Papers written by Australian Maths Software

SEMESTER TWO

MATHEMATICS SPECIALIST

REVISION 1

UNIT 3-4

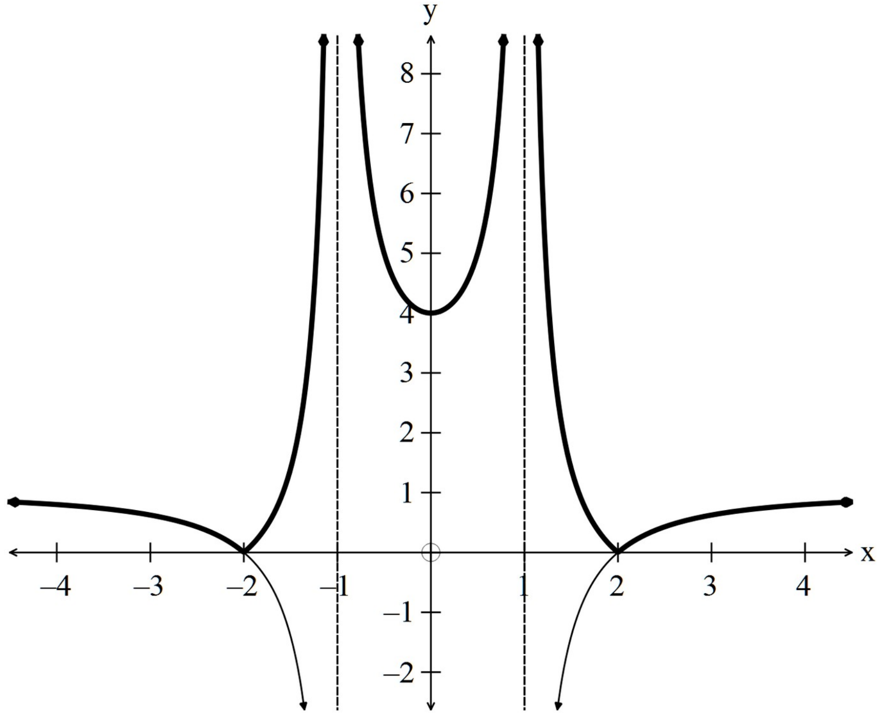
2016

SOLUTIONS

Section One

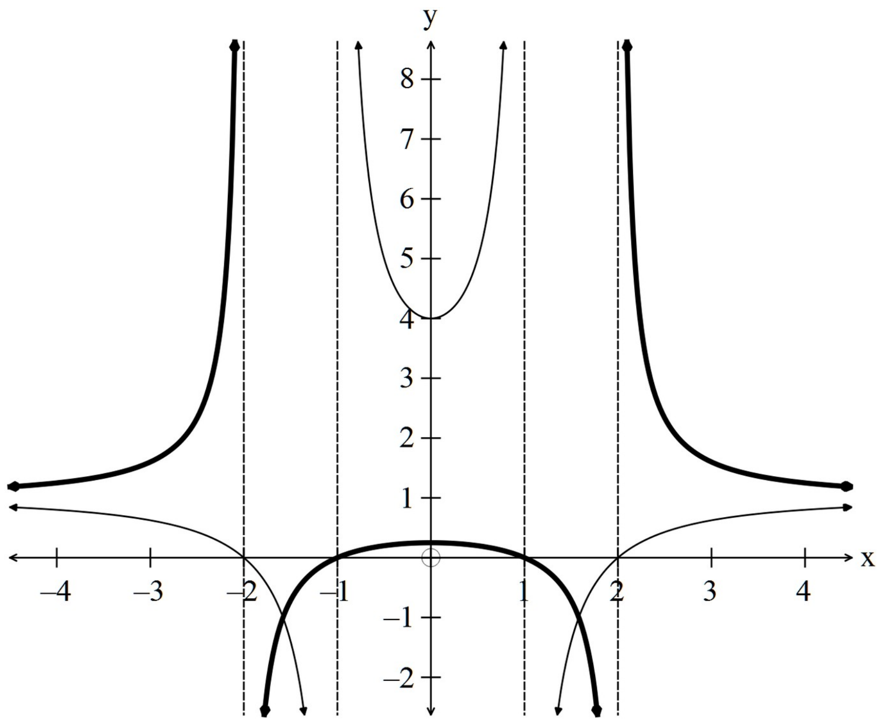
1. (5 marks)

(i) $y = \left| f(x) \right|$



(2)

(ii) $y = \dfrac{1}{f(x)}$



(3)

2. (4 marks)

$$z^4 = -16$$
$$= 2^4(-1)$$
$$= 2^4\left(cis\left(\pi + n(2\pi)\right)\right)$$

$$z = 2\left(cis\left(\pi + n(2\pi)\right)\right)^{\frac{1}{4}}$$

$$z = 2\left(cis\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)\right)$$

$n = 0 \qquad z = 2\left(cis\left(\frac{\pi}{4}\right)\right) = 2\left(cos\left(\frac{\pi}{4}\right) + i\,sin\left(\frac{\pi}{4}\right)\right) = \sqrt{2}(1+i)$

