PERTH MODERN SCHOOL
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INDEPENDENT PUBLIC SCHOOL

Semester One Examination, 2023

Question/Answer booklet

MATHEMATICS METHODS
UNIT 3
Section One:
Calculator-free

Your Name: _____

Your Teacher's Name: _____

Time allowed for this section
Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Marks	Max	Question	Marks	Max
1		7	4		9
2		6	5		8
3		15	6		6

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	6	6	50	51	35
Section Two: Calculator-assumed	10	10	100	90	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2019*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

See next page

Additional working space

Question number: _____

Additional working space

Question number: _____

Section One: Calculator-free **(53 marks)**

This section has **six** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 50 minutes.

Question 1 **(7 marks)**

A function $f(x)$ has the derivative $f'(x)=3x^2+12x$.

(3 marks)

The graph has an x-intercept at $x=-1$.

(a) Find $f(x)$.

$f(x)=x^3+6x^2+c$
$f(-1)=0=-1+6+c$
$c=-5$
$f(x)=x^3+6x^2-5$

Specific behaviours	Mark allocation
Integrates correctly	1
Substitutes $x=-1$	1
States correct equation	1

(b) The graph has a point of inflection at $x=p$. Find p.

(2 marks)

$f''(x)=6x+12$
$0=6(p+2)$
$p=-2$

Specific behaviours	Mark allocation
Uses second derivative	1
Solves equation correctly	1

(c) Find the value(s) of x for which the graph $y=f(x)$ is concave down.

(2 marks)

$f''(x)=6x+12<0$
Concave down $x<-2$

Specific behaviours	Mark allocation
Shows use of second derivative or sign test	1
States correct inequality	1

Question 2 **(6 marks)**

(a) Determine:

(i) $\int 3+\cos(3-2x)\,dx$

(1 mark)

¿ $3x-\frac{1}{2}\sin(3-2x)+c$

Specific behaviours	Mark allocation
Integrates correctly	1

(ii) $\int_3^4 \left((2x-5)^{-2}\right)dx$

(3 marks)

¿ $\left[\frac{(2x-5)^{-1}}{2(-1)}\right]_3^4$

¿ $\frac{-1}{2}\left[\frac{1}{(2x-5)}\right]_3^4$

¿ $\left(\frac{-1}{2}\right)\left[\left(\frac{1}{3}\right)-1\right]=\frac{1}{3}$

Specific behaviours	Mark allocation
Integrates correctly	1
Substitutes to find definite integral	1
Finds correct answer	1

(b) (i) Determine the derivative of $y=e^{3x}(3x+5)$. Simplify your answer.

(2 marks)

$\frac{dy}{dx}=e^{3x}(3)+(3x+5)e^{3x}(3)$

¿ $3e^{3x}(3x+6)$

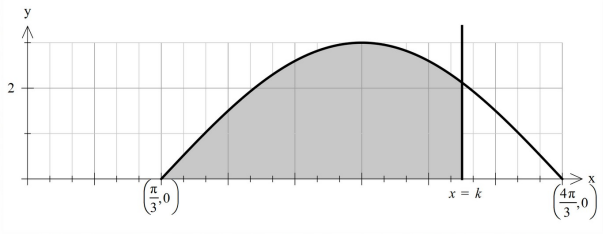
¿ $9e^{3x}(x+2)$

Specific behaviours	Mark allocation
Finds derivative using product rule	1
Simplifies answer correctly	1

See next page

Question 6 **(6 marks)**

The region bounded by the x-axis and the graph of $y=\sin\left(x-\frac{\pi}{3}\right)$, $0\le x\le\pi$ shown below is divided into two regions by the line $x=k$.



If the area of the region for $\frac{\pi}{3}\le x\le k$ is 3 times the size of the area of the region for $k\le x\le\frac{4\pi}{3}$, determine k.

