Math 272A homework 5

Note: Please read the collaboration policy on the syllabus.

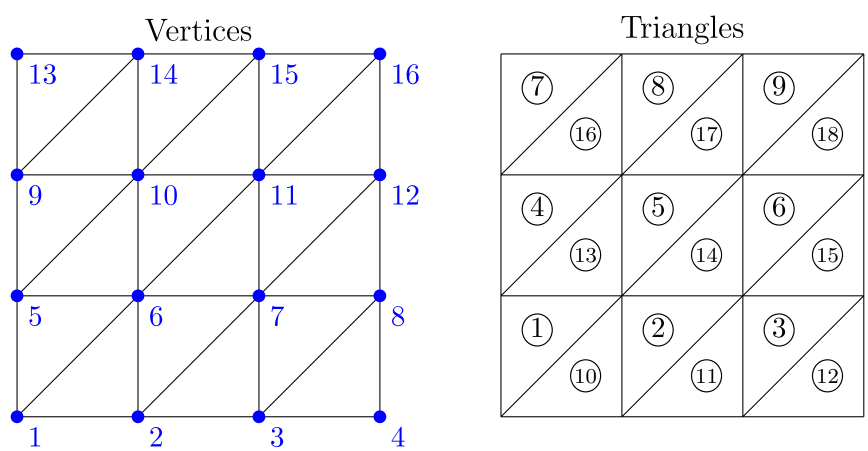
In this homework set, our main goal is to solve the 2D Poisson equation

$$(1) \qquad \begin{cases} -\Delta u = f & \text{on } \Omega = (0,1)^2 \\ u = g & \text{on } \partial\Omega \end{cases}$$

using finite element method with continuous piecewise linear elements.

1. (3pts) Write a code that generates automatically a uniform finite element mesh for Ω with $2n^2$ triangles for a given positive integer n. The data structure corresponding to the mesh should include:
(1) A *node* matrix of size (total number of vertices) $\times$ 2
(2) An *element* matrix of size (total number of triangle) $\times$ 3
(3) A *bdNode* vector of size (total number of vertices) which specifies which vertices are on the boundary (give a value 1 if a vertex is on the boundary, otherwise 0).

You may number the vertices and triangles in the way that is shown in the following picture (take $n = 3$ for an example)



To test your code, take $n = 5$ and print out the 7th row of the *node* matrix, the 8th row of the *element* matrix and the 12th component of the *bdNode* vector.

2. (3pts) Write a code to compute the local stiffness matrix $A^K = (a_{ij}^K)_{i,j=1}^3$ and the local load vector $F^K = (f_j^K)_{j=1}^3$ for a given triangle K with three vertices $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$. To test your code, print out A^K and F^K for $\boldsymbol{v}_1 = (0,0)^T$, $\boldsymbol{v}_2 = (0.5, 0.5)^T$ and $\boldsymbol{v}_3 = (0,1)^T$.

3. (4pts) Implement the finite element method with continuous piecewise linear elements (Algorithm 2) to solve equation (1). Use the exact solution $u(x_1, x_2) = (x_1 + x_2)^2 \cos(x_1 + 2x_2)$ (with f and g computed accordingly) to test your code. Test your code for $h = 1/4, 1/8, 1/16, 1/32$ (corresponding to $n = 4, 8, 16, 32$). You may plot the numerical solution u_h to see whether it looks like the exact solution u. For a quantitative study on the order of convergence, we want to

compare the difference between u_h and u in the two norms:

$$|u - u_h|_{H^1} = \left[\int_\Omega (\nabla u - \nabla u_h) \cdot (\nabla u - \nabla u_h) d\boldsymbol{x} \right]^{1/2}$$

and

$$||u - u_h||_{L^2} = \left[\int_\Omega |u(\boldsymbol{x}) - u_h(\boldsymbol{x})|^2 d\boldsymbol{x} \right]^{1/2}.$$

To evaluate these norms approximately, we can approximate the integral $\int_K v(\boldsymbol{x}) d\boldsymbol{x}$ on each triangle K by the following rule

$$\int_K v(\boldsymbol{x}) d\boldsymbol{x} \approx |K| \frac{1}{3} \sum_{i=1}^{3} v(\boldsymbol{m}_i)$$

where $|K|$ is the area of K and $\{\boldsymbol{m}_i\}_{i=1}^3$ are the three midpoints of the edges of K.