PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Test 1 Year 12

Student name: _______________ Teacher name: _______________

Task type: **Response/Investigation**

Reading time for this test : 5 mins

Working time allowed for this task: 40 mins

Number of questions: 7

Materials required: No cals allowed!!

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, NO notes allowed!

Marks available: **41 marks**

Task weighting: **13%**

Formula sheet provided: no, but formulae stated on page 2

Note: All part questions worth more than 2 marks require working to obtain full marks.

Useful formulae

Complex numbers

Cartesian form	
$z = a + bi$	$\bar{z} = a - bi$
$\text{Mod}(z) = \lvert z\rvert = \sqrt{a^2 + b^2} = r$	$\text{Arg}(z) = \theta, \quad \tan\theta = \frac{b}{a}, \quad -\pi < \theta \le \pi$
$\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$	$\left\lvert \frac{z_1}{z_2} \right\rvert = \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$
$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$	$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
$z\bar{z} = \lvert z\rvert^2$	$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{\lvert z\rvert^2}$
$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$	$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
Polar form	
$z = a + bi = r(\cos\theta + i\sin\theta) = r\,cis\,\theta$	$\bar{z} = r\,cis(-\theta)$
$z_1 z_2 = r_1 r_2\,cis(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,cis(\theta_1 - \theta_2)$
$cis(\theta_1 + \theta_2) = cis\,\theta_1\,cis\,\theta_2$	$cis(-\theta) = \frac{1}{cis\,\theta}$
De Moivre's theorem	
$z^n = \lvert z\rvert^n\,cis(n\theta)$	$(cis\,\theta)^n = \cos n\theta + i\sin n\theta$
$z^{\frac{1}{q}} = r^{\frac{1}{q}}\left(\cos\frac{\theta + 2\pi k}{q} + i\sin\frac{\theta + 2\pi k}{q}\right)$, for k an integer	

$$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

No cals allowed!!

Q1 (2, 2, 2 & 2 = 8 marks)

If $z = 5 - 4i$ and $w = 2 + 3i$ determine the following:

a) zw

Solution
$(5-4i)(2+3i) = 10 + 12 - 8i + 15i$ $= 22 + 7i$
Specific behaviours
✓ real part ✓ Imaginary part

b) $\dfrac{1}{w}$

Solution
$\dfrac{1}{2+3i}\dfrac{2-3i}{2-3i} = \dfrac{2-3i}{13}$
Specific behaviours
✓ uses conjugate ✓ express answer

c) $\dfrac{\overline{z}}{w}$

Solution
$\dfrac{5+4i}{2+3i}\dfrac{2-3i}{2-3i} = \dfrac{22-7i}{13}$
Specific behaviours
✓ numerator ✓ denominator

d) $z^2\overline{w}$

Solution

$(5-4i)^2(2-3i)=(25-16-40i)(2-3i)$ $(9-40i)(2-3i)$ $=18-120-80i-27i$ $=-102-107i$
Specific behaviours
✓ evaluates square term ✓ determines answer

Q2 (2 & 3 = 5 marks)

a) Determine the complex roots of $3z^2+z+2=0$.

Solution
$3z^2+z+2=0$ $z=\frac{-1\pm\sqrt{1-24}}{6}$ $z=\frac{-1\pm\sqrt{23}i}{6}$
Specific behaviours
✓ uses quadratic formula ✓ has two complex roots

b) Use the quadratic equation to prove that if a quadratic equation with real coefficients has any non-real roots then it must have two complex roots and they must be conjugates of each other.

