PERTH MODERN SCHOOL
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INDEPENDENT PUBLIC SCHOOL

Semester One Examination, 2021

Question/Answer booklet

MATHEMATICS SPECIALIST
UNIT 3
Section One:
Calculator-free

Your Name

Your Teacher's Name

Time allowed for this section
Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Mark	Max	Question	Mark	Max
1			5		
2			6		
3			7		
4			8		

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	50	34
Section Two: Calculator-assumed	14	14	100	96	66
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

See next page

Section One: Calculator-free **(50 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 50 minutes.

Question 1 **(6 marks)**

Consider the plane $3x - 2y + 5z = 10$ which contains point A $(1, -1, 1)$

(a) Write a vector equation for this plane. (3 marks)

Solution
$r.\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 10$ $r.\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = 10$
Specific behaviours
✓ identifies a normal vector ✓ shows dot product to determine scalar constant ✓ uses correct format for vector equation

Consider the line $r = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$

(b) Determine the coordinates of where the line above meets the plane. (3 marks)

Solution

$$\begin{pmatrix} 2+5\lambda \\ -3+\lambda \\ 1-2\lambda \end{pmatrix} . \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = 10$$

$$6+15\lambda+6-2\lambda+5-10\lambda=10$$

$$3\lambda=-7$$

$$\lambda=\frac{-7}{3}$$

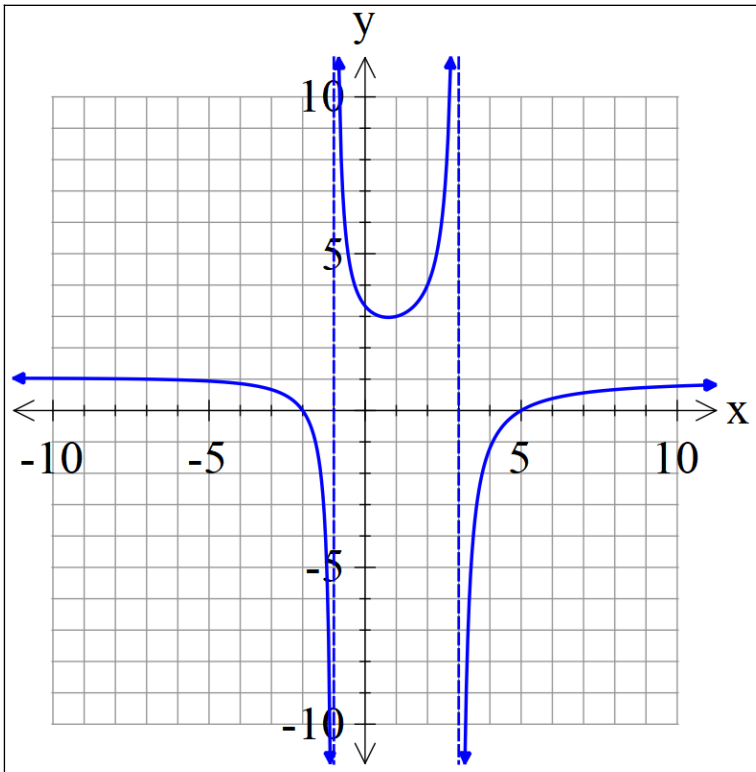
$$r=\begin{pmatrix} \frac{-29}{3} \\ \frac{-16}{3} \\ \frac{17}{3} \end{pmatrix}$$

Specific behaviours
✓ sets up dot product equation ✓ solves for parameter ✓ subs parameter back into equation (no need to simplify)

Question 2 **(6 marks)**

Sketch the graph $y=f(x)$ where $f(x)=\frac{(x+2)(x-5)}{(x+1)(x-3)}$. Clearly show the major features of the graph.

Solution



Specific behaviours
✓ shows both vertical asymptotes ✓ shows both x intercepts(exact) ✓ shows approx. y intercept ✓ shows horizontal asymptote ✓ two of the three parts have correct shape ✓ all parts have correct shape

Question 3 (6 marks)

Consider the plane Ψ that contains the following three points $A(1,4,-1), B(1,1,2) \& C(3,-1,2)$. Using vector methods, determine the distance of point $D(6,-7,1)$ from the plane Ψ. Show all working and reasoning.

