Student Name: ____

Methodist Ladies' College Semester 2, 2010

3CD MATHEMATICS: SPECIALIST

Question/Answer Booklet – Section 2 – Calculator-assumed

Teacher's Name: __________**SOLUTIONS**________________________________

Time allowed for this paper

Section	Reading	Working
~~**Calculator-free**~~	~~5 minutes~~	~~50 minutes~~
Calculator-assumed	10 minutes	100 minutes

Materials required/recommended for this paper

Section One (Calculator-assumed): 80 marks

To be provided by the supervisor
Section Two Question/Answer booklet Formula sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this course.

Important Note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

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Instructions to candidates

1. **All** questions should be attempted.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare answer pages may be found at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued (i.e. give the page number).

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

Section Two: Calculator-assumed **(80 Marks)**

This section has **eleven (11)** questions. Answer **all** questions. Write your answers in the space provided.

Suggested working time for this section is 100 minutes.

Question 8 **(6 marks)**

(a) Express in polar form the cube roots of $\frac{-\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$. [3]

Solution

Let $z^3=\frac{-\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$

$$¿\,cis\left(\frac{-3\pi}{4}+2k\pi\right)$$

$$z=cis\left(\frac{-3\pi}{12}+\frac{2k\pi}{3}\right)$$

$$z_0=cis\left(\frac{-\pi}{4}\right)$$

$$z_1=cis\left(\frac{5\pi}{12}\right)$$

$$z_2=cis\left(\frac{13\pi}{12}\right)$$

Specific behaviours

✓ express $\frac{-\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$ in polar form

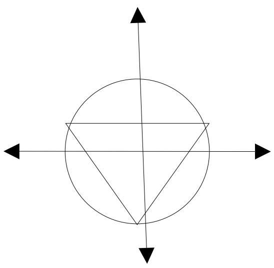
□ apply De Moivre's theorem when taking cube root

□ 3 cube roots in polar form

(b) An equilateral triangle is to have its vertices on an Argand diagram on a circle, centre (0,0), radius 2. One vertex represents a complex number with argument $\frac{\pi}{6}$.

Find, in polar form, the numbers represented by the vertices, and the Cartesian equation which has these three numbers for roots. [3]

Solution



Vertices are: $2\,cis\left(\frac{\pi}{6}\right), 2\,cis\left(\frac{5\pi}{6}\right), 2\,cis\left(\frac{3\pi}{2}\right)$

Equation is $z^3=\left(2\,cis\left(\frac{\pi}{6}\right)\right)^3$

i.e. $z^3=8i$

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Specific behaviours
✓ Correct vertices ✓ Equation in polar form ✓ Equation in Cartesian form

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Question 9 **(8 marks)**

(a) Let $A=\begin{bmatrix}-4 & 20 & 2\\ -3 & 15 & -3\\ 7 & -17 & 1\end{bmatrix}$ and $B=\begin{bmatrix}2 & 3 & 5\\ 1 & 1 & 1\\ 3 & -4 & 0\end{bmatrix}$.

Determine $C=A\times B$. [1]

Solution
$$C=\begin{bmatrix}18 & 0 & 0\\ 0 & 18 & 0\\ 0 & 0 & 18\end{bmatrix}$$
Specific behaviours
✓ Correct calculation

(b) Tickets to a concert cost \$2 for children, \$3 for teenagers and \$5 for adults. 570 people attended the concert and the total ticket receipts were \$1950. The ratio of teenagers to children attending was 3 to 4.

Use your answer to (a) to determine how many children, teenagers and adults attended the concert. [7]

Solution

x: number of children that attended the concert
y: number of teenagers that attended the concert
z: number of adults that attended the concert

$$2x+3y+5z=1950x+y+z=5703x-4y=0$$

In matrix form,

$$\begin{bmatrix}2 & 3 & 5\\ 1 & 1 & 1\\ 3 & -4 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1950\\ 570\\ 0\end{bmatrix}$$

i.e. **B X** = **D**

∴ **ABX** = **AD**

$$18\mathbf{I}\ \mathbf{X}=\begin{bmatrix}-4 & 20 & 2\\ -3 & 15 & -3\\ 7 & -17 & 1\end{bmatrix}\begin{bmatrix}1950\\ 570\\ 0\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}200\\ 150\\ 220\end{bmatrix}$$

Hence, there were 200 children, 150 teenagers and 220 adults at the concert.

