PERTH MODERN SCHOOL
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**Semester Two Examination, 2023**

**Question/Answer booklet**

# 12 SPECIALIST MATHEMATICS
# UNIT 3
## Section Two: Calculator-assumed

Your Name ______________________________

Your Teacher's Name ________________________

### Time allowed for this section
Reading time before commencing work: ten minutes
Working time: one hundred minutes

### Materials required/recommended for this section
***To be provided by the supervisor***
This Question/Answer booklet
Formula sheet (retained from Section One)

***To be provided by the candidate***
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

### Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

| Question | Marks | Max | Question | Marks | Max |
|---|---|---|---|---|---|
| 7 | | 4 | 16 | | 7 |
| 8 | | 10 | 17 | | 6 |
| 9 | | 11 | 18 | | 8 |
| 10 | | 5 | 19 | | 9 |
| 11 | | 5 | | | |
| 12 | | 13 | | | |
| 13 | | 6 | | | |
| 14 | | 6 | | | |
| 15 | | 7 | | | |

## Structure of this paper

| Section | Number of questions available | Number of questions to be answered | Working time (minutes) | Marks available | Percentage of examination |
|---|---|---|---|---|---|
| Section One: Calculator-free | 6 | 6 | 50 | 50 | 34 |
| Section Two: Calculator-assumed | 13 | 13 | 100 | 97 | 66 |
| | | | | **Total** | 100 |

## Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

**Section Two: Calculator-assumed** **(97 Marks)**

This section has **13** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 100 minutes.

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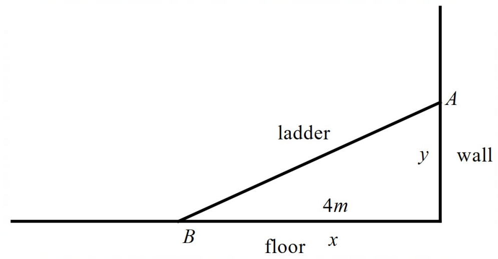
**Question 7** **(4 marks)**

Consider a ladder placed with one end, point A, on a wall and the other, point B, on the floor as shown below. The ladder has a length of 5 metres and point B is moving towards the wall at a speed of 3 metres per minute. When point B is 4 metres from the base of the wall, determine the speed of point A which is moving up the wall.



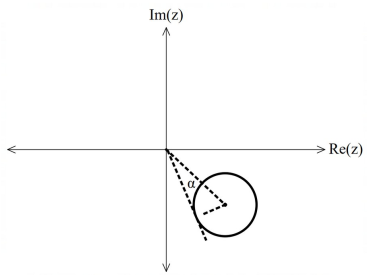

| c |
|---|
| $x^2 + y^2 = 25$<br>$2x\dot{x} + 2y\dot{y} = 0$<br>$2(4)(-3) + 2(3)\dot{y} = 0$<br>$\dot{y} = 4m / \min s$ |
| **Specific behaviours** |
| ✓ introduces two variables<br>✓ states equation linking both variables<br>✓ uses implicit diff and subs known quantities<br>✓ states required rate with units |

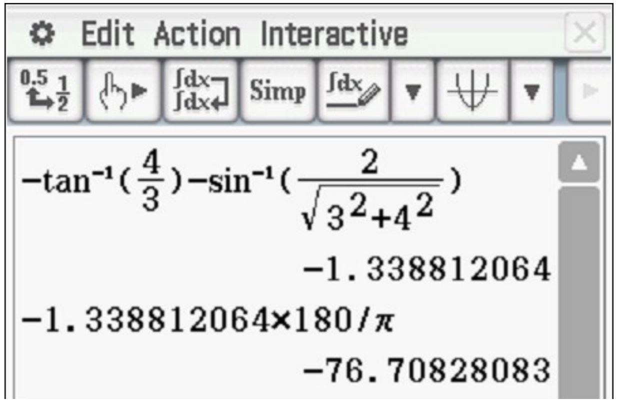
**Question 8** **(10 marks)**

a) Consider the locus $|z - 3 + 4i| = 2$ in the complex plane.
Determine the following:

i) Minimum Arg(z). (3 marks)

| c |
|---|




| Specific behaviours |
|---|
| ✓ determines argument of centre of circle<br>✓ uses tangent and acute angle of right angled triangle<br>✓ states min argument in radians or degrees in fourth quadrant |

ii) Maximum $|z|$. (2 marks)