Solution

See next page

- $AB = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}$
- $AC = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$
- $\overrightarrow{AB} \times AC = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
- $AD = \begin{pmatrix} 6 \\ -7 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -11 \\ 2 \end{pmatrix}$

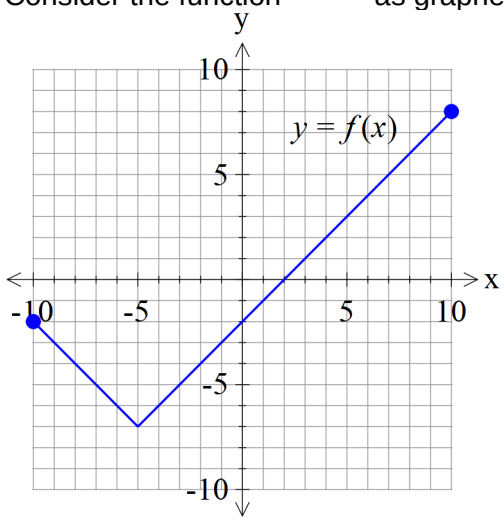
$$dist = \begin{pmatrix} 5 \\ -11 \\ 2 \end{pmatrix} . \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \left| \frac{5 - 11 + 2}{\sqrt{3}} \right| = \frac{4\sqrt{3}}{3}$$

Specific behaviours
✓ determines two vectors in plane ✓ uses cross product to find a normal ✓ Determines a vector from D to any point on plane OR uses a line through D parallel to normal ✓ uses dot product or finds intersection of line & plane ✓ uses unit normal or solves for parameter of line ✓determines exact distance(accept irrational denominator)

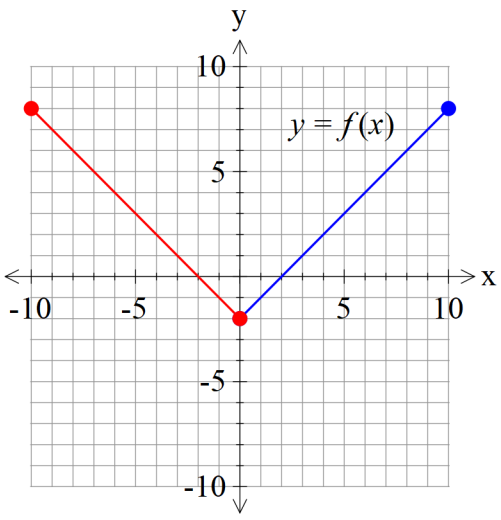
See next page

Question 4 **(5 marks)**

Consider the function $f(x)$ as graphed below.



a) Graph $y = f(|x|)$ on the axes below. (2 marks)

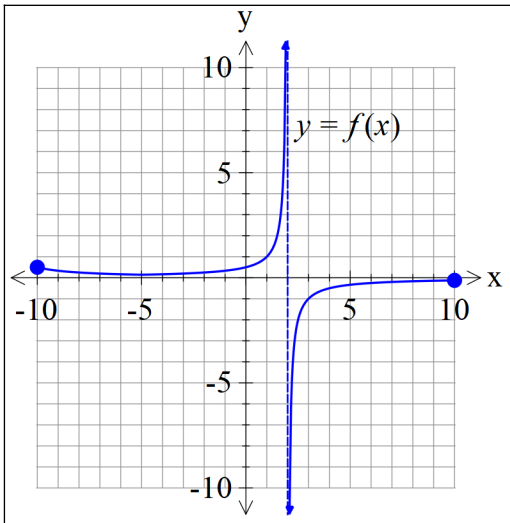
Solution


Specific behaviours
✓ reflects one side ✓ correct graph

b) Graph $y = \frac{-1}{f(x)}$ on the axes below. (3 marks)

Solution

See next page



Specific behaviours
✓ shows asymptote ✓ approx. y intercept and min turning pt at x=-5 (do not accept max) ✓ correct shape on both sides of asymptote

Question 5 **(7 marks)**

Consider the function $f(x) = 4x^2 - 8x + 2$ with domain $x \leq 1$

(a) Determine $f^{-1}(x)$ and its domain. (4 marks)

