ALL SAINTS' COLLEGE

WA Exams Practice Paper D, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	53	35
Section Two: Calculator-assumed	12	12	100	98	65
			Total	151	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Additional working space

Question number: _________

Additional working space

Question number: _________

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 **(5 marks)**

Fuel was observed to leak from a damaged tank at a rate of $\frac{224}{5e^{0.1t}}$ litres per minute, where t is the number of minutes that have elapsed since the tank was ruptured.

(a) How much fuel leaked from the tank during the first two minutes? (2 marks)

$$V = \int_0^2 \frac{224}{5e^{0.1t}}\ dt$$

$$= 81.2 \text{ L}$$

(b) If the tank initially contained 350 litres of fuel, determine the time taken, to the nearest second, for the tank to empty. (3 marks)

$$\int_0^t \frac{224}{5e^{0.1t}}\ dt = 350$$

$$448 - \frac{448}{e^{-0.1t}} = 350$$

$$t = 15.1982$$

$$= 15 \text{ m } 12 \text{ s}$$

Question 10 **(7 marks)**

A manufacturer makes a certain item and was investigating the proportion of faulty items coming off the production line.

(a) From a random sample of 500 items taken off the production line, it was found that 15 were faulty and the remainder good. Use this data to determine the probability that the next item off the production line will be faulty. (1 mark)

$$p = \frac{15}{500} = 0.03$$

(b) the mean and standard deviation of a Bernoulli distribution with the above probability of success. (2 marks)

$$E(X) = p = 0.03$$

$$Var(X) = p(1-p) = 0.0291$$

$$sd = \sqrt{0.0291} = 0.1706$$

(c) the probability that the next faulty item off the production line will be the 20th. (2 marks)

$$p = 0.97^{19} \times 0.03$$

$$= 0.0168$$

(d) at least one of the next 20 items off the production line will be faulty. (2 marks)

$$X \sim B(20,\ 0.03)$$

$$P(X \geq 1) = 0.4562$$

Question 20 **(8 marks)**

A polynomial function $f(x) = ax^4 + bx^2 + c$, where a, b and c are real constants, has the following features:

- $f(x) = 0$ **only** for $x = -2$ and $x = 2$
- $f'(x) = 0$ **only** for $x = -1$, $x = 0$ and $x = 1$
- $f'(x) > 0$ **only** for $-1 < x < 0$ and $x > 1$
- $f''(0) < 0$

(a) At the point where the curve intersects the y-axis, is it concave up or concave down? Explain your answer. (2 marks)

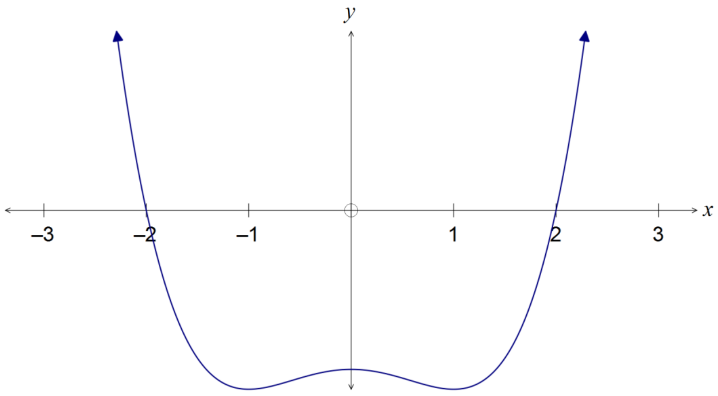
Concave down, since $f''(0) < 0$.

(b) Is c positive or negative? Explain your answer. (2 marks)

c is y-intercept and must be negative.

Only two roots, so between $x = -2$ and $x = 2$ function must always be below x-axis, as continuous and gradient at $x = -2$ is -ve and at $x = 2$ is +ve.

(c) Sketch a possible graph of the function on the axes below. (4 marks)



- two roots at correct x-values
- three turning points at correct x-values
-concave down at y-intercept
-'w' shape with smooth continuous curve

Question 19 **(9 marks)**

The probability distribution of the discrete random variable X is shown in the table below.

x	0	1	2	3	4
$P(X=x)$	$\frac{a^2}{3}$	$\frac{a}{3}$	$\frac{a}{2}$	$\frac{1}{3}$	$\frac{7}{16}$

(a) Determine the value of a. (3 marks)

$$\frac{a^2}{3}+\frac{a}{3}+\frac{a}{2}+\frac{1}{3}+\frac{7}{16}=1$$

$$16a^2+40a-11=0$$

$$(4x+11)(4x-1)=0$$

$$a=\cancel{-\frac{11}{4}},\ \frac{1}{4}$$

(b) Determine $P(X<3\mid X<4)$. (2 marks)

