MATH 272A: Numerical PDE
HW6

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1. (a) Recall the weak formulation of Poisson's equation:

$$-\Delta u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega.$$

Show that this weak formulation is equivalent to the variational formulation:

$$u = \arg\min_{w \in H^1(\Omega)} \left\{ \frac{1}{2}\int_\Omega |\nabla w|^2\, dx - \int_\Omega f w\, dx - \int_{\partial\Omega} g w\, dS \right\}. \tag{1}$$

Solution. Let $E(w) = \left\{ \frac{1}{2}\int_\Omega |\nabla w|^2\, dx - \int_\Omega f w\, dx - \int_{\partial\Omega} g w\, dS \right\}$. The first-order variational derivative is defined as

$$\begin{aligned}
\langle \delta E(w), v\rangle &= \lim_{\epsilon \to 0} \frac{E(w+\epsilon v) - E(w)}{\epsilon} && (2)\\
&= \lim_{\epsilon\to 0} \frac{\epsilon \int_\Omega \nabla w \cdot \nabla v dx + \frac{1}{2}\epsilon^2 \int_\Omega |\nabla v|^2 dx - \epsilon \int_\Omega f v dx - \epsilon \int_{\partial\Omega} g v dS}{\epsilon} && (3)\\
&= \int_\Omega \nabla w \cdot \nabla v dx - \int_\Omega f v dx - \int_{\partial\Omega} g v dS && (4)\\
&= (\nabla w, \nabla v)_{L^2(\Omega)} - \langle f, v\rangle - \int_{\partial\Omega} g v dS && (5)
\end{aligned}$$

Since the functional is convex $u = \arg\min_{w\in H^1(\Omega)} \iff \langle \delta E(u), v\rangle = 0 \quad \forall v \in H^1(\Omega)$ i.e.

$$(\nabla u, \nabla v)_{L^2(\Omega)} = \langle f, v\rangle + \int_{\partial\Omega} g v dS \quad \forall v \in H^1(\Omega) \tag{6}$$

which is the definition of weak formulation.

(b) Derive the weak formulation for Poisson's equation:

$$-\Delta u = f \quad \text{in } \Omega,$$

with the Robin boundary condition:

$$a\frac{\partial u}{\partial n} + bu = g \quad \text{on } \partial\Omega,$$

where a, b are nonzero real numbers.

Solution. By Integration by Parts, given test function v, we have

$$\begin{aligned}\int_\Omega f v dx &= \int_\Omega -\Delta u v dx & (7)\\ &= \int_\Omega \nabla u \cdot \nabla v dx - \int_{\partial\Omega} (\nabla u \cdot n) v dS & (8)\\ &= \int_\Omega \nabla u \cdot \nabla v dx - \int_{\partial\Omega} (\frac{g - bu}{a}) v dS & (9)\end{aligned}$$

Therefore, we have the weak formulation. $u \in H^1(\Omega)$ is a weak solution if

$$(\nabla u, \nabla v)_{L^2(\Omega)} = \langle f, v\rangle + \int_{\partial\Omega} (\frac{g - bu}{a}) v dS \quad \forall v \in H^1(\Omega) \tag{10}$$

(c) Show that the weak formulation derived in 1b) is equivalent to the variational formulation:

$$u = \arg \min_{w \in H^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega |\nabla w|^2 \, dx - \int_\Omega f w \, dx - \int_{\partial\Omega} \frac{1}{a} g w \, dS + \int_{\partial\Omega} \frac{b}{2a} w^2 \, dS \right\}.$$

Solution. Like in part (a), we derive the first order variational derivative of the convex energy functional $E(w) = \left\{ \frac{1}{2} \int_\Omega |\nabla w|^2 \, dx - \int_\Omega f w \, dx - \int_{\partial\Omega} \frac{1}{a} g w \, dS + \int_{\partial\Omega} \frac{b}{2a} w^2 \, dS \right\}$.

