

## SOLUTIONS PART A – MATHEMATICAL INDUCTION INVESTIGATION

1. Required to prove  $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$  for all positive integers  $n$ .

Step 1 Verify the statement is true when  $n = 1$

$$\text{L.H.S.} = 1$$

$$\text{R.H.S.} = \frac{1 \cdot 2 \cdot 3}{6} = 1$$

$\Rightarrow$  Statement is true for  $n = 1$

Step 2 Assume the statement is true for  $n = k$

$$\text{That is,} \quad 1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Step 3 Prove statement true for  $n = k + 1$

$$\text{That is, prove } 1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\begin{aligned} \text{LHS} &= 1^2 + 2^2 + \dots + k^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{from Step 2} \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)((k(2k+1) + 6(k+1)))}{6} \\ &= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(2k+3)(k+2)}{6} \\ &= \frac{(k+1)(2k+3)(k+2)}{6} \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\ &= \text{RHS} \end{aligned}$$

$\Rightarrow$  The statement is true for  $n = k + 1$  if it is true for  $n = k$ .

Step 4 As the statement is true for  $n = 1$  it must be true for  $n = 2$ .

As the statement is true for  $n = 2$  it must be true for  $n = 3$  and so on.

$$\text{Hence, } 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ is true for all positive integers } n.$$

2. Required to prove  $1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$  for all positive integers  $n$ .

Step 1 Verify the statement is true when  $n = 1$

$$\text{L.H.S.} = 1$$

$$\text{R.H.S.} = \frac{1 \cdot 2^2}{4} = 1$$

$\Rightarrow$  Statement is true for  $n = 1$

Step 2 Assume the statement is true for  $n = k$

$$\text{That is, } 1^3 + 2^3 + \dots + k^3 = \frac{k^2(k+1)^2}{4}$$

Step 3 Prove statement true for  $n = k + 1$

$$\text{That is, prove } 1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

$$\begin{aligned} \text{LHS} &= 1^3 + 2^3 + \dots + k^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \text{from Step 2} \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\ &= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \frac{(k+1)^2((k+1)+1)^2}{4} \\ &= \text{RHS} \end{aligned}$$

$\Rightarrow$  The statement is true for  $n = k + 1$  if it is true for  $n = k$ .

Step 4 As the statement is true for  $n = 1$  it must be true for  $n = 2$ .

As the statement is true for  $n = 2$  it must be true for  $n = 3$  and so on.

$$\text{Hence, } 1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4} \text{ is true for all positive integers } n.$$

3. Required to prove  $1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$  for all positive integers  $n$ .

Step 1 Verify the statement is true when  $n = 1$

$$\text{L.H.S.} = 2$$

$$\text{R.H.S.} = \frac{1.2.3}{3} = 2$$

$\Rightarrow$  Statement is true for  $n = 1$

Step 2 Assume the statement is true for  $n = k$

$$\text{That is, } 1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

Step 3 Prove statement true for  $n = k + 1$

$$\text{That is, prove } 1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + k(k+1) + (k+1)((k+1)+1) = \frac{(k+1)((k+1)+1)((k+1)+2)}{3}$$

$$\begin{aligned} \text{LHS} &= 1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + k(k+1) + (k+1)((k+1)+1) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{from Step 2} \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3} \\ &= \frac{(k+1)((k+1)+1)((k+1)+2)}{3} \end{aligned}$$

=RHS

$\Rightarrow$  The statement is true for  $n = k + 1$  if it is true for  $n = k$ .

Step 4 As the statement is true for  $n = 1$  it must be true for  $n = 2$ .

As the statement is true for  $n = 2$  it must be true for  $n = 3$  and so on.

Hence,  $1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$  is true for all positive integers  $n$ .

