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PERTH MODERN SCHOOL Exceptional schooling. Exceptional students. Independent Public School	Year 12 Specialist TEST 3 2018 TIME: 45 minutes working Classpads **allowed!** 3~~8~~9 Marks 7 Questions

Name: Marking Key

Teacher: ______________________

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (2 & 2 = 4 marks)

Consider a line with parametric equations $\begin{aligned} x &= 3 - 5\lambda \\ y &= -7 + 2\lambda \end{aligned}$

i) Determine a vector equation

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 2 \end{pmatrix}$$

✓ uses $\underset{\sim}{r}$ + λ
✓ obtains vector eqn

ii) Determine a cartesian equation.

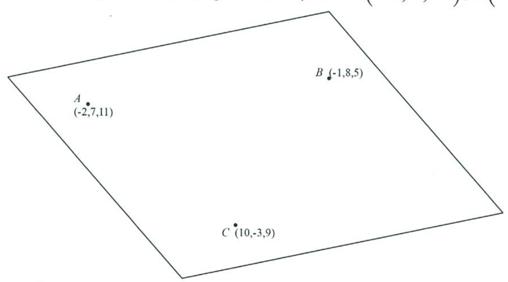
$$\lambda = \frac{x-3}{-5} \qquad y = -7 + \frac{2(x-3)}{-5}$$

$$y = \frac{-2x}{5} - \frac{29}{5}$$

✓ Expresses λ in terms of one variable
✓ obtains cartesian eqn

Q2 (3 & 2 = 5 marks)

Consider a plane containing the three points A$(-2,7,11)$, B$(-1,8,5)$ & C$(10,-3,9)$.



i) Determine the vector equation of the plane.

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} 12 \\ -10 \\ -2 \end{pmatrix} \qquad \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -62 \\ -70 \\ -22 \end{pmatrix} = -2 \begin{pmatrix} 31 \\ 35 \\ 11 \end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 31 \\ 35 \\ 11 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 31 \\ 35 \\ 11 \end{pmatrix} = 304$$

<u>OR</u> $\underset{\sim}{r} = \begin{pmatrix} -2 \\ 7 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 12 \\ -10 \\ -2 \end{pmatrix}$

✓ obtains two vectors in plane
✓ uses cross product to find normal
✓ finds vector eqn of plane

Continued-

ii) Determine the cartesian equation of the plane. (simplified)

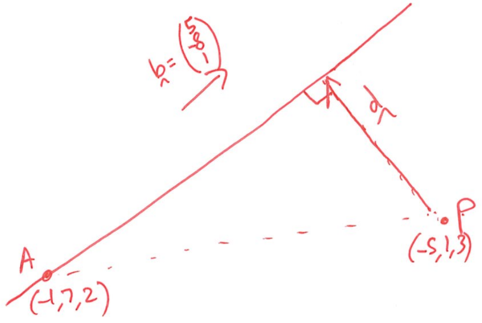
$31x + 35y + 11z = 304$

✓ uses dot product with $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

✓ simplified co-efficients

Q3 (4 marks)

Determine the distance of point P $(-5,1,3)$ from the line $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -8 \\ 1 \end{pmatrix}$



$\hat{d} = \overrightarrow{PA} + \lambda \hat{b}$

$= \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix} - \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -8 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 4 + 5\lambda \\ 6 - 8\lambda \\ -1 + \lambda \end{pmatrix}$

$\hat{d} \cdot \begin{pmatrix} 5 \\ -8 \\ 1 \end{pmatrix} = 0$

$\begin{pmatrix} 4 + 5\lambda \\ 6 - 8\lambda \\ -1 + \lambda \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -8 \\ 1 \end{pmatrix} = 5(4 + 5\lambda) - 8(6 - 8\lambda) - 1 + \lambda = 0$

$\lambda = \frac{29}{90}$

$|\hat{d}| = \frac{\sqrt{39290}}{30}$ or ≈ 6.607

using classpad

VECTORS

✓ sets up a displacement vector $\hat{d}$

✓ uses dot product equated to zero

✓ solves for parameter λ

✓ determines $|\hat{d}|$

CALCULUS

OR ✓ determines $\hat{r} - \overrightarrow{OP}$

✓ obtain expression for magnitude

✓ minimises distance using calculus

✓ determines distance.

