MATH 272A: Numerical PDE
HW3

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1. (4pt) Recall the four properties of A_h discussed in class. Prove the following results.

   (a) Show that if a matrix A satisfies properties (P1) and (P2), then

$$Ax \geq 0 \implies x \geq 0.$$

Hint: Prove by contradiction. You may mimic the proof of the Discrete Maximum Principle given in class.

Proof. Assume not and let $x_i = \min_j x_j < 0$. Consider the i-th entry of Ax

$$
\begin{aligned}
0 \leq (Ax)_i &= \sum_{j=1}^{N} a_{ij} x_j & (1)\\
&= a_{ii} x_i + \sum_{j=1, j\neq i}^{N} a_{ij} x_j & (2)\\
&\leq a_{ii} x_i + \sum_{j=1, j\neq i}^{N} a_{ij} x_i & (3)\\
&= (a_{ii} + \sum_{j=1, j\neq i}^{N} a_{ij}) x_i & (4)\\
&\leq 0 & (5)
\end{aligned}
$$

Note that in inequality in line 3 we use the fact that $a_{ij} < 0$ for $i \neq j$ (P1) and $x_i = \min_j x_j < 0$; inequality in line 5 uses the fact that A is weakly diagonally dominant. The above inequality shows $(Ax)_i = 0$ i.e. all inequalities above are equalities. This implies x is a constant vector and $x < 0$. By definition of irreducibly diagonally dominant, there exists i^* such that $a_{i^*i^*} > \sum_{j=1, j\neq i^*} |a_{i^*j}|$. Given property (P2), we have $(a_{i^*i^*} + \sum_{j=1, j\neq i^*} a_{i^*j}) > 0$. However, this further implies

$$(Ax)_{i^*} = \sum_{j=1}^{N} a_{i^*j} x_j = (a_{i^*i^*} + \sum_{j=1, j\neq i^*} a_{i^*j}) x_i < 0 \qquad (6)$$

which contradicts to the assumption. It follows that $Ax \geq 0 \implies x \geq 0$. □

   (b) Use (a) to show that A is non-singular, i.e., the null space of A is trivial. This means:

$$Ax = 0 \implies x = 0.$$

Proof. If $Ax = 0$, with (a), we have $x \geq 0$. However, we also have $A(-x) = 0$, which means

$-x \geq 0 \iff x \leq 0$. This further implies the only possible case is $x = 0$. $\square$

(c) Prove that the following three statements are equivalent:

i. $Ax \geq 0 \implies x \geq 0$;

ii. $b \geq 0 \implies A^{-1}b \geq 0$;

iii. A^{-1} is element-wise nonnegative.

Proof. We show equivalence between these 3 arguments by showing (i) $\iff$ (ii) and (ii) $\iff$ (iii).

- (i) $\iff$ (ii)
  - ($\implies$) With part (b) we have A^{-1} is well-defined. Therefore, $b = A(A^{-1}b) \geq 0$ implies $A^{-1}b \geq 0$ by (i).
  - ($\impliedby$) If $Ax \geq 0$, we then have $x = A^{-1}(Ax) \geq 0$ with (ii).
- (ii) $\iff$ (iii)
  - ($\implies$)If not, assuming $A^{-1}_{ij} < 0$. Let $b = [0 \dots 1 \dots 0]^T$ where the only 1 is on j-th entry, we will have $(A^{-1}b)_i = A^{-1}_{ij} < 0$.
  - ($\impliedby$) This is almost trivial. If $b \geq 0$, then $(A^{-1}b)_i = \sum_{j=1}^{N} A^{-1}_{ij} b_j \geq 0$ for all i since every term here is nonnegative.

$\square$

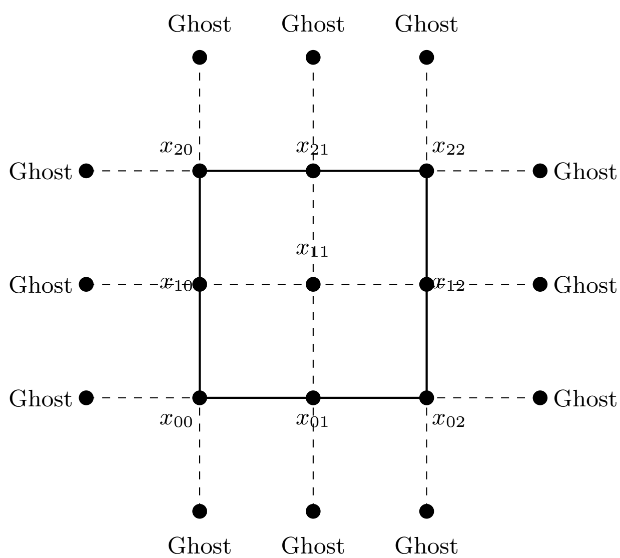
2. (4pt) (Pure Neumann boundary value problem) Consider the pure Neumann boundary value problem:

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega = (0,1)^2, \\ \frac{\partial u(x)}{\partial n} = 0, & x \in \partial\Omega \end{cases}$$

where f satisfies $\int_\Omega f(x)dx = 0$.

