SEMESTER TWO

MATHEMATICS SPECIALIST UNITS 1 & 2

2018

SOLUTIONS

Calculator-free Solutions

1. (a) (i) $^{8}P_5$ or $^{8}C_5 \times 5!$ ✓

(ii) $^{4}C_3 \times {}^{4}C_2 \times 5!$ ✓✓

(b) (i) $6^4 \times 5^2 \times 6!$ ✓

(ii) '9' from first set only: $^{5}C_3 \times {}^{4}C_2 \times 6!$ ✓

'9' from second set only: $^{5}C_4 \times {}^{4}C_1 \times 6!$ ✓

no '9' chosen: $^{5}C_4 \times {}^{4}C_2 \times 6!$ ✓

total ¿ $\left({}^{5}C_3 \times {}^{4}C_2 + {}^{5}C_4 \times {}^{4}C_1 + {}^{5}C_4 \times {}^{4}C_2\right) \times 6!$ ✓ [8]

2. (a) (i) $i^{n+2} = i^n \times i^2 = -i \times -1 = i$ ✓

(ii) $i^{2n+1} = \left(i^n\right)^2 \times i = (-i)^2 \times i = -i$ ✓✓

(b) $\dfrac{1-i}{i+\dfrac{2}{i}} \times \dfrac{i}{i} = \dfrac{i-i^2}{i^2+2} = \dfrac{1+i}{-1+2} = 1+i$ ✓✓

(c) $(1+i)^4 - (1-i)^4$

¿ $\left[(1+i)^2 + (1-i)^2\right] \times \left[(1+i)^2 - (1-i)^2\right]$

¿ $[1+2i-1+1-2i-1] \times [1+2i-1-1+2i+1]$

¿ $0 \times 4i = 0$ ✓✓ [7]

3. (a) $5^3 = 5 + 1 \times 10 \times 6 + 1 \times 4 \times 15 = 5 + 60 + 60 = 125$ ✓

(b) $^{5}C_2, {}^{4}C_1$ ✓✓

(c) $n^3 = n + {}^{n-1}C_0 \times {}^{n+1}C_1 \times {}^{n}C_2 + {}^{n}C_0 \times {}^{n-1}C_1 \times {}^{n+1}C_2$ ✓✓ [5]

4. (a) (i) $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓✓

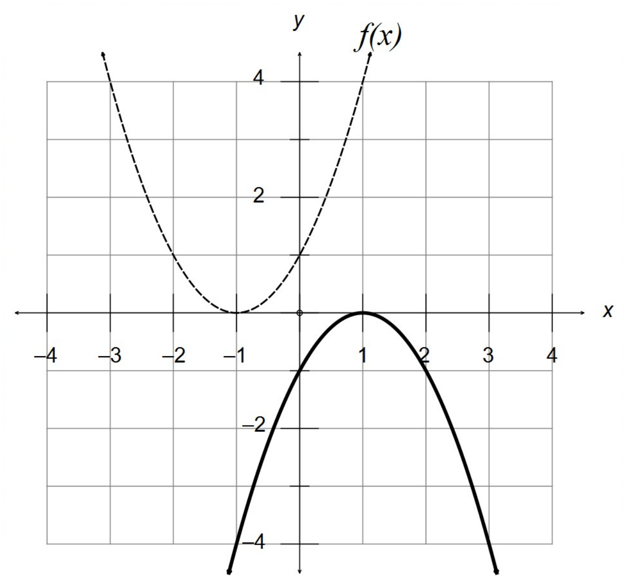
A and B are inverses of each other. ✓

(ii) $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \\ 3 \end{bmatrix}$ ✓

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \times \begin{bmatrix} -6 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} -6 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$ ✓✓

