WATP

SEMESTER TWO

MATHEMATICS SPECIALIST UNITS 3 & 4

2019

SOLUTIONS

Calculator-Free Solutions

1. $\bar{z}+\frac{z}{i}=(a-bi)+\frac{a+bi}{i}=(a+b)-(a+b)i$ ✓✓

i.e. $\Re\left(\bar{z}+\frac{z}{i}\right)=-\Im\left(\bar{z}+\frac{z}{i}\right)$

i.e. the complex number lies on the line $y=-x$ ✓

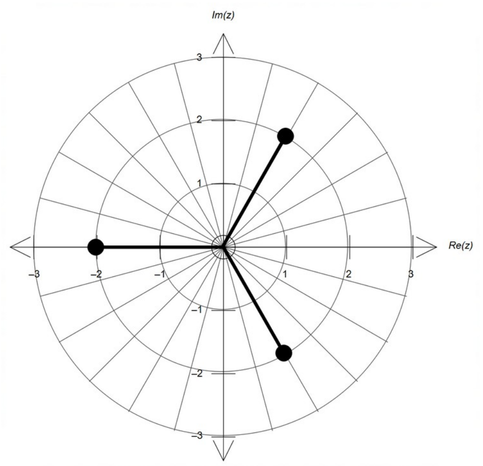
$\therefore arg\left(\bar{z}+\frac{z}{i}\right)=\frac{-\pi}{4}\vee\frac{3\pi}{4}$ ✓✓ [5]

2. (a) $z^3=-8=8\,cis(\pi+2k\pi)$

$\therefore z=2\,cis\left(\frac{\pi+2k\pi}{3}\right) for\, k=0,\pm1$ ✓

$z_0=2\,cis\left(\frac{\pi}{3}\right), z_1=2\,cis(\pi), z_3=2\,cis\left(\frac{-\pi}{3}\right)$ ✓✓

(b)



✓ magnitude = 2

✓ $\frac{2\pi}{3}$ radians

(c) $P(z)=(z^3+8)(z^2+bz+c)=z^5-z^4-2z^3+8z^2-8z-16$

$\therefore 8c=-16\rightarrow c=-2$

expanding with $c=-2$ gives:

$P(z)=z^5+bz^4-2z^3+8z^2+8bz-16$

and therefore $b=-1$ and $c=-2$ ✓

(division of polynomials is also possible)

Hence, $Q(z)=z^2-z-2=(z-2)(z+1)$ ✓

(d) $\therefore P(z)=(z^3+8)(z-2)(z+1)$ ✓

$\therefore z = -1, \pm 2 \wedge 1 \pm \sqrt{3}i$ ✓✓ [10]

3. (a) $x=2\tan(\theta)\rightarrow\dfrac{dx}{d\theta}=\dfrac{2}{\cos^2\theta}\rightarrow dx=\dfrac{2d\theta}{\cos^2\theta}$ ✓

$\theta(x)=\tan^{-1}\left(\dfrac{x}{2}\right)\rightarrow\theta(0)=0\wedge\theta(2)=\dfrac{\pi}{4}$

$\therefore a=0\wedge b=\dfrac{\pi}{4}$ ✓

$\sqrt{x^2+4}=\sqrt{4\tan^2\theta+4}=\sqrt{4(\tan^2\theta+1)}=\sqrt{\dfrac{4}{\cos^2\theta}}$

$\therefore\sqrt{x^2+4}=\dfrac{2}{\cos\theta}$ ✓

$$\therefore\int_0^2\frac{x}{\sqrt{x^2+4}}dx=\int_0^{\frac{\pi}{4}}\frac{2\tan\theta}{\left(\frac{2}{\cos\theta}\right)}\times\frac{2d\theta}{\cos^2\theta}=\int_0^{\frac{\pi}{4}}\frac{2\tan\theta}{\cos\theta}d\theta$$

(b) Using the substitution recommended in (a):

$$\int_0^{\frac{\pi}{4}}\frac{2\tan\theta}{\cos\theta}d\theta=2\int_0^{\frac{\pi}{4}}\frac{\sin\theta}{\cos^2\theta}d\theta=2\int_0^{\frac{\pi}{4}}\sin\theta\cos^{-2}\theta\,d\theta$$

