Test 4
(Matrices, Exponentials and Logarithms, Functions)

This assessment contributes 5% towards the final year mark.
45 minutes are allocated for this task.

MARKING KEY and SOLUTIONS

Part A **The use of a CAS calculator is assumed.**
(12 minutes permitted)

Do NOT turn over this page until you are instructed to do so.

1. A native reptile of MathMagic Isle called the hypottentot has a generation change every 2 years. The table below shows the survival rates, breeding rates and the initial population profile for 5 of the age groups :

Age (years)	0-2	2-4	4-6	6-8	8-10
Survival Rate	0.6	0.8	0.7	0.4	0
Breeding Rate	0.1	0.9	1.4	0.5	0.4
Initial Population	10	12	15	20	10

a. State the Leslie matrix L for this colony of hypottentots.

$$\mathbf{L} = \begin{bmatrix} 0.1 & 0.9 & 1.4 & 0.5 & 0.4 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix}$$

✓ **5 x 5 matrix with breeding rates in Row 1**

✓ **Other elements correct**

[2]

b. Use matrix L to determine the population profile after 10 years.

10 years is equivalent to 5 generations.
P(5) = L^5 x P(0)

✓ **5 generations or transitions**

✓ **expression for P(5)**

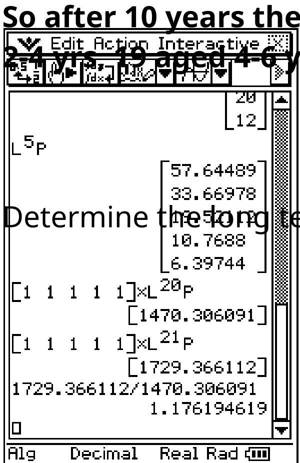
$$\mathbf{P(5)} = \begin{bmatrix} 0.1 & 0.9 & 1.4 & 0.5 & 0.4 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix}^5 \begin{bmatrix} 10 \\ 12 \\ 15 \\ 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 57.16... \\ 33.29... \\ 19.31... \\ 10.74... \\ 6.30... \end{bmatrix}$$

✓ **sensible conclusion using whole values**

So after 10 years there will be approximately 57 aged 0-2 yrs, 33 aged 2-4 yrs, 19 aged 4-6 yrs, 11 aged 6-8 yrs and 6 aged 8-10 yrs.

[3]

c. Determine the long term inter-generational growth rate, as a percentage.



From CAS calculator, inter-generational growth rate is approx 17.6%.

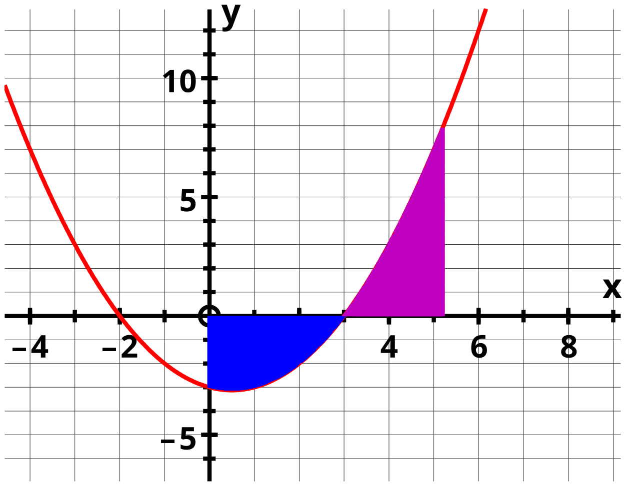
✓ **percentage growth conclusion**

[3]

✓ **explores consecutive generations for a LARGE number of transitions.**
✓ **determines totals for each generation.**

2. The graph of function $f(x) = 0.5(x + 2)(x - 3)$ is shown below.

Region A is bounded by the curve, the x axis, and the lines $x = 0$ and $x = 3$.
Region B is bounded by the curve, the x axis, and the lines $x = 3$ and $x = \mathbf{a}$.

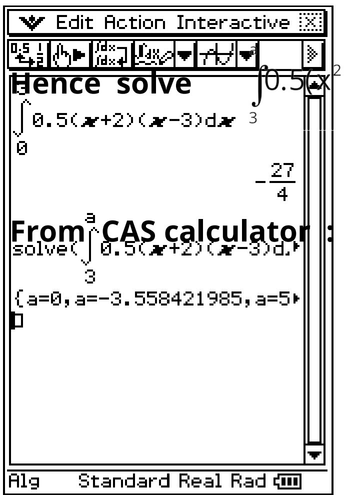


If the areas of regions A and B have the same area, determine the value of the constant **a** correct to 0.001.

