PERTH MODERN SCHOOL Exceptional schooling. Exceptional students. Independent Public School	Year 12 Specialist TEST 4 Weds 28 Aug 2019 TIME: 50 minutes working Classpads allowed No notes allowed 45 marks 8 Questions

Name:________________________________

Teacher: _____________________________

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (3, 3 & 3 = 9 marks)
Determine the following integrals using the given substitutions.

a) $\int 3x\left(5x^2+1\right)^7 dx \qquad u=5x^2+1$

Solution
$\int 3x\left(5x^2+1\right)^7 dx \qquad u=5x^2+1$ $\int 3xu^7 \frac{1}{10x} du = \frac{3}{10}\int u^7 du = \frac{3u^8}{80} = \frac{3}{80}\left(5x^2+1\right)^8 + C$
Specific behaviours
✓ subs du ✓ integrates wrt u ✓ expresses answer in terms of x only with a constant

b) $\int (5x-2)\sqrt{2x-1}\,dx \qquad u = 2x-1$

Solution
$\int (5x-2)\sqrt{2x-1}\,dx \qquad u = 2x-1 \quad x = \dfrac{u+1}{2}$ $\int [5(\dfrac{u+1}{2}) - \dfrac{4}{2}]u^{\frac{1}{2}}\dfrac{1}{2}du = \int \dfrac{5u+1}{4}u^{\frac{1}{2}}du = \int \dfrac{5u^{\frac{3}{2}}+u^{\frac{1}{2}}}{4}du$ $\dfrac{3}{6}u^{\frac{5}{2}} + \dfrac{1}{6}u^{\frac{3}{2}} + c = \dfrac{1}{2}(2x-1)^{\frac{5}{2}} + \dfrac{1}{6}(2x-1)^{\frac{3}{2}} + c$
Specific behaviours
✓ subs du   ✓ integrates wrt u   ✓ expresses answer in terms of x only (no need for constant)

c) $\int \sec^2 x \tan^8 x\,dx \qquad u = \tan x$

Solution
$\int \sec^2 x \tan^8 x\,dx \qquad u = \tan x$ $\int \sec^2 x u^8 \dfrac{1}{\sec^2 x}du = \int u^8 du = \dfrac{u^9}{9} + C = \dfrac{\tan^9 x}{9} + C$
Specific behaviours
✓ subs du   ✓ integrates wrt u   ✓ expresses answer in terms of x only (no need for constant)

Q2 (3 marks)

Identical twins Sherry and Mary were both given the following integral to solve. $\int 2\sin x\cos x\,dx$
Sherry's solution was as follows.

$\int 2\sin x\cos x\,dx \quad u = \sin x$

$\int 2u\cos x\frac{du}{\cos x}$

$\int 2u\,du = u^2 = \sin^2 x$

While Mary's solution was to:

$\int 2\sin x\cos x\,dx = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x$

Explain why the solutions differ and state which is the correct answer. Show your reasoning.

Solution
Both missing constants Constant differ Both answers correct as $\frac{-1}{2}\cos 2x = \frac{-1}{2}\left(1-2\sin^2 x\right) = \sin^2 x + C$
Specific behaviours
✓ mentions that constants missing ✓ states that constants are different ✓ shows that both expressions differ by an added constant

Q3 (3 & 4 = 7 marks)
Determine the following integrals showing all working.

a) $\int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x - \sin x}\,dx$

Solution
$\int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x - \sin x}\,dx = \left[-\ln\lvert\cos x - \sin x\rvert\right]_0^{\frac{\pi}{2}} = (\ln 1) - (-\ln 1) = 0$
Specific behaviours
✓ integrates using ln ✓ uses absolute value ✓ determines result

Q3 cont-

b) $\displaystyle\int \frac{6x^3+11x^2+15x+20}{(x+1)^2(x^2+4)}\,dx$ (4 marks)

Solution

$$\frac{6x^3+11x^2+15x+20}{(x+1)^2(x^2+4)}=\frac{a}{x+1}+\frac{b}{(x+1)^2}+\frac{cx+d}{x^2+4}$$

