SCHOOL

Trial WACE Examination, 2011

Question/Answer Booklet

MATHEMATICS
SPECIALIST 3C/3D

Section Two:
Calculator-assumed

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination.

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	6	6	50	40	33
Section Two: Calculator-assumed	13	13	100	80	67
			Total	120	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2011*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

Section Two: Calculator-assumed **(80 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 7 **(6 marks)**

The temperature, I °C, of a liquid in an insulated flask at any time t seconds can be described by the differential equation $\frac{dI}{dt} = -0.003I$.

(a) How long will it take for the liquid in the flask to fall by 10%? (2 marks)

$0.9 = e^{-0.003t}$

$t = 35.120$

Take just over 35 seconds.

The temperature of a liquid in another, uninsulated, flask is falling exponentially at a percentage decay rate of 0.75%.

(b) If the initial temperatures of the liquids in the insulated and uninsulated flasks are 65°C and 95°C respectively, determine when the difference in temperature between the two liquids is 10°C. (4 marks)

$I = 65e^{-0.003t}$ and $U = 95e^{-0.0075t}$

$95e^{-0.0075t} - 65e^{-0.003t} = 10$ when $t = 47.99$ seconds

$65e^{-0.003t} - 95e^{-0.0075t} = 10$ when $t = 144.55$ and $t = 587.34$ seconds

Difference is 10 seconds after 48, 145 and 587 seconds.

Question 8 **(5 marks)**

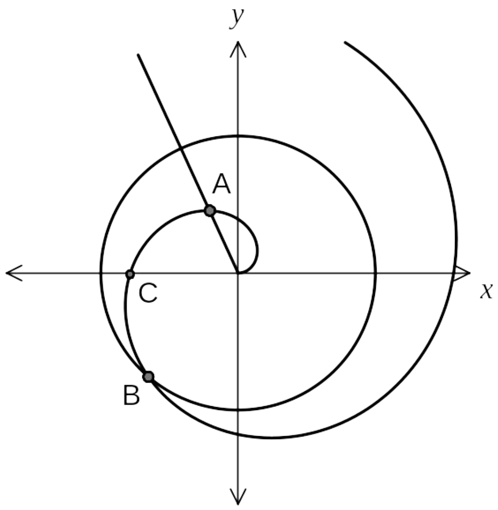
(a) Find the distance between the points with polar coordinates $\left(5,\ \frac{2\pi}{3}\right)$ and $\left(12,\ -\frac{5\pi}{6}\right)$, where distances are in centimetres and angles in radians. (2 marks)

$2\pi - \left(\frac{2\pi}{3} + \frac{5\pi}{6}\right) = 2\pi - \frac{3\pi}{2} = \frac{\pi}{2} \Rightarrow$ At right angles.

$d^2 = 5^2 + 12^2$

$d = 13$ cm.

(b) The graphs of $\theta = \alpha$, $r = b$ and $r = n\theta$ are shown below together with the points A and B which have polar coordinates of $(1,\ 2)$ and $(b,\ 4)$. Find the values of $\alpha, b,\ n$ and the polar coordinates of point C. (3 marks)



Using A, $\alpha = 2$ and $1 = n \times 2 \Rightarrow n = 0.5$

Using B, $b = 0.5 \times 4 = 2$

$\alpha = 2$, $b = 2$, n $= 0.5$, $C\left(\frac{\pi}{2}, \pi\right)$

Question 9 **(8 marks)**

The point A has position vector $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

(a) Find the value of a if the vectors OA and $a\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ are perpendicular. (1 mark)

$$\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} a \\ 3 \\ -3 \end{bmatrix} = 0$$

$$3a = 18$$

$$a = 6$$

(b) Find the size of the angle between OA and the z-axis, to the nearest degree. (2 marks)

