Fixed Effects Interacting Across Two Categories

Liu, Minsheng

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Introduction

Here we discuss fixed effects interacting across two categories.

For instance, consider Alice, Bob, and Cathy work for Macrohard and Redhole. Their performance can be linearly related to the multiplication of some known factor of each individual, such as age, and some unknown factor of each company.

Some notation. We use (A, F) to denote some factor F of some category A. We use $C = ((A, F), (B, G))$ to denote a fixed effect interacting across two categories A and B through F and G, where F is known and G is to estimate. Thus, a model concerned here looks like

$$Y = X\beta + \langle F, G \rangle + \epsilon\,,$$

where $\langle \cdot, \cdot \rangle$ is defined as follows. For each observation (row) i, with $m \in A$ and $n \in B$ being the row's corresponding individual in those two categories, we have

$$\langle F, G \rangle_i = F_m G_n\,.$$

Clearly, given any dataset ϕ, we can represent $\langle F, G \rangle$ by matrix multiplication, $F_\phi G_\phi$. As a concrete example, consider the following fabricated data:

Individual	Age
Alice	21
Bob	24
Cathy	27
Company	Factor
Macrohard	0.3
Redhole	0.4

If our dataset is in the order of AM, BM, CM, AR, BR, CR, we have

$$F_\phi = \begin{bmatrix} 21 & 0 \\ 24 & 0 \\ 27 & 0 \\ 0 & 21 \\ 0 & 24 \\ 0 & 27 \end{bmatrix}$$

and

$$G_\phi = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}\,.$$

Solution

We solve general models of this kind by applying the Frisch–Waugh–Lovell theorem. In other words, we project everything but those fixed effects interacting across two categories onto the orthogonal complement of the projection matrix $P_F = F(F^\top F)^{-1}F^\top$, or equivalently, $1 - P$.

For multiple fixed effects of this kind, instead of MAP, we choose to apply FWL in iterations. For instance, consider

$$Y = X\beta + F_1 G_1 + F_2 G_2 + F_3 G_3 + \epsilon\,.$$

We have

$$\begin{aligned} M_3 Y &= M_3 X\beta + (M_3 F_1)G_1 + (M_3 F_2)G_2 + \epsilon \\ M_2 M_3 Y &= M_2 M_3 X\beta + (M_2 M_3 F_1)G_1 + \epsilon \\ M_1 M_2 M_3 Y &= M_1 M_2 M_3 X\beta + \epsilon\,, \end{aligned}$$

where

$$\begin{aligned} M_3 &= 1 - P_{F_3} \\ M_2 &= 1 - P_{M_3 F_2} \\ M_1 &= 1 - P_{M_2 M_3 F_1}\,. \end{aligned}$$

There are two benefits not to use MAP here. First, the original reason we use demean and MAP is to avoid solving a very large matrix. Since for fixed effects discussed here, the projection has to be solved explicitly anyway, FWL seems to be the way to go. Second, in this manner, estimation can be easy.

Estimation can be done as follows. Consider first the system

$$Y = X\beta + FG + \epsilon\,.$$

After β is estimated, we can subtract $X\beta$ on both sides to get

$$Y - X\beta = FG + \epsilon\,.$$

OLS can be used to estimate for G. Since $F(F^\top F)^{-1}F^\top$ has already been computed, the process does not involve too much computation overhead. The algorithm extends easily to systems with multiple fixed effects.

If our system further includes simple fixed effects, we would estimate them first, demeaning both X and fixed effects interacting across two categories. After that, everything is the same as before. The existing algorithm to estimate those simple fixed effects still works, but we need to remove the effect of demeaning not only X but also other fixed effects interacting across two categories.