PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Test 1 Year 12

Student name: _______________ Teacher name: _______________

Task type: **Response/Investigation**

Reading time for this test : 5 mins

Working time allowed for this task: 40 mins

Number of questions: **7**

Materials required: No cals allowed!!

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, NO notes allowed!

Marks available: **41 marks**

Task weighting: **13%**

Formula sheet provided: **no, but formulae stated on page 2**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Useful formulae

Complex numbers

Cartesian form	
$z = a + bi$	$\bar{z} = a - bi$
$\text{Mod}(z) = \lvert z\rvert = \sqrt{a^2 + b^2} = r$	$\text{Arg}(z) = \theta, \quad \tan\theta = \frac{b}{a}, \quad -\pi < \theta \le \pi$
$\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$	$\left\lvert\frac{z_1}{z_2}\right\rvert = \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$
$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$	$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
$z\bar{z} = \lvert z\rvert^2$	$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{\lvert z\rvert^2}$
$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$	$\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$
Polar form	
$z = a + bi = r(\cos\theta + i\sin\theta) = r\,cis\,\theta$	$\bar{z} = r\,cis(-\theta)$
$z_1 z_2 = r_1 r_2\,cis(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,cis(\theta_1 - \theta_2)$
$cis(\theta_1 + \theta_2) = cis\,\theta_1\,cis\,\theta_2$	$cis(-\theta) = \frac{1}{cis\,\theta}$
De Moivre's theorem	
$z^n = \lvert z\rvert^n cis(n\theta)$	$(cis\,\theta)^n = \cos n\theta + i\sin n\theta$
$z^{\frac{1}{q}} = r^{\frac{1}{q}}\left(\cos\frac{\theta + 2\pi k}{q} + i\sin\frac{\theta + 2\pi k}{q}\right)$, for k an integer	

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

No cals allowed!!

Q1 (2, 2, 2 & 2 = 8 marks)

If $z = 5 - 4i$ and $w = 2 + 3i$ determine the following:

a) zw

Solution
$(5-4i)(2+3i) = 10 + 12 - 8i + 15i$ $= 22 + 7i$
Specific behaviours
✓ real part ✓ Imaginary part

b) $\dfrac{1}{w}$

Solution
$\dfrac{1}{2+3i}\dfrac{2-3i}{2-3i} = \dfrac{2-3i}{13}$
Specific behaviours
✓ uses conjugate ✓ express answer

c) $\dfrac{\overline{z}}{w}$

Solution
$\dfrac{5+4i}{2+3i}\dfrac{2-3i}{2-3i} = \dfrac{22-7i}{13}$
Specific behaviours
✓ numerator ✓ denominator

d) $z^2\overline{w}$

Solution

$(5-4i)^2(2-3i)=(25-16-40i)(2-3i)$ $(9-40i)(2-3i)$ $=18-120-80i-27i$ $=-102-107i$
Specific behaviours
✓ evaluates square term ✓ determines answer

Q2 (2 & 3 = 5 marks)

a) Determine the complex roots of $3z^2+z+2=0$.

Solution
$3z^2+z+2=0$ $z=\frac{-1\pm\sqrt{1-24}}{6}$ $z=\frac{-1\pm\sqrt{23}i}{6}$
Specific behaviours
✓ uses quadratic formula ✓ has two complex roots

b) Use the quadratic equation to prove that if a quadratic equation with real coefficients has any non-real roots then it must have two complex roots and they must be conjugates of each other.

Solution
$ax^2+bx+c=0$ $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $b^2-4ac=-n^2=i^2n^2$ $x=\frac{-b\pm\sqrt{i^2n^2}}{2a}=\frac{-b\pm in}{2a}$

Specific behaviours
✓ sets up equation with a negative discriminant ✓ uses $i^2 = -1$ with discriminant ✓ derives two complex roots which are conjugates of each other

