Perth Modern School

Semester One Examination, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 1
Section One: Calculator-free

SOLUTIONS

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section
Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	49	35
Section Two: Calculator-assumed	13	13	100	102	65
			Total	151	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section One: Calculator-free **35% (49 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(7 marks)**

(a) Evaluate

(i) $\dfrac{8!}{2!3!4!}$. (2 marks)

Solution
$\dfrac{8\times7\times6\times5}{2\times6}=140$
Specific behaviours
✓ expands and cancels ✓ simplifies

(ii) $\dfrac{{}^{20}P_6}{{}^{21}C_{14}}$. (3 marks)

Solution
$\dfrac{20!}{14!}\div\dfrac{21!}{14!7!}=\dfrac{\cancel{20!}}{\cancel{14!}}\times\dfrac{\cancel{14!}\times7!}{21\times\cancel{20!}}$ $=\dfrac{\cancel{7}\times6\times5\times4\times\cancel{3}\times2}{\cancel{21}}=240$
Specific behaviours
✓ uses P and C notation correctly ✓ cancels like factorials ✓ simplifies answer

(b) Determine the values of *a* and *b* given $8!+9!+10!=a\times b!$ (2 marks)

Solution
$8!+9!+10!=8!(1+9+10\times9)$ $=100\times8!\ \Rightarrow\ a=100,\ b=8$
Specific behaviours
✓ factors out 8! to obtain *a* ✓ simplifies rest of expression to obtain *b*

Question 2 **(8 marks)**

Given $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$, determine

(a) $5\mathbf{a} + 10\mathbf{b}$. (1 mark)

Solution
$10\mathbf{i} - 25\mathbf{j} + 10\mathbf{i} + 10\mathbf{j} = 20\mathbf{i} - 15\mathbf{j}$
Specific behaviours
✓ determines vector

(b) $4(\mathbf{b} - 2\mathbf{a})$. (1 mark)

Solution
$4(\mathbf{i} + \mathbf{j} - 4\mathbf{i} + 10\mathbf{j}) = -12\mathbf{i} + 44\mathbf{j}$
Specific behaviours
✓ determines vector

(c) $\lvert\mathbf{a} + 6\mathbf{b}\rvert$. (2 marks)

Solution
$2\mathbf{i} - 5\mathbf{j} + 6\mathbf{i} + 6\mathbf{j} = 8\mathbf{i} + \mathbf{j}$ $\lvert 8\mathbf{i} + \mathbf{j}\rvert = \sqrt{65}$
Specific behaviours
✓ determines vector ✓ determine magnitude

(d) a unit vector in the same direction as $\mathbf{a} + \mathbf{b}$. (2 marks)

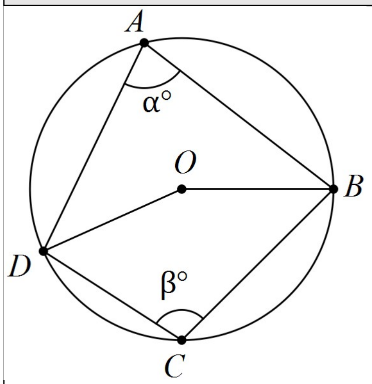
Solution
$\mathbf{u} = 2\mathbf{i} - 5\mathbf{j} + \mathbf{i} + \mathbf{j} = 3\mathbf{i} - 4\mathbf{j}$ $\hat{\mathbf{u}} = \frac{1}{5}(3\mathbf{i} - 4\mathbf{j}) = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$
Specific behaviours
✓ determines magnitude of vector ✓ determines unit vector

(e) the scalar projection of **b** onto $\mathbf{a} + \mathbf{b}$. (2 marks)

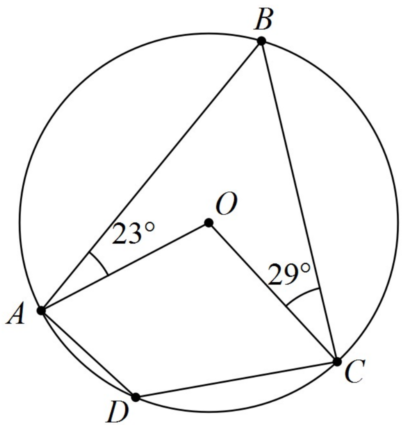
Solution
$\frac{\mathbf{b} \cdot (\mathbf{a} + \mathbf{b})}{\lvert\mathbf{a} + \mathbf{b}\rvert} = (\mathbf{i} + \mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)$ $= \frac{3}{5} - \frac{4}{5} = -\frac{1}{5}$
Specific behaviours
✓ uses scalar product ✓ determines magnitude

