PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Year 12 Test Four 2022

Student name: _______________ Teacher name: _______________

Task type: **Response**

Time allowed for this task: **_____40______ mins**

Number of questions: **_____6______**

Materials required: Upto 3 Calculators with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **__40____ marks**

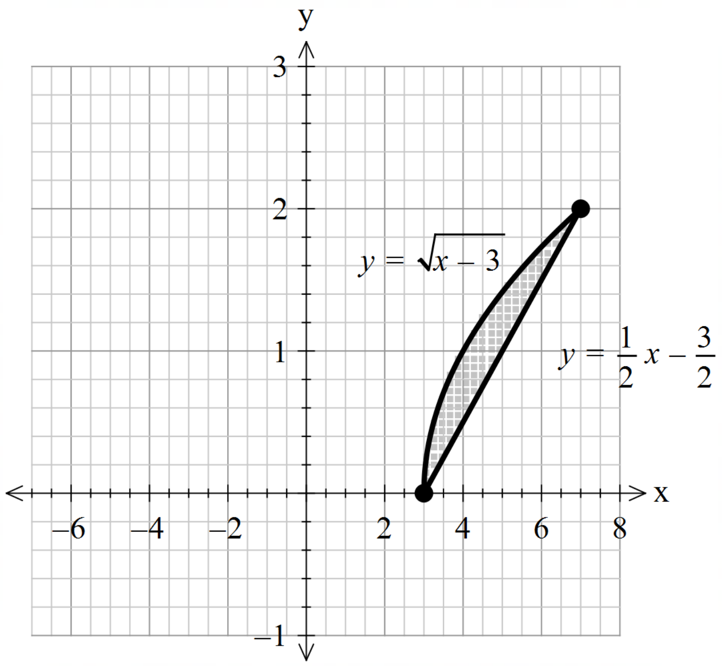
Task weighting: **_10___%**

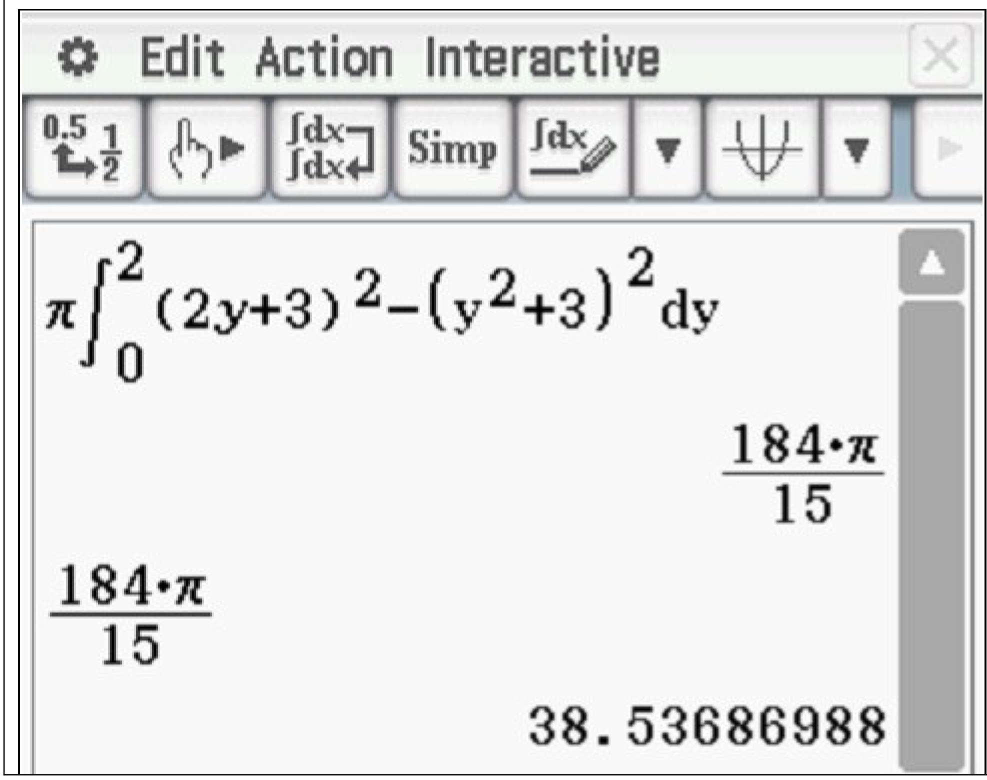
Formula sheet provided: **Yes**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (5 marks)