$n = 1 \qquad z = 2\left(cis\left(\frac{\pi}{4} + \frac{\pi}{2}\right)\right) = 2\left(cos\left(\frac{3\pi}{4}\right) + i\,sin\left(\frac{3\pi}{4}\right)\right) = \sqrt{2}(-1+i)$

$n = 2 \qquad \cancel{z = 2\left(cis\left(\frac{5\pi}{4}\right)\right)}$

$n = -1 \qquad z = 2\left(cis\left(-\frac{\pi}{4}\right)\right) = \sqrt{2}(1-i)$

$n = -2 \qquad z = 2\left(cis\left(-\frac{3\pi}{4}\right)\right) = \sqrt{2}(-1-i)$

$$z = \sqrt{2}(1+i),\ \sqrt{2}(-1+i),\ \sqrt{2}(1-i),\ \sqrt{2}(-1-i)$$

✓✓✓ -1/error

(4)

3. (6 marks)

(a) $e^{cos(x)} + e^{sin(y)} = e + 1.$

$- sin(x)e^{cos(x)} + cos(y)e^{sin(y)}\dfrac{dy}{dx} = 0$ ✓✓✓ -1/error

$\dfrac{dy}{dx} = \dfrac{sin(x)e^{cos(x)}}{cos(y)e^{sin(y)}}$

(4)

(b) At (0,0) $\dfrac{dy}{dx} = \dfrac{sin(0)e^{cos(0)}}{cos(0)e^{sin(0)}} = \dfrac{0 \times e^1}{1 \times e^0} = 0$ ✓✓ (2)

4. (7 marks)

(a) Prove that "If $(x-a)$ is a factor of a polynomial, then $P(a)=0$."

Assume $P(x)$ is a polynomial of degree n.

Then $P(x)=x^n+ax^{n-1}+bx^{n-1}+.....+k$.

$$
\begin{aligned}
P(x)&=x^n+ax^{n-1}+bx^{n-1}+.....+k\\
&=(x-a)(x^{n-1}+px^{n-1}+.....+l)
\end{aligned}
$$

since $(x-a)$ is a factor.

$\therefore\ P(a)=(a-a)(a^{n-1}+pa^{n-1}+.....+l)$

$\therefore\ P(a)=0$

Therefore, if $(x-a)$ is a factor, then $P(a)=0$

(4)

(b) $x^3+x^2+x-3=0$

$P(x)=x^3+x^2+x-3$

$P(1)=1^3+1^2+1-3=0$

$\therefore\ x=1$

$$
\begin{array}{r|rrrr}
1 & 1 & 1 & 1 & -3\\
 & \downarrow & 1 & 2 & 3\\
\hline
 & 1 & 2 & 3 & 0
\end{array}
$$

$\therefore P(x)=(x-1)(x^2+2x+3)$

$x=1$ or $x^2+2x+3=0$

$$x=\frac{-2\pm\sqrt{4-12}}{2} \qquad -8=8i^2$$

$x=1$ or $x=\dfrac{-2\pm i2\sqrt{2}}{2}$

$x=1$ or $x=-1\pm i\sqrt{2}$

(3)

5. (10 marks)

(a) $\int_{\pi/6}^{\pi/3} \left(\sin^2(x) + 2x\right) dx$

$$\cos(2x) = 1 - 2\sin^2(x)$$

$$\therefore \sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$= \int_{\pi/6}^{\pi/3} \left(\frac{1 - \cos(2x)}{2} + 2x \right) dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2x)}{2} \right]_{\pi/6}^{\pi/3} + \left[x^2 \right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left(\left(\pi/3 - \frac{\sin(2\pi/3)}{2} \right) - \left(\pi/6 - \frac{\sin(2\pi/6)}{2} \right) \right) + \left((\pi/3)^2 - (\pi/6)^2 \right)$$

$$= \frac{1}{2}\left(\pi/3 - \pi/6 - \cancel{\sqrt{3}/2} + \cancel{\sqrt{3}/2} \right) + \pi^2/9 - \pi^2/36$$

$$= \pi/6 - \pi/12 + \pi^2/9 - \pi^2/36$$

$$= \pi/12 - \pi^2/12$$

(3)

(b) $\int_0^1 \sqrt{1 - x^2}\ dx$

put $x = \sin(\theta)$

$$\frac{dx}{d\theta} = \cos(\theta)$$

$$dx = \cos(\theta)\,d\theta$$

$$\sqrt{1 - x^2} = \sqrt{1 - \sin^2(\theta)}$$

$$= \cos(\theta)$$

$$If\ x = 1,\ 1 = \sin(\theta),\ \theta = \frac{\pi}{2}$$

$$If\ x = 0,\ 0 = \sin(\theta),\ \theta = 0$$

$$= \int_0^{\frac{\pi}{2}} \cos(\theta)\cos(\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1 + \cos(2\theta)}{2}\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + 0 - 0 \right)$$

$$= \frac{\pi}{4}$$

(5)

