PERTH MODERN SCHOOL
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Independent Public School

Course ________Specialist_______ Year __12______

Student name: _______________ Teacher name: _______________

Date: 24 Feb

Task type: **Response**

Time allowed for this task: **_____45______ mins**

Number of questions: **_____8______**

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **__50____ marks**

Task weighting: **_10___%**

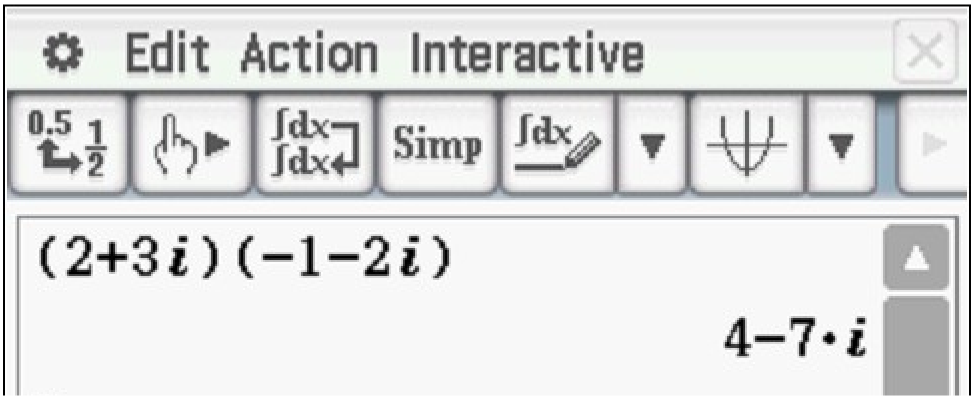
Formula sheet provided: **Yes/No**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (3.1.1, 3.1.2, 3.1.3) (2, 2, 3 & 3 = 10 marks)

If $z = 2 + 3i$ and $w = -1 + 2i$ determine exactly the following. (Simplify)

a) $z\overline{w}$

b) Solution

Specific behaviours
✓ determines conjugate of w ✓ determines product

b) $w\overline{w}$

Solution
$1^2 + 2^2 = 5$
Specific behaviours
✓ gives a real result ✓ determines product

c) $w \div \overline{w}$

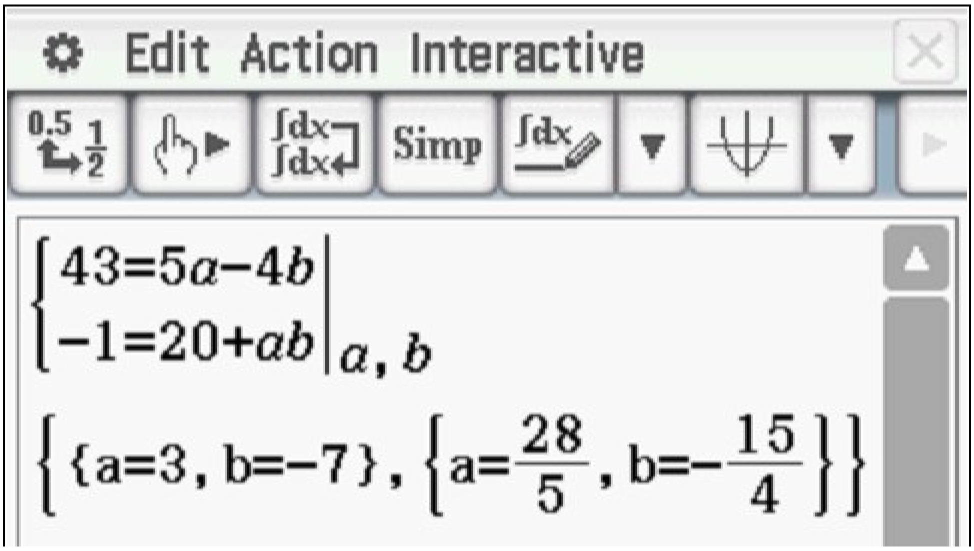
Solution
$\dfrac{-1+2i}{-1-2i} \times \dfrac{-1+2i}{-1+2i} = \dfrac{1-4-4i}{1+4} = \dfrac{-3-4i}{5}$
Specific behaviours
✓ multiplies by conjugate over conjugate ✓ evaluates numerator ✓evaluates denominator