Determine the volume of the solid formed by rotating the area enclosed between $y=\sqrt{x-3}$ & $y=\frac{1}{2}x-\frac{3}{2}$ about the y axis, as shown below.



Solution
$\int_0^2 \pi(2y+3)^2\,dy - \int_0^2 \pi(y^2+3)^2\,dy$ 

Specific behaviours
✓ uses correct integral type ✓ determines x as the subject for each graph ✓ sets up integrals for both functions with limits ✓ uses subtraction after squaring ✓states approx. volume of solid

Q2 (5, 3 & 2= 10 marks)

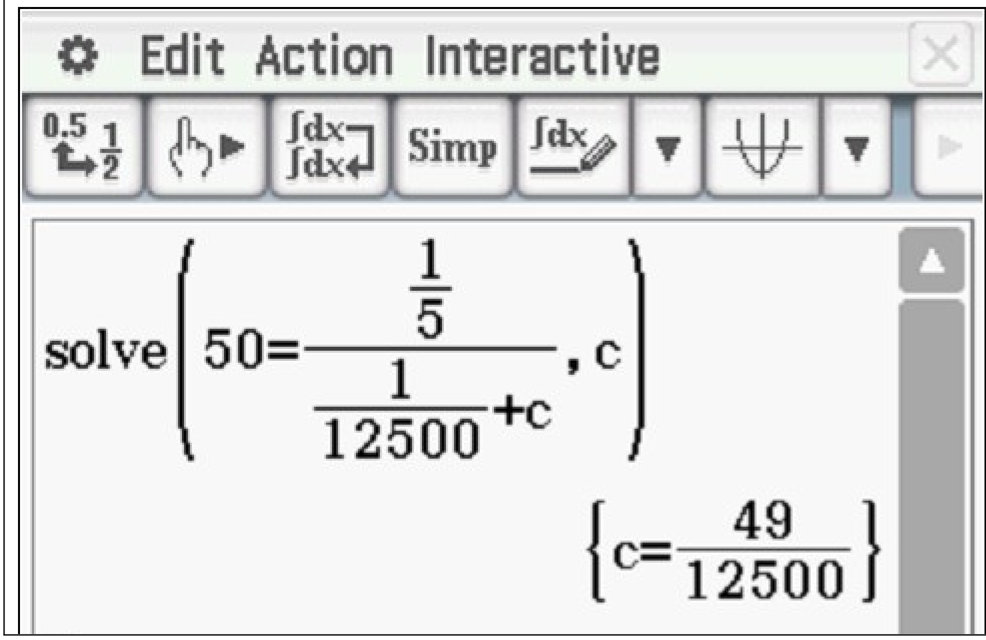
a) By using integration and partial fractions, show how to derive $N = \frac{a}{b + Ce^{-at}}$ from the differential equation $\frac{dN}{dt} = aN - bN^2$ ($a, b > 0$) and c is a constant