Solution
$ax^2+bx+c=0$ $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $b^2-4ac=-n^2=i^2n^2$ $x=\frac{-b\pm\sqrt{i^2n^2}}{2a}=\frac{-b\pm in}{2a}$

Specific behaviours
✓ sets up equation with a negative discriminant ✓ uses $i^2 = -1$ with discriminant ✓ derives two complex roots which are conjugates of each other

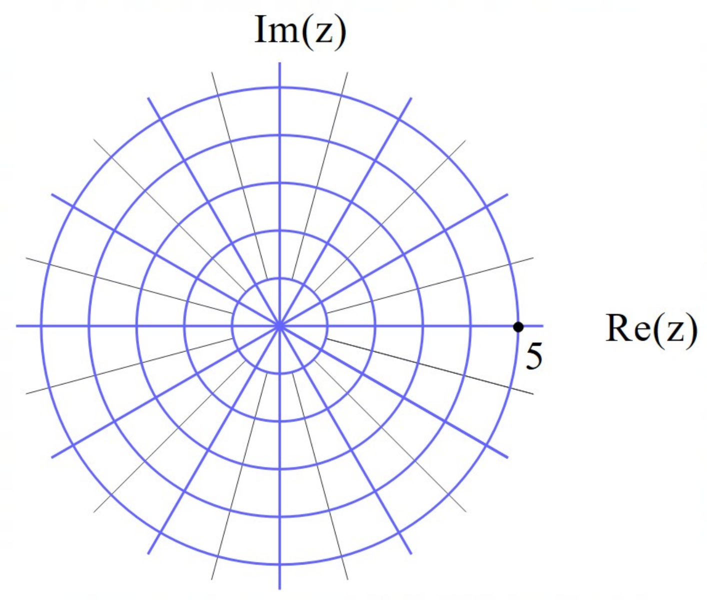
Q3 (4 marks)

Determine all possible real number pairs $a \,\&\, b$ such that $\frac{31-29i}{a+2i} = 3+bi$.

Solution
$\frac{31-29i}{a+2i} = 3+bi$ $31-29i = (3+bi)(a+2i) = 3a-2b+i(ab+6)$ $31 = 3a-2b$ $-29 = ab+6$ $ab = -35, \quad a = \frac{-35}{b}$ $31 = \frac{-105}{b} - 2b$ $31b = -105-2b^2$ $2b^2+31b+105 = 0$ $(2b+21)(b+5) = 0$ $b = -5, \frac{-21}{2}$ $a = 7, \frac{70}{21}$ $(7,-5) \& \left(\frac{70}{21}, \frac{-21}{2}\right)$
Specific behaviours
✓ sets up equation and equates real and imaginary ✓ obtains two simultaneous equations ✓ solves for one pair of values ✓ solves for two pairs of values

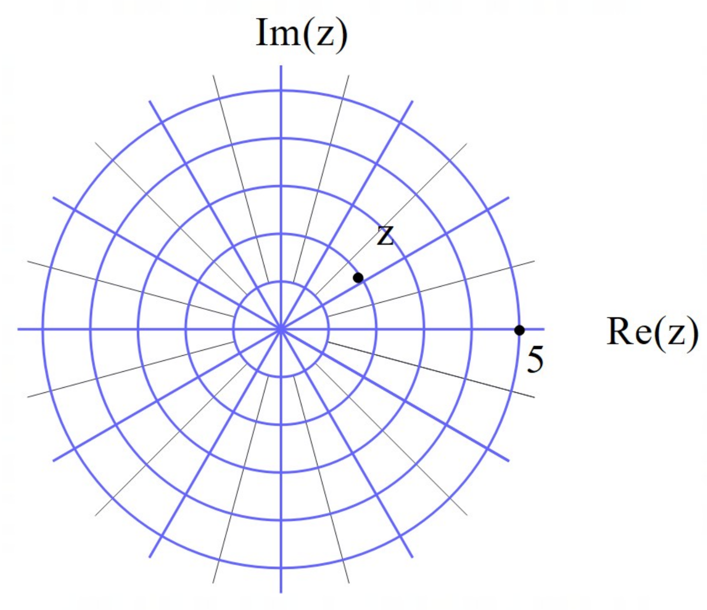
Q4 (2, 2, 2 & 2 = 8 marks)

Consider the complex number $z = \sqrt{3} + i$.



