Papers written by
Australian Maths
Software

SEMESTER ONE

MATHEMATICS METHODS

UNIT 3

2017

SOLUTIONS

SECTION ONE

1. (8 marks)

(a) (i) $y = e^{2x} \times cos(2x)$

$$\frac{dy}{dx} = 2e^{2x}(cos(2x)) - 2(sin(2x))e^{2x}$$

$$\frac{dy}{dx} = 2e^{2x}(cos(2x) - sin(2x))$$

✓ -1/error

(ii) $y = \frac{(x^3 - 3x)}{e^{3x}}$

$$\frac{dy}{dx} = \frac{(3x^2 - 3)e^{3x} - 3e^{3x}(x^3 - 3x)}{(e^{3x})^2}$$ ✓ ✓

$$\frac{dy}{dx} = \frac{3(x^2 - 1 - x^3 + 3x)}{e^{3x}}$$

(b) $g(f(x)) = g(x^2 - x)$

$= sin(x^2 - x)$

$$\frac{d}{dx}(g(f(x))) = (cos(x^2 - x)) \times (2x^2 - 1)$$

$= (2x^2 - 1) \times cos(x^2 - x)$

.

2. (9 marks)

(a) $\int_{\pi/6}^{\pi/2} cos(2x)dx$

$$= \left[\frac{sin(2x)}{2}\right]_{\pi/6}^{\pi/2}$$

$$= \frac{1}{2}\left(sin(\pi) - sin\left(\frac{\pi}{3}\right)\right)$$

$$= \frac{1}{2}\left(0 - \frac{\sqrt{3}}{2}\right)$$

$$= -\frac{\sqrt{3}}{4}$$

(b) $\int_{1}^{3}\left(x^2 - 4x^3\right)dx$

$$= \left[\frac{x^3}{3} - x^4\right]_1^3$$

$$= (9 - 81) - \left(\frac{1}{3} - 1\right)$$

$$= -71\frac{1}{3}$$

(c) $\int e^{0.5x}\,dx = \frac{e^{0.5x}}{0.5} + c = 2e^{0.5x} + c$

(d) $\int_{-3\pi/4}^{\pi/3}(1 - sin(x))dx$

$$= \left[x + cos(x)\right]_{-3\pi/4}^{\pi/3}$$

$$= \left(\frac{\pi}{3} + cos\left(\frac{\pi}{3}\right)\right) - \left(-\frac{3\pi}{4} + cos\left(-\frac{3\pi}{4}\right)\right)$$

$$= \frac{\pi}{3} + \frac{1}{2} - \left(-\frac{3\pi}{4} - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{13\pi}{12} + \frac{1}{2} + \frac{1}{\sqrt{2}}$$

3. (7 marks)

(a) (i)

x	10	11	12
$P(X=x)$	**0.36**	**0.36**	**0.28**

✓✓

(ii) $0.72 \times 0.72 = 0.5184$
✓ ✓

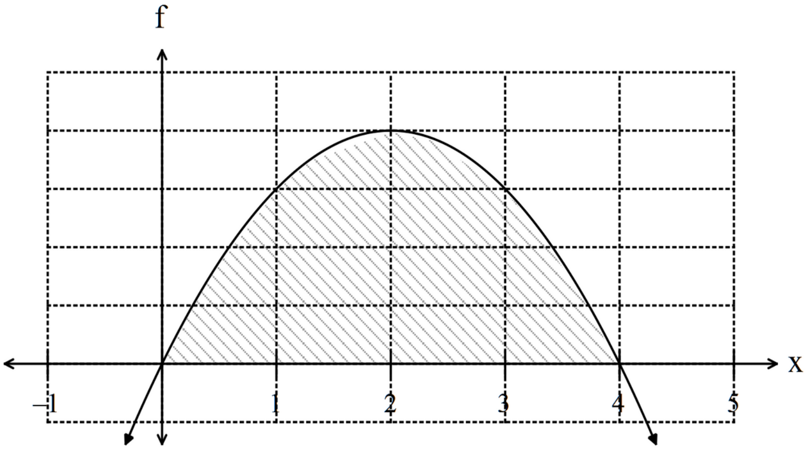
(b) (i) Can represent a probability density function as the probabilities add to one. ✓

