PERTH MODERN SCHOOL
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Mathematics Specialist
Unit 3

TEST 1 2016

Student name: ______________________ **Teacher name:** ______________________

Class: ________

Time allowed for this task:	50 minutes, in class, under test conditions	
	Section One – calculator-free section – 30 minutes	(26 marks)
	Section Two – calculator-assumed section – 20 minutes	(20 marks)
Materials required:	Calculator with CAS capability (to be provided by the student)	
Standard items:	Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlights	
Special items:	Drawing instruments, templates, notes on one unfolded sheets of A4 paper, and up to three calculators approved for use in WACE examinations	
Marks available:	46 marks	
Task weighting:	**7%**	

Section One – calculator-free section **(26 marks)**

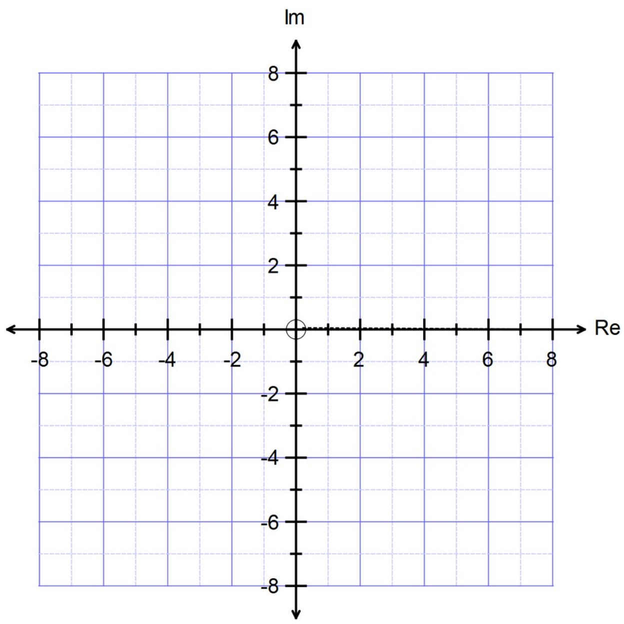
Question 1 **(4 marks)**

(a) Given $z = \sqrt{3} + i$ evaluate z^6 giving the answer in Cartesian form. (2 marks)

(b) Solve $x^2 - 6x + 13 = 0$ for $x \in \mathrm{Im}$ in exact form. (2 marks)

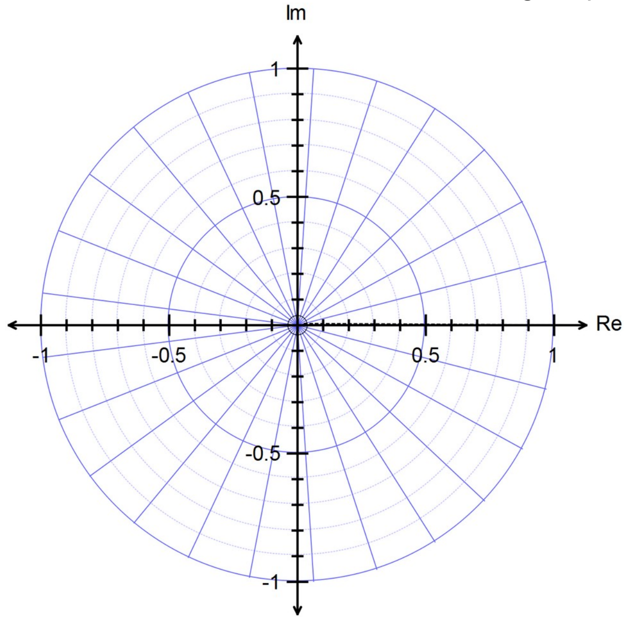
Question 2 **(6 marks)**

(a) Sketch the set of points defined by $\left|z - (2 + 3i)\right| = \sqrt{13}$.



Question 3 **(8 marks)**

Determine and locate all solutions in the Argand plane to the equation $z^5 = 1$.



