MATHEMATICS METHODS UNITS 3 & 4

Semester Two

2018

SOLUTIONS

Calculator-free Solutions

1. (a) $\log_x 9 + \log_x x^2$ ✓

$= \dfrac{\log_3 9}{\log_3 x} + 2\log_x x$ ✓

$= \dfrac{2}{p} + 2$ or $\dfrac{2+2p}{p}$ ✓

(b) $3^{2\log_3 3} = 3^2$ ✓

$= 9$ ✓

(c) $3^{2x^2}(2x) = 2x\left(3^{2x^2}\right)$ ✓✓ [7]

2. (a) $\dfrac{dV}{dt} = 4\pi(12t - t^2)^2(12 - 2t)$ ✓

$\dfrac{dV}{dt} = 0$ when $t(12 - t) = 0$ or $(12 - 2t) = 0$ ✓

$\therefore$ $t = 0$ or 6 or 12 ✓

Using the sign table yields Maximum occurs at $t = 6$ ✓

(b) (i) $V = \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi(12t - t^2)^3$ ✓

$\therefore$ $r = 12t - t^2$ ✓

(ii) When $t = 6$ then $r = 36$ ✓

r =36 is the maximum value of r, so max Volume. ✓ [8]

3. (a) $p = 0{\cdot}5[(2) + (1{\cdot}5) + (1{\cdot}2)] = 0{\cdot}5(4{\cdot}7)$ ✓

$= 2.35$ ✓

$q = 0{\cdot}5[(3) + (2) + (1{\cdot}5)] = 0{\cdot}5(6{\cdot}5)$ ✓

$= 3.25$ ✓

(b) $\displaystyle\int_1^{2\cdot5} \frac{3}{x}\,dx = \Big[3\ln x\Big]_1^{2\cdot5}$ ✓

$= 3\ln 2{\cdot}5 - 3\ln 1 = 3\ln 2{\cdot}5$ ✓ [6]

4. (a) 0.07 ✓

(b) 0.25 ✓

(c) $P(1) = (0.1)^2 = 0.01$

$P(2) = (0.15)^2 = 0.0225$ ✓

$\therefore$ $P(1) + P(2) = 0.0325$ ✓

(d) Unbiased would be uniform ✓ [5]

5. (a) $\cos 5x - 5x\sin 5x$ ✓✓

(b) $\int \left(\frac{d}{dx}(x\cos 5x)\right) dx = \int (\cos 5x - 5x\sin 5x)\ dx$ ✓

$\therefore \int (\cos 5x)dx - \int (5x\sin 5x)dx = x\cos 5x + c$ ✓

$\therefore \int (5x\sin 5x)dx = \int (\cos 5x)dx - x\cos 5x + c$ ✓

$\therefore \int (x\sin 5x)dx = \frac{1}{25}\sin 5x - \frac{1}{5}x\cos 5x + c$ ✓ [6]

6. $4e^{2x} - 9e^{x} - 9 = 0$

Let $y = e^x$

$\therefore 4y^2 - 9y - 9 = 0$ ✓

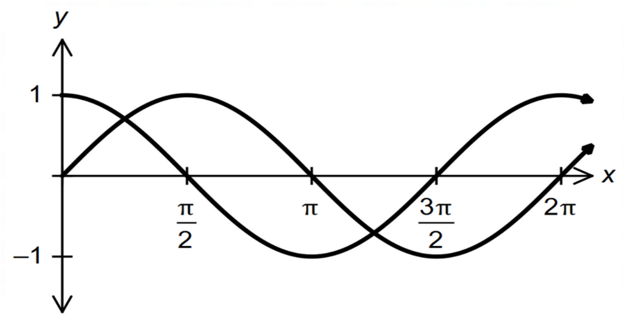
$\therefore (4y + 3)(y - 3) = 0$ ✓

$\therefore y = -\frac{3}{4}$ or $y = 3$ ✓

$\therefore e^x = -\frac{3}{4} \rightarrow$ no solution ✓

and $e^x = 3 \rightarrow x = \ln 3$ ✓ [5]

7. (a)