$\therefore x=2, y=1, z=-5$

4. (b) (i) T_1 performs a rotation of 180°



$\therefore\ y=-(x-1)^2$ ✓✓

(ii) Reflection of the line $y=x$ ✓

T_2¿$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ ✓

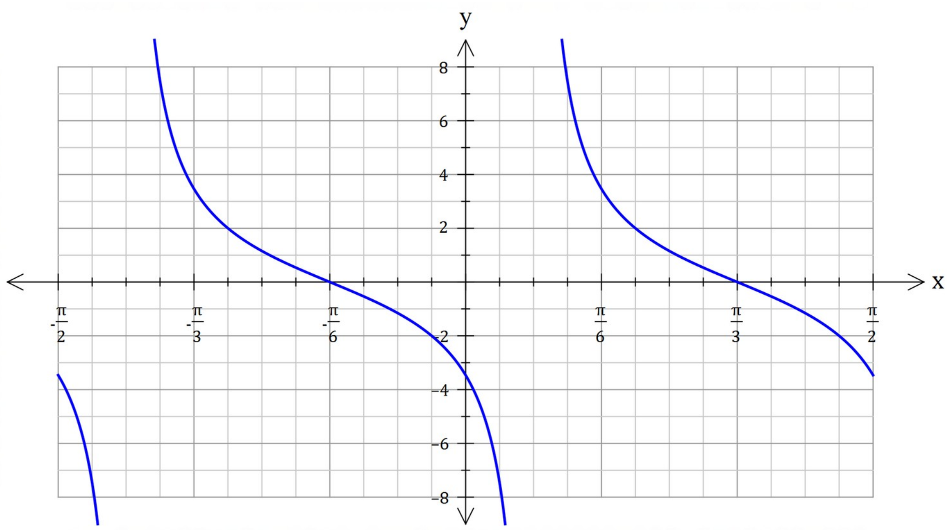
(iii) New Area ¿$|T_3|\times 10\frac{2}{3}=6\times\frac{32}{3}=64\ units^2$ ✓✓ [12]

5. (a) (i) $y=-3\cos\left[4\left(x-\frac{\pi}{12}\right)\right]=-3\cos\left[4x-\frac{\pi}{3}\right]$

$\therefore\ A=-3\ \omega=4\ \theta=\frac{-\pi}{3}$ ✓✓✓

(ii) $y=-3\sin\left(4x+\frac{\pi}{6}\right)$ ✓✓

(b)



✓ Scale factor ($y=2$ at $x=\frac{5\pi}{24}$)

✓ Period of $\frac{\pi}{2}$

✓ Vertical Asymptote

[8]

6. (a) (i) $\sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$ ✓

(ii) "If ab is irrational, then both a and b must be irrational" ✓

(b) A $\Rightarrow$ B: If the triangle has two equal sides, then it is isosceles, and therefore it has two congruent sides.

$\therefore$ A $\Rightarrow$ B is valid and True. ✓

B $\Rightarrow$ A: If the triangle has two congruent sides, then it is isosceles, and therefore it has two equal sides.

$\therefore$ B $\Rightarrow$ A is valid and True. ✓

$\therefore$ A $\Leftrightarrow$ B

(c) $\forall\, x \in P, \exists\, y \in P : xy \in Q$ ✓✓ [6]

7. Assume n is even and that n^3 is odd. ✓

Let $m \in N : n = 2m$ is even. ✓

$n^3 = (2m)^3 = 8m^3 = 2(4m^3)$ ✓

Since n^3 cannot be both even and odd simultaneously, then this is a contradiction. And therefore n must be odd. ✓ [4]

Calculator-assumed Solutions

8. (a) $z=\dfrac{4\pm\sqrt{16-24}}{4}=1\pm\dfrac{\sqrt{2}}{2}i$ ✓

$P(z)=\left(z-1-\dfrac{\sqrt{2}}{2}i\right)\left(z-1+\dfrac{\sqrt{2}}{2}i\right)$ ✓

(b) Since $a,b,c\in R$ then $\bar{z}=2-i$ is also a solution ✓

$\therefore R(z)=(z+1)(z-2-i)(z-2+i)$ ✓

¿ z^3-3z^2+z+5

$\therefore a=-3b=1c=5$ ✓ [5]

9. LHS ¿ $\dfrac{\tan\theta+\tan\dfrac{\pi}{4}}{1-\tan\theta\times\tan\dfrac{\pi}{4}}$ ✓

¿ $\dfrac{\tan\theta+1}{1-\tan\theta}$ ✓

¿ $\dfrac{\dfrac{\sin\theta}{\cos\theta}+1}{1-\dfrac{\sin\theta}{\cos\theta}}$ ✓

¿ $\dfrac{\dfrac{\sin\theta}{\cos\theta}+1}{1-\dfrac{\sin\theta}{\cos\theta}}\times\dfrac{\cos\theta}{\cos\theta}$ ✓