(6 marks)

$\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \sin\left(x-\frac{\pi}{3}¿\right)dx=-\cos(\pi)+\cos(0)=2¿$

Area between $\frac{\pi}{3}$ and k is three quarters the total area ¿ $\frac{3}{4}(2)=1.5$

$\int_{\frac{\pi}{3}}^{k} \sin\left(x-\frac{\pi}{3}¿\right)dx=-\cos\left(k-\frac{\pi}{3}\right)+\cos 0=1.5¿$

$-\cos\left(k-\frac{\pi}{3}\right)=0.5$

$\cos\left(k-\frac{\pi}{3}\right)=-0.5$

$k-\frac{\pi}{3}=\frac{2\pi}{3}$

$k=\pi$

Or

$\int_{\frac{\pi}{3}}^{k} \sin\left(x-\frac{\pi}{3}¿\right)dx=-\cos\left(k-\frac{\pi}{3}\right)+\cos 0=1-\cos$

$\int_{k}^{\frac{4\pi}{3}} \sin\left(x-\frac{\pi}{3}¿\right)dx=-\cos(\pi)+\cos\left(k-\frac{\pi}{3}¿\right)=1+$

$LHS=3\times RHS$

$\therefore 1-\cos\left(k-\frac{\pi}{3}\right)=3¿$

$-2=4\cos\left(k-\frac{\pi}{3}¿\right)¿$

$\cos\left(k-\frac{\pi}{3}¿\right)=-0.5¿$

$k-\frac{\pi}{3}=\frac{2\pi}{3}$

$k=\pi$

Specific behaviours	Marks	Specific behaviours	Marks
Integrates to find Area under curve	2	Integrates both regions correctly	2
Recognises area required is 1.5	1	Uses LHS is 3 times RHS	1
Creates correct integral	1	Creates correct integral equation	1
Creates simple equation	1	Creates simple equation	1
States correct answer	1	States correct answer	1
Note: No follow through if trig equation is incorrect.		Note: No follow through if trig equation is incorrect.	

See next page

Question 5 **(8 marks)**

The probability function for the random variable X is $P(X=x)=\begin{cases} k^2-k+x, & x=0 \\ 5k^2x, & x=1 \\ 0, & \text{otherwise.} \end{cases}$

(a) Determine the value of the constant k. (4 marks)

Solution
$P(X=0)+P(X=1)=1$ $k^2-k+5k^2=1$ $6k^2-k-1=0$ $(3k+1)(2k-1)=0$ $k=\frac{-1}{3}, k=\frac{1}{2}$ $k=\frac{-1}{3} \Rightarrow P(X=0)=\frac{4}{9}, P(X=1)=\frac{5}{9}$ $k=\frac{1}{2} \Rightarrow P(X=1)=\frac{5}{4}, P(X=0)=\frac{-1}{4}$ Ignore $k=\frac{1}{2}$ as we require $0 \le p \le 1$ and hence $k=\frac{-1}{3}$.
Specific behaviours
✓ sums probabilities to 1 and forms quadratic equation
ü solves for both values of k
ü indicates check of both values of k
ü correct value of k

(b) Determine the mean and variance of X. (2 marks)

Solution
$E(X)=p=\frac{5}{9}, Var(X)=p(1-p)=\frac{5}{9}\times\frac{4}{9}=\frac{20}{81}$
Specific behaviours
✓ mean
ü variance

(c) The random variable $Y=3X+1$. Determine the mean and variance of Y. (2 marks)

Solution
$E(Y)=3E(X)+1=\frac{8}{3}, Var(Y)=3^2\times Var(X)=\frac{20}{9}$
Specific behaviours
✓ mean
ü variance

See next page

Question 3 **(15 marks)**

At a mathematics camp, a student selects a counter from a bag containing 10 counters.

The counters are numbered from 1 to 10 inclusive. Depending on the outcome, the student wins Maths Dollars that can be spent on prizes.

(a) Complete the table showing the possible outcomes and the probability. (2 marks)

Outcomes	Probability	Maths Dollars Won
A multiple of 3	$\frac{3}{10}$	$\$3$
A multiple of 4	$\frac{2}{10}$	$\$9$
A multiple of 10	$\frac{1}{10}$	$\$15$
All other numbers	$\frac{4}{10}$	$\$x$

Solution	Specific behaviours
See table.	✓ Determines at least two entries. ✓ Determines all entries.