✓✓

(b) $A = \int_{\frac{\pi}{2}}^{\pi} (\sin x - \cos x)\ dx$ ✓

$= \left[-\cos x - \sin x\right]_{\frac{\pi}{2}}^{\pi}$ ✓

$= (0 + 1) -(-1 - 0) = 2$ ✓ [5]

8. $E_1 = 1{\cdot}645\sqrt{\frac{\hat{p}(1-\hat{p})}{n_1}}$

$E_2 = 1{\cdot}645\sqrt{\frac{\hat{p}(1-\hat{p})}{9n_1}}$ ✓

$\therefore \frac{E_2}{E_1} = \frac{\sqrt{n_1}}{\sqrt{9n_1}}$ ✓

$= \frac{1}{3}$ ✓ [3]

9. (a) $\ln|\sin x| + c$ ✓✓

(b) $\left[\ln|\sin x|\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ ✓

$= \ln(1) - \ln\left(\sin\frac{\pi}{4}\right) = 0 - \ln\left(\frac{1}{\sqrt{2}}\right) = \ln\sqrt{2}$ ✓

$= \frac{1}{2}\ln 2$ ✓ [5]

Calculator-assumed Solutions

10. (a) $p + q = 0{\cdot}4$ ✓

$0{\cdot}2 + 4p + 2{\cdot}7 + 16q + 2{\cdot}5 = 8{\cdot}2$ ✓

$\therefore$ From CAS p = 0.3 and q = 0.1 ✓

(b) E(X) = 0.2 + 0.6 + 0.9 + 0.4 + 0.5 = 2.6 ✓

(c) E(Y) = 1 - 5.2 = -4.2 ✓ [5]

11. (a) $\hat{p} = \frac{86}{200} = 0{\cdot}43$ ✓

(b) $\text{CI} = 0{\cdot}43 \pm 1{\cdot}645\sqrt{\frac{(0{\cdot}43)(0{\cdot}57)}{200}}$ ✓✓

$= 0{\cdot}43 \pm 0{\cdot}0576$ ✓

$\therefore\ 0{\cdot}372 \leq p \leq 0{\cdot}488$ ✓

(c) The mean suggested by the CI is $\frac{0{\cdot}62 + 0{\cdot}82}{2} = 0{\cdot}72$ ✓

This does not lie within the 90% CI as calculated in (b) ✓

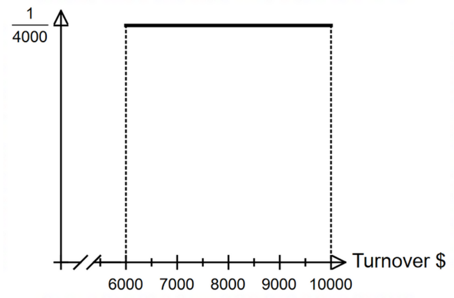
$\therefore$ Evidence doesn't support the claim. ✓

(d) (i) Binomial requires independent events and a constant probability of success. Neither criteria apply here. ✓

(ii) $P(X = 1) = \frac{{}^{7}\mathbf{C}_1 \times {}^{3}\mathbf{C}_3}{{}^{10}\mathbf{C}_4}$ ✓✓

$= 0.1$ ✓ [12]

12. (a)



$$f(x) = \begin{cases} \dfrac{1}{4000} & 6000 < x < 1000 \\ 0 & \text{otherwise} \end{cases}$$

✓✓

(b) (i) $P(8000 \le X \le 9000) = 0{\cdot}25$ ✓

(ii) $P(X = 8500) = 0$ ✓

(c)

$$E(X) = \int_{6000}^{10\,000} \left(x \times \frac{1}{4000} \right) dx = 8000$$

✓✓

(d) (i)

$$VAR(X) = \int_{6000}^{10\,000} \left((x - 8000)^2 \times \frac{1}{4000} \right) dx = 1\,333\,333{\cdot}33$$

✓✓

$\therefore$ st dev = $1154{\cdot}7$ = \$1155 ✓

(ii) Standard deviation = \$1155
Not affected by change of origin ✓ [10]

