First Semester Examination REVISION MATERIAL

ANSWERS

Complex Numbers

1. [11 marks]

Given that z and w^{-1} are complex numbers such that :

$$zw^{-1} = 0.4 - 0.7i$$
$$\overline{w} = 4 - 2i$$

Determine :

a. $w = 4 + 2i$ ✓

[1]

b. $z = zw^{-1} . w$
$= (0.4 - 0.7i)(4 + 2i)$ ✓
$= 3 - 2i$ ✓

[2]

c. $z^{-1} = \dfrac{\overline{z}}{|z|^2}$ ✓ $= \dfrac{3 + 2i}{13}$ ✓

[2]

Another complex number v is given by $v = r \text{ cis } \theta$. Write expressions, in terms of r and θ, for the following complex numbers :

d. v^2 in polar form $v^2 = (r \text{ cis } \theta)^2$
$= r^2 \text{ cis } 2\theta$ ✓ ✓

[2]

e. $-2v$ in polar form $-2v = 2 \text{ cis } \pi . r \text{ cis } \theta$
$= 2r . \text{ cis } (\theta + \pi)$ ✓ ✓

[2]

f. $v + 2$ in Cartesian form i.e. $a + bi$

$v + 2 = r \text{ cis } \theta + 2$
$= (r \cos\theta + 2) + (r \sin\theta) i$ ✓ ✓

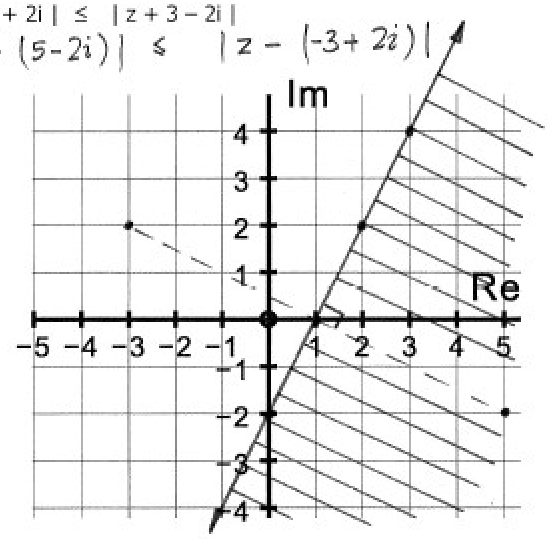
[2]

5. [11 marks]

Sketch on the following Argand diagrams, the locus for :

a. $|z - 5 + 2i| \leq |z + 3 - 2i|$

i.e. $|z - (5-2i)| \leq |z - (-3+2i)|$

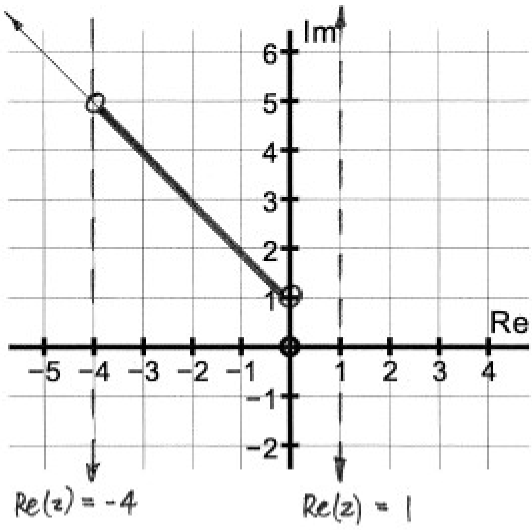


✓ Boundary

✓ Shading

[2]

b. $\text{Arg}(z - i) = \frac{3\pi}{4}$ and $-4 < \text{Re}(z) < 1$



✓ Open circles at $x = -4$, $x = 0$

✓ Line from $(0, 1)$, $m = -1$

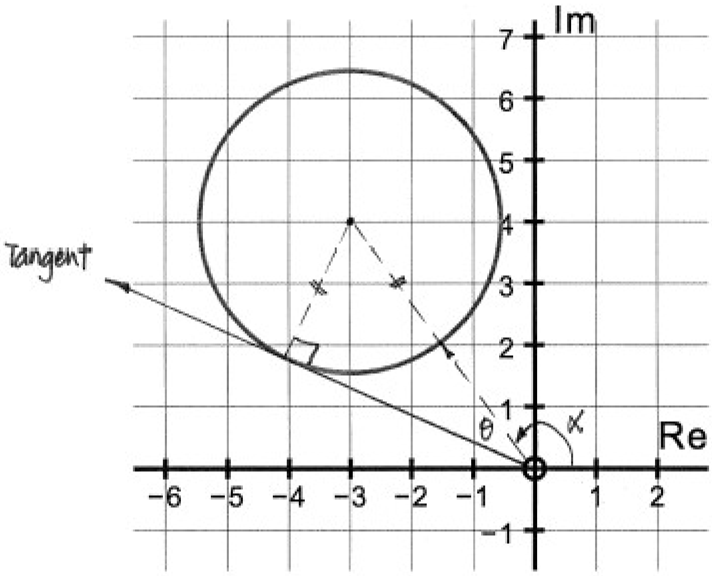
✓ Line segment intersection

[3]

