Question 21 **(6 marks)**

The discrete random variable X is defined by

$$P(X = x) = \begin{cases} \dfrac{4k}{e^{1-x}} & x = 0, 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Show that $k = \dfrac{e}{4+4e}$. (3 marks)

Solution
$\dfrac{4k}{e} + \dfrac{4k}{1} = 1$ $k\left(\dfrac{4+4e}{e}\right) = 1$ $k = \dfrac{e}{4+4e}$
Specific behaviours
✓ indicates $P(X = 0)$ and $P(X = 1)$ ✓ sums probabilities to 1   ✓ factors out k and rearranges

(b) Determine, in simplest form, the exact mean and standard deviation of X. (3 marks)

Solution
NB Bernoulli distribution. $E(X) = 4k = \dfrac{e}{1+e}$ $\text{Var}(X) = \dfrac{4k}{e} \times 4k = \dfrac{4^2k^2}{e}$ $SD = \sqrt{\dfrac{4^2k^2}{e}} = \dfrac{4}{\sqrt{e}}\left(\dfrac{e}{4+4e}\right) = \dfrac{\sqrt{e}}{1+e}$
Specific behaviours
✓ simplified $E(X)$ ✓ correct expression for variance   ✓ simplified expression for standard deviation

• students did not fully simplify E(X).

End of questions

Makes Comments

Trinity College

Semester One Examination, 2018

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section One: Calculator-free

SOLUTIONS

Student number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(5 marks)**

A particle travels in a straight line so that its distance x cm from a fixed point O on the line after t seconds is given by

$$x = \frac{t^2}{2t+1}, t \geq 0.$$

Calculate the acceleration of the particle when $t = 1$.

$$x'(t) = \frac{2t(2t+1) - 2t^2}{(2t+1)^2}$$

Done well.

Some use product rule (often making errors)

$$x'(t) = \frac{2t^2 + 2t}{(2t+1)^2}$$

Most used quotient.

Some didn't simplify vel which made next step more difficult.

$$x''(t) = \frac{2}{(2t+1)^3}$$

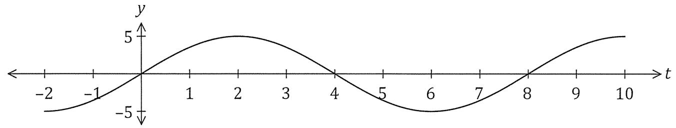
$$x''(1) = \frac{2}{(2(1)+1)^3}$$

$$= \frac{2}{27}$$

Specific behaviours
✓ correct form of quotient rule
✓ simplifies expression for v
✓ correct use of chain rule in second derivative
✓ correct expression for acceleration
✓ substitutes and simplifies

Question 20 **(7 marks)**

The graph of $y = f(t)$ is shown below, where $f(t) = 5\sin\left(\frac{\pi t}{4}\right)$.



(a) Determine the exact area between the horizontal axis and the curve for $0 \leq t \leq 4$. (2 marks)

Solution
$\int_0^4 5\sin\left(\frac{\pi t}{4}\right) = \frac{40}{\pi}$
Specific behaviours
✓ writes integral
✓ evaluates

12.732

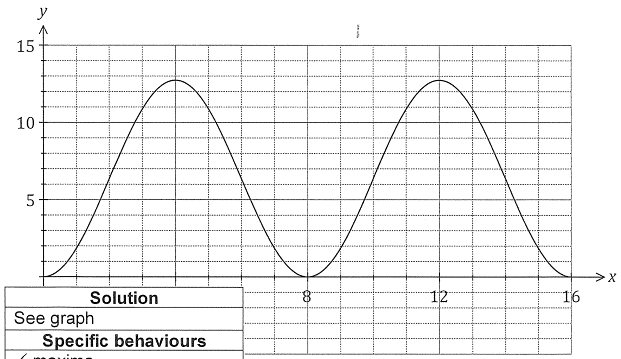
• Many students did not provide the exact area.

Another function, F, is defined as $F(x) = \int_0^x f(t)\,dt$ over the domain $0 \leq x \leq 16$.

