Math 272A homework 1
Note: Please read the collaboration policy on the syllabus.

1. (2pt) (*Multi-index notation*) Taylor's theorem for a given function $f : \mathbb{R}^d \to \mathbb{R}$ is given by

$$f(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}| \le n} \frac{D^{\boldsymbol{\alpha}} f(\boldsymbol{c})}{\boldsymbol{\alpha}!} (\boldsymbol{x} - \boldsymbol{c})^{\boldsymbol{\alpha}} + R_n(\boldsymbol{x}),$$

where the first term on the right hand side of the above equation is the nth-degree Taylor polynomial of f at $\boldsymbol{c} \in \mathbb{R}^d$. Notice that we have used the multi-index notation. Find out the 2nd-degree Taylor's polynomial of $f(x, y) = xe^y + 1$ at $\boldsymbol{c} = (0, 1)$.

2. (4pt) (*Invariance of the type under coordinate transformations*) Second-order linear elliptic PDEs in two variables have the form

$$au_{xx} + 2bu_{xy} + cu_{yy} + I(u_x, u_y, u, x, y) = 0 \tag{1}$$

where $b^2 - ac < 0$ and $I(u_x, u_y, u, x, y)$ contains all the lower order terms. Define the new variables

$$\begin{cases} \xi = \xi(x, y) \\ \eta = \eta(x, y) \end{cases}$$

with a non-singular Jacobian matrix $S = \frac{\partial(\xi,\eta)}{\partial(x,y)}$. Rewrite Equation (1) in the new variables (ξ, η) and show that the type of equation under the new variables is still elliptic.

More generally, linear elliptic PDEs in d-dimensions have the form

$$-\sum_{i,j=1}^{d} a_{ij}(\boldsymbol{x}) \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) + (\text{lower order terms}) = 0.$$

where $A := (a_{ij})_{i,j=1}^d$ is positive definite (i.e., $\boldsymbol{z}^T A \boldsymbol{z} > 0$ for any nonzero vector $\boldsymbol{z} \in \mathbb{R}^d$). Let $\boldsymbol{\xi} = \Phi(\boldsymbol{x})$ denote a coordinate transform where $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ and the Jacobian matrix $S = \partial\Phi/\partial\boldsymbol{x}$ is non-singular. Show that the elliptic equation does not change type if the equation is written in the new coordinates $\boldsymbol{\xi}$. [Hint: matrix $A = (a_{ij})$ becomes SAS^T after transformation].

3. (4pt) (*Programming problem*) Use the 5-point finite difference method to solve the 2d Poisson's equation

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) & \boldsymbol{x} \in \Omega := (0, 1)^2 \\ u(\boldsymbol{x}) = 0 & \boldsymbol{x} \in \partial\Omega. \end{cases}$$

Test your codes using the exact solution

$$u(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2) + \sin(\pi x_1)\sin(2\pi x_2)$$

with $f(\boldsymbol{x})$ computed accordingly. Take $h = 1/10, 1/20, 1/40, 1/80$. Find out $\|u_h - u\|_\infty$ for the corresponding values of h, where $\|u_h - u\|_\infty$ denotes the maximum error between u_h and u at the grid points. Based on the numerical results, generate a table or a plot to see that the convergence rate of the error $\|u_h - u\|_\infty$ is second order in h.