5. [11 marks]

Radius

The locus for $|z + 3 - 4i| = \sqrt{6}$ is shown in the Argand diagram below.



c. Give the minimum value for $|z|$.

$$\text{Min } |z| = |-3 + 4i| - \sqrt{6} \quad \checkmark \text{ idea}$$
$$= 5 - \sqrt{6} \quad \checkmark$$

[2]

d. Give the maximum value for Arg(z), correct to 0.01 radians.

$$\tan \alpha = -\frac{4}{3}$$
$$\therefore \alpha = 2.214\ldots \checkmark$$
$$\sin \theta = \frac{\sqrt{6}}{5}$$
$$\therefore \theta = 0.511\ldots \checkmark$$
$$\text{Maximum } \operatorname{Arg}(z) = \theta + \alpha \checkmark$$
$$= 2.73 \checkmark (2 \text{ d.p.})$$

[4]

15. [12 marks]

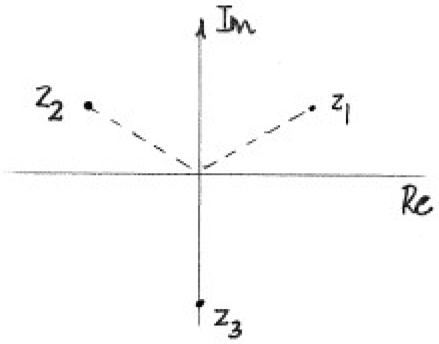
b. i. Express $\text{cis}\,\frac{5\pi}{2}$ in exact Cartesian form.

$\text{cis}\,\frac{5\pi}{2} = \text{cis}\,\frac{\pi}{2} = i$ ✓

[1]

ii. Given that $\text{cis}\,3\theta = \text{cis}\,\frac{\pi}{2}$, determine the 3 possible exact values for $\text{cis}\,\theta$, in Cartesian form.

[Hint : consider an Argand diagram]



Let $z = \text{cis}\,\theta$

$\therefore \text{cis}\,3\theta = i$

$\therefore z^3 = \text{cis}\,\frac{\pi}{2}$ ✓

✓ 3 solutions evident in diagram.

$\therefore z_1 = \text{cis}\,\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2} + \frac{i}{2}$ ✓

$z_2 = \text{cis}\,\frac{5\pi}{6}$

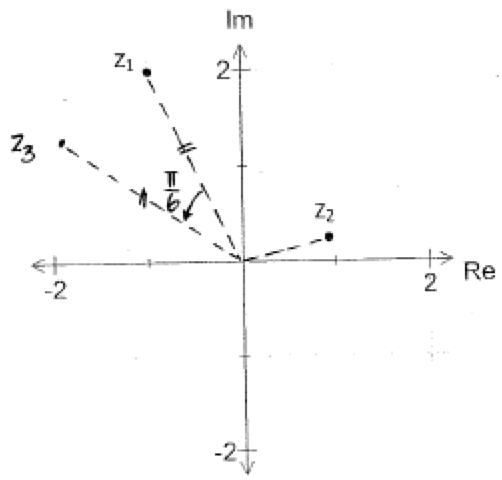
$= -\frac{\sqrt{3}}{2} + \frac{i}{2}$ ✓

$z_3 = \text{cis}\,-\frac{\pi}{2}$

$= -i$ ✓

[5]

5. The Argand diagram below shows z_1 as representative of ANY comp umber, with $z_2 = \text{cis}\left(\dfrac{\pi}{12}\right)$:



a. Show the position for the complex number $z_3 = z_1(z_2)^2$ on the Argand diagram above.

$z_3 = z_1 . \text{cis}\, \frac{\pi}{6}$

$\therefore$ Rotate z_1 by $\frac{\pi}{6}$ ✓ anti-clockwise about origin.

$|z_3| = |z_1|$ ✓ [2]

b. Simplify the expression $z_1(z_2)^{12}$.

$z_1\,(z_2)^{12}$

$= z_1 . \text{cis}\,\pi$ ✓

$= -z_1$ ✓ [2]

c. Solve for the value(s) of n, in the complex equation $(z_2)^n = -i$, where n is a positive integer.

Solve $\left(\text{cis}\,\frac{\pi}{12}\right)^n = -i$ ✓ Convert $-i$ to cis form

$\therefore \text{cis}\,\frac{n\pi}{12} = \text{cis}\left(-\frac{\pi}{2} + 2\pi k\right)$

$\therefore \frac{n\pi}{12} = -\frac{\pi}{2} + 2\pi k$ $\quad k = 0, 1, 2, \ldots$

$\therefore n = -6 + 24k$

$\therefore n = 18, 42, 66, \ldots$ ✓

✓ Infinite set of values [3]

20/ [3 marks]

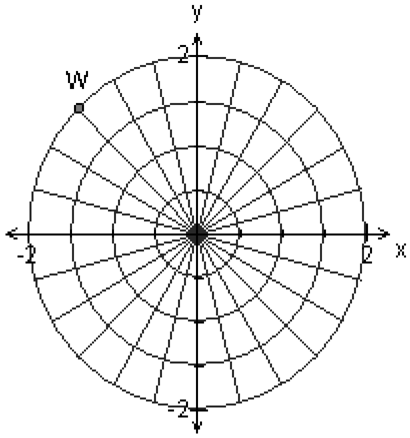
The point W on the Argand diagram below represents the complex number w where $|w| = 2$.