(ii) Cannot represent a probability density function as the probabilities add to 1.1 ✓

(iii) Cannot represent a probability density function as one of the probabilities is negative. ✓

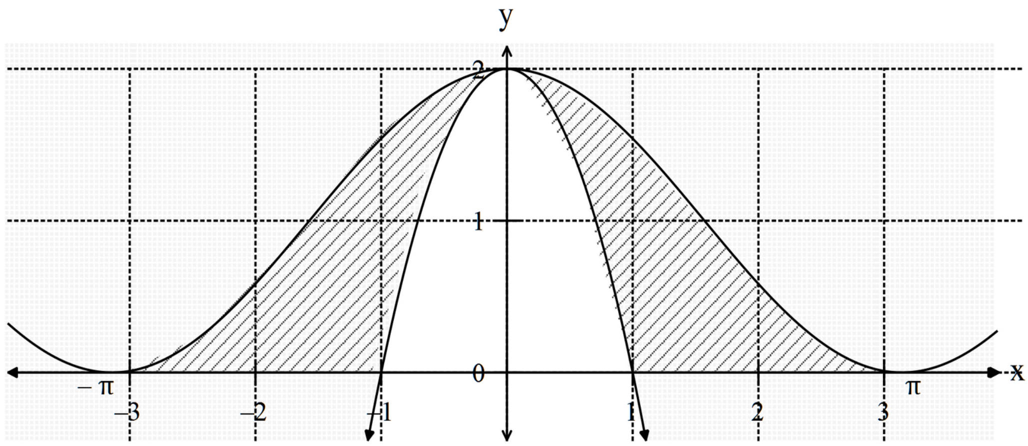
4. (6 marks)

(a) $f(x) = -(x-2)^2 + 4$



$$\begin{aligned}
Area &= \int_0^4 \left(-(x-2)^2 + 4\right) dx \\
&= \int_0^4 \left(-x^2 + 4x - 4 + 4\right) dx \\
&= \int_0^4 \left(-x^2 + 4x\right) dx \\
&= \left[\frac{-x^3}{3} + 2x^2\right]_0^4 \\
&= \left(\frac{-64}{3} + 32\right) - (0) \\
&= 10\frac{2}{3} unit^2
\end{aligned}$$

(b) $y = 2x^2 - 2$ and $y = 1 + \cos(x)$



$$\text{Area} = 2\left\{ \int_0^{\pi} (1 + \cos(x))\,dx \quad - \quad \int_0^{1} (2 - 2x^2)\,dx \right\}$$

5. (9 marks)

(a) $a = -6t + 6 \ m\,s^{-2}$

$v = \int(-6t + 6)\,dt$

$v = -3t^2 + 6t + c_1$

but $v_0 = -3 \ m\,s$

$-3 = 0 + 0 + c_1$

$\therefore \ v = -3t^2 + 6t - 3$

$x = \int(-3t^2 + 6t - 3)\,dt$

$x = -t^3 + 3t^2 - 3t + c_2$

but $x_0 = 4 \ m \ \Rightarrow \ c_2 = 4$

$\therefore \ x = -t^3 + 3t^2 - 3t + 4$

(b) At $t = 2$

$v = -3t^2 + 6t - 3$

$v_2 = -12 + 12 - 3$

$v_2 = -3\,m\,s^{-1}$

$x = -t^3 + 3t^2 - 3t + 4$

$x_2 = -8 + 12 - 6 + 4$

$x_2 = 2\,m$

(c) Particle changes direction when the velocity is equal to 0 m s^{-1}.

