Perth Modern School

Semester One Examination, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 1

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	49	35
Section Two: Calculator-assumed	13	13	100	102	65
			Total	151	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

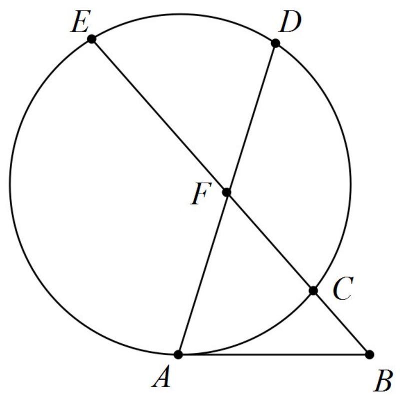
Section Two: Calculator-assumed **65% (102 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(5 marks)**

In the diagram below, AB is a tangent to the circle and $AB = BF = FD$.



If $BC = 4$ and $CE = 32$, determine the lengths of

(a) AB. (2 marks)

Solution
$AB^2 = BC \times BE$ $= 4 \times (4 + 32)$ $AB = 12$
Specific behaviours
✓ uses secant-tangent relationship   ✓ calculates length

(b) CF. (1 mark)

Solution
$CF = BF - BC = 12 - 4 = 8$
Specific behaviours
✓ calculates length

(c) AF. (2 marks)

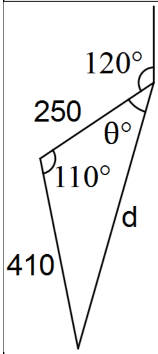
Solution
$AF \times FD = CF \times EF$ $AF \times 12 = 8 \times (32 - 8) \Rightarrow AF = 16$
Specific behaviours
✓ uses product of chord intervals   ✓ calculates length

Question 9 **(7 marks)**

Two forces act on body. The first has magnitude 250 N and acts in direction 240° and the second has magnitude 410 N and acts in direction 170°.

(a) Determine the resultant of the two forces. (4 marks)

Solution



$$d = \sqrt{250^2 + 410^2 - 2(250)(410)\cos 110} = 548.37$$

$$\frac{410}{\sin\theta} = \frac{548.37}{\sin 110} \Rightarrow \theta = 44.6^\circ$$

$$360 - 120 - 44.6 = 195.4^\circ$$

Resultant has magnitude 548.4 N in direction 195.4°.

Specific behaviours

✓ diagram
✓ uses cosine rule to determine magnitude
✓ uses sine rule to determine angle
✓ states resultant as magnitude and direction

(b) The work done, in joules, by a force in moving a body is the scalar product of the force, in newtons, and the displacement, in metres. Determine the total work done by the two forces, to the nearest 100 joules, if the body moves 45 metres in direction 215°. (3 marks)

Solution

$$\theta = 215 - 195.4 = 19.6$$

$$\begin{aligned} W &= 548.37 \times 45 \times \cos 19.6 \\ &= 23246.8 \\ &\approx 23\ 200 \text{ joules} \end{aligned}$$

Specific behaviours

✓ determines angle between directions
✓ uses scalar product
✓ rounds correctly

Question 10 **(8 marks)**

(a) Use a counterexample to demonstrate that each of following statements are false.

(i) $ab = ac,\ \{a, b, c \in \mathbb{R}\} \Rightarrow b = c$. (2 marks)

Solution
If $a = 0,\ b = 1,\ c = 2$, then $ab = ac$ but $b \neq c$ and so statement is false.
Specific behaviours
✓ supplies values for counterexample ✓ uses values to show statement is false

(ii) If $f(n) = n^2 + n + 1$ then $f(n)$ is always prime. (2 marks)

Solution
$f(5) = 5^2 + 5 + 5$ $= 5(5 + 1 + 1)$ Since f(5) is clearly not prime, then the statement is false
Specific behaviours
✓ supplies value of *n* so that $f(n)$ is not prime ✓ shows why supplied value is not prime

(b) The statement 'if a natural number is a multiple of 4 and 5 then the natural number is a multiple of 20' is true.