Mark on the Argand diagram the following complex numbers clearly labelling each point.

a) $\frac{w}{2}$ √

b) w^{-1} √

c) $-iw$ √



Functions

3. [12 marks]

Find $\frac{dy}{dx}$ given :

a. $y = 2\cos^3(4x)$

$$\frac{dy}{dx} = 2 \cdot 3\cos^2(4x) \cdot -\sin(4x) \cdot 4 \quad \checkmark\checkmark$$

$$= -24\cos^2(4x)\sin(4x) \quad \checkmark \text{ simplify}$$

[3]

b. $xy^2 + 4y = \sin 2x$

$$x \cdot 2y \cdot \frac{dy}{dx} + 1 \cdot y^2 + 4 \cdot \frac{dy}{dx} = \cos 2x \cdot 2 \quad \checkmark\checkmark$$

$$\frac{dy}{dx} \cdot (2xy + 4) = 2\cos 2x - y^2 \quad \checkmark$$

$$\therefore \frac{dy}{dx} = \frac{2\cos 2x - y^2}{2(xy + 2)} \quad \checkmark$$

[4]

c. $y = (2x)^{\sin x}$

$$\ln y = \sin x \cdot \ln 2x \quad \checkmark$$

$$\therefore \frac{1}{y} \cdot \frac{dy}{dx} \checkmark = \sin x \cdot \frac{1}{2x} \cdot 2 + \ln 2x \cdot \cos x \quad \checkmark$$

$$= \frac{\sin x}{x} + \cos x \cdot \ln 2x \quad \checkmark$$

$$\therefore \frac{dy}{dx} = \left(\frac{\sin x}{x} + \cos x \cdot \ln 2x\right) \cdot (2x)^{\sin x} \quad \checkmark$$

5. [3, 3, 2, 4 = 12 marks]

(a) Write $f(x) = |x+a| - |x-a|$ where $a > 0$ as a piecewise defined function.

$$\begin{cases} -2a & x \le -a & \checkmark \\ 2x & -a < x < a & \checkmark \\ 2a & x \ge a & \checkmark \end{cases}$$

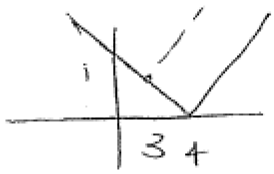
(b) $\{x \in R : x \le 3\}$ is the set of values for which $|x-4| = |x+a| + b$

Determine a possible set of solutions for a and b

$a = -3$ ✓

$b = 1$ ✓

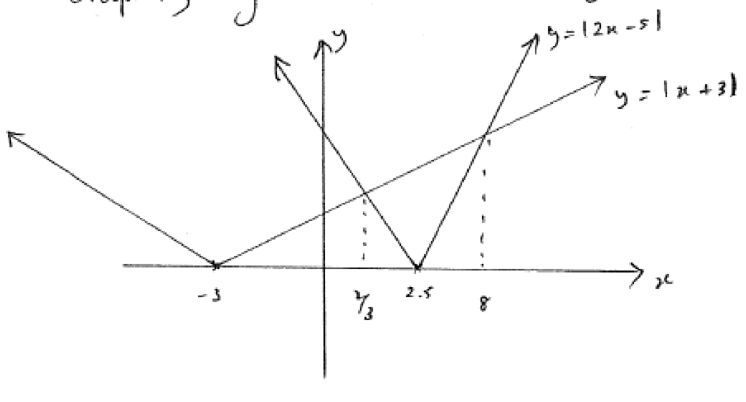
graphical approach



(d) Solve exactly for x

$|2x-5| < |x+3|$

Graphing $y = |x+3|$ and $y = |2x-5|$ gives



✓

Intersections occur when

$2x - 5 = x + 3$ or $-2x + 5 = x + 3$

$x = 8$ ✓ or $3x = 2$

$x = 2/3$ ✓

Solution is $\underline{\underline{\frac{2}{3} < x < 8}}$ ✓

6. [8 marks]

Solve the following equations exactly :

a. $|2x + 2| = x^2 - 2$

For $x < -1$: $-(2x+2) = x^2 - 2$ ✓ Case 1

$\therefore x^2 + 2x = 0$

$\therefore x(x+2) = 0$

$\therefore x = 0$ or $x = -2$

But $x < -1$ $\therefore x = -2$.

