PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Test 3 Year 12

Student name: _____ Teacher name: _____

Task type: **Response**

Time allowed for this task: **____40_____ mins**

Number of questions: **_____7_____**

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **_44_____ marks**

Task weighting: **_10___%**

Formula sheet provided: **Yes**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (6 marks)

a) Solve the following system of linear equations. (3 marks)

$$x + 2y - 3z = -28$$
$$2x - 7y + 5z = 76$$
$$3x - 4y + 6z = 71$$

Solution
$\begin{bmatrix} 1 & 2 & -3 & -28 \\ 2 & -7 & 5 & 76 \\ 3 & -4 & 6 & 71 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -3 & -28 \\ 0 & 11 & -11 & -132 \\ 0 & 10 & -15 & -155 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & -3 & -28 \\ 0 & 11 & -11 & -132 \\ 0 & 0 & 55 & 385 \end{bmatrix}$ $55z = 385$ $z = 7$ $y = -5$ $x = 3$
Specific behaviours
✓ eliminates one variable from two equations ✓ eliminates two variables ✓ solves for all variables

b) Determine all possible values of $p \& q$ for the three scenarios below. (3 marks)

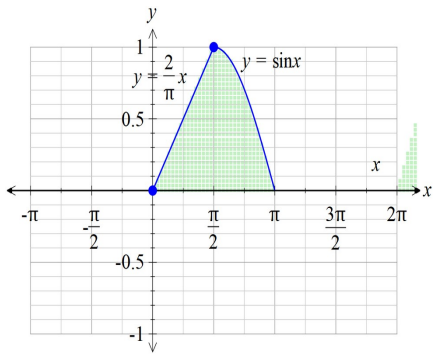
$$x + 2y - 3z = q$$
$$2x - 7y + 5z = 76$$
$$3x - 4y + pz = 71$$

i) No solutions

ii) One solution

✓ obtains correct volume -must be numeric (no need to round) (Max 1 out of 4 if revolved around x axis- too easy)

r

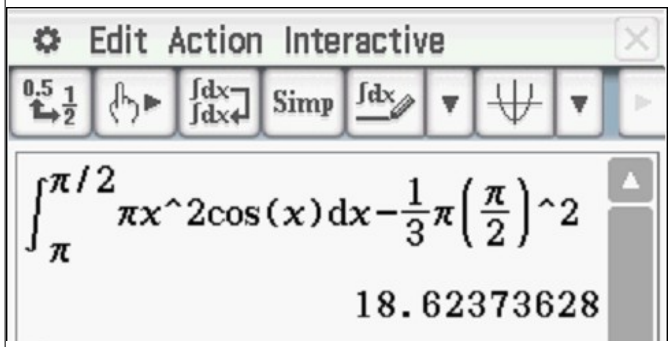


If the shaded area above is revolved around the y axis, determine the volume of the 3D object created.

Solution

$\int_{y=0}^{y=1} \pi x^2 dy = \int_{x=\pi}^{x=\frac{\pi}{2}} \pi x^2 \frac{dy}{dx} dx \quad , y = \sin x$

$\int_{\pi}^{\frac{\pi}{2}} \pi x^2 \cos x \, dx - \frac{1}{3}\pi\left(\frac{\pi}{2}\right)^2$



(Note- use of inverse sine function without a translation is incorrect as sine x is a many to one function over 0 to pi domain)

Specific behaviours

✓ uses correct integral with appropriate limits for area under sin x
✓ uses change of variable with correct order of limits
✓ determines volume of cone

iii) Infinite solutions

Solution

$\left[\begin{array}{cccc} 1 & 2 & -3 & q \\ 2 & -7 & 5 & 76 \\ 3 & -4 & p & 71 \end{array}\right]$

$\left[\begin{array}{cccc} 1 & 2 & -3 & q \\ 0 & 11 & -11 & 2q-76 \\ 0 & 10 & -9-p & 3q-71 \end{array}\right]$

$\left[\begin{array}{cccc} 1 & 2 & -3 & -14 \\ 0 & 11 & -11 & 2q-76 \\ 0 & 0 & -11+11p & 21-13q \end{array}\right]$

$i)\, p = 1 \,\&\, q \neq \frac{21}{13}$

$ii)\, p \neq 1$

$iii)\, p = 1 \,\&\, q = \frac{21}{13}$

Specific behaviours

✓ derives equation with two variables eliminated
✓ states values for uniqueness
✓ states values for no solution and infinite (follow through)

Q2 (9 marks)

A particle moves with acceleration $a=\begin{pmatrix} t^3 \\ \sqrt{t} \end{pmatrix} m/s^2$ at time t seconds. The initial velocity is $\begin{pmatrix} 3 \\ -2 \end{pmatrix} m/s$ and initial position $\begin{pmatrix} 4 \\ -1 \end{pmatrix} m$.