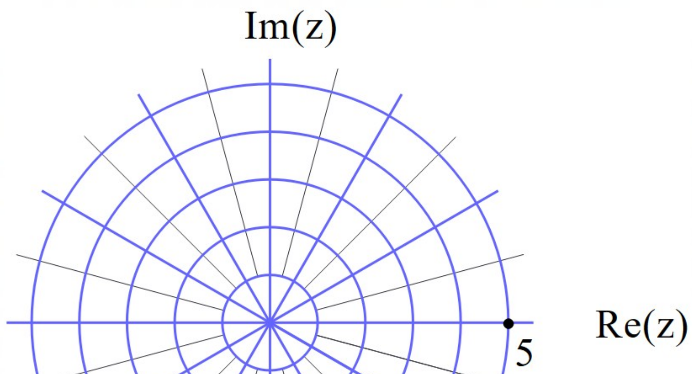
Q3 (4 marks)

Determine all possible real number pairs $a \& b$ such that $\dfrac{27-3i}{a-5i} = 3+bi$.

Solution
$\dfrac{27-3i}{a-5i} = 3+bi$ $27-3i = (a-5i)(3+bi) = 3a+5b+i(ab-15)$ $27 = 3a+5b$ $-3 = ab-15,\ ab = 12,\ a = \dfrac{12}{b}$ $27 = 3\dfrac{12}{b}+5b$ $27b = 36+5b^2$ $5b^2-27b+36 = 0$ $(5b-12)(b-3) = 0$ $b = 3, a = 4$ $b = \dfrac{12}{5}, a = 5$
Specific behaviours
✓ sets up equation and equates real and imaginary ✓ obtains two simultaneous equations ✓ solves for one pair of values ✓ solves for two pairs of values

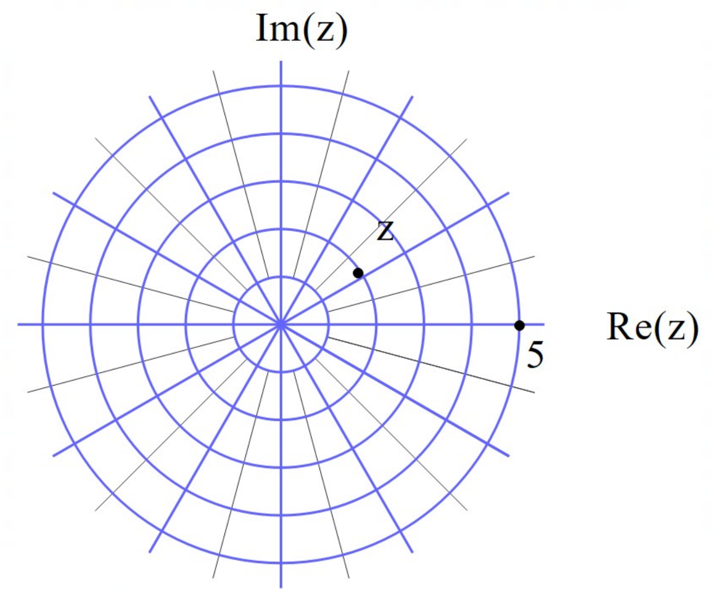
Q4 (2, 2, 2 & 2 = 8 marks)

Consider the complex number $z = \sqrt{3}+i$.



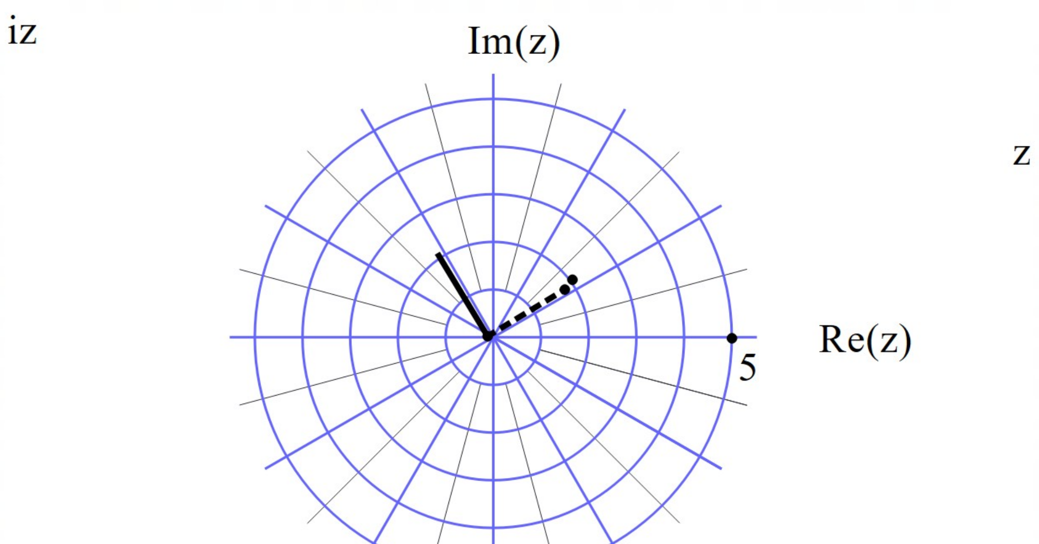
Plot the following on the axes above.

