2021 Year 12 ViSN Mathematics Specialist Unit 3

Test 3 – Vectors & Vector Calculus

Section One – Calculator Free

Mr Daniel Comtesse

Mandurah Catholic College

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Calculator Free:_____/20

Calculator Assumed:_____/31

Result: _____/ _____%

Student Name: Solutions

School: _____

Time allowed: Section One - 20 minutes

Section Two – 30 minutes

Assessment Date: **2021**

Material required/recommended

To be provided by the supervisor

This Question/Answer Paper

SCSA Formula Sheet

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Submission Details

Timed Assessments are to be returned to the ViSN teacher by the ViSN mentor (scan completed assessment and email to teacher above) within 24 hours of assessment date (above).

Instructions to Students

1. **ALL** questions should be attempted.
2. Write your answers in the spaces provided in this Question/Answer Booklet.
3. **SHOW ALL YOUR WORKING CLEARLY**. Your working should be sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Correct answers given without supporting reasoning may not be allocated full marks. Incorrect answers given without supporting reasoning cannot be allocated any marks.
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5. It is recommended that you **do not use pencil**, except in diagrams.

Question 1 **[4, 2 = 6 marks]**

A system of equations, where b is a real constant, is as follows:

$$x-y+3z=11 \quad x+2y+2z=32 \quad x+by-4z=8$$

(a) Solve the system when $b=3$.

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 11 \\ 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 8 \end{array}\right]$$

$\therefore z = 2$ ✓

$y = -2$

$x = 3$ ✓x,y

$R_2 - R_1$, $R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 11 \\ 0 & 3 & -1 & -8 \\ 0 & 5 & -2 & -14 \end{array}\right]$$

✓reduced

$3R_3 - 5R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 11 \\ 0 & 3 & -1 & -8 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

✓reduced

(b) Interpret the system of equations geometrically when $b=4$.

when $b=4$, equation (3) is $2\times$(2), ✓identifies multiple except for RHS constant.

Therefore system is two parallel planes cut by a third plane ✓correct interpretation

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Question 2 **[2, 4 = 6 marks]**

The equation of line L is

$$\frac{x-2}{2}=\frac{y+1}{-3}=\frac{z-1}{6}.$$

(a) Show that the vector equation of the line is $r=\begin{pmatrix}2\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\6\end{pmatrix}$

① $\lambda=\frac{x-2}{2}$ ② $\lambda=\frac{y+1}{-3}$ ③ $\lambda=\frac{z-1}{6}$ ✓ state for λ

$2\lambda+2=x$ $-3\lambda-1=y$ $6\lambda+1=z$

$\therefore r=\begin{pmatrix}2\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\6\end{pmatrix}$ ✓ shows correctly

(b) The diameter of sphere S is the segment of line L between $x=2$ and $x=6$. Determine the equation of the sphere.

When $x=2$, $\lambda=0$

When $x=6$, $\lambda=1$ ✓ λ

$\therefore$ Centre occurs when $\lambda=1$.

$r=\begin{pmatrix}4\\-4\\7\end{pmatrix}$ ✓ centre

Radius $=\left|\begin{pmatrix}2\\-3\\6\end{pmatrix}\right|=7$ ✓ radius

$\therefore$ Equation $\left|r-\begin{pmatrix}4\\-4\\7\end{pmatrix}\right|=7$ ✓ correct form

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Question 3 **[5, 3 = 8 marks]**

Points A, B, C and D have position vectors $\overrightarrow{OA}=\begin{pmatrix}-1\\2\\4\end{pmatrix}, \overrightarrow{OB}=\begin{pmatrix}-3\\1\\7\end{pmatrix}, \overrightarrow{OC}=\begin{pmatrix}3\\4\\-2\end{pmatrix}$ and $\overrightarrow{OD}=\begin{pmatrix}0\\0\\4\end{pmatrix}$.

Note that $|u \times v|=|u||v|\sin\theta$, where θ is the angle between u and v.

(a) Determine $\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$ and use the result to explain why A, B and C are collinear.

$\overrightarrow{AB}=\begin{pmatrix}-2\\-1\\-3\end{pmatrix}$ $\therefore |\overrightarrow{AB}\times\overrightarrow{AC}|=\left|\begin{pmatrix}0\\0\\0\end{pmatrix}\right|=0$ ✓ cross product

$\overrightarrow{AC}=\begin{pmatrix}4\\2\\-6\end{pmatrix}$ ✓ $\overrightarrow{AB}, \overrightarrow{AC}$

As $\overrightarrow{AB}\times\overrightarrow{AC}=|\overrightarrow{AB}||\overrightarrow{AC}|\sin\theta$

$\therefore 0=|\overrightarrow{AB}||\overrightarrow{AC}|\sin\theta$

$\therefore 0=\sin\theta$ as $|\overrightarrow{AB}|\neq 0, |\overrightarrow{AC}|\neq 0$

Hence, $\theta=0$, indicating they are parallel. ✓

As the vectors are parallel and have a common starting point A, B & C are collinear ✓

(b) Determine the Cartesian equation of the plane containing all four points.

