PERTH MODERN SCHOOL
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Semester Two Examination, 2019

Question/Answer booklet

Yr 12 SPECIALIST
UNIT 3 & 4

Section Two:
Calculator-assumed

Your Name

Your Teacher's Name

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Marks	Max	Question	Marks	Max
8		4	15		9
9		8	16		8
10		9	17		8
11		7	18		9
12		7	19		7
13		7	20		7
14		10			

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	51	34
Section Two: Calculator-assumed	13	13	100	100	66
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **(100 Marks)**

This section has **13** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 100 minutes.

Question 8 **(4 marks)**

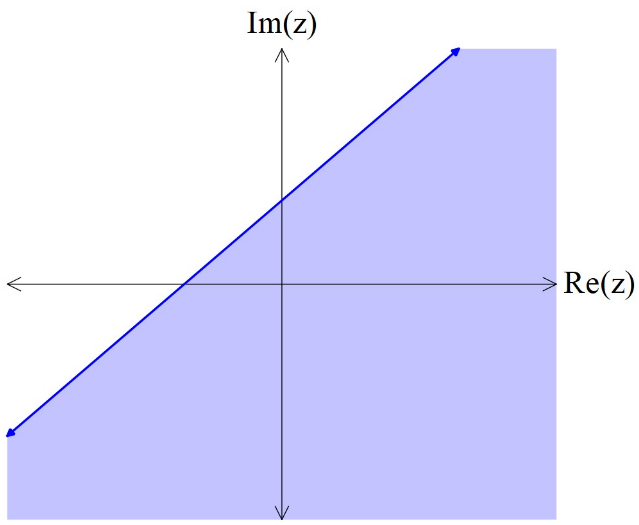
Consider the complex number $z = cis\theta$. By using De Moivre's theorem show that $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

Solution
$(cis\theta)^2 = (\cos\theta + i\sin\theta)^2$ $cis2\theta = \cos^2\theta + 2\cos\theta\sin\theta i - \sin^2\theta = \cos 2\theta + i\sin 2\theta$ $equate\ reals$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$
Specific behaviours
✓ sets up equation for cis ✓ uses De Moivre's for one side ✓ expands binomial expression for other side ✓ equates real parts of both sides

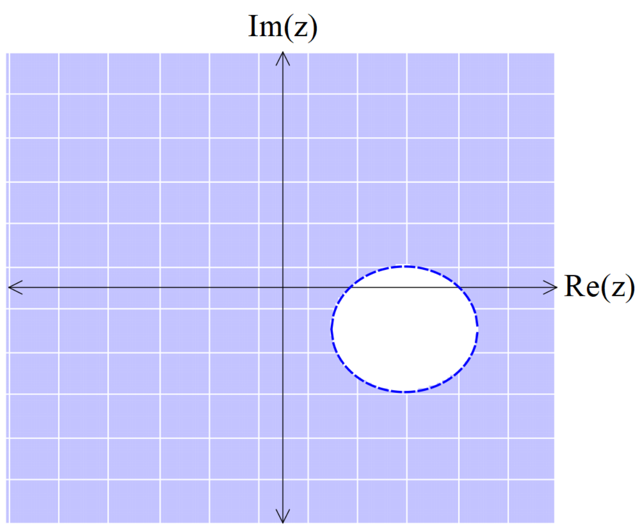
Question 9 **(8 marks)**

Sketch the following regions in the complex plane.

a) $\text{Im}(z) \leq \text{Re}(z) + 4$ (2 marks)

Solution

Specific behaviours
✓ draws line firmed in ✓ shades region

b) $\lvert z - 5 + 2i \rvert > 3$ (2 marks)

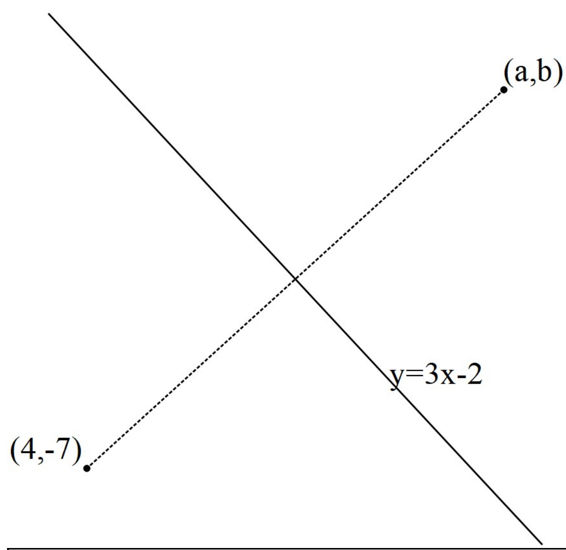
Solution

Specific behaviours
✓ dotted circle of correct centre and radius ✓ shaded region outside circle

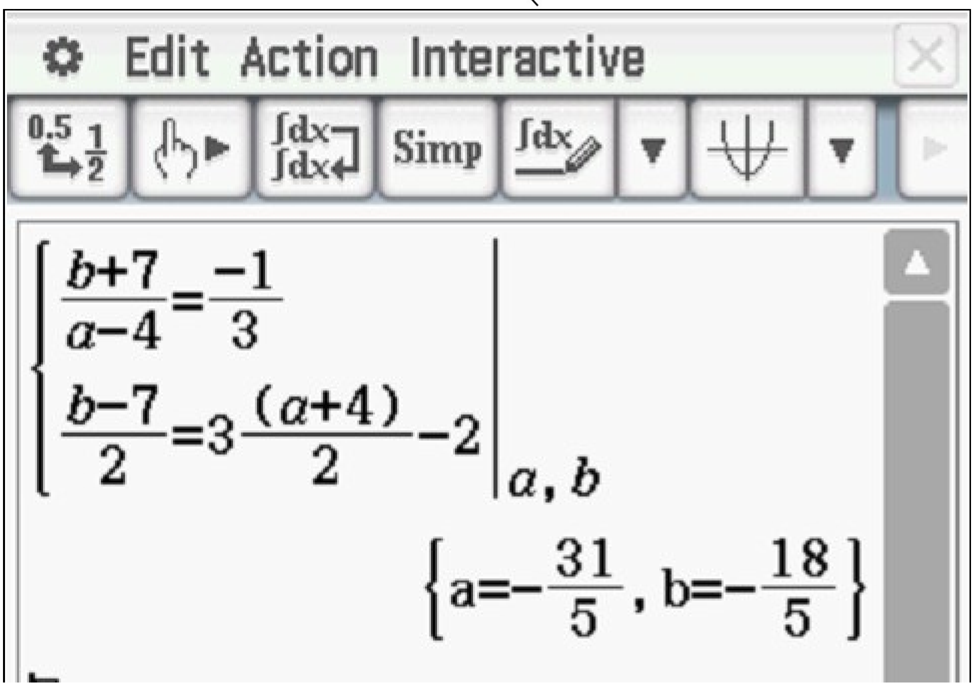
Q9 Cont-

The solution to $|z-4+7i|=|z-a-bi|$, where $a\,\&\,b$ are real constants, is given by $\operatorname{Im}(z)=3\operatorname{Re}(z)-2$.

c) Determine the exact values of $a\,\&\,b$. (4 marks)

Solution





Specific behaviours

✓ sets up equation for unknowns using gradient of perpendicular
✓ sets up equation for midpoint using given line
✓ solves for a exactly
✓ solves for b exactly

