Math 272A homework 3

Note: Please read the collaboration policy on the syllabus.

1. (4pt) Recall the four properties of A_h discussed in class. Prove the following results.

(1a) Show that if a matrix A satisfies (P1) and ($P2$), then

$$A\boldsymbol{x} \geq \mathbf{0} \implies \boldsymbol{x} \geq \mathbf{0}.$$

[*Hint*: Prove by the argument of contradiction. You may mimic the proof of the Discrete Maximum Principle given in class.]

(1b) Use (1a) to show that A is non-singular, i.e., the null space of A is trivial, and this means

$$A\boldsymbol{x} = \mathbf{0} \implies \boldsymbol{x} = \mathbf{0}.$$

(1c) Prove that the three statements are equivalent:

(1) $A\boldsymbol{x} \geq \mathbf{0} \implies \boldsymbol{x} \geq \mathbf{0}$;
(2) $\boldsymbol{b} \geq \mathbf{0} \implies A^{-1}\boldsymbol{b} \geq \mathbf{0}$;
(3) A^{-1} is element-wise nonnegative.

2. (4pt) (*Pure Neumann boundary value problem*) Consider the pure Neumann boundary value problem

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) & \boldsymbol{x} \in \Omega = (0,1)^2, \\ \dfrac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) = 0 & \boldsymbol{x} \in \partial\Omega. \end{cases}$$

where f satisfies $\int_\Omega f = 0$.

(2a) Use the finite difference method with "ghost point" technique (central difference for Nuemann boundary) to discretize the problem and write down the linear system $A_h U_h = F_h$ corresponding to $h = 1/2$. [*Hint*: A_h is a 9×9 matrix.]

(2b) Show that A_h is weakly diagonally dominant.

(2c) Show that

$$A_h\boldsymbol{x} \geq \mathbf{0} \implies \boldsymbol{x} \geq \mathbf{0} \text{ or } \boldsymbol{x} \text{ is a constant vector.}$$

(2d) Use (2c) to show that

$$A_h\boldsymbol{x} = \mathbf{0} \implies \boldsymbol{x} \text{ is a constant vector.}$$

This says that the null space of A_h consists of constant vectors.

3. (2pt) (*Cauchy-Schwartz inequality*) Let $(X, \|\cdot\|_X)$ be a Hilbert space equipped the inner product $(\cdot,\cdot)_X : X \times X \to \mathbb{R}$. Prove the Cauchy-Schwartz inequality:

$$|(u,v)_X| \leq \|u\|_X \|v\|_X \quad \forall u, v \in X.$$

[*Hint*: Let $w = \frac{u}{\|u\|_X} - \frac{v}{\|v\|_X}$ and use $(w,w)_X \geq 0$.]