MATHEMATICS METHODS

MAWA Semester 1 (Unit 3) Examination 2019
Calculator-assumed

Marking Key

The release date for this exam and marking scheme is 14th June.

Section Two: Calculator-assumed (100 Marks)

Question 8(a) (4 marks)

Solution
$\int_2^6 \frac{f(x)}{3}\,dx = 4$ (i) $\int_2^6 f(x)\,dx = 3\times 4 = 12$ (ii) $\int_2^6 \frac{3f(x)-1}{2}\,dx = \int_2^6 \frac{3f(x)}{2}\,dx - \int_2^6 \frac{1}{2}\,dx$ $= \frac{3}{2}\int_2^6 f(x)\,dx - \int_2^6 \frac{1}{2}\,dx$ $= \left(\frac{3}{2}\times 12\right) - \left[\frac{1}{2}x\right]_2^6$ $= 18 - [3-1]$ $= 16$

Mathematical behaviours	Marks
• states $\int_2^6 f(x)\,dx = 12$	1
• uses linearity and additivity to deduce $\int_2^6 \frac{3f(x)-1}{2}\,dx = \int_2^6 \frac{3f(x)}{2}\,dx - \int_2^6 \frac{1}{2}\,dx$	1
• anti-differentiates $\frac{1}{2}$	1
• determines correct result of 16	1

Question 8(b) (3 marks)

Solution
$\int_{-\frac{1}{4}}^{0} e^{4x+1}\,dx = \frac{1}{4}\int_{-\frac{1}{4}}^{0} 4e^{4x+1}\,dx$ $= \frac{1}{4}\left[e^{4x+1}\right]_{-\frac{1}{4}}^{0}$ $= \frac{1}{4}\left[e^{1} - e^{0}\right]$ $= \frac{1}{4}[e-1]$

Mathematical behaviours	Marks
• anti-differentiates correctly	1
• substitutes limits of integration correctly	1
• determines exact result	1

Question 13(b) **(5 marks)**

Solution
(i) $V=\frac{\pi h}{3}(R^2+r^2+Rr)$ $V=\frac{\pi(15)}{3}(5^2+3^2+5\times3)$ $\approx 769.69\,cm^3 \approx 770\,cm^3$ (ii) $V=\frac{\pi 15}{3}(R^2+3^2+3R)$ $\frac{dV}{dR}=5\pi(2R+3)$ $\frac{dV}{dR}\approx\frac{\delta V}{\delta R}, R=5, \delta R=-0.2$ $\delta V\approx 5\pi(2\times5+3)(-0.2)$ $\delta V\approx -40.84\,cm^3 \approx -41\,cm^3$ *ie a decrease in capacity of approximately 41 millilitres*

Mathematical behaviours	Mark
(i)	
• states correct volume to the nearest cubic centimetre	1
(ii)	
• states V in terms of R	1
• uses incremental formula to obtain expression for small change in V	1
• substitutes, $R=5$ and $\delta R=-0.2$	1
• states the decrease in capacity	1

Question 14(a) **(3 marks)**

Solution

Total number of cars in sample is $27+13+11+4+14=69$

Proportions of the various colours, and rounded to a whole multiple of 0.05:

White	Black	Red	Blue	Other
$\frac{27}{69}\approx 0.391\approx$	$\frac{13}{69}\approx 0.188\approx$	$11/69\approx 0.15$	$4/69\approx 0.058$ 7	$14/69\approx 0.20$

Mathematical behaviours	Marks
• obtains total sample size	1
• calculates all fractions correctly	1
• rounds all answers correctly	1

Question 14(b) **(3 marks)**

Solution

Expected number of points per car ¿ $2\times0.4+4\times0.2+7\times0.15+9\times0.05+5\times0.2=4.1$

So expected number of points per 100 cars ¿ $100\times4.1=410$

Mathematical behaviours	Marks
• obtains correct expression for expected value	1
• calculates expected value (per car) correctly	1
• obtains correct answer	1

Question 14(c) **(2 marks)**

Solution

Expected number of points per car (by colour)

White	Black	Red	Blue	Other
$2\times0.4=0.8$	$4\times0.2=0.8$	$7\times0.15=1.05$	$9\times0.05=0.45$	$5\times0.2=1$

Since the expected points per car is greatest for Rodney's red cars, Rodney is most likely to accumulate points fastest.

