PERTH MODERN SCHOOL
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INDEPENDENT PUBLIC SCHOOL

Semester Two Examination, 2022

Question/Answer booklet

MATHEMATICS SPECIALIST
UNITs 3&4
Section One:
Calculator-free

Your Name

Your Teacher's Name

Time allowed for this section
Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Mark	Max	Question	Mark	Max
1			5		
2			6		
3			7		
4			8		

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	49	35
Section Two: Calculator-assumed	14	14	100	97	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section One: Calculator-free **(49 Marks)**

This section has 8 questions. Answer **all** questions. Write your answers in the spaces provided.

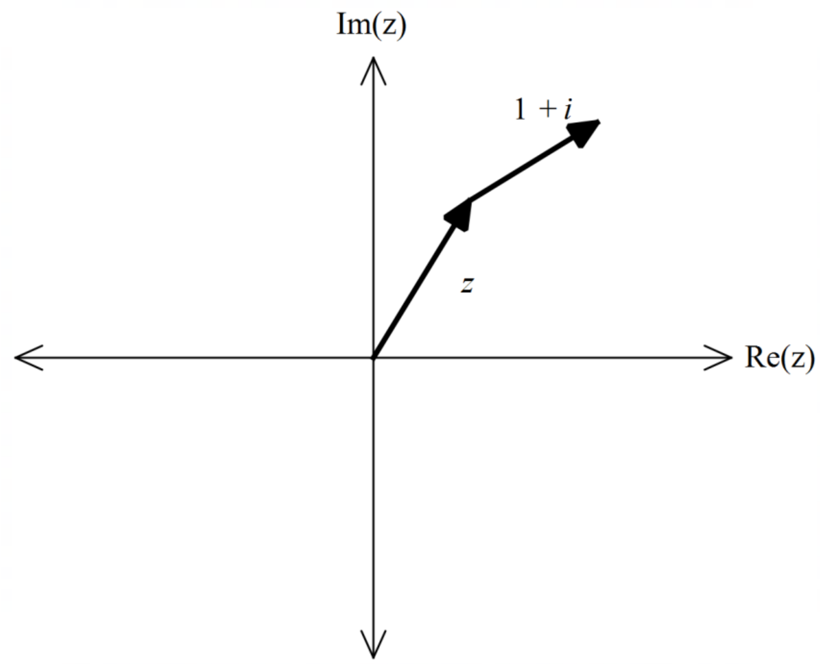
Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 50 minutes.

Question 1 **(5 marks)**

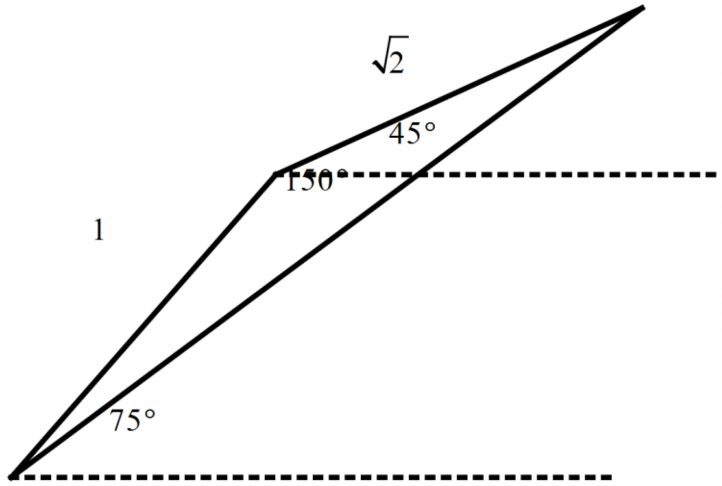
Consider the complex numbers $z = cis75^\circ$ and $1+i$ which are plotted below.



a) Express $1+i$ in polar form. (1 mark)

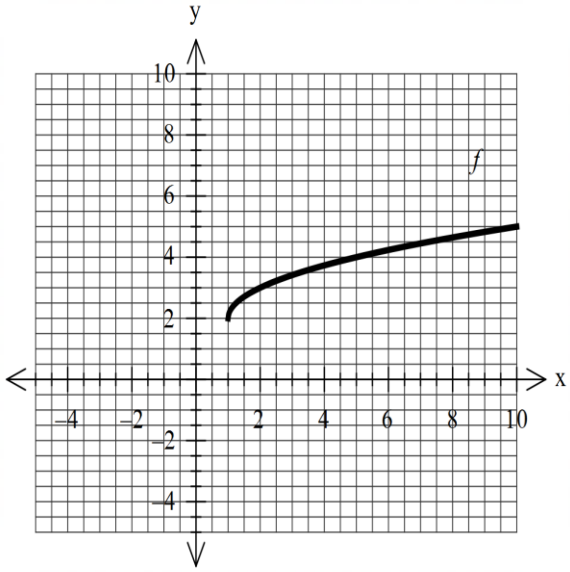
Solution
$1+i=\sqrt{2}cis45^\circ$
Specific behaviours
✓ states polar form

