Rossmoyne Senior High School

Semester One Examination, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ____________________________

Your name ____________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	53	35
Section Two: Calculator-assumed	12	12	100	98	65
			Total	151	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(5 marks)**

Consider the function $f(x) = x^2 - 4x$.

(a) Explain why it is necessary to restrict the natural domain of f in order that its inverse is also a function. (1 mark)

Solution
f is not a one-to-one function for $x \in \mathbb{R}$
Specific behaviours
✓ explains function is not one-to-one over natural domain

(b) State a minimal restriction to the domain of f that includes $x = 3$, and then use this restriction to show that $f^{-1}(x) = 2 + \sqrt{x+4}$. (4 marks)

Solution
f has a turning point at (2, -4) and so minimal restriction is $x \geq 2$ to include $x = 3$ $y + 4 = x^2 - 4x + 4$ $= (x-2)^2$ $x - 2 = \pm\sqrt{y+4}$ but choose $+\sqrt{\ }$ to ensure range of f^{-1} = domain of f $x = 2 + \sqrt{y+4}$ $f^{-1}(x) = 2 + \sqrt{x+4}$
Specific behaviours
✓ states minimal restriction to domain that includes $x = 3$ ✓ completes square on RHS, adjusting LHS   ✓ chooses, with reason, +ve root   ✓ states inverse

Question 9 **(5 marks)**

(a) Let z be a non-zero complex number located in the complex plane. Describe the linear transformation(s) required to transform z to each of the following locations:

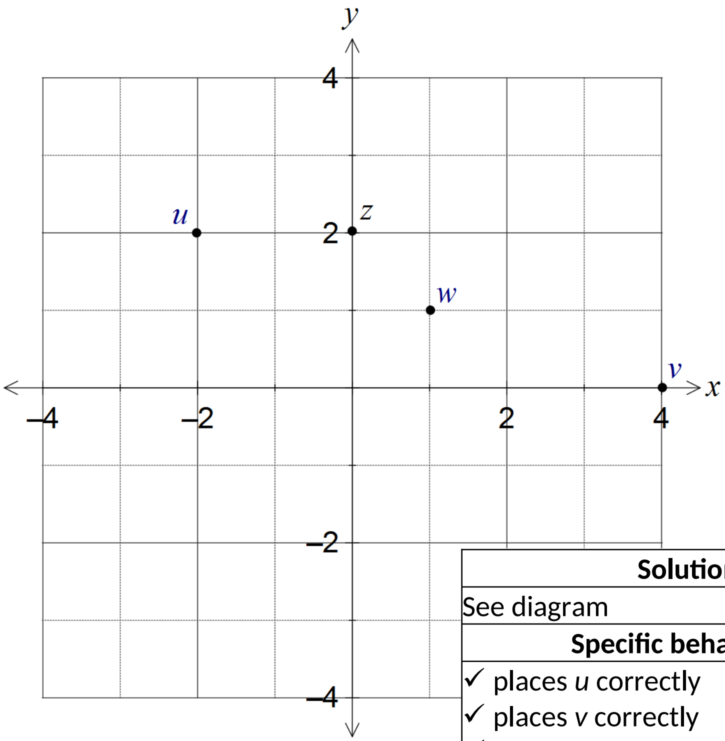
(i) $2z$. (1 mark)

Solution
Dilation of scale factor 2 about the origin.
Specific behaviours
✓ states dilation with scale factor and centre

(ii) $i^3 z$. (1 mark)

Solution
Rotation of 270° anticlockwise about origin.
Specific behaviours
✓ states rotation with angle and centre

(b) Consider the complex number z shown in the Argand diagram below. Add to the diagram the location of u, v and w where $u = (1+i)z$, $v = z \cdot \bar{z}$ and $w = \sqrt{z}$. (3 marks)



Solution
See diagram
Specific behaviours
✓ places u correctly
✓ places v correctly
✓ places w correctly

Question 10 **(8 marks)**

Two functions are given by $f(x) = 2\sqrt{x+1}$ and $g(x) = x^2 - 2x$.

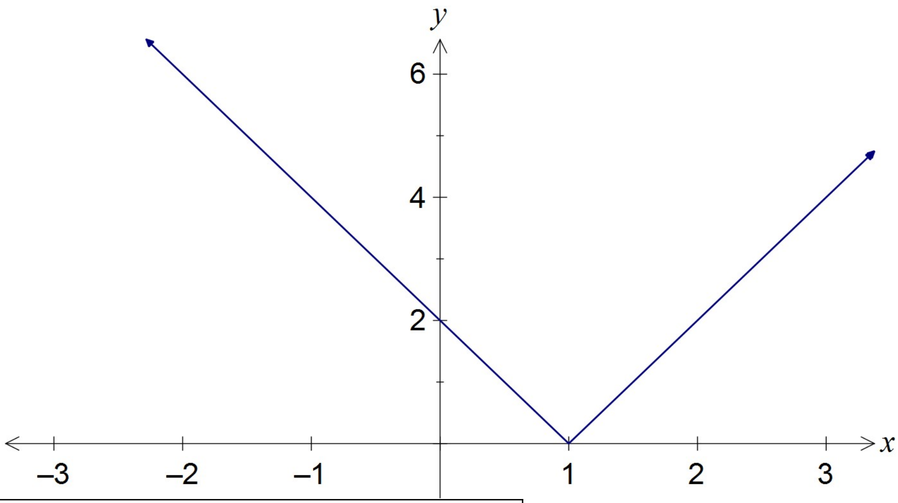
(a) Determine $g \circ f(x)$ and state the domain and range of this composite function. (3 marks)