Question 3 **(7 marks)**

(a) Prove that the opposite angles of a cyclic quadrilateral are supplementary. (4 marks)

Solution

$\angle DOB = 2\alpha$ (angle on arc DCB at centre is twice angle on circumference) $\angle DOB = 2\beta$ (angle on arc DAB at centre is twice angle on circumference) $2\alpha + 2\beta = 360$ (angle sum of circle) $\alpha + \beta = 180$ - opposite angles are supplementary
Specific behaviours
✓ labelled diagram ✓ uses angle at centre twice circumference ✓ uses angle sum at centre ✓ completes proof

(b) Determine, with reasons, the size of $\angle ADC$ in the diagram below. (3 marks)



Solution
$\angle OBA = 23^\circ$ (isosceles triangle) $\angle OBC = 29^\circ$ (isosceles triangle) $\angle ABC = \angle OBA + \angle OBC$ $= 23 + 29 = 52^\circ$ $\angle ADC = 180 - 52$ (opp angles in cyc quad) $= 128^\circ$
Specific behaviours
✓ uses isosceles triangles ✓ determines $\angle ABC$ ✓ determines $\angle ADC$ with reason

Question 4 **(7 marks)**

Consider the vectors $\mathbf{a} = 21\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} - 3\mathbf{j}$ and $\mathbf{c} = 18\mathbf{i} + 24\mathbf{j}$.

(a) Determine the vector projection of $\mathbf{a}$ onto $\mathbf{b}$. (3 marks)

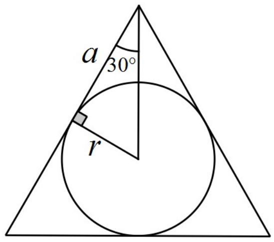
Solution
$\frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{b}\rvert}\times\hat{\mathbf{b}} = \frac{21(4)+3(-3)}{5}\times\frac{1}{5}\begin{bmatrix}4\\-3\end{bmatrix}$ $= 3\begin{bmatrix}4\\-3\end{bmatrix} = \begin{bmatrix}12\\-9\end{bmatrix}$
Specific behaviours
✓ calculates scalar product   ✓ calculates magnitude of b   ✓ determines vector

(b) Express $\mathbf{c}$ in the form $x\mathbf{a} + y\mathbf{b}$. (4 marks)

Solution
$\begin{bmatrix}18\\24\end{bmatrix} = x\begin{bmatrix}21\\3\end{bmatrix} + y\begin{bmatrix}4\\-3\end{bmatrix}$ $21x + 4y = 18$ $3x - 3y = 24$ $63x + 12y = 54$ $12x - 12y = 96$ $75x = 150$ $x = 2$ $y = \frac{6-24}{3} = -6$ $\mathbf{c} = 2\mathbf{a} - 6\mathbf{b}$
Specific behaviours
✓ writes simultaneous equations   ✓ eliminates one variable and solves   ✓ solves for other variable   ✓ writes in required form

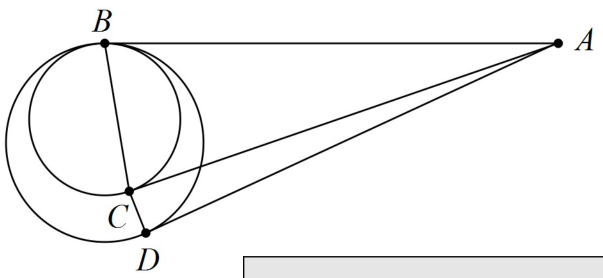
Question 5 **(8 marks)**

(a) An equilateral triangle of side $2a$ circumscribes a circle, as shown in the diagram below. Express the exact radius of the circle in terms of a. (4 marks)



Solution
$r = a\tan 30$ $r = \dfrac{a\sqrt{3}}{3}$
Specific behaviours
✓ draws right triangle using tangent and radius ✓ places angle and variables on diagram ✓ determines relationship between *r* and *a* ✓ expresses *r* in terms of *a*

(b) Two circles touch internally at B, as shown below. AB, AC and AD are tangents, $\angle ABC = 76^\circ$ and $\angle BAD = 38^\circ$. Determine, with reasons, the size of $\angle CDA$. (4 marks)



