

Question	Max	Marks	Question	Max	Marks
12			10		
11			10		
10			8	16	
9			8	15	
8			15	14	
7			8	13	
				11	12
					8

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material if you have any unauthorised material with you, hand it to the supervisor before reading any further.

Important note to candidates

Special items: to three calculators approved for use in this examination

Standard items: drawing instruments, empire, ruler, highlighters

To be provided by the candidate

Formula sheet (extracted from Section One)

This Question/Answer booklet

Materials required/recommended for this section

Working time: one hundred minutes
Reading time before commencing work: ten minutes

Your Teacher's Name:

Your Name:

Calculator-assumed

Section Two:

UNIT 3

MATHEMATICS METHODS

Question/Answer booklet

Semester One Examination, 2022

INDEPENDENT PUBLIC SCHOOL

Exceptional schooling. Exceptional students.



Question number: _____

Additional working space

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	6	6	50	53	35
Section Two: Calculator-assumed	10	10	100	100	65
Total					100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2019*. Sitting this examination implies that you agree to abide by these rules.
2. Write your answers in this Question/Answer booklet.
3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.
4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.
5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.
6. It is recommended that you **do not use pencil**, except in diagrams.
7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Additional working space

Question number: _____

- (c) Evaluate the derivative of $\frac{g'(x)}{f(x)}$ at $x=0$. (3 marks)

Solution

$$\frac{d}{dx} \left(\frac{g'(x)}{f(x)} \right)_{x=0} = \frac{g''(0)f(0) - g'(0)f'(0)}{(f(0))^2} \cdot \frac{[0](3) - (-2)(1)}{3^2} \cdot \frac{2}{9}$$

Specific behaviours

ü Indicates correct application of quotient rule

ü Indicates correct value of $g''(0)$

ü Correctly evaluates derivative

- (a) Determine the height of the sculpture
mark) (1

Solution

$$-3\pi \leq x \leq 3\pi, x=0, -3\pi, 3\pi$$

(b)

$$x=0, y=0$$

$$x=\pm 3\pi, y=6$$

Hence the maximum value √ height is 6 m

Specific Behaviours

✓ States the height with units

- Determine the volume of the material making the sculpture. (3 marks)

Solution

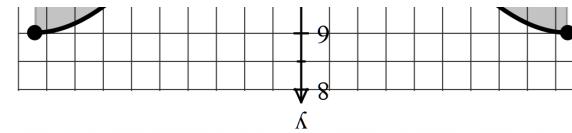
$$\int_{-3\pi}^{3\pi} 3 - 3\cos\left(\frac{x}{3}\right) dx = \left[3x - 9\sin\left(\frac{x}{3}\right) \right]_{-3\pi}^{3\pi} = 18\pi$$

$$\begin{aligned} \text{Volume} &= \text{Area of the cross-section} \times \text{width} \\ &= 18\pi \times 3 \\ &= 54\pi \text{ m}^3 \text{ (or } 169.65 \text{ m}^3 \text{ approximately)} \end{aligned}$$

Specific Behaviours

- ✓ Sets up the integral with correct boundary points
✓ Determines the correct area
✓ Determines volume (approx. or exact) no need for units

- c) Determine the coordinates of the steepest point(s) on the cross-section. Justify. (4 marks)



A sculpture has a cross-section shown below (indicated by the shaded region). All measurements are in metres. The equation of the curve is $y = 3 - 3 \cos\left(\frac{\pi}{6}x\right)$ for $x \in [-3, 3]$.

Question 16

(8 marks)

The sculpture has a cross-section shown below (indicated by the shaded region). All measurements are in metres. The equation of the curve is $y = 3 - 3 \cos\left(\frac{\pi}{6}x\right)$ for $x \in [-3, 3]$.

