Penrhos College
Semester 2 Examination, 2011

Question/Answer Booklet

MATHEMATICS SPECIALIST: 3C/3DMAS

Section Two: Calculator-assumed

Student Name:____SOLUTIONS_____

Time allowed for this section

Reading time before commencing work: 10 minutes
Working time for section: 100 minutes

Material required/recommended for this section

To be provided by the supervisor
Question/answer booklet for Section Two. Candidates may use the removable formula sheet from Section One

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler
Special items: drawing instruments, templates, notes on up to two unfolded sheets of A4 paper, and up to three calculators, CAS, graphic or scientific, which satisfy the conditions set by the Curriculum Council for this course.

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

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Instructions to candidates

1. **All** questions should be attempted.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare answer pages may be found at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued (i.e. give the page number).

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

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D O N O T W R I T E I N T H I S A R E A

Section Two: Calculator-assumed **(80 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the space provided.

Suggested working time for this section is 100 minutes.

Question 9 **(4 marks)**

Use proof by exhaustion to prove that 127 is a prime number.

Solution
If 127 is a prime number it is not divisible by another prime number. $127 \div 2 = 63.5$ $127 \div 3 = 42.3$ $127 \div 5 = 25.4$ $127 \div 7 = 18.14$ $127 \div 11 = 11.55$ As $11 > \sqrt{127}$ there can be no other prime factors of 127. $\therefore$ 127 has no other factors other than 1 and itself i.e. 127 is prime
Specific behaviours
✓ States that a prime number is not divisible by another prime number Follows process of exhaustion for dividing 127 by all possible prime numbers ✓ States why the process is exhausted ✓ establishes correct conclusion

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Question 10 **(9 marks)**

The triangle *ABC* with vertices *A*(0, 0), *B*(3, 1) and *C*(-1, 2) is transformed by the matrix $\mathbf{M} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ on to the triangle *A*'*B*'*C*'.

(a) Determine the coordinates of *A*' and *C*'.

[2]

Solution
A'(0, 0). *C*'(4, 1)
Specific behaviours
✓ correctly determines *A*' ✓ correctly determines C'

(b) The area of triangle *A*'*B*'*C*' is 7 square units. What is the area of triangle *ABC*?

[2]

Solution
$\lvert\mathbf{M}\rvert = 0 - (-2)$ $= 2$ Area of ABC = 7 ÷ $\lvert 2 \rvert$ = 3.5 units^2
Specific behaviours
✓ correctly calculates the value of the determinant ✓ correct area of triangle

(c) Matrix ***M*** represents a combination of transformation ***X*** followed by transformation ***Y***. If the matrix for transformation $\boldsymbol{X} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, determine the matrix for transformation ***Y*** and describe the geometric transformation ***Y*** represents.

[3]

Solution
$\boldsymbol{Y}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ $\boldsymbol{Y} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$ $\boldsymbol{Y} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ ***Y*** represents a dilation factor 2 parallel to x-axis
Specific behaviours
✓ correctly acknowledges ***YX*** = ***M*** ✓ correctly calculates matrix ***Y*** ✓ correctly describes matrix ***Y*** geometrically

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(d) The triangle *A*'*B*'*C*' then undergoes a shear of factor k parallel to the *y*-axis such that the image of the coordinate *C*' is (4, -3). Determine the value of k.

[2]

Solution
$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ $4k + 1 = -3$ $k = -1$
Specific behaviours
✓ obtains an expression to solve involving *k* ✓ correctly calculates *k*

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Question 11 **(6 marks)**

The complex number $z = x + yi$ satisfies the inequality $|(\bar{z})^2 - z^2| \le 16$.

