

Section Two: Calculator – 80 marks

This section has **eleven (11)** questions. Attempt **all** questions.

Suggested working time: **100 minutes**

Question 9: [4 marks]

Vectors \mathbf{a} and \mathbf{b} are as follows: $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ t \\ -6 \end{pmatrix}$. Determine the value of t if \mathbf{a} and \mathbf{b} are:

- a)** Parallel to each other.

Parallel if $\lambda \mathbf{a} = \mathbf{b}$

$$\lambda \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ t \\ -6 \end{pmatrix}$$

$$\lambda = 3$$

$$\therefore t = -9 \quad \checkmark$$

[1]

- b)** Perpendicular to each other.

Perpendicular when: $\mathbf{a} \cdot \mathbf{b} = 0$

$$\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ t \\ -6 \end{pmatrix} = 0 \quad \checkmark$$

$$(1)(3) + (-3)(t) + (-2)(-6) = 0$$

$$3 - 3t + 12 = 0 \quad \checkmark$$

$$3t = 15$$

$$\therefore t = 5 \quad \checkmark$$

[3]

Question 10: [6 marks]

Prove that the square of an integer NOT divisible by 5 leaves a remainder of 1 or 4 when divided by 5.

Let integer $n = 5x + a$, where $a = 0, 1, 2, 3, 4$. ✓

Case 1: $n = 5x$

$$n^2 = (5x)^2$$

$$n^2 = 25x^2$$

$$n^2 = 5(5x^2)$$

Multiple of 5 not considered ✓
(or if $n = 5x$ is not included)

