

SOLUTIONS

2016

UNITS 3-4

MATHEMATICS METHODS

REVISION 1

SEMESTER TWO

Papers written by
Australian Maths
Software

SECTION ONE

1. (8 marks)

$$(a) \int (2x+4)^6 dx = \frac{(2x+4)^7}{7 \times 2} + c = \frac{(2x+4)^7}{14} + c$$

$$\begin{aligned} (b) \quad & \int_{\pi/4}^{\pi/2} 2 \sin(x) - \cos(x) dx \\ &= [-\cos(x) - \sin(x)]_{\pi/4}^{\pi/2} \\ &= -\left(\left(\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\right) - \left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right)\right) \\ &= -1 + \sqrt{2} \end{aligned}$$

$$(c) \quad \int \left(x^4 + e^{2x} + \frac{2}{x}\right) dx = \frac{x^5}{5} + \frac{e^{2x}}{2} + 2 \ln(x) + c \quad \checkmark \checkmark \quad -1/\text{error}$$

2. (16 marks)

$$\begin{aligned} (a) \quad (i) \quad f(x) &= \ln\left(\frac{x^2 - 3}{1+x}\right) = \ln(x^2 - 3) - \ln(1+x) \\ f'(x) &= \frac{2x}{(x^2 - 3)} - \frac{1}{(1+x)} \end{aligned}$$

$$\begin{aligned} (ii) \quad g(x) &= \frac{e^{\sin(x)}}{\cos(x)} \\ g'(x) &= \frac{(e^{\sin(x)} \cos(x)) \cos(x) - (-\sin(x)) e^{\sin(x)}}{(\cos(x))^2} \\ g'(x) &= \frac{e^{\sin(x)} (\cos^2(x) + \sin(x))}{\cos^2(x)} \end{aligned}$$

$$(iii) \quad h(x) = e^x \times \ln(x^2) = 2e^x \times \ln(x)$$

$$\begin{aligned} h'(x) &= 2 \left(e^x \times \ln(x) + \frac{e^x}{x} \right) \quad \checkmark \checkmark \\ h'(x) &= 2e^x \left(\ln(x) + \frac{1}{x} \right) \end{aligned}$$

$$\begin{aligned}
 \frac{d\theta}{dr} &= -\frac{\cos(\theta)}{\sin(\theta)} \\
 \frac{d\theta}{dr} &= -\frac{4\cos(\theta)}{2\sin(\theta)} \\
 \frac{d\theta}{dr} &= -\frac{\sqrt{t}}{2\sin(\theta)} \\
 \frac{d\theta}{dr} &= \frac{2}{1-t} \times 4 \times (-\sin(\theta)) \\
 \frac{\theta}{xp} &= \frac{dt}{dp} \times \frac{dx}{dt} = \frac{\theta}{\theta} = 4 \cos(\theta) \quad (\text{p}) \\
 &= 2(4 + 8.7) \\
 &= 2(4 - (6.4 - 2.3)) \\
 &= 25.4 \\
 &= 2 \times 12.7 \\
 &= 25.4
 \end{aligned}$$

(c) Given $\int_0^x f(x) dx = 6.4$ and $\int_0^x f(x) dx = 2.3$.

$$\begin{aligned}
 &\int_0^x \sin(x) dx + C \\
 &= \int_0^x \frac{\sin(x)}{\cos(x)} d\cos(x) \\
 &= \int_0^x \frac{\sin(x)}{\cos(x)} dx \\
 &= \int_0^x \frac{\sin(x)}{\cos(x)} dx
 \end{aligned}$$

(b) (i) Given $y(x) = \sqrt{\sin(x)}$ show that $y'(x) = \frac{\cos(x)}{\sin(x)}$.