Angle between OA and the line $\mathbf{r} = \mathbf{k}$

```
angle([3,-2,4],[0,0,1])
                42.03111377
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(c) Find the value of b if the point (7, b, 2) lies in the plane containing the point (-1, 2, 5) and with normal vector OA. (2 marks)

Plane $3x - 2y + 4z = k$

$$3(-1) - 2(2) + 4(5) = 13 \Rightarrow k = 13$$

$$3(7) - 2(b) + 4(2) = 13$$

$$b = 8$$

(d) Find the value of c if the point (15, -14, c) lies on the straight line through A and the point (-1, 2, 5). (3 marks)

Direction of line given by

$$\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -14 \\ c \end{bmatrix}$$

$$3 + 4\lambda = 15 \Rightarrow \lambda = 3$$

$$c = 4 + 3(-1)$$

$$= 1$$

Question 10 **(5 marks)**

When an object is at a distance u cm from a lens of focal length f cm, an image is formed at a distance of v cm from the lens.

The variables are related by the formula $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$.

An object is moving with a constant speed of 2 cm/s towards a lens of focal length 20 cm.

At the instant when the image is 30 cm from the lens, in what direction and with what speed is it moving?

Given $\frac{du}{dt}=-2$ find $\frac{dv}{dt}$ when $v=30$

$$\frac{1}{20}=\frac{1}{u}+\frac{1}{30}\Rightarrow u=60$$

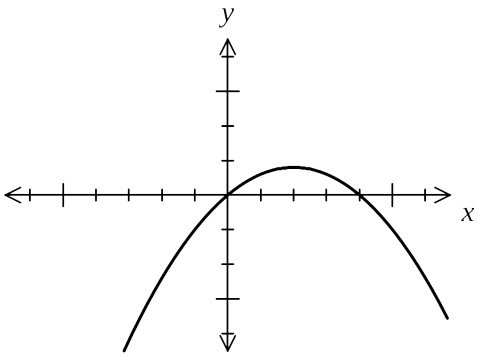
$$\frac{1}{20}=\frac{1}{u}+\frac{1}{v}\Rightarrow v=\frac{20u}{u-20}$$

$$\frac{dv}{du}=\frac{-400}{(u-20)^2}$$

$$\begin{aligned}\frac{dv}{dt}&=\frac{dv}{du}\times\frac{du}{dt}\\&=\frac{-400}{(60-20)^2}\times-2\\&=\frac{1}{2}\text{ cm/s away from the lens (since +ve)}\end{aligned}$$

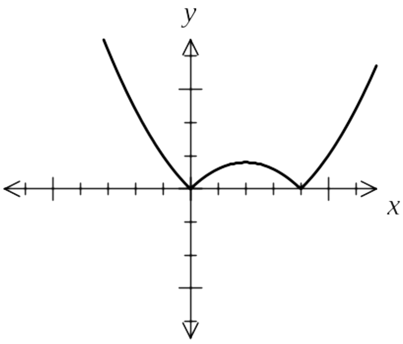
Question 11 **(5 marks)**

(a) The graph of $y = f(x)$ is shown below.

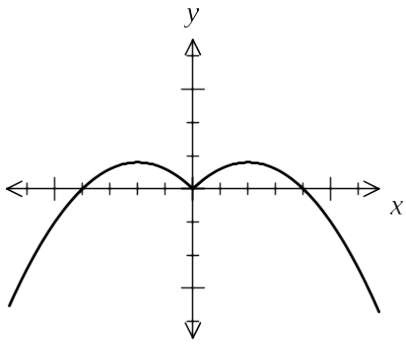


Sketch the graphs of (2 marks)

(i) $y = |f(x)|$



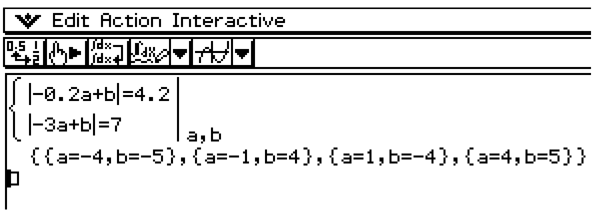
(ii) $y = f(|x|)$



(b) The equation $|ax + b| = |x - 4|$ has solutions $x = -0.2$ and $x = -3$. Find the values of a and b. (3 marks)

Solutions when $|-0.2a + b| = 4.2$ (using $x = -0.2$)

and when $|-3a + b| = 7$ (using $x = -3$)



Question 12 **(4 marks)**

Prove by deduction that $\dfrac{1+\sin 2\theta-\cos 2\theta}{1+\sin 2\theta+\cos 2\theta}=\tan\theta$.

$$
\begin{aligned}
LHS &= \frac{1+2\sin\theta\cos\theta-(1-2\sin^2\theta)}{1+2\sin\theta\cos\theta+(2\cos^2\theta-1)} \\
&= \frac{2\sin^2\theta+2\sin\theta\cos\theta}{2\cos^2\theta+2\sin\theta\cos\theta} \\
&= \frac{\sin\theta(\sin\theta+\cos\theta)}{\cos\theta(\sin\theta+\cos\theta)} \\
&= \tan\theta
\end{aligned}
$$

Question 13 **(6 marks)**

