PERTH MODERN SCHOOL
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INDEPENDENT PUBLIC SCHOOL

Semester One Examination, 2022

Question/Answer booklet

MATHEMATICS SPECIALIST
UNIT 3

Section One:
Calculator-free

Your Name

Your Teacher's Name

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Mark	Max	Question	Mark	Max
1			5		
2			6		
3			7		
4			8		

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	34
Section Two: Calculator-assumed	14	14	100	97	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section One: Calculator-free **(52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

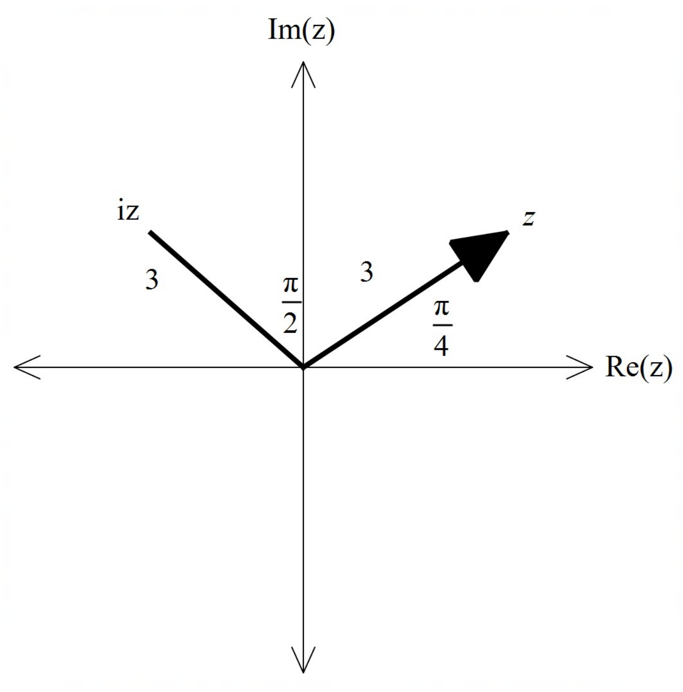
- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 50 minutes.

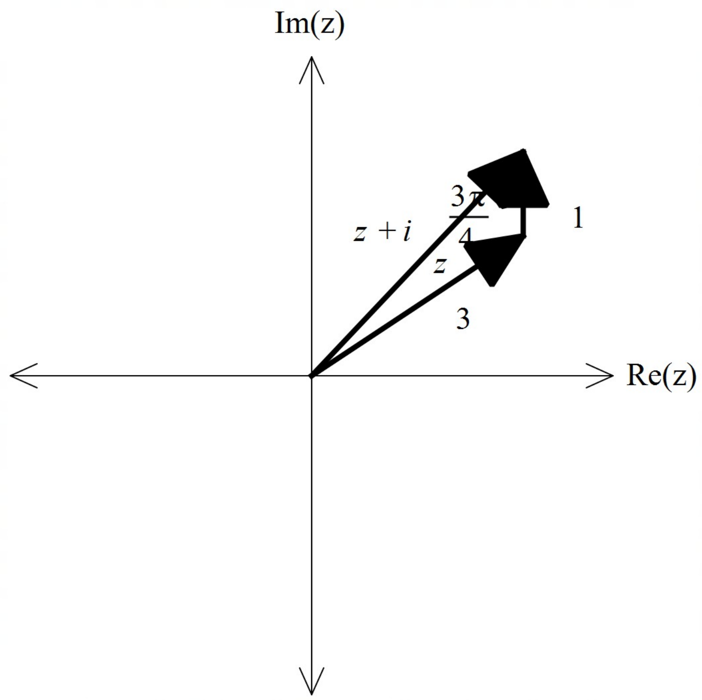
Question 1 **(4 marks)**

Consider the complex number z as plotted on the complex plane below.

a) Determine the exact value of $Arg(iz)$ and plot on the axes above. (2 marks)