Case 3: $n = 5x + 2$

$$n^2 = (5x + 2)^2$$

$$n^2 = 25x^2 + 20x + 4$$

$$n^2 = 5(5x^2 + 4x) + 4$$

Remainder of 4, when
divisible by 5

Case 5: $n = 5x + 4$

$$n^2 = (5x + 4)^2$$

$$n^2 = 25x^2 + 40x + 16$$

$$n^2 = 5(5x^2 + 8x + 3) + 1$$

Remainder of 1, when
divisible by 5

Case 2: $n = 5x + 1$

$$n^2 = (5x + 1)^2$$

$$n^2 = (25x^2 + 10x + 1)$$

$$n^2 = 5(5x^2 + 2x) + 1$$

Remainder of 1, when
divisible by 5

Case 4: $n = 5x + 3$

$$n^2 = (5x + 3)^2$$

$$n^2 = (25x^2 + 30x + 9)$$

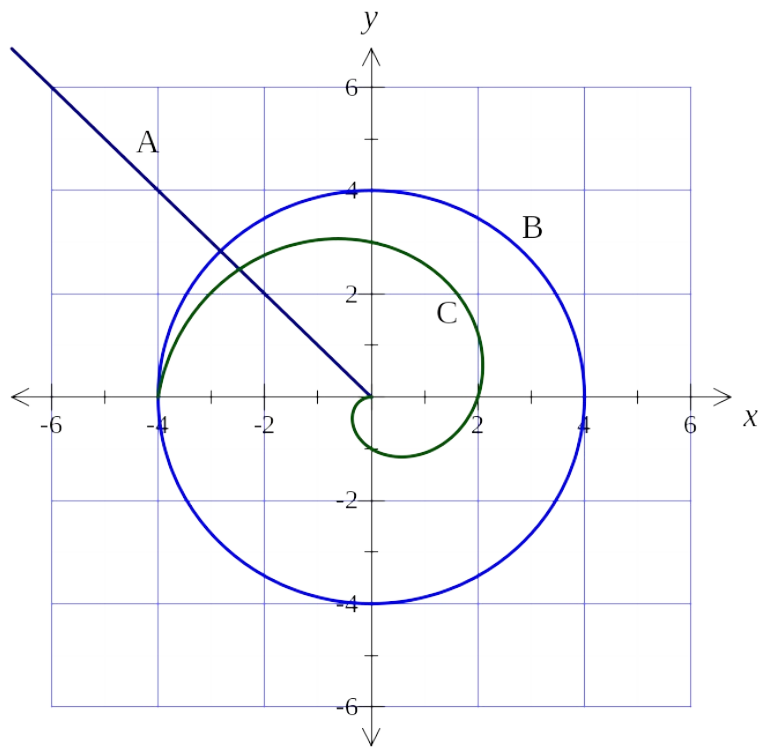
$$n^2 = 5(5x^2 + 6x + 1) + 4$$

Remainder of 4, when
divisible by 5

- ✓ Logical presentation
- ✓ All cases considered
- ✓ Working
- ✓ Statement about remainders and
divisibility (at end of each or overall)

Question 11: [6 marks]

Determine the equations of A, B and C from the graph below:



$$A = \theta = \frac{3\pi}{4} \checkmark \checkmark$$

or

$$y = -x \checkmark \text{ for } x \leq 0 \checkmark$$

$$B = r = 4 \checkmark \checkmark$$

or

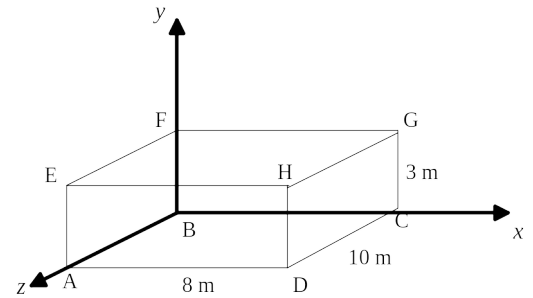
$$\checkmark x^2 + y^2 = 4^2 \checkmark$$

$$C = r = \frac{-2\theta}{\pi} \checkmark \checkmark$$

Question 12: [9 marks]

The diagram on the right shows a rectangular prism.

Determine:



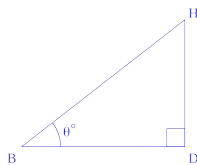
a) $\vec{BH} = \langle 8, 3, 10 \rangle$ ✓

[1]

b) $\vec{BD} = \langle 8, 0, 10 \rangle$ ✓

[1]

c) $\angle HBD$ (in degrees, to 1 decimal place).



$$\begin{aligned} \cos \theta &= \frac{\vec{BH} \cdot \vec{BD}}{|\vec{BH}| |\vec{BD}|} \\ &= \frac{8^2 + (3 \times 0) + 10^2}{\sqrt{8^2 + 3^2 + 10^2} \sqrt{8^2 + 0^2 + 10^2}} \\ &= \frac{164}{\sqrt{173} \sqrt{164}} \quad \checkmark \checkmark \\ \theta &\approx 13.2^\circ \quad \checkmark \end{aligned}$$

[3]

d) The acute angle between the skew lines BD and FE.

$$\begin{aligned} \vec{FE} &= \vec{BA} \quad \checkmark \\ \vec{BA} &= \langle 0, 0, 10 \rangle \quad \checkmark \\ \cos \theta &= \frac{\vec{BD} \cdot \vec{BA}}{|\vec{BD}| |\vec{BA}|} \\ &= \frac{(8)(0) + 0^2 + 10^2}{\sqrt{164} \sqrt{100}} \\ &= \frac{100}{20\sqrt{41}} \quad \checkmark \\ \theta &\approx 38.7^\circ \quad \checkmark \end{aligned}$$

[4]

Question 13: [4 marks]

Prove the following:

$$\frac{\sin A}{\cos B} + \frac{\cos A}{\sin B} = \frac{2 \cos (A - B)}{\sin 2B}.$$

$$\begin{aligned} \text{RHS} &= \frac{2 \cos (A - B)}{\sin 2B} \\ &= \frac{2 (\cos A \cos B + \sin A \sin B)}{2 \sin B \cos B} \quad \checkmark \checkmark \\ &= \frac{\cancel{\cos A \cos B}}{\cancel{\sin B \cos B}} + \frac{\cancel{\sin A \sin B}}{\cancel{\sin B \cos B}} \quad \checkmark \\ &= \frac{\cos A}{\sin B} + \frac{\sin A}{\cos B} \\ &= \frac{\sin A}{\cos B} + \frac{\cos A}{\sin B} \quad \checkmark \\ &= \text{LHS} \\ &\therefore \text{Proved} \end{aligned}$$

[4]

Question 14: [4 marks]Use calculus techniques to determine: $\int 5^x dx$.

