Rossmoyne Senior High School

Semester Two Examination, 2022

Question/Answer booklet

MATHEMATICS METHODS UNITS 3&4

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words

Your name

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	55	35
Section Two: Calculator-assumed	12	12	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Supplementary page

Question number: _________

End of questions

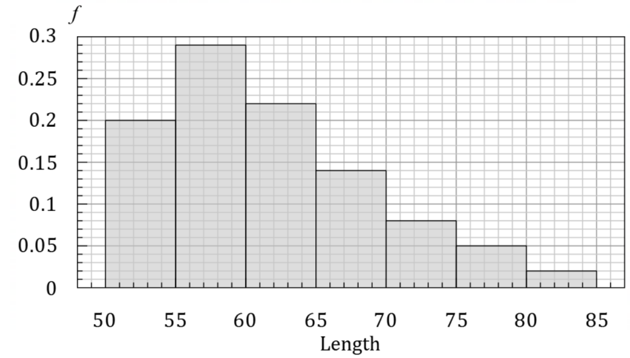
Section One: Calculator-free **35% (55 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(7 marks)**

(a) The relative frequency histogram below shows the distribution of the lengths in centimetres of a large sample of fish bred in an offshore fish farm.



Use the distribution to determine the probability that

(i) a randomly selected fish will be longer than 70 cm. (1 mark)

Solution
$P(X > 70) = 0.08 + 0.05 + 0.02 = 0.15$
Specific behaviours
✓ correct probability

(ii) a randomly selected fish will be exactly 71 cm long. (1 mark)

Solution
$P(X = 71) = 0$
Specific behaviours
✓ correct probability

(iii) when two fish are randomly selected, one is shorter than 55 cm and the other is not. (2 marks)

Solution
$p = 0.2 \times 0.8 \times 2 = 0.32$ OR $\frac{1}{5} \times \frac{4}{5} \times 2 = \frac{8}{25}$ =0.32
Specific behaviours
✓ correct probabilities for each fish ✓ correct probability

See next page

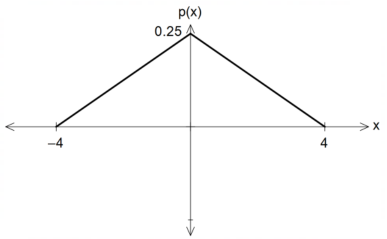
(b) Determine whether the following represent or do not represent a probability distribution.

Justify each answer.

(i) $f(x) = \dfrac{x}{x+2}$, $x = 0,\ 1,\ 2.$ (1 mark)

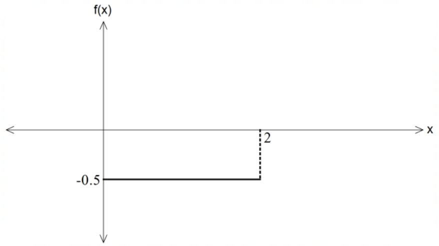
Solution
Not a probability distribution, sum of probabilities $\neq$ 1
Specific behaviours
✓

(ii) (1 mark)



Solution
Does represent a probability distribution; Area = 1 and $p(x)$ is always positive
Specific behaviours
All three points need to stated ✓

(iii) (1 mark)



Solution
Not a probability distribution. $f(x) < 0$ (negative)
Specific behaviours
✓

(b) The speed, s cm per second, of model car B at time t seconds is given by $s = e^{\sqrt{4t+2}}$, so that when $t = 3.5$, its speed was 54.6 cm per second. Use the increments formula to determine a decimal approximation for the speed of this car when $t = 3.6$.