Mathematical behaviours	Marks
• evaluates expected values correctly	1
• correct answer	1

Question 14(d) **(2 marks)**

Solution

$P=0.4^2+0.2^2+0.15^2+0.05^2+0.2^2=0.265$

Mathematical behaviours	Marks
• uses correct formula	1
• evaluates correctly	1

Question 15(a) **(4 marks)**

Solution

(i) none (consecutive selections are not independent so not binomial)
(ii) uniform
(iii) binomial
(iv) binomial

Mathematical behaviours	Marks
i)	
• states none	1
(ii)	
• states uniform	1
(iii)	
• states binomial	1
(iv)	
• states binomial	1

Question 9(a) **(1 mark)**

Solution

$f'(x) = (x-1)^2(4x-1) = 4x^3 + bx^2 + cx + d + e$

$hence\ f(x) = x^4 + ...\qquad ie\quad a > 0$

Mathematical behaviours	Mark
• states $a > 0$ justifies answer using anti-differentiation	1

Question 9(b) **(1 mark)**

Solution

For stationary points, $f'(x) = 0$

$ie\,(x-1)^2(4x-1) = 0 \Rightarrow x = 1, \frac{1}{4}$

Mathematical behaviours	Marks
• states x coordinates of stationary points	1

Question 9(c) **(3 marks)**

Solution

$f'(1) = 0$ and $f''(1) = 0$

$f''(x) = 6(x-1)(2x-1)$

$f''(1^-) = -ve \times +ve = -ve$

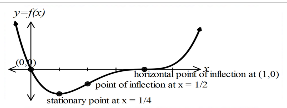
$f''(1^+) = +ve \times +ve = +ve$

Hence there is a change in concavity at $x = 1$ and $f'(1) = 0$ so there is a horizontal point of inflection at $x = 1$. Hence *m* = 1.

Mathematical behaviours	Marks
• states $f'(1) = 0$ and $f''(1) = 0$	1
• demonstrates change in concavity at $x = 1$	1
• states that horizontal point of inflection occurs at *m* = 1.	1

Question 9(d) **(3 marks)**

Solution



Mathematical behaviours	Marks
• sketch shows $x \to \pm\infty,\ f(x) \to \infty$ and roots	1
• clearly shows x coordinate of minimum turning point	1
• graphs correct shape and clearly labels points of inflection	1

Question 20(b) **(5 marks)**

Solution

$A = \pi\left(\frac{l}{2\pi}\right)^2 + \left(\frac{100-l}{4}\right)^2$

$\frac{dA}{dl} = \frac{2\pi l}{4\pi^2} - \frac{1}{8}(100-l) = \frac{l}{2\pi} - \frac{1}{8}(100-l) = l\left(\frac{1}{2\pi} + \frac{1}{8}\right) - \frac{25}{2} = l\left(\frac{4+\pi}{8\pi}\right) - \frac{25}{2}$

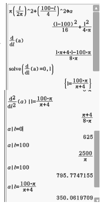
$\frac{dA}{dl} = 0 \Rightarrow l = \frac{25}{2}\left(\frac{8\pi}{4+\pi}\right) \approx 43.99\,cm$

$\frac{d^2A}{dl^2} = \frac{4+\pi}{8\pi} > 0 \Rightarrow min$

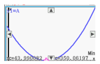
$A\big|_{l=43.99} = 350.06$

$A\big|_{l=0} = 25^2 = 625$

$A\big|_{l=100} = \pi\left(\frac{100}{2\pi}\right)^2 \approx 795.77$



Or, to establish minimum has been achieved at $l = 43.99\,cm$, states coefficient of l^2 is positive, hence minimum turning point or demonstrates with graph



Hence the minimum total area is obtained when $l = 43.99\,cm$

Mathematical behaviours	Marks
• determines $\frac{dA}{dl}$	1
• equates $\frac{dA}{dl} = 0$ and solves	1
• establishes $\left.\frac{d^2A}{dl^2}\right\|_{l=43.99} > 0$ hence a minimum	1
• determines A for $l = 0$ and $l = 100$ OR demonstrates through graph or coefficient of l^2 that A is a quadratic with a minimum turning point	1
• concludes minimum area is when $l = 43.99\,cm$	1