¿ $2\left[\dfrac{-\cos^{-1}\theta}{-1}\right]_0^{\frac{\pi}{4}}=2\left[\dfrac{1}{\cos\theta}\right]_0^{\frac{\pi}{4}}=2\left[\dfrac{1}{\frac{1}{\sqrt{2}}}-\dfrac{1}{1}\right]=2(\sqrt{2}-1)$ ✓✓

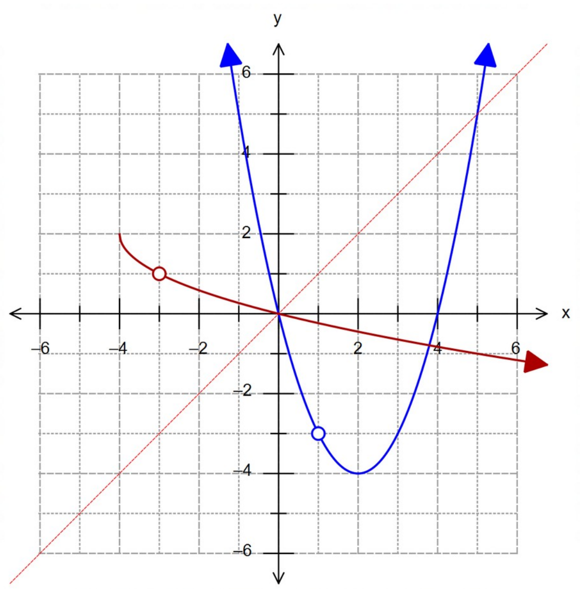
(other method is possible by setting $u=x^2$)

4. (a) $f(x)=\dfrac{x(x^2-5x+4)}{(x-1)}=\dfrac{x(x-4)(x-1)}{(x-1)}=x(x-4)$ provided $x\neq1$ ✓

$\therefore D_x=\{x\in R: x\neq1\}$ ✓

$R_y=\{y\in R: y\geq-4\wedge y\neq-3\}$ ✓✓

(b)



(b)

✓ parabola with roots at x=0 and 4, with tp at (2,-4)

(d)

✓ square root function with tp at (-4,2)

✓ discontinuity at (-3,1)

4. (c) Turning point at $(2,-4) \rightarrow \therefore x \le 2 \rightarrow k=2$ ✓

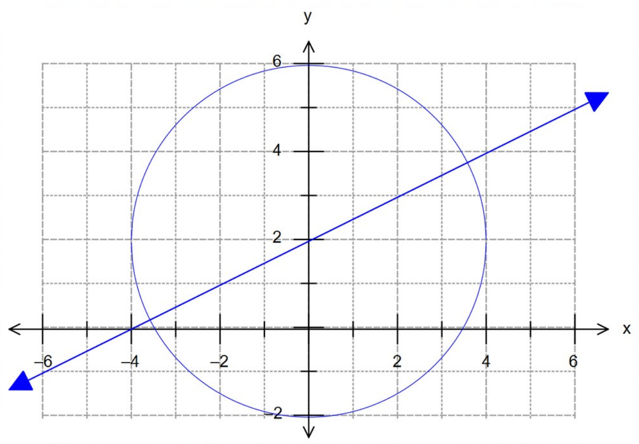
Algebraically by rearranging to make x the subject

gives $f^{-1}(x)=2-\sqrt{x+4}$ ✓

(d) Shown on the graph in (b). ✓✓ [10]

5. (a) $|r-2j|=4$ ✓

(b) The line can be obtained from two points, by choosing any two different values of λ.



✓✓ line $y=\frac{x}{2}+2$

$$\left|\begin{pmatrix}2\lambda-2\\ \lambda+1\end{pmatrix}-\begin{pmatrix}0\\2\end{pmatrix}\right|=\left|\begin{matrix}2(\lambda-1)\\ \lambda-1\end{matrix}\right|=4$$ ✓