Solution
$g(f(x)) = \left(2\sqrt{x+1}\right)^2 - 2\left(2\sqrt{x+1}\right)$ $= 4(x+1) - 4\sqrt{x+1}$ $D_{gf} : x \in \mathbb{R},\ x \geq -1$ $R_{gf} : y \in \mathbb{R},\ y \geq -1$
Specific behaviours
✓ writes simplified composite function ✓ states domain ✓ states range

(b) Show that the composite function $f \circ g(x)$ is defined for $x \in \mathbb{R}$. (3 marks)

Solution
$f(g(x)) = 2\sqrt{x^2 - 2x + 1}$ $= 2\sqrt{(x-1)^2}$ $= 2\lvert x-1 \rvert$ $= \begin{cases} 2 - 2x, & x < 1 \\ 2x - 2, & x \geq 1 \end{cases}$
Specific behaviours
✓ substitutes *g* into *f* ✓ simplifies root of square as absolute value ✓ shows piecewise definition for all real *x*

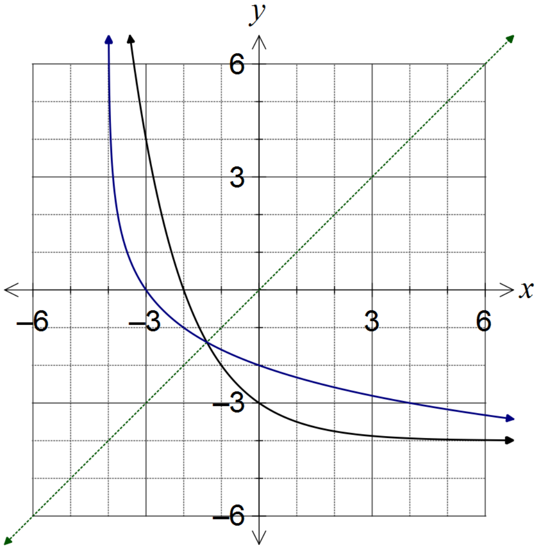
(c) Sketch the graph of $y = f \circ g(x)$ on the axes below. (2 marks)



Solution
See diagram
Specific behaviours
✓ all axes intercepts ✓ shows two parts of composite function

Question 11 **(12 marks)**

(a) The graph of $y = f(x)$ is shown below.



Solution
(ii) See diagram
Specific behaviours
✓ both axes intercepts shown correctly ✓ curve approaches vertical asymptote $x = -4$ ✓ smooth curve with $y = x$ as mirror line

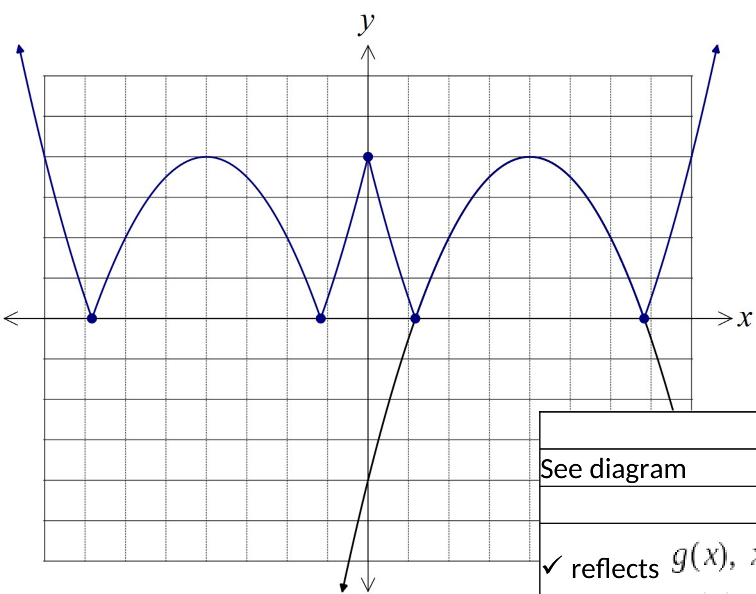
(i) What feature of the graph suggests that the inverse of *f* is a function? (1 mark)

Solution
The part of the graph shown is clearly one-to-one using the horizontal line test.
Specific behaviours
✓ describes function as one-to-one using graph feature

(ii) On the same axes, sketch the graph of the inverse of *f*, $y = f^{-1}(x)$. (3 marks)

(b) The graph of $y = g(x)$ is shown below.