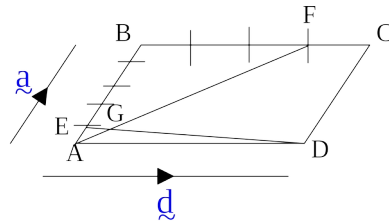
$$\begin{aligned} &= \int e^{(\ln 5) x} dx \quad \checkmark \checkmark \\ &= \frac{1}{\ln 5} \times e^{(\ln 5) x} + c \quad \checkmark \\ &= \frac{5^x}{\ln 5} + c \quad \checkmark \end{aligned}$$

[4]

Question 15: [11 marks]

ABCD is a parallelogram with points E and F such that $\overrightarrow{AE} : \overrightarrow{EB} = 1 : 4$ and $\overrightarrow{BF} : \overrightarrow{FC} = 3 : 1$. \overrightarrow{ED} and \overrightarrow{AF} intersect each other at G. Let: $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{d}$.

- a) Complete the diagram below with the information given above.



✓✓

(-1 each mistake)

[2]

- b) Determine the ratios in which \overrightarrow{AF} and \overrightarrow{ED} intersect each other, if the intersection point is at G.

$$\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF}$$

$$= \mathbf{a} + \frac{3}{4} \overrightarrow{BC}$$

$$= \mathbf{a} + \frac{3}{4} \mathbf{d} \quad \checkmark$$

$$\overrightarrow{ED} = \overrightarrow{EA} + \overrightarrow{AD}$$

$$= -\frac{1}{5} \mathbf{a} + \mathbf{d} \quad \checkmark$$

$$\overrightarrow{AG} = h \overrightarrow{AF}$$

$$= h \left(\mathbf{a} + \frac{3}{4} \mathbf{d} \right) \quad \checkmark$$

$$\overrightarrow{EG} = k \overrightarrow{ED}$$

$$= k \left(-\frac{1}{5} \mathbf{a} + \mathbf{d} \right) \quad \checkmark$$

$$\overrightarrow{AG} = \overrightarrow{AE} + \overrightarrow{EG}$$

$$h \mathbf{a} + \frac{3}{4} h \mathbf{d} = \frac{1}{5} \mathbf{a} - \frac{1}{5} k \mathbf{a} + k \mathbf{d} \quad \checkmark$$

$$\mathbf{a} : h = \frac{1}{5} - \frac{1}{5} k \quad \text{(Equation 1)} \quad \checkmark$$

$$\mathbf{d} : \frac{3}{4} h = k \quad \text{(Equation 2)} \quad \checkmark$$

$$\therefore h = \frac{4}{23} \text{ and } k = \frac{3}{23} \quad \checkmark$$

$$\therefore G \text{ divides } \overrightarrow{AF} \text{ in the ratio } 4 : 19 \quad \checkmark$$

$$\therefore G \text{ divides } \overrightarrow{ED} \text{ in the ratio } 3 : 20 \quad \checkmark$$

[9]

Question 16: [15 marks]

If: $\mathbf{a} = \langle 8, -6, 0 \rangle$ and $\mathbf{b} = \langle -2, 4, -1 \rangle$, determine:

a) $2\mathbf{b} - \mathbf{a}$

$$= 2 \langle -2, 4, -1 \rangle - \langle 8, -6, 0 \rangle$$

$$= \langle -12, 14, -2 \rangle \quad \checkmark$$

[1]

b) A vector in the same direction as \mathbf{a} but equal in magnitude to \mathbf{b} .

