PERTH MODERN SCHOOL
Exceptional schooling. Exceptional students.
Independent Public School

Year 12 Methods
TEST 2
Monday 8 April 2019
TIME: 45 minutes working
No notes allowed
Calculator Assumed
~~4~~3 marks 5 Questions

Name: Solutions Teacher: ____________

Note: All part questions worth more than 2 marks require working to obtain full marks.

Question 1 **(7 marks)**

(a) Differentiate $\frac{x}{e^x}$ and simplify your answer if necessary. (3 marks)

$$\frac{d}{dx}\frac{x}{e^x} = \frac{1 \times e^x - e^x \cdot x}{(e^x)^2}$$ ✓✓ Uses quotient rule (−1 per error)

$$= \frac{e^x(1-x)}{e^x \cdot e^x}$$

$$= \frac{1-x}{e^x}$$ ✓ simplifies $\frac{dy}{dx}$

(b) Using your result from (a) above and **without the use** of a classpad, show how to determine the definite integral $\int_0^1 \frac{1-x}{2e^x}\,dx$. (4 marks)

$$\int_0^1 \frac{1-x}{2e^x} = \frac{1}{2}\int_0^1 \frac{1-x}{e^x}$$ ✓ Takes out $\frac{1}{2}$.

$$= \frac{1}{2}\int_0^1 \left(\frac{d}{dx}\frac{x}{e^x}\right)$$

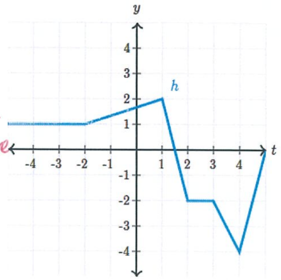
$$= \frac{1}{2}\left[\frac{x}{e^x}\right]_0^1$$ ✓ Uses F.T.C.

$$= \frac{1}{2}\left[\frac{1}{e} - \frac{0}{e^0}\right]$$ ✓ Substitution

$$= \frac{1}{2e}$$ ✓ Correct answer.

Question 2 **(8 marks)**

The graph of $h(x)$ is shown on the right.



(a) Evaluate the following definite integrals

(i) $\int_{-2}^{2} h(x)\,dx$ (2 marks)

$= \left(\frac{3}{2}+3\right)+\left(\frac{1}{4}\right)-\left(\frac{1}{4}\right)$ ✓ Uses area under the curve

$= 4.5$ or $\frac{9}{2}$ ✓ correct answer

(ii) $\int_{-2}^{2} h'(x)\,dx$ (2 marks)

$= h(2) - h(-2)$ ✓ Uses F. T. C.

$= (-2) - (1)$

$= -3$ ✓ Correct answer

(b) Determine the area bounded by the graph of $h'(x)$ and the x axis between $x = -2$ and $x = 2$. Justify your answer. (4 marks)

$A = \int_{-2}^{2} |f'(x)|\,dx$ ✓ Writes the expression for area in terms of absolute value

where $f'(x) > 0$ for $-2 \le x \le 1$ ✓ determines the interval where $f'(x)$ is "+" & "-"

$f'(x) < 0$ for $1 \le x \le 2$

So $A = \left|\int_{-2}^{1} f'(x)\,dx\right| + \left|\int_{1}^{2} f'(x)\,dx\right|$ ✓ breaks the integral over the correct intervals

$= |f(1) - f(-2)| + |f(2) - f(1)|$

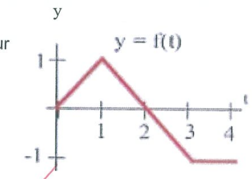
$= |2 - 1| + |(-2) - 2|$

$= 1 + 4$

$= 5$ ✓ calculates the correct area

Question 3 **(10 marks)**

$F(x) = \int_0^x f(t)\,dt$ for $f(x)$ in the picture on the right.



(a) Determine the value of x for a maximum of $F(x)$. Briefly explain your reasons. (3 marks)

Uses F.T.C to determine $F'(x) = f(x)$ ✓

Optimal value occurs at horizontal intercept ✓

Selects $x = 2$ as $F''(x) = f'(x) < 0$ ∴ local max ✓

OR

Area under $f(t)$ is positive for $1 \le t \le 2$ ✓

After $t = 2$, area change is negative ✓

Selects $t = 2$ as maximum area ✓

(b) Evaluate $F(3)$. (2 marks)

$F(3) = \int_0^3 f(t)\,dt$

$= \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt$ ✓ breaks the integrals over the correct intervals

$= 1 + \left(-\frac{1}{2}\right)$

$= \frac{1}{2}$ ✓ Calculates the correct area.