| c |
|---|
| |

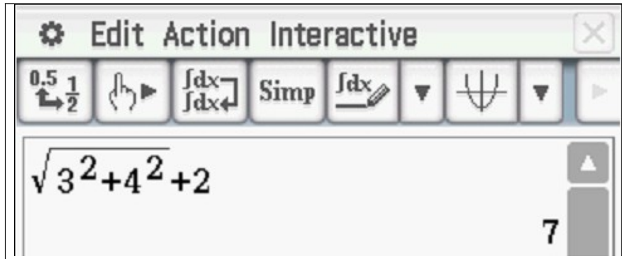
**Additional working space**

Question number: _________

**Additional working space**

Question number: ________






| Specific behaviours |
|---|
| ✓ determines modulus of centre<br>✓ adds radius |

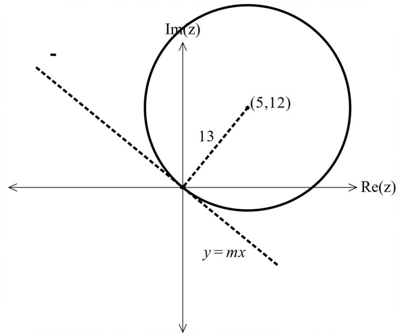
b) Sketch the following locus $|z - 5 - 12i| = 13$ on the axes below. The Arguments in this locus lie between the following $b < Arg(z) < c$. Determine the values of $b \,\&\, c$.

(5 marks)

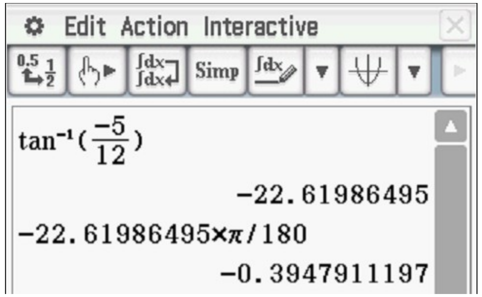
$$\frac{12}{5}m = -1$$

$$m = \frac{-5}{12}$$

$$\tan\theta = \frac{-5}{12}$$




| c |
|---|
| |

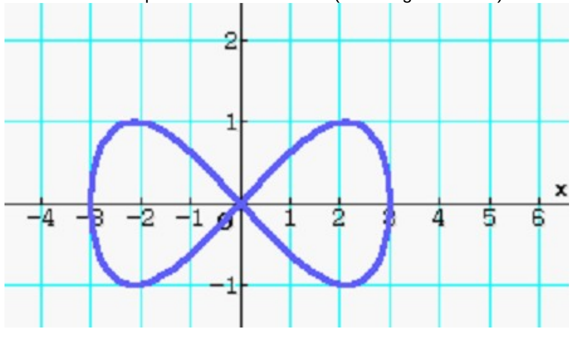



b=-22.6 degrees (-0.395 rads)

| c= 157.4 degrees (2.75 rads) |
|---|
| **Specific behaviours** |
| ✓ sketches circle going through origin<br>✓ determines gradient of radius line<br>✓ determines gradient of tangent through origin<br>✓ states lower argument (non inclusive)<br>✓ states upper argument (non inclusive) |

**Question 9** **(11 marks)**

Consider a racing car that travels in a racecourse with velocity $v = \begin{pmatrix} -3\sin t \\ 2\cos 2t \end{pmatrix} km/hr$ at time $t$ hours. The initial position is $r = \begin{pmatrix} 3 \\ 0 \end{pmatrix} km$. (See diagram below).




a) Determine the acceleration at $t = \pi$ hours. (2 marks)

| c |
|---|
| $v = \begin{pmatrix} -3\sin t \\ 2\cos 2t \end{pmatrix} km/hr$<br>$a = \begin{pmatrix} -3\cos t \\ -4\sin 2t \end{pmatrix}$<br>$t = \pi$<br>$a = \begin{pmatrix} 3 \\ 0 \end{pmatrix} km/hr^2$ |
| **Specific behaviours** |
| ✓ diff velocity<br>✓ subs t value |