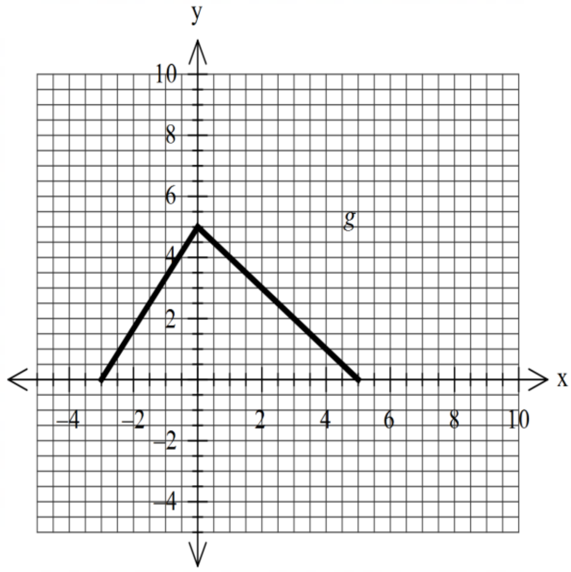
b) Determine an exact expression for $|z+1+i|$. (4 marks)

Solution
 $|z+1+i|^2 = 1+2-2\sqrt{2}\cos 150 = 3+2\sqrt{2}\left(\frac{\sqrt{3}}{2}\right) = 3+\sqrt{6}$ $|z+1+i| = \sqrt{3+\sqrt{6}}$
Specific behaviours
✓ uses triangle with angles stated as above ✓ uses cosine rule with correct lengths ✓ evaluates cos 150 ✓ states exact answer as above without cosine

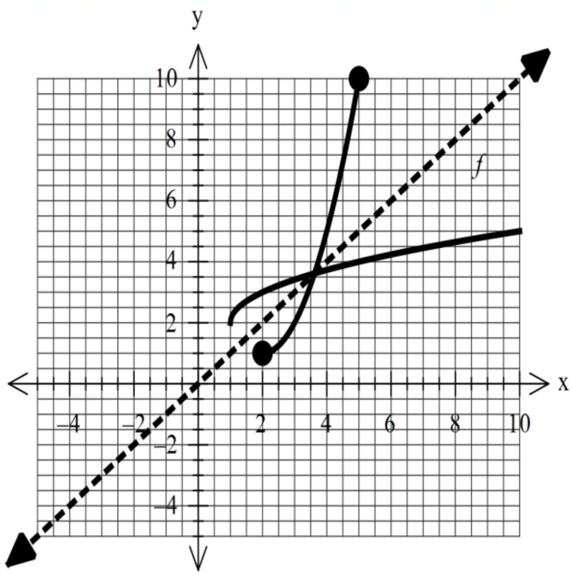
Question 2 **(13 marks)**

Consider the two functions f & g as shown below.





a) Sketch the inverse of f on the same set of axes. (2 marks)

Solution

Specific behaviours
✓ reflected in y=x ✓ both endpoints correct

b) Explain why g does not have an inverse. (1 mark)

Solution
Many to one function
Specific behaviours

✓ states reason

The defining rule for f is as follows $f(x) = 2 + \sqrt{x-1} \quad , 1 \le x \le 10$

c) Determine the rule for $f^{-1}(x)$ and state its domain. (3 marks)

Solution
$f(x) = 2 + \sqrt{x-1} \quad , 1 \le x \le 10, 2 \le y \le 5$ $x = 2 + \sqrt{y-1}, \; 2 \le x \le 5$ $x - 2 = \sqrt{y-1}, \; 2 \le x \le 5$ $(x-2)^2 = y - 1$ $f^{-1}(x) = (x-2)^2 + 1, \quad 2 \le x \le 5$
Specific behaviours
✓ swaps variables or solves for x ✓ states inverse rule ✓ states domain

d) Determine $f \circ g(0)$. (2 marks)

Solution
$f \circ g(0) = f(5) = 4$
Specific behaviours
✓ uses g(0) ✓ states final result

e) Does $f \circ g(x)$ exist for all values of the natural domain of g ? Explain. (2 marks)

Solution
$f \circ g(x)$ $to\ exist\ r_g \subseteq d_f$ $[0,5] \not\subset [1,10]$ $\therefore not\ exist$

Specific behaviours
✓ states relevant domain and range ✓ states no with a reason

f) Determine the domain and range of $g \circ f(x)$. Justify.

