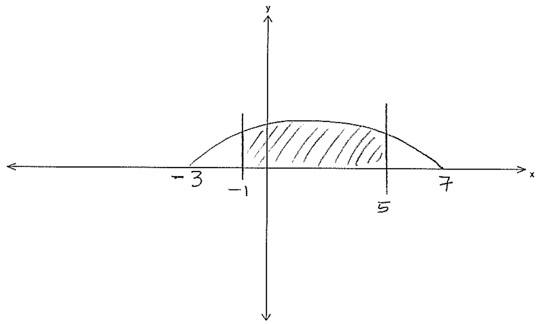
Question 21 **(6 marks)**

The internal contours of a barrel are defined by the rotation about the x axis of the curve

$y = 0.1(x+3)(7-x)$ between $x = -1$ and $x = 5$

Each unit on both the x and y axes represents 15 cm.

(a) Sketch and label this situation on the axes below. (1 mark)



(b) When the barrel is filled to its brim, how many litres can it hold? Give your answer correct to one decimal place. (5 marks)

$V = \pi \int_{-1}^{5} \left(0.1(x+3)(7-x)\right)^2 dx$

$= 92.589 \text{ units}^3$

each unit = 15 cm so

$V = 92.589 \times 15^3$

$= 312487.9381 \text{ cm}^3$

Barrel can hold 312.5 L

END OF QUESTIONS

Section One: Calculator-free **(40 Marks)**

This section has **nine (9)** questions. Answer **all** questions. Write your answers in the space provided or on the spare pages included at the end of this booklet.

Working time for this section is 50 minutes.

Question 1 **(4 marks)**

Determine $\frac{dy}{dx}$ for the following functions:

(a) $y = \frac{5}{(4x+2)^3}$ (2 marks)

$y = 5(4x+2)^{-3}$

$\frac{dy}{dx} = -15(4x+2)^{-4} \times 4$

$= \frac{-60}{(4x+2)^4}$

(b) $y = \frac{3x^5}{e^{2x}}$ (2 marks)

$\frac{dy}{dx} = \frac{15x^4e^{2x} - 2e^{2x}3x^5}{e^{4x}}$

$= \frac{15x^4 - 6x^5}{e^{2x}}$

OR $y = 3x^5e^{-2x}$

$= 15x^4e^{-2x} + (-2e^{-2x})(3x^5)$

$= \frac{15x^4 - 6x^5}{e^{2x}}$

or $\frac{3x^4(5-2x)}{e^{2x}}$

Question 2 **(3 marks)**

R and S are events where $P(R) = \frac{1}{3}$, $P(S) = \frac{1}{4}$, and $P(R \cup S) = \frac{1}{2}$.

(a) Find $P(S|R)$. (2 marks)

$P(R \cap S) = \frac{1}{3} + \frac{1}{4} - \frac{1}{2}$

$= \frac{1}{12}$

$P(S/R) = \frac{P(R \cap S)}{P(R)}$

$= \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{4}$

(b) Are R and S independent? Give a reason. (1 mark)

Yes $P(S/R) = P(S) = \frac{1}{4}$

Question 3 **(3 marks)**

Determine the gradient of $y = (1-2x)^4$ at the point ~~(2, 81)~~ $(1, 1)$.

$\frac{dy}{dx} = 4(1-2x)^3(-2)$

$= -8(1-2x)^3$

at $x = 1$

$m = \frac{dy}{dx} = -8(1-2)^3$

$= -8 \times -1$

$= 8$

Question 20 **(~~7~~ 8 marks)**

A particle is moving under rectilinear motion with velocity $v(t) = -2t + 9t^2$ m/s. Answer the following questions for the movement of the particle over the time interval $0 \le t \le 6$.