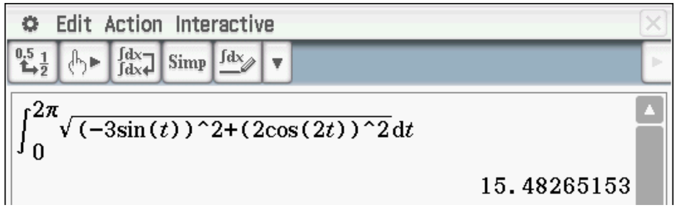
(ii) Determine, with justification, the exact location of the centre of the circle. (2 marks)

| Solution |
|---|
| When $arg\left(z-(2+7i)\right)=0$, then $arg\left(z-(2+i)\right)=\frac{\pi}{3}$ and a right-triangle is formed in the circle.<br>The midpoint of the hypotenuse of this triangle must be the centre of the circle. Hence the centre is at<br>$\left(2+\sqrt{3}\right)+4i$ |
| **Specific behaviours** |
| ✓ indicates adoption of suitable method<br>ü correct centre, fully justified |

b) Determine $\int_0^{\frac{3\pi}{2}} v\,dt$. (3 marks)

| c |
|---|
| $\int_0^{\frac{3\pi}{2}} \begin{pmatrix} -3\sin t \\ 2\cos 2t \end{pmatrix} dt$<br>$\left[\begin{pmatrix} 3\cos t \\ \sin 2t \end{pmatrix}\right]_0^{\frac{3\pi}{2}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} km$ |
| **Specific behaviours** |
| ✓ integrates<br>✓ subs limits<br>✓ states position with units |

c) Determine the length of one track of the racecourse. (3 marks)

| c |
|---|
| <br>Length = 15.5 km |
| **Specific behaviours** |
| ✓ determines period of one cycle<br>✓ sets up integral<br>✓ states distance with units |

d) Determine the cartesian equation of the path of the race car. (3 marks)

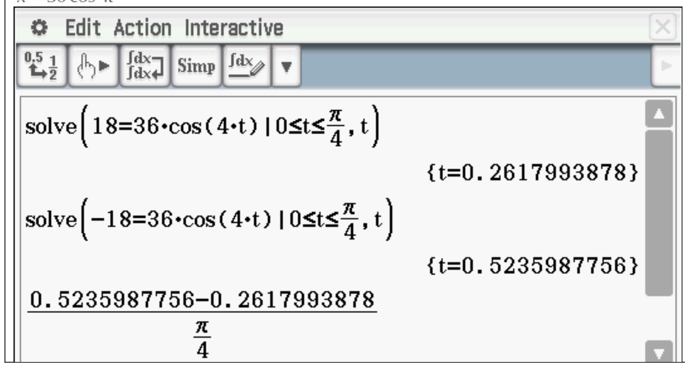
| c |
|---|

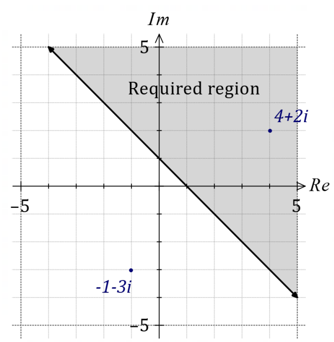
| |
|---|
| $x = 3\cos t$<br>$y = \sin 2t = 2\sin t\cos t = \frac{2x}{3}\sqrt{1-\cos^2 t}$<br>$y = \frac{2x}{3}\sqrt{1-\frac{x^2}{9}}$<br>*or*<br>$y = -\frac{2x}{3}\sqrt{1-\frac{x^2}{9}}$<br>*alternative*<br>$y^2 = \frac{4x^2}{9}\left(1-\frac{x^2}{9}\right)$ |
| **Specific behaviours** |
| ✓ uses double angle formula for sine<br>✓ uses Pythagorean identity<br>✓ states at least one possible cartesian equation |

**Question 10** **(5 marks)**

Consider a particle that undergoes motion defined by $\ddot{x} = -16x$ with $x$ ,metres being the displacement at time, $t$ seconds. The velocity is zero when $x = 9$ metres.
Determine the percentage of time that the particle has a speed less than half of its maximum speed.

| c |
|---|
| *let* $x = 9\sin 4t$<br>$\dot{x} = 36\cos 4t$ |

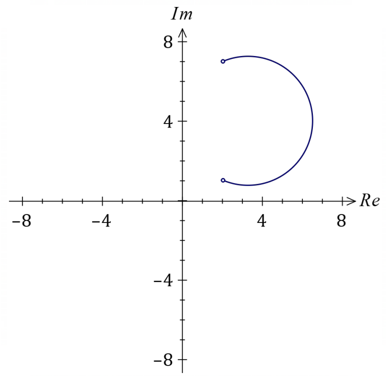



| **Solution (b)(ii)** |
|---|
| See shading on diagram |
| **Specific behaviours** |
| ✓ correct shading |

(c) The locus of points that satisfy $arg\left(\frac{z-2-i}{z-2-7i}\right) = \frac{\pi}{3}$ is an arc of a circle.