	<ul style="list-style-type: none"> □ Substitutes $x = 50$ into the correct equation for marginal profit □ Substitutes $x = 50$ into the result which indicates the profit of making one more item □ Determines the marginal profit when $x = 50$ and explain what this value predicts.
	<p>The profit made on the 51st item produced was \$3950.</p>
	$P(x) = -3x^2 + 230x - 50$
	<p>Solution</p>
c)	<p>Find the marginal profit when $x = 50$ and explain what this value predicts. (3 marks)</p>
	<ul style="list-style-type: none"> □ Equates $P'(x)$ to zero and solves for both values of x □ Uses an appropriate Calculus method to determine the nature of both values of x □ Determines the maximum profit
	<p>The maximum profit is \$215 986.93</p>
	$\therefore P(76.449) = -(76.449)^3 + 115(76.449)^2 - 50(76.449) - 5500 = 215986.93$
	$P''(76.449) = -228.694 \text{ (ve)} \text{ so, this is a MAXIMUM}$
	$P''(0.218) = 228.692 \text{ (+ve)} \text{ so, this is a MINIMUM}$
	$P'(x) = -6x + 230$
	$\therefore x = 0.218 \text{ and } x = 76.449$
	$-3x^2 + 230x - 50 = 0$
	$P(x) = -3x^2 + 230x - 50$
	<p>Solution</p>
b)	<p>Use Calculus methods to determine the maximum profit. (4 marks)</p>
	<ul style="list-style-type: none"> □ Interprets the value as a loss of \$5500 when zero items are produced □ Interprets the value as a loss of \$5500 when zero items are produced.
	<p>Solution</p>
a)	<p>Interpret the value of $P(0)$ in this context. (1 mark)</p>
	<p>The profit function, $P(x)$ in \$, of a company producing x items, is given by:</p>
	$P(x) = -x^3 + 115x^2 - 50x - 5500$

Question 8

(15 marks)

d) State the maximum marginal profit and when this occurs.

(3 marks)

Solution

$$P''(x) = -6x + 230 = 0 \\ x = \frac{115}{3} \approx 38$$

$$P'(38) = -3(38)^2 + 230(38) - 50 = 4358$$

Thus the maximum marginal profit is \$4358 when the 38th item is produced

Specific Behaviours

- ü Equates the correct equation to zero and solves for x
- ü Determines the marginal profit
- ü Explains when it occurs (explain the value of x)

e) How many items must be produced to ensure a profit?

(2 marks)

Solution

$$-x^3 + 115x^2 - 50x - 5500 = 0$$

CAS Solve: $x = -6.5$, $x = 7.4$ and $x = 114.14$
Reject $x = -6.5$

Between $x=8$ and $x=114$ items, **inclusive**, is needed to ensure a profit is made.

Specific Behaviours

- ü Equates $P(x)$ to zero and solve for 3 values of x
- ü Reject the negative solution, and gives the correct range of values for x in integers

f) Determine the average profit when $x=50$.

(2 marks)

Solution

$$\text{Average profit} = \frac{P(x)}{x}$$

$$\frac{P(x)}{x} = -x^2 + 115x - 50 - \frac{5500}{x}$$

$$-(50)^2 + 115(50) - 50 - \frac{5500}{50} = 3090$$

The average profit when $x=50$ is \$3090

Specific Behaviours

- ü Divides $P(x)$ by x to obtain the average profit function
- ü Solves for the average profit when $x=50$

(a) Determine the velocity function, $v(t)$, for the particle.

(2 marks)

Solution

$$v(t) = x'(t) = \frac{7t^2 - 56t + 84}{2}$$

Specific behaviours

- ü relates velocity to first derivative of displacement wrt t
- ü determines the first derivative

(b) Determine the displacement of the particle at the instant it is stationary.

(3 marks)

Solution

$$\text{solve}\left(\frac{7t^2 - 56t + 84}{2} = 0, t\right)$$

$$\{t=2, t=6\}$$

$$\frac{7t \times (t^2 - 12t + 36)}{6} |_{t=2}$$

$$\frac{112}{3}$$

$$\frac{7t \times (t^2 - 12t + 36)}{6} |_{t=6}$$

$$0$$

Specific behaviours

- ü solves $v=0$ for two values
- ü determines displacement for one value
- ü both values

(c) Determine the acceleration function, $a(t)$, for the particle.