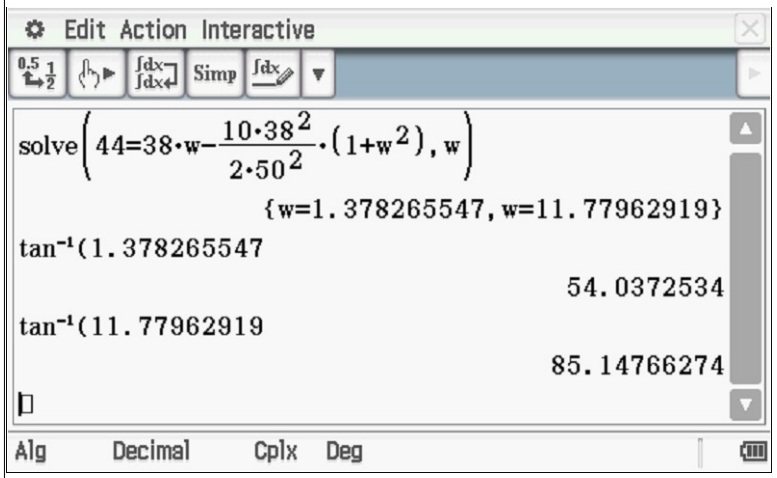
a) Determine the velocity at time t seconds. (2 marks)

Solution
$a=\begin{pmatrix} t^3 \\ \sqrt{t} \end{pmatrix}$ $v=\begin{pmatrix} \frac{t^4}{4} \\ \frac{2t^{\frac{3}{2}}}{3} \end{pmatrix}+c$ $\begin{pmatrix} 3 \\ -2 \end{pmatrix}=c$ $v=\begin{pmatrix} \frac{t^4}{4}+3 \\ \frac{2t^{\frac{3}{2}}}{3}-2 \end{pmatrix} m/s$
Specific behaviours
✓ anti-differentiates ✓ solves for constant

b) Determine the position vector at time $t=5$ seconds. (2 marks)

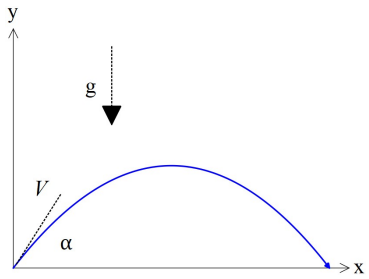
Solution

b) Given that $V=50\,m/s$ and that $y=44\,m$ when $x=38\,m$, determine possible value(s) for α. (3 marks)

Solution
$y=x\tan\alpha-\frac{gx^2}{2V^2}(1+\tan^2\alpha)$ $44=38\tan\alpha-\frac{10(38^2)}{2(50^2)}(1+\tan^2\alpha)$ 
Specific behaviours
✓ subs all knowns into cartesian equation ✓ solves for one angle ✓ solves for two angles

Q7 (4 marks)

Consider the area between $y=\sin x$, $y=\frac{2}{\pi}x$ and the x axis with $0\le x\le\pi$, as shown below.



a) Using vector calculus and starting with the acceleration, show how to derive the cartesian equation of the path in terms of $V, g \& \alpha$. (4 marks)

Solution
$\ddot{r} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$
$\dot{r} = \begin{pmatrix} 0 \\ -gt \end{pmatrix} + c$
$\begin{pmatrix} V\cos\alpha \\ V\sin\alpha \end{pmatrix} = c$
$\dot{r} = \begin{pmatrix} V\cos\alpha \\ V\sin\alpha - gt \end{pmatrix}$
$r = \begin{pmatrix} Vt\cos\alpha \\ Vt\sin\alpha - \frac{gt^2}{2} \end{pmatrix} + c$
$c = 0$
$t = \frac{x}{V\cos\alpha}$
$y = \frac{xV\sin\alpha}{V\cos\alpha} - \frac{gx^2}{2V^2\cos^2\alpha} = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$

Specific behaviours
✓ integrates to find velocity and solves for constant
✓ integrates to find r
✓ subs x expression into y by eliminating t
✓ obtains cartesian expression in terms of constants

$$v = \begin{pmatrix} \frac{t^4}{4} + 3 \\ \frac{2t^{\frac{3}{2}}}{3} - 2 \end{pmatrix}$$

$$r = \begin{pmatrix} \frac{t^5}{20} + 3t \\ \frac{4t^{\frac{5}{2}}}{15} - 2t \end{pmatrix} + k$$

$$\begin{pmatrix} 4 \\ -1 \end{pmatrix} = k$$

$$r = \begin{pmatrix} \frac{t^5}{20} + 3t + 4 \\ \frac{4t^{\frac{5}{2}}}{15} - 2t - 1 \end{pmatrix}$$

$$r = \begin{pmatrix} \frac{701}{4} \\ \frac{20\sqrt{5}}{3} - 11 \end{pmatrix} \text{ or } \begin{pmatrix} 175.25 \\ 3.91 \end{pmatrix} m$$

