Rossmoyne Senior High School

Year 12 Trial WACE Examination, 2014

Question/Answer Booklet

MATHEMATICS 3C/3D

Section Two:
Calculator-assumed

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination.

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	50	33⅓
Section Two: Calculator-assumed	13	13	100	100	66⅔
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2013*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

Section Two: Calculator-assumed **(100 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 **(6 marks)**

Among the blood cells of an animal species, 42% of the cells are of type A and 0.5% of the cells are of type B and the remaining 57.5% of the cells are neither of these types.

(a) Calculate the probability that in a random sample of nine blood cells, exactly three cells will be of type A. (2 marks)

$X \sim B(9, 0.42)$

$P(X = 3) = 0.2369$

(b) The random variable Y is the number of type B cells in a random sample of n blood cells. If the mean of the distribution of Y is 0.8, determine the standard deviation of the distribution of Y. (2 marks)

$0.005n = 0.8 \Rightarrow n = 160$

$sd = \sqrt{160 \times 0.005 \times (1 - 0.005)} = 0.8922$

(c) Determine the probability that in a random sample of 40 blood cells, the total number of type A and type B cells is at least 24. (2 marks)

$W \sim B(40, 0.425)$

$P(W \geq 24) = 0.0194$

Question 10 **(6 marks)**

In this question, the units on the x and y-axis are in centimetres. You should give your answers rounded to three significant figures.

(a) Find the volume of the solid of revolution formed when the line $3x + 4y = 36$ between the limits $y = 1$ and $y = 7$ is rotated about the y-axis. (3 marks)

$$V = \pi \int_1^7 \left(12 - \tfrac{4}{3}y\right)^2 \, dy$$
$$= \frac{896\pi}{3}$$
$$\approx 938 \text{ cm}^3 \text{ to 3sf}$$

(b) When the same line, $3x + 4y = 36$, is rotated about the x-axis between the limits $x = 1$ and $x = a$, the volume of the solid of revolution formed is 750 cm^2. Determine the value of a, given that $a > 1$. (3 marks)

$$750 = \pi \int_1^a \left(9 - \tfrac{3}{4}x\right)^2 \, dx$$
$$750 = \frac{4\pi\left(\tfrac{3}{4}a - 9\right)^3}{9} + \frac{3993\pi}{16}$$
$$a = 8.134$$

Question 11 **(7 marks)**

The value, V, of a school photocopier that was bought new for \$1350 is changing at a rate given by $\frac{dV}{dt} = -0.22V$, where t is the time in years since the copier was bought.

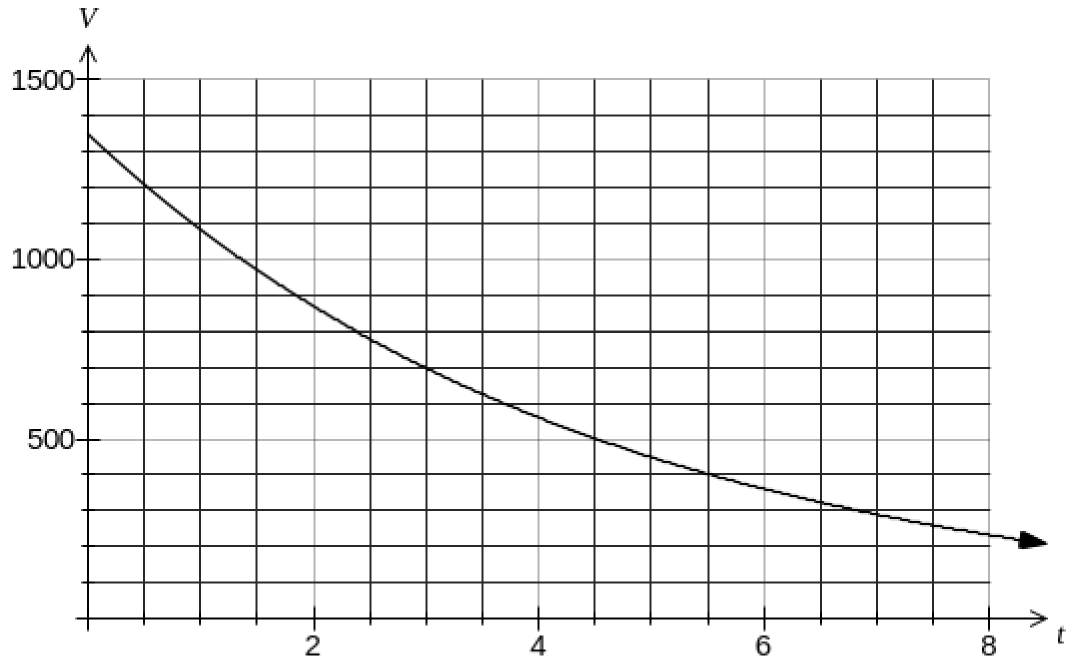
(a) State an equation for V in terms of t. (1 mark)

