MATH 272B: Problem Set 1

Numerical Partial Differential Equations II

Winter 2025

Problem 1: Analysis of Gauss-Seidel Method (50 points)

Consider the two-dimensional Poisson equation on the unit square $\Omega = [0, 1]^2$:

$$-\Delta u = f \quad \text{in } \Omega \; u \qquad = 0 \quad \text{on } \partial\Omega$$

Using a uniform grid with mesh size $h = \frac{1}{n+1}$ where n is the number of interior points in each direction: a) (15 points) For the discretized system using the 5-point Laplacian stencil, write the matrix A as $A = L + D + U$ where L is strictly lower triangular, D is diagonal, and U is strictly upper triangular. For n = 3, write out these matrices explicitly and explain the structure of the Gauss-Seidel iteration matrix $G = -(L + D)^{-1}U$

b) (15 points) Calculate the spectral radius $\rho(G)$.

c) (20 points) Using this result, derive the asymptotic convergence rate of the Gauss-Seidel method. Show that the number of iterations required for convergence grows as $O(h^{-2})$.

Problem 2: Implementation and Analysis of Iterative Methods (50 points)

Implement and compare the Jacobi and Gauss-Seidel methods for solving the Poisson equation:

$$-\Delta u = \sin(\pi x)\sin(\pi y) \quad \text{in } \Omega = [0, 1]^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

a) (15 points) Implement both the Jacobi and Gauss-Seidel methods in Python. Your code should: 1) Accept arbitrary grid size n, 2) Monitor the residual norm at each iteration, 3) Implement a suitable stopping criterion

b) (20 points) For grid sizes n = 16, 32, and 64:

- Compare the convergence rates of both methods

- Plot the logarithm of the residual norm versus iteration number
- Calculate and tabulate the observed convergence rates
- Compare with the theoretical predictions from Problem 1

c) (15 points) How does the convergence behavior change if you modify the right-hand side to:

$$f(x,y) = \begin{cases} 1 & \text{if } 0.4 \leq x, y \leq 0.6 \\ 0 & \text{otherwise} \end{cases}$$

Explain any differences you observe in the convergence behavior.

Submission Guidelines

1. Submit a PDF containing: - Complete solutions to Problem 1 - Analysis and results for Problem 2 - All theoretical derivations and proofs - Plots and tables of numerical results

2. Submit your code as separate files, including: - Clear documentation - Function headers explaining inputs/outputs - Comments explaining key steps - Example usage