(a) If the particle was initially 2 m to the right of the origin, what is the displacement from the origin after 2 seconds? (2 marks)

$s = -t^2 + 3t^3 + 2$

at $t = 2$

$s = 22m$
(to the right)

(b) How far did the particle travel in the first 2 seconds? (2 marks)

$\int_0^2 |-2t + 9t^2| dt = 20\ 03\ m$

OR $v(t) = 0$ at $t = 0$ and $t = \frac{2}{9}$

at $t = 0$ $s = 2$
at $t = \frac{2}{9}$ $s = 1 \cdot 9835$
at $t = 2$ $s = 22$

distance $= 0.0165 + 20.0165$
$= 20.03\ m$

(c) When was the particle moving fastest? (~~1~~ 2 mark)

test at tp and beginning + end of interval

at $t = 0 \cdot \dot{1}$ $v = -0 \cdot \dot{1}$
at $t = 0$ $v = 0$
at $t = 6$ $v = 312$

(or look at graph)

At $t = 6$ seconds

(d) For what subset(s) of the given time interval is the acceleration negative? (2 marks)

$a(t) = -2 + 18t$

$-2 + 18t < 0$

$t < \frac{1}{9}$

ie $0 \le t < \frac{1}{9}$

or just $t < \frac{1}{9}$

Question 19 **(7 marks)**

The mass of tahini in a particular type of jar is a normally distributed random variable, whose standard deviation is 2.75 g and whose mean μ g is unknown.

In studying a random sample of n such jars, call the average mass of tahini per jar for the sample, $\bar{x}$ g.

(a) What should the sample size n be, if we want to be 99% confident that $\bar{x}$ differs from μ by less than 1.5 g? (3 marks)

$$\frac{2.576\sigma}{\sqrt{n}} < 1.5$$

$$n > 22.3$$

Need a sample size of 23 (or more)

The average mass of tahini in one random sample of 25 such jars is 250.2 g.

(b) Calculate a 95% confidence interval for μ. (2 marks)

$$250.2 - \frac{1.96 \times 2.75}{\sqrt{25}} < \mu < 250.2 + \frac{1.96 \times 2.75}{\sqrt{25}}$$

$$249.12 < \mu < 251.28$$

(c) For customer satisfaction, the desired amount of tahini in a jar is to be the amount stated on the label (which is 250g) or perhaps a little over rather than under that amount. Explain what the tahini producers can infer from the confidence interval obtained in your answer to part (b) and what choices they might then decide to make. (2 marks)

They can infer that 95% of the jars lie within a narrow range of 2.16g, or that only 2.5% of them are at least 0.89g below the labelled amount. 0.89g is a very small amount of error (0.356%) so they may choose to ignore the results. 97.5% have a mass at or above the labelled amount but they could have good PR if they calibrated the machines to have the lower end of the confidence interval above 250g.

See next page

Question 4 **(5 marks)**

A function is defined by the rule $y = f(g(x))$, where $f(x) = \cancel{\sqrt{x-2}}$ 2^{x+2} and $g(x) = \cancel{\frac{2}{x+1}}$ $\sqrt{x+1}$.

(a) Determine the domain and range of:

(i) $f(x)$ ($\cancel{1}$ 2 mark)

domain $\{x : x \in \mathbb{R}\}$
range $\{y : y > 0,\ y \in \mathbb{R}\}$

(ii) $f(g(x))$ (2 marks)

domain: $\{x : x \geq -1,\ x \in \mathbb{R}\}$
range: $\{y : y \geq 4,\ y \in \mathbb{R}\}$

(b) Without substituting any values in f(g(x)), italics determine whether or not the point ~~(1,1)~~ (0,1) lies on the curve defined by $f(g(x))$ and justify your answer. (1 mark)

No if $x = 0$, $y = 8$

or if $y = 1$ then not in the range of $y \geq 4$

(* Note point should have been (0, 8)!)

Question 5 **(5 marks)**

Solve the inequality below:

$$\frac{1}{2x-1} \geq \frac{2}{x+2}$$

$$\frac{1}{2x-1} - \frac{2}{x+2} \geqslant 0$$

$$\frac{x+2-2(2x-1)}{(2x-1)(x+2)} \geqslant 0 \qquad x \neq -2 \quad x \neq \frac{1}{2}$$

$$\frac{-3x+4}{(2x-1)(x+2)} \geqslant 0$$

Critical points

$-3x+4=0$ $\quad 2x-1=0$ $\quad x+2=0$

$x = \frac{4}{3}$ $\quad x=\frac{1}{2}$ $\quad x=-2$

$+ \quad - \quad + \quad -$

$-2 \quad \frac{1}{2} \quad \frac{4}{3}$

$\therefore \quad x < -2$

$\frac{1}{2} < x \leq \frac{4}{3}$

Question 6 **(4 marks)**

(a) Determine the indefinite integral: ($\cancel{2}$ 1 marks)