Solution
$f(x) = 4x^2 - 8x + 2$ $x = 1, y = -2$ $x = 4y^2 - 8y + 2$ $4y^2 - 8y + 2 - x = 0$ $y = \frac{8 \pm \sqrt{64 - 4(4)(2 - x)}}{8} = \frac{8 \pm \sqrt{32 + 16x}}{8} = \frac{8 \pm 4\sqrt{2 + x}}{8} = 1 \pm \frac{1}{2}\sqrt{2 + x}$ $y \leq 1$ $f^{-1}(x) = 1 - \frac{1}{2}\sqrt{2 + x}$ $x \geq -2$

Specific behaviours
✓ swaps x & y OR solves for x as subject ✓ uses a quadratic formula expression OR completes the square ✓ uses minus and states domain of inverse (no need to simplify) ✓ states rule for inverse

See next page

(b) Consider $g(x) = x^2 + bx + c$ with $x \leq \frac{-b}{2}$ and $b \& c$ real constants. Given that $g(x)$ has an inverse which intersects graphically with $g^{-1}(x)$ at one point only, determine a possible exact solution for x in terms of $b \& c$ and an equation that $b \& c$ must satisfy.

(3 marks)

Solution
$g(x) = x^2 + bx + c = x$ $x^2 + bx - x + c = 0$ $x^2 + (b-1)x + c = 0$ $x = \frac{-(b-1)}{2} and\ (b-1)^2 - 4c = 0$
Specific behaviours
✓ equates function to x ✓ uses quadratic formula to give one answer only ✓ uses discriminant equaling zero

Question 6 **(7 marks)**

Consider the following system of linear equations.

$$5x + y + 2z = 19$$
$$x - y + z = 8$$
$$2x - 3y + 4z = 27$$

a) Solve for $x, y \& z$. (3 marks)

Solution
$5x + y + 2z = 19$ $x - y + z = 8$ $2x - 3y + 4z = 27$ $\begin{bmatrix} 1 & -1 & 1 & 8 \\ 2 & -3 & 4 & 27 \\ 5 & 1 & 2 & 19 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -2 & -11 \\ 0 & -6 & 3 & 21 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -2 & -11 \\ 0 & 0 & -9 & -45 \end{bmatrix}$ $-9z = -45$ $z = 5$ $y - 2z = -11$ $y = -1$ $x - y + z = 8$ $x = 2$ $(2, -1, 5)$
Specific behaviours
✓ eliminates one variable for two equations ✓ eliminates two variables for one equation ✓ solves for all three variables

Q6 cont-

See next page

b) If we modify the equations to the following with $p \& q$ being constants, solve for the following values of $p \& q$ such that there are:

i) no solutions

ii) infinite solutions (Give a geometrical interpretation of this situation) (4 marks)

$$5x + y + pz = 19$$
$$x - y + z = 8$$
$$2x - 3y + 4z = q$$

Solution
$\begin{bmatrix} 1 & -1 & 1 & 8 \\ 2 & -3 & 4 & q \\ 5 & 1 & p & 19 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -2 & 16-q \\ 0 & -6 & 5-p & 21 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -2 & 16-q \\ 0 & 0 & -7-p & 117-6q \end{bmatrix}$ No solns p=-7 and $q \neq \frac{117}{6}$ Infinite p=-7 and $q = \frac{117}{6}$ Line of common points that lie on all 3 planes
Specific behaviours
✓ eliminates two variables ✓ states all values for no solns ✓ states values for infinite solns ✓with geometric explanation