Area A = $-\int_0^3 0.5(x^2 - x - 6)\, dx = \frac{27}{4}$

Area B = $\int_3^a 0.5(x^2 - x - 6)\, dx$

Hence solve $\int_3^a 0.5(x^2 - x - 6)\, dx = \frac{27}{4}$



From CAS calculator : **a = 5.058 (3 d.p.)**

✓ **value for Region A**

✓ expression for Region B
✓ equation for equal areas
✓ solves for a correct to 3 dp

[4]

Year 12 3CD Mathematics Specialist | Term 2, 2010

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Part B **No calculator to be used.**
(33 minutes permitted)

MARKING KEY and SOLUTIONS

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3. Consider the following transformation matrices in the co-ordinate plane :

R rotates 180° about the origin
D dilates vertically about y = 0 with factor 1.5
S downward shear parallel to the vertical axis with factor 1

a. Give matrices R, D and S.

$$\mathbf{R} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

✓ **each matrix correct**

[3]

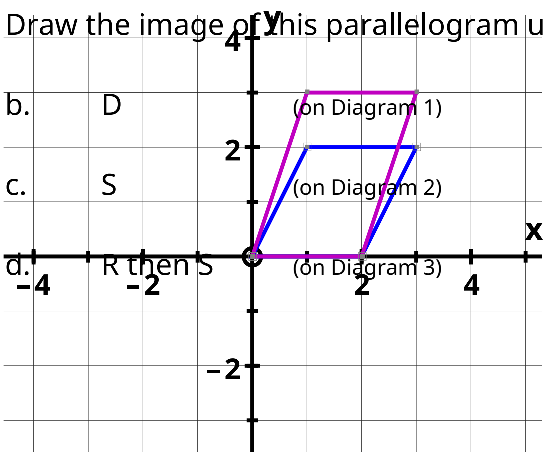
The 3 diagrams below show a parallelogram.

Draw the image of this parallelogram under the action of transformation :

b. D (on Diagram 1) [1]

c. S (on Diagram 2) [2]

d. R then S (on Diagram 3) [2]



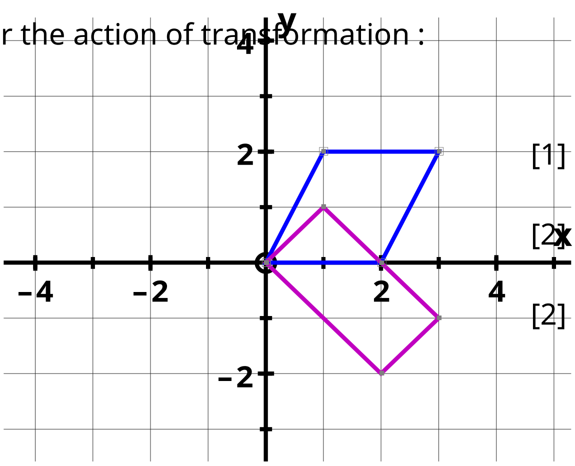


Diagram 1

✓ shows the dilation correctly

Diagram 2

✓ shows image of (3,2) as (3,-1)

✓ shows the TOTAL image correctly as a rectangle

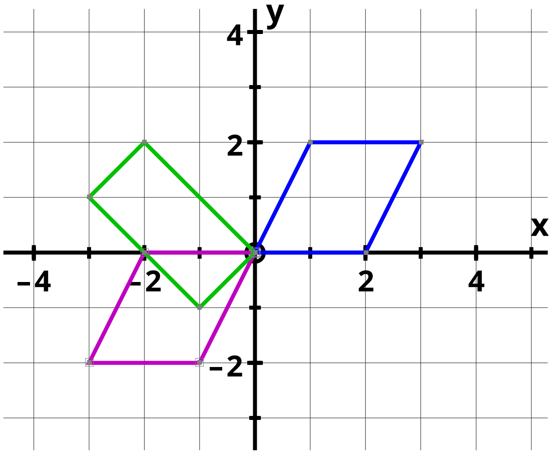


Diagram 3

✓ shows the rotation correctly

✓ shows the shear correctly

3. e. Does transformation matrix S change the area of any object it transforms ? Explain.

S = $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ **det(S) = 1(1) - (-1)(0) = 1**

Hence since |det(S)| = 1, then the AREA of the image will NOT be any different from the area of the object.

✓ **calculates the determinant**

✓ **makes the correct conclusion**

[2]

f. If the parallelogram is transformed by matrix D then S, what matrix will return the resultant image back to the original parallelogram ?