For $x \geq -1$: $2x + 2 = x^2 - 2$

$\therefore x^2 - 2x - 4 = 0$ ✓ Case 2

$(x-1)^2 = 5$

$\therefore x = 1 \pm \sqrt{5}$

But $x > -1$ $\therefore x = 1 + \sqrt{5}$

$\therefore$ Two solutions $x = -2,\ 1+\sqrt{5}$ ✓ ✓

[4]

b. $\left|\dfrac{x}{x+3}\right| < 4$

$\therefore \left(\dfrac{x}{x+3}\right)^2 < 16$ ✓

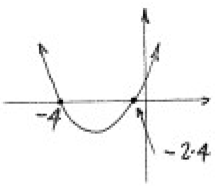
$\therefore x^2 < 16(x+3)^2$

$\therefore 0 < 15x^2 + 96x + 144$

$0 < 3(5x^2 + 32x + 48)$ ✓ Technique

$0 < 3(5x+12)(x+4)$

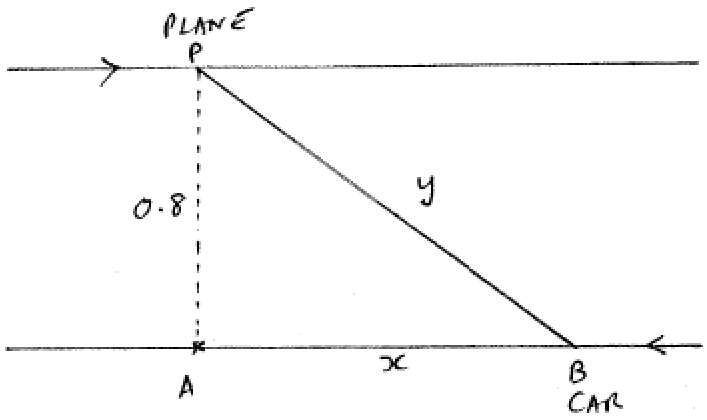
$\therefore x < -4,\ x > -\dfrac{12}{5}$ ✓ ✓



5. [7 marks]

A police plane flies 800 metres above a level straight road at a steady speed of 200 km per hour. The pilot sees an oncoming car and tries to determine whether the car is speeding. Using radar he measures the direct distance from the plane to the car as 2.8 km and determines that this value is decreasing at 300 km per hour.

Use calculus to determine whether or not the car is speeding and if so by how much. You may assume that the speed limit is 110 km per hour.



✓

Let y = distance between plane and car

x = horizontal distance between plane and car

$$y^2 = x^2 + 0.8^2 \quad ✓$$

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt} \quad ✓$$

$$\frac{y}{x}\frac{dy}{dt} = \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{2.8}{\sqrt{2.8^2 - 0.8^2}} \times {}^{-}300 \quad ✓$$

$$= {}^{-}313.05 \text{ km/hr} \quad ✓$$

But plane is moving at 200 km/hr

∴ Car's speed is 113 km/hr ✓

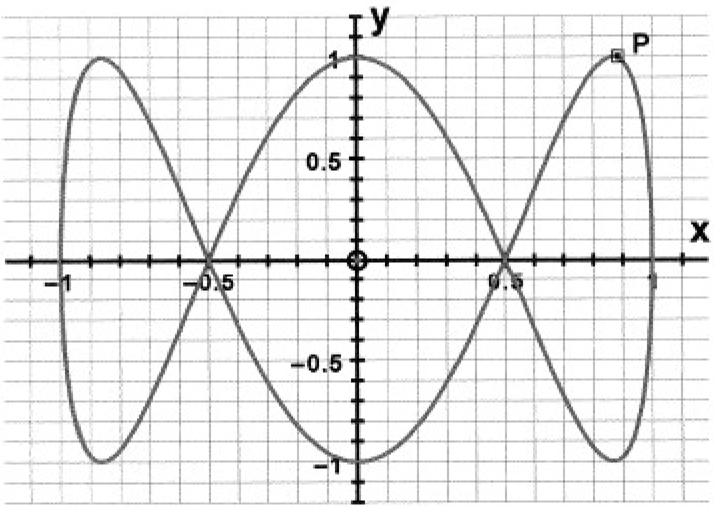
This is greater than 110 km/hr ∴ speeding ✓

by 3 km/hr

7

13. [9 marks]

The well known 'ABC logo' is parametrically defined by the equations : $x = \cos t$
$y = \sin 3t$



$\frac{dx}{dt} = -\sin t$ ✓

$\frac{dy}{dt} = 3\cos 3t$ ✓

a. Find an expression for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= -\frac{3\cos 3t}{\sin t} \checkmark$$

[3]

b. Find the slope of one of the tangents to the curve at the point (0.5, 0).

At (0.5, 0) $y = 0$
$\sin 3t = 0$ ✓ $\therefore t = \frac{\pi}{3}$ ✓ (or $\frac{5\pi}{3}$)

$$\frac{dy}{dx} = -\frac{3\ (-1)}{\frac{\sqrt{3}}{2}} = 2\sqrt{3}$$

$\therefore$ Slope of tangents are $\pm 2\sqrt{3}$. ✓

[3]

c. At point P, the tangent to the curve is horizontal.
Determine the exact x co-ordinate for point P.