(b) Determine the value(s) of x for which $F(x)$ has a maximum and state the value of $F(x)$ at this location. (2 marks)

Solution
$x = 4, x = 12, \quad F(4) = F(12) = \frac{40}{\pi}$
Specific behaviours
✓ values of x
✓ value of $F(x)$

(c) Sketch the graph of $y = F(x)$ on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ maxima
✓ minima
✓ smooth continuous curve

Question 19 **(7 marks)**

The hourly cost of fuel to run a train is proportional to the square of its speed and is $\$64$ per hour when the train moves at a speed of 25 kmh^{-1}. Other costs amount to $\$100$ per hour, regardless of speed.

(a) Show that when the train moves at a steady speed of x kmh^{-1}, where $x > 0$, the total cost per kilometre, C, is given by (3 marks)

$$C = \frac{64x}{625} + \frac{100}{x}.$$

Solution
Fuel cost, f, is $f = kx^2 \Rightarrow k = \frac{64}{25^2} = \frac{64}{625}$ Total cost per hour, t, is $t = \frac{64x^2}{625} + 100$ Cost per km, C, is $C = \frac{t}{x} = \frac{64x}{625} + \frac{100}{x}$
Specific behaviours
✓ expression for hourly cost of fuel ✓ expression for total cost per hour ✓ indicates derivation of cost per km

(b) Use calculus to determine the minimum cost for the train to travel 180 km, assuming that the train travels at a constant speed for the entire journey. (4 marks)

Solution
$\frac{dC}{dx} = \frac{64x^2 - 62500}{625x^2}$ $\frac{dC}{dx} = 0 \Rightarrow x = 32.25 \ (x > 0)$ $C = \frac{64(32.25)}{625} + \frac{100}{32.25} = 6.4$ $\text{Journey cost} = 6.4 \times 180 = \$1\,152$
Specific behaviours
✓ obtains first derivative ✓ indicates critical point ✓ indicates optimum cost per km ✓ correct minimum cost

many students did not answer the question entirely stopping after obtaining the stationary pt.

Question 2 **(5 marks)**

A function defined by $f(x) = 13 + 18x - 6x^2 - 2x^3$ has stationary points at $(1, 23)$ and $(-3, -41)$.

(a) Use the second derivative to show that one of the stationary points is a local maximum and the other a local minimum. (3 marks)

Solution
$f'(x) = 18 - 12x - 6x^2$ $f''(x) = -12x - 12$ $f''(1) = -12(1) - 12 = -24 < 0 \Rightarrow (1, 23)$ is a maximum $f''(-3) = -12(-3) - 12 = 24 > 0 \Rightarrow (-3, -41)$ is a minimum
Specific behaviours
✓ differentiates twice ✓ shows $f''(1) < 0$ and interprets ✓ shows $f''(-3) > 0$ and interprets

Done well

(b) Determine the coordinates of the point of inflection of the graph of $y = f(x)$. (2 marks)

Solution
$f''(x) = 0 \Rightarrow x = -1$ $f(-1) = 13 - 18 - 6 + 2 = -9$ At $(-1, -9)$
Specific behaviours
✓ correct x-coordinate ✓ correct y-coordinate

Done well
Only a few minor calculation errors
A small number of students forgot to answer as coordinates.

Question 3 **(6 marks)**

A box contains five balls numbered $1, 3, 5, 7$ and 9. Three balls are randomly drawn from the box at the same time and the random variable X is the largest of the three numbers drawn.

(a) By listing all possible outcomes $(135,\ 137$, etc.), determine $P(X \leq 7)$. (2 marks)

Solution
$\{135, 137, 139, 157, 159, 179, 357, 359, 379, 579\}$ $P(X \leq 7) = \frac{4}{10}$
Specific behaviours
✓ lists outcomes   ✓ correct probability

Mixed response
Some saw the combinations
others used arrangements.
ft was possible when
working was shown.