(c) Determine the value of x for a maximum of $F'(x)$. Briefly explain your reasons. (3 marks)

$F'(x) = \frac{d}{dx}\int_0^x f(t)\,dt$ ✓ differentiates both sides

$= f(x)$ ✓ Uses F.T.C

maximum $f(1) = 1$

∴ $x = 1$ ✓ state x.

(d) Evaluate $F'(4)$. (2 marks)

$F'(4) = f(4)$ ✓ Uses F.T.C

$= -1$ ✓ Correct answer

Question 4 **(10 marks)**

A new substance labelled **XX** is found to decay by the rule $N = 1200e^{-0.116t}$, where N equals the mass of the substance in kilograms at time t minutes.

Determine the following:

a) the initial mass of **XX**. (1 mark)

$N = 1200 \times e^{0} = 1200$ kg ✓ (Correct answer with units)

b) the time taken for half of the mass to decay away to the nearest minute. (3 marks)

$1200e^{-0.116t} = 600$

$e^{-0.116t} = \frac{1}{2}$ ✓ Recognises $e^{-0.116t}$ is half

$t = 5.9754 \approx 6$ min. ✓. correct answer ✓ correct rounding

The radiation is ~~dangerous~~ safe to humans when the rate of decay is ~~greater~~ less than 100kg per minute.

c) Determine ~~after~~ at what time the radiation will be safe for humans. (3 marks)

$N'(t) = -0.116 \times 1200e^{-0.116t}$ ✓. determines $N(t)$

$= -100$ ✓ equates $N(t) = -100$

$t \approx 2.85$ minutes. ✓ correct value for t.

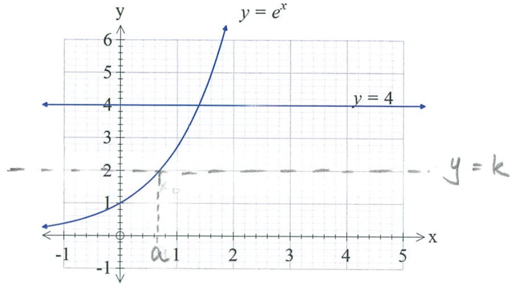
A different substance **YY** has a rate of decay given by $\frac{dN}{dt} = -50e^{-0.447t}$, where N equals the mass of the substance in kilograms at time t minutes.

d) Determine the total change in the mass from $t = 3$ to $t = 7$ minutes. (3 marks)

$\int_3^7 -50e^{-0.447t}\, dt$ ✓ Uses correct integral ✓ Uses correct limits 3 & 7

$= -24.36$ kg. ✓ Evaluates correct answer.

Question 5 **(8 marks)**



a) Using the solve facility on your classpad, determine to 2 decimal places the x value where the two graphs above intersect. (2 marks)

$y = e^x = 4$ $x = 1.39$ ✓✓ Correct answer

b) Determine to two decimal places the area bounded by $y = e^x$, $y = 4$ and the y axis. (3 marks)

$\int_0^{1.39} (4 - e^x)\, dx$ ✓ correct boundary points ✓ correct difference of functions

$= 2.55$ ✓ correct value for area.

c) Let $y = k$ where $1 \le k \le 4$, determine the value of k, to two decimal places, such that the area between $y = 4, y = k, y = e^x$ and the y axis equals 1.5 sq units. (3 marks)

let a be the x coordinate for the point of intersection between $y = k$ and $y = 4$

$\int_0^a (e^a - e^x) = 2.55 - 1.5 = 1.05$ ✓ Sets up the integral using difference of area with correct limits

$\left[e^a x - e^x\right]_0^a = 1.05$

$(e^a \cdot a - e^a) - (e^a \cdot 0 - e^0) = 1.05$

$e^a(a-1) + 1 = 1.05$

$a = 1.018$ ✓ Solves for a

$k = e^a = 2.77$ ✓ Subs. for k.