If $\frac{dy}{dx} = 0$ $\therefore$ $\cos 3t = 0$ ✓
$3t = \frac{\pi}{2}$
$\therefore t = \frac{\pi}{6}$ ✓

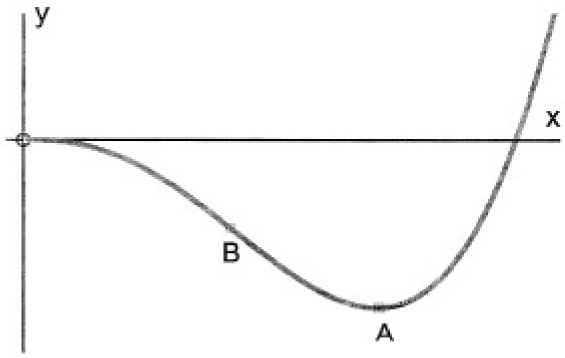
$$\therefore x_P = \cos\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{2} \checkmark$$

[3]

Σ 9m

14. [9 marks]

The graph of $f(x) = x^3 \ln x$ is shown below where $x > 0$.



a. Determine the exact global minimum value for function f .

$$f'(x) = x^3 \cdot \frac{1}{x} + 3x^2 \cdot \ln x$$
$$= x^2 (1 + 3 \ln x) \checkmark$$

Require $f'(x) = 0 \checkmark$ $\therefore$ $\ln x = -\frac{1}{3}$

$\therefore x = e^{-\frac{1}{3}} \checkmark$

$$\therefore \text{Minimum value} = f(e^{-\frac{1}{3}}) \checkmark$$
$$= (e^{-\frac{1}{3}})^3 \cdot (-\tfrac{1}{3})$$
$$= -\frac{e^{-1}}{3}$$
$$= -\frac{1}{3e} \checkmark$$

[5]

b. Give the exact x co-ordinate for point B, the point of inflection.

$$f''(x) = 2x + 3x^2 \cdot \frac{1}{x} + 6x \cdot \ln x$$
$$= x(5 + 6 \ln x) \checkmark\checkmark$$

Require $f''(x) = 0 \checkmark$

$\therefore \ln x = -\frac{5}{6}$

$\therefore x = e^{-\frac{5}{6}} \checkmark$

$\therefore$ x co-ordinate of B is $\frac{1}{\sqrt[6]{e^5}}$.

[4]

7/ [6+5=11 marks]

a) Find the exact value(s) of k for which $|k(4-k)| = |k+2|$

$$K(4-K) = \pm (K+2)$$

$$4K - K^2 = K + 2 \quad ✓$$

$$0 = K^2 - 3K + 2$$

$$0 = (K-2)(K-1)$$

$$K = 2, \quad K = 1$$

✓ ✓

$$K(4-K) = -K - 2 \quad ✓$$

$$0 = K^2 - 5K - 2$$

$$K = \frac{5 \pm \sqrt{25 + 4\times 2}}{2}$$

$$K = \frac{5 \pm \sqrt{33}}{2} \quad ✓✓$$

b) The inequality $|kx - 6| < |3x + 1|$ has a solution $0.625 < x < 3.5$. Find the value(s) of k.

$$0.625 - 6 = 3 \times 0.625 + 1$$

$$K = 14.2 \checkmark$$

$$0.625 - 6 = -3 \times 0.625 - 1$$

$$K = 5 \checkmark$$

$$3.5K - 6 = 10.5 + 1$$

$$K = 5 \checkmark$$

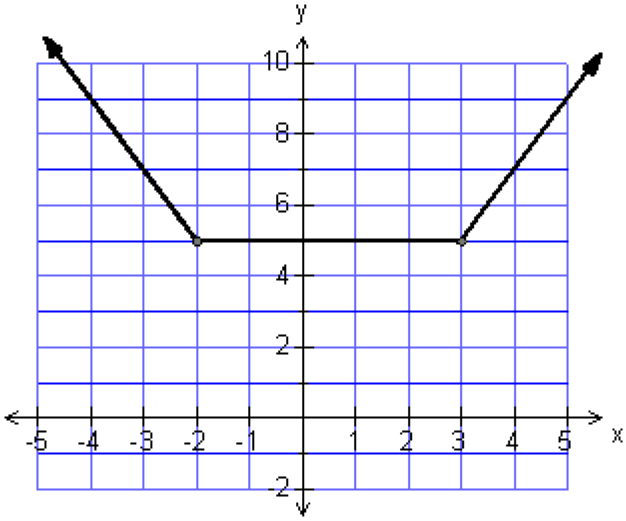
$$3.5K - 6 = -10.5 - 1$$

$$K = -1.57 \text{ - - - -} \checkmark$$

$\therefore$ $K = 5$ since both inequalities need to be satisfied. $\checkmark$

8/ [2+3=5 marks]

The function f, defined for all real x by $f(x) = |x - a| + |x + b|$, where a and b are positive integers, has the following graph.



a) Find the values of a and b.