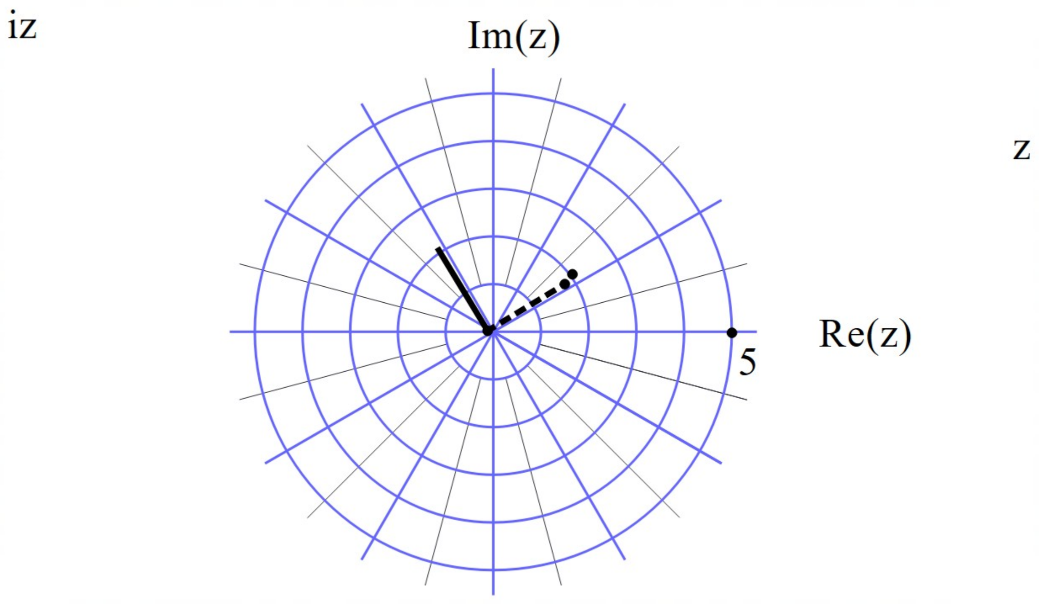
Plot the following on the axes above.