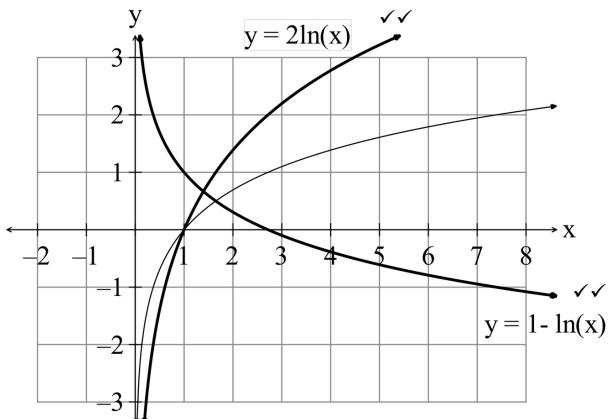
3. (6 marks)

$$\begin{aligned}
 (a) & \frac{\log_{10}(4 \times 3^2) - \log_{10}(3 \times 6) - 3\log_{10}2}{-2\log_{10}2} \\
 &= \frac{\log_{10}\left(\frac{4 \times 3^2}{3 \times 6 \times 8}\right)}{-2\log_{10}2} \\
 &= \frac{\log_{10}\left(\frac{1}{4}\right)}{-2\log_{10}2} \\
 &= \frac{-2\log_{10}2}{-2\log_{10}2} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 (b) & (\log_3(x)-1)(\ln(x)-1)=0 \\
 & \log_3(x)-1=0 \quad \text{or} \quad \ln(x)-1=0 \\
 & \log_3(x)=1 \quad \text{or} \quad \ln(x)=1 \\
 & x=3 \quad \text{or} \quad x=e
 \end{aligned}$$

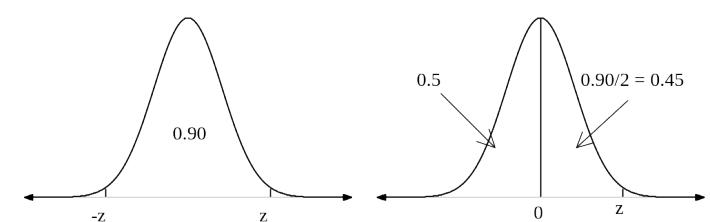
4. (6 marks)

(a) (i)



$$(ii) f(x) = \ln(x) \Rightarrow f^{-1}(x) = e^x \quad \text{for } x \in \mathbb{R}$$

(d)



$$\begin{aligned}
 P(X < z) &= 0.95 \\
 z &= 1.645
 \end{aligned}$$

Use $p = 0.5$ as the maximum value as p is unknown.

$$\begin{aligned}
 \text{So with } p = 0.5 \quad sd &= \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.25}{n}} \\
 E &= z \times s \quad \text{but } E = 0.05
 \end{aligned}$$

Therefore

$$\begin{aligned}
 0.05 &= 1.645 \times \sqrt{\frac{0.25}{n}} \\
 n &= 270.6
 \end{aligned}$$

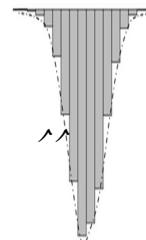
You need to survey 271 people to have an error margin of 5% at a confidence level of 90%.

END OF SECTION TWO

END OF SECTION ONE

- (c) (i) $p = \frac{180}{200} = 0.9$
- (a) (i) Not a probability density functions as you cannot have negative probabilities.
- (ii) Is a probability density functions as the probabilities add to one.
- (b) The shape will be a highly clustered histogram, close to a normal curve.
5. (16 marks)

(iii) Is a probability density functions as the probabilities add to one.



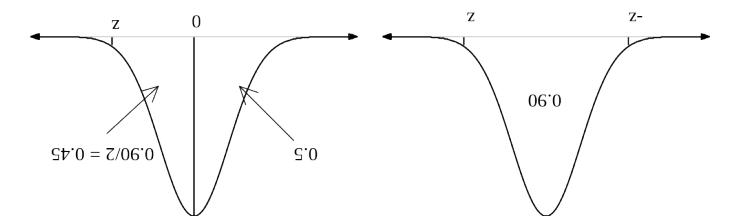
(b)

(iii) Is a probability density functions as the probabilities add to one.

