Math 272A homework 2
Note: Please read the collaboration policy on the syllabus.

1. (3pt) Use the *Discrete Maximum Principle* to prove the following result (do not use the discrete L^∞ stability directly). Let w_h be the solution to the problem

$$\begin{cases} -\Delta_h w_h = 0 & \text{on } \Omega_h \\ w_h = g & \text{on } \partial\Omega_h. \end{cases}$$

Then

$$\|w_h\|_{L^\infty(\Omega_h)} \leq \|g\|_{L^\infty(\partial\Omega_h)}.$$

(The notation follows what we have used in class.)

2. (3pt) (*Exactness of discrete Laplacian on quadratic functions*) Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a quadratic polynomial, i.e.,

$$\phi(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$$

for constants a, b, c, d, e, f. Show that

$$\Delta\phi(x, y) = \Delta_h \phi(x, y),$$

where Δ_h is defined by the 5-point finite difference operator. [Note: in fact the result is also true if ϕ is a cubic polynomial.]

Use the above fact to show $(-\Delta_h)\phi(x, y) = 1$ if $\phi(x, y) = \frac{1}{8} - \frac{1}{4}\left[(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2\right]$.

3. (4pt) (*Programming problem*) Consider the 1d Poisson's equation with mixed boundary conditions:

$$\begin{cases} -u''(x) = f(x) & x \in (0, 1) \\ u(0) = a \\ u'(1) = b. \end{cases}$$

Use the two different approaches discussed in class (one-sided difference approximation to $u'(1)$ v.s. symmetric difference approximation to $u'(1)$) to solve the problem and compare the results. Test your codes using the exact solution

$$u(x) = \sin(x)\exp(x^2 + 1) - 1$$

with $f(x)$, $u(0)$ and $u'(1)$ computed accordingly. Take $h = 1/10, 1/20, 1/40, 1/80, 1/160$. Find out $\|u_h - u\|_\infty$ for the corresponding values of h, where $\|u_h - u\|_\infty$ denotes the maximum error between u_h and u at the grid points. Generate tables or plots to study the convergence rate in h.