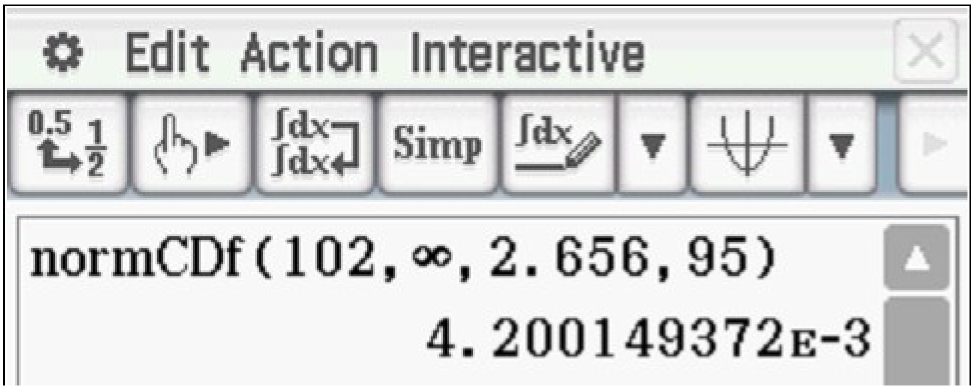
Question 10 **(9 marks)**

Consider an electronics company that manufactures transistors with weights that forms a Normal distribution of mean 95 milligrams and a standard deviation of 23 milligrams. A sample of 75 transistors is taken and the sample mean weight $\overline{X}$ of this sample of 75 is examined.

a) State the distribution $\overline{X}$ with its mean and standard deviation. (3 marks)

Solution
$\overline{X} \sim N\left(95, \left[\frac{23}{\sqrt{75}}\right]^2\right)$ OR $N(95, 2.656^2)$
Specific behaviours
✓ states Normal ✓gives mean ✓ gives standard deviation (un-simplified)

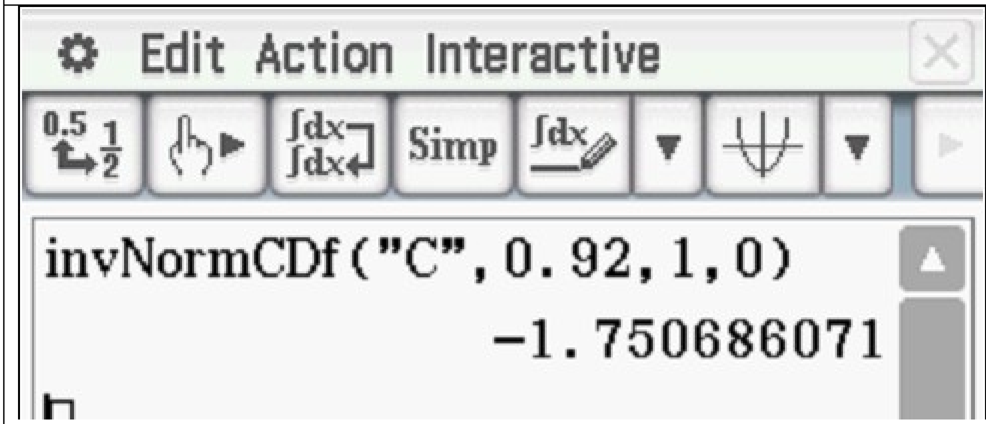
b) Determine the probability that the sample mean is greater than 102 milligrams. (2 marks)

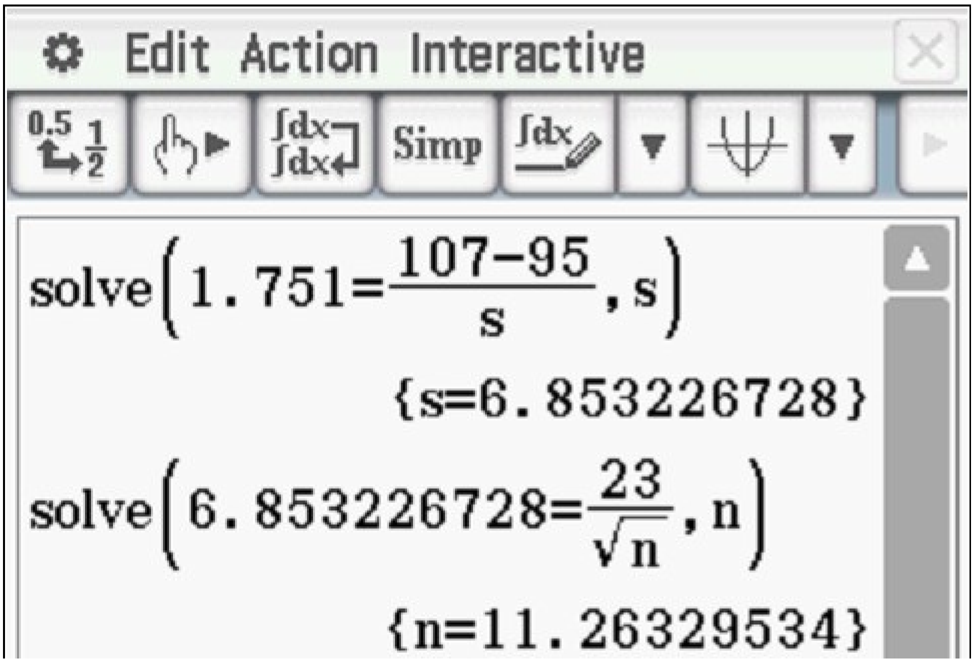
Solution

Specific behaviours
✓ uses correct parameters ✓ states probability

c) A new sample size is chosen such that the probability that the sample mean is no more than 12 milligrams from 95 milligrams is 92%. Determine the new sample size.

(4 marks)

Solution





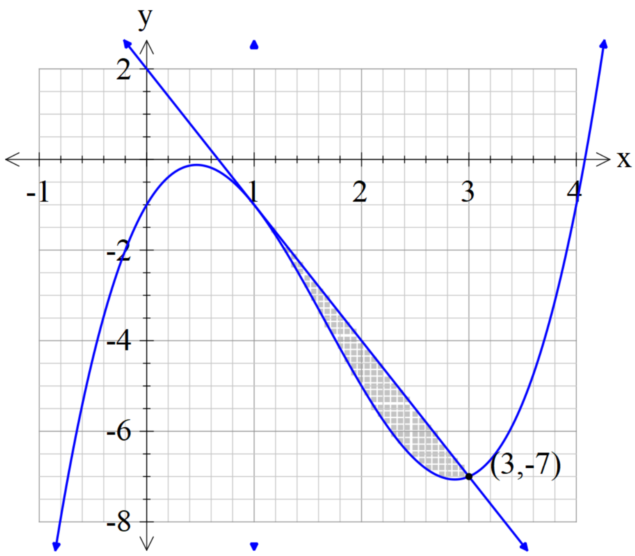
Sample size is 12

Specific behaviours

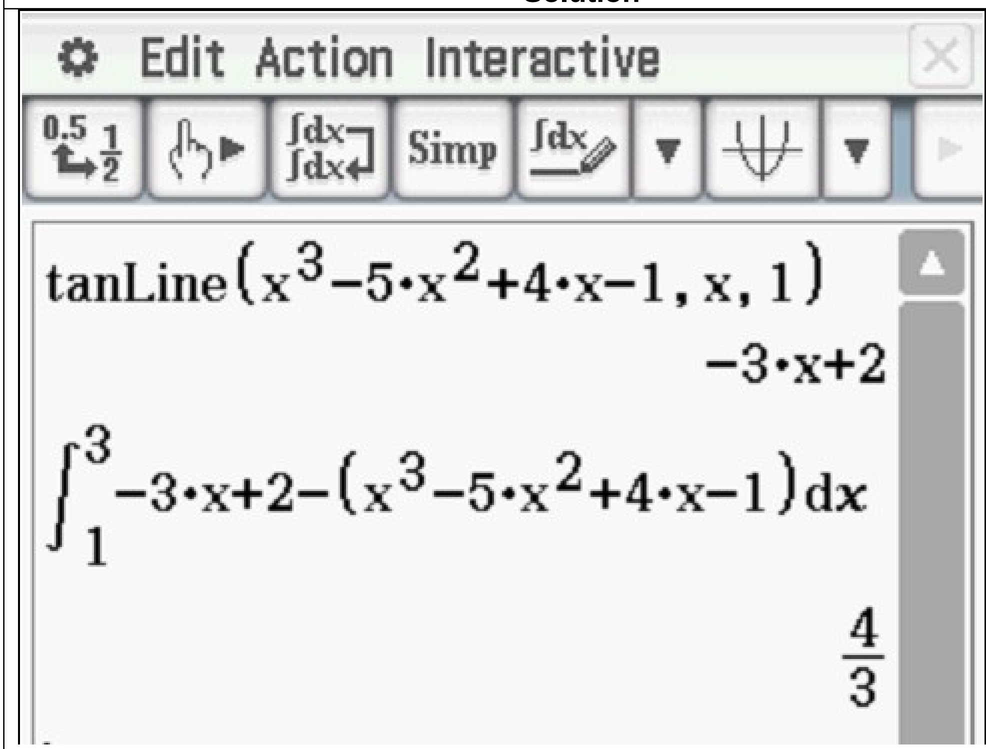
✓ determines z percentile
✓ equates z score with 107
✓ solves for standard deviation
✓ gives rounded up value for sample size