$6x^3+11x^2+15x+20=a(x+1)(x^2+4)+b(x^2+4)+(cx+d)(x+1)^2$

$x=-1$

$10=5b \quad b=2$

$x=0$

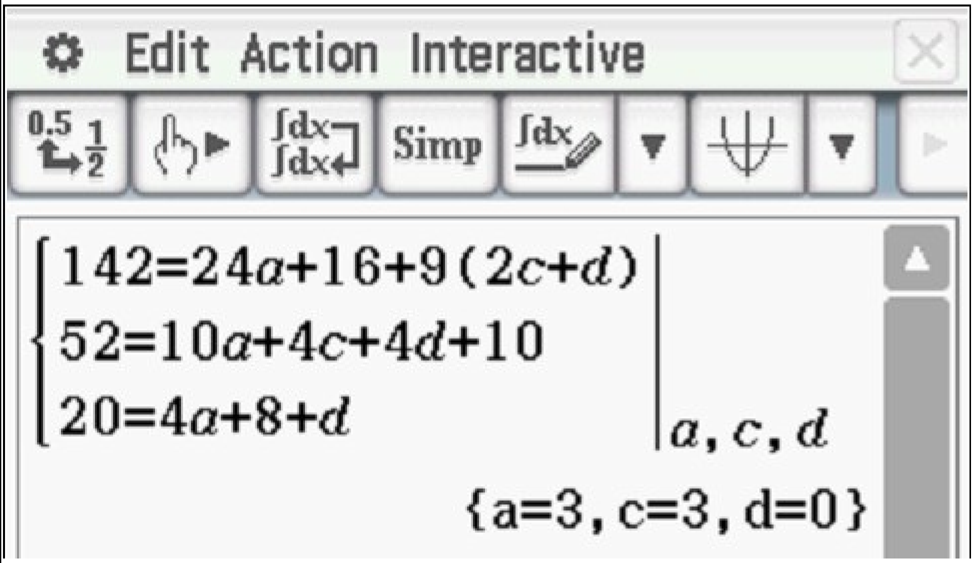
$20=4a+8+d$

$x=1$

$52=10a+4c+4d+10$

$x=2$

$142=24a+16+9(2c+d)$



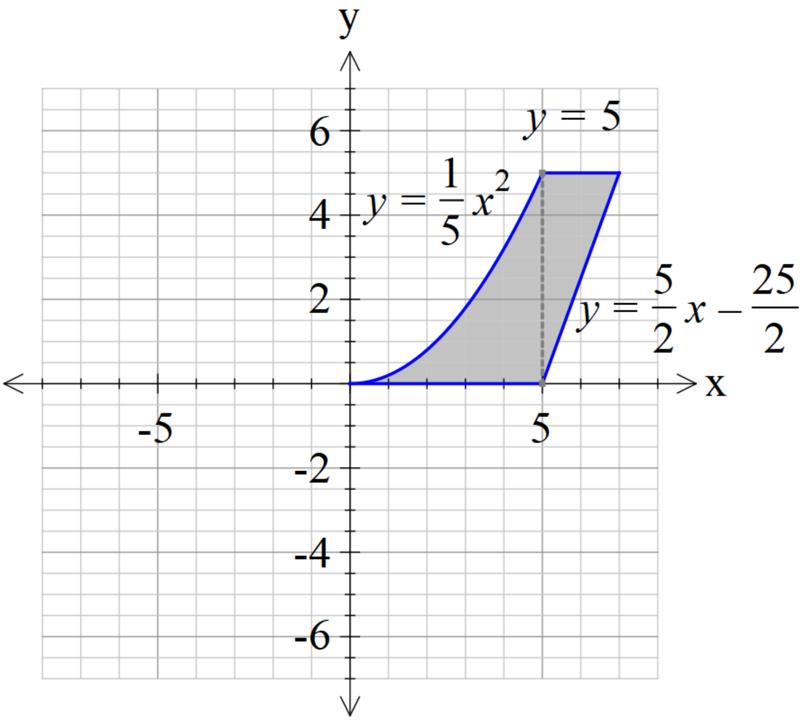
$\displaystyle\int \frac{3}{x+1}+\frac{2}{(x+1)^2}+\frac{3x}{x^2+4}\,dx$

