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Mathematics Specialist
Unit 3
2017

TEST 4: Differentiation and Integration

Student name: ______________________________ **Teacher name:** ____________________________

SOLUTIONS

Time allowed for this task:	***45 minutes***, in class, under test conditions Calculator-Assumed
Materials required:	
Standard items:	Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters, SCSA Formula Sheet. Classpad Calculator and Scientific Calculator.
Special items:	Drawing instruments, templates
Marks available:	**44 *marks***
Task weighting:	**8%**

Question 1 **(6 marks)**

Determine $\frac{dy}{dx}$ for each of the following:

$y=\log_5\left(x^2+9\right)$ (3 marks)

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Solution
$y = \log_5(x^2+9)$ $= \frac{\ln(x^2+9)}{\ln 5}$ $\frac{dy}{dx} = \frac{1}{\ln 5} \times \frac{2x}{x^2+9}$
Specific behaviours
✓ changes the function base to e correctly ✓ differentiates the natural log function correctly ✓ uses the chain rule correctly

(b)□ $x=e^{sint}$ and $y=e^{cost}$ simplifying in terms of t (3marks)

Solution
$x = e^{\sin t}$ $\frac{dx}{dt} = \cos t \times e^{\sin t}$ $y = e^{\cos t}$ $\frac{dy}{dt} = -\sin t \times e^{\cos t}$ $\frac{dy}{dx} = \frac{-\sin t \times e^{\cos t}}{\cos t \times e^{\sin t}} = -\tan t \times e^{\cos t - \sin t}$
Specific behaviours
✓ determines the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ correctly ✓ forms the derivative for $\frac{dy}{dx}$ ✓ fully simplifies $\frac{dy}{dx}$ in terms of t

Question 2 **(7 marks)**

Evaluate exactly:

(a) $\int_0^1 \frac{1-x}{x+1}\,dx$ ¿) (4 marks)

(b) $\int_0^{\frac{1}{4}} \cos^2(\pi x)\,dx$ (3 marks)

Solution
$\int_0^{\frac{1}{4}} \cos^2(\pi x)\,dx = \int_0^{\frac{1}{4}} \frac{(1+\cos(2\pi x))}{2}\,dx$
$= \left[\frac{x}{2}+\frac{\sin(2\pi x)}{4\pi}\right]_0^{\frac{1}{4}}$
$= \left(\frac{1}{8}+\frac{\sin\left(\frac{\pi}{2}\right)}{4\pi}\right)-\left(0+\frac{\sin(0)}{4\pi}\right)$
$= \frac{1}{8}+\frac{1}{4\pi}$ or $\frac{\pi+2}{8\pi}$
Specific behaviours
✓ re-writes the integrand correctly using a double angle identity ✓ determines the correct anti-derivative ✓ evaluates the integral correctly in terms of π

Question 3 **(4 marks)**

Use the substitution $x = 2\left(1 + \cos^2\theta\right)$ to show that $\int_2^3 \sqrt{\frac{x-2}{4-x}}\,dx = \frac{\pi}{2} - 1.$

Solution
$x = 2\left(1+\cos^2\theta\right)$ $\frac{dx}{d\theta} = -4\cos\theta\sin\theta \Rightarrow dx = -4\cos\theta\sin\theta\ d\theta$ $x = 2,\quad \theta = \frac{\pi}{2}$ $x = 3,\quad \theta = \frac{\pi}{4}$ $\int_2^3 \sqrt{\frac{x-2}{4-x}}\,dx = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \sqrt{\frac{2\cos^2\theta}{2\left(1-\cos^2\theta\right)}}\cdot\frac{\left(-4\sin\theta\cos\theta\right)}{1}\ d\theta$ $= -4\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \cos^2\theta\ d\theta$ $= -4\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{\cos 2\theta + 1}{2}\ d\theta$ $= -2\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \cos 2\theta + 1\ \ d\theta$ $= -2\left(\frac{\sin 2\theta}{2} + \theta\right)_{\frac{\pi}{2}}^{\frac{\pi}{4}}$ $= \frac{\pi}{2} - 1$
Specific behaviours
✓expresses in terms of θ ✓ correct substitution ✓ correct values of θ ✓ integrates and evaluates correctly

Question 4 **(3 marks)**

(a) Find the equation of the tangent to the curve $x^3 - 4xy + y^3 = 1$ at the point (1, -2).
(3 marks)

$$3x^2 - 4y - 4x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$x = 1, y = -2$$

$$3 + 8 + (-4 + 12)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{11}{8}$$

$$y + 2 = -\frac{11}{8}(x - 1)$$

$$11x + 8y + 5 = 0$$

Question 5 **(4 marks)**

(a) Using partial fractions, or otherwise, determine $\int \frac{x-19}{(x+1)(x-4)}dx$.

Solution
$A(x-4)+B(x+1)=x-19 \Rightarrow A+B=1,\ B-4A=-19$ Solving gives $A=4, B=-3$ $$\int \frac{x-19}{(x+1)(x-4)}dx = \int \frac{4}{x+1} - \frac{3}{x-4}dx$$ $$= 4\ln\lvert x+1\rvert - 3\ln\lvert x-4\rvert + c$$
Specific behaviours
✓ writes equations for A and B ✓ determines A and B ✓ integrates both fractions correctly ✓ includes constant of integration

Calc assumed

Question 6 **(7 marks)**

The graph of $y=f(x)$ is shown below, where $f(x)=4\cos^4(2x)$ and A is the smallest root of $f(x), x>0$
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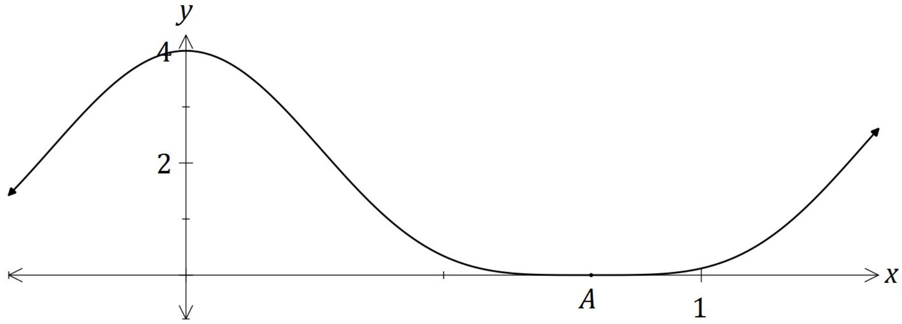


(a) Show that $4\cos^4(2x)=\frac{3+4\cos(4x)+\cos(8x)}{2}$. (3 marks)

Solution
$4\cos^4(2x)=4$¿¿¿¿$4\left(\frac{1+\cos 4x}{2}\right)^2$ ¿ $1+2\cos 4x+\cos^2 4x$ ¿ $1+2\cos 4x+\frac{1+\cos 8x}{2}$ ¿ $\frac{3+4\cos(4x)+\cos(8x)}{2}$
Specific behaviours
✓ uses double angle identity ✓ expands and uses identity again

(b) Hence determine $\int 4\cos^4(2x)\,dx$. (2 marks)

Solution
$\int \frac{3+4\cos(4x)+\cos(8x)}{2}dx=\frac{3x}{2}+\frac{1}{2}\sin 4x+\frac{1}{16}\sin 8x+c$
Specific behaviours
✓ uses result from (a) to integrate ✓ obtains correct result, including constant

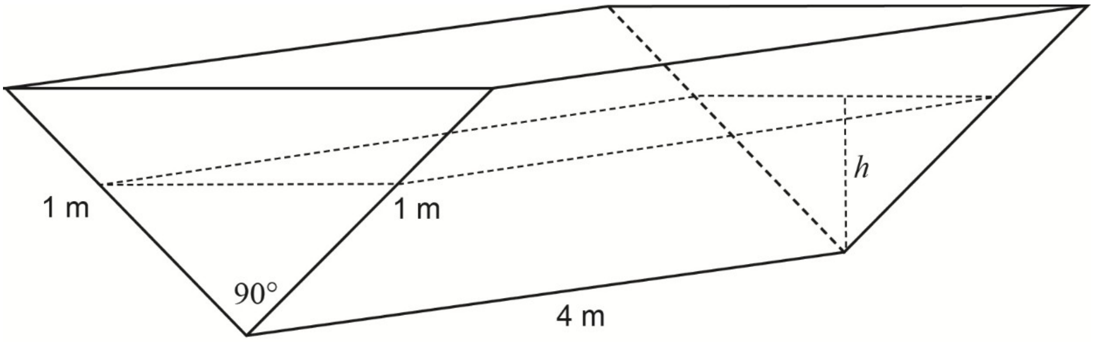
(c) Determine the exact volume of the solid generated when the region bounded by ¿ $f(x)$, $y=0$, $x=0$ and $x=A$ is rotated through 360° about the x-axis.

(2 marks)

Solution
$\int_0^{\frac{\pi}{4}} \pi\left(4\cos^4(2x)\right)^2 dx = \frac{35\pi^2}{32}$ cubic units
Specific behaviours

Question 7 **(5 marks)**

A four-metre-long water tank, open at the top, is in the shape of a triangular prism. The triangular face is a right isosceles triangle with congruent sides of one metre length.



Initially the tank is completely full with water, but it develops a leak and loses water at a constant rate of 0.08 cubic metres per hour.

Let h = the depth of water, in metres, in the tank after t hours.

(a) Show that the volume of water in the tank V cubic metres, is given by the expression

$V(h) = 4h^2$. (2 marks)

Solution
$h = x\cos 45^o$ i.e. $x = \sqrt{2}h$ Volume $V = \frac{1}{2}(x^2)(4) = 2x^2 = 2\left(\sqrt{2}h\right)^2$ i.e. $V(h) = 4h^2$ **or** Area of triangle $A = \frac{1}{2}bh$ where $b = 2h$ Volume $V = \frac{1}{2}(2h)(h)(4) = 4h^2$
Specific behaviours
✓ uses an appropriate method to relate dimensions ✓ forms the area of the triangular base correctly

(b) Determine the rate of change of the depth, correct to the nearest 0.01 metres per hour, when the depth is 0.6 metres. (3 marks)

Solution
$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ i.e. $-0.08 = 8h \times \frac{dh}{dt}$ $-0.08 = 8(0.6) \times \frac{dh}{dt}$ $\therefore \frac{dh}{dt} = -0.02$ m/hr (two decimal places) Hence the depth is decreasing at approximately 2 cm per hour when the depth is 0.6 metres.
Specific behaviours
✓ uses the chain rule correctly to relate the volume and depth rates ✓ substitutes the values for $\frac{dV}{dt}$ and h correctly ✓ calculates the depth rate with the correct units (no penalty for incorrect rounding)

Question 8 **(10 marks)**

13. (a) $V = k\left(\frac{1}{2}r\right)^4$

$= \frac{1}{16}kr^4$ (2 marks)

$\therefore$ Halving the radius of the artery decreases the volume of blood flow to $\frac{1}{16}$ of the original.

(b) $\frac{\delta V}{\delta r} \approx \frac{dV}{dr}$

$\therefore \delta V \approx \frac{dV}{dr}\delta r$

$\delta V \approx 4kr^3\delta r$

$-0.1V \approx 4kr^3\delta r$

$\therefore \delta r \approx \frac{-0.1V}{4kr^3}$

$\approx \frac{-0.1kr^4}{4kr^3}$ (5marks)

$\approx -0.025r$

$\therefore$ A 2.5% decrease in the radius of the artery will produce a 10% decrease in blood flow.

(c) $\delta r = -0.5r$

$\delta V \approx 4kr^3\delta r$

$\delta V \approx 4kr^3(-0.5r)$

$\approx -0.5 \times 4kr^4$

$\approx -2 \times kr^4$

$\approx -2V$ (3 marks)

-2V is a 200% decrease in volume which is impossible.