(c) $\int \frac{ln(x)}{x} dx$

put $u = ln(x)$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{dx}{x}$$

$$\equiv \int u \, du$$

$$= \frac{u^2}{2} + c$$

$$\equiv \frac{(ln(x))^2}{2} + c$$

(2)

6. (12 marks)

(a) (i) At (-1, -1)

$-1 = sin(t) + cos(t)$ and $-1 = cos(t)$

$-1 = sin(t) - 1$ $\quad t = \pi$

$sin(t) = 0$

$\therefore t = \pi$

(2)

(ii) At $t = 0$, $(1, 1)$ (1)

(iii) $\boldsymbol{r}(t) = (sin(t) + cos(t))\boldsymbol{i} + (cos(t))\boldsymbol{j}$

$\boldsymbol{v}(t) = (cos(t) - sin(t))\boldsymbol{i} - (sin(t))\boldsymbol{j}$

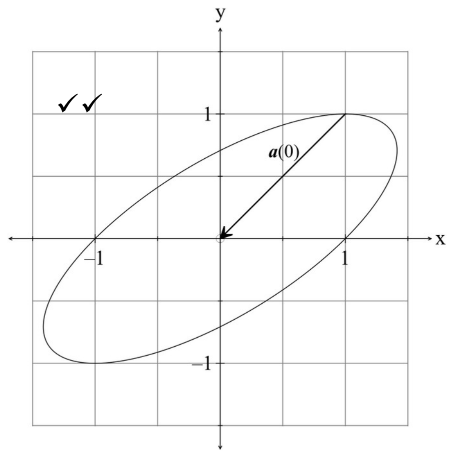
$\boldsymbol{a}(t) = (-sin(t) - cos(t))\boldsymbol{i} - (cos(t))\boldsymbol{j}$

$\therefore \boldsymbol{v}(0) = \boldsymbol{i}$

$\boldsymbol{a}(0) = -\boldsymbol{i} - \boldsymbol{j}$

(3)

(iv)



(v) $\boldsymbol{r}(t) = (\sin(t)+\cos(t))\boldsymbol{i} + (\cos(t))\boldsymbol{j}$

$\boldsymbol{a}(t) = (-\sin(t)-\cos(t))\boldsymbol{i} - (\cos(t))\boldsymbol{j}$

$= -((\sin(t)+\cos(t))\boldsymbol{i} + (\cos(t))\boldsymbol{j})$

$= -\boldsymbol{r}(t)$

True for all values of t. ✓ (1)

(b) $\mathbf{r}(t) = \begin{pmatrix}1\\2\\-1\end{pmatrix} + t\begin{pmatrix}3\\0\\2\end{pmatrix}$ and the point $P(-1,2,-4)$.

Does P belong to the line?

$If\ x = -1$, then $t = -\frac{2}{3}$

$If\ z = -4$, then $t = -\frac{3}{2} \neq -\frac{2}{3}$

$\therefore$ the point P does not belong to the line.

$A(1,2,-1)$ then $AP = \begin{pmatrix}-2\\0\\-3\end{pmatrix}$

Equation of the plane is

$r(t) = \begin{pmatrix}1\\2\\-1\end{pmatrix} + t\begin{pmatrix}3\\0\\2\end{pmatrix} + s\begin{pmatrix}-2\\0\\-3\end{pmatrix}$

(3)

7. (3 marks)

$DC = 6 \quad BC = 6$

The diameter is 6 and the radius is 3.

Midpoint of AD is $(0,-2,3)$.

$x^2 + (y+2)^2 + (z-3)^2 = 9$

(3)

8. (5 marks)

(a) $|x-1+iy| = 2|x+i(y-1)|$

$\sqrt{(x-1)^2+y^2} = 2\sqrt{x^2+(y-1)^2}$

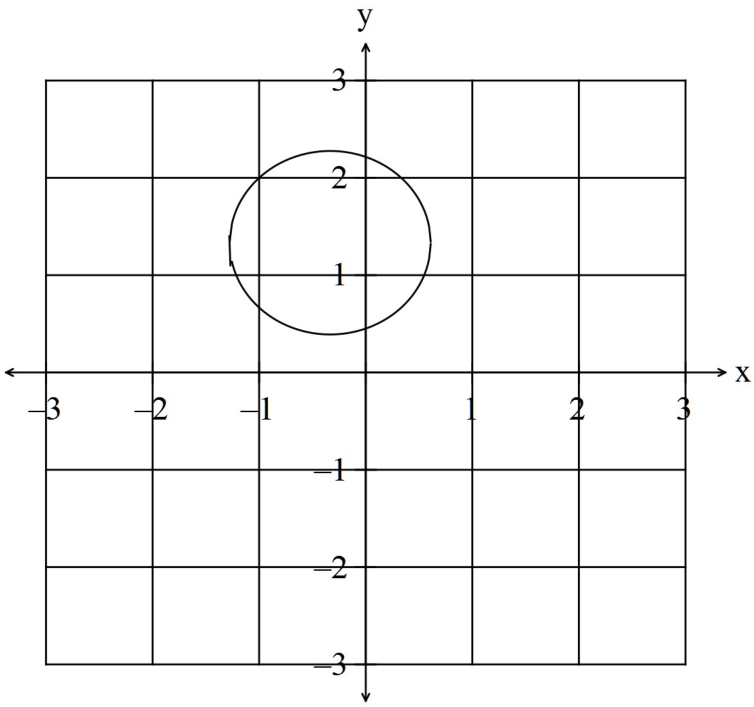
$(x-1)^2+y^2 = 4\left(x^2+(y-1)^2\right)$