4. Required to prove  $\cos(n\pi + x) = (-1)^n \cos x$  for all positive integers  $n$ .

Step 1 Verify the statement is true when  $n = 1$

$$\begin{aligned} \text{L.H.S} &= \cos(\pi + x) \\ &= \cos \pi \cos x - \sin \pi \sin x \\ &= -1 \cdot \cos x - 0 \cdot \sin x \\ &= -\cos x \end{aligned}$$

$$\text{R.H.S.} = (-1)^1 \cos x$$

$$= -\cos x$$

⇒ Statement is true for  $n = 1$

Step 2 Assume the statement is true for  $n = k$

That is,  $\cos(k\pi + x) = (-1)^k \cos x$

Step 3 Prove statement true for  $n = k + 1$

That is, prove  $\cos((k + 1)\pi + x) = (-1)^{k+1} \cos x$

$$\begin{aligned} \text{LHS} &= \cos((k + 1)\pi + x) \\ &= \cos(k\pi + \pi + x) \\ &= \cos((k\pi + x) + \pi) \\ &= \cos(k\pi + x)\cos\pi - \sin(k\pi + x)\sin\pi \\ &= (-1)^k \cos x \cdot (-1) - \sin(k\pi + x) \cdot 0 && \text{from Step 2} \\ &= (-1)^{k+1} \cos x \\ &= \text{RHS} \end{aligned}$$

⇒ The statement is true for  $n = k + 1$  if it is true for  $n = k$ .

Step 4 As the statement is true for  $n = 1$  it must be true for  $n = 2$ .

As the statement is true for  $n = 2$  it must be true for  $n = 3$  and so on.

Hence,  $\cos(n\pi + x) = (-1)^n \cos x$  is true for all positive integers  $n$ .

5. Required to prove  $n(n^2 + 5)$  is divisible by 6 for all positive integers  $n$ .

Step 1 Verify the statement is true when  $n = 1$

$$1(1 + 5) = 6$$

Divisible by 6

⇒ Statement is true for  $n = 1$

Step 2 Assume the statement is true for  $n = k$

That is,  $k(k^2 + 5)$  is divisible by 6

Step 3 Prove statement true for  $n = k + 1$

That is, prove  $(k + 1)((k + 1)^2 + 5)$  is divisible by 6

$$\begin{aligned}
 & (k + 1)((k + 1)^2 + 5) \\
 &= (k + 1)(k^2 + 2k + 1 + 5) \\
 &= (k + 1)(k^2 + 2k + 6) \\
 &= k^3 + 2k^2 + 6k + k^2 + 2k + 6 \\
 &= k^3 + 3k^2 + 8k + 6 \\
 &= k^3 + 5k + 3k^2 + 3k + 6 \\
 &= k(k^2 + 5) + 3(k^2 + k + 2)
 \end{aligned}$$

$k(k^2 + 5)$  is divisible by 6 from Step 2

$3(k^2 + k + 2)$  is divisible by 6 if  $(k^2 + k + 2)$  is even

If  $k$  is an odd number  $k^2$  is odd so  $(k^2 + k + 2) = \text{odd} + \text{odd} + 2 = \text{even}$

If  $k$  is an even number  $k^2$  is even so  $(k^2 + k + 2) = \text{even} + \text{even} + 2 = \text{even}$

So  $(k^2 + k + 2)$  is even for all positive integers  $k$  so  $3(k^2 + k + 2)$  is divisible by 6

$k(k^2 + 5) + 3(k^2 + k + 2)$  is divisible by 6

$\Rightarrow$  The statement is true for  $n = k + 1$  if it is true for  $n = k$ .

Step 4 As the statement is true for  $n = 1$  it must be true for  $n = 2$ .

As the statement is true for  $n = 2$  it must be true for  $n = 3$  and so on.

Hence,  $n(n^2 + 5)$  is divisible by 6 is true for all positive integers  $n$ .

6. Required to prove that the  $n$ th derivative of  $y = x \ln x, x > 0$  is  $\frac{d^n y}{dx^n} = \frac{(-1)^n (n-2)!}{x^{n-1}}$  for all positive integers  $n \geq 2$ .

