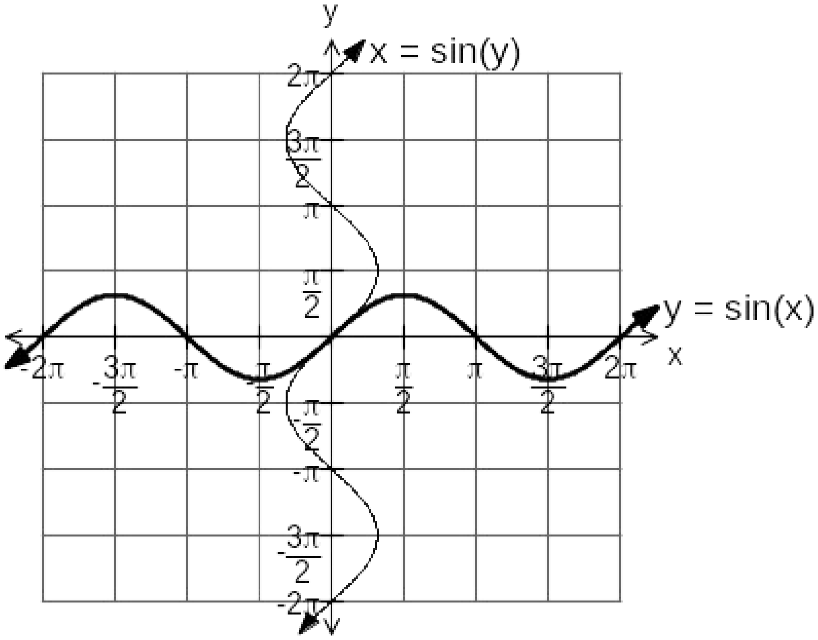
THE INVERSE TRIGONOMETRIC FUNCTIONS

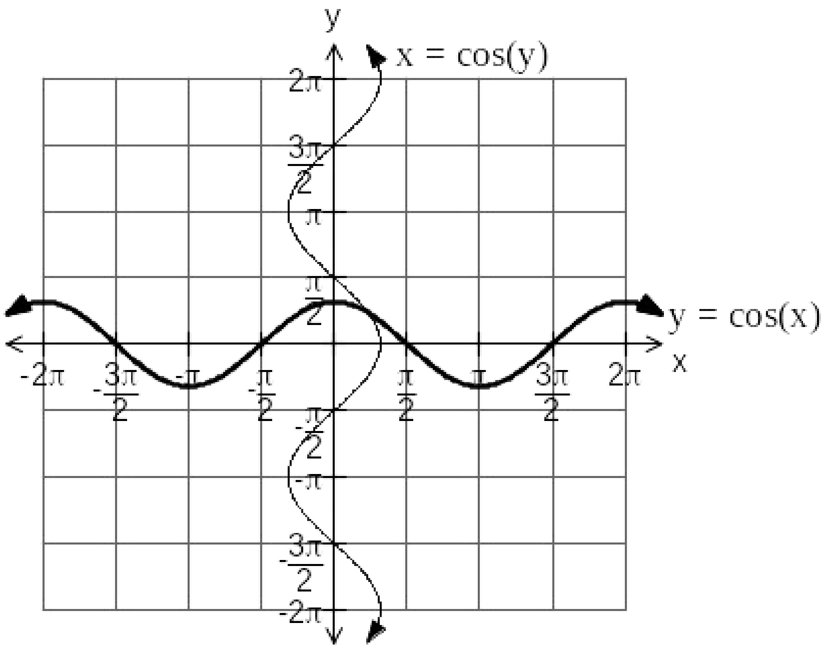
Consider the trigonometric functions and their inverse relationships below:

y = sin(x) and x = sin(y)



