PERTH MODERN SCHOOL

UNIT 3CD MAS – 2015

TEST 1: SOLUTIONS

POLAR COORDINATES, COMPLEX NUMBERS & VECTORS

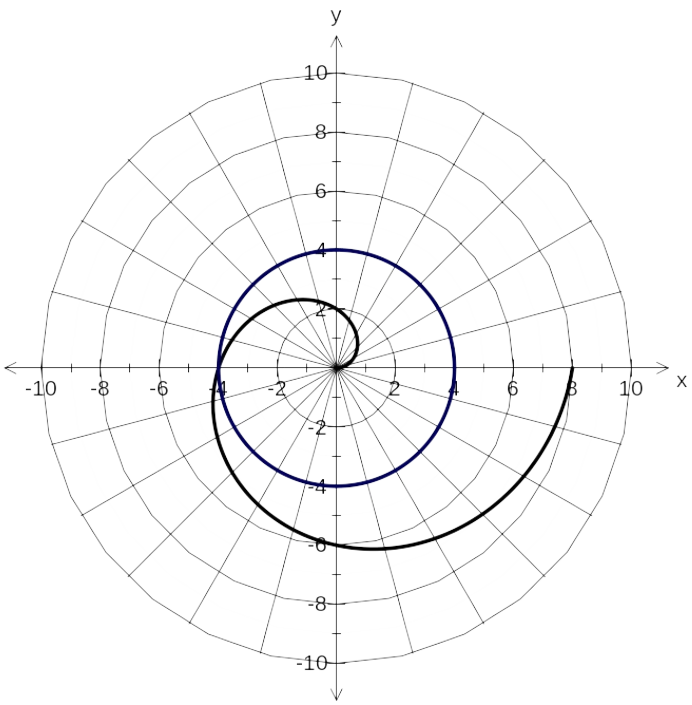
NAME: ______________________________ DATE: Feb.

Total: 45 marks Time: 50 min.

Question 1. **(5 marks)**

(i) Point P has polar coordinates (4,17°) and it lies on the line $y = -x + 5$
Point Q also lies on the line and is 12cm away from P.

Find the polar coordinates of Q $[r,\theta]$ where $90^0 < \theta < 180^0$. (3 marks)



Solution
$OQ^2 = 12^2 + 4^2 - 2(12)(4)\cos 62^0$ $OQ = 10.72$ $12^2 = 4^2 + 10.72^2 - 2(4)(10.72)\cos\angle POQ$ $\angle POQ = 98.77^0$ Polar Coordinates of Q = $\left[10.72, 115.77^0\right]$
Specific behaviours
✓angle of 62^0, ✓✓ $\left[10.72, 115.77^0\right]$

(ii) Sketch the graph of $r = \frac{4}{\pi}\theta$ on the axes above and hence state where it intersects $r = 4$ (2 marks)

Question 2 **(9 marks)**

On a 3D computer game, Chris, a keen cyclist leaves from *position* $(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ metres is travelling at $(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$ m/s while his mate Dave leaves from position $(a\mathbf{i} + \mathbf{j} + b\mathbf{k})$ metres running at $(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ m/s.

(i) Although they do not collide, their paths do intersect at the point with coordinates (*a*, 1, *b*).

Determine the values of *a* and *b.* (4 marks)

Solution
$\underset{\sim}{r}_C = (1+2\lambda)\underset{\sim}{i} + (-1+3\lambda)\underset{\sim}{j} + (2-6\lambda)\underset{\sim}{k}$ As paths meet at (a, 1, b), then (a, 1, b) lies on $\underset{\sim}{r}_C$ i.e. $(1+2\lambda) = a$, $(-1+3\lambda) = 1$, $(2-6\lambda) = b$ ✓ Solving these equations simultaneously, $\lambda = \frac{2}{3},\ a = \frac{7}{3},\ b = -2$ ✓✓✓
Specific behaviours
As allocated

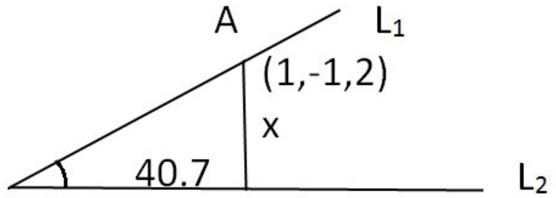
(ii) Find the acute angle between these two paths. (2 marks)