Transformation T : D then S i.e. T = SD

We require T^{-1}.

$$T = SD = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1.5 \end{bmatrix}$$

$$T^{-1} = \frac{1}{1.5 - 0} \begin{bmatrix} 1.5 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

✓ **interprets matrix SD**
✓ **calculates matrix SD**
✓ **calculates the INVERSE to return image to the original**

[3]

See next page

4. Find the following indefinite integrals, using an appropriate Calculus technique :
[ONE mark will be given for a correct answer only]

a. $$\int \frac{4x + 8}{(x^2 + x)^5}\, dx = \int (4x + 8)(x^2 + 4x)^{-5}\, dx$$

$$= (4x + 8) \cdot \frac{(x^2 + 4x)^{-4}}{(-4)(2x + 4)} + c$$

$$= \frac{(x^2 + 4x)^{-4}}{(-2)} + c$$

$$= -\frac{1}{2(x^2 + 4x)^4} + c$$

✓ **integrates power (-5)**
✓ **derivative factor cancels**
✓ **simplifies expression**

[3]

b. $\displaystyle\int \frac{e^{2x}}{e^{2x} + 1}\,dx = \int e^{2x}(e^{2x} + 1)^{-1}\,dx$

✓ recognises reciprocal function
✓ ln anti-derivative

$$= \frac{e^{2x}\ \ln|e^{2x} + 1|}{e^{2x}\ .\ 2} + c$$

$$= \frac{1}{2}\ln(e^{2x} + 1) + c$$

✓ divide by derivative factor to simplify correctly

[3]

c. $\displaystyle\int 30x\sqrt{x + 2}\,dx$ Put $u = x + 2$ $\quad \dfrac{du}{dx} = 1$

$$= \int 30(u - 2)\sqrt{u}\ du$$

✓ express x in terms of u

$$= \int 30(u^{\frac{3}{2}} - 2u^{\frac{1}{2}})\ du$$

✓ multiply integrand

$$= \frac{30u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{60u^{\frac{3}{2}}}{\frac{3}{2}} + c$$

✓ anti-derivative

$$= 12(x + 2)^{\frac{5}{2}} - 40(x + 2)^{\frac{3}{2}} + c$$

✓ express in terms of x

$$= 12\sqrt{(x + 2)^5} - 40\sqrt{(x + 2)^3} + c$$

✓ ONE mark penalty in Question 4 for lack of an integration constant

[4]

5. Evaluate the following definite integrals, using the given substitution and an anti-derivative technique : [ONE mark will be given for a correct answer only]

a. $\displaystyle\int_{\frac{1}{6}}^{\frac{1}{3}} \frac{dx}{\sqrt{1 - 9x^2}}$ Put $3x = \cos\theta$

x	$\frac{1}{6}$	$\frac{1}{3}$
θ	$\frac{\pi}{3}$	0

$$3\ .\ \frac{dx}{d\theta} = -\sin\theta$$

$$dx = -\frac{\sin\theta}{3}\ d\theta$$

$$= \int_{\frac{\pi}{3}}^{0} - \frac{\sin\theta}{3\sqrt{1 - \cos^2\theta}} \, d\theta$$

$$= \int_{0}^{\frac{\pi}{3}} \frac{\sin\theta}{3 \sin\theta} \, d\theta \qquad \text{since for } 0 < \theta < \frac{\pi}{2}, \quad \sin\theta > 0$$

$$= \int_{0}^{\frac{\pi}{3}} \frac{1}{3} \, d\theta$$

$$= \left[\frac{\theta}{3}\right]_{0}^{\frac{\pi}{3}}$$

$$= \frac{\pi}{9}$$

✓ **express dx in terms of $d\theta$**
✓ **change limits of integration**
✓ **simplifies integrand using the identity for $1 - \cos^2\theta$**
✓ **evaluates correctly**

[4]

5. b. Given that $\frac{d}{du}\left[\tan^{-1}(u)\right] = \frac{1}{1 + u^2}$ evaluate $\int_{0}^{\frac{\pi}{4}} \tan^2\theta \, d\theta$

by using $u = \tan\theta$.

$$\frac{du}{d\theta} = \sec^2\theta \qquad \therefore \quad d\theta = \frac{du}{\sec^2\theta}$$

θ	0	$\frac{\pi}{4}$
u	0	1

$$\int_0^{\frac{\pi}{4}} \tan^2\theta \, d\theta = \int_0^1 \frac{u^2 \, du}{\sec^2\theta}$$

$$= \int_0^1 \frac{u^2 \, du}{u^2 + 1}$$

$$= \int_0^1 \frac{u^2 + 1}{u^2 + 1} \, du - \int_0^1 \frac{1}{u^2 + 1} \, du$$

$$= \int_0^1 1 \, du - \int_0^1 \frac{1}{u^2 + 1} \, du$$

$$= [u]_0^1 - \left[\tan^{-1}(u)\right]_0^1$$

$$= 1 - \tan^{-1}(1) + \tan^{-1}(0)$$

$$= 1 - \frac{\pi}{4}$$

- ✓ expresses $d\theta$ in terms of du
- ✓ change limits of integration
- ✓ uses indentity for $\sec^2\theta = 1 + \tan^2\theta$
- ✓ splits integrand into 2 parts
- ✓ anti-derivatives correct (using the given result for inverse tangent)
- ✓ evaluates correctly

[6]

End of Assessment Task