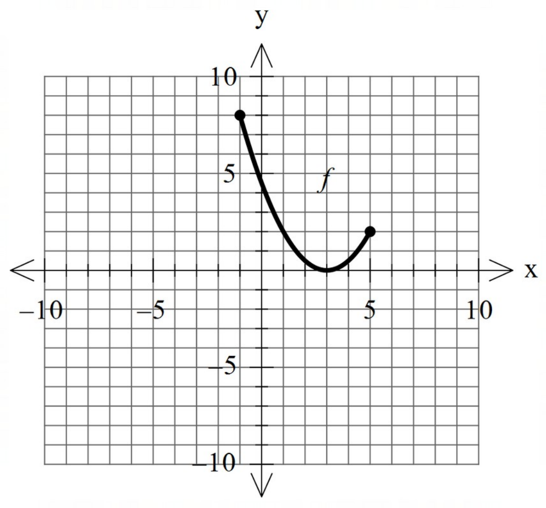
Solution
 $Arg(iz) = \frac{3\pi}{4}$
Specific behaviours
P uses rotation of 90 degrees P states argument

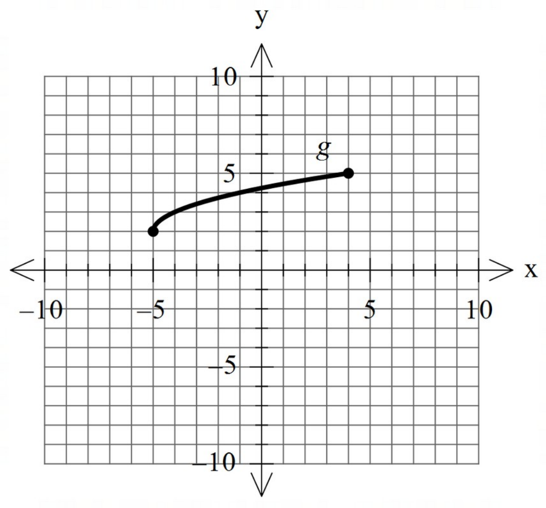
b) Determine the exact value of $|z+i|$. (2 marks)

Solution
 $|z+i|^2 = 3^2 + 1^2 - 2(3)(1)\cos\frac{3\pi}{4} = 10 + 6\frac{1}{\sqrt{2}} = 10 + 3\sqrt{2}$ $|z+i| = \sqrt{10+3\sqrt{2}}$
Specific behaviours
P uses triangle rule P states exact value that must be simplfied

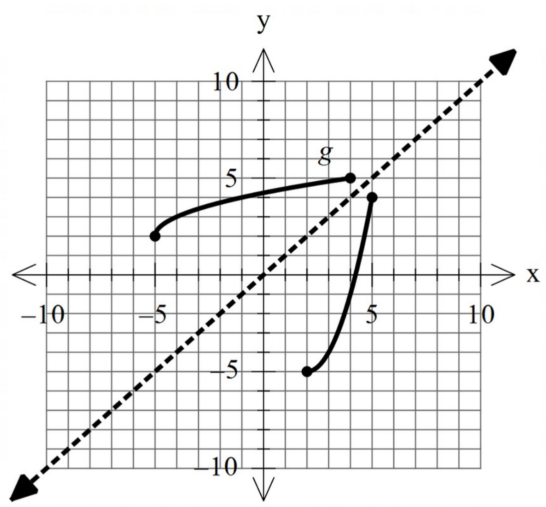
Question 2 **(8 marks)**

Consider the functions f & g with domains shown in the graphs below.





a) Sketch the inverse functions above.
If they exist. (2 marks)

Solution

Specific behaviours
P states that f does not exist P sketches inverse of g with correct endpoints

b) Which of the following exist over natural domains, $f \circ g$, $g \circ f$? Explain.
State the natural domain and corresponding ranges for those that exist. (3 marks)

Solution
$d_f : -1 \le x \le 5 \quad r_f : 0 \le y \le 8$ $d_g : -5 \le x \le 4 \quad r_g : 2 \le y \le 5$ $f \circ g \quad r_g \subseteq d_f \therefore exists$ $d : -5 \le x \le 4 \quad r : 0 \le y \le 2$ $g \circ f \quad r_f \not\subset d_g \therefore not\ exist$
Specific behaviours
P states domain and ranges of both functions in argument P shows that fog exists with condition P shows that gof does not exist with condition