$$= |\mathbf{b}| \times \hat{\mathbf{a}}$$

$$= \sqrt{2^2 + 4^2 + 1^2} \times \frac{1}{\sqrt{8^2 + 6^2 + 0^2}} \langle 8, -6, 0 \rangle$$

$$= \frac{\sqrt{21}}{10} \langle 8, -6, 0 \rangle \quad \checkmark$$

[3]

c) The acute angle that vector \mathbf{a} makes with the y-axis.

$$\mathbf{a} \cdot \mathbf{j} = \langle 8, -6, 0 \rangle \cdot \langle 0, -6, 0 \rangle \quad \checkmark$$

$$= (8)(0) + (-6)^2 + 0^2$$

$$= 36 \quad \checkmark$$

$$\cos \theta = \frac{36}{10 \times 6} \quad \checkmark$$

$$\therefore \theta \approx 53^\circ \quad \checkmark$$

[4]

Question 16 cont...

- d) Point P lies on the line AB (from **a** to **b**) and it is known that: $\vec{AP} : \vec{AB} = 2 : 3$. Determine the position vector of the point P.



$$\begin{aligned}\vec{AB} &= \vec{AO} + \vec{OB} \\ &= -\langle 8, -6, 0 \rangle + \langle -2, 4, -1 \rangle \\ &= \langle -10, 10, -1 \rangle \quad \checkmark\end{aligned}$$

$$\begin{aligned}\vec{OP} &= \vec{OA} + \vec{AP} \\ &= \vec{OA} + \frac{2}{3}\vec{AB} \\ &= \langle 8, -6, 0 \rangle + \frac{2}{3}\langle -10, 10, -1 \rangle \quad \checkmark \\ &= \langle 8, -6, 0 \rangle + \langle -\frac{20}{3}, \frac{20}{3}, -\frac{2}{3} \rangle \quad \checkmark \\ &= \langle \frac{4}{3}, \frac{2}{3}, -\frac{2}{3} \rangle \quad \checkmark\end{aligned}$$

[4]

- e) If: $\mathbf{c} = \langle 6, 5, -2 \rangle$ and $\mathbf{d} = \langle -10, -38, 11 \rangle$, express \mathbf{d} in the form: $\lambda\mathbf{a} + \mu\mathbf{b} + \eta\mathbf{c}$, and hence determine λ , μ , and η . (Hint: Use the vectors **a** and **b** from the start of the question).

$$\begin{pmatrix} 8 \\ -6 \\ 0 \end{pmatrix} \lambda + \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} \mu + \begin{pmatrix} 6 \\ 5 \\ -2 \end{pmatrix} \eta = \begin{pmatrix} -10 \\ -38 \\ 11 \end{pmatrix} \quad \checkmark$$

Simultaneous equations:

$$\begin{aligned}8\lambda - 2\mu + 6\eta &= -10 \\ -6\lambda + 4\mu + 5\eta &= -38 \\ -\mu - 2\eta &= 11\end{aligned}$$

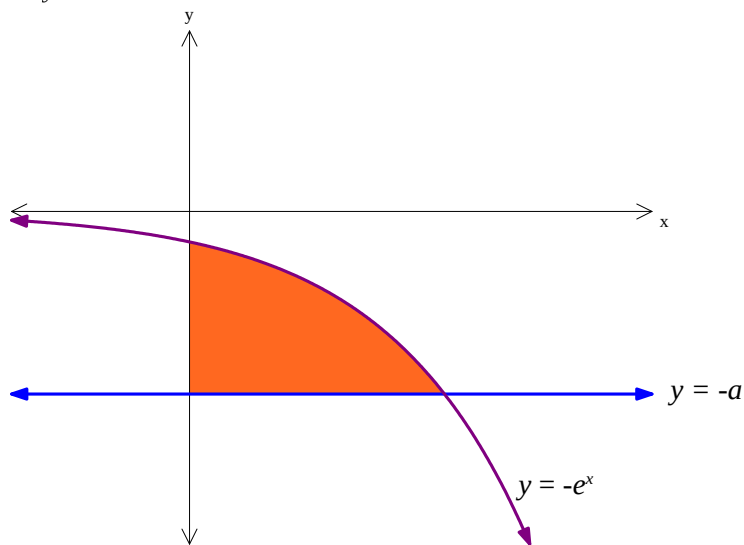
$$\mu = 1, \mu = -3, \eta = -4 \quad \checkmark$$

$$\therefore \mathbf{d} = \mathbf{a} - 3\mathbf{b} - 4\mathbf{c} \quad \checkmark$$