$a = 3$ ✓
$b = 2$ ✓

b) Express $f(x)$ as a piecewise function.

$$f(x) = \begin{cases} x - 3 + x + 2 = 2x - 1 & x \geq 3 \checkmark \\ -x + 3 + x + 2 = 5 & -2 < x < 3 \checkmark \\ -x + 3 - x - 2 = -2x + 1 & x \leq -2 \checkmark \end{cases}$$

10/ [7 marks]

$y = e^{2x}\cos x$ is a solution of the differential equation

$\frac{d^2y}{dx^2} + k\frac{dy}{dx} + y = -2e^{2x}\sin x$ where $k \in R$. Find the value of k.

$$\frac{dy}{dx} = 2e^{2x}\cos x - e^{2x}\sin x \checkmark\checkmark$$

$$= e^{2x}(2\cos x - \sin x)$$

$$\frac{d^2y}{dx^2} = e^{2x}(-2\sin x - \cos x) + 2e^{2x}(2\cos x - \sin x) \checkmark\checkmark$$

$$= e^{2x}(-4\sin x + 3\cos x)$$

$$e^{2x}(-4\sin x + 3\cos x + 2k\cos x - k\sin x + \cos x) \checkmark$$

$$= -2e^{2x}\sin x$$

$$\therefore 4\cos x + 2k\cos x = 0 \checkmark$$

$$4 = -2k$$

$$k = -2 \checkmark$$

$$(-4\sin x - k\sin x = -2\sin x$$

$$-k\sin x = 2\sin x$$

$$k = -2)$$

14/ [2+3+4=9 marks]

Find $\frac{dy}{dx}$ for the following, simplifying your answer.

a) $y = e^{2x}\ln\left(\frac{1}{x^2}\right)$

$$\frac{dy}{dx} = 2e^{2x}\ln\left(\frac{1}{x^2}\right) - \frac{2e^{2x}}{x}$$

✓ ✓

b) $y = 3\cos^2 2x - 4\sin(x^2)$ $= 3(\cos 2x)^2 - 4\sin(x^2)$

$$\frac{dy}{dx} = -12\cos 2x \sin 2x - 8x\cos(x^2)$$

✓ ✓

$$= -6\sin 2x - 8x\cos(x^2)$$ ✓

c) $x = \frac{t}{1+t^2}$ and $y = 1+t^2$

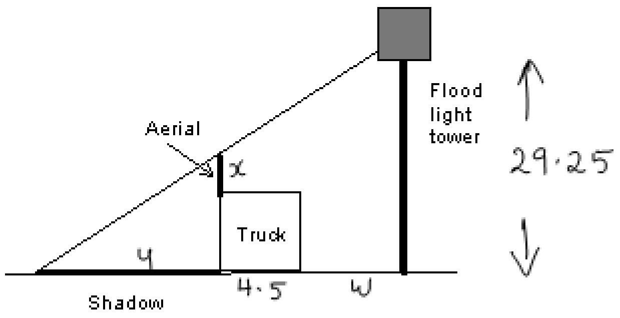
$$\frac{dx}{dt} = \frac{1\times(1+t^2) - t\times 2t}{(1+t^2)^2}$$ ✓ $$\frac{dy}{dt} = 2t$$ ✓

$$= \frac{-t^2+1}{(1+t^2)^2}$$ ✓

$$\frac{dy}{dx} = \frac{2t(1+t^2)^2}{1-t^2}$$ ✓

27/ [8+2=10 marks]

A truck is moving towards a flood light tower at a constant speed of 7.5 m/s. The truck has an aerial attached to it which can be raised and lowered. The light from the tower causes the aerial to cast a shadow on the ground behind the truck. Not including the aerial, the truck is 3.2 m in height and 4.5 m long. The flood light tower is 29.25 m tall.



When the front of the truck is 20 m from the base of the tower, the length of the shadow is decreasing at the rate of 2.5 m/s and the aerial is 4 metres tall.

a) Determine the rate of change of the length of the aerial at this instant.

$$\frac{x+3\cdot2}{29\cdot25} = \frac{y}{4\cdot5+y+w} \checkmark \checkmark$$

$$\frac{dx}{dt} = ?$$

$$\frac{dy}{dt} = -2\cdot5\,m/s$$

$$\frac{dw}{dt} = -7\cdot5\,m/s$$

$$x = \frac{29\cdot25y}{4\cdot5+y+w} - 3\cdot2 \checkmark$$

$$\frac{dx}{dt} = 29\cdot25\frac{dy}{dt}(4\cdot5+y+w)^{-1} \checkmark$$
$$+ 29\cdot25(4\cdot5+y+w)^{-1}\left(\frac{dy}{dt}+\frac{dw}{dt}\right) \checkmark$$

3D Vectors

2. **[2 + 2 + 4 + 3 + 3 + 3 + 4 = 21 marks]**

Consider the following vectors in space :

$$\mathbf{a} = -2i + j - 4k$$
$$\mathbf{b} = xi + 3j - 2k$$
$$\mathbf{c} = 5i + 2j - zk$$
$$\mathbf{d} = 3i - 7k$$

Determine :

(a) $4\mathbf{a} - \mathbf{d}$

2m

$$= \begin{pmatrix} -8 \\ 4 \\ -16 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -7 \end{pmatrix} = \begin{pmatrix} -11 \\ 4 \\ -9 \end{pmatrix} \quad ✓✓$$

(b) vector **e** such that **e** is parallel to **d** and double its length.

2m

$$\underset{\sim}{e} = 2\underset{\sim}{d}$$
$$= \begin{pmatrix} 6 \\ 0 \\ -14 \end{pmatrix} \quad ✓✓$$

(c) the acute angle between vectors **a** and **d** (to nearest degree).

