# MATHEMATICS METHODS

# MAWA Semester 2 (Unit 3&4) Examination 2018
# Calculator-assumed

# Marking Key



The release date for this exam and marking scheme is

- **the end of week 1 of term 4, Fri October 12th 2018**

**Section Two: Calculator-assumed (100 Marks)**

**Question 8 (a) (2 marks)**

| Solution | |
|---|---|
| Bernoulli distribution with parameter<br>$p = \frac{1}{300}$ | |
| Mathematical behaviours | Marks |
| • states Bernoulli distribution | 1 |
| • states parameter of $p = \frac{1}{300}$ | 1 |

**Question 8 (b) (2 marks)**

| Solution | |
|---|---|
| $E(X) = \frac{1}{300}$<br><br>Variance($X$)<br>$= \frac{1}{300}\left(1 - \frac{1}{300}\right)$<br>$= \frac{1}{300}\left(\frac{299}{300}\right)$<br>$= \frac{299}{90000}$ | |
| Mathematical behaviours | Marks |
| • states correct mean | 1 |
| • states variance | 1 |

**Question 21 (c) (iii)** **(1 mark)**

Solution

$E_{95\%} < E_{\text{simple interval}}$

Hence the interval $\hat{p} - E < p < \hat{p} + E$ is a subset of the interval $\hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$

and hence the confidence interval of $\hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$ is at least 95%

| Mathematical behaviours | Marks |
|---|---|
| • uses the relationship between the errors to draw valid conclusion | 1 |

**Question 21 (d)** **(3 marks)**

Solution

True margin of error $E = z_\alpha \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

If $E = \frac{1}{\sqrt{n}}$ and $\hat{p} = 0.7$, $\frac{1}{\sqrt{n}} = z_\alpha \sqrt{\frac{(0.7)(0.3)}{n}}$

*ie* $\frac{1}{\sqrt{n}} = z_\alpha \frac{\sqrt{0.21}}{\sqrt{n}} \Rightarrow z_\alpha \approx 2.18$

$P(-2.18 < z < 2.18) = 0.9707$ Hence level of confidence is approximately 97%

| Mathematical behaviours | Marks |
|---|---|
| • substitute $\hat{p} = 0.7$ and equates true $E$ to $E$ for simple interval | 1 |
| • solves inequality to determine $z$ | 1 |
| • states 97% level of confidence | 1 |

**Question 21 (e)** **(1 mark)**

Solution

$\frac{1}{\sqrt{n}} = 0.04 \Rightarrow n = 625$

Hence a sample size of 700 is appropriate

| Mathematical behaviours | Marks |
|---|---|
| • calculates $n$ and deduces that sample size is appropriate | 1 |





**Question 9 (a)** **(2 marks)**

Solution

$$E(X) = \mu = \int_{-\infty}^{\infty} x \times p(x)\,dx = \int_2^5 x \times \frac{1}{3}\,dx = 3.5$$

$$\sigma^2 = \int_2^5 (x-3.5)^2 \times \frac{1}{3}\,dx = 0.75$$

$\int_2^5 x \times \frac{1}{3}dx$ 3.5

$\int_2^5 (x-3.5)^\wedge 2 \times \frac{1}{3}dx$ 0.75

| Mathematical behaviours | Marks |
|---|---|
| • determines the expected value | 1 |
| • determines the variance | 1 |

**Question 9 (b)** **(2 marks)**

Solution

$$P(x \le 4 \mid x > 3) = \frac{P(3 < x \le 4)}{P(x > 3)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

| Mathematical behaviours | Marks |
|---|---|
| • determines numerator | 1 |
| • determines denominator | 1 |

**Question 9 (c)** **(3 marks)**

Solution

$$F(x) = \begin{cases} 0 & x < 2 \\ \int_2^x \frac{1}{3}\,dx = \frac{1}{3}(x-2) & 2 \le x \le 5 \\ 1 & x > 5 \end{cases}$$

| Mathematical behaviours | Marks |
|---|---|
| • identifies need to integrate $f(x)$ | 1 |
| • determines definite integral using correct limits of integration $(2, x)$ | 1 |
| • determines $F(x)$ and states it as a piecewise function | 1 |

**Question 9 (d)** **(2 marks)**

Solution

$F(x) = 0.75 \Rightarrow 0.75 = \frac{1}{3}(x-2)$

ie $x = 4.25$

Hence the upper quartile is 4.25.