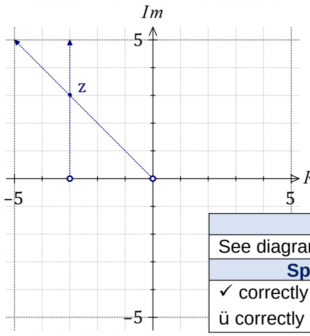
(i) Sketch the locus of $z$ in the complex plane. (2 marks)




| **Solution** |
|---|
| $arg(z-(2+i)) = arg(z-(2+7i)) + \frac{\pi}{3}$<br>Anticlockwise major arc from $2+i$ to $2+7i$.<br>*NB Marks for location of major arc rather than neatness/curvature* |
| **Specific behaviours** |
| ✓ major arc of a circle drawn anywhere<br>ü correctly locates endpoints and major arc drawn to their right |

**Question 19** **(9 marks)**

(a) Plot the complex number that satisfies the conditions $arg(z)=\frac{3\pi}{4}$ and $arg(z+3)=\frac{\pi}{2}$ on the Argand diagram below. (2 marks)



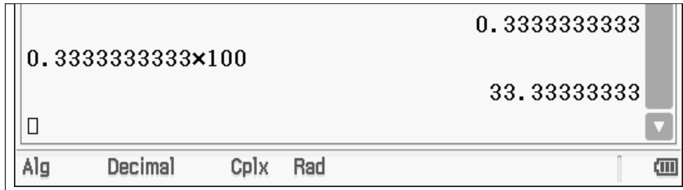

| Solution |
|---|
| See diagram |
| **Specific behaviours** |
| ✓ correctly indicates at least one ray |
| ü correctly plots complex number |

(b) Let $z_1=4+2i$ and $z_2$ be another complex number. The locus of a complex number $z$ satisfies the condition $|z-z_1|=¿z-z_2\vee¿$ and is shown in the diagram below.

(i) Determine the complex number $z_2$. (2 marks)

| Solution |
|---|
| $z_2=-1-3i$ |
| **Specific behaviours** |
| ✓ indicates point lies on perpendicular to locus through $z_1$ |
| ü correct complex number |

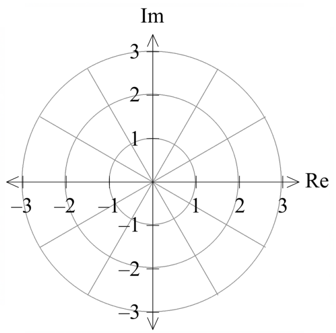
(ii) On the same diagram, indicate the locus of a complex number $z$ that satisfies the condition $|z-z_1|\leq\vee z-z_2\vee¿$. (1 mark)




| Specific behaviours |
|---|
| ✓ recognizes that n=4 and A=9 |
| ✓ states an expression for velocity as function of time |
| ✓ solves for times less than half of max speed in one cycle(Half cycle) |
| ✓ divides by length of cycle(half cycle) |
| ✓ states a percentage |

**Question 11** **(5 marks)**

(a) Determine the solutions to $z^6 - 64 = 0$ in polar form and plot them on the Argand plane below. Label the solutions $z_1$, $z_2$, $z_3$, $z_4$, $z_5$ and $z_6$ in an **anti-clockwise** direction, starting from $z_1$ which is on the positive real axis. (3 marks)




| Solution | Specific behaviours |
|---|---|
| $z^6 = 64$<br>$z_1 = 2\,cis\,0,\, z_2 = 2\,cis\,\frac{\pi}{3}$<br>$z_3 = 2\,cis\,\frac{2\pi}{3},\, z_4 = 2\,cis\,\pi = -2$<br>$z_5 = 2\,cis\left(\frac{-2\pi}{3}\right),\, z_6 = 2\,cis\left(\frac{-\pi}{3}\right)$ | ✓ Determines at least half of the solutions correctly in polar form.<br>✓ Determines all solutions in polar form.<br>✓ Plots and labels all solutions. |