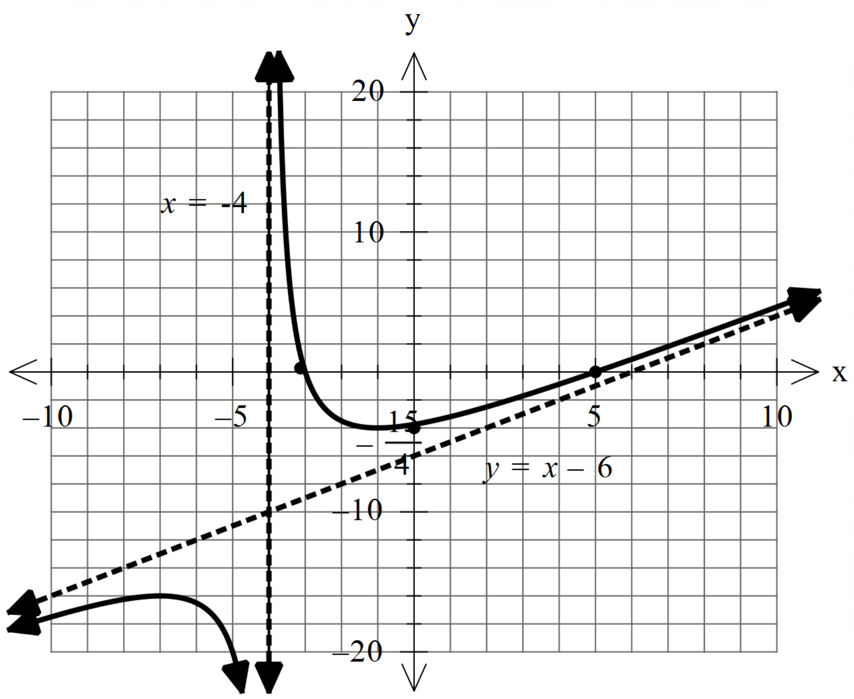
c) The rule for g is $2+\sqrt{x+5}$. State the inverse rule g^{-1} and its domain. (3 marks)

Solution
$d_{g^{-1}} = r_g = 2 \le x \le 5$ $x = 2+\sqrt{y+5}$ $(x-2)^2 = y+5$ $g^{-1}(x) = (x-2)^2 - 5$
Specific behaviours
P swaps x & y P solves for inverse P states domain

Question 3 **(5 marks)**

Consider the function $f(x)=\frac{(x+3)(x-5)}{(x+4)}=x-6+\frac{9}{x+4}$.

Plot on the axes below labelling intercepts and asymptotes.

Solution


Specific behaviours
P shows and labels vertical asymptote
P shows and labels oblique asymptote (accept y=x)
P shows and labels x intercepts
P shows and labels y intercept
P shape is correct

Question 4 **(8 marks)**

Consider the polynomial $P(z) = 3z^4 - 18z^3 - 3z^2 + 258z + 510$ for the complex variable z.

a) Given that $P(5+3i) = 0$, determine a quadratic factor of $P(z)$. (3 marks)

Solution
$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$ $\alpha = 5+3i$ $\beta = 5-3i$ $(x^2 - 10x + 34)\ or\ 3(x^2 - 10x + 34)$
Specific behaviours
P uses conjugate P correct coefficient of x P correct quadratic factor

b) If $P(5+3i) = 0 = P(-2-i)$, determine all solutions to $3z^4 - 18z^3 - 3z^2 + 258z + 510 = 0$.
(2 marks)

Solution
$5 \pm 3i, -2 \pm i$
Specific behaviours
P states 2 solns P states all 4 roots

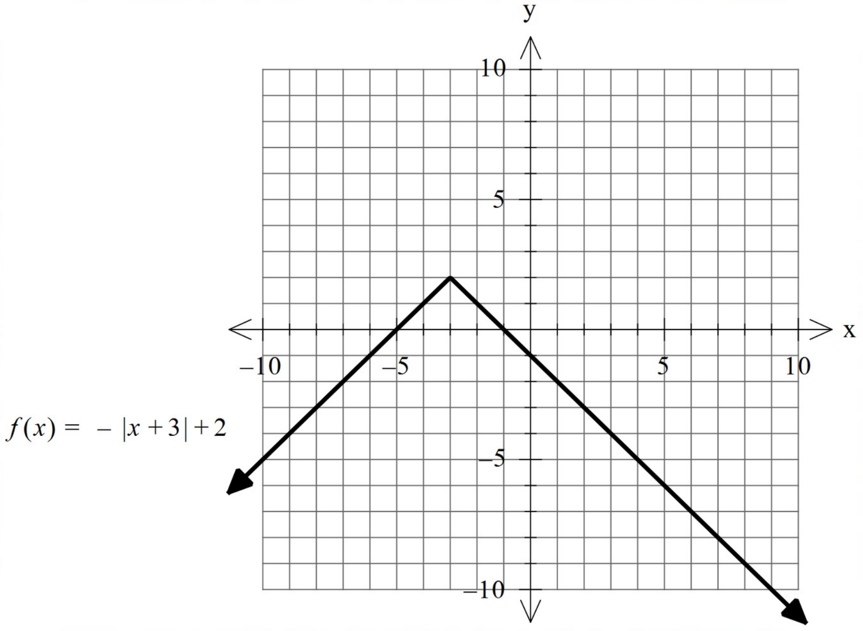
c) $P(z)$ can be expressed as $a(z^2+bz+c)(z^2+dz+e)$ where $a, b, c, d\ \&\ e$ are real integers. Determine the values of $a, b, c, d\ \&\ e$. (3 marks)