(x^3-3x^2+1)

$\int \cancel{(x+2)^2(x-1)}dx$

$$= \frac{x^4}{4} - x^3 + x + c$$

(b) Evaluate the definite integral: (in terms of e). ($\cancel{2}$ 3 marks)

$$\int_0^2 \frac{3x}{2}e^{x^2}dx = \frac{3}{4}\int_0^2 2xe^{x^2}dx$$

$$= \frac{3}{4}\left[e^{x^2}\right]_0^2$$

$$= \frac{3}{4}(e^4-1)$$

See next page

Question 18 **(7 marks)**

In a quiet country town, the waiting time to get onto a roundabout through a particular entry point during the afternoon peak hour can be represented by a random variable in a normal distribution. The mean waiting time is 75 seconds, with a standard deviation of 25 seconds. Give answers to 4 decimal places.

(a) What is the probability that a motorist has a wait of less than 5 seconds? (1 mark)

$P(t<5) = 0.0026$

(b) Evaluate the probability that a motorist has to wait more than a minute and a half. (1 mark)

$P(t>90) = 0.2743$

(c) What is the probability that the wait is between 1 and 2 minutes? (1 mark)

$P(60<t<120) = 0.6898$

(d) Given that a motorist has already been waiting behind one other car for 30 seconds, what is the probability that once he gets onto the roundabout he will have waited at most a total of 125 seconds before being able to enter it? (2 marks)

$$P(t<125 \mid t>30) = \frac{P(30<t<125)}{P(t>30)} = \frac{0.9413}{0.9641}$$

$$= 0.9764$$

(e) A resident of the town passes through this intersection during the afternoon peak hour every day from Monday to Friday in a particular week. What is the probability that this motorist has to wait between 1 and 2 minutes on at least 3 of these days? (2 marks)

$Y \sim B(5, 0.6898)$

$P(Y \geqslant 3) = 0.8232.$

See next page

Question 17 **(7 marks)**

(a) Show that the surface area and volume of any cube are related to each other by deriving the formula $S.A. = 6V^{\frac{2}{3}}$ where $S.A.$ is surface area and V is volume. (2 marks)

let ℓ be side length of cube

$SA = 6\ell^2$

$V = \ell^3 \rightarrow \ell = \sqrt[3]{V} = V^{\frac{1}{3}}$

$SA = 6\left(V^{\frac{1}{3}}\right)^2$

$= 6V^{\frac{2}{3}}$

(b) Use calculus to find the approximate percentage change in surface area if the volume increases by 12%. (5 marks)

$\delta V = 0.12V$

$\delta SA \approx \dfrac{dSA}{dV} \times \delta V$

$= 6 \times \frac{2}{3} V^{-\frac{1}{3}} \times 0.12V^1$

$= 0.48V^{\frac{2}{3}}$

$\dfrac{\delta SA}{SA} = \dfrac{0.48V^{\frac{2}{3}}}{6V^{\frac{2}{3}}}$

$= 0.08$

so approx % change in SA is 8%

Question 7 **(5 marks)**

Solve the system of equations:

$2x + 3y - z = 15$ (1)

$4x + 5y + 2z = 4$ (2)

$2x - 4y - 3z = 13$ (3)

(1) × 2 $\quad 4x + 6y - 2z = 30$

(2) $\quad 4x + 5y + 2z = 4$

(1) + (2) $\quad 8x + 11y = 34$ (4)

(1) × 3 $\quad 6x + 9y - 3z = 45$

(3) $\quad 2x - 4y - 3z = 13$

(1) − (3) $\quad 4x + 13y = 32$ (5)

