Hale School

MATHEMATICS
SPECIALIST
3CD

Semester Two Examination 2010

MARKING KEY and SOLUTIONS

Section Two

Calculator-Assumed

Question 8 **[8 marks]**

Evaluate the following definite integrals **exactly** using Calculus and algebraic techniques :

a. $\displaystyle\int_{1}^{2} \sqrt{4 - x^2}\, dx$ Put $x = 2 \sin u$

[4 marks]

Solution
$\displaystyle\int_{1}^{2} \sqrt{4 - x^2}\, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4 - 4\sin^2 u}\ .\ 2\cos u\, du$ $\displaystyle = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\cos^2 u\, du = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2(\cos 2u + 1)\, du$ $\displaystyle = \left[\sin 2u + 2u\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$
Specific Behaviours
✓ Change the limits of integration   ✓ Express integrand in terms of u using the COSINE DOUBLE ANGLE result   ✓ Anti-differentiate correctly   ✓ Determines the exact value

b. $\displaystyle\int_{0}^{\ln 2} \frac{e^x}{e^x + 2}\, dx$

[4 marks]

Solution
$\displaystyle\int_{0}^{\ln 2} \frac{e^x}{e^x + 2}\, dx = \Big[\ln\left(e^x + 2\right)\Big]_{0}^{\ln 2}$ $= \ln(e^{\ln 2} + 2) - \ln(1 + 2)$ $= \ln 4 - \ln 3$ $= \ln(4/3)$
Specific Behaviours
✓ ✓ ln Anti-derivative   ✓ evaluate $e^{\ln 2} = 2$ and $e^0 = 1$ ✓ express as an exact value using natural logarithm values

Question 9 **[14 marks]**

Consider the following transformation matrices in the co-ordinate plane :

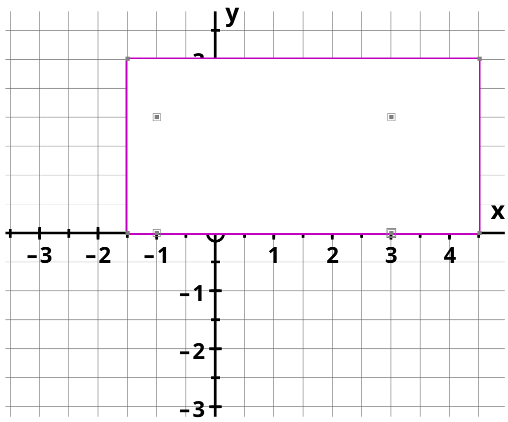
R	rotate 90° clockwise about the origin
D	dilate vertically about the origin with dilation factor 1.5
S	horizontal shear parallel to the x axis with shear factor -1.

(a) Give matrices R, D and S.

[3 marks]

Solution
$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$ $S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$
Specific Behaviours
✓✓✓ One mark for each correct matrix

The 3 diagrams below show a rectangle.



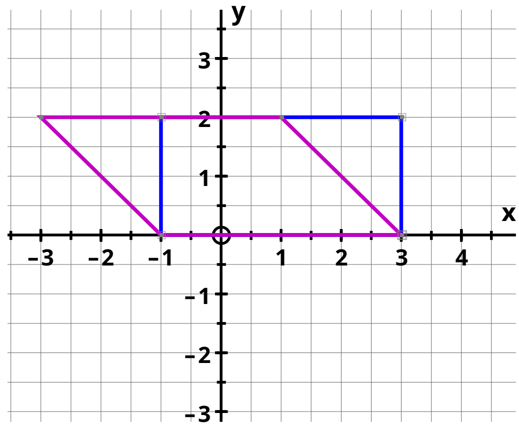


Diagram 1

Diagram 2

Draw the image of this rectangle under the action of transformation :

(b)	D	(on Diagram 1)	✓ enlarged rectangle	[1 mark]
(c)	S	(on Diagram 2)	✓ parallelogram drawn ✓ shear to the LEFT	[2 marks]
(d)	R then S	(on Diagram 3)	✓ rectangle rotated ✓ correct parallelogram shear effect	[2 marks]