| Mathematical behaviours | Marks |
|---|---|
| • states $0.75 = \frac{1}{3}(x - 2)$<br>• solves for $x$ | 1<br>1 |

**Question 10 (a)** **(2 marks)**

| Solution | |
|---|---|
| $P = P_o e^{kt}$<br>ie $P = 35e^{0.03t}$<br>$P = 50 \Rightarrow 50 = 35e^{0.03t}$<br>ie $t = 11.88$<br>ie the population will reach 50 million in roughly 11 years and 11 months | |
| **Mathematical behaviours** | **Marks** |
| • substitutes $P_0 = 35$ and $P = 50$ to obtain required equation<br>• solves equation to obtain $t$ | 1<br>1 |

**Question 10 (b)** **(2 marks)**

| Solution | |
|---|---|
| In 15 years the country's population = $35e^{15\times 0.03}$<br>In 15 years the city's population = $(0.22)35e^{c\times 15}$ where $c$ represents its growth rate<br>Hence $(0.22)35e^{c\times 15} = (0.4)35e^{15\times 0.03}$<br>Solving gives $c \approx 0.0699$<br>Hence the continuous growth rate is approximately 7%. | |
| **Mathematical behaviours** | **Marks** |
| • equates city's population to 40% of country's population in 15 years<br>• solves equation and states percentage growth rate | 1<br>1 |





**Question 21 (a)** **(3 marks)**

| Solution | |
|---|---|
| $\hat{p} = \frac{465}{700} \approx 0.6643, \quad E = 1.96\sqrt{\frac{(0.6643)(1 - 0.6643)}{700}} \approx 0.0350$<br>CI is $0.6643 - 0.0350 < p < 0.6643 + 0.0350$<br>ie $0.6293 < p < 0.6993$<br>So the 95% confidence interval is $0.629 < p < 0.699$ | |
| **Mathematical behaviours** | **Marks** |
| • calculates sample proportion correctly<br>• calculate standard error correctly<br>• calculates interval correctly | 1<br>1<br>1 |

**Question 21 (b)** **(2 marks)**

| Solution | |
|---|---|
| Because the old satisfaction rate (65%) lies within the new confidence interval, the recent survey does not provide conclusive evidence that the satisfaction rate has improved. | |
| **Mathematical behaviours** | **Marks** |
| • states that survey is not conclusive<br>• states a valid reason | 1<br>1 |

**Question 21 (c) (i)** **(3 marks)**

| Solution | |
|---|---|
| Since $\hat{p}(1 - \hat{p}) \le \frac{1}{4}$<br>$E_{95\%} = 1.96\sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} \le 1.96\sqrt{\frac{1}{4n}} = \frac{1.96}{2}\left(\frac{1}{\sqrt{n}}\right) < \frac{1}{\sqrt{n}}$ | |
| **Mathematical behaviours** | **Marks** |
| • substitutes $\hat{p}(1 - \hat{p}) = 0.25$<br>• expresses error as a constant $\times\frac{1}{\sqrt{n}}$<br>• deduces error $< \frac{1}{\sqrt{n}}$ | 1<br>1<br>1 |

**Question 21 (c) (ii)** **(1 mark)**

| Solution | |
|---|---|
| $E = \frac{1}{\sqrt{n}}$ | |
| **Mathematical behaviours** | **Marks** |
| • states Margin of error | 1 |

**Question 20 (c)** **(3 marks)**

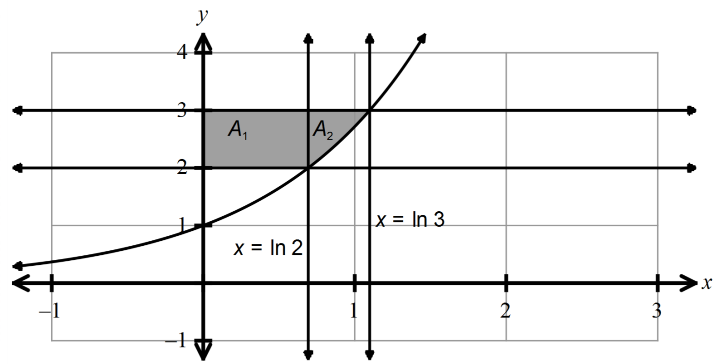
| Solution | |
|---|---|
| The mass travels 120 cm in the first 9 seconds.<br>In the next 9-second period the mass travels $120r$ cm,<br>where from part (b) $r = e^{-\pi b/c} \cong e^{-\pi \times \frac{0.0616}{0.349}} \cong 0.574$<br>Hence distance travelled between its first and second return to the origin is<br>$0.574 \times 120 = 68.88\text{cm} \approx 69\text{cm}$ | |
| **Mathematical behaviours** | **Marks** |
| • deduces that the mass travels 120 m in the first 9 seconds<br>• evaluates $r$<br>• obtains correct answer to the nearest centimetre | 1<br>1<br>1 |