$$-3t^2+6t-3=0$$

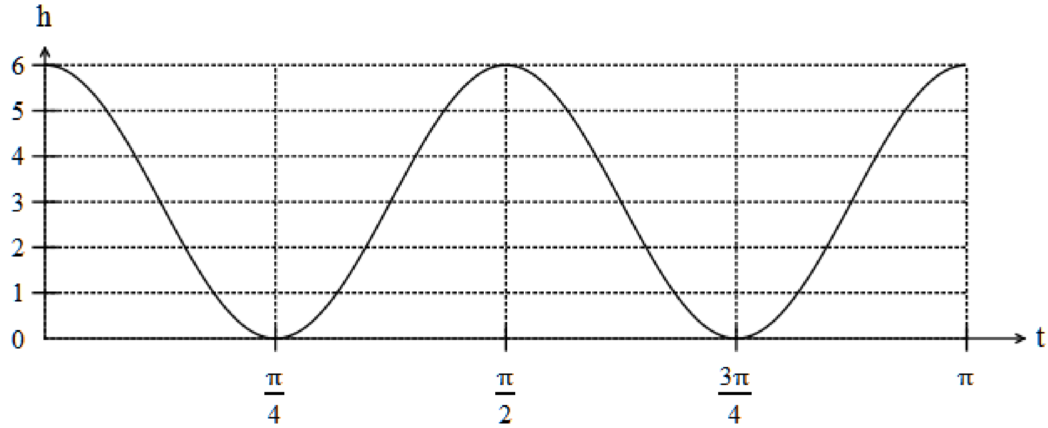
$$-3(t^2-2t+1)=0$$

$$(t-1)^2=0$$

$$t=1s$$

6. (7 marks)

(a) $h(t)=3\cos(4t)+3$



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(b) $h(t)=3\cos(4t)+3$

$$h'(t)=-12\sin(4t)$$

At $t=\dfrac{3\pi}{8}$

$$h'\left(\frac{3\pi}{8}\right)=-12\sin\left(4\left(\frac{3\pi}{8}\right)\right)$$

$$=-12\sin\left(\frac{3\pi}{2}\right)$$

$$=12\ m\,s^{-1}$$

The piston is rising at 12 m s^{-1}.

(c) $t=\dfrac{3\pi}{8}$ is exactly between $t=\dfrac{\pi}{4}$ and $t=\dfrac{\pi}{2}$. This is at the point of inflection where the gradient is at its highest as it has been increasing from zero at $t=\dfrac{\pi}{4}$ at the minimum turning point and starts to decrease after $t=\dfrac{3\pi}{8}$. This means the rate of rise of the of the piston is greatest at $t=\dfrac{3\pi}{8}$.

(d) $v = h'(t) = -12\,sin(4t)$

$$\frac{dv}{dt} = h''(t) = -48\,sin(4t)$$

At $t = \frac{3\pi}{8}$

$$h''\left(\frac{3\pi}{8}\right) = -48\,cos\left(4\left(\frac{3\pi}{8}\right)\right)$$

$$h''\left(\frac{3\pi}{8}\right) = -48\,cos\left(\frac{3\pi}{2}\right)$$

$$= 0 \;\; m\,s^{-2}$$

So at $t = \frac{3\pi}{8}$ the rate the velocity of the piston is changing at 0 m s^{-2}.

7. (4 marks)

Show that $\lim_{x \to 0}\left(\frac{e^h - 1}{h}\right) = 1.$

Let $f(x) = e^x$

$$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$f'(x) = \lim_{h \to 0}\left(\frac{e^{x+h} - e^x}{h}\right)$$

$$f'(x) = e^x \times \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right)$$

but $f'(x) = e^x$

$$e^x = e^x \times \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right)$$

