SCHOOL

Semester One Examination, 2013

Question/Answer Booklet

MATHEMATICS SPECIALIST 3C

SOLUTIONS

Section Two: Calculator-assumed

Student Number:	In figures	
	In words	______________________________
	Your name	______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination.

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	50	33
Section Two: Calculator-assumed	12	12	100	100	67
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2013*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

Section Two: Calculator-assumed **(100 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

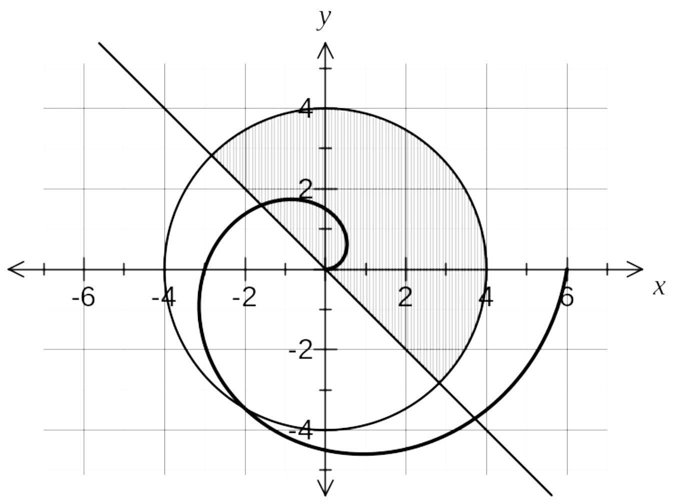
Question 8 **(4 marks)**

Determine the derivative of $y = x^2 - 6x + 5$ from first principles.

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - 6(x+h) + 5 - x^2 + 6x - 5}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 6x - 6h + 5 - x^2 + 6x - 5}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2 - 6h}{h} \\
&= \lim_{h \to 0} 2x - 6 + h \\
&= 2x - 6
\end{aligned}$$

Question 9 **(8 marks)**

A circle and line intersect as shown.



(a) Use polar inequalities to describe the shaded region in the diagram. (2 marks)

$$r \le 4$$

$$-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$$

(b) The curve $r = k\theta$, where k is a positive constant, passes through the point A$\left(\frac{3}{2}, \frac{\pi}{2}\right)$.
Add the graph of $r = k\theta$ on the axes above, for $0 \le \theta \le 2\pi$. (3 marks)

$$\frac{3}{2} = k \times \frac{\pi}{2} \Rightarrow k = \frac{3}{\pi}$$

(c) Another point, B, is located at the intersection of the curve $r = k\theta$ with the circle.
Find the distance between points A and B. (3 marks)

$$4 = \frac{3}{\pi}\theta \Rightarrow \theta = \frac{4\pi}{3}$$

$$\frac{4\pi}{3} - \frac{\pi}{2} = \frac{5\pi}{6}$$

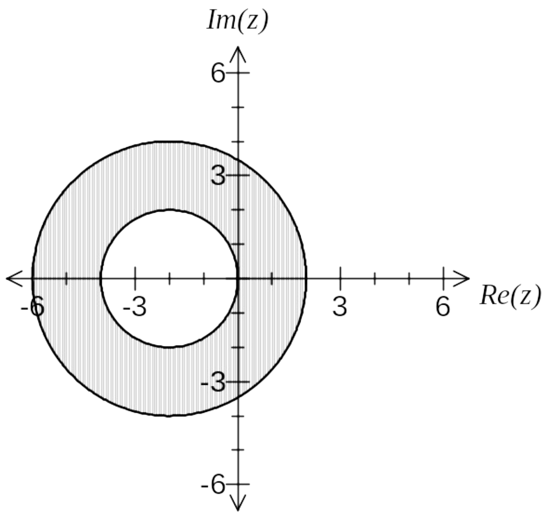
$$d^2 = 4^2 + 1.5^2 - 2 \times 4 \times 1.5 \times \cos\frac{5\pi}{6}$$

$$d = 5.35 \text{ units}$$

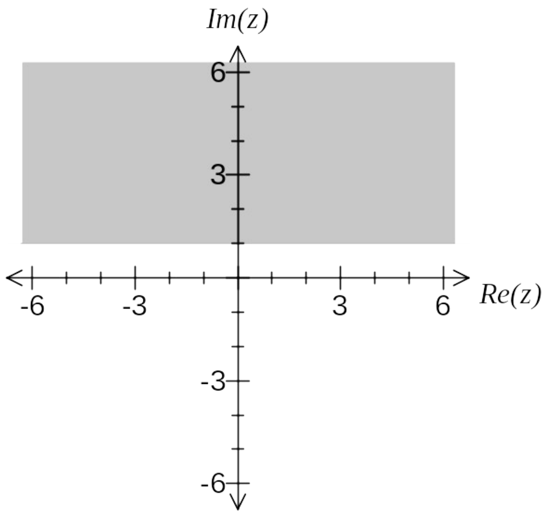
Question 10 **(8 marks)**

(a) Sketch the following regions in the complex plane.

(i) $2 \leq |z+2| \leq 4$ (3 marks)



(ii) $z - \bar{z} > 2i$ (3 marks)



(b) The set of points in the complex plane that satisfy $z - \bar{z} > 2i$ can also be described by $|z - 2 + i| > |z + a + bi|$, where a and b are real constants. State the values of a and b.

(2 marks)

$a = -2$
$b = -3$

Question 11 **(7 marks)**

A curve is defined parametrically by $x(t) = t + \cos 2t$ and $y(t) = 5 + 3\sin t$ for $0 \le t \le 2\pi$ seconds.

(a) Find an expression for the gradient function $\frac{dy}{dx}$. (2 marks)

$$x'(t) = 1 - 2\sin 2t$$
$$y'(t) = 3\cos t$$
$$\frac{dy}{dx} = \frac{3\cos t}{1 - 2\sin 2t}$$

(b) Find the coordinates of all points on the curve where the gradient is zero. (3 marks)

$$\frac{dy}{dx} = 0 \text{ when } \cos t = 0 \Rightarrow t = \tfrac{\pi}{2}, \tfrac{3\pi}{2}$$

$$\left.\begin{aligned} x(\tfrac{\pi}{2}) &= \frac{\pi}{2} - 1 \\ y(\tfrac{\pi}{2}) &= 8 \end{aligned}\right\} \Rightarrow \left(\frac{\pi - 2}{2},\ 8\right)$$

$$\left.\begin{aligned} x(\tfrac{3\pi}{2}) &= \frac{3\pi}{2} - 1 \\ y(\tfrac{3\pi}{2}) &= 2 \end{aligned}\right\} \Rightarrow \left(\frac{3\pi - 2}{2},\ 2\right)$$

(c) At what time is the tangent to the curve first parallel to the y-axis? (2 marks)

$$1 - 2\sin 2t = 0$$
$$\sin 2t = \tfrac{1}{2}$$
$$2t = \tfrac{\pi}{6}$$
$$t = \tfrac{\pi}{12} \text{ seconds}$$

Question 12 **(9 marks)**

A function is defined as $f(x) = e^{(-1+\sqrt{x})}$ for $x \geq 0$.

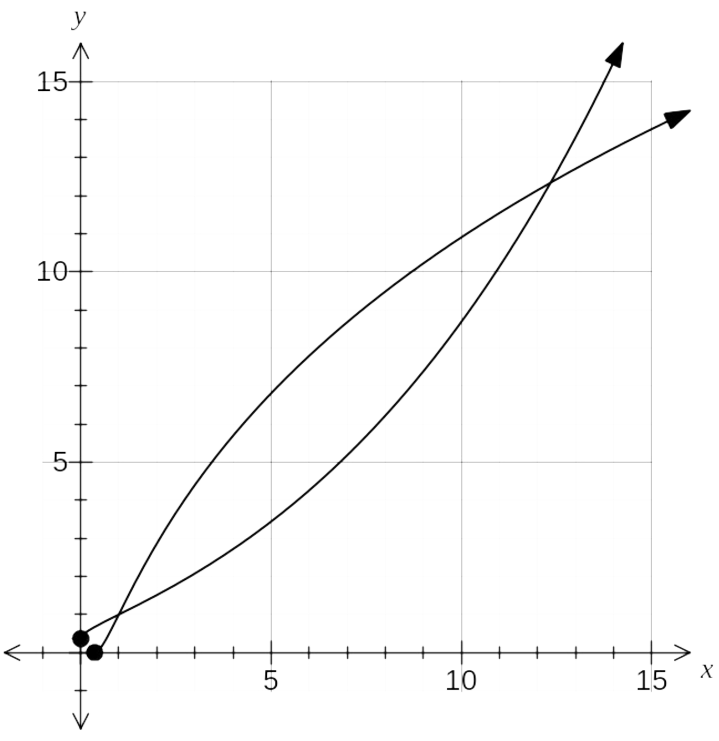
(a) State the range of $f(x)$. (1 mark)

$$\boxed{y \geq \frac{1}{e}}$$

(b) Find $f^{-1}(x)$. (2 marks)

$$x = e^{\sqrt{y}-1}$$
$$\ln x = \sqrt{y} - 1$$
$$\sqrt{y} = 1 + \ln x$$
$$f^{-1}(x) = (1 + \ln x)^2$$