• uses the point $\left(-\frac{1}{2}, 1\right)$ to determine the value of c • states the correct equation of the function	1

Question 15(b) **(4 marks)**

Solution

(i) $f(x) = \frac{x}{6}$, where $x = -1, 1, 2, 4.$ No, since $f(-1) = \frac{-1}{6}$ represents a probability and probability cannot be negative

(ii)

x	4	6	8	10
$f(x)$	0.05	0.30	0.25	0.4

Yes as $0 \le f(x) \le 1\ \forall\ x$ and the sum of the probabilities is 1.

Mathematical behaviours	Marks
(i)	
• states no	1
• recognises negative probability	1
(ii)	
• states yes	1
• states both reasons	1

Question 16(a) **(3 marks)**

Solution

$v = \int 8\,dt$

$v = \frac{ds}{dt} = 8t + c$

$v(0) = p \Rightarrow \quad p = c$

$\therefore v = \frac{ds}{dt} = 8t + p$

$s = 4t^2 + pt + k$

$s(1) = q \Rightarrow \quad q = 4 + p + k \ \Rightarrow \quad k = q - 4 - p$

$\therefore s = 4t^2 + pt + q - 4 - p$

or $s = 4t^2 + pt + q - p - 4$ as required

Mathematical behaviours	Marks
• anti-differentiates $a(t)$ to obtain $v(t)$ and uses $v(0) = p$ to get correct expression for c.	1
• anti-differentiates $v(t)$ to obtain $s(t)$ and uses $s(1) = q$ to get correct expression for k	1
• states required answer	1

Question 19(c) **(1 mark)**

Solution	
$A=\int_{-1}^{4}\lvert f(x)\rvert dx=5+16+11+27=59$	
Mathematical behaviours	Marks
• states correct answer	1

Question 19(d) **(2 marks)**

Solution	
Shaded area marked M $=(16\times 3)-21=27$	
Marking key/mathematical behaviours	Marks
• recognises area of rectangle subtract $\int_{-1}^{2} f(x)dx$ • states correct answer	1 1

Question 19(e) **(2 marks)**

(i) Correct statement is $\int_{k}^{4} f(x)dx=48$ (ii) Use CAS and solve for k: Solve ($\left(\int_{k}^{4}(12x^2-4x^3)dx=48,k\right)$ $\Rightarrow k=-2$	
Mathematical behaviours	Marks
(i) • chooses correct statement (ii) • solves for k	 1 1

Question 20(a) **(3 marks)**

Solution	
For the circle, $l=2\pi r\Rightarrow r=\frac{l}{2\pi}$ $\therefore A_c=\pi\left(\frac{l}{2\pi}\right)^2$ For the square, $x=\left(\frac{100-l}{4}\right)$ $\therefore A_s=\left(\frac{100-l}{4}\right)^2$ Hence, $A=\pi\left(\frac{l}{2\pi}\right)^2+\left(\frac{100-l}{4}\right)^2$	
Mathematical behaviours	Mark
• demonstrates that $r=\frac{1}{2\pi}$ and states expression for the area of the circle • demonstrates that side length = $\frac{100-l}{4}$ and states expression for the area of the square • concludes formula for A	1 1 1

Question 10(a) **(1 mark)**

Solution	
X has a binomial distribution with parameters n and $p=0.5$ ie $X\sim Bin(n,0.5)$	
Mathematical behaviours	Marks
• identifies binomial distribution and states parameters	1

Question 10(b) **(1 mark)**

Solution	
$E(X)=\mu=np=\frac{n}{2}$	
Mathematical behaviours	Marks
• states correct answer	1

Question 10(c) **(3 marks)**

Solution	
$n=20$: $P_1=P(5\le X\le 15)\cong 0.988$ $n=1000$: $P_1=P(495\le X\le 505)\cong 0.272$ $n=10\,000$: $P_1=P(4995\le X\le 5005)\cong 0.088$ (from calculator)	
Mathematical behaviours	Marks
• states a probability inequality relevant to one of the n values • calculates one probability correctly • calculates all probabilities correctly	1 1 1