See next page

Question 7 **(10 marks)**

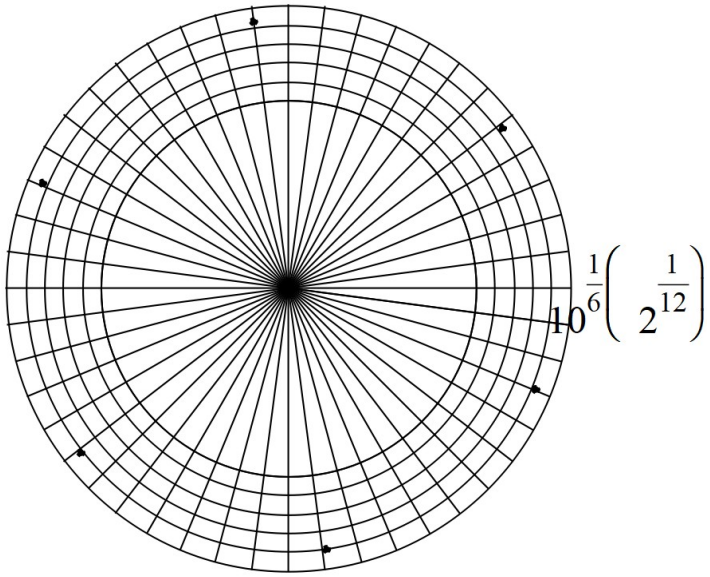
a) Solve for all solutions to the following $z^6 = -5(2+2i)$ in the form $rcis\theta$ with $-\pi < \theta \leq \pi$. (4 marks)

Solution
$z^6 = -5(2+2i) = 10\sqrt{2}cis\left(\frac{-3\pi}{4} + 2n\pi\right) \quad , n = 0, \pm1, \pm2, \pm3...$ $z = 10^{\frac{1}{6}} 2^{\frac{1}{12}} cis\left(\frac{-3\pi}{24} + 2n\pi\frac{1}{6}\right) \quad , n = 0, \pm1, \pm2, \pm3...$ $z = 10^{\frac{1}{6}} 2^{\frac{1}{12}} cis\left(\frac{-3\pi}{24} + \frac{8n\pi}{24}\right) \quad , n = 0, \pm1, \pm2, \pm3...$ $z_1 = 10^{\frac{1}{6}} 2^{\frac{1}{12}} cis\left(\frac{5\pi}{24}\right)$ $z_2 = 10^{\frac{1}{6}} 2^{\frac{1}{12}} cis\left(\frac{13\pi}{24}\right)$ $z_3 = 10^{\frac{1}{6}} 2^{\frac{1}{12}} cis\left(\frac{-3\pi}{24}\right)$ $z_4 = 10^{\frac{1}{6}} 2^{\frac{1}{12}} cis\left(\frac{21\pi}{24}\right)$ $z_5 = 10^{\frac{1}{6}} 2^{\frac{1}{12}} cis\left(\frac{-11\pi}{24}\right)$ $z_6 = 10^{\frac{1}{6}} 2^{\frac{1}{12}} cis\left(\frac{-19\pi}{24}\right)$
Specific behaviours
✓ determines argument of right hand side (third quadrant) ✓ uses De Moivre's Theorem with arguments in increments of $\pi/3$ ✓ six roots with correct modulus ✓ six roots, all with correct principal arguments

b) Plot the above roots on the diagram below, labelling the axes. (3 marks)