$\boxed{V = 1350e^{-0.22t}}$

(b) Calculate the value of the photocopier after 8 years. (1 mark)

$\boxed{V(8) = \$232.26}$

(c) Draw the graph of V against t on the axes below. (2 marks)



(d) At the same time the photocopier was purchased, the school also bought a computer for \$2350. One year later it was valued at \$1690. The value of this computer after t years is given by $2350e^{-kt}$, where k is a positive constant.

(i) Determine the value of k. (1 mark)

$\boxed{1690 = 2350e^{-k} \Rightarrow \quad = 0.3297}$

(ii) After how long did the value of the computer first fall below that of the photocopier? (2 marks)

$$2350e^{-0.3297t} = 1350e^{-0.22t} \Rightarrow t = 5.05$$

After just over 5 years.

Question 12 **(11 marks)**

Records from a dental practice show that the number of minutes per visit spent in the dentist's chair by a patient are normally distributed with a mean 16.5 minutes and standard deviation 3.9 minutes.

Assume that on any given day, patient's times in the dentist's chair are independent of each other.

(a) On a day when the dentist has 16 patients, how many of these are expected to spend at least 20 minutes in the chair? (2 marks)

$P(X > 20) = 0.1847$

$16 \times 0.1847 = 2.96 \approx 3$ patients

(b) If a patient has already spent 15 minutes in the chair, what is the probability that they will spend less than 20 minutes in the chair? (2 marks)

$$\frac{P(15 < X < 20)}{P(X > 15)} = \frac{0.4650}{0.6497} = 0.7157$$

(c) On a day when the dentist has 16 patients, what is the probability that no more than five of them spend less than 15 minutes in the chair? (3 marks)

$P(X < 15) = 0.3502$

Y = Number out of 16 patients who spend <15 minutes in chair

$Y: \ B(16, 0.3502)$

$P(Y \le 5) = 0.4891$

(d) A random sample of 12 consultations from recent records gave the following times in minutes.

17.3	18.9	21.8	23.2	18.9	16.3
21.5	17.3	15.2	12.5	18.7	21.1

Use this sample to calculate a 95% confidence interval for the mean length of time spent in the dentist's chair and explain whether there is reason to doubt that the mean is 16.5 minutes. (4 marks)

Sample mean, $\bar{x} = 18.56$

Standard error $= \dfrac{3.9}{\sqrt{12}} = 1.126$

95% confidence interval is $18.56 \pm 1.96 \times 1.126$

$= (16.35,\ 20.76)$

No reason to doubt existing mean of 16.5, as this value lies within CI.

Question 13 **(5 marks)**

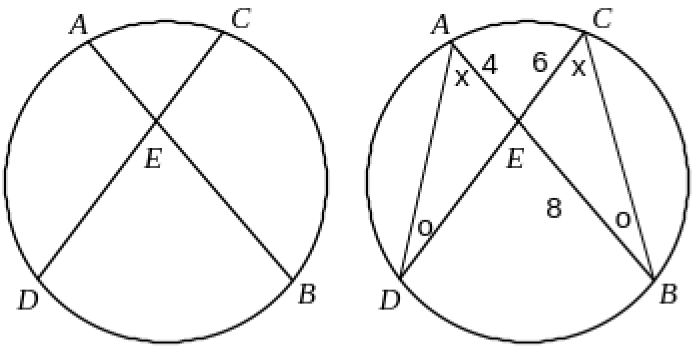
(a) In triangles ABC and DEF, $AC \cong DF$ and $\angle A \cong \angle D$. Is the additional fact that $BC \cong EF$ enough to prove that triangle ABC is congruent with triangle DEF? Justify your answer.

