

Note: For both Section 1 and section 2, working out must be shown for full marks to be awarded.

Section 1: [ / 17 marks] Section 2: [ / 31 marks] Total: [ / 48 marks] = %

Section 1: Calculator and Resource Free Time: 20 minutes

1. [3,2 = 5 marks]

Differentiate the following with respect to x.

a)  $f(x) = \frac{-x}{x^2+1}$  { Express numerator in simplest form }

$$f'(x) = \frac{-(x^2+1) - (-x)(2x)}{(x^2+1)^2} = \frac{-x^2-1+2x^2}{(x^2+1)^2} = \frac{x^2-1}{(x^2+1)^2}$$

✓

b)  $y = (1-x)^3 \left(1 + \frac{x}{2}\right)^2$  { Apply the product rule but do not simplify }

$$\frac{dy}{dx} = \left[ 3(1-x)^2(-1) \left(1 + \frac{x}{2}\right)^2 + \left[ 2 \left(1 + \frac{x}{2}\right) \left(-\frac{1}{2}\right) \right] (1-x)^3 \right]$$

$$= -3(1-x)^2 \left(1 + \frac{x}{2}\right)^2 - \frac{4}{2} \left(1 + \frac{x}{2}\right) (1-x)^3$$

2. [2,2= 4 marks ]

A particle moves in a straight line such that its velocity,  $v$  m/s, depends upon displacement,  $x$  m, from some fixed point O according to the rule  $v = 5x - 4$

a) Find an expression in terms of  $x$  for the acceleration of the particle.

$$a = \frac{dv}{dt}, \quad v = \frac{dx}{dt}, \quad \frac{dv}{dx} = 5$$

$$\begin{aligned} \frac{dv}{dt} &= \frac{dv}{dx} \times \frac{dx}{dt} \\ &= 5(5x - 4) \\ &= \underline{25x - 20} \end{aligned}$$

✓ only  
if  $5\text{ms}^{-2}$

b) Determine the displacement and the acceleration of the particle when  $v = 6\text{m/s}$ .

$$5x - 4 = 6$$

$$x = 2$$

When  $x = 2$  ✓

EL

2m.

$$a = 25x - 20$$

$$= 25(2) - 20$$

$$\underline{a = 30\text{ m/s}^2} \quad \checkmark$$

$$a = 5\text{ms}^{-2} \checkmark$$

3. [8 marks]  
 The equation of the tangent to the curve  $y = ax^3 - bx^2 + 2$  where  $x = -1$  is  $y = 18x + c$ .  
 The curve has a point of inflection when  $x = 1$ .  
 Find the values of  $a$ ,  $b$  and  $c$ .  
 [Note: Working out must be shown]

$$y = ax^3 - bx^2 + 2$$

$$\frac{dy}{dx} = 3ax^2 - 2bx \quad \checkmark$$

$$\text{when } x = -1, \frac{dy}{dx} = 18$$

$$3a(-1)^2 - 2b(-1) = 18$$

$$3a + 2b = 18 \quad \text{--- (1) } \checkmark$$

Point of inflection when  $\frac{d^2y}{dx^2} = 0$ .

$$\frac{d^2y}{dx^2} = 6ax - 2b$$

$$\text{when } x = 1, \frac{d^2y}{dx^2} = 0$$

$$6a(-1) - 2b = 0$$

$$\text{--- (2) } \checkmark$$

Solve simultaneously

$$\text{--- (1) } \checkmark$$

$$\text{--- (2) } \checkmark$$

$$\text{--- (1) + (2) } \checkmark$$

$$a = 2$$

$$6a - 2b = 0$$

$$12 - 2b = 0$$

$$b = 6 \quad \checkmark$$

$$\therefore a = 2$$

$$b = 6$$

$$c = 12$$

$$\text{when } x = -1, y = -6$$

$$y = 18x + c$$

$$-6 = -18 + c$$

$$c = 12 \quad \checkmark$$

$$\left\{ \begin{array}{l} y = 2x^3 - 6x^2 + 2 \\ y = 2(-1)^3 - 6(-1)^2 + 2 \\ y = -2 - 6 + 2 \\ y = -6 \end{array} \right. \quad \checkmark$$

Name : \_\_\_\_\_

Marks:  $\frac{\quad}{31}$ **Section 2 : Calculator and Resource Assumed.****Time Allowed : 35 minutes****Note : Show working for full marks to be awarded.**

1. [ 2,2= 4 marks ]

A company produces  $n$  items of a certain product.The cost function  $\$C$  is given by  $C(n) = 1200 + 5n^{1/3}$ Each item sells for  $\$52$ .