Solution
As *B* is common to both circles, then as tangents from external point, $AB = AC$ and $AB = AD \Rightarrow AC = AD$. Hence triangles *ABC* and *ACD* are both isosceles. $\angle BAC = 180 - 76 - 76 = 28^\circ$ (isoceles) $\angle DAC = 38 - 28 = 10^\circ$ $\angle CDA = (180 - 10) \div 2 = 85^\circ$ (isoceles)
Specific behaviours
✓ explains why $AB = AC = AD$ ✓ determines $\angle BAC$ ✓ determines $\angle DAC$

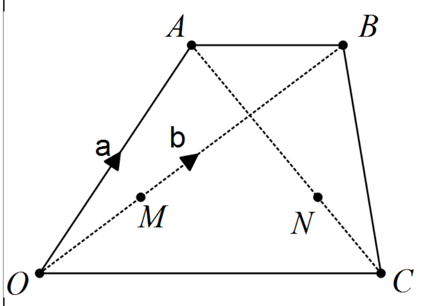
✓ determines

Question 6 **(6 marks)**

OABC is a trapezium with $\mathbf{OC} = 2\mathbf{AB}$. *M* lies on the diagonal *OB* so that $\mathbf{OM} = \frac{1}{3}\mathbf{OB}$ and *N* lies on the diagonal *CA* so that $\mathbf{CN} = \frac{1}{3}\mathbf{CA}$. Let $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$.

By determining a vector for **MN**, or otherwise, prove that *ABNM* is a parallelogram.

Solution



$$\begin{aligned}
\mathbf{OM} &= \tfrac{1}{3}\mathbf{b} \\
\mathbf{ON} &= \mathbf{OC} + \tfrac{1}{3}\mathbf{CA} \\
&= 2\mathbf{AB} + \tfrac{1}{3}(\mathbf{CO} + \mathbf{OA}) \\
&= 2\mathbf{b} - 2\mathbf{a} + \tfrac{1}{3}(2\mathbf{a} - 2\mathbf{b} + \mathbf{a}) \\
&= \tfrac{4}{3}\mathbf{b} - \mathbf{a} \\
\mathbf{MN} &= \mathbf{ON} - \mathbf{OM} \\
&= \tfrac{4}{3}\mathbf{b} - \mathbf{a} - \tfrac{1}{3}\mathbf{b} \\
&= \mathbf{b} - \mathbf{a} \\
&= \mathbf{AB}
\end{aligned}$$

Hence ABNM is a parallelogram, as it has a pair of opposite sides that are parallel and congruent.

Specific behaviours

✓ draws diagram
✓ determines **OM** in terms of **b**
✓ determines **ON** in terms of other vectors
✓ determines **ON** in terms of **a** and **b**
✓ determines **MN** in terms of **a** and **b**
✓ shows **MN=AB** and concludes proof with reasons

Question 7 **(6 marks)**

(a) Show that $\frac{x}{x+1} < \frac{x+1}{x+2}$ when $x = 1.5$ but not when $x = -1.5$. (2 marks)

Solution
$x = 1.5: \ \frac{1.5}{2.5} < \frac{2.5}{3.5} \ \Rightarrow \ \frac{3}{5} - \frac{5}{7} < 0 \ \Rightarrow \ \frac{-4}{35} < 0 \ \Rightarrow$ True $x = -1.5: \ \frac{-1.5}{-0.5} < \frac{-0.5}{0.5} \ \Rightarrow \ 3 < -1 \ \Rightarrow$ False
Specific behaviours
✓ clearly demonstrates first case is true ✓ demonstrates second case is false

(b) Prove by contradiction that, for every positive real number x, $\frac{x}{x+1} < \frac{x+1}{x+2}$. (5 marks)

Solution
Assume there exists a positive real number x such that $\frac{x}{x+1} \geq \frac{x+1}{x+2}$. Since $x > 0$, then $x+1 > 0$ and $x+2 > 0$, and so inequality can be multiplied by $(x+1)(x+2)$ without reversing inequality direction. Hence $x(x+2) \geq (x+1)^2$ $x^2 + 2x \geq x^2 + 1 + 2x$ $0 \geq 1$ This results in the contradiction that $0 \geq 1$ and so we must conclude that no such positive real number x exists so that $\frac{x}{x+1} \geq \frac{x+1}{x+2}$, hence proving that $\frac{x}{x+1} < \frac{x+1}{x+2}$.
Specific behaviours
✓ writes contra of proof ✓ cross multiplies ✓ notes no need to reverse inequality ✓ simplifies inequality

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________