This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

A box contains five balls numbered $1, 3, 5, 7$ and 9. Three balls are randomly drawn from the box at the same time and the random variable X is the largest of the three numbers drawn.

(a) By listing all ten possible outcomes $(135, 137$, etc.), determine $P(X \leq 7)$. (2 marks)

135 ✓ 157 ✓ 179
137 ✓ 159
139

357 ✓ 379 579
359

$P(X \leq 7) = \frac{4}{10}$

✓ correct list

✓ probability based on correct list

(b) Construct a table to show the probability distribution of X. (2 marks)

x	5	7	9
$P(X=x)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{6}{10}$

✓ correct values of x
(1 & 3 ok if P=0)
✓ correct values for P(X) based on X,
(not on incorrect lists from (a))

(c) Calculate $\mathrm{E}(X)$. (2 marks)

$$E(X) = 5\left(\frac{1}{10}\right) + 7\left(\frac{3}{10}\right) + 9\left(\frac{6}{10}\right)$$
$$= \frac{5}{10} + \frac{21}{10} + \frac{54}{10}$$
$$= 8$$

✓ indicate $x \times P(X)$

✓ $E(X)$

Question 2 **(6 marks)**

A function defined by $f(x) = 39 + 24x - 3x^2 - x^3$ has stationary points at $(-4, -41)$ and $(2, 67)$.

(a) Use the second derivative to show that one of the stationary points is a local maximum and the other a local minimum. (3 marks)

$f'(x) = 24 - 6x - 3x^2$

$f''(x) = -6 - 6x$

$f''(-4) = 18$

$18 > 0 \Rightarrow$ local minimum at $(-4, -41)$

$f''(2) = -18$

$-18 < 0 \Rightarrow$ local maximum at $(2, 67)$

✓ $f''(x)$

✓ show $f''(-4)$ and interpret

✓ show $f''(2)$ and interpret.

(b) Determine the coordinates of the point of inflection of the graph of $y = f(x)$ and Justify whether it is a horizontal or oblique point of inflection. (3 marks)

Inflection when $f''(x) = 0$

$-6 - 6x = 0$

$x = -1$

$f'(-1) = 27 \Rightarrow$ oblique

(horizontal have $f'(x) = 0$ when $f''(x) = 0$).

$f(-1) = 39 - 24 - 3 + 1$

$= 13$

$(-1, 13)$ is an oblique point of inflection

✓ x value for $f''(x) = 0$

✓ y value.

✓ justifies oblique.

Question 3 **(9 marks)**

Determine $\frac{dy}{dx}$ for the following, simplifying each answer.

(a) $y = \sqrt{3-4x}$. (2 marks)

$= (3-4x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(3-4x)^{-\frac{1}{2}}(-4)$

$= -2(3-4x)^{-\frac{1}{2}}$

✓ use of chain rule

✓ correct simplified derivative.

(b) $y = 4x^3\sin(3x)$. (3 marks)

$\frac{dy}{dx} = (12x^2)(\sin(3x)) + (4x^3)(3\cos(3x))$

$= 12x^2\sin(3x) + 12x^3\cos(3x)$

or $12x^2(\sin(3x) + x\cos(3x))$

(no part factorisation)

✓ product rule

✓ clearly shows trig derivative with separation of parts.

✓ correct simplified derivative

(c) $y = \int_x^1 t\sqrt{t}\,dt$. (2 marks)

$= -\int_1^x t\sqrt{t}\,dt$

$\frac{dy}{dx} = -x\sqrt{x}$

✓ reverse bounds

✓ correct derivative

(d) $y = \frac{2x^3}{3x+1}$ (2 marks)

$\frac{dy}{dx} = \frac{(6x^2)(3x+1) - (2x^3)(3)}{(3x+1)^2}$

$= \frac{6x^2(3x+1-x)}{(3x+1)^2}$

$= \frac{6x^2(2x+1)}{(3x+1)^2}$

✓ quotient rule

✓ simplified derivative.

Question 4 **(6 marks)**

The height, in metres, of a lift above the ground t seconds after it starts moving is given by

$$h = 2\cos^2\left(\frac{t}{5}\right).$$

Use the increments formula to estimate the change in height of the lift from $t = \frac{5\pi}{6}$ to $t = \frac{17\pi}{20}$.