Q7 (2, 3 & 3 = 8 marks)

Consider the function $f(x) = ax^4 + bx^3 + cx^2 + dx$ where $a, b, c \,\&\, d$ are constants.

The graph has a stationary point ($f' = 0$) at $(1,1)$ and passes through the point $(-1,4)$.

i) Write down three linear equations satisfied by $a, b, c \,\&\, d$.

$$\left.\begin{array}{l} 1 = a + b + c + d \quad \cdots (1) \\ 4 = a - b + c - d \quad \cdots (2) \end{array}\right\} \checkmark$$

$$f'(x) = 4ax^3 + 3bx^2 + 2cx + d$$

$$0 = 4a + 3b + 2c + d \quad \cdots (3) \quad \checkmark$$

ii) Express $a, b \,\&\, c$ in terms of d.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1-d \\ 1 & -1 & 1 & 4+d \\ 4 & 3 & 2 & -d \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1-d \\ 0 & 2 & 0 & -3-2d \\ 0 & 1 & 2 & 4-3d \end{array}\right] \begin{array}{l} \\ R_1 - R_2 \\ 4R_1 - R_3 \end{array}$$

✓ Obtains an equation with only one variable from a, b, c

✓ solves for one of a, b, c in terms of d

✓ Solves all three of a, b, c in terms of d

$$2b = -3 - 2d$$
$$b = -\frac{3}{2} - d$$
$$b + 2c = 4 - 3d$$
$$2c = 4 - 3d + \frac{3}{2} + d$$
$$c = \frac{11}{4} - d$$

$$a + b + c = 1 - d$$
$$a = 1 - d + \frac{3}{2} + d + d - \frac{11}{4}$$
$$a = d - \frac{1}{4}$$
$$a = -\frac{1}{4} + d$$
$$b = -\frac{3}{2} - d$$
$$c = \frac{11}{4} - d$$

iii) Determine the value of d for which the graph has a stationary point where $x = 4$

$$f'(x) = 4ax^3 + 3bx^2 + 2cx + d$$
$$0 = 256a + 48b + 8c + d$$
$$0 = 256\left(-\frac{1}{4} + d\right) + 48\left(-\frac{3}{2} - d\right) + 8\left(\frac{11}{4} - d\right) + d$$

Solve on classpad

$$d = \frac{38}{67} \quad \text{or} \quad \approx 0.567$$

✓ obtains equation for a, b, c, d using $f' = 0$ at $x = 4$

✓ solves all variables in terms of d

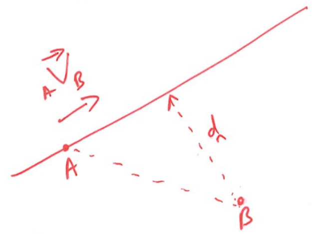
✓ Solves for d

Q4 (4 marks)

Consider two particles A and B whose position at $t = 0$ is recorded as below moving with constant velocities $v_A \,\&\, v_B$. Determine the distance of closest approach and the time that this occurs.

$$r_A = \begin{pmatrix} 2 \\ -5 \\ 9 \end{pmatrix} \quad v_A = \begin{pmatrix} 11 \\ -5 \\ 7 \end{pmatrix}$$

$$r_B = \begin{pmatrix} 1 \\ -1 \\ 9 \end{pmatrix} \quad v_B = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix}$$



$${}_A\vec{V}_B = \begin{pmatrix} 11 \\ -5 \\ 7 \end{pmatrix} - \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 5 \end{pmatrix}$$

$$\underset{\sim}{d} = \vec{BA} + t\begin{pmatrix} -1 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 9 \end{pmatrix} + t\begin{pmatrix} -1 \\ 5 \\ 5 \end{pmatrix}$$

$$\underset{\sim}{d} = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 1-t \\ -4+5t \\ 5t \end{pmatrix}$$

$$\underset{\sim}{d} \cdot {}_A\vec{V}_B = 0 \qquad \begin{pmatrix} 1-t \\ -4+5t \\ 5t \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 5 \end{pmatrix} = 0$$

$$-(1-t) + 5(-4+5t) + 25t = 0$$
$$t = \frac{7}{17}$$

$|\underset{\sim}{d}|$ when $t = \frac{7}{17}$ hr is $\sqrt{\frac{2414}{17}}$ or ≈ 2.890 Km

VECTOR

✓ uses relative velocity

✓ obtains expression for separation vector d

✓ uses dot product and solves for t

✓ obtains distance.

CALCULUS ✓ sets up vector eqn for displacement of each

✓ subtracts to find separation & finds distance

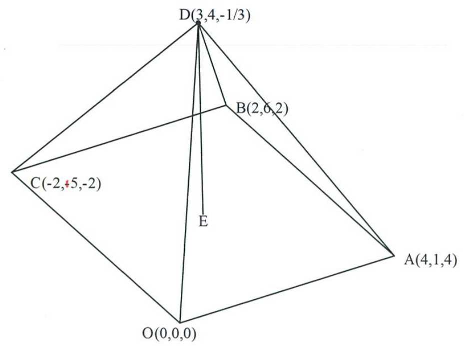
✓ minimises distance expression & solves for time

✓ obtains distance (minimum)

(2, 4, 3 = 9)

Q5 (2, ~~3 & 3 = 8~~ marks)

OABCD is a pyramid. The height of the pyramid is the length of DE, where E is the point on the base OABC such that DE is perpendicular to the base.



i) Show that the base OABC is a rhombus. $\overrightarrow{OC} = \overrightarrow{AB}$

LHS $= \begin{pmatrix} -2 \\ +5 \\ -2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ RHS $= \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix}$ ✓

$|CB| = |OC|$ ✓

The unit vector $p\underset{\sim}{i} + q\underset{\sim}{j} + r\underset{\sim}{k}$ is perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

ii) Show that $q = 0$ and determine the exact values of $p \,\&\, r$.

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix} = 0$ $\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} = 0$

$-2p + 5q - 2r = 0$ ... (1)

$4p + q + 4r = 0$ ... (2)

$2 \times (1) + (2)$ .... $11q = 0$

$q = 0$

✓ uses dot product equate to zero

✓ sets up two linear eqns

✓ shows $q = 0$

✓ solves for $p + r$ using cross product

iii) Hence determine the exact height of the pyramid.

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \parallel \overrightarrow{OA} \times \overrightarrow{OC} = \begin{pmatrix} -22 \\ 0 \\ 22 \end{pmatrix}$

$\therefore p = -r$

$\left|\begin{pmatrix} p \\ 0 \\ -p \end{pmatrix}\right| = 1$ $p^2 + p^2 = 1$

$2p^2 = 1$

$p = \pm\frac{1}{\sqrt{2}}$ $q = 0$ $r = \mp\frac{1}{\sqrt{2}}$

iii) height $= \left|\overrightarrow{OD} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\right| = \left|\begin{pmatrix} 3 \\ 4 \\ -\frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\right| = \frac{3}{\sqrt{2}} + \frac{1}{3\sqrt{2}}$

$= \frac{10}{3\sqrt{2}}$ ✓

$= \frac{5\sqrt{2}}{3}$

✓ uses dot product

✓ expresses dot with normal

No need to rationalise for height

Q6 (5 marks)

Consider a sphere of centre $(-3, 2, 7)$ and radius of a units, where a is a constant.

The line $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$ is a tangent to the above sphere.

Determine the possible value(s) of a

$\left|\underset{\sim}{r} - \begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix}\right| = a$

$\left|\begin{pmatrix} 2+4\lambda \\ 1+\lambda \\ -8-3\lambda \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix}\right| = a$

$\left|\begin{pmatrix} 5+4\lambda \\ \lambda - 1 \\ -15-3\lambda \end{pmatrix}\right| = a$

$(5+4\lambda)^2 + (\lambda-1)^2 + (-15-3\lambda)^2 = a^2$

$16\lambda^2 + 40\lambda + 25 + \lambda^2 - 2\lambda + 1 + 9\lambda^2 + 90\lambda + 225 = a^2$

$26\lambda^2 + 128\lambda + 251 - a^2 = 0$

One solution for λ $\therefore \Delta = 0$

$128^2 - 4(26)(251 - a^2) = 0$

$a = \pm\frac{9\sqrt{195}}{13}$ but $a > 0$

$\therefore a = \frac{9\sqrt{195}}{13}$ or ≈ 9.6675

✓ subs line into vector eqn of sphere.

✓ uses magnitude of 3D vector equated to a

✓ obtains a quadratic eqn for λ in terms of a

✓ uses $b^2 - 4ac = 0$ to solve for a values

✓ states one positive value of a and discards negative.