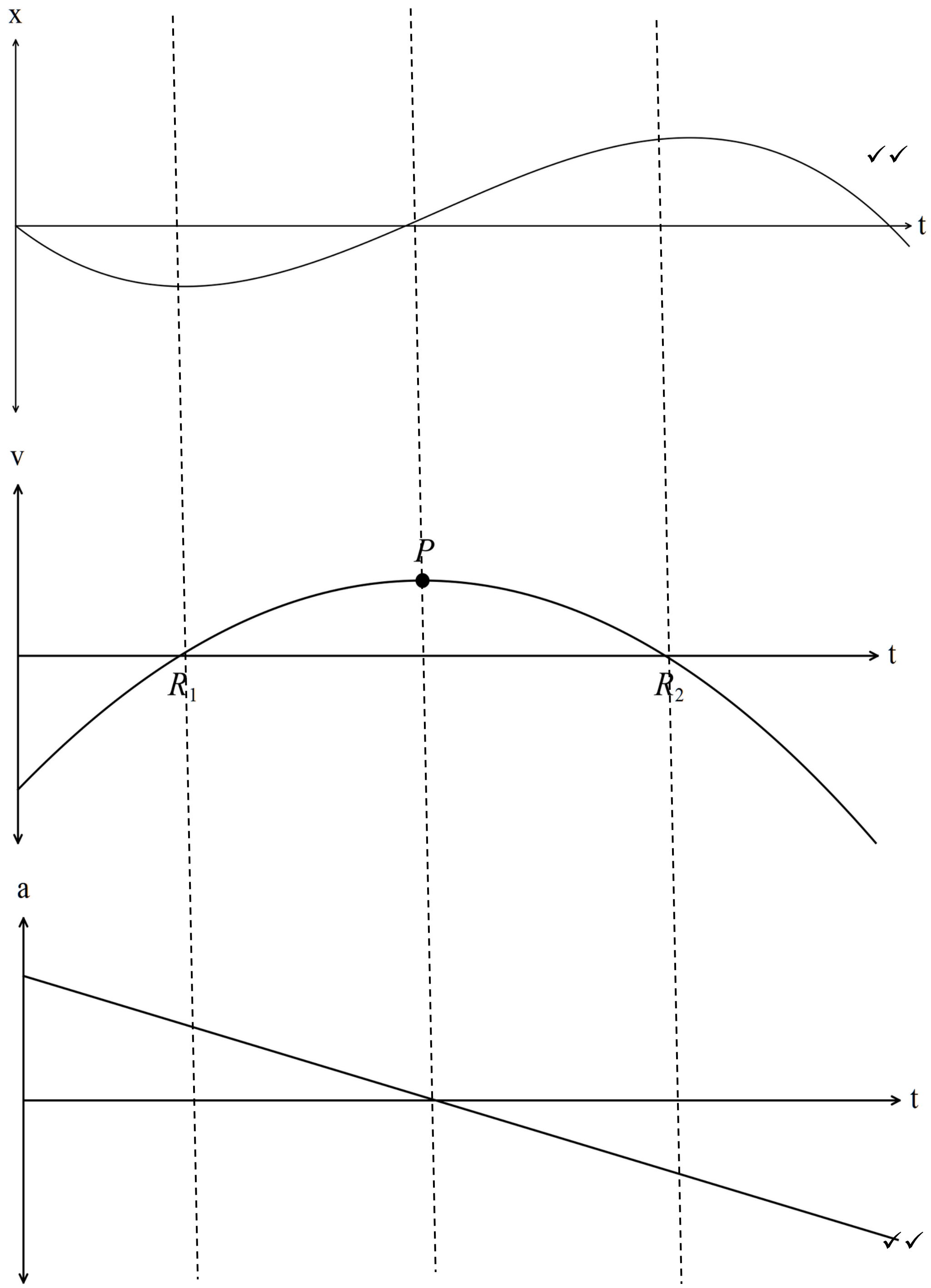
$$\therefore \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right) = 1$$

END OF SECTION ONE

SECTION TWO

8. (7 marks)

(a) (i)



(b) The roots of $y = v(t)$ occur at the same t value as the turning points on $y = x(t)$.
At R_1, $v(R_1^-) < 0$, $v(R_1) = 0$ and $v(R_1^+) > 0$. , i.e. the turning point in $y = v(t)$ is a minimum.
At R_2, $v(R_2^-) > 0$, $v(R_2) = 0$ and $v(R_2^+) < 0$. , i.e. the turning point in $y = v(t)$ is a maximum.

The turning point of $y = v(t)$, P, has a zero gradient so its derivative, $y = a(t)$ has a zero value at $t = P$.
The gradient of $y = v(t)$ is positive for $t < P$ and is negative for $t > P$, so the linear function $y = a(t)$ is a decreasing value with an x intercept at $t = P$.