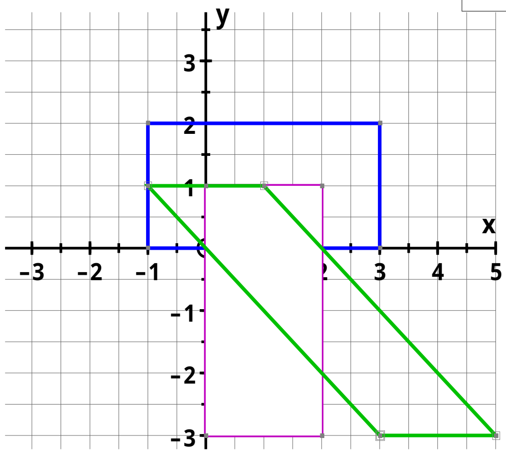


Diagram 3

next page

Question 9 **[14 marks]**

(e) If the rectangle is transformed by matrix R then S, what matrix will return the resultant image back to the original rectangle ?

[3 marks]

Solution
The transformation was SR = $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$.
$\therefore$ Require transformation $(SR)^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$
Specific Behaviours
✓ Determine matrix SR ✓ Recognise inverse matrix required ✓ Determine the inverse matrix

A student observes a change in the area of the image when working with matrix D. The student writes a conjecture about the area of the resultant image, where n is the number of times the original rectangle has been dilated by matrix D.

(f) Suggest a conjecture for the Area(Image) when the original rectangle has been dilated n times by matrix D.

Also test your conjecture.

[3 marks]

Solution
Conjecture : Area(Image) = Area(Object) x $(2.25)^n$ = 8 (2.25^n)
Test : For n = 1 Area(Image) = 8 x 2.25 = 18 From diagram Image measures 6 x 3 = 18 square units Hence the conjecture is correct.
Specific Behaviours
✓ Use of determinant to find area ratio ✓ Correct conjecture ✓ Test of conjecture using a known result

Question 10 [7 marks]

Three rival supermarkets compete for customers in the suburb of Innaloo. Market research surveys show that given a customer shops at a supermarket one week, there is a chance that they will change and shop at another supermarket the next week. The transition matrix T shown below summarises this :

$$T = \begin{array}{c} \text{Current Supermarket} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.1 \\ 0.1 & 0.2 & 0.8 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \end{array}$$

Supermarket for NEXT week

(a) Explain in words what element t_{23} of matrix T represents.

[1 mark]

Solution
t_{23} is the probability that given a customer shops at supermarket C they will be shopping at supermarket B next week.
Specific Behaviours
✓ Correct description (given shopping at C then shop at B next week)

(b) Given that a customer is shopping at supermarket B this week, give the probabilities that the customer will be shopping at each of A, B or C in 2 weeks time.

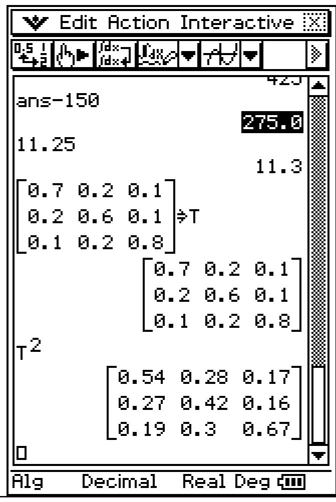
[2 marks]

Solution

$$^{2} = \begin{bmatrix} 0.54 & 0.28 & 0.17 \\ 0.27 & 0.42 & 0.16 \\ 0.19 & 0.3 & 0.67 \end{bmatrix}$$

Require matrix T
So given a customer is shopping at B (column 2),
Then the probabilities of :

Shopping at A = 0.28
Shopping at B = 0.42
Shopping at C = 0.3



Specific Behaviours

✓ Recognises that second column of T^2 is required
✓ Correct probabilities given (all 3)

Question 10 **[7 marks]**

(c) If there are currently 40% shopping at supermarket A, 30% at B and 30% at C, give the proportions shopping at each supermarket in 4 weeks time.

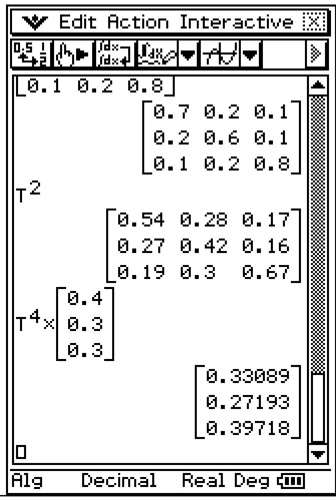
[2 marks]

Solution

Require matrix $T^4 \begin{bmatrix} 0.4 \\ 0.3 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.33089 \\ 0.27193 \\ 0.39718 \end{bmatrix}$

So in 4 weeks time, there are approx. :

33.1% shopping at A
27.2% shopping at B
40.0% shopping at C.