**Question 11** **(6 marks)**

Solution




$$\text{Total area} = A_1 + A_2$$
$$= (\ln 2)(3-2) + 3(\ln 3 - \ln 2) - \int_{\ln 2}^{\ln 3} e^x\,dx$$
$$= \ln 2 + 3\ln 3 - 3\ln 2 - \left[e^x\right]_{\ln 2}^{\ln 3}$$
$$= 3\ln 3 - 2\ln 2 - (3-2)$$
$$= \ln 3^3 - \ln 2^2 - 1$$
$$= \ln \frac{27}{4} - 1$$

| Mathematical behaviours | Marks |
|---|---|
| • indicates an appropriate expression involving an integral to determine required area | 1 |
| • indicates $x$ axis values, ln 2 and ln 3 | 1 |
| • determines $A_1$ | 1 |
| • substitutes correct bounds to determine $A_2$ | 1 |
| • evaluates integral and $A_2$ | 1 |
| • rearranges expression using log laws and simplifies | 1 |

**Question 12 (a)** **(1 mark)**

| Solution | |
|---|---|
| Confidence interval is $(\hat{p}-E,\hat{p}+E)=(0.53,0.61)$<br>So $\hat{p}=\frac{0.53+0.61}{2}=0.57$ | |
| **Mathematical behaviours** | **Marks** |
| • Obtains correct answer | 1 |

**Question 12 (b)** **(2 marks)**

| Solution | |
|---|---|
| $E=z_\alpha\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$<br>ie $0.04=1.96\sqrt{\frac{0.57\times0.43}{n}}$<br>Solving for $n$ gives $n\cong588.5$<br>So the sample size was 589 (approximately) | |
| **Mathematical behaviours** | **Marks** |
| • Uses $Z_\alpha\approx1.96$<br>• Solves for $n$ and rounds | 1<br>1 |

**Question 12 (c)** **(3 marks)**

| Solution | |
|---|---|
| $E=z_\alpha\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$<br>$0.07=z_\alpha\sqrt{\frac{0.57(0.43)}{589}}$<br>$z_\alpha=3.43$<br>$P(-3.43<Z<3.43)=0.9993964$<br>Hence the level of confidence is 99.94% | |
| **Mathematical behaviours** | **Marks** |
| • Substitutesvalues into error equation<br>• Solves for $z_\alpha$<br>• States level of confidence to at least 1 decimal place | 1<br>1<br>1 |

**Question 12 (d)** **(2 marks)**

| Solution | |
|---|---|
| From the confidence interval in 10(c), there is a 99.94% probability that $p$ lies between 0.5 and 0.64, and in particular $p>0.5$. So the claim is justified. | |
| **Mathematical behaviours** | **Marks** |
| • States the claim is justified<br>• Gives a valid reason | 1<br>1 |



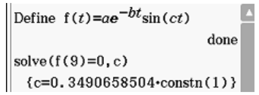


**Question 20 (a)** **(6 marks)**

Solution

$y(9)=0\Rightarrow ae^{-bt}\sin 9c=0\Rightarrow 9c=\pi$

$\Rightarrow c=\frac{\pi}{9}\approx0.349$



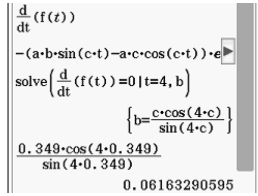

$\frac{dy}{dt}=-abe^{-bt}\sin ct+ace^{-bt}\cos ct=ae^{-bt}(c\cos ct-b\sin ct)$

max at $t=4$, $\Rightarrow \frac{dy}{dt}\Big|_{t=4}=0$

ie $-abe^{-bt}\sin ct+ace^{-bt}\cos ct=0$

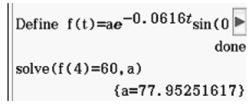
ie $-ae^{-bt}(b\sin ct-c\cos ct)=0$

ie $b\sin ct-c\cos ct=0$




ie $b\sin(4\times0.349)-0.439\cos(4\times0.349)=0\Rightarrow b=0.0616$

$y(4)=60\Rightarrow ae^{-(0.0616\times4)}\sin(4\times0.349)=60\Rightarrow a=78.0.$




| Mathematical behaviours | Marks |
|---|---|
| • uses (9,0) to conclude $9c=\pi$<br>• solves for $c$<br>• determines derivative function<br>• equates derivative function to 0 at $t=4$ and solves for $b$<br>• uses (4,60) to solve for $a$<br>• gives all answers correct to 3 significant figures | 1<br>1<br>1<br>1<br>1<br>1 |

**Question 20 (b)** **(3 marks)**

| Solution | |
|---|---|
| $y\left(t+\frac{\pi}{c}\right)=ae^{-b\left(t+\frac{\pi}{c}\right)}\sin\left(c\left(t+\frac{\pi}{c}\right)\right)$<br>$¿ae^{-bt}e^{\frac{-b\pi}{c}}\sin(ct+\pi)$<br>$¿-ae^{-bt}e^{\frac{-b\pi}{c}}\sin ct=-ry(t)$ | |
| **Mathematical behaviours** | **Marks** |
| • replaces $t$ with $t+\frac{\pi}{c}$ to obtain required function<br>• uses indices laws to factor out $ae^{-bt}$<br>• uses $\sin(ct+\pi)=-\sin ct$ to complete argument | 1<br>1<br>1 |

**Question 19 (a)** **(2 marks)**

| Solution | |
|---|---|
| $dB = 10\log_{10}(\frac{I}{I_0})$<br>$60 = 10\log_{10}(\frac{I}{1\times10^{-12}})$<br>$I = 10^{-6}$ Watts/m$^2$ | |
| Mathematical behaviours | Marks |
| • substitutes $dB = 60$<br>• solves the equation | 1<br>1 |

**Question 19 (b)** **(2 marks)**

| Solution | |
|---|---|
| $k = 10\log_{10}(\frac{I}{I_0}) \Rightarrow 0.1k = \log_{10}(\frac{I}{I_0}) \Rightarrow 10^{(0.1k)} = \frac{I}{I_0} \Rightarrow I = 10^{(0.1k)}I_0$<br>$k = 85 \Rightarrow I_g = 10^{(0.1\times85)}I_0 \Rightarrow I_g = 10^{8.5}I_0$ | |
| Mathematical behaviours | Marks |
| • rearranges log expression correctly<br>• substitutes $k = 85$ and states $I_g$ in terms of $I_o$ | 1<br>1 |

**Question 19 (c)** **(1 mark)**

| Solution | |
|---|---|
| $\frac{I_g}{I_c} = \frac{10^{8.5}I_0}{10^6 I_0} = 10^{2.5} \approx 316$<br>ie the gunshot is approximately 316 times louder than the restaurant conversation | |
| Mathematical behaviours | Marks |
| • evaluates the ratio $\frac{I_g}{I_c}$ correctly | 1 |





**Question 13 (a)** **(3 marks)**

| Solution | |
|---|---|
| $v(t) = 3t^2 + 2t - 9$<br>$v(0) = 9$<br>$a(t) = 6t - 2$<br>$a(0) = -2$<br>Since velocity and acceleration are opposing one another, the particle is slowing down | |
| Mathematical behaviours | Marks |
| • calculates $v(0)$<br>• differentiates $v(t)$ to obtain $a(t)$ and $a(0)$<br>• states particle is slowing down | 1<br>1<br>1 |

**Question 13 (b)** **(2 marks)**

| Solution | |
|---|---|
| $v(t) = 3t^2 - 2t + 9,$<br>$\therefore s(t) = t^3 - t^2 + 9t + c$<br>$(0, -1) \Rightarrow c = -1$<br>$\therefore s(t) = t^3 - t^2 + 9t - 1$<br>$s(5) = 5^3 - 5^2 + 9(5) - 1 = 144$<br>Hence its final position is 144m from the origin. | |
| Mathematical behaviours | Marks |
| • integrates velocity equation to determine displacement equation including $c = -1$<br>• substitutes $t = 5$, calculates and states final position, with unit | 1<br>1 |

**Question 14** **(4 marks)**

**Solution**

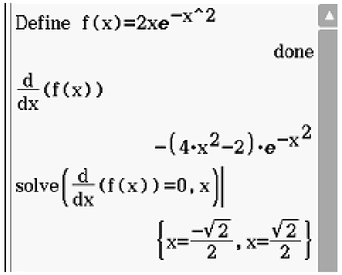
$A(x) = 2x.e^{-x^2}$

$\frac{dA}{dx} = 2\left(-2x^2e^{-x^2} + e^{-x^2}\right) = 2e^{-x^2}(-2x^2+1)$

$\frac{dA}{dx} = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

$x = \frac{1}{\sqrt{2}} \Rightarrow y = e^{-\frac{1}{2}}$

Hence $C$ has coordinates $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$




| Mathematical behaviours | Marks |
|---|---|
| • states $A(x)$ | 1 |
| • differentiates $A(x)$ | 1 |
| • solves $\frac{dA}{dx} = 0$ | 1 |
| • states co-ordinates of $C$ | 1 |





**Question 18 (a)** **(4 marks)**

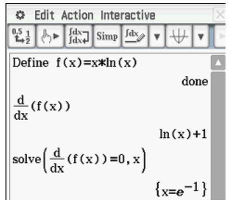
**Solution**

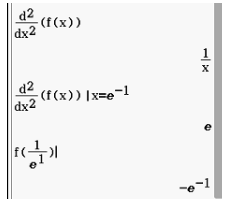
$y = x\ln x$

$\frac{dy}{dx} = (\ln x)(1) + x.\frac{1}{x}$

$\frac{dy}{dx} = \ln x + 1$

$\frac{d^2y}{dx^2} = \frac{1}{x}$




For stationary points,

$\ln x + 1 = 0$

ie $\ln x = -1$

ie $x = e^{-1}$

$x = e^{-1} \Rightarrow y = -\frac{1}{e}$

$\frac{d^2y}{dx^2} = \frac{1}{e^{-1}} = e > 0$, hence a minimum turning point

Thus exact coordinates of minimum turning point = $\left(\frac{1}{e}, \frac{-1}{e}\right)$

| Mathematical behaviours | Marks |
|---|---|
| • differentiates correctly using the product rule | 1 |
| • equates first derivative to zero to determine $x$ co-ordinate of stationary points | 1 |
| • uses second derivative test to determine nature of turning point | 1 |
| • determines correct coordinates of turning point | 1 |

**Question 18 (b)** **(1 mark)**

**Solution**

Since

$\frac{d^2y}{dx^2} = \frac{1}{x} \neq 0$ for any $x$ value.

Hence the curve cannot have a point of inflection.

| Mathematical behaviours | Marks |
|---|---|
| • states second derivative is never zero hence no P.O.I. | 1 |

| Solution | |
|---|---|
| Let $T = 25X - 0.15$<br>$E(T) = 25E(X) - 0.15 = 25(0.2305) - 0.15 = 5.6125 \approx \$5.61$<br>$Std\ Dev(T) = 25 \times Std\ Dev(X) = 25 \times 0.0294 = 0.735 \approx 74c$ | |
| **Mathematical behaviours** | **Marks** |
| • states linear transformation required | 1 |
| • determines mean | 1 |
| • determines standard deviation | 1 |





**Question 15 (a)** **(1 mark)**

| Solution | |
|---|---|
| Probability of winning a prize = 0.1 + 0.001<br>= 0.101 | |
| Mathematical behaviours | Marks |
| • uses Addition Principle to calculate the correct probability | 1 |

**Question 15 (b)** **(2 marks)**

| Solution | |
|---|---|
| $X \sim Bin(20, 0.101)$ | |
| Mathematical behaviours | Marks |
| • states Binomial distribution | 1 |
| • states correct parameters | 1 |

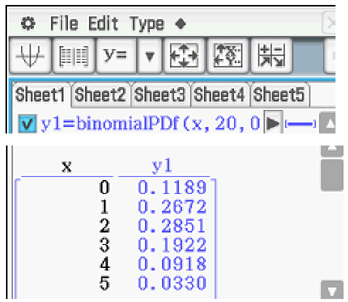
**Question 15 (c)** **(2 marks)**

| Solution | |
|---|---|
| $P(X \le 3) = \binom{20}{0}(0.101)^0(0.899)^{20} + \binom{20}{1}(0.101)^1(0.899)^{19} + \binom{20}{2}(0.101)^2(0.899)^{18} + \binom{20}{3}(0.101)^3(0.899)^{17}$ | |
| Mathematical behaviours | Marks |
| • identifies that "no more than 3 prizes" means "can win 0,1,2, or 3 prizes" | 1 |
| • states correct expression for the probability | 1 |

