Hale School

MATHEMATICS
SPECIALIST
3CD

Semester Two Examination 2010

MARKING KEY and SOLUTIONS

Section One
Calculator-Free

MARKING KEY and SOLUTIONS

Question 1 [6 marks]

Find the anti-derivative :

(a) $\int xe^{2x^2}dx$

[2 marks]

Solution
$\int xe^{2x^2}dx = x \cdot \dfrac{e^{2x^2}}{4x} + c = \dfrac{e^{2x^2}}{4} + c$
Specific Behaviours
✓ Anti-derivative of exponential is itself ✓ Recognises the derivative factor to divide by 4x

(b) $\int \sqrt{\sin x}\ \cos x\,dx$

[2 marks]

Solution
$\int \sqrt{\sin x}\ \cos x\,dx = \int (\sin x)^{\frac{1}{2}}\ \cos x\,dx$ $= \dfrac{(\sin x)^{\frac{3}{2}}\ \cos x}{\frac{3}{2}\cos x} + c$ $= \dfrac{2\sqrt{\sin^3 x}}{3} + c$
Specific Behaviours
✓ Integrates power of sine function ✓ Recognises the derivative factor to divide by cos x

(c) $\int \dfrac{\ln x}{x}\,dx$

Solution
$\int \dfrac{\ln x}{x}\,dx = \int \ln x \cdot \dfrac{1}{x}\,dx$ $= \dfrac{(\ln x)^2}{2} + c$
Specific Behaviours

✓ Integrates the power of $\ln x$ ✓ Recognises the derivate factor $1/x$

[2 marks]

Note : **If there is NO use of integration constants in Q1, then ONE mark is to be deducted from Q1c.**

Question 2 **[9 marks]**

Given that $z = 2e^{ix}$ and $w = 2e^{-ix}$:

(a) express iz in complex exponential form.

[2 marks]

Solution
$iz = e^{\frac{i\pi}{2}} . 2e^{ix} = 2e^{i(x + \frac{\pi}{2})}$
Specific Behaviours
✓ Converts i into exponential form ✓ adds indices to give argument $x + \pi/2$ and real part 2

(b) express $\dfrac{\text{cis } 5x}{z}$ in complex exponential form.

[2 marks]

Solution
$\dfrac{\text{cis } 5x}{z} = \dfrac{e^{5ix}}{2e^{ix}} = \dfrac{1}{2}e^{4ix}$
Specific Behaviours
✓ Expresses cis 5x in exponential form ✓ subtracts indicies to give argument 4x and real part 0.5

(c) simplify $z^2 + w^2$

[2 marks]

Solution
$z^2 + w^2 = 4e^{2ix} + 4e^{-2ix} = 4(e^{2ix} + e^{-2ix})$ $= 4 . 2\cos 2x$ $= 8\cos 2x$
Specific Behaviours
✓ Multiply indices by 2 ✓ Express as twice the real part cos 2x

(d) solve for x given that $z^3 + 8 = 0$

[3 marks]

Solution
$z^3 = -8 = 8 \text{ cis } \pi$ $\therefore \quad z = 8^{\frac{1}{3}} \text{ cis}\left(\dfrac{\pi + 2\pi k}{3}\right) \qquad k = 0, 1, 2$

$z = 2\text{ cis}\left(\frac{\pi}{3}\right),\ 2\text{ cis}(\pi),\ 2\text{ cis}\left(-\frac{\pi}{3}\right)$ $\therefore\ x = \pi/3,\ \pi,\ -\pi/3$
Specific Behaviours
✓ Expression for cube roots using De Moivre's Theorem ✓ Give roots in cis form ✓ Solve for x

Question 3 **[4 marks]**

Points A and B have respective position vectors given by :

$\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$
$\mathbf{b} = x\mathbf{i} + \mathbf{j} + \mathbf{k}$

Determine the value of x given that vectors **a** and **b** are at an angle of 60°.

Solution
$\mathbf{a}\cdot\mathbf{b} = 2x + 1 - 1 = \sqrt{6}\,.\,\sqrt{x^2+2}\,.\,\cos 60^\circ$ $\therefore\ 2x = \sqrt{6(x^2+2)}\,.\,\frac{1}{2}$ so $x > 0$ (square root gives a positive) $\therefore\ (4x)^2 = 6x^2 + 12$ $10x^2 = 12$ $x = \pm\sqrt{\frac{6}{5}}$ But $x > 0$ $\therefore\ x = \sqrt{\frac{6}{5}}$
Specific Behaviours
✓ expresses dot product using components AND using magnitudes ✓ uses $\cos 60^0 = 0.5$ to form an equation in x^2 ✓ squares both sides to eliminate the square root ✓ solves to find 2 values for x but only accepts the POSITIVE solution

