MATH 272B: Numerical PDE
HW3

University of California, San Diego

Zihan Shao

Code source: Github Link

Some of the coding are assisted by generative AI (plotting especially).

Problem 1: FDM Discretization and Residual Computation

I'm not sure how to use FEM here. I think if we use weak form derivation, then the Jacobian should be based on that weak form derivation instead of the one with FDM in Problem 2. For simplicity, I use FDM here. Also, I'm not too sure how to impose natural boundary condition here.

Consider the Swift-Hohenberg equation in 1D with natural boundary conditions:

$$F(u) = ru - (1 + \partial_x^2)^2 u - u^3 \tag{1}$$

Using FDM, we approximate the second and fourth derivatives as follows:

$$\partial_x^2 u_i \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2},$$

$$\partial_x^4 u_i \approx \frac{u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}}{h^4}.$$

Substituting these into the equation, the discrete residual at each grid point i is:

$$F_i = ru_i - \left(u_i + 2\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \frac{u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}}{h^4} \right) - u_i^3.$$

Natural boundary conditions are enforced by assuming $\partial_x^2 u = 0$ and $\partial_x^4 u = 0$ at the boundaries (I searched the lecture notes and did not find a definition. This is what google tells me.) Basically, we set $F = 0$ at the boundary grid points.

Use the following parameters:

- $N = 100$
- $L = 20\pi$
- $r = 0$
- $u_1 = 0, u_2 = \sin(2\pi x/L)$

Analytically, we have $F_1 = 0$ and $F_2 = -(\frac{99}{100})^2 \sin(2\pi x/L) - \sin(2\pi x/L)^3$. We plot the functions, analytical/numerical solutions:

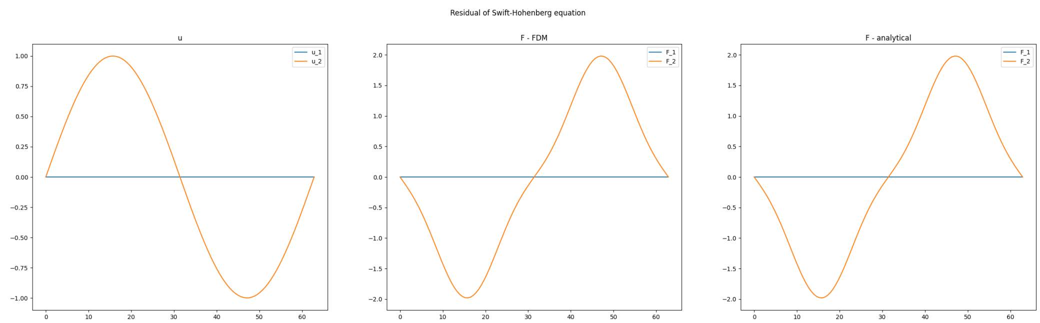


Figure 1: Residual Visualization

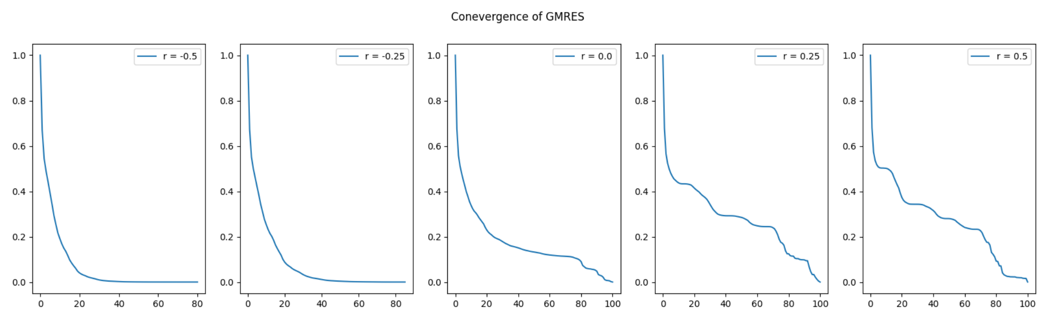


Figure 2: Convergence history of GMRES

Problem 2: GMRES

Implementation of GMRES method can be found in .
We test GMRES at $u = 0$. We ask the algorithm to solve for $Jv = b$ (b is randomly selected) for $r = -0.5, -0.25, 0, 0.25, 0.5$. The convergence hist are shown in Figure 2.
We see that, as we stepping into the unstable regime, convergence of GMRES becomes slower and bizzare. This is because the Jacobian is becoming ill-conditioned.

Problem 3: Newton-Krylov Solution Finding

Stopping criterion

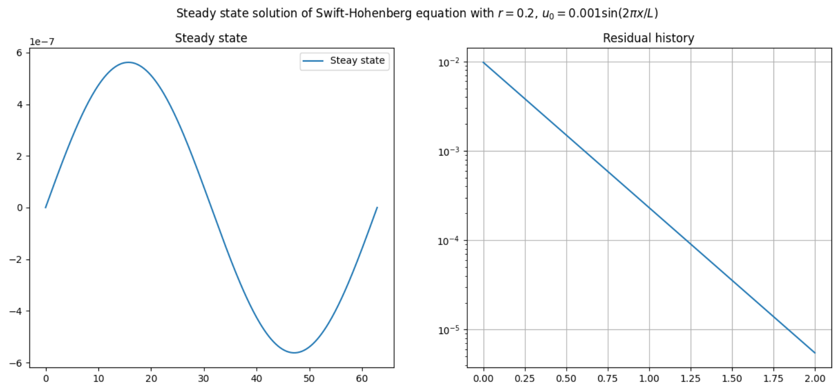
- GMRES: $\|Ax - b\| < 1e - 8$
- Newton: $\|F(u)\| < 1e - 4$.