a) z

Solution

Specific behaviours
✓ argument

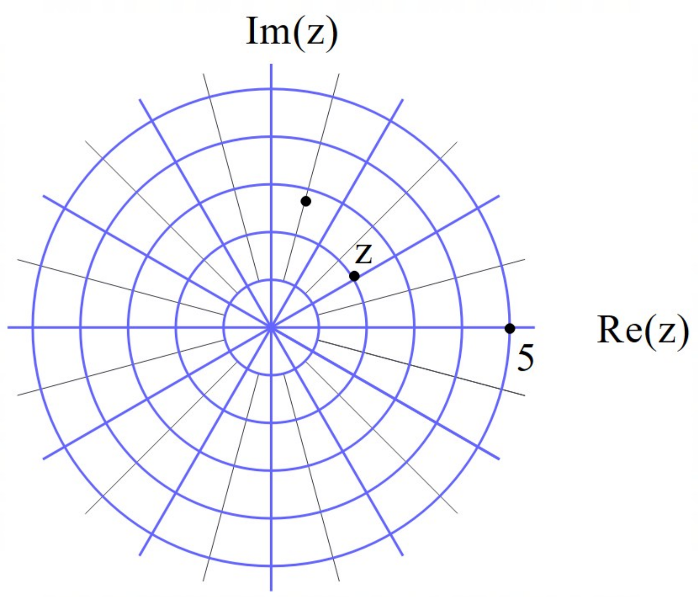
✓ length of 2 units

b) iz

Solution


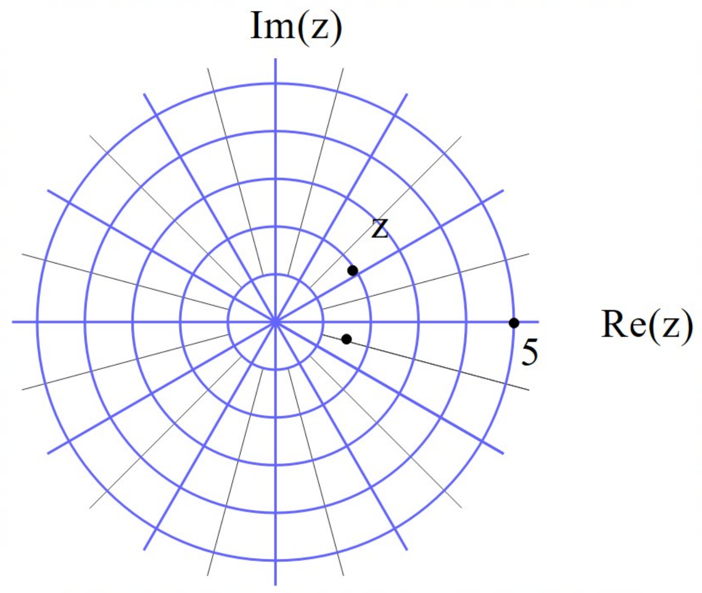
Specific behaviours
✓ uses right angle ✓ rotates anticlockwise with unchanged length

c) $(1+i)z$

Solution


Specific behaviours
✓ argument ✓ modulus

d) $\frac{z}{(1+i)}$

Solution


Specific behaviours
✓ argument ✓ modulus

Q5 (5 marks)

Consider the polynomial $f(z) = az^4 + bz^3 + cz^2 + dz + e$ where $a, b, c, d \;\&\; e$ are real numbers.

Given that $f(1+i) = 0 = f(2-3i)$

and $f(0) = 52$

Determine the values of $a, b, c, d \;\&\; e$.

(Note: answers without working will receive zero marks)

Solution

$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$

$-(\alpha+\beta) = -2\,\mathrm{Re}\,al, \alpha\beta = |z|^2$

$f(z) = a(z^2 - 2z + 2)(z^2 - 4z + 13)$

$z = 0, f(z) = 52 \therefore a = 2$

$f(z) = 2(z^4 - 6z^3 + 23z^2 - 34z + 26)$

$a = 2$

$b = -12$

$c = 46$

$d = -68$

$e = 52$

Specific behaviours

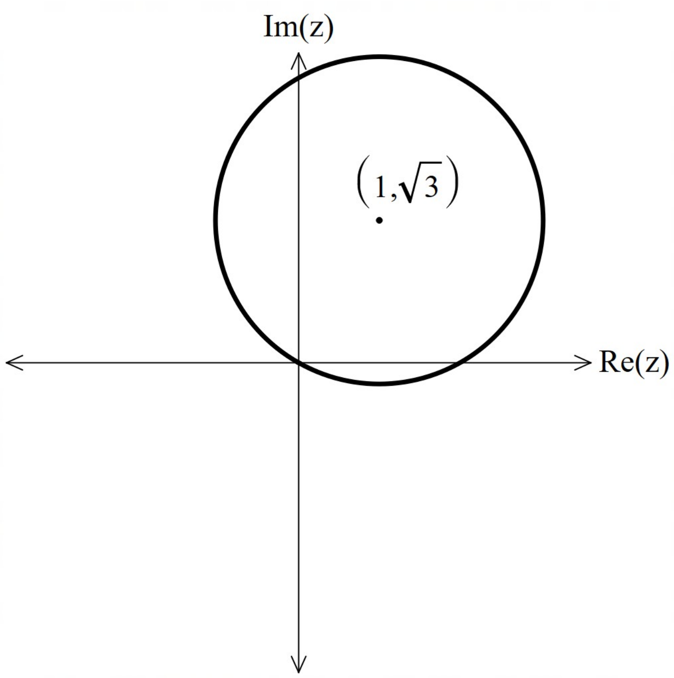
✓ shows reasoning for determining value of a
✓ uses one quadratic factor
✓ uses two quadratic factors
✓ shows reasoning in determining quadratic factors (i.e roots in brackets0
✓ shows reasoning on how to determine quartic polynomial.
Note: Any statement of values without reasoning will NOT receive any marks!