Solution
$a(z^2+bz+c)(z^2+dz+e) = 3(x^2 - 10x + 34)(x^2 + 4x + 5)$
Specific behaviours
P uses conjugate of -2-i

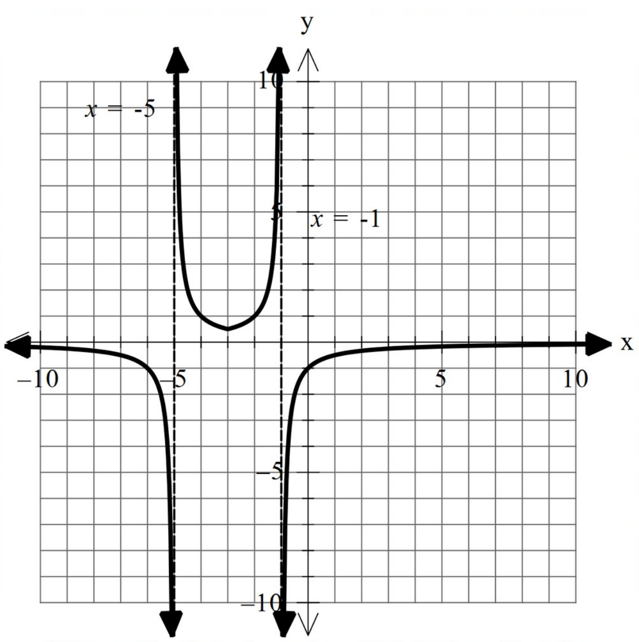
P uses a=3
P states both quadratic factors

Question 5 **(7 marks)**

Consider the function $f(x) = -|x+3|+2$.



a) Sketch $y = \frac{1}{f(x)}$ on the axes below, labeling important features. (4 marks

Solution

Specific behaviours
P shows and labels both asymptotes P y intercept is -1, (no need to label) P x axis appears to be an asymptote P correct shape with turning point (-3,0.5)

b) Hence determine the natural domain and range of $g(x)=\dfrac{1}{3|x+3|-6}$ (3 marks)

Solution
$g(x)=\dfrac{1}{-3f(x)}$ $d_g: x \neq -5, -1$ $r_g: R \setminus (\frac{-1}{6} < y < 0)$ i.e Range of g is all Real numbers **excluding** $\frac{-1}{6} < y \leq 0$ i.e $y \leq -\frac{1}{6}, y > 0$
Specific behaviours
P shows use of factor -1/3 and graph in (a) or other stated reasoning P states domain P states range

Question 6 **(6 marks)**

Consider the linear equations

$$\begin{aligned} 3x+2y+z&=9\\ x+4y+3z&=5\\ 2x-5y+2z&=15 \end{aligned}$$

a) Solve for $x, y \& z$. (3 marks)

Solution

$\begin{bmatrix}1 & 4 & 3 & 5\\ 2 & -5 & 2 & 15\\ 3 & 2 & 1 & 9\end{bmatrix}$ $\begin{bmatrix}1 & 4 & 3 & 5\\ 0 & 13 & 4 & -5\\ 0 & 10 & 8 & 6\end{bmatrix}$ $\begin{bmatrix}1 & 4 & 3 & 5\\ 0 & 13 & 4 & -5\\ 0 & 16 & 0 & -16\end{bmatrix}$ $16y = -16 \quad y = -1$ $-13 + 4z = -5 \;\; z = 2$ $x - 4 + 6 = 5 \cdots x = 3$
Specific behaviours
P eliminates one variable from two equations P eliminates two variables from one equation P solves for all unknowns

b) Consider the following system $\begin{aligned}3x + 2y + z &= 9\\ x + 4y + pz &= 5\\ 2x - 5y + 2z &= q\end{aligned}$ where $p \,\&\, q$ are constants. (3 marks)

Solve for all possible values of $p \,\&\, q$ for the following scenarios:

i) Unique solution
ii) Infinite solutions
iii) No solutions.