Specific behaviours

✓ define variables
▯ ▯ ✓ determine equations
✓ express in matrix form using matrix **B**

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✓ use matrix **A** to determine matrix **X** ✓ correct conclusion

Question 10 **(9 marks)**

Water flows into and leaks out of a container such that $\frac{dx}{dt}=-k(2x-1)$, where the depth of the water is x metres at time t seconds. At $t=0$, the depth is 0.75 m and is decreasing at a rate of 0.01 ms^{-1}.

(a) Show that $k=0.02$. [2]

Solution
At $t=0$, $x=0.75$, $\frac{dx}{dt}=-0.01$ Hence, $-0.5k=-0.01k=0.02$
Specific behaviours
✓ substitutes for x and dx/dt when $t=0$ correctly deduces that $k=0.02$.

(b) Determine the time at which the depth will be 0.55 m. [7]

Solution
$\int \frac{dx}{2x-1}=\int -0.02\,dt$ $\frac{1}{2}\ln\lvert 2x-1\rvert=-0.02t+c\lvert 2x-1\rvert=e^{-0.04t+2c}$ $2x-1=\pm e^{2c}.e^{-0.04t}$ $t=0, x=0.75, 0.5=\pm e^{2c}2x-1=0.5e^{-0.04t}x=0.5+0.25e^{-0.04t}$ $x=0.55, 0.55=0.5+0.25e^{-0.04t}$using calculator, $t\approx 40.24$ The depth will be 0.55 m after approx 40 seconds.
Specific behaviours
✓ separates variables correctly determines antiderivatives applies log definition ✓ substitutes to calculate constant ✓ correctly determines x as a function of t ✓ substitutes to determine time for given depth ✓ correctly calculates time

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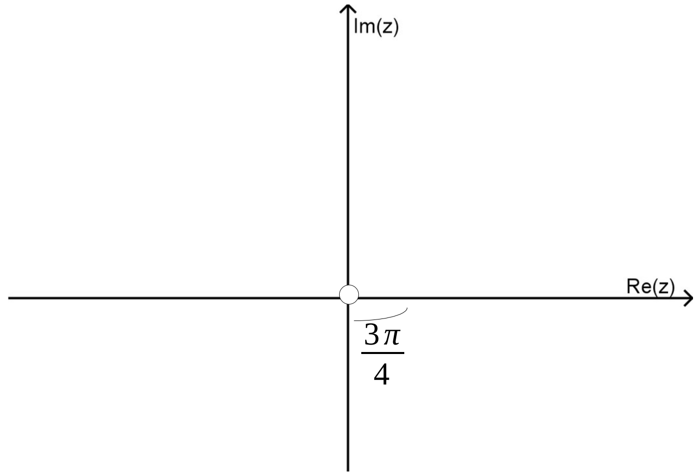
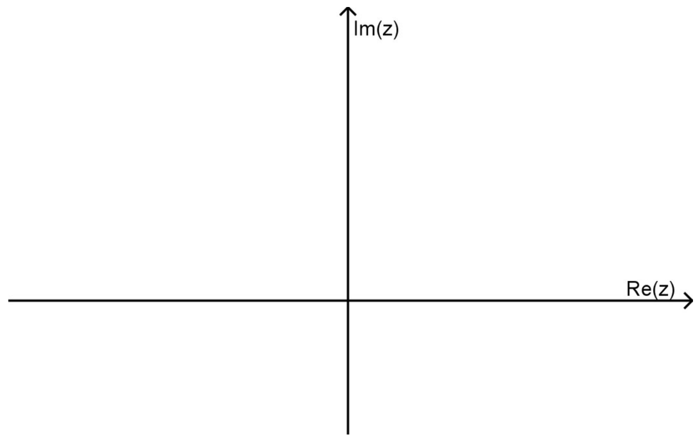
Question 11 **(10 marks)**

(a) Sketch in the complex plane the graph of $\arg\left(\frac{1}{z}\right)=\frac{3\pi}{4}$. [3]

Solution
$\arg\left(\frac{1}{z}\right)=\frac{3\pi}{4}$ $\arg 1 - \arg z=\frac{3\pi}{4}$ $\arg z=0-\frac{3\pi}{4}=\frac{-3\pi}{4}$ 
Specific behaviours
✓ applies property of argument ✓ establishes $\arg z=\frac{-3\pi}{4}$ ✓ draws correct ray (no penalty if open circle at origin missing)

