PRESBYTERIAN LADIES' COLLEGE
A COLLEGE OF THE UNITING CHURCH IN AUSTRALIA

NAME:____

TEST 5: **Sampling, Vector Calculus and Vectors**

Date: 26th August 2016

SECTION ONE: CALCULATOR FREE

TOTAL: 23 marks

EQUIPMENT: Pens, pencils, pencil sharpener, highlighter, eraser, ruler, SCSA formula sheet.

WORKING TIME: 25 minutes maximum

SECTION TWO: CALCULATOR ASSUMED

TOTAL: 35 marks

EQUIPMENT: Pens, pencils, pencil sharpener, highlighter, eraser, ruler, drawing instruments, templates, up to 3 Calculators,

1 A4 page of notes (one side only), SCSA formula sheet.

WORKING TIME: 35 minutes minimum

Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

SECTION 1 Question	Marks available	Marks awarded	SECTION 2 Question	Marks available	Marks awarded
1	12		4	5	
2	5		5	14	
3	6		6	11	
			7	5	
Total	23			35	
		Overall	Total	58	

Calculator-free **25 minutes** **[23 marks]**

This paper has **Three (3)** questions. Answer **all** questions. Write your answers in the spaces provided

Question 1 [12 marks]

The position in metres from an origin of a particle after t seconds can be modelled by the position vector:

$\underset{\sim}{r}(t) = cosec^2(t)\underset{\sim}{i} + cot(t)\underset{\sim}{j}$ for $0 \le t \le \pi$

(Note: $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$)

(a) Find the Cartesian equation of the curve defined by the position vector. (4)

$x = \underbrace{cosec^2 t \qquad y = \cot(t)}$ ✓

$x = \cot^2 t + 1$ ✓ ✓

$x = y^2 + 1$ ✓

(b) Sketch the position vector and indicate the direction of motion using $t = 0,\ \dfrac{\pi}{3},$ and $\dfrac{\pi}{2}.$ (3)

$y = \sqrt{x - 1}$

$\underline{t = 0}$

x = undefined

y = undefined

$t = \frac{\pi}{3}$

$x = \dfrac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} = \dfrac{4}{3}$

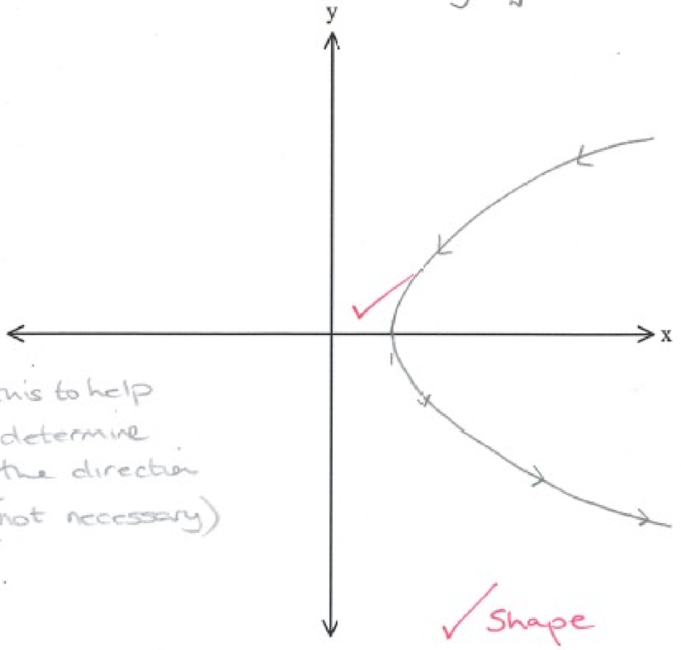
$y = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{1}{\sqrt{3}}$

$t = \frac{\pi}{2}$

$x = 1$

$y = 0$

this to help determine the direction (not necessary)



✓ Shape

✓ direction

4

(c) Determine the velocity vector $\underset{\sim}{v}(t)$. (3)

$$\underset{\sim}{v}(t) = r'(t) = \frac{d}{dt}\left(\left\langle \frac{1}{\sin^2 t}, \frac{\cos t}{\sin t} \right\rangle\right)$$

$$= \left\langle \frac{-2\cos t}{\sin^2 t}, \frac{-\sin^2 t - \cos^2 t}{\sin^2 t} \right\rangle$$

$$= -\frac{2\cos t}{\sin^2 t}\underset{\sim}{i} - \frac{1}{\sin^2 t}\underset{\sim}{j}$$