Specific Behaviours

✓ Recognises that T^4 x P(0) is required
✓ Correct percentages (or probabilities)

(d) Determine the long run proportions of customers shopping at each of the supermarkets A, B and C.

[2 marks]

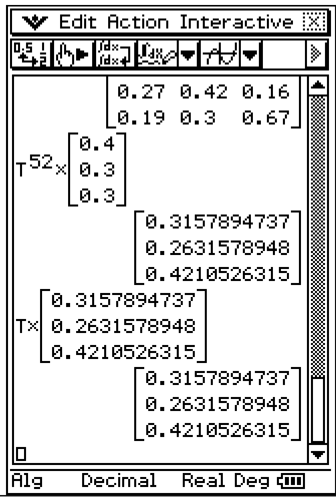
Solution

Consider $T^n \times \begin{bmatrix} 0.4 \\ 0.3 \\ 0.3 \end{bmatrix}$ as n gets large

Hence in the long run there will be approx.

31.6% shopping at A
26.3% shopping at B
42.1% shopping at C.

OR solve for P (3 x 1 matrix) such that
T x P = P

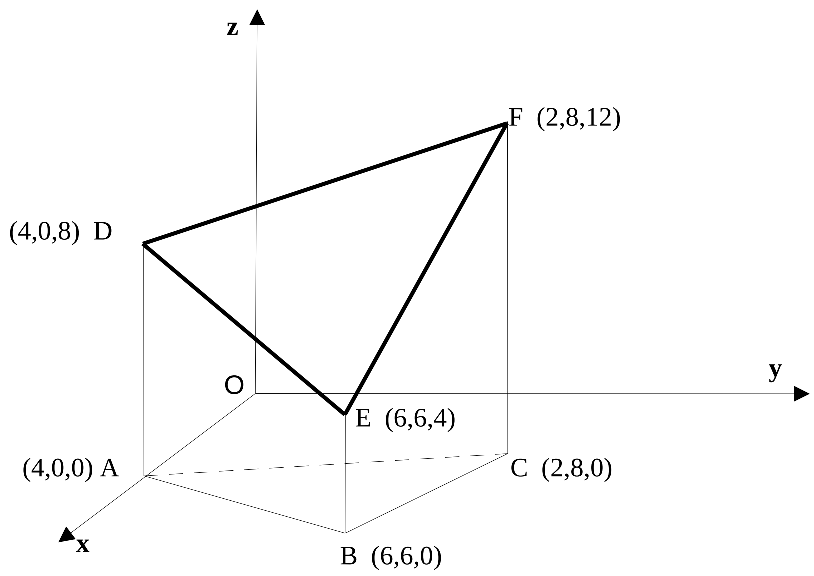


Specific Behaviours

✓ Consider behaviour as n is large OR the matrix P that is unchanged by T
✓ Correct percentages (or probabilities)

Question 11 **[11 marks]**

The end of a solid triangular prism is sawn off so that its top face Δ DEF is NOT parallel to the bottom face Δ ABC. Point O is the origin and **i**, **j** and **k** are unit vectors in the direction of the positive x, positive y and positive z directions respectively.



(a) Determine the unit vector in the direction of vector EF.

[2 marks]

Solution
Vector EF = $\begin{bmatrix} -4 \\ 2 \\ 8 \end{bmatrix}$ $\therefore$ Unit vector = $\frac{1}{\sqrt{84}}(-4i + 2j + 8k)$
Specific Behaviours
✓ Determine vector EF ✓ Divide by magnitude of EF to obtain the UNIT vector

(b) Give the vector equation for the line containing points E and F.

[2 marks]

Solution

$$\mathbf{r} = \begin{bmatrix} 6 \\ 6 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} -4 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 6-4\lambda \\ 6+2\lambda \\ 4+8\lambda \end{bmatrix}$$

Specific Behaviours
✓ Form of vector equation using parameter
✓ Use of the direction vector EF

Question 11 **[11 marks]**

(c) Determine the intercept of the line containing points E and F with the xy plane.