$x^2-2x+1+y^2 = 4\left(x^2+y^2-2y+1\right)$

$3x^2+3y^2+2x-8y+3=0$

$3\left(x^2+y^2+\frac{2x}{3}-\frac{8y}{3}+1\right)=0$

$C\left(-\frac{1}{3},\frac{4}{3}\right) \quad r=\sqrt{\frac{1}{9}+\frac{16}{9}-1}=\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3}$



(3)

(b) $z = \frac{1+i}{1-i} \times (3+3i) \qquad \bar{z} = ?$

$$z = 3\frac{1+i}{1-i} \times (1+i) \times \frac{1+i}{1+i}$$

$$= 3\frac{(1+3i+3i^2+i^3)}{1-i^2}$$

$$= \frac{3}{2}(1+3i-3-i)$$

$$z = \frac{3}{2}(-2+2i)$$

$$z = 3(-1+i)$$

$$\therefore \bar{z} = 3(-1-i)$$

(2)

END OF SECTION ONE

Section Two

9. (4 marks)

(a) $\boldsymbol{r}(t)=\begin{pmatrix}1\\-2\\0\end{pmatrix}+t\begin{pmatrix}2\\3\\-1\end{pmatrix}$ as $\begin{pmatrix}2\\3\\-1\end{pmatrix}$ is perpendicular to the plane. ✓ (1)

(b) Determine if point $(1,2,1)$ belongs to P_2.

Determine if point $(-2,1,2)$ belongs to P_1. ✓

Find a point at random on P_1 by say letting $s=0$ and $t=1$. ✓

Substitute this point into P_2 .

If the point belongs to the plane, you have three non-linear points in two planes.
To check collinearity of the points find the equation of the line through two points and see if the third point belongs to the line. ✓

ALTERNATIVELY find three points on one of the planes and test them in the other plane. You need to test they are not collinear. (3)

10. (5 marks)

Boy: $\boldsymbol{r}(t)=\begin{pmatrix}0\\0\\0.5\end{pmatrix}+t\begin{pmatrix}0.5\\0.5\\0\end{pmatrix}$ from $t=0$.

(a) Kookaburra

$$\boldsymbol{r}_k(t)=\begin{pmatrix}-5.5\\-1.5\\4.5\end{pmatrix}+(t-1)\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$\boldsymbol{r}_k(t)=\begin{pmatrix}-5.5\\-1.5\\4.5\end{pmatrix}+t\begin{pmatrix}2\\1\\-1\end{pmatrix}-1\times\begin{pmatrix}2\\1\\-1\end{pmatrix}$$ ✓

$$\boldsymbol{r}_k(t)=\begin{pmatrix}-7.5\\-2.5\\5.5\end{pmatrix}+t\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

(1)

(b) Speed of kookaburra = $\left\|\begin{pmatrix}2\\1\\-1\end{pmatrix}\right\| = \sqrt{4+1+1} = 2.45\,\text{m s}^{-1}$ (1)

(c) $\begin{pmatrix}0\\0\\0.5\end{pmatrix} + t\begin{pmatrix}0.5\\0.5\\0\end{pmatrix} = \begin{pmatrix}-7.5\\-2.5\\5.5\end{pmatrix} + t\begin{pmatrix}2\\1\\-1\end{pmatrix}$ ✓

$z: 0.5 = 5.5 - t \Rightarrow t = 5$

Check:

$y: 0.5t = -2.5 + t$

$2.5 = 0.5t \Rightarrow t = 5$

$x: 0.5t = -7.5 + 2t$

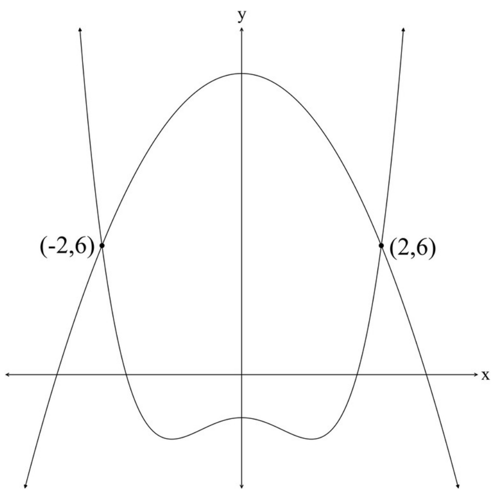
$7.5 = 1.5t \Rightarrow t = 5$

The kookaburra takes 4 seconds to steal the sandwich. (He waits for one second). (3)

