Additional working space

Question number: _____

Christ Church Grammar School

2016
UNIT TEST 3

MATHEMATICS METHODS Year 12

Section One:
Calculator-free

Student name Solution

Teacher name _____

Time and marks available for this section

Reading time before commencing work:	2 minutes
Working time for this section:	15 minutes
Marks available:	15 marks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. Write your answers in this Question/Answer Booklet.

2. Answer all questions.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that **you do not use pencil**, except in diagrams.

Question 11 **(4 marks)**

A dice game is played according to the following rules.

Each game costs \$2 to play.

Two dice are rolled and the total of the numbers on the upper faces is noted.

If the total is less than 5 or more than 8, the player wins \$12. Otherwise he loses \$6.

(a) What is the expected payout (amount received) for a player in each game? (3 marks)

$P(T<5 \text{ or } T>8) = \frac{16}{36}$ ✓

$\therefore E(\text{payout}) = 12\times\frac{16}{36} - 6\times\frac{20}{36}$ ✓

$= \$2$ ✓

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(b) Is this a fair game? Give a reason. (1 mark)

Yes, since $E(\text{winning}) = \$2 - \$2$

$= 0$. ✓

Question 10 **(5 marks)**

15% of a large batch of iPhones contain a hardware fault. Let X be the number of randomly inspected iPhones until a faulty iPhone is found.

(a) Find the probability for the following values of X. (1 mark)

x	1	2	3	4
$P(X = x)$	0.15	0.1275	0.1084	0.0921

(b) Determine a rule for $P(X = k)$ where k is a positive integer. (1 mark)

$P(X = k) = (0.85)^{k-1}(0.15)$

(c) Find $P(X = 16)$. (1 mark)

$= 0.0131$

(d) Find the smallest value of k such that for $P(X = k) < 0.01$. (2 marks)

$P(X = 17) = 0.0111$

$P(X = 18) = 0.0095$

$\therefore$ Smallest $k = 18$

Question 1 **(3 marks)**

Evaluate the following.

(a) $\frac{d}{dx}\int_2^{x^2} e^t \sin t \, dt$ (1 mark)

$= 2xe^{x^2}\sin x^2$

(b) $\int_{-1}^{1} \frac{d}{dx} x^3 e^{x^2} \, dx$ (2 marks)

$= \left[x^3 e^{x^2} \right]_{-1}^{1}$

$= e - (-e)$

$= 2e$

Question 2 **(4 marks)**

State the limit of each expression.

(a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ (1 mark)

$= e$ ✓

(b) $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n$ (1 mark)

$= e^{0.5}$ ✓

(c) $\lim_{h\to 0}\left(\frac{e^h-1}{h}\right)$ (1 mark)

$= 1$ ✓

(d) $\lim_{h\to 0}\left(\frac{\sin(2x+h)-\sin 2x}{h}\right)$ (1 mark)

$= \frac{1}{2}\lim_{\frac{h}{2}\to 0}\left(\frac{\sin 2(x+\frac{h}{2})-\sin 2x}{\frac{h}{2}}\right)$

$= \frac{1}{2}\frac{d}{dx}\sin 2x$

$= \cos 2x$ ✓

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Question 9 **(6 marks)**

A box contains 100 iPads, of which $\frac{1}{5}$ are known to be defective.

A sample of 5 iPads is selected at random.

(a) Find the probability that exactly 2 iPads will be defective. (2 marks)

No. of defectives $= \frac{1}{5}\times 100 = 20$ ✓

$P(\text{exactly 2 defectives}) = \frac{\binom{20}{2}\binom{80}{3}}{\binom{100}{5}}$

$= 0.2073$ ✓

(b) Find the probability that at least 2 iPads will be defective. (2 marks)

$P(\text{at least 2 defectives})$

$= 1 - P(0 \text{ def.}) - P(1 \text{ def.})$

$= 1 - \frac{\binom{20}{0}\binom{80}{5}}{\binom{100}{5}} - \frac{\binom{20}{1}\binom{80}{4}}{\binom{100}{5}}$ ✓

$= 0.2605$ ✓

(c) If it is known that at least 2 iPads are defective, find the probability that 4 iPads will be defective. (2 marks)

$P(4 \text{ defectives} \mid \text{at least 2 defectives})$

$= \frac{P(4 \text{ defectives})}{P(\text{at least 2 defectives})}$

$= \frac{\frac{\binom{20}{4}\binom{80}{1}}{\binom{100}{5}}}{0.2605}$ ✓

$= 0.01976$ ✓

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Question 8 **(3 marks)**

Indicate if each of the following **is** or **is not** a discrete probability distribution. Give a reason if it is not.