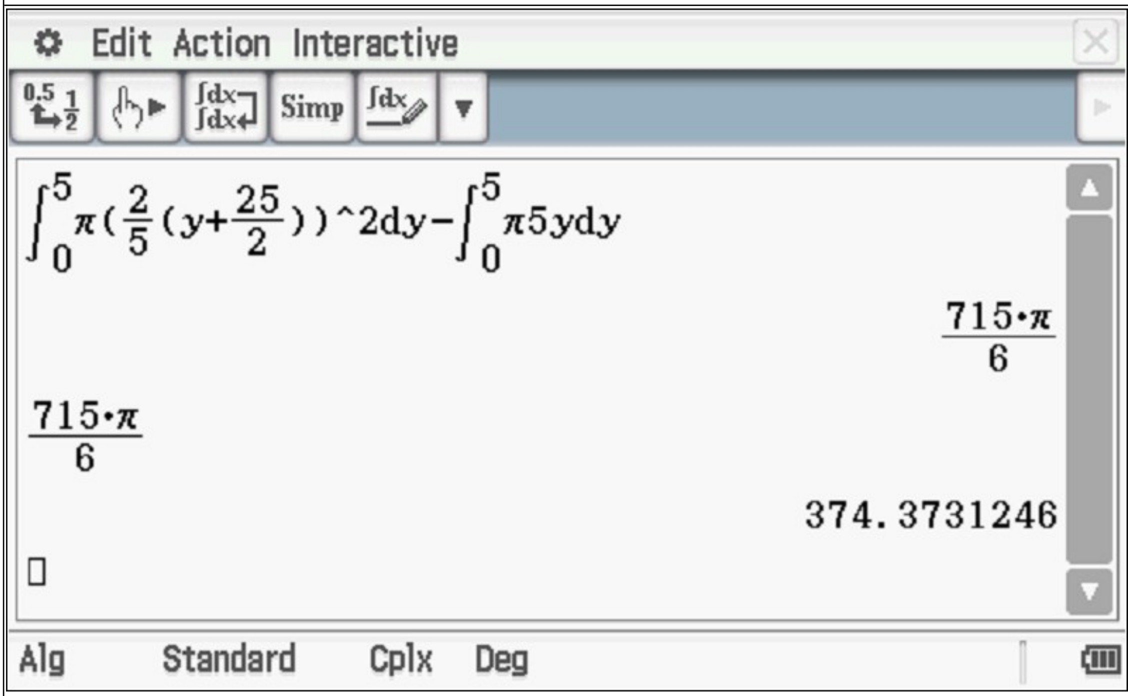
$\displaystyle=3\ln|x+1|-2(x+1)^{-1}+\frac{3}{2}\ln|x^2+4|+c$

Specific behaviours

✓ uses correct partial fractions with 4 constants
✓ solves for at least one constant
✓ sets up simultaneous equations for other constants
✓ integrates correctly (no need to add c)

Q4 (5 marks)
The shaded region is rotated about the y axis. Determine the volume of the resulting solid.



Solution

Specific behaviours
✓ uses correct integral around y axis ✓ uses correct limits ✓ sets up a difference calculation with volumes ✓ states correct approx. volume

Q5 (1 & 4 = 5 marks)

The mass, N grams, of a gas produced in a factory at time t seconds can be modelled by the logistical formula $\frac{dN}{dt} = 9N - 5N^2$ with an initial mass of 0.1 grams.

a) Determine the limiting mass as $t \to \infty$.

Solution
$0 = 9N - 5N^2 = N(9 - 5N)$ $N = \frac{9}{5}$
Specific behaviours
✓ states limiting value

b) Show that $N = \frac{9}{5 + ce^{-9t}}$ and determine the constant.