Specific behaviours
✓ determines r
✓ approx. at t=5 (maybe exact or 2 dp)

c) Determine $\frac{dy}{dx}$ on the cartesian path at time $t = 5$ seconds. (2 marks)

Solution
$v = \begin{pmatrix} \frac{t^4}{4} + 3 \\ \frac{2t^{\frac{3}{2}}}{3} - 2 \end{pmatrix} = \begin{pmatrix} 159.25 \\ 5.45 \end{pmatrix}$
$\frac{dy}{dx} \approx \frac{5.45}{159.25} \approx 0.03$

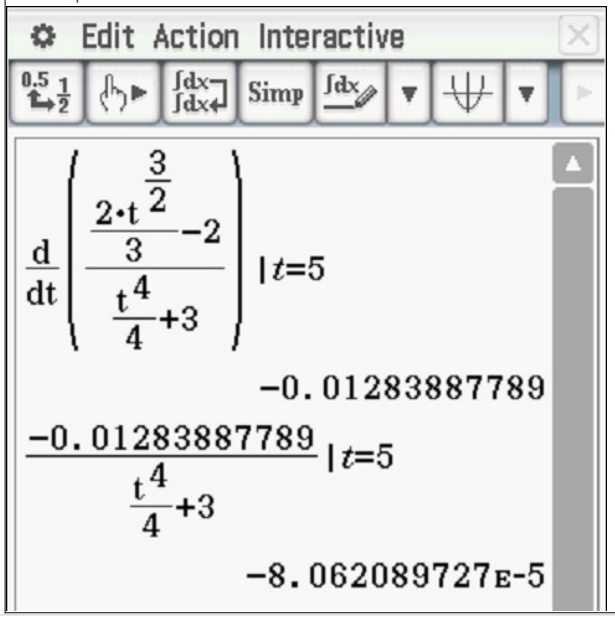
Specific behaviours
✓ uses v at t=5 ✓ determines rate

d) Determine $\frac{d^2y}{dx^2}$ on the cartesian path at time $t=5$ seconds. (3 marks)

Solution

$$v=\begin{pmatrix}\frac{t^4}{4}+3\\ \frac{2t^{\frac{3}{2}}}{3}-2\end{pmatrix}$$

$$\frac{dy}{dx}=\frac{\frac{2t^{\frac{3}{2}}}{3}-2}{\frac{t^4}{4}+3}$$



✓ uses Pythagorean identity ✓ breaks into two terms with sinx ✓ anti- diffs both terms ✓ subs both limits to give final result

Q5 cont-

b) $\int\frac{2x+1}{(x-3)(x+5)}dx$ (4 marks)

Solution

$$\frac{2x+1}{(x-3)(x+5)}=\frac{A}{x-3}+\frac{B}{x+5}$$

$$2x+1=A(x+5)+B(x-3)$$

$$x=3$$

$$7=8A \quad A=\frac{7}{8}$$

$$x=-5$$

$$-9=-8B \quad B=\frac{9}{8}$$

$$\int\frac{2x+1}{(x-3)(x+5)}dx=\frac{7}{8}\ln|x-3|+\frac{9}{8}\ln|x+5|+c$$

Specific behaviours
✓ uses partial fractions ✓ solves for constants ✓ integrates using logs ✓ states answer with a plus constant

Q6 (7 marks).

Consider a projectile that leaves with speed $V\,m/s$ at an angle α to the horizontal, see diagram. Assume that the constant acceleration is $-g\,m/s^2$.

Q4 (3 marks)

Show using logarithmic differentiation how to differentiate $y = x^{\sin(2x)}$.

Solution
$\ln y = \ln x^{\sin(2x)} = \sin 2x \ln x$ $\frac{1}{y} y' = \sin 2x \frac{1}{x} + 2\cos 2x \ln x$ $y' = \left(\sin 2x \frac{1}{x} + 2\cos 2x \ln x \right) y$
Specific behaviours
✓ tales natural log of both sides ✓ implicit diff of both sides and uses product rule ✓ expresses in terms of x&y

Q5 (8 marks)

Show how to evaluate the following without any use of the classpad. Show all working.

