MATH 272A: Numerical PDE
HW4

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1. Note that the space $H_0^1(\Omega)$ is defined as

$$H_0^1(\Omega) := \left\{ u \in H^1(\Omega) : \exists u_n \in C_c^\infty(\Omega) \text{ and } \|u_n - u\|_{H^1(\Omega)} \to 0 \right\}.$$

Use this definition to show that if u is a classical solution to Poisson's equation (with homogeneous Dirichlet boundary conditions), then it is a weak solution.

Proof. The definition of weak solutions says $u \in H_0^1(\Omega)$ is a weak solution to Poisson's equation with homogeneous Dirichlet boundary condition if

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx \tag{1}$$

for all $v \in H_0^1$. First of all, the classical solution u lies in H_0^1 trivially[1]. We then follow the hint to complete the proof i.e. first show (1) holds for C_c^∞ functions and then generalize to H_0^1 with density.

- With integration by parts,

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_{\partial\Omega} (\nabla u \cdot \vec{n}) v dS - \int_\Omega (\nabla \cdot \nabla u) v dx \tag{2}$$

$$= 0 + \int_\Omega (-\Delta u) v dx \tag{3}$$

$$= \int_\Omega f v dx \tag{4}$$

  Note that in (2) we use the fact that $v = 0$ on $\partial\Omega$.

- For simplicity, we use $(\cdot, \cdot)$ to denote inner product in $L^2(\Omega)$. Let $v \in H_0^1$ and $\{v_n\}_{n=1}^\infty$ such that $\|v_n - v\|_H \to 0$ i.e. we have both $\|v_n - v\|_{L^2(\Omega)} \to 0$ and $\|\nabla v_n - \nabla v\|_{L^2(\Omega)} \to 0$. Therefore, we have

$$|\int_\Omega \nabla u \cdot \nabla v dx - \int_\Omega f v dx| = |(\nabla u, \nabla v) - (f, v)| \tag{5}$$

$$\leq |(\nabla u, \nabla v) - (\nabla u, \nabla v_n)| + |(\nabla u, \nabla v_n) - (f, v_n)| + |(f, v_n) - (f, v)| \tag{6}$$

  Note that the second term vanishes as is proved in step 1. Moreover, by Cauchy-Schwartz, we have

$$|\int_\Omega \nabla u \cdot \nabla v dx - \int_\Omega f v dx| \leq |(\nabla u, \nabla v - \nabla v_n)| + |(f, v_n - v)| \tag{7}$$

$$\leq \|\nabla u\|_{L^2(\Omega)} \|v_n - v\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \|v_n - v\|_{L^2(\Omega)} \tag{8}$$

[1] $u \in C^1$ vanishes on boundary

Take limit of the right hand side, we have

$$0 \leq |\int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\Omega} f v dx| \leq 0 \tag{9}$$

and conclusion follows.

$\square$

(a) Use integration by parts to show that for all $v \in C_c^\infty(\Omega)$, we have

$$(\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}.$$

(b) Use Step 1 to show that

$$(\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}$$

for all $v \in H_0^1(\Omega)$. In this step, you may take $v_n \in C_c^\infty(\Omega)$ with $\|v - v_n\|_{H^1(\Omega)} \to 0$ and use the Cauchy-Schwarz inequality to reach the conclusion.

2. (Poincaré inequality in 1D) Poincaré inequality is the key ingredient to show the coercivity of the bilinear form corresponding to Poisson's equation. The Poincaré inequality in 1D is stated as

$$\int_a^b |u(x)|^2 \, dx \leq \frac{(b-a)^2}{2} \int_a^b |u'(t)|^2 \, dt, \tag{10}$$

for all $u \in H_0^1(\Omega)$ where $\Omega = (a, b)$. Use the definition of $H_0^1(\Omega)$ to prove this inequality.

Proof. We follow the given hint: first prove the conclusion for C_c^∞ functions and then generalize it to H_0^1 with density.