$$\therefore 4(\lambda-1)^2+(\lambda-1)^2=16$$

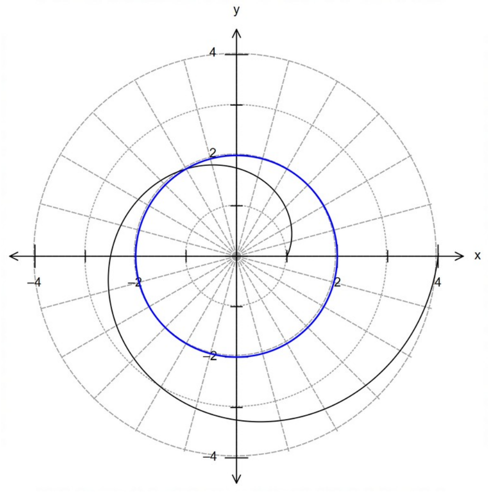
$$5(\lambda-1)^2=16 \rightarrow \lambda=1\pm\frac{4}{\sqrt{5}}$$ ✓

$$\therefore r\left(1\pm\frac{4}{\sqrt{5}}\right)=\begin{pmatrix}2\left(1\pm\frac{4}{\sqrt{5}}\right)-2\\ 1\pm\frac{4}{\sqrt{5}}+1\end{pmatrix}=\begin{pmatrix}\pm\frac{8}{\sqrt{5}}\\ 2\pm\frac{4}{\sqrt{5}}\end{pmatrix}$$ ✓ [6]

6. (a) Choosing the polar point $(4,2\pi)^P$:

$arg(z)=2\pi \rightarrow |z|=k(2\pi)+1=4 \therefore k=\dfrac{3}{2\pi}$ ✓

(b)



✓ circle cented at O with radius 2

(c) Using $|z|=2$:

$2=\dfrac{3}{2\pi}arg(z)+1 \rightarrow arg(z)=\dfrac{2\pi}{3}$ ✓✓

$\therefore w=2cis\left(\dfrac{2\pi}{3}\right)=2\cos\left(\dfrac{2\pi}{3}\right)+2i\sin\left(\dfrac{2\pi}{3}\right)=-1+\sqrt{3}i$ ✓✓ [6]

7. (a) Area of a segment: $A=\dfrac{r^2}{2}(\theta-\sin\theta)$

$\therefore A=\dfrac{\left(\frac{1}{2}\right)^2}{2}(2\theta-\sin(2\theta))=\dfrac{1}{8}(2\theta-\sin 2\theta)$

$\therefore V=A\times l=\dfrac{1}{8}(2\theta-\sin 2\theta)\times 8=2\theta-\sin 2\theta$ ✓

(b) $\dfrac{dV}{d\theta}=2-2\cos 2\theta$ ✓

(c) $\cos\theta=\dfrac{\frac{1}{2}-x}{\frac{1}{2}}=1-2x$ ✓

$\therefore x=\dfrac{1}{2}-\dfrac{1}{2}\cos\theta$ ✓

$\dfrac{dx}{dt}=0-\dfrac{1}{2}\times-\sin\theta\times\dfrac{d\theta}{dt}=\dfrac{1}{2}\dfrac{\sin\theta\, d\theta}{dt}$ ✓

(d) $\dfrac{d\theta}{dt}=\dfrac{d\theta}{dV}\times\dfrac{dV}{dt}=\dfrac{0.1}{2-2\cos 2\theta}$

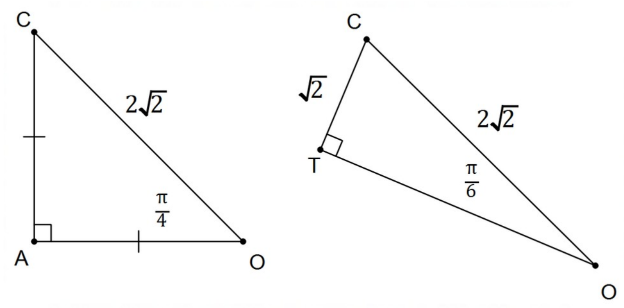
$$\therefore \frac{dx}{dt} = \frac{1}{2}\sin\theta \times \frac{0.1}{2-2\cos 2\theta} = \frac{\sin\theta}{40(1-\cos 2\theta)}$$ ✓