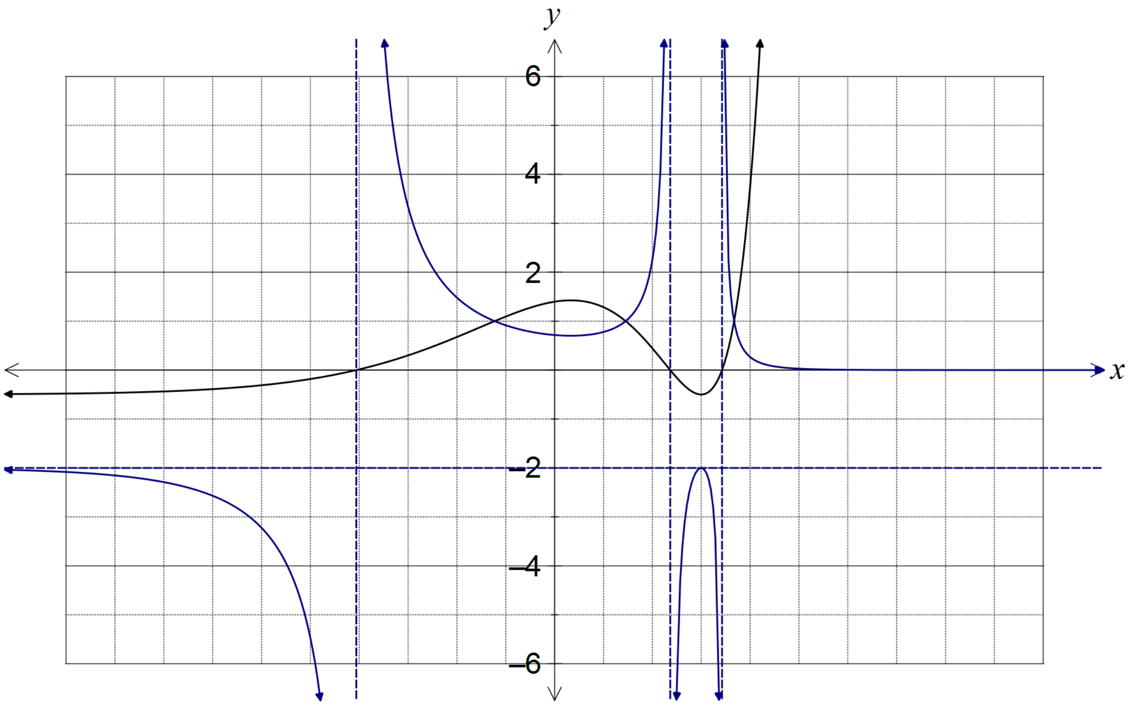
On the same axes, sketch the graph of $y = |g(|x|)|$. (3 marks)



Solution
See diagram
Specific behaviours
✓ reflects $g(x),\ x \geq 0$, in vertical axis to get $g(\lvert x\rvert)$ ✓ reflects $g(\lvert x\rvert),\ y \leq 0$ in horizontal axis ✓ all five indicated points in correct position

(c) The graph of $y = h(x)$ is shown below. As $x \to -\infty,\ h(x) \to -0.5$. On the same axes, sketch the graph of $y = \dfrac{1}{h(x)}$, clearly indicating all vertical and horizontal asymptotes.

(5 marks)



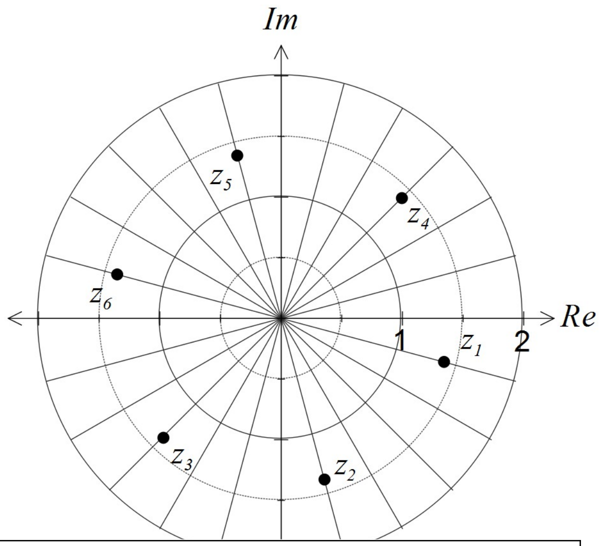
Solution
See diagram
Specific behaviours
✓ correctly shows two parts of curve approaching horizontal asymptotes ✓ correctly shows parts of curve approaching vertical asymptotes ✓ correctly shows *h* and its reciprocal intersect three times when $x = 1$ ✓ uses *y*-axis scale to locate min and max correctly ✓ smooth curves used throughout

Question 12 **(8 marks)**

(a) Determine all roots of the equation $z^6 + 8i = 0$, expressing them in exact polar form $r cis\theta$ where $r > 0$ and $-\pi < \theta \leq \pi$. (5 marks)