[2 marks]

Solution

$$\mathbf{r} = \begin{bmatrix} 6-4\lambda \\ 6+2\lambda \\ 4+8\lambda \end{bmatrix}$$ xy plane has equation $z = 0$

$\therefore$ $4 + 8\lambda = 0$ gives $\lambda = -0.5$

Hence intercept with xy plane is (8, 5, 0)

Specific Behaviours
✓ solve for parameter λ to give z = 0
✓ give co-ordinates for intercept

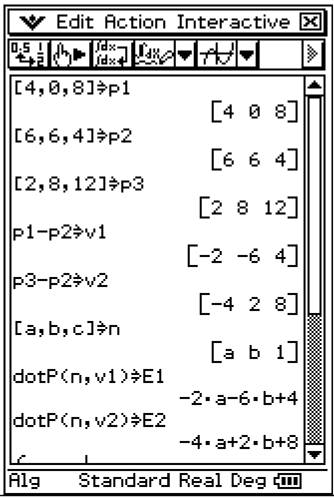
(d) Give the vector equation for plane DEF.

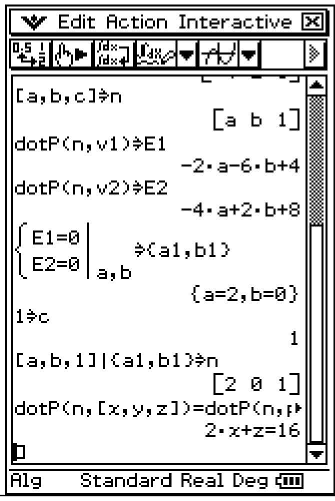
[2 marks]

Solution

Using CAS :
Equation plane is $2x + z = 16$

Vector equation $\mathbf{r} \cdot (2\mathbf{i} + 0\mathbf{j} + 1\mathbf{k}) = 16$





✓ ✓ vector equation form

(e) State the normal vector for plane DEF.

[1 mark]

Solution

See next page

Normal vector is **n** = (2**i** + 0**j** + 1**k**)
Specific Behaviours
✓ Use vector equation to deduce normal vector

Question 11 **[11 marks]**

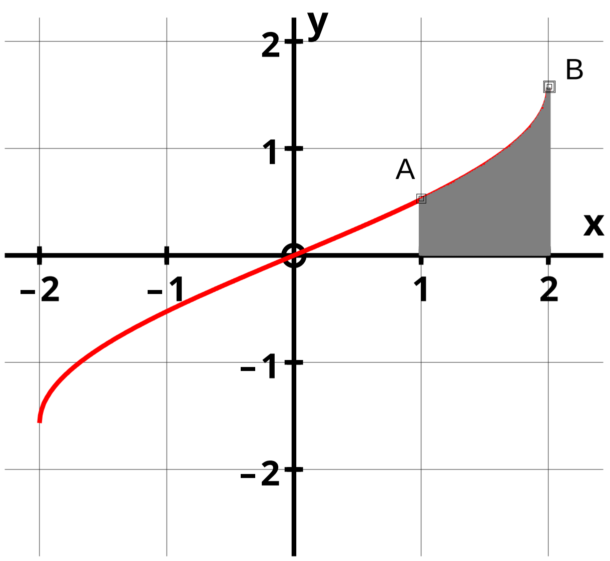
(f) Give the angle, correct to the nearest degree, between plane DEF and the xy plane.

[2 marks]

Solution
Need to find the angle between the normals of each plane. Normal vector for xy plane is **n** = (0**i** + 0**j** + 1**k**) $\therefore$ 2(0) + 0(0) + 1(1) = $\sqrt{5}$. 1 . cos θ Cos θ = $1/\sqrt{5}$ $\therefore$ θ = 63.43 … ^{o} i.e. angle between planes is approx. 63^{o}.
Specific Behaviours
✓ Angle between normal vectors ✓ Deduce angle using the dot product

Question 12 **[10 marks]**

The graph of $y = \sin^{-1}\left(\frac{x}{2}\right)$ is shown below, where $-2 \le x \le 2$.



(a) Determine the co-ordinates for points A and B.

[2 marks]

Solution
Point A $x = 1$, $y = \sin^{-1}(1/2) = \pi/6$ Hence A $(1, \pi/6)$ Point B $x = 2$, $y = \sin^{-1}(1) = \pi/2$ Hence B $(2, \pi/2)$
Specific Behaviours
✓ Evaluates inverse sine function correctly in terms of π to find y values ✓ Gives co-ordinates for A and B

(b) Find the equation of the tangent to the curve at point A.