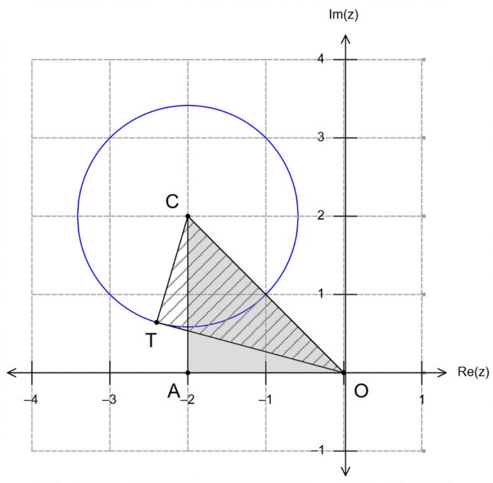
when $x = \frac{1}{4}$ m we obtain $\theta = \frac{\pi}{3}$ ✓

$$\therefore \left.\frac{dx}{dt}\right|_{\theta=\frac{\pi}{3}} = \frac{\sin\left(\frac{\pi}{3}\right)}{40\left(1-\cos\left(\frac{2\pi}{3}\right)\right)} = \frac{\frac{\sqrt{3}}{2}}{40\left(1+\frac{1}{2}\right)} = \frac{\sqrt{3}}{120}\left[\frac{m^3}{min}\right]$$ ✓ [8]

Calculator-assumed Solutions

8. From the diagram we obtain two right angled triangles:





And therefore we have:

$|z|_{min}=2\sqrt{2}-\sqrt{2}=\sqrt{2}$ ✓

$|z|_{max}=2\sqrt{2}+\sqrt{2}=3\sqrt{2}$ ✓

$arg(z)_{min}=\frac{3\pi}{4}-\frac{\pi}{6}=\frac{7\pi}{12}$ ✓✓

$arg(z)_{max}=\frac{3\pi}{4}+\frac{\pi}{6}=\frac{11\pi}{12}$ ✓ [5]

9. (a) $\overline{T}\ \ N\left(15,\frac{300}{30}\right)$ ✓

$\therefore \sigma^2(\overline{T})=10 \rightarrow \sigma(\overline{T})=\sqrt{10}\approx 3.16$ ✓

$\therefore P(5\leq\overline{T}\leq 15)=0.8862\approx 0.89$ (2dp.) ✓

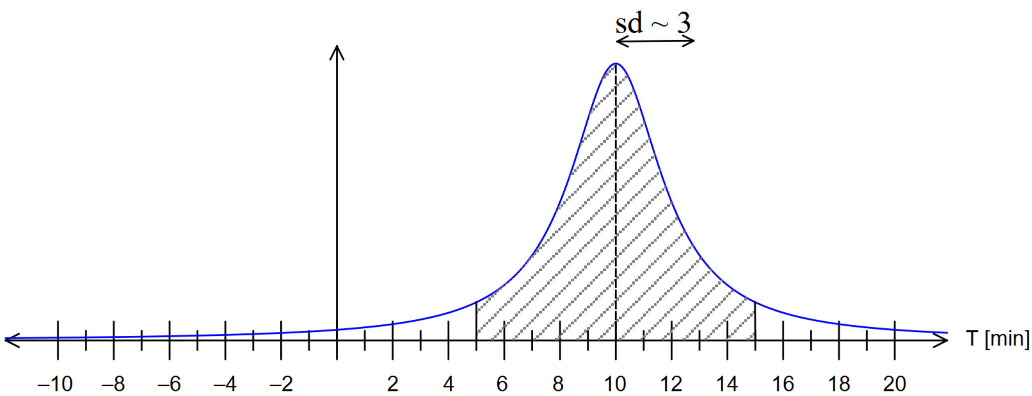
(b)

✓ Normal distribution centred at $\mu(\overline{T})=10$

ndard

pprox. 3 min

with area

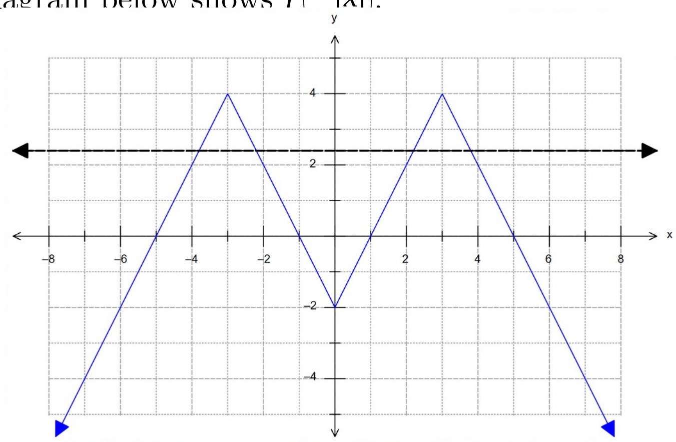


[6]