a) z

Solution

Specific behaviours
✓ argument ✓ length of 2 units

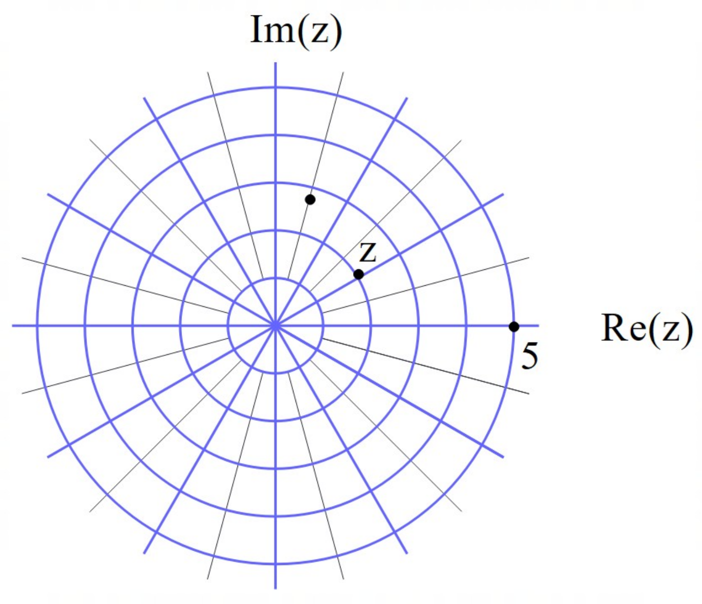
b) iz

Solution

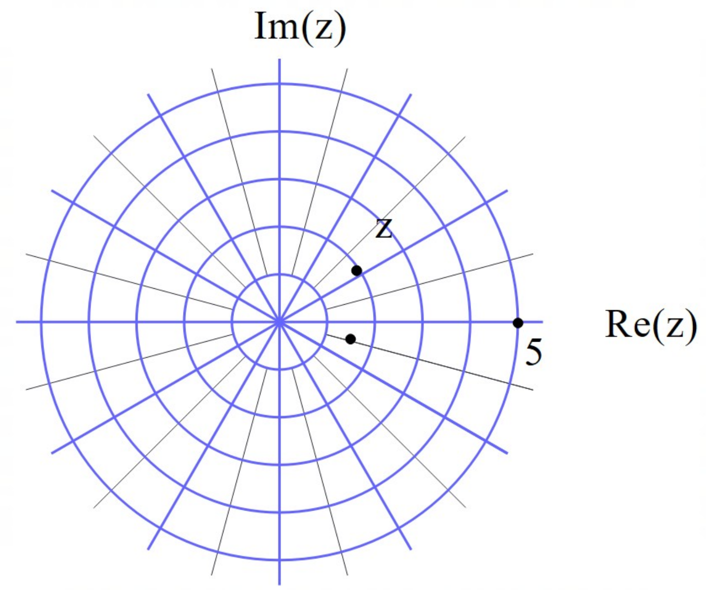


Specific behaviours
✓ uses right angle ✓ rotates anticlockwise with unchanged length

c) $(1+i)z$

Solution

Specific behaviours
✓ argument ✓ modulus

d) $\dfrac{z}{(1+i)}$

Solution

Specific behaviours
✓ argument ✓ modulus

Q5 (5 marks)

Consider the polynomial $f(z) = az^4 + bz^3 + cz^2 + dz + e$ where $a, b, c, d \,\&\, e$ are real numbers.