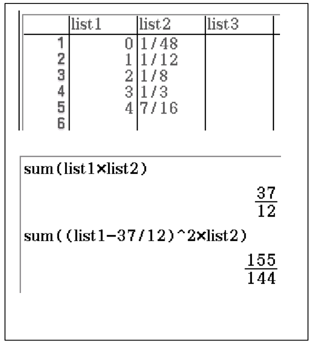
$$P(X<3\mid X<4)=\frac{P(0\le X\le 2)}{P(0\le X\le 3)}$$

$$=\frac{1-\left(\frac{1}{3}+\frac{7}{16}\right)}{1-\frac{7}{16}}=\frac{11/48}{9/16}=\frac{11}{27}$$

(c) Calculate the exact values of

(i) $E(X)$. (2 marks)

$$\boxed{E(X)=\frac{37}{12}}$$



(ii) $Var(X)$. (2 marks)

$$\boxed{Var(X)=\frac{155}{144}}$$

Question 11 **(9 marks)**

A small body is moving in a straight line with velocity $v=2t^2-19t+30$ m/s, where t is the time, in seconds, since the body first passed through the origin, O.

(a) Determine an expression for $x(t)$, the displacement of the body at time t. (2 marks)

$$x(t)=\int v(t)dt$$

$$=\int 2t^2-19t+30\ dt$$

$$=\tfrac{2}{3}t^3-\tfrac{19}{2}t^2+30t\cancel{+c}\quad (t=0,\ x=0)$$

(b) Show that the body is stationary twice and find the change in displacement of the body between these two instants. (4 marks)

$$v=0\ \Rightarrow\ 2t^2-19t+30=0$$

$(t-2)(2t-15)=0$ when $t=2,\ t=7.5$ seconds.

$$D=\int_2^{7.5} 2t^2-19t+30dt$$

$$=-\frac{1331}{24}$$

≈ -55.46 metres.

(c) Determine the position of the body when it's velocity is a minimum. (3 marks)

$$v'(t)=4t-19$$

$$4t-19=0\ \Rightarrow\ t=4.75$$

$$x(4.75)=-\frac{19}{48}\text{ m}$$

≈ -0.396 m

Question 12 **(10 marks)**

Atmospheric pressure, P kPa, decreases exponentially with increasing height, h m, above sea level according to the relationship $P = 101.3e^{-kh}$, where k is a positive constant.

(a) What is the the atmospheric pressure at sea level? (1 mark)

101.3 kPa

(b) Given that atmospheric pressure halves with every 5 800 m increase in height, determine the value of k, rounded to four significant figures. (2 marks)

$0.5 = e^{-5800k}$

$k = 0.0001195$

(c) Calculate the atmospheric pressure at the top of a mountain of height 3 785 m. (1 mark)

$P = 101.3e^{-0.0001195(3785)}$

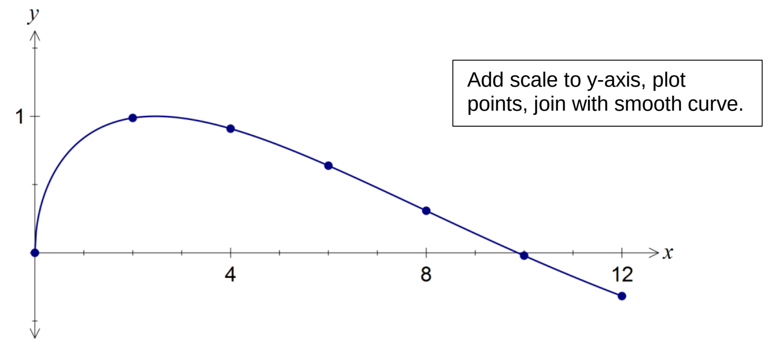
$= 64.44$ kPa

(d) Atmospheric pressure at a camp site at the base of a mountain is 43 kPa. Determine the height of the camp site. (2 marks)

$43 = 101.3e^{-0.0001195h}$

$h = 7170$ m

(d) Use your graph from (c) to sketch the graph of $y = A'(x)$ for $0 \le x \le 12$. (2 marks)

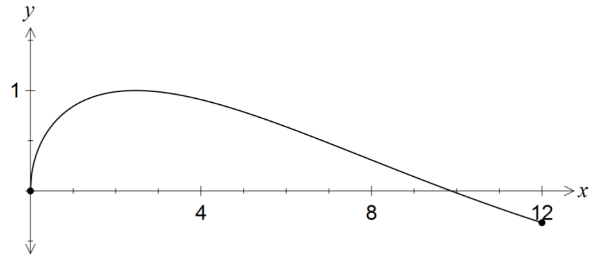


(e) Suggest a defining rule for $A'(x)$. (1 mark)

$A'(x) = \sin\sqrt{x}$

Question 18 **(9 marks)**

The graph of the function $y = f(x)$, where $f(x) = \sin\sqrt{x}$ for $0 \le x \le 12$, is provided below.



