MATH 272B: Numerical PDE
HW1

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1. Consider the two-dimensional Poisson equation on the unit square $\Omega = [0,1]^2$:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

Using a uniform grid with mesh size $h = \frac{1}{n+1}$ where n is the number of interior points in each direction:

(a) (15 points) For the discretized system using the 5-point Laplacian stencil, write the matrix A as $A = L + D + U$ where L is strictly lower triangular, D is diagonal, and U is strictly upper triangular. For $n = 3$, write out these matrices explicitly and explain the structure of the Gauss-Seidel iteration matrix

$$G = -(L+D)^{-1}U.$$

Solution.

Constructing linear system. We label the interior points by $(x_{i,j})_{i=1,j=1}^{n}$ and indicate boundary points by subscripts with $0, n+1$. We abbreviate $u(x_{i,j})$ by $u_{i,j}$. This gives the discretized LHS as the following

$$-\Delta_h u(x_{i,j}) = \frac{4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}}{h^2} \tag{1}$$

This results in a $(n^2 \times n^2)$ linear system $AU = F$. In the linear system, we have

$$A \in \mathbb{R}^{n^2 \times n^2}, \quad A = \begin{bmatrix} M & -I & & & \\ -I & M & -I & & \\ & -I & M & -I & \\ & & \ddots & \ddots & \ddots \\ & & & -I & M \end{bmatrix} \tag{2}$$

in which I denotes a $n \times n$ identity matrix and M is of the form

$$M \in \mathbb{R}^{n \times n}, \quad M = \begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & 4 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 4 \end{bmatrix} \tag{3}$$

F and U are labeled boundary value given and interior values to solve.

Gauss-Seidel. We let L, D, U be the strict lower triangle, diagonal, strict upper diagonal of A, respectively. $G = -(L+D)^{-1}U$ will be the Gauss-Seidel iteration matrix.

Case $n = 3$. For $n = 3$, we have

$$A = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \tag{4}$$

where the green cells indicates L, red cells indicates D, blue cells indicates U; $G = -(L+D)^{-1}(U)$.

(b) (15 points) Calculate the spectral radius $\rho(G)$.

Solution. We will use the fact, without justification[1] that $\lambda_{i,j}(A) = 4(\sin^2(\frac{i\pi}{2(n+1)}) + \sin^2(\frac{j\pi}{2(n+1)}))$. Then, we have

$$\rho(G) = \rho(-(L+D)^{-1}(U)) = \rho((L+D)^{-1}(L+D-A)) = |(1 - \rho((L+D)^{-1}A))| \tag{5}$$

We may estimate the eigenvalues of $(L+D)^{-1}$ by Gershgorin's theorem: spectrum of $L+D$ lies in $[2, 6]$ so the spectrum of its inverse lies in $[1/6, 1/2]$ (assuming it is real). Also, as n gets large, we have $4(\sin^2(\frac{i\pi}{2(n+1)}) + \sin^2(\frac{j\pi}{2(n+1)})) \approx \frac{(i^2+j^2)\pi^2}{(n+1)^2}$ [2]. We finally get the following estimate when n gets large, by combining the previous results,

$$\rho(G) \leq 1 - \frac{1}{6}\frac{2\pi^2}{(n+1)^2} = 1 - \frac{\pi^2}{3(n+1)^2} \tag{6}$$

We verify this by numerical tests

n	ρ(actual)	ρ(estimated)
16	0.9833	0.9886
32	0.9955	0.9970
64	0.9988	0.9992

Table 1: Comparison of actual and estimated spectral radii for different grid sizes n.

which shows above is a good estimate.

(c) (20 points) Using this result, derive the asymptotic convergence rate of the Gauss-Seidel method. Show that the number of iterations required for convergence grows as $O(h^{-2})$.

Solution. Denote the number of iterations by N and the tolerance by $\epsilon << 1$. WLOG, we assume $\|e^{(0)}\| = 1$. For convergence, we need

$$\rho^N \leq \epsilon \tag{7}$$

Thus the number of iterations for convergence should satisfy

$$N \geq \log(\epsilon)/\log(\rho) \tag{8}$$

[1] proved in 272A

[2] It is quite sloppy, we have to assume we only consider i, j to be small

Note that here we use the fact $\log \rho < 0$. We have $N = O(-1/\log(\rho))$. However, we have $\rho(G_n) = 1 - O(h^2)$. As $\rho(G_n)$ is close to 1, we have $\log(\rho(G_n)) \approx -O(h^2)$. It follows that $N = O(h^2)$.

2. Implement and compare the Jacobi and Gauss-Seidel methods for solving the Poisson equation:

$$-\Delta u = \sin(\pi x)\sin(\pi y) \quad \text{in } \Omega = [0,1]^2$$

$$u = 0 \quad \text{on } \partial\Omega$$

(a) (15 points) Implement both the Jacobi and Gauss-Seidel methods in Python. Your code should:

i. Accept arbitrary grid size n.

ii. Monitor the residual norm at each iteration.

iii. Implement a suitable stopping criterion.