(i) Write the contrapositive of the statement and explain whether or not the contrapositive is also true. (2 marks)

Solution
If a natural number is not a multiple of 20 then it is not a multiple of 4 and 5. This is true, as contrapositive always true if original statement true.
Specific behaviours
✓ writes contrapositive ✓ states with reason that it is true

(ii) Write the converse of the statement and explain whether or not the converse is also true. (2 marks)

Solution
If a natural number is a multiple of 20 then it is a multiple of 4 and 5. This is true, as if a number has a factor of 20 then it will also have factors of 4 and 5.
Specific behaviours
✓ writes converse ✓ explains converse is true

Question 11 **(8 marks)**

Fifteen children at a summer camp are to be divided into two groups of nine and six.

(a) Determine the number of different groupings. (2 marks)

Solution
${}^{15}C_9 = 5005$ groupings
Specific behaviours
✓ uses combinations ✓ calculates correct number

(b) Determine how many groupings are possible if the two youngest children must be in the same group. (3 marks)

Solution
Go into larger group: ${}^{2}C_2 \times {}^{13}C_7 = 1716$ Go into smaller group: ${}^{2}C_2 \times {}^{13}C_4 = 715$ $1716 + 715 = 2431$
Specific behaviours
✓ calculates ways using larger group ✓ calculates ways using smaller group ✓ calculates total number of groupings

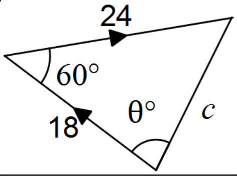
(c) If ten of the fifteen were girls, in how many of the different groupings do both groups contain more girls than boys? (3 marks)

Solution
Large group must have 6G3B or 5G4B (so that other has 4G2B or 5G1B). ${}^{10}C_6 \times {}^{5}C_3 + {}^{10}C_5 \times {}^{5}C_4 = 2100 + 1260$ $= 3360$
Specific behaviours
✓ determines required groupings ✓ calculates first grouping ✓ calculates second grouping and total

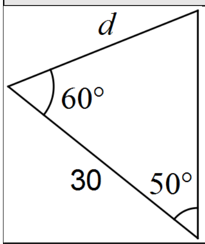
Question 12 **(7 marks)**

Vector $\mathbf{a}$ has magnitude 6 units and acts on a bearing of 310°. Vector $\mathbf{b}$ has magnitude 12 units and acts on a bearing of 070°.

(a) Determine the magnitude and direction of $3\mathbf{a} + 2\mathbf{b}$. (4 marks)

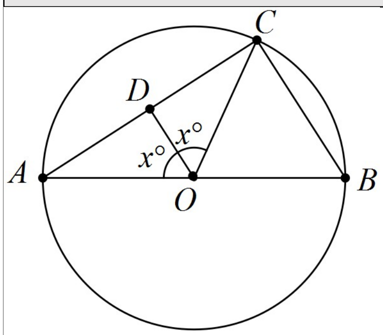
Solution

$c^2 = 18^2 + 24^2 - 2(18)(24)\cos 60^\circ$
$c = 21.63 \Rightarrow$ magnitude is 21.6 units
$\dfrac{24}{\sin\theta} = \dfrac{21.63}{\sin 60} \Rightarrow \theta = 73.9^\circ$
$73.9 - 50 = 23.9^\circ \Rightarrow$ bearing is 024
Specific behaviours
✓ sketches diagram to show addition of multiples of vectors
✓ uses cosine rule to determine magnitude
✓ uses sine rule to determine angle
✓ states direction as bearing

(b) Determine the value of the constant k if the direction of $5\mathbf{a} + k\mathbf{b}$ is due north. (3 marks)

Solution

$\dfrac{d}{\sin 50} = \dfrac{30}{\sin 70}$
$d = 24.456$
$k = 24.456 \div 12 \approx 2.04$
Specific behaviours
✓ sketches diagram
✓ determines required magnitude
✓ determines k