10. (a) $y = -2|x+3| + 4$

$\therefore a = -2, b = 3, c = 4$ ✓✓✓

(b) The diagram below shows $f(-|x|)$:



Thefore, a horizontal line $y = d$ would intersect the graph

for $-2 < d < 4$. ✓✓✓ [6]

(graphical explanation can be accepted)

11. (a) $\overrightarrow{OM} = \begin{pmatrix} 120 \\ -80 \\ 40 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = ¿$ location of ground collision ✓

$\therefore 40 - t = 0 \rightarrow t = 40$ seconds ✓

$\therefore \overrightarrow{OM}(40) = \begin{pmatrix} 120 \\ -80 \\ 40 \end{pmatrix} + 40\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 200 \\ -40 \\ 0 \end{pmatrix}$ ✓

and $\left|\overrightarrow{OM}(40)\right| = \begin{vmatrix} 200 \\ -40 \\ 0 \end{vmatrix} = 40\begin{vmatrix} 5 \\ -1 \\ 0 \end{vmatrix} = 40\sqrt{26} \approx 203.96\,km$ from O ✓

(b) $\overrightarrow{OM} = \begin{pmatrix} 120 \\ -80 \\ 40 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 10 \end{pmatrix} = ¿$ point 10 km above ground

$\therefore 40 - t = 10 \rightarrow t = 30$ sec from detection time ✓

$\therefore \overrightarrow{OM}(30) = \begin{pmatrix} 120 \\ -80 \\ 40 \end{pmatrix} + 30\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 180 \\ -50 \\ 10 \end{pmatrix}$ ✓

$\begin{pmatrix} 180 \\ -50 \\ 10 \end{pmatrix} - \begin{pmatrix} 20 \\ 160 \\ 0 \end{pmatrix} = \begin{pmatrix} 160 \\ -210 \\ 10 \end{pmatrix}$ displacement vector for ABM ✓

$$\therefore speed = \frac{1}{30}\left|\begin{matrix}160\\-210\\10\end{matrix}\right| = \frac{\sqrt{698}}{3} \approx 8.81\, km/s$$ ✓

11. (c) (Using the conditions for collision will show that they do not collide. If no other work is shown then award two marks for this attempt)

Using closest-approach method with CAS: f_{min}

$$\overrightarrow{OM}=\begin{pmatrix}2t+120\\ t-80\\ 40-t\end{pmatrix} \text{ and } \overrightarrow{ABM}=\begin{pmatrix}20+4.6t\\ 160-5.25t\\ 0+0.042t\end{pmatrix}$$

$$\therefore {}_{ABM}r_{OM}=\overrightarrow{ABM}-\overrightarrow{OM}=\begin{pmatrix}2.6t-100\\ 240-6.25t\\ 1.042t-40\end{pmatrix}$$ ✓✓

$$\therefore \left|{}_{ABM}r_{OM}\right|=\sqrt{(2.6t-100)^2+(240-6.25t)^2+(0.042t-40)^2}$$ ✓

CAS: $f_{min} \rightarrow \left|{}_{ABM}r_{OM}\right|_{min}\approx 0.15\,km$ for $t=38.41\,s$ ✓✓ [13]

12. (a) $\left.\dfrac{dy}{dx}\right|_{(0,2)}=\dfrac{1}{2(0)+2}=\dfrac{1}{2}$ ✓✓

(b) The slope is undefined at $x=-1$ ✓

(c) $\dfrac{dy}{dx}=\dfrac{1}{2(x+1)}\rightarrow 2\,dy=\dfrac{dx}{x+1}$

$$\therefore \int 2\,dy=\int \frac{dx}{x+1}$$

$\therefore 2y=\ln|x+1|+C_1$ ✓

$y=\dfrac{1}{2}\ln|x+1|+C_2=\ln\sqrt{x+1}+C_2$

$(0,2)\rightarrow 2=\ln\sqrt{1}+C_2\rightarrow C_2=2$ ✓

$\therefore y=\ln\sqrt{x+1}+2=\ln e^2\sqrt{x+1}$ ✓

13. (a) $\overline{X}$ is approximately normally distributed as the sample size $n=40>30$