Question 4 **(8 marks)**

Given $H(z) = z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8$

(a) Evaluate $H(i)$, $H(-i)$ and $H(2)$. (3 marks)

(b) Hence, find all roots of the equation $z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8 = 0$. (5 marks)

Section Two – calculator-assumed section **(20 marks)**

Question 5 **(10 marks)**

(a) Expand and simplify the expression $F(\theta) = (\cos\theta + i\sin\theta)^5$. (2 marks)

(b) Hence, express the $\text{Re}(F)$ in terms of $\cos\theta$. (3 marks)

(c) Use $\text{Re}(F)$ to solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$ and express the solutions in trigonometric form. (5 marks)

Question 6 **(10 marks)**

Given $z = \cos\theta + i\sin\theta$:

(a) Express $\dfrac{\left(z - \dfrac{1}{z}\right)}{i\left(z + \dfrac{1}{z}\right)}$ in trigonometric form. (4 marks)

(b) Show $z^2 + \dfrac{1}{z^2} = 2\cos 2\theta$ and hence prove $\cos 2\theta = 2\cos^2\theta - 1$. (6 marks)

Solutions and marking key for Test 1 for concurrent Unit 3 and Unit 4 program

Section One – calculator-free section (30 marks)

Question 1 (3.1.6, 3.1.15) (8 marks)

(a) Given $z=\sqrt{3}+i$ evaluate z^6 giving the answer in Cartesian form. (2 marks)

Given $z=\sqrt{3}+i$ evaluate z^6

$$z^6=\left(\sqrt{3}+i\right)^6=\sqrt{3}^6+6.\left(\sqrt{3}\right)^5.(i)^1+15.\left(\sqrt{3}\right)^4.(i)^2+20.\left(\sqrt{3}\right)^3.(i)^3+15.\left(\sqrt{3}\right)^2.(i)^4+6.\left(\sqrt{3}\right)^1.(i)^5+(i$$

$$=27-135+45-1+\left(54\sqrt{3}-60\sqrt{3}+6\sqrt{3}\right)i$$

$$=-64$$

OR $\quad z=2cis\left(\frac{\pi}{6}\right)\Rightarrow z^6=64cis(\pi)=-64$

Specific behaviours	Mark	Item
Expresses the Cartesian form of z^6 Simplifies correctly	1 1	simple simple
Or Expresses z^6 in polar form Expresses the answer in Cartesian form	 1 1	 simple simple

(b) (4 marks)

Given $Z_1=cis\left(\frac{\pi}{3}\right)$ and $Z_2=cis\left(\frac{\pi}{4}\right)$ evaluate the following in exact Cartesian form:

(i) $\overline{Z_1}$ $\quad$ (ii) iZ_2 $\quad$ (iii) $cis\left(\frac{\pi}{12}\right)$

Given $Z_1=cis\left(\frac{\pi}{3}\right)$ and $Z_2=cis\left(\frac{\pi}{4}\right)$ evaluate the following in exact Cartesian form:

(i) $\overline{Z_1}=\dfrac{1-\sqrt{3}i}{2}$ $\qquad$ (ii) $iZ_2=\dfrac{-\sqrt{2}+\sqrt{2}i}{2}$

(iii) $$cis\left(\frac{\pi}{12}\right)=\frac{Z_1}{Z_2}=\frac{1+\sqrt{3}i}{2}\times\frac{2}{\sqrt{2}+\sqrt{2}i}=\left(\frac{\left(\sqrt{2}+\sqrt{6}\right)+\left(\sqrt{6}-\sqrt{2}\right)i}{4}\right)$$

Specific behaviours	Mark	Item
Writes the Cartesian form of $\overline{Z_1}$ correctly Writes the Cartesian form of iZ_2 correctly Expresses polar term for $\overline{Z_3}$ in Cartesian form Simplifies the Cartesian form correctly	1 1 1 1	simple simple complex complex

(c) Solve $x^2 - 6x + 13 = 0$ for $x \in \mathrm{Im}$ in exact form. (2 marks)