(b) There is a cubic polynomial with real coefficients whose roots are $z_3$, $z_4$ and $z_5$. Write down this cubic polynomial in the form $a x^3 + b x^2 + cx + d$. (2 marks)

| Solution | Specific behaviours |
|---|---|
| $2\,cis\,\frac{2\pi}{3} = -1 + \sqrt{3} i$<br>*Factorised form*:<br>$(x + 1 - \sqrt{3} i)(x + 1 + \sqrt{3} i)(x + 2)$<br>*Expanding on calculator*:<br>$x^3 + 4x^2 + 8x + 8$ | ✓ Writes polynomial in factorised form (using rectangular or polar form of $z_3, z_4$ and $z_5$).<br>✓ Writes down cubic polynomial. |

| |
|---|
| ✓ states that sample mean will be normally distributed |
| ü states the mean of the distribution |
| ü states the variance or standard deviation of the distribution |

(b) Determine the probability that the mean weight of a random sample of 76 bags of sugar is at least 502 grams, given that the sample mean is less than 505 grams. (2 marks)

| Solution |
|---|
| $P(\overline{X} > 502 \mid \overline{X} < 505) = \frac{P(502 < \overline{X} < 505)}{P(\overline{X} < 505)} = \frac{0.438}{0.5} = 0.876$ |
| **Specific behaviours** |
| ✓ forms correct probability statement |
| ü correct probability |

(c) Occasionally, the inspector only has enough time to take a random sample of 50 bags. In the long run, 80% of sample means derived from samples with this smaller size will lie in the range $505 \pm k$ grams. Determine the value of $k$. (3 marks)

| Solution |
|---|
| The new standard deviation of $\overline{X}$ is $\frac{17}{\sqrt{50}} = 2.404$ grams (variance¿ 5.78).<br>$\overline{X} \quad N(505, 5.78)$<br>$P(505 - k < \overline{X} < 505 + k) = 0.8\, k = 3.08\,\mathrm{g}$ |
| **Specific behaviours** |
| ✓ states new parameters of distribution of sample mean |
| ü writes correct probability statement |
| ü correct value of $k$ |

(i) equation of its horizontal asymptote. (1 mark)

| Solution |
|---|
| $y=\frac{1}{2}$ |
| **Specific behaviours** |
| ✓ correct equation |

(ii) $x$-axis intercepts. (1 mark)

| Solution |
|---|
| Vertical asymptotes → roots: $x=\pm 2$. |
| **Specific behaviours** |
| ✓ correct intercepts |

(iii) equations of its vertical asymptotes. (1 mark)

| Solution |
|---|
| Roots → vertical asymptotes: $2x^2-4x-30=0 \Rightarrow x=-3$ and $x=5$. |
| **Specific behaviours** |
| ✓ correct equations |

Question 18 (8 marks)

A machine fills bags with sugar. The mean and standard deviation of the weight of sugar it delivers into a bag is $505$ and $17$ grams respectively. An inspector routinely takes a random sample of $76$ bags filled by the machine.

(a) For repeated random sampling of $76$ bags of sugar filled by this machine, state the approximate distribution of the sample mean that the inspector should expect. (3 marks)

| Solution |
|---|
| Let $\overline{X}$ be the sample mean. Since the sample size is large then the distribution of $\overline{X}$ will be approximately normal with mean $505$ g.<br>The standard deviation of $\overline{X}$ is $\frac{17}{\sqrt{76}}=1.950$ grams (variance$\approx 3.8$)<br>Hence $\overline{X} \sim N(505, 1.95^2)$. |
| **Specific behaviours** |



**Question 12** **(13 marks)**

Given the points $A(1,-3,0)$, $B(3,2,-1)$, $C(7,1,2)$ and $D(5,-4,3)$.

(a) Determine the vector equation of the line through the points $A$ and $B$. (1 mark)

| Solution | Specific behaviours |
|---|---|
| $d=\overrightarrow{AB}=\begin{pmatrix}2\\5\\-1\end{pmatrix}$<br>$r=\begin{pmatrix}1\\-3\\0\end{pmatrix}+\mu\begin{pmatrix}2\\5\\-1\end{pmatrix}$ | ✓ Determines vector equation of the line. |

The vector equation of the line through $A$ and $C$ is $r=\begin{pmatrix}1\\-3\\0\end{pmatrix}+\mu\begin{pmatrix}6\\4\\2\end{pmatrix}$.