Solution
$z^6 = 8cis\left(-\frac{\pi}{2}\right)$ $z = \sqrt[6]{8}cis\left(-\frac{\pi}{2}\times\frac{1}{6}+\frac{2\pi n}{6}\right),\ n = \ldots,\ -1,\ 0,\ 1,\ 2,\ \ldots$ $z = \sqrt{2}cis\left(-\frac{\pi}{12}+\frac{\pi n}{3}\right)$ $z_1 = \sqrt{2}cis\left(-\frac{\pi}{12}\right),\ z_2 = \sqrt{2}cis\left(-\frac{5\pi}{12}\right),\ z_3 = \sqrt{2}cis\left(-\frac{3\pi}{4}\right)$ $z_4 = \sqrt{2}cis\left(\frac{\pi}{4}\right),\ z_5 = \sqrt{2}cis\left(\frac{7\pi}{12}\right),\ z_6 = \sqrt{2}cis\left(\frac{11\pi}{12}\right)$
Specific behaviours
✓ expresses equation in polar form ✓ expresses first root with correct modulus ✓ expresses first root with correct argument ✓ determines argument between roots ✓ lists remaining five roots

(b) Show all solutions of the equation on the Argand diagram below. (3 marks)



Solution
See diagram - six equally spaced points on circle
Specific behaviours
✓ locates roots with $r \approx 1.4$ ✓ locates first root with correct argument ✓ correctly spaces other five roots

Question 13 **(7 marks)**

Two small bodies, A and B, simultaneously leave their initial positions of $\mathbf{i}+4\mathbf{j}-25\mathbf{k}$ and $16\mathbf{i}+\mathbf{j}-2\mathbf{k}$, and move with constant velocities of $4\mathbf{i}+\mathbf{j}+5\mathbf{k}$ and $-\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ respectively.

(a) Determine whether the paths of the bodies cross or if the bodies meet. (4 marks)

Solution
$\mathbf{r}_A = \langle 1+4s, 4+s, -25+5s \rangle$ $\mathbf{r}_B = \langle 16-t, 1+2t, -2-3t \rangle$ $\mathbf{r}_A = \mathbf{r}_B \Rightarrow \begin{array}{r} 1+4s=16-t \\ 4+s=1+2t \\ -25+5s=-2-3t \end{array}$ $\left.\begin{cases} 1+4s=16-t \\ 4+s=1+2t \\ -25+5s=-2-3t \end{cases}\right\vert_{s,t}$ No Solution System has no solution Paths of bodies do not cross, so bodies do not meet.
Specific behaviours
✓ describe paths as vector equations ✓ equate coefficients ✓ states equations inconsistent/have no solutions ✓ interprets that paths do not cross

(b) At the same time, a third small body, C, leaves its initial position, passes through the origin and crosses the path of body A. If C moves with a steady velocity of $5a\mathbf{i}+5\mathbf{j}+a\mathbf{k}$, determine the value of the constant *a*. (3 marks)

Solution
$\mathbf{r}_C = t\langle 5a, 5, a \rangle$ at time t after pass thru O $\mathbf{r}_A = \mathbf{r}_C \Rightarrow \begin{array}{r} 1+4s=5at \\ 4+s=5t \\ -25+5s=at \end{array}$ $\left.\begin{cases} 1+4s=5a\times t \\ 4+s=5\times t \\ -25+5s=a\times t \end{cases}\right\vert_{s,a,t}$ $\{a=2.5, s=6, t=2\}$ $t=2,\ s=6,\ a=2.5$
Specific behaviours
✓ describe path of C as vector equation ✓ equate coefficients ✓ states value of *a* is 2.5

Question 14 **(9 marks)**

The function f is defined by $f(x) = \dfrac{x^2 - 1}{x^2 - 3x + 2}$.

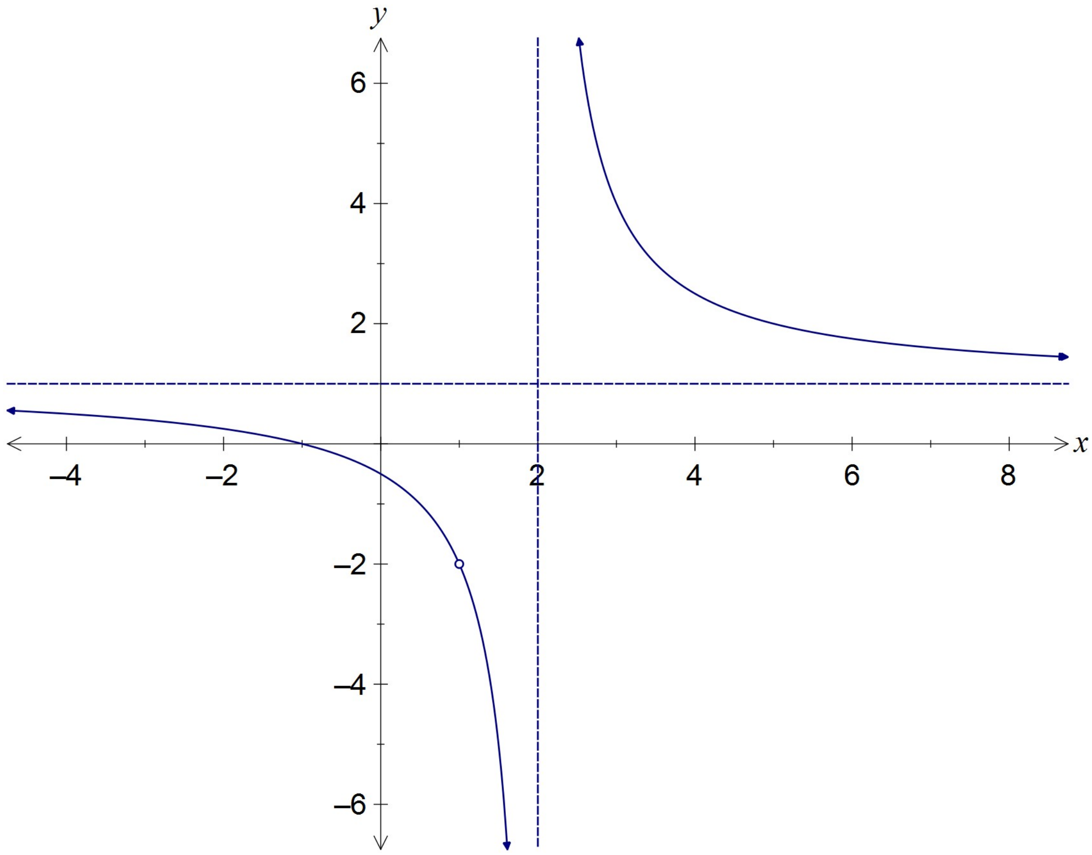
(a) Determine the natural domain and range of $f(x)$. (4 marks)