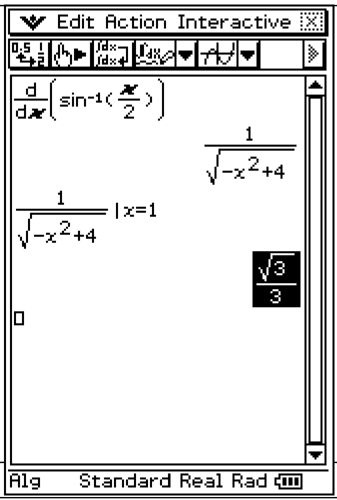
[4 marks]

Solution
Graph is given by $y = \sin^{-1}(x/2)$ i.e. $x/2 = \sin y$

$\therefore \quad \frac{1}{2} = \cos y \,.\, \frac{dy}{dx}$ At point A $y = \pi/6$ $\therefore$ m = $1/\sqrt{3}$ Equation of tangent $y - \pi/6 = 1/\sqrt{3}(x - 1)$ i.e. $y = x/\sqrt{3} + (\pi/6 - 1/\sqrt{3})$
Specific Behaviours
✓ Express curve in terms of sin y   ✓ Differentiatte implicitly to find dy/dx   ✓ Find slope of tangent   ✓ Determines equation of tangent

Question 12 **[10 marks]**

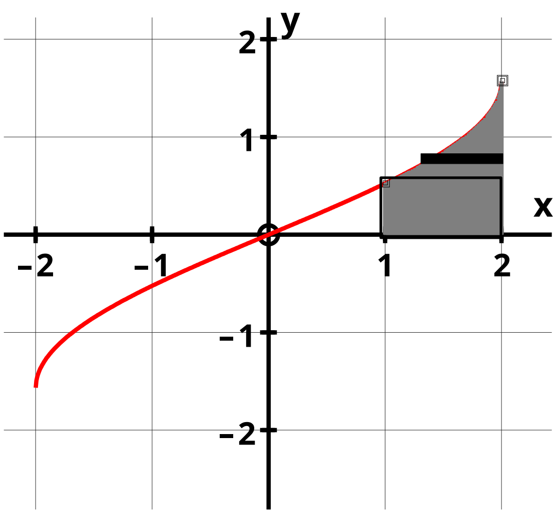
(b) OR

Solution	
	CAS solution : At point A $x = 1$ $\therefore$ m = $1/\sqrt{3}$ = 0.5773…. Equation of tangent $y - \pi/6 = 1/\sqrt{3}(x - 1)$ i.e. $y = x/\sqrt{3} + (\pi/6 - 1/\sqrt{3})$ OR $y = 0.5773x - 0.0538$
Specific Behaviours	
✓✓ Differentiatte to find dy/dx. ✓ Find slope of tangent ✓ Determines equation of tangent	

Question 12 **[10 marks]**

(c) Show, **using the anti-derivative of the sine function**, that the exact area of the region shaded is equal to $\frac{5\pi}{6} - \sqrt{3}$ square units.

[4 marks]

Solution
 dA = (difference in x values). dy Using horizontal slices : Area = $1(\pi/6) + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2 - 2\sin y)\, dy$ $= \frac{\pi}{6} + \left[2y + 2\cos y\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ $= \pi/6 + (\pi + 0) - (\pi/3 + \sqrt{3})$ $= 5\pi/6 - \sqrt{3}$ square units
Specific Behaviours
✓ Express curve with x as subject to obtain integrand 2 – 2 sin y ✓ Expression of the required area using horizontal slices ✓ Anti-derivative of integrand ✓ Use of exact values to obtain required answer

See next page

Question 13 **[8 marks]**

A cyclist moves according to the velocity time graph shown below over a period of 60 seconds. Initially the cyclist is 150 metres to the left of a sign post.

(a) Give the acceleration of the cyclist at t = 5 seconds.

[1 mark]

Solution
Acceleration is given by the slope of the tangent = v'(5) = 1.5 ms^{-2}
Specific Behaviours
✓ Angle between normal vectors ✓ Deduce angle using the dot product

(b) Determine the position of maximum displacement of the cyclist from the sign post over the 60 second period.