¿ $\dfrac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}$ ✓

¿ $\dfrac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}\times\dfrac{\cos\theta+\sin\theta}{\cos\theta+\sin\theta}$ ✓

¿ $\dfrac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}$ ✓

¿ $\dfrac{1+\sin 2\theta}{\cos 2\theta}=$ ¿RHS ✓ [8]

10. (a) (i) $\begin{pmatrix}3\\1\end{pmatrix}=k\begin{pmatrix}\alpha\\-2\end{pmatrix}\rightarrow k=\dfrac{-1}{2}\rightarrow\alpha=-6$ ✓✓

(ii) $a+c=\begin{pmatrix}3+\alpha\\-1\end{pmatrix}\rightarrow\begin{pmatrix}3+\alpha\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-3\end{pmatrix}=0\rightarrow\alpha=-6$ ✓✓

(b) $\begin{pmatrix}1\\5\end{pmatrix}\cdot\begin{pmatrix}6\\4\end{pmatrix}=\begin{vmatrix}1\\5\end{vmatrix}\times\begin{vmatrix}6\\4\end{vmatrix}\times\cos\theta\rightarrow\cos\theta=\dfrac{1}{\sqrt{2}}$ ✓

$$|m| = \left|\begin{matrix}1\\5\end{matrix}\right| = \sqrt{26}$$ ✓

$$\hat{n} = \frac{1}{2\sqrt{13}}\begin{pmatrix}6\\4\end{pmatrix}$$ ✓

$$_{m}proj_{n} = |m|\cos\theta\,\hat{n} = \sqrt{26}\times\frac{1}{\sqrt{2}}\times\frac{1}{2\sqrt{13}}\begin{pmatrix}6\\4\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} = 3i + 2j$$ ✓ [8]

11. (a) $S_3 = \frac{1}{2!} + \frac{2}{3!} = \frac{1}{2} + \frac{2}{6} = \frac{5}{6}$ ✓

$S_4 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{1}{2} + \frac{2}{6} + \frac{3}{24} = \frac{23}{24}$ ✓

(b) $\frac{3!-1}{3!} = \frac{6-1}{6} = \frac{5}{6}$ ✓

$\frac{4!-1}{4!} = \frac{24-1}{24} = \frac{23}{24}$ ✓

(c) $S_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n-1}{n!} = \frac{n!-1}{n!}$ ✓

(d) For $n=2$: $S_2 = \frac{1}{2}$ and $\frac{2!-1}{2!} = \frac{1}{2}$ $\therefore$ True for $n=2$ ✓

Assume true for $n=k$:

$S_k = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{k-1}{k!} = \frac{k!-1}{k!}$ ✓

For $n=k+1$:

$S_{k+1} = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{k-1}{k!} + \frac{(k+1)-1}{(k+1)!}$

¿ $S_k + \frac{k}{(k+1)!} = \frac{k!-1}{k!} + \frac{k}{(k+1)!}$ ✓

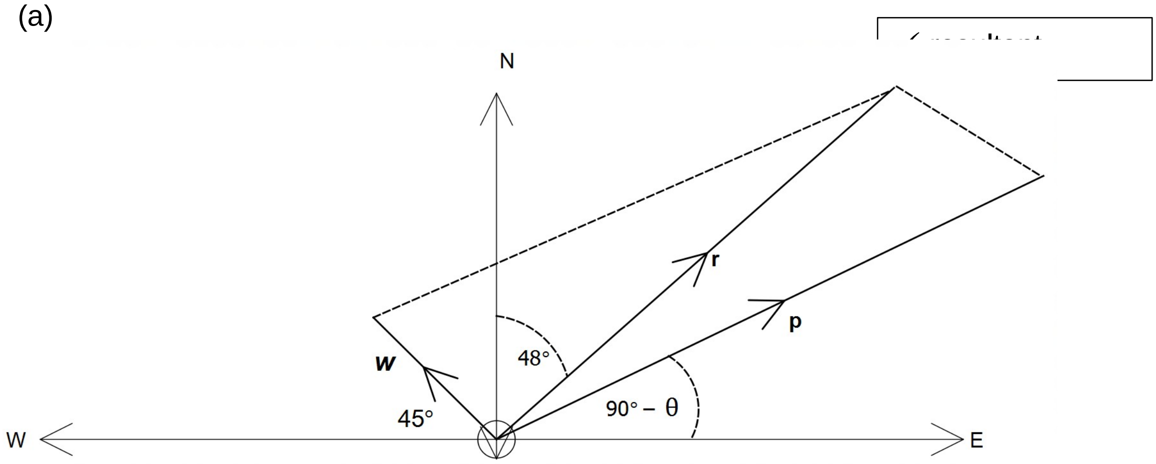
¿ $\frac{k!-1}{k!} \times \frac{k+1}{k+1} + \frac{k}{(k+1)!}$ ✓