- Let $u \in C_c^\infty$, with Fundamental theorem of calculus and the fact that $u(a) = 0$

$$\int_a^b |u(x)|^2 dx = \int_a^b (\int_a^x u'(t))^2 dt dx \tag{11}$$

We further relax the integrand, with Cauchy-Schwartz inequality,

$$\begin{aligned}(\int_a^x u'(t)dt)^2 &= (\int_a^x 1 \cdot u'(t)dt)^2 &(12)\\ &\leq (x-a)\int_a^x |u'(t)|^2 dt &(13)\end{aligned}$$

Then we have

$$\begin{aligned}\int_a^b |u(x)|^2 dx &\leq \int_a^b (x-a) \int_a^x |u'(t)|^2 dt dx &(14)\\ &= \int_a^b \int_t^b (x-a)|u'(t)|^2 dx dt &(15)\\ &= \int_a^b \frac{(b-a)^2 - (t-a)^2}{2} |u'(t)|^2 dt &(16)\\ &\leq \frac{(b-a)^2}{2} \int_a^b |u'(t)|^2 dt &(17)\end{aligned}$$

Note that in (15) we interchange 2 integrals, which is legitimate due to Fubini-Tonelli theorem (the integrand is bounded and the domain is bounded).

Alternatively, I think we may directly cite the following result

Theorem 0.1 (Wirtinger's Inequality). *If $f \in C^1([a,b])$ and $f(a) = f(b) = 0$, then $\int_a^b |f(x)|^2 dx \leq (\frac{b-a}{\pi})^2 \int_a^b |f'(x)|^2 dx$*

Clearly, C_c^∞ functions satisfy the requirements. This also implies we got strict inequality in the claim of the problem if $u_n \neq 0$, which is consistent with our above proof — Equality holds only when u' is a constant, which contradicts to compact support property unless $u = 0$.

- We then complete the proof with density of C_c^∞ functions in H_0^1. Let $u \in H_0^1$ and $u_n \in C_c^\infty$ such that $\|u_n - u\|_{H^1} \to 0$. By definition, we have both $\|u_n - u\|_{L^2(a,b)} \to 0$ and $\|u_n' - u'\|_{L^2(a,b)} \to 0$, which further implies $\int_a^b |u_n(x)|^2 \to \int_a^b |u(x)|^2$ and $\frac{(b-a)^2}{2} \int_a^b |u_n'(t)|^2 \, dt \to \frac{(b-a)^2}{2} \int_a^b |u'(t)|^2 \, dt$.

  For each u_n, we have

$$\int_a^b |u_n(x)|^2 \, dx - \frac{(b-a)^2}{2} \int_a^b |u_n'(t)|^2 \, dt \leq 0 \tag{18}$$

  Take limit, and we have the result

$$\int_a^b |u(x)|^2 \, dx \leq \frac{(b-a)^2}{2} \int_a^b |u'(t)|^2 \, dt \tag{19}$$

□

3. (Discontinuous functions in Sobolev spaces $W^{1,p}$) Recall the definition of $W^{1,p}(\Omega)$ for $1 \leq p < \infty$. It is true that in one dimension, any function $u \in W^{1,p}(\Omega)$ has a continuous representative in the same equivalent class (i.e., by modifying u on a set of measure zero, we can get a continuous function). However, in higher dimensions, this result is no longer true. Show that in 2D, $u(x) = \log(|x|) \in W^{1,1}(\Omega)$, where $x = (x_1, x_2)$ and $\Omega = \{x_1^2 + x_2^2 < 1\}$ is the unit disk.

Proof. We prove $u \in W^{1,1}(\Omega)$ by showing $u \in L^1(\Omega)$ and $\frac{\partial u}{\partial x_i} \in L^1(\Omega)$.

- $u \in L^1(\Omega)$.

