PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Test 3 Year 12

Student name: ________________ Teacher name: ________________

Task type: **Response**

Time allowed for this task: **____40_______ mins**

Number of questions: **_____7______**

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **_44_____ marks**

Task weighting: **_10___%**

Formula sheet provided: **Yes**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (6 marks)

a) Solve the following system of linear equations. (3 marks)

$$\begin{aligned} x+2y-3z&=-28\\ 2x-7y+5z&=76\\ 3x-4y+6z&=71 \end{aligned}$$

Solution
$\begin{bmatrix} 1 & 2 & -3 & -28 \\ 2 & -7 & 5 & 76 \\ 3 & -4 & 6 & 71 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -3 & -28 \\ 0 & 11 & -11 & -132 \\ 0 & 10 & -15 & -155 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -3 & -28 \\ 0 & 11 & -11 & -132 \\ 0 & 0 & 55 & 385 \end{bmatrix}$ $55z=385$ $z=7$ $y=-5$ $x=3$
Specific behaviours
✓ eliminates one variable from two equations   ✓ eliminates two variables   ✓ solves for all variables

b) Determine all possible values of $p\ \&\ q$ for the three scenarios below. (3 marks)

$$\begin{aligned} x+2y-3z&=q\\ 2x-7y+5z&=76\\ 3x-4y+pz&=71 \end{aligned}$$

i) No solutions

ii) One solution

iii) Infinite solutions

Solution
$\begin{bmatrix} 1 & 2 & -3 & q \\ 2 & -7 & 5 & 76 \\ 3 & -4 & p & 71 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -3 & q \\ 0 & 11 & -11 & 2q-76 \\ 0 & 10 & -9-p & 3q-71 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -3 & -14 \\ 0 & 11 & -11 & 2q-76 \\ 0 & 0 & -11+11p & 21-13q \end{bmatrix}$ $i)\ p=1\ \&\ q \neq \frac{21}{13}$ $ii)\ p \neq 1$ $iii)\ p=1\ \&\ q = \frac{21}{13}$
Specific behaviours
✓ derives equation with two variables eliminated ✓ states values for uniqueness ✓ states values for no solution and infinite (follow through)

Q2 (9 marks)

A particle moves with acceleration $a = \begin{pmatrix} t^3 \\ \sqrt{t} \end{pmatrix} m/s^2$ at time t seconds. The initial velocity is $\begin{pmatrix} 3 \\ -2 \end{pmatrix} m/s$ and initial position $\begin{pmatrix} 4 \\ -1 \end{pmatrix} m$.

a) Determine the velocity at time t seconds. (2 marks)

Solution
$a = \begin{pmatrix} t^3 \\ \sqrt{t} \end{pmatrix}$ $v = \begin{pmatrix} \frac{t^4}{4} \\ \frac{2t^{\frac{3}{2}}}{3} \end{pmatrix} + c$ $\begin{pmatrix} 3 \\ -2 \end{pmatrix} = c$ $v = \begin{pmatrix} \frac{t^4}{4} + 3 \\ \frac{2t^{\frac{3}{2}}}{3} - 2 \end{pmatrix} m/s$
Specific behaviours
✓ anti-differentiates ✓ solves for constant

b) Determine the position vector at time $t = 5$ seconds. (2 marks)

Solution

$$v = \begin{pmatrix} \frac{t^4}{4} + 3 \\ \frac{2t^{\frac{3}{2}}}{3} - 2 \end{pmatrix}$$

$$r = \begin{pmatrix} \frac{t^5}{20} + 3t \\ \frac{4t^{\frac{5}{2}}}{15} - 2t \end{pmatrix} + k$$

$$\begin{pmatrix} 4 \\ -1 \end{pmatrix} = k$$

$$r = \begin{pmatrix} \frac{t^5}{20} + 3t + 4 \\ \frac{4t^{\frac{5}{2}}}{15} - 2t - 1 \end{pmatrix}$$

$$r = \begin{pmatrix} \frac{701}{4} \\ \frac{20\sqrt{5}}{3} - 11 \end{pmatrix} or \begin{pmatrix} 175.25 \\ 3.91 \end{pmatrix} m$$