(2 marks)

No. SAS can only be used when angle is included by the two sides.

(b) In the circle shown below, not to scale, AB and CD are chords that intersect at E. If $AE = 4$ cm, $BE = 8$ cm and $CE = 6$ cm, determine the length of DE. Justify your answer.

(3 marks)



$\angle DAB \cong \angle DCB$ (angles on common chord). Similarly $\angle ADC \cong \angle ABC$. Hence ΔAED : ΔCEB (AA) $\frac{4}{6} = \frac{DE}{8} \Rightarrow DE = 5\frac{1}{3}$ cm

Question 14 **(8 marks)**

A store accepts credit card payments from customers using American Express, Mastercard or VISA cards. Records indicate that 65% of customers use a credit card, and of these customers, 20% use American Express, 35% Mastercard and the rest VISA. Further analysis shows that the male to female ratio for users of each type of card is 5:3 for American Express, 2:3 for Mastercard and 3:2 for VISA.

(a) Calculate the probability that a randomly selected customer from the records will be a female who uses an American Express credit card. (2 marks)

$$0.65\times 0.2\times \tfrac{3}{8}=0.04875$$

(b) Given that a randomly selected customer used a credit card, what is the probability that they are male? (3 marks)

$$\begin{aligned}P(\text{male}) &= 0.2\times\tfrac{5}{8}+0.35\times\tfrac{2}{5}+0.45\times\tfrac{3}{5}\\ &= 0.125+0.14+0.27\\ &= 0.535\end{aligned}$$

(c) What is the probability that a randomly selected female customer who used a credit card used a VISA card? (3 marks)

$$\begin{aligned}P(\text{Female} \mid \text{used card}) &= 1-0.535\\ &= 0.465\end{aligned}$$

$$\begin{aligned}P(\text{Female and VISA} \mid \text{used card}) &= 0.45\times\tfrac{2}{5}\\ &= 0.18\end{aligned}$$

$$\begin{aligned}P &= \frac{0.18}{0.465}\\ &= \frac{12}{31}\\ &\approx 0.3871\end{aligned}$$

Question 15 **(11 marks)**

A beautician is planning to use old stock to make two types of promotional packs - Pamper packs, containing a defoliating scrub, a sachet of face cleanser and two sachets of skin cream, and Youthful packs, containing a defoliating scrub, three sachets of face cleanser and three sachets of skin cream.

The beautician's supplies to make these packs are limited to 70 defoliating scrubs, 135 sachets of face cleanser and 150 sachets of skin cream.

The beautician sells the Pamper packs for $7 and the Youthful packs for $12.

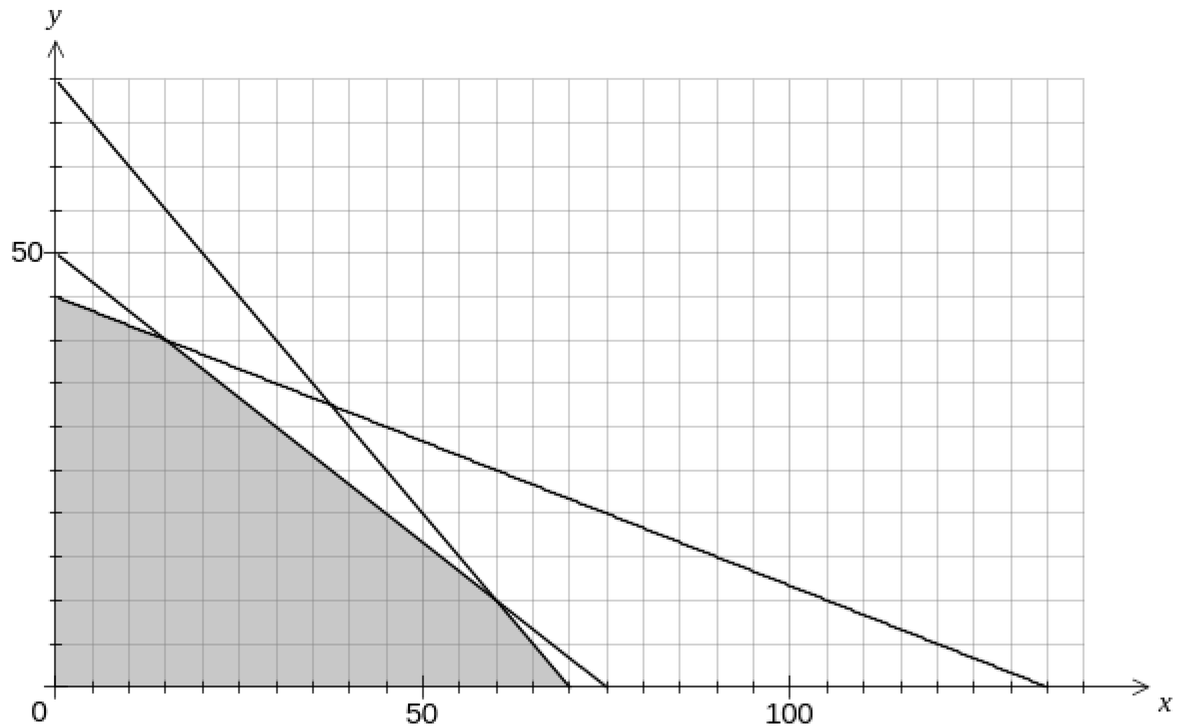
If x is the number of Pamper packs and y the number of Youthful packs the beautician prepares, then one constraint arising from the above information is $x + y \leq 70$.

(a) Determine another two constraints, in terms of x and y, that restrict the number of packs that can be made (other than $x \geq 0$ and $y \geq 0$). (2 marks)

$x + 3y \leq 135$

$2x + 3y \leq 150$

(b) Add the constraints from (a) on the axes below and indicate the feasible region. (3 marks)



(c) Assuming that all packs are sold, how many of each type of pack should the beautician make in order to maximise income from their sale and what is the maximum income? (2 marks)

$P = 7x + 12y$

$(0, 45) \Rightarrow P = 540$
$(15, 40) \Rightarrow P = 585$
$(60, 10) \Rightarrow P = 540$
$(70, 0) \Rightarrow P = 490$

Maximum income from selling
15 Pamper and 40 Youthful packs
for total income of $585

(d) If the beautician makes and sells the optimum number of packs to maximise income, some stock will be left over. State which product will be left over, and how many units of this product remain. (2 marks)

$70 - (15 + 40) = 15$

15 defoliating scrubs will remain.

(e) By how much can the beautician decrease the price of Youthful packs without changing the optimum solution found in (c)? (2 marks)

Let new price be k.

$15(7) + 40k = 60(7) + 10k$
$30k = 45(7)$
$k = 10.5$

Decrease price by up to $1.50.

Question 16 **(8 marks)**

For events A and B, $P(A)=a$, $P(B)=b$ and $P(\bar{A}\cap\bar{B})=0.1$.

(a) Determine an expression for $P(A\cap B)$ in terms of a and b. (2 marks)

$$\begin{aligned}P(A\cap B)&=P(A)+P(B)-P(A\cup B)\\&=a+b-(1-0.1)\\&=a+b-0.9\end{aligned}$$

It is also known that $P(A|B)=0.4$.

(b) Determine an expression for a in terms of b. (2 marks)

$$P(A|B)=\frac{a+b-0.9}{b}$$
$$\frac{a+b-0.9}{b}=0.4$$
$$a=0.9-0.6b$$

(c) Determine the values of a and b under each of the following conditions.

(i) $P(A\cap\bar{B})=0.3$. (2 marks)

$$\begin{aligned}P(A\cap\bar{B})&=a-(a+b-0.9)\\0.3&=0.9-b\\b&=0.6\end{aligned}$$

$$\begin{aligned}a&=0.9-0.6(0.6)\\a&=0.54\end{aligned}$$

(ii) A and B are independent. (2 marks)

$$\begin{aligned}P(A)&=P(A|B)\\a&=0.4\end{aligned}$$

$$\begin{aligned}0.4&=0.9-0.6b\\b&=\tfrac{5}{6}\end{aligned}$$

Question 17 **(9 marks)**

The points A, B, C, D, E, F, G and H lie on the graph of the continuous function $y = f(x)$.
The table below contains information about the sign of $f(x)$, $f'(x)$ and $f''(x)$ at these points.