[3]

Question 17: [7 marks]

The graphs of $y = -a$ and $y = -e^x$ are shown below:



- a) Write an expression for the shaded area (above), in the form: $\int_0^b c \, dx$.
- $\therefore \int_0^{\ln a} (-e^x + a) \, dx$

b: $-e^x = -a$

$e^x = a$

$x \ln e = \ln a$

$x = \ln a$

[3]

- b) Determine the values of a and b , if it is known that: $-e^x = -a$, when $x = 1.7918$.

$a = e^{1.7918}$

$a \approx 6$

$b = \ln a$

$= \ln 6$

$\therefore b = 1.7918$

[3]

- c) Determine the area of the shaded region (to two decimal places).

$\therefore \int_0^{1.7918} (-e^x + 6) \, dx$

$\approx 5.75 \text{ units}^2$

(-1 overall if no units)

[1]

Question 18: [14 marks]

The two countries Zedlandia and Xenutia are at war. Zedlandia (z) fired a ground to air missile in order to intercept a missile coming in from Xenutia (x). When z was launched, the position vectors (in metres), relative to the army base were:

$$r_z = \begin{pmatrix} 720 \\ 0 \\ 0 \end{pmatrix} \text{ and } r_x = \begin{pmatrix} 3000 \\ 4800 \\ 650 \end{pmatrix}.$$

Both z and x have constant velocities (m/s):

$$v_z = \begin{pmatrix} -200 \\ 208 \\ 40 \end{pmatrix} \text{ and } v_x = \begin{pmatrix} -255 \\ 127 \\ 3 \end{pmatrix}.$$

a) Prove that the two missiles *did not* intercept.

$$r_z(t) = \begin{pmatrix} 720 - 200\lambda \\ 208\lambda \\ 40\lambda \end{pmatrix}$$

✓

$$r_x(t) = \begin{pmatrix} 3000 - 255\mu \\ 4800 + 127\mu \\ 650 + 3\mu \end{pmatrix}$$

$$i: 720 - 200\lambda = 3000 - 255\mu$$

✓

$$j: 208\lambda = 4800 + 127\mu$$

$$\lambda = \frac{37839}{691} \text{ and } \mu = \frac{35856}{691}$$

✓

$$k: 40\lambda = 650 + 3\mu$$

$$\frac{15135600}{691} \neq \frac{556718}{691}$$

✓

\therefore Missiles do not intersect

Question 18 cont...

- b) Determine by how much (distance and time) the two missiles missed each other (Note: Enough working must be shown in order to gain full marks).

$$\begin{aligned}
 {}_Z V_x &= V_Z - V_x \\
 &= \begin{pmatrix} -200 \\ 208 \\ 40 \end{pmatrix} - \begin{pmatrix} -255 \\ 127 \\ 3 \end{pmatrix} \\
 &= \begin{pmatrix} 55 \\ 81 \\ 37 \end{pmatrix} \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \vec{ZX} &= \vec{ZO} + \vec{OX} \\
 &= - \begin{pmatrix} 720 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3\,000 \\ 4\,800 \\ 650 \end{pmatrix} \\
 &= \begin{pmatrix} 2\,280 \\ 4\,800 \\ 650 \end{pmatrix} \quad \checkmark
 \end{aligned}$$

Let P be the point that is on Z and forms the shortest distance between the missiles.