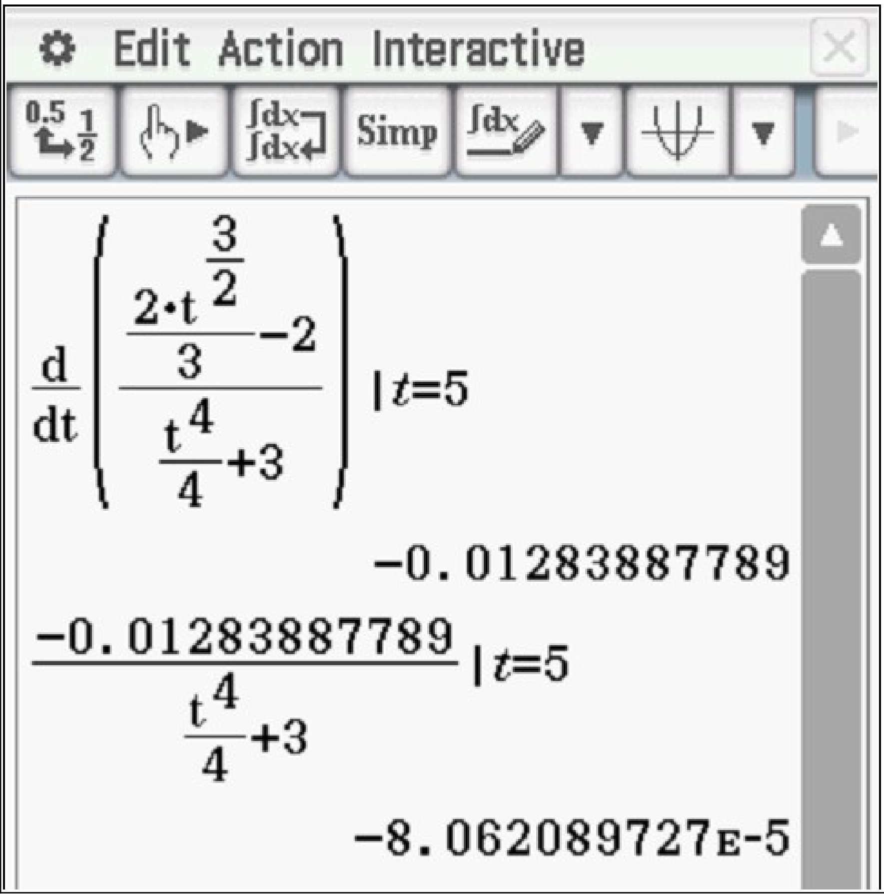
Specific behaviours
✓ determines r ✓ approx. at t=5 (maybe exact or 2 dp)

c) Determine $\frac{dy}{dx}$ on the cartesian path at time $t = 5$ seconds. (2 marks)

Solution
$v = \begin{pmatrix} \frac{t^4}{4} + 3 \\ \frac{2t^{\frac{3}{2}}}{3} - 2 \end{pmatrix} = \begin{pmatrix} 159.25 \\ 5.45 \end{pmatrix}$ $\frac{dy}{dx} \approx \frac{5.45}{159.25} \approx 0.03$

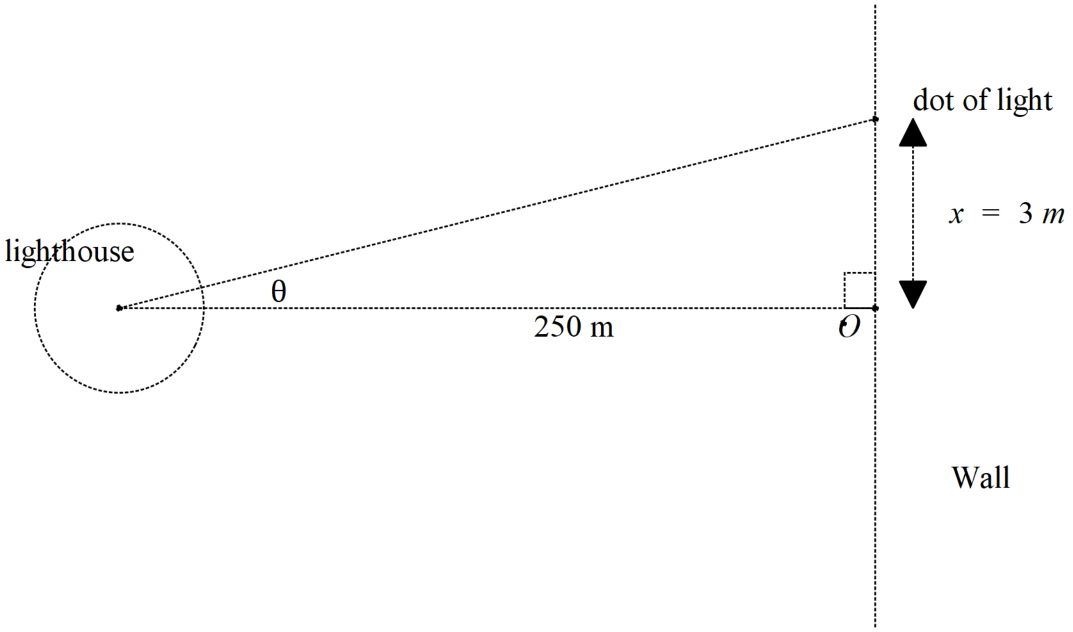
Specific behaviours
✓ uses v at t=5 ✓ determines rate

d) Determine $\frac{d^2y}{dx^2}$ on the cartesian path at time $t = 5$ seconds. (3 marks)

Solution
$v = \begin{pmatrix} \frac{t^4}{4} + 3 \\ \frac{2t^{\frac{3}{2}}}{3} - 2 \end{pmatrix}$ $\frac{dy}{dx} = \frac{\frac{2t^{\frac{3}{2}}}{3} - 2}{\frac{t^4}{4} + 3}$ 

Rate =0.00
Specific behaviours
✓ time diff dy/dx ✓ divides by dx/dt ✓ determines approx. rate (do not penalise if not 2dp)

Q3 (7 marks)
Consider an artificial island that contains a revolving light that is 250 metres from shore. There is a long wall on the shore and the light from the lighthouse can be seen as a moving dot of light on the wall. The angular speed of the light is 24 radians/second, ($\frac{d\theta}{dt}=24$).



a) Determine the speed of the dot of light on the wall when the dot is 3 metres away from the closest point to the lighthouse, pt O, see diagram above. (4 marks)

Solution
$\tan\theta = \frac{x}{250}$ $250\sec^2\theta\frac{d\theta}{dt} = \frac{dx}{dt}$ $\cos\theta = \frac{250}{\sqrt{250^2+3^2}}$

$$\frac{250(250^\wedge2+3^\wedge2)}{250^\wedge2}24$$ $$\frac{750108}{125}$$ $$\frac{750108}{125}$$ $$6000.864$$
Specific behaviours
✓ states relationship between x and angle ✓ implicit diff wrt time (or related rates(✓ subs values and uses derivative of angle ✓ determines approx. speed (no need for units)

b) If the artificial island containing the lighthouse is moving towards the shore, pt O, at a speed of 5 metres per second, determine the speed of the dot when 3 metres away from pt O and the lighthouse being 170 metres from the shore, pt O. (3 marks)

Solution
$\tan\theta = \frac{x}{L} \quad , \dot{L} = -5m/s$ $x = L\tan\theta$ $\dot{x} = L\sec^2\theta(\dot{\theta}) + \dot{L}\tan\theta$ $\tan\theta = \frac{3}{170} \quad \cos\theta = \frac{170}{\sqrt{170^2+3^2}}$ $$170\frac{170^\wedge2+3^\wedge2}{170^\wedge2}24-5\frac{3}{170}$$ $$4081.182353$$
Specific behaviours
✓ uses three variables ✓ uses time implicit with product or quotient and all time rates with correct signs ✓ determines speed

Q4 (3 marks)

Show using logarithmic differentiation how to differentiate $y = x^{\sin(2x)}$.

Solution
$\ln y = \ln x^{\sin(2x)} = \sin 2x \ln x$ $\frac{1}{y} y' = \sin 2x \frac{1}{x} + 2\cos 2x \ln x$ $y' = \left(\sin 2x \frac{1}{x} + 2\cos 2x \ln x \right) y$
Specific behaviours
✓ tales natural log of both sides ✓ implicit diff of both sides and uses product rule ✓ expresses in terms of x&y

Q5 (8 marks)
Show how to evaluate the following without any use of the classpad. Show all working.

a) $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx$ (4 marks)

Solution
$\int_0^{\frac{\pi}{2}} \sin x \sin^2 x \, dx$ $\int_0^{\frac{\pi}{2}} \sin x (1 - \cos^2 x) \, dx$ $\int_0^{\frac{\pi}{2}} \sin x - \sin x \cos^2 x \, dx$ $\left[-\cos x + \frac{1}{3}\cos^3 x \right]_0^{\frac{\pi}{2}} = (0) - \left(-1 + \frac{1}{3} \right) = \frac{2}{3}$
Specific behaviours

✓ uses Pythagorean identity ✓ breaks into two terms with sinx ✓ anti- diffs both terms ✓ subs both limits to give final result

Q5 cont-

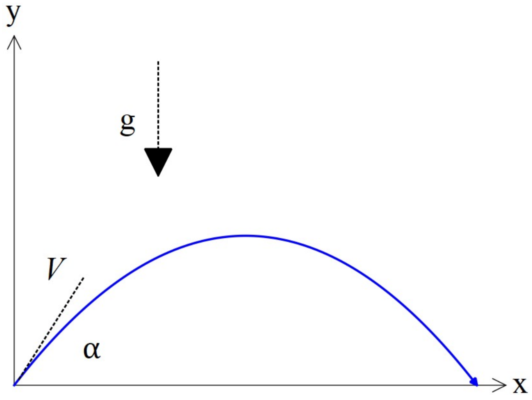
b) $\int \frac{2x+1}{(x-3)(x+5)}dx$ (4 marks)

Solution
$\frac{2x+1}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$ $2x+1 = A(x+5) + B(x-3)$ $x = 3$ $7 = 8A \quad A = \frac{7}{8}$ $x = -5$ $-9 = -8B \quad B = \frac{9}{8}$ $\int \frac{2x+1}{(x-3)(x+5)}dx = \frac{7}{8}\ln\lvert x-3\rvert + \frac{9}{8}\ln\lvert x+5\rvert + c$
Specific behaviours
✓ uses partial fractions ✓ solves for constants ✓ integrates using logs ✓ states answer with a plus constant

Q6 (7 marks).

Consider a projectile that leaves with speed $V\ m/s$ at an angle α to the horizontal, see diagram.

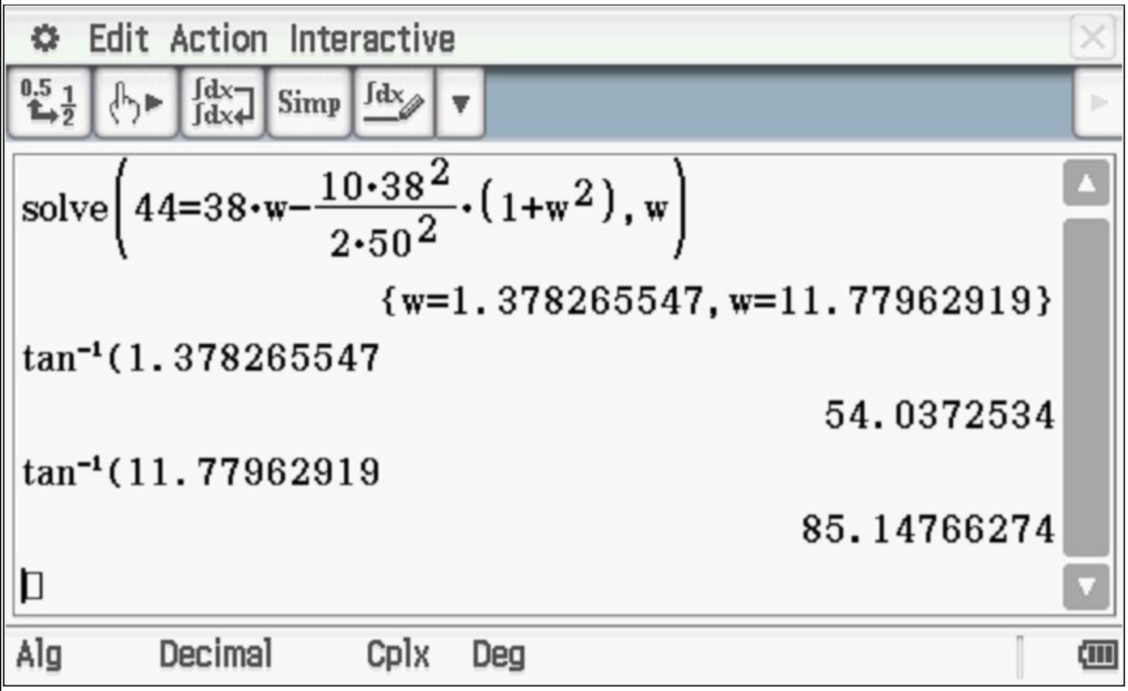
Assume that the constant acceleration is $-g\ m/s^2$.



a) Using vector calculus and starting with the acceleration, show how to derive the cartesian equation of the path in terms of $V, g \,\&\, \alpha$. (4 marks)

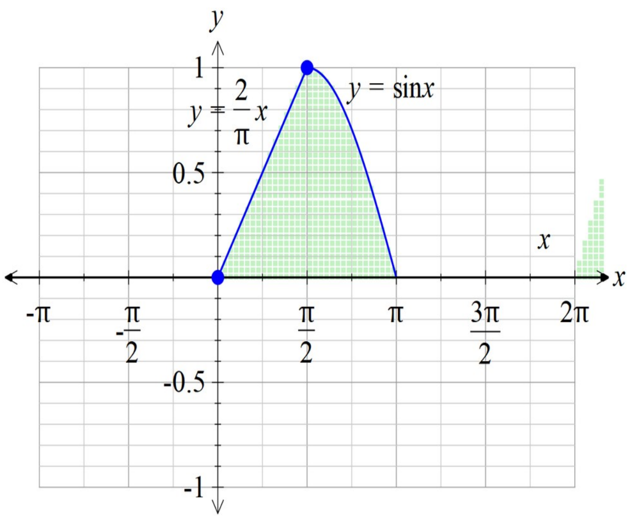
Solution
$\ddot{r} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$ $\dot{r} = \begin{pmatrix} 0 \\ -gt \end{pmatrix} + c$ $\begin{pmatrix} V\cos\alpha \\ V\sin\alpha \end{pmatrix} = c$ $\dot{r} = \begin{pmatrix} V\cos\alpha \\ V\sin\alpha - gt \end{pmatrix}$ $r = \begin{pmatrix} Vt\cos\alpha \\ Vt\sin\alpha - \frac{gt^2}{2} \end{pmatrix} + c$ $c = 0$ $t = \frac{x}{V\cos\alpha}$ $y = \frac{xV\sin\alpha}{V\cos\alpha} - \frac{gx^2}{2V^2\cos^2\alpha} = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$
Specific behaviours
✓ integrates to find velocity and solves for constant   ✓ integrates to find r   ✓ subs x expression into y by eliminating t   ✓ obtains cartesian expression in terms of constants

b) Given that $V = 50\,m/s$ and that $y = 44\,m$ when $x = 38\,m$, determine possible value(s) for α.
(3 marks)

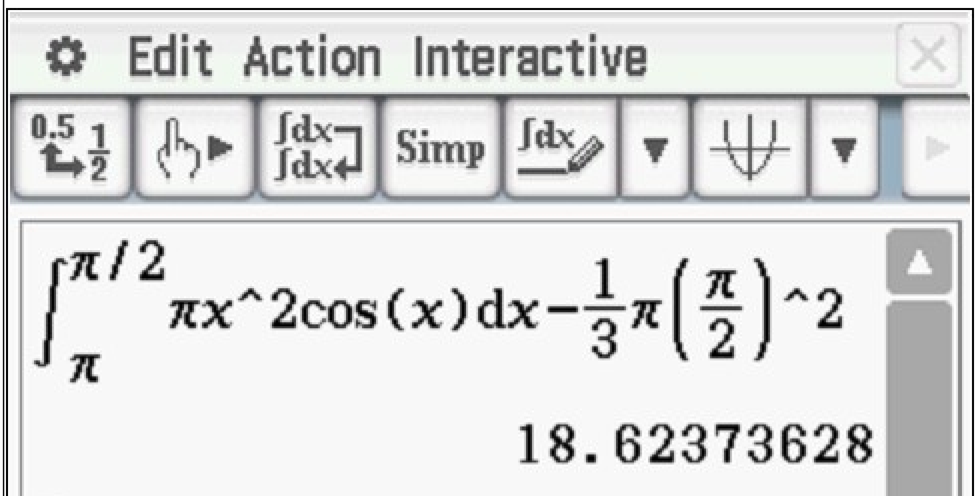
Solution
$y = x\tan\alpha - \frac{gx^2}{2V^2}\left(1+\tan^2\alpha\right)$ $44 = 38\tan\alpha - \frac{10\left(38^2\right)}{2\left(50^2\right)}\left(1+\tan^2\alpha\right)$ 
Specific behaviours
✓ subs all knowns into cartesian equation ✓ solves for one angle ✓ solves for two angles

Q7 (4 marks)

Consider the area between $y = \sin x$, $y = \frac{2}{\pi}x$ and the x axis with $0 \le x \le \pi$, as shown below.



If the shaded area above is revolved around the y axis, determine the volume of the 3D object created.

Solution
$\int_0^1 \pi x^2 dy = \int_{y=0}^{y=1} \pi x^2 \frac{dy}{dx} dx \quad , y = \sin x$ $\int_{\pi}^{\frac{\pi}{2}} \pi x^2 \cos x\, dx - \frac{1}{3}\pi\left(\frac{\pi}{2}\right)^2$    (Note- use of inverse sine function without a translation is incorrect as sine x is a many to one function over 0 to pi domain)
Specific behaviours
✓ uses correct integral with appropriate limits for area under sin x   ✓ uses change of variable with correct order of limits   ✓ determines volume of cone

✓ obtains correct volume -must be numeric (no need to round) (Max 1 out of 4 if revolved around x axis- too easy)

r