PERTH MODERN SCHOOL
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INDEPENDENT PUBLIC SCHOOL

Semester Two Examination, 2023

Question/Answer booklet

12 SPECIALIST MATHEMATICS

Section One:
Calculator-free

Your Name ________________________________

Your Teacher's Name ___________________________

Time allowed for this section
Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Mark	Max	Question	Mark	Max
1		9	5		9
2		7	6		8
3		8			
4		9			

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	6	6	50	50	34
Section Two: Calculator-assumed	13	13	100	97	66
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section One: Calculator-free **(50 Marks)**

This section has **six (6)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 50 minutes.

Question 1 **(9 marks)**

Consider the functions $g(x) = \frac{1}{x^2 - 1}$ and $h(x) = \sqrt{x - 4}$.

a) Determine the natural domain and range of $g(x)$. (3 marks)

c
$d_g : R \setminus x = -1, 1$ $r_g : R \setminus (-1 < y \leq 0)$
Specific behaviours
✓ states domain ✓ identifies endpoints of y=-1&0 in the range ✓ uses correct inequalities in excluded/included range

b) Does $g \circ h(x)$ exist over the natural domain of $h(x)$? Explain. (3 marks)

c
$r_h : y \geq 0$ $d_g : x \neq -1, 1$ $r_h \not\subset d_g$ $\therefore not\ exist$
Specific behaviours
✓ states relevant domain and range ✓ states does not exist ✓ states clearly reason why

c) State $h \circ g(x)$ and its natural domain. (3 marks)

c

$h \circ g(x) = \sqrt{\frac{1}{x^2 - 1} - 4} = \sqrt{\frac{5 - 4x^2}{x^2 - 1}}$ $5 - 4x^2 \geq 0 \rightarrow x^2 \leq \frac{5}{4} \rightarrow \frac{-\sqrt{5}}{2} \leq x \leq \frac{\sqrt{5}}{2}$ $x^2 - 1 > 0 \rightarrow x < -1, x > 1$ $d: \frac{-\sqrt{5}}{2} \leq x < -1 \cup 1 < x \leq \frac{\sqrt{5}}{2}$
Specific behaviours
✓ states rule simplified ✓ states conditions for numerator ✓ states domain for composite

Question 2 **(7 marks)**

Determine the following integrals.

a) $\int \frac{2x}{\sqrt{1-3x}}dx$. (3 marks)

c
$let\, u = 1 - 3x$
$\frac{du}{dx} = -3$
$x = \frac{1-u}{3}$
$\int \frac{2x}{\sqrt{1-3x}}\frac{dx}{du}du = \int \frac{2(1-u)}{3}u^{\frac{-1}{2}}\frac{1}{-3}du$
$\frac{-2}{9}\int u^{\frac{-1}{2}} - u^{\frac{1}{2}}du = \frac{-2}{9}\left[2u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right] + c$
$\frac{-4}{9}\sqrt{1-3x} + \frac{4}{27}(1-3x)^{\frac{3}{2}} + c$
Specific behaviours
✓ uses an appropriate substitution which is stated ✓ rearranges integral in terms of new variable ✓ integrates and adds a constant

b) $\int_0^{\frac{\pi}{2}} 5\sin^3(2x)\,dx$. (4 marks)

c
$\int_0^{\frac{\pi}{2}} 5\sin^3(2x)\,dx = 5\int_0^{\pi} \sin^2(2x)\sin 2x\,dx$
$5\int_0^{\frac{\pi}{2}}(1-\cos^2(2x))\sin 2x\,dx$
$5\int_0^{\frac{\pi}{2}}(\sin 2x - \cos^2(2x)\sin 2x)\,dx$
$5\left[\frac{-1}{2}\cos 2x + \frac{1}{6}\cos^3(2x)\right]_0^{\frac{\pi}{2}}$
$= \frac{10}{3}$
Specific behaviours
✓ uses Pythagorean identity ✓ rearranges integral into two parts ✓ integrates ✓ subs limiting values

Question 3 **(8 marks)**

$$f(x) = \frac{2x^2 + 11x + 7}{(x+2)(x^2+6x+1)}$$

a) The function can be expressed in the form $\frac{a}{x+2} + \frac{bx+c}{x^2+6x+1}$ where $a, b \& c$ are constants.