¿ $\frac{k!(k+1)-(k+1)}{(k+1)!} + \frac{k}{(k+1)!}$ ✓

¿ $\frac{(k+1)!-1}{(k+1)!}$ as required

Therefore, True for $n = k + 1$, and since true for $n = 2$,
by induction the conjecture is true for all whole numbers. [10]

12. (a)



(b) $w=\begin{pmatrix}-80\cos 45°\\ 80\sin 45°\end{pmatrix}=\begin{pmatrix}-40\sqrt{2}\\ 40\sqrt{2}\end{pmatrix}$ ✓

$r=\begin{pmatrix}r\cos 42°\\ r\sin 42°\end{pmatrix}$ ✓

$p=\begin{pmatrix}870\cos(90°-\theta)\\ 870\sin(90°-\theta)\end{pmatrix}=\begin{pmatrix}870\sin\theta\\ 870\cos\theta\end{pmatrix}$ ✓

(c) $p+\begin{pmatrix}-40\sqrt{2}\\ 40\sqrt{2}\end{pmatrix}=\begin{pmatrix}r\cos 42°\\ r\sin 42°\end{pmatrix}$

$\therefore |p|=\begin{vmatrix}40\sqrt{2}+r\cos 42°\\ -40\sqrt{2}+r\sin 42°\end{vmatrix}=870$ ✓

CAS: $r=862.14$kmh^{-1} ✓

$\theta=53.27°$ ✓

(d) $t=\dfrac{d}{v}=\dfrac{878}{862.14}=1.018\,hrs=1\,hour\wedge 1\,minute$ ✓✓ [11]

13. (a) $P(Q-I)=Q+I$

$\therefore P=(Q+I)(Q-I)^{-1}$ ✓

$¿\begin{bmatrix}2&3\\-2&2\end{bmatrix}\times\begin{bmatrix}0&3\\-2&0\end{bmatrix}^{-1}=\begin{bmatrix}2&3\\-2&2\end{bmatrix}\times\frac{1}{6}\begin{bmatrix}0&-3\\2&0\end{bmatrix}$ ✓

$\therefore P=\begin{bmatrix}1&-1\\ \frac{2}{3}&1\end{bmatrix}$ ✓

(b) $A^4=A^2\times A^2=(3A-2I)\times(3A-2I)$

$¿9A^2-12A+4I$ ✓

¿ $9(3A-2I)-12A+4I$ ✓

¿ $27A-18I-12A+4I$

$\therefore A^4=15A-14I$ ✓ [6]

14. (a) $f(x)=R\cos(2x+\theta)$

¿ $R\cos 2x\cos\theta - R\sin 2x\sin\theta = \cos 2x - \sqrt{3}\sin 2x$ ✓

$\therefore R\sin\theta=\sqrt{3}R\cos\theta=1$ ✓

$\Rightarrow R=2\theta=60°$ ✓

$f(x)=2\cos(2x+60°)$ ✓

(b) $f(x)=2\cos(2x+60°)=-1$

$\cos(2x+60°)=\dfrac{-1}{2}$

$\therefore 2x+60°=120°,240°,480°,600°$ ✓

$x=30°,90°,210°,270°$ ✓ [6]

15. (a) (i) $P=\begin{bmatrix}-2 & 3\\ 2 & -3\end{bmatrix}\times\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}^{-1}=\begin{bmatrix}-1 & 2\\ 1 & -2\end{bmatrix}$ ✓