Solve

$$x^2 - 6x + 12 = 0$$

$$\Leftrightarrow \quad x^2 - 6x + 9 = -3$$

$$\Leftrightarrow \quad (x - 3)^2 = -3$$

$$\Leftrightarrow \quad (x - 3)^2 = 3i^2$$

$$\Leftrightarrow \quad x = 3 \pm \sqrt{3}\, i$$

Or

Solve

$$x^2 - 6x + 12 = 0$$

$$\Leftrightarrow \quad a = 1, b = -6 \text{ and } c = 12$$

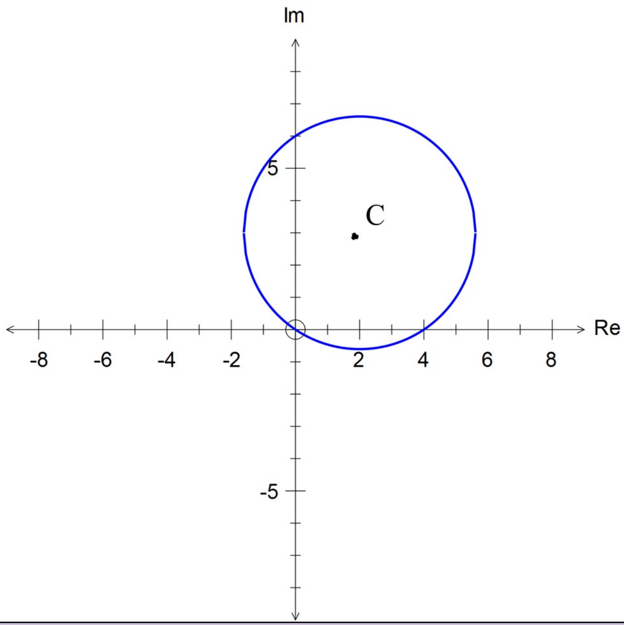
$$\Leftrightarrow \quad x = \frac{6 \pm \sqrt{36 - 48}}{2}$$

$$\Leftrightarrow \quad x = \frac{6 \pm \sqrt{-12}}{2} = 3 \pm \sqrt{3}i$$

Specific behaviours	Mark	Item
Completes the square correctly	1	simple
Solves the equation using the exact form	1	simple
Or		
Uses the quadratic formula	1	simple
Simplifies the expressions to the correct exact form	1	simple

Question 2 (1.1.7) **(6 marks)**

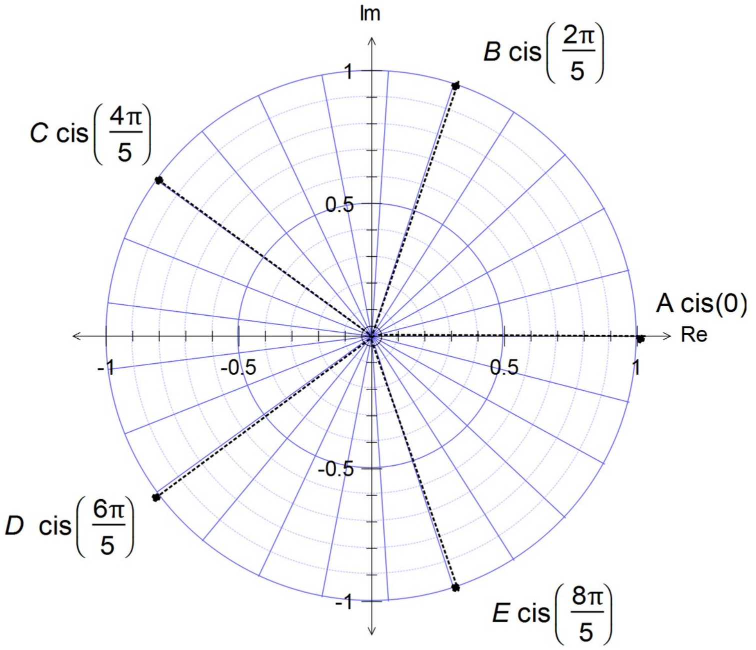
(a) Sketch the set of points defined by $|z - (2 + 3i)| = \sqrt{13}$.