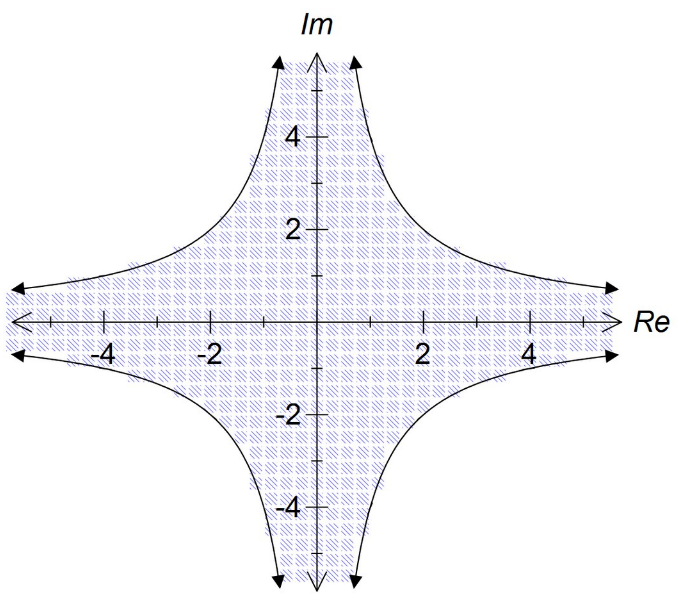
(a) Show that $|xy| \le 4$

[3]

Solution
$\left\|(x - yi)^2 - (x + yi)^2\right\| \le 16$ $\left\|x^2 - 2xyi - y^2 - x^2 - 2xyi + y^2\right\| \le 16$ $\left\|-4xyi\right\| \le 16$ $\left\|xy\right\| \le 4$
Specific behaviours
✓ substitutes $x – yi$ and $x + yi$ into the inequality ✓ expands and simplifies expression ✓ deduces the required result

(b) Hence sketch the set of all complex numbers z that satisfy the inequality $|(\bar{z})^2 - z^2| \le 16$ on the axes below.

[3]

Solution

Specific behaviours
✓ correct plotting of $xy = 4$ ✓ correct plotting of $xy = -4$ ✓ solid line and correct shading

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Question 12 **(8 marks)**

(a) According to research carried out by a company the proportion of households switching between oil, gas and electric heat in the United States after 1 year is shown in the table below.

		To		
		Oil	Gas	Electric
From	Oil	70%	30%	0
	Gas	10%	80%	10%
	Electric	20%	0	80%

If the pattern of switching types of heating continues

(i) determine the proportion of current households using gas who will be using gas in 5 years time.

[2]

Solution
$\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0.2 & 0 & 0.8 \end{bmatrix}^5 = \begin{bmatrix} 0.3242 & 0.5366 & 0.1392 \\ 0.2717 & 0.5031 & 0.2253 \\ 0.3577 & 0.2783 & 0.3639 \end{bmatrix}$ 50.3%
Specific behaviours
✓Sets up transition matrix from table and raises to the power of 5   ✓ correct proportion of households

(ii) determine in the long term what proportion, to 1 decimal place, of households will be using each of the three forms of heating?

[2]

Solution
As $n \to \infty$ $\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0.2 & 0 & 0.8 \end{bmatrix}^n = \begin{bmatrix} 0.3077 & 0.4615 & 0.2308 \\ 0.3077 & 0.4615 & 0.2308 \\ 0.3077 & 0.4615 & 0.2308 \end{bmatrix}$ Oil 30.8%, gas 46.2% and electric 23.1%
Specific behaviours
✓determines steady state matrix   ✓ correct proportions for each method of heating

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(b) A farmer is breeding marron in one of his dams. He has collected the following data on their breeding and survival rates in 2003.

Age (years)	1	2	3	4
Population	750	1200	900	600
Birth Rate	0	0.7	1.4	0.5
Survival Rate	0.7	0.6	0.5	0

(i) Construct a Leslie matrix, ***L***, to represent this population.

[1]

Solution

$$\begin{bmatrix} 0 & 0.7 & 1.4 & 0.5 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{bmatrix}$$

Specific behaviours

✓correctly sets up matrix from information in the table

(ii) What is the total population in 2009?

[1]

Solution

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \boldsymbol{L}^6 \begin{bmatrix} 750 \\ 1200 \\ 900 \\ 600 \end{bmatrix} = 5123.11$$

The total population in 2009 is approximately 5123 marron.