$\overrightarrow{AB}=\begin{pmatrix}1\\-2\\0\end{pmatrix}$ ✓ triangle method

$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{pmatrix}6\\3\\5\end{pmatrix}$ ✓ normal

$\therefore \mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$

$=\begin{pmatrix}0\\0\\4\end{pmatrix}\cdot\begin{pmatrix}6\\3\\5\end{pmatrix}$

$=20$

$\therefore 6x+3y+5z=20$ ✓

End of Section One

Extra working page

Question number: _____

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Test 3 – Vectors & Vector Calculus

Section Two – Calculator Assumed

Mr Daniel Comtesse

Mandurah Catholic College Calculator Assumed:_____/

daniel.comtesse@cewa.edu.au

Student Name: _____

School: _____

Time allowed: Section One - minutes

Section Two – minutes

Assessment Date: **2021**

Material required/recommended

To be provided by the supervisor

This Question/Answer Paper

SCSA Formula Sheet

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: scientific and/or CAS calculator, one A4 (one sided) page of notes.

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Question 3 **[3, 2 = 5 marks]**

Find the Cartesian equation of the following vector functions:

(a) $r(t)=(1-2\cos(t))i+3\sin(t)\,j$

$x = 1 - 2\cos(t) \Rightarrow \frac{1-x}{2} = \cos(t)$ ①

$y = 3\sin(t) \Rightarrow \frac{y}{3} = \sin t$ ②

✓ parametric

$①^2 + ②^2$

$1 = \frac{(1-x)^2}{4} + \frac{y^2}{9}$ ✓

(b) $r(t)=2\sec(t)i+\tan(t)\,j$

$x = 2\sec(t) \Rightarrow \frac{x}{2} = \sec(t)$ ①

$y = \tan(t)$ ②

✓ parametric

$①^2 - ②^2$

$\sec^2(t) - \tan^2(t) = \frac{x^2}{4} - y^2$

$1 = \frac{x^2}{4} - y^2$ ✓

Question 4 **[3, 3, 2 = 8 marks]**

Drone A and drone B move with constant velocities and relative to the origin O have initial positions $(-4,22,2)$ and $(5,15,3)$ respectively, where distances are in metres.

One second later, the position of A is $(-1,20,3)$ and the position of B is $(1,14,8)$.

(a) Determine a position vector relative to the origin for each drone after t seconds.

$\underline{r}_A = \begin{pmatrix}-4\\22\\2\end{pmatrix} + t\begin{pmatrix}-1-(-4)\\20-22\\3-2\end{pmatrix}$ $\underline{r}_B = \begin{pmatrix}5\\15\\3\end{pmatrix} + t\begin{pmatrix}1-5\\14-15\\8-3\end{pmatrix}$

$= \begin{pmatrix}-4\\22\\2\end{pmatrix} + t\begin{pmatrix}3\\-2\\1\end{pmatrix}$ ✓ $= \begin{pmatrix}5\\15\\3\end{pmatrix} + t\begin{pmatrix}-4\\-1\\5\end{pmatrix}$ ✓

requires how to determine velocity

(b) Determine an expression for the distance between the two drones at any time $t, t \geq 0$.

$\underline{r}_B - \underline{r}_A = \begin{pmatrix}9-7t\\-7+t\\1+4t\end{pmatrix}$ ✓

$\therefore$ distance $= \sqrt{(9-7t)^2 + (-7+t)^2 + (1+4t)^2}$ ✓

$= \sqrt{66t^2 - 152t + 131}$ ✓

(c) Determine the minimum distance between the drones.

Using CAS to determine minimum of function in (b) :

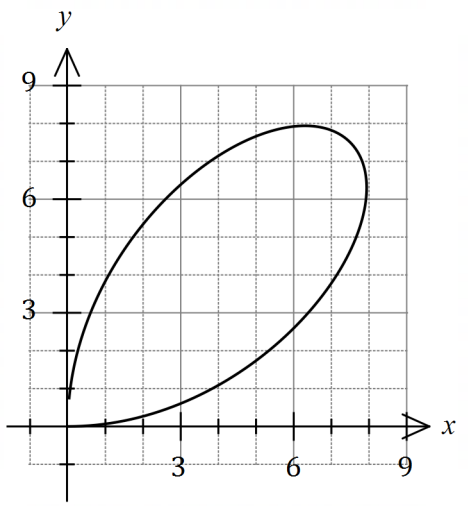
min dist $= \sqrt{65}$ m

≈ 8.06 m //

(8)

Question 5 **[2, 2, 2 = 6 marks]**

The path of a particle with position vector $r(t)=\frac{15t}{1+t^3}i+\frac{15t^2}{1+t^3}j$ metres is shown below, where t is the time in seconds and $t\geq 0$.



(a) Determine the initial velocity of the particle.

$$\underline{v}(t)=\frac{15-30t^3}{(1+t^3)^2}\underline{i}+\frac{30t-15t^4}{(1+t^3)^2}\underline{j}$$ ✓

$$\therefore \underline{v}(0)=15\underline{i}\ \text{m/s}$$ ✓

(b) Determine the velocity of the particle at the instant, $t>0$, when it is moving parallel to the x-axis.