Question 11 **(7 marks)**

Consider the graph of $f(x) = x^3 - 5x^2 + 4x - 1$ and the tangent line drawn at $x = 1$.
The area between the graph and the tangent is shaded as seen below.



a) Determine the shaded area. (Exact) (3 marks)

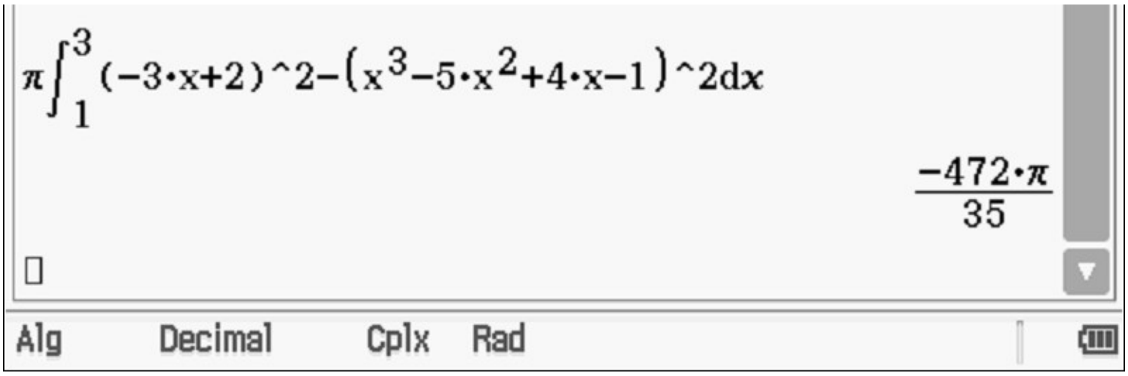
Solution

Specific behaviours
✓ determines tangent line ✓ sets up integral for area

✓ determines exact area

The shaded area is then revolved around the x axis.

b) Determine the exact volume of the resulting solid. (4 marks)

Solution



Absolute value $\frac{472\pi}{35}$

Specific behaviours
✓ uses correct integral ✓ determines volume of outer curve ✓ determines volume of inner curve ✓ determines exact volume (no need for units)

Question 12 **(7 marks)**

A super-heated metal rod cools according to the differential equation $\frac{dT}{dt} = k(T - T_o)$ where T_o is a constant representing the room temperature and k is a constant. $T(t)$ represents the temperature of the rod in degrees at time t seconds that the rod has been left in the room,

a) Determine an expression for the temperature $T(t)$ at **any time** in terms of t and the constants k & T_o. (4 marks)

Solution
$\frac{dT}{dt} = k(T - T_o)$ $\int \frac{dT}{(T - T_o)} = \int k\,dt$ $\ln\lvert T - T_o\rvert = kt + c$ $\lvert T - T_o\rvert = Ce^{kt}$ $T > T_o \quad , T = Ce^{kt} + T_o$ $T < T_o \quad , T = -Ce^{kt} + T_o$
Specific behaviours
✓ uses separation of variables ✓ uses ln(absolute value) ✓ examines 2 cases compared to room temp ✓ gives two expressions for T

It is known that the room temperature is 18 degrees and that the initial temperature is 65 degrees and $k = -0.5$.

b) Determine the time taken for the temperature of the rod to cool to 32 degrees. (3 marks)

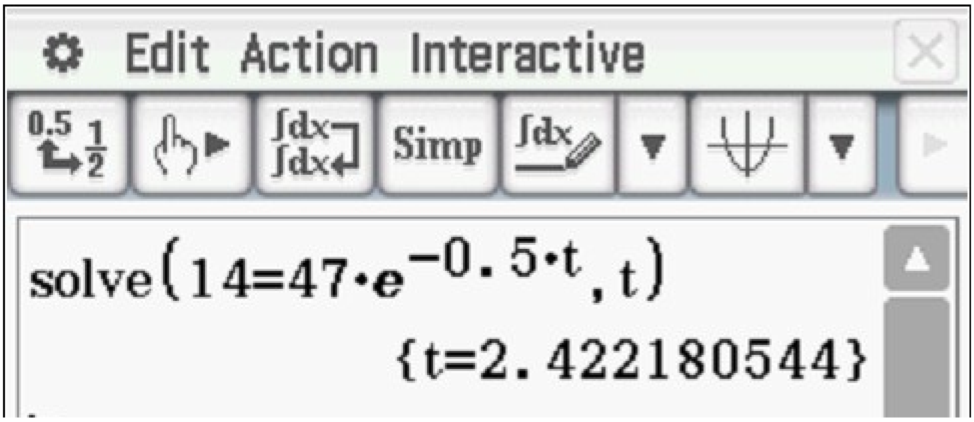
Solution

$\ln|T - T_o| = kt + c$

$|T - T_o| = Ce^{-0.5t}$

$|65 - 18| = C = 47$

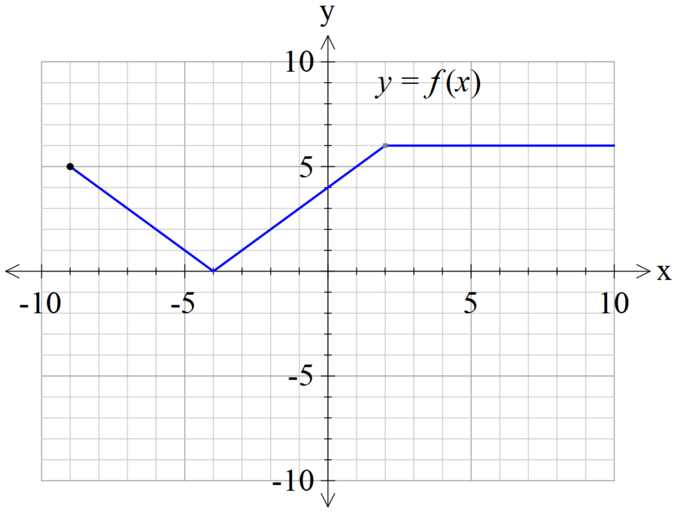
$|32 - 18| = 47e^{-0.5t}$



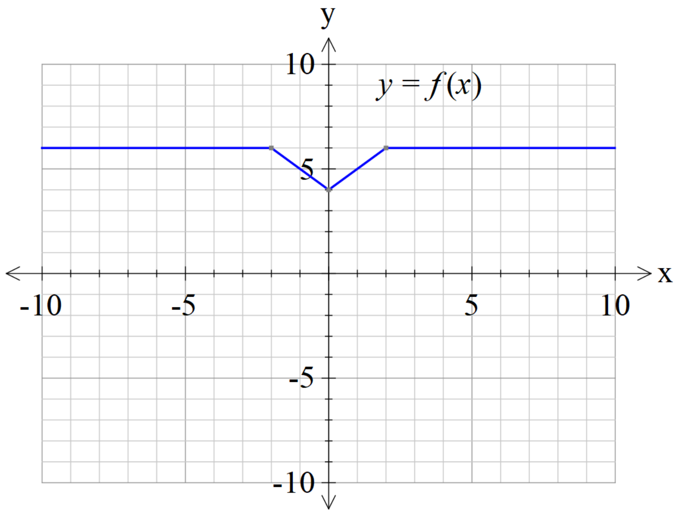
Specific behaviours
✓ solves for constant C ✓ sets up equation for t ✓ solves for t (no need for units)

Question 13 **(7 marks)**

Consider the graph of the function $y = f(x)$ as shown below.