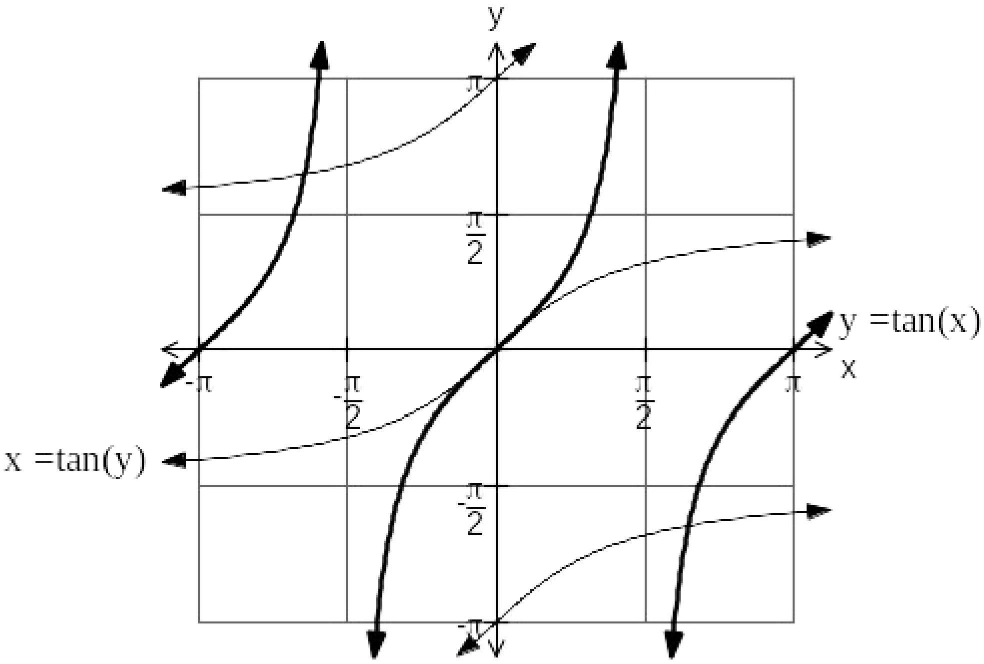
$x = \sin(y)$ can also be written $y = \sin^{-1}(x)$ or $y = \arcsin(x)$

y = cos(x) and **x = cos(y)**



$x = \cos(y)$ can also be written $y = \cos^{-1}(x)$ or $y = \arccos(x)$

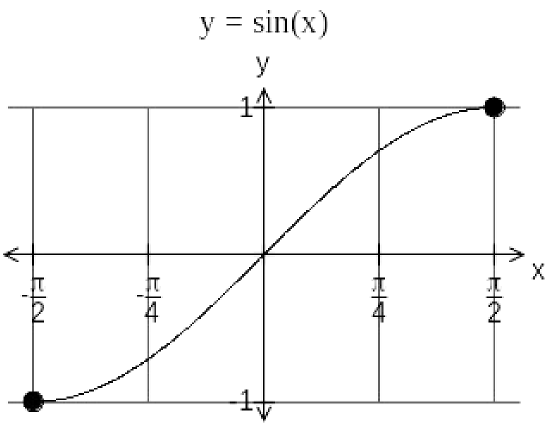
y = tan(x) and **x = tan(y)**

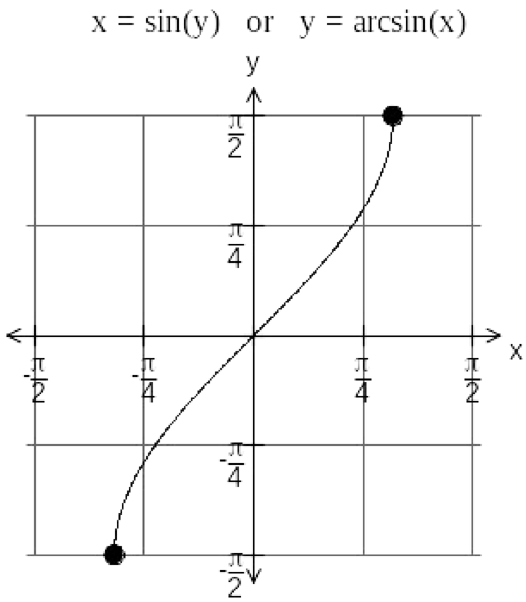


x = tan y) can also be written y = $\tan^{-1}(x)$ or y = ar c tan(x)