$\underline{j}$ component of $\underline{v}=0$.

$$30t-15t^4=0$$ ✓

$$t=\sqrt[3]{2}$$

$$\underline{v}(\sqrt[3]{2})=-5\underline{i}\ \text{m/s}$$ ✓

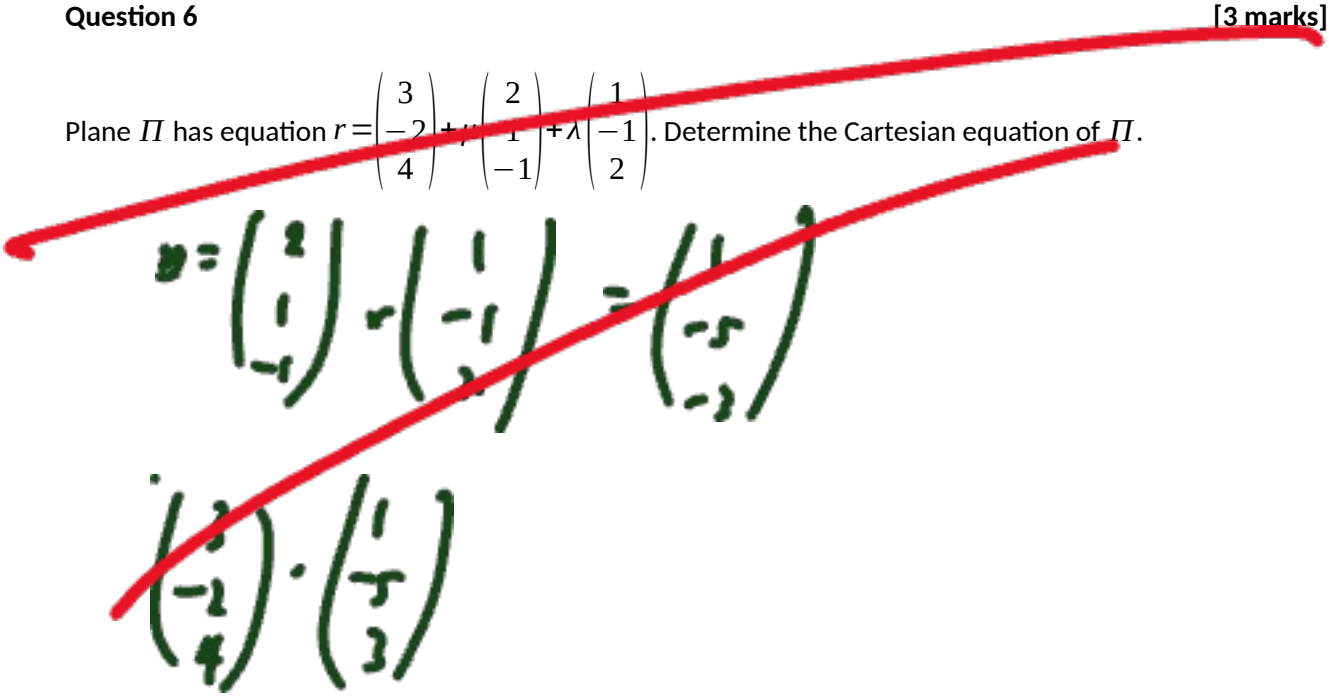
(c) Explain whether the particle will return to its initial position.

No. Initial position is at (0,0). ✓

As $t \to \infty$ each component will approach 0, but never ✓

Question 6 **[3 marks]**

Plane Π has equation $r = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. Determine the Cartesian equation of Π.



Question 7 **[2, 3, 4 = 9 marks]**

The position vectors of particles A and B (in centimetres) at time t seconds, $t \geq 0$, are

$$r_A = 7i + 15j + t(0.5i - 2j) \text{ and } r_B = 5i + 2j + t((t-6)i - j).$$

(a) Show that A is moving with constant speed and determine this speed.

$v_A = \begin{pmatrix} 0.5 \\ -2 \end{pmatrix}$ which is independent of t, and hence, constant. ✓

$|v_A| = \frac{\sqrt{17}}{2} \approx 2.06$ cm/s ✓

(b) Determine the Cartesian path of B.

$x = 5 + t^2 - 6t$ ①

$y = 2 - t \Rightarrow 2 - y = t$ ②

Sub. ② into ①

$x = 5 + (2-y)^2 - 6(2-y)$ ✓

$x = y^2 + 2y - 3$, $y \leq 2$ ✓ ✓ simplifies, domain

(c) Determine the position vector of the point where the paths of the particles cross and state whether a collision occurs.

$7 + 0.5t_1 = 5 + t_2^2 - 6t_2$ ①

$15 - 2t_1 = 2 - t_2$ ② ✓ solve by eqn

solving ① & ② on CAS:

$t_1 = 10$, $t_2 = 7$ ✓ solve

∴ Paths cross at $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$, but no collision occurs ✓ position ✓ collision

End of Assessment

Extra working page

Question number: _____