Question 10(d) **(1 mark)**

Solution	
$P_1\to 0$ as $n\to\infty$	
Mathematical behaviours	Marks
• obtains correct answer	1

Question 10(e) **(3 marks)**

Solution	
$n=20$: $P_2=P(9.5\le X\le 10.5)=P(X=10)\cong 0.176$ $n=200$: $P_2=P(95\le X\le 105)\cong 0.563$ $n=1000$: $P_2=P(475\le X\le 525)\cong 0.893$	
Mathematical behaviours	Marks
• states probability inequality relevant to $n=20$ • calculates one probability correctly • calculates all probabilities correctly	1 1 1

Question 10(f) **(1 mark)**

Solution	
$P_2\to 1$ as $n\to\infty$	
Mathematical behaviours	Marks

Question 16(b) **(1 mark)**

Solution	
Distance travelled = $\int_{0}^{3}\lvert 8t+p\rvert dt$	
Mathematical behaviours	Marks
• states the integral of the **absolute** velocity function from $t=0$ to $t=3$	1

Question 17(a) **(2 marks)**

Solution	
Define the random variable, X as the number of batteries that last for less than 2000 hours. Hence, $X\sim Bin(120,0.1)$ $P(X=15)\approx 0.0742$	
Mathematical behaviours	Marks
• recognizes Binomial nature • obtains correct answer	1 1

Question 17(b) **(2 marks)**

Solution	
$X\sim Bin(120,0.1)$ $P(X\le 15)\approx 0.8560$	
Mathematical behaviours	Marks
• recognizes binomial nature • obtains correct answer	1 1

Question 17(c) **(2 marks)**

Solution	
From part (b) we can conclude that there is an 85.6% chance that no more than 15 batteries out of 120 last less than 2000hrs. This would imply that there is only a 14.4% chance that more than 15 out of 120 batteries last less than 2000hrs. Hence the test does not imply compelling evidence that the manufacturer's claim is false.	
Mathematical behaviours	Marks
• obtains correct answer • gives valid reason	1 1

Question 18(a) **(3 marks)**

Solution

Outcome	Death	Permanent Disability	No payout
Profit	-49000	-9000	1000
Probability	0.01	0.02	0.97

Mathematical behaviours	Marks
• completes Probability row of table correctly	1

Question 13(a) **(5 marks)**

Solution

Stationary Points: $\frac{dy}{dx}=0$

i.e. $(6x-1)\left(x+\frac{1}{2}\right)=0$

$x=\frac{1}{6}$ or $x=-\frac{1}{2}$

```
(6x−1)(x+0.5)⇒y
                    (x+1/2)·(6·x−1)
solve(y=0,x)
                    {x=−1/2, x=1/6}
d/dx(y)|x=−0.5
                                 −4
d/dx(y)|x=1/6
                                  4
```

Now $\frac{dy}{dx}=(6x-1)\left(x+\frac{1}{2}\right)$

$=6x^2+2x-\frac{1}{2}$

$\frac{d^2y}{dx^2}=12x+2$

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y1=2·x^3+x^2−x/2
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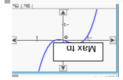
At $x=\frac{1}{6}$, $\frac{d^2y}{dx^2}=4\Rightarrow$ min At $x=-\frac{1}{2}$, $\frac{d^2y}{dx^2}=-4\Rightarrow$ max



$\therefore$ max turning pt at $\left(-\frac{1}{2},1\right)$

```
∫ydx
                    2·x^3+x^2−x/2
2·x^3+x^2−x/2|x=−1/2
                                1/4
```

Now $y=2x^3+x^2-\frac{1}{2}x+c$

$\left(-\frac{1}{2},1\right)\Rightarrow 1=2\left(-\frac{1}{2}\right)^3+\left(-\frac{1}{2}\right)^2-\frac{1}{2}.\left(-\frac{1}{2}\right)+c$

$1=\frac{1}{4}+c\Rightarrow c=\frac{3}{4}$

$\therefore$ equation of the function is $y=2x^3+x^2-\frac{1}{2}x+\frac{3}{4}$

Mathematical behaviours	Marks
• uses $\frac{dy}{dx}=0$ to find stationary points • substitutes into $\frac{d^2y}{dx^2}$, $x=\frac{1}{6}$ and $x=-\frac{1}{2}$ to find which x value gives a local maximum turning point or clearly shows on sketch location of maximum and confirms maximum using 2nd derivative test • integrates the derivative function correctly	1 1 1 1