$$(iii) p = \sqrt{\frac{p(1-p)}{0.9 \times 0.1}} = \sqrt{\frac{200}{200}} = 1$$

$$sd_p = \sqrt{\frac{n}{p(1-p)}} = \sqrt{\frac{200}{0.9 \times 0.1}} = 0.45$$

$$sd_p = \sqrt{\frac{200}{0.9 \times 0.1}} = 0.45$$



$$P(X < z) = 0.95$$

$$z = 1.645$$

$$E = z \times s = 1.645 \times 0.0212132$$

$$E \approx 0.0349$$

The 90% confidence limit are 0.9 ± 0.03 i.e. $(0.90, 0.96)$

$$(d) F(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{2} \left[\operatorname{erf} \left(\frac{x}{\sqrt{2}} \right) + 1 \right] = \frac{1}{2} \operatorname{erf} \left(\frac{x}{\sqrt{2}} \right)$$

$$(e) p(x) = \begin{cases} 0.2 & \text{for } 5 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

$$E(x) = \int_5^{10} x \times 0.2 dx = 0.2 \left[\frac{x^2}{2} \right]_5^{10} = 0.1(100 - 25) = 7.5$$

$$Var(x) = \int_5^{10} 0.2(x - 7.5)^2 dx$$

$$Var(x) = \int_5^{10} p(x)(x - \bar{x})^2 dx$$

$$Var(x) = \int_0^5 0.2(x - 7.5)^2 dx$$

$$E(x) = \int_0^5 x \times 0.2 dx = 0.2 \left[\frac{x^2}{2} \right]_0^5 = 0.1(25) = 7.5$$

$$The 90\% confidence limit are 0.9 \pm 0.03 i.e. (0.90, 0.96)$$

SECTION TWO

6. (7 marks)

(a) $A = \int_0^a e^x dx = [e^x]_0^a = e^a - e^0 = e^a - 1$

(b) $A = e^a - 1$

$\frac{dA}{da} = e^a$

$\frac{\delta A}{\delta a} \approx \frac{dA}{da}$

$\delta A \approx \frac{dA}{da} \times \delta a$

At $a = 3, \delta a = 0.1$

$\delta A \approx e^3 \times 0.1$

$\delta A \approx 2.0086$

7. (6 marks)

(a) $v = 10t - 1 \text{ ms}^{-1}$.

$a = 10 \text{ ms}^{-2}$

$x = \int (10t - 1) dt$

$x = 5t^2 - t + c$

At $t = 0, x = 3$

$x = 5t^2 - t + 3$

(b) Changes direction when $v = 0$ i.e. at $t = 0.1$ (c) At $t = 0, x = 3$

At $t = 0.1, x = 0.05 - 0.1 + 3 = 2.95$

At $t = 5, x = 123$

Distance travelled = $123 - 2.95 + 0.05 = 120.1 \text{ m}$

$p = 0.2 \quad q = 0.98 \Rightarrow np = 100 \times 0.2 = 20 > 5$

 $nq = 100 \times 0.8 = 80 > 5$ so can use normal distribution.

Mean $= p = 0.2$

$sd_{\hat{p}} = \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} = \sqrt{\frac{0.2 \times 0.8}{100}}$

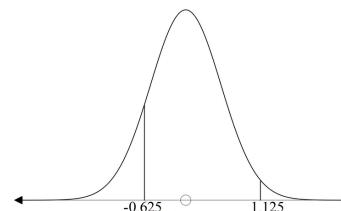
$sd_{\hat{p}} = 0.04$

Standardised score (using 17.5 to 24.5)

$z = \frac{X - \mu}{\sigma}$

$z_{0.075} = \frac{17.5 - 0.2}{0.04} = -0.625$

$z_{0.145} = \frac{24.5 - 0.2}{0.04} = 1.125$



$P(-0.625 \leq z \leq 1.125) = 0.6037199538$

The probability that between 18 and 24 of the wine tasters should not drive is 0.604.