$\delta h \simeq \left.\frac{dh}{dt}\right|_{t=\frac{5\pi}{6}} \delta h$

✓ increment formula with correct variables

$\delta t = \frac{17\pi}{20} - \frac{5\pi}{6}$

$= \frac{51\pi - 50\pi}{60}$

$= \frac{\pi}{60}$

✓ δt

$\frac{dh}{dt} = 2\left(2\cos\frac{t}{5}\right)\left(\frac{1}{5}\right)\left(-\sin\frac{t}{5}\right)$

$= \frac{-4}{5}\cos\frac{t}{5}\sin\frac{t}{5}$

✓ chain rule used for arguement

✓ derivative of cos

$\left.\frac{dh}{dt}\right|_{t=\frac{5\pi}{6}} = \frac{-4}{5}\cos\frac{\pi}{6}\sin\frac{\pi}{6}$

$= \left(\frac{-4}{5}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{-\sqrt{3}}{5}$

✓ $\left.\frac{dh}{dt}\right|_{t=\frac{5\pi}{6}}$ using exact values

$\delta h \simeq \frac{-\sqrt{3}}{5} \times \frac{\pi}{60}$

$\simeq \frac{-\sqrt{3}\pi}{300}$

✓ δh

height decreases by $\frac{\sqrt{3}\pi}{300}$ metres.

Question 21 **(7 marks)**

A fuel storage tank, initially containing 550 L, is being filled at a rate given by

$$\frac{dV}{dt} = \frac{t^2(60-t)}{250}, \qquad 0 \le t \le 60$$

where V is the volume of fuel in the tank in litres and t is the time in minutes since filling began. The tank will be completely full after one hour.

(a) Calculate the volume of fuel in the tank after 10 minutes. (3 marks)

$\int_0^{10} \frac{t^2(60-t)}{250}\,dt$

$= 70$ L

Vol $= 550 + 70$

$= 620$ L

✓ Indicates use of integral of rate of change

✓ calculates increase

✓ States Volume

(b) Determine the time taken for the tank to fill to one-half of its maximum capacity. (4 marks)

$V = \int_0^{60} \frac{t^2(60-t)}{250}\,dt$

$\doteq 4320$ L

After 1 hour

$V = 550 + 4320$

$= 4870$ L

✓ Calculates Max Volume

✓ Indicates $V(T)$

✓ Indicates Equation

$V(T) = \int_0^T \frac{t^2(60-t)}{250}\,dt + 550 = 2435$

$\int_0^T \frac{t^2(60-t)}{250}\,dt = 1885$

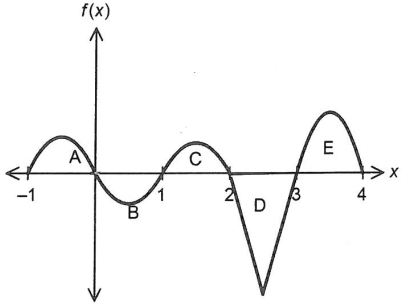
$T = 34.64$

It takes 34.64 minutes

✓ Solves for time

Question 20 **(7 marks)**

Consider the graph of $y = f(x)$ for $-1 \le x \le 4$.



It is known that:

- $\int_{-1}^{1} f(x) = 0$
- Areas C, D and E are 1, 5 and 4 units^2 respectively.

(a) Determine :

(i) $\int_{-1}^{4} f(x)\,dx = 0+1-5+4$ (1 mark)

$= 0$

✓ determines correct integral

(ii) $\int_{0}^{4} f(x)\,dx$ given that Area A = 3 units^2 (1 mark)

$-3+1-5+4$

$= -3$

✓ determines correct integral

(iii) the area enclosed by the graph of f and the x–axis between 1 and 4. (1 mark)

$1+|-5|+4$

$= 10$ square units

✓ determines correct area.

(b) Determine the values of:

(i) $\int_{2}^{4} [f(x)+7]\,dx = \int_2^4 f(x)\,dx + \int_2^4 7\,dx$ (2 marks)

$= (-5+4) + [7x]_2^4$

$= -1 + 14$

$= 13$

✓ correct use of linearity propertie

✓ determines correct integral

(ii) $\int_{3}^{4} 2f(x)\,dx$ (2 marks)

$= 2\int_3^4 f(x)\,dx$

$= 2(4)$

$= 8.$

✓ correct use of linearity property

✓ correct integral

Question 5 **(9 marks)**

The function g is such that $g'(x) = ax^2 - 12x + b$, it has a non-horizontal point of inflection at $(2, 7)$ and a stationary point at $(-2, 135)$.