(c) Sketch $f(x)$ and $f^{-1}(x)$ on the axes below. (4 marks)

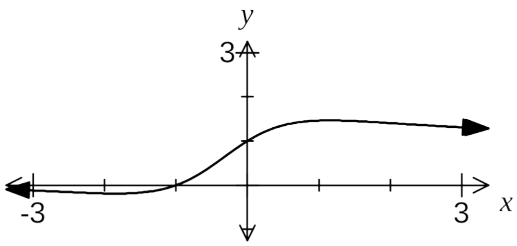


(d) Determine the coordinates of all point(s) of intersection of $f(x)$ and $f^{-1}(x)$ to two decimal places. (2 marks)

$\boxed{(1, 1) \text{ and } (12.34, 12.34)}$

Question 13 **(7 marks)**

The graph of the function $f(x) = \dfrac{e^x(1+x)}{1+xe^x}$ is shown.



(a) Determine the equation of the tangent to $f(x)$ where the function cuts the y-axis.

(2 marks)

$$\boxed{y = x + 1}$$

(b) Find, in simplest form, the exact value of $\int_{-1}^{0} f(x)\,dx$, using the substitution $u = xe^x$.

(5 marks)

When $x = -1,\ u = -\dfrac{1}{e}$ and when $x = 0,\ u = 0$.

$$du = xe^x + e^x dx = e^x(1+x)dx$$

$$\int_{-\frac{1}{e}}^{0} \frac{1}{1+u}du = \left[\ln|1+u|\right]_{-\frac{1}{e}}^{0}$$

$$= \ln 1 - \ln\left(1 - \tfrac{1}{e}\right)$$

$$= -\ln\left(\frac{e-1}{e}\right)$$

$$= 1 - \ln(e-1)$$

Question 14 **(8 marks)**

(a) Prove by contradiction that if n is a natural number, then $\frac{n}{n+1} > \frac{n}{n+2}$. (3 marks)

Suppose the opposite is true, that $\frac{n}{n+1} \le \frac{n}{n+2}$.

Then $n(n+2) \le n(n+1)$

so that $n^2 + 2n \le n^2 + n$

and $n \le 0$

But this contradicts the original statement that n is a natural number.

Hence the original statement must be true.

(b) Let $f(n) = 5n^2 + 3n + 1$, where n is an integer.

(i) Examine $f(n)$ and make a conjecture about the parity of $f(n)$. (1 mark)

..., -27, -9, 9, 27, 55, 93, 141, 191, ...

$f(n)$ is always odd.

(ii) Prove your conjecture. (4 marks)

Consider case when n is odd and then case when n is even:

Let $n = 2x + 1$

$$\begin{aligned} 5n^2 + 3n + 1 &= 5(2x+1)^2 + 3(2x+1) + 1 \\ &= 20x^2 + 26x + 9 \\ &= 2(10x^2 + 13x + 4) + 1 \text{ which is odd} \end{aligned}$$

Let $n = 2x$

$$\begin{aligned} 5n^2 + 3n + 1 &= 5(2x)^2 + 3(2x) + 1 \\ &= 20x^2 + 6x + 1 \\ &= 2(10x^2 + 3x) + 1 \text{ which is odd} \end{aligned}$$

Hence $5n^2 + 3n + 1$ is always odd.

Question 15 **(11 marks)**

A is the plane with equation $2x - y + 3z = 1$.

(a) Find the equation of the line that is perpendicular to A and passes through (-3, 6, 2). Give your answer in parametric form. (3 marks)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$x = 2\lambda - 3$$
$$y = 6 - \lambda$$
$$z = 2 + 3\lambda$$

(b) Find the point of intersection of the line in part (a) and the plane A. (3 marks)

$$2(2\lambda - 3) - (6 - \lambda) + 3(2 + 3\lambda) = 1$$
$$14\lambda = 7$$
$$\lambda = \frac{1}{2}$$

$$x = -2$$
$$y = 5.5$$
$$z = 3.5$$

Intersect at (-2, 5.5, 3.5)

(c) Give an equation for the plane that contains (15, 11, -5) and is parallel to A. (2 marks)

$$2(15) - 11 + 3(-5) = c$$
$$c = 4$$

$$2x - y + 3z = 4$$

(d) Calculate the size of the acute angle between A and the line $\mathbf{r} = \mathbf{j} - \mathbf{k} + \lambda(3\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$.