Solution
$f(x) = \dfrac{(x+1)(x-1)}{(x-2)(x-1)}$ $= \dfrac{(x+1)}{(x-2)},\ x \neq 1 \quad \Rightarrow$ $= 1 + \dfrac{3}{x-2},\ x \neq 1$ Domain: $x \in \mathbb{R},\ x \neq 1,\ x \neq 2$ Range: $y \in \mathbb{R},\ y \neq -2,\ y \neq 1$
Specific behaviours
✓ factorises and simplifies f ✓ states domain   ✓ states range using asymptote   ✓ includes 'hole' at (1, -2) in range

(b) Show that the function has no stationary points. (2 marks)

Solution
$f'(x) = \dfrac{-3}{(x-2)^2}$ $-3 \neq 0 \Rightarrow f'(x) \neq 0$
Specific behaviours
✓ states $f'(x)$ ✓ shows cannot be zero

(c) Sketch the graph of $y = f(x)$ on the axes below. (3 marks)



Solution
See diagram
Specific behaviours
✓ axes intercepts ✓ smooth curves approach asymptotes correctly ✓ indicates undefined point

Question 15 **(8 marks)**

Given the two complex numbers $w = r(\cos\theta + i\sin\theta)$ and $z = s(\cos\phi + i\sin\phi)$, determine the following in terms of the non-zero constants r, s, θ and ϕ:

(a) $\arg(\bar{z})$. (1 mark)

Solution
$\bar{z} = r \cdot cis(-\phi) \Rightarrow arg(\bar{z}) = -\phi$
Specific behaviours
✓ determines argument

(b) $\left\|\dfrac{i}{w^2}\right\|$. (2 marks)

Solution
$\dfrac{i}{w^2} = \dfrac{cis\left(\frac{\pi}{2}\right)}{r^2 \cdot cis2\theta} = \dfrac{1}{r^2} cis\left(\dfrac{\pi}{2} - 2\theta\right) \Rightarrow \left\|\dfrac{i}{w^2}\right\| = \dfrac{1}{r^2}$
Specific behaviours
✓ simplifies into $acisb$ form ✓ states modulus

(c) $\|(1-i)wz\|$. (2 marks)

Solution
$(1-i)wz = \sqrt{2}cis\left(-\dfrac{\pi}{4}\right) \cdot r \cdot cis\theta \cdot s \cdot cis\phi = \sqrt{2}rs \cdot cis\left(\theta + \phi - \dfrac{\pi}{4}\right)$ $\|(1-i)wz\| = \sqrt{2}rs$
Specific behaviours
✓ simplifies into $acisb$ form ✓ determines modulus

(d) $\arg\left(-\dfrac{z}{iw}\right)$. (3 marks)

Solution
$-\dfrac{z}{iw} = \dfrac{cis(\pi) \cdot s \cdot cis\phi}{cis\left(\frac{\pi}{2}\right) \cdot r \cdot cis\theta} = \dfrac{s}{r} \cdot cis\left(\dfrac{\pi}{2} + \phi - \theta\right)$ $\arg\left(-\dfrac{z}{iw}\right) = \dfrac{\pi}{2} + \phi - \theta$
Specific behaviours
✓ writes -1 and i in cis form ✓ simplifies into $acisb$ form ✓ determines argument

Question 16 **(7 marks)**

Consider the three vectors $\mathbf{a} = \langle 2, 1, -3 \rangle$, $\mathbf{b} = \langle -3, 5, -2 \rangle$ and $\mathbf{c} = \langle 2, -4, 1 \rangle$.

(a) Prove that the three vectors do not lie in the same plane. (4 marks)

Solution
If vectors lie in the same plane, then a vector perpendicular to **a** and **b** will also be perpendicular to **c**. Vector perpendicular to **a** and **b** is **d**: $\mathbf{d} = \langle 2, 1, -3 \rangle \times \langle -3, 5, -2 \rangle$ $= \langle 13, 13, 13 \rangle$ Consider scalar product of **c** and **d**: $\langle 2, -4, 1 \rangle \cdot \langle 13, 13, 13 \rangle = -13$ Since this is not zero, then **c** and **d** are not perpendicular, and so we conclude that the three vectors cannot lie in the same plane.
Specific behaviours
✓ chooses cross product to find a perpendicular ✓ calculates perpendicular correctly ✓ chooses scalar product to show perpendicular not perpendicular to other vector ✓ shows scalar product is not 0

(b) Determine the value(s) of the constant a if the vector $\langle a^2, a, a-3 \rangle$ lies in the same plane as vectors $\mathbf{a}$ and $\mathbf{b}$. (3 marks)

Solution
$\langle 1, 1, 1 \rangle \cdot \langle a^2, a, a-3 \rangle = a^2 + a + a - 3$ $a^2 + 2a - 3 = 0$ $(a+3)(a-1) = 0 \Rightarrow a = -3,\ 1$
Specific behaviours
✓ calculates scalar product ✓ solves scalar product equal to zero ✓ determines all values of *a*

Question 17 **(9 marks)**

Let the complex number $z = \cos\theta + i\sin\theta$.

(a) Show that $\dfrac{1}{z} = \cos\theta - i\sin\theta$. (2 marks)

Solution
$\dfrac{1}{z} = z^{-1}$ $= cis(-\theta)$ $= \cos(-\theta) + i\sin(-\theta)$ $= \cos\theta - i\sin\theta$
Specific behaviours
✓ uses De Moivre's theorem to obtain $cis(-\theta)$ ✓ uses trig identity to obtain result

(b) Show that $z^3 - \dfrac{1}{z^3} = 2i\sin 3\theta$. (2 marks)

Solution
$z^3 - \dfrac{1}{z^3} = cis(3\theta) - cis(-3\theta)$ $= \cos 3\theta + i\sin 3\theta - \cos 3\theta + i\sin 3\theta$ $= 2i\sin 3\theta$
Specific behaviours
✓ uses De Moivre's theorem to obtain triple angles   ✓ simplifies result

(c) Determine $\mathrm{Im}\left(z^3 - \dfrac{1}{z^3}\right)$ in terms of $\sin\theta$ and $\cos\theta$.

(3 marks)

Solution
$z^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$ $= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$ $\dfrac{1}{z^3} = \cos^3\theta - 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta - i^3\sin^3\theta$ $= \cos^3\theta - 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta + i\sin^3\theta$ $\mathrm{Im}\left(z^3 - \dfrac{1}{z^3}\right) = 3\cos^2\theta\sin\theta - \sin^3\theta - \left(-3\cos^2\theta\sin\theta + \sin^3\theta\right)$ $= 6\cos^2\theta\sin\theta - 2\sin^3\theta$
Specific behaviours
✓ expands z^3 ✓ expands z^{-3} ✓ simplifies imaginary part

(d) Express $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$. (2 marks)

Solution
Using results from (b) and (c): $\text{Im}\left(z^3 - \frac{1}{z^3}\right) = 2\sin 3\theta = 6\cos^2\theta\sin\theta - 2\sin^3\theta$ $2\sin 3\theta = 6(1-\sin^2\theta)\sin\theta - 2\sin^3\theta$ $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$
Specific behaviours
✓ equates and eliminates $\cos\theta$ from results in (b) and (c)   ✓ simplifies and rearranges for required result

Question 18 **(13 marks)**

The velocity vector of a particle at time *t* seconds is $\mathbf{v}(t) = 3\mathbf{i} - \frac{3}{t^2}\mathbf{j}$, for $t \geq 1$. When $t = 1$, the particle has position vector $2\mathbf{j}$.

(a) Calculate the exact speed of the particle when $t = 2$. (2 marks)

Solution
$\mathbf{v}(2) = 3\mathbf{i} - \frac{3}{4}\mathbf{j}$ $\lvert\mathbf{v}(2)\rvert = \frac{3\sqrt{17}}{4}$
Specific behaviours
✓ determines the velocity vector ✓ calculates the exact speed

(b) Determine the acceleration vector of the particle and comment on its direction. (2 marks)

Solution
$\mathbf{a}(t) = \frac{6}{t^3}\mathbf{j}$ Acceleration has no **i** component, so always acts parallel to *y*-axes.
Specific behaviours
✓ differentiates velocity vector ✓ states always acts parallel to y-axis, or vertically

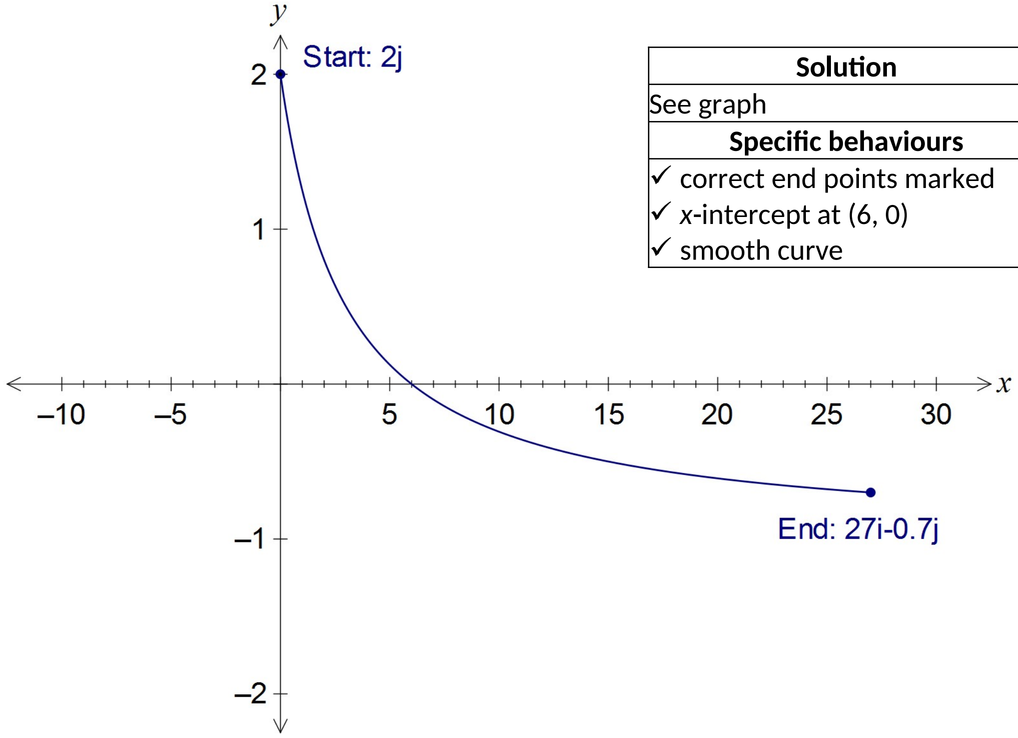
(c) Determine the position vector of the particle for $t \geq 1$. (2 marks)

Solution
$\mathbf{r}(t) = (3t + c_1)\mathbf{i} + \left(\frac{3}{t} + c_2\right)\mathbf{j}$ $\mathbf{r}(0) = 2\mathbf{j} \Rightarrow c_1 = -3,\ c_2 = -1$ $\mathbf{r}(t) = (3t - 3)\mathbf{i} + \left(\frac{3}{t} - 1\right)\mathbf{j}$
Specific behaviours
✓ integrates velocity vector ✓ evaluates constants and includes in position vector

(d) Derive the Cartesian equation of the path of the particle in the form $y = f(x)$. (2 marks)

Solution
$x = 3t - 3 \Rightarrow t = \frac{x+3}{3}$ $y = \frac{3}{t} - 1 \Rightarrow t = \frac{3}{y+1}$ $\Rightarrow \frac{x+3}{3} = \frac{3}{y+1} \Rightarrow y = \frac{9}{x+3} - 1$
Specific behaviours
✓ expresses *t* in terms of *x* and *y* ✓ eliminates parameter *t* and re-arranges for *y*

(e) On the axes below, sketch the path taken by the particle for $1 \le t \le 10$, clearly indicating the position of the particle at the start and end of this interval. (3 marks)



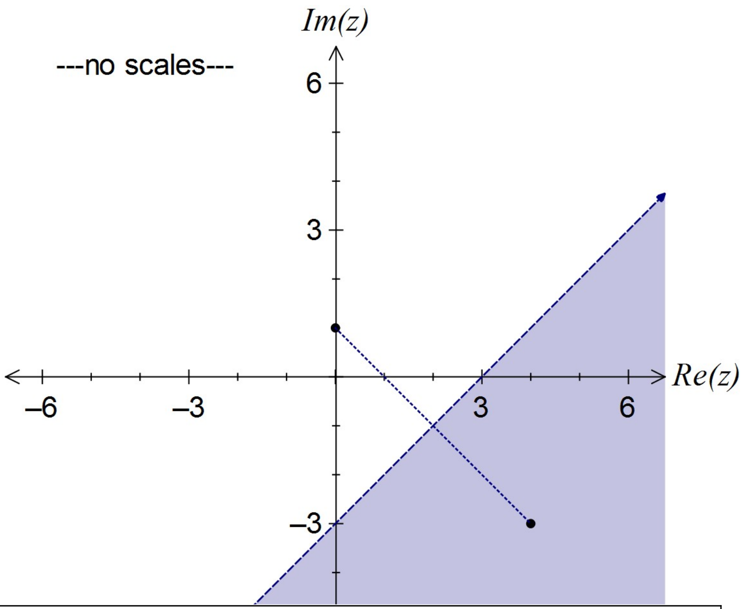
Solution
See graph
Specific behaviours
✓ correct end points marked ✓ *x*-intercept at (6, 0) ✓ smooth curve