(a) Determine $g(1)$. (5 marks)

$g''(x) = 2ax - 12$

$g''(2) = 0 \Rightarrow a = 3$ ✓ a

$g'(x) = 3x^2 - 12x + b$

$g'(-2) = 0 \Rightarrow b = -36$ ✓ b

$g(x) = \int 3x^2 - 12x - 36\,dx$

$= x^3 - 6x^2 - 36x + c$ ✓ indefinite integal

$g(2) = 7 \Rightarrow c = 95$ ✓ c

$g(1) = 54$ ✓ g(1)

(b) Determine

(i) $\int_{-2}^{2} g'(x)\,dx.$ (2 marks)

$= g(2) - g(-2)$ ✓ use of total change

$= 7 - 135$

$= -128$ ✓ value

(ii) $\int_{-2}^{2} 2g'(x) + 6\,dx.$ (2 marks)

$= 2\int_{-2}^{2} g'(x)\,dx + \int_{-2}^{2} 6\,dx$ ✓ linear property of integrals

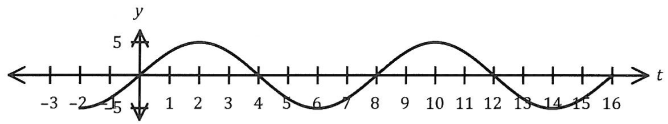
$= 2(-128) + [6x]_{-2}^{2}$

$= -256 + 24$

$= -232$ ✓ value

Question 6 **(8 marks)**

(a) The graph of $y = f(t)$ is shown below, where $f(t) = 5\sin\left(\frac{\pi t}{4}\right)$.



(i) Determine the exact area between the horizontal axis and the curve for $0 \le t \le 4$. (4 marks)

$$\int_0^4 5\sin\left(\tfrac{\pi}{4}t\right) dt$$
$$= \left[\frac{-5\cos\left(\frac{\pi}{4}t\right)}{\frac{\pi}{4}}\right]_0^4$$
$$= \frac{-20}{\pi}\left[\cos\pi - \cos 0\right]$$
$$= \frac{-20}{\pi}(-1-1)$$
$$= \frac{40}{\pi}$$

✓ sin → −cos
✓ divide by $\frac{\pi}{4}$
✓ correct exact values
✓ area

Another function, F, is defined as $F(x) = \int_0^x f(t)\,dt$ over the domain $0 \le x \le 16$.

(ii) Determine the value(s) of x for which $F(x)$ has a maximum and state the value of $F(x)$ at this location. (2 marks)

Critical points when $\frac{d}{dx}\int_0^x f(t)\,dt = 0$

$$\frac{d}{dx}\int_0^x 5\sin\left(\tfrac{\pi}{4}t\right)dt = 5\sin\left(\tfrac{\pi}{4}x\right)$$

$5\sin\left(\frac{\pi}{4}x\right) = 0$ when $x = 4k,\ k \in \mathbb{Z}$

From the graph, we can see that $x = 0, 8$ are minima and $x = 4, 12$ are maxima.

✓ both x values
✓ correct y value.

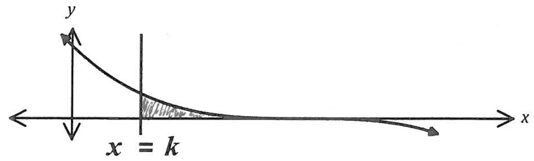
(b) $\int \left(e^x - \frac{1}{e^x}\right)^2 dx$ (2 marks)

$$= \int e^{2x} - 2 + e^{-2x}\,dx$$
$$= \frac{e^{2x}}{2} - 2x - \frac{e^{-2x}}{2} + C$$

✓ expands binomial
✓ integral

Question 19 **(7 marks)**

(a) The graph of $y = 6(2-3x)^3$ is shown below.



(i) Determine the area of the region enclosed by the curve and the coordinates axes. (2 marks)

$$6(2-3x)^3 = 0$$
$$x = \tfrac{2}{3}$$
$$\int_0^{\frac{2}{3}} 6(2-3x)^3\,dx$$
$$= 8 \text{ sq units.}$$

✓ determines correct integral limits $\int_0^{\frac{2}{3}}$
✓ determines integral

(ii) Given that the area of the region bounded by the curve, the x-axis and the line $x = k$ is 2 square units, determine the value of k, where $0 < k < \frac{2}{3}$. (2 marks)

$$\int_k^{\frac{2}{3}} 6(2-3x)^3\,dx = 2$$
$$k \approx 0.1953$$

$-\frac{\sqrt{2}}{3} + \frac{2}{3}$

✓ writes equation with correct antiderivative
✓ determines correct value of k.

(b) Given the function $y = xe^x - e^x$

(i) Determine $\frac{dy}{dx}$. (1 marks)

$$\frac{dy}{dx} = x.e^x + e^x(1) - e^x$$
$$= xe^x$$

✓ determines $\frac{dy}{dx}$

(ii) Using part (i), determine the exact value of $\int_0^1 \left(xe^x + x^3\right)dx$. (2 marks)

$$\int_0^1 xe^x\,dx + \int_0^1 x^3\,dx$$
$$= \left[xe^x - e^x\right]_0^1 + \left[\frac{x^4}{4}\right]_0^1$$
$$= 1\tfrac{1}{4}$$

✓ Demonstrates use of $\frac{dy}{dx}$
✓ determines definite integral

Question 18 **(7 marks)**

A random sample of n components are selected at random from a factory production line. The proportion of components that are defective is p and the probability that a component is defective is independent of the condition of any other component.

The random variable X is the number of faulty components in the sample. The mean and standard deviation of X are 8.46 and 2.82 respectively.

(a) Determine the values of n and p. (4 marks)

$X \sim B(n, p)$

$np = 8.46$

$np(1-p) = (2.82)^2$

$8.46(1-p) = 7.9524$

$p = 0.06$

$n = 141$

✓ Indicates Binomial Distribution

✓ Writes equation using mean

✓ writes equation using standard deviation

✓ solves for p and n correctly

(b) After changes are made to the manufacturing process, the proportion of defective components is now 2%. Determine the smallest sample size required to ensure that the probability that the sample contains at least one defective component is at least 0.8. (3 marks)

$X \sim B(n, 0.02)$.

$P(X \geq 1) \geq 0.8$

$1 - P(X<1) \geq 0.8$

$P(X<1) \leq 0.2$

$P(X=0) \leq 0.2$

${}^nC_0 (0.02)^0 (0.98)^n \leq 0.2$

$(0.98)^n = 0.2$

$n = 79.66$

$n \geq 80$

Smallest sample size 80.

✓ indicates required Binomial Distribution

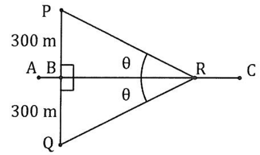
✓ uses $P(X=0)$ to create correct inequality

OR lists trials from calculator

✓ Solves and rounds to obtain n

Question 7 **(7 marks)**

Two houses, P and Q, are 600 m apart on either side of a straight railway line AC. AC is the perpendicular bisector of PQ and the midpoint of PQ is B. A small train, R, leaves station C and travels towards B, 1000 m from C.



Let $\angle PRB = \angle QRB = \theta$, where $0 < \theta < 90°$, and $X = PR + QR + CR$, the sum of the distances of the train from the houses and station.

(a) By forming expressions for PR, BR and CR, show that $X = 1000 + \dfrac{300(2-\cos\theta)}{\sin\theta}$. (3 marks)

$PR = \dfrac{300}{\sin\theta} \; (= QR)$

$BR = \dfrac{300}{\tan\theta}$

$CR = 1000 - \dfrac{300}{\tan\theta}$

$X = \dfrac{300}{\sin\theta} \times 2 + \left(1000 - \dfrac{300}{\tan\theta}\right)$

$= 1000 + \dfrac{600}{\sin\theta} - \dfrac{300\cos\theta}{\sin\theta}$

$= 1000 + \dfrac{600 - 300\cos\theta}{\sin\theta}$

$= 1000 + \dfrac{300(2-\cos\theta)}{\sin\theta}$

✓ PR in terms of θ

✓ BR and CR in terms of θ

✓ Shows use of expressions to get X

(b) Use a calculus method to determine the minimum value of X. (4 marks)

