Rossmoyne Senior High School

Semester Two Examination, 2022

Question/Answer booklet

MATHEMATICS METHODS UNITS 3&4

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures

In words

Your name

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	55	35
Section Two: Calculator-assumed	12	12	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 8 **(8 marks)**

The launch speed of a small projectile fired from a catapult was measured and found to be normally distributed with a mean of 15.8 ms^{-1} and a standard deviation of 0.17 ms^{-1}.

(a) Determine the probability that the projectile is launched with a speed exceeding 16 ms^{-1}. (1 mark)

Solution
$P(X > 16) = 0.1197$
Specific behaviours
✓ correct probability

(b) Determine the probability that the projectile is launched with a speed exceeding 15.7 ms^{-1} given that its launch speed is less than 16 ms^{-1}. (2 marks)

Solution
$P(X > 15.7 \mid X < 16) = \dfrac{P(15.7 < X < 16)}{P(X < 16)} = \dfrac{0.6021}{1 - 0.1197} = 0.6840$
Specific behaviours
✓ indicates both probabilities required ✓ correct probability working required – answer only 1 mark

(c) The projectile is expected to have a speed exceeding v ms once in every 200 launches. Determine the value of v. (1 mark)

Solution
$P(X > v) = 0.005, \quad v = 16.238 \text{ ms}^{-1}$.
Specific behaviours
✓ correct speed (at least 2 dp)

ClassPad
$invNormcdf(R, 0.005, 0.17, 15.8)$

(d) In a series of 20 launches, determine the probability that the speed of the projectile exceeds 16 ms^{-1} in no more than 3 of these launches. (2 marks)

Solution
$Y \sim B(20, 0.1197), \quad P(Y \le 3) = 0.7886$
Specific behaviours
✓ indicates binomial distribution with parameters ✓ correct probability- correct answer only– 2 marks

(e) The instrument used to measure the launch speed was suspected to overestimate the speed of the projectile by 0.03 ms^{-1}. If this was the case, state the true mean and standard deviation of the distribution of launch speeds for the projectile. (2 marks)

Solution
Mean: $\mu = 15.8 - 0.03 = 15.77 \text{ ms}^{-1}$. SD unchanged: $\sigma = 0.17 \text{ ms}^{-1}$.
Specific behaviours
✓ correct mean ✓ correct sd

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Question 9 **(7 marks)**

Naltrexone is useful in managing heroin-dependent patients who find it difficult to shift away from dependent use patterns. The blood naltrexone level N of a patient who has received a naltrexone implant was observed to halve every 33 days, from an initial level of 7.7 ng/ml. The level can be modelled by an equation of the form $N = ae^{kt}$, where t is the time in days since the implant was received.

(a) State the value of the constant a and the determine the value of the constant k. (3 marks)

Solution
$a = 7.7$
$0.5 = e^{33k} \Rightarrow k = -0.0210$ (at least 3 d.p.)
Specific behaviours
✓ value of a
✓ writes equation using half-life
✓ value of k (accept $-\frac{\ln(2)}{33}$)

The treatment is effective whilst the naltrexone level remains above 1.6 ng/ml.

(b) Determine the number of days that the implant will be effective. (2 marks)

Solution
$1.6 = 7.7e^{-0.021t}$
$t = 74.8$
Implant effective for 75 days.
Specific behaviours
✓ writes equation
✓ number of days (accept 74.8 or 75 days)

(c) Determine the rate at which the naltrexone level is decreasing 15 days after the implant is received. (2 marks)

Solution
$\frac{dN}{dt} = -0.021(7.7e^{-0.021t})\vert_{t=15}$
$= -0.118$
Hence decreasing at 0.118 ng/ml/day.
Specific behaviours
✓ indicates correct method
✓ correct rate of decrease

Units question

See next page

Question 10 **(9 marks)**

In a random sample of 250 adult female Australians, 65 were born overseas. This data is to be used to construct a 90% confidence interval for the proportion of adult male Australians born overseas.

(a) Determine the margin of error for the 90% confidence interval. (3 marks)

Solution
$p = 65 \div 250 = 0.26, \quad \sigma = \sqrt{\dfrac{0.26(1-0.26)}{250}} = 0.0277$ $z_{0.9} = 1.645, \quad E = 1.645 \times 0.0277 = 0.0456$
Specific behaviours
✓ correct proportion   ✓ correct standard deviation of sample proportion   ✓ correct margin of error

Alternative Solution
$\hat{p} = \dfrac{65}{250}$ ✓ $E = 1.645\sqrt{\dfrac{0.26(1-0.26)}{250}}$ ✓ $= 0.0456$ ✓

(b) State the 90% confidence interval. (1 mark)

Solution
$p \pm E \to (0.2144, 0.3056)$
Specific behaviours
✓ correct interval (at least 3 d.p.)

(c) If 7 similar samples are taken and each used to construct a 90% confidence interval, determine the probability that no more than 5 of the intervals will contain the true proportion of adult female Australians who were born overseas. (2 marks)

Solution
$X \sim B(7, 0.9)$ $P(X \leq 5) = 0.1497$
Specific behaviours
✓ indicates binomial distribution with parameters   If $p \neq 0.9$, no FT   ✓ correct probability (answer only 2 marks)

Accuracy question

(d) The 90% confidence interval for the proportion of adult male Australians born overseas constructed from another random sample was $(0.195, 0.295)$. Determine the number of adult males who were born overseas in this sample. (3 marks)

Solution
$E = (0.295 - 0.195) \div 2 = 0.05, \quad p = 0.195 + 0.05 = 0.245$ $\sqrt{\dfrac{0.245(1-0.245)}{n}} = \dfrac{0.05}{1.645} \to n = 200$ $X = 200 \times 0.245 = 49$ males.
Specific behaviours
✓ calculates p and E ✓ calculates sample size n ✓ correct number of males

Error
If rounding n to 201 $\Rightarrow X = 50$ Penalty -1

Treatment began at $1{:}15$ pm in tank A, and at $1{:}30$ pm in tank B.

(e) Determine the time at which the turbidity indices of the water in the tanks first become the same. (2 marks)

Solution
Using $t = 0$ at $1{:}15$ pm: $8e^{-0.2t} = e^{1.86-0.14(t-0.25)} \to t = 3.074\text{h} = 3\text{h }4\text{m}$. or Using $t = 0$ at $1{:}30$ pm: $8e^{-0.2(t+0.25)} = e^{1.86-0.14(t)} \to t = 2.824\text{h} = 2\text{h }49\text{m}$. Hence turbidity indices the same $3\text{h }4\text{m}$ after $1{:}15$ pm, at $4{:}19$ pm.
Specific behaviours
✓ correct equation for t ✓ correct time of day Only FT if time t=0 is stated or inferred Using 15 instead of 0.25 - no marks

Question 11 **(7 marks)**

(a) A polynomial function is defined by $f(x) = (kx-1)^3$, where k is a constant. The area under the curve $y = f(x)$ between $x = 3$ and $x = 9$ is 12 square units.

Determine the area under the curve $y = f(x)$ between $x = 3$ and $x = 6$. (3 marks)

Solution
$\int_3^9 (kx-1)^3\,dx = \left[\frac{1}{4k}(kx-1)^4\right]_3^9 = \frac{(9k-1)^4-(3k-1)^4}{4k}$ But $\frac{(9k-1)^4-(3k-1)^4}{4k} = 12$ $k = \frac{1}{3}$ Hence $\int_3^6 f(x)\,dx = \frac{3}{4} = 0.75$ sq units
Specific behaviours
✓ integral for area under curve ✓ forms equation in k using given area value of k ✓ correct area

Alternative Solution
$Solve \int_3^9 (kx-1)^3 dx = 12, k$ ✓ $\Rightarrow k = \frac{1}{3}$ ✓ $\therefore \int_3^6 f(x)dx = \frac{3}{4}$ ✓

(b) The graph of another polynomial $y = g(x)$ has a point of inflection at $(3, 7)$ and a stationary point when $x = -1$.

If $g'(x) = 3x^2 + ax + b$, where a and b are constants, determine $g(x)$. (4 marks)

Solution
Since $g''(3) = 0$ then $6(3) + a = 0 \Rightarrow a = -18$. Since $g'(-1) = 0$ then $3(-1)^2 - 18(-1) + b = 0 \Rightarrow b = -21$. $g(x) = \int 3x^2 - 18x - 21\,dx$ $= x^3 - 9x^2 - 21x + c$ Since $\text{g}(3) = 7$ then $27 - 81 - 63 + c = 7 \Rightarrow c = 124$. Hence $g(x) = x^3 - 9x^2 - 21x + 124$.
Specific behaviours
✓ value of a ✓ value of b ✓ antiderivative of $g'(x)$ ✓ evaluates constant of integration and states $g(x)$ If no marks awarded , award 1 mk for $a+b=-3$

Question 12 **(7 marks)**

The owners of a shopping mall wanted to confirm their estimate that 35% of local school students visited their mall at least once a week. The owners considered the following three ways of selecting a sample:

A Ask students who turn up to the mall after school.

B Create an online survey and publish a link to it in the local newspaper.

C Visit local homes chosen at random and ask students who live there.

(a) Briefly discuss a source of bias in each sampling method and suggest a better sampling procedure. (4 marks)

Solution
A: Non-response, students might not want to divulge information when asked. A: Undercoverage, will not sample students who don't visit mall after school. A: Convenience, only sample students who visit mall after school. B: Undercoverage, will not sample students who don't see link in newspaper. B: Self-selection, only sample students who volunteer to take survey. C: Non-response, students might not want to divulge information when asked.
Specific behaviours
✓ discusses a source of bias in A ✓ discusses a source of bias in B (B does not include non-response) ✓ discusses a source of bias in C ✓ describes procedure involving random sampling from whole(school) population- must mention both

(b) It was found that 105 out of a random sample of 375 students visited the mall at least once a week. Determine the 95% confidence interval for the proportion based on this data and use it to comment on the owner's estimate. (3 marks)

Solution
$p = \dfrac{105}{375} = 0.28, \quad 0.28 \pm 1.96\sqrt{\dfrac{0.28(1-0.28)}{375}} \approx (0.2346, 0.3254)$ ✓ ✓ The 95% confidence interval does not contain the owner's estimate of 0.35, and it suggests that the true value of the proportion is likely to be less than 35%. Or As the CI does not contain (capture) the estimate of 0.35, based on this sample there is no evidence to support the claim
Specific behaviours
✓ indicates correct method to construct confidence interval ✓ correct confidence interval (to at least 2 dp) ✓ uses interval to dispute owner's estimate Don't penalise for words such as "wrong" or "incorrect" provided the correct reasons are given – teaching point

(c) Determine the turbidity index of the water in tank B when $t = 4$. (1 mark)

Solution
$\log_e(I) = 1.3 \to I = e^{1.3} = 3.67$
Specific behaviours
✓ correct index (accept exact and approx. values)

(d) Determine the equation of the linear relationship shown in the graph in the form $\log_e(I) = at + b$, where a and b are constants and hence express the turbidity index I as a function of time t for the water being treated in tank B. (3 marks)

Solution
Equation of line using $y = mx + c$: $m = \dfrac{1.3 - 0.6}{4 - 9} = -\dfrac{7}{50} = -0.14$ $y - 1.3 = -0.14(x - 4)$ $y = -0.14x + 1.86 \to b = 1.86 = \dfrac{93}{50}$ Hence $\log_e(I) = 1.86 - 0.14t$ Turbidity index: $I(t) = e^{1.86 - 0.14t} \quad [= 6.4237e^{-0.14t}]$
Specific behaviours
✓ calculates slope and intercept (possibly using CAS) ✓ correct equation for $\log_e(I)$ ✓ correct function for I

Alt Solution 1
$1.3 = 4m + c \qquad 0.6 = 9m + c$ Solv sim $m = -0.14$ ✓ $c = 1.86$ $\Rightarrow \ln I = -0.14t + 1.86$ ✓ $\Rightarrow I = e^{-0.14t + 1.86}$ ✓

Alt Solution 2
Statistics – Linear regression $\Rightarrow \ln I = -0.14t + 1.86$ ✓✓ $\Rightarrow I = e^{-0.14t + 1.86}$ ✓

error
If $\ln 1.3 = 4a + b \quad and \quad \ln 0.6 = 9a + b$ $\Rightarrow I = e^{-0.15t + 0.88}$ Only award 1/3

Question 19 **(10 marks)**

The turbidity index I (a measure of purity) of water being treated in tank A can be modelled by the relationship $I = 8e^{-0.2t}$, where t is the time in hours since treatment began.

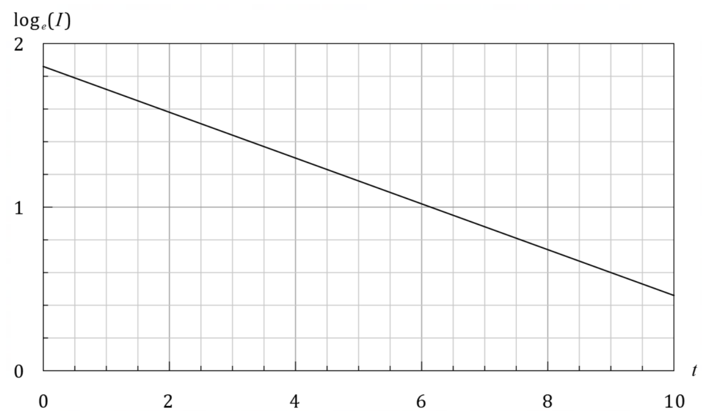
(a) Express this relationship in the form $t = p\log_e(kI)$, where p and k are constants. (2 marks)

Solution
$\frac{1}{8}I = e^{-0.2t}$ $\log_e\left(\frac{1}{8}I\right) = -0.2t$ $t = -5\log_e\left(\frac{1}{8}I\right) \quad \left[= -5\log_e\left(\frac{I}{8}\right) = -5\log_e(0.125I)\right]$
Specific behaviours
✓ correctly converts from exponential to natural log form ✓ simplifies into required form

(b) Determine the time taken, to the nearest minute, for the turbidity index of the water in tank A to halve. (2 marks)

Solution
$I_0 = 8$ $t = -5\log_e\left(\frac{4}{8}\right) = 3.4657\text{h} = 3\text{h } 28\text{m}$ = 208 min
Specific behaviours
✓ correct expression for time ✓ correct time, to nearest minute

Readings of water being treated in tank B were used to construct the graph below, where a linear relationship between $\log_e(I)$ and time t exists. The line passes through the points $(4, 1.3)$ and $(9, 0.6)$.

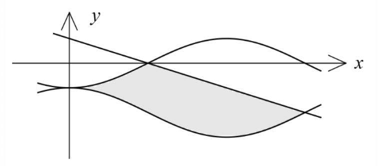


Question 13 **(7 marks)**

Functions f, g and h are defined by

$$f(x) = 10\cos\left(\frac{\pi x}{5}\right) - 20$$
$$g(x) = -10\cos\left(\frac{\pi x}{5}\right)$$
$$h(x) = 10 - 4x.$$

The graphs of these functions are shown to the right.



(a) Determine the area between $y = f(x)$, the x-axis, $x = 3.75$ and $x = 5$. (3 marks)

Solution
$I = \int_{3.75}^{5} f(x)\,dx$ $= -\frac{25\sqrt{2}}{\pi} - 25$ Hence area is $\frac{25\sqrt{2}}{\pi} + 25 \approx 36.3$ sq units.
Specific behaviours
✓ writes integral (may preface with negative sign - see last mark) ✓ evaluates integral ✓ clearly deals with negative value of integral to obtain area

Alternative Solution
$Area = \int_{3.75}^{5} \lvert f(x)\rvert\,dx$ ✓✓ $\Rightarrow = 36.3$ sq units ✓

(b) Determine the area of the shaded region enclosed by the three functions. (4 marks)

Solution
Using CAS, $f = h$ when $x = 2.5$ and $g = h$ when $x = 7.5$. $A = \int_0^{2.5} g(x) - f(x)\,dx + \int_{2.5}^{7.5} h(x) - f(x)\,dx$ $= \left(50 - \frac{100}{\pi}\right) + \left(50 + \frac{100}{\pi}\right)$ $= 100$ sq units
Specific behaviours
✓ writes correct integral for area between $x = 0$ and $x = 2.5$ ✓ evaluates first integral ✓ writes correct integral for area between $x = 2.5$ and $x = 7.5$ ✓ evaluates second integral and states area of shaded region

Alt Solution 1
Solve $h(x) = g(x)$ $\Rightarrow x = 2.5$ Solve $h(x) = f(x)$ $\Rightarrow x = 7.5$ ✓ ✓ ✓ $A = \int_0^{2.5} g(x) - f(x)dx + \int_{2.5}^{7.5} h(x) - f(x)dx$ $= 100\,sq\,units$ ✓

Alt Solution 2
Solve $h(x) = g(x) \quad \Rightarrow x = 2.5$ Solve $h(x) = f(x) \quad \Rightarrow x = 7.5$ ✓ $A = -\left[\int_0^{7.5} f(x)dx - \int_0^{2.5} g(x)dx - \int_{2.5}^{7.5} h(x)dx\right]$ ✓✓ $= 100\,sq\,units$ ✓

Question 14 **(10 marks)**

An online retailer of auto parts knows that on average, 18.5% of parts sold will be returned.

(a) Let the random variable X be the number of parts returned when a batch of 88 parts are sold.

(i) Describe the distribution of X. (2 marks)

Solution
X is binomially distributed with parameters $n = 88$ and $p = 0.185$.   or $X \sim B(88, 0.185)$
Specific behaviours
✓ states binomial   ✓ states correct parameters

(ii) Determine the probability that less than 15% of the parts sold in this batch will be returned. (2 marks)

Solution
$0.15 \times 88 = 13.2$ $P(X \leq 13) = 0.2264$
Specific behaviours
✓ indicates correct binomial probability to calculate   ✓ correct probability   No FT from (i) if X is not a Binomial

The retailer takes a large number of random samples of 150 parts from its sales data and records the proportion $\hat{p}$ of returned parts in each sample. Under certain circumstances, the distribution of $\hat{p}$ will approximate normality.

(b) Explain why the retailer can expect the distribution of $\hat{p}$ to closely approximate normality in this case. (3 marks)

Solution
The sampling is random (each observation is independent).   The sample size is sufficiently large (typically 30 or more).   At least* 15 returns and 15 non-returns can be expected in each sample.   or $n\hat{p} \geq 15$ and $n(1-\hat{p}) \geq 15$.   **15 seems to be currently accepted practice, but also accept 5 (or more).*
Specific behaviours
✓ states samples are randomly selected   ✓ states sample size sufficiently large   ✓ states least number of successes and failures required

Question 18 **(8 marks)**

A small body moves along the x-axis with acceleration t seconds after leaving the origin given by $a(t) = 3.6 + kt$ cm/s^2, where k is a constant. The initial velocity of the body is -10 cm/s, and its change in displacement during the fifth second is 3.76 cm.

(a) Determine the maximum velocity of the body. (6 marks)

Solution
Expression for velocity $v(t) = \int 3.6 + kt\,dt$ $= 3.6t + \frac{kt^2}{2} + c$ $v(0) = -10 \Rightarrow c = -10$ Change in displacement $\Delta x = \int_4^5 3.6t + \frac{kt^2}{2} - 10\,dt$ $= \left[1.8t^2 + \frac{kt^3}{6} - 10t\right]_4^5$ $= \frac{61k}{6} + \frac{31}{5}$ Hence $\frac{61k}{6} + \frac{31}{5} = 3.76 \Rightarrow k = -\frac{6}{25} = -0.24$ Maximum velocity when no acceleration $3.6 - 0.24t = 0 \Rightarrow t = 15$ s   Maximum velocity $v(15) = 17$ cm/s
Specific behaviours
✓ obtains correct expression for velocity   ✓ correct integral for change in displacement   ✓ obtains linear expression for change in displacement   ✓ obtains correct value of k ✓ obtains time of maximum velocity   ✓ correct maximum velocity

alternative
$\Delta x = \int_4^5 v(t)dt$ ✓ $Solve \int_4^5 v(t)dt = 3.76, k$ ✓ $\Rightarrow k = -\frac{6}{25}$ ✓

(b) Determine, to the nearest centimetre, the distance travelled by the body between $t = 0$ and the instant it reaches its maximum velocity. (2 marks)

Solution
$d = \int_0^{15} \lvert v(t) \rvert\,dt$ $\approx 149.8 \approx 150$ cm
Specific behaviours
✓ indicates correct method to determine distance travelled   ✓ correct distance travelled (needs units)   FT from (a)   No absolute value , 0/2

Question 17 **(10 marks)**

The number of points awarded each time an online game is played is the random variable X, where $E(X) = 1.8$ and X has the following probability distribution.

x	0	1	2	4	c=7
$P(X=x)$	k=0.2	0.35	0.25	0.15	0.05

(a) Determine the value of the constant c and the value of the constant k. (3 marks)

Solution

$$k = 1 - (0.35 + 0.25 + 0.15 + 0.05) = 0.2$$

$$0 \times k + 1 \times 0.35 + 2 \times 0.25 + 4 \times 0.15 + 0.05c = 1.8$$
$$c = 7$$

Specific behaviours

✓ value of k
✓ expression for $E(X)$
✓ value of c

(b) Calculate the variance of Y, where $Y = 5X - 3$. (3 marks)

Solution

$$\text{Var}(X) = 0.2(0-1.8)^2 + 0.35(1-1.8)^2 + 0.25(2-1.8)^2 + 0.15(4-1.8)^2 + 0.05(7-1.8)^2$$
$$= 2.96$$

Hence $\text{Var}(Y) = 5^2 \times 2.96 = 74$.

Specific behaviours

✓ indicates appropriate method to determine variance or standard deviation of X
✓ variance of X
✓ variance of Y

When playing a set of 7 games, the points awarded in each game is independent of other games and a player wins a prize if the total number of points scored in the set is at least 28.

(c) A player has completed 5 games in a set and has been awarded a total of 20 points. Determine the probability that they win a prize on completion of the set. (4 marks)

Solution

Ways of getting at least 8 points from next two games:

$$P(7,1|1,7) = 2 \times 0.05 \times 0.35 = 0.035$$
$$P(7,2|2,7) = 2 \times 0.05 \times 0.25 = 0.025$$
$$P(7,4|4,7) = 2 \times 0.05 \times 0.15 = 0.015$$
$$P(7,7) = 0.05^2 = 0.0025$$
$$P(4,4) = 0.15^2 = 0.0225$$
$$\sum P = 0.1 = {}^{1}/_{10}$$

Hence probability of winning a prize is 0.1.

Specific behaviours

✓✓ ~~identifies all required point combinations~~
(all 8 comb ✓✓, 5-7 comb ✓, 0-4 comb – bad luck)
✓ correctly calculates probability of two combinations
✓ correct probability of winning a prize =0.1 (no FT)

(c) State the parameters of the normal distribution that $\hat{p}$ approximates and use this distribution to determine the probability that the proportion of returns in a random sample of 150 parts is less than 15%. (3 marks)

Solution

$$\hat{p} \sim N(\mu_{\hat{p}}, \sigma_{\hat{p}}^2)$$

$$\mu_{\hat{p}} = p = 0.185$$

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.185(1-0.185)}{150}} \approx 0.0317, \qquad \sigma_{\hat{p}}^2 \approx 0.0010052$$

Hence normally distributed with mean 0.185 and standard deviation 0.0317.

$$P(\hat{p} < 0.15) = 0.1348$$

Specific behaviours

✓ states mean of distribution
✓ states standard deviation or variance of distribution
✓ correct probability
If Binomial is used , 0/3

Question 15 **(8 marks)**

Steel ingots are cast by a metal recycling machine with masses of X kg, where X is a continuous random variable with cumulative distribution function

$$F(x)=\begin{cases}0 & x<2\\ ax^2-bx & 2\le x\le 3\\ 1 & x>3\end{cases}$$

(a) Deduce from the cumulative distribution function that the values of the constants a and b are $a=\frac{1}{3}$ and $b=\frac{2}{3}$. (3 marks)

Solution
When $x=2$ then $4a-2b=0$ and when $x=3$ then $9a-3b=1$.
Solving these equations simultaneously gives $a=\frac{1}{3}$ and $b=\frac{2}{3}$.
(NB No not accept substitution as *deduction* is required, not *show*.)
Specific behaviours
✓ correctly uses lower bound to form first equation
✓ correctly uses upper bound to form second equation
✓ solutions stated (No need to state method if two clear equations are given and the solutions are stated.)

(b) Determine the probability that a randomly selected ingot cast by the machine has a mass less than 2.2 kg. (1 mark)

Solution
$F(2.2)=\frac{11}{75}=0.14\bar{6}$
Specific behaviours
✓ correct probability

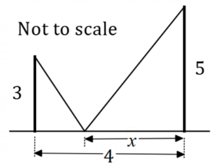
(c) Determine the mean and standard deviation of the masses of ingots cast by the machine. (4 marks)

Solution
$f(x)=F'(x)=\frac{2x}{3}-\frac{2}{3}$
$E(X)=\int_2^3 x\,f(x)\,dx=\frac{23}{9}=2.\bar{5}$ kg
$\mathrm{Var}(X)=\int_2^3\left(x-\frac{23}{9}\right)^2 f(x)\,dx=\frac{13}{162}\approx 0.0802$
$\sigma_X=\sqrt{\mathrm{Var}(X)}=\frac{\sqrt{26}}{18}\approx 0.2833$ kg
Specific behaviours
✓ obtains probability density function
✓ correct mean
✓ indicates correct integral for variance
✓ correct standard deviation

Error
If $F(x)$ is used instead of $f(x)$
$\mu_X=1.1944$
$\sigma_X=1.3441$
award 2/4

Question 16 **(7 marks)**

Two thin vertical posts, one 5 m and the other 3 m tall, stand 4 m apart on horizontal ground. A small stake is positioned directly between the bases of the posts at a distance of x m from the base of the taller post.



A length of thin wire runs in a straight line from the top of one post, to the stake, and then to the top of the other post.

(a) Calculate the length of the wire when the stake is positioned midway between the bases. (1 mark)

Solution
$L=\sqrt{5^2+2^2}+\sqrt{3^2+2^2}=\sqrt{29}+\sqrt{13}\approx 8.99$ m
Specific behaviours
✓ correct length (exact or at least 2 dp)

(b) Use a calculus method to determine where the stake should be positioned to minimise the length of wire, state what this minimum length is and justify that the length is a minimum. (6 marks)

Solution
$L=\sqrt{5^2+x^2}+\sqrt{3^2+(4-x)^2}$
$\frac{dL}{dx}=\frac{1}{2}\frac{2x}{\sqrt{25+x^2}}+\frac{1}{2}\frac{2(4-x)(-1)}{\sqrt{9+(4-x)^2}}$ $=\frac{x}{\sqrt{25+x^2}}+\frac{x-4}{\sqrt{x^2-8x+25}}$
$\frac{dL}{dx}=0\Rightarrow x=\frac{5}{2}=2.5$ m
$L(2.5)=4\sqrt{5}\approx 8.944$ m
Justify minimum using sign test
$L'(2.4)\approx -0.04,\quad L'(2.6)\approx 0.04$
Hence $4\sqrt{5}$ is the minimum length as the gradient changes from $-$ve to 0 to +ve as x increases through 2.5.
Or using second derivative
$L''(2.5)\approx 0.38>0$ (0.38 and >0 required)
Hence $4\sqrt{5}$ is the minimum length as the function is stationary and concave up when $x=2.5$.
Specific behaviours
✓ expression for length
✓ writes first derivative (in any form)
✓ equates first derivative to 0 and obtains solution for x
✓ states minimum length (exact or at least 2 dp)
✓ indicates use of second derivative / sign test
✓ ~~justifies length minimum~~ uses words to state min length