  Integration by polar coordinates, we have

$$\int_\Omega |u(x)| dx = \int_0^{2\pi} \int_0^1 |\log(r)| r dr d\theta \tag{20}$$

$$= 2\pi \int_0^1 |\log(r)| r dr \tag{21}$$

$$= -2\pi \int_0^1 \log(r) r dr \tag{22}$$

$$= -2\pi [(\log(r) \frac{1}{2} r^2)|_0^1 - \int_0^1 \frac{r^2}{2} \frac{1}{r} dr] \tag{23}$$

$$= \frac{\pi}{2} < \infty \tag{24}$$

- $\frac{\partial u}{\partial x_i} \in L^1(\Omega)$.

  Note that $\frac{\partial u}{\partial x_i} = \frac{1}{|x|} \frac{1}{2|x|} 2x_i = \frac{x_i}{|x|^2}$. Hence $\nabla u = \frac{1}{|x|^2}(x_1, x_2)$ and $|\nabla u(x)| = \frac{1}{|x|}$.

Integration with polar coordinates indicates

$$\int_\Omega |\frac{\partial u}{\partial x_i}| dx = \int_0^{2\pi} \int_0^1 |\frac{r\cos(\theta)}{r^2}| dr d\theta \tag{25}$$

$$\leq \int_0^{2\pi} \int_0^1 \frac{1}{r} \cdot r dr d\theta \tag{26}$$

$$= 2\pi < \infty \tag{27}$$

It follows $\nabla u \in L^1(\Omega)$

$\square$

4. Solve the 1D Poisson's equation on $\Omega = (0, 1)$ with homogeneous Dirichlet boundary conditions using the finite element method.

$$-u''(x) = f(x) \quad \text{on} \quad \Omega = (0, 1), \quad u(0) = u(1) = 0 \tag{28}$$

Implement the method on your computer using a uniform grid with $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}$. Test your code using the exact solution $u(x) = x(x-1)\exp(x^2+1)$.

Solution. We implement the finite element method with finite element method and compute L_2 norm, energy norm and H^1 norm. We have the following result

h	$\|u_h - u\|_2$	$\|u_h' - u'\|_2$	$\|u_h - u\|_{H^1}$
1/10	0.012856610	0.407757533	0.407960168
1/20	0.003249950	0.205696630	0.205722302
1/40	0.000814766	0.103079749	0.103082969
1/80	0.000203835	0.051530700	0.051531104
1/160	0.000050968	0.025799014	0.025799064

Table 1: Errors in L^2 norm, energy norm, and H^1 norm.

From the table we see that energy norm decreases slower than L^2 norm and also dominates in H^1 norm. We test at a broader range and obtained the following plot (See Fig.1) Note that we omit the line for H^1 norm since that one overlaps with energy norm.

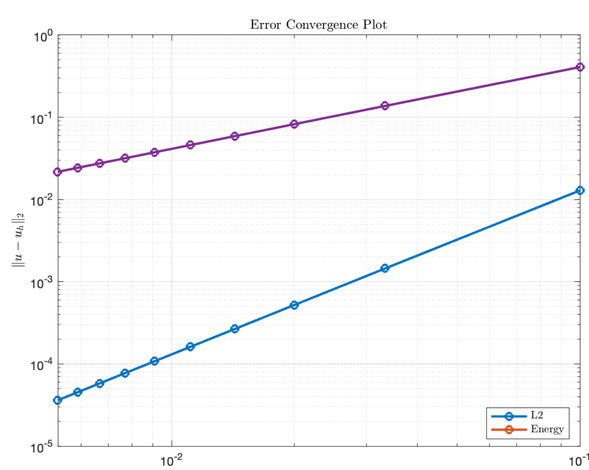

Figure 1: Convergence in L^2 and energy norm

With a simple least square estimation, we obtain the following convergence rate[2]:

$$\|u - u_h\|_{L^2} = O(h^2)$$
$$\|u' - u_h'\|_{L^2} = O(h)$$

Code source: Codes for above numerical results can be found at https://github.com/EddyShao/272-numericalPDE.

[2] fitted slope are 1.9958 and 0.9967