13. (a) (i)

$$\ln V = \frac{7{\cdot}22 - 6{\cdot}91}{4} t + 6{\cdot}91$$ ✓

$\therefore \quad \ln V = 0{\cdot}0775t + 6{\cdot}91$ ✓

(ii) $V = e^{0{\cdot}0775t + 6{\cdot}91}$ ✓

$\therefore \quad V = 1002{\cdot}25e^{0{\cdot}0775t}$ ✓

(b) $2000 = 1002{\cdot}25e^{0{\cdot}0775t}$ ✓

$\therefore \quad t = 8{\cdot}91$ years ✓

(c) $V = V_0 e^{kt}$

$\therefore \quad 2V_0 = V_0 e^{\frac{rt}{100}}$

$\therefore \quad e^{\frac{rt}{100}} = 2$ ✓

$\therefore \quad \dfrac{rt}{100} = \ln 2 \rightarrow t = \dfrac{100\ln 2}{r}$ ✓

$\therefore \quad t = \dfrac{69}{r}$ ✓ [9]

14. (a) $f'(x) = 2x\ln x + x^2\left(\frac{1}{x}\right) = 2x\ln x + x$ ✓

$\therefore$ $2x\ln x + x = 0$ when $x(2\ln x + 1 = 0)$

$\therefore$ $x = 0$ or $\ln x = -\frac{1}{2}$ ✓

ie $x \neq 0$ or $x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}} = 0{\cdot}61$ ✓

$f''(x) = 3 + 2\ln x$ ✓

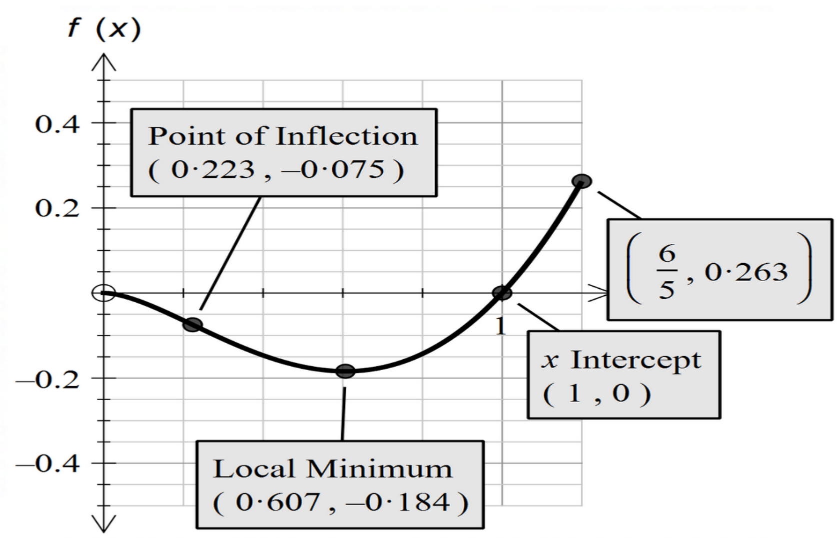
$\therefore$ $f''(0{\cdot}61) > 0$ $\therefore$ Min at $(0{\cdot}61,\ -0{\cdot}18)$ ✓

(b) $f''(x) = 3 + 2\ln x$

$3 + \ln x = 0$ when $x = e^{-\frac{3}{2}}$ ✓

$\therefore$ Oblique POI at $(0{\cdot}22,\ -0{\cdot}08)$ ✓✓

(c)



✓✓✓✓ [12]

15. (a) (i) $v(t) = \int (3t + 5)\ dt = \frac{3t^2}{2} + 5t + 20$ ✓✓

$\therefore$ $v(3) = 48{\cdot}5$ m/sec ✓

(ii) $x(t) = \int \left(\frac{3t^2}{2} + 5t + 20\right) dt = \frac{t^3}{2} + \frac{5t^2}{2} + 20t - 10$ ✓✓

$\therefore$ $x(3) = 86$ m ✓

(b) $v = \frac{3t^2}{2} + 5t + 20 = 0$ when stopped ✓

However $\frac{3t^2}{2} + 5t + 20 \neq 0$ so never stops ✓

(c) Distance travelled = $x(3) - x(0)$ ✓

$\therefore$ $86 - (-10) = 96$ m ✓ [10]

