PERTH MODERN SCHOOL
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INDEPENDENT PUBLIC SCHOOL

Semester Two Examination, 2021

Question/Answer booklet

SPECIALIST MATHEMATICS
UNITs 3 & 4
Section One:
Calculator-free

Your Name

Your Teacher's Name

Time allowed for this section
Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Mark	Max	Question	Mark	Max
1			5		
2			6		
3			7		
4			8		

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	51	35
Section Two: Calculator-assumed	13	13	100	101	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section One: Calculator-free **(50 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 50 minutes.

Question 1 **(4 marks)**

Evaluate $\int_0^{\frac{\pi}{4}} \cos 3x + \sin^2 x \, dx$

Solution
$\int_0^{\frac{\pi}{4}} \cos 3x + \sin^2 x \, dx = \int_0^{\frac{\pi}{4}} \cos 3x + \frac{1}{2}(1 - \cos 2x) dx$ $\left[\frac{1}{3}\sin 3x + \frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}}$ $= \left(\frac{1}{3\sqrt{2}} + \frac{\pi}{8} - \frac{1}{4}\right) - (0)$
Specific behaviours
P uses double angle formula P integrates all terms P evaluates upper limit P states exact value

Question 2 **(6 marks)**

Consider a plane that contains the following points $A(3,-2,5), B(7,-1,-2) \& C(4,4,-3)$.

a) Determine a normal vector to the plane. (4 marks)

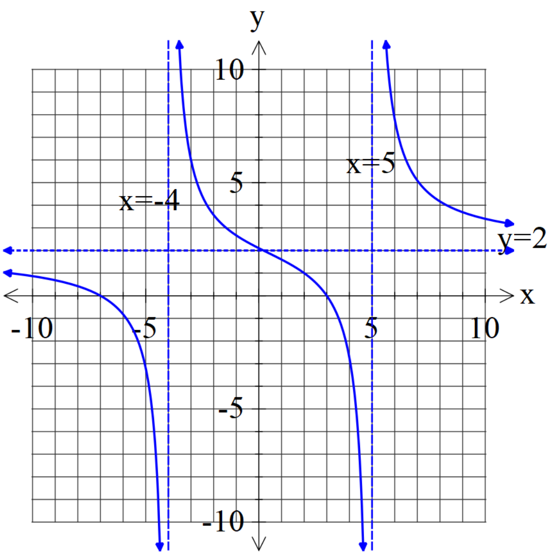
Solution
$AB = \begin{pmatrix} 7 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -7 \end{pmatrix}$ $AC = \begin{pmatrix} 4 \\ 4 \\ -3 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ -8 \end{pmatrix}$ $\begin{pmatrix} 4 \\ 1 \\ -7 \end{pmatrix} \times \begin{pmatrix} 1 \\ 6 \\ -8 \end{pmatrix} = \begin{pmatrix} 34 \\ 25 \\ 23 \end{pmatrix}$
Specific behaviours
P determines one vector in plane P determines two vectors in plane P uses cross product P gives a normal vector

b) Determine a cartesian equation for the plane. (2 marks)

Solution
$r.\begin{pmatrix} 34 \\ 25 \\ 23 \end{pmatrix} = \begin{pmatrix} 34 \\ 25 \\ 23 \end{pmatrix}.\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = 167$ $34x + 25y + 23z = 167$
Specific behaviours
P determines vector equation P states cartesian equation

Question 3 **(6 marks)**

Sketch the function $f(x)=\dfrac{2x^2+8x-42}{(x-5)(x+4)}$ on the axes below, labelling important features.

Solution

Specific behaviours
P vertical asymptotes dotted and labelled with equation P horizontal asymptote dotted and labelled with equation P approx. y intercept correct P both x intercepts correct exactly P shape correct between vertical asymptotes Poverall shape is correct for all x values

Question 4 **(6 marks)**

a) Solve the following system of linear equations. (3 marks)

$$\begin{aligned} -5x + y + 2z &= 12 \\ x + 2y + 3z &= -4 \\ 2x - y + z &= -18 \end{aligned}$$

Solution
$\begin{bmatrix} 1 & 2 & 3 & -4 \\ -5 & 1 & 2 & 12 \\ 2 & -1 & 1 & -18 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 11 & 17 & -8 \\ 0 & 5 & 5 & 10 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 1 & 1 & 2 \\ 0 & 11 & 17 & -8 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -6 & 30 \end{bmatrix}$ $z = -5$ $y + -5 = 2, y = 7$ $x + 14 - 15 = -4, x = -3$
Specific behaviours
P eliminates one variable in two equations   P eliminates two variables in one equation   P solves for all variables

b) Solve for all possible values of $p \,\&\, q$ for the system below for each of the following scenarios. (3 marks)