(4 marks)

Solution
Let $t = 3.5, \delta t = 3.6 - 3.5 = 0.1$ and $s = e^u$ where $u = \sqrt{4t+2}$. Then $\dfrac{du}{dt} = \dfrac{4}{2\sqrt{4t+2}}$ Hence $\dfrac{ds}{dt} = \dfrac{2}{\sqrt{4t+2}} e^{\sqrt{4t+2}}\Big\|_{t=3.5}$ $= \dfrac{2}{4}(54.6) = 27.3$ Using increments formula $\delta s \approx \dfrac{ds}{dt}\delta t$ $\approx 27.3 \times 0.1$ ≈ 2.73 Hence approximate speed of car is $54.6 + 2.73 = 57.33$ cm/s.
Specific behaviours
✓ indicates correct derivative for u wrt to t ✓ indicates correct derivative for s wrt to t ✓ shows correct use of increments formula ✓ obtains speed of car

Alternative Solution
Let $t = 3.5, \delta t = 3.6 - 3.5 = 0.1$ $\dfrac{ds}{dt} = e^{(4t+2)^{\frac{1}{2}}}.\tfrac{1}{2}(4t+2)^{-\frac{1}{2}}.(4)$ $= \dfrac{2e^{\sqrt{4t+2}}}{\sqrt{4t+2}}$ ✓ Using increments formula $\delta s \approx \dfrac{ds}{dt}\delta t$ $\approx \dfrac{2e^{\sqrt{4t+2}}}{\sqrt{4t+2}}\Big\|_{t=3.5} \times (0.1)$ ✓ $\approx \dfrac{2}{4}(54.6)(0.1)$ $\approx 27.3 \times 0.1$ ≈ 2.73 ✓ Hence approximate speed of car is $54.6 + 2.73 = 57.33$ cm/s. ✓

Question 7 **(8 marks)**

(a) The velocity, v cm per second, of electrically powered model car A at time t seconds is given by $v = \sqrt{4t+2}$. Determine the change in displacement of this car between $t = 0.5$ and $t = 3.5$ seconds. (4 marks)

Solution

$$\Delta x = \int_{0.5}^{3.5} (4t+2)^{\frac{1}{2}}\,dt$$
$$= \left[\frac{2}{3\times 4}(4t+2)^{\frac{3}{2}}\right]_{0.5}^{3.5}$$
$$= \left[\frac{1}{6}(16)^{\frac{3}{2}}\right] - \left[\frac{1}{6}(4)^{\frac{3}{2}}\right]$$
$$= \frac{1}{6}(64-8)$$
$$= \frac{28}{3}\text{ cm}$$

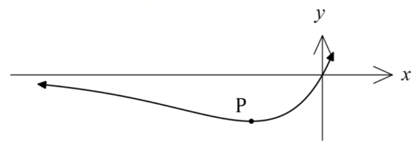
Specific behaviours

✓ writes integral for change in displacement
✓ obtains antiderivative
✓ substitutes upper and lower bounds and starts simplification
✓ correct change in displacement (accept $\frac{56}{6}$)

Question 2 **(7 marks)**

Let $f(x) = 5xe^{(0.2x+1)}$.

The graph of $y = f(x)$ is shown. It has one stationary point, at P, and one point of inflection.



(a) Clearly show that $f'(x) = (x+5)e^{(0.2x+1)}$. (2 marks)

Solution

$$f'(x) = (5)\left(e^{(0.2x+1)}\right) + (5x)\left(0.2e^{(0.2x+1)}\right)$$
$$= (5 + 5x\times 0.2)e^{(0.2x+1)}$$
$$= (x+5)e^{(0.2x+1)}$$

Specific behaviours

✓ correctly differentiates exponential term
✓ shows correct use of product rule
First line correct plus last line – full marks

(b) Determine the coordinates of point P. (2 marks)

Solution

$f'(x) = 0$ when $x + 5 = 0 \rightarrow x = -5$, and $f(-5) = -25$.

$\therefore P(-5, -25)$

Specific behaviours

✓ solves $f'(x) = 0$ that is $x = -5$ one mark
✓ correctly states coordinates

(c) Determine the values of x for which the curve $y = f(x)$ is concave down. (3 marks)

Solution

$$f''(x) = (1)\left(e^{(0.2x+1)}\right) + (x+5)\left(0.2e^{(0.2x+1)}\right)$$
$$= (0.2x+2)e^{(0.2x+1)}$$

$f''(x) = 0$ when $0.2x + 2 = 0 \rightarrow x = -10$.

From the graph, the curve is concave down to the left of the point of inflection and so the values of x are $x < -10$.

Specific behaviours

✓ correctly obtains $f''(x)$ - simplified or unsimplified form
✓ indicates x-coordinate of point of inflection
✓ correct inequality for x

Question 3 **(11 marks)**

Determine the following:

(a) $\int 6e^{3x-2}\,dx$. (1 mark)

Solution
$2e^{3x-2}+c$
Specific behaviours
✓ correct antiderivative, with constant of integration

(b) $\int_0^{\frac{\pi}{6}} \cos(3x)\,dx$. (2 marks)

Solution
$\left[\frac{1}{3}\sin(3x)\right]_0^{\pi/6}=\frac{1}{3}-0=\frac{1}{3}$
Specific behaviours
✓ correct antiderivative ✓ correct value

(c) $f'\left(\frac{\pi}{2}\right)$ when $f(x)=\frac{\sin(4x)}{1+\cos(x)}$. (3 marks)

Solution
$f'(x)=\frac{4\cos(4x)\,(1+\cos(x))-\sin(4x)\,(-\sin(x))}{(1+\cos(x))^2}$ $f'\left(\frac{\pi}{2}\right)=\frac{4(1)-0}{(1+0)^2}$ $=4$
Specific behaviours
✓ correctly uses quotient rule ✓ correctly differentiates all trig terms ✓ correctly evaluates

(b) The current, I amps, flowing through component B reaches a peak very quickly and then declines as time goes on, as modelled by $I(t)=\frac{2+\ln(t)}{4t}$. Determine, in simplest form, the maximum current that flows through this component. (4 marks)

Solution
$I'(t)=\frac{\left(\frac{1}{t}\right)(4t)-(2+\ln t)(4)}{(4t)^2}$ $=\frac{4-4(2+\ln t)}{4\times 4t^2}$ $=\frac{-1-\ln t}{4t^2}$ $I'(t)=0\Rightarrow \ln t=-1$ $t=e^{-1}$ $I(e^{-1})=\frac{2-1}{4e^{-1}}=\frac{e}{4}$ Maximum current is $\frac{e}{4}$ amps.
Specific behaviours
✓ uses quotient rule correctly ✓ obtains derivative that is, 2 mks for the correct un-simplified derivative ✓ obtains root of derivative ✓ calculates maximum current in simplified form

Question 6 **(8 marks)**

Components A and B form part of an electronic circuit, and properties of these components are measured t seconds after the circuit is turned on.

(a) The rate of change of temperature, T °C, of component A is given by $\frac{dT}{dt} = \frac{18t}{3t^2+8}$.
Determine, in simplest form, the increase in temperature of this component during the first 4 seconds. (4 marks)

Solution
$\Delta T = \int_0^4 \frac{18t}{3t^2+8}\,dt$ $= 3\int_0^4 \frac{6t}{3t^2+8}\,dt$ $= 3[\ln(3t^2+8)]_0^4$ $= 3(\ln(56) - \ln(8))$ $= 3\ln(7)$ °C
Specific behaviours
✓ writes integral to evaluate total change   ✓ integrates rate of change   ✓ substitutes limits of integral   ✓ correct increase, simplified (also accept $\ln(343)$)   Penalise for failing to simplify $\frac{56}{8}$ *or* $\frac{28}{4}$ to 7

(d) $\frac{d}{dx}\int_x^e \sin(t-1)\,dt.$ (1 mark)

Solution
$-\sin(x-1)$
Specific behaviours
✓ correct result   -1 for +C (only penalise once in Q3)

(e) $\int_0^2 \frac{d}{dx}(xe^{5x})\,dx.$ (1 mark)

Solution
$[xe^{5x}]_0^2 = 2e^{10}$
Specific behaviours
✓ correct result

(f) $\frac{d^2y}{dx^2}$ if $y = \int \ln(\sin 2x)\,dx$ (3 marks)

Solution
$\frac{dy}{dx} = \ln(\sin 2x)$ ✓ $\therefore$ $\frac{d^2y}{dx^2} = \frac{1}{\sin 2x}(2\cos 2x)$ ✓✓
Specific behaviours
✓ correct $\frac{dy}{dx}$ ✓ correct denominator   ✓ correct numerator

