

PERTH MODERN SCHOOL

UNIT 3CD MAS – 2015

TEST 1: SOLUTIONS

POLAR COORDINATES, COMPLEX NUMBERS & VECTORS

NAME: _____

DATE: Feb.

Total: 45 marks

Time: 50 min.

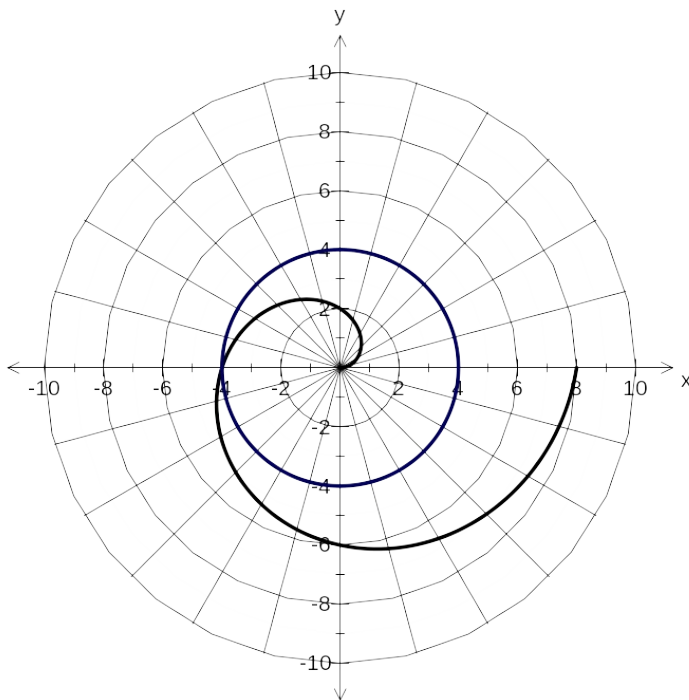
Question 1.

(5 marks)

- (i) Point P has polar coordinates $(4, 17^\circ)$ and it lies on the line $y = -x + 5$. Point Q also lies on the line and is 12cm away from P.

Find the polar coordinates of Q $[r, \theta]$ where $90^\circ < \theta < 180^\circ$.

(3 marks)



Solution
$OQ^2 = 12^2 + 4^2 - 2(12)(4)\cos 62^\circ$ $OQ = 10.72$ $12^2 = 4^2 + 10.72^2 - 2(4)(10.72)\cos \angle POQ$ $\angle POQ = 98.77^\circ$ Polar Coordinates of Q = $[10.72, 115.77^\circ]$
Specific behaviours
✓ angle of 62° , ✓ ✓ $[10.72, 115.77^\circ]$

- (ii) Sketch the graph of $r = \frac{4}{\pi} \theta$ on the axes above and hence state where it intersects $r = 4$

(2 marks)

Question 2

(9 marks)

On a 3D computer game, Chris, a keen cyclist leaves from *position* $(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ metres is travelling at $(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$ m/s while his mate Dave leaves from position $(a\mathbf{i} + \mathbf{j} + b\mathbf{k})$ metres running at $(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ m/s.

- (i) Although they do not collide, their paths do intersect at the point with coordinates $(a, 1, b)$.

Determine the values of a and b .

(4 marks)

Solution
$\underline{r}_C = (1 + 2\lambda)\underline{i} + (-1 + 3\lambda)\underline{j} + (2 - 6\lambda)\underline{k}$ <p>As paths meet at $(a, 1, b)$, then $(a, 1, b)$ lies on \underline{r}_C i.e. $(1 + 2\lambda) = a$, $(-1 + 3\lambda) = 1$, $(2 - 6\lambda) = b$ ✓</p> <p>Solving these equations simultaneously, $\lambda = \frac{2}{3}$, $a = \frac{7}{3}$, $b = -2$ ✓✓✓</p>
Specific behaviours
As allocated

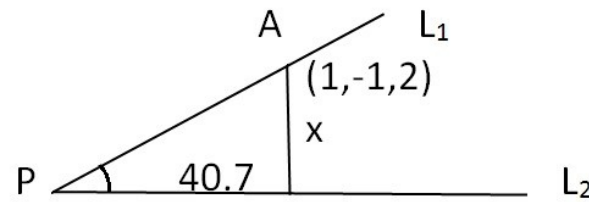
- (ii) Find the acute angle between these two paths.