		Solutions	Solutions	Mathematics Methods Units 3-4, 2016, Semester Two
14. (6 marks)	(a) (i)	$P(X=x)$	$\begin{array}{ c c c c c c c } \hline x & 0 & 1 & 2 & 3 & 4 & 5 & \geq 6 \\ \hline P(X=x) & 0.1465 & 0.366 & 0.3945 & 0.068 & 0.015 & 0.0075 & 0.0025 \\ \hline \end{array}$	
14. (ii)	(a)	$f(i) = \sqrt{\sin(\pi i)}$		
14. (7 marks)	8.			
15. (7 marks)	(a)	$E(x) = 1.4715$		
	(b)	$P(\text{a given battery will last between } 320 \text{ and } 380 \text{ hours}) = 0.8663855975 \approx 0.866$		
	(c)	$P(\text{Jenny's battery will run out in a 3 hour exam if it has been used for } 340 \text{ hours})$		$P(x \geq 340) = \frac{0.6914624613}{0.0546318101} = 0.07900907592$
	(d)	$F(x) = \frac{1}{\pi} \int_0^x \frac{dt}{\sqrt{1-t^2}}$		$\int_2^x F(x) dx = \int_2^x \frac{1}{\pi} dt = \left[\ln(t) \right]_2^x = \ln(x) - \ln(2)$
16. (6 marks)	9.	(8 marks)		
17. (22 marks)	(a)	$P(\text{there are 5 girls and 5 boys}) = 0.24609375$		$P(\text{exactly 4 girls}) = 0.2734375$
	(b)	$y = f''(x) > 0$ which suggests that the concavity is concave upwards for all x values.		
	(c)	$f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$		
	(d)	$P(\text{being selected})$		$(i) \quad A \text{ biased sample is one in which not every sample point has equal chance of being selected.}$
				$(ii) \quad A \text{ skewed distribution is a set of data where a large proportion of the data is clustered at one end.}$
				$(iii) \quad A \text{ sample that is chosen in such a way that each point has an equal chance of being selected. The process requires each possible selection point being numbered and then the sample is generated using random numbers.}$

10. (7 marks)

(a) $0.693 \quad \checkmark \checkmark \quad (= \ln(2))$

(b) (i) True as $+C$ can have any value

(ii) No, this is bounded and has only one solution.

$$\int_1^2 3x^2 dx = [x^3]_1^2 = 8 - 1 = 7$$

(iii) True.

(iv) $\ln(f(x)) = \ln(3) + 2\ln(x)$

Not valid as initially, x can be negative as it is squared.Need $2\ln|x| \quad \checkmark \checkmark$

11. (8 marks)

(a) $t=0, P=23$

$78.9 = 23e^{60k}$

$k = 0.02054478$

(b) $t=66, P=89.25$ (million)

(c) The actual population is smaller than 89.25 so the growth rate is slowing down (minimally!)

(d) 2010 $78.9 \times 10^6 \quad t=0$

2016 87238973

$P=78.9$

At $t=6, 87238973 = 78.9 \times 10^6 e^{6k}$

$k = 0.01674499021$

$P = 78.9 \times 10^6 \times e^{0.01674499021 \times t}$

$P = 100 \times 10^6, t=?$

$100 = 78.9 \times e^{0.01674499021 \times t}$

$t = 14.15$

The population is expected to reach 100 million just into 2025.

12. (7 marks)

(a) Area $= \int_{0.05}^{1.47} (\ln(x) - (e^x - 4)) dx = 1.68$ units²

(b) (i) $P(0) = 22 \ln(3) = 24.169 \approx 24$

(ii) $100 = 22 \ln(t+3)$

$t = 91.203$

$2002 + 91 = 2093$

The population will reach 100 in 2094 or just into 2094.

13. (7 marks)

(a) $A = x \times y \quad y^2 = 1 - x^2$
 $A = x\sqrt{1 - x^2}$

(b) Maximum area when $\frac{dA}{dx} = 0$ and $\frac{d^2 A}{dx^2} < 0$

$$\frac{dA}{dx} = -\frac{2x^2 - 1}{\sqrt{1 - x^2}}$$

$$\frac{d^2 A}{dx^2} = -\frac{2x^3\sqrt{1-x^2} + 4x\sqrt{(1-x^2)^3} - x\sqrt{1-x^2}}{(x^2 - 1)^2}$$

If $\frac{dA}{dx} = 0, 2x^2 - 1 = 0$

$x^2 = 0.5$

$x = \frac{1}{\sqrt{2}}, x > 0$

Max or min?

$$\frac{d^2 A}{dx^2} = -\frac{4\left(\frac{1}{2\sqrt{2}}\right)\sqrt{0.5} + 4\left(\frac{1}{2\sqrt{2}}\right) - \frac{1}{\sqrt{2}}(\sqrt{0.5})}{2} = -\frac{\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)\sqrt{0.5} + 4\left(\frac{1}{2\sqrt{2}}\right)}{2} < 0$$

\therefore max

$$x = \frac{1}{\sqrt{2}}, y = \sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}}$$

Therefore $P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$