Solution

End of questions

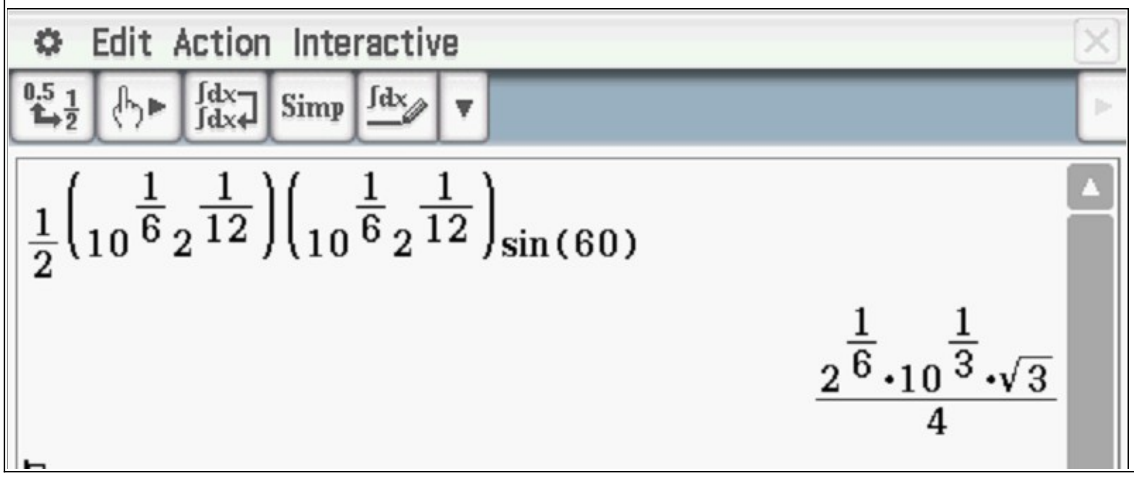


Specific behaviours

✓ one point at correct position
✓ scale indicated
✓ six points equally spaced

c) If these points are joined, forming a polygon, determine the exact area of this polygon.
(3 marks)

Solution



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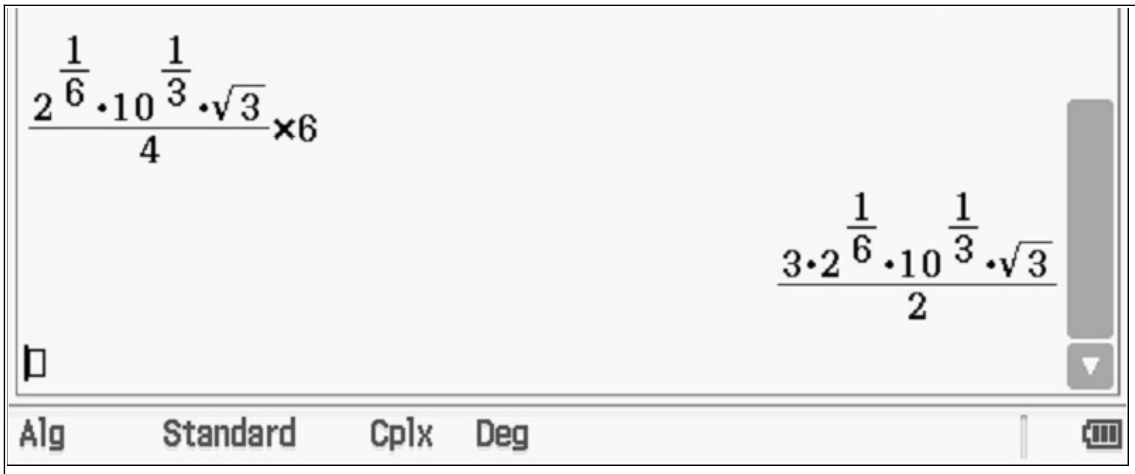


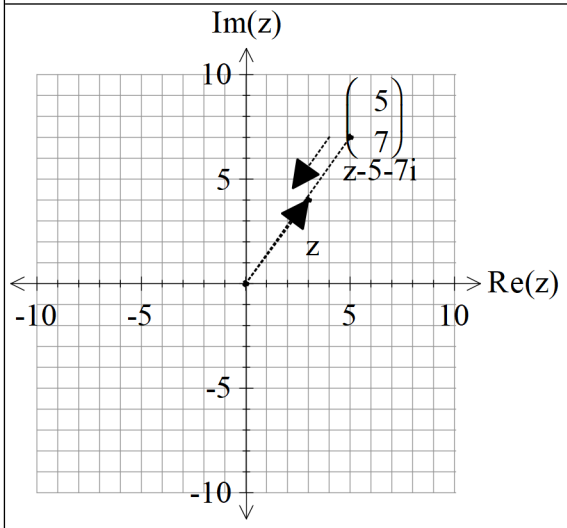
Specific behaviours

✓ uses equilateral triangles with side length equaled to modulus of roots
✓ determines area of one triangle
✓ states total area as an exact expression. (no need to simplify)

End of questions

Question 8 **(3 marks)**

Sketch the locus of points that satisfy $Arg(z-5-7i)+\pi=Arg(z)$ on the complex plane below and explain your reasoning.

Solution

Specific behaviours
✓ uses a line segment ✓ correct endpoints ✓ indicates the two arguments on diagram showing a difference of pi radians

Additional working space

Question number: _____

Additional working space

Question number: _____

Additional working space

Question number: _____

Acknowledgements