Given that $(7+i) = 0 = f(2-2i)$

and $f(0) = 40$

Determine the values of $a, b, c, d \,\&\, e$.

(Note: answers without working will receive zero marks)

Solution

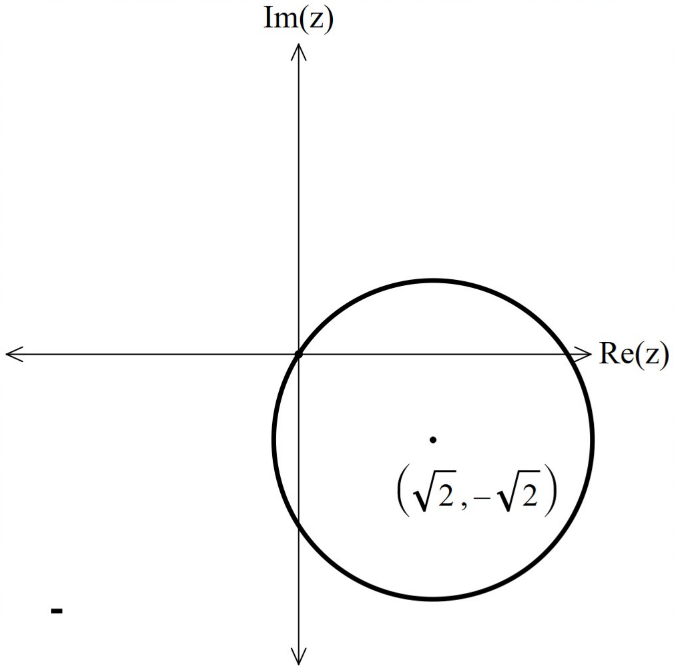
$(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta$ $-(\alpha+\beta)=-2\operatorname{Re} al, \alpha\beta=\lvert z\rvert^2$ $f(z)=a(z^2-14z+50)(z^2-4z+8)$ $z=0, f(z)=40 \therefore a=\frac{1}{10}$ $f(z)=\frac{1}{10}(z^4-18z^3+114z^2-312z+400)$ $a=\frac{1}{10}$ $b=-\frac{18}{10}$ $c=11.4$ $d=-31.2$ $e=40$
Specific behaviours
✓ shows reasoning for determining value of a ✓ uses one quadratic factor ✓ uses two quadratic factors ✓ shows reasoning in determining quadratic factors (i.e roots in brackets) ✓ shows reasoning on how to determine quartic polynomial. Note: Any statement of values without reasoning will NOT receive any marks!

Q6 (2, 1, 2 & 2 = 7 marks)

Consider the locus of complex numbers z that satisfy $\left|z-\sqrt{2}+\sqrt{2}i\right|=2$.

a) Sketch the locus on the axes below.

Solution

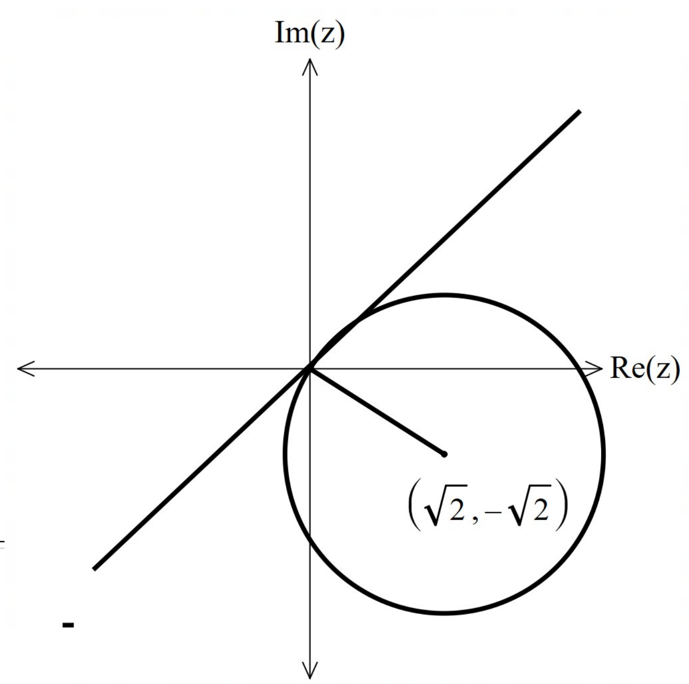


Specific behaviours
✓ circle with centre coordinates stated ✓ goes through origin