Question 4 **[4 marks]**

Give the following transformation matrices, describe their effect on some object in the co-ordinate plane :

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

[1 mark]

Solution
Horizontal dilation about $x = 0$ with factor 2
Specific Behaviours

✓ Uses term dilation about x = 0 and mentions factor 2

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

[1 mark]

Solution
Reflection about y = x
Specific Behaviours
✓ Uses term reflection and states the position of the mirror y = x

Question 4 **[4 marks]**

(c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

[2 marks]

Solution
Vertical shear with factor 2 AND THEN an anti-clockwise rotation of 90o about origin
Specific Behaviours
✓ States 2 transformations and gives the correct order (shear then rotate) ✓ Describes each transformation correctly.

Question 5 **[6 marks]**

Prove using the method of mathematical induction that, for all values of the positive integer n, $7^n + 2$ is always divisible by 3.

Solution
For n = 1, 71 + 2 = 9 which is divisible by 3. Hence true for n = 1. Assume true for n = k i.e. 7k + 2 = 3m where m is some integer Consider n = k + 1 : 7k+1 + 2 = 7.7k + 2 = 7(3m – 2) + 2 = 21m – 12 = 3(7m – 4) which is divisible by 3 Hence true for n = k + 1 Hence true ∀ n.
Specific Behaviours
✓ Show true for n = 1 ✓ Assumes true for n = k

✓ Expresses divisibility by 3 using some multiple of 3 ✓ Expresses 7^{k+1} in terms of previous result ✓ Shows that result for n = k+1 has a factor of 3 ✓ Concludes that true for all values n

Question 6 **[7 marks]**

A particle's position is given by $x(t)$ cm after t seconds and moves according to the differential equation :

$$\frac{d^2x}{dt^2} = -\pi^2 x$$

It is known that $x(0) = k$ cm, and its velocity $v(0) = -\sqrt{3}\pi k$ cm s^{-1} where k is some positive constant. Write an expression (in terms of the constant k) for :

(a) the displacement $x(t)$.

[5 marks]

Solution
Motion is SHM from the differential equation with T = 2 seconds Hence suggest $x(t) = A\cos(\pi t + \alpha)$ Given $x(0) = k$ then $k = A\cos\alpha$ …… (1) $v(t) = -\pi A\sin(nt + \alpha)$ Given $v(0) = -\sqrt{3}\pi k$ then $-\sqrt{3}\pi k = -\pi A\sin\alpha$ ….. (2) From (1) $\cos\alpha = k/A$ From (2) $\sin\alpha = -\sqrt{3}k/A$ ∴ $\cos2\alpha + \sin2\alpha = 1$ $\frac{k^2}{A^2} + \frac{3k^2}{A^2} = 1$ ∴ i.e. $A = 2k$ $(k > 0)$

$\therefore$ $\cos\alpha = 0.5$ i.e. $\alpha = \pi/3$ Hence $x(t) = 2k\cos(\pi t + \pi/3)$
Specific Behaviours
✓ States motion as SHM and determines the period T ✓ Writes $x(t)$ of form $A\cos(nt + \alpha)$ i.e. a trigonometric function with phase shift ✓ Obtains relationships between k and A using $x(0)$ and $v(0)$ ✓ Solves for A and α. ✓ Concludes with expression for $x(t)$ in terms of k

(b) the distance travelled in the first 4 seconds.

[2 marks]

Solution
Since the period $T = 2$ sec, then over 4 seconds, the particle does 2 complete cycles. Hence Distance $= 2(4A) = 16k$
Specific Behaviours
✓ Recognises 2 cycles of motion ✓ Expresses distance in terms of k

Question 7 **[4 marks]**

It is known that $z = \text{cis}\left(\frac{3\pi}{4}\right)$ is a solution to the equation $z^n = i$. Determine the set of possible values for the positive integer n.

Solution
$z^n = \text{cis}\,\pi/2$ $\therefore\ z = \text{cis}\left(\dfrac{\frac{\pi}{2} + 2\pi k}{n}\right)$ $k = 0, 1, 2, \ldots, n-1$ $\therefore\ \dfrac{3\pi n}{4} = \dfrac{\pi}{2} + 2\pi k$ for some integer value of k and n $\therefore\ 3n = 2 + 8k$ $\therefore\ 3n = 2, 10, \mathbf{18}, 26, 34, \mathbf{42}, \ldots$ $\therefore\ n = 6, 14, 22, \ldots$ Hence n can be any integer that is 2 less than a multiple of 8
Specific Behaviours
✓ Expression for the n-th roots using De Moivre's Theorem ✓ Express cis($3\pi/4$) as one of the solutions ✓ Develops equation to determine n

✓ Obtains the entire set of values for n