$X = 1000 + 300\left(\dfrac{2-\cos\theta}{\sin\theta}\right)$

$\dfrac{dX}{d\theta} = 300\left(\dfrac{\sin\theta\sin\theta - (2-\cos\theta)(\cos\theta)}{\sin^2\theta}\right)$

$= 300\left(\dfrac{\sin^2\theta - 2\cos\theta + \cos^2\theta}{\sin^2\theta}\right)$

$= 300\left(\dfrac{1-2\cos\theta}{\sin^2\theta}\right)$

$\dfrac{dX}{d\theta} = 0 \implies 1 - 2\cos\theta = 0$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

$X|_{\theta=\frac{\pi}{3}} = 1000 + 300\left(\dfrac{2-\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}}\right)$

$= 1000 + 300\left(\dfrac{2-\frac{1}{2}}{\frac{\sqrt3}{2}}\right)$

$= 1000 + 300\left(\frac{3}{2} \times \frac{2}{\sqrt3}\right)$

$= 1000 + 300\sqrt3$

✓ derivative

✓ Pythagorean

✓ $\dfrac{dX}{d\theta} = 0$ and solve

✓ minimum value

End of questions

Section Two: Calculator-assumed **(97 Marks)**

This section has **fourteen (14)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 8 **(7 marks)**

A large inland lake contains an abundance of fish. 78% of the fish in the lake are known to be trout. Ten fish are caught at random from the lake every day.

(a) Describe, with parameters, a suitable probability distribution to model the number of trout in a day's catch. (2 marks)

$X \sim B(10, 0.78)$

✓ Selects suitable distribution

✓ Gives correct parameters.

(b) Justify your choice of distribution. (1 marks)

Large population ∴ P(Success) is approximately same for each trial

✓ suitable justification

(c) Determine the probability that less than half of the catch is trout in a day's catch. (2 marks)

$P(X \leq 4) = 0.01039$

$P(X < 5)$

✓ writes $P(X \leq 4)$ or $P(X<5)$

✓ determines correct probability.

(c) Calculate the probability that over two consecutive days, a total of exactly 19 trout are caught. (2 marks)

$Y \sim (20, 0.78)$

$P(X = 19) = 0.0392$

✓ defines a new distribution

✓ determines the probability.

Question 17 **(4 marks)**

The cost of producing x items of a product is given by $\$[6x + 1000e^{-0.01x}]$. Each item is sold for \$21.80.

(a) Write an equation to describe $R(x)$, the revenue from selling the product. (1 mark)

$R(x) = 21.8x$

✓ Determines $R(x)$ equation

(b) Write an equation for $P(x)$, the profit function. (1 mark)

$P(x) = 21.8x - 6x - 1000e^{-0.01x}$

$= 15.8x - 1000e^{-0.01x}$

✓ Determines $P(x)$ equation

(c) Determine $P'(500)$ and interpret this value. (2 marks)

$P'(x) = 15.8 + 10e^{-0.01x}$

$P'(500) = 15.87$

\$15.87 profit produced by the sale of 501st item. (one extra item)

✓ Determines $P'(500)$

✓ Gives valid interpretation of $P'(500)$

Question 16 **(11 marks)**

A particle starts from rest at O and travels in a straight line.

Its velocity v ms^{-1}, at time t seconds, is given by $v = 15t - 3t^2$ for $0 \le t \le 2$ and $v = 72t^{-2}$ for $t > 2$.

(a) Determine the initial acceleration of the particle. (2 marks)

$\frac{dV}{dt} = 15 - 6t$

$t = 0 \quad \frac{dV}{dt} = 15 \text{ MS}^{-2}$

✓ determines a as $\frac{dV}{dt}$
✓ determines initial acceleration

(b) Calculate the change in displacement of the particle during the first two seconds. (2 marks)

$\int_0^2 15t - 3t^2 \, dt$

$= 22 \text{ m}$

✓ $\int_0^2 v \, dt$ ~~area~~
✓ determines correct definite integral

(c) Justify that the displacement of the particle at $x = 2$ seconds is the same as the change of displacement in the first two seconds calculated in (b). (2 marks)

$15t - 3t^2 = 0$

$t = 0, \ t = 5$

✓ identifies change in displacement.
✓ Gives valid reason

Particle does not change direction in first five seconds.
$\therefore t = 2$, displacement same as change in displacement in first 2 second.