$\therefore k=-2$ ✓

(ii) $|P|=k(k+1)-2=k^2+k-2\neq 0$ ✓

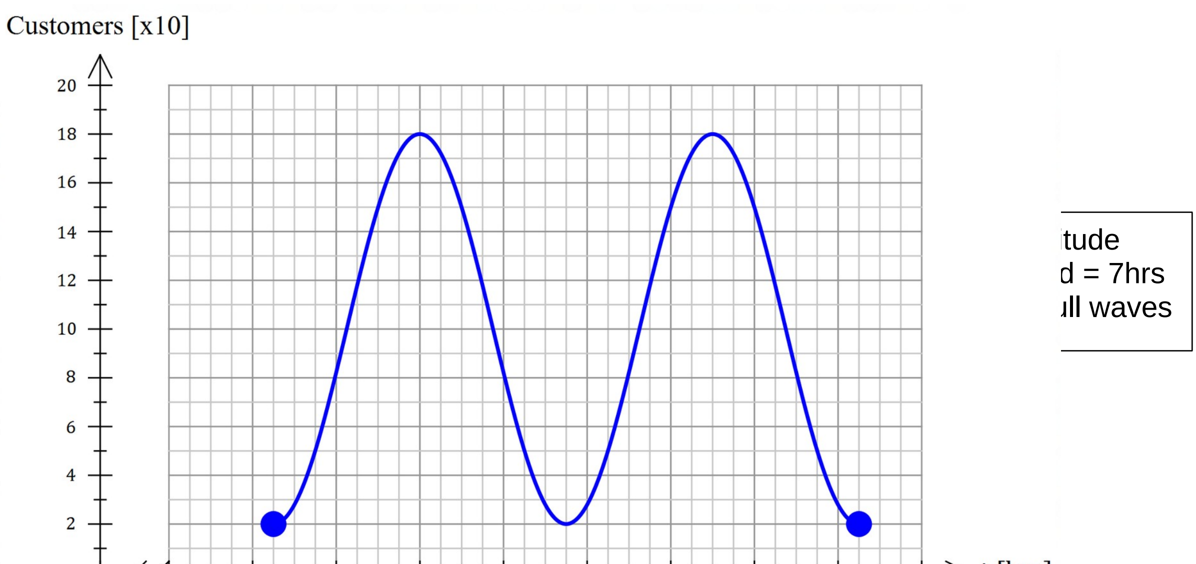
$\therefore k\neq -2,1$ ✓

(b) (i) $M^{2n+1}=(M^2)^n\times M=I^n\times M=M$ ✓

(ii) $M^{-2n}=(M^2)^{-n}=I^{-n}=I$ ✓

(c) $A^n=\begin{bmatrix}1 & 0\\ 2^n-1 & 2^n\end{bmatrix}$ ✓✓ [8]

16. (a)