(4) $\quad 8x + 11y = 34$

(5) × 2 $\quad 8x + 26y = 64$

(4) − (5) $\quad -15y = -30$

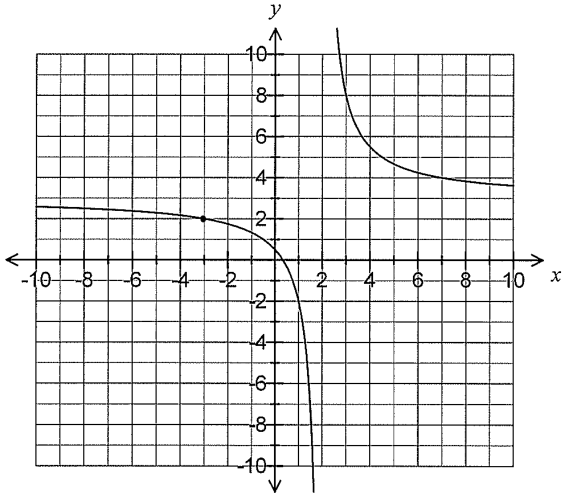
$y = 2$

$\therefore x = 1.5$

$z = -6$

Question 8 **(3 marks)**

The graph of the hyperbola $y = \frac{a}{x+b} + c$ is shown below. The point (-3, 2) lies on the curve, $y \to 3$ as $x \to \pm\infty$ and $y \to \pm\infty$ as $x \to 2$.



Evaluate a, b and c showing any working.

$b = -2$
$c = 3$

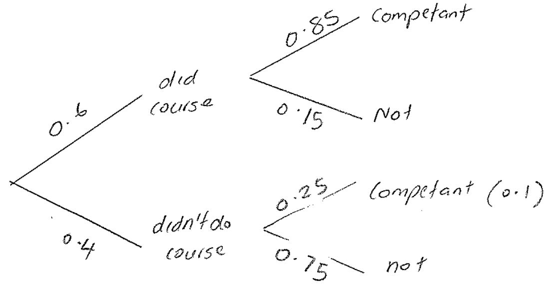
$2 = \frac{a}{-3-2} + 3$

$a = 5$

Question 16 **(6 marks)**

When new lathes were bought for the fabrication section of a large manufacturer of steel products, 60% of the employees attended a special training course on how to use them. Of these 85% passed the competency test for working on them.

(a) If 10% of the employees didn't go on the course because they already had the competency qualifications to use them, what percentage of employees is now qualified to use the new lathes? (4 marks)



% qualified $= 0.6 \times 0.85 + 0.4 \times 0.25$
$= 0.61$
or 61%

(b) What is the probability that a randomly chosen employee attended the training course, given that they are found <u>not</u> to be qualified to use the new lathes? (2 marks)

P (did course / not qualified)

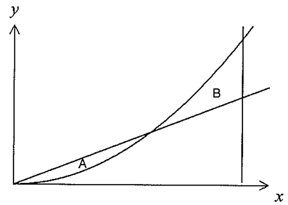
$= \frac{0.6 \times 0.15}{0.6 \times 0.15 + 0.4 \times 0.75}$

$= \frac{0.09}{0.39}$

$= \frac{3}{13}$ or 0.2308

Question 15 **(7 marks)**

The graph below, not to scale, shows the functions $f(x)=\frac{x}{10}$, $g(x)=\frac{x^2}{10}$ and the line $x=2$.



Region A is the area trapped by f and g.
Region B is the area trapped by f, g and the line $x=2$.

(a) Find the areas of regions A and B. (3 marks)

$f(x)$ + $g(x)$ intersect at $x=1$

Area $A = \int_0^1 (f-g)\,dx = \frac{1}{60}$ units^2

Area $B = \int_1^2 (g-f)\,dx = \frac{1}{12}$ units^2

(b) $f(x)$ is modified to become the line $f(x)=kx$, so that the area of region A is exactly the same as the area of region B. Determine the value of k. (4 marks)

$kx = \frac{x^2}{10}$

$x = 10k$
$(x=0)$

$\int_0^{10k}\left(kx - \frac{x^2}{10}\right)dx = \int_{10k}^{2}\left(\frac{x^2}{10} - kx\right)dx$