Specific behaviours

✓correct total population

(iii) Over a period of time the population growth reaches a steady state of 6.5%. If in the long term the farmer wishes to maintain a stable population level in the dam what culling rate of each age group will the farmer need to set?

[2]

Solution

Let k be the proportion surviving after the cull

$k \times 1.065 = 1$

$k = 93.9\%$

∴ The culling rate will need to be 6.1%

Specific behaviours

✓ correctly determines k
✓ correctly determines the culling rate

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Question 13 **(6 marks)**

The velocity of a particle that travels in a straight line is given by $v = 1 - \sqrt{2}\sin t$, $0 \le t \le 2\pi$ where v is in m/s and t is in seconds.

(a) Determine the times when the particle is at rest.

[2]

Solution
$0 = 1 - \sqrt{2}\sin t$ $t = \frac{\pi}{4}$ sec or $t = \frac{3\pi}{4}$ sec
Specific behaviours
✓ recognises that $t = 0$ when the velocity is at rest ✓ correctly determines the two times the particle is at rest for $0 \le t \le 2\pi$

(b) If the particle was initially at the origin determine an expression for its displacement.

[2]

Solution
$x = t + \sqrt{2}\cos t + c$ At $t = 0$ $x = 0$ $\therefore c = \sqrt{2}$ $x = t + \sqrt{2}\cos t + \sqrt{2}$
Specific behaviours
✓ integrates velocity to determine displacement ✓ correctly determines the value of the constant and hence states the expression

(c) Determine the distance the particle travelled in the third second.

[2]

Solution
$\int_2^3 \left\lvert 1 - \sqrt{2}\sin t \right\rvert dt$ $= 0.30\text{m}$
Specific behaviours
✓ integrates the absolute value of the velocity over the appropriate lower and upper limits ✓ correctly determines the distance travelled in the third second

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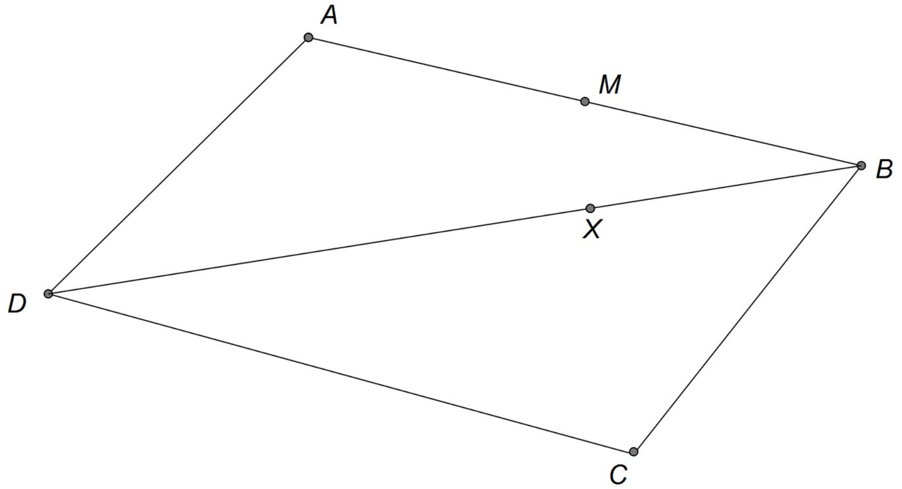
Question 14 **(5 marks)**

The diagram below shows parallelogram *ABCD* where $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{BC} = \boldsymbol{b}$.
Point *X* divides *DB* internally in the ratio 2:1.
Point *M* is the midpoint of *AB*.



(a) Show that $\overrightarrow{DX} = \frac{2}{3}\boldsymbol{a} - \frac{2}{3}\boldsymbol{b}$

[1]

Solution
$\overrightarrow{DX} = \frac{2}{3}\overrightarrow{DB}$ $= \frac{2}{3}(\mathbf{a} - \mathbf{b})$
Specific behaviours
✓ correctly shows $\overrightarrow{DX}$

(b) Find $\overrightarrow{CX}$ in terms of ***a*** and ***b***.