(b) (i) Sketch the locus represented by $\lvert z-2i\rvert\leq 4$. [2]

Solution

Specific behaviours
✓ draws circle centre (0,2) and radius 4

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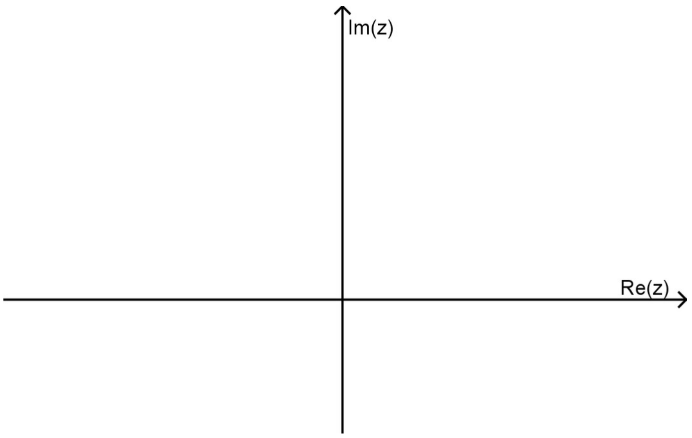
includes inner region of circle

(ii) Hence, find the exact greatest and least values of ¿ $z-3+4i\vee$ ¿ given that $|z-2i|\leq 4$. [3]

Solution



(3,-4)

$\left|2i-(3-4i)\right|=\sqrt{9+36}=\sqrt{45}$

Min value of ¿ $z-3+4i\vee$ ¿ is $\sqrt{45}-4$
Max value of ¿ $z-3+4i\vee$ ¿ is $\sqrt{45}+4$

Specific behaviours
✓ calculates distance between centre and (3,-4)
correctly states min value
correctly states max value

(c) The complex number $x+iy$ is such that $arg(x+iy)=\theta$, where $x>0,\ y>0$. Find, in terms of θ and π, the values of

(i) $arg(-x+iy)$ [1]

Solution
$arg(-x+iy)=\pi-\theta$ because $-x+iy$ is reflection of $x+iy$ in the vertical axis
Specific behaviours
✓ correctly determines argument

(ii) $arg(-y+ix)$ [1]

Solution
$arg(-y+ix)=\frac{\pi}{2}+\theta$ because $-y+ix$ is 90° anticlockwise rotation of $x+iy$ about the origin
Specific behaviours

✓ correctly determines argument

Question 12 **(6 marks)**

The table below shows the details of a population of kangaroos in a region of Western Australia in 2000.

Age (years)	0 – 2	2 – 4	4 – 6	6 - 8	8 - 10
Initial population	1200	1400	1600	810	425
Breeding Rate	0	0.1	3.5	2.5	0.5
Survival Rate	0.4	0.5	0.7	0.2	0

(a) Write down the Leslie matrix, L, for this population. [1]

Solution
$L=\begin{bmatrix} 0 & 0.1 & 3.5 & 2.5 & 0.5 \\ 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 \end{bmatrix}$
Specific behaviours
✓ sets up matrix from information in table

(b) What is the total population in 2006? [1]

Solution
$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} L^3 \begin{bmatrix} 1200 \\ 1400 \\ 1600 \\ 810 \\ 425 \end{bmatrix} = \begin{bmatrix} 6509.2 \end{bmatrix}$ The total population in 2006 is approx 6509 kangaroos.
Specific behaviours
✓ correct total population in 2006

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(c) Find the percentage growth rate between the 3rd and 4th generation. [2]

Solution

$$\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix} L^4 \begin{bmatrix}1200\\1400\\1600\\810\\425\end{bmatrix} = \begin{bmatrix}9493.1\end{bmatrix}$$

Percentage growth rate = $\dfrac{9493-6509}{6509}\times 100\,\% = 45.8\,\%$

Specific behaviours

✓ correct total population in 2008
✓ correct percentage growth rate

Due to complaints from local property owners, it is decided that culling is necessary.