Point	A	B	C	D	E	F	G	H
x	-2	-1	0	2	4	6	7	8
$f(x)$	+	0	+	+	+	+	0	-
$f'(x)$	-	0	+	0	-	-	0	-
$f''(x)$	+	+	0	-	0	+	0	-

There are no other points at which $f(x)$, $f'(x)$ and $f''(x)$ are equal to zero.

(a) For the graph of this function, state all points that are

(i) roots. (1 mark)

B and G

(ii) points of inflection but not stationary points. (1 mark)

C and E

(b) Describe the nature of the graph of this function at point

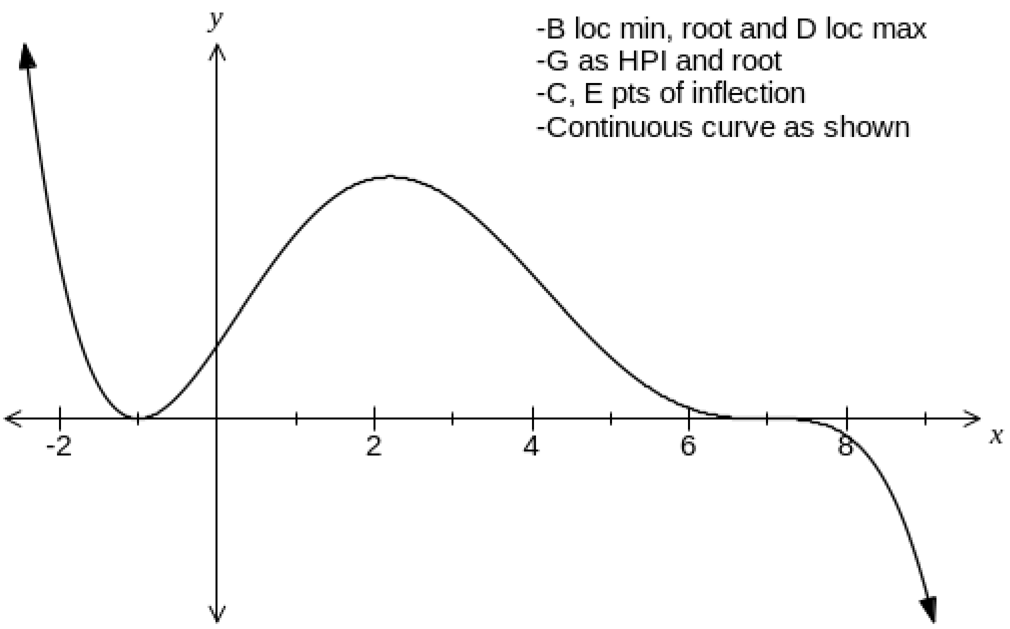
(i) B. (1 mark)

Local minimum

(ii) G. (2 marks)

Horizontal point of inflection

(c) Sketch a possible graph of $y = f(x)$ on the axes below. (4 marks)



Question 18 **(7 marks)**

A budget of $225 is available to buy 4 mm thick steel sheeting to construct an open water tank in the shape of a rectangular prism of height h cm, that is twice as long ($2x$ cm) as it is wide (x cm). Cut to size, the sheeting costs 1.5 cents per square centimetre.

(a) Show that the total cost of the steel, in dollars, is given by $\frac{3x^2+9xh}{100}$. (1 mark)

$$C=\frac{1.5\left((x)(2x)+(x+2x+x+2x)(h)\right)}{100}$$
$$=\frac{3x^2+9xh}{100}$$

(b) Assuming the whole budgeted amount is used to buy steel sheeting, show that volume of the tank in cubic centimetres is given by $V=5000x-\frac{2}{3}x^3$. (2 marks)

$$\frac{3x^2+9xh}{100}=225 \Rightarrow h=\frac{22500-3x^2}{9x}$$

$$V=LWH$$
$$=(x)(2x)\left(\frac{22500-3x^2}{9x}\right)$$
$$=5000x-\tfrac{2}{3}x^3$$

(c) Use calculus methods to determine the dimensions of the water tank that maximises the volume, and state this volume. (4 marks)

$$\frac{dV}{dx}=5000-2x^2$$
$$=0 \text{ when } \cancel{x=-50}, x=50$$

$x=50$ cm

$2x=100$ cm

$$h=\frac{22500-3(50)^2}{9(50)}=\frac{100}{3} \text{ cm}$$

Dimensions are 50 by 100 by $33\frac{1}{3}$ cm

Volume $=(50)(100)(\frac{100}{3})=\frac{500000}{3}\approx 166667$ cm^3

Question 19 **(7 marks)**

The volume of a raindrop, assumed to be spherical in shape, increases at a steady rate from 1.05 mm^3 to 1.50 mm^3 over a period of 15 seconds.