16. (b) $C(t) = -80\cos\left[\frac{2\pi}{7}(t-8.5)\right]+100$

$A=-80, \omega=\frac{2\pi}{7}, \phi=\frac{17\pi}{7}, v=100$ ✓✓✓✓

OR $A=80, \omega=\frac{2\pi}{7}, \phi=\frac{24\pi}{7}, v=100$

(c) $-80\cos\left[\frac{2\pi}{7}t-\frac{17\pi}{7}\right]+100=140$ ✓

CAS: $t_1=10.83, t_2=13.17, t_3=17.83, t_4=20.18$ ✓✓

Between 10:50am and 1:11pm ✓

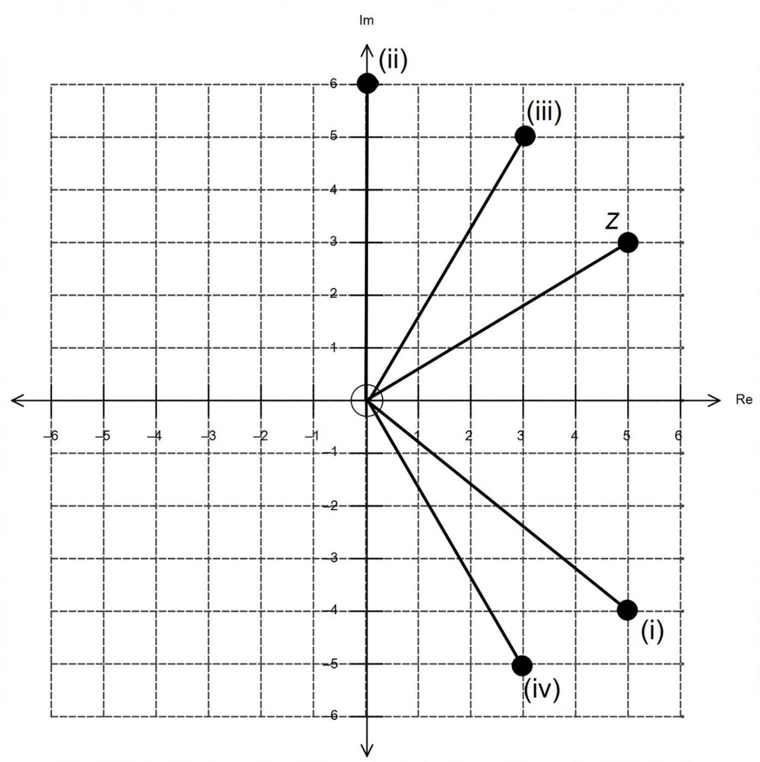
and 5:50pm and 8:11pm ✓ [12]

17. (a) (i) $5-4i$ ✓✓

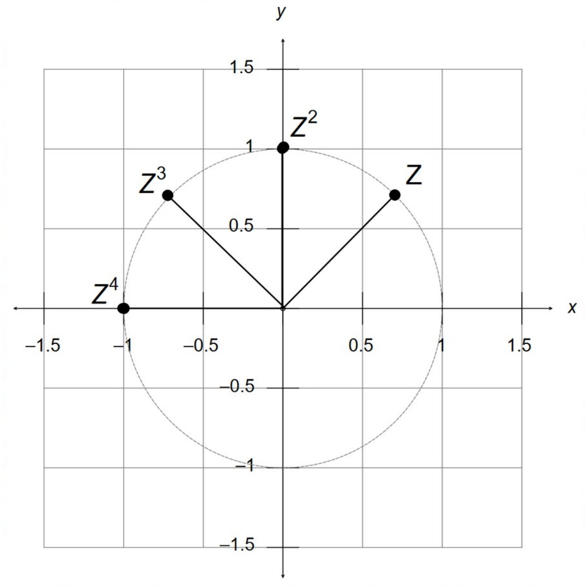
(ii) $6i$ ✓✓

(iii) $3+5i$ ✓✓

(iv) $3-5i$ ✓✓



17. (b) (i)



✓✓✓

(ii) Rotation ✓ of 45° anticlockwise ✓

(iii) Powers that are multiples of 8 give $z^{8n}=1$ ✓ [14]

18. (a) $|T|=\cos\theta\cos\phi+\sin\theta\sin\phi=0$ ✓

$\therefore \cos(\theta-\phi)=0$ ✓

$\therefore \theta-\phi=\pm 90°$ ✓

(b) (i) $T=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

Rotation of 90° anti-clockwise ✓✓

(ii) $T=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$

Reflection over the line $y=-x$ ✓✓

(c) i.e. $T^2=I$

$$\begin{bmatrix}\cos\theta & -\sin\phi\\ \sin\theta & \cos\phi\end{bmatrix}\times\begin{bmatrix}\cos\theta & -\sin\phi\\ \sin\theta & \cos\phi\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}\cos^2\theta-\sin\theta\sin\phi & -\sin\phi(\cos\theta+\cos\phi)\\ \sin\theta(\cos\theta+\cos\phi) & \cos^2\phi-\sin\theta\sin\phi\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$ ✓

$\sin\theta(\cos\theta+\cos\phi)=-\sin\phi(\cos\theta+\cos\phi)=0$

$\therefore \sin\theta=\sin\phi=0$ and $\cos^2\theta=\cos^2\phi=1$ ✓✓

OR $\cos\theta+\cos\phi=0$ and $\sin\theta=\sin\phi$ ✓✓ [12]