(3 marks)

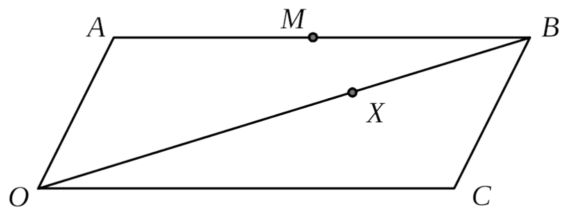
Require complement of angle between

$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ (normal to plane) and $\begin{bmatrix} 3 \\ -5 \\ -2 \end{bmatrix}$ (direction of line).

From CAS, angle is $90^\circ - 77.5^\circ = 12.5^\circ$

Question 16 **(11 marks)**

In the parallelogram $OABC$, where $\vec{OA} = \mathbf{a}$ and $\vec{OC} = \mathbf{c}$, point X divides OB internally in the ratio $2:1$ and point M is the midpoint of AB.



(a) Show that $\vec{OX} = \frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{c}$ (1 mark)

$$\begin{aligned}\vec{OX} &= \frac{2}{1+2}\vec{OB}\\ &= \frac{2}{3}(\mathbf{a}+\mathbf{c})\\ &= \frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{c}\end{aligned}$$

(b) Find $\vec{CX}$ in terms of **a** and **c**. (2 marks)

$$\begin{aligned}\vec{CX} &= \vec{CO} + \vec{OX}\\ &= -\mathbf{c} + \frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{c}\\ &= \frac{2}{3}\mathbf{a} - \frac{1}{3}\mathbf{c}\end{aligned}$$

(c) Use vector methods to prove that points C, X and M are collinear. (3 marks)

$$\begin{aligned}\vec{XM} &= \vec{XO} + \vec{OA} + \vec{AM}\\ &= -\frac{2}{3}\mathbf{a} - \frac{2}{3}\mathbf{c} + \mathbf{a} + \frac{1}{2}\mathbf{c}\\ &= \frac{1}{3}\mathbf{a} - \frac{1}{6}\mathbf{c}\\ &= \frac{1}{2}\left(\frac{2}{3}\mathbf{a} - \frac{1}{3}\mathbf{c}\right)\\ &= \frac{1}{2}\vec{CX}\end{aligned}$$

Hence the points C, X and M are collinear since $\vec{XM}$ and $\vec{CX}$ are parallel and both have point X in common.

(d) If $\mathbf{a} = 3\mathbf{i} - 6\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 6\mathbf{i} - 8\mathbf{k}$, find the area of triangle MXB. (5 marks)

$$\overrightarrow{MB} = \frac{1}{2}\mathbf{c} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$\left|\overrightarrow{MB}\right| = 5$$

$$\overrightarrow{MX} = -\left(\frac{1}{3}\mathbf{a} - \frac{1}{6}\mathbf{c}\right) = \frac{1}{6}\begin{bmatrix} 6 \\ 0 \\ -8 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 3 \\ -6 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

$$\left|\overrightarrow{MX}\right| = \sqrt{5}$$

Angle between $\overrightarrow{MB}$ and $\overrightarrow{MX}$ is 69.04° (CAS)

Area $= \frac{1}{2} \times 5 \times \sqrt{5} \times \sin 69.04 = 5.22$ sq units

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A simpler solution (not in MAS course) is to use vector (cross) product:

$$\frac{1}{2}\left|\overrightarrow{MB} \times \overrightarrow{MX}\right| = \frac{1}{2}\left|\begin{bmatrix} 8 \\ 3 \\ 6 \end{bmatrix}\right| = \frac{1}{2} \times 10.44 = 5.22$$

Question 17 **(10 marks)**

Initially a large tank contains 1 000 litres of pure water. Each minute thereafter, five litres of an alcohol-water mix is added, and five litres of the mixture in the tank is drained. The alcohol-water mix being added is 80 percent alcohol. Throughout this process, the mixture in the tank is thoroughly and uniformly mixed.

Let t be the number of minutes since alcohol was first added to the tank and y be the litres of pure alcohol in the tank at time t.