(d) To reduce the population growth, 10 % of kangaroos aged between 4 and 8 years are culled at the start of every second year. What will the population be in 2016? [2]

Solution

$$L=\begin{bmatrix}0 & 0.1 & 0.9\times3.5 & 0.9\times2.5 & 0.5\\0.4 & 0 & 0 & 0 & 0\\0 & 0.5 & 0 & 0 & 0\\0 & 0 & 0.9\times0.7 & 0 & 0\\0 & 0 & 0 & 0.9\times0.2 & 0\end{bmatrix}=\begin{bmatrix}0 & 0.1 & 3.15 & 2.25 & 0.5\\0.4 & 0 & 0 & 0 & 0\\0 & 0.5 & 0 & 0 & 0\\0 & 0 & 0.63 & 0 & 0\\0 & 0 & 0 & 0.18 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix} L^8 \begin{bmatrix}1200\\1400\\1600\\810\\425\end{bmatrix} = \begin{bmatrix}7730.6\end{bmatrix}$$

The total population in 2016 would be approx 7730 kangaroos.

Specific behaviours

✓ Correctly states new Leslie matrix
✓ Correct total population for 2016

Question 13 **(7 marks)**

The lines l_1 and l_2 have equations

$$r=\begin{pmatrix}3\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\4\end{pmatrix} \text{ and } r=\begin{pmatrix}1\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

respectively, where λ and μ are parameters.

(a) Find the acute angle between l_2 and the line joining the points P$(1,-1,1)$ and Q $(2,-1,-4)$, giving your answer correct to the nearest degree. [2]

Solution
$\overrightarrow{PQ}=\begin{pmatrix}1\\0\\-5\end{pmatrix}$ Using calculator, angle between l_2 and $\overrightarrow{PQ}$ is 56° to the nearest degree.
Specific behaviours
✓ uses vector methods to determine $\overrightarrow{PQ}$ □ correct angle

(b) Determine the position vector of the point R that lies on the line joining P$(1,-1,1)$ and Q$(2,-1,-4)$ such that PR : RQ = 1 : 2. [3]

Solution
$\vec{¿}=\overrightarrow{OP}+\frac{1}{3}\overrightarrow{PQ}$ $¿\begin{pmatrix}1\\-1\\1\end{pmatrix}+\frac{1}{3}\begin{pmatrix}1\\0\\-5\end{pmatrix}¿\frac{1}{3}\begin{pmatrix}4\\-3\\-2\end{pmatrix}$
Specific behaviours
✓ correct proportion of $\overrightarrow{PQ}$ □ vector equation for $\vec{¿}$ □ correct position vector for $\vec{¿}$

(c) Find an equation of the plane Π through Q$(2,-1,-4)$ and perpendicular to l_1, in the form $r\cdot n=\rho$. [2]

Solution
$r\cdot\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}2\\-1\\-4\end{pmatrix}\cdot\begin{pmatrix}1\\2\\4\end{pmatrix}r\cdot\begin{pmatrix}1\\2\\4\end{pmatrix}=-16$
Specific behaviours
✓ uses direction of l_1 as normal   □ correct equation in normal form

Question 14 **(7 marks)**

The path of a particle is defined by the parametric equations

$$x=120t-4t^2,\quad y=60t-6t^2.$$

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The path crosses the $x-¿$axis at $t=a$ and at $t=b$.

(a) Determine the value of a and b. [2]

Solution
$60t-6t^2=0t=0$ or $t=10$ Hence, $a=0$ and $b=10$.
Specific behaviours
✓ solves $y=0$ values for a and b

(b) Determine the value of $\frac{dy}{dx}$ at $t=a$ and at $t=b$ [5]

Solution
$\frac{dx}{dt}=120-8t, \frac{dy}{dt}=60-12t \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}¿\frac{60-12t}{120-8t}$ when $t=0, \frac{dy}{dx}=\frac{1}{2}$ when $t=10, \frac{dy}{dx}=\frac{-3}{2}$
Specific behaviours
✓ differentiates correctly for $\frac{dx}{dt}$ differentiates correctly for $\frac{dy}{dt}$ uses chain rule ✓ finds $\frac{dy}{dx}$ when $t=0$ ✓ finds $\frac{dy}{dx}$ when $t=10$

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Question 15 **(6 marks)**

(a) Use de Moivre's Theorem to show that $\cos n\theta = \frac{1}{2}\left(z^n + z^{-n}\right)$ and

$\sin n\theta = \frac{1}{2i}\left(z^n - z^{-n}\right)$, where $z = \cos\theta + i\sin\theta$. [3]