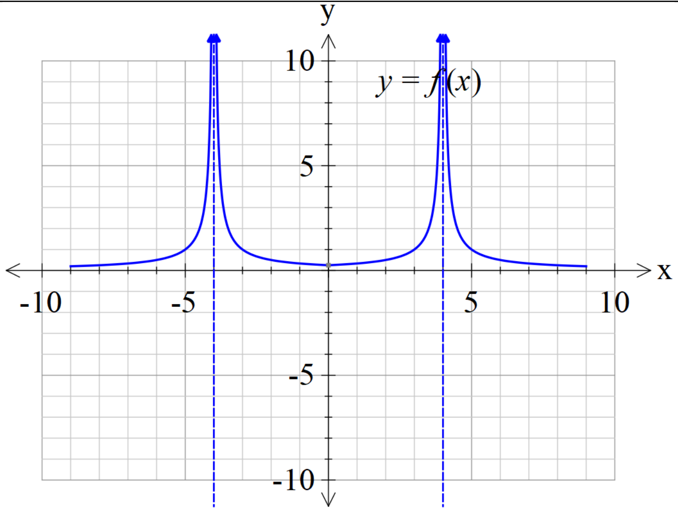


a) Sketch the graph $y = f(|x|)$ on the axes below. (3 marks)

Solution

Specific behaviours
✓ right side unchanged ✓ y intercept correct ✓ reflection of right side

b) Sketch the graph $y = \dfrac{1}{f(-|x|)}$ on the axes below. (4 marks)

Solution



Specific behaviours
✓ asymptote at x=4 ✓ asymptote at x=-4 ✓ x axis as asymptote and **only** drawn between x=-9 and x=9 ✓ symmetry about y axis

Question 14 **(10 marks)**

An object with speed v and displacement x at time t is moving with the following accelerations.

a) $a=(v+3)^2$ with $v=1$ at $t=2$. Determine the speed at $t=10$. (3 marks)

Solution

$$\frac{dv}{dt}=(v+3)^2$$

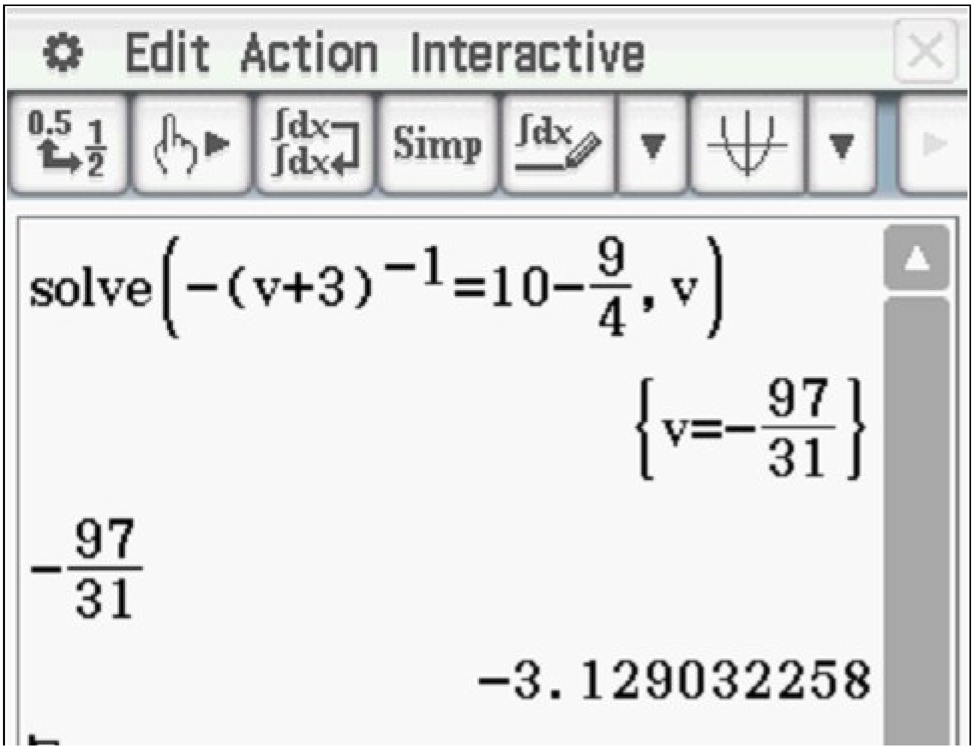
$$\int\frac{dv}{(v+3)^2}=\int dt$$

$$-(v+3)^{-1}=t+c$$

$$-\frac{1}{4}=2+c$$

$$c=-\frac{9}{4}$$

$$-(v+3)^{-1}=10-\frac{9}{4}$$

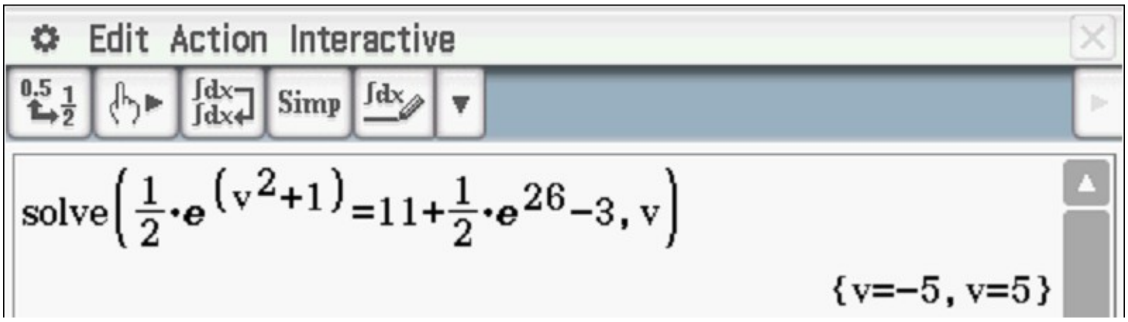


Speed is approx. 3.13 m/s

Specific behaviours

✓ uses separation of variables
✓ solves for constant
✓ solves for and states positive speed (approx.)

b) $a = e^{-(v^2+1)}$ with $v = 5$ at $x = 3$. Determine the speed at $x = 11$. (3 marks)

Solution
$v\dfrac{dv}{dx} = e^{-(v^2+1)}$ $\int ve^{(v^2+1)}\,dv = \int dx$ $\dfrac{1}{2}e^{(v^2+1)} = x + c$ $\dfrac{1}{2}e^{(25+1)} = 3 + c$ $\dfrac{1}{2}e^{(25+1)} - 3 = c$ $\dfrac{1}{2}e^{(v^2+1)} = 11 + \dfrac{1}{2}e^{(25+1)} - 3$  Speed = 5
Specific behaviours
✓ uses separation of variables ✓ integrates exponential term ✓ solves for constant and speed(states positive only)

An object is known to be moving with **speed** v given by the equation $v = 3\sqrt{(25 - x^2)}$
.

c) If initially at the origin, determine the displacement from the origin, x, at any time t.
(Hint- use the substitution $x = 5\sin u$) (4 marks)

Solution

$\frac{dx}{dt} = 3\sqrt{(25 - x^2)}$ $\int \frac{dx}{\sqrt{(25 - x^2)}} = \int 3dt$ $\int \frac{1}{\sqrt{(25 - x^2)}} \frac{dx}{du} du = \int 3dt$ $\int \frac{1}{5\cos u} 5\cos u \, du = \int 3dt$ $\int du = 3t + c$ $\text{Sin}^{-1}(\frac{x}{5}) = 3t + c$ $x = 5\sin(3t + c)$ $x = 0 \quad , t = 0 \therefore c = 0$ $x = 5\sin(3t)$
Specific behaviours
✓ uses separation of variables ✓ uses u substitution ✓ solves for constant ✓ expresses x in terms of trig function of t.

Question 15 **(9 marks)**

A particle moves according to the following parametric equations.