Specific behaviours	Mark	Item
Draws a circle	1	simple
Has the correct centre *(2 + 3i)*	1	simple
Has the correct radius	1	simple
Circumference passes through *(0, 0) (4 + 0i)* and *(0 + 6i)*	3	simple

Question 3 (3.1.11, 3.1.12) (8 marks)

Determine and locate all solutions in the Argand plane to the equation $z^5 = 1$.



$$= 1 \times cis\left(0 + \frac{2\pi \times 0}{5}\right) = 1 + 0i$$

$$= 1 \times cis\left(0 + \frac{2\pi \times 1}{5}\right) = cis\left(\frac{2\pi}{5}\right)$$

$$= 1 \times cis\left(0 + \frac{2\pi \times 2}{5}\right) = cis\left(\frac{4\pi}{5}\right)$$

$$= 1 \times cis\left(0 + \frac{2\pi \times 3}{5}\right) = cis\left(\frac{6\pi}{5}\right)$$

$$= 1 \times cis\left(0 + \frac{2\pi \times 4}{5}\right) = cis\left(\frac{8\pi}{5}\right)$$

Specific behaviours	Mark	Item
Expresses the five solutions correctly	5	simple
Locates the solutions accurately on a polar graph	3	simple

Question 4 (3.1.13, 3.1.15) (8 marks)

Given $H(z) = z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8$:

(a) Evaluate $H(i)$, $H(-i)$ and $H(2)$

(3 marks)

Given $H(z) = z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8$

(a) Evaluate $H(i)$, $H(-i)$ and $H(2)$

$H(i) = i - 2 - 5i + 10 + 4i - 8 = 0$

$H(-i) = -i - 2 + 5i + 10 - 4i - 8 = 0$

$H(2) = 32 - 32 + 40 - 40 + 8 - 8 = 0$

Specific behaviours	Mark	Item
Evaluates each of the three terms $H(i)$, $H(-i)$ and $H(2)$	3	simple

(b) Hence, find all roots of the equation $z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8 = 0$. (5 marks)

Given $H(z) = z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8$ from part (a)

$H(i) = 0 \quad \Leftrightarrow (z - i)$ is a factor of $H(z)$

$H(-i) = 0 \quad \Leftrightarrow (z + i)$ is a factor of $H(z)$

And $\quad \Leftrightarrow (z^2 + 1)$ is a factor of $H(z)$

$H(2) = 0 \quad \Leftrightarrow (z - 2)$ is a factor of $H(z)$

Since $\quad z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8 \div (z^2 + 1) \quad = (z^3 - 2z^2 + 4z - 8)$

and $\quad (z^3 - 2z^2 + 4z - 8) \div (z - 2) \quad = (z^2 + 4)$

then $\quad z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8 \quad = (z + i)(z - i)(z - 2)(z + 2i)(z - 2i)$

Hence the roots to $z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8 = 0$ are $z = \{\pm i, \pm 2i, 2\}$

Specific behaviours	Mark	Item
Uses the factor theorem to give factors $(z+i)(z-i)(z-2)$	1	simple
Determines the remaining factors $(z+2i)(z-2i)$	2	complex
Correctly writes all the roots	2	complex

Section Two – calculator-assumed section (20 marks)

Question 5 (3.1.7) (10 marks)

(a) Expand and simplify the expression $F(\theta) = (\cos\theta + i\sin\theta)^5$. (2 marks)

$F(\theta) = (\cos\theta + i\sin\theta)^5$

$= (\cos^5\theta - 10\cos^3\theta.\sin^2\theta + 5\cos\theta.\sin\theta^4) + (\sin^5\theta + 5\cos^4\theta.\sin\theta - 10\cos^2\theta.\sin^3\theta)i$

Specific behaviours	Mark	Item
Shows the real and imaginary terms correctly	2	simple

(b) Hence, express the $\mathrm{Re}(F)$ in terms of $\cos\theta$. (3 marks)

$\mathrm{Re}(F) = \cos^5\theta - 10\cos^3\theta.\sin^2\theta + 5\cos\theta.\sin\theta^4$