Determine the values of $a, b \& c$. (4 marks)

c
$f(x) = \frac{2x^2+11x+7}{(x+2)(x^2+6x+1)} = \frac{a}{x+2} + \frac{bx+c}{x^2+6x+1}$ $2x^2+11x+7 = a(x^2+6x+1) + (bx+c)(x+2)$ $x=-2$ $-7=-7a, \quad a=1$ $x=0$ $7=1+2c, \ c=3$ $x=1$ $20=8+(b+3)3, \ b=1$
Specific behaviours
✓ sets up equation to solve for constants ✓ solves for a ✓ solves for b ✓ solves for c

b) Hence determine $\int \frac{4x^2+22x+14}{(x+2)(x^2+6x+1)} dx$. (4 marks)

c
$\int \frac{4x^2+22x+14}{(x+2)(x^2+6x+1)} dx = \int \frac{2}{x+2} + \frac{2x+6}{x^2+6x+1} dx$ $= 2\ln\lvert x+2\rvert + \ln\lvert x^2+6x+1\rvert + c$
Specific behaviours
✓ multiplies by factor 2 ✓ integrates first term ✓ integrates second term

✓ adds a constant

Question 4 (9 marks)

Consider a herd of 25 horses, N in an isolated habitat such that the growth rate after t years is given by $\frac{dN}{dt} = \frac{N}{4} - \frac{N^2}{1000}$.

a) By using separation of variables and partial fractions, derive $N(t)$ showing all working. (5 marks)

c

$$\frac{dN}{dt}=\frac{250N}{1000}-\frac{N^2}{1000}=\frac{N(250-N)}{1000}$$

$$t\to\infty,\ \frac{dN}{dt}\to 0,\ 250-N=0,\ N<250$$

$$\int\frac{dN}{N(250-N)}=\int\frac{dt}{1000}$$

$$\frac{1}{N(250-N)}=\frac{a}{N}+\frac{b}{250-N}$$

$$1=a(250-N)+bN$$

$$N=0$$

$$1=a250,\quad a=\frac{1}{250}$$

$$N=250$$

$$1=250b,\quad b=\frac{1}{250}$$

$$\int\frac{dN}{N(250-N)}=\frac{1}{250}\int\frac{1}{N}+\frac{1}{250-N}dN=\frac{1}{250}\left(\ln|N|-\ln|250-N|\right)=\frac{1}{250}\left(\ln N-\ln(250\right.$$

$$\ln N-\ln(250-N)=250t+c$$

$$\ln\frac{N}{250-N}=\frac{1}{4}t+c$$

$$\frac{N}{250-N}=Ce^{\frac{1}{4}t}$$

$$\frac{250-N}{N}=Ce^{-\frac{1}{4}t}$$

$$250-N=NCe^{-\frac{1}{4}t}$$

$$250=N+NCe^{-\frac{1}{4}t}=N\left(1+Ce^{-\frac{1}{4}t}\right)$$

$$N=\frac{250}{1+Ce^{-\frac{1}{4}t}}$$

$$25=\frac{250}{1+C},C=9$$

$$N=\frac{250}{1+9e^{-\frac{1}{4}t}}$$

Specific behaviours

✓ explains limit for N and hence no need for absolute value when integrating
✓ separates variables
✓ uses partial fractions and then shows integration
✓ rearranges to give function with all constants solved

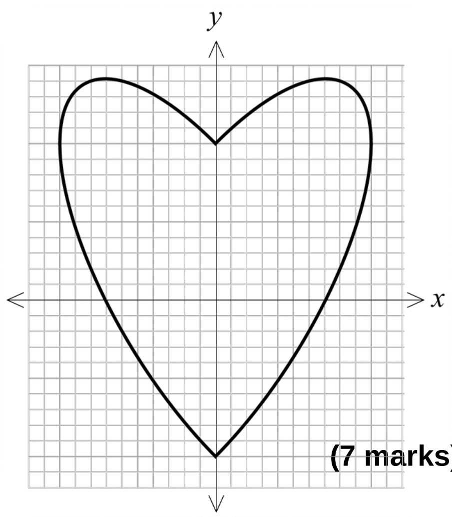
Q4 continued

b) Determine the limiting value of the number of horses. (2 marks)

c
N=250
Specific behaviours
✓ uses t approaching infinity ✓ states limit

c) Set up an equation, but do not solve, that will allow the time to be calculated where the growth rate is a maximum. (2 marks)

c
$\frac{250}{2}=\frac{250}{1+9e^{-\frac{1}{4}t}}$
Specific behaviours
✓ selects half the limiting value for N ✓ states equation for t



