Additional working space

Question number: ________

PERTH MODERN SCHOOL
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INDEPENDENT PUBLIC SCHOOL

Semester Two Examination, 2023

Question/Answer booklet

12 SPECIALIST MATHEMATICS
UNIT 3

Section Two:
Calculator-assumed

Your Name ______________________________

Your Teacher's Name ________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Marks	Max	Question	Marks	Max
7		4	16		7
8		10	17		6
9		11	18		8
10		5	19		9
11		5			
12		13			
13		6			
14		6			
15		7			

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	6	6	50	48	33
Section Two: Calculator-assumed	13	13	100	97	67
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Additional working space

Question number: ________

Working out space

Section Two: Calculator-assumed **(97 Marks)**

This section has **13** questions. Answer **all** questions. Write your answers in the spaces provided.

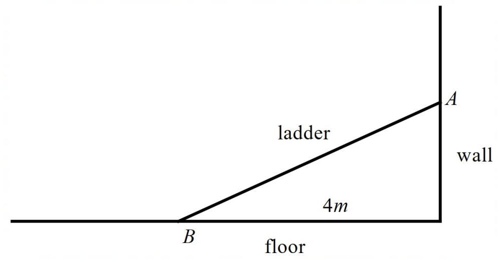
Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 100 minutes.

Question 7 **(4 marks)**

Consider a ladder placed with one end, point A, on a wall and the other, point B, on the floor as shown below. The ladder has a length of 5 metres and point B is moving towards the wall at a speed of 3 metres per minute. When point B is 4 metres from the base of the wall, determine the speed of point A which is moving up the wall.



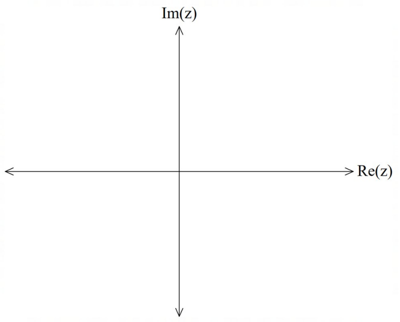
Question 8 **(10 marks)**

a) Consider the locus $|z - 3 + 4i| = 2$ in the complex plane.
Determine the following:

i) Minimum Arg(z). (3 marks)

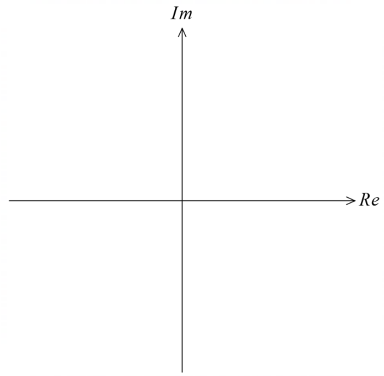
ii) Maximum $|z|$. (2 marks)

b) Sketch the following locus $|z - 5 - 12i| = 13$ on the axes below. The Arguments in this locus lie between the following $b < Arg(z) < c$. Determine the values of $b \,\&\, c$.
(5 marks)



(c) The locus of points that satisfy $arg\left(\frac{z-2-i}{z-2-7i}\right) = \frac{\pi}{3}$ is an arc of a circle.

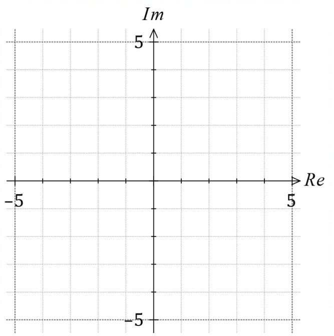
(i) Sketch the locus of z in the complex plane. (2 marks)



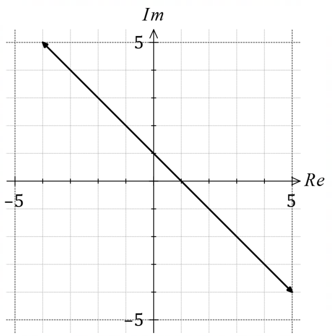
(ii) Determine, with justification, the exact location of the centre of the circle. (2 marks)

Question 19 **(9 marks)**

(a) Plot the complex number that satisfies the conditions $arg(z)=\frac{3\pi}{4}$ and $arg(z+3)=\frac{\pi}{2}$ on the Argand diagram below. (2 marks)



(b) Let $z_1=4+2i$ and z_2 be another complex number. The locus of a complex number z satisfies the condition $|z-z_1|=¿z-z_2\vee¿$ and is shown in the diagram below.