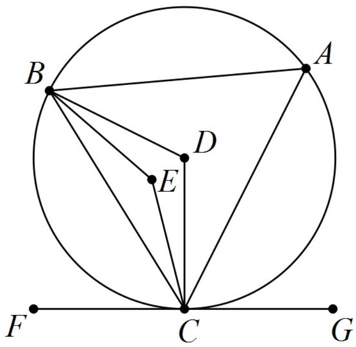
Question 13 **(9 marks)**

(a) AB is a diameter of a circle centre O. C is a point on the circumference. D is a point on AC such that OD bisects $\angle AOC$. Prove that OD is parallel to BC. (4 marks)

Solution
 $\angle ABC = x$ (angle at circumference half that at centre $\angle AOC$) $\angle BCO = x$ (isoceles triangle, $OB = OC$ = radius) $\angle DOC = x = \angle BCO \Rightarrow OD \parallel BC$ (alternate angles are equal)
Specific behaviours
✓ shows information using diagram ✓ determines angle ABC ✓ determines angle BCO ✓ uses alternate angles to explain parallel

Also $\angle$AOD = 90 (mid chord)
$\angle$ACB = 90 Angle in semi Circle
Therefore OD ll BC Cointerior angles supplementary

(b) In the diagram below, FCG is a tangent to the circle ABC. BD bisects $\angle ABC$, CD bisects $\angle ACB$, BE bisects $\angle DBC$ and CE bisects $\angle DCB$. If $AB = AC$ and $\angle ACG = 48^\circ$, determine the ratio, in simplest form, of $\angle BAC : \angle BDC : \angle BEC$. (5 marks)



Solution
ABC is isosceles and angle bisectors will intersect on line of symmetry $\Rightarrow$ $DB = DC$ and $EB = EC$.
$\angle ABC = \angle ACG = 48$ (angle in opp segement)
$\angle DBC = 48 \div 2 = 24$ (bisector)
$\angle EBC = 24 \div 2 = 12$ (bisector)
$\angle BAC = 180 - (2 \times 48) = 84^\circ$ (isosceles)
$\angle BDC = 180 - (2 \times 24) = 132^\circ$ (isosceles)
$\angle BEC = 180 - (2 \times 12) = 156^\circ$ (isosceles)
$\angle BAC : \angle BDC : \angle BEC = 84 : 132 : 156 = 7 : 11 : 13$
Specific behaviours
✓ explains *ABC* isosceles and results ✓ uses angle in opposite segment ✓ determines *DBC* and *EBC* ✓ determines *BAC*, *BDC* and *BEC* ✓ determines simplified ratio

Question 14 **(9 marks)**

Points O, P, Q and R have position vectors $(0,0)$, $(15,y)$, $(x,-1)$ and $(2,-5)$.

(a) Determine the value of y if $|OP| = 17$. (2 marks)

Solution
$15^2 + y^2 = 17^2 \Rightarrow y^2 = 64 \Rightarrow y = \pm 8$
Specific behaviours
✓ writes magnitude equation ✓ states both +ve and -ve solutions

(b) Determine the value of x if $\overrightarrow{OQ} \perp \overrightarrow{QR}$. (3 marks)

Solution
QR = <(2-x),(-5 - -1)> = <(2-x),(-4)> <x,-1> •<(2-x),(-4)> = 0 2x − x^2 + 4 = 0 x = -1.236, 3.236
Specific behaviours
✓ determines $\overrightarrow{QR}$ ✓ solves scalar product equal to zero ✓ states both values of x

(c) Determine the values of x and y if R lies on the line between P and Q such that $PR : RQ = 1 : 2$. (4 marks)