Solution

$$\frac{dN}{dt}=9N-5N^2=N(9-5N)$$

$$\int\frac{dN}{N(9-5N)}=\int dt$$

$$\frac{1}{N(9-5N)}=\frac{a}{N}+\frac{b}{9-5N}$$

$$1=a(9-5N)+bN$$

$$N=0$$

$$1=9a \quad a=\frac{1}{9}$$

$$N=\frac{9}{5}$$

$$1=b\frac{9}{5} \quad b=\frac{5}{9}$$

$$\int\frac{\frac{1}{9}}{N}+\frac{\frac{5}{9}}{9-5N}\,dN=\frac{1}{9}\ln|N|-\frac{1}{9}\ln|9-5N|=t+c$$

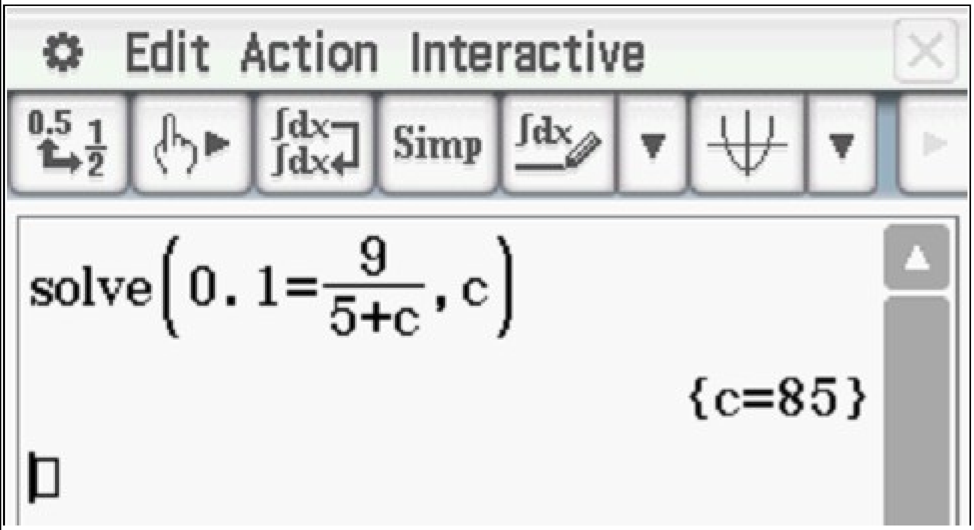
$$-\ln\left|\frac{N}{9-5N}\right|=-9t+c \quad as\, N<\frac{9}{5}\therefore 9-5N>0$$

$$\frac{9-5N}{N}=Ce^{-9t}$$

$$9-5N=NCe^{-9t}$$

$$9=(5+Ce^{-9t})N$$

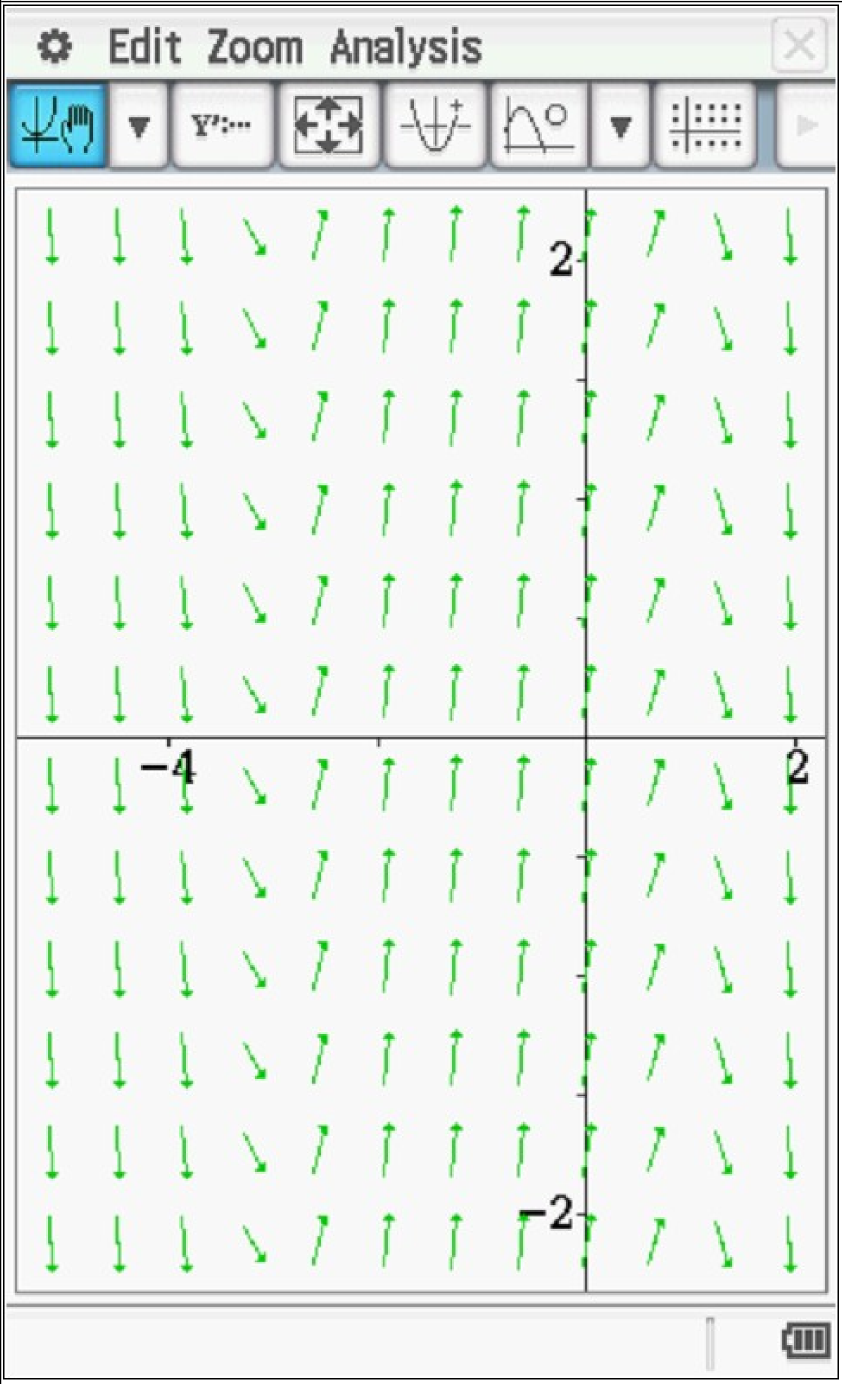
$$N=\frac{9}{5+Ce^{-9t}}$$



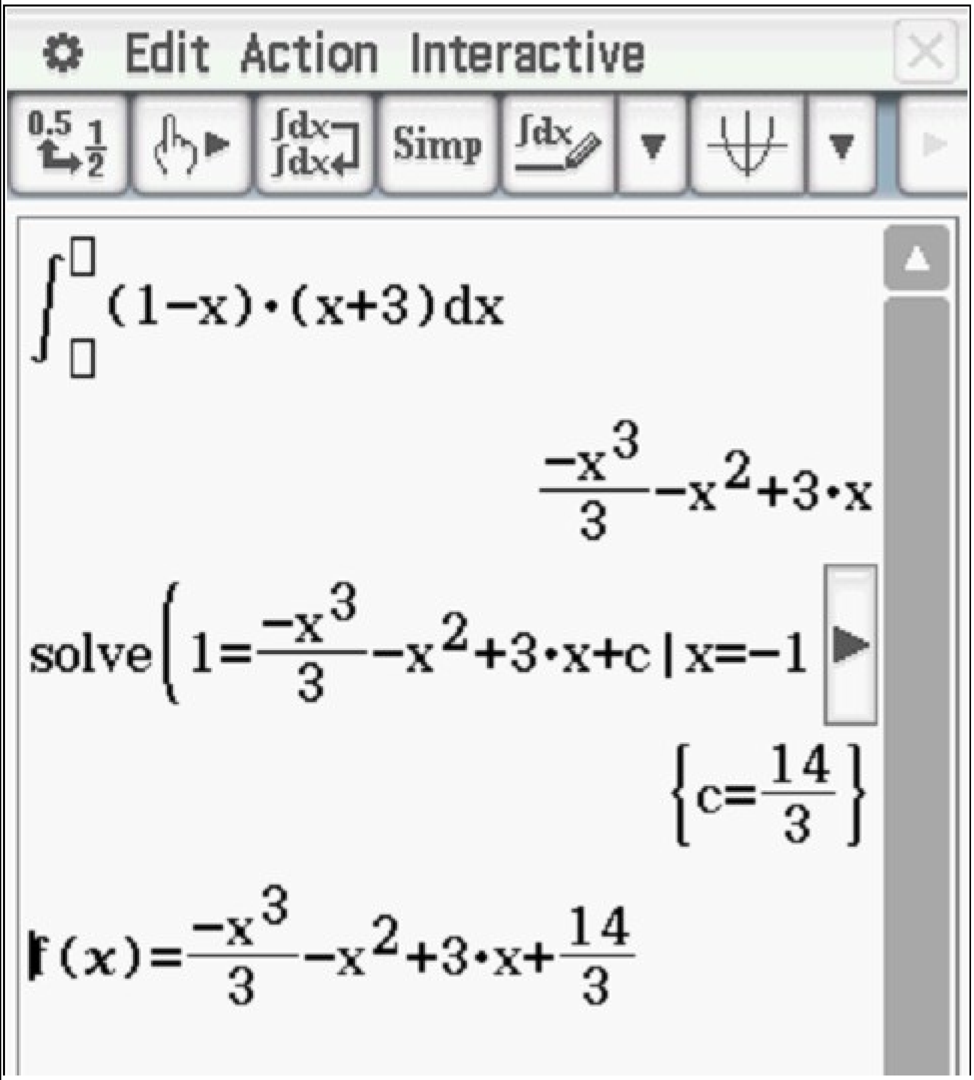
Specific behaviours
✓ separates variables ✓ sets up partial fractions ✓ integrates and shows why absolute value not needed ✓ solves for constant

Q6 (3 & 3 = 6 marks)

a) Sketch the slope field for $\frac{dy}{dx} = (1 - x)(x + 3)$ on the axes below.

Solution

Specific behaviours
✓ shows one zero ✓ shows both zeros ✓ pattern agrees with above