9. (6 marks)

(a) (i) $\int_0^4 f(x)dx = 11.73 - 20.27 + 41.87 = 33.33$ ✓

(ii) $\int_3^1 f(x)dx = -\int_1^3 f(x)dx = -(-20.27) = 20.27$

(iii) $\int_{-4}^4 |f(x)|dx = 2(11.73 + 20.27 + 41.87) = 147.74$ ✓ ✓

(b) $\int_{-4}^4 |f(x)|dx$ ✓

10. (6 marks)

(a) $\frac{dW}{dt} = 2.457e^{0.491t}$

$W = \int 2.457e^{0.491t}dt$

$W = \frac{2.457e^{0.491t}}{0.491} + c$

At $t = 7, W = 35.7$

$35.7 = \frac{2.457e^{0.491\times 7}}{0.491} + c$

$c = -119.9$

$W = 5.004e^{0.491t} - 119.9$

(b) $W = \int_{2}^{15} 2.457e^{0.491t}\,dt$

$W = \left[5.00407e^{0.491t}\right]_{12}^{15}$

$W = 7905.011819 - 1812.119158$

$W \approx 6093\,grams$

The expected total change in the weight of a halibut between 12 and 15 months old is 6093 grams.

11. (11 marks)

(a) $P = e^{0.2t}\,sin(t)$

$$\frac{dP}{dt} = 0.2e^{0.2t}\,sin(t) + e^{0.2t}\,cos(t)$$

$$\frac{dP}{dt} = e^{0.2t}\left(0.2\,sin(t) + cos(t)\right) \quad ✓ \quad ✓$$

$$\frac{d^2P}{dt^2} = 0.2e^{0.2t}\left(0.2\,sin(t) + cos(t)\right) + e^{0.2t}\left(0.2\,cos(t) - sin(t)\right)$$

$$\frac{d^2P}{dt^2} = e^{0.2t}\left(0.04\,sin(t) + 0.2\,cos(t) + 0.2\,cos(t) - sin(t)\right)$$

$$\frac{d^2P}{dt^2} = e^{0.2t}\left(-0.96\,sin(t) + 0.4\,cos(t)\right) \quad ✓ \quad ✓$$

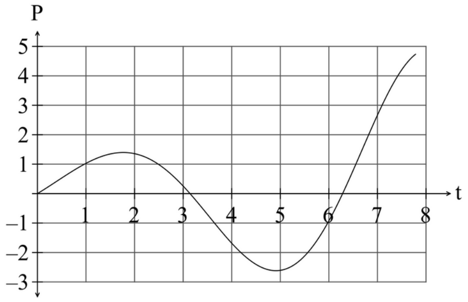
Find where $\frac{dP}{dt} = 0$ to find the turning points then use $\frac{d^2P}{dt^2}$ to identify the types of turning points.

If $\frac{d^2P}{dt^2} < 0$ then maximum turning point. If $\frac{d^2P}{dt^2} > 0$ then minimum turning point.

If $\frac{d^2P}{dt^2} = 0$ then you have the t value so you can find the points of inflection.

✓✓

(b)



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(c) The profit started to increase again at $t = 4.9$ months. ✓

(d) The break even point was reached at $t = 6.28$ months. ✓

12. (10 marks)

(a) (i) $\sum_{i=0.5}^{2} f(x)\delta x = 0.5\times 3.75 + 0.5\times 3 + 0.5\times 1.75 + 0.5\times 0$ ✓✓

$$= 0.5\times 8.5$$
$$= 4.25$$

(ii) $\lim_{\delta x\to 0}\left(\sum_i f(x_i)\delta x_i\right) = \int_a^b f(x)dx$ for $a \le x \le b$ ✓✓

(b) (i)

$$F(x) = \int_a^x f(t)dt$$
$$= \int_a^x \sqrt{e^t}\, dt$$
$$= \int_a^x e^{t/2}\, dt$$
$$= 2\left[e^{t/2}\right]_a^x$$
$$F(x) = 2\left(e^{x/2} - e^{a/2}\right)$$
$$\text{so } F'(x) = 2\left(\frac{1}{2}\times e^{x/2} - 0\right)$$
$$= e^{x/2}$$
$$= \sqrt{e^x}$$
$$F'(x) = \sqrt{e^x} = f(x)$$
$$F'(x) = f(x)$$
$$\therefore\ F'(t) = f(t)$$

(ii) $\frac{d}{dx}\left(\int_1^x tan^2(t)dt\right) = tan^2(x)$ ✓✓

13. (5 marks)

