WA Exams Practice Paper A, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work:	ten minutes
Working time for section:	one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet

Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items:	pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters
Special items:	drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	99	65
			Total	151	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Additional working space

Question number: _________

Question 21 **(9 marks)**

(a) A discrete random variable X can only assume the values 0, 1, 2 and 3.
Given $P(X \geq 2) = 0.6$, $P(X \leq 2) = 0.85$ and $E(X) = 1.7$, determine

(i) $P(X = 1)$. (3 marks)

$P(X \leq 2) = 0.85 \Rightarrow P(X = 3) = 0.15$

$P(X = 2) + P(X = 3) = 0.6 \Rightarrow P(X = 2) = 0.45$

$0 \times P(X = 0) + 1 \times P(X = 1) + 2 \times 0.45 + 3 \times 0.15 = 1.7$
$P(X = 1) + 0.9 + 0.45 = 1.7 \Rightarrow P(X = 1) = 0.35$

(ii) $Var(X)$. (2 marks)

$P(X = 0) = 0.05$

Using CAS, $Var(X) = 0.61$.

(b) A school caretaker locked up all 25 classrooms at a primary school at the end of each day. He also turned off any lights that had been left on in each classroom, and recorded the number of classrooms, N, where he had to do this over a 20-day period. The results are summarised in the table below.

N	0	1	2	3	4	5	6	7	8
Frequency	0	0	1	3	7	6	2	0	1

(i) Use this information to determine an estimate for the proportion of classrooms that had lights left on each day. (2 marks)

$2 + 9 + 28 + 30 + 12 + 8 = 89$
$20 \times 25 = 500$
$p = 89 / 500 = 0.178$

(ii) Calculate the probability that on a randomly chosen day, more than eight classrooms have lights left on. (2 marks)

$N \sim B(25,\ 0.178)$

$P(N \geq 9) = 0.0236$

End of questions

Section Two: Calculator-assumed **65% (99 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 **(5 marks)**

The concentration of a pollutant in a water reservoir can be expressed by the equation $C = 82e^{kt}$, where C is the concentration in parts per million t days after observations began and k is a constant.

After two weeks the concentration was 35 ppm.

(a) Determine the value of k. (2 marks)

$35 = 82e^{14k}$

$k \approx -0.0608$

(b) Determine the concentration of the pollutant after three weeks. (1 mark)

$C = 82e^{-0.0608 \times 21}$
$= 22.9$ ppm

(c) The water can be used for drinking once the concentration of the pollutant falls below 5 parts per million. Determine how long it will take for the concentration to reach this level. (2 marks)

$5 = 82e^{-0.0608 \times t}$

$t = 45.999$
≈ 46 days

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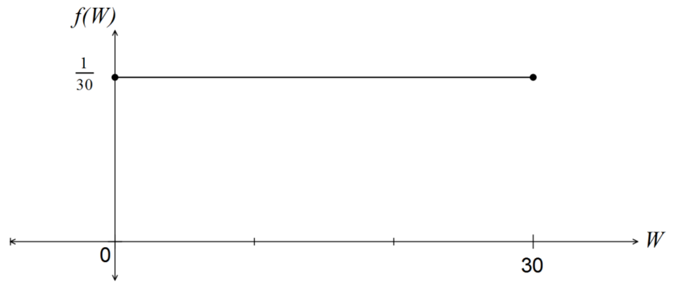
Question 10 **(8 marks)**

A bus service departs from a terminus every 30 minutes throughout the day. If a passenger arrives at the terminus at a random time to catch the bus, their waiting time in minutes, W, until the next bus departs is a uniformly distributed random variable.

(a) Sketch the graph of the probability density function of W. (2 marks)



(b) What is the probability that a passenger who arrives at the terminus at a random time has to wait no more than 25 minutes for the bus to depart? (1 mark)

$$\frac{25-0}{30}=\frac{5}{6}$$

(c) What is the probability that fewer than four passengers, out of a random selection of ten, have to wait at least 25 minutes for the bus to depart? (2 marks)

$X \sim B(10, \frac{1}{6})$

$P(X \le 3) = 0.9303$

(d) Determine the probability that a passenger who waits for at least 12 minutes has to wait for no more than 20 minutes. (1 mark)

$$\frac{20-12}{30-12}=\frac{8}{18}=\frac{4}{9}$$

(e) Determine the value of w for which $P(W < w) = P(W > 3w)$. (2 marks)

$w = 30 - 3w$

$4w = 30 \Rightarrow w = 7.5$

(c) State the maximum volume of the cone, and the dimensions to achieve this maximum. (3 marks)

$\sin^2\theta = \frac{1}{3} \Rightarrow \sin\theta = \frac{1}{\sqrt{3}}$

$h = \frac{2\sqrt{6}}{\sqrt{3}} = 2\sqrt{2}$ (≈ 2.828) cm

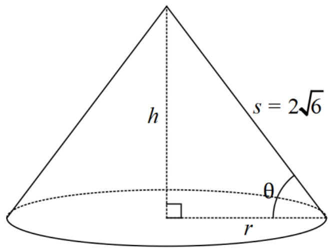
$r = \frac{2\sqrt{6}\sqrt{2}}{\sqrt{3}} = 4$ cm

$V = \frac{1}{3}\pi \times 4^2 \times 2\sqrt{2}$

$= \frac{32\sqrt{2}\pi}{3}$ (≈ 47.39) cm^3

Question 20 **(9 marks)**

A cone has a slant height of $2\sqrt{6}$ cm. The sloping edge makes an angle of θ with the base, where $0 < \theta < \frac{\pi}{2}$.



(a) State expressions for the height and radius of the cone in terms of θ and hence show that the volume of the cone is given by $V = 16\sqrt{6}\pi(\sin\theta - \sin^3\theta)$. (3 marks)

$h = 2\sqrt{6}\sin\theta$

$r = 2\sqrt{6}\cos\theta$

$$V = \frac{1}{3}\pi\left(2\sqrt{6}\cos\theta\right)^2\left(2\sqrt{6}\sin\theta\right)$$
$$= 16\sqrt{6}\pi\cos^2\theta\sin\theta$$
$$= 16\sqrt{6}\pi\left(\sin\theta - \sin^3\theta\right)$$

(b) Determine the value of θ that maximises the volume of the cone, rounding your answer to four decimal places. (3 marks)

$$\frac{dV}{d\theta} = 16\sqrt{6}\pi\left(\cos\theta - 3\cos\theta\sin^2\theta\right)$$

$$\frac{dV}{d\theta} = 0 \Rightarrow \cos\theta\left(1 - 3\sin^2\theta\right) = 0$$

$$\sin^2\theta = \frac{1}{3} \Rightarrow \theta = 0.6155$$

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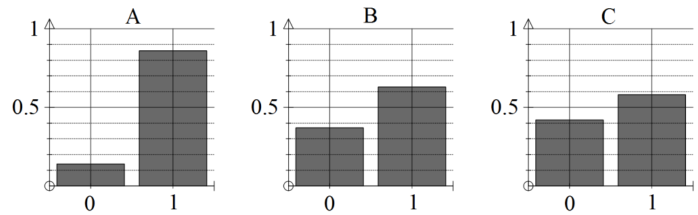
Question 11 **(7 marks)**

A random variable that can only take the values of 0 or 1 is modelled by a Bernoulli distribution such that $P(1) = 0.65$.

(a) Three random samples were taken from this distribution. One sample was of size 25, one of size 100 and the other size 500. The following three relative frequency graphs each show the distribution of one of the random samples.



(i) State, with reasons, the letter of the graph that is most likely to be that for the sample of size 25. (2 marks)

A

Very unlikely to have a relative frequency of 0.85 for 1 when P(1)=0.65 unless sample size is small.

(ii) State, with reasons, the letter of the graph that is most likely to be that for the sample of size 500. (2 marks)

B

Can't be A, and the relative frequency of B (0.63) is closer to population proportion of 0.65 than C (0.58) so more likely sample size of B is larger than C.

(b) An experiment consists of recording the proportion of 1's obtained when 100 random samples are taken from this distribution. Describe the expected features of a frequency graph showing the distribution of the proportion of 1's obtained by repeating this experiment a large number of times. (3 marks)

Would expect frequency graph to show

- distribution of proportions to be normal
- distribution would be centred on $p = 0.65$
- distribution would have standard deviation of $\sqrt{\frac{0.65 \times 0.35}{100}} \approx 0.048$

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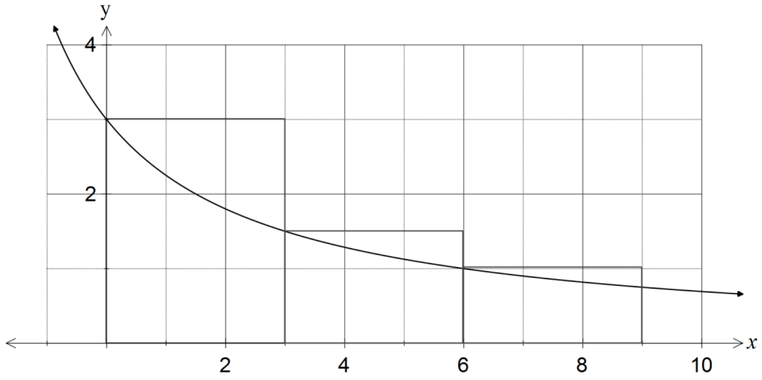
Question 12 **(6 marks)**

The graph below shows the function $f(x) = \dfrac{9}{x+3}$.



An estimate for the area under the curve between $x = 0$ and $x = 9$ is required.

(a) Three left-rectangles are shown on the diagram. Use these rectangles to calculate an over-estimate for the area. (2 marks)

$$f(\{0,\ 3,\ 6,\ 9\}) = \{3,\ 1.5,\ 1,\ 0.75\}$$

$$A = 3 \times (3 + 1.5 + 1)$$
$$= 16.5 \text{ sq u}$$

(b) Use three right-rectangles to calculate an under-estimate for the area. (2 marks)

$$A = 3 \times (1.5 + 1 + 0.75)$$
$$= 9.75 \text{ sq u}$$

(c) Use your over- and under- estimates to calculate a better estimate for the area under the curve between $x = 0$ and $x = 9$. (1 mark)

$$A = \frac{16.5 + 9.75}{2} = 13.125$$

(d) The exact area is $18\log_e 2$. Calculate the error in the best estimate above as a percentage of the exact area. (1 mark)

$$\frac{13.125 - 18\ln 2}{18\ln 2} \times 100 = 5.2\%$$

Question 19 **(8 marks)**

In a recent survey carried out in a suburb, 52 people out of a random sample of 92 said they regularly shopped in supermarket A. A similar survey in the same suburb found that 62 people out of a random sample of 90 said they regularly shopped in supermarket B.

(a) Compare the proportion of people in these surveys who regularly shop in supermarket A to the proportion of people who regularly shop in supermarket B. (2 marks)

$$A:\ \frac{52}{92} \approx 0.5652$$
$$B:\ \frac{62}{90} = 0.6889$$

The surveys show a higher proportion of people shop in B.

(b) Calculate a 95% confidence interval for the proportion of people in the suburb who regularly shop in supermarket A. (2 marks)

$$0.5652 \pm \sqrt{\frac{0.5652(1-0.5652)}{92}} = (0.464, 0.667)$$

(c) Calculate a 95% confidence interval for the proportion of people in the suburb who regularly shop in supermarket B. (2 marks)

$$0.6889 \pm \sqrt{\frac{0.6889(1-0.6889)}{90}} = (0.593, 0.785)$$

(d) On the basis of these surveys, is there any evidence that a higher proportion of people living in the suburb regularly shop in supermarket B than supermarket A? Justify your answer. (2 marks)

No evidence that B has a higher proportion than A, as the two confidence intervals overlap. If the surveys were repeated, it is possible that we find that A has a higher proportion than B.

Question 18 **(6 marks)**

The magnitude of an earthquake on the Richter scale, R, is given by $R = \log I$, $I \geq 1$, where I is the relative intensity of the earthquake compared to the smallest seismic activity that can be measured. The location, year and magnitude of three large earthquakes are given in this table.

Earthquake	Meeberrie, 1941	Sumatra, 2005	Valdivia, 1960
Magnitude	7.2	8.6	9.5

(a) Calculate the magnitude of the 1968 earthquake in Meckering, 130 km inland from Perth, which had a relative intensity of 7 940 000. (1 mark)

$$R = \log(7940000)$$
$$= 6.9$$

(b) Determine the relative intensity of the 2005 earthquake in Sumatra, rounded to 3 significant figures? (2 marks)

$$8.6 = \log(I)$$
$$I = 10^{8.6}$$
$$= 398\ 000\ 000$$

(c) The strongest earthquake ever recorded in the world was at Valdivia, Chile, in 1960. The strongest onshore earthquake ever recorded in Australia was at Meeberrie in 1941.

Express the ratio of the relative intensities of the Meeberrie to the Valdivia earthquake in the form $1:n$. (3 marks)

$I_m = 10^{7.2}$ $\quad I_v = 10^{9.5}$

Ratio is $1:\dfrac{10^{9.5}}{10^{7.2}}$

$$= 1:10^{2.3}$$
$$= 1:199.53$$
$$= 1:200$$

Question 13 **(6 marks)**

An interval estimate for the proportion of germinating seeds from a particular plant supplier is to be constructed.

(a) State, with reasons, whether a 90% or a 95% confidence interval will have a larger margin of error. (2 marks)

95% will be larger.

To be more confident, the margin of error must increase, reflected in the fact that the z-score for 0.95 (~1.96) is larger than the z-score for 0.9 (~1.645).

(b) State, with reasons, whether a confidence interval constructed from a random sample of 100 seeds will have a margin of error four times that of an interval constructed from a random sample of 400 seeds. (2 marks)

No, it won't.

$E \propto \dfrac{1}{\sqrt{n}}$ and $\dfrac{1}{\sqrt{100}} = \dfrac{1}{10}$, $\dfrac{1}{\sqrt{400}} = \dfrac{1}{20}$

Margin of error will be twice as wide.

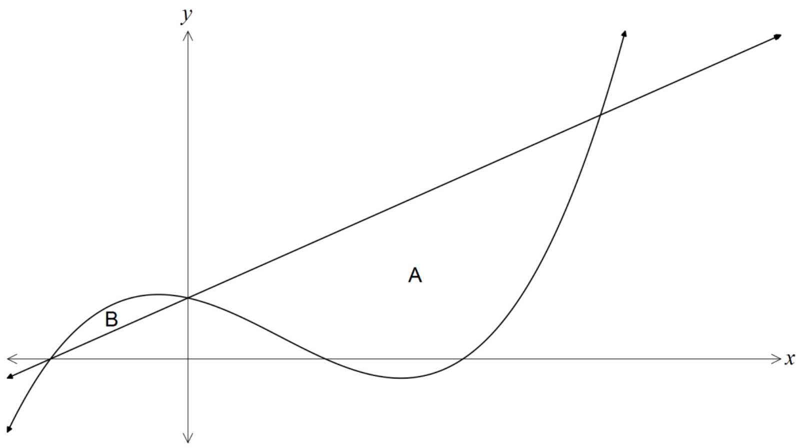
(c) State, with reasons, whether a larger or smaller sample size is required to maintain a constant margin of error as the proportion of germinating seeds increases from 60% to 90%. (2 marks)

Smaller.

As p moves away from 0.5, $p(1\text{-}p)$ will decrease and so a smaller sample is required to maintain a constant margin of error.

Question 14 **(9 marks)**

The graphs of $y = 2x + 2$ and $y = f(x)$ are shown below, where $f(x) = x^3 - 2x^2 - x + 2$.



(a) Use calculus techniques to show and justify that a point of inflection exists on the graph of $y = f(x)$ when $x = \frac{2}{3}$. (4 marks)

$f'(x) = 3x^2 - 4x - 1$
$f''(x) = 6x - 4$

$f''(x) = 0 \Rightarrow x = \dfrac{2}{3}$

$f''(0) = -4 \Rightarrow$ curve is concave down
$f''(1) = 2 \Rightarrow$ curve is concave up

As x increases from 0 through $\frac{2}{3}$ to 1, so the curve changes from concave down to concave up. Hence a point of inflection exists when $x = \frac{2}{3}$.

(c) Calculate the displacement of the body when it has a velocity of $\frac{41}{3}$ m/s. (5 marks)

$v = \dfrac{41}{3} \Rightarrow \dfrac{t^2}{4} - 2t + \dfrac{5}{3} = \dfrac{41}{3}$

$t = \cancel{-4},\ 12$

$x = \int \dfrac{t^2}{4} - 2t + \dfrac{5}{3}\,dt$
$= \dfrac{t^3}{12} - t^2 + \dfrac{5t}{3} + k$

$t = 0,\ x = 8 \Rightarrow k = 8$
$x = \dfrac{t^3}{12} - t^2 + \dfrac{5t}{3} + 8$

$x(12) = 28$ m.

Question 17 **(10 marks)**

A small body with an initial displacement and velocity of 8 m and $\frac{5}{3}$ m/s respectively, travels in a straight line with acceleration at time t of $0.5t-2$ m/s², $t \geq 0$.

(a) Determine an expression for the velocity, v, of the body at time t. (2 marks)

$$v = \int 0.5t - 2dt = \frac{t^2}{4} - 2t + c$$

$$t = 0,\ v = \frac{5}{3} \Rightarrow c = \frac{5}{3}$$

$$v = \frac{t^2}{4} - 2t + \frac{5}{3}$$

(b) Determine minimum velocity of the body for $t \geq 0$ and apply the second derivative test to justify that it is a minimum. (3 marks)

$$v_{\min} \Rightarrow 0.5t - 2 = 0 \Rightarrow t = 4$$

$$v(4) = -\frac{7}{3} \text{ m/s.}$$

$$v'' = 0.5 \Rightarrow +ve \Rightarrow \text{minimum.}$$

See next page

(b) Two regions are trapped between the linear and cubic functions, marked A and B on the diagram. Show that the difference in the areas of these two regions is $10\frac{2}{3}$ square units. (5 marks)

$$x^3 - 2x^2 - x + 2 = 2x + 2$$

$$x = -1,\ 0,\ 3$$

$$A: \int_0^3 (2x+2) - (x^3 - 2x^2 - x + 2)\ dx = \frac{45}{4}$$

$$B: \int_{-1}^0 (x^3 - 2x^2 - x + 2) - (2x+2)\ dx = \frac{7}{12}$$

$$\frac{45}{4} - \frac{7}{12} = \frac{32}{3} = 10\tfrac{2}{3} \text{ sq u}$$

See next page

Question 15 **(8 marks)**

A bakery packages a loaf of bread as a Standard if it weighs between 450 g and 500 g. The weights of all loaves produced by the bakery are normally distributed with a mean of 470 g and a standard deviation of 16 g.

(a) What is the probability that a randomly selected loaf produced by the bakery

(i) weighs 450 g? (1 mark)

$$P(450 < X < 450) = 0$$

(ii) is a Standard loaf? (1 mark)

$$P(450 < X < 500) = 0.8640$$

(b) In a batch of 250 loaves, how many would be expected to weigh less than a Standard loaf? (2 marks)

$$P(X < 450) = 0.1056$$

$$250 \times 0.1056 = 26.4$$
$$\approx 26 \text{ loaves}$$

(c) Determine the probability that a randomly selected Standard loaf weighs less than 470 g. (2 marks)

$$\begin{aligned} &P(X < 470 \mid 450 < X < 500) \\ &= \frac{P(450 < X < 470)}{P(450 < X < 500)} \\ &= \frac{0.3944}{0.8640} \\ &= 0.4565 \end{aligned}$$

(d) A bakery worker randomly selects loaves of bread until the loaf chosen is not a Standard. Determine the probability that the sixth loaf chosen is the first that is not a Standard. (2 marks)

$$P = 0.8640^5 \times (1 - 0.8640) = 0.0655$$

Question 16 **(8 marks)**

It is known that 15% of households in a large city own a cat.

(a) Let X be the random variable that represents the number of cat owning households chosen in a random sample of 60 households.

(i) Describe the distribution of X and state its mean and standard deviation. (3 marks)

$$X \sim B(60, 0.15)$$

$$\bar{x} = 60 \times 0.15 = 9$$

$$sd = \sqrt{60 \times 0.15 \times 0.85} = 2.766$$

(ii) Determine the probability that 12 or more cat owning households will be chosen in a random sample of 60. (1 mark)

$$P(X \geq 12) = 0.1806$$

(b) A large number of random samples of 60 households are taken and each sample is used to calculate a point estimate for the proportion of cat owning households in the city.

(i) Describe the distribution of these sample proportions and state the mean and standard deviation of the distribution. (3 marks)

$$X \sim N(p, s^2)$$

$$p = 0.15$$

$$s = \sqrt{\frac{0.15 \times 0.85}{60}} = 0.0461$$

(ii) Determine the probability that a point estimate for the proportion of cat owning households in the city exceeds 0.2. (1 mark)

$$P(X > 0.2) = 0.1390$$