$x = 3\cos(2t)$

$y = 4 - \sin t$ at time t seconds, $x \,\&\, y$ in metres.

a) Determine the cartesian equation. (3 marks)

Solution
$x = 3\cos(2t) = 3(1 - 2\sin^2 t) = 3(1 - 2(4 - y)^2) = 3 - 6(4 - y)^2$ $\sin t = 4 - y$
Specific behaviours
✓ uses double angle formula for cosine ✓ expresses sint in terms of y ✓ obtains quadratic equation

b) Determine the equation of the tangent when $t = \frac{\pi}{6}$. (3 marks)

Solution
$\frac{dy}{dx} = \frac{-\cos t}{-6\sin 2t} = \frac{\frac{\sqrt{3}}{2}}{\frac{6\sqrt{3}}{2}} = \frac{1}{6}$ $y = \frac{1}{6}x + c$ $t = \frac{\pi}{6} \quad (\frac{3}{2}, \frac{7}{2})$ $\frac{7}{2} = \frac{3}{12} + c$ $c = \frac{13}{4}$ $y = \frac{1}{6}x + \frac{13}{4}$
Specific behaviours
✓ uses chain rule to find dy/dx ✓ solves for constant

✓ determines equation of tangent

c) Determine $\frac{d^2y}{dx^2}$ when $t=\frac{\pi}{6}$. (Simplify) (3 marks)

Solution
$\frac{dy}{dx}=\frac{\cos t}{6\sin 2t}$ $\frac{d^2y}{dx^2}=\frac{(6\sin 2t)(-\sin t)-\cos t(12\cos 2t)}{(6\sin 2t)^2}/(-6\sin 2t)$ $=\frac{(6\frac{\sqrt{3}}{2})(\frac{-1}{2})-\frac{\sqrt{3}}{2}(6)}{(6\frac{\sqrt{3}}{2})^2}/(-6\frac{\sqrt{3}}{2})=\frac{-9\frac{\sqrt{3}}{2}}{27}/(-6\frac{\sqrt{3}}{2})=\frac{-\sqrt{3}}{6}/(-6\frac{\sqrt{3}}{2})=\frac{1}{18}$
Specific behaviours
✓ diff dy/dx with wrt t   ✓ divides by dx/dt   ✓ simplifies result

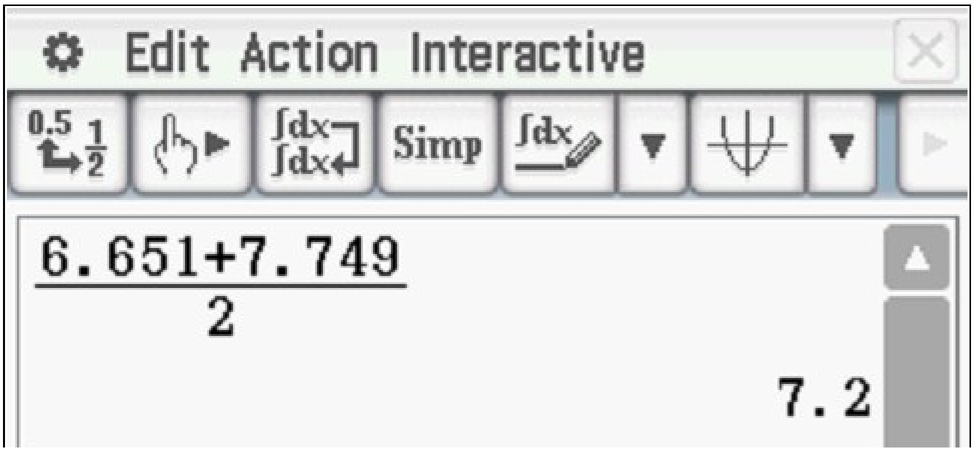
d)

Question 16 **(8 marks)**

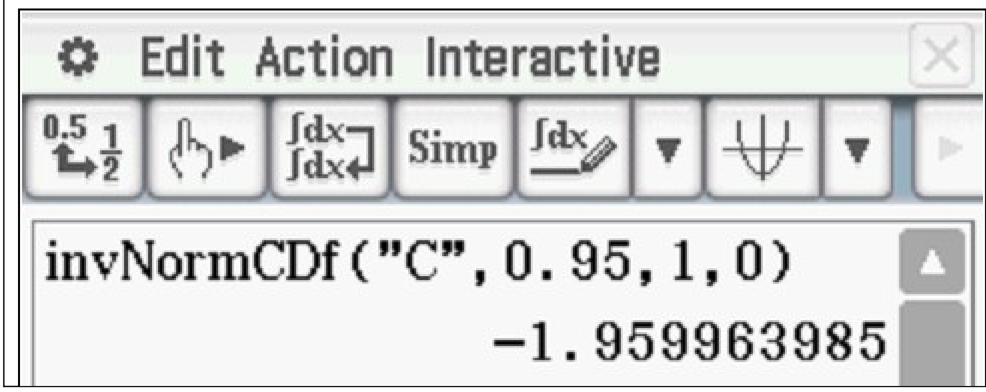
A sample of 25 tyres are used to determine the population mean weight of the type of tyre.

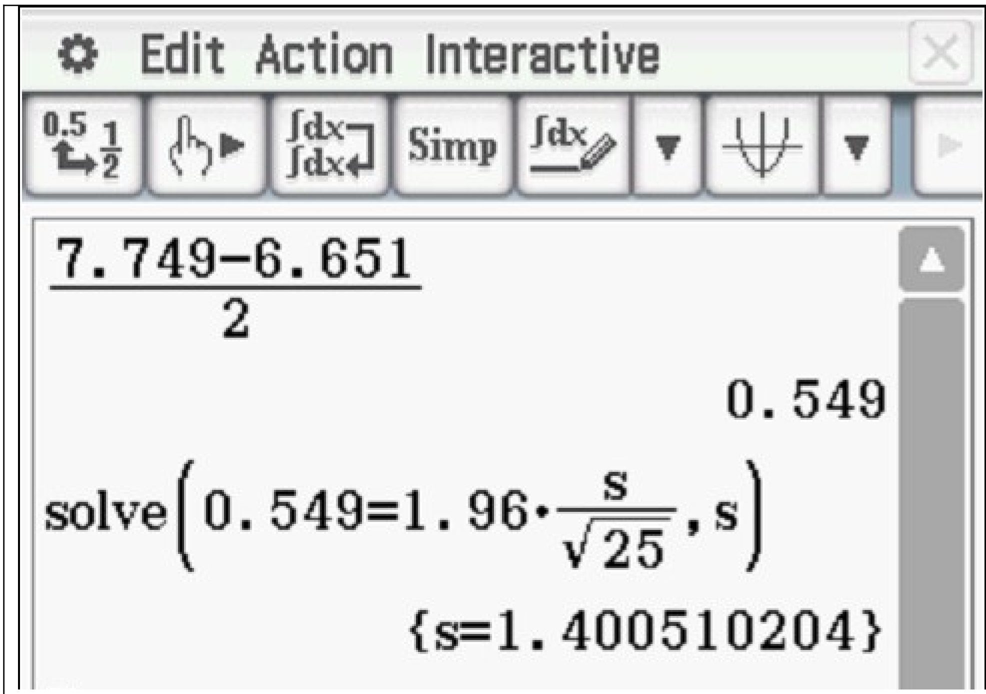
The following 95% confidence interval was calculated $(6.651, 7.749)$ kg.

a) Determine the sample mean. (1 mark)

Solution

Specific behaviours
✓ determines midpoint of interval

b) Determine the sample standard deviation. (3 marks)

Solution




Specific behaviours
✓ determines z precentile ✓ sets up equation for standard deviation ✓ solves for standard deviation

State whether the following changes would increase or decrease the width of the confidence interval and give a reason.