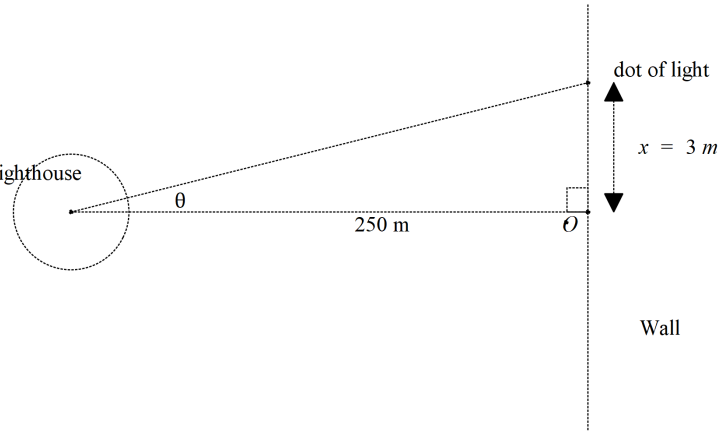
a) $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx$ (4 marks)

Solution
$\int_0^{\frac{\pi}{2}} \sin x \sin^2 x \, dx$ $\int_0^{\frac{\pi}{2}} \sin x (1 - \cos^2 x) \, dx$ $\int_0^{\frac{\pi}{2}} \sin x - \sin x \cos^2 x \, dx$ $\left[-\cos x + \frac{1}{3}\cos^3 x \right]_0^{\frac{\pi}{2}} = (0) - \left(-1 + \frac{1}{3} \right) = \frac{2}{3}$
Specific behaviours

Rate =0.00
Specific behaviours
✓ time diff dy/dx ✓ divides by dx/dt ✓ determines approx. rate (do not penalise if not 2dp)

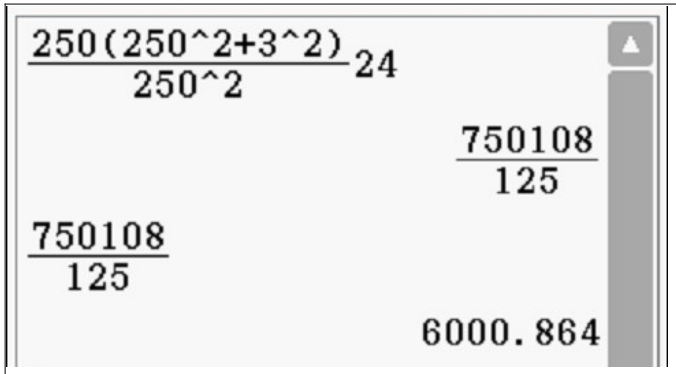
Q3 (7 marks)

Consider an artificial island that contains a revolving light that is 250 metres from shore. There is a long wall on the shore and the light from the lighthouse can be seen as a moving dot of light on the wall. The angular speed of the light is 24 radians/second, ($\frac{d\theta}{dt} = 24$).



a) Determine the speed of the dot of light on the wall when the dot is 3 metres away from the closest point to the lighthouse, pt O, see diagram above. (4 marks)

Solution
$\tan\theta = \frac{x}{250}$ $250 \sec^2 \theta \frac{d\theta}{dt} = \frac{dx}{dt}$ $\cos\theta = \frac{250}{\sqrt{250^2 + 3^2}}$



Specific behaviours
✓ states relationship between x and angle ✓ implicit diff wrt time (or related rates(✓ subs values and uses derivative of angle ✓ determines approx. speed (no need for units)

b) If the artificial island containing the lighthouse is moving towards the shore, pt O, at a speed of 5 metres per second, determine the speed of the dot when 3 metres away from pt O and the lighthouse being 170 metres from the shore, pt O. (3 marks)

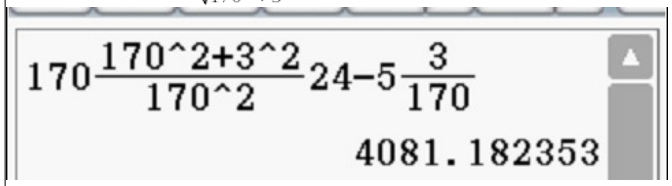
Solution

$\tan\theta = \frac{x}{L}$, $\dot{L} = -5 m/s$

$x = L\tan\theta$

$\dot{x} = L\sec^2\theta(\dot{\theta}) + \dot{L}\tan\theta$

$\tan\theta = \frac{3}{170}$ $\cos\theta = \frac{170}{\sqrt{170^2+3^2}}$



Specific behaviours
✓ uses three variables ✓ uses time implicit with product or quotient and all time rates with correct signs ✓ determines speed