d) $\dfrac{1}{z} + \dfrac{1}{w}$

Solution
$\frac{1}{2+3i}\,\frac{2-3i}{2-3i}+\frac{1}{-1+2i}\,\frac{-1-2i}{-1-2i}$ $\frac{2-3i}{13}+\frac{-1-2i}{5}$ $\frac{10-15i}{65}+\frac{-13-26i}{65}=\frac{-3-41i}{65}$
Specific behaviours
✓ uses idea of conjugates ✓ expresses both reciprocals with real denominator ✓determines correct simplified sum

Q2 (3.1.3) (3 marks)

Determine all possible real values of $a\,\&\,b$ such that $\frac{43-i}{a+4i}=5+bi$

Solution
$(43-i)=(5+bi)(a+4i)=5a-4b+i(20+ab)$ $43=5a-4b$ $-1=20+ab$ 
Specific behaviours
✓ sets up one equation involving a or b ✓ sets up two equations for a & b

✓solves for two pairs of real values for a & b

Q3 (3.1.14, 3.1.15) (3& 3 = 6 marks)

Consider the quadratic equation $x^2+bx+c=0$ where $b\,\&\,c$ are real.

a) If one root of the above equation is $x=4-2i$, determine $b\,\&\,c$.

Solution
$(4-2i)^2+b(4-2i)+c=0$ $16-4-16i+4b-2bi+c=0$ $12+4b+c=0$ $-16-2b=0$ $b=-8$ $c=-12+32=20$
Specific behaviours
✓ sets up one equation involving c or b ✓ sets up two equations for c & b ✓solves for one pair of real values for c & b

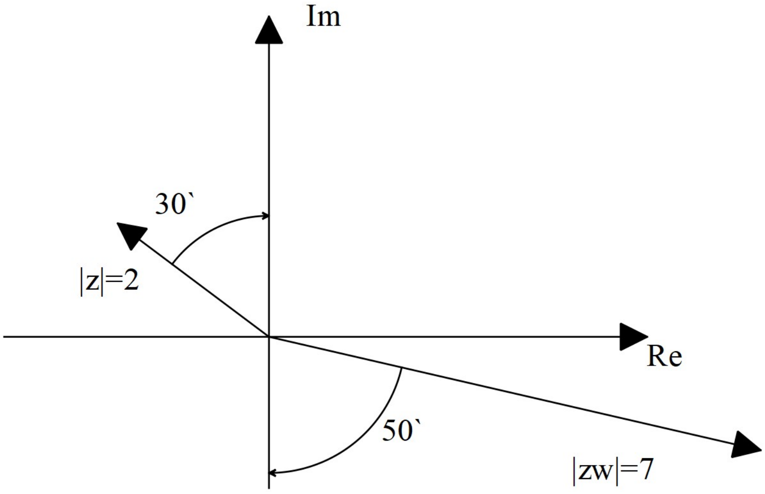
Consider the equation $x^3+px^2+qx+w=0$ where $p,q\,\&\,w$ are real.

b) If the cubic equation above has roots $x=2\,\&\,x=\sqrt{3}i$, determine $p,q\,\&\,w$.