Step 1 Verify the statement is true when  $n = 2$

L.H.S.  $y = x \ln x$

$$\begin{aligned}
 \frac{dy}{dx} &= 1 \cdot \ln x + \frac{1}{x} \cdot x \\
 &= \ln x + 1
 \end{aligned}$$

$$\frac{d^2 y}{dx^2} = \frac{1}{x}$$

$$\text{R.H.S.} = \frac{(-1)^2 0!}{x} = \frac{1}{x} = \text{LHS}$$

$\Rightarrow$  Statement is true for  $n = 2$

Step 2 Assume the statement is true for  $n = k$

$$\text{That is, } \frac{d^k y}{dx^k} = \frac{(-1)^k (k-2)!}{x^{k-1}}$$

Step 3 Prove statement true for  $n = k + 1$

$$\text{That is, prove } \frac{d^{k+1} y}{dx^{k+1}} = \frac{(-1)^{k+1} ((k+1)-2)!}{x^{(k+1)-1}}$$

$$\begin{aligned} \text{LHS} &= \frac{d^{k+1} y}{dx^{k+1}} \\ &= \frac{d}{dx} \left( \frac{d^k y}{dx^k} \right) \\ &= \frac{d}{dx} \left( \frac{(-1)^k (k-2)!}{x^{k-1}} \right) \quad \text{from Step 2} \\ &= \frac{d}{dx} \left( (-1)^k (k-2)! x^{-(k-1)} \right) \\ &= (-1)^k (k-2)! (- (k-1)) x^{-(k-1)-1} \\ &= (-1)^k (k-2)! (- (k-1)) x^{-(k-1)-1} \\ &= (-1)^k (k-2)! (-1)(k-1) x^{-k} \\ &= (-1)^{k+1} (k-1)(k-2)! x^{-k} \\ &= \frac{(-1)^{k+1} (k-1)!}{x^k} \\ &= \frac{(-1)^{k+1} ((k+1)-2)!}{x^{(k+1)-1}} \\ &= \text{RHS} \end{aligned}$$

$\Rightarrow$  The statement is true for  $n = k + 1$  if it is true for  $n = k$ .

Step 4 As the statement is true for  $n = 2$  it must be true for  $n = 3$ .

As the statement is true for  $n = 3$  it must be true for  $n = 4$  and so on.

Hence, the  $n$ th derivative of  $y = x \ln x, x > 0$  is  $\frac{d^n y}{dx^n} = \frac{(-1)^n (n-2)!}{x^{n-1}}$  for all positive integers  $n \geq 2$ .

Required to prove  $\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \dots + \frac{1}{(n+1)(n+2)} = \frac{n}{2(n+2)}$  for  $n \geq 1$

Step 1 Conjecture is true for  $n = 1 \Leftrightarrow \frac{1}{(2) \times (3)} = \frac{1}{2(3)}$  True

Step 2 Assume conjecture is true for  $n = k$

$$\Leftrightarrow \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \dots + \frac{1}{(k+1)(k+2)} = \frac{k}{2(k+2)}$$

Step 3 Conjecture is true for  $n = k + 1$

$$\Leftrightarrow \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \dots + \frac{1}{(k+2)(k+3)} = \frac{k+1}{2(k+3)}$$

$$\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \dots + \frac{1}{(k+2)(k+3)} =$$

$$= \left( \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \dots + \frac{1}{(k+1)(k+2)} \right) + \frac{1}{(k+2)(k+3)} \dots \text{Associative property}$$

$$= \left( \frac{k}{2(k+2)} \right) + \left( \frac{1}{(k+2)(k+3)} \right) \dots \text{Assumption in step 2}$$

$$= \frac{k(k+3)}{2(k+2)(k+3)} + \frac{2}{2(k+2)(k+3)} \dots \text{Common denominators}$$

$$= \frac{k^2 + 3k + 2}{2(k+2)(k+3)} \dots \text{Adding fractions}$$

$$= \frac{(k+1)(k+2)}{2(k+2)(k+3)} \dots \text{Factorising numerator}$$

$$= \frac{k+1}{2(k+3)} \dots \text{Simplifying factors}$$

$$\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \dots + \frac{1}{(k+2)(k+3)} = \frac{k+1}{2(k+3)} \dots \text{QED}$$

Step 4 As the statement is true for  $n = 1$  it must be true for  $n = 2$ .

As the statement is true for  $n = 2$  it must be true for  $n = 3$  and so on.

Hence  $\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \dots + \frac{1}{(n+1)(n+2)} = \frac{n}{2(n+2)}$  is true for all positive integers  $n$ .