Question 5 **(7 marks)**

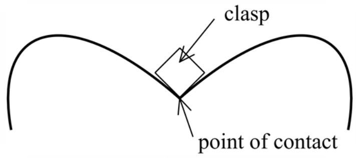
A designer creates a heart-shaped pendant for Valentine's Day shown on the right, using the function

$$2|x|^2-2|x|\,y+y^2-1=0$$

For $x\geq 0$ this equation becomes

$$2x^2-2xy+y^2-1=0$$

(a) Show that $\dfrac{dy}{dx}=\dfrac{2x-y}{x-y}$. $x\geq 0$ (3 marks)



Solution	Specific behaviours
$\frac{d}{dx}(2x^2-2xy+y^2-1)=\frac{d}{dx}(0)$ $4x-2y-2x\frac{dy}{dx}+2y\frac{dy}{dx}=0$ $\frac{dy}{dx}(2y-2x)=2y-4x$ $\frac{dy}{dx}=\frac{2y-4x}{2y-2x}=\frac{2x-y}{x-y}$	✓ Implicitly differentiates left hand side. ✓ Takes out $\frac{dy}{dx}$ as common factor. ✓ Rearranges to get required result.

Q5 continued-
The jewellery plans to attached a small square shaped clasp to the pendant. One corner of the square will sit in the cusp on the curve at the point of contact. The situation is illustrated on the right.

(b) (i) Determine the coordinates of the point of contact. (1 mark)

Solution	Specific behaviours
$Substitute\ x=0$ $y^2-1=0 \Rightarrow point\ is\ (0,1)$	✓ Determines coordinates of point of contact.

(ii) At the point of contact, will the gradient of the heart match that of the clasp? Justify your answer. (3 marks)

Solution	Specific behaviours
$For\ x\geq 0\ \left.\frac{dy}{dx}\right\|_{(0,1)}=1$ $\Rightarrow angle\ of\ inclination\ is\ 45°$	✓ Determines gradient of tangent to the curve at $(0,1)$. ✓ Determines angle of inclination.

As curve is symmetrical about the $y-$axis, then angle at cusp is $90°$, ¿ hence square clasp will meet at the point of contact.	✓ Explains why the clasp will meet at the point of contact.

Question 6 (8 marks)

Let $v=1+\sqrt{3}i$.

(a) Determine the three cube roots of v. (3 marks)

Solution
$$v=2\,cis\left(\frac{\pi}{3}\right)v^{\frac{1}{3}}=2^{\frac{1}{3}}cis\left(\frac{\pi}{9}+\frac{2n\pi}{3}\right)$$ Hence the cube roots are: $$\sqrt[3]{2}\,cis\left(\frac{7\pi}{9}\right),\sqrt[3]{2}\,cis\left(\frac{\pi}{9}\right),\sqrt[3]{2}\,cis\left(\frac{-5\pi}{9}\right)$$
Specific behaviours
✓ writes v in polar form ü obtains one correct root ü correctly states all roots

(b) Consider the polynomial $P(z)=z^4-8z^3+kz^2-46z+44$, where k is a real constant.

Given that $P(v)=0$, solve the equation $P(z)=0$. (5 marks)

Solution
P has real coefficients and so v and $\bar{v}$ are factors: $$(z-v)(z-\bar{v})=(z-1+\sqrt{3}i)(z-1-\sqrt{3}i)¿z^2-2z+4$$ Hence $z^4-8z^3+kz^2-46z+44=(z^2-2z+4)(z^2+az+11)$, where a is a real constant. Comparing coefficients of z then $-46=4a-22\Rightarrow a=-6$. Zeros of second quadratic factor: $$z^2-6z+11=0(z-3)^2=-2=2i^2z=3\pm\sqrt{2}i$$ Solutions are $z=1\pm\sqrt{3}i,=3\pm\sqrt{2}i$. *Note that it is possible to deduce $k=27$, but this is not required.*
Specific behaviours
✓ indicates $P(\bar{v})=0$ or $z-\bar{v}$ is a factor of P ü correctly determines quadratic factor of P ✓ determines second quadratic factor

ü obtains a zero from second quadratic factor ü states all solutions

Working out space

Working out space

Working out space.