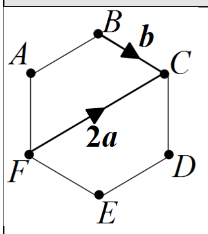
Solution
$\mathbf{OR} = \mathbf{OP} + \frac{1}{3}\mathbf{PQ}$ $= \mathbf{OP} + \frac{1}{3}(\mathbf{OQ} - \mathbf{OP})$ $\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 15 \\ y \end{bmatrix} + \frac{1}{3}\left(\begin{bmatrix} x \\ -1 \end{bmatrix} - \begin{bmatrix} 15 \\ y \end{bmatrix}\right)$ $2 = 15 + \frac{1}{3}(x - 15) \Rightarrow x = -24$ $-5 = y + \frac{1}{3}(-1 - y) \Rightarrow y = -7$
Specific behaviours
✓ writes vector equation ✓ substitutes position vectors ✓ solves for x using **i**-coefficients ✓ solves for y using **j**-coefficients

Question 15 **(9 marks)**

(a) $ABCDEF$ is a regular hexagon in which $\overrightarrow{BC}$ represents $\mathbf{b}$ and $\overrightarrow{FC}$ represents $2\mathbf{a}$.

Express the vectors $\overrightarrow{CD}$, $\overrightarrow{EA}$ and $\overrightarrow{BE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. (3 marks)

Solution



$\overrightarrow{CD} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{EA} = \mathbf{a} - 2\mathbf{b}$ and $\overrightarrow{BE} = 2\mathbf{b} - 2\mathbf{a}$.

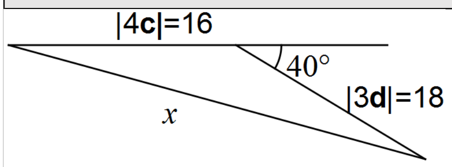
Specific behaviours

✓ determines $\overrightarrow{CD}$
✓ determines $\overrightarrow{EA}$
✓ determines $\overrightarrow{BE}$

(b) Given $\mathbf{c}$ and $\mathbf{d}$ are vectors such that $|\mathbf{c}| = 4$, $|\mathbf{d}| = 6$ and the angle between their directions is 40°, determine

(i) $|4\mathbf{c} + 3\mathbf{d}|$. (3 marks)

Solution



$x^2 = 16^2 + 18^2 - 2(16)(18)\cos 140$

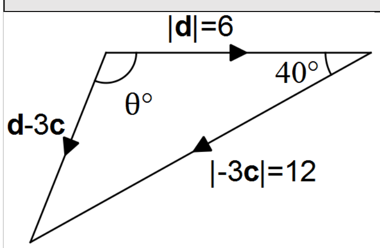
$x = 31.96$ units

Specific behaviours

✓ diagram to show vectors
✓ uses cosine rule
✓ determines magnitude

(ii) the angle between $\mathbf{d}$ and $\mathbf{d} - 3\mathbf{c}$. (3 marks)

Solution



$l = \sqrt{6^2 + 12^2 - 2(6)(12)\cos 40} = 8.34803$

$\dfrac{12}{\sin\theta} = \dfrac{8.34803}{\sin 40} \Rightarrow \theta \approx 112.5^\circ$

Specific behaviours

✓ diagram to show vectors
✓ determines magnitude
✓ determines angle

Question 16 **(10 marks)**

(a) If a room contains 75 adults, use the pigeonhole principle to explain why a group containing at least 11 of these people could be chosen so that all were born on the same day of the week. (3 marks)

Solution
There are 7 days of week (pigeonholes) in which to place 75 people (pigeons). If *n* pigeons are assigned to *m* pigeonholes, then one of the pigeonholes must contain at least $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons. $\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{75-1}{7} \right\rfloor + 1 = 10 + 1 = 11$. So at least 11 share same day of week.
Specific behaviours
✓ describes weekdays as pigeonholes ✓ describes adults as pigeons ✓ explains why at least 11 pigeons in one pigeonhole.