(3 marks)

Solution
$g \circ f(x)$ $d_f : 1 \le x \le 10$ $r_f : 2 \le y \le 5$ $d_{gof} : 1 \le x \le 10$ $r_{gof} : 0 \le y \le 3$
Specific behaviours
✓ uses domain of f ✓ states range of f ✓ states range of gof

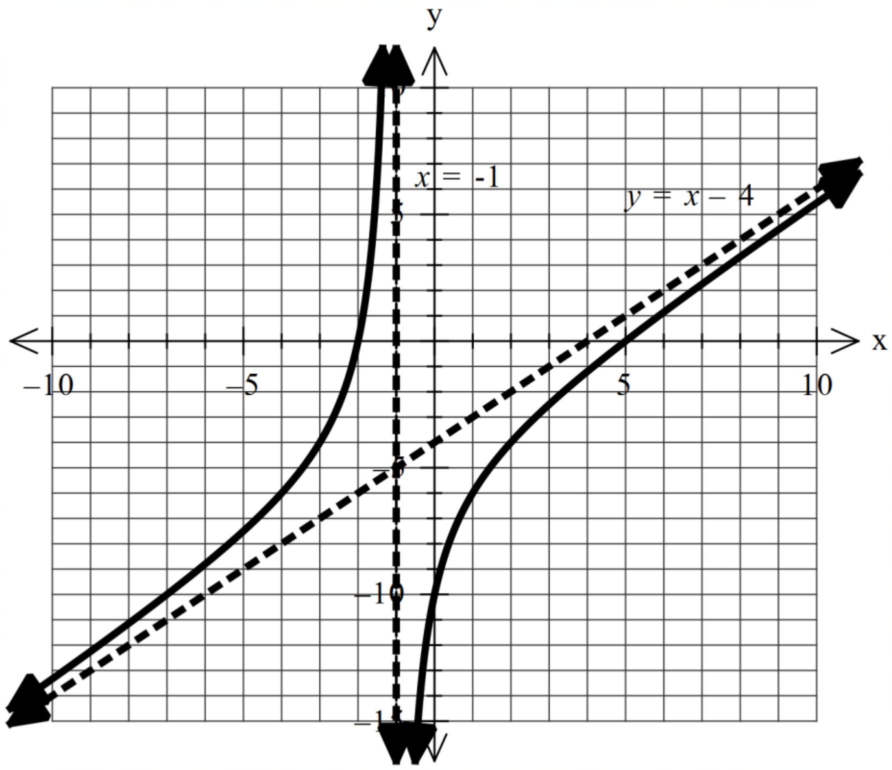
Question 3 **(5 marks)**

Using an appropriate substitution, determine the following integral $\int_0^1 \frac{x+5}{(2x+3)^{\frac{3}{2}}}\,dx$.

Solution
$let\ u = 2x+3 \quad \frac{du}{dx} = 2\ ,\ x = \frac{u-3}{2}$ $\int_0^1 \frac{x+5}{(2x+3)^{\frac{3}{2}}} \frac{dx}{du} du$ $\int_3^5 \frac{\frac{u-3}{2}+5}{u^{\frac{3}{2}}}\left(\frac{1}{2}\right)du = \int_3^5 \frac{u+7}{u^{\frac{3}{2}}}\left(\frac{1}{4}\right)du = \frac{1}{4}\int_3^5 u^{\frac{-1}{2}} + 7u^{\frac{-3}{2}}du$ $= \frac{1}{4}\left[2u^{\frac{1}{2}} - 14u^{\frac{-1}{2}}\right]_3^5 = \frac{1}{4}\left[2\sqrt{5} - \frac{14}{\sqrt{5}} - 2\sqrt{3} + \frac{14}{\sqrt{3}}\right]$
Specific behaviours
✓ defines a subs variable u ✓ changes integral from dx to du ✓ changes limits to u values ✓ integrates expression ✓obtains a value for integral in terms of surds

Question 4 **(5 marks)**

Consider the function $f\left(x\right)=x-4-\dfrac{6}{x+1}$. Sketch the function on the axes below showing all asymptotes and major features.



Solution
Specific behaviours
✓ shows and labels vertical asymptote ✓ shows and labels oblique asymptote (accept y=x) ✓ x intercepts correct ✓ y intercept correct ✓ shape is correct

Question 5 **(6 marks)**

a) Given that $\dfrac{7x^2+4x+9}{(x+1)(x^2+3)} = \dfrac{a}{x+1} + \dfrac{bx+c}{x^2+3}$, determine the values of the constants $a, b \,\&\, c$. (3 marks)

Solution
$\dfrac{7x^2+4x+9}{(x+1)(x^2+3)} = \dfrac{a}{x+1} + \dfrac{bx+c}{x^2+3}$ $7x^2+4x+9 = a(x^2+3)+(bx+c)(x+1)$ $x=-1$ $12 = a4 \quad ,a=3$ $x=0$ $9=9+c, \quad c=0$ $x=1$ $20=12+2b, \quad b=4$
Specific behaviours
✓ sets up quadratic equation with three constants ✓ solves for a ✓ solves for b & c

b) Hence determine $\displaystyle\int_0^1 \frac{7x^2+4x+9}{(x+1)(x^2+3)}\,dx$ and express answer as one log term.