[3 marks]

Solution
Maximum x(t) occurs when x'(t) = 0 and x"(t) < 0 i.e. When v(t) = 0 From graph t = 30 and v'(30) < 0 ∴ x(30) = x(0) + Δx

$\int_0^{30} x(t)\,dt$ = -150 +   = -150 + 0.5(5 + 20)(10) + 10(20) + 0.5(10)(20)   = -150 + 425   x(30) = 275 m   ∴ Maximum displacement from sign post is 275 m
Specific Behaviours
✓ Deduce t = 30 for maximum displacement   ✓ Expression for x(30) using x(0)   ✓ Evaluates areas under graph correctly

Question 13 **[8 marks]**

(c) Determine when, correct to the nearest 0.1 seconds, the cyclist will be cycling past the signpost.

[4 marks]

Solution
Let T = the time when the cyclist moves past the signpost x = 0   As x(0) = -150, we require Δx = 150 to be moving past x = 0   ∴ 150 = 125 + 20(T – 10) or 150 = 425 - 0.5(5)(10) - 10(T – 35)   25 = 20(T – 10) -250 = -10(T – 35)   ∴ T = 11.25 sec T = 60 sec   ∴ Cyclist is moving past the signpost after 11.3 sec and after 60.0 sec (1 d.p.)
Specific Behaviours
✓ Require Δx = 150   ✓ Recognises there are 2 points in time when this occurs   ✓ Writes an equation using T for the required Δx   ✓ Solves equation correctly to 0.1 sec

Question 14 **[10 marks]**

(a) i. Using $z = e^{i\theta}$, show that $2i \sin\theta = e^{i\theta} - e^{-i\theta}$

[1 mark]

Solution
$e^{i\theta} - e^{-i\theta} = \text{cis}(\theta) - \text{cis}(-\theta)$ $= \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta) = 2i\sin\theta$
Specific Behaviours
✓ Uses polar form to express cis(θ) and cis(-θ) correctly OR conjugates

ii. Hence prove that : $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

[4 marks]

Solution
As $2i\sin\theta = e^{i\theta} - e^{-i\theta}$ Then $(2i\sin\theta)^5 = (e^{i\theta} - e^{-i\theta})^5$ $= e^{5i\theta} - 5e^{4i\theta}e^{-i\theta} + 10e^{3i\theta}e^{-2i\theta} - 10e^{2i\theta}e^{-3i\theta} + 5e^{i\theta}e^{-4i\theta} - e^{-5i\theta}$ $= e^{5i\theta} - 5e^{3i\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - e^{-5i\theta}$ $= (e^{5i\theta} - e^{-5i\theta}) - 5(e^{3i\theta} - e^{-3i\theta}) + 10(e^{i\theta} - e^{-i\theta})$ $32i^5\sin^5\theta = 2i\sin 5\theta - 5.2i\sin 3\theta + 10.2i\sin\theta$ $32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$ $\therefore 16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$
Specific Behaviours
✓ Raises $(e^{i\theta} - e^{-i\theta})$ to power of 5 ✓ Correctly expands ✓ Re-groups terms in pairs ✓ Divides both sides by 2i to isolate the $16\sin^5\theta$ term.

Question 14 **[10 marks]**

(b) i. Solve, in exact polar form, the complex equation $z^5 + 1 = 0$.

[3 marks]

Solution
$Z^5 = -1 = \text{cis}\,\pi$ $\therefore\ z = \text{cis}\left(\frac{\pi + 2\pi k}{5}\right)$ k = 0, 1, 2, 3, 4 $\therefore\ z = \text{cis}\left(\frac{\pi}{5}\right), \text{cis}\left(\frac{3\pi}{5}\right), \text{cis}\left(\frac{5\pi}{5}\right), \text{cis}\left(\frac{7\pi}{5}\right), \text{cis}\left(\frac{9\pi}{5}\right)$ $\therefore\ z = \text{cis}\left(\frac{\pi}{5}\right), \text{cis}\left(\frac{3\pi}{5}\right), -1, \text{cis}\left(-\frac{3\pi}{5}\right), \text{cis}\left(-\frac{\pi}{5}\right)$
Specific Behaviours
✓ Expression for 5th roots using De Moivre's Theorem using k   ✓ Give 5 roots in cis form   ✓ Express solutions using $-\pi < \theta < \pi$

ii. Hence or otherwise prove that $\cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right) = \frac{1}{2}$.