(a) $A = \frac{1}{2} \times qr\, sin(QPR)$

$A = \frac{1}{2} \times 14 \times 10 \times sin(\theta)$

$A = 70 \times sin(\theta)$

(b) $A = 70\, sin(\theta)$

$\frac{dA}{d\theta} = 70\, cos(\theta)$

$\frac{dA}{d\theta} \approx \frac{\delta A}{\delta\theta}$

$\therefore\ \delta A \approx \frac{dA}{d\theta} \times \delta\theta$

$\delta A \approx 70\, cos(\theta) \times \delta\theta$

At $\theta = 0.84$ and $\delta\theta = 0.02$

$\delta A \approx 70\, cos(0.84) \times 0.02$

$\delta A \approx 0.93\, m^2$

14. (7 marks)

(a) $\int_0^{\pi/4} \left(\frac{1 - e^x}{tan(x)} \right) dx = -0.895$ ✓✓✓ (3 decimal places)

(b) $\int_a^b (2 - 2e^x) dx = \int_a^b (2) dx - 2 \int_a^b (e^x) dx$

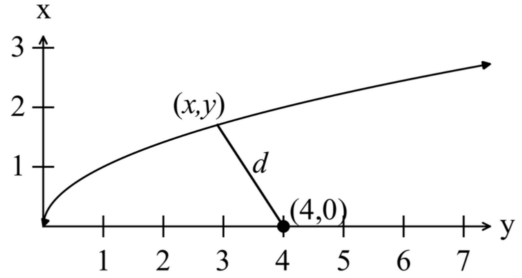
$= [2x]_a^b - 2 \times 4.5$

$= 2(b - a) - 9$

$= 2b - 2a - 9$

(c) $a = 4$ and $b = 1$ ✓✓

15. (8 marks)



$d^2 = (x-4)^2 + y^2 \qquad y = \sqrt{x}$

$d^2 = (x-4)^2 + x = x^2 - 8x + 16 + x$

$d = \sqrt{x^2 - 7x + 16}$

$\dfrac{dd}{dx} = \dfrac{0.5(2x-7)}{(x^2-7x+16)^{0.5}}$

If $\dfrac{dd}{dx} = 0,\ x = 3.5$

Max or min?

$\dfrac{d^2d}{dx^2} = \dfrac{d}{dx}\left(\dfrac{0.5(2x-7)}{(x^2-7x+16)^{0.5}}\right)$

At $x = 3.5$ $\dfrac{d^2d}{dx^2} = 0.516 > 0 \rightarrow$ min

Need point $y = \sqrt{3.5} = 1.87$

$(x, y) = (3.5, 1.87)$

$d^2 = (x-4)^2 + y^2 \qquad y = \sqrt{x}$

$d^2 = (x-4)^2 + x = x^2 - 8x + 16 + x$

$d^2 = x^2 - 7x + 16$

$\dfrac{d(d^2)}{dx} = 2x - 7$

If $\dfrac{dd}{dx} = 0,\ x = 3.5$

Max or min?

$\dfrac{d^2(d^2)}{dx^2} = \dfrac{d}{dx}(2x-7) = 2$

At $x = 3.5$ $\dfrac{d^2(d^2)}{dx^2} = 2 > 0 \rightarrow$ min

Need point $y = \sqrt{3.5} = 1.87$

$(x, y) = (3.5, 1.87)$

16. (5 marks)

(a) $123\ 202\ 624 = 50\ 189\ 209e^{50k}$

$k = 0.0179606$

$P = 50\ 189\ 209e^{0.0179606t}$

(b) $e^{0.0179606} = 1.018123$

The annual rate of growth of the population is 1.8123%

(c) $P = 123\,202\,624e^{0.01170761165t}$

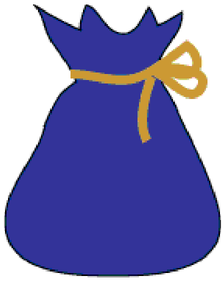
$e^{0.01170761165} = 1.011776414$

The annual rate of growth of the population is now 1.1776414% so the rate of growth of the population has slowed down considerably.

(d) $P_{2016} = 123\ 202\ 624e^{0.01170761165\times 86}$

$P_{2016} = 337\ 202\ 942$

17. (4 marks)



(a) $\frac{8}{20}\times\frac{8}{20}\times\frac{8}{20}\times\frac{12}{20}=0.0384$

(b) $P(\text{no yellow})=P(RRRR)$

$=\left(\frac{8}{20}\right)^4$

$=0.0256$

OR

$P(\text{a yellow})=P(Y)+P(RY)+P(RRY)+P(RRRY)$

$=\frac{12}{20}+\frac{8}{20}\times\frac{12}{20}+\left(\frac{8}{20}\right)^2\times\frac{12}{20}+\left(\frac{8}{20}\right)^3\times\frac{12}{20}$

$=0.9744$

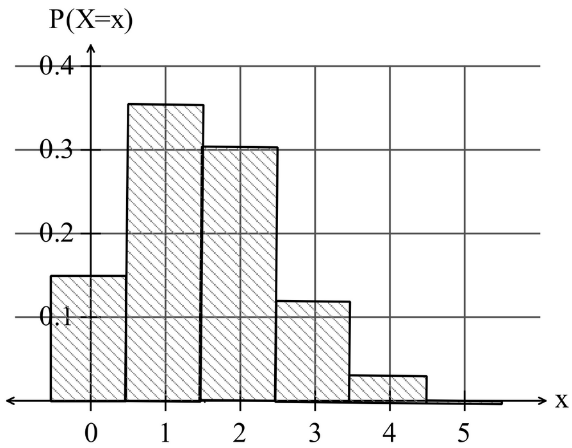
$P(\text{no yellow})=1-0.9744$

$=0.0256$

18. (5 marks)

(a)

x	0	1	2	3	4	5
$P(X=x)$	0.17	0.36	0.31	0.13	0.03	0.00



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(b) p = 0.7

19. (10 marks)

(a) (i)

x	1	2	3	4
$P(X = x)$	0.3	0.2	0.2	0.3

$$E(X) = \sum x \times p(x)$$
$$= 1\times 0.3 + 2\times 0.2 + 3\times 0.2 + 4\times 0.3$$
$$= 0.3 + 0.4 + 0.6 + 1.2$$
$$E(X) = 2.5$$

$$Var(X) = E(X^2) - (E(X))^2$$
$$E(X^2) = 1^2\times 0.3 + 2^2\times 0.2 + 3^2\times 0.2 + 4^2\times 0.3 \checkmark\checkmark$$
$$= 0.3 + 0.8 + 1.8 + 4.8$$
$$= 7.7$$

$$Var(X) = 7.7 - 2.5^2$$
$$Var(X) = 1.45$$

(ii) $Y = 2X + 1.$

$E(Y) = 2.5\times 2 + 1$ $\quad Var(Y) = 1.45\times 2^2$

$E(Y) = 6$ $\quad Var(Y) = 5.8$

(b) Sam's average payout

$= 0\times 0.3 + 2\times 0.2 + 2\times 0.2 + 0\times 0.3 - 1$ ✓ ✓

$= 0 + 0.4 + 0.4 + 0 - 1$

$= -0.2$

Sam loses an average of 20 cents per spin.

20. (14 marks)

(a) $X \sim Bin(10, 0.6)$

$E(X) = n \times p = 10 \times 0.6 = 6$ ✓

(b) $p = 0.6, n = 10$ $P(x = 4) = 0.11148$ ✓✓

(c) $P(x \geq 4) = 0.94524$ ✓✓

(d) $Y \sim Bin(10, 0.4)$

$P(y \geq 4) = 0.61772$ ✓✓

(e) $P(\text{alternating colours}) = 2 \times (0.6 \times 0.4 \times 0.6 \times 0.4 \times 0.6 \times 0.4 \times 0.6 \times 0.4 \times 0.6 \times 0.4)$

$= 2 \times (0.6 \times 0.4)^5$

$= 0.00159$

(f) $P(x \geq 2) = 0.95$

$i.e.\ P(x < 2) = 0.05$

$P(x < 2) = P(x = 0) + P(x = 1)$

$P(x = 0) + P(x = 1) = 0.05$

Consider

${}^nC_0 (0.4)^0 (0.6)^n + {}^nC_1 (0.4)^1 (0.6)^{n-1} = 0.05$

$(0.6)^n + n(0.4)^1 (0.6)^{n-1} = 0.05$

$n = 9.82$

Ten (or more) seedlings must be planted to have a 95% chance of obtaining at least two white flowering plants.

END OF SECTION TWO