$\therefore \overline{X} \sim N\left(8,\dfrac{8^2}{40}\right)=N(8,1.6)$ ✓✓

i.e. $\sigma(\overline{X})=\sqrt{1.6}\approx 1.2649$ ✓

(b) $P(5<\overline{X}<11)=0.9823$ ✓✓

(c) No, there is no change. ✓

Because the sample size $(40 > 30)$ provides a normal distribution for the sample means, despite the shape of the parent distribution. ✓

13. (d) $P(\overline{X}>10)=0.05$ ✓

and $\overline{X}\quad N\left(8,\frac{8^2}{n}\right)$, i.e. $\sigma(\overline{X})=\frac{8}{\sqrt{n}}$

If $P(z>k)=0.05 \rightarrow k=1.6448$ ✓

$\therefore \frac{10-8}{\left(\frac{8}{\sqrt{n}}\right)}=1.6448 \rightarrow n=43.29 \approx 44$ ✓✓

14. (a) (i) $\frac{d(x\cos x)}{dx}=\cos x - x\sin x$ ✓

(ii) $d(x\cos x)=\cos x\,dx - x\sin x\,dx$

$$\therefore \int_0^\pi d(x\cos x)=\int_0^\pi \cos x\,dx - \int_0^\pi x\sin x\,dx$$

$$\int_0^\pi x\sin x\,dx=\int_0^\pi \cos x\,dx - \int_0^\pi d(x\cos x)$$

¿$[\sin x]_0^\pi - [x\cos x]_0^\pi$ ✓

¿$[\sin\pi - \sin 0]-[\pi\cos\pi - 0]$ ✓

¿$-[-\pi]=\pi$

(b) $x^2-x-2=(x-2)(x+1)$

Let $\frac{3}{x^2-x-2}=\frac{A}{x-2}+\frac{B}{x+1}$

then: $3=A(x+1)+B(x-2)$

for $x=2 \rightarrow 3=3A+0 \rightarrow A=1$ ✓

for $x=-1 \rightarrow 3=0-3B \rightarrow B=-1$ ✓

$$\int \frac{3}{x^2-x-2}\,dx=\int \frac{dx}{x-2}-\int \frac{dx}{x+1}$$

¿$\ln|x-2|-\ln|x+1|+C$ ✓

¿$\ln\left|\frac{x-2}{x+1}\right|+C$

15. (a) $\frac{dQ}{dt}=k(100-Q)=-k(Q-100)$

$$\therefore \int \frac{dQ}{Q-100}=-\int k\,dt$$

$\ln|Q-100|=-kt+C$ ✓

$Q-100=e^{-kt+C}=e^{C}\times e^{-kt}=A\,e^{-kt}$ ✓

$\therefore Q(t)=A\,e^{-kt}+100$

15. (b) $Q(0)=A+100=900 \rightarrow A=800$ ✓

$2.5\,hrs=150\,min$

$\therefore Q(150)=800e^{-150k}+100=450$

$\rightarrow e^{-150k}=\dfrac{350}{800} \rightarrow k=\dfrac{\ln\left(\frac{7}{16}\right)}{-150}\approx 0.0055$ ✓

$\therefore Q(t)=800e^{-0.0055t}+100$

Since exponential decay never crosses the x-axis, we must choose the first value that would round to zero.
i.e. $Q(t)=100.49$ (other values less than 100.5 are acceptable)

$Q(t)=800e^{-0.0055t}+100=100.49$ ✓

$e^{-0.0055t}=\dfrac{0.49}{800} \rightarrow t=\dfrac{\ln\left(\frac{0.49}{800}\right)}{-0.0055}\approx 1345.08\approx 1345\,min$ ✓