(2 marks)

Solution
<p>Let angle be θ</p> $(2, 3, -6) \cdot (1, -1, 2) = \sqrt{4 + 9 + 36} \cdot \sqrt{1 + 1 + 4} \cdot \cos \theta$ <p>$\theta = 139.3^\circ$ Hence acute angle is 40.7° ✓</p>
Specific behaviours
As allocated

- (iii) Hence, determine the perpendicular distance from the point (1,-1,2) to (3 marks)

$$r = \begin{pmatrix} \frac{7}{3} \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Solution	
 <p> $\begin{pmatrix} \frac{7}{3}, 1, -2 \end{pmatrix}$ $PA = \begin{pmatrix} -\frac{7}{3}, -1, 2 \end{pmatrix} + (1, -1, 2)$ $= \begin{pmatrix} -\frac{4}{3}, -2, 4 \end{pmatrix} \checkmark$ $\therefore \vec{PA} = \frac{14}{3} \checkmark$ Perpendicular distance = $\frac{14}{3} \sin 40.7^\circ = 3.04 \text{ m} \checkmark$ </p>	
Specific behaviours	
As allocated	

Question 3**(5 marks)**

- (a) The vectors $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + a\mathbf{k}$ are perpendicular. Determine the value of a .
(1marks)

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ a \end{bmatrix} = 0 \Rightarrow 3 - 2 + 2a = 0 \Rightarrow a = -\frac{1}{2}$$

- (b) Determine whether the two lines
 $\mathbf{r} = 8\mathbf{i} - \mathbf{j} - 8\mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{k})$ and $\mathbf{r} = \mathbf{j} - 3\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ intersect.
If they do intersect, state the position vector of their point of intersection.

If they do not intersect, justify your answer.

(4 marks)

$$\begin{aligned} \mathbf{i}: 8 + 2\lambda &= \mu \\ \mathbf{j}: -1 &= 1 - \mu \Rightarrow \mu = 2, \lambda = -3 \\ \mathbf{k}: -8 - 3(-3) &= -3 + 2(2) \Rightarrow 1 = 1 \Rightarrow \text{intersect} \\ \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \text{intersect at } 2\mathbf{i} - \mathbf{j} + \mathbf{k} \end{aligned}$$

Question 4

(7 marks)

(a) If $z = 3 - 4i$, determine the reciprocal, $\frac{1}{z}$

(2 marks)

$$\begin{aligned}\frac{1}{z} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{3 + 4i}{3^2 + 4^2} \\ &= \frac{3}{25} + \frac{4i}{25}\end{aligned}$$

(b) Let the non-zero complex number $z = a + bi$. Show that $\frac{1}{a + bi} = \frac{\bar{z}}{|z|^2}$

(3 marks)

$$\begin{aligned}LHS &= \frac{1}{a + bi} \\ &= \frac{1}{a + bi} \times \frac{a - bi}{a - bi} \\ &= \frac{a - bi}{a^2 + b^2} \\ &= \frac{\bar{z}}{|z|^2} \\ &= RHS\end{aligned}$$

(c) Describe the geometrical relationship between any non-zero complex number and its reciprocal.

(2marks)

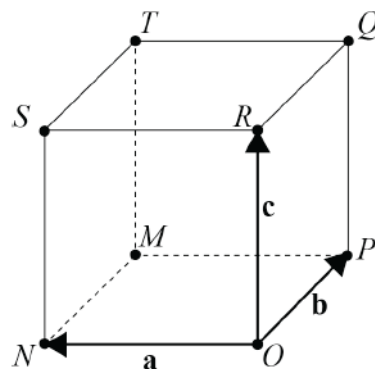
The reciprocal z^{-1} is the conjugate of z but multiplied by scale factor of $\frac{1}{|z|}$.

So the reciprocal z^{-1} will be the reflection of z in the real axis and of length $\frac{1}{|z|}$

Question 5

(5 marks)

Let $MNOPQRST$ be a rectangular prism with sides \overrightarrow{ON} , \overrightarrow{OP} and \overrightarrow{OR} denoted by vectors \mathbf{a} , \mathbf{b} and \mathbf{c} respectively, as shown in the diagram below.



Suppose that A is the midpoint of \overrightarrow{MN} , B is the midpoint of \overrightarrow{MT} , C is the midpoint of \overrightarrow{QR} and D is the midpoint of \overrightarrow{OR} .