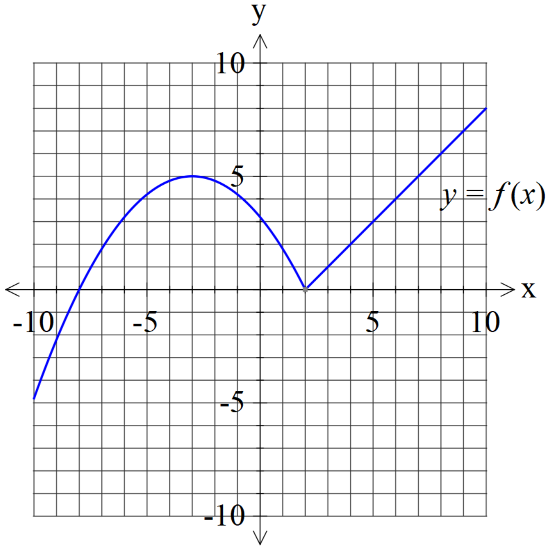
$$-5x + y + 2z = 12$$
$$x + 2y + pz = -4$$
$$2x - y + z = q$$

i) Unique solution.
ii) Infinite solutions
iii) No solutions.

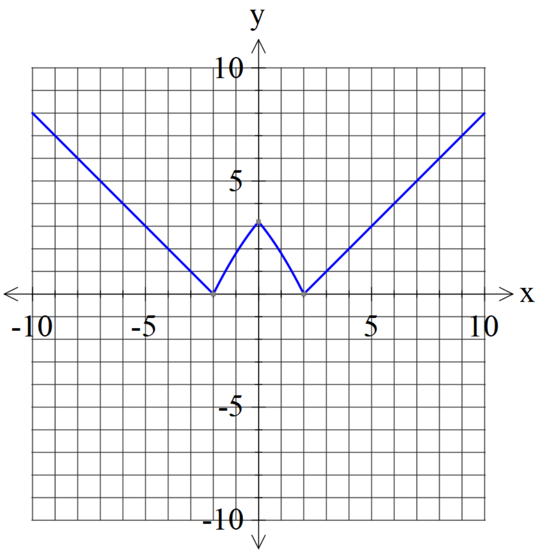
Solution
$\begin{bmatrix} 1 & 2 & p & -4 \\ 2 & -1 & 1 & q \\ -5 & 1 & 2 & 12 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & p & -4 \\ 0 & 5 & 2p-1 & -8-q \\ 0 & 11 & 5p+2 & -8 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & p & -4 \\ 0 & 5 & 2p-1 & -8-q \\ 0 & 0 & -3p-21 & -48-11q \end{bmatrix}$ i) $p \neq -7$ ii) $p = -7 \, and \, q = \frac{-48}{11}$ iii) $p = -7 \, and \, q \neq \frac{-48}{11}$
Specific behaviours
P eliminates two variables   P states values for uniqueness   P states values for infinite and no solns

Question 5 **(6 marks)**

Consider the function $f(x)$ drawn below.

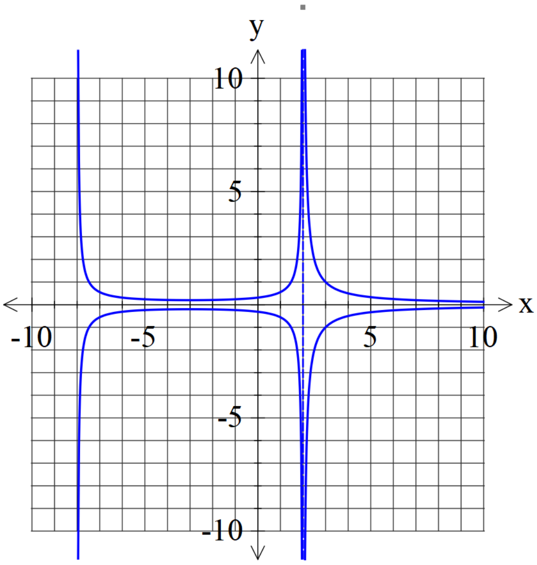


a) Plot $y = f(|x|)$ on the axes below. (2 marks)

Solution

Specific behaviours
P reflects right side P x & y intercepts correct

b) Plot $|y| = \dfrac{1}{f(x)}$ on the axes below. (4 marks)

Solution



Specific behaviours
P both asymptotes for y>0
P approx y intercept and turning pt for y>0
P shape for y>0
P shape for y<0

Question 6 **(9 marks)**

Consider the function $f(x)=\sin x$ with domain $0 \le x \le \frac{\pi}{2}$.