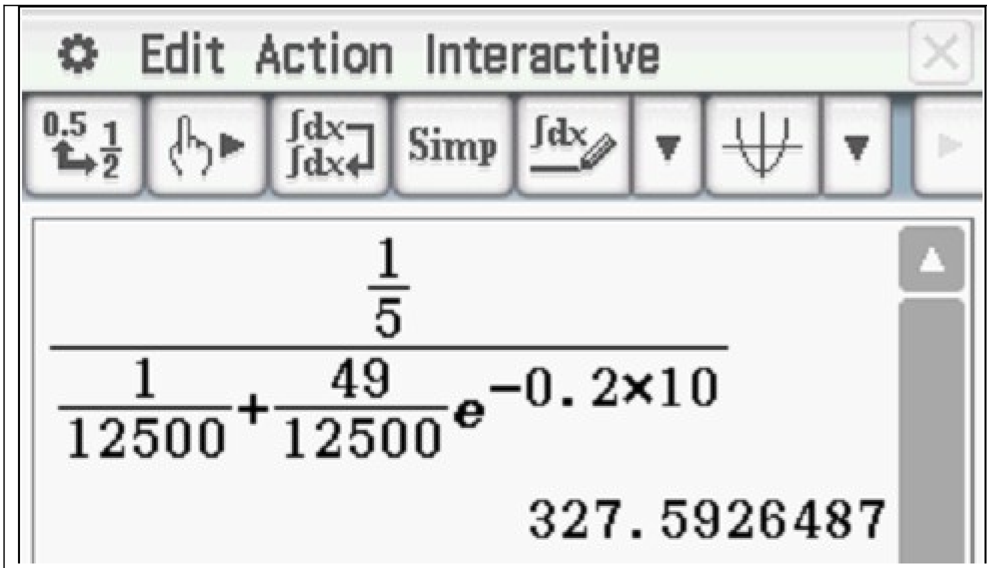
Solution
$\frac{dN}{dt} = aN - bN^2 = N(a - bN)$ $\frac{dN}{dt} \to 0, a - bN = 0 \therefore N < \frac{a}{b} \to a - bN > 0$ $\int \frac{dN}{N(a - bN)} = \int dt$ $\frac{1}{N(a - bN)} = \frac{c}{N} + \frac{d}{a - bN}$ $1 = c(a - bN) + dN$ $N = 0$ $1 = ca, c = \frac{1}{a}$ $N = \frac{a}{b}$ $1 = d\frac{a}{b}, d = \frac{b}{a}$ $\int \frac{\frac{1}{a}}{N} + \frac{\frac{b}{a}}{a - bN} dn = t + c$ $\frac{1}{a}\ln\lvert N\rvert - \frac{1}{a}\ln\lvert a - bN\rvert = t + c$ $As\ a - bN > 0$ $\ln N - \ln(a - bN) = at + c$ $\ln\frac{N}{a - bN} = at + c$ $\frac{N}{a - bN} = e^{at+c} = Ce^{at}, \frac{a - bN}{N} = Ce^{-at}$ $a - bN = NCe^{-at}$ $a = N(b + Ce^{-at}), N = \frac{a}{b + Ce^{-at}}$

Specific behaviours
✓ explains limit of N and sign of a-bN ✓ separates dN & dt and integrates ✓ uses partial fractions ✓ uses logs and obtains expression of N in terms of t ✓shows derivation of final rule

Q2 continued

b) Let N equal the number of kangaroos living in a habitat after t years and $\frac{dN}{dt}=\frac{1}{5}N-\frac{1}{12500}N^2$ .If initially there are 50 kangaroos, determine the number in 10 years time.

