MATH 272: Numerical PDE
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1. The multi-index notation for Taylor's theorem is given by:

$$f(x) = \sum_{|\alpha|\le n} \frac{D^\alpha f(c)}{\alpha!}(x-c)^\alpha + R_n(x),$$

where the first term on the right-hand side is the n-th degree Taylor polynomial of f at $c \in \mathbb{R}^d$. Find out the 2nd-degree Taylor's polynomial of $f(x,y) = xe^y + 1$ at $c = (0,1)$.

Solution. For the function $f(x,y) = xe^y + 1$, and the point, the 2nd-degree ($|\alpha| \le 2$) Taylor polynomial is:

$$\begin{aligned} f(x,y) &= f(0,1) + \partial_x f(0,1)(x-0)^1(y-1)^0 + \partial_y f(0,1)(x-0)^0(y-1)^1 \\ &\quad + \frac{1}{2}(\partial_{xx} f(0,1)(x-0)^2(y-1)^0 + \partial_{yy} f(0,1)(x-0)^2(y-1)^0) \\ &\quad + \partial_{xy} f(0,1)(x-0)^1(y-1)^1 + R_3(x,y) \\ &= 1 + e(x-0) + 0 + 0 + e(x-0)(y-1) \\ &= 1 + exy \end{aligned}$$

2. Second-order linear elliptic PDEs in two variables have the form:

$$au_{xx} + 2bu_{xy} + cu_{yy} + I(u_x, u_y, u, x, y) = 0$$

where $b^2 - ac < 0$ and $I(u_x, u_y, u, x, y)$ contains all the lower order terms. Define the new variables $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ with a non-singular Jacobian matrix $S = \frac{\partial(\xi,\eta)}{\partial(x,y)}$. Rewrite Equation (1) in the new variables (ξ, η) and show that the type of equation under the new variables is still elliptic.

More generally, linear elliptic PDEs in d-dimensions have the form:

$$-\sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + (\text{lower order terms}) = 0,$$

where $A := (a_{ij})_{i,j=1}^d$ is positive definite (i.e., $z^T A z > 0$ for any nonzero vector $z \in \mathbb{R}^d$). Let $\xi = \Phi(x)$ denote a coordinate transform where $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ and the Jacobian matrix $S = \frac{\partial \Phi}{\partial x}$ is non-singular. Show that the elliptic equation does not change type if the equation is written in the new coordinates ξ.

Hint: Matrix $A = (a_{ij})$ becomes SAS^T after transformation.

Solution.

(a) For the 2 variable case, we may directly solve for $u_{xx}, u_{x,y}, u_{yy}$

$$
\begin{aligned}
u_x &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \\
u_y &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} \\
u_{xx} &= \partial_x(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}) \\
&= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} \\
u_{yy} &= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} \\
u_{xy} &= \partial_y(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}) \\
&= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y}\right) + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}
\end{aligned}
$$

In that case, we have the PDE in new variable is

$$a' u_{\xi\xi} + 2b' u_{\xi\eta} + c' u_{\eta\eta} + I' = 0$$

where

$$
\begin{aligned}
a' &= a(\partial_x \xi)^2 + 2b\partial_x\xi\partial_y\xi + c(\partial_y\xi)^2 \\
b' &= a\partial_x\xi\partial_x\eta + c\partial_y\xi\partial_y\eta + b(\partial_x\xi\partial_y\eta + \partial_y\xi\partial_x\eta) \\
c' &= a(\partial_x\eta)^2 + 2b\partial_x\xi\partial_y\eta + c(\partial_y\eta)^2
\end{aligned}
$$

We may directly compute $b'^2 - a'c'$ and get the result, but that is somehow too tedious. With the hint let, $A \in \mathbb{R}^2, A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ We have

$$
\begin{aligned}
a' &= [SAS^T]_{11} \\
b' &= [SAS^T]_{12} = [SAS^T]_{21} \\
c' &= [SAS^T]_{22}
\end{aligned}
$$

Therefore,

$$b'^2 - ac = \det(SAS^T) = \det(S)\det(A)\det(S^T) = \det(S)^2(b^2 - a'c') > 0$$

which shows the PDE under new variables is still elliptic.