(a) The function A is defined as $A(x) = \int_0^x f(t)\, dt$ for $0 \le x \le 12$. Determine the value of x when $A(x)$ starts to decrease. (2 marks)

$\sin\sqrt{x} = 0$

$\sqrt{x} = \cancel{0},\ \pi$

$x = \pi^2$

(b) Complete the table below. (2 marks)

x	0	2	4	6	8	10	12
$A(x)$	0	1.53	3.48	**5.05**	**6.00**	**6.28**	**5.94**

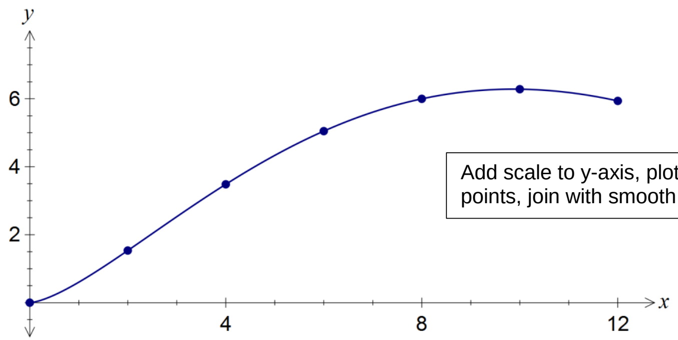
$\int_0^6 \sin(\sqrt{t})\,dt$

5.048067646

$\int_0^8 \sin(\sqrt{t})\,dt$

5.997866039

(c) On the axes below, sketch the graph of $y = A(x)$ for $0 \le x \le 12$. (2 marks)



Add scale to y-axis, plot points, join with smooth curve.

See next page

(e) Use the increments formula $\delta y = \frac{dy}{dx} \times \delta x$ to estimate the change in pressure as a climber descends 250 m from the top of a mountain of height 3 785 m. (4 marks)

$\delta h = -250$

$\delta P \approx \frac{dP}{dh} \times \delta h$

$\approx kP \times \delta h$

$\approx -0.0001195 \times 64.44 \times -250$

≈ 1.93 kPa

(An increase in pressure)

See next page

Question 13 **(10 marks)**

A student designed a game of chance in which two fair tetrahedral dice (both with faces numbered 1, 2, 3 and 4) were thrown and then the score, X, was calculated from the product of the numbers on which the dice fall.

(a) Complete the table below to show the probability distribution for the random variable X. (3 marks)

x	1	2	3	4	6	8	9	12	16
$P(X = x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

The player paid $3 for each game, winning a prize of $10 if the score was two and $5 if the score was odd. All other scores won nothing.

(b) Calculate the expected gain or loss of a person who played the game once. (5 marks)

Let Y be gain or loss per game.

Y ($)	7	2	-3
$P(Y = y)$	$\frac{2}{16}$	$\frac{4}{16}$	$\frac{10}{16}$

$$E(Y) = \frac{14}{16} + \frac{8}{16} - \frac{30}{16}$$

$$= -\frac{8}{16} = -0.5$$

loss of 50c per game

(c) If the student doubled the cost of the game but otherwise made no changes, determine the new expected gain or loss per game for a player. (2 marks)

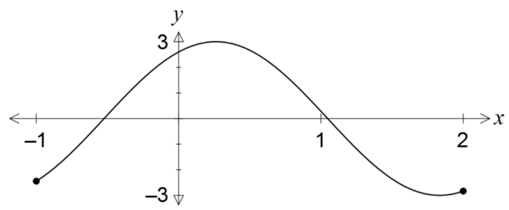
Game now costs $6, so $Y \to Y - 3$.

$E(Y-3) = -0.5 - 3$

$= -3.50 \Rightarrow$ loss of $3.50 per game

Question 17 **(7 marks)**

The graph of $y = 3\sin\left(2x + \frac{\pi}{3}\right)$ is given below for $-1 \le x \le 2$.



(a) Calculate the area under the curve between the two roots shown. (3 marks)

$$3\sin\left(2x + \frac{\pi}{3}\right) = 0 \Rightarrow x = -\frac{\pi}{6}, \frac{\pi}{3}$$

$$A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} 3\sin\left(2x + \frac{\pi}{3}\right) dx$$

$= 3$ sq u

(b) Let the area under the curve between the left-hand root and the y-axis be P, and the area under the curve between the y-axis and $x = a$ be Q, where $0 < a < 1$. Determine the value of a such that $P = Q$. (4 marks)

$$\int_{-\frac{\pi}{6}}^{0} 3\sin\left(2x + \frac{\pi}{3}\right) dx = \frac{3}{4}$$

$$\int_{0}^{a} 3\sin\left(2x + \frac{\pi}{3}\right) dx = \frac{3}{4}$$

$$\frac{3}{4} - \frac{3}{2}\cos\left(2a + \frac{\pi}{3}\right) = \frac{3}{4}$$

$$a = \frac{\pi}{12}$$

Question 16 **(8 marks)**

A manufacturer of chocolate produces 3 times as many soft centred chocolates as hard centred ones. The chocolates are randomly packed in boxes of 20.