✓

11. (4 marks)

$f(x) = -2x^2 + 14$ and $g(x) = x^4 - 2x^2 - 2$



The x values of the intersection points are -2 and 2 ✓

Area = $\int_{-2}^{2} \left(-2x^2 + 14\right) - \left(x^4 - 2x^2 - 2\right) dx = 51.2$

✓✓ ✓

12. (8 marks)

(a) Prove that $cos(3\theta) = 4cos^3(\theta) - 3cos^3(\theta)$

De Moivre's theorem states that

$(cos(x) + i\,sin(x))^n = cos(nx) + i\,sin(nx)$

For $n = 3$

$(cos(x) + i\,sin(x))^3 = cos(3x) + i\,sin(3x)$

so $cos(3x) = Re(cos(x) + i\,sin(x))^3$

$(cos(x) + i\,sin(x))^3 = cos^3(x) + 3i\,sin(x)cos^2(x) + 3i^2\,sin^2(x)cos(x) + i^3\,sin^3(x)$

$(cos(x) + i\,sin(x))^3 = (cos^3(x) - 3sin^2(x)cos(x)) + i(3sin(x)cos^2(x) - sin^3(x))$

as $cos(3x) = Re(cos(x) + i\,sin(x))^3$

we have $cos(3x) = Re(cos^3(x) - 3sin^2(x)cos(x)) + i(3sin(x)cos^2(x) - sin^3(x))$

so $cos(3x) = cos^3(x) - 3sin^2(x)cos(x)$

$= cos^3(x) - 3cos(x)\left[1 - cos^2(x)\right]$

$= cos^3(x) - 3cos(x) + 3cos^3(x)$

Therefore $cos(3x) = 4cos^3(x) - 3cos(x)$

(6)

(b) $(1 - i)^{20} = -1024$ ✓✓ (2)

13. (17 marks)

(a) (i) $y = f(g(x)) = f(\sqrt{x}) = ln\sqrt{x} = \frac{1}{2}ln(x)$ ✓

$ln(x)$ is defined for $x > 0$ and this is the given domain so

$y = f(g(x))$ is defined. ✓ (2)

(ii) $y = f(g(x)) = \frac{1}{2}ln(x)$ is monotonically ✓ increasing, so $y = f(g(x))$ has an inverse. (1)

(iii) $y = \frac{1}{2}ln(x)$

To obtain the inverse, switch x and y

$x = \frac{1}{2}ln(y) \Rightarrow 2x = ln\,y$

$\therefore\ y = e^{2x}$ or if $f(g(x)) = f \circ g$ then $(f \circ g)^{-1}(x) = e^{2x}$

(3)

(b) (i) $g(f(x)) = g(ln(x)) = \sqrt{ln(x)}$

$\sqrt{ln(x)}$ is defined for $ln(x) \geq 0$

$ln(1) = 0$

$\therefore$ $g(f(x))$ is defined for $x \geq 1$

(3)

(ii) $y = g(h(x)) = g\left(e^{-x^2}\right) = \sqrt{e^{-x^2}}$

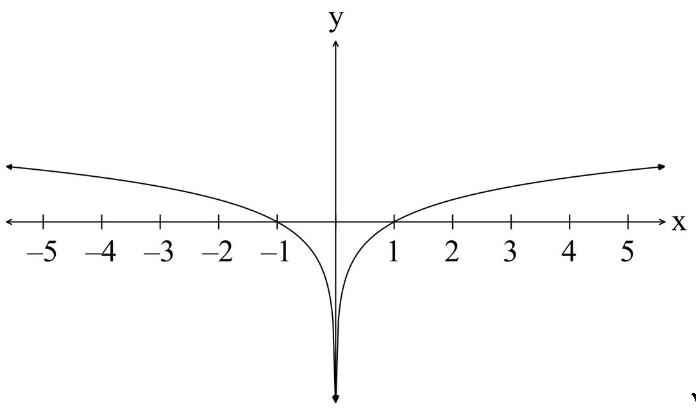
$g(h(1)) = \sqrt{e^{-(1)^2}} = \sqrt{e^{-1}} = \dfrac{1}{\sqrt{e}}$

$g(h(-1)) = \sqrt{e^{-(-1)^2}} = \sqrt{e^{-1}} = \dfrac{1}{\sqrt{e}}$

$\therefore g(h(1)) = g(h(-1))$

The function $y = g(h(x))$ is not a 1-1 function. (3)

(c) (i) $y = f(p(x)) = f(|x|) = ln|x|$ ✓



✓

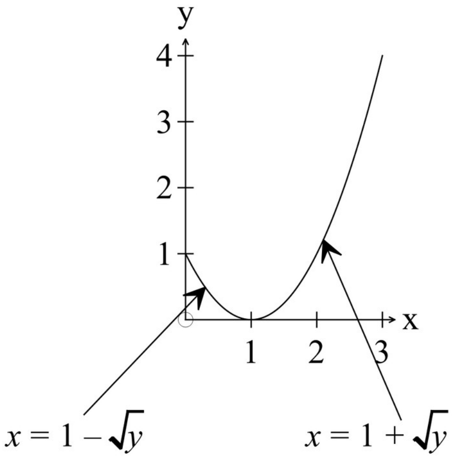
(2)