16. (a) $\dfrac{dB}{dt}=rB(k-B)$ from formula sheet ✓

(b) $B(t)=\dfrac{5000}{1+Ae^{-kt}}$ from formula sheet, with 5000 as limit ✓✓

$\therefore B(15)=\dfrac{5000}{1+Ae^{-15k}}=2278$ and $B(25)=\dfrac{5000}{1+Ae^{-25k}}=4304$

solving simultaneously: CAS $\rightarrow A=24, k=0.2$ ✓✓

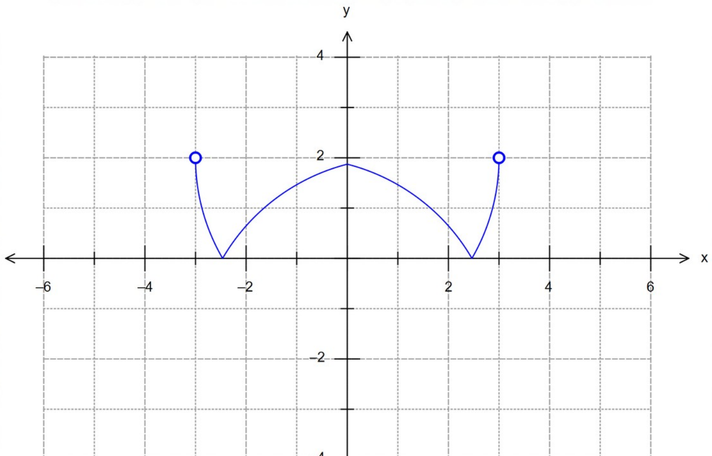
$\therefore B(t)=\dfrac{5000}{1+24e^{-0.2t}}$

(c) Since the line $y=5000$ is the upper asymptote, it will never actually reach 5000. Therefore we must choose the first value that would round to 5000:

$B(t)=\dfrac{5000}{1+24e^{-0.2t}}=4999.5$ ✓

CAS $\rightarrow t=61.94\,hrs=61:56\,hrs$ ✓

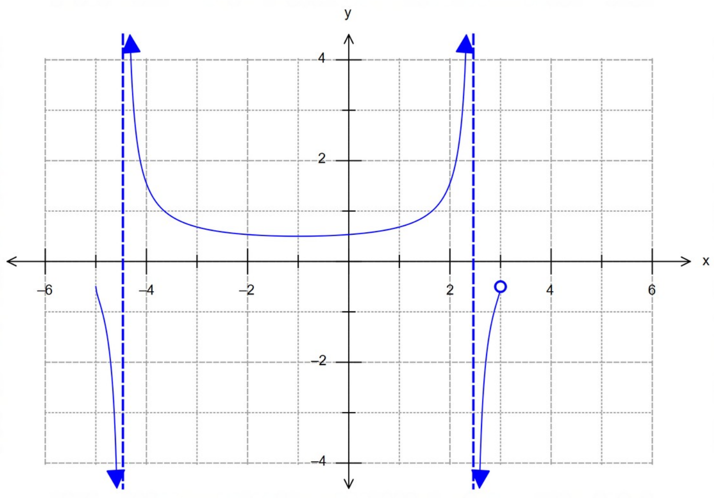
17. (a) (i)



✓ reflection over x axis

✓ reflection over y axis

17. (a) (ii)



✓ vertical asymptotes at the roots of the circle ✓ crosses the original function at points $y=\pm 1$ ✓ behaviour on either side of the asymptotes

(b) (i) $h \circ f(x) = h(f) = \dfrac{1}{(4-f)^2} = \dfrac{1}{\left(4-4+4\sqrt{x-1}\right)^2} = \dfrac{1}{16(x-1)}$ ✓

(ii) For $h(x)$: $(4-x)^2 \neq 0$

$\therefore (4-f)^2 \neq 0$ ✓

$4-f \neq 0 \rightarrow \sqrt{x-1} \neq 0 \rightarrow x \neq 1$

Domain of $f(x) = \{x \in R : x > 1\}$ ✓

Range of $h \circ f(x) = \{y \in R : y > 0\}$ ✓ [9]