We use a finer grid with $N = 300$, otherwise it is hard for the method to converge.

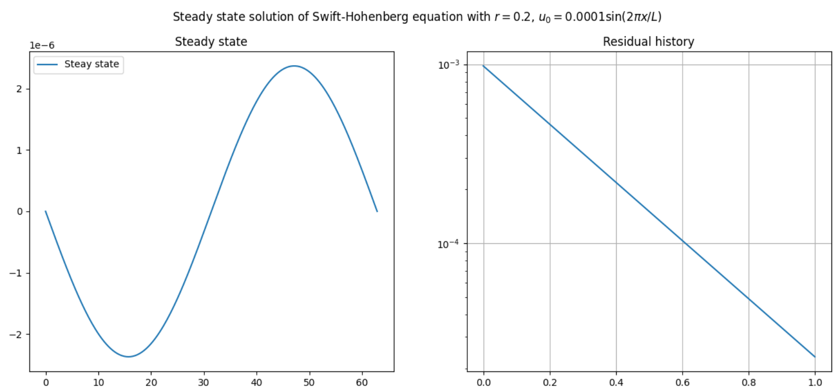
Unstable: $r = 0.2$

By using $u_0 = 0.001 \sin(2\pi x/L)$, we are able to find a nonzero steady state solution (see solution obtained and convergence plot in Figure 3)
we have the following observations:

- The convergence is fast (second order convergence).



(a) $u_0 = 0.001\sin(2\pi x/L)$



(b) $u_0 = 0.0001\sin(2\pi x/L)$

Figure 3: Comparison of two plots

- The solution is very sensitive to the initial guess. For $u_0 = \sigma\sin(2\pi x/L)$. Roughly speaking, for $\sigma = 0.001 - 0.1$, we got the steady state shown in Figure 3a. For even smaller σ, we obtain the steadt state shown in Figure 3b. There are other steady states, some of which requires even finer grid to reach stopping criterion.

Stable: $r = -0.2$

By using $u_0 = \sin(2\pi x/L)$, we are able to obtain a nonzero steady state solution (see solution obtained and convergence plot in Figure 4)
Here are some observations

- We observe second order convergence for early epochs.
- Line search is necessary for later epochs. This is due to the truncation error when estimating the Jacobian-vector product.
- Even for stable regime, we see sensitivity on initial guess. When using $u_0 = \sin(2\pi x/L) + 0.1\cos(2\pi x/L)$ (this means we shift a little bit), the algorithm cannot converge (line search fails).

However, I'm not sure how to relate it to the bifurcation analysis.

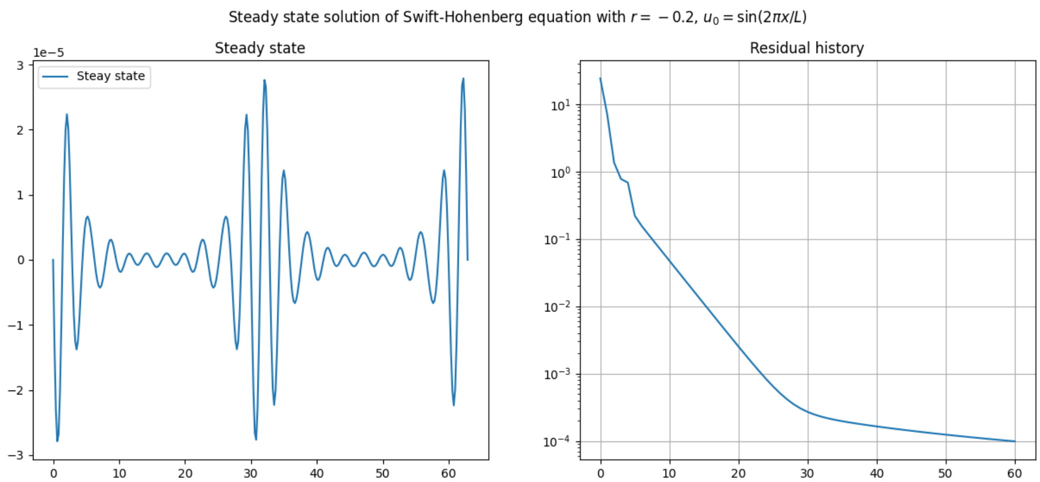


Figure 4: $r = -0.2$

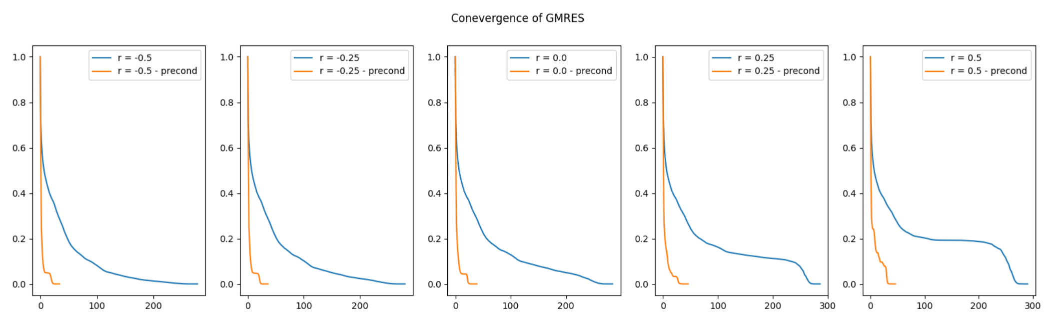


Figure 5: Caption

Problem 4: Preconditioner

By using preconditioner, it does not change the convergence of Newton-krylov iterations much so that we do not put a figure here (since it does not change each u). However, it took shorter time since it makes each GMRES iteration faster. We here compare the convergence of GMRES in the following figure (see Fig.5)
The plots justify the power of preconditioning.