(b) Construct a table to show the probability distribution of X. (2 marks)

Solution			
x	5	7	9
$P(X = x)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{6}{10}$
Specific behaviours			
✓ values of x ✓ values of $P(X = x)$			

Done ok.
It had to ft
from part (a)

(c) Calculate $\mathrm{E}(X)$. (2 marks)

Solution
$\mathrm{E}(X) = \frac{5 + 21 + 54}{10} = 8$
Specific behaviours
✓ indicates products $x \cdot P(X = x)$ ✓ correct value

Done well.

(d) Determine $\frac{dh}{dt}$ when the height of the balloon is 2.08 km. (3 marks)

Solution
$h(t) = 2.08 \Rightarrow t = 20$ $\frac{dh}{dt} = \frac{300t - 3t^2}{25000}$ $= \frac{300(20) - 3(20)^2}{25000} = \frac{24}{125} = 0.192\ \mathrm{km/m}$
Specific behaviours
✓ determines time   ✓ indicates derivative   ✓ determines rate of change

(e) Determine $\frac{dP}{dt}$ when the height of the balloon is 2.08 km. (3 marks)

Solution
$\frac{dP}{dh} = -0.138 \times 101.3e^{-0.138(2.08)}$ $= -10.53$ $\frac{dP}{dt} = \frac{dP}{dh} \times \frac{dh}{dt}$ $= -10.53 \times 0.192$ $= -2.02\ \mathrm{kPa/m}$
Specific behaviours
✓ rate of change of P wrt h ✓ indicates use of chain rule   ✓ correct rate of change

- students not reading the question or using the wrong formulas.
- (e) not many students used the chain rule
- (b) students did not mention key words rate w.r.t height.

Question 17 **(11 marks)**

The air pressure, $P(h)$ in kPa, experienced by a weather balloon varies with its height above sea level h km and is given by

$$P(h) = 101.7e^{-0.138h}, 0 \le h \le 20\,.$$

(a) Determine $\frac{dP}{dh}$ when the height of the balloon is 0.9 km. (2 marks)

Solution
$\frac{dP}{dh} = -0.138 \times 101.7e^{-0.138(0.9)}$ $= -12.4\ \text{kPa/km}$
Specific behaviours
✓ uses derivative   ✓ correct rate of change

(b) What is the meaning of your answer to (a). (1 mark)

Solution
The rate of change of pressure with respect to height when the height is 0.9 km.
Specific behaviours
✓ meaning *(must include wrt h and refer to height)*

The height of the balloon above sea level varies with time t minutes and is given by

$$h(t) = \frac{t^2(150 - t)}{25000}, 0 \le t \le 100\,.$$

(c) Determine the air pressure experienced by the balloon when $t = 75$. (2 marks)

Solution
$h(75) = 16.875$ km $P(16.875) = 9.907$ kPa
Specific behaviours
✓ determines height   ✓ determines pressure

Question 4 **(6 marks)**

(a) Determine $\int 5(2x-1)^3 dx$. (2 marks)

$f(x) = 2x - 1$
$f'(x) = 2$

$$= \frac{5}{2}\int \frac{2}{5}(5)(2x-1)^3\,dx$$

$$= \frac{5}{2} \times \frac{(2x-1)^4}{4} + c$$

$$= \frac{5(2x-1)^4}{8} + c$$

Done well.
Most common mistake was to forget the $+c$

(b) Determine $\frac{d}{dx} e^{1-3x} - 2e$. (2 marks)

$$= -3e^{1-3x}$$

Done well
Had to make both alternatives
$\frac{d}{dx}(e^{1-3x} - 2e)$ or $\frac{d}{dx}(e^{1-3x}) - 2e$
Some use the product rule! why!

(c) Determine $\frac{dy}{dx}$ given $y = \int_x^1 t\sqrt{t}\,dt$. (2 marks)

$$y' = \frac{d}{dx}\int_x^1 t\sqrt{t}\,dt$$

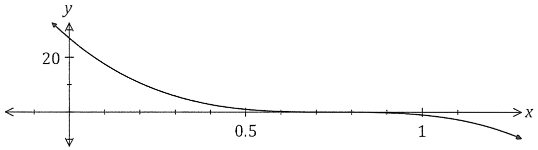
$$= -\frac{d}{dx}\int_1^x t\sqrt{t}\,dt$$

$$= -x\sqrt{x}$$

Done well.
Only a few dropped the $-$ve sign.