Solution
$z^n = \cos\theta + i\sin\theta z^{-n} = \cos\theta - i\sin\theta z^n + z^{-n} = 2\cos\theta,$ hence $\cos\theta = \frac{1}{2}\left(z^n + z^{-n}\right)$ $z^n - z^{-n} = 2i\sin\theta,$ hence $\sin\theta = \frac{1}{2i}\left(z^n - z^{-n}\right)$
Specific behaviours
✓ applies de Moivre's theorem to obtain expressions for z^n and z^{-n} □ adds to obtain expression for $\cos n\theta$ □ subtracts to obtain expression for $\sin n\theta$

(b) Use (a) to prove the identity, $\sin 3\theta \cos 2\theta = \frac{1}{2}\left(\sin 5\theta + \sin\theta\right)$ [3]

Solution
$\sin 3\theta \cos 2\theta = \frac{1}{2i}\left(z^3 - z^{-3}\right) \times \frac{1}{2}\left(z^2 + z^{-2}\right) \text{¿} \frac{1}{4i}\left(z^5 - z^{-5} + z - z^{-1}\right) \text{¿} \frac{1}{4i} \text{¿¿} \frac{1}{2}\left(\sin 5\theta + \sin\theta\right)$
Specific behaviours
✓ uses (a) to rewrite $\sin 3\theta \cos 2\theta$ in terms of z □ expands and regroups □ establishes correct conclusion

Question 16 **(5 marks)**

Use mathematical induction to show that

$$n! > 2^n$$

for all positive integers $n \geq 4$.

Note: $n! = 1 \times 2 \times 3 \times \ldots \times n$

Solution

Try $n=4$, LHS = 4! = 24, RHS = $2^4 = 16$
Since 24 > 16, true for $n=4$

Assume true for $n=k$, i.e. $k! > 2^k, k \geq 4$

Try $n=k+1$, LHS = $(k+1)!$, RHS = 2^{k+1}
$¿(k+1) \times k!$
$¿(k+1) \times 2^k$

Since $k \geq 4, k+1 > 4 > 2$
Hence, $(k+1)! > 2 \times 2^k$
i.e. $(k+1)! > 2^{k+1}$

Hence, if true for $n=k$, then it is true for $n=k+1$ and it is true for $n=4$, therefore, by induction, $n! > 2^n$ for all positive integers $n \geq 4$.

Specific behaviours

✓ proves proposition for $n=4$
□ defines proposition for $n=k$
□ assumes proposition for $n=k$ is true
✓ defines proposition for $n=k+1$
✓ each step correct in proof

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Question 17 **(10 marks)**

A rocket ship leaves space station A which is located at $\begin{pmatrix} -20 \\ 40 \\ 20 \end{pmatrix}$ km at 9 am with a constant velocity of $\begin{pmatrix} 60 \\ 120 \\ 360 \end{pmatrix}$ km h^{-1}. It is supposed to reach the neighbouring space station B, which is located at $\begin{pmatrix} 80 \\ 160 \\ 2020 \end{pmatrix}$ km.

(a) Determine whether or not this rocket ship reaches space station B. If not, find the closest distance between the rocket ship and space station B and when this occurs to the nearest minute. [6]

Solution
t hours after 9 am, the location of the rocket ship is given by $r(t) = \begin{pmatrix} -20 \\ 40 \\ 20 \end{pmatrix} + t\begin{pmatrix} 60 \\ 120 \\ 360 \end{pmatrix}$ km If the rocket ship reaches space station B, $\begin{pmatrix} -20 \\ 40 \\ 20 \end{pmatrix} + t\begin{pmatrix} 60 \\ 120 \\ 360 \end{pmatrix} = \begin{pmatrix} 80 \\ 160 \\ 2020 \end{pmatrix}$. Equating components, $-20+60t=80, t=\frac{5}{3}$ and $40+120t=160, t=1$ Contradiction Hence, the rocket ship does not reach space station B. **EITHER**, closest distance occurs when $\left\lvert r(t) - \begin{pmatrix} 80 \\ 160 \\ 2020 \end{pmatrix} \right\rvert$ is a minimum. Using calculator, minimum distance is approximately 557 km when $t \approx 5.02$. This occurs at 2.01 pm. **OR**, closest distance occurs when $\left(r(t) - \begin{pmatrix} 80 \\ 160 \\ 2020 \end{pmatrix} \right) \cdot \begin{pmatrix} 60 \\ 120 \\ 360 \end{pmatrix} = 0$ i.e. when $60(-20+60t-80)+120(40+120t-160)+360(20+360t-2020)=0$ i.e. when $t \approx 5.02$ Minimum distance is approximately 557 km at 2.01 pm.