Find

a) An expression for the marginal profit  $P'(n)$ 

$$C(n) = 1200 + 5n^{1/3}$$

$$R(n) = 52n$$

$$P(n) = 52n - 1200 - 5n^{1/3} \quad \checkmark$$

$$P'(n) = 52 - \frac{5}{3}n^{-2/3}$$

$$P'(n) = 52 - \frac{5}{3n^{2/3}} \quad \checkmark$$

b) A value for  $P'(64)$  and comment on its meaning.

$$P'(64) = 52 - \frac{5}{3(64^{2/3})}$$

$$= 51.90 \quad \checkmark$$

Profit of  $65^{\text{th}}$  unit is  $\$51.90$ .  $\checkmark$ 

must have both  
to get, mark.

c) Find the depth of the drinking trough to the nearest mm, if the amount of stainless steel is to be kept to a minimum. Justify your answer by using Calculus techniques.

$$A = \frac{60}{h} - h + 2h\sqrt{2} + 120$$

$$A = 60h^{-1} - h + 2h\sqrt{2} + 120$$

$$\frac{dA}{dh} = -60h^{-2} - 1 + 2\sqrt{2} \quad \checkmark$$

$$\text{For minimum value } \frac{dA}{dh} = 0$$

$$-60h^{-2} - 1 + 2\sqrt{2} = 0 \quad \checkmark$$

$$h = \frac{5.728445657}{\quad} \approx 5.728 \text{ m (3dp)}$$

$$\frac{d^2A}{dh^2} = 120h^{-3} = \frac{120}{h^3}$$

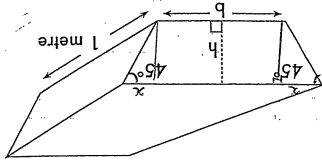
$$\frac{d^2A}{dh^2} \Big|_{h=5.728} > 0 \quad \therefore \text{Rel. Min}$$

$$\therefore \text{Depth} = 5.728 \text{ m}$$

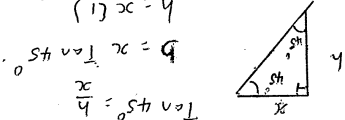
End of Test

5. [3,2,3 = 8 marks]

An animal drinking trough is constructed from stainless steel in the shape of a trapezoidal prism, with height 'h' metres and length of 1 metre.  
The cross section of the prism is an isosceles trapezium with acute angle of  $45^\circ$ , base 'b' metres and area of  $60 \text{ m}^2$ .



a) Show that  $b = \frac{60}{h} - h$



$$\tan 45^\circ = \frac{h}{x} \Rightarrow x = h$$

$$b = x + x = 2x$$

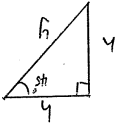
$$h = x \quad \checkmark$$

$$h = x \text{ (isos } \Delta)$$

b) Show that the surface area 'A' in  $\text{m}^2$  is:  $A = \frac{60}{h} - h + 2h\sqrt{2} + 120$

$$\begin{aligned} \text{Area} &= \frac{1}{2}(b+b+2h) \cdot x \cdot h = 60 \\ \frac{2b+2h}{2} \cdot h &= 60 \\ b+h &= \frac{60}{h} \\ b &= \frac{60}{h} - h \end{aligned}$$

$$\begin{aligned} \text{Area} &= A = 2 \times 60 + b(1) + 2\sqrt{2}h \times 1 \\ A &= 120 + b + 2\sqrt{2}h \\ b &= \frac{60}{h} - h \\ A &= 120 + \frac{60}{h} - h + 2\sqrt{2}h \end{aligned}$$



$$\begin{aligned} y^2 &= h^2 + h^2 \\ y &= \sqrt{2}h \end{aligned}$$

2. [2,4,4 = 10 marks]

a) It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth 'y' of fluid in the tank 't' hours after the valve is opened is given by  $y = 6 \left(1 - \frac{t}{12}\right)^2$  metres.  
Show, with full working out, the rate  $\frac{dy}{dt}$  m/hour at which the tank is draining at time t is  $\frac{t}{12} - 1$

$$\begin{aligned} \frac{dy}{dt} &= 12 \left(1 - \frac{t}{12}\right) \left(-\frac{1}{12}\right) = -\left(1 - \frac{t}{12}\right) \\ &= \frac{t}{12} - 1 \end{aligned}$$

iii) a) When is the fluid in the tank falling fastest and slowest?  
Slowest: when  $t = 12 \text{ h}$   
Fastest: when  $t = 0 \text{ h}$

b) What are the values of  $\frac{dy}{dt}$  at these times?