$\frac{50k^3}{3} = \frac{50k^3}{3} - 2k + \frac{4}{15}$

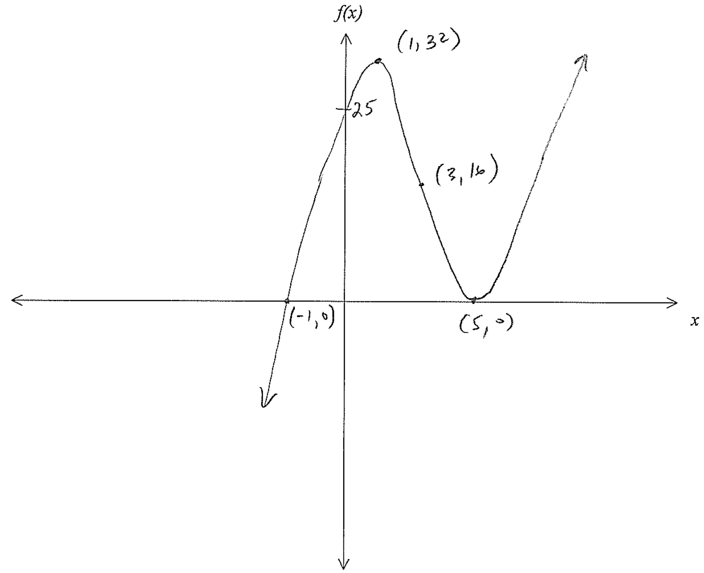
$k = \frac{2}{15}$

(or just $\int_0^1 \left(kx - \frac{x^2}{10}\right)dx = \int_1^2 \left(\frac{x^2}{10} - kx\right)dx$)

See next page

Question 9 **(8 marks)**

On the axes below, sketch the function $f(x) = x^3 - 9x^2 + 15x + 25$ showing any turning points, points of inflection and intercepts on axes.



$f(0) = 25$ y-intercept

$f'(x) = 3x^2 - 18x + 15 = 0$ stationary pts

ie $x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0$

$x = 5$ or $x = 1$

$f(5) = 0$
$f(1) = 32$

$(5,0)$ $(1,32)$ are turning points

$f''(x) = 6x - 18 = 0$ pointof inflection
$x = 3$

$f(3) = 16$

End of Questions

$(3,16)$ point of inflection

$f(-1) = 0$ shape of graph

Section Two: Calculator-assumed **(80 Marks)**

twelve (12)

This section has ~~thirteen~~ **(11)** questions. Answer **all** questions. Write your answers in the space provided.

Suggested working time for this section is 100 minutes.

Question 10 **(4 marks)**

A spherical balloon is being blown up. When its radius is increasing at a rate of 1.5 cm per second, its volume is 905 cm^3.
What is the rate at which the volume is increasing at this instant?

$\frac{dr}{dt} = 1.5$ cm/sec

$V = \frac{4}{3}\pi r^3$ $\qquad \frac{dV}{dr} = 4\pi r^2$

$905 = \frac{4}{3}\pi r^3$

$r = 6.000489$

or $r = 6$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$= 4\pi r^2 \times 1.5$

$= 678.7$

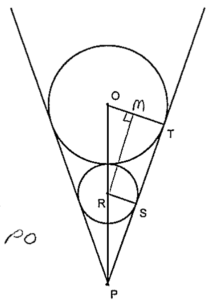
Volume is increasing at a rate of $678.7\ cm^3/s$

Question 14 **(7 marks)**

Two spheres fit inside an inverted cone as shown in the diagram below. Circle centre O has radius 4 cm and circle centre R has radius 2 cm.

(a) Prove that Δ PRS is congruent to Δ ROM. Hint: draw a perpendicular from R ~~to~~ to m, m a point on OT. (4 marks)

(b) Hence, or otherwise, calculate the height that the centre of the larger sphere is above the vertex of the cone, citing any theorems or axioms or previously deduced facts used in the logical steps needed to get the answer. (3 marks)



a) In Δ OMR and Δ RSP

∠OMR = 90° given

∠PSR = 90° SR ⊥ PT radius contact with tangent

∴ ∠OMR = ∠PSR

∠MOR = ∠SRP corresponding angles OT // RS, transversal PO

$\overline{OM} = \overline{RS}$ both 2cm (*)

∴ Δ OMR ≅ Δ RSP AAS

(*) om = 2cm actually needs proving. However allow 1 mk for the statement

b) PR = OR corresponding parts of congruent triangles

OR = 6cm sum of the 2 radii

PO = PR + OR

= 12 units

(*)

Rm ⊥ to OT but not given that it is 90°

Prove mTSR is a rectangle

$\overline{MR}$ // $\overline{TS}$ ∠OMR = 90° given, ∠MTS = 90° tangent ⊥ radius
corresponding angles equal ∴ parallel lines