b) Given that point A (-1,1) is a known point on our solution, show this curve on the slope field above and give the equation.

Solution


Specific behaviours
✓ shows curve on slope field going through pt A ✓ integrates slope field ✓ solves for constant

Q7 (2, 3 & 2 = 7 marks)

A particle with displacement, x metres from the origin at time t seconds, moves such that $x = 5\sin\left(2t + \frac{\pi}{3}\right)$.

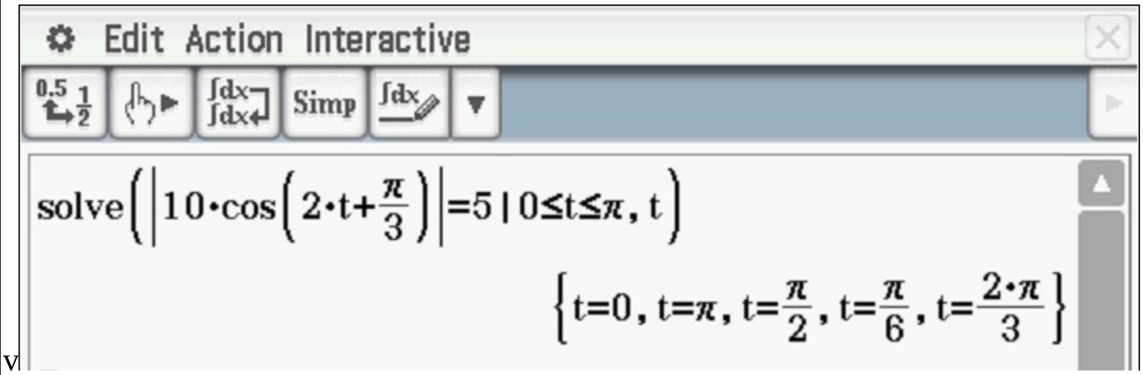
a) Show that the motion is simple harmonic.

Solution
$x = 5\sin\left(2t + \frac{\pi}{3}\right)$ $\dot{x} = 10\cos\left(2t + \frac{\pi}{3}\right)$ $\ddot{x} = -20\sin\left(2t + \frac{\pi}{3}\right) = -4x \quad \therefore SHM$
Specific behaviours
✓ obtains acceleration function ✓ shows correct differential equation for SHM

b) Determine the first two times that the speed is exactly half of the maximum speed.

Solution

t=0 v=5m/s



v

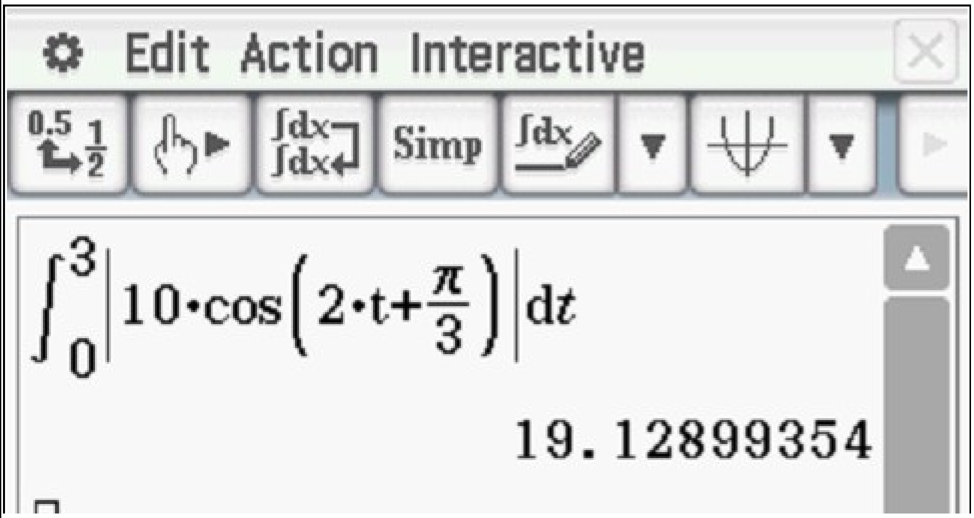
First two times are $0 \,\&\, \frac{\pi}{6}$

Specific behaviours

✓ states initial time
✓ uses negative velocity for second time
✓ solves for second time, approx

c) Determine the distance travelled in the first 3 seconds.

Solution



Specific behaviours

✓ uses correct integral with absolute velocity
✓ states distance travelled

Q8 (4 marks)

A particle with displacement, x metres from the origin at time t seconds, has an acceleration given by $\ddot{x} = -n^2 x$. The amplitude of the motion is given by A metres.

Show by using integration that the speed, v metres per second, is given by $v^2 = n^2\left(A^2 - x^2\right)$.

Solution

$$v\frac{dv}{dx} = -n^2 x$$

$$\int v\,dv = \int -n^2 x\,dx$$

$$\frac{v^2}{2} = -n^2\frac{x^2}{2} + c \quad v^2 = -n^2x^2 + c$$

$$x = A, v = 0 \quad c = n^2A^2$$

$$v^2 = -n^2x^2 + n^2A^2 = n^2\left(A^2 - x^2\right)$$

Specific behaviours

✓ uses alternative expression for accleration

✓ uses separation of variables
✓ integrates correctly
✓ solves for constant in terms of A & n