(b) Twelve senior students each brought at least one toy to school to donate to the pre-primary centre. Given that a total of 75 toys were donated, prove that at least two of the senior students brought in the same number of toys.
Hint: Think of a contradiction. (3 marks)

Solution
Contradiction: Assume they all brought a different number of toys. Smallest possible number of toys donated would then be $1 + 2 + \ldots + 12 = 78$ toys. 78 is more than 75, which contradicts assumption and so at least two must have brought the same number of toys.
Specific behaviours
✓ states assumption that contradicts ✓ calculates smallest number of toys ✓ shows contradiction

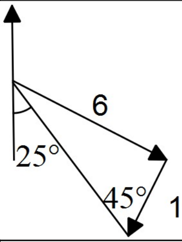
(c) Determine how many numbers between 1 and 150 inclusive are multiples of 3, 4 or 5. (4 marks)

Solution
Multiples of 3: 50 Multiples of 4: 37 Multiples of 5: 30 Multiples of 3 & 4: 12 Multiples of 3 & 5: 10 Multiples of 4 & 5: 7 Multiples of 3 & 4 & 5: 2 $n = 50 + 37 + 30 - 12 - 10 - 7 + 2 = 90$
Specific behaviours
✓ calculates individual multiples ✓ calculates paired multiples ✓ calculates triple multiples ✓ calculates total

Question 17 **(8 marks)**

A boat with a constant speed of 6 ms^{-1} is required to leave its mooring and motor directly to a jetty located 1 045 metres away on a bearing of 155°. A current of 1.5 ms^{-1} is running on a bearing of 200°.

(a) Sketch a diagram to that can be used to determine the direction the boat should steer. (2 marks)

Solution

Specific behaviours
✓ shows directions ✓ shows magnitudes

(b) Determine the bearing that the boat should steer. (3 marks)

Solution
$\frac{6}{\sin 45} = \frac{1.5}{\sin\theta} \Rightarrow \theta = 10.2^\circ$ Bearing: $180 - 25 - 10.2 \approx 145^\circ$
Specific behaviours
✓ uses sine rue ✓ determines angle ✓determines bearing

(c) Determine the time taken for the boat to reach the jetty. (3 marks)

Solution
$s^2 = 6^2 + 1.5^2 - 2 \times 6 \times 1.5 \times \cos(180 - 45 - 10.2) \Rightarrow s = 6.966$ $t = 1045 \div 6.966 \approx 150$ seconds
Specific behaviours
✓ uses cosine rule ✓ determines speed over ground ✓ determines time

Question 18 **(7 marks)**

(a) Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j}$.

Prove that the scalar product is distributive over vector addition: $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$.

(4 marks)

Solution
$$\begin{aligned} LHS &= \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) \\ &= (a_1, a_2)\cdot(b_1 + c_1, b_2 + c_2) \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 \\ &= a_1b_1 + a_2b_2 + a_1c_1 + a_2c_2 \\ &= (a_1, a_2)\cdot(b_1, b_2) + (a_1, a_2)(c_1, c_2) \\ &= \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} \\ &= RHS \end{aligned}$$
Specific behaviours
✓ sums components of **b** and **c** ✓ uses scalar product ✓ expands result ✓ re-writes as scalar product using components

(b) Given $\mathbf{p} = 1.5\mathbf{i} + 4.5\mathbf{j}$, $\mathbf{q} = 3.5\mathbf{i} - 1.5\mathbf{j}$, $\mathbf{r} = -3.5\mathbf{i} + 5.5\mathbf{j}$ and $\mathbf{s} = -1.5\mathbf{i} + 1.5\mathbf{j}$, determine $\mathbf{p}\cdot\mathbf{r} - \mathbf{p}\cdot\mathbf{s} + \mathbf{q}\cdot\mathbf{r} - \mathbf{q}\cdot\mathbf{s}$.

(3 marks)

Solution
$$\begin{aligned} \mathbf{p}\cdot\mathbf{r} - \mathbf{p}\cdot\mathbf{s} + \mathbf{q}\cdot\mathbf{r} - \mathbf{q}\cdot\mathbf{s} &= \mathbf{p}\cdot(\mathbf{r}-\mathbf{s}) + \mathbf{q}\cdot(\mathbf{r}-\mathbf{s}) \\ &= (\mathbf{p}+\mathbf{q})\cdot(\mathbf{r}-\mathbf{s}) \\ &= \begin{bmatrix} 5 \\ 3 \end{bmatrix}\cdot\begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ &= 2 \end{aligned}$$
Specific behaviours
✓ factorises expression ✓ substitutes and simplifies ✓ calculates scalar product