(d) Determine, in terms of t, an expression for the displacement, x m, of the particle from O for $t > 2$. (2 marks)

$x = \int 72t^{-2} \, dt$

$= -72t^{-1} + c$

$t = 2 \quad x = 22$

$22 = -\frac{72}{t} + c$

$c = 58$

$x = -\frac{72}{t} + 58$

✓ integrates velocity
✓ evaluates c

(e) Determine the distance of the particle from O when its acceleration is -2.25 ms^{-2}. (3 marks)

$a = -\frac{144}{t^3}$

$-\frac{144}{t^3} = -2.25$

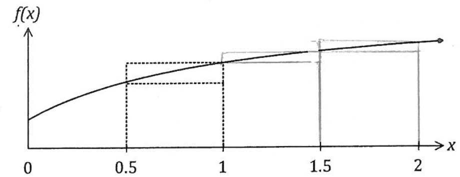
$x = -\frac{72}{t} + 58$

$t = 4 \quad x = 40 \text{ m}$

✓ determines acceleration $t > 2$
✓ solves for time
✓ calculates distance

Question 9 **(6 marks)**

The graph of $f(x) = \dfrac{5x+1}{x+1}$ is shown below.



Two rectangles are also shown on the graph, with dotted lines, and they both have corners just touching the curve. The smaller is called the inscribed rectangle and the larger is called the circumscribed rectangle.

(a) Complete the missing values in the table below. (1 mark)

x	0	0.5	1	1.5	2
$f(x)$	1	$\frac{7}{3}$	3	$\frac{17}{5}$	$\frac{11}{3}$

✓ determines BOTH missing values.

(b) Complete the table of areas below and use the values to determine a lower and upper bound for $\int_0^2 f(x)\,dx$. (4 marks)

x interval	0 to 0.5	0.5 to 1	1 to 1.5	1.5 to 2
Area of inscribed rectangle	$\frac{1}{2} \times 1$ $\frac{1}{2}$	$\frac{7}{3} \times \frac{1}{2}$ $\frac{7}{6}$	$3 \times \frac{1}{2}$ $\frac{3}{2}$	$\frac{17}{5} \times \frac{1}{2}$ $\frac{17}{10}$
Area of circumscribed rectangle	$\frac{7}{6}$	$3 \times \frac{1}{2}$ $\frac{3}{2}$	$\frac{17}{5} \times \frac{1}{2}$ $\frac{17}{10}$	$\frac{11}{3} \times \frac{1}{2}$ $\frac{11}{6}$

$L = \frac{1}{2} + \frac{7}{6} + \frac{3}{2} + \frac{17}{10} \approx 4.8667 \quad \frac{73}{15}$

$U = \frac{7}{6} + \frac{3}{2} + \frac{17}{10} + \frac{11}{6} \approx 6.2 \quad \frac{31}{5}$

✓ determines inscribed areas
✓ determines circumscribed areas.
✓ States lower bound
✓ States upper bound

(c) Explain how the bounds you found in (b) would change if a larger number of smaller intervals were used. (1 mark)

The lower bound would increase and the upper bound decrease

✓ Gives description of change mentioning BOTH bounds

Question 10 **(8 marks)**

The population of a city can be modelled by $P = P_0 e^{kt}$, where P is the number of people living in the city, in millions, t years after the start of the year 2000.

At the start of years 2007 and 2012 there were 2 245 000 and 2 521 000 people respectively living in the city.

(a) Determine the value of the constant k. (2 marks)

$2.245 = P_0 e^{7k}$
$2.521 = P_0 e^{12k}$ } Solve simultaneously for k

OR $2.521 = 2.245e^{5k}$ Solve for k

$k = 0.02319$

✓ choose suitable equation/s
✓ solve for k.