(b) Determine the Cartesian equation of the plane, $\Pi$, containing the lines passing through $AB$ and $AC$. (2 marks)

| Solution | Specific behaviours |
|---|---|
| $n=\begin{pmatrix}2\\5\\-1\end{pmatrix}\times\begin{pmatrix}6\\4\\2\end{pmatrix}=\begin{pmatrix}14\\-10\\-22\end{pmatrix}$<br>$r\cdot n=a\cdot n$<br>$14x-10y-22z=44$<br>$7x-5y-11z=22$ | ✓ Recognises cross product of direction vectors gives the normal to the plane.<br>✓ Determines Cartesian equation of plane. |

(c) (i) Show that $A$, $B$, $C$ and $D$ are coplanar. (2 marks)

| Solution | Specific behaviours |
|---|---|
| $Substitute\ D\ into\ \Pi$<br>$LHS=7(5)-5(-4)-11(3)¿22=RHS$<br>$Hence\ D\ is\ on\ the\ plane\ containing\ A, B\wedge C.$ | ✓ Substitutes $D$ into plane<br>✓ Explains why $A$, $B$, $C$ and $D$ are coplanar. |

(ii) Prove that $ABCD$ is a rectangle. (2 marks)

| Solution | Specific behaviours |
|---|---|
| $\overrightarrow{AB}=\begin{pmatrix}2\\5\\-1\end{pmatrix}, \overrightarrow{DC}=\begin{pmatrix}2\\5\\-1\end{pmatrix}, \overrightarrow{AD}=\begin{pmatrix}4\\-1\\3\end{pmatrix}$<br>$\overrightarrow{DC}\cdot\overrightarrow{AD}=0$ | ✓ Determines at least one additional side of the rectangle.<br>✓ Shows that two sides are |

| | |
|---|---|
| $As\ \overrightarrow{AB}=\overrightarrow{DC} \wedge \overrightarrow{AB} \perp \overrightarrow{AD}$, then $ABCD$ is a rectangle | perpendicular and proves that $ABCD$ is a rectangle. |

A sphere is constructed with its centre on plane $\Pi$ from part (b).

(d) Determine the vector equation of this sphere if $A$, $B$, $C$ and $D$ lie on the surface. (3 marks)

| Solution | Specific behaviours | Point |
|---|---|---|
| $Centre=(4,-1,1)$<br>$r=\frac{\lvert\overrightarrow{AC}\rvert}{2}=\sqrt{14}$<br>$\left\lvert r-\begin{pmatrix}4\\-1\\1\end{pmatrix}\right\rvert=\sqrt{14}$ | ✓ Determines coordinates of centre of sphere.<br>✓ Determines radius.<br>✓ States vector equation of sphere. | 3.3.3 |

A set of three planes is given as follows:

$$6x+5y+2z=21$$
$$3x-3y+3z=18$$
$$6x+5y+2a^2z=a+20$$

(e) Determine the value of $a$ such that the above planes only intersect at the centre of the sphere found in part (d). (3 marks)

| Solution | Specific behaviours |
|---|---|
| $Substitute\ (4,-1,1)\ into$<br>$6x+5y+2a^2z=a+20$<br>$2a^2+19=a+20$<br>$2a^2-a-1=0$<br>$a=1,-\frac{1}{2}$<br>$Reject\ a=1$, as this gives infinite solutions<br>$Hence\ a=\frac{-1}{2}$ | ✓ Substitutes in $(4,-1,1)$ and forms a quadratic equation.<br>✓ Solves for $a$.<br>✓ Rejects $a=1$ with reason, and states final value of $a$. |



Determine the area enclosed by the curve and the line $3x-y+4=0$. (3 marks)

| Solution |
|---|
| $x=\frac{y}{\sqrt{2y+5}},x=\frac{y-4}{3}$<br>Lines intersect when $y=-2,y=10$.<br>$A=\int_{-2}^{10}\left(\frac{y}{\sqrt{2y+5}}\right)-\left(\frac{y-4}{3}\right)dy¿\frac{32}{3}=10.\overline{6}$ |
| **Specific behaviours** |
| ✓ obtains bounds of integral<br>ü writes correct integral for area<br>ü correct area |

**Question 17** **(6 marks)**

Consider the function $f(x)=\frac{ax^2-2ax-b}{x^2-c}$, where $a,b$ and $c$ are positive constants.

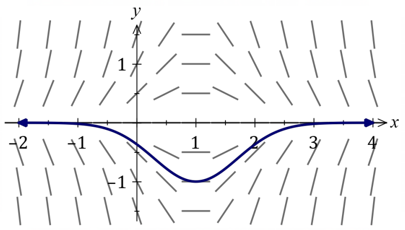
The graph of $y=f(x)$ cuts the $x$-axis at $x=-3$, has a horizontal asymptote with equation $y=2$ and has a vertical asymptote with equation $x=-2$.