Question 5 **(8 marks)**

The graph of $y = (3 - 4x)^3$ is shown below.



(a) Determine the area of the region enclosed by the curve and the coordinates axes. (4 marks)

Solution
$3 - 4x = 0 \Rightarrow x = 0.75$ $A = \int_0^{0.75} (3 - 4x)^3\,dx$ $= \left[\frac{(3-4x)^4}{-4 \times 4}\right]_0^{0.75}$ $= (0) - \left(\frac{81}{-16}\right)$ $= \frac{81}{16}$ sq units
Specific behaviours
✓ writes integral with limits ✓ antidifferentiates ✓ expression with both limits substituted ✓ correct area

OK.

Some had issues with the upper boundary of 3/4

Others would not integrate (missing the -4

(b) Given that the area of the region bounded by the curve, the x-axis and the line $x = k$ is 4 square units, determine the value of k, where $0 < k < 0.75$. (4 marks)

Solution
$A = \int_k^{0.75} (3 - 4x)^3\,dx \Rightarrow 8 = \left[\frac{(3-4x)^4}{-16}\right]_k^{0.75}$ $4 = (0) - \left(\frac{(3-4k)^4}{-8}\right)$ $(3 - 4k)^4 = 64$ $3 - 4k = \sqrt{8}$ $4k = 3 - 2\sqrt{2} \Rightarrow k = \frac{3 - 2\sqrt{2}}{4}$
Specific behaviours
✓ equation with antiderivative ✓ equation with both limits substituted ✓ simplifies equation ✓ value of k

Not done well.

Most student used the lower side and not the upper side

ie $\int_k^{3/4} f(x)\,dx$

Question 18 **(7 marks)**

A random sample of n components are selected at random from a factory production line. The proportion of components that are defective is p and the probability that a component is defective is independent of the condition of any other component.

The random variable X is the number of faulty components in the sample. The mean and standard deviation of X are 30.6 and 5.1 respectively.

(a) Determine the values of n and p. (4 marks)

Solution
$X \sim B(n, p)$ $np = 30.6$ $np(1 - p) = 5.1^2$ $n = 204, \quad p = 0.15$
Specific behaviours
✓ indicates binomial distribution ✓ equation using mean ✓ equation using standard deviation ✓ solves correctly for n and p

Some students forgot st. dev was $\sqrt{np(1-p)}$

$n = 36.72$
$p = 0.83$

(b) After changes are made to the manufacturing process, the proportion of defective components is now 3%. Determine the smallest sample size required to ensure that the probability that the sample contains at least one defective component is at least 0.95. (3 marks)

Solution
$X \sim B(n, 0.03)$ $P(X \geq 1) \geq 0.95$ $1 - P(X = 0) \geq 0.95$ $P(X = 0) < 0.05$ $0.97^n < 0.05$ $n > 98.4 \Rightarrow n \geq 99$
Specific behaviours
✓ indicates required binomial probability ✓ uses $P(X = 0)$ to create inequality ✓ solves and rounds to obtain n

question was done well.

Question 16 **(9 marks)**

A particle starts from rest at O and travels in a straight line.

Its velocity v ms^{-1}, at time t s, is given by $v = 14t - 3t^2$ for $0 \le t \le 4$ and $v = 128t^{-2}$ for $t > 4$.

(a) Determine the initial acceleration of the particle. (2 marks)

Solution
$a = \frac{dv}{dt} = 14 - 6t \Rightarrow a(0) = 14 \text{ ms}^{-2}$
Specific behaviours
✓ differentiates velocity ✓ acceleration

(b) Calculate the change in displacement of the particle during the first four seconds. (2 marks)

Solution
$x = \int_0^4 14t - 3t^2\,dt = 48 \text{ m}$
Specific behaviours
✓ integrates velocity ✓ change in displacement

(c) Determine, in terms of t, an expression for the displacement, x m, of the particle from O for $t > 4$. (2 marks)