$\overline{mT}$ // $\overline{RS}$ ∠mTS = 90°, ∠RSP = 90° both tangent ⊥ radius
corresponding angles equal ∴ parallel lines

2 pairs parallel sides, all 90° angles → rectangle

∴ $\overline{RS} = \overline{mT}$ = 2cm. (∴ $\overline{om}$ = 2cm)

(c) How many of each pack should the apprentice buy to minimise the purchase cost and what is the minimum cost? (3 marks)

Point	Cost = $3.5x + 6.5y$
(0, 16)	104
(5, 6)	56.5
(9, 4)	57.5
(21, 0)	73.5

Buy 5 Best Buys and 6 chef choice for a min cost of $56.50

(d) By how much can the price of a 'Best Buy' pack rise without changing the optimum number of packs found in your answer to (c)? (~~2~~ 3 marks)

$C = kx + 6.5y$

$(0, 16) \geq (5, 6)$

$0x + 104 \geq 5x + 39$

$65 \geq 5x$

$x \leq 13$

Can increase by $9.50

Question ~~12~~ 11 **(7 marks)**

When an unfair coin is tossed, it has an 80% chance of landing heads up. Assign the value 1 to X if it lands heads up and 0 to X if it lands tails up.

(a) Use this information to complete the table below which shows all possible sample outcomes for experiments of tossing the coin 4 times. Calculate their corresponding probabilities, assign X values and calculate the mean $\overline{X}$ each time. (3 marks)

Outcome	Probability	$\overline{X}$
HHHH	0.4096	1
HHHT	0.1024	$\frac{3}{4}$
HHTH	0.1024	$\frac{3}{4}$
HTHH	0.1024	$\frac{3}{4}$
THHH	0.1024	$\frac{3}{4}$
HHTT	0.0256	$\frac{1}{2}$
HTHT	0.0256	$\frac{1}{2}$
HTTH	0.0256	$\frac{1}{2}$
THTH	0.0256	$\frac{1}{2}$
THHT	0.0256	$\frac{1}{2}$
TTHH	0.0256	$\frac{1}{2}$
HTTT	0.0064	$\frac{1}{4}$
THTT	0.0064	$\frac{1}{4}$
TTHT	0.0064	$\frac{1}{4}$
TTTH	0.0064	$\frac{1}{4}$
TTTT	0.0016	0

(b) Use your data from part (a) to create a sampling distribution in the table below. Let $\bar{x}$ be the possible values of the means obtained in table (a). (1 mark)

$\bar{x}$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$P(\overline{X} = \bar{x})$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

(c) What type of statistical distribution does that in part (b) resemble? (1 mark)

Binomial

(d) If the sample size (number of times the coin is tossed) were to be increased from 4, describe how the type of distribution you have could change. Name any rules or theorems that may apply. (2 marks)

The bigger the sample size the distribution approximates a normal distribution.

The central limit theorem tell us this

Question 12 **(3 marks)**

A new drug can kill off a live mould in a petri dish according to the rule $\frac{dM}{dt} = -0.5M$, where M is the amount of live mould in the petri dish when the drug was added to it and t is the time in hours since it was added.

When will the size of the mould have reduced to 20% of the original amount? Give your answer correct to 2 significant figures.

$A = A_0 e^{-0.5t}$

$0.2 \cancel{A_0} = \cancel{A_0} e^{-0.5t}$

Any method

$t = 3.2$ hours

Question 13 (~~10~~ 11 marks)

Every weekday the chef at a restaurant sends out an apprentice to the local market to spend as little as possible and at the same time come back with at least 16kg of onions, at least 17kg of carrots and at least 21kg of potatoes.

One stall at the market sells 'Best Buy' packs consisting of 2kg of onions, 1kg of carrots and 1kg of potatoes for $3.50 each. Another stall sells 'Chefs Choice' packs consisting of 1kg of onions, 2kg of carrots and 3kg of potatoes for $6.50 each.

The apprentice buys x 'Best Buy' packs and y 'Chefs Choice packs.

(a) Write down three inequalities to represent the above constraints, apart from $x \geq 0$ and $y \geq 0$. (2 marks)

$2x + y \geq 16$

$x + 2y \geq 17$

$x + 3y \geq 21$

(b) Complete the constraints on the graph below and indicate the feasible region. (3 marks)