b) State the maximum value of $|z|$

Solution
$\|z\| = 4$
Specific behaviours
✓ states maximum

c) State the minimum value of $Arg(z)$

Solution


$m\dfrac{-\sqrt{2}}{\sqrt{2}}=-1$ $m=1=\tan\theta$ $\theta=\dfrac{\pi}{4},\dfrac{-3\pi}{4}$
Specific behaviours
✓ determines gradient of tangent ✓ determines min argument

d) State the maximum value of $Arg(z)$

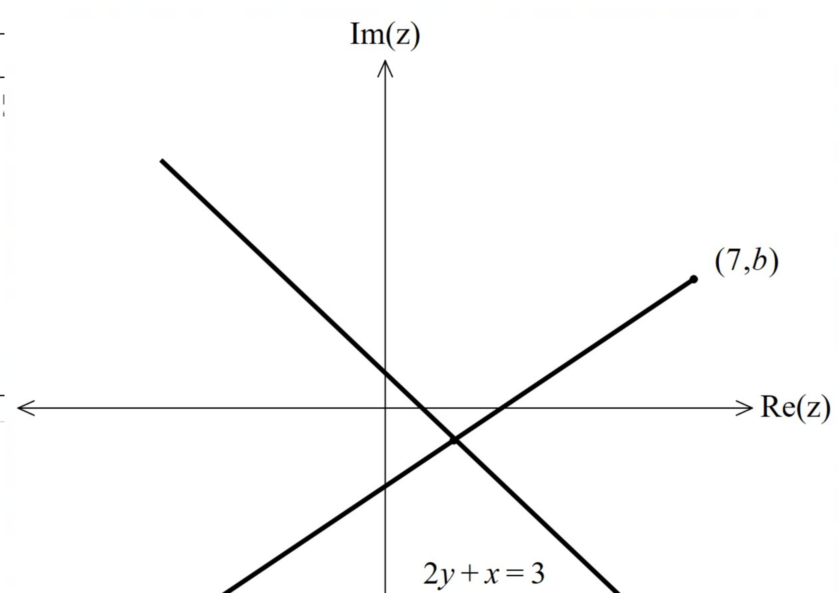
Solution
Max = $\dfrac{\pi}{4}$ See above
Specific behaviours
✓ determines gradient of tangent ✓ determines max argument

Q7 (4 marks)

Consider the locus defined by $\left|z-a+5i\right|=\left|z-7-bi\right|$ where $a\,\&\,b$ are constants. This locus can also be defined by $2\,\mathrm{Im}(z)+\mathrm{Re}(z)=3$.

Determine the values of $a\,\&\,b$

$|z-a+$



$m = \frac{b - -5}{7 - a} = \frac{b+5}{7 - a}$ $m_{\perp} = \frac{-1}{2}$ $\frac{b+5}{7-a} = 2$ $midpo\text{int}(\frac{7+a}{2}, \frac{b-5}{2})$ $y = \frac{-x}{2} + \frac{3}{2}$ $\frac{b-5}{2} = -\frac{7+a}{4} + \frac{3}{2}$ $2b - 10 = -7 - a + 6$ $a + 2b = 9$ $b + 5 = 14 - 2a$ $b + 5 = 14 - 2(9 - 2b)$ $b + 5 = 14 - 18 + 4b$ $3b = 9$ $b = 3$ $a = 3$
Specific behaviours
✓ uses perpendicular bisector ✓ sets up one equation for a& b ✓ sets up two equations for a & b ✓ solves simultaneous eqs for a&b NOTE: any statement that is not supported receives zero marks)