Solution
$x = \int \frac{128}{t^2}\,dt = -\frac{128}{t} + c$ $x(4) = 48 = -\frac{128}{4} + c \Rightarrow c = 80$ $x = -\frac{128}{t} + 80$
Specific behaviours
✓ integrates velocity ✓ evaluates c

many students determined $c = 48$

(d) Determine the distance of the particle from O when its acceleration is -0.5 ms^{-2}. (3 marks)

Solution
$a = -\frac{256}{t^3}$ $-\frac{256}{t^3} = -0.5 \Rightarrow t = 8$ $x(8) = 64 \Rightarrow \text{Distance from } O = 64 \text{ m}$
Specific behaviours
✓ acceleration for $t > 4$ ✓ solves for time ✓ calculates distance

Question 6 **(9 marks)**

The function g is such that $g'(x) = ax^2 + 18x + b$, it has a point of inflection at $(-1, 29)$ and a stationary point at $(1, -19)$.

(a) Determine $g(2)$. (5 marks)

Solution
$g''(x) = 2ax + 18$ $g''(-1) = -2a + 18 = 0 \Rightarrow a = 9$ $g'(1) = 0 \Rightarrow 9 + 18 + b = 0$ $b = -27$ $g'(x) = 9x^2 + 18x - 27$ $g(x) = 3x^3 + 9x^2 - 27x + c$ $g(1) = -19 \Rightarrow 3 + 9 - 27 + c = -19$ $c = -4$ $g(2) = 24 + 36 - 54 - 4 = 2$
Specific behaviours
✓ value of a ✓ value of b ✓ antiderivative ✓ constant of integration ✓ value

Done well.

Some got mixed up with equating it.

ie $f'(x) = 0$

not $f(x) = -19$

Some did this the long way!

(b) Determine

(i) $\int_1^2 g'(x)\,dx$. (2 marks)

Solution
$\int g'(x) = \Delta y = 2 - (-19) = 21$
Specific behaviours
✓ uses total change ✓ correct value

(ii) $\int_1^2 4g'(x) + 16\,dx$. (2 marks)

Solution
$4\int_1^2 g'(x)\,dx + \int_1^2 16\,dx = 4(21) + 16 = 100$
Specific behaviours
✓ uses linearity ✓ correct value

Done well

Most saw the link to part (b) (i)

Question 7 **(5 marks)**

The height, in metres, of a lift above the ground t seconds after it starts moving is given by

$$h = 8\cos\left(\frac{t}{7}\right)$$

Use the increments formula to estimate the change in height of the lift from $t = \frac{7\pi}{4}$ to $t = \frac{176\pi}{100}$.

$$\frac{dh}{dt} = -8\sin\left(\frac{t}{7}\right) \times \frac{1}{7}$$

$$\Delta h \approx \frac{dh}{dt} \times \Delta t$$

$$\Delta t = \frac{176\pi}{100} - \frac{175\pi}{100} = \frac{\pi}{100}$$

$$= \frac{-8}{7}\sin\left(\frac{\pi}{4}\right) \times \frac{\pi}{100}$$

$$= \frac{-8}{7} \times \frac{\sqrt{2}}{2} \times \frac{\pi}{100}$$

$$= \frac{\sqrt{2}\pi}{100}$$

Done OK.

Some struggles with the exact value of $\sin\left(\frac{\pi}{4}\right)$ and then could not simplify.

Specific behaviours
✓ correctly uses chain rule
✓ correct derivative
✓ increment of time
✓ substitutes correctly into increments formula
✓ fully simplifies

Question 15 **(7 marks)**

A fuel storage tank, initially containing 550 L, is being filled at a rate given by

$$\frac{dV}{dt} = \frac{t^2(60-t)}{250}, \qquad 0 \le t \le 60$$

where V is the volume of fuel in the tank in litres and t is the time in minutes since filling began. The tank will be completely full after one hour.