16. $X \sim Bin(36,\ 0{\cdot}8)$

(a) $P(X \leq 30) = 0{\cdot}7536$ ✓✓

(b) (i) $E(\hat{p}) = 0{\cdot}8$ ✓

(ii) $VAR(\hat{p}) = \sqrt{\dfrac{(0{\cdot}8)(0{\cdot}2)}{36}} = \dfrac{1}{15}$ or $0{\cdot}067$ ✓✓

(c) $(0.9)^{10} = 0.349$ ✓✓ [7]

17. (a) Take proportions as follows:

$20 - 29 : \dfrac{727}{3100} \times 100 = 23$

$30 - 39 : \dfrac{1050}{3100} \times 100 = 34$

$40 - 49 : \dfrac{800}{3100} \times 100 = 26$

$50 - 59 : \dfrac{523}{3100} \times 100 = 17$ ✓✓✓

(b) (i) Normal distribution ✓

(ii) $\mu = 0{\cdot}3387$ ✓

Standard deviation = $\sqrt{\dfrac{(0{\cdot}3387)(0{\cdot}6613)}{100}} = 0{\cdot}0473$ ✓

(c) $Y \sim N(0.3387,\ 0{\cdot}0473^2)$ ✓

$\therefore$ $P(X \geq 40) = P(Y \geq 0{\cdot}4) = 0{\cdot}0975$ ✓✓ [9]

18. (a) $P(7) = \dfrac{1}{6}$ ✓

(b) $P(<7) = \dfrac{15}{36}$ ✓

(c) $P(>7) = \dfrac{15}{36}$ ✓

(d) \$12 ✓

(e)

Game	7s	Unders	Overs
P(X)	$\frac{6}{36}$	$\frac{15}{36}$	$\frac{15}{36}$
Return	20	1	1

$\therefore$ $E(X) = \dfrac{6}{36} \times 20 + \dfrac{15}{36} \times 1 + \dfrac{15}{36} \times 1 = \dfrac{150}{36} = 4{\cdot}16$ ✓✓

$\therefore$ Club makes \$5 - \$4.16 = 84 cents per roll ✓ [7]

19. (a) $SA = (3x)(x) + (3x)(h)(2) + (x)(h)(2)$

$= 3x^2 + 8xh$ ✓

$V = 3x^2h = 18 \rightarrow h = \dfrac{6}{x^2}$ ✓

$SA = 3x^2 + 8x\left(\dfrac{6}{x^2}\right) = 3x^2 + \dfrac{48}{x}$ ✓

(b) $\dfrac{d(SA)}{dx} = 6x - \dfrac{48}{x^2}$ ✓

$\therefore$ Max occurs when $6x - \dfrac{48}{x^2} = 0 \rightarrow x = 2$ ✓

$\therefore$ $h = 1.5$ and $SA = 36\text{ m}^2$ ✓

(c) $\dfrac{d(SA)}{dx} \approx \dfrac{\delta SA}{\delta x}$

$\delta SA = \left(\dfrac{d(SA)}{dx}\right)(\delta x) = \left(6x - \dfrac{48}{x^2}\right)(0{\cdot}1)$ ✓

When $x = 2$ then $\delta SA = 0 \times 0{\cdot}1 = 0$ ✓

(d) $\dfrac{dSA}{dt} = \dfrac{dSA}{dx} \times \dfrac{dx}{dh} \times \dfrac{dh}{dt}$

$= \left(6x - \dfrac{48}{x^2}\right)\left(-\dfrac{1}{12x^{-3}}\right)(0{\cdot}1)$ ✓✓

When $x = 1$, $\dfrac{dSA}{dt} = 0{\cdot}35\text{ cm}^3/\text{sec}$ ✓ [11]

20. (a) $X \sim N(1{\cdot}6,\ 0{\cdot}4^2)$

$P(X > 2) = 0.1587$ ✓✓

(b) $P(X < k) = 0{\cdot}9 \rightarrow k = 2{\cdot}11$ ✓✓

(c) (i) $X \sim Bin(4, 0.1587)$ ✓

$\therefore$ $P(X = 1) = 0.378$ ✓

(ii) $0{\cdot}1587 \times 0{\cdot}5^3 \times ({}^4\mathbf{C}_1)$ ✓

$= 0.0794$ ✓ [8]