- (a) Express \overrightarrow{OA} , \overrightarrow{OB} , \overrightarrow{OC} and \overrightarrow{OD} in terms of \mathbf{a} , \mathbf{b} and \mathbf{c} . (2 marks)

Solution	
$\overrightarrow{OA} = \mathbf{a} + \frac{\mathbf{b}}{2}$	
$\overrightarrow{OB} = \mathbf{a} + \mathbf{b} + \frac{\mathbf{c}}{2}$	
$\overrightarrow{OC} = \mathbf{c} + \frac{\mathbf{b}}{2}$	
$\overrightarrow{OD} = \frac{\mathbf{c}}{2}$	
Specific behaviours	
✓ correctly determines \overrightarrow{OA} and \overrightarrow{OC} from the diagram	
✓ correctly determines \overrightarrow{OB} and \overrightarrow{OD} from the diagram	

(b) Prove that the quadrilateral $ABCD$ is a parallelogram.

(3 marks)

Solution	
$\begin{aligned}\overrightarrow{BC} &= \mathbf{c} + \frac{\mathbf{b}}{2} - \mathbf{a} - \mathbf{b} - \frac{\mathbf{c}}{2} \\ &= \frac{\mathbf{c}}{2} - \frac{\mathbf{b}}{2} - \mathbf{a} \\ \overrightarrow{AD} &= \frac{\mathbf{c}}{2} - \mathbf{a} - \frac{\mathbf{b}}{2} \\ &= \overrightarrow{BC} \\ \therefore \overrightarrow{BC} \text{ is parallel to } \overrightarrow{AD}\end{aligned}$	
$\begin{aligned}\overrightarrow{AB} &= \mathbf{a} + \mathbf{b} + \frac{\mathbf{c}}{2} - \mathbf{a} - \frac{\mathbf{b}}{2} \\ &= \frac{\mathbf{b}}{2} + \frac{\mathbf{c}}{2} \\ \overrightarrow{DC} &= \mathbf{c} + \frac{\mathbf{b}}{2} - \frac{\mathbf{c}}{2} \\ &= \frac{\mathbf{b}}{2} + \frac{\mathbf{c}}{2} \\ &= \overrightarrow{AB} \\ \therefore \overrightarrow{DC} \text{ is parallel to } \overrightarrow{AB}\end{aligned}$	
i.e. Opposite sides of the quadrilateral $ABCD$ are parallel	
$\therefore ABCD$ is a parallelogram	
Specific behaviours	
<ul style="list-style-type: none">✓ correctly determines \overrightarrow{BC} and \overrightarrow{AD} in terms of \mathbf{a}, \mathbf{b} and \mathbf{c}✓ correctly determines \overrightarrow{AB} and \overrightarrow{DC} in terms of \mathbf{a}, \mathbf{b} and \mathbf{c}✓ states that two pairs of sides are parallel to deduce the result	

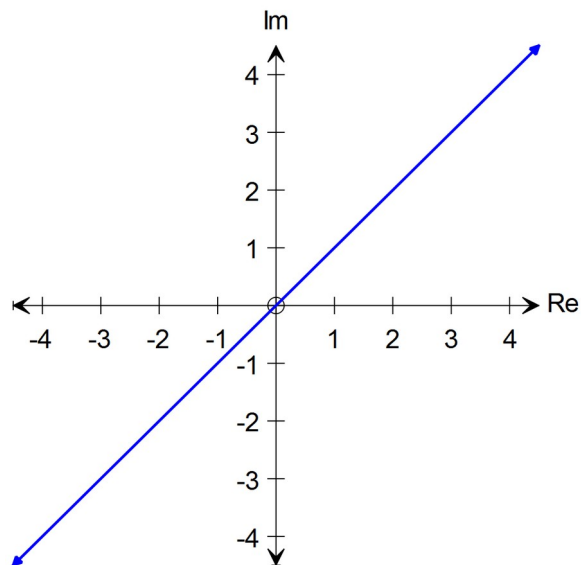
Question 6**(5 marks)****(5 marks)**

- (a) Change the complex equation $|Z - i| = |Z - 1|$ into its Cartesian equivalent. (3 marks)

$$\begin{aligned} |Z - i| &= |Z - 1| \\ |x + yi - i| &= |x + yi - 1| \\ |x + i(y - 1)| &= |(x - 1) + yi| \\ x^2 + (y - 1)^2 &= (x - 1)^2 + y^2 \\ x^2 + y^2 - 2y + 1 &= x^2 - 2x + 1 + y^2 \\ 2x - 2y &= 0 \end{aligned}$$

$$\therefore \quad \mathbf{x - y = 0}$$

- (b) Hence identify, the locus of all points Z satisfying the equation in (a). (2 marks)



Question 7

An equilateral triangle has vertices A , B and C , where A is the point $\sqrt{3} - i$ in the Argand plane.

The circumcircle is drawn that passes through vertices A , B and C and has a centre inside the triangle, called the circumcentre.