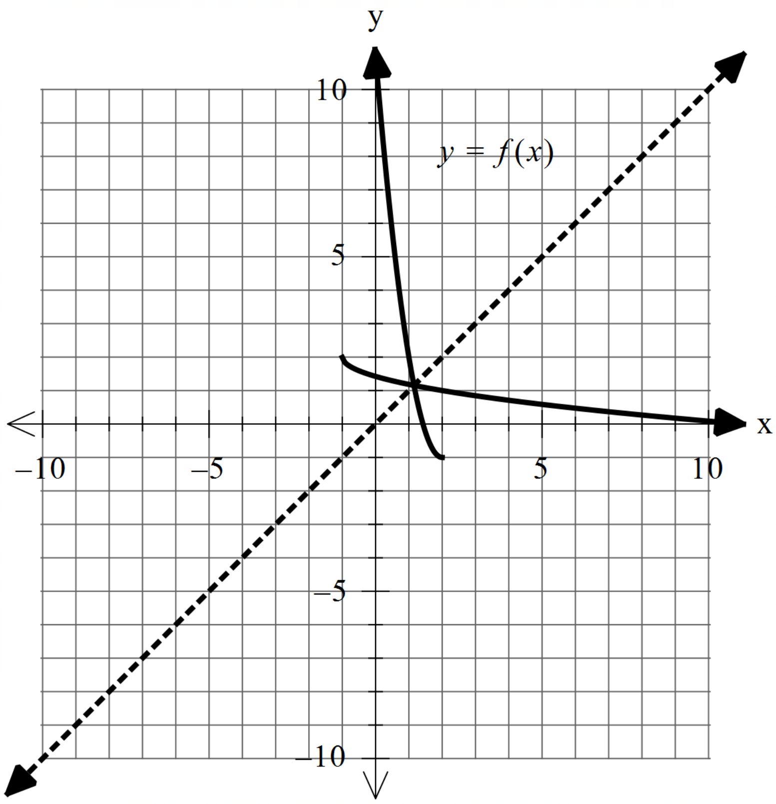
Solution
$\begin{bmatrix}1 & 4 & p & 5\\ 2 & -5 & 2 & q\\ 3 & 2 & 1 & 9\end{bmatrix}$ $\begin{bmatrix}1 & 4 & p & 5\\ 0 & 13 & 2p-2 & 10-q\\ 0 & 10 & 3p-1 & 6\end{bmatrix}$ $\begin{bmatrix}1 & 4 & 3 & 5\\ 0 & 13 & 2p-2 & 10-q\\ 0 & 0 & 19p+7 & 10q-22\end{bmatrix}$ $unique: p \neq -\frac{7}{19}$ $\inf inite: p = \frac{-7}{19} \; and \; q = \frac{22}{10}$ $none: p = \frac{-7}{19} \; and \; q \neq \frac{22}{10}$

Specific behaviours
P obtains an equation with two variable eliminated and in terms of p&q P states requirement for uniqueness P states requirements for infinite and no solns

Question 7 **(10 marks)**

Consider the function $f(x) = 3x^2 - 12x + 11$ for $x \le 2$ which is plotted below.

a) Sketch the inverse function $f^{-1}(x)$ on the axes above. (2 marks)



Specific behaviours
P appears reflected in line y=x P correct endpoint

b)Determine the rule for $f^{-1}(x)$ and state its domain. (4 marks)

Solution

$d_{f^{-1}} = r_f = x \ge -1$

$r_{f^{-1}} = d_f = y \le 2$

$x = 3y^2 - 12y + 11$

$0 = 3y^2 - 12y + 11 - x$

$y = \dfrac{12 \pm \sqrt{144 - 4(3)(11 - x)}}{6} = \dfrac{12 \pm \sqrt{12 + 12x}}{6} = 2 \pm \dfrac{\sqrt{3(x+1)}}{3}$

$f^{-1}(x) = 2 - \dfrac{\sqrt{3(x+1)}}{3}$

Specific behaviours
P states domain P swaps x & y of solves for x

P states inverse rule with two possibilities P discards positive to give only one rule