Question 4 **(8 marks)**

A computer program scans selected text messages passing through a network to see if the message contains a particular keyword. The random variable X takes the value 0 if the keyword is not found, the value 1 if it is found, and has probability distribution

$$P(X=x)=\begin{cases}\dfrac{e^{kx}}{4} & x=0,1\\ 0 & \text{elsewhere.}\end{cases}$$

(a) Complete the table for the probability distribution of X (1 mark)

Solution

x	0	1
$P(X=x)$	$\frac{1}{4}$	$\frac{3}{4}$

Specific behaviours

✓ correct result - ($\frac{e^k}{4}$ no marks)

(b) Show that the value of the constant k is $\log_e(3)$. (2 marks)

Solution

$$P(x=0)+P(x=1)=1\rightarrow\frac{1}{4}+\frac{e^k}{4}=1$$

$$e^k=3\Rightarrow k=\log_e(3)$$

Specific behaviours

✓ correctly substitutes $x=0$ and $x=1$

✓ uses sum of probabilities to form equation and derive value of k

(c) Determine the mean and standard deviation of X. (2 marks)

Solution

$$\mu=P(X=1)=\frac{3}{4}$$

$$\sigma=\sqrt{p(1-p)}=\sqrt{\frac{3}{4}\times\frac{1}{4}}=\frac{\sqrt{3}}{4}$$

Specific behaviours

✓ correct mean

✓ correct standard deviation - accept $\sqrt{\frac{3}{16}}$

errors

$$\mu=\frac{e^k}{4}$$

Specific behaviours

✓ no marks , no FT

(d) Determine the probability that the program finds the keyword in exactly three of the next four randomly selected text messages that it scans. (3 marks)

Solution

$$Y\sim B\left(4,\frac{3}{4}\right)$$

$$P(Y=3)=\binom{4}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^1$$

$$=\frac{4\times3^3}{4^3\times4}=\frac{27}{64}$$

Specific behaviours

3 mks – (all three lines) or (one of the first two lines and final correct answer)

2 mks – first two lines only

1 mk - Ist or 2nd or 3rd line (on their own)

Question 5 **(6 marks)**

Let $f(x)=k\log_6(x+6)+c$, where k and c are constants.

The graph of $y=f(x)$ intersects line L with equation $5y+2x+15=0$ when $x=0$ and $x=-5$.

(a) Determine the value of the constant c and the value of the constant k. (3 marks)

Solution

$$x=0,y=-\frac{0+15}{5}=-3,\qquad x=-5,y=-\frac{-10+15}{5}=-1$$

Using $(-5,-1)$: $-1=k\log_6(1)+c\rightarrow c=-1$

Using $(0,-3)$: $-3=k\log_6(6)-1\rightarrow k=-2$

Specific behaviours

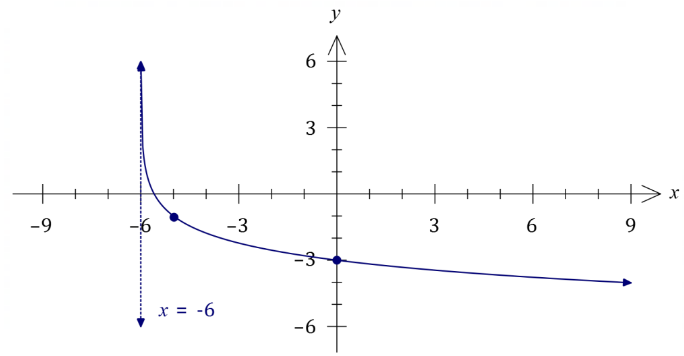
✓ calculates two points on curve

✓ value of c

✓ value of k

If no marks can be awarded, give one mark if they state $k+c=-3$

(b) Sketch the graph of $y=f(x)$ on the axes below. (3 marks)



Solution

See graph

Specific behaviours

✓ through two points from (b) , No FT, must be those 2 points, and will be awarded this mark even if a graph is not drawn.

✓ asymptote, correct curvature nearby

✓ smooth curve, concave up throughout

No penalty for missing arrows on asymptote or graph