Let $g(x)=f^{-1}(x)$.

a) Determine the domain and range of $g(x)$. (2 marks)

Solution
$d_g : 0 \le x \le 1$ $r_g : 0 \le y \le \frac{\pi}{2}$
Specific behaviours
P domain P range

b) By using implicit differentiation show that $g'(x)$ is of the form $\frac{1}{\sqrt{a^2 - x^2}}$ where a is a constant. (4 marks)

Solution
$g(x) = y$ $x = \sin y$ $1 = \cos y \dfrac{dy}{dx}$ $\dfrac{dy}{dx} = g' = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1 - \sin^2 y}} = \dfrac{1}{\sqrt{1 - x^2}}$ (Note: accept $\pm$)
Specific behaviours
P replaces x & y P implicit diff wrt x both sides P uses Pythagorean identity P expresses in required form

c) Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{4 - x^2}}\,dx$ with substitution $x = 2\sin u$. (3 marks)

Solution
$\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{4 - x^2}} \frac{dx}{du}\,du$ $\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{4 - 4\sin^2 u}} 2\cos u\,du$ $\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{2\cos u} 2\cos u\,du$ $\displaystyle\int_0^{\frac{\pi}{6}} 1\,du = [u]_0^{\frac{\pi}{6}} = \frac{\pi}{6}$
Specific behaviours
P changes limits P simplifies in terms of u P integrates and states final result

Question 7 **(7 marks)**

a) Given that $\frac{x^3+6x^2-3x+8}{(x^2+2)(x^2+2x-3)}=\frac{Ax+B}{x^2+2}+\frac{C}{x-1}+\frac{D}{x+3}$ with $A,B,C\,\&\,D$ constants. Solve for $A,B,C\,\&\,D$. (4 marks)

Solution
$\frac{x^3+6x^2-3x+8}{(x^2+2)(x^2+2x-3)}=\frac{Ax+B}{x^2+2}+\frac{C}{x-1}+\frac{D}{x+3}$ $x^3+6x^2-3x+8=(Ax+B)(x^2+2x-3)+C(x^2+2)(x+3)+D(x^2+2)(x-1)$ $x=1$ $12=12C, C=1$ $x=-3$ $44=-44D, D=-1$ $x=0$ $8=-3B+6+2, B=0$ $x=-1$ $16=4A+12, A=1$
Specific behaviours
P sets up am equation with all constants P shows working on solving for 2 constants P solves for 3 constants P solves for 4 constants

b) Hence determine an expression for $\int\frac{x^3+6x^2-3x+8}{(x^2+2)(x^2+2x-3)}dx$. (3 marks)

Solution
$\int\frac{x}{x^2+2}+\frac{1}{x-1}-\frac{1}{x+3}dx=\frac{1}{2}\ln\left\|x^2+2\right\|+\ln\left\|x-1\right\|-\ln\left\|x+3\right\|+c$
Specific behaviours
P integrates one term P integrates two terms P integrates all terms and adds a constant

End of questions

Question 8 **(7 marks)**

Evaluate the following integrals.

a) $\int_0^2 (1-3x)\sqrt{5+2x}\,dx$ (3 marks)

Solution
$\int_0^2 (1-3x)\sqrt{5+2x}\,dx \cdots let \ldots u = 5+2x$ $\int_5^{11} (1-3x)\sqrt{u}\,\frac{dx}{du}\,du$ $\int_5^{11} \left(1-3\frac{(u-5)}{2}\right)\sqrt{u}\,\frac{1}{2}\,du$ $\int_5^{11} \left(\frac{17u^{\frac{1}{2}}-3u^{\frac{3}{2}}}{4}\right)du = \frac{1}{4}\left[\frac{34}{3}u^{\frac{3}{2}}-\frac{6}{5}u^{\frac{5}{2}}\right]_5^{11} = \frac{1}{4}\left(\frac{34}{3}11^{\frac{3}{2}}-\frac{6}{5}11^{\frac{5}{2}}\right)-\frac{1}{4}\left(\frac{34}{3}5^{\frac{3}{2}}-\frac{6}{5}5^{\frac{5}{2}}\right)$
Specific behaviours
P uses a change of variable P expresses integral in terms of new variable and integrates P changes limits and **subs** into final expression (no need to simplify)

b) $\int \frac{6\sec^2 x}{1-\tan^2 x}\,dx$ (4 marks)

Solution
$\int \frac{6\sec^2 x}{1-\tan^2 x}\,dx \ldots let\, u = \tan x \ldots \frac{du}{dx} = \sec^2 x$ $\int \frac{6\sec^2 x}{1-u^2}\frac{dx}{du}\,du$ $\int \frac{6}{1-u^2}\,du$ $\frac{1}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u}$ $1 = A(1+u) + B(1-u)$ $u = 1$ $1 = 2A,\ A = 1/2$ $u = -1$ $1 = 2B,\ B = 1/2$ $\int \frac{6}{1-u^2}\,du = \int \frac{3}{1-u} + \frac{3}{1+u}\,du = -3\ln\lvert 1-u\rvert + 3\ln\lvert 1+u\rvert + c$ $= 3\ln\frac{1+\tan x}{1-\tan x} + c$

Specific behaviours
P uses u=tanx P uses partial fractions and shows working for constants P integrates wrt new variable P expresses in terms of x (unsimplified) Note: Follow through for last two marks only if partial fractions used

End of questions

Additional working space

Question number: ________

Additional working space

Question number: ________

Additional working space

Question number: ________

Acknowledgements