The inverse relation can be inverse functions if the domain is restricted,

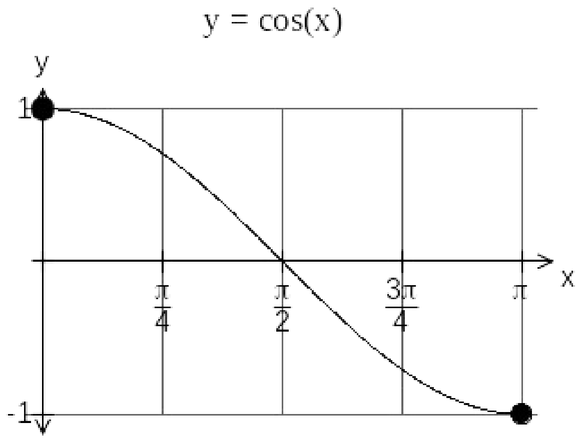
It is conventional to define the inverse functions as follows:

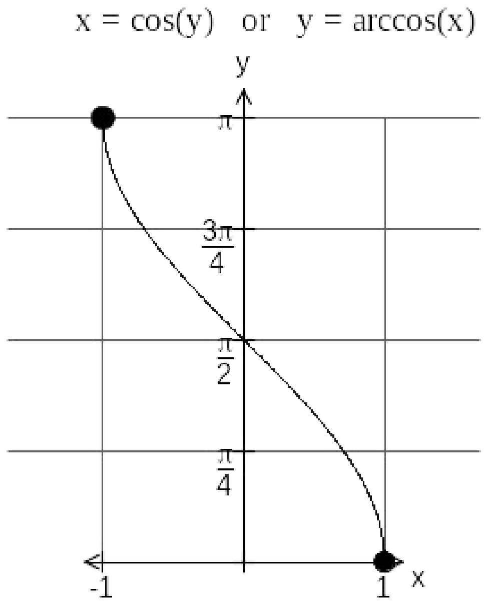




Domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and range [-1, 1]

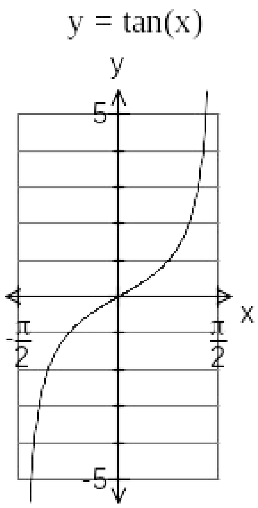
Domain [-1, 1] and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

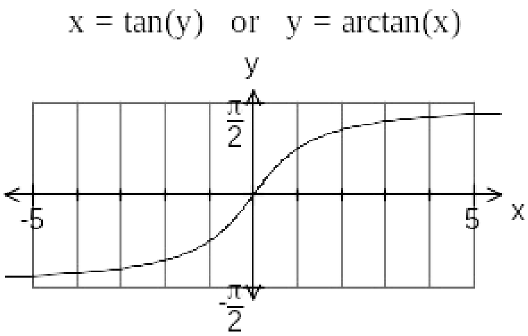




Domain $[0,\pi]$ and range [-1, 1]

Domain [-1, 1] and range $[0,\pi]$





Domain $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and range (-∞, ∞)

Domain (-∞, ∞) and range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Summary

Function	Domain	Range

y = $\sin^{-1}(x)$ = arc sin(x)	[-1, 1]	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
y = $\cos^{-1}(x)$ = arc cos(x)	[-1, 1]	$\left[0,\pi\right]$
y = $\tan^{-1}(x)$ = arc tan(x)	(-∞, ∞)	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

How to differentiate the inverse trig functions:

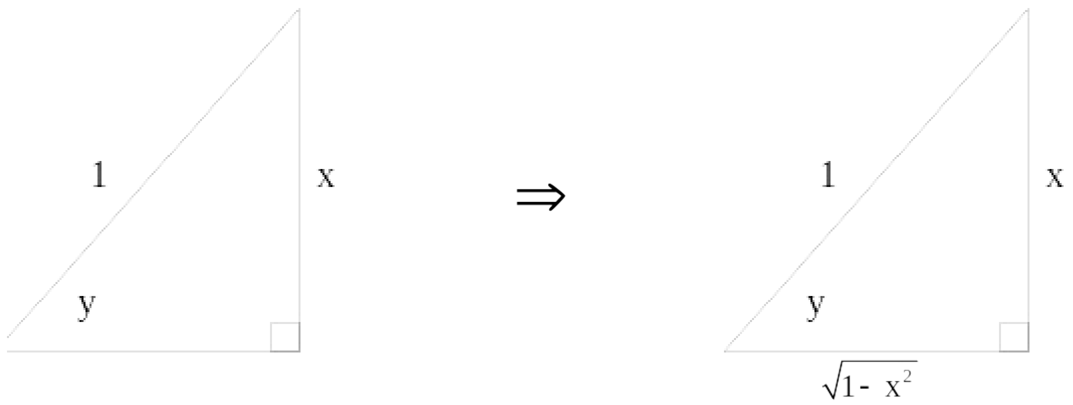
Given y = $\sin^{-1}(x)$ then x = sin (y). $\frac{dy}{dx}$ = ?

$$\frac{dx}{dy} = \cos(y) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2(y)}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Alternatively, we can consider the triangle where sin (y) = x



The third side is $\sqrt{1-x^2}$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} \quad \text{then} \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

so if .

Likewise given $y = \cos^{-1}(x)$ then $x = \cos(y)$. $\frac{dy}{dx} =$?

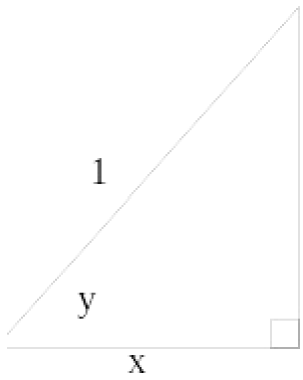
$$\frac{dx}{dy} = -\sin(y) \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin(y)}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\cos^2(y)}}$$

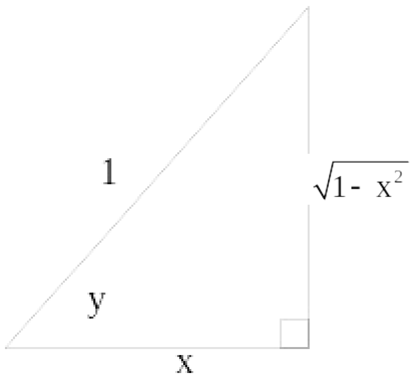
$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$x = \cos(y)$

OR using the triangle



$\Rightarrow$



If $\frac{dy}{dx} = -\frac{1}{\sin(y)}$ then $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

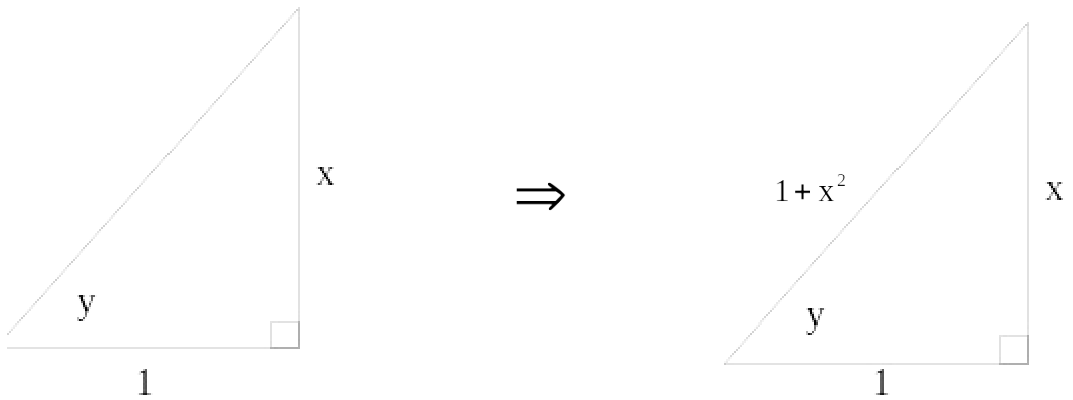
Given y = $\tan^{-1}(x)$ then x = tan (y). $\frac{dy}{dx} =$?