Solution
$N=\dfrac{\frac{1}{5}}{\frac{1}{12500}+ce^{-0.2t}}$ 



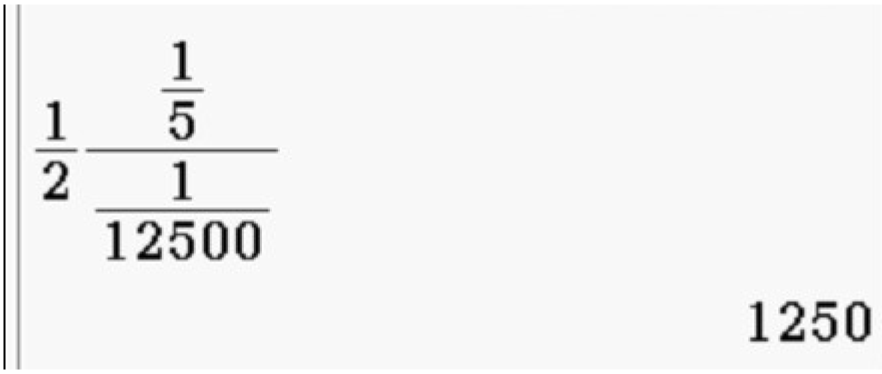
Specific behaviours

✓ solves for constant
✓ subs t=10 into correct expression
✓ states population (accept decimal)

c) Determine the size of the population at the maximum growth rate.

Solution

$N = \frac{1}{2}\frac{a}{b}$

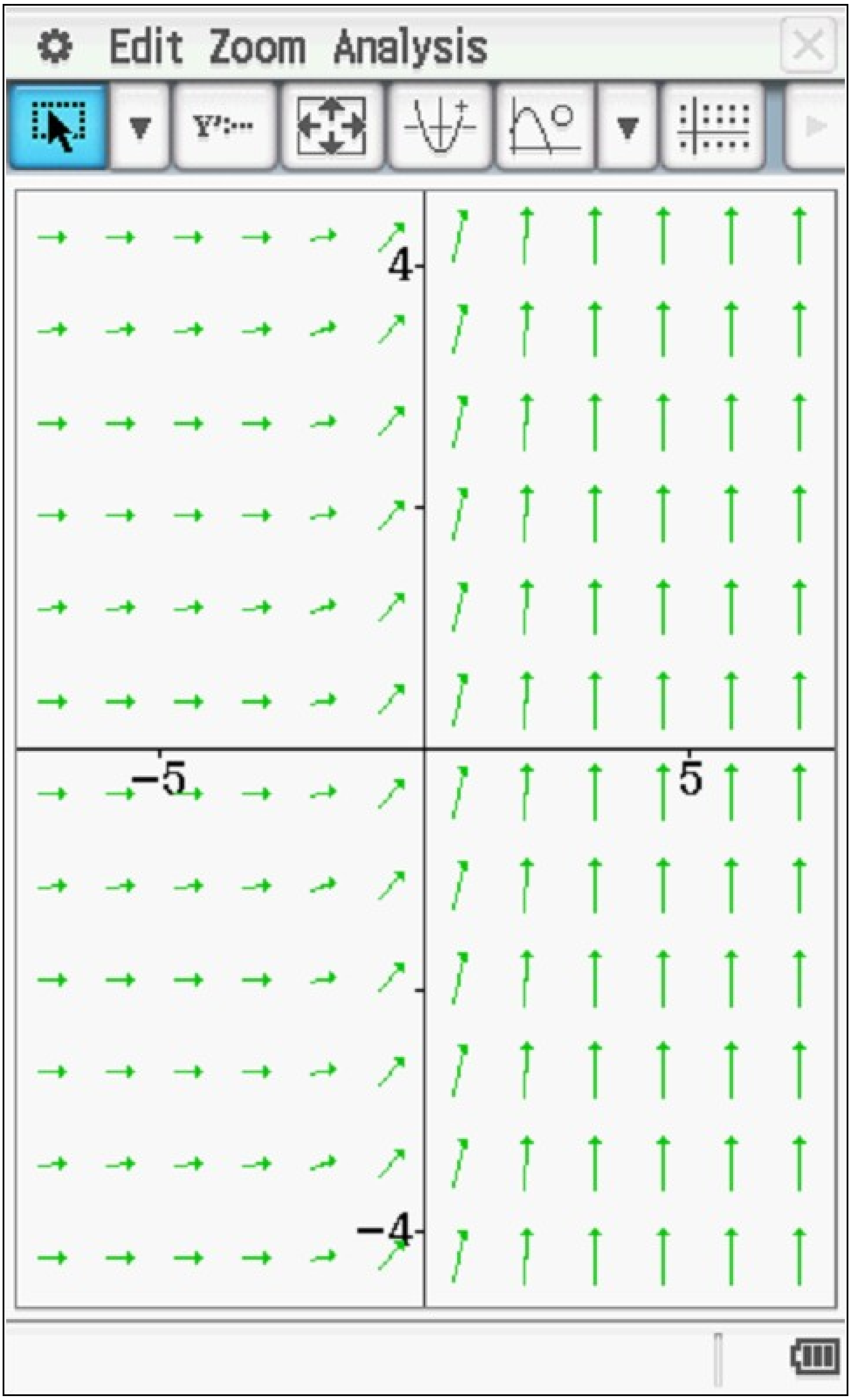


Specific behaviours

✓ uses half of maximum
✓ States population

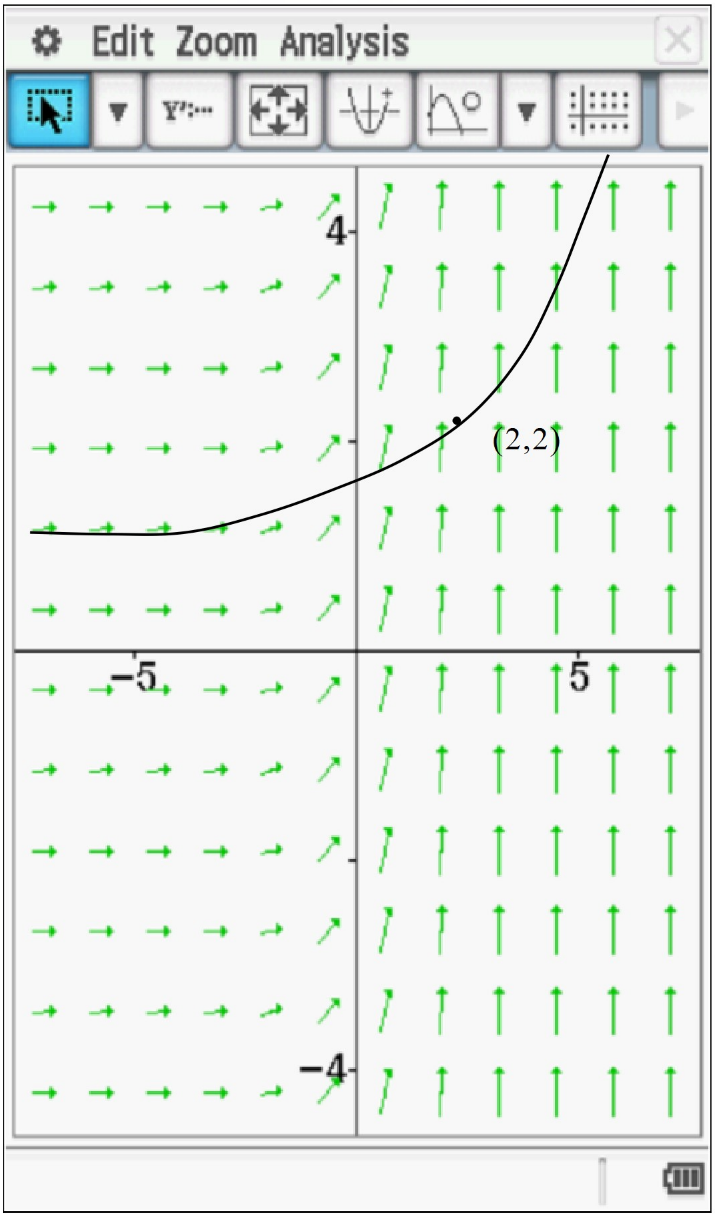
Q3 (3, 2 & 3 = 8 marks)

a) Sketch the slope field on the axes below for $\frac{dy}{dx} = 3^x$

Solution


Specific behaviours
✓ left side near zero gradients ✓ 45 degrees on y axis i.e 1 ✓ right side approaches vertical lines, i.e infinite

b) Show the solution curve on the axes above that passes through point (2,2).

Solution

Specific behaviours
✓ follows contours ✓ passes through (2,2)

c) Determine in cartesian form the solution curve for b above **without using a classpad**.

Hint – use logarithmic differentiation. Show all working.

Solution
$y=3^x$ $\ln y=\ln 3^x=x\ln 3$ $\frac{1}{y}y'=\ln 3$ $y'=y\ln 3=3^x\ln 3$ $\int 3^x dx=\frac{1}{\ln 3}3^x+c$ $\frac{dy}{dx}=3^x$ $y=\frac{1}{\ln 3}3^x+c$ $2=\frac{9}{ln3}+c$ $c=2-\frac{9}{\ln 3}$ $y=\frac{1}{\ln 3}3^x+2-\frac{9}{\ln 3}$
Specific behaviours
✓ uses log diff to diff exponential ✓ uses integration ✓ solves for exact constant **Note max 1 out of 3 if log diff not shown**

Q4 (5 marks)

Determine expressions in terms of $x \,\&\, y$ only for $\frac{dy}{dx} \,\&\, \frac{d^2y}{dx^2}$ in terms of $x, y \,\&\, y'$ using the following equation $x^3y^2 = 5 - xy$

Solution
$x^3y^2 = 5 - xy$ $x^3(2yy') + y^2 3x^2 = -xy' - y$ $y'(2x^3y + x) = -y(1 + 3x^2y)$ $y' = \dfrac{-y(1 + 3x^2y)}{(2x^3y + x)}$ $y'(2x^3y' + 6x^2y + 1) + (2x^3y + x)y'' = -y(3x^2y' + 6xy) - y'(1 + 3x^2y)$ $y'' = \dfrac{-y(3x^2y' + 6xy) - y'(1 + 3x^2y) - y'(2x^3y' + 6x^2y + 1)}{(2x^3y + x)}$
Specific behaviours
✓ uses product rule on both sides for first derivative ✓ uses implicit diff in terms of x ✓ uses product/quotient rule for second drivative ✓ obtains an expression with second derivative ✓makes second derivative subject in terms of x,y and first derivative

Q5 (3 & 3 = 6 marks)

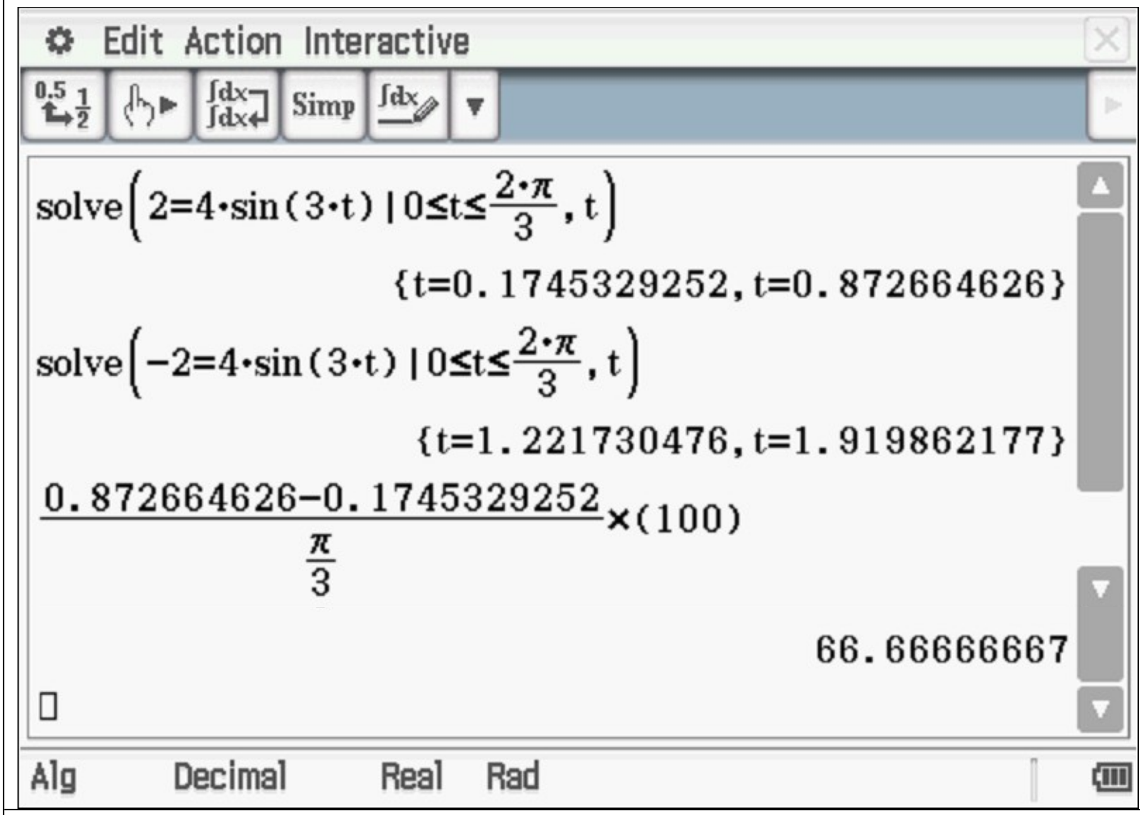
Consider a particle that is moving with SHM such that $\ddot{x} = -9x$ with a maximum speed of 12 m/s.

a) Determine the exact speed when the particle is half of an amplitude from the origin.

Solution
$\ddot{x} = -9x$ $n = 3$ $12 = nA = 3A$ $A = 4$ $v^2 = n^2\left(A^2 - x^2\right) = 9(16 - 4)$ $v = \sqrt{108}$
Specific behaviours
✓ determines n & A ✓ uses correct formula ✓ states exact speed

b) Determine the percentage of the time that the particle is more than half an amplitude from the centre.

Solution
Let $x = A\sin nt = 4\sin 3t$



Specific behaviours

✓ determines a model for displacement and period ✓ solves for times at half an amplitude in one cycle/(half cycle) ✓ determines percentage of time

Q6 (4 & 2 = 6 marks)

The motion of a bullet through a wall is modelled by the equation $a = -25(v+75)^2, \quad v > 0$ where $a\ m/s^2$ is its acceleration and $v\ m/s$ its velocity t seconds after impact. Initially the speed is 300 and is at the origin ($x=0$ metres)

a) Determine x in terms of v.

Solution
$v\dfrac{dv}{dx} = -25(v+75)^2, \quad v>0$ $\int \dfrac{v}{(v+75)^2}dv = \int -25dx$ $let\ y = v+75$ $\int \dfrac{y-75}{y^2}dy = \int y^{-1} - 75y^{-2}\,dy = \ln\lvert y\rvert + 75y^{-1} = \ln\lvert v+75\rvert + \dfrac{75}{v+75} = -25x + c$ $x=0, v=300$ $\ln 375 + \dfrac{75}{375} = c$ $x = \dfrac{1}{-25}\left(\ln\lvert v+75\rvert + \dfrac{75}{v+75} - \ln 375 - \dfrac{75}{375}\right)$
Specific behaviours
✓ uses dv & dx and separates variables v & x ✓Integrates both sides ✓ changes variable to integrate dv ✓ solves for exact constant

Q6 continued-

b) Determine how far the bullet penetrates the wall before coming to rest to the nearest mm.

Solution
$x = \dfrac{1}{-25}\left(\ln\lvert 75\rvert + 1 - \ln 375 - \dfrac{75}{375}\right)$ $\approx 0.0324m$ $\approx 32mm$

Specific behaviours
✓ subs v=0 ✓ rounds to nearest mm