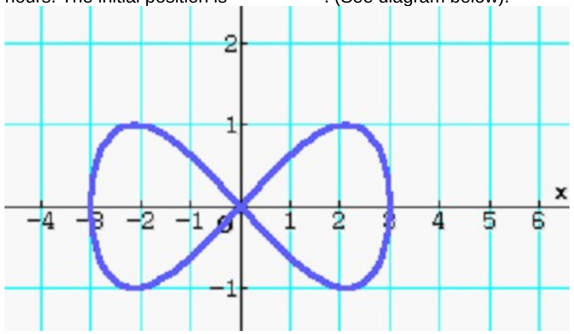


(i) Determine the complex number z_2. (2 marks)

(ii) On the same diagram, indicate the locus of a complex number z that satisfies the condition $|z-z_1|\leq\vee z-z_2\vee¿$. (1 mark)

Question 9 **(11 marks)**

Consider a racing car that travels in a racecourse with velocity $v=\begin{pmatrix}-3\sin t\\ 2\cos 2t\end{pmatrix} km/hr$ at time t hours. The initial position is $r=\begin{pmatrix}3\\0\end{pmatrix}km$. (See diagram below).



a) Determine the acceleration at $t=\pi$ hours. (2 marks)

b) Determine $\int_0^{\frac{3\pi}{2}} v\,dt$. (3 marks)

Q9 continued on next page

c) Determine the length of one track of the racecourse. (3 marks)

d) Determine the cartesian equation of the path of the race car. (3 marks)

Question 10 **(5 marks)**

Consider a particle that undergoes motion defined by $\ddot{x} = -16x$ with x,metres being the displacement at time, t seconds. The velocity is zero when $x = 9$ metres.
Determine the percentage of time that the particle has a speed less than half of its maximum speed.

Question 18 **(8 marks)**

A machine fills bags with sugar. The mean and standard deviation of the weight of sugar it delivers into a bag is 505 and 17 grams respectively. An inspector routinely takes a random sample of 76 bags filled by the machine.

(a) For repeated random sampling of 76 bags of sugar filled by this machine, state the approximate distribution of the sample mean that the inspector should expect. (3 marks)

(b) Determine the probability that the mean weight of a random sample of 76 bags of sugar is at least 502 grams, given that the sample mean is less than 505 grams. (2 marks)

(c) Occasionally, the inspector only has enough time to take a random sample of 50 bags. In the long run, 80% of sample means derived from samples with this smaller size will lie in the range $505 \pm k$ grams. Determine the value of k. (3 marks)

Question 17 **(6 marks)**

Consider the function $f(x)=\frac{ax^2-2ax-b}{x^2-c}$, where a, b and c are positive constants.

The graph of $y=f(x)$ cuts the x-axis at $x=-3$, has a horizontal asymptote with equation $y=2$ and has a vertical asymptote with equation $x=-2$.

(a) Determine $f(0)$. (3 marks)

(b) Now consider the graph of $y=\frac{1}{f(x)}$. State the

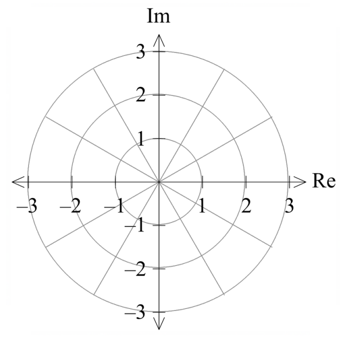
(i) equation of its horizontal asymptote. (1 mark)

(ii) x-axis intercepts. (1 mark)

(iii) equations of its vertical asymptotes. (1 mark)

Question 11 **(5 marks)**

(a) Determine the solutions to $z^6-64=0$ in polar form and plot them on the Argand plane below. Label the solutions z_1, z_2, z_3, z_4, z_5 and z_6 in an **anti-clockwise** direction, starting from z_1 which is on the positive real axis. (3 marks)



(b) There is a cubic polynomial with real coefficients whose roots are z_3, z_4 and z_5. Write down this cubic polynomial in the form ax^3+bx^2+cx+d. (2 marks)

Question 12 **(13 marks)**

Given the points $A(1,-3,0)$, $B(3,2,-1)$, $C(7,1,2)$ and $D(5,-4,3)$.