Solution
Let angle be θ $(2,3,-6)\cdot(1,-1,2) = \sqrt{4+9+36}.\sqrt{1+1+4}.\cos\theta$ ✓ θ = 139.3^0 Hence acute angle is 40.7^0✓
Specific behaviours
As allocated

(iii) Hence, determine the perpendicular distance from the point (1,-1,2) to (3 marks)

$$r = \begin{pmatrix} \frac{7}{3} \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Solution



$\left(\frac{7}{3}, 1, -2\right)$

,

$PA = \left(-\frac{7}{3}, -1, 2\right) + (1, -1, 2)$

$= \left(-\frac{4}{3}, -2, 4\right)$ ✓

$\therefore \left|\overrightarrow{PA}\right| = \frac{14}{3}$ ✓

Perpendicular distance = $\frac{14}{3}\sin 40.7^0$ = 3.04 m ✓

Specific behaviours

As allocated

Question 3 **(5 marks)**

(a) The vectors $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + a\mathbf{k}$ are perpendicular. Determine the value of **a**.
(1marks)

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 2 \\ a \end{bmatrix} = 0 \Rightarrow 3 - 2 + 2a = 0 \Rightarrow a = -\tfrac{1}{2}$$

(b) Determine whether the two lines
$\mathbf{r} = 8\mathbf{i} - \boldsymbol{j} - 8\mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{k})$ and $\mathbf{r} = \boldsymbol{j} - 3\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ intersect.
If they do intersect, state the position vector of their point of intersection.

If they do not intersect, justify your answer. (4 marks)

$\mathbf{i}:\ 8 + 2\lambda = \mu$

$\mathbf{j}:\ -1 = 1 - \mu \Rightarrow \mu = 2,\ \lambda = -3$

$\mathbf{k}:\ -8 - 3(-3) = -3 + 2(2) \Rightarrow 1 = 1 \Rightarrow$ intersect

$$\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \text{intersect at } 2\mathbf{i} - \mathbf{j} + \mathbf{k}$$

Question 4 (7 marks)

(a) If $z = 3 - 4i$, determine the reciprocal, $\frac{1}{z}$ (2 marks)

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{3+4i}{3^2+4^2} = \frac{3}{25} + \frac{4i}{25}$$

(b) Let the non-zero complex number $z = a + bi$. Show that $\frac{1}{a+bi} = \frac{\overline{Z}}{|Z|^2}$ (3 marks)

$$\begin{aligned} LHS &= \frac{1}{a+bi} \\ &= \frac{1}{a+bi} \times \frac{a-bi}{a-bi} \\ &= \frac{a-bi}{a^2+b^2} \\ &= \frac{\overline{z}}{|z|^2} \\ &= RHS \end{aligned}$$

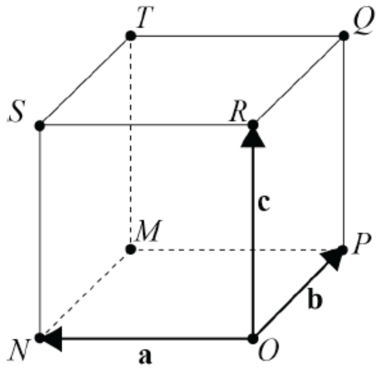
(c) Describe the geometrical relationship between any non-zero complex number and its reciprocal. (2marks)

The reciprocal z^{-1} is the conjugate of z but multiplied by scale factor of $\frac{1}{|z|}$.

So the reciprocal z^{-1} will be the reflection of z in the real axis and of length $\frac{1}{|z|}$

Question 5 **(5 marks)**

Let $MNOPQRST$ be a rectangular prism with sides $\overrightarrow{ON}$, $\overrightarrow{OP}$ and $\overrightarrow{OR}$ denoted by vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, as shown in the diagram below.



Suppose that A is the midpoint of $\overrightarrow{MN}$, B is the midpoint of $\overrightarrow{MT}$, C is the midpoint of $\overrightarrow{QR}$ and D is the midpoint of $\overrightarrow{OR}$.