(3 marks)

Solution
$\displaystyle\int_0^1 \frac{3}{x+1} + \frac{4x}{x^2+3}\,dx$ $\left[3\ln\lvert x+1\rvert + 2\ln\lvert x^2+3\rvert\right]_0^1$ $3\ln 2 + 2\ln 4 - 3\ln 1 - 2\ln 3$ $\ln 8 + \ln 16 - 0 - \ln 9$ $\ln\dfrac{128}{9}$

Specific behaviours
✓ integrates all terms ✓ subs limits and subtracts ✓ expresses as one log term

Question 6 (5 marks)

Consider the polynomial $P(z) = z^4 - 8z^3 + 56z^2 - 128z + 340$ with $P(3+5i) = 0$.

a) Determine a quadratic factor of $P(z)$. (2 marks)

Solution
$(z-\alpha)(z-\beta) = z^2 - (\alpha+\beta)x + \alpha\beta$ $z^2 - 6z + 34$ $\beta = 3 - 5i$
Specific behaviours
✓ uses conjugate ✓ states quadratic factor

b) Hence solve the equation $z^4 - 8z^3 + 56z^2 - 128z + 340 = 0$ (3 marks)

Solution
$z^4 - 8z^3 + 56z^2 - 128z + 340 = (z^2 - 6z + 34)(z^2 + cz + d)$ $z = 0$ $340 = 34d, \quad d = 10$ $-8 = c - 6, \quad c = -2$ $(z^2 - 6z + 34)(z^2 + 2z + 10) = 0$ $\dfrac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3i$ $3 \pm 5i, 1 \pm 3i$
Specific behaviours
✓ solves for second quadratic factor ✓ uses quadratic equation ✓ states all four roots

Question 7 **(5 marks)**

By using $\sqrt{1+i}$ and De Moivre's theorem, determine exact values for $\sin\frac{\pi}{8} \& \cos\frac{\pi}{8}$.

Solution
$(a+bi)^2 = 1+i = a^2 - b^2 + 2abi$ $2ab = 1$ $a^2 - \left(\frac{1}{2a}\right)^2 = 1$ $4a^4 - 4a^2 - 1 = 0$ $a^2 = \frac{4 \pm \sqrt{32}}{8} = \frac{1+\sqrt{2}}{2}, \quad b^2 = \frac{1}{2+2\sqrt{2}}$ $(a+bi)^2 = 1+i = \sqrt{2}cis\frac{\pi}{4}$ $a+bi = 2^{\frac{1}{4}}cis\frac{\pi}{8}$ $\cos\frac{\pi}{8} = \left(\frac{1+\sqrt{2}}{2}\right)^{\frac{1}{2}} 2^{\frac{-1}{4}} \quad ,\sin\frac{\pi}{8} = \left(\frac{1}{2+2\sqrt{2}}\right)^{\frac{1}{2}} 2^{\frac{-1}{4}}$
Specific behaviours
✓ sets up equation to find cartesian roots   ✓ determines polar square roots   ✓ solves for a & b   ✓ equates real and imaginary parts of both forms of roots   ✓ states exact surd expressions for cosine and sine

Question 8 **(5 marks)**

Determine the integral $\int 4\sin^5(3x)\cos^{10}(3x)\,dx$ showing full working.

Solution
$\int 4\sin^5(3x)\cos^{10}(3x)\,dx = \int_0^{\frac{\pi}{6}} 4\sin^4(3x)\cos^{10}(3x)\sin(3x)dx$ $\int 4(1-\cos^2(3x))^2\cos^{10}(3x)\sin(3x)dx$ $\int 4(1-2\cos^2(3x)+\cos(3x)^4)\cos^{10}(3x)\sin(3x)dx$ $\int 4\cos^{10}(3x)\sin(3x)-8\cos^{12}(3x)\sin(3x)+4\cos^{14}(3x)\sin(3x)dx$ $=\frac{-4}{33}\cos^{11}(3x)+\frac{8}{39}\cos^{13}(3x)-\frac{4}{45}\cos^{15}(3x)+c$
Specific behaviours
✓ takes out one sin expression ✓ uses Pythagorean to convert rest to cos functions ✓ expands brackets into 3 terms ✓ integrates all terms by adding one to cosine power ✓determines all correct factors (no need for plus constant)

Working out space