$$\frac{dx}{dy} = \sec^2(y) \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2(y)} \quad \text{but} \quad \sec^2(y) = \tan^2(y) + 1$$

$$\therefore \quad \sec^2(y) = x^2 + 1$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

OR



If $\frac{dy}{dx} = \frac{1}{\sec^2 x}$ then $\frac{dy}{dx} = \frac{1}{1+x^2}$

SUMMARY

Function	Derivative

y = $\sin^{-1}(x)$	$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$
y = $\cos^{-1}(x)$	$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$
y = $\tan^{-1}(x)$	$\frac{dy}{dx} = \frac{1}{1 + x^2}$

Also

Function	Derivative
y = $\sin^{-1}f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1 - (f(x))^2}}$
y = $\cos^{-1}f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1 - (f(x))^2}}$
y = $\tan^{-1}f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1 + (f(x))^2}$

Examples

Find the derivative of each of the following:

(a) $y = \tan^{-1}(x)$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

(b) $y = \tan^{-1}(5x)$

$$\frac{dy}{dx} = \frac{1}{1+(5x)^2} \times 5$$

$$\frac{dy}{dx} = \frac{5}{1+25x^2}$$

(c) $y = \tan^{-1}(4+x)$

$$\frac{dy}{dx} = \frac{1}{1+(4+x)^2}$$

$$\frac{dy}{dx} = \frac{1}{17+8x+x^2}$$

OR

$$\text{Put} \quad u = 4 + x$$

$$y = \tan^{-1} u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{1+u^2} \times 1$$

$$\frac{dy}{dx} = \frac{1}{1+(4+x)^2} \times 1$$

$$\frac{dy}{dx} = \frac{1}{17+8x+x^2}$$

(d) $y = \arcsin(x^2 - 3)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^2-3)^2}} \times 2x$$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{6x^2-8-x^4}}$$

In general

If $y = \arcsin(x^n)$ then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-(x^n)^2}} \times nx^{n-1}$

Also

Function	Derivative
$y = \sin^{-1}f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-(f(x))^2}}$
$y = \cos^{-1}f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-(f(x))^2}}$
$y = \tan^{-1}f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+(f(x))^2}$

More examples:

(e) $y = \cos^{-1}(3x-2)$

$$\frac{dy}{dx} = -\frac{3}{\sqrt{1-(3x-2)^2}}$$

$$\frac{dy}{dx} = -\frac{3}{\sqrt{-9x^2+12x-3}}$$

(f) $y = \tan^{-1}(\sqrt{x})$

$$\frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{1+\left(\sqrt{x}\right)^2}$$

$$\frac{dy}{dx} = \frac{1}{1+x} \times \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}\,(1+x)}$$

(g) $y = \sin^{-1}(4x^2)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(4x^2)^2}} \times 8x$$

$$\frac{dy}{dx} = \frac{8x}{\sqrt{1-(4x^2)^2}}$$

(h) $y = x^2\cos^{-1}(x)$ Using the product rule:

$$\frac{dy}{dx} = 2x\left(\cos^{-1}x\right) + \left(-\frac{1}{\sqrt{1-(x)^2}}\right)x^2$$

$$\frac{dy}{dx} = 2x\left(\cos^{-1}x\right) - \frac{x^2}{\sqrt{1-(x)^2}}$$

(i) If $f(x) = \sin^{-1}x + \cos^{-1}x$, show that $f'(x) = 0$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$f'(x) = 0$$

Mental work: <u>State</u> the derivatives of

$y = \sin^{-1}(6x)$	$y = \cos^{-1}(3x)$	$y = \tan^{-1}(2x)$
$y = \sin^{-1}(10x)$	$y = \cos^{-1}(8x)$	$y = \tan^{-1}(3x)$
$y = \sin^{-1}(5x)$	$y = \cos^{-1}(2x)$	$y = \tan^{-1}(6x)$
$y = \sin^{-1}(bx)$	$y = \cos^{-1}(px)$	$y = \tan^{-1}(kx)$
$y = \sin^{-1}(x^2)$	$y = \cos^{-1}(x^4)$	$y = \tan^{-1}(x^2)$
$y = \sin^{-1}(x^7)$	$y = \cos^{-1}(x^3)$	$y = \tan^{-1}(x^7)$
$y = \sin^{-1}(3x^4)$	$y = \cos^{-1}(2x^6)$	$y = \tan^{-1}(3x^3)$
$y = \sin^{-1}(10x^3)$	$y = \cos^{-1}(10x^3)$	$y = \tan^{-1}(8x^3)$
$y = \sin^{-1}(3x+4)$	$y = \cos^{-1}(2x+3)$	$y = \tan^{-1}(5x+2)$
$y = \sin^{-1}(-x+2)$	$y = \cos^{-1}(5-2x)$	$y = \tan^{-1}(2-3x)$