[2 marks]

Solution
Sum of the roots = 0 (= $w + w^3 + w^5 + w^7 + w^9$)   Hence $\text{Re}(w + w^3 + w^5 + w^7) = 0$ $\therefore\ -1 + \cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right) + \cos\left(-\frac{\pi}{5}\right) + \cos\left(-\frac{3\pi}{5}\right) = 0$ $\therefore\ -1 + \cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right) = 0$

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$\therefore \quad -1 + 2(\cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right)) = 0$ $\therefore \quad \cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right) = \frac{1}{2}$
Specific Behaviours
✓ Use the SUM of the roots is ZERO ✓ obtain 2 multiples of cos(π/5) and cos(3π/5)

Question 15 **[12 marks]**

In this era of electronic communications, information is quickly shared. A rumour is started in a small country town with a population of 1 000. A small group of 5 people decide to start a rumour that "the town's pub is about to permanently close".

Let $N(t)$ = the number of people who have heard the rumour after t hours

The rate of spread of the rumour is proportional to both the number of people who have already heard the rumour AND the number of people who have NOT heard the rumour. The rate of spread of the rumour is given by :

$$\frac{dN}{dt} = 0.0005\,N(1000 - N)$$

(a) Using the above differential equation, explain what happens to the rate of spread of the rumour as the number of people who have heard the rumour rises.

[2 marks]

Solution
Rate of spread will rise to a maximum and then decrease to ZERO as the number of people who have heard the rumour approaches 1000.
Specific Behaviours
✓ Describe rise to a maximum ✓ Describe fall to ZERO as N → 1000

(b) Using the result $\frac{1000}{N(1000 - N)} = \frac{1}{N} + \frac{1}{1000 - N}$ and the method of separation of variables, show that :

$$\ln\left(\frac{N}{1000 - N}\right) = 0.5t + k \qquad \text{where } k \text{ is some constant}$$

[4 marks]

Solution

Separation of variables : $\int \frac{dN}{N(1000 - N)} = \int 0.0005 \, dt$

Multiply each side by 1000 :

$$\int \frac{1000 \, dN}{N(1000 - N)} = \int 0.5 \, dt$$

$$\int \left(\frac{1}{N} + \frac{1}{1000 - N} \right) dN = \int 0.5 \, dt$$

Hence :

ln N - ln(1000 – N) = 0.5t + k

$$\therefore \ln\left(\frac{N}{1000 - N} \right) = 0.5t + k$$

Specific Behaviours

- ✓ Separate variables on each side and integrate
- ✓ Use the given result to write a sum of reciprocal functions
- ✓ Anti-differentiate each side correctly
- ✓ Use logarithm property

Question 15 [12 marks]

(c) Deduce the value of the constant k and hence determine the expression for N(t) in terms of t.

[4 marks]

Solution

Since N(0) = 5, $\ln\left(\frac{5}{995} \right) = 0.5(0) + k$

i.e. $k = \ln\left(\frac{1}{199} \right) = -5.2933 \ldots$

$$\therefore \ln\left(\frac{N}{1000 - N} \right) = 0.5t + \ln\left(\frac{1}{199} \right)$$

$$\frac{N}{1000 - N} = e^{0.5t + \ln\left(\frac{1}{199} \right)}$$

$$\frac{N}{1000 - N} = \frac{1}{199} e^{0.5t}$$

$$199N = (1000 - N) e^{0.5t}$$

$$199N = 1000e^{0.5t} - N e^{0.5t}$$

$$N\left(e^{0.5t} + 199\right) = 1000e^{0.5t}$$

$$N(t) = \frac{1000e^{0.5t}}{e^{0.5t} + 199}$$

Specific Behaviours

- ✓ Determine the value of k

✓	Express as a power of e using the value of k
✓	Manipulate expression to obtain N terms on one side
✓	Obtain an equivalent expression for N(t) as a quotient

It is considered that "everyone" in the town has heard the rumour when 99% of the town's population have heard it.

(d) Determine how long (correct the nearest minute) it takes for "everyone" to have heard the rumour ?

[2 marks]

Solution
Require N(t) = 990 From CAS, t = 19.7768 … hrs i.e. will take 19 hours 47 minutes for "everyone" to have heard the rumour.
Specific Behaviours
✓ Equation to solve for t using 990 people ✓ Solve for t correct to nearest minute