(a) Express $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. (2 marks)

Solution
$\overrightarrow{OA} = \mathbf{a} + \dfrac{\mathbf{b}}{2}$ $\overrightarrow{OB} = \mathbf{a} + \mathbf{b} + \dfrac{\mathbf{c}}{2}$ $\overrightarrow{OC} = \mathbf{c} + \dfrac{\mathbf{b}}{2}$ $\overrightarrow{OD} = \dfrac{\mathbf{c}}{2}$
Specific behaviours
✓ correctly determines $\overrightarrow{OA}$ and $\overrightarrow{OC}$ from the diagram ✓ correctly determines $\overrightarrow{OB}$ and $\overrightarrow{OD}$ from the diagram

(b) Prove that the quadrilateral $ABCD$ is a parallelogram. (3 marks)

Solution
$\overrightarrow{BC} = \mathbf{c} + \frac{\mathbf{b}}{2} - \mathbf{a} - \mathbf{b} - \frac{\mathbf{c}}{2}$ $= \frac{\mathbf{c}}{2} - \frac{\mathbf{b}}{2} - \mathbf{a}$ $\overrightarrow{AD} = \frac{\mathbf{c}}{2} - \mathbf{a} - \frac{\mathbf{b}}{2}$ $= \overrightarrow{BC}$ $\therefore \overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$ $\overrightarrow{AB} = \mathbf{a} + \mathbf{b} + \frac{\mathbf{c}}{2} - \mathbf{a} - \frac{\mathbf{b}}{2}$ $= \frac{\mathbf{b}}{2} + \frac{\mathbf{c}}{2}$ $\overrightarrow{DC} = \mathbf{c} + \frac{\mathbf{b}}{2} - \frac{\mathbf{c}}{2}$ $= \frac{\mathbf{b}}{2} + \frac{\mathbf{c}}{2}$ $= \overrightarrow{AB}$ $\therefore \overrightarrow{DC}$ is parallel to $\overrightarrow{AB}$ i.e. Opposite sides of the quadrilateral $ABCD$ are parallel $\therefore ABCD$ is a parallelogram
Specific behaviours
✓ correctly determines $\overrightarrow{BC}$ and $\overrightarrow{AD}$ in terms of **a**, **b** and **c** ✓ correctly determines $\overrightarrow{AB}$ and $\overrightarrow{DC}$ in terms of **a**, **b** and **c** ✓ states that two pairs of sides are parallel to deduce the result

Question 6 **(5 marks)**

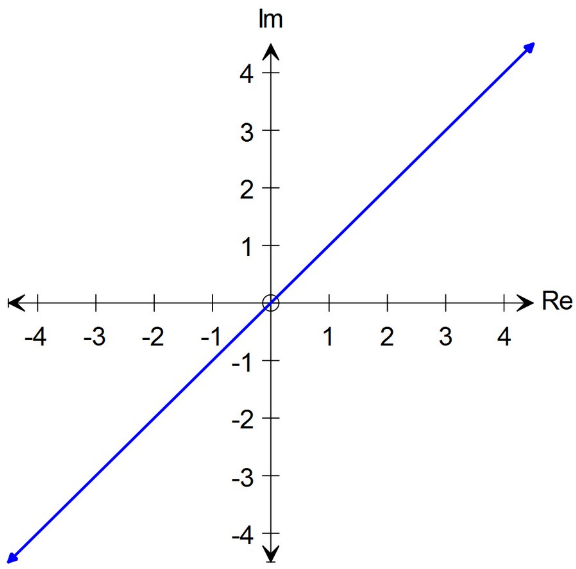
(5 marks)

(a) Change the complex equation $|Z - i| = |Z - 1|$ into its Cartesian equivalent. (3 marks)

$$|Z - i| = |Z - 1|$$
$$|x + yi - i| = |x + yi - 1|$$
$$|x + i(y - 1)| = |(x - 1) + yi|$$
$$x^2 + (y - 1)^2 = (x - 1)^2 + y^2$$
$$x^2 + y^2 - 2y + 1 = x^2 - 2x + 1 + y^2$$
$$2x - 2y = 0$$

$\therefore$ $\boldsymbol{x - y = 0}$

(b) Hence identify, the locus of all points Z satisfying the equation in (a). (2 marks)



Question 7

An equilateral triangle has vertices A, B and C, where A is the point $\sqrt{3}-i$ in the Argand plane.

The circumcircle is drawn that passes through vertices A, B and C and has a centre inside the triangle, called the circumcentre.

The circumcentre of the triangle is located at the origin.