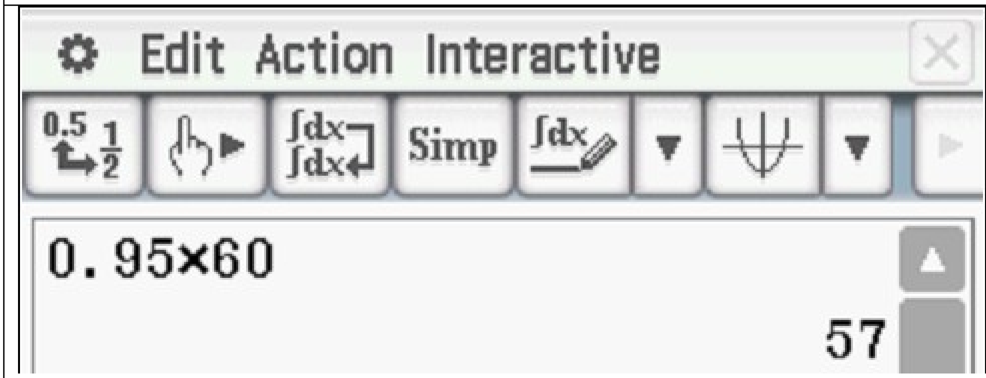
i) Have a sample size greater than 25 tyres. (1 mark)

ii) Calculate a 90% confidence interval. (1 mark)

iii) Using a smaller sample standard deviation. (1 mark)

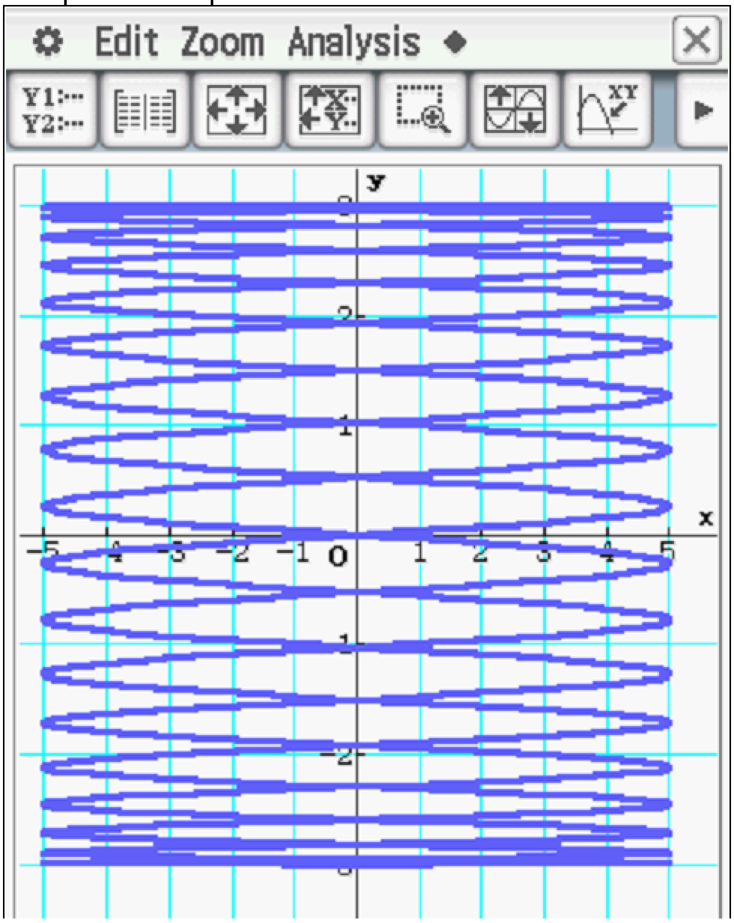
Solution
i) Decrease as width inversely prop to root n ii) Decrease as z percentile decreases iii) Decrease as width directly proportional
Specific behaviours
✓ States decrease only for two points with no reason ✓ States reason for two points ✓ States decrease with an appropriate reason for all three points

c) If 60 lots of 95% confidence interval were calculated, what number would you expect to contain the true population mean? (1 mark)

Solution

Specific behaviours
✓ states correct number

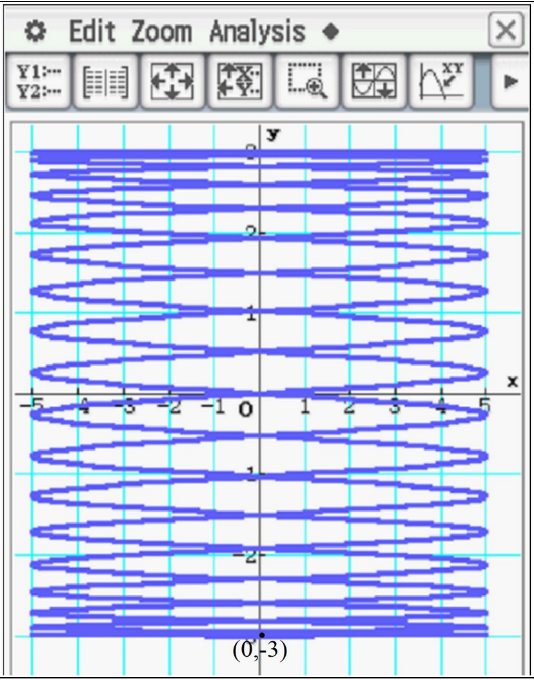
Question 17 **(8 marks)**

The position vector of a particle at time, t seconds, is given by $\underset{\sim}{r} = \begin{pmatrix} 5\sin 3t \\ -3\cos \frac{t}{6} \end{pmatrix}$ metres.
The path of the particle is shown as follows.



a) State the initial position and label on the path above. (1 mark)

Solution
$\underset{\sim}{r} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$



Specific behaviours

✓ Plots point on diagram

b) Determine the acceleration when $t = \pi$ seconds. (3 marks)

Solution

$$\underset{\sim}{r} = \begin{pmatrix} 5\sin 3t \\ -3\cos \frac{t}{6} \end{pmatrix}$$

$$\dot{\underset{\sim}{r}} = \begin{pmatrix} 15\cos 3t \\ \frac{1}{2}\sin \frac{t}{6} \end{pmatrix}$$

$$\ddot{\underset{\sim}{r}} = \begin{pmatrix} -45\sin 3t \\ \frac{1}{12}\cos \frac{t}{6} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{24} \end{pmatrix} m/s^2$$

Specific behaviours

✓ determines velocity function
✓ determines acceleration function
✓ subs correct value of t

c) Explain why the time of one complete circuit is 12π seconds. (2 marks)

Solution

$$\underset{\sim}{r} = \begin{pmatrix} 5\sin 3t \\ -3\cos\dfrac{t}{6} \end{pmatrix} \quad x\ period\ \frac{2\pi}{3} \quad y\ period\ 12\pi\ LCM\ 12\pi$$

Specific behaviours

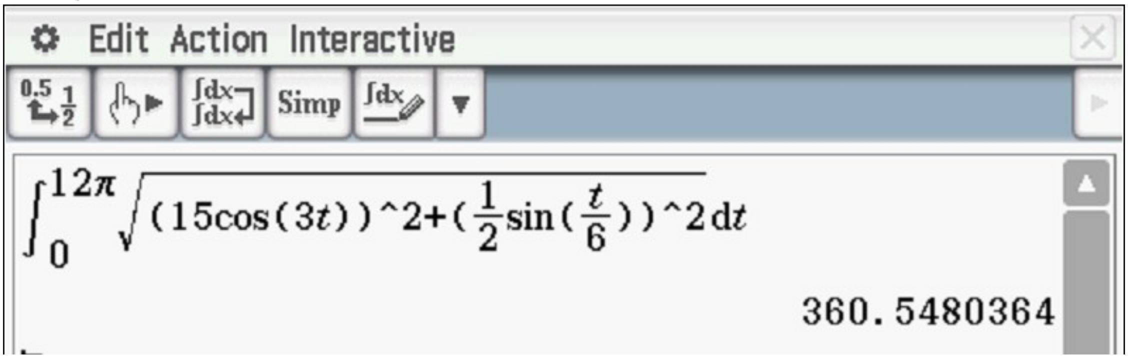
✓ states period of each dimension
✓ states LCM

d) Determine the distance travelled in one circuit. (2 marks)

Solution

$$\dot{\underset{\sim}{r}} = \begin{pmatrix} 15\cos 3t \\ \dfrac{1}{2}\sin\dfrac{t}{6} \end{pmatrix}$$

$$\left|\dot{\underset{\sim}{r}}\right| = \sqrt{(15\cos 3t)^2 + \left(\frac{1}{2}\sin\frac{t}{6}\right)^2}$$