(a) Use the finite difference method with the "ghost point" technique (central difference for Neumann boundary) to discretize the problem and write down the linear system $A_h U_h = F_h$ corresponding to $h = 1/2$. *Hint*: A_h is a 9×9 matrix.

Solution. Our linear system is assembled under the following grid boxes:



With the above labeling, the linear system $A_h U_h = F_h$ is in the following way

$$\frac{1}{h^2}\begin{pmatrix} 4 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 & -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -2 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -2 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 & 4 \end{pmatrix}\begin{bmatrix} u_{00} \\ u_{01} \\ u_{02} \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{20} \\ u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} f_{00} \\ f_{01} \\ f_{02} \\ f_{10} \\ f_{11} \\ f_{22} \\ f_{20} \\ f_{21} \\ f_{22} \end{bmatrix} \tag{7}$$

Explanation of notation: $f_{ij} = f(x_{ij})$ and $u_{ij} = u(x_{ij})$.

Explanation of the matrix: We create ghost node $u_{i-1,j}$ $u_{i+1,j}, u_{i,j-1}, u_{i,j+1}$ for indices ij with $i = 0, 2$ or $j = 0, 2$. With central difference, we have either $\frac{u_{i+1,j} - u_{i-1,j}}{2h} = \frac{\partial u}{\partial n} = 0$ or with similar process with j so that we can represent the value of ghost node (note that for corner node we do both i and j). We then assmble the matrix accordingly.

(b) Show that A_h is weakly diagonally dominant.

Proof. According to the definition and direct computation, we see A_h is weakly diagonally dominant. Note that A_h is indeed very "weak" here since every row the abs sum of off-diagonal elements exactly equal to the diagonal element. This leads to the property in (c). □

(c) Show that:

$$A_h x \geq 0 \implies x \geq 0 \text{ or } x \text{ is a constant vector.}$$

Proof. The proof is almost identical to 1(a); actually I think the conclusion should only **be x is a constant vector**.

If x is not a constant vector:

Let $x_i = \min_j x_j < 0$, we have

$$0 \leq (A_h x)_i = \sum_{j=1}^{N} a_{ij} x_j \tag{8}$$

$$= a_{ii} x_i + \sum_{j=1, j\neq i}^{N} a_{ij} x_j \tag{9}$$

$$< a_{ii} x_i + \sum_{j=1, j\neq i}^{N} a_{ij} x_i \tag{10}$$

$$= (a_{ii} + \sum_{j=1, j\neq i}^{N} a_{ij}) x_i = 0 \tag{11}$$

Note that line 10 uses the fact that x is not a constant vector (so we got strict inequality) and $a_{ii} > 0$ and $a_{ij} < 0 \quad \forall i \neq j$. The last line uses the fact that $a_{ii} + \sum_{j=1, j\neq i}^{N} a_{ij} = 0$ which can be observed directly. The above computation clearly leads to contradiction, and the result follows.

□

(d) Use (c) to show that:

$$A_h x = 0 \implies x \text{ is a constant vector.}$$

This says that the null space of A_h consists of constant vectors.

Proof. Since $A_h x \geq 0$ and $A_h(-x) = 0$, this turns out that both x and $-x$ should be constant or ≥ 0. This turns out that x can only be a constant vector or a zero vector (which is also a constant vector), and the result follows. □

3. (2pt) (Cauchy-Schwarz inequality) Let $(X, \|\cdot\|_X)$ be a Hilbert space equipped with the inner product $(\cdot,\cdot)_X : X \times X \to \mathbb{R}$. Prove the Cauchy-Schwarz inequality:

$$|(u,v)_X| \leq \|u\|_X \|v\|_X \quad \forall u, v \in X.$$

[**Hint:** Let $w = \frac{u}{\|u\|} - \frac{v}{\|v\|}$ and use $(w,w)_X \geq 0$.]

Proof. WLOG, we may assume $\|u\|, \|v\| > 0$ (since the case where at least one of them is 0 is trivial). Let $w = u \pm v$ we have

$$0 \leq (w,w)_X = (\frac{u}{\|u\|} \pm \frac{v}{\|v\|}, \frac{u}{\|u\|} \pm \frac{v}{\|v\|}) \tag{12}$$

$$= \frac{\|u\|^2}{\|u\|^2} + \frac{\|v\|^2}{\|v\|^2} \pm 2\frac{(u,v)}{\|u\|\|v\|} \tag{13}$$

$$= 2(1 \pm \frac{(u,v)}{\|u\|\|v\|}) \tag{14}$$

It follows that

$$\frac{|(u,v)|}{\|u\|\|v\|} \leq 1 \iff |(u,v)| \leq \|u\|\|v\| \tag{15}$$

□