Find the complex numbers z_1 and z_2 corresponding to the vertices B and C, expressing your answer in exact Cartesian form.

Solution
The centre is at the origin so the other two points can be found by rotating the point A 120°.   A rotation of 120° is equivalent to multiplying by $z = 1\,\text{cis}\left(\frac{2\pi}{3}\right)$ $z_A = 2\,\text{cis}\left(\frac{-\pi}{6}\right)$, so $z_B = 1\,\text{cis}\left(\frac{2\pi}{3}\right) \times 2\,\text{cis}\left(\frac{-\pi}{6}\right) = 2\,\text{cis}\left(\frac{3\pi}{6}\right) = 2\,\text{cis}\left(\frac{\pi}{2}\right)$ $z_C = 1\,\text{cis}\left(\frac{2\pi}{3}\right) \times 2\,\text{cis}\left(\frac{\pi}{2}\right) = 2\,\text{cis}\left(\frac{7\pi}{6}\right)$ In Cartesian form: $z_B = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 2i$ $z_C = 2\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) = -\sqrt{3} - i$
Specific behaviours
✓ uses separation between vectors as 120° ✓ uses a magnitude of 2 for all the vectors   ✓ deduces the correct position for vertex B ✓ deduces the correct position for vertex C ✓ states the correct Cartesian coordinates for both B and C

Question 11 **(5 marks)**

An equilateral triangle has vertices A, B and C, where A is the point $\sqrt{3}-i$ in the Argand plane.

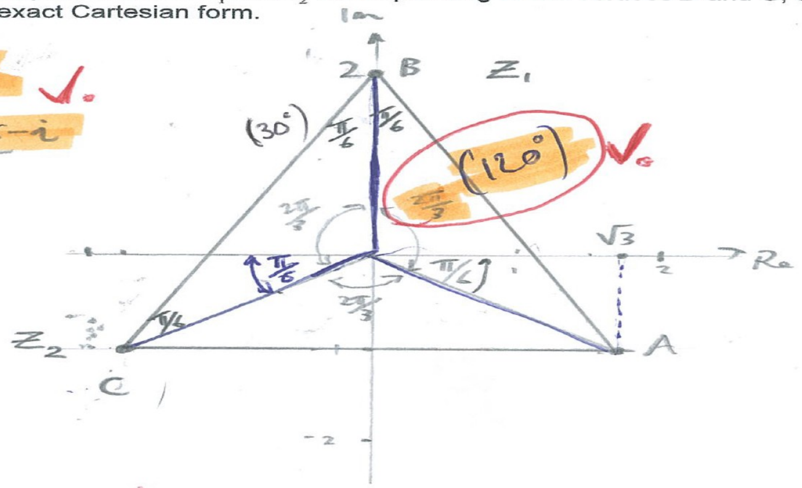
The circumcircle is drawn that passes through vertices A, B and C and has a centre inside the triangle, called the circumcentre.

The circumcentre of the triangle is located at the origin.

Find the complex numbers z_1 and z_2 corresponding to the vertices B and C, expressing your answer in exact Cartesian form.

$z_1 = 2i$ ✓

$z_2 = -\sqrt{3}-i$



$z_A = \sqrt{3}-i$

$|z_A| = \sqrt{\sqrt{3}^2+(-1)^2}$

$= 2$ ✓

arg: $\tan^{-1}\frac{1}{\sqrt{3}}$

$= 30° = \frac{\pi}{6}$

Positions

Vertex B : $2\operatorname{cis}\frac{\pi}{2}$ ✓

Vertex C : $2\operatorname{cis}\left(\pi+\frac{\pi}{6}\right)$

$2\operatorname{cis}\left(\frac{7\pi}{6}\right) = 2\operatorname{cis}\left(\frac{-5\pi}{6}\right)$ ✓

Question 8 **(4 marks)**

Find two numbers which have a product of 2 and a sum of 2.

Let the numbers be x and y

$$xy = 2 \qquad ①$$

and $x + y = 2$ ②

rearrange ② $y = 2 - x$ and subst. into ①

$$x(2 - x) = 2$$

$$2x - 2x^2 = 2$$

$$x^2 - 2x + 2 = 0$$

(*Using Classpad*)

$$x = 1 \pm i$$

$$y = 1 \pm i$$

∴ **the two numbers are 1 + i and 1 - i**