newspaper and hence comment on the validity of the newspaper's claim. (4 marks)

| Solution |
|---|
| Calculation: $$0.72 \pm 2.576 \times \sqrt{\frac{0.72(1-0.72)}{625}}$$ Interval: $$(0.674, 0.766)$$ The claimed proportion of $0.75$ made by the newspaper is contained in the $99\%$ confidence interval and hence the claim is likely to be valid. |
| **Specific behaviours** |
| ✓ indicates correct calculation<br>ü correct interval<br>ü states claimed proportion in interval<br>ü states claim valid |

(b) The research group carried out the same sampling task in different city, from which the $95\%$ confidence interval $(0.448, 0.516)$ was constructed. Determine the number of people in this sample who trusted their local newspaper. (4 marks)

| Solution |
|---|
| $$z_{0.95} = 1.96$$ $$E = (0.516 - 0.448) \div 2 = 0.034$$ $$p = 0.448 + 0.034 = 0.482$$ $$n = \frac{1.96^2 \times 0.482 \times (1-0.482)}{0.034^2} = 830$$ $$x = 830 \times 0.482 = 400 \text{ people}$$ |
| **Specific behaviours** |
| ✓ uses correct $z$-score<br>ü indicates margin of error and proportion<br>ü calculates sample size |

ü calculates number of people



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**Question 18** **(6 marks)**

A player throws a regular tetrahedral die whose faces are numbered $1, 2, 3$ and $4$. If the player throws a three, the die is thrown a second time, and in this case the score is the sum of $3$ and the second number; otherwise, the score is the number obtained. The player has no more than two throws. Let $X$ be the random variable denoting the player's score.

(a) Write down the probability distribution of $X$. (3 marks)

**Solution**

| $x$ | 1 | 2 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $P(X=x)$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{5}{16}$ | $\frac{1}{16}$ | $\frac{1}{16}$ | $\frac{1}{16}$ |

**Specific behaviours**

✓ correct $x$ values

ü probabilities sum to $1$

ü correct probabilities



(b) Determine the mean and standard deviation of $X$. (2 marks)

**Solution**

$$E(X)=\frac{25}{8}=3.125$$

$$Var(X)=\frac{215}{64} \Rightarrow sd=\frac{\sqrt{215}}{8} \approx 1.833$$

**Specific behaviours**

✓ mean

ü standard deviation

(c) Determine $P(X=4 \vee X \geq E(X))$. (1 mark)

**Solution**

$$P=\frac{\frac{5}{16}}{\frac{1}{2}}=\frac{5}{8}$$

**Specific behaviours**

✓ correct probability

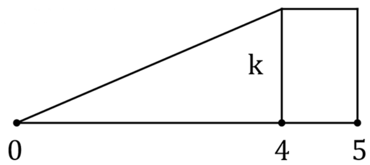
**Question 19** **(8 marks)**

An electronic device is powered by an AAA battery that will always last for a minimum of 12 hours. The random variable $T$ is the number of hours exceeding 12 for which the device will continue to operate, and it has probability density function $f$ shown below:

$$f(t)=\begin{cases} \frac{kt}{4} & 0\le t\le 4 \\ k & 4<t\le 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the value of the constant $k$. (3 marks)

**Solution**




$$\text{Area}=\frac{1}{2}\times 4\times k+k=3k$$

Since $f(t)$ is a pdf then $3k=1\Rightarrow k=\frac{1}{3}$.