(b) Now we generalize the above to computation to d dimension. We have

$$\partial_{x_j} u = \sum_{k=1}^{d} \frac{\partial u}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_j}$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{l=1}^{d} \sum_{k=1}^{d} \frac{\partial^2 u}{\partial \xi_k \partial \xi_l} \frac{\partial \xi_l}{\partial x_i} \frac{\partial \xi_k}{\partial x_j} + \frac{\partial u}{\partial \xi_k} \frac{\partial^2 \xi_k}{\partial x_i \partial x_j}$$

Therefore, after transformation, the coefficient for $\frac{\partial^2 u}{\partial \xi_k \partial \xi_l}$ is

$$\begin{aligned} a'_{kl}(\xi) &= \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij}(\xi^{-1}(\xi)) \frac{\partial \xi_l}{\partial x_i} \frac{\partial \xi_k}{\partial x_j} \\ &= SA(\xi^{-1}(\xi)) S^T \end{aligned}$$

Since A is always positive definite, and S is nonsingular, we have A' is positive definite as $z^T SAS^T z = (z^T S) A (x^T S)^T > 0$ for all $z \neq 0$. It follows that the PDE after transformation is still ellipctic.

3. Use the 5-point finite difference method to solve the 2D Poisson's equation:

$$-\Delta u(x) = f(x), \quad x \in \Omega := (0,1)^2,$$

$$u(x) = 0, \quad x \in \partial\Omega.$$

Test your codes using the exact solution

$$u(x) = \sin(\pi x_1)\sin(\pi x_2) + \sin(\pi x_1)\sin(2\pi x_2),$$

with $f(x)$ computed accordingly. Take $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}$. Find out $\|u_h - u\|_\infty$ for the corresponding values of h, where $\|u_h - u\|_\infty$ denotes the maximum error between u_h and u at the grid points. Based on the numerical results, generate a table or a plot to see that the convergence rate of the error $\|u_h - u\|_\infty$ is second order in h.

Solution. With given u, we have

$$f(x_1, x_2) = -\Delta u(x_1, x_2) = 2\pi^2 \sin(\pi x_1)\sin(\pi x_2) + 5\pi^2 \sin(\pi x_1)\sin(2\pi x_2)$$

With 5-points method, we have the following error table (see Tab.1)

Table 1: Error Table

h	$\|u_h - u\|_\infty$
1/10	0.033696
1/20	0.008474
1/40	0.002119
1/80	0.000530

To prove that $\|u_h - u\|_\infty = O(h^2)$, we first plot a log-log plot for $(1/h) = N$ from 10 to 300 (See Fig.1). This plot shows linear(affine) relation i.e.

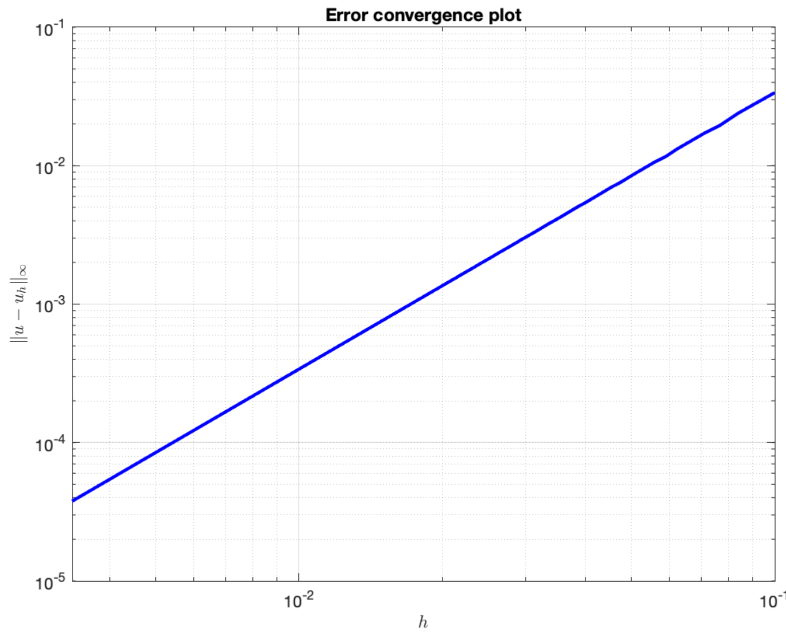

Figure 1: Convergence of error.

$$\log(\|u - u_h\|_\infty) = m \log(h) + b \iff \|u - u_h\|_\infty = e^b h^m.$$

With least square estimation, we may estimate find slope (m) and intercept (b). The results are

$$m = 1.994, \quad b = 1.2184$$

Hence, we shows the convergence rate of the error $\|u - u_h\|_\infty$ is second order in h.