(a) Determine $f(0)$. (3 marks)

| Solution |
|---|
| Horizontal asymptote $y=2\Rightarrow a=2$.<br>$f(-3)=0\Rightarrow 2(-3)^2-2(2)(-3)-b=0\Rightarrow b=30$<br>Vertical asymptote $x=-2\Rightarrow(-2)^2-c=0\Rightarrow c=4$<br>$f(0)=\frac{-30}{-4}=\frac{15}{2}$ |
| **Specific behaviours** |
| ✓ obtains value of one constant<br>ü obtains value of second constant<br>ü correct value of $f(0)$ |

(b) Now consider the graph of $y=\frac{1}{f(x)}$. State the

(c) Sketch the solution curve that contains the point $B(1,-1)$ on the slope field. (1 mark)



| Solution (c) |
|---|
| See graph |
| **Specific behaviours** |
| ✓ 'normal' curve thru' $(1,-1)$ |

**Question 16** **(7 marks)**

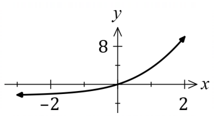
(a) Use the substitution $u^2=2y+5$ to show that $\int \frac{y}{\sqrt{2y+5}}dy=\frac{(y-5)\sqrt{2y+5}}{3}+c$,

where $c$ is a constant of integration. (4 marks)

| Solution |
|---|
| $u^2=2y+5\Rightarrow dy=u\,du,\ y=\frac{u^2-5}{2}$ <br> $\int \frac{y}{\sqrt{2y+5}}dy=\int \frac{u^2-5}{2u}\times u\,du¿\frac{1}{2}\int u^2-5\,du¿\frac{1}{2}\left(\frac{1}{3}u^3-5u\right)+c¿\frac{u}{6}(u^2-15)+c$ <br> $¿\frac{\sqrt{2y+5}(2y+5-15)}{6}+c¿\frac{(y-5)\sqrt{2y+5}}{3}+c$ |
| **Specific behaviours** |
| ✓ obtains $y$ and $dy$ in terms of $u$ and $du$ <br> ü obtains simplified integral in terms of $u$ <br> ü obtains correct antiderivative <br> ü shows step(s) that clearly lead to required result |

(b) The equation of the curve shown is

$y=x\sqrt{2y+5}$.





**Question 13** **(6 marks)**

(a) By using partial fractions, show that:

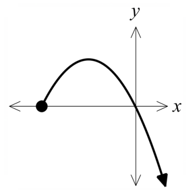
$$\int \frac{4}{4-y^2}dy=-\ln|2-y|+\ln|2+y|+c$$

(2 marks)

| Solution | Specific behaviours |
|---|---|
| $\frac{4}{4-y^2}=\frac{A}{2-y}+\frac{B}{2+y}$ <br> $4\equiv A(2+y)+B(2-y)$ <br> $Let\ y=-2\Rightarrow B=1$ <br> $Let\ y=2\Rightarrow A=1$ <br> $\int \frac{4}{4-y^2}dy=\int\left(\frac{1}{2-y}+\frac{1}{2+y}\right)dy$ <br> $¿-\ln|2-y|+\ln|2+y|+c$ | ✓ Correctly forms partial fractions. <br> ✓ Determines constants, and integrates to give result. Must contain absolute values. |

On a coordinate plane, a point $P$ moves along a path, such that after $t$ seconds $(t\geq 0)$, the position of the point is defined by

$$x=\frac{t}{2}-1$$
$$\frac{dy}{dt}=1-t$$



The direction of motion is shown in the diagram on the right.

(b) Determine when the angle between the direction of motion and the positive direction of the $x$-axis is $\pm 45°$. (4 marks)

| Solution | Specific behaviours |
|---|---|
| $\frac{dy}{dx}=\frac{\frac{dy}{dt}/dx}{dt}¿(1-t)\times 2=2-2t$ <br> $2-2t=1\Rightarrow t=\frac{1}{2}$ <br> $2-2t=-1\Rightarrow t=\frac{3}{2}$ <br> $After\ \frac{\frac{1}{2}\wedge 3}{2}\ seconds$ | ✓ Uses related rates to determine $\frac{dy}{dx}$. <br> ✓ Determines $\frac{dy}{dx}$. <br> ✓ Determines at least one time when angle between the direction of motion and $x$-axis is $45°$. <br> ✓ Determines second time. |