[1]

Solution
$\overrightarrow{CX} = \overrightarrow{CD} + \overrightarrow{DX}$ $= -\mathbf{a} + \frac{2}{3}(\mathbf{a} - \mathbf{b})$ $= -\frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$
Specific behaviours
✓ determines correctly $\overrightarrow{CX}$

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(c) Prove that points *M*, *X* and *C* are collinear.

[3]

Solution
$\overrightarrow{CM} = \overrightarrow{CB} + \overrightarrow{BM}$ $= -\mathbf{b} + \frac{1}{2}(-\mathbf{a})$ $= -\frac{1}{2}\mathbf{a} - \mathbf{b}$ $= \frac{3}{2}\overrightarrow{CX}$ Since $\overrightarrow{CM} = k\overrightarrow{CX}$ and point C is common, C, M and X are collinear.
Specific behaviours
✓ determines an expression for $\overrightarrow{CM}$ in terms of $\overrightarrow{CB}$ and $\overrightarrow{BM}$ ✓ shows is $\overrightarrow{CM}$ a scalar of $\overrightarrow{CX}$ ✓ deduces the required result

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Question 15 **(5 marks)**

A certain type of electronic circuit will remain in a stable state if the values of two variable resistors, x and y, satisfy the equation $\frac{1}{x}+\frac{1}{y}=0.005$.
In a particular circuit, the value of y is increasing at a rate of 15 units per second. At what rate must x be changing when $y = 1000$ for the circuit to remain stable?

Solution	
$\frac{1}{x}+\frac{1}{y}=0.005$ $-\frac{1}{x^2}-\frac{1}{y^2}\frac{dy}{dx}=0$ $\frac{dy}{dx}=-\frac{y^2}{x^2}$ When $y=1000$ $x=250$ $\frac{dx}{dt}=\frac{dx}{dy}\cdot\frac{dy}{dt}$ $=-\frac{x^2}{y^2}.15$ $=-\frac{250^2}{1000^2}(15)$ $=-\frac{15}{16}$	OR $\frac{1}{x}+\frac{1}{y}=0.005$ $-\frac{1}{x^2}\frac{dx}{dt}-\frac{1}{y^2}\frac{dy}{dt}=0$ $\frac{dx}{dt}=-\frac{x^2}{y^2}\frac{dy}{dt}$ When $y=1000$ $x=250$ $\frac{dx}{dt}=-\frac{250^2}{1000^2}(15)$ $=-\frac{15}{16}$
x is decreasing at a rate of 0.9375 units/sec	
Specific behaviours	
✓ correctly determines expression for $\frac{dy}{dx}$ ✓ correctly determines x when $y = 1000$ ✓ uses chain rule with $\frac{dy}{dt}$ ✓ correctly calculates $\frac{dx}{dt}$ ✓statement correctly interpreting the rate is decreasing	OR ✓ correctly differentiates with respect to t ✓ correctly determines expression for $\frac{dx}{dt}$ ✓ correctly determines x when $y = 1000$ ✓ correctly calculates $\frac{dx}{dt}$ ✓statement correctly interpreting the rate is decreasing

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Question 16 **(6 marks)**

A line and a plane are given by $\mathbf{r}=\begin{pmatrix}-1\\1\\4\end{pmatrix}+\lambda\begin{pmatrix}0\\-2\\4\end{pmatrix}$ and $\mathbf{r}\bullet\begin{pmatrix}1\\2\\2\end{pmatrix}=29$.

(a) Given that the point (2, *c*, -2) lies on the plane determine *c*.

[2]

Solution
$\begin{pmatrix}2\\c\\-2\end{pmatrix}\bullet\begin{pmatrix}1\\2\\2\end{pmatrix}=29$ $c=15.5$
Specific behaviours
✓ substitutes point into equation of plane   □ correct value of *c*

(b) Find the position vector of the intersection between the line and the plane.