(a) Calculate the volume of fuel in the tank after 10 minutes. (3 marks)

Solution
$\Delta V = \int_0^{10} V'(t)\,dt$ $= 70$ $V = 550 + 70 = 620$ L
Specific behaviours
✓ indicates use of integral of rate of change   ✓ calculates increase   ✓ states volume

(b) Determine the time taken for the tank to fill to one-half of its maximum capacity. (4 marks)

Solution
$V = 550 + \int_0^{60} V'(t)\,dt$ $= 550 + 4320 = 4870$ $V(T) = \int_0^T V'(t)\,dt = \frac{2T^3}{25} - \frac{T^4}{1000} + 550$ $\frac{2T^3}{25} - \frac{T^4}{1000} + 550 = \frac{4870}{2}$ $T = 34.6$ minutes
Specific behaviours
✓ calculates V_{MAX} ✓ indicates $V(T)$ ✓ indicates equation   ✓ solves for time

Question 14 **(8 marks)**

The discrete random variable X has a mean of 5.28 and the following probability distribution.

x	3	4	5	6	7
$P(X = x)$	0.15	a	b	0.2	0.2

(a) Determine the values of the constants a and b. (3 marks)

Solution
$a + b + 0.55 = 1$ $4a + 5b + 3.05 = 5.28$ $a = 0.02, \quad b = 0.43$
Specific behaviours
✓ equation using sum of probabilities ✓ equation using mean ✓ values of a and b

(b) Determine $P(X < 4 \mid X < 7)$. (2 marks)

Solution
$P(X < 4 \mid X < 7) = \frac{0.15}{0.8} = 0.1875 = \frac{3}{16}$
Specific behaviours
✓ denominator ✓ numerator and expresses as decimal or fraction

(c) Determine

(i) $\mathrm{Var}(X)$. (1 mark)

Solution
$\mathrm{Var}(X) = 1.5416$ (using CAS)
Specific behaviours
✓ correct variance

(ii) $\mathrm{E}(100 - 15X)$. (1 mark)

Solution
$\mathrm{E}(100 - 15X) = 100 - 15 \times 5.28 = 20.8$
Specific behaviours
✓ correct mean

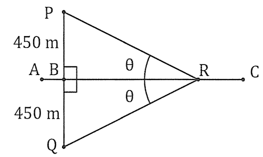
(iii) $\mathrm{Var}(12 - 5X)$. (1 mark)

Solution
$\mathrm{Var}(12 - 5X) = (-5)^2 \times 1.5416 = 38.54$
Specific behaviours
✓ correct variance

See next page

Question 8 **(7 marks)**

Two houses, P and Q, are 900 m apart on either side of a straight railway line AC. AC is the perpendicular bisector of PQ and the midpoint of PQ is B. A small train, R, leaves station C and travels towards B, 1200 m from C.



Let $\angle PRB = \angle QRB = \theta$, where $0 < \theta < 90°$, and $X = PR + QR + CR$, the sum of the distances of the train from the houses and station.

(a) By forming expressions for PR, BR and CR, show that $X = 1200 + \frac{450(2 - \cos\theta)}{\sin\theta}$. (3 marks)

Solution
$PR = \frac{450}{\sin\theta}, \quad BR = PR\cos\theta = \frac{450\cos\theta}{\sin\theta}, \quad CR = 1200 - BR = 1200 - \frac{450\cos\theta}{\sin\theta}$ $X = 2 \times \frac{450}{\sin\theta} + 1200 - \frac{450\cos\theta}{\sin\theta} = 1200 + \frac{450(2 - \cos\theta)}{\sin\theta}$
Specific behaviours
✓ expression for PR in terms of θ ✓ expressions for BR and CR in terms of θ ✓ expression for X in terms of θ

OK. Some got started. Others did not show enough step.