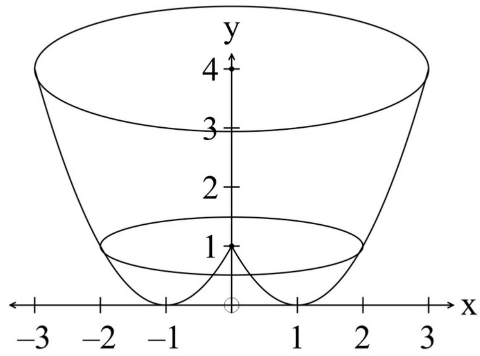
(ii) $f\left(p\left(-e^{-3}\right)\right) = f\left(\left|-e^{-3}\right|\right) = f\left(e^{-3}\right) = ln\left(e^{-3}\right) = -3\,ln\,e = -3$ (2)

✓ ✓

14. (10 marks)

The function $f(x) = (x-1)^2$ for $0 \le x \le 3$ is rotated about the y axis.





(a) $V = \int_a^b \pi x^2 dy$

$f(3) = 4$

$$V = \int_0^4 \pi\left(1+\sqrt{y}\right)^2 dy - \int_0^1 \pi\left(1-\sqrt{y}\right)^2 dy$$

$$= 71.20943348 - 0.5235987756$$

$$V \approx 70.686 \text{ units}^3$$

(5)

(b) $V = \int_0^1 \pi\left(1+\sqrt{y}\right)^2 dy - \int_0^1 \pi\left(1-\sqrt{y}\right)^2 dy = 8.38 \text{units}^3$ (1)

✓

(c) $22.5 = \int_0^h \pi\left(1+\sqrt{y}\right)^2 dy - \int_0^1 \pi\left(1-\sqrt{y}\right)^2 dy$

$$22.5 = \int_0^h \pi\left(1+\sqrt{y}\right)^2 dy - 0.5235987756$$

$$\therefore \int_0^h \pi\left(1+\sqrt{y}\right)^2 dy = 23.0235987756$$

$$\frac{23.0235987756}{\pi} = \int_0^h \left(1+\sqrt{y}\right)^2 dy$$

$$\int_0^h \left(1+\sqrt{y}\right)^2 dy = \int_0^h \left(1+2\sqrt{y}+y\right) dy$$

$$= \left[y + \frac{4}{3}\sqrt{y^3} + \frac{y^2}{2}\right]_0^h$$

$$\frac{23.0235987756}{\pi} = h + \frac{4}{3}\sqrt{h^3} + \frac{h^2}{2}$$

$$h \approx 1.92$$

(4)

15. (12 marks)

(a) $a=-4x\,\text{cm s}^{-2} \Rightarrow n^2=4, n=2$

Assume that

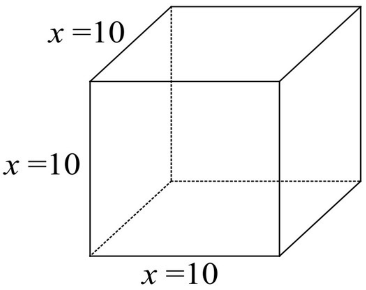
$x=5\cos(2t+\varepsilon)$

$v=-10\sin(2t+\varepsilon)$

Therefore the maximum velocity is 10 cm s^{-1}

(which occurs when $\sin(2t+\varepsilon)=-1$). (3)

(b) $SA=6x^2$



$\frac{dSA}{dx}=12x$

$x=10,\ \delta x=0.1,\ \delta SA=?\ \frac{dSA}{dx}\approx\frac{\delta SA}{\delta x}$

$\therefore\ \delta SA\approx\frac{dSA}{dx}\times\delta x$

$\delta SA\approx(12\times10)0.1$

$\delta SA\approx12$

The increase in surface area, when the side increases from 10 to 10.1 cm, is 12 cm^2.

(3)

(c) $v(t)=t^2-4\ \text{cm s}^{-1}$

$x(t)=\int(t^2-4)dt$ cm

$x(t)=\frac{t^3}{3}-4t+c$

At $t=0, x=3\Rightarrow c=3$

$\therefore\ x(t)=\frac{t^3}{3}-4t+3$

$t=?$ when $x=8\frac{1}{3}$cm

$\frac{25}{3}=\frac{t^3}{3}-4t+3$

$t=-2\ or\ t=4$ but $t\ge0$ so $t=4$

$a(t)=2t$

$a(4)=8\ \text{cm s}^{-2}$

(3)