(b) (20 points) For grid sizes $n = 16, 32$, and 64:

- Compare the convergence rates of both methods.
- Plot the logarithm of the residual norm versus iteration number.
- Calculate and tabulate the observed convergence rates.
- Compare with the theoretical predictions from Problem 1.

(c) (15 points) How does the convergence behavior change if you modify the right-hand side to:

$$f(x,y) = \begin{cases} 1, & \text{if } 0.4 \le x, y \le 0.6 \\ 0, & \text{otherwise} \end{cases}$$

Explain any differences you observe in the convergence behavior.

Solution.

For coding part, please see attached code files or use Github Link.

We first show the results of part (b) and part (c) in Figure 1 and Table 2. We have the following interpretations.

(a) Since we are only investigating the convergence of solving linear system, changing right-hand side function will not incur any significant difference.

(b) We denote $e^{(k)} = Ax^{(k)} - b$ and we observe exponential convergence in the log-scale plot, which is depicted by the following

$$\|e^{(k)}\| \approx \rho^k \|e_{(0)}\| \tag{9}$$

$$\log \|e^k\| \approx \log \|e_{(0)}\| + k \log \rho \tag{10}$$

This aligns with theoretical analysis of iterative methods. According to our previous analysis (as well as suggested by notation ρ), ρ should be qual to $\rho(G)$ ($\rho(J)$) since we are using 2 norms here[3]. We verifying this by fitting the plot with least square estimation. See Table 2 for details.

[3] ρ is equivalent to 2-norm in this case

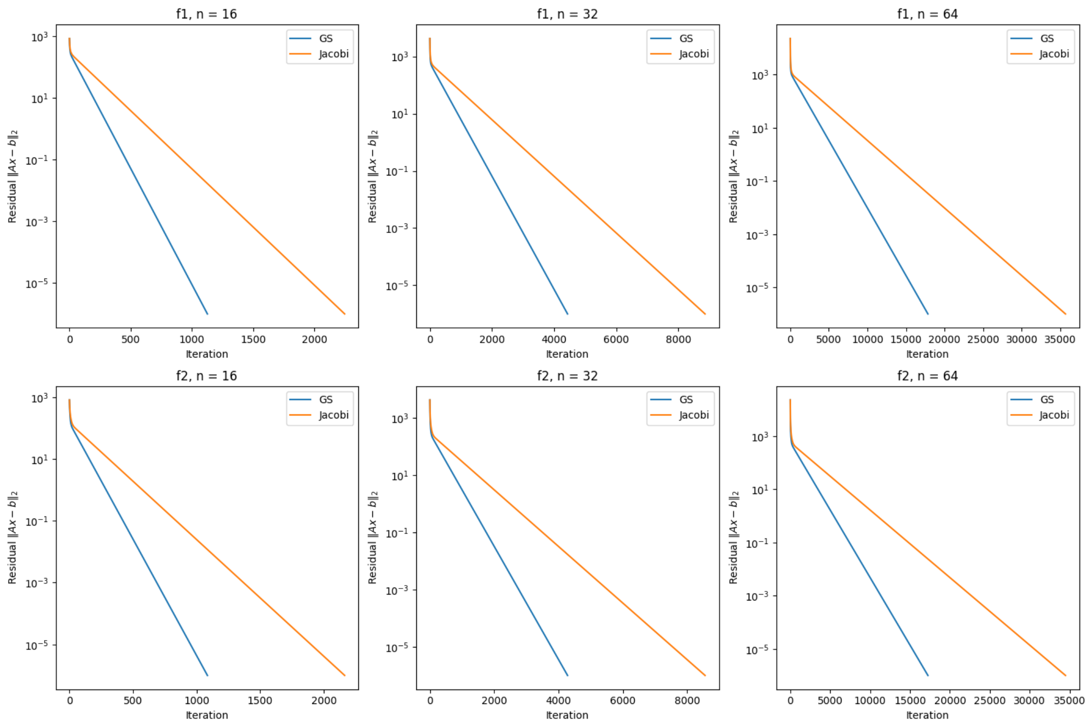

Figure 1: Convergence plot for Gauss-Seidel and Jacobi method with different n and different right-hand side (First row for f_1; second row for f_2). Note that we are using 2-norms here, and the iteration is stopped if residual is below $1e^{-6}$.

Table 2: Iterations for Convergence, Slope, and $\log(\rho)$ for Gauss-Seidel and Jacobi Methods

Method	**n**	**#Iters**	**Slope** ($\log(\rho)$)	**Log spectral radius**
Gauss-Seidel	16	1127	-1.7347e-02	-1.6844e-02
	32	4435	-4.5534e-03	-4.5003e-03
	64	17842	-1.1697e-03	-1.1638e-03
Jacobi	16	2247	-8.6903e-03	-8.6903e-03
	32	8863	-2.2779e-03	-2.2779e-03
	64	35676	-5.8492e-04	-5.8492e-04

(c) Based on table above, we see the alignment between slope and log spectral radius. We also see that the log of spectral radius is of $O(1/n^2) = O(h^2)$, which agrees with our previous analysis. The scaling of number of iterations for convergence also reveal this.

(d) We also observe that Gauss-Seidel outperform Jacobi in this particular case.