(d) Calculate the speed of the particle at $t = \frac{\pi}{2}$. (2)

$$v\left(\frac{\pi}{2}\right) = \langle 0, 1 \rangle$$

Speed =

$$\left|v\left(\frac{\pi}{2}\right)\right| = \sqrt{0^2 + 1^2}$$

$$= \underline{\underline{1 ms^{-1}}}$$

Question 2 [5 marks]

Determine the equation of the plane consisting of all points that are equidistant from $P(-1, 2, -3)$ and $Q(4, -2, 2)$.

$$\vec{PQ} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 5 \end{pmatrix} \checkmark \longrightarrow$$ This is the normal to the plane.

$$\text{Midpoint of } PQ = \frac{1}{2}\left[(-1,2,-3)+(4,-2,2)\right]$$

$$= \left(\frac{3}{2}, 0, -\frac{1}{2}\right) \checkmark$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 5 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 0 \\ -1/2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -4 \\ 5 \end{pmatrix} \checkmark$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 5 \\ -4 \\ 5 \end{pmatrix} = 5 \quad \Longrightarrow 5x - 4y + 5z = 5 \checkmark$$

Question 3 [6 marks]

A plane Π has vector equation $\underset{\sim}{r} = \left(-2\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right) + \lambda\left(2\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right) + \mu\left(6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}\right)$.

(a) Show that the Cartesian equation of the plane Π is $3x + 2y - 6z = 12$. (3)

normal to plane $\begin{pmatrix}2\\3\\2\end{pmatrix} \times \begin{pmatrix}6\\-3\\2\end{pmatrix} \Rightarrow \begin{vmatrix}2 & 3 & 2\\6 & -3 & 2\end{vmatrix} = 12\underset{\sim}{i} + 8\underset{\sim}{j} - 24\underset{\sim}{k}$

normal $\Rightarrow \begin{pmatrix}3\\2\\-6\end{pmatrix}$

$\underset{\sim}{r} \cdot \begin{pmatrix}3\\2\\-6\end{pmatrix} = \begin{pmatrix}-2\\3\\-2\end{pmatrix} \cdot \begin{pmatrix}3\\2\\-6\end{pmatrix}$

$\underset{\sim}{r} \cdot \begin{pmatrix}3\\2\\-6\end{pmatrix} = 12$

$3x + 2y - 6y = 12$ as required

(b) The plane Π meets the x, y and z axes at A, B and C respectively. Find the coordinates of A, B and C. (2)

* Meets the x-axis when y and z equal zero $\Rightarrow A(4,0,0)$

* " y-axis when x and z " " $\Rightarrow B(0,6,0)$

* " z-axis when x and y " " $\Rightarrow C(0,0,-2)$

(c) Find the volume of the pyramid OABC. (1)

Area of $\Delta OAB = \frac{1}{2} 4 \times 6 = 12$

Volume $\frac{1}{3} \times 12 \times 2 = \underline{\underline{8 \text{ units}^3}}$

NAME:____

Calculator Allowed **35 minutes** **[35 marks]**

This paper has **Four (4)** questions. Answer **all** questions. Write your answers in the spaces provided

Question 4 [5 marks]

The length of time, T, in months, that a football manager stays in his job before he is removed can be approximately modelled by a normal distribution with population mean μ and population variance $\sigma^2 = 16$.

An independent sample of 25 managers is taken. The mean time for this sample is 13 months.

(a) Determine a point estimate for μ. (1)

13 months

(b) Find the 95% and 99% confidence interval for μ. (2)

95% C.I $11.43 \leq \mu \leq 14.57$

99% C.I $10.99 \leq \mu \leq 15.06$

(c) The mean time for a second sample is exactly 15 months. Using the confidence intervals from part b) determine with reasons if the managers in this sample stayed in the job longer than those in the first sample. (2)

The mean time lies within the 99% C.I but outside the 95% C.I. Hence if the 90% C.I is used we would conclude the managers in this sample stayed in the position longer. With the 99% we would conclude there is no evidence to suggest they stayed longer.

7

Question 5 [14 marks]

The velocity vector of a particle P at time t hours is given by $\underset{\sim}{v}_p(t) = 2\cos t\underset{\sim}{i} - 3\sin t\underset{\sim}{j}$ kilometres per hour. It is known that the initial position of the particle is $0\underset{\sim}{i} + 7\underset{\sim}{j}$.