Solution
$(x-2)(x-\sqrt{3}i)(x+\sqrt{3}i)$ $(x-2)(x^2+3)$ $x^3+3x-2x^2-6$ x^3-2x^2+3x-6 $p=-2,q=3\,\&\,w=-6$
Specific behaviours
✓ recognises that conjugate also a root ✓ expands at least two linear factors ✓solves for p,q&w

Q4 (3.1.3, 3.1.3, 3.1.3) (2 marks)

Determine $z \& w$ in the form $r cis\theta$ with $-\pi < \theta \leq \pi$. (Note: diagram not drawn to scale)



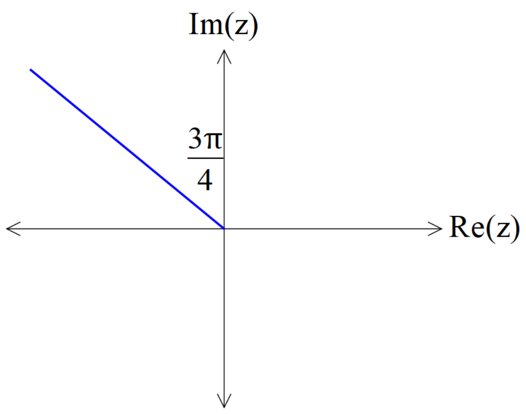
Solution
$z = 2cis120 \; or \; 2cis\frac{2\pi}{3}$ $w = \frac{7}{2}cis(-160) \; or \; \frac{7}{2}cis\left(\frac{-8\pi}{9}\right)$ Accept radians or degrees
Specific behaviours
✓ determines z with principal argument ✓ determines w with principal argument

Q5 (3.1.10) (2, 2 & 3 = 7 marks)

Sketch the following regions in the complex plane showing major features.

a) $Arg(z) = \dfrac{3\pi}{4}$

Solution

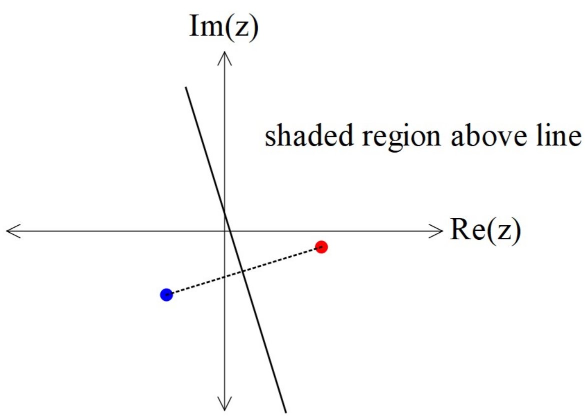


Specific behaviours

✓ uses line y=-x
✓ only drawn for Re(z)<0 (allow zero)

b) $|z + 3 + 4i| \geq |z - 5 + i|$

Solution

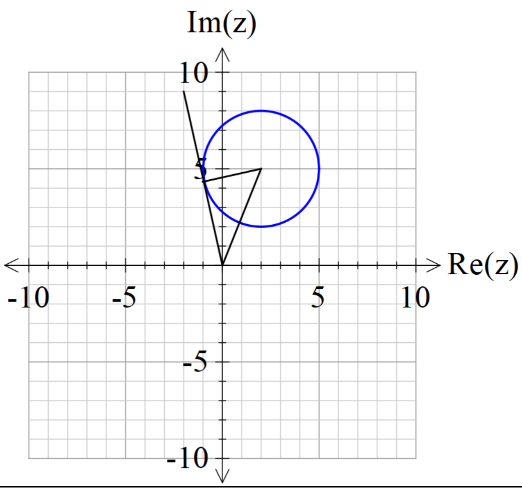


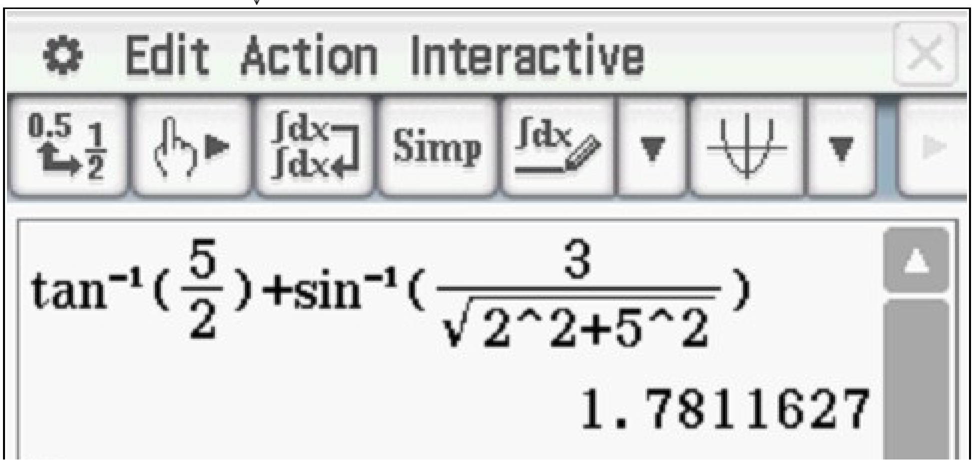
Specific behaviours

✓ plots correct points
✓ shades above perpendicular bisector

c) Consider all the complex numbers z that satisfy $|z-(2+5i)|=3$, determine the maximum possible value of $Arg(z)$, giving your answer in radians correct to two decimal places.

Solution





Specific behaviours

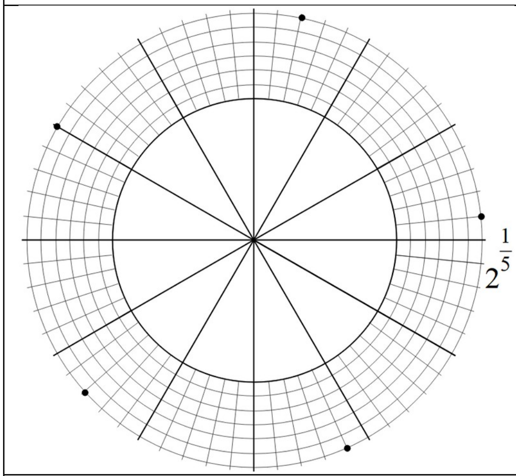
✓ determines argument of centre of circle
✓ uses left tangent idea for max argument
✓solves for max argument in radians(no need to round to 2 dp)

Q6 (3.1.7, 3.1.12) (4 & 3=7 marks)

a) Determine all the roots of $z^5 = \sqrt{3}+i$ expressing in the form $rcis\theta$ with $-\pi < \theta \le \pi$.

Solution
$z^5 = \sqrt{3}+i = 2cis\left(\frac{\pi}{6}+2n\pi\right) \quad , n = 0, \pm 1, \pm 2 ...$ $z = 2^{\frac{1}{5}} cis\left(\frac{\pi}{30}+\frac{2n\pi}{5}\right) = 2^{\frac{1}{5}} cis\left(\frac{\pi}{30}+\frac{12n\pi}{30}\right)$ $z_1 = 2^{\frac{1}{5}} cis\left(\frac{\pi}{30}\right)$ $z_2 = 2^{\frac{1}{5}} cis\left(\frac{13\pi}{30}\right)$ $z_3 = 2^{\frac{1}{5}} cis\left(-\frac{11\pi}{30}\right)$ $z_4 = 2^{\frac{1}{5}} cis\left(\frac{-23\pi}{30}\right)$ $z_5 = 2^{\frac{1}{5}} cis\left(\frac{25\pi}{30}\right) or\, 2^{\frac{1}{5}} cis\left(\frac{5\pi}{6}\right)$
Specific behaviours
✓ expresses right hand side into polar form ✓ uses De Moivre's theorem ✓ obtains five distinct roots in polar form ✓uses principal arguments for all roots

b) Plot all of these roots on the diagram below.

Solution

Specific behaviours
✓ shows scale ✓ plots one root correctly

✓all five roots equally spaced

Q7 (3.2.1, 3.2.2) (1, 2, 2 & 2 = 7 marks)

Consider the functions $f(x)=\sqrt{x-8}$ & $g(x)=x^3$.

a) Give the defining rule for $f\circ g(x)$.

Solution
$f\circ g(x)=\sqrt{x^3-8}$
Specific behaviours
✓ states rule

a) Does $f\circ g(x)$ exist over the natural domain of $g(x)$? Explain

Solution
$r_g : R$ $d_f : x\geq 8$ $r_g \not\subset d_f \quad \therefore does\ not\ exist$
Specific behaviours
✓ determine appropriate domain and range ✓ shows that condition not meet for natural domain of g

b) State the natural domain and range for $f\circ g(x)$.

Solution
$f\circ g(x)=\sqrt{x^3-8}$ $x^3\geq 8$ $x\geq 2$ $y\geq 0$
Specific behaviours
✓ states natural domain ✓ states range

Consider the function $h(x) = x - 8$.

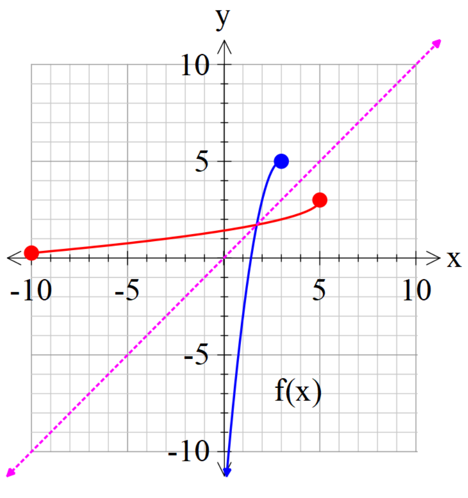
c) Does the function $\left[f(x)\right]^2 = h(x)$? Justify your answer.

Solution
$\left[f(x)\right]^2$ $d : x \geq 8$ $h(x)$ $d : R$ $not\ equal\ as\ different\ domains$
Specific behaviours
✓ states not equal with a reason ✓ shows that domains are different

Q8 (3.2.3, 3.2.4) (2 & 3, 3 = 8 marks)

Consider the function $f(x)$ drawn below.

a) Sketch $y = f^{-1}(x)$ on the axes above.

b) Solution

Specific behaviours
✓ reflects in line y=x ✓ inverse contains pt (5,3)

b) Given that $f(x) = -2x^2 + 12x - 13, \quad x \le 3$, determine the defining rule for $y = f^{-1}(x)$.

Solution
$y \le 3$ $x = -2y^2 + 12y - 13$ $-2y^2 + 12y - 13 - x = 0$ $y = \frac{-12 \pm \sqrt{144 - 4(-2)(-13 - x)}}{-4} = \frac{-12 \pm \sqrt{144 - 104 - 8x}}{-4}$ $= \frac{-12 \pm 2\sqrt{10 - 2x}}{-4} = \frac{-6 \pm \sqrt{2(5 - x)}}{-2}$ $f^{-1}(x) = 3 - \sqrt{\frac{5 - x}{2}}$
Specific behaviours
✓ interchanges x and y to solve for inverse ✓ states possible rules for inverse ✓states correct rule with negative only

c) Consider the function $h(x) = ax^3$ where a is a positive constant. Solve in terms of a, the solution(s) to $h(x) = h^{-1}(x)$.

c) Solution
$h(x) = h^{-1}(x) = x \quad or\ ax^3 = \sqrt[3]{\frac{x}{a}} \quad \Rightarrow a^3x^9 = \frac{x}{a} \Rightarrow a^4x^9 - x = 0 \Rightarrow x\left(a^4x^8 - 1\right) = 0$ $ax^3 = x$ $x(ax^2 - 1) = 0$ $x = 0, \frac{1}{\sqrt{a}}, -\frac{1}{\sqrt{a}}$
Specific behaviours
✓ sets up equation in terms of x & a ✓factorises equation ✓states all three x values in terms of a.

Working out space