$$\begin{aligned}
 \vec{PX} &= -V_x + \vec{ZX} \\
 &= -t \begin{pmatrix} 55 \\ 81 \\ 37 \end{pmatrix} + \begin{pmatrix} 2\,280 \\ 4\,800 \\ 650 \end{pmatrix} \\
 &= \begin{pmatrix} 2\,280 - 55t \\ 4\,800 - 81t \\ 650 - 37t \end{pmatrix} \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 {}_Z V_x \cdot \vec{PX} &= 0 \\
 &= \begin{pmatrix} 55 \\ 81 \\ 37 \end{pmatrix} \cdot \begin{pmatrix} 2\,280 - 55t \\ 4\,800 - 81t \\ 650 - 37t \end{pmatrix} = 0
 \end{aligned}$$

$$538\,250 - 10\,955\,t = 0$$

$$t \approx \frac{538\,250}{10\,955}$$

$$t \approx 49.13 \text{ secs} \quad \checkmark$$

$$\vec{PX} \approx \begin{pmatrix} -422.3 \\ 820.2 \\ -1167.9 \end{pmatrix} \quad \checkmark$$

$$|\vec{PX}| \approx 1\,488.3 \text{ m} \quad \checkmark$$

(-1 overall if no units)

Question 18 cont...

- c) In order for the two missiles to collide, the z missile changed its flight plan after 15 seconds. The x continued on its path and then interception occurred 25 seconds later. Determine the constant velocity that z maintained in this second stage for interception to occur.

Method 1:

$$\begin{aligned} \underline{r}_z(15) &= \begin{pmatrix} 20 - 20 \times 15 \\ 208 \times 15 \\ 40 \times 15 \end{pmatrix} \\ &= \begin{pmatrix} -2\,280 \\ 3\,120 \\ 600 \end{pmatrix} \quad \checkmark \end{aligned}$$

$$\begin{aligned} \underline{r}_x(40) &= \begin{pmatrix} 3\,000 - 255 \times 40 \\ 4\,800 + 127 \times 40 \\ 650 + 3 \times 40 \end{pmatrix} \\ &= \begin{pmatrix} -7\,200 \\ 9\,880 \\ 770 \end{pmatrix} \quad \checkmark \end{aligned}$$

$$\underline{ZX} = \underline{ZO} + \underline{OX}$$

$$= -\underline{r}_z(15) + \underline{r}_x(40)$$

$$= -\begin{pmatrix} -2\,280 \\ 3\,120 \\ 600 \end{pmatrix} + \begin{pmatrix} -7\,200 \\ 9\,880 \\ 770 \end{pmatrix}$$

$$= \begin{pmatrix} -4\,920 \\ 6\,760 \\ 170 \end{pmatrix} \quad \checkmark$$

$$V = \frac{s}{t}$$

$$= \frac{< -4\,920, 6\,760, 170 >}{25}$$

$$\approx \begin{pmatrix} -196.8 \\ 270.4 \\ 6.8 \end{pmatrix} \text{ m/s} \quad \checkmark$$

Method 2:

$$\begin{aligned} \underline{r}_z(40) &= \begin{pmatrix} 720 \\ 0 \\ 0 \end{pmatrix} + 15 \begin{pmatrix} -200 \\ 208 \\ 40 \end{pmatrix} + 25 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} -2\,280 \\ 3\,120 \\ 600 \end{pmatrix} + 25 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \checkmark \end{aligned}$$

$$\begin{aligned} \underline{r}_x(40) &= \begin{pmatrix} 3\,000 \\ 4\,800 \\ 650 \end{pmatrix} + 40 \begin{pmatrix} -255 \\ 127 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -7\,200 \\ 9\,880 \\ 770 \end{pmatrix} \quad \checkmark \end{aligned}$$

$$\text{At collision } \underline{r}_z(40) = \underline{r}_x(40)$$

$$\begin{pmatrix} -2\,280 \\ 3\,120 \\ 600 \end{pmatrix} + 25 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7\,200 \\ 9\,880 \\ 770 \end{pmatrix} \quad \checkmark$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -196.8 \\ 270.4 \\ 6.8 \end{pmatrix}$$

$$\therefore \underline{v}_z = \begin{pmatrix} -196.8 \\ 270.4 \\ 6.8 \end{pmatrix} \text{ m/s} \quad \checkmark$$

(-1 overall if no units)