4m

$$\underset{\sim}{a} \cdot \underset{\sim}{d} = |\underset{\sim}{a}|\,|\underset{\sim}{d}| \cos\theta$$
$$22 ✓ = (\sqrt{21})(\sqrt{58})\cos\theta \quad ✓$$
$$\therefore \theta = \cos^{-1}\left(\frac{22}{(\sqrt{21})(\sqrt{58})}\right) \quad ✓$$
$$\theta = 51° \text{ (nearest degree)} \quad ✓$$

(d) the value of z such that **c** is perpendicular to **a**.

3m

$$\underset{\sim}{c} \cdot \underset{\sim}{a} = 0 \quad ✓$$
$$-10 + 2 + 4z = 0 \quad ✓$$
$$z = 2 \quad ✓$$

(e) the value of x such that **a** is parallel to **b**.

3m

$$\begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} = k \begin{pmatrix} x \\ 3 \\ -2 \end{pmatrix}$$ ✓

$3k = 1$ and $-2k = -4$

$k = \frac{1}{3}$ $k = 2$ ✓

Since k is not unique, there exists no value of x that makes $\underset{\sim}{a}$ and $\underset{\sim}{b}$ parallel. ✓

(f) a unit vector **v** in the direction of **a**.

3m

$$\underset{\sim}{v} = \frac{a}{|a|} = \frac{1}{\sqrt{21}} \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$$ ✓ ✓ ✓

$$= \begin{pmatrix} -2/\sqrt{21} \\ 1/\sqrt{21} \\ -4/\sqrt{21} \end{pmatrix} \quad \text{OR} \quad \begin{pmatrix} -2\sqrt{21}/21 \\ \sqrt{21}/21 \\ -4\sqrt{21}/21 \end{pmatrix}$$

Suppose that points A and D are determined by their position vectors **a** and **d** respectively.

(g) Determine the position vector **p** for the point P in space such that P divides $\overrightarrow{AD}$ internally in the ratio 2:1.

4m

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AD}$$ ✓

$$= \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 3+2 \\ 0-1 \\ -7+4 \end{pmatrix}$$ ✓

$$= \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$$ ✓

$$= \begin{pmatrix} 4/3 \\ 1/3 \\ -6 \end{pmatrix}$$ ✓

10

5. **[6 marks]**

6m

Two jets, A and B, have velocities of $v_A = \begin{pmatrix} 200 \\ 350 \\ 450 \end{pmatrix}$ m/s and $v_B = \begin{pmatrix} -300 \\ -450 \\ 250 \end{pmatrix}$ m/s.

Their positions at $t = 0$ are $r_A = \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$ km and $r_B = \begin{pmatrix} 3 \\ 7 \\ k \end{pmatrix}$ km.

It is known that the two jets are closest to each other after approximately 8 seconds. Determine the value of k.

$${}_A\underset{\sim}{V}_B = \begin{pmatrix} 500 \\ 800 \\ 200 \end{pmatrix} \text{ m/s} \checkmark = \begin{pmatrix} 0.5 \\ 0.8 \\ 0.2 \end{pmatrix} \text{ km/s} \checkmark$$

Note : units are written in km !!

$${}_A\underset{\sim}{r}_B = \begin{pmatrix} 4+0.2t \\ -5+0.35t \\ 6+0.45t \end{pmatrix} - \begin{pmatrix} 3-0.3t \\ 7-0.45t \\ k+0.25t \end{pmatrix}$$

$$= \begin{pmatrix} 1+0.5t \\ -12+0.8t \\ (6-k)+0.2t \end{pmatrix} \checkmark$$

given that ${}_A\underset{\sim}{V}_B \cdot {}_A\underset{\sim}{r}_B = 0$ when $t=8$, $\checkmark$

$$\begin{pmatrix} 0.5 \\ 0.8 \\ 0.2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -5.6 \\ 7.6-k \end{pmatrix} = 0 \checkmark$$

$$2.5 - 4.48 + 1.52 - 0.2k = 0$$

$$k = \frac{0.46}{-0.2}$$

$$\underline{k = -2.3} \checkmark$$

Alternative Solution to Q5:

$$\vec{AB} = \begin{pmatrix} -1 - 0.5t \\ 12 - 0.8t \\ (k-6) - 0.2t \end{pmatrix} \text{ km} \quad \checkmark$$

Closest approach when $\left|\vec{AB}\right|^2$ is minimum. ✓

$$\left|\vec{AB}\right|^2 = (-1-0.5t)^2 + (12-0.8t)^2 + ((k-6)-0.2t)^2$$ needs to be minimised. ✓

$$\frac{d}{dt}\left|\vec{AB}\right|^2 = 2(-1-0.5t)(-0.5) + 2(12-0.8t)(-0.8) + 2(k-6-0.2t)(-0.2) = 0 \quad \checkmark$$

substitute $t = 8$,

$$5 - 8.96 - 0.4k + 3.04 = 0 \quad \checkmark$$

$$-0.4k = 0.92$$

$$\underline{\underline{k = -2.3}} \quad \checkmark$$

7. [4,3]
From an observation base, position (0,0,0) in Afghanistan, an Australian Captain observes an enemy Taliban military jet fighter, travelling at velocity (36**i** – 72 **j** + 8**k**) passing into a "no fly Zone". At this instant she fires a missile at velocity (-130**i** +185 **j** + 66**k**) m/s, which after three seconds collides with the enemy jet and destroys in on impact.

a. At what position, relative to the observation base, did the military jet cross into the "no fly Zone".

$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ position when cross.

$$\underset{\sim}{r}\,(\text{Jet fighter}) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t\begin{pmatrix} 36 \\ -72 \\ 8 \end{pmatrix} \checkmark$$

$$\underset{\sim}{r}\,(M) = t\begin{pmatrix} -130 \\ 185 \\ 66 \end{pmatrix} \checkmark$$

when $t = 3$, at same position

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 3\begin{pmatrix} -130-36 \\ 185+72 \\ 66-8 \end{pmatrix} = \begin{pmatrix} -498 \\ 771 \\ 174 \end{pmatrix}$$

✓ ✓

b. How close was the military jet to the observation base, when it was destroyed?

✓ ✓

Position when destroyed $= 3\cdot\begin{pmatrix} -130 \\ 185 \\ 66 \end{pmatrix}$

Distance $= 706.6$

8. [2,3,3]

a.

Let the position vectors of points P and Q be
p = **i** + 4 **j** –**k** and **q** =-**i** +3 **j** – 4**k**

i. Find the magnitude of the vector **PQ**

$$\overrightarrow{PQ} = \underline{q} - \underline{p} \checkmark$$
$$= -2\underline{i} - \underline{j} - 3\underline{k} \checkmark$$

(2)

ii Find the **vector** projection of **p** on **q**

$$(p\cos\theta)\,\hat{q} = \frac{\underline{p}\cdot\underline{q}}{q}\frac{\underline{q}}{q} = \frac{(\underline{p}\cdot\underline{q})\,\underline{q}}{q^2} \checkmark$$

$$\text{Vector proj} = \frac{(\underline{p}\cdot\underline{q})\underline{q}}{q^2} = \frac{\checkmark 15}{\sqrt{26}}\cdot\underline{q} = \frac{15}{26}\begin{pmatrix}-1\\3\\-4\end{pmatrix}$$

(3)

iii. Find a vector parallel to **q** with the same magnitude as **p**

$$\hat{\underline{q}} = \frac{1}{\sqrt{26}}\underline{q} \checkmark$$

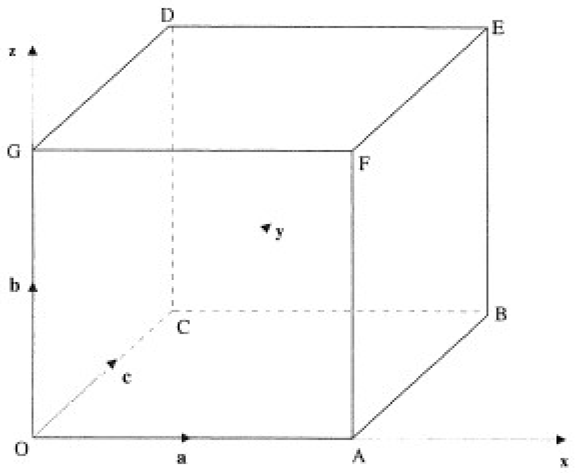
$$|\underline{p}| = \sqrt{18} \checkmark$$

(3)

$$\text{Vector} \parallel \underline{q}\text{, magnitude } \underline{p} = \frac{\sqrt{18}}{\sqrt{26}}\begin{pmatrix}-1\\3\\-4\end{pmatrix} \checkmark$$

$$= \frac{3\sqrt{13}}{13}\begin{pmatrix}-1\\3\\-4\end{pmatrix}$$

9. [2,2,2]



(**a** is the position vector of point A from O, etc.)

find

(a) a vector expression for **a**, **b** and **c** given |a|, |b| and |c| = 1.

$$\underline{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \underline{c} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(b) EG and EB.

$$EG \text{ magnitude} = \left\| \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right\| = \sqrt{2}$$

$$\overrightarrow{EG} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{EB} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Both ok

$$EB \text{ magnitude} = \left\| \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\| = 1.$$

(c) Prove EG is perpendicular to EB.

$$\overrightarrow{EG} \cdot \overrightarrow{EB} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$= 0$$

$\therefore$ Perp, since neither vector has zero magnitude

17. a. $P = \begin{bmatrix} 300 & 100 & 200 \\ 400 & 500 & 300 \end{bmatrix}$ b. $C = \begin{bmatrix} 5 & 8 \\ 3 & 4 \\ 2 & 2 \end{bmatrix}$ c. $A = \begin{bmatrix} 2200 & 3200 \\ 2750 & 4000 \end{bmatrix}$

d. Cost of removing the pollutants at each plant from the production of Alpha and Beta.
$a_{21} = 2750$ i.e. it costs $2 750 to remove the pollutants from product Beta at plant X.

e. $B = \begin{bmatrix} 5\,400 \\ 6\,750 \end{bmatrix}$ f. The daily cost of removing pollutants Alpha and Beta.

g. $T = [12\,150]$ h. The total cost of removing the pollutants.

i. $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2200 & 3200 \\ 2750 & 4000 \end{bmatrix}$ i.e. $\begin{bmatrix} 1 & 1 \end{bmatrix} A$