Specific behaviours

✓ uses correct integral with speed
✓ determines approx. distance in one interval

Question 18 **(9 marks)**

a) Determine all positive values of the constant m for the function $f(x) = e^{mx}$ so that $f(x)$ will satisfy the differential equation $15\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} - 4y = 0$. (3 marks)

Solution

$$15\frac{d^2y}{dx^2}+7\frac{dy}{dx}-4y=0$$ $$15m^2e^{mx}+7me^{mx}-4e^{mx}=0$$ $$15m^2+7m-4=0$$ $$(5m+4)(3m-1)=0$$ $$m=\frac{1}{3}$$
Specific behaviours
✓ sets up equation with exponentials ✓ sets up quadratic equation for m ✓ states positive value only as solution

b) The section of the curve of the function $f(x)=e^{mx}$ in the interval $0\le x\le a$ is rotated about the x axis. Show that for the value of m found in part a above, the volume of the solid produced after one rotation is $V=\frac{3\pi}{2}\left(e^{\frac{2a}{3}}-1\right)$. (3 marks)

Solution
$$\int_0^a \pi\left(e^{2mx}\right)dx=\pi\left[\frac{1}{2m}e^{2mx}\right]_0^a=\frac{\pi}{2m}\left(e^{2ma}-1\right)=\frac{3\pi}{2}\left(e^{\frac{2a}{3}}-1\right)$$
Specific behaviours
✓ uses volume of revolution integral ✓ integrates correctly ✓ determines correct expression for required m value

c) Show that if A the area under the curve $f(x)$ in the interval $0 \le x \le a$, then

$$V = \frac{3\pi}{2}\left[\left(\frac{A}{3}+1\right)^2 - 1\right].$$

(3 marks)

Solution
$A = \int_0^a \left(e^{mx}\right)dx = \left[3e^{\frac{1}{3}x}\right]_0^a = 3\left(e^{\frac{a}{3}} - 1\right)$ $e^{\frac{a}{3}} = \frac{A}{3}+1$ $V = \frac{3\pi}{2}\left((\frac{A}{3}+1)^2 - 1\right)$
Specific behaviours
✓ integrates to determine area ✓ obtains expression for exponential term in terms of A ✓ obtains required expression.

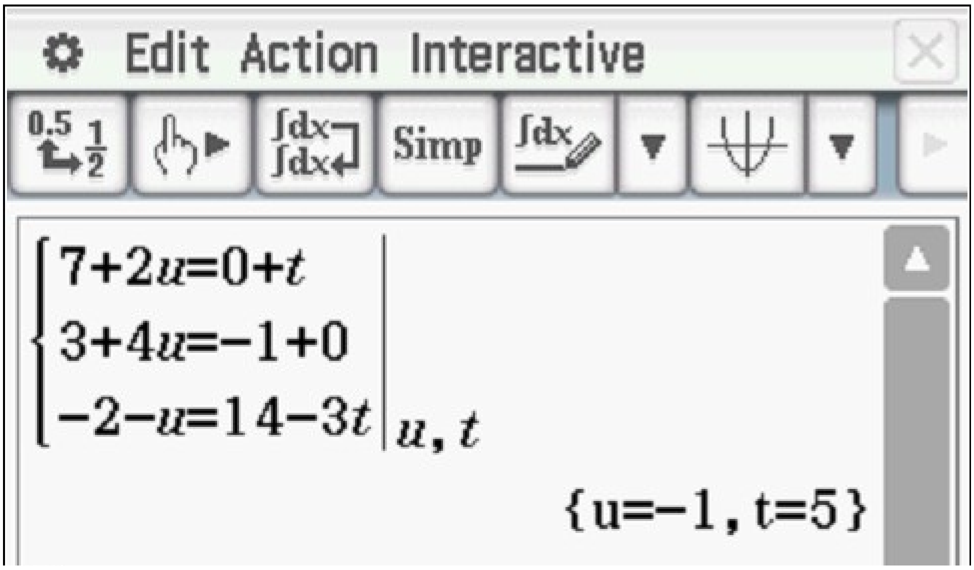
Question 19 **(7 marks)**

Two rockets A & B have initial position's $r_A = \begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} \& r_B = \begin{pmatrix} 0 \\ -1 \\ 14 \end{pmatrix}$ km at noon. They both move with constant velocities $v_A = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \& v_B = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ km/h.

a) The two rockets leave a smoke trail that stays in the air for a long period of time. Determine the point (if any) where the smoke trails cross. (3 marks)

Solution

$$\begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 14 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$



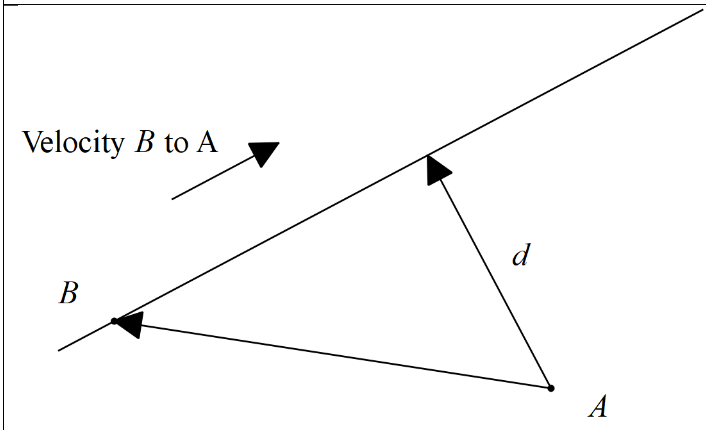
Point (5,-1,-1)
But as this involves a negative time for one rocket, the smoke trials do not cross.

Specific behaviours

✓ uses two variables
✓ sets up simultaneous equations and solves for both variables
✓ determines common point on both lines OR states that they do not cross

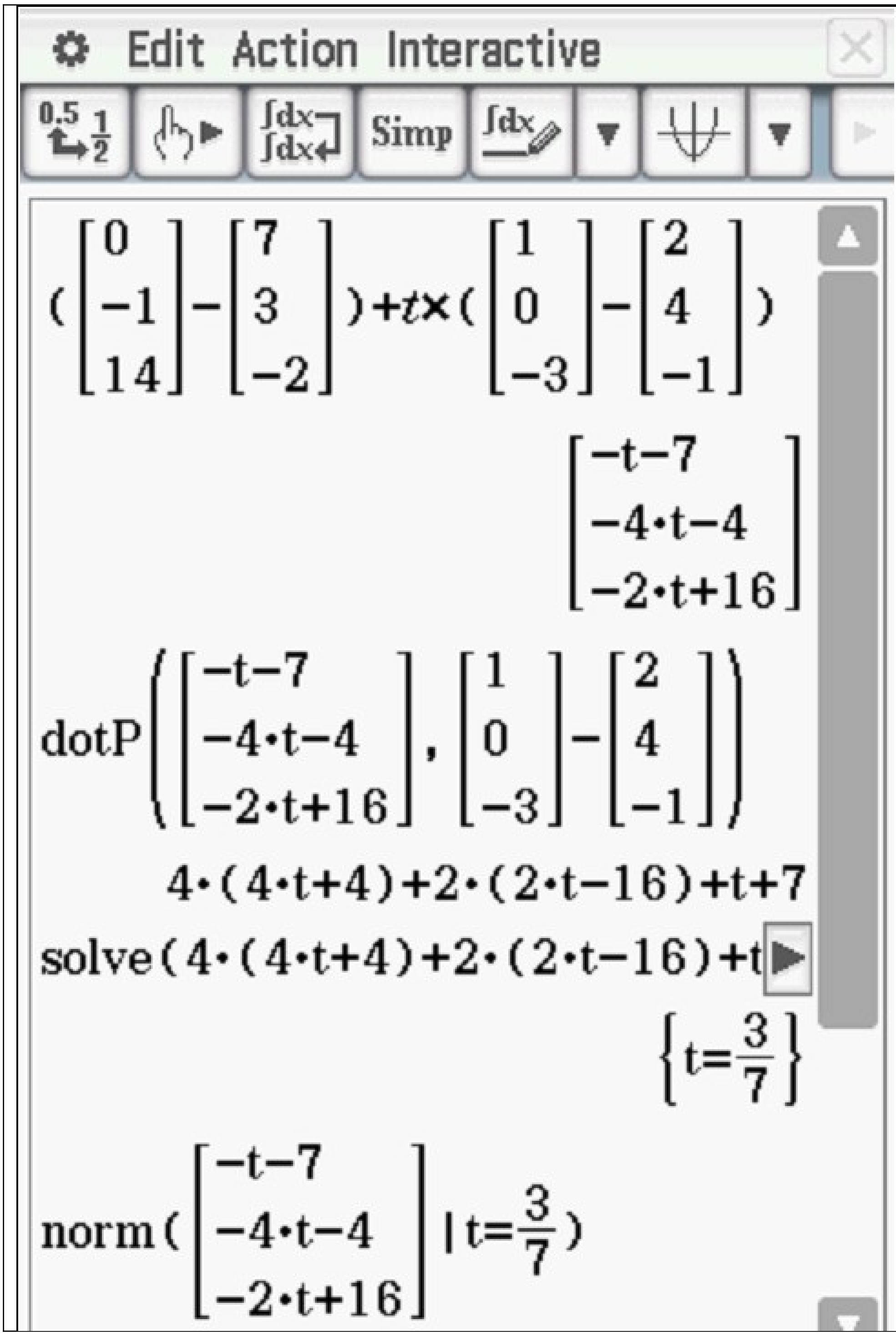
b) Determine the shortest distance between the two rockets and the time that this occurs. (4 marks)