A:

x	-1	0	1	2
$P(X=x)$	0.1	0.2	0.3	0.4

Yes ✓

B:

x	3	6	9	10.5
$P(X=x)$	$\frac{1}{3}$	$\frac{1}{6}$	0	0.5

Yes ✓

C:

x	3	6	9	12
$P(X=x)$	0.2	0.4	-0.1	0.5

No.
$P(X=9) = -0.1$ ✓ is invalid
since $0 \leq p \leq 1$

Question 3 **(4 marks)**

The random variable X has only two outcomes, 6 and 3, with the probability of each outcome equal to $\frac{1}{3}$ and $\frac{2}{3}$ respectively.

(a) Find $E(X)$. (1 mark)

$E(x) = \frac{1}{3} \times 6 + \frac{2}{3} \times 3$

$= 4$ ✓

(b) Find $E(3X-5)$. (1 mark)

$E(3X-5) = 3E(X) - 5$

$= 12 - 5$

$= 7$ ✓

(c) Find $Var(3X-5)$ given that $Var(X) = 2$. (2 marks)

$Var(3X-5) = Var(3X)$

$= 3^2 Var(X)$ ✓

$= 18$ ✓

Question 4 **(4 marks)**

Let $F(x) = \int_0^x f(t)dt$, where $F(5) = e^5$ and $F''(x) = e^x$.

Find the function $f(x)$.

$F'(x) = \frac{d}{dx}\int_0^x f(t)\,dt$

$= f(x)$ ✓ using FTC

$F''(x) = f'(x)$

$\therefore f'(x) = e^x$ ✓

$f(x) = \int e^x\,dx$

$= e^x + c$ ✓

$F(x) = \int_0^x e^t + c\,dt$

$F(5) = \int_0^5 e^t + c\,dt = e^5$

$[e^t + ct]_0^5 = e^5$

$e^5 + 5c - 1 = e^5$

$c = \frac{1}{5}$

$\therefore f(x) = e^x + \frac{1}{5}$ ✓

End of questions

Question 7 **(4 marks)**

A radioactive substance with a half-life of 80 days has been decaying for 20 days. Currently, 180 g of the substance remains. Determine how much of the substance was present initially, giving your answer to the nearest grams.

$P = P_0 e^{kt}$

$\frac{P}{P_0} = e^{kt}$

$\frac{1}{2} = e^{80k}$ ✓

$k \approx -0.008664$ ✓

$180 = P_0 e^{20(-0.008664)}$ ✓

$P_0 = 214.0558$

$\approx 214\,g$ ✓

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Question 6 **(4 marks)**

A biologist is researching a newly-discovered species of bacteria. At time $t = 0$ hours, he places 100 bacteria into what he has determined to be a favourable growth medium. Six hours later, he measures 450 bacteria. Assuming exponential growth where $\frac{dP}{dt} = kP$,

(a) determine the growth rate k for the bacteria rounded to two decimal places. (2 marks)

$P = 100e^{kt}$ ✓

$450 = 100e^{6t}$

$k \approx 0.25$ ✓

(b) using the growth rate k found in (a), determine the time taken for the colony to increase to 10 000 bacteria, giving your answer to the nearest hour. (2 marks)

$10000 = 100e^{0.25t}$ ✓

$t = 18.42$

≈ 19 hours ✓

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Christ Church Grammar School

2016
UNIT TEST 3

MATHEMATICS METHODS Year 12

Section Two:
Calculator-assumed

Student name _____

Teacher name _____

Time and marks available for this section

Reading time before commencing work:	3 minutes
Working time for this section:	30 minutes
Marks available:	30 marks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, and up to three calculators approved for use in the WACE examinations

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Question 5 **(4 marks)**

A water tank has developed a leak and is losing water at the rate given by the equation $\frac{dV}{dt} = 3 + 2e^{0.5t}$ in litres per hour, where V is the volume in litres of the water at time t hours since the leak was discovered.

(a) If the water leakage was discovered after an hour, how much water to the nearest millilitre has been leaked? (2 marks)

$V = \int_0^1 3 + 2e^{0.5t}\, dt$ ✓

$= 5.594885$ l

≈ 5595 ml ✓

(b) If there were initially 1 000 litres of water in the tank, how long does it take for the water to completely drain out of the tank?
Give your answer to the nearest hour. (2 marks)

$\int_0^T 3 + 2e^{0.5t}\, dt = 1000$ ✓

$T = 10.984$

≈ 11 hours ✓