Math 272A homework 4

Note: Please read the collaboration policy on the syllabus.

1. (2pt) Note that the space $H_0^1(\Omega)$ is defined as

$$H_0^1(\Omega) := \{u \in H^1(\Omega) : \exists u_n \in C_c^\infty(\Omega) \text{ and } \|u_n - u\|_{H^1(\Omega)} \to 0.\}$$

Use this definition to show that if u is a classical solution to Poisson's equation (with homogeneous Dirichlet boundary conditions), then it is a weak solution.

[*Hint*: Step 1: Use integration by parts to show that for all $v \in C_c^\infty(\Omega)$, we have $(\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}$.
Step 2: Use Step 1 to show that $(\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}$ for all $v \in H_0^1(\Omega)$. In this step, you may take $v_n \in C_c^\infty(\Omega)$ with $\|v - v_n\|_{H^1(\Omega)} \to 0$ and use the Cauchy-Schwartz inequality to reach the conclusion.]

2. (3pt) (*Poincaré inequality in 1D*) Poincaré inequality is the key ingredient to show the coercivity of the bilinear form corresponding to Poisson's equation. Poincaré inequality in 1D is stated as

$$\int_a^b |u(x)|^2 dx \le \frac{(b-a)^2}{2} \int_a^b |u'(t)|^2 dt,$$

for all $u \in H_0^1(\Omega)$ where $\Omega = (a, b)$. Use the definition of $H_0^1(\Omega)$ to prove this inequality.

[*Hint*: Again, we show this in two steps. Step 1 is to show the inequality for all $u \in C_c^\infty(\Omega)$ and Step 2 is to prove the inequality for all $u \in H_0^1(\Omega)$ through approximation of smooth functions. In Step 1, there are two ingredients you may use:
(1) The fundamental theorem of calculus gives $u(x) = u(a) + \int_a^x u'(t)dt = \int_a^x u'(t)dt$ if $u \in C_c^\infty(\Omega)$.
(2) The Cauchy-Schwartz inequality for integrals in 1D:

$$\int_a^b h(x)g(x)dx \le \left(\int_a^b |h(x)|^2 dx\right)^{1/2} \left(\int_a^b |g(x)|^2 dx\right)^{1/2}.$$

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3. (2pt) (*Discontinuous functions in Sobolev spaces $W^{1,p}$*) Recall the definition of $W^{1,p}(\Omega)$ for $1 \le p < \infty$. It is true that in one dimension, any function $u \in W^{1,p}(\Omega)$ has a continuous representative in the same equivalent class (i.e., by modifying u on a set of measure zero, we can get a continuous function). However, in higher dimensions, this result is no longer true. Show that in 2d, $u(\boldsymbol{x}) = \log(|\boldsymbol{x}|) \in W^{1,1}(\Omega)$ where $\boldsymbol{x} = (x_1, x_2)$ and $\Omega = \{x_1^2 + x_2^2 < 1\}$ is the unit disk.

[*Hint*: Compute $\frac{\partial u}{\partial x_i}$ $(i = 1, 2)$ first and show $\frac{\partial u}{\partial x_i} \in L^1(\Omega)$ using polar coordinates. You may need to use the inequality $|x_1| \le \sqrt{x_1^2 + x_2^2}$.]

4. (3pt) (*Programming problem*) Solve the 1d Poisson's equation on $\Omega = (0, 1)$ with homogeneous Dirichlet boundary condition using the finite element method.

Implement the method on your computer with a uniform grid with $h = 1/10, 1/20, 1/40, 1/80, 1/160$. Test your code using the exact solution

$$u(x) = x(x-1)\exp(x^2+1).$$

Let u_h denote the finite element solution corresponding to h. Investigate the order of convergence of your solver through a convergence study of the error using the L^2 norm:

$$||u - u_h||_{L^2} = \left[\int_0^1 |u(x) - u_h(x)|^2 dx\right]^{1/2}$$

and the H^1 seminorm (also called the energy norm):

$$|u - u_h|_{H^1} = \left[\int_0^1 |u'(x) - u_h'(x)|^2 dx\right]^{1/2}$$

and the

To evaluate the two integrals in the above (as well as $\int_0^1 f(x)v(x)dx$ for a given v) approximately, use the composite Gaussian quadrature rule on the mesh with two points per mesh cell (*Hint*: two point Gaussian quadrature rule is given as $\int_{-1}^1 v(x) \approx v(-1/\sqrt{3}) + v(1/\sqrt{3})$, and rescale it to each mesh cell $[x_j, x_{j+1}]$ gives the composite Gaussian quadrature rule).