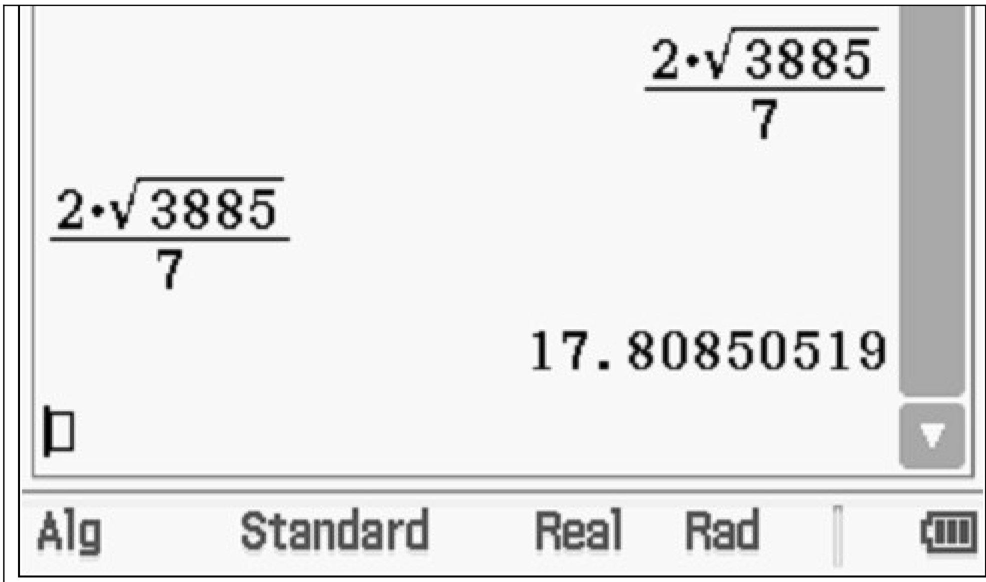
Solution



$d = AB + t_B v_A$

$d.v = 0$

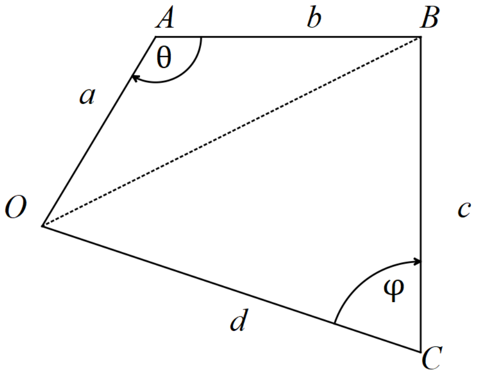

Edit Action Interactive
Simp
$\left(\begin{bmatrix}0\\-1\\14\end{bmatrix}-\begin{bmatrix}7\\3\\-2\end{bmatrix}\right)+t\times\left(\begin{bmatrix}1\\0\\-3\end{bmatrix}-\begin{bmatrix}2\\4\\-1\end{bmatrix}\right)$
$\begin{bmatrix}-t-7\\-4\cdot t-4\\-2\cdot t+16\end{bmatrix}$
$\mathrm{dotP}\left(\begin{bmatrix}-t-7\\-4\cdot t-4\\-2\cdot t+16\end{bmatrix},\begin{bmatrix}1\\0\\-3\end{bmatrix}-\begin{bmatrix}2\\4\\-1\end{bmatrix}\right)$
$4\cdot(4\cdot t+4)+2\cdot(2\cdot t-16)+t+7$
solve$(4\cdot(4\cdot t+4)+2\cdot(2\cdot t-16)+t$
$\left\{t=\frac{3}{7}\right\}$
norm$\left(\begin{bmatrix}-t-7\\-4\cdot t-4\\-2\cdot t+16\end{bmatrix}\mid t=\frac{3}{7}\right)$



Specific behaviours
✓ uses relative velocity ✓ obtains an expression for closest distance ✓ uses dot product to solve for time ✓ states both time and approx. closest distance

Question 20 **(7 marks)**

Consider the quadrilateral $OABC$ with fixed side lengths $a, b, c \,\&\, d$. Let $\theta \,\&\, \varphi$ be opposite angles.



a) Show that the area of the quadrilateral is $A = \frac{1}{2}ab\sin\theta + \frac{1}{2}cd\sin\varphi$. (1 mark)

Solution
Uses sine rule for area for both triangles
Specific behaviours
✓ uses areas sine rule for both triangles

b) By considering the common side $\overline{OB}$ to both triangles above, show that $\frac{d\varphi}{d\theta} = \frac{ab\sin\theta}{cd\sin\varphi}$.

(3 marks)

Solution
$a^2 + b^2 - 2ab\cos\theta = OB^2 = c^2 + d^2 - 2cd\cos\varphi$ $diff\ both\ sides\ wrt\ \theta$ $2ab\sin\theta = 2cd\sin\varphi\frac{d\varphi}{d\theta}$ $\frac{d\varphi}{d\theta} = \frac{ab\sin\theta}{cd\sin\varphi}$
Specific behaviours
✓ uses cosine rule for diagonal length OB ✓ implicit diff of both sides wrt to one angle ✓ obtains required expression

c) Hence show **using calculus** that the area of the quadrilateral is optimal, $\frac{dA}{d\theta}=0$, when opposite angles are supplementary, $\theta+\varphi=\pi$. (3 marks)

Solution
$A=\frac{1}{2}ab\sin\theta+\frac{1}{2}cd\sin\varphi$ $\frac{dA}{d\theta}=\frac{1}{2}ab\cos\theta+\frac{1}{2}cd\cos\varphi\frac{d\varphi}{d\theta}$ $\frac{dA}{d\theta}=\frac{1}{2}ab\cos\theta+\frac{1}{2}cd\cos\varphi\frac{ab\sin\theta}{cd\sin\varphi}=0$ $\frac{\sin\theta}{\sin\varphi}=\frac{-\cos\theta}{\cos\varphi}$ $\sin\theta\cos\varphi+\sin\varphi\cos\theta=0$ $\sin(\theta+\varphi)=0$ $as\ \theta\ \&\ \varphi\ are\ less\ than\ \pi$ $\theta+\varphi=\pi$
Specific behaviours
✓ obtains first derivative using expression in part b ✓ equates to zero and obtains an expression in terms of angles only ✓ shows using compound formula for sine that angles must be supplementary

Additional working space

Question number: __________

Additional working space

Question number: _________

Additional working space

Question number: ________

Additional working space

Question number: __________