(d) $\frac{dy}{dx}=\frac{y}{2x+1}$

$\int\frac{dy}{y}=\int\frac{dx}{2x+1}$

$ln(y)=\frac{ln(2x+1)}{2}+c$

$c=ln\sqrt{(2x+1)}-ln(y)$

$c=ln\left(\frac{\sqrt{(2x+1)}}{y}\right)$

$e^c=\left(\frac{\sqrt{(2x+1)}}{y}\right)\quad Let\ e^c=A$

$At\ (1,3)\quad A=\frac{\sqrt{3}}{3}$

$\frac{\sqrt{(2x+1)}}{y}=\frac{1}{\sqrt{3}}$

$y=\sqrt{3(2x+1)}$

(3)

16. (5 marks)

(a)

$x+y+z=2$

$x-2y+3z=8$

$2x-y+4z=10$

$$\begin{bmatrix}1&1&1&2\\1&-2&3&8\\2&-1&4&10\end{bmatrix}$$

$$\begin{bmatrix}1&1&1&2\\0&-3&2&6\\0&-3&2&6\end{bmatrix}\quad\begin{array}{l}\\R_2-R_1\\R_3-2R_1\end{array}$$

$$\begin{bmatrix}1&1&1&2\\0&-3&2&6\\0&0&0&0\end{bmatrix}$$

There are two dependant equations. None of the planes are parallel or identical planes, so the three planes intersect in a line. (2)

(b) $\boldsymbol{OA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\boldsymbol{OB} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

$Area_{\Delta} = \frac{1}{2} a \times b \times sin(C)$

$Area_{\Delta} = \frac{1}{2} |\boldsymbol{OA} \times \boldsymbol{OB}|$

$= \frac{1}{2} \left| \begin{pmatrix} 1 \\ 13 \\ -9 \end{pmatrix} \right|$

$= \frac{1}{2} \sqrt{1 + 169 + 81}$

$= \frac{1}{2} \sqrt{251}$

$Area_{\Delta} = 7.92$ units^2

(3)

17. (12 marks)

(a) $\mu_{\bar{x}} = 23.245$ and $\sigma_{\bar{x}} = 0.819$ ✓ ✓ (2)

(b) $\mu = 23.245$ ✓

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$

$\sigma = \sqrt{10} \times 0.819$

$\sigma \approx 2.59$

(2)

(c) 99% confidence limits $\Rightarrow z = 1.96$

$\mu = 23.245 \pm z \times \sigma_{\bar{x}}$

$\mu = 23.245 \pm 1.96 \times 0.819$

$\mu = 23.245 \pm 1.106$

Therefore we are 95% sure that $21.64 \le \mu \le 24.85$ ✓ ✓(3)

(d) 99% confidence limits $\Rightarrow z = 2.5758293$

$\mu = 23.245 \pm z \times \sigma_{\bar{x}}$

$\mu = 23.245 \pm 2.5758 \times 0.819$

$\mu = 23.245 \pm 2.1107$

$\therefore 21.13 \le \mu \le 25.36$ $\quad so \ P(21.13 \le \mu \le 25.36) = 0.99$

The 99% confidence limits are broader as you need to be MORE confident your results are correct. There is only room for 1% error so a wider range is given for the estimate of the mean. (3)

18. (5 marks)

(i) Each of the sampling distribtions will have a mean of 20.
The sampling distribution with have a much smaller standard deviation than the original population as each of the points is an estimate of the mean of the population.

The standard deviation of the sampling distributions is found by $\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$.

$P_1 \quad \sigma_{\bar{x}} = \dfrac{12}{\sqrt{18}} = 2.83$

$P_2 \quad \sigma_{\bar{x}} = \dfrac{6}{\sqrt{18}} = 1.41$

$P_3 \quad \sigma_{\bar{x}} = \dfrac{18}{\sqrt{18}} = 4.24$ ✓✓

It can be seen that the bigger the population standard deviation, the bigger the standard deviation of the sampling distributions.

(ii) Each of the sampling distributions are a tightly clustered normal distribution.
✓ (1)

19. (12 marks)

(a) $\frac{dN}{dt} = kN\left(\frac{K - N}{K}\right)$ for $r = 0.1$ and $K = 100$

$$\frac{dN}{dt} = 0.1N\left(1 - \frac{N}{100}\right) = \frac{0.1N}{100}(100 - N)$$

$$so\ \frac{dt}{dN} = \frac{100}{0.1}\left(\frac{1}{N(100 - N)}\right)$$

$$0.001t = \int\left(\frac{1}{N(100 - N)}\right)dN$$

Using partial fractions,

$$\frac{1}{N(100 - N)} = \frac{a}{N} + \frac{b}{100 - N}$$

$$= \frac{a(100 - N) + bN}{N(100 - N)}$$

$$\frac{0 \times N + 1}{N(100 - N)} = \frac{N(b - a) + 100a}{N(100 - N)}$$

$0 = b - a$ and $1 = 100a$

$a = b$ and $a = 0.01 = b$

$$\frac{1}{N(100 - N)} = \frac{1}{100N} + \frac{1}{100(100 - N)}$$

so $0.001t = \int\left(\frac{1}{N(100 - N)}\right)dN$ becomes

$$\frac{0.1t}{100} = \int \frac{1}{100N} + \frac{1}{100(100 - N)}dN$$

$$\frac{0.1}{\cancel{100}}t = \frac{1}{\cancel{100}}\left[\int \frac{1}{N}dN + \int \frac{1}{(100 - N)}dN\right]$$