(b) Use a calculus method to determine the minimum value of X. (5 marks)

Solution
$\frac{dX}{d\theta} = 450\left(\frac{\sin\theta \times \sin\theta - (2 - \cos\theta)(\cos\theta)}{\sin^2\theta}\right)$ $= 450\left(\frac{\sin^2\theta + \cos^2\theta - 2\cos\theta}{\sin^2\theta}\right)$ $= 450\left(\frac{1 - 2\cos\theta}{\sin^2\theta}\right)$ $\frac{dX}{d\theta} = 0 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$ $X_{MIN} = 1200 + \frac{450(2 - \cos\theta)}{\sin\theta} = 1200 + 450\left(\frac{3}{2}\right) \times \frac{2}{\sqrt{3}} = 1200 + 450\sqrt{3}$ m
Specific behaviours
✓ uses quotient rule ✓ simplifies derivative ✓✓ roots of derivative ✓ minimum value of X_{MIN}

Not done well. Most could not integrate correctly. Those that did often could not simplify. Those able to solve the eqn then sometimes forgot to calculate the distance!

End of questions

Trinity College

Semester One Examination, 2018

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section Two:
Calculator-assumed

SOLUTIONS

Student number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question 13 **(8 marks)**

A fairground shooting range charges customers $\$6$ to take 9 shots at a target. A prize of $\$20$ is awarded if a customer hits the target three times and a prize of $\$40$ is awarded if a customer hits the target more than three times. Otherwise no prize money is paid.

Assume that successive shots made by a customer are independent and hit the target with the probability 0.15.

(a) Calculate the probability that the next customer to buy 9 shots wins

(i) a prize of $\$20$. (2 marks)

Solution
$X \sim B(9, 0.15)$ $P(X = 3) = 0.1069$
Specific behaviours
✓ defines distribution   ✓ calculates probability

(ii) a prize of $\$40$. (1 mark)

Solution
$P(X \geq 4) = 0.0339$
Specific behaviours
✓ calculates probability

(b) Calculate the expected profit made by the shooting range from the next 30 customers who pay for 9 shots at the target. (3 marks)

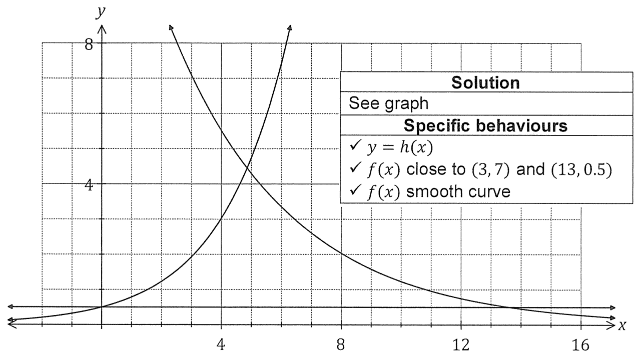
Solution
Let Y be the profit per customer $P(Y = 6) = 0.8592$ $P(Y = -14) = 0.1069$ $P(Y = -34) = 0.0339$ $\mathrm{E}(Y) = 2.504$ $\text{Expected profit} = 30 \times 2.504 = \$75.13$
Specific behaviours
✓ indicates probability distribution   ✓ calculates expected value for one customer   ✓ calculates expected value

(c) Determine the probability that more than 6 out of the next 8 customers will not win a prize. (2 marks)

Solution
$X \sim B(8, 0.8592)$ $P(X \geq 7) = 0.6862$
Specific behaviours
✓ defines distribution   ✓ calculates probability

Question 12 **(8 marks)**

Three functions are defined by $f(x) = 14e^{-0.25x}, g(x) = 0.5e^{0.45x}$ and $h(x) = 0.5$.



Solution
See graph
Specific behaviours
✓ $y = h(x)$ ✓ $f(x)$ close to $(3, 7)$ and $(13, 0.5)$ ✓ $f(x)$ smooth curve

(a) One of the functions is shown on the graph above. Add the graphs of the other two functions. (3 marks)

(b) Working to three decimal places throughout, determine the area of the region enclosed by all three functions. (5 marks)

Solution
$f(x) = g(x)$ when $x = 4.760$ $\int_0^{4.760} g(x) - h(x)\,dx = 5.972$ $g(x) = h(x)$ when $x = 13.329$ $\int_{4.760}^{13.329} f(x) - h(x)\,dx = 10.752$ Area $= 5.972 + 10.752 = 16.724$ sq units
Specific behaviours
✓ writes first integral ✓ evaluates first integral ✓ writes second integral ✓ evaluates second integral ✓ total area *(Rounding instruction supplied for guidance only)*

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(6 marks)**

65% of the fish in a large inland lake are known to be trout. Eight fish are caught at random from the lake every day.