(a) Find the probability that in a box there are

(i) an equal number of soft centred and hard centred chocolates (3 marks)

Let the rv X be the number of hard centred chocolates per box of 20.

Then $X \sim B(20, 0.25)$

$P(X = 10) = 0.00992$

(ii) fewer than 5 hard centred chocolates. (1 mark)

$P(X < 5) = P(X \le 4) = 0.41484$

(b) A random sample of 5 boxes is taken from the production line. Find the probability that exactly 3 of them contain fewer than 5 hard centred chocolates. (2 marks)

Let the rv Y be the number of boxes out of 5 with fewer than 5 hard centres.

Then $Y \sim B(5, 0.41484)$

$P(Y = 3) = 0.24445$

(c) Determine the mean and standard deviation of the number of hard centred chocolates in a box of 20. (2 marks)

Mean: $np = 20 \times 0.25 = 5$

SD: $\sqrt{np(1-p)} = \sqrt{20 \times 0.25 \times 0.75} = \sqrt{3.75} = 1.9365$

Question 14 **(8 marks)**

A particle is moving along the x-axis so that at time t, in seconds, its position is given by $x(t) = 2\pi t + \cos(2\pi t)$, $t \ge 0$.

(a) State the initial position of the particle. (1 mark)

$x(0) = 1$

(b) Determine the velocity function for this particle. (2 marks)

$v(t) = x'(t)$

$= 2\pi - 2\pi \sin(2\pi t)$

(c) At what time does the particle first come to rest? (2 marks)

$2\pi - 2\pi \sin(2\pi t) = 0$

$\sin(2\pi t) = 1$

$2\pi t = \frac{\pi}{2} \Rightarrow t = \frac{1}{4}$

(d) At what time does the particle first reach its maximum velocity? (3 marks)

$v'(t) = 0$

$-4\pi^2 \cos(2\pi t) = 0$

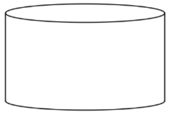
$2\pi t = \frac{\pi}{2}, \frac{3\pi}{2}$

$t = \cancel{\frac{1}{4}}, \frac{3}{4}$

(NB When t=0.25, v is min)

Question 15 **(8 marks)**

A cylindrical oil drum, of radius r m and height h m, has circular ends constructed from material costing \$75 per square metre and sides constructed from material costing \$40 per square metre.



(a) Explain why the cost of construction C, in dollars, is given by $C = 150\pi r^2 + 80\pi rh$. (1 mark)

TSA of cylinder given by ends plus side:

$$C = 75 \times 2\pi r^2 + 40 \times 2\pi rh$$
$$= 150\pi r^2 + 80\pi rh$$

(b) If the oil drum must be constructed for \$250, show that the volume of the oil drum is given by $V = \dfrac{25r - 15\pi r^3}{8}$. (3 marks)

$$250 = 150\pi r^2 + 80\pi rh$$
$$h = \frac{250 - 150\pi r^2}{80\pi r}$$

$$V = \pi r^2 h$$
$$= \pi r^2 \frac{250 - 150\pi r^2}{80\pi r}$$

$$= \frac{25r - 15\pi r^3}{8}$$

(c) Use calculus methods to determine the dimensions that maximise the volume of the oil drum, and state this maximum volume. (4 marks)

$$V = \frac{25r - 15\pi r^3}{8}$$
$$\frac{dV}{dr} = \frac{25 - 45\pi r^2}{8}$$

$$\frac{dV}{dr} = 0 \text{ when } r^2 = \frac{25}{45\pi} \Rightarrow r = \frac{\sqrt{5}}{3\sqrt{\pi}} \approx \underline{0.4205 \text{ m}}$$

$$h = \left.\frac{250 - 150\pi r^2}{80\pi r}\right|_{r=0.4205}$$
$$= \frac{5\sqrt{5}}{4\sqrt{\pi}} \approx \underline{1.577 \text{ m}}$$

$$V = \left.\frac{25r - 15\pi r^3}{8}\right|_{r=0.4205}$$
$$= \frac{25\sqrt{5}}{36\sqrt{\pi}} \approx \underline{0.8761 \text{ m}^3}$$