$= \cos^5\theta - 10\cos^3\theta(1 - \cos^2\theta) + 5\cos\theta(1 - \cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Specific behaviours	Mark	Item
Writes the real part of $F(\theta)$	1	simple
Substitutes for $\sin^2\theta = 1 - \cos^2\theta$	1	simple
Gives the correct expression for $\mathrm{Re}(F)$	1	simple

(c) Use $\text{Re}(F)$ to solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$ and express the solutions in trigonometric form. (5 marks)

$F(\theta) = (\cos\theta + i\sin\theta)^5$

$= \cos 5\theta + i\sin 5\theta$ {DeMoivre}

$\Leftrightarrow \text{Re}(F) = \cos 5\theta$

$\text{Re}(F) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ from part (b)

Hence $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$..........(1)

To solve $16x^5 - 20x^3 + 5x - 1 = 0$

$\Leftrightarrow$ $16x^5 - 20x^3 + 5x = 1$ {Let $x = \cos\theta$}

$\Leftrightarrow$ $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 1$

$\Leftrightarrow$ $\cos 5\theta = 1$

$\Leftrightarrow$ $5\theta = 0, 2\pi, \ldots$

$\Leftrightarrow$ $\theta = \frac{2\pi n}{5}, n = 0,1,2,3,4$

$\Leftrightarrow$ $x = \cos\left(\frac{2\pi n}{5}\right), n = 0,1,2,3,4$

Specific behaviours	Mark	Item
Uses De Moivre to state $\text{Re}(F) = \cos 5\theta$	1	complex
Makes the substitution $x = \cos\theta$ in polynomial	1	complex
Replaces the polynomial in $\cos\theta$ with $\cos 5\theta$	1	complex
Solves $\cos 5\theta = 1$ in terms of θ	1	complex
Gives all five solutions in terms of *x*	1	complex

Question 6 (3.1.7) (10 marks)

Given $z = \cos\theta + i\sin\theta$:

(a) Express $\dfrac{\left(z - \frac{1}{z}\right)}{i\left(z + \frac{1}{z}\right)}$ in trigonometric form. (4 marks)

$$\frac{\left(z - \frac{1}{z}\right)}{i\left(z + \frac{1}{z}\right)} = \frac{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{i((\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta))}$$

$$= \frac{2i\sin\theta}{2i\cos\theta}$$

$$= \tan\theta$$

Specific behaviours	Mark	Item
Rewrites the complex numbers z and $\frac{1}{z}$ in trig form	2	simple
Simplifies both numerator and denominator Writes the correct final term	2	simple

(b) Show $z^2 + \frac{1}{z^2} = 2\cos 2\theta$ and hence prove $\cos 2\theta = 2\cos^2\theta - 1$ (6 marks)

$$z^2 + \frac{1}{z^2} = (\cos 2\theta + i\sin 2\theta) + (\cos 2\theta - i\sin 2\theta)$$
$$= 2\cos 2\theta \ldots\ldots\ldots\ldots(1)$$
$$z^2 + \frac{1}{z^2} = (\cos\theta + i\sin\theta)^2 + (\cos\theta - i\sin\theta)^2$$
$$= (\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta) + (\cos^2\theta - 2i\sin\theta\cos\theta - \sin^2\theta)$$
$$= 2(\cos^2\theta - \sin^2\theta) \ldots\ldots\ldots(2)$$
$$(1) = (2) \Leftrightarrow 2\cos 2\theta = 2(\cos^2\theta - \sin^2\theta)$$
$$\Leftrightarrow \cos 2\theta = 2\cos^2\theta - 1$$

Specific behaviours	Mark	Item
Rewrites z^2 and $\frac{1}{z^2}$ using double angle form 2θ	1	complex
Gathers terms and simplifies	1	complex
Rewrites z^2 and $\frac{1}{z^2}$ using single angle form θ	1	complex
Gathers terms and simplifies	1	complex
Equates both equations	1	complex
Writes correct final expression	1	complex

Question	1	2	3	4	5	6	Total
Simple	8	6	8	4	5	4	35
Complex	0	0	0	4	5	6	15
	8	6	8	8	10	10	50