Two complex numbers are given by $u = 3i$ and $v = \dfrac{3\sqrt{3} - 3i}{2}$.

(a) Express $u^3 v$ in the form $r(\cos\theta + i\sin\theta)$ where $-\pi \le \theta \le \pi$ and $r \ge 0$. (2 marks)

3i⇒u

$$3\cdot i$$

$\dfrac{3\sqrt{3}-3i}{2}$⇒v

$$\frac{3\cdot\sqrt{3}-3\cdot i}{2}$$

u^3×v

$$\frac{-27\cdot(3\cdot\sqrt{3}-3\cdot i)\cdot i}{2}$$

compToTrig(

$$81\cdot\left(\cos\left(\frac{-2\cdot\pi}{3}\right)+\sin\left(\frac{-2\cdot\pi}{3}\right)\cdot i\right)$$

(b) Find all solutions for z in the form $re^{i\theta}$, given that $z^4 = u^3 v$. (2 marks)

z^4=$\dfrac{-27\cdot(3\cdot\sqrt{3}-3\cdot i)\cdot i}{2}$

$$z^4=\frac{-27\cdot(3\cdot\sqrt{3}-3\cdot i)\cdot i}{2}$$

solve(ans, z

$$\left\{z=\left(-\frac{1}{2}-\frac{i}{2}\right)\cdot(162+162\cdot\sqrt{3}\cdot i)^{\frac{1}{4}}, z=\left(-\frac{1}{2}+\frac{i}{2}\right)\cdot(162+162\cdot\sqrt{3}\cdot i)\right.$$

compToPol(

$$\left\{z=3\cdot e^{\pi\cdot\frac{-2\cdot i}{3}}, z=3\cdot e^{\frac{\pi\cdot 5\cdot i}{6}}, z=3\cdot e^{\pi\cdot\frac{-i}{6}}, z=3\cdot e^{\frac{\pi\cdot i}{3}}\right\}$$

(c) Show that the sum of all the solutions from part (b) is 0. (2 marks)

$3e^{-\frac{2\pi}{3}}$ and $3e^{\frac{\pi}{3}}$ are equal in magnitude and opposite in direction and sum to 0.

$3e^{-\frac{\pi}{6}}$ and $3e^{\frac{5\pi}{6}}$ are equal in magnitude and opposite in direction and sum to 0.

Hence sum of all roots is 0.

Question 14 **(5 marks)**

At a school with 108 boarders, boarders can either eat breakfast or not. The canteen manager estimates that of those boarders who eat breakfast one morning, 5% of them will not eat breakfast the next morning and of those boarders who do not eat breakfast one morning, 55% of them eat breakfast the following morning.

(a) If 55 boarders eat breakfast on Monday, how many boarders should the canteen manager expect to eat breakfast on Wednesday? (3 marks)

$T = \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} \qquad P = \begin{bmatrix} 55 \\ 53 \end{bmatrix}$

$T^2 P = \begin{bmatrix} 91.96 \\ 16.04 \end{bmatrix}$

Expect 92 students for breakfast

(b) In the long term, what proportion of boarders can be expected to eat breakfast? (2 marks)

$\dfrac{0.55}{0.55 + 0.05} = \dfrac{11}{12}$

or

$T^n P \to \begin{bmatrix} 99 \\ 9 \end{bmatrix}$ as n increases $\Rightarrow \dfrac{99}{99+9} = \dfrac{11}{12}$

Question 15 **(7 marks)**

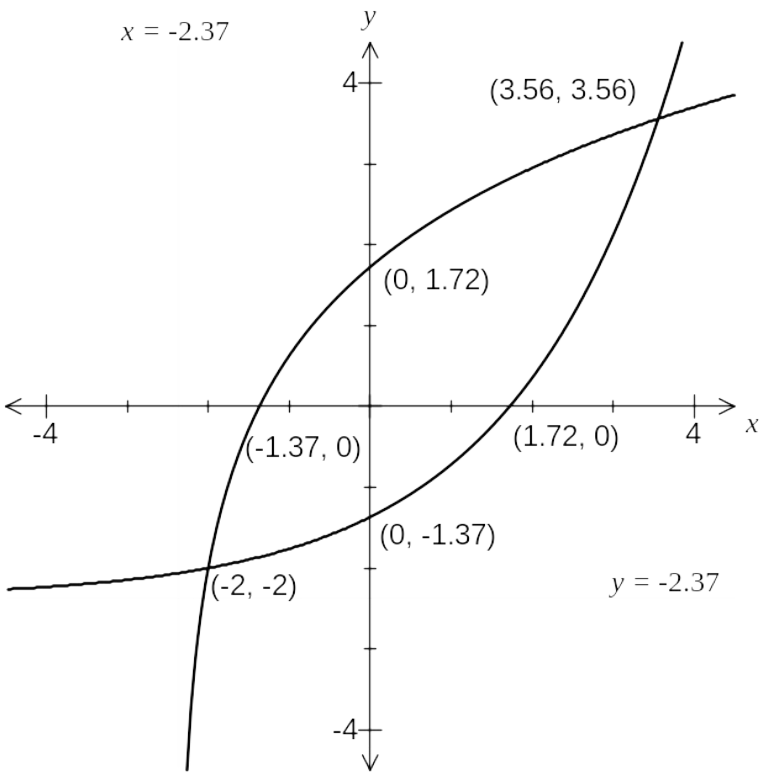
The graphs of the function $f(x) = 2\log_e(x+k)$, where k is a constant, and its inverse $f^{-1}(x)$, intersect where $x = -2$ and at one other point.

(a) Find the exact value of k. (2 marks)