(a) Determine $\underset{\sim}{r}_p(t)$, the position vector of P at time t hours. (2)

$\underset{\sim}{r}_p(t) = \int \underset{\sim}{v}_p(t)\,dt = 2\sin t\,\underset{\sim}{i} + 3\cos t\,\underset{\sim}{j} + \underset{\sim}{c}$

$\underset{\sim}{r}_p(0) = 0\underset{\sim}{i} + 7\underset{\sim}{j} \Rightarrow \underset{\sim}{c} = 0\underset{\sim}{i} + 4\underset{\sim}{j}$

$\underset{\sim}{r}_p(t) = 2\sin t\,\underset{\sim}{i} + (3\cos t + 4)\underset{\sim}{j}$

(b) Show that the speed of P at time t hours is given by $\sqrt{4+5\sin^2 t}$ (2)

Speed $= |\underset{\sim}{v}_p(t)| = \sqrt{4\cos^2 t + 9\sin^2 t}$

$= \sqrt{4(1-\sin^2 t) + 9\sin^2 t}$

$= \sqrt{4 - 4\sin^2 t + 9\sin^2 t}$

$= \sqrt{4 + 5\sin^2 t}$ $\square$

(c) Find the position vector of P when its speed is maximised for the first time. (2)

Max speed when $\sin(t) = \pm 1$

$\boxed{t = \frac{\pi}{2}}$, $\cancel{\frac{3\pi}{2}}$

$$\underset{\sim}{r}_p\left(\frac{\pi}{2}\right) = 2\sin\left(\frac{\pi}{2}\right)\underset{\sim}{i} + \left(3\cos\left(\pi/2\right) + 4\right)\underset{\sim}{j}$$

$$= 2\underset{\sim}{i} + 4\underset{\sim}{j}$$

A second particle Q has position vector at time t hours is given by $\underset{\sim}{r}_Q(t) = 5\cos\frac{\pi t}{12}\underset{\sim}{i} + (2\sin\frac{\pi t}{12} + 1)\underset{\sim}{j}$ kilometres per hour. P and Q start moving at the same time.

(d) Sketch on the axes below the paths of particles P and Q. (3)

$\underset{\sim}{r}_P(t)$

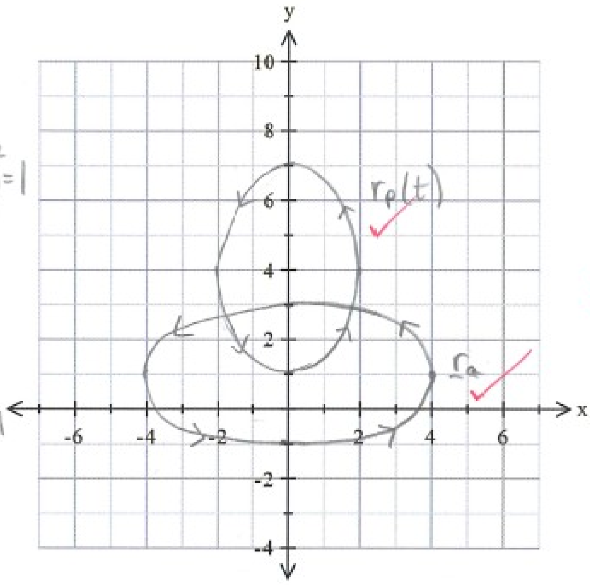
$x = 2\sin t$ $\left(\frac{x}{2}\right)^2 + \left(\frac{y-4}{3}\right)^2 = 1$

$y = 3\cos t + 4$

$\underset{\sim}{r}_Q(t)$

$x = 5\cos\frac{\pi t}{12}$ $\left(\frac{x}{5}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$

$y = 2\sin\frac{\pi t}{12} + 1$



directions ✓

(e) Determine the period of particle Q. (1)

$\frac{2\pi}{\pi/12} = 24$ hours ✓

(f) Show that whilst the paths of the two particles cross twice, the two particles do not collide. (4)

$\underset{\sim}{i}$ components equal:

Solve $\left(2\sin t = 5\cos\frac{\pi t}{12}\right)$

$t = 6.17$ ✓, 16.70 ✓ (ignore all other values outside of $0 \le t \le 24$)

① when $t = 6.17$

$\underset{\sim}{j}$ components $\underset{\sim}{r}_P(6.17) = 6.98$, $\underset{\sim}{r}_Q(6.17) = 3.00$ } not equal ✓

② when $t = 16.7$

$\underset{\sim}{j}$ components

$\underset{\sim}{r}_P(16.7) = 6.87$, $\underset{\sim}{r}_Q(16.7) = 1.15$ } Not equal ✓

Hence Not a collision

9

Question 6 [11 marks]

A person standing on top of a 35 m high cliff throws a stone into the water with velocity 16 ms^{-1} at an angle of 30° to the horizontal.

(a) Using the origin as the base of the cliff, derive the vector equations for the velocity and position of the stone (let g = 10 m s^{-2} and neglect air resistance). (5)

$\underset{\sim}{a}(t) = 0\underset{\sim}{i} - 10\underset{\sim}{j}$

$\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\,dt = -10t\,\underset{\sim}{j} + \underset{\sim}{c}$ ✓

when $t=0$ $\theta = 30$ $\underset{\sim}{v}(0) = (16\cos 30)\underset{\sim}{i} + (16\sin 30)\underset{\sim}{j}$

$= 8\sqrt{3}\,\underset{\sim}{i} + 8\underset{\sim}{j}$ ✓

$\underset{\sim}{v}(t) = 8\sqrt{3}\,\underset{\sim}{i} + (8-10t)\underset{\sim}{j}$ ✓

$\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\,dt = 8\sqrt{3}t\,\underset{\sim}{i} + (8t-5t^2)\underset{\sim}{j} + \underset{\sim}{k}$ ✓

$\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 35\underset{\sim}{j}$

$\underset{\sim}{r}(t) = 8\sqrt{3}t\,\underset{\sim}{i} + (35+8t-5t^2)\underset{\sim}{j}$ ✓

(b) Derive the Cartesian equation of the stone's flight as a function of y in terms of x. (3)

$x = 8\sqrt{3}t$

$t = \frac{x}{8\sqrt{3}}$ ✓

$y = 35 + 8t - 5t^2$ ✓

$y = 35 + 8\left(\frac{x}{8\sqrt{3}}\right) - 5\left(\frac{x}{8\sqrt{3}}\right)^2$

$y = 35 + \frac{x}{\sqrt{3}} - \frac{5x^2}{192}$ ✓

(c) Calculate the time that the stone will take to land in the water and its distance from the foot of the cliff at that time. (3)

y component $= 0$

$0 = -5t^2 + 8t + 35$

$t = \cancel{-1.96}, 3.56$ ✓

$x = 8\sqrt{3}t$

$= 49.38$ ✓

The stone lands about 49.4m from the base of the cliff after 3.6 seconds ✓

10

Question 7 [5 marks]

Jack and Ben are athletes specialising in the long jump. When Jack jumps, the length of his jumps are normally distributed with a mean 5.2 metres and standard deviation 0.1 metres. When Ben jumps, the lengths of his jumps are normally distributed with a mean 5.1 metres and standard deviation 0.12 metres. For both athletes, the length of a jump is independent of all other jumps. During a training session, Jack makes four jumps and Ben makes three jumps.

(a) Let $\overline{J}$ and $\overline{B}$ denote the mean of Jack's and Ben's jumps respectively. The distributions of $\overline{J}$ and $\overline{B}$ can be described as $\overline{J} \sim N(a, b^2)$ and $\overline{B} \sim N(c, d^2)$. Determine the values of a, b, c and d. (2)

$\overline{J} \sim N\left(5.2, \left(\frac{0.1}{\sqrt{4}}\right)^2\right)$ $\overline{B} \sim N\left(5.1, \left(\frac{0.12}{\sqrt{3}}\right)^2\right)$

$\therefore a = 5.2 \quad b = 0.05$ $c = 5.1 \quad d = 0.069$

✓ a & c values ✓ b & d values

(b) Using the fact that $(\overline{J} - \overline{B}) \sim N(a - c,\ b^2 + c^2)$, calculate the probability that the mean length of Jack's four jumps is less than the mean length of Ben's three jumps. (3)

$(\overline{J} - \overline{B}) \sim N(0.1, 0.0073)$ ✓

(note the s.d is $\sqrt{0.0073}$)

$P(\overline{J} < \overline{B}) = P(\overline{J} - \overline{B} < 0) = \underline{\underline{0.121}}$ ✓ ✓

END OF QUESTIONS