The circumcentre of the triangle is located at the origin.

Find the complex numbers z_1 and z_2 corresponding to the vertices B and C , expressing your answer in exact Cartesian form.

Solution

The centre is at the origin so the other two points can be found by rotating the point A 120° .

A rotation of 120° is equivalent to multiplying by $z = 1 \operatorname{cis}\left(\frac{2\pi}{3}\right)$

$$z_A = 2 \operatorname{cis}\left(\frac{-\pi}{6}\right), \text{ so}$$

$$z_B = 1 \operatorname{cis}\left(\frac{2\pi}{3}\right) \times 2 \operatorname{cis}\left(\frac{-\pi}{6}\right) = 2 \operatorname{cis}\left(\frac{3\pi}{6}\right) = 2 \operatorname{cis}\left(\frac{\pi}{2}\right)$$

$$z_C = 1 \operatorname{cis}\left(\frac{2\pi}{3}\right) \times 2 \operatorname{cis}\left(\frac{\pi}{2}\right) = 2 \operatorname{cis}\left(\frac{7\pi}{6}\right)$$

In Cartesian form:

$$z_B = 2 \left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right) = 2i$$

$$z_C = 2 \left(\cos\left(\frac{7\pi}{6}\right) + i \sin\left(\frac{7\pi}{6}\right) \right) = -\sqrt{3} - i$$

Specific behaviours

- ✓ uses separation between vectors as 120°
- ✓ uses a magnitude of 2 for all the vectors
- ✓ deduces the correct position for vertex B
- ✓ deduces the correct position for vertex C
- ✓ states the correct Cartesian coordinates for both B and C

Question 11

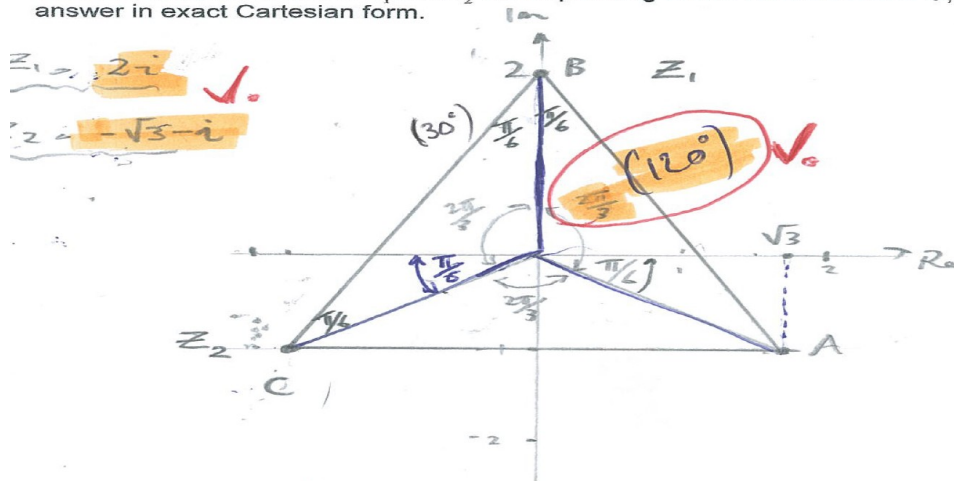
(5 marks)

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The circumcentre of the triangle is located at the origin.

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$$\begin{aligned}
 z_A &= \sqrt{3} - i \\
 |z_A| &= \sqrt{(\sqrt{3})^2 + (-1)^2} \\
 &= 2 \quad \checkmark \\
 \arg &= \tan^{-1} \frac{-1}{\sqrt{3}} \\
 &= 30^\circ = \frac{\pi}{6}
 \end{aligned}$$

Positions

Vertex B : $2 \operatorname{cis} \frac{\pi}{2}$ ✓

Vertex C : $2 \operatorname{cis} \left(\pi + \frac{\pi}{6} \right)$ ✓
 $2 \operatorname{cis} \left(\frac{7\pi}{6} \right) = 2 \operatorname{cis} \left(-\frac{5\pi}{6} \right)$

Question 8**(4 marks)**

Find two numbers which have a product of 2 and a sum of 2.

Let the numbers be x and y

$$xy = 2 \quad (1)$$

and $x + y = 2 \quad (2)$

rearrange (2) $y = 2 - x$ and subst. into (1)

$$x(2 - x) = 2$$

$$2x - 2x^2 = 2$$

$$x^2 - 2x + 2 = 0$$

(Using Classpad)

$$x = 1 \pm i$$

$$y = 1 \pm i$$

\therefore the two numbers are $1 + i$ and $1 - i$