$f(x)$ and inverse intersect along $y = x$
Solve $f(-2) = -2$

$-2 = 2\ln(-2 + k)$
$k = \dfrac{1}{e} + 2$

(b) Sketch the graphs of $f(x)$ and its inverse $f^{-1}(x)$ on the axes below, giving equations of any asymptotes and showing the coordinates of all points of intersection and axes-intercepts correct to 2 decimal places. (5 marks)



Question 16 **(5 marks)**

(a) The matrix equation $AX = B$ could be used to solve the following system of equations.

$$2a + 3b = c - 5$$
$$b - 2c - 4a = 1$$
$$5 = c + b$$

Write down suitable matrices for A, X and B. (DO NOT SOLVE YOUR EQUATIONS)

(2 marks)

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -4 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$B = \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix}$$

(b) If $PQ = 3P + I$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 7 & -9 \\ 5 & -8 \end{bmatrix}$, find matrix P. (3 marks)

$$PQ - 3P = I$$
$$P(Q - 3I) = I$$
$$P = (Q - 3I)^{-1}$$
$$P = \begin{bmatrix} -11 & 9 \\ -5 & 4 \end{bmatrix}$$

Question 17 **(7 marks)**

The displacement $x(t)$ metres, of a small particle undergoing simple harmonic motion is given by $x(t) = A\cos\omega t + B\sin\omega t$, where A, B and ω are positive constants.

(a) Show that $x''(t) + \omega^2 x(t) = 0$. (2 marks)

$$x'(t) = \omega(-A\sin\omega t + B\cos\omega t)$$
$$x''(t) = -\omega^2(A\cos\omega t + B\sin\omega t)$$
$$= -\omega^2 x(t)$$
$$x''(t) + \omega^2 x(t) = 0$$

The body passes through the origin (where $x(t) = 0$) five times per second, $x(0) = 1.5$ m and $x'(0) = 7.5$ ms^{-1}.

(b) Find the exact values of the constants A, B and ω. (3 marks)

Passing origin 5 times per sec means frequency is 2.5 cycles/sec.
$\omega = 2\pi \times 2.5 = 5\pi$

$1.5 = A\cos 0 + B\sin 0$
$A = 1.5$

$7.5 = 5\pi(-1.5\sin 0 + B\cos 0)$
$B = \dfrac{3}{2\pi}$

(c) What is the amplitude of motion, correct to the nearest millimetre? (2 marks)

$$\sqrt{1.5^2 + \left(\frac{3}{2\pi}\right)^2} = 1.57416$$
≈ 1.574 m to nearest mm.

Question 18 **(8 marks)**

Every odd integer I can be written as $I = 10n + c$, where $c = 1, 3, 5, 7, 9$.

(a) Show how the integers 237, 3 and -35 can be written this way. (1 mark)

$$237 = 10 \times 23 + 7$$
$$3 = 10 \times 0 + 3$$
$$-35 = 10 \times (-4) + 5$$

(b) By considering the five different cases for c, or otherwise, prove that the square of every odd integer ends in 1, 5 or 9. (7 marks)

$c = 1$

$I = 10n + 1 \Rightarrow I^2 = 100n^2 + 20n + 1 = 10(10n^2 + 2n) + 1$

This number must end in 1, as for any number in form $10p + q$, q is the units digit.

$c = 3$