Q7 continued.

c) Determine $f \circ f^{-1}(x)$ (1 mark)

Solution
$f \circ f^{-1}(x) = x$
Specific behaviours
P states x

d) Determine the exact coordinates, if any, where $f(x) = f^{-1}(x)$ (3 marks)

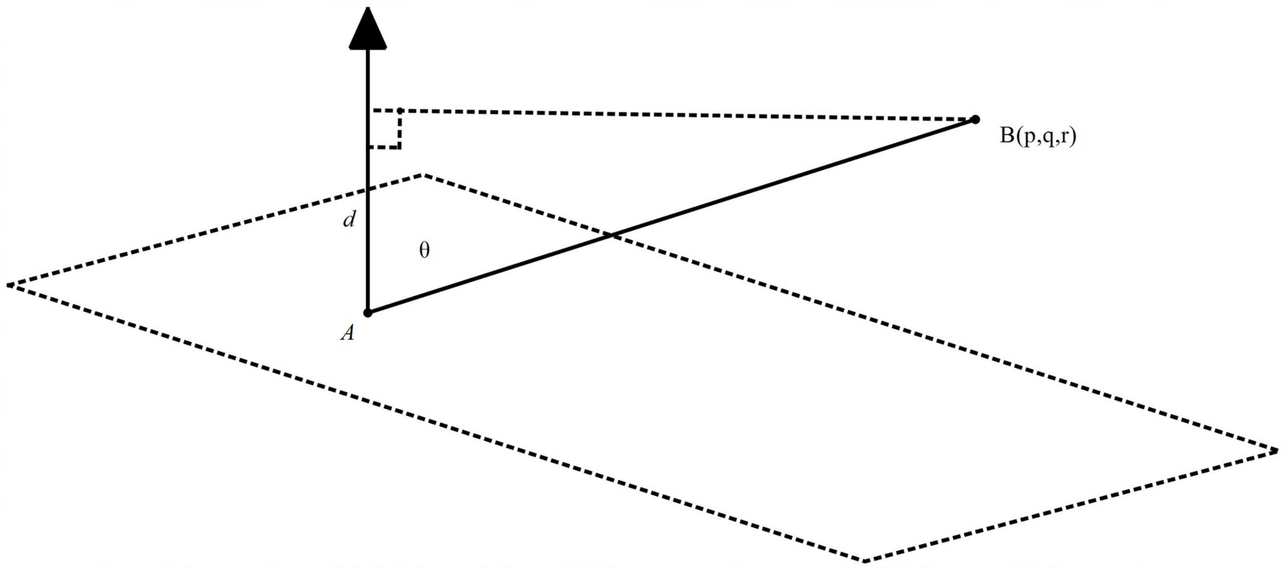
Solution
$f(x) = 3x^2 - 12x + 11 = x$ $3x^2 - 13x + 11 = 0$ $x = \frac{13 \pm \sqrt{169 - 4(3)(11)}}{6} = \frac{13 \pm \sqrt{37}}{6}$ $x = \frac{13 - \sqrt{37}}{6} \quad as\ x \leq 2$ $(\frac{13 - \sqrt{37}}{6}, \frac{13 - \sqrt{37}}{6})$
Specific behaviours
P equates to x and solves for two values P discards the positive value P states **simplified** exact coordinates for point

Question 8 **(4 marks)**

Consider two parallel planes $\Lambda \,\&\, \Psi$. Plane Λ is given by $r.\begin{pmatrix}1\\1\\-2\end{pmatrix}=4$ and plane Ψ contains the point (p,q,r).

Show that the distance between the two planes is given by $\left|\dfrac{p+q-2r-4}{\sqrt{6}}\right|$

Solution



$A(0,0,-2)$ $B(p,q,r)$

$$d = |AB|\cos\theta = \left|\vec{AB}\hat{n}\right| = \left|\begin{pmatrix}p\\q\\r+2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}\right|\frac{1}{\sqrt{1+1+4}} = \left|\frac{p+q-2r-4}{\sqrt{6}}\right|$$

Specific behaviours

P finds a point on plane
P uses dot product with **normal and separation** vector
P uses unit normal vector
P shows derivation of final expression
Note: max of 1 mark if separation vector or line equation not used
Note: (p,q,r) is not on first plane and therefore cannot be subs into vector eqn

Additional working space

Question number: