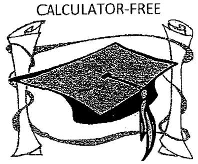


Semester 1 Examination 2012

Question/Answer Booklet

MATHEMATICS 3C

Section One:
Calculator-free

SOLUTIONS

Name of Student: ______________________________

Time allowed for this section
Reading time before commencing work: 5 minutes
Working time for this section: 50 minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the student
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters
Special items: nil

Important note to students
No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Section One: Calculator-free **(50 marks)**

This section has **six (6)** questions. Answer all questions. Write your answers in the spaces provided.

Working time: 50 minutes

Question 1 **(8 marks)**

(a) Simplify

(i) $\frac{x^2+5x-14}{5x^2-20} \div \frac{x^2+12x+35}{4x+20}$ (ii) $\frac{1}{x^2+x}+\frac{2}{x^2+2x}$ (4)

$= \frac{(x+7)(x-2)}{5(x^2-4)} \times \frac{4(x+5)}{(x+7)(x+5)}$ ✓

$= \frac{\cancel{(x+7)}\cancel{(x-2)}}{5\cancel{(x-2)}(x+2)} \times \frac{4\cancel{(x+5)}}{\cancel{(x+7)}\cancel{(x+5)}}$

$= \frac{4}{5(x+2)}$ ✓

$\frac{1}{x(x+1)} + \frac{2}{x(x+2)}$

$= \frac{1(x+2)+2(x+1)}{x(x+1)(x+2)}$ ✓

$= \frac{x+2+2x+2}{x(x+1)(x+2)}$

$= \frac{3x+4}{x(x+1)(x+2)}$ ✓

(b) The functions $f(x)$ and $g(x)$ are defined as follows

$f(x)=x^2-4$ and $g(x)=\sqrt{x-5}$

(i) Determine expressions for $f[g(x)]$ and $g[f(x)]$. (2)

$f(g(x)) = (\sqrt{x-5})^2 - 4$

$= x-5-4$

$= x-9$ ✓

$g f(x) = \sqrt{(x^2-4)-5}$

$= \sqrt{x^2-9}$ ✓

(ii) Determine the range of $f[g(x)]$. (1)

$R_{f(g(x))} = \{y : y \geq -4,\ y \in \mathbb{R}\}$ ✓

either ✓ or ✗

(iii) Determine the domain of $g[f(x)]$. (1)

$D_{g(f(x))} = \{x : x \geq 3 \text{ or } x \leq -3,\ x \in \mathbb{R}\}$

✓

either ✓ or ✗

Question 2 **(9 marks)**

(a) Differentiate the following with respect to x.

(i) $f(x) = \dfrac{-x}{x^2+1}$ (express in simplest form) (3)

$f'(x) = \dfrac{(x^2+1)(-1) - (-x)(2x)}{(x^2+1)^2}$ ✓

$f'(x) = \dfrac{-x^2 - 1 + 2x^2}{(x^2+1)^2}$ ✓

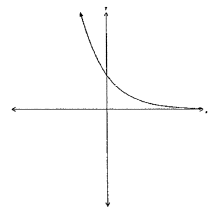
$f'(x) = \dfrac{x^2 - 1}{(x^2+1)^2}$ ✓

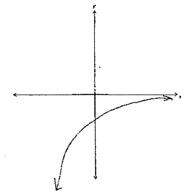
(ii) $g(x) = (2x+1)^2 e^{x^2}$ (do not simplify) (2)

$g'(x) = (2x+1)^2 2x e^{x^2} + e^{x^2} 2(2x+1)^1 (2)$ ✓✓

or $2x e^{x^2}(2x+1)^2 + 4e^{x^2}(2x+1)$

(b) For the graph already drawn, sketch the derivative function on the axes below. (2)

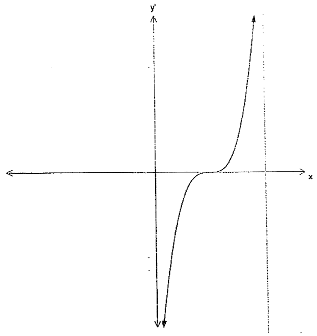


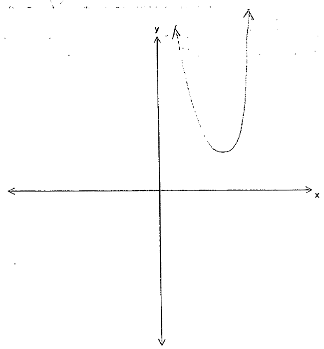


✓✓ shape
all under x-axis

Question 2 (continued)

(c) Given the **derivative function**, sketch a possible graph of the function. (2)





✓ Shape
✓ Turning point

Question 6 (continued)

(c) (ii) $\int_{-3}^{2}(4f(x)+3)\,dx$ (2)

$= \int_{-3}^{2} 4f(x)\,dx + \int_{-3}^{2} 3\,dx$

$= 4\int_{-3}^{2} f(x)\,dx + \int_{-3}^{2} 3\,dx$ ✓

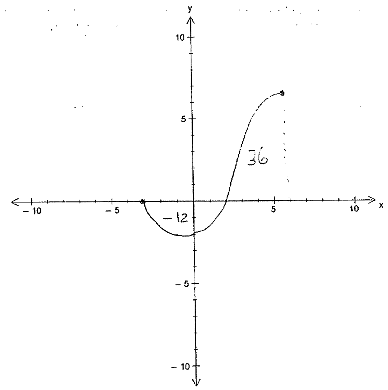
$= 4(-12) + \left[3x\right]_{-3}^{2}$

$= -48 + (6-(-9))$

$= -48 + 15$

$= -33$ ✓

(iii) Sketch a possible graph of $y=f(x)$ for $-3 \le x \le 6$. Your graph should display the relative areas of important regions but you do not need to draw this graph to scale. (1)



✓

✓ or x

Question 6 **(8 marks)**

(a) Differentiate $y = \sqrt{3x^2 + 4}$ by letting $u = 3x^2 + 4$ and using the chain rule.

Show your working. (2)

$y = u^{\frac{1}{2}}$ $\quad \frac{du}{dx} = 6x$

$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{2u^{\frac{1}{2}}} \times 6x$ ✓

$= \frac{6x}{2u^{\frac{1}{2}}}$

$= \frac{3x}{(3x^2+4)^{\frac{1}{2}}}$ ✓

(b) Determine $\int 3x^3(2x^4-5)^8\,dx$ (2)

$= \frac{3}{8}\int 8x^3(2x^4-5)^8\,dx$ ✓

$= \frac{3}{8}\left[\frac{(2x^4-5)^9}{9}\right] + c$

$= \frac{3(2x^4-5)^9}{72} + c$

$= \frac{(2x^4-5)^9}{24} + c$ ✓

(c) $f(x)$ is defined such that $\int_{-3}^{6} f(x)\,dx = 24$ and $\int_{2}^{6} f(x)\,dx = 36$

Find

(i) $\int_{-3}^{2} f(x)dx$. (1)

$= \int_{-3}^{6} f(x)\,dx - \int_{2}^{6} f(x)\,dx$

$= 24 - 36$

$= -12$ ✓

Question 3 **(7 marks)**

(a) It is claimed that the tangent line to the curve $y = x^3 - 2x^2 - 4x + 3$ at x=1 passes through the point (3,8). Is this claim valid? Justify your answer. (5)

$\frac{dy}{dx} = 3x^2 - 4x - 4$ ✓

$\left.\frac{dy}{dx}\right|_{(1,-2)} = 3 - 4 - 4 = -5$ ✓

Equation of tangent line at $(1,-2)$ is $y = -5x + c$

$-2 = -5(1) + c$

$-2 = -5 + c$

$3 = c$ ✓

Thus equation of tangent is $y = -5x + 3$

Substitute $(3,8)$ into equation to see if it results in true statement

$8 = -5(3) + 3$

$8 = -15 + 3$

$8 = -12$ (false) ✓

Claim is not valid as the tangent at $(1,-2)$ to the curve does not pass through $(3,8)$ ✓

(b) Two identical biased coins are tossed together, and the outcome is recorded. After a large number of trials it is observed that the probability that both coins land showing heads is 0.36.

What is the probability that both coins land showing tails? (2)

$P(2H) = 0.36 \Rightarrow P(1\text{ head}) = 0.6$

$\therefore P(1\text{ Tail}) = 0.4$ ✓

Hence $P(2\text{ Tails}) = 0.4 \times 0.4 = 0.16$ ✓

Question 4 **(8 marks)**

The volume of a certain rectangular box is given by the equation $V = x^3 - 5x^2 - 8x + 48$

(i) The height of the box is $(4-x)$ units. (2)

Show that $\frac{x^3-5x^2-8x+48}{4-x}$ results in the expression $-x^2+x+12$.

$$\begin{array}{r} -x^2 + x + 12 \\ -x+4 \overline{)\, x^3 - 5x^2 - 8x + 48} \\ -\ x^3 - 4x^2 \qquad\qquad \\ \hline -x^2 - 8x + 48 \\ -\ -x^2 + 4x \qquad \\ \hline -12x + 48 \\ -12x + 48 \\ \hline 0 \end{array}$$

✓✓

alternatively

$(4-x)(-x^2+x+12)$

$= -4x^2 + 4x + 48 + x^3 - x^2 - 12x$

$= x^3 - 5x^2 - 8x + 48$ ✓✓

(ii) In the context of this question, what does $-x^2+x+12$ represent? (1)

$-x^2+x+12$ represents the area of the base ✓

(iii) Calculate the value of x for which the volume is a maximum. (5)

$V = x^3 - 5x^2 - 8x + 48$

$\frac{dV}{dx} = 3x^2 - 10x - 8$ ✓

Let $\frac{dV}{dx} = 0$

$0 = 3x^2 - 10x - 8$

$0 = (3x+2)(x-4)$

$x = -\frac{2}{3}$ or $x = 4$

✓ ✓

$\frac{d^2V}{dx^2} = 6x - 10$

✓ when $x = -\frac{2}{3}$ $\frac{d^2V}{dx^2} = -14 \Rightarrow$ Max tpt

when $x = 4$, $\frac{d^2V}{dx^2} = 14 \Rightarrow$ Min tpt

Thus value of x when volume is a maximum is $x = -\frac{2}{3}$ ✓

Question 5 **(10 marks)**

(a) Find the global maximum and minimum values over the interval $\frac{1}{2} \le x \le 2$ of the function $y = x + \frac{1}{2x^2}$ (5)

$y = x + \frac{x^{-2}}{2}$

$y' = 1 - x^{-3}$

$y' = 1 - \frac{1}{x^3}$ ✓

Let $y' = 0$ to find stationary points

$0 = 1 - \frac{1}{x^3}$

$\frac{1}{x^3} = 1$

$x^3 = 1$

$\Rightarrow x = 1$ ✓

$y'' = 3x^{-4} = \frac{3}{x^4}$

$y''(1) = 3 \Rightarrow$ Min tpt ✓

Check function values at $x = \frac{1}{2}, 1, 2$

✓ $f(\frac{1}{2}) = \frac{1}{2} + \frac{1}{\frac{1}{2}} = \frac{1}{2} + 2 = 2\frac{1}{2}$

$f(1) = 1 + \frac{1}{2} = 1\frac{1}{2}$

$f(2) = 2 + \frac{1}{8} = 2\frac{1}{8}$

✓ Thus minimum value over interval is $1\frac{1}{2}$

Maximum value over interval is $2\frac{1}{2}$

(b) Events A and B are such $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\overline{A \cup B}) = \frac{1}{4}$

(i) Show that event A and B are **NOT** mutually exclusive. (3)

$P(A \cup B) = 1 - P(\overline{A \cup B}) = 1 - \frac{1}{4} = \frac{3}{4}$

$P(A) + P(B) = \frac{1}{2} + \frac{7}{12} = \frac{13}{12}$ ✓

As $P(A \cup B) \neq P(A) + P(B)$ ✓

A & B are not mutually exclusive

alternatively

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$ ✓

$= \frac{1}{2} + \frac{7}{12} - \frac{3}{4}$

$= \frac{6}{12} + \frac{7}{12} - \frac{9}{12}$

✓ $= \frac{4}{12} = \frac{1}{3}$

Since $P(A \cap B) \neq 0 \Rightarrow$ not mutually exclusive

(ii) Hence find $P(A \cap B)$. (2)

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= \frac{1}{2} + \frac{7}{12} - \frac{3}{4}$ ✓

$= \frac{6}{12} + \frac{7}{12} - \frac{9}{12}$

$= \frac{4}{12} = \frac{1}{3}$ ✓