$I = 10n + 3 \Rightarrow I^2 = 100n^2 + 60n + 9 = 10(10n^2 + 6n) + 9$, which ends in 9

$c = 5$

$I = 10n + 5 \Rightarrow I^2 = 100n^2 + 100n + 25 = 10(10n^2 + 10n + 2) + 5$, which ends in 5

$c = 7$

$I = 10n + 7 \Rightarrow I^2 = 100n^2 + 140n + 49 = 10(10n^2 + 14n + 4) + 9$, which ends in 9

$c = 9$

$I = 10n + 9 \Rightarrow I^2 = 100n^2 + 180n + 81 = 10(10n^2 + 18n + 8) + 1$, which ends in 1

Hence in all possible cases, the square ends in 1, 5 or 9.

Question 19 **(9 marks)**

Relative to itself, an anti-ballistic missile (ABM) launch site detects a ballistic missile at $11\mathbf{i} - 18\mathbf{j} + 10\mathbf{k}$ km headed at constant velocity for a target at $35\mathbf{i} + 14\mathbf{j} + \mathbf{k}$ km. The ballistic missile is expected to hit the target in 50 seconds.

(a) How close does the ballistic missile come to the ABM launch site? (5 marks)

Let missile M be at A and target T at B initially. Then

$$\overrightarrow{AB} = \begin{bmatrix} 35 - 11 \\ 14 - (-18) \\ 1 - 10 \end{bmatrix} = \begin{bmatrix} 24 \\ 32 \\ -9 \end{bmatrix} \text{ km and } V_m = \frac{1}{50}\begin{bmatrix} 24 \\ 32 \\ -9 \end{bmatrix} = \begin{bmatrix} 0.48 \\ 0.64 \\ -0.18 \end{bmatrix} \text{ km/s}$$

Closest when $(\overrightarrow{OA} + t \times V_m) \bullet V_m = 0$

$$\begin{bmatrix} 11 + 0.48t \\ -18 + 0.64t \\ 10 - 0.18t \end{bmatrix} \bullet \begin{bmatrix} 0.48 \\ 0.64 \\ -0.18 \end{bmatrix} = 0 \text{ when } t = 11.957 \text{ seconds.}$$

$$\overrightarrow{OM} = \begin{bmatrix} 11 + 0.48t \\ -18 + 0.64t \\ 10 - 0.18t \end{bmatrix}_{t=11.957} = \begin{bmatrix} 16.739 \\ -10.347 \\ 7.848 \end{bmatrix} \text{ and } \left|\overrightarrow{OM}\right| = 21.186 \text{ km}$$

(b) The launch site plans to fire an ABM to hit the ballistic missile. The hit is timed to take place at the instant the ballistic missile comes within 8km of the target. Assuming the ABM instantly achieves a constant velocity of 1150 ms^{-1} as it is launched, how long from the time of detection should the defence site fire it? (4 marks)

$\left|\overrightarrow{AB}\right| = 41$ km. Collide at C, 8 km from B and 33 km from A after $\frac{33}{41} \times 50 \approx 40.244$ seconds.

$$\overrightarrow{OC} = \begin{bmatrix} 11 \\ -18 \\ 10 \end{bmatrix} + \frac{33}{41}\begin{bmatrix} 24 \\ 32 \\ -9 \end{bmatrix} = \begin{bmatrix} 30.317 \\ 7.756 \\ 2.756 \end{bmatrix} \text{ and } \left|\overrightarrow{OC}\right| = 31.415 \text{ km.}$$

Time for ABM to hit M $= 31.415 \div 1.15 = 27.317$ seconds.

Must launch after $40.244 - 27.317 = 12.93$ seconds.

End of questions

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