Question 19 **(8 marks)**

(a) Determine the number of ways that ten different coloured cubes can be placed in a line. (1 mark)

Solution
$10! = 3628800$
Specific behaviours
✓ determines number of ways

(b) Determine the number of ways that six different coloured cubes can be placed in a line if the blue, red and green cubes must be together. (2 marks)

Solution
$4! \times 3! = 144$
Specific behaviours
✓ groups blue, red and green as one item and arranges within ✓ arranges all four items

(c) Eight identical cubes, coloured red, red, red, blue, blue, green, orange and yellow, are arranged in a line.

(i) Determine the number of arrangements in which the red cubes are all adjacent. (2 marks)

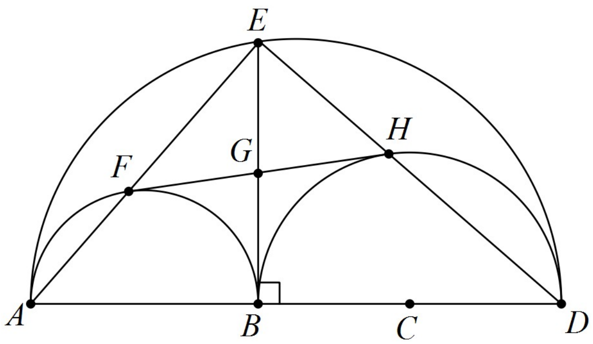
Solution
Treat 3 reds as one item, leaving 6 items, 2 alike: $\frac{6!}{2!} = 360$
Specific behaviours
✓ treats reds as one item ✓ arranges remaining items correctly

(ii) Determine the number of arrangements in which no two red cubes are adjacent. (3 marks)

Solution
Remove all reds and arrange remaining cubes, 2 alike: $\frac{5!}{2!} = 60$ With five cubes, there are six spaces to insert a red cube, so choose any three of these six: ${}^6C_3 = 20$. Since choices for position and arrangements of cubes independent, then total number of ways is: $60 \times 20 = 1200$
Specific behaviours
✓ arranges remaining 5 cubes ✓ calculates number of spaces ✓ calculates total ways

Question 20 **(7 marks)**

The diagram shows three semicircles with diameters AD, AB and BD, where B is a point on the diameter AD. Point C is the centre of the semicircle with diameter BD. Line BE is perpendicular to diameter AD and meets the largest semicircle at E. Points F and H are the intersections of lines AE and DE with the smaller semicircles. Point G is the intersection of lines FH and BE.



(a) Explain why $BFEH$ is a rectangle. (2 marks)

Solution
$\angle AED = 90^\circ$ (Angle in semicircle)
$\angle AFB = 90^\circ \Rightarrow \angle EFB = 90^\circ$ (Angle in semicircle and supplementary angle)
Similarly, $\angle DHB = 90^\circ \Rightarrow \angle EHB = 90^\circ$
Hence $BFEH$ is rectangle as all angles are 90°
Specific behaviours
✓ uses angle in semicircle ✓ explains three angles are right, and so is a rectangle

(b) Prove that $\triangle CBG$ and $\triangle CHG$ are congruent. (3 marks)

Solution
$CB = CH$ (radii in same circle)
CG is common to both triangles
$BG = HG$ ($BFEH$ is rectangle and diagonals bisect each other)
Hence $\triangle CBG \equiv \triangle CHG$ (SSS)
Specific behaviours
✓ uses radii ✓ uses diagonals bisecting ✓ uses common side

(c) Deduce that line FH is a tangent to the semicircle with diameter BD. (2 marks)

Solution
$\angle CHG = \angle CBG = 90^\circ$ (corresponding angles in congruent triangles)
Hence $FH \perp CH \Rightarrow FH$ is a tangent
Specific behaviours
✓ uses corresponding angles ✓ deduces must be tangent

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________