$$0.1t = \ln N + (-1)\ln(100 - N) + C$$

$$0.1t - C = \ln\left(\frac{N}{100 - N}\right)$$

$$\frac{N}{100-N}=e^{0.1t-C}$$

$$\frac{100-N}{N}=Ae^{-0.1t} \quad where\ A=e^{C}$$

$$At\ t=0, N=N_0$$

$$\frac{100-N_0}{N_0}=Ae^{0}=A \quad \text{so} \quad A=\frac{100-N_0}{N_0}$$

$$\frac{100-N}{N}=Ae^{-0.1t}$$

Rearrange the formula to get N

$$100-N=NAe^{-0.1t}$$

$$100=N\left(1+Ae^{-0.1t}\right)$$

$$\therefore\ N=\frac{100}{1+Ae^{-0.1t}} \text{ with } A=\frac{100-N_0}{N_0}$$

(5)

(b) (i) $K=610 \Rightarrow N=\dfrac{610}{1+Ae^{-kt}}$ with $A=\dfrac{610-N_0}{N_0}$

$$\text{At } t=0,\ N_0=300 \ \text{(Jan 2015)}$$

$$A=\frac{610-300}{300}=1.0\dot{3}$$

$$N=\frac{610}{1+1.0\dot{3}e^{-kt}} \quad k=?$$

$$\text{At } t=12,\ N_{12}=400 \ \text{(Jan 2016)}$$

$$400=\frac{610}{1+1.0\dot{3}e^{-k\times12}}$$

$$1+1.0\dot{3}e^{-k\times12}=\frac{610}{400}\Rightarrow k=0.05642890327$$

$$\therefore N=\frac{610}{1+1.0\dot{3}e^{-0.05642890324t}}$$

$$\text{At } t=24,\ N_{24}=? \ \text{(Jan 2017)}$$

$$N=\frac{610}{1+1.0\dot{3}e^{-0.05642890324\times24}}$$

$$N=481.55$$

The number of trout in the dam in January 2017 is expected to be close to 482.

(5)

(ii) $N = 600, \quad t = ?$

$$600 = \frac{610}{1 + 1.03e^{-0.05642890324 \times t}}$$

$$1 + 1.03e^{-0.05642890324 \times t} = \frac{610}{600}$$

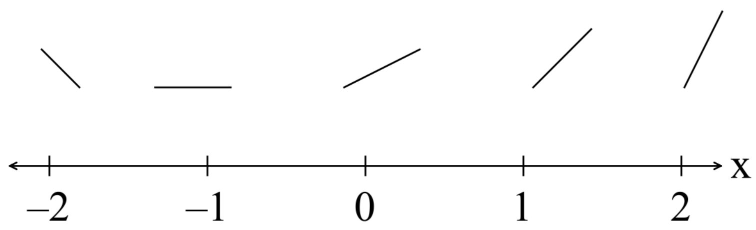
$$t = 73.14$$

i.e. expect 600 trout in the dam in February 2021. (2)

20. (6 marks)

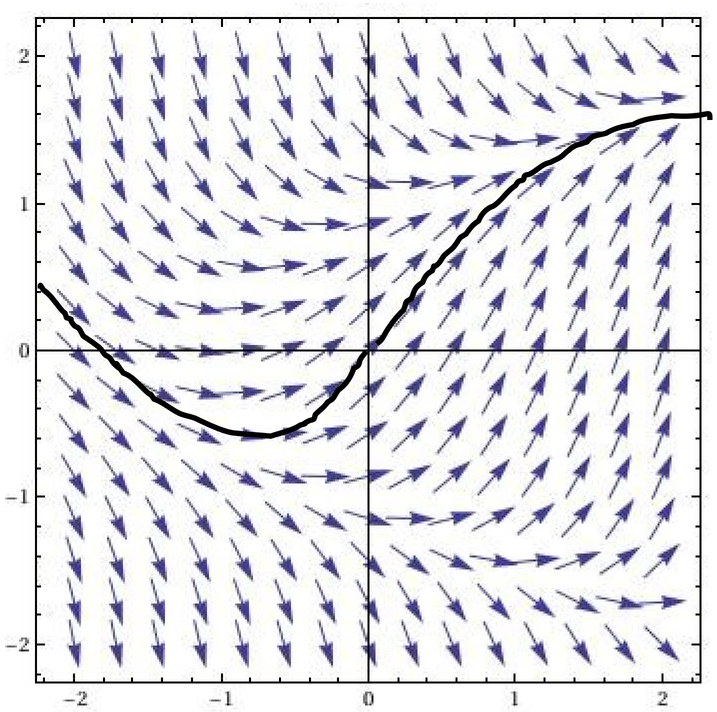
(a)

x	-2	-1	0	1	2	
y	0	0	0	0	0	
$\frac{dy}{dx}$	-1	0	1	2	3	✓✓



✓✓

(b)



✓✓

(6)

END OF PAPER