PERTH MODERN SCHOOL
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Independent Public School

Mathematics Specialist Year 11

Student name: ________________________ Teacher name: _______________

Date: Friday 24 September 2021

Task type: **Response**

Time allowed: **40 mins**

Number of questions: **7**

Materials required: Notes on two unfolded sheets of paper (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates and up to three calculators approved for use in the WACE examinations

Marks available: **40 marks**

Task weighting: **10%**

Formula sheet provided: **Yes**

Scientific Calculator and CAS: **Not Permitted**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Question 1 (2.2.1, 2.2.2) (6 marks)

Given that A, B and C are 2×2 matrices, $X=\begin{bmatrix}2\\1\end{bmatrix}$, $Y=\begin{bmatrix}3 & 4\end{bmatrix}$, $Z=\begin{bmatrix}3 & -1\\0 & 2\end{bmatrix}$ and I is the 2×2 identity matrix, find the following where possible

(a) XY (1 mark)

$\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}3 & 4\end{bmatrix} = \begin{bmatrix}6 & 8\\3 & 4\end{bmatrix}$ ✓ multiplies correctly

(b) YX (1 mark)

$\begin{bmatrix}3 & 4\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}10\end{bmatrix}$ ✓ multiplies correctly

(c) Matrix W given that $3Z-W=I$ (2 marks)

$-W = I - 3Z$

vivi

$W = 3Z - I$ ✓ rearranges

$= 3\begin{bmatrix}3 & -1\\0 & 2\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$= \begin{bmatrix}8 & -3\\0 & 5\end{bmatrix}$ ✓ correct matrix

(d) An expression for matrix V in terms of other matrices given that $V-ABV=C$ (2 marks)

$(I-AB)V = C$ ✓ factorises correctly

$(I-AB)^{-1}(I-AB)V = (I-AB)^{-1}C$

$V = (I-AB)^{-1}C$ ✓ correct expression

Question 2 **(2.2.3, 2.2.11)** **(5 marks)**

(a) For what values of a is the matrix $\begin{bmatrix} a & 5 \\ 3 & a \end{bmatrix}$ singular? (2 marks)

$\det\left(\begin{bmatrix} a & 5 \\ 3 & a \end{bmatrix}\right) = 0$ when $a^2 - 15 = 0$

$a = \pm\sqrt{15}$

✓ equation for determinant

✓ both values of a

(b) Use matrices to find the point of intersection of the lines given by the equations $3x + y = 2$ and $5x + 2y = 1$. (3 marks)

$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ✓ matrix equation

$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

✓ pre-multiply by inverse

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6-5}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

✓ correct values for x and y

$x = 3,\ y = -7$

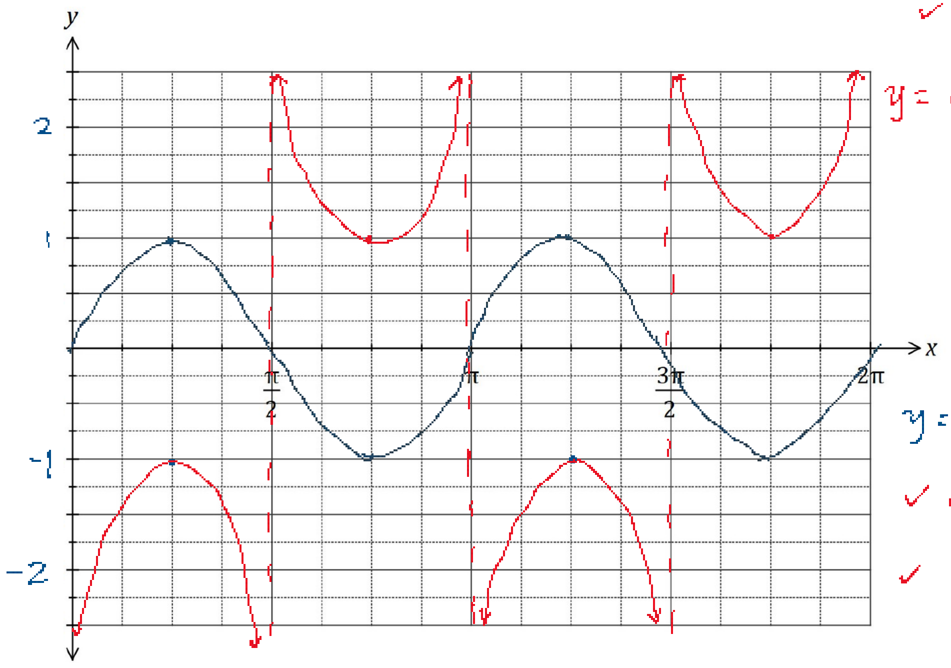
Question 3 **(2.1.4)** **(5 marks)**

Using the same scale, sketch the graphs of $y = \sin(2x)$ and $y = cosec(2x + \pi)$ on the grid below for $0 \le x \le 2\pi$

$\frac{1}{\sin x} = \operatorname{cosec} x$

$\operatorname{cosec}(2x+\pi) = \operatorname{cosec}\left[2\left(x+\frac{\pi}{2}\right)\right]$

✓ shape
✓ asymptotes
✓ stat. pts.