[2]

Solution
Intersect when $\begin{bmatrix}-1\\1-2\lambda\\4+4\lambda\end{bmatrix}\bullet\begin{bmatrix}1\\-2\\2\end{bmatrix}=29$ $\therefore\lambda=2$ Hence point of intersection at $\begin{bmatrix}-1\\-3\\12\end{bmatrix}$
Specific behaviours
✓ correctly determines value of λ   □ correct point of intersection

(c) Find the acute angle between the line and the plane.

[2]

Solution
Find angle between line and normal to plane: $\cos\theta = \dfrac{\begin{bmatrix}0\\-2\\4\end{bmatrix} \bullet \begin{bmatrix}1\\2\\2\end{bmatrix}}{\left\|\begin{bmatrix}0\\-2\\4\end{bmatrix}\right\| \times \left\|\begin{bmatrix}1\\2\\2\end{bmatrix}\right\|}$ $\theta = 72.654$ Hence angle between line and plane $= 90 - 72.654 = 17^\circ$
Specific behaviours
✓ correctly determines θ correctly determines the acute angle between the line and plane

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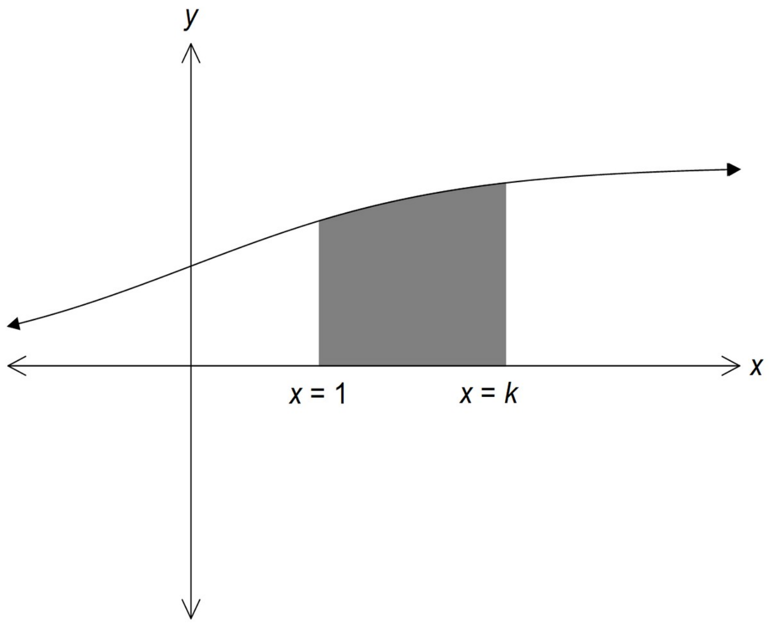
Question 17 **(6 marks)**

The graph of $f(x)=\dfrac{e^x}{1+e^x}$ is shown below

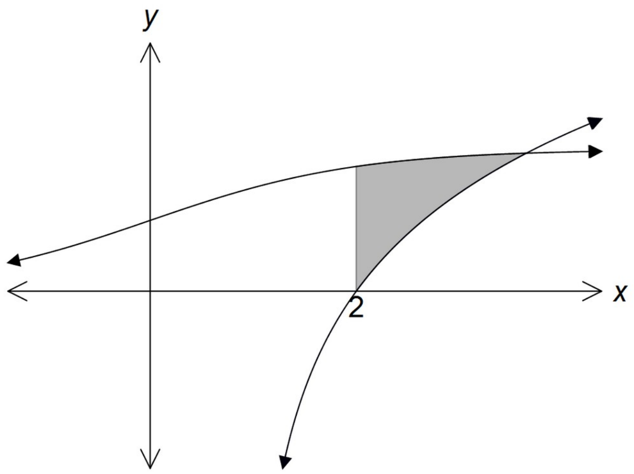


(a) Show that the area enclosed between the curve $f(x)$ and the x-axis between $x = 1$ and $x = k$, is $\ln\left(\dfrac{e^k+1}{e+1}\right)$.