(a) How much pure alcohol is added to the tank every minute? (1 mark)

$$\boxed{5 \times 0.8 = 4 \text{ litres/minute}}$$

(b) Explain why $\frac{dy}{dt} = 4 - \frac{y}{200}$ litres per minute. (2 marks)

Rate alcohol lost from tank is 5 (L/min drained) $\times y \div 1000$ (fraction of alcohol in tank)

$\frac{dy}{dt} =$ rate alcohol added - rate alcohol lost $= 4 - \frac{y}{400}$ litres/minute

(c) Express y as a function of t. (5 marks)

$$\frac{dy}{dt} = \frac{800 - y}{200}$$

$$\int \frac{1}{800 - y}\, dy = \int \frac{1}{200}\, dt$$

$$\ln|800 - y| = -0.005 + c$$

$$800 - y = y_0 e^{-0.005t}$$

$$y = 800 - y_0 e^{-0.005t}$$

$$y(0) = 0 \Rightarrow y_0 = 800$$

$$y = 800\left(1 - e^{-0.005t}\right)$$

(d) How long, to the nearest minute, until the tank contains 50% alcohol? (2 marks)

$$800\left(1 - e^{-0.005t}\right) = 0.50 \times 1000$$

$$t = 196.17 \approx 196 \text{ minutes}$$

Question 18 **(7 marks)**

Let the vector $\mathbf{a} = \mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ and the vector $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

(a) Determine, in exact form, a unit vector parallel to $\mathbf{a} - 6\mathbf{b}$. (2 marks)

$\mathbf{a} - 6\mathbf{b} = \begin{bmatrix} -11 \\ 0 \\ 2 \end{bmatrix}$ and $|\mathbf{a} - 6\mathbf{b}| = 5\sqrt{5}$

Hence unit vector is $\dfrac{1}{5\sqrt{5}}\begin{bmatrix} -11 \\ 0 \\ 2 \end{bmatrix}$

(b) Resolve **a** into two vector components, one that is parallel to **b** and one that is perpendicular to **b**. (5 marks)

Let perpendicular vector be **c**. Then $\mathbf{a} = k\mathbf{b} + \mathbf{c}$.

$\mathbf{c} = \begin{bmatrix} 1 \\ 6 \\ -4 \end{bmatrix} - k\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-2k \\ 6-k \\ k-4 \end{bmatrix}$

But $\mathbf{b} \bullet \mathbf{c} = 0$

$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1-2k \\ 6-k \\ k-4 \end{bmatrix} = 0 \Rightarrow k = 2$

So parallel vector is $\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$ and perpendicular vector is $\begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix}$

Question 19 **(10 marks)**

The functions f and g are defined for $0 < x < \frac{\pi}{2}$ by $f(x) = \sin x - x\cos x$ and $g(x) = x - \sin x$.

(a) Over the given domain,

(i) Show that $f'(x) > 0$. (2 marks)

$$f'(x) = \cos x - (\cos x + x(-\sin x))$$
$$= x\sin x$$

Over domain, both $x > 0$ and $\sin x > 0 \Rightarrow f'(x) > 0$

(ii) Hence explain why $f(x) > 0$. (1 mark)

Since $f'(x) > 0$, then $f(x)$ is an increasing function and as $f(0) = 0$ then $f(x) > 0$.

(b) Over the given domain,

(i) Show that $g'(x) > 0$. (2 marks)

$$g'(x) = 1 - \cos x$$

Over domain, $\cos x < 1 \Rightarrow 1 - \cos x > 0 \Rightarrow g'(x) > 0$

(ii) Hence explain why $g(x) > 0$. (1 mark)

Since $g'(x) > 0$, then $g(x)$ is an increasing function and as $g(0) = 0$ then $g(x) > 0$.

(c) Over the given domain, use your results from (a) and (b) to deduce that

(i) $\dfrac{\sin x}{x} > \cos x$. (1 mark)

From (a)

$$f(x) > 0$$
$$\sin x - x\cos x > 0$$
$$\sin x > x\cos x$$
$$\frac{\sin x}{x} > \cos x$$

(ii) $\dfrac{\sin x}{x} < 1$. (1 mark)

From (b)

$$g(x) > 0$$
$$x - \sin x > 0$$
$$x > \sin x$$
$$1 > \frac{\sin x}{x}$$

(d) Use your results from (c) to show that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} \to 1$. (2 marks)

$$\cos x < \frac{\sin x}{x} < 1$$

As $x \to 0$, $\cos x \to 1 \Rightarrow 1 < \dfrac{\sin x}{x} < 1$

Hence $\lim\limits_{x \to 0} \dfrac{\sin x}{x} \to 1$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

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