$y = \operatorname{cosec}\left[2\left(x+\frac{\pi}{2}\right)\right]$

$y = \sin(2x)$

✓ amplitude/shape
✓ period

Question 4 **(2.1.5, 2.1.6, 2.1.8)** **(5 marks)**

Prove the identity below

$$\frac{1-\sin(2\theta)}{\sin\theta-\cos\theta}=\sin\theta-cos\theta$$

$$\text{LHS} = \frac{1-\sin 2\theta}{\sin\theta-\cos\theta}$$

$$= \frac{1-\sin 2\theta}{\sin\theta-\cos\theta} \times \frac{\sin\theta-\cos\theta}{\sin\theta-\cos\theta}$$

✓ multiplies

$$= \frac{(1-\sin 2\theta)(\sin\theta-\cos\theta)}{\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta}$$

✓ expands denominator

$$= \frac{(1-\sin 2\theta)(\sin\theta-\cos\theta)}{1-2\sin\theta\cos\theta}$$ ✓

$$= \frac{(1-\sin 2\theta)(\sin\theta-\cos\theta)}{1-\sin 2\theta}$$ ✓

uses Pythagorean identity and double-angle formula

$$= \sin\theta-\cos\theta$$

✓ simplifies

$= \text{RHS}$ as required

accept other valid proofs

Question 5 **(2.2.5, 2.2.7, 2.2.10)** **(5 marks)**

(a) Find the matrices that produce each of the transformations described below

i. A reflection in the line $y = x$ (1 mark)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ ✓ correct matrix

ii. A rotation clockwise about the origin by $90°$ (2 mark)

$$\begin{bmatrix} \cos(-90°) & -\sin(-90°) \\ \sin(-90°) & \cos(-90°) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

✓ substitutes $-90°$ ✓ correct matrix

(b) Find and describe a single transformation matrix T that is a result of a reflection in the line $y = x$ followed by a $90°$ clockwise rotation about the origin. (2 marks)

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ ✓ correct matrix

reflection in the x-axis

✓ correct description

Question 6 **(2.2.6, 2.2.9)** **(9 marks)**

(a) Find the matrix of the linear transformation such that $(1,2) \to (12,7)$ and $(-3,1) \to (-1,0)$ (4 marks)

Let the matrix be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix}$$ ✓ forms matrix equation

✓ $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1}$$

post-multiply by inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix} \times \frac{1}{7}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$$ ✓ correct inverse

$$= \frac{1}{7}\begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 14 & 35 \\ 7 & 21 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$ ✓ correct matrix

(b) The matrix $\begin{bmatrix} t & t \\ 1 & t \end{bmatrix}$ maps the unit square into a parallelogram of area 2 square units. Find the possible value(s) of t (5 marks)

Area of image $= \left|\det\left(\begin{bmatrix} t & t \\ 1 & t \end{bmatrix}\right)\right| \times$ area of original

$2 = |t^2 - t| \times 1$ ✓ equation using abs. and det.

either $t^2 - t = 2$ ✓ or $t^2 - t = -2$ ✓ forms two equations

$t^2 - t - 2 = 0$ or $t^2 - t + 2 = 0$

$(t-2)(t+1) = 0$

$\Delta = (-1)^2 - 4(1)(2)$

$= 1 - 8 = -7$

$\therefore$ no real solutions

$t = -1$ ✓ or $t = 2$ ✓

Finds both values

Question 7 **(2.1.7)** **(5 marks)**

Find the general solution of $3\cos(x)-\sqrt{3}\sin(x)=3$

$r=\sqrt{9+3}$
$=\sqrt{12}$
$=2\sqrt{3}$

$\cos\alpha = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$

$\sin\alpha = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}$

$\therefore \alpha = \frac{\pi}{6}$

$3\cos x - \sqrt{3}\sin x = 3$

$\Rightarrow 2\sqrt{3}\cos\left(x+\frac{\pi}{6}\right) = 3$ ✓ ✓ forms an expression using r and θ

$\cos\left(x+\frac{\pi}{6}\right) = \frac{3}{2\sqrt{3}}$

$\cos\left(x+\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\left(x+\frac{\pi}{6}\right) = 2n\pi \pm \cos^{-1}\left(\frac{\sqrt{3}}{2}\right), \quad n\in\mathbb{Z}$

$= 2n\pi \pm \frac{\pi}{6}$ ✓ obtains an expression for $x+\frac{\pi}{6}$ or x

$x = 2n\pi \pm \frac{\pi}{6} - \frac{\pi}{6}$

$x = 2n\pi + \frac{\pi}{6} - \frac{\pi}{6}$ or $x = 2n\pi - \frac{\pi}{6} - \frac{\pi}{6}$

$x = 2n\pi$ ✓ or $2n\pi - \frac{\pi}{3}$ ✓, $n\in\mathbb{Z}$

obtains both expressions