[3]

Solution
$\int_1^k \dfrac{e^x}{1+e^x}\,dx$ $=\left[\ln\lvert 1+e^x\rvert\right]_1^k$ $=\ln\lvert 1+e^k\rvert - \ln\lvert 1+e^1\rvert$ $=\ln\left(\dfrac{1+e^k}{1+e}\right)$
Specific behaviours
✓recognises $\dfrac{f'(x)}{f(x)}$ and integrates correctly ✓substitutes upper/lower limits ✓uses log laws to simplify expression and deduce required result

(b) The graphs of $f(x)$ and $g(x) = \ln(x - 1)$ are shown below. Determine the area bound by the two curves and the line $x = 2$.



[3]

Solution
Value of x at point of intersection $x \approx 3.650$ $\int_2^{3.650} \left(\frac{e^x}{1+e^x} - \ln(x-1) \right) dx$ $= 0.62$ units^2
Specific behaviours
✓determines x value at point of intersection of functions ✓determines an expression for the area between the curves ✓ correctly determines the area of the required region

Question 18 **(10 marks)**

An object, *P*, has an initial position of $\begin{pmatrix} 5 \\ 2 \\ -9 \end{pmatrix}$ metres and is moving with a constant velocity of $\begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}$ metres per second.

(a) A second object, *Q*, is moving with constant velocity of $\begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix}$ metres per second and collides with object *P* after six seconds.

Determine the initial distance apart of object *P* and object *Q*.

[4]

Solution

P and *Q* collide at

$$\begin{pmatrix} 5 \\ 2 \\ -9 \end{pmatrix} + 6\begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -25 \\ 20 \\ -3 \end{pmatrix}$$

Hence initial position of Q is at

$$\boldsymbol{r}_Q + 6\begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -25 \\ 20 \\ -3 \end{pmatrix}$$

$$\boldsymbol{r}_Q = \begin{pmatrix} -7 \\ 8 \\ 21 \end{pmatrix}$$

Distance between *P* and *Q* is

$$\left|\begin{pmatrix} -7 \\ 8 \\ 21 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ -9 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} -12 \\ 6 \\ 30 \end{pmatrix}\right|$$

$= 32.9$ m

Specific behaviours

✓ determines point of collision
□ determines $\mathbf{r}_Q$
✓ determines ${}_Q\mathbf{r}_P$
✓ determine correct distance between *P* and *Q*

(b) A third object, R, is initially located at $\begin{pmatrix} 4 \\ -11 \\ 2 \end{pmatrix}$ metres and is also moving with a constant velocity $\begin{pmatrix} 3 \\ x \\ -2 \end{pmatrix}$ metres per second. Determine the value of x such that after 5 seconds the distance between objects P and R is minimised. State the minimum distance at this time.

[6]

Solution
$\boldsymbol{r}_P(5) = \begin{pmatrix} -20 \\ 17 \\ -4 \end{pmatrix} \qquad \boldsymbol{r}_R(5) = \begin{pmatrix} -31 \\ -11+5x \\ 12 \end{pmatrix}$ ${}_P\boldsymbol{r}_R = \begin{pmatrix} 11 \\ 28-5x \\ -16 \end{pmatrix}$ fmin(norm[11,28-5x,-16]),x] {minvalue = 19.41648784, x = 5.6} Hence for the distance to be minimised after 5 seconds x = 5.6. The minimum distance is 19.4 m
Specific behaviours
✓ determines $\mathbf{r}_P$ when t = 5 □ determines $\mathbf{r}_R$ when t = 5 ✓ determines ${}_P\mathbf{r}_R$ when t = 5 ✓ Recognise need to find minimum of $\lvert {}_P\mathbf{r}_R \rvert$ ✓ correct value of x ✓ correct minimum distance