$$\begin{aligned}\langle \delta E(w), v\rangle &= \lim_{\epsilon\to 0} \frac{\frac{1}{2}\epsilon^2(\cdots) + \epsilon(\int_\Omega \nabla w \cdot \nabla v dx - \int_\Omega f v dx - \int_{\partial\Omega} \frac{1}{a} g v \, dS + \int_{\partial\Omega} \frac{b}{a} w v \, dS)}{\epsilon} & (11)\\ &= \int_\Omega \nabla w \cdot \nabla v dx - \int_\Omega f v dx - \int_{\partial\Omega} \frac{1}{a} g v \, dS + \int_{\partial\Omega} \frac{b}{a} w v \, dS & (12)\end{aligned}$$

Hence, $u = \arg\min_{w\in H^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega |\nabla w|^2 \, dx - \int_\Omega f w \, dx - \int_{\partial\Omega} \frac{1}{a} g w \, dS + \int_{\partial\Omega} \frac{b}{2a} w^2 \, dS \right\}$ is equivalent to

$$(\nabla u, \nabla v)_{L^2(\Omega)} = \langle f, v\rangle + \int_{\partial\Omega} (\frac{g - bu}{a}) v dS \quad \forall v \in H^1(\Omega) \tag{13}$$

which is exactly the definition of weak solution.

2. Recall (WF1) and (WF2) for the Stokes equation

   (a) Show that (WF1) is equivalent to (WF2).

   Solution. (WF1 $\Longrightarrow$ WF2) is obvious.

   (WF2) $\Longrightarrow$ (WF1) Set $v = 0$, we have $b(u, q) = 0 \quad \forall q \in Q$. Insert this back to the formula of (WF2), we have $a(u, v) + b(v, p) = \langle f, v\rangle \quad \forall (v, q) \in V \times Q$. We then get (WF1).

   (b) Show that the bilinear form $B(\cdot,\cdot)$ on $H_0^1(\Omega;\mathbb{R}^d) \times L_0^2(\Omega)$ is not coercive, i.e., there does not exist $r > 0$ such that

   $$B((v, q), (v, q)) \geq r\|(v, q)\|_{H_0^1(\Omega;\mathbb{R}^d)\times L_0^2(\Omega)}.$$

   Solution. Set $v = 0$. We have $B((v, q), (v, q)) = a(v, v) + b(v, q) + b(v, q) = 0$. However, q is free here i.e. $\|(v, q)\|$ can be any positive real number. Hence, there does not exists r such that $B((v, q), (v, q)) \geq r\|(v, q)\|_{H_0^1(\Omega;\mathbb{R}^d)\times L_0^2(\Omega)}$ for all $(v, q) \in V \times Q$.

3. Modify the FEM code for Poisson's equation and implement the linear FEM for the convection-diffusion equation:

$$-D\Delta u + \mathbf{b} \cdot \nabla u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega,$$

where $D > 0$ and $\mathbf{b} = [1, 0]^T$.

(a) Use the exact solution $u(x_1, x_2) = (x_1 + x_2)^2 \cos(x_1 + 2x_2)$ (with f and g computed accordingly) to test your code. Test your code for $D = 1$ and $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ (corresponding to $n = 4, 8, 16, 32$) and study the order of convergence using the two norms $|u - u_h|_{H^1}$ and $\|u - u_h\|_{L^2}$ as before.

Solution. Below is the results.

Table 1: Convergence Study ($D = 1$)

n	h	$\|u - u_h\|_{L^2}$	$\lvert u - u_h\rvert_{H^1}$
4	$\frac{1}{4}$	0.098943	1.079279
8	$\frac{1}{8}$	0.025544	0.546966
16	$\frac{1}{16}$	0.006440	0.274454
32	$\frac{1}{32}$	0.001613	0.137350

We also obtained the following convergence plot (See Fig.1), which aligns with theory of qualsi-optimal approximation ($\|u - u_h\|_{L^2} = O(h^2)$ and $|u - u_h|_{H^1} = O(h)$).