A student pays $\$5$ to play the game

(b) Find the value of x if the expected return on the game is $\$0$. (3 marks)

Solution	Specific behaviours
$E(X)=\frac{3}{10}\times 3+\frac{2}{10}\times 9+\frac{1}{10}\times 15+\frac{4x}{10}$	✓ Determines expected value.
$5=\frac{9+18+15+4x}{10}$	✓ Expected value = \$5.
$50=42+4x$	
$x=2$	✓ Solves for x.

(c) Is the selection of a counter a Bernoulli trail? Justify your answer. (2 marks)

Solution	Specific behaviours
No. As there is no success $\vee$ *failure.*	✓ States no. ✓ States there is no success or failure.

See next page

Question 3 (continued)

The random variable W is a Bernoulli distribution that models the situation from (a).

(d) Given that $Var(W)=0.16$, determine the probability of success and use the information in part (a) to what outcome is therefore considered to be a success. (2 marks)

Solution	Specific behaviours
$0.16=p(1-p)$ *By inspection*: $p=0.2$	✓ Substitutes into formula for variance and determines p.
W is whether a multiple of 4 is drawn ¿ not ∨ whether $9 is won ∨ not .	✓ Determines what a success is.

The counter selected is returned to the bag, then the bag is shaken before the next student selects a counter. A student is interested in whether they win $\$15$ or not. They pay to play 10 games.
Let Y be the number of times the student wins $\$15$.

(e) State the distribution that would be used to determine the probability of Y. (2 marks)

Solution	Specific behaviours	Point
$Y \quad Bin\left(10,\frac{1}{10}\right)$	✓ Identifies the distribution as binomial. ✓ Includes the correct parameters.	3.3.13 3.3.14

(f) The student performs the calculations given. Determine the number of times the student wins $15 in each of the following:

(i) $\binom{10}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^8$ (1 mark)

Solution	Specific behaviours
2	✓ States 2 times.

(ii) $\left(\frac{9}{10}\right)^{10}$ (1 mark)

Solution	Specific behaviours
0	✓ States 0 times

(iii) $\binom{10}{4}\left(\frac{1}{10}\right)^4\left(\frac{9}{10}\right)^6+\binom{10}{5}\left(\frac{1}{10}\right)^5\left(\frac{9}{10}\right)^5$ (1 mark)

Solution	Specific behaviours
4 ∨ 5	✓ States 4 or 5 times.

(iv) $1-\left(\frac{9}{10}\right)^{10}$ (1 mark)

See next page

Solution	Specific behaviours
At least once	✓ States at least once (or similar).

Question 4 **(9 marks)**

The function $f(x)$ is defined for $x>1.5$, has derivative $f'(x)=\frac{8}{(2x-3)^2}$, and passes through the point $(4,1)$.

(a) Determine the rate of change of $f'(x)$ when $x=3$. (3 marks)

Solution

$f'(x)=8(2x-3)^{-2}$ $f''(x)=8(-2)(2)(2x-3)^{-3}$

¿ $-32(2x-3)^{-3}$

$f''(3)=-32(3)^{-3}=\frac{-32}{27}$

Specific behaviours

✓ indicates correct use of chain rule
ü obtains correct derivative
ü substitutes and obtains correct value

(b) Determine $f(x)$. (4 marks)

Solution

$f(x)=\int 8(2x-3)^{-2}dx$ ¿ $\frac{8}{(-1)(2)}(2x-3)^{-1}+c$ ¿ $-4(2x-3)^{-1}+c$

$f(4)=1\Rightarrow -4(2\times 4-3)^{-1}+c=1\Rightarrow c=1+\frac{4}{5}=\frac{9}{5}$

$f(x)=\frac{-4}{2x-3}+\frac{9}{5}$

Specific behaviours

✓ attempts to obtain antiderivative, with constant
ü correct antiderivative
ü indicates use of point to evaluate constant
ü correct function

(c) Determine $\frac{d}{dt}\int_t^3 (f'(x)-4x)dx$. (2 marks)

Solution

$\frac{d}{dt}\int_t^3 (f'(x)-4x)dx=\frac{-d}{dt}\int_3^t (f'(x)-4x)dx$ ¿ $4t-f'(t)$

¿ $4t-\frac{8}{(2t-3)^2}$

Specific behaviours

✓ adjusts integral so that variable is upper bound
ü applies fundamental theorem to obtain correct result

See next page