Q6 (2, 1, 2 & 2 = 7 marks)

Consider the locus of complex numbers z that satisfy $\left|z - 1 - \sqrt{3}i\right| = 2$.

a) Sketch the locus on the axes below.

Solution

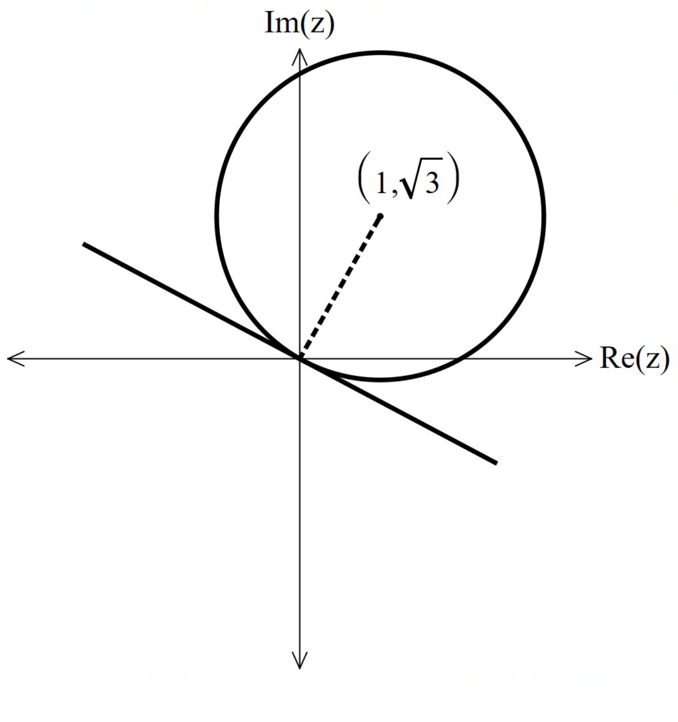


Specific behaviours
✓ circle with centre coordinates stated ✓ goes through origin

b) State the maximum value of $|z|$

Solution
$\|z\| = 4$
Specific behaviours
✓ states maximum

c) State the minimum value of $Arg(z)$

Solution


$m\dfrac{\sqrt{3}}{1}=-1$ $m=-1/\sqrt{3}=\tan\theta$ $\theta=\dfrac{5\pi}{6},\dfrac{-\pi}{6}$
Specific behaviours
✓ determines gradient of tangent ✓ determines min argument

d) State the maximum value of $Arg(z)$

Solution
Max = $\dfrac{5\pi}{6}$ See above
Specific behaviours
✓ determines gradient of tangent ✓ determines max argument

Q7 (4 marks)

In the following simultaneous equations, $a\,\&\,b$ are real numbers.

$a^3=3ab^2-13\sqrt{2}$

$b^3=3a^2b-\sqrt{5}$

In order to determine the value of a^2+b^2 from these equations, it is useful to consider the complex expansion for $(a+bi)^3$. Hence or otherwise, determine the exact value of a^2+b^2.

(Note: answers without working will receive zero marks)

Solution

$(a+bi)^3 = a^3 + 3a^2bi + 3a(-b^2) - b^3i = a^3 - 3ab^2 + i(3a^2b - b^3)$ $= -13\sqrt{2} + \sqrt{5}i$ $\left\|-13\sqrt{2} + \sqrt{5}i\right\| = \|a+bi\|^3$ $\sqrt{169(2)+5} = \left(\sqrt{a^2+b^2}\right)^3$ $a^2 + b^2 = (343)^{\frac{1}{3}} = 7$
Specific behaviours
✓ expands cubic (no need to simplify) ✓ determines real and imaginary parts of z cubed ✓ rearranges to obtain expression of a squared plus b squared ✓ shows that 7 is cube root of 343 NOTE: any statement that is not supported receives zero marks)