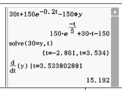
• obtains correct answer	1

Question 11(a) **(1 mark)**

Solution	
$y=30t+150e^{-0.2t}+k$ $t=0, y=0 \Rightarrow 0=150+k \Rightarrow k=-150$	
Mathematical behaviours	Mark
• evaluates k	1

Question 11(b) **(3 marks)**

Solution	
$y=30t+150e^{-0.2t}-150$ $y=30 \Rightarrow t=3.53s$ $v=30-30e^{-0.2t}$ $v_{t=3.53}\rvert=15.19m/s$	
Mathematical behaviours	Marks
• equates $y=30$ and determines time taken to hit the ground • differentiates to obtain v • calculates the speed	1 1 1

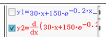


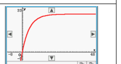
Question 11(c) **(2 marks)**

Solution	
$v=30-30e^{-0.2t}$ $\Rightarrow a=6e^{-0.2t}m/s^2>0$ Since $v>0$ and $a>0$ the ball is speeding up.	
Mathematical behaviours	Marks
• differentiates v to determine a and states $a>0$ • draws conclusion noting the same sign of both v and a.	1 1

Question 11(d) **(1 mark)**

Solution	
$v=30-30e^{-0.2t}$, $a=6e^{-0.2t}$ $t\to\infty, v\to 30, a\to 0$ Hence constant speed is attained.	
Mathematical behaviours	Marks
• states $v\to 30\,m/s$ ie is constant	1





Question 11(e) **(1 mark)**

Solution	
A restriction on the domain is needed. ie $0\le t\le 3.53$	
Mathematical behaviours	Marks

• states restriction required on the domain	1

Question 12(a) **(2 marks)**

Solution	
$\mu=\frac{49\times 63.3+38\times 54.1}{87}=59.28$	
Mathematical behaviours	Marks
• uses correct expression • obtains correct answer	1 1

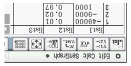
Question 12(b) **(4 marks)**

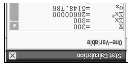
Solution	
If $Y=aX+b$, then $E(Y)=aE(X)+b$and $St.Dev(Y)=aSt.Dev(X)$ So $59.28=a\times 63.3+b$ and $9=a\times 7.6$ So $a=1.18$ and $b=-15.68$	
Mathematical behaviours	Marks
• expresses $E(Y)$ in terms of $E(X)$ • expresses $Std\ Dev\ (Y)$ in terms of $Std\ Dev(X)$ • calculates a • calculates b	1 1 1 1

• completes exactly 2 entries of Profit row of table correctly • completes table correctly	1 1

Question 18(b) **(2 marks)**

Solution	
$E(X)=0.01\times(-49000)+0.02\times(-9000)+0.97\times 1000=300$ Hence the expected profit is $300	
Mathematical behaviours	Marks
• states correct formula for $E(X)$ • obtains correct answer	1 1





Question 18(c) **(3 marks)**

Solution	
$Var(X)=(-49000-300)^2\times 0.01+(-9000-300)^2\times 0.02+(1000-300)^2\times 0.97=26510000$ $Std\ Dev=\sqrt{26510000}\approx 5149$ or $E(X^2)=(-49000)^2\times 0.01+(-9000)^2\times 0.02+(1000)^2\times 0.97=26600000$ $Var(X)=E(X^2)-(E(X))^2$ $=26600000-(300)^2=26510000$ $Std\ Dev(X)=\sqrt{26510000}\approx 5149$ Note: CAS screen above shows $E(X^2)=26600000$	
Mathematical behaviours	Marks
• demonstrates calculations required to obtain variance • obtains variance • obtains standard deviation	1 1 1

Question 19(a) **(1 mark)**

Solution	
$\int_{-1}^{2} f(x)\,dx=5+16=21$	
Mathematical behaviours	Marks
• states correct answer	1

Question 19(b) **(1 mark)**

Solution	
$\int_{-1}^{4} f(x)\,dx=5+16+11-27=5$	
Mathematical behaviours	Marks
• states correct answer	1