$$\begin{aligned} \text{Slowest: } \frac{dy}{dt} &= 0 \\ \text{Fastest: } \frac{dy}{dt} &= -1 \end{aligned}$$

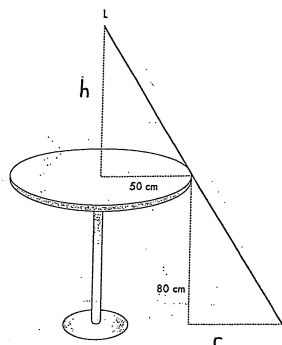
$$\frac{dy}{dt} = -1 \text{ m/h}$$

$$\text{Fastest: } \frac{dy}{dt} = -1 \text{ m/h}$$

b) If the volume of a cylinder is given by  $V = 2\pi r^3$ , find the approximate percentage change in V when r changes by  $\frac{1}{2}\%$ .

$$\begin{aligned} \frac{\delta V}{V} &\approx 6\pi r^3 \times \delta r \\ \frac{\delta V}{V} &\approx \frac{6\pi r^3 \delta r}{2\pi r^3} \\ \frac{\delta V}{V} &\approx 3\delta r \\ \frac{\delta V}{V} &\approx 3 \times 0.005 \\ &\approx 0.015 \\ &= 1.5\% \end{aligned}$$

3. [ 1,3 = 4 marks ]



A table has a radius of 50 cm and a height of 80 cm.

A light ( L ) is lowered vertically downwards from a point above the centre of the table at a constant rate of 0.2 cm per second.

When the light is h cm above the table it casts a shadow that extends r cm from the edge of the table.

a) Show that  $r = \frac{4000}{h}$

As triangles are similar, corresponding sides are in proportion.

$$\frac{r}{50} = \frac{80}{h} \quad \checkmark$$

$$\therefore r = \frac{4000}{h}$$

b) Find the rate at which r is changing when h = 60

$$\frac{dr}{dt} = \frac{dr}{dh} \cdot \frac{dh}{dt}$$

$$= \frac{-4000}{h^2} \times \frac{dh}{dt} \quad \checkmark$$

$$\frac{dh}{dt} = -0.2 \quad \checkmark$$

When h = 60

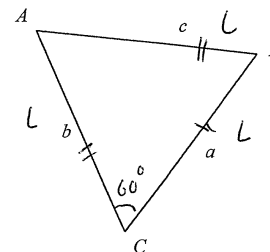
$$\frac{dr}{dt} = \frac{-4000}{60^2} \times (-0.2)$$

$$= \frac{2}{9} \text{ or } 0.2 \text{ cm/sec} \quad \checkmark$$

$\therefore$  Radius is increasing at  $\frac{2}{9} \text{ cm/sec}$

4. [ 5 marks ]

The area of a triangle can be found by the formula :  $\text{Area} = \frac{ab \sin C}{2}$



Using the **incremental formula**, determine the approximate change in area ( to 3 decimal places ) of an **equilateral triangle** with each side of 10 cm, when each side increases by 0.1 cm.

[ Hint : Use exact value for  $60^\circ$  ]

$\triangle ABC$  is an equilateral  $\triangle$

Let  $a = b = c = L \text{ cm}$

$$\angle C = 60^\circ = \frac{\sqrt{3}}{2}$$

$$\text{Area} = \frac{1}{2} ab \sin \theta$$

$$= \frac{1}{2} (L)(L) \sin \frac{\pi}{3}$$

$$= \frac{1}{2} L^2 \cdot \frac{\sqrt{3}}{2} \quad \checkmark$$

$$A = \frac{\sqrt{3}}{4} L^2$$

$$\frac{dA}{dL} = \frac{2L \cdot \sqrt{3}}{4} \quad \checkmark$$

$$L = 10 \text{ cm}$$

$$\delta L = 0.1 \text{ cm}$$

$$\frac{\delta A}{\delta L} \approx \frac{dA}{dL} \approx \frac{2L \cdot \sqrt{3}}{4}$$

$$\delta A \approx 2L \cdot \frac{\sqrt{3}}{4} \cdot \delta L \quad \checkmark$$

$$\delta A \approx 2(10) \cdot \frac{\sqrt{3}}{4} \cdot (0.1) \quad \checkmark$$

$$\delta A \approx 0.8660254038$$

$$\delta A \approx 0.866 \text{ sq cm (3dp)}$$

$\therefore$  Approximate Change in area of 0.866 sq cm.