Or using integral: $1=\int_0^4 \frac{kt}{4}\,dt+\int_4^5 k\,dt$

**Specific behaviours**

✓ sketch of function or correct integral

ü calculates area in terms of or correct limits

(b) Calculate the mean of $T$. (2 marks)

**Solution**

$$E(T)=\int_0^4 \frac{t^2}{12}\,dt+\int_4^5 \frac{t}{3}\,dt$$

$$¿\frac{16}{9}+\frac{3}{2}=\frac{59}{18}=3.2\overline{7}\text{h}$$

**Specific behaviours**

✓ correct expression for mean

ü calculates mean

(c) Given that $P(T\ge a)=0.865$, determine the value of the constant $a$. (3 marks)

**Solution**

$$P(T\ge a)=0.865\Rightarrow\int_0^a \frac{t}{12}\,dt=1-0.865$$

$$\frac{a^2}{24}=0.135,\ a^2=3.24,\ a=1.8$$

**Specific behaviours**

✓ integral with $a$ as a bound / indicates use of triangle area

ü evaluates integral $a$ / forms equation

ü positive value of



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**Question 20** **(7 marks)**

A popcorn container of capacity 660 mL is made from paper and has the shape of an open inverted cone of radius $r$ and height $h$.

Determine the least area of paper required to make the container.

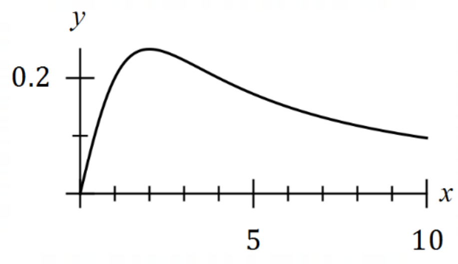


| Solution |
|---|
| $A=\pi rs=\pi r\sqrt{r^2+h^2}$<br>$V=\frac{1}{3}\pi r^2 h \Rightarrow h=\frac{3V}{\pi r^2}$<br>$A=\pi r\sqrt{r^2+\left(\frac{3(660)}{\pi r^2}\right)^2}$<br>$\frac{dA}{dr}=\frac{2r^6\pi^2-3920400}{r^2\sqrt{r^6\pi^2+3920400}}$<br>$\frac{dA}{dr}=0\ \text{when}\ r=7.638\ \text{cm}$<br>$A_{MIN}=317.5\ \text{cm}^2$ |
| **Specific behaviours** |
| ✓ expresses $A$ in terms of $r$ and $h$<br>✓ expresses $h$ in terms of $r$<br>✓ expresses $A$ in terms of $r$<br>✓ differentiates $A$<br>✓ finds positive zero of derivative<br>✓ substitutes to find minimum area<br>✓ uses second derivative or sign test to check min |

**Question 21** **(7 marks)**

The graph of $y=f(x)$ is shown, where

$$f(x)=\frac{x}{4+x^2}, x\geq 0.$$




$f(x)$ is concave down for $0<x<2\sqrt{3}$.

(a) Determine the area bounded by the graph of $f$ and the line $y=\frac{x}{20}$. (3 marks)

| Solution |
|---|
| $\frac{x}{4+x^2}=\frac{x}{20}\Rightarrow x=0,4$ <br> $\int_0^4 \frac{x}{4+x^2}-\frac{x}{20}\,dx=\frac{1}{2}\ln(5)-\frac{2}{5}\approx 0.405\text{ sq units}$ |
| **Specific behaviours** |
| ✓ solves for bounds <br> ü writes integral <br> ü calculates area |

The line $y=mx$ and the graph of $f$ enclose a finite region $R$.

(b) Determine the values of the slope $m$ for which $R$ exists. (2 marks)

| Solution |
|---|
| $f'(0)=\frac{1}{4}$ <br> Hence <br> $0<m<\frac{1}{4}$ |
| **Specific behaviours** |
| ✓ slope at origin <br> ü correct inequality |

(c) Determine the area of $R$ in terms of $m$. (2 marks)

| Solution |
|---|
| $\frac{x}{4+x^2}=mx\Rightarrow x=\sqrt{\frac{1-4m}{m}}$ <br> $\int_0^{\sqrt{\frac{1-4m}{m}}} \frac{x}{4+x^2}-mx\,dx=\frac{4m-\ln(m)-2\ln(2)-1}{2}$ |
| **Specific behaviours** |
| ✓ integral, with bounds <br> ü simplified expression for area |

**End of questions**

**End of questions**

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Supplementary page

Question number: ____________________

Supplementary page

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