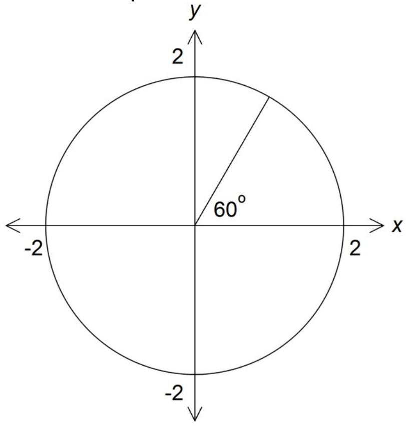
Question 19 **(5 marks)**

The solution to the differential equation $\frac{dy}{dx} = 3y - 6xy$ passes through the point where $x = 0$ and $y = 2$. Determine y when $x = 1$.

Solution
$\frac{dy}{dx} = 3y(1 - 2x)$ $\int \frac{1}{y}\, dy = 3\int (1 - 2x)\, dx$ $\ln\lvert y\rvert = 3x - 3x^2 + c$ Given $(0, 2)$ $c = \ln 2$ $\ln y = x - x^2 + \ln 2$ $y = e^{3x - 3x^2 + \ln 2}$ $= e^{\ln 2} . e^{3x - 3x^2}$ $= 2e^{3x - 3x^2}$
Specific behaviours
✓ separates variables □ correctly determines antiderivatives ✓ applies log definition ✓ substitutes (0, 2) to calculate constant ✓ deduce required result

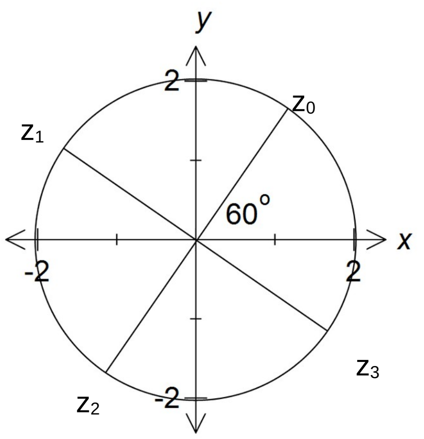
Question 20 **(10 marks)**

(a) One of the solutions to the equation $z^4 - k = 0$ is shown on the graph below.



(i) Make a sketch of the remaining roots on the axes above.

[1]

Solution

Specific behaviours
✓ correctly places remaining 3 roots on axes

(ii) Determine algebraically the value of *k* in Cartesian form.

[2]

Solution
$k = \left(2cis\frac{\pi}{3}\right)^4$ $= 16cis\frac{4\pi}{3}$ $= -8 - 8\sqrt{3}i$
Specific behaviours
✓ determines *k* in polar form ✓ determines *k* in Cartesian form

(b) (i) Solve $z^3 + 27 = 0$ algebraically, using de Moivre's theorem giving your answers in Cartesian form.

[5]

Solution
$z_k = 27^{\frac{1}{3}} cis\left(\pi + \frac{2k\pi}{3}\right)$ $z_0 = 3cis(\pi) = -3$ $z_1 = 3cis\left(\frac{5\pi}{3}\right) = \frac{3}{2} - \frac{3\sqrt{3}}{2}i$ $z_2 = 3cis\left(\frac{\pi}{3}\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$
Specific behaviours
✓ Expresses z^3 in polar form ✓ Expresses z in polar form ✓✓ three roots in polar form (only ✓ for 2 roots in polar form and 0 for 1 root) ✓ each root in Cartesian form

(ii) Solve $(z+1)^3 + 27 = 0$ algebraically.

[2]

Solution
$z_k = 27^{\frac{1}{3}} cis\left(\pi + \frac{2k\pi}{3}\right) - 1$ $z_0 = -4$ $z_1 = \frac{1}{2} - \frac{3\sqrt{3}}{2}i$ $z_2 = \frac{1}{2} + \frac{3\sqrt{3}}{2}i$
Specific behaviours
✓ recognises solutions are translation on $z^4 - k = 0$, real component by -1 ✓ correct four solutions in Cartesian form

Additional working space

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Additional working space

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