(2 marks)

Solution

$$\frac{d}{dt} \left(\frac{7t^2 - 56t + 84}{2} \right)$$

$$7t-28$$

Specific behaviours

- ü relates acceleration to first derivative of velocity wrt t
- ü determines acceleration function

(d) How far has the particle travelled before its acceleration is zero?

(3 marks)

Solution

$$\text{solve}(7t-28=0, t)$$

$$\{t=4\}$$

$$\int_0^4 \left| \frac{7t^2 - 56t + 84}{2} \right| dt$$

Question 15

The displacement in metres, $x(t)$, of a particle t seconds after it was launched is given by:

(10 marks)

$$x(t) = \frac{7t(t^2 - 12t + 36)}{6}, \quad t \geq 0.$$

<input type="checkbox"/> States the distribution of the new random variable with correct parameters
Specific Behaviours

Question 9**(8 marks)**

Ryan and Oliver play a game where two normal six-sided dice are rolled, the uppermost number noted and then a third six-sided die is rolled. To win the game the number rolled on the third die must fall between the numbers rolled on the first two dice. For example, if a 2 and a 5 are rolled on the first two dice, to win the game a 3 or a 4 must be rolled on the third dice.

- a) Determine the probability that a player has no chance of winning before even rolling the third die.

Solution

If the first two dice are rolled, and their numbers are noted, the table below shows the number of values that sit between the noted numbers from each die:

	1	2	3	4	5	6
1	0	0	1	2	3	4
2	0	0	0	1	2	3
3	1	0	0	0	1	2
4	2	1	0	0	0	1
5	3	2	1	0	0	0
6	4	3	2	1	0	0

The probability that a player has no chance of winning before the third die is rolled is $\frac{16}{36}$

Specific Behaviours

üü Shows the sample space for the first two dice rolled (-1 for each error)

ü Determines the probability

(HINT: Showing the sample space is helpful) (3 marks)

- b) Let the random variable, X , be the number of numbers between the first two dice. Complete the probability distribution table below. (2 marks)

$X = x$	0	1	2	3	4
$P(X = x)$	$\frac{16}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

Specific Behaviours

üü Correct probabilities (-1 for each error)

- c) Determine the probability that a player wins the game. (3 marks)

Solution

$$P(\text{Win})=0+\left(1 \times \frac{8}{36} \times \frac{1}{6}\right)+\left(2 \times \frac{6}{36} \times \frac{1}{6}\right)+\left(3 \times \frac{4}{36} \times \frac{1}{6}\right)+\left(4 \times \frac{2}{36} \times \frac{1}{6}\right)$$

$$P(\text{Win})=\frac{40}{216}=\frac{5}{27}$$

Specific Behaviours

ü Multiplies each score by its probability

ü Multiplies by 1/6

ü Determines the probability

- The probability of a light failing is constant for successive trials of the same experiment.

Specific Behaviours

ü States a binomial distribution

ü States the parameters of the distribution

ü Gives one appropriate reason for the use of the distribution

ü Gives two appropriate reasons for the use of the distribution

- b) If the variance of X is 0.3384,

- i. Determine the number of lights that have been used. (2 marks)

Solution

$$\text{Var}(X)=np(1-p)$$

$$0.3384=0.06n(1-0.06)$$

$$\therefore n=6$$

Specific Behaviours

ü Substitutes values into the formula for variance

ü Determines the correct value of n

- ii. What is the probability that less than half the lights fail given more than 1 light failed? (2 marks)

marks)

Solution

$$P(X < 3 \vee X > 1) = \frac{P(X=2)}{P(X>1)} = \frac{0.04216}{0.04592} = 0.91803$$

Specific Behaviours

ü Shows the correct numerator

ü Calculates the probability

- iii. What is the probability that the system fails? (2 marks)

Solution

The system will fail if 4 or more lights fail (as the system continues to work if at least three lights are working). Thus, the probability that the system fails is:

$$P(X \geq 4) = 0.000176$$

Specific Behaviours

ü Shows the correct numerator

ü Calculates the probability

- c) One night, Dezz removes two of the lights so that they can be repaired. The lights are not replaced for the next night. What is the probability that the system works for that night? (2 marks)

marks)

Solution

For the system to continue working, at most two lights can fail.