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Specific behaviours
✓ establish position vector for rocket ship t hours after 9 am ✓ identify relationship that must hold if rocket ship does reach space station B ✓ establish that rocket ship does not reach space station B EITHER ✓ express the distance between the rocket ship and space station B using modulus ✓ obtain minimum distance and corresponding value of t ✓ state time to nearest minute OR ✓ recognise that perpendicular implies closest point ✓ use dot product to solve for t and hence calculate min distance ✓ state time to nearest minute

A second rocket ship is also launched from space station B at 9 am with constant velocity and is aimed to collide with the first rocket at exactly 1 pm.

(b) Determine the velocity of the second rocket ship that will ensure collision takes place at the required time. [4]

Solution
t hours after 9 am, second rocket ship with velocity **v** is located at, $$r(t)=\begin{pmatrix}80\\160\\2020\end{pmatrix}+t\,v$$ For collision to occur at 1 pm, $$\begin{pmatrix}80\\160\\2020\end{pmatrix}+4v=\begin{pmatrix}-20\\40\\20\end{pmatrix}+4\begin{pmatrix}60\\120\\360\end{pmatrix}$$ i.e. $v=\begin{pmatrix}35\\90\\-140\end{pmatrix}$ km h^{-1}
Specific behaviours
✓ establish position vector for second rocket ship t hours after 9 am ✓ identify location of each rocket ships at 1 pm ✓ equate position vectors ✓ solve for **v**

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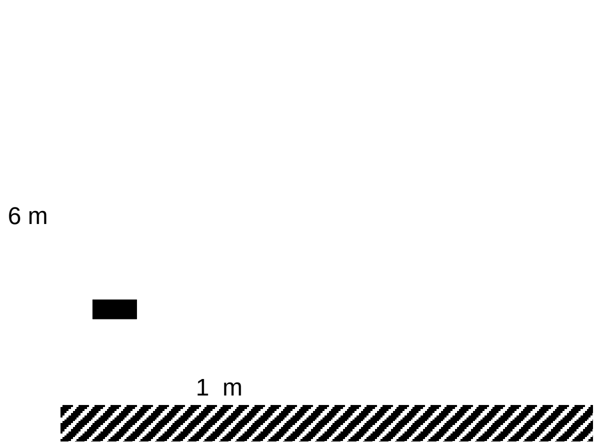
Question 18 **(6 marks)**

A weight W is attached to a rope 16 m long that passes over a pulley at point P, 6 m above the ground. The other end of the rope is attached to a truck at a point A, 1 m above the ground, as shown in the diagram.

(a) Show that $y=\sqrt{25+x^2}-11$ represents the distance in metres the weight is above point B, given x m represents the horizontal distance from point B to the truck. [2]

Solution



$5-y+\sqrt{25+x^2}=16y=\sqrt{25+x^2}-11$

Specific behaviours

✓ Recognises distance from W to P to A is 16 m
 establishes distance from P to A is $\sqrt{25+x^2}$ and hence shows that $y=\sqrt{25+x^2}-11$

(b) If the truck moves away at the rate of 3 ms^{-1}, how fast is the weight rising when it is 2 m above the ground? [4]

Solution

Given $\frac{dx}{dt}=3$ ms^{-1} , $\frac{dy}{dx}=\frac{x}{\sqrt{25+x^2}}$

$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}=\frac{x}{\sqrt{25+x^2}}\times 3$ ms^{-1}

When $y=1,\sqrt{25+x^2}=12,\ \ x=\sqrt{119}$ and $\frac{dy}{dt}=2.73$ ms^{-1}

The weight is rising at the rate of 2.73 ms^{-1}.

Specific behaviours

✓ correctly differentiates to obtain $\frac{dy}{dx}$
 uses chain rule with $\frac{dx}{dt}=3$
 establishes $y=1$ and $x=\sqrt{119}$ when the weight is 2 m above the ground
✓ correct calculation for $\frac{dy}{dt}$

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End of questions