(b) Determine the value of the constant P_0. (2 marks)

$P_0 = \frac{2.245}{e^{7k}}$ ✓ states suitable equation

$P_0 = 1.9086$ (million) ✓ value of P_0 determined (in millions)

(c) Use the model to determine during which year the population of the city will first exceed 3 000 000. (2 marks)

$1.9086e^{0.02319t} = 3$

$t = 19.5$ ✓ Solves for t

Exceeds 3 million during 2019 ✓ correct year given

(d) Determine the rate of change of the city's population at the start of 2007. (2 marks)

$\frac{dP}{dt} = kP$

$= 0.02319 \times 2.245$ million

$= 52\ 061.6$ people/year

✓ uses rate of change correctly
✓ correct rate WITH units.

Question 15 **(6 marks)**

The discrete random variable X is defined by

x	0	1
$P(X=x)$	$\frac{4k}{e}$	$\frac{4k}{1}$

$$P(X = x) = \begin{cases} \frac{4k}{e^{1-x}} & x = 0, 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Show that $k = \frac{e}{4 + 4e}$. (3 marks)

PDF ∴ $\frac{4k}{e^1} + \frac{4k}{e^0} = 1$ ✓ Indicates P(0) and P(1)

$\frac{4k}{e} + 4k = 1$ ✓ equates sum of probabilities to 1

$\frac{4k + 4ek}{e} = 1$ ✓ shows re-arrangement to obtain k=

$\frac{k(4+4e)}{e} = 1$

$k = \frac{e}{4+4e}$

(b) Use your value of k to determine, in simplest form, the exact mean and standard deviation of X. (3 marks)

$E(X) = \frac{4k}{e^1}\cdot(0) + \frac{4k}{e^0}\cdot(1)$

$= 4k$

$= \frac{e}{1+e}$

✓ Determines simplified E(X)

Bernoulli ∴ $p = 4k$

$\text{Var } X = p(1-p)$

$= 4k(1-4k)$

$= \frac{e}{1+e}\left(1 - \frac{e}{1+e}\right)$

$= \frac{e}{1+e}\left(\frac{1+e-e}{1+e}\right)$

$= \frac{e}{(1+e)^2}$

$SD(X) = \frac{\sqrt{e}}{1+e}$

✓ Determines correct expression for variance

✓ Determines expression for standard deviation

USING FORMULA:
$\text{VAR } X = E(X^2) - (E(X))^2$
$= 1^2(4k) - (4k)^2$
$= \frac{e}{1+e} - \left(\frac{e}{1+e}\right)^2$
$= \frac{e}{(1+e)^2}$
$SD = \frac{\sqrt{e}}{1+e}$

Question 14 **(5 marks)**

One thousand people were asked the following question:
Do you regularly use social media?

The responses were classified by age of the respondents, as shown in the table below.

Response	Age ≤30 (years)	Age > 30 (years)	Total
Yes	400	250	650
No	50	300	350
Total	450	550	1000

(a) A random variable X is defined to be the probability that a respondent regularly uses social media. Define the probability distribution, in tabular form, for the random variable X (2 marks)

$X=x$	1	0
$P(X=x)$	$\frac{650}{1000}$	$\frac{350}{1000}$

✓ Provides correct 1/0 $X=x$ values
✓ Provides $P(1)$ and $P(0)$ correctly

(b) State the type of probability distribution that underlies the random variable X. (1 mark)

Bernoulli

✓ States the distribution is Bernoulli

(c) If one of the respondents is selected at random, determine:

(i) The probability that the respondent was over 30 years of age and regularly used social media. (1 mark)

$\frac{250}{1000} = 0.25$

✓ Determines correct probability

(ii) The probability that the respondent was over 30 years of age given that he/she regularly used social media. (1 mark)

$\frac{250}{650}$

✓ Determines correct probability

Question 11 **(8 marks)**

A fairground shooting range charges customers \$6 to take 9 shots at a target. A prize of \$20 is awarded if a customer hits the target three times and a prize of \$40 is awarded if a customer hits the target more than three times. Otherwise no prize money is paid.

Assume that successive shots made by a customer are independent and hit the target with the probability 0.15.