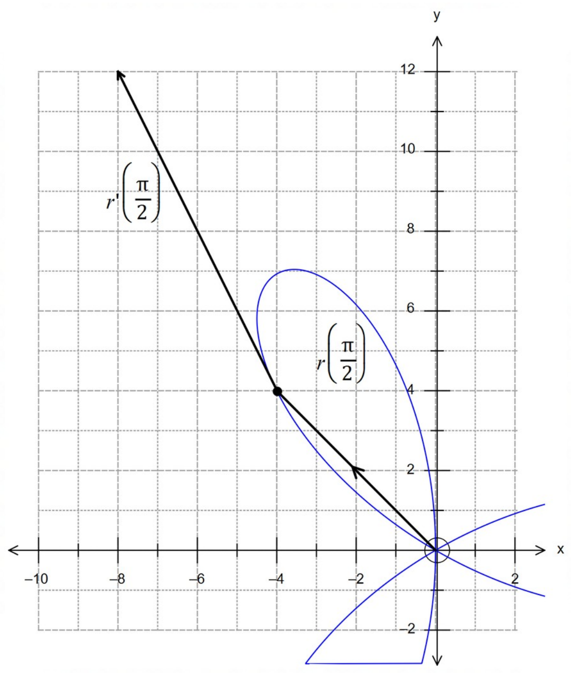
18. (a) $t = 2\pi \approx 6.28$ seconds ✓

(b) $r\left(\dfrac{\pi}{2}\right) = 4\begin{pmatrix} \cos\left(\dfrac{\pi}{2}\right) + \cos\pi \\ \sin\left(\dfrac{\pi}{2}\right) - \sin\pi \end{pmatrix} = 4\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -4i + 4j$ [m] from O ✓

$\dot{r}(t) = 4\begin{pmatrix} -\sin t - 2\sin(2t) \\ \cos t - 2\cos(2t) \end{pmatrix}$ ✓

$\dot{r}\left(\dfrac{\pi}{2}\right) = 4\begin{pmatrix} -\sin\left(\dfrac{\pi}{2}\right) - 2\sin\pi \\ \cos\left(\dfrac{\pi}{2}\right) - 2\cos\pi \end{pmatrix} = 4\begin{pmatrix} -1 \\ 2 \end{pmatrix} = -4i + 8j$ [m/s] ✓

18. (b) Continued.



✓ position vector (-4, 4) drawn from O.

✓ velocity vector (-4, 8) relative to (-4,4) [i.e. drawn from (-4 ,4)]

(c) $|v|_{max} = 4\sqrt{5+4\times 1} = 12$ m/s ✓

for $\sin(3t) = 1 \to 3t = \frac{\pi}{2} \to t = \frac{\pi}{6}$ ✓

$$r\left(\frac{\pi}{6}\right) = 4\begin{pmatrix} \cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} 2\sqrt{3}+2 \\ 2-2\sqrt{3} \end{pmatrix} \approx \begin{pmatrix} 5.46 \\ -1.46 \end{pmatrix}$$ ✓ [9]

19. (a) the apparent "root" is in fact a discontinuity which occurs at $\tan\left(\frac{\pi}{2}\right) \to x + \frac{\pi}{6} = \frac{\pi}{2} \to x = \frac{\pi}{3}$ ✓

$$\therefore \int_0^{\frac{\pi}{3}} \frac{1}{\tan\left(x+\frac{\pi}{6}\right)}\,dx = \int_0^{\frac{\pi}{3}} \frac{\cos\left(x+\frac{\pi}{6}\right)}{\sin\left(x+\frac{\pi}{6}\right)}\,dx$$

$$¿\left[\ln\left|\sin\left(x+\frac{\pi}{6}\right)\right|\right]_0^{\frac{\pi}{3}}$$ ✓

$$¿\ln\left|\sin\left(\frac{\pi}{2}\right)\right| - \ln\left|\sin\left(\frac{\pi}{6}\right)\right|$$

$$¿\ln(1) - \ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2}\right)^{-1} = \ln 2$$ ✓

(b)

$$V = \pi\int_0^h \left(\frac{r}{h}x\right)^2 dx = \frac{\pi r^2}{h^2}\int_0^h x^2\,dx$$

$$¿\frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2}{h^2} \times \frac{h^3}{3} = \frac{\pi}{3} r^2 h$$ ✓✓ [9]