(a) Describe, with parameters, a suitable probability distribution to model the number of trout in a day's catch. (2 marks)

Solution
Binomial, with $n = 8$ and $p = 0.65$
Specific behaviours
✓ binomial ✓ parameters

(b) Determine the probability that there are fewer trout than fish of other species in a day's catch. (2 marks)

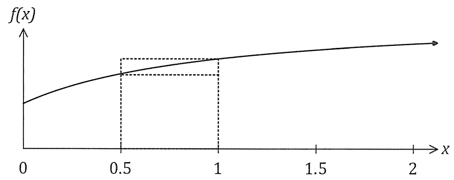
Solution
$P(X \leq 3) = 0.1061$
Specific behaviours
✓ writes $P(X \leq 3)$ or $P(X < 4)$ ✓ probability, to at least 3dp

(c) Calculate the probability that over two consecutive days, a total of exactly 15 trout are caught. (2 marks)

Solution
$X \sim B(16, 0.65)$ $P(X = 15) = 0.0087$
Specific behaviours
✓ defines new distribution ✓ probability

Question 10 **(6 marks)**

The graph of $f(x) = \dfrac{6x+2}{x+1}$ is shown below.



Two rectangles are also shown on the graph, with dotted lines, and they both have corners just touching the curve. The smaller is called the inscribed rect… the circumscribed rectangle.

Solution (a)
See table
Specific behaviours
✓ missing values

(a) Complete the missing values in the table below. (1 mark)

x	0	0.5	1	1.5	2
$f(x)$	2	$\frac{10}{3}$	4	$\frac{22}{5}$	$\frac{14}{3}$

(b) Complete the table of areas below and use the values to determine a lower and upper bound for $\int_0^2 f(x)\,dx$. (4 marks)

x interval	0 to 0.5	0.5 to 1	1 to 1.5	1.5 to 2
Area of inscribed rectangle	1	$\frac{5}{3}$	2	$\frac{11}{5}$
Area of circumscribed rectangle	$\frac{5}{3}$	2	$\frac{11}{5}$	$\frac{7}{3}$

Solution
Lower bound: $L = 1 + \frac{5}{3} + 2 + \frac{11}{5} = \frac{103}{15} \approx 6.867$ Upper bound: $U = \frac{5}{3} + 2 + \frac{11}{5} + \frac{7}{3} = \frac{41}{5} = 8.2$
Specific behaviours
✓ inscribed areas ✓ circumscribed areas ✓ states lower bound ✓ states upper bound

(c) Explain how the bounds you found in (b) would change if a smaller number of larger intervals were used. (1 mark)

Solution
The lower bound would decrease and the upper bound increase.
Specific behaviours
✓ describes changes to both bounds

Question 11 **(8 marks)**

The population of a city can be modelled by $P = P_0e^{kt}$, where P is the number of people living in the city, in millions, t years after the start of the year 2000.

At the start of years 2007 and 2012 there were 2 245 000 and 2 521 000 people respectively living in the city.

(a) Determine the value of the constant k. (2 marks)

Solution
$2.521 = 2.245e^{5k}$ $k = 0.02319$
Specific behaviours
✓ equation ✓ value of k to at least 3sf

(b) Determine the value of the constant P_0. (2 marks)

Solution
$2.521 = P_0e^{0.02319(12)}$ $P_0 = 1.909$
Specific behaviours
✓ equation ✓ value of P_0 (in millions)

(c) Use the model to determine during which year the population of the city will first exceed 3 000 000. (2 marks)

Solution
$3 = 1.909e^{0.02319t}$ $t = 19.5 \Rightarrow$ during 2019
Specific behaviours
✓ value of t ✓ correct year

(d) Determine the rate of change of the city's population at the start of 2007. (2 marks)

Solution
$\frac{dP}{dt} = 0.02319 \times 2\,245\,000$ $= 52\,100$ people per year
Specific behaviours
✓ substitutes into rate of change ✓ correct rate with units