(a) Calculate the probability that the next customer to buy 9 shots wins

(i) a prize of \$20. (2 marks)

$X \sim B(9, 0.15)$

$P(X=3) = 0.10692$

✓ defines distribution
✓ Calculates Probability

(ii) a prize of \$40. (1 mark)

$P(X>3) = P(X \geq 4) = 0.03393$

✓ calculates probability

(b) Calculate the expected profit made by the shooting range from the next 30 customers who pay for 9 shots at the target. (3 marks)

Let X = PROFIT/customer

$(X=x)$	\$ −14	\$ −34	\$ 6
$P(X=x)$	0.10692	0.03393	0.85915

✓ indicates probability distribution of some kind

$E(X) = -14(0.10692) + (-34)(0.03393) + 6(0.85915)$
$= \$2.5044$ / customer / 9 shots

Expected Profit for 30 customers = \$75.13

✓ Calculates E(Profit) for 1 customer
✓ calculates expected value

(c) Determine the probability that more than 6 out of the next 8 customers will not win a prize. (2 marks)

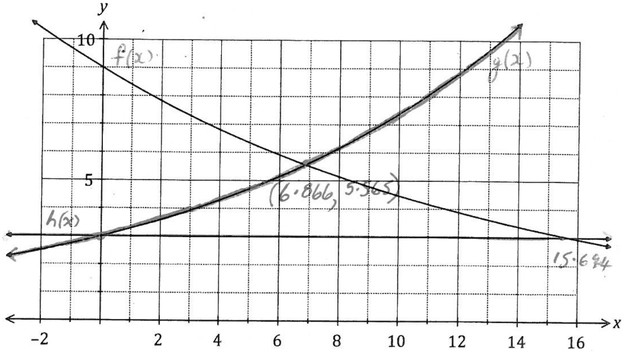
$X \sim B(8, 0.85915)$

$P(X \geq 7) = 0.6862$

✓ Indicates using P(not winning) 0.8592
✓ calculates Probability

Question 12 **(8 marks)**

Three functions are defined by $f(x) = 9e^{-0.07x}$, $g(x) = 3e^{0.09x}$ and $h(x) = 3$.



(a) One of the functions is shown on the graph above. Add the graphs of the other two functions. (3 marks)

✓ Graphs $y = h(x)$ correctly.
✓ $g(x)$ correct shapes and goes through $(0,3)$ and close to $(12,9)$
✓ smooth curve for $g(x)$

(b) Working to three decimal places throughout, determine the area of the region enclosed by all three functions. (5 marks)

$\int_0^{6.866} g(x) - h(x)\, dx =$ ✓ writes $\int_0^{6.866}$ integral

$= 7.905$ ✓ evaluates integral

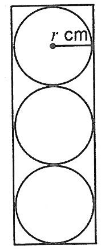
$\int_{6.866}^{15.694} f(x) - h(x)\, dx$ ✓ writes $\int_{6.866}^{15.694}$ integral

$= 10.166$ ✓ evaluates integral

Area $= 18.071$ sq. units. ✓ Total Area to 3 dp.

Question 13 **(5 marks)**

Three tennis balls of radius r cm fit snugly into a closed cylindrical can. A cross section of the metal can with the tennis balls inside is shown in the diagram below.



(a) Determine an expression for the surface area of the can in terms of r. (1 mark)

$S.A = 2\pi r h + 2\pi r^2$

$= 2\pi r(6r) + 2\pi r^2$ ✓ determines expression

$= 12\pi r^2 + 2\pi r^2$

$S.A = 14\pi r^2$

International standards state that the diameter of a tennis ball must be at least 6.541 cm. The maximum allowable diameter (6.858 cm) is roughly 4.8% larger than this.

(b) Use the incremental formula to determine the approximate percentage change in the metal required per can if the radius of each ball is increased 4.8% from the minimum allowable size. (4 marks)

$\frac{\delta r}{r} = \frac{4.8}{100}$ $A = 14\pi r^2$

$\frac{\delta A}{A} \approx \frac{\frac{dA}{dr} \cdot \delta r}{A}$ $\frac{dA}{dr} = 28\pi r$

$\approx \frac{\cancel{28\pi r}\, \delta r}{\cancel{14\pi r^2}}$ ✓ uses $\frac{\delta r}{r} = \frac{4.8}{100}$

$\approx 2\frac{\delta r}{r}$ ✓ demonstrates use of incremental formula for $\frac{\delta A}{A}$.

$\approx 2 \cdot \frac{4.8}{100}$ ✓ Obtains $2\frac{\delta r}{r}$

$\approx 9.6\%$

9.6% increase in metal required. ✓ determines approx % increase in metal required