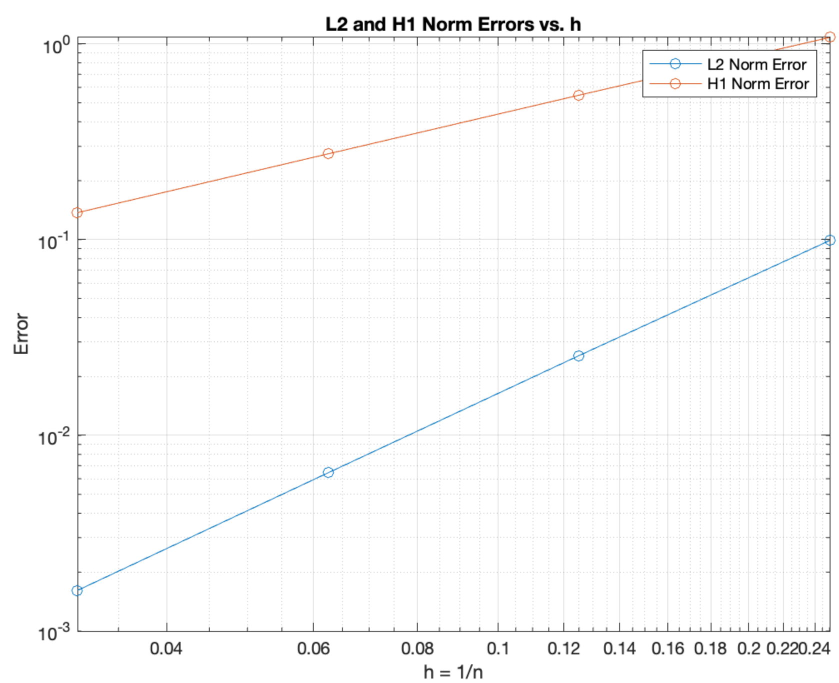


Figure 1: Convergence plot when $D = 1$

(b) We study the behavior of the method for small $D > 0$.

- Use the exact solution $u(x_1, x_2) = (x_1 + x_2)^2 \cos(x_1 + 2x_2)$ and repeat the same test as in 3a) for $D = 1 \times 10^{-7}$. Compare the convergence curve with that in 3a). What differences do you observe? Visualize the numerical solutions. What do you observe?

  Solution. The results are shown below (see Fig.2 and Table 2). Notice that when h is big the quality of solution is of very low quality.

  From the convergence plot, we still observe exponential decay in error and the order of convergence is relatively big. From the comparison of solved numerical solution and exact solution (See Fig.3),

Table 2: Convergence Study for $D = 1e - 7$

n	h	$\|u - u_h\|_2$	$\|u - u_h\|_{H^1}$
4	$\frac{1}{4}$	2239.154350	25333.706484
8	$\frac{1}{8}$	140.555553	3180.677440
16	$\frac{1}{16}$	8.788595	397.838316
32	$\frac{1}{32}$	0.549373	49.737355
64	$\frac{1}{64}$	0.034778	6.236259
128	$\frac{1}{128}$	0.002388	0.857766