**Question 14** **(6 marks)**

(a) By letting $w=u+iv$ and $z=x+iy$, prove $\overline{w}+\overline{z}=\overline{w+z}$. (1 mark)

| Solution | Specific behaviours |
|---|---|
| $LHS=\overline{w}+\overline{z}=u-iv+x-iy$<br>$=u+x-i(v+y)=\overline{w+z}$ | ✓ Correctly proves result. |

(b) By letting $z=r\,cis\,\theta$, use De Moivre's theorem to prove that $\left(\overline{z^n}\right)=\overline{z}^n$. (1 mark)

| Solution | Specific behaviours |
|---|---|
| $LHS=\left(\overline{z^n}\right)=\left(\overline{r^n cis\,n\theta}\right)=r^n cis(-n\theta)$<br>$=\left(r\,cis(-\theta)\right)^n=\overline{z}^n$ | ✓ Correct proves result using De Moivre's theorem. |

(c) A polynomial $P(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x+a_0$ is divided by $(x-z)$, where $z$ is a complex number, leaving a remainder of $1-i$.

(i) Using parts (a) and (b), show that the remainder when $P(x)$ is divided by $(x-\overline{z})$ is $1+i$. (3 marks)

| Solution | Specific behaviours |
|---|---|
| $By\,remainder\,theorem: P(z)=2+i$ | ✓ Correctly uses remainder theorem. |
| $P(z)=a_n z^n+a_{n-1}z^{n-1}+\ldots+a_0$<br>$P(\overline{z})=a_n\overline{z}^n+a_{n-1}\overline{z}^{n+1}+\ldots+a_0$<br>$P(\overline{z})=a_n\overline{z^n}+a_{n-1}\overline{z^{n+1}}+\ldots+a_0$<br>$P(\overline{z})=\overline{a_n z^n+a_{n-1}z^{n+1}+\ldots+a_0}$<br>$P(\overline{z})=\overline{P(z)}P(\overline{z})=1+i$ | ✓ Substitutes in $\overline{z}$ and uses part (b).<br>✓ Uses part (a), and shows how to obtain required result. |

(ii) If for all solutions $z_n$ of it is known that $P(\overline{z_n})=0$, where $z_n$ is a complex number what can be said about the coefficients of $P(x)$? (1 mark)

| Solution | Specific behaviours |
|---|---|
| $All\,coefficients\,are\,real.$ | ✓ States coefficients are real. |

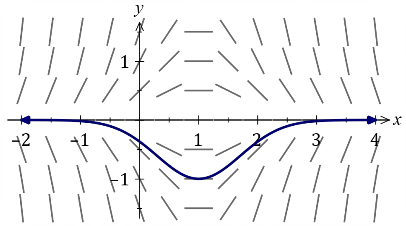


**Question 15**
**(7 marks)**

The slope field for the differential equation

$$\frac{dy}{dx}+y(2x-k)=0$$

where $k$ is a constant, is shown at right.




(a) Use a feature of the slope field to explain why $k=2$ and hence determine the slope at the point $A(-2,1)$. (2 marks)

| Solution |
|---|
| $y'=-y(2x-k)$<br>When $x=1$ and $y\neq 0$ it can be seen that $y'=0$ and so $2(1)-k=0\Rightarrow k=2$.<br>$A(-2,1)\rightarrow y'=-(1)(2(-2)-2)=6$. Slope at $A(-2,1)$ is 6. |
| **Specific behaviours** |
| ✓ explains using $y'=0$ at $x=1$ |
| ü correct slope at $A$ |

(b) Determine the solution of the differential equation that contains the point $B(1,-1)$ in the form $y=f(x)$. (4 marks)

| Solution |
|---|
| $\frac{dy}{dx}=-y(2x-2)$<br>$\int\frac{1}{y}dy=-\int 2x-2\,dx\ln\|y\|=-x^2+2x+c$<br>At $B(1,-1)$, $y<0$ and so require $\ln(-y)=-x^2+2x+c$<br>$-(1)^2+2(1)+c=0\Rightarrow c=-1$<br>$y=-e^{-x^2+2x-1}=e^{-(x-1)^2}(\approx -0.368\,e^{-x^2+2x})$ |
| **Specific behaviours** |
| ✓ separates variables and antidifferentiates |
| ü recognises that $y<0$ to replace $\|y\|\rightarrow(-y)$ |
| ü evaluates constant |
| ü correctly expresses $y$ as a function of $x$ |