(f) Determine the length of the path travelled by the particle between $t = 1$ and $t = 10$. (2 marks)

Solution
$L = \int_1^{10} \lvert v(t) \rvert \, dt$ $= \int_1^{10} \sqrt{(3)^2 + \left(-\frac{3}{t^2}\right)^2} \, dt$ $= 27.46$ units
Specific behaviours
✓ writes correct integral ✓ evaluates integral

Question 19 **(7 marks)**

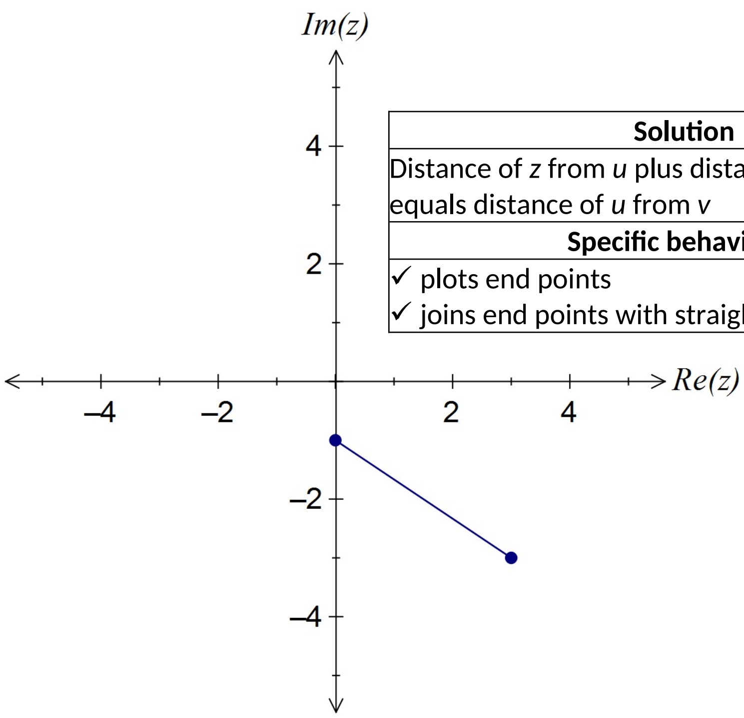
(a) Shade the region satisfying the complex inequality $|z - i| > |z - 4 + 3i|$ on the Argand diagram below. (3 marks)



Solution
See diagram $|z - (0 + i)| > |z - (4 - 3i)|$
Specific behaviours
✓ adds scale and locates two points ✓ draws perpendicular bisector of points using dotted line ✓ shades correct region

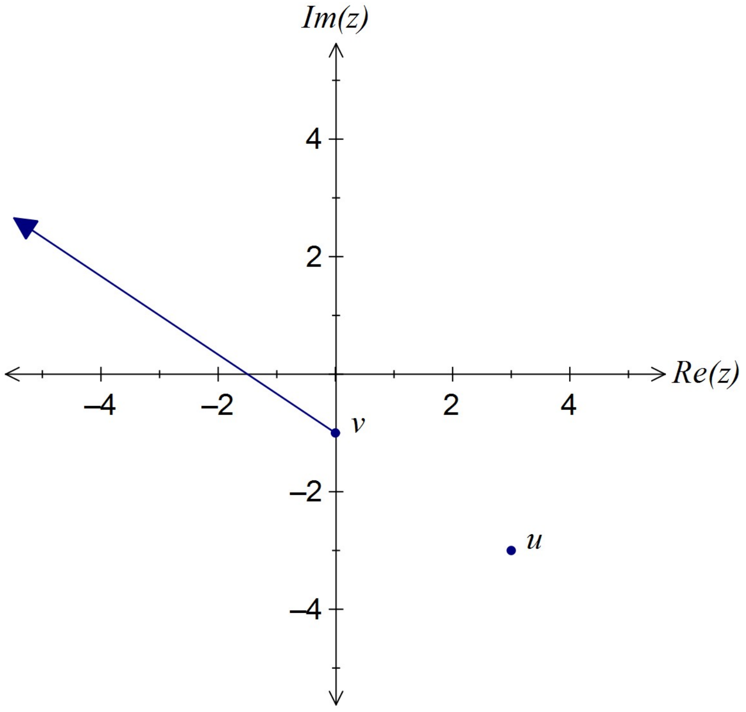
(b) Consider the two complex numbers given by $u = 3 - 3i$ and $v = -i$. Sketch each of the following sets of points in the complex plane.

(i) $|z - u| + |z - v| = |u - v|$. (2 marks)



Solution
Distance of *z* from *u* plus distance of *z* from *v* equals distance of *u* from *v*
Specific behaviours
✓ plots end points ✓ joins end points with straight line

(ii) $|z-v|+|u-v|=|z-u|$. (2 marks)



Solution
Distance of *z* from *v* plus distance of *u* from *v* equals distance of *z* from *u*
Specific behaviours
✓ constructs straight line using points ✓ indicates solution extends from *v*

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

Additional working space

Question number: _________

Additional working space

Question number: _________