The properties of Pascal's triangle

ACARA

- derive and use simple identities associated with Pascal's triangle. (ACMSM009)

Pascal's triangle

Pascal's triangle is an arrangement of numbers. In general the n^{th} row consists of the binomial Coefficients

^nC_r or $\binom{n}{r}$ with the $r = 0, 1, \ldots, n$

				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

In Pascal's triangle any term is the sum of the two terms 'above' it.

For example 10 = 4 + 6.

Identities include:

$$^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$$

The recurrence relation,.

$$^nC_k = \frac{n}{k}\,{}^{n-1}C_{k-1}$$

http://www.australiancurriculum.edu.au/Glossary?a=SSCMSM&t=Pascal%E2%80%99s%20triangle

Prove $\quad {}^{n}C_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$

$$ {}^{n-1}C_{k-1} + {}^{n-1}C_k = \frac{(n-1)!}{(k-1)!\left((n-1)-(k-1)\right)!} + \frac{(n-1)!}{k!\left((n-1)-k\right)!} $$

$$ = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} $$

$$ = (n-1)!\left(\frac{1}{(k-1)!(n-k)!} + \frac{1}{k!(n-k-1)!}\right) $$

$$ = (n-1)!\left(\frac{1}{(k-1)!(n-k)(n-k-1)!} + \frac{1}{k(k-1)!(n-k-1)!}\right) $$

$$ = \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{(n-k)} + \frac{1}{k}\right) $$

$$ = \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{k+(n-k)}{(n-k)k}\right) $$

$$ = \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{n}{(n-k)k}\right) $$

$$ = \frac{n \times (n-1)!}{k \times (k-1)!(n-k) \times (n-k-1)!} $$

$$ = \frac{n!}{k!(n-k)!} $$

$$ = {}^{n}C_k $$

Prove $\quad {}^{n}C_k = \frac{n}{k}\,{}^{n-1}C_{k-1}$

$$ \frac{n}{k}\,{}^{n-1}C_{k-1} = \frac{n}{k} \times \left(\frac{(n-1)!}{(k-1)!\left((n-1)-(k-1)\right)!}\right) $$

$$=\frac{n}{k}\times\left(\frac{(n-1)!}{(k-1)!(n-k)!}\right)$$

$$=\frac{n\times(n-1)!}{k\times(k-1)!(n-k)!}$$

$$=\frac{n!}{k!(n-k)!}$$

$$={}^{n}C_{k}$$

Some other basic properties:

Pascal's triangle can be used to find the coefficients of the binomial expansion $(a + b)^n$.

For example $(a + b)^6 = a^6 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + a^6$

where the coefficients are the elements of the 7th row of Pascal's triangle 1 6 15 20 15 6 1

Given the 6 factors of $(a + b)^6$ are $(a + b)(a + b)(a + b)(a + b)(a + b)(a + b)$ then, for example, to get the $15a^4b^2$ term, we have to select which of the terms contributes the four "a"s and the remaining terms contribute the "b"s. This can be done in ${}^6C_4$ (or ${}^6C_2$) ie 15 ways.

so we have

1 6 15 20 15 6 1 is the same as ${}^6C_0$ ${}^6C_1$ ${}^6C_2$ ${}^6C_3$ ${}^6C_4$ ${}^6C_5$ ${}^6C_6$

Verify that this is true!

so $(a + b)^6 = {}^6C_0a^6 + {}^6C_1a^5b^1 + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5a^1b^5 + {}^6C_6a^6$.

What is the sum of the elements in a row in Pascal's triangle?

If we put a = b = 1 then we obtain

$(1 + 1)^6 = {}^6C_0 + {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$

so $2^6 = {}^6C_0 + {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$

In general

$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots {}^nC_{n-2} + {}^nC_{n-1} + {}^nC_n$

i.e. $\sum_{i=1}^{n} {}^nC_i = 2^n - {}^nC_0$ or $\sum_{i=0}^{n} {}^nC_i = 2^n$

Application: In how many ways can a selection be made from 10 different lollies?

${}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + \ldots {}^{10}C_9 + {}^{10}C_{10} = 2^{10} - {}^{10}C_0$

$= 2^{10} - 1$

Another property that follows is

${}^6C_0 + {}^6C_2 + {}^6C_4 + {}^6C_6 = {}^6C_1 + {}^6C_3 + {}^6C_5$

i.e. **1** 6 **15** 20 **15** 6 **1**

$\Rightarrow$ **1 + 15 + 15 + 1** = 6 + 20 + 6

i.e. the sum of the "odd" terms is equal to the sum of the "even" terms.

Check that this works for any row in Pascal’s triangle.

```
                      1
                    1   1
                  1   2   1
                1   3   3   1
              1   4   6   4   1
            1   5  10  10   5   1
          1   6  15  20  15   6   1
        1   7  21  35  35  21   7   1
      1   8  28  56  70  56  28   8   1
    1   9  36  84 126 126  84  36   9   1
  1  10  45 120 210 252 210 120  45  10   1
1  11  55 165 330 462 462 330 165  55  11   1
```

There are many patterns in Pascal’s triangle that are well known.

Note: If in any row in Pascal’s triangle, the number after the one is prime, then all of the elements in that row with the exception of one, are multiples of the prime.

Explain using n and k.

The following diagram may be useful in the next few investigations:

										1										
									1		1									
								1		2		1								
							1		3		3		1							
						1		4		6		4		1						
					1		5		10		10		5		1					
				1		6		15		20		15		6		1				
			1		7		21		35		35		21		7		1			
		1		8		28		56		70		56		28		8		1		
	1		9		36		84		126		126		84		36		9		1	
1		10		45		120		210		252		210		120		45		10		1

										$^{0}C_{0}$										
									$^{1}C_{0}$		$^{1}C_{1}$									
								$^{2}C_{0}$		$^{2}C_{1}$		$^{2}C_{2}$								
							$^{3}C_{0}$		$^{3}C_{1}$		$^{3}C_{2}$		$^{3}C_{3}$							
						$^{4}C_{0}$		$^{4}C_{1}$		$^{4}C_{2}$		$^{4}C_{3}$		$^{4}C_{4}$						
					$^{5}C_{0}$		$^{5}C_{1}$		$^{5}C_{2}$		$^{5}C_{3}$		$^{5}C_{4}$		$^{5}C_{5}$					
				$^{6}C_{0}$		$^{6}C_{1}$		$^{6}C_{2}$		$^{6}C_{3}$		$^{6}C_{4}$		$^{6}C_{5}$		$^{6}C_{6}$				
			$^{7}C_{0}$		$^{7}C_{1}$		$^{7}C_{2}$		$^{7}C_{3}$		$^{7}C_{4}$		$^{7}C_{5}$		$^{7}C_{6}$		$^{7}C_{7}$			
		$^{8}C_{0}$		$^{8}C_{1}$		$^{8}C_{2}$		$^{8}C_{3}$		$^{8}C_{4}$		$^{8}C_{5}$		$^{8}C_{6}$		$^{8}C_{7}$		$^{8}C_{8}$		
	$^{9}C_{0}$		$^{9}C_{1}$		$^{9}C_{2}$		$^{9}C_{3}$		$^{9}C_{4}$		$^{9}C_{5}$		$^{9}C_{6}$		$^{9}C_{7}$		$^{9}C_{8}$		$^{9}C_{9}$	
$^{10}C_{0}$		$^{10}C_{1}$		$^{10}C_{2}$		$^{10}C_{3}$		$^{10}C_{4}$		$^{10}C_{5}$		$^{10}C_{6}$		$^{10}C_{7}$		$^{10}C_{8}$		$^{10}C_{9}$		$^{10}C_{10}$

1. Find an expression for T_n i.e. the nth triangular number in terms of $^{n}C_{k}$. Hence find T_{30}. Write down a much simpler algebraic expression for T_n and explain how it relates to the expression in terms of $^{n}C_{k}$.

2. Find a summation expression in terms of $^{n}C_{k}$ for
   (a) the sum of the set of counting numbers from 1 to 10.
   (b) the sum of the set of triangular numbers from 1 to n.

3. Interpret $^{n}C_{k} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k}$ referring to cells in Pascal's triangle.

ANSWERS

1. $T_n = {}^{n+1}C_2 \quad T_{30} = {}^{31}C_2 = 465$

$T_n = \frac{(n+1)n}{2}$ This term is the simplification of ${}^{n+1}C_2$.

2. (a) $\sum_{i=1}^{10} {}^{i}C_1$ (b) $\sum_{i=2}^{10} {}^{i}C_2$

3. ${}^8C_3 + {}^8C_4 = {}^9C_4$

Add two adjacent cells in Pascal's triangle and you get the cell under and betweenthem.

SQUARE NUMBERS

Consider the triangular numbers 1, 3, 6, 10, 15, 21 etc

Any two adjacent triangular numbers add to give a square number. eg 3 + 6 = 9, 15 + 21 = 36

By using the formula for triangular numbers in terms of ${}^nC_k$, find and simplify $T_n + T_{n+1}$ to prove that the sum of two adjacent triangular numbers is in fact a square number.

ANSWER

$T_n + T_{n+1} = {}^{n+1}C_2 + {}^{n+2}C_2$

$$= \frac{(n+1)!}{2!(n-1)!} + \frac{(n+2)!}{2!n!}$$

$$= \frac{(n+1)!}{2!(n-1)!}\left(1 + \frac{n+2}{n}\right)$$

$$= \frac{(n+1)n(n-1)!}{2(n-1)!}\left(\frac{2n+2}{n}\right)$$

$$= \frac{(n+1)n}{2}\left(\frac{2n+2}{n}\right)$$

$$= \frac{(n+1)2(n+1)}{2}$$

$$= (n+1)^2$$

which is a square number.

POWERS of 11

Row number	Powers of 11	Pascal's triangle
0	$11^0 = 1$	1
1	$11^1= 11$	1 1
2	$11^2=121$	1 2 1
3	$11^3= 1331$	1 3 3 1
4	$11^4=14641$	1 4 6 4 1
5	$11^5=161051$	1 5 10 10 5 1
6	$11^6=1771561$	1 6 15 20 15 61
7	$11^7 =$	1 7 21 3535 21 7 1
8	$11^8 =$	1 8 28 56 70 56 28 8 1

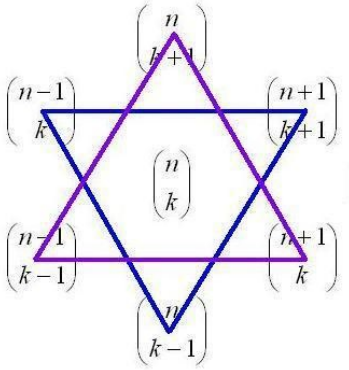
Write down the relationship between the power of 11 and the numbers in the corresponding row of Pascal's triangle.

Can you see how this works for 11^5 and 11^6?

Use this to determine the values of 11^7 and 11^8.

STAR of DAVID

inspired by http://threesixty360.wordpress.com/2008/12/21/star-of-david-theorem/



Draw the Star of David  on the Pascal's triangle below using the $\binom{n}{k} = {}^nC_k$ expressions in the diagram above. Note : You will have to rotate the vertices 60º to the right or left.

1

1 1

1 2 1

1 3 3 1

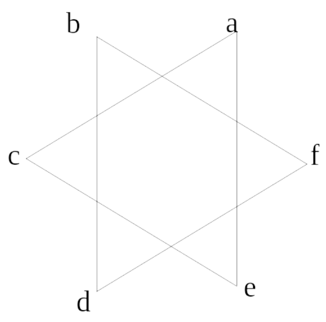
1 4 6 4 1

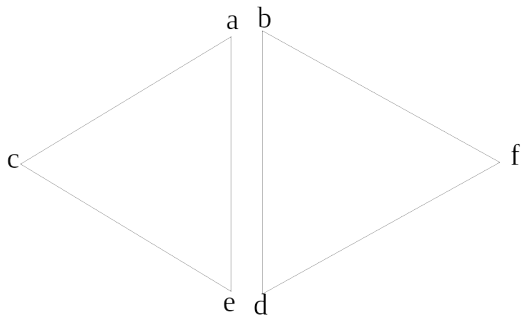
1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Consider the triangles





Confirm that a×c×e = b×d ×f . Check with another Star of David on Pascal's triangle.

The products of the numbers at the vertices of each triangle on the Star of David pattern are the same.

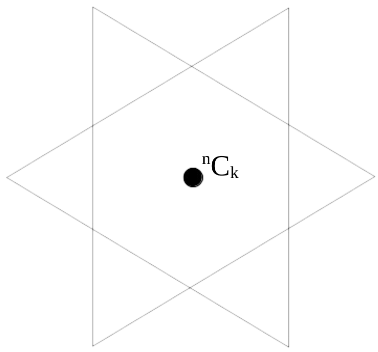
Prove this is true for all positions on Pascal's triangle where the Star of David can be drawn.

Hint: Let the point in the middle of the Star of David be represented by ^nC_k.

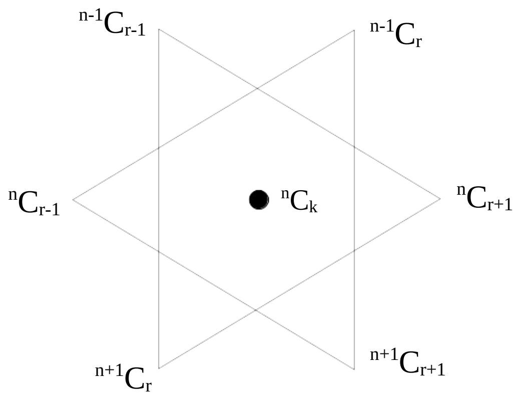
To make it easier:

Steps to consider:

Find the relevant corners of the star in terms of n and k



Further hint



Show $\quad {}^{n-1}C_{r-1} \times {}^{n+1}C_{r} \times {}^{n}C_{r+1} = {}^{n}C_{r-1} \times {}^{n-1}C_{r} \times {}^{n+1}C_{r+1}$

ANSWER

Show $^{n-1}C_{r-1} \times {}^{n+1}C_r \times {}^{n}C_{r+1} = {}^{n}C_{r-1} \times {}^{n-1}C_r \times {}^{n+1}C_{r+1}$

$$\text{LHS} = {}^{n-1}C_{r-1} \times {}^{n+1}C_r \times {}^{n}C_{r+1}$$

$$= \frac{(n-1)!}{((n-1)-(r-1))!(r-1)!} \times \frac{(n+1)!}{((n+1)-r)!r!} \times \frac{n!}{(n-(r+1))!(r+1)!}$$

$$= \frac{(n-1)!}{(n-r)!(r-1)!} \times \frac{(n+1)!}{(n-(r-1))!r!} \times \frac{n!}{((n-1)-r)!(r+1)!} \qquad \text{but} \quad n-r = (n+1)-(r+1)$$

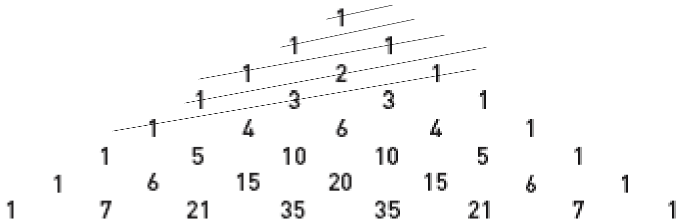
$$= \frac{(n-1)!}{((n+1)-(r+1))!(r-1)!} \times \frac{(n+1)!}{(n-(r-1))!r!} \times \frac{n!}{((n-1)-r)!(r+1)!}$$

$$= \frac{(n-1)!}{((n-1)-r)!r!} \times \frac{n!}{(n-(r-1))!(r-1)!} \times \frac{(n+1)!}{((n+1)-(r+1))!(r+1)!}$$

$$= {}^{n-1}C_r \times {}^{n}C_{r-1} \times {}^{n+1}C_{r+1}$$

as required.

The Fibonacci sequence is nested in Pascal's triangle.



It is a lot easier to see if Pascal's triangle is drawn differently as below and coloured appropriately.

1												
1	1											
1	2	1										
1	3	3	1									
1	4	6	4	1								
1	5	10	10	5	1							
1	6	15	20	15	6	1						
1	7	21	35	35	21	7	1					
1	8	28	56	70	56	28	8	1				
1	9	36	84	126	126	84	36	9	1			
1	10	45	120	210	252	210	120	45	10	1		
1	11	55	165	330	462	462	330	165	55	11	1	
1	12	66	220	495	792	924	792	495	220	66	12	1

	Sum
	1
	1
	2
	3
	5
	8
	13
	21
	34
	55
	89
	144
	233

Find an expression for the nth term of the Fibonacci sequence in terms of ${}^{n}C_{k}$.

Hence find F_{15}.

NB Recursive formulae depend on knowing the previous terms. This expression enables you to calculate any terms of the Fibonacci sequence knowing just the term number n, so although awkward, it has advantages!

ANSWER

${}^{0}C_0$							
${}^{1}C_0$	${}^{1}C_1$						
${}^{2}C_0$	${}^{2}C_1$	${}^{2}C_2$					
${}^{3}C_0$	${}^{3}C_1$	${}^{3}C_2$	${}^{3}C_3$				
${}^{4}C_0$	${}^{4}C_1$	${}^{4}C_2$	${}^{4}C_3$	${}^{4}C_4$			
${}^{5}C_0$	${}^{5}C_1$	${}^{5}C_2$	${}^{5}C_3$	${}^{5}C_4$	${}^{5}C_5$		
${}^{6}C_0$	${}^{6}C_1$	${}^{6}C_2$	${}^{6}C_3$	${}^{6}C_4$	${}^{6}C_5$	${}^{6}C_6$	
${}^{7}C_0$	${}^{7}C_1$	${}^{7}C_2$	${}^{7}C_3$	${}^{7}C_4$	${}^{7}C_5$		
${}^{8}C_0$	${}^{8}C_1$	${}^{8}C_2$	${}^{8}C_3$	${}^{8}C_4$			
${}^{9}C_0$	${}^{9}C_1$	${}^{9}C_2$	${}^{9}C_3$				
${}^{10}C_0$	${}^{10}C_1$	${}^{10}C_2$					
${}^{11}C_0$	${}^{11}C_1$						
${}^{12}C_0$							

$F_1 = 1 = {}^{0}C_0$

$F_2 = 1 = {}^{1}C_0$

$F_3 = 2 = {}^{2}C_0 + {}^{1}C_1$

$F_4 = 3 = {}^{3}C_0 + {}^{2}C_1$

$F_5 = 5 = {}^{4}C_0 + {}^{3}C_1 + {}^{2}C_2$

$F_6 = 8 = {}^{5}C_0 + {}^{4}C_1 + {}^{3}C_2$

$F_7 = 13 = {}^{6}C_0 + {}^{5}C_1 + {}^{4}C_2 + {}^{3}C_3$

$F_8 = 21 = {}^{7}C_0 + {}^{6}C_1 + {}^{5}C_2 + {}^{4}C_3$

If n is even $F_n = \sum_{i=0}^{n/2} {}^{n-1-i}C_i$

odd $F_n = \sum_{i=0}^{\frac{1}{2}(n-1)} {}^{n-1-i}C_i$

If n is

ANSWER

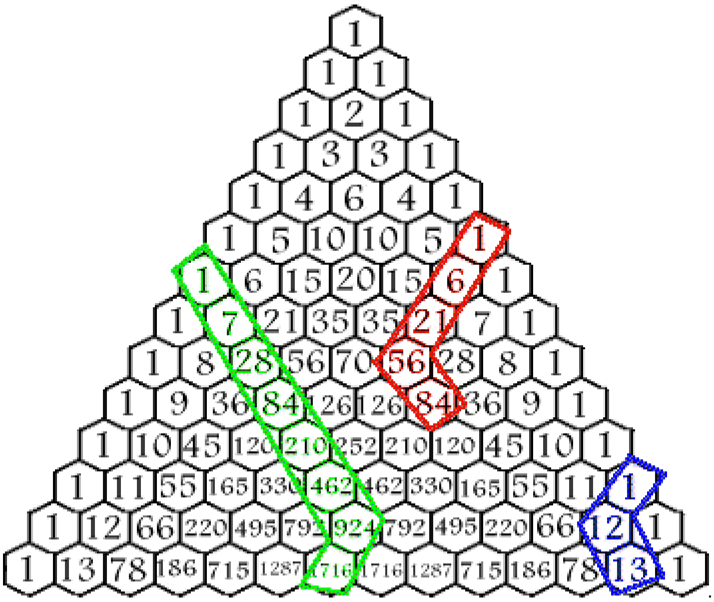
$F_{15} = ?$

n is odd so $F_{15} = \sum_{i=0}^{7} {}^{14-i}C_i = {}^{14}C_0 + {}^{13}C_1 + {}^{12}C_2 + {}^{11}C_3 + {}^{10}C_4 + {}^{9}C_5 + {}^{8}C_6 + {}^{7}C_7$

$= 1 + 13 + 66 + 165 + 210 + 126 + 28 + 1$

= 610

The Hockey Stick Pattern



http://britton.disted.camosun.bc.ca/pascal/pascal.html

The hockey stick pattern is formed by starting at any edge, going down any number of cells then turning to form the foot of the hockey stick.

It has been conjectured that the sum of the stem of the hockey stick (including the heel) is equal to the number in the toe cell. Check using the examples in the diagram above.

Prove the conjecture using cells in terms such as ${}^{n}C_{k}$.

Hint : $\mathbf{{}^{n}C_{k} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k}}$

ANSWER

1+7+28+84+210+462+924 = 1716

1+12 = 13

1+6+21+456 = 84 It works!

and now to prove it works for all cases!

The sum of the stem = ${}^{n}C_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \cdots + {}^{r-1}C_k$

The toe = ${}^{r}C_k$

Remember ${}^{n}C_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$

The three hockey sticks illustrated consist of the patterns:

(a) ${}^{6}C_0$, ${}^{7}C_1$, ${}^{8}C_2$, ${}^{9}C_3$, ${}^{10}C_4$, ${}^{11}C_5$, and ${}^{12}C_6$ and the length of the toe is ${}^{13}C_6$
which is equivalent to
${}^{6}C_6$, ${}^{7}C_6$, ${}^{8}C_6$, ${}^{9}C_6$, ${}^{10}C_6$, ${}^{11}C_6$, and ${}^{12}C_6$ and the length of the toe is ${}^{13}C_7$

(b) ${}^{11}C_{11}$, and ${}^{12}C_{11}$ and the length of the toe is ${}^{13}C_{12}$

(c) ${}^{5}C_5$, ${}^{6}C_5$, ${}^{7}C_5$ and ${}^{8}C_5$ and the length of the toe is ${}^{9}C_6$.

Considering the pattern of the hockey sticks, and if the toe is defined to be ${}^{n}C_r$

then the stem elements are ${}^{n-1}C_{r-1}$, ${}^{n-2}C_{r-1}$, ${}^{n-3}C_{r-1} \ldots {}^{r-1}C_{r-1}$

But ${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$.

Start with the toe : ${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$.

Using the rule again, ${}^{n-1}C_r = {}^{n-2}C_{r-1} + {}^{n-2}C_r$, and substituting into the stem equation

$$ {}^{n}C_r = {}^{n-1}C_{r-1} + ({}^{n-1}C_r) = {}^{n-1}C_{r-1} + ({}^{n-2}C_{r-1} + {}^{n-2}C_r). $$

Keep using the rule until you obtain

${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-2}C_{r-1} + {}^{n-3}C_{r-1} + \ldots + {}^{r+1}C_{r-1} + {}^{r}C_{r-1} + {}^{r}C_r$.

But = ${}^{r}C_r = 1 = {}^{r-1}C_{r-1}$

So ${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-2}C_{r-1} + {}^{n-3}C_{r-1} + \ldots + {}^{r+1}C_{r-1} + {}^{r}C_{r-1} + {}^{r-1}C_{r-1}$.

Therefore the sum of the cells in the stem is equal to the cell that represents the toe.

A delightful site supplied by Dr Dennis Ireland MLC is

https://theconversation.com/the-12-days-of-pascals-triangular-christmas-21479

MERSENNE PRIMES

A Mersenne prime is a prime number of the form $M_n = 2^n - 1$.

Mersenne primes are named after the French monk Marin Mersenne who studied them in the early 17th century.
The first four Mersenne numbers are 1, 3, 7, 15.

(a) Determine a way to find these numbers on the Pascal Triangle below:

$$
\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&1&&1&&&&&& \\
&&&&&1&&2&&1&&&&& \\
&&&&1&&3&&3&&1&&&& \\
&&&1&&4&&6&&4&&1&&& \\
&&1&&5&&10&&10&&5&&1&& \\
&1&&6&&15&&20&&15&&6&&1& \\
1&&7&&21&&35&&35&&21&&7&&1
\end{array}
$$

(b) Hence determine the next three Mersenne prime numbers.

(c) Find an expression in terms of ${}^nC_r$ for the fifth Mersenne number.

(d) Explain why 255 is a Mersenne prime.

(e) (i) Find a prime number between 1000 and 2000

(ii) Explain a method to find the prime number you found in (i) using Pascal's triangle.

(f) Prove that any Mersenne prime M_n may be expressed by $1 + 2 + 2^2 + 2^3 + ... + 2^{n-1}$

(g) Prove that $\sum_{i=0}^{n-1} 2^i$ is prime.

ANSWER

(a) To find the nth Mersenne prime number, add up all the terms of Pascal's triangle for the first n rows.

(b) 31, 63, 127

(c) $M_5 = {}^oC_o + {}^1C_o + {}^1C_1 + {}^2C_o + {}^2C_1 + {}^2C_2 + {}^3C_o + {}^3C_1 + {}^3C_2 + {}^3C_3$

$+ {}^4C_o + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 + {}^5C_o + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

(d) $255 = 256 – 1 = 2^8 -1$

$\therefore$ 255 = M_6, the 6th Mersenne prime.

(e) (i) $2^{10} = 1024$

$\therefore 1023 = M_{10}$

(ii) To find the 10th Mersenne prime number, add up all the terms of Pascal's triangle for the first 10 rows.

(f) <u>Method 1</u>

$1 + 2 + 2^2 + 2^3 + ... + 2^{n-1}$ is the sum of n terms of a geometric progression.

Therefore $1 + 2 + 2^2 + 2^3 + ... + 2^{n-1} = \frac{a(1 - r^n)}{1 - r}$ where a = 1, r = 2 and n – n

$1 + 2 + 2^2 + 2^3 + ... + 2^{n-1} = \frac{1(1 - 2^n)}{1 - 2} = \frac{2^n - 1}{1} = 2^n - 1$ which is the nth Mersenne prime.

<u>Method 2</u>

M_n is equal to the sum of the first n lines of Pascal's triangle.

$M_n = ({}^oC_o) + ({}^1C_o + {}^1C_1) + ({}^2C_o + {}^2C_1 + {}^2C_{2)} + ...+ ({}^{n-1}C_o + {}^{n-1}C_1 + {}^{-1}C_2 +... {}^{n-1}C_{n-1})$

$= 2^0 + 2^1 + 2^2 +... + 2^3 + ... + 2^{n-1}$

$= 1 + 2 + 2^2 + 2^3 + ... + 2^{n-1}$

(g) $\sum_{i=0}^{n-1} 2^i = 1 + 2 + 2^2 + 2^3 + ... + 2^{n-1} = M_n$ (from (f))

which is a Mersenne prime so is a prime number.

FINAL FUN

Colour in black all the odd numbers in the Pascal’s triangle below:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1

Extend to 15 lines.

You will have produced a portion of the Sierpinski Triangle.
Look up fractals and Sierpinski Triangle on the internet.

Kindly proofed and extra suggestions by Dr Dennis Ireland and his staff at MLC.