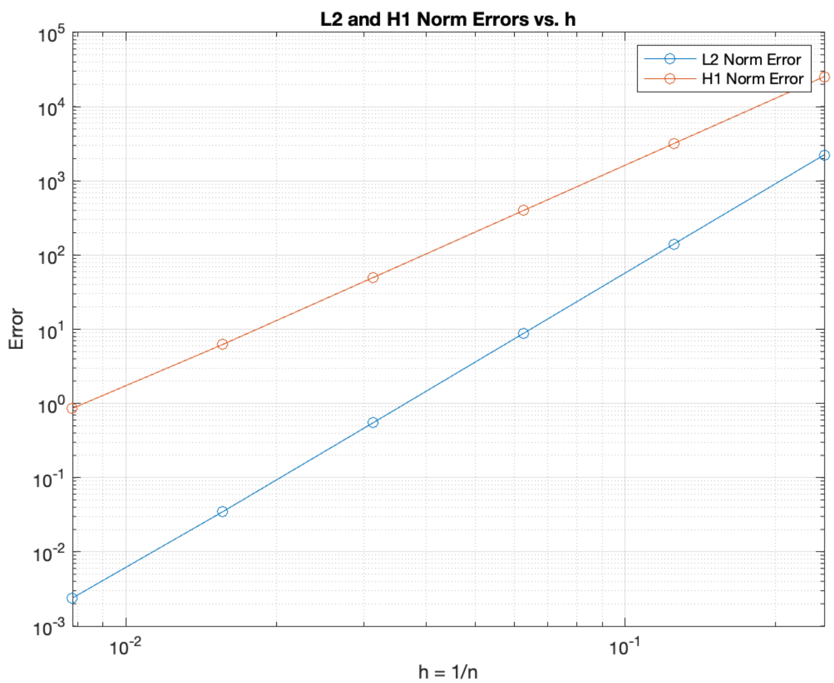


Figure 2: Convergence plot when $D = 1e - 7$

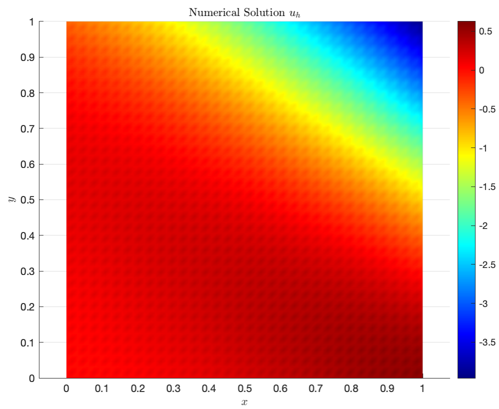
we observe that the numerical solution, though qualitatively good, is less regular than the exact solution (there's some dark red patterns indicating non-smooth solutions). This aligns with the big H_1 norm error shown in the table.

Also, we also notice that even though D is very small i.e. the system is very illly-posed, we are still able to get a good result when the mesh size is small. This is becasue the exact solution of this equation is smooth enough.

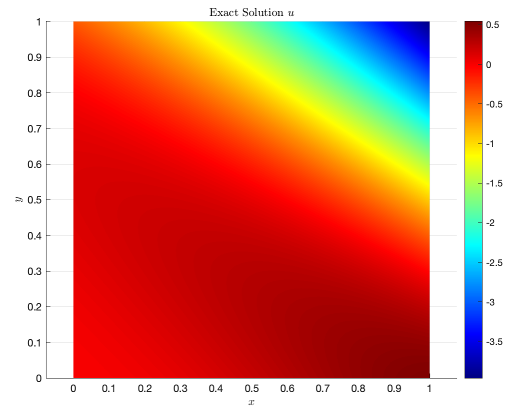
– Perform a test where the exact solution is not explicit. Choose $f \equiv 1$, $g \equiv 0$, and $D = 1 \times 10^{-3}$. By elliptic regularity theory, the exact solution should be smooth. Test your code for $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ (corresponding to $n = 4, 8, 16, 32$) and visualize the numerical solutions. Do you think the numerical results can be trusted? Try with an even larger n to visualize the numerical solution. What do you think is the reason it is harder to approximate the exact solution well in this case?

Solution. Numerical solutions are visualized in Fig.4.

We should not trust these results since they are oscillating, which contradicts with the regularity



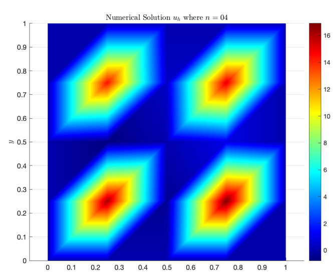
(a) u_h (numerical result $n = 64$)



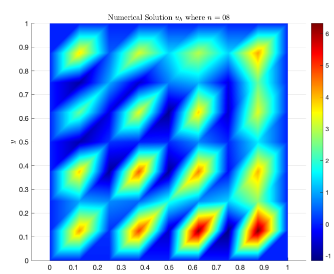
(b) u (exact result)

Figure 3: Comparison of numerical solution and exact solution when D is small.

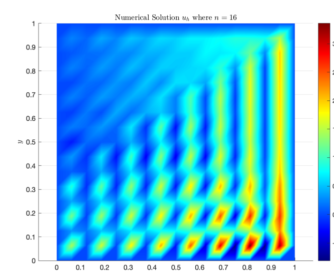
theory. When D is small, we gradually lose the covercivity (coercive constant is proportional to D), which makes the system illly-posed.