Let $Y \sim \text{Bin}(4, 0.06)$

$$P(Y \geq 2) = 0.01991088 \text{ (this is the probability that the system fails)}$$

$$1 - 0.01991088 = 0.9801$$

Thus the probability that the system works for that night is 0.9801.

• The probability of any light failing is independent.

We can assume a binomial distribution if:

$$X \sim Bi(n, 0.06)$$

Solution

use this distribution.

- a) State the distribution of X , including its parameters, and state two assumptions that were required to use this distribution. (4 marks)

Dezz installs a outdoor security lights that are connected to a system which has been configured so that all lights will turn on if their sensors detect motion. The system will continue to work if at least three of the lights are working. There is a 6% chance that any light fails. If the random variable X is the number of lights that fail, then the equation to predict its population F in t years, time (from the beginning of 2022) is:

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Question 14

(12 marks)

- i) What is the current population of Scampton (at the beginning of 2022)? (1 mark)

$$F(t) = 35000 - 2500e^{-0.015t}$$

- c) The nearby town of Scampton has also been growing, but its population growth has been such that the equation to predict its population F in t years, time (from the beginning of 2022) is:

Dezz installs a outdoor security lights that are connected to a system which has been configured so that all lights will turn on if their sensors detect motion. The system will continue to work if at least three of the lights are working. There is a 6% chance that any light fails. If the random variable X is the number of lights that fail, then the equation to predict its population F in t years, time (from the beginning of 2022) is:

Note: Accept 13063 people and 13058 people (followed through $k = 0.0148$)

$$\begin{aligned} & \text{i) Solves for the number of people at the beginning of 2030} \\ & P = 13062.44 \approx 13062 \text{ people} \\ & P = 11600 e^{0.014842(8)} \end{aligned}$$

Solution

- b) Hence predict the population of Clipalmeron at the beginning of 2030. (1 mark)

Note: Accept $k = 0.0148$

$$\begin{aligned} & \text{i) Finds the population for 2022} \\ & \text{ii) Solves for the value of } k \\ & \text{iii) Shows the correct model for } P \text{ from the year 2022} \\ & \text{iv) Solves for the value of } k \end{aligned}$$

Note: Accept $k = 0.0148$

$$\begin{aligned} & \text{The model for } P \text{ is } P = 11600 e^{0.014842t} \\ & 10000 = 11600 e^{-10k} \\ & P = P_0 e^{kt} \\ & \therefore k = 0.014842 \end{aligned}$$

Solution

- writing an equation to predict the population of Clipalmeron t years from the beginning of 2022. (3 marks)

- a) Assuming the growth rate of the population P remains the same in the future, use this information to

- The population of the town 10 years ago was 10 000, and there are now (at the beginning of 2022) an extra 1600 people living in the town.

- The mining town of Clipalmeron has been experiencing population exponential growth over the last decade.

- where P is the population after t years)

- (i.e. $\frac{dp}{dt} = kp$)

- Question 10 (8 marks)

• Sets up an appropriate integral to calculate the area

• Determines the correct area (Must be exact)

$$? 4.6 \text{ units}^2$$

$$\frac{3}{2(4)^{1/2}} - \frac{2(1)^{1/2}}{3}$$

$$\text{Area} = \int_{-1}^1 x dx = \left[\frac{3}{2}x^{1/2} \right]_{-1}^1$$

$$? 4.6 \text{ units}^2$$

- Use integration to evaluate the exact area estimated in (i). (2 marks)

• Determines the average of the two estimations

• Determines the correct overestimate

• Determines the correct underestimate

- ii) During which years will the population of Scomotown be greater than the population of Clipalmerton? (3 marks)

Solution

$$35000 - 25000 e^{-0.015t} > 11600 e^{0.014842t}$$

CAS Solve : $t = 9.74, t = 42.71$

The function $f(x) = \sqrt{x}$ is graphed below.

b)

$\therefore 9.74 < t < 42.71$ is where the population of Scomotown will be greater than the population of Clipalmerton. Thus this occurs between the years **2032 to 2065**

Specific Behaviours

- ü Solves for 1 value of t
- ü Solves for both t values
- ü States between which years Scomotown will have a greater population than Clipalmerton

Note: Accept beginning in 2031 and/or ending in 2064

Solution

$$\int_a^b f(x) dx = \int_0^b f(x) dx - \int_0^a f(x) dx$$

$$\int_a^b f(x) dx = 3.9 - 5.4 = -1.5$$

Specific Behaviours

- ✓ Uses linearity to evaluate the integral
- ✓ Correctly evaluates the integral

- ii) Determine the area bounded by the function, the x -axis and y -axis on the interval $[0, b]$. (2 marks)

Solution

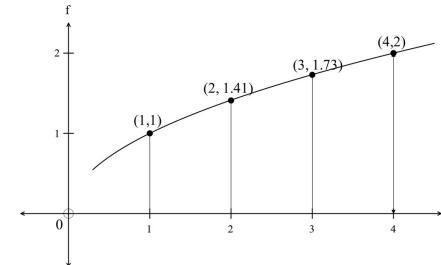
$$\text{Area} = \int_0^a f(x) dx + \left| \int_a^b f(x) dx \right| = 5.4 + 1.5 = 6.9 \text{ units}^2$$

Specific Behaviours

- ✓ Recognises that the region from a to b needs to be positive
- ✓ Determines the correct area

The function $f(x) = \sqrt{x}$ is graphed below.

b)



- (i) Using rectangles from above and from below, find an estimate for the area between the function $y = \sqrt{x}$, $x = 1, x = 4$ and the x axis. Use $x = 1$ as the width of the interval. (5 marks)

Solution

From below:

$$1 \times 1 + 1 \times 1.41 + 1 \times 1.73 = 4.14$$

From above:

$$1 \times 1.41 + 1 \times 1.73 + 1 \times 2 = 5.14$$

The average of the two estimates is 4.64 units^2

Specific Behaviours

- ✓ Shows the calculation for the underestimation

(20 Trials)

(a) Given the variance of a Bernoulli distribution is 0.276, determine the mean.
(2 marks)

The probability of success in a Bernoulli trial is p . Given that it is repeated n times, the expected value and variance of the resulting distribution of the number of successes are 7.52 and 3.856 .

$p(1-p) = 0.2176p = 0.32, 0.68$ Solution	<u>Specific Behaviours</u> p mean is either 0.32 or 0.68	<u>Sets up the equation for variance</u> $\sum \text{States both values of } p \text{ are possible means}$
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$$\begin{aligned}n &= 16 \\d &= 0.47 \\1-d &= 0.53 \\np(1-p) &= 3.9856 \\np &= 7.52 \\X &\sim \text{Bin}(n, p)\end{aligned}$$

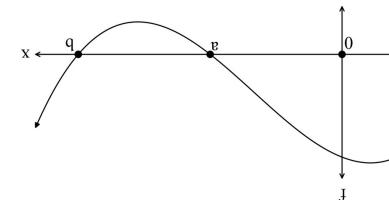
Specific Behaviours	
u Identifies distribution of successes as binomial	
u States equation for mean	
u States equation for variance	
u Solves equations for n and p	

<p>The probability of Jeremy being late to his Maths class is 0.3, and the probability that he is late to his Maths class on any day is independent of whether he was late on the previous day.</p>	<p>Over five consecutive weekdays, what is the probability that Jeremy is only late to his Maths class on Tuesday?</p>	<p>(i) Is only late to his Maths class on Tuesday? (1 mark)</p>
		<p>Is only late to his Maths class on Tuesday? (1 mark)</p>
	<p>Solution</p>	<p>Correct probability ✓</p>
	<p>0.3 × 0.7 = 0.2103 Specific behaviours</p>	

(2 marks)

Evaluate $\int x dx$. (i)

a) Given $\int_0^a f(x)dx = 5.4$ and $\int_b^0 f(x)dx = 3.9$



Consider the graph below

(11 marks)

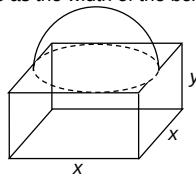
- ✓ Correctly determines $A'(x)$
- ✓ Equates $A'(x)$ to zero and solves for x
- ✓ Uses the sign test or second derivative test to justify x is a minimum
- ✓ Clearly states the value of x and the minimum surface area

Specific behaviours
✓ identifies binomial situation for other two days
✓ evaluates probability of being late on two other days
✓ determines probability
Note: Max 1 mark f.t. for binomial $Cdf(2, 5, 5, 0.3)=0.16308$

Question 12

(10 marks)

A square based prism as shown in the diagram has a hemisphere added to its top in such a way that the diameter of the hemisphere is the same as the width of the box. The volume (V) of the object is 800 cm^3 .



- (a) Determine V in terms of x and y .
mark)

(1)

Solution

$$V = x^2 y + \frac{2\pi \left(\frac{x}{2}\right)^3}{3}$$

$$V = x^2 y + \frac{\pi x^3}{12}$$

Specific Behaviours

- ✓ Correct answer

- (b) Show that the value of y is given by $y = \frac{800}{x^2} - \frac{\pi x}{12}$ (2 marks)

Solution

$$800 = x^2 y + \frac{\pi x^3}{12}$$

$$800 = x^2 \left(y + \frac{\pi x}{12}\right)$$

$$y = \frac{800}{x^2} - \frac{\pi x}{12}$$

Specific Behaviours

- ✓ Substitutes the volume as 800
✓ Shows rearranging and simplifies to get y

- (c) Show that the surface area of the shape, $A(x)$, is given by $A(x) = 2x^2 - \frac{\pi x^2}{12} + \frac{3200}{x}$ (3 marks)

Solution

$$A(x) = 2x^2 + 4xy + 2\pi \left(\frac{x}{2}\right)^2 - \pi \left(\frac{x}{2}\right)^2 = 2x^2 + 4xy + \frac{\pi x^2}{4}$$

$$A(x) = 2x^2 + 4x \left(\frac{800}{x^2} - \frac{\pi x}{12}\right) + \frac{\pi x^2}{4}$$

$$A(x) = 2x^2 + \left(\frac{\pi}{4} - \frac{\pi}{3}\right)x^2 + \frac{3200}{x}$$

$$A(x) = 2x^2 - \frac{\pi x^2}{12} + \frac{3200}{x}$$

Specific Behaviours

- ✓ Correct formula for surface area in terms of x and y
✓ Substitutes y from part b)
✓ Simplifies to obtain the correct equation for $A(x)$

- (d) Using calculus to justify your answer, determine the minimum possible surface area of the prism and the value of x for which it occurs. (4 marks)

Solution

$$A'(x) = 4x - \frac{\pi x}{6} - \frac{3200}{x^2}$$

$$4x - \frac{\pi x}{6} - \frac{3200}{x^2} = 0$$

Using CAS Solve:
 $x=9.7276 \text{ cm}$

$$A''(x) = 4 - \frac{\pi}{6} + \frac{6400}{x^3}$$

$$A''(9.7276) = 10.429 \text{ (+ve, so we have a minimum)}$$

$$A(9.7276) = 493.44 \text{ cm}^2$$

Thus the minimum surface area is 493.44 cm^2 at $x=9.7276 \text{ cm}$

Other Solutions