**Question 15 (d)** **(1 mark)**

| Solution | |
|---|---|
| $P(X \le 3)$<br>Bincdf(0,3,20,0.101)=0.8634 | |
| Mathematical behaviours | Marks |
| • states probability | 1 |

**Question 15 (e)** **(1 mark)**

| Solution | |
|---|---|
| Using CAS,<br>$k$=3   | |
| Mathematical behaviours | Marks |
| • states $k$=3 | 1 |

**Question 16 (a) (i)** **(2 marks)**

| Solution | |
|---|---|
| $\int_{-3}^{2} f(x)\,dx$ <br> $= -3 + 4 - 4 = -3$ | |
| **Mathematical behaviours** | **Marks** |
| • indicates addition of signed areas | 1 |
| • determines result | 1 |

**Question 16 (a) (ii)** **(2 marks)**

| Solution | |
|---|---|
| Area = 3+4+4=11 | |
| **Mathematical behaviours** | **Marks** |
| • expresses the area as a sum of areas | 1 |
| • determines result | 1 |

**Question 16 (b)** **(3 marks)**

| Solution | |
|---|---|
| $\int_{2}^{0}(x-2f(x))dx=\int_{2}^{0}x\,dx+2\int_{0}^{2}f(x)\,dx$ <br> $=\left[\frac{x^2}{2}\right]_2^0+2(-4)$ <br> $=\frac{1}{2}(0-4)-8=-10$ | |
| **Mathematical behaviours** | **Marks** |
| • applies the additivity of integrals to split the integral | 1 |
| • applies the linearity of integrals to deduce $\int_{2}^{0}(-2f(x))dx=2\int_{0}^{2}f(x)\,dx$ | 1 |
| • determines result | 1 |

**Question 16 (c)** **(3 marks)**

| Solution | |
|---|---|
| Maximum value of $F(x)$ occurs where $F'(x)=0$ | |





| Solution (continued) | |
|---|---|
| $F'(x)=\frac{d}{dx}\int_{-3}^{x}f(t)\,dt=f(x)$ <br> $f(x)=0\Rightarrow x=-2,0,2$ <br> $F(-2)=-3, F(0)=1, F(2)=-3$ <br> Hence, max is 1. | |
| **Mathematical behaviours** | **Marks** |
| • applies the Fundamental Theorem | 1 |
| • solves $f(x)=0$, stating $x=-2,0,2$ | 1 |
| • determines maximum value | 1 |

**Question 17 (a)** **(1 mark)**

Solution

$D \sim N(65, 4.9^2)$

| | small | medium | large |
|---|---|---|---|
| Proportion of peaches | $P(d<57)=0.0513$ | $P(57\le d\le 70)=0.7950$ | 0.1538 |

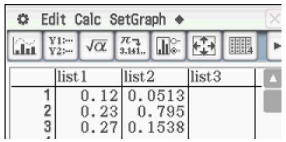
| Mathematical behaviours | Marks |
|---|---|
| • determines both probabilities | 1 |

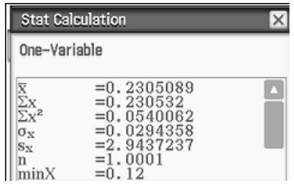
**Question 17 (b)** **(3 marks)**

Solution

Let $X$ = profit

$$\begin{aligned}\mu &= \Sigma xp(x)\\ &=0.12\times0.0513+0.23\times0.795+0.27\times0.1538\\ &=0.2305\\ &\approx 23c\\ \sigma^2 &= \Sigma(x-\mu)^2p(x)\\ &=(0.12-0.23)^2\times0.0513+(0.23-0.23)^2\times0.795+(0.27-0.23)^2\times0.1538\\ &=0.00087\\ \sigma &= 0.0294\\ &\approx 2.9c\end{aligned}$$




| Mathematical behaviours | Marks |
|---|---|
| • states a calculation to determine the mean or variance | 1 |
| • determines mean | 1 |
| • determines standard deviation | 1 |

**Question 17 (c)** **(3 marks)**

Solution