(a) Determine the rate of increase of the radius of the raindrop at the instant the volume reaches 1.50 mm^3. (5 marks)

$$\frac{dV}{dt} = \frac{1.5 - 1.05}{15}$$
$$= 0.03 \text{ mm}^3/\text{s}$$

$$V = \tfrac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$$

$$1.50 = \tfrac{4}{3}\pi r^3 \Rightarrow r = 0.710 \text{ mm}$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$
$$= \frac{1}{4\pi r^2} \times 0.03$$

$$= \frac{1}{4\pi(0.710)^2} \times 0.03 = 0.0047 \text{ mm/s}$$

(b) Assuming the volume continues to increase at the same steady rate, will the rate you calculated in (a) increase, stay the same or decrease after one more second? Justify your answer. (2 marks)

Decrease, as the rate of change is inversely proportional to the square of the radius. Hence, as the volume increases, the radius increases and so the rate of change will decrease.

Question 20 **(9 marks)**

The time to process orders received by a company is a uniformly distributed random variable with minimum and maximum values of 30 seconds and 110 seconds. Processing times can be assumed to be independent of each other. The mean and standard deviation of the times is 70 and 23 seconds respectively.

(a) Determine the probability that a randomly chosen order takes

(i) less than one minute to process. (1 mark)

$$\boxed{\frac{60-30}{110-30}=\frac{3}{8}}$$

(ii) more than 80 seconds, given that it has already taken 50 seconds. (2 marks)

$$\boxed{\frac{110-80}{110-50}=\frac{1}{2}}$$

(b) Determine the probability that at least half of the next 10 orders take less than one minute to process. (2 marks)

Y = number out of next ten orders taking less than one minute

$Y \sim B\left(10, \frac{3}{8}\right)$

$P(Y \geq 5) = 0.3057$

(c) A random sample of 200 times are taken from the order processing log kept by the company. Determine the probability that

(i) the sample mean is no more than 67 seconds. (2 marks)

$\hat{T} \sim N\left(70, \frac{23^2}{200}\right)$

$P(\hat{T} < 67) = 0.0325$

(ii) the total of the 200 times is longer than four hours. (2 marks)

$\hat{T} = \frac{4 \times 60 \times 60}{200} = 72$ seconds

$P(\hat{T} > 72) = 0.1094$

Question 21 **(6 marks)**

Let $f(x) = \dfrac{e^x - e^{-x}}{2} - x$.

(a) Show that $f''(x) > 0$ for all $x > 0$. (2 marks)

$$f'(x) = \frac{e^x + e^{-x}}{2} - 1$$
$$f''(x) = \frac{e^x - e^{-x}}{2}$$

When $x > 0,\ e^x > e^{-x} \Rightarrow f''(x) > 0$

(b) Using your result from (a), or otherwise, show that $f'(x) > 0$ for all $x > 0$. (2 marks)

$$f'(0) = \frac{1+1}{2} - 1 = 0.$$

For $x > 0$, $f'(x)$ is always increasing ($f''(x) > 0$) and as $f'(0) = 0$, then $f'(x) > 0$.

(c) Hence, or otherwise, show that $\dfrac{e^x - e^{-x}}{2} > x$ for all $x > 0$. (2 marks)

$$f(0) = \frac{1-1}{2} - 0 = 0.$$

For $x > 0$, $f(x)$ is always increasing ($f'(x) > 0$) and as $f(0) = 0$, then $f(x) > 0$.

$$\therefore\ \frac{e^x - e^{-x}}{2} - x > 0 \Rightarrow \frac{e^x - e^{-x}}{2} > x \text{ for } x > 0.$$

End of questions

Additional working space

Question number: __________

Additional working space

Question number: _________

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