(a) Determine the vector equation of the line through the points A and B. (1 mark)

The vector equation of the line through A and C is $r=\begin{pmatrix}1\\-3\\0\end{pmatrix}+\mu\begin{pmatrix}6\\4\\2\end{pmatrix}$.

(b) Determine the Cartesian equation of the plane, Π, containing the lines passing through AB and AC. (2 marks)

(c) (i) Show that A, B, C and D are coplanar. (2 marks)

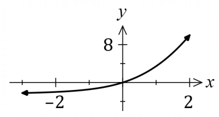
(ii) Prove that $ABCD$ is a rectangle. (2 marks)

Question 16 **(7 marks)**

(a) Use the substitution $u^2=2y+5$ to show that $\int \frac{y}{\sqrt{2y+5}}dy=\frac{(y-5)\sqrt{2y+5}}{3}+c$, where c is a constant of integration. (4 marks)

(b) The equation of the curve shown is

$y=x\sqrt{2y+5}$.



Determine the area enclosed by the curve and the line $3x-y+4=0$. (3 marks)

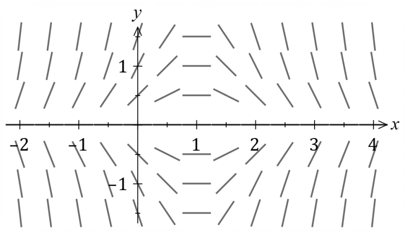
Question 15

(7 marks)

The slope field for the differential equation

$$\frac{dy}{dx}+y(2x-k)=0$$

where k is a constant, is shown at right.



(a) Use a feature of the slope field to explain why $k=2$ and hence determine the slope at the point $A(-2,1)$. (2 marks)

(b) Determine the solution of the differential equation that contains the point $B(1,-1)$ in the form $y=f(x)$. (4 marks)

(c) Sketch the solution curve that contains the point $B(1,-1)$ on the slope field. (1 mark)

A sphere is constructed with its centre on plane Π from part (b).

(d) Determine the vector equation of this sphere if A, B, C and D lie on the surface. (3 marks)

A set of three planes is given as follows:

$$6x+5y+2z=21$$
$$3x-3y+3z=18$$
$$6x+5y+2a^2z=a+20$$

(e) Determine the value of a such that the above planes only intersect at the centre of the sphere found in part (d). (3 marks)

Question 13 **(6 marks)**

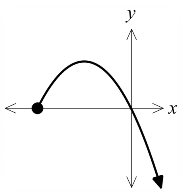
(a) By using partial fractions, show that:

$$\int \frac{4}{4-y^2}\,dy = -\ln|2-y| + \ln|2+y| + c$$

(2 marks)

On a coordinate plane, a point P moves along a path, such that after t seconds $(t \geq 0)$, the position of the point is defined by

$$x = \frac{t}{2} - 1$$
$$\frac{dy}{dt} = 1 - t$$



The direction of motion is shown in the diagram on the right.

(b) Determine when the angle between the direction of motion and the positive direction of the x-axis is $\pm 45°$. (4 marks)

Question 14 **(6 marks)**

(a) By letting $w = u + iv$ and $z = x + iy$, prove $\overline{w} + \overline{z} = \overline{w + z}$. (1 mark)

(b) By letting $z = r\,cis\,\theta$, use De Moivre's theorem to prove that $\overline{(z^n)} = \overline{z}^n$. (1 mark)

(c) A polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is divided by $(x - z)$, where z is a complex number, leaving a remainder of $1 - i$.

(i) Using parts (a) and (b), show that the remainder when $P(x)$ is divided by $(x - \overline{z})$ is $1 + i$. (3 marks)

(ii) If for all solutions of $P(z_n) = 0$, it is known that $P(\overline{z_n}) = 0$, where z_n is a complex number what can be said about the coefficients of $P(x)$? (1 mark)