THE INTEGRATION OF TRIGONOMETRIC FUNCTIONS.

Given

Function	Derivative
y = $\sin^{-1}(x)$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$
y = $\cos^{-1}(x)$	$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$
y = $\tan^{-1}(x)$	$\frac{dy}{dx} = \frac{1}{1+x^2}$

It follows that

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + c$
$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \cos^{-1}(x) + c$
$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + c$

and

likewise

$\int \frac{f'(x)}{\sqrt{1-(f(x))^2}}\,dx = \sin^{-1}(f(x)) + c$
$\int -\frac{f'(x)}{\sqrt{1-(f(x))^2}}\,dx = \cos^{-1}(f(x)) + c$
$\int \frac{f'(x)}{1+(f(x))^2}\,dx = \tan^{-1}(f(x)) + c$

Examples

(a) $\int \frac{2}{\sqrt{1-(2x)^2}}\,dx = \sin^{-1}(2x) + c$ Experiment with the substitution $u = 2x$

(b) $\int \frac{-3}{\sqrt{1-(3x)^2}}\,dx = \cos^{-1}(3x) + c$

(c) $\int \frac{4}{1+(4x)^2}\,dx = \tan^{-1}(4x) + c$

(d) $\int \frac{3}{\sqrt{1-9x^2}}\,dx = \sin^{-1}(3x) + c$

(e) $\int \frac{-5}{\sqrt{1-25x^2}}\,dx = \cos^{-1}(5x) + c$

(f) $\int \frac{6}{\sqrt{1-9x^2}}\,dx = 2\int \frac{3}{\sqrt{1-9x^2}}\,dx = 2\sin^{-1}(3x) + c$

(g) $$\int \frac{1}{\sqrt{1-9x^2}}\,dx = \frac{1}{3}\int \frac{3}{\sqrt{1-9x^2}}\,dx = \frac{1}{3}\sin^{-1}(3x) + c$$

(h) $$\int \frac{-3}{\sqrt{1-16x^2}}\,dx = \frac{3}{4}\int \frac{-4}{\sqrt{1-16x^2}}\,dx = \frac{3}{4}\cos^{-1}(4x) + c$$

(i) $$\int \frac{6}{1+x^2}\,dx = 6\int \frac{1}{1+x^2}\,dx = 6\tan^{-1}(x) + c$$

(j) $$\int \frac{4}{1+(4x)^2}\,dx = \tan^{-1}(4x) + c$$

(k) $$\int \frac{6}{1+(6x)^2}\,dx = \tan^{-1}(6x) + c$$

(l) $$\int \frac{3}{1+9x^2}\,dx = \tan^{-1}(3x) + c$$

(m) $$\int \frac{4}{1+x^2}\,dx = 4\tan^{-1}(x) + c$$

(n) $$\int \frac{4}{1+4x^2}\,dx = 2\int \frac{2}{1+4x^2}\,dx = 2\tan^{-1}(x) + c$$

Sometimes the transformation required before we integrate is a little more complicated.

For example

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = ?$$

$$\frac{1/2}{1/2} \times \frac{1}{\sqrt{4-x^2}} = \frac{1}{2}\frac{1}{\sqrt{\frac{1}{4}(4-x^2)}} = \frac{1}{2}\frac{1}{\sqrt{\frac{4}{4}-\frac{x^2}{4}}} = \frac{1}{2}\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

$$\therefore \int \frac{1}{\sqrt{4-x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx$$

$$= \frac{1}{2}\frac{\sin^{-1}\left(\frac{x}{2}\right)}{\frac{1}{2}} + c$$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right) + c$$

The rule illustrated here is

Prove this rule:

$$\boxed{\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a^2}\tan^{-1}\left(\frac{x}{a}\right) + c}$$

Examples

(a) $\displaystyle\int \frac{1}{4^2 + x^2}\,dx = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + c$ because $\displaystyle\int \frac{1}{4^2 + x^2}\,dx = \int \frac{\frac{1}{16}\times 1}{1 + \frac{x^2}{16}}\,dx$

$$= \frac{1}{16}\int \frac{1}{1 + \left(\frac{x}{4}\right)^2}\,dx$$

$$= \frac{1}{16}\,\frac{\tan^{-1}\left(\frac{x}{4}\right)}{\frac{1}{4}} + c$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + c$$

(b) $\displaystyle\int \frac{1}{4 + x^2}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$ because $\displaystyle\int \frac{1}{2^2 + x^2}\,dx = \int \frac{\frac{1}{4}\times 1}{1 + \frac{x^2}{4}}\,dx$

$$= \frac{1}{4}\int \frac{1}{1 + \left(\frac{x}{2}\right)^2}\,dx$$

$$= \frac{1}{4}\,\frac{\tan^{-1}\left(\frac{x}{2}\right)}{\frac{1}{2}} + c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

(c) $\int \frac{1}{9+x^2}dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$ because $\int \frac{1}{3^2+x^2}dx = \int \frac{\frac{1}{9}\times 1}{1+\frac{x^2}{9}}dx$

$$= \frac{1}{9}\int \frac{1}{1+\left(\frac{x}{3}\right)^2}dx$$

$$= \frac{1}{9}\frac{\tan^{-1}\left(\frac{x}{3}\right)}{\frac{1}{3}}+c$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$$

(d) $\int \frac{3}{25+x^2}dx = \frac{3}{5}\tan^{-1}\left(\frac{x}{5}\right)+c$ because $\int \frac{3}{5^2+x^2}dx = 3\int \frac{\frac{1}{25}\times 1}{1+\frac{x^2}{25}}dx$

$$= \frac{3}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}dx$$

$$= \frac{3}{25}\frac{\tan^{-1}\left(\frac{x}{5}\right)}{\frac{1}{5}}+c$$

$$= \frac{3}{5}\tan^{-1}\left(\frac{x}{5}\right)+c$$

(e) $\int \frac{3}{25+9x^2}dx = \int \frac{3}{5^2+(3x)^2}dx$

$$= 3\int \frac{\frac{1}{25}\times 1}{1+\frac{9x^2}{25}}dx$$

$$= \frac{3}{25}\int \frac{1}{1+\left(\frac{3x}{5}\right)^2}dx$$

$$= \frac{3}{25}\int \frac{1}{1+\left(\frac{x}{5/3}\right)^2}dx$$

$$= \frac{3}{25}\frac{\tan^{-1}\left(\frac{x}{5/3}\right)}{\frac{3}{5}}+c$$

$$= \frac{1}{5}\tan^{-1}\left(\frac{3x}{5}\right)+c$$

1. Show that

(a) $$\int \frac{4}{16+4x^2}\,dx = 0.5\tan^{-1}(0.5x) + c$$

(b) $$\int \frac{5}{9+4x^2}\,dx = \frac{5}{6}\tan^{-1}\left(\frac{2x}{3}\right) + c$$

(c) $$\int \frac{-7}{25+9x^2}\,dx = -\frac{7}{15}\tan^{-1}\left(\frac{3x}{5}\right) + c$$

2. Use the substitution $u = x^{10}$ to show $\int \frac{10x^9}{\sqrt{1-x^{20}}}\,dx = \sin^{-1}(x^{10}) + c$.

3. Show that

(a) $$\int \frac{3x^5}{\sqrt{1-x^{12}}}\,dx = 0.5\sin^{-1}(x^6) + c$$

(b) $$\int \frac{1}{\sqrt{4-4x^2}}\,dx = 0.5\sin^{-1}(x) + c$$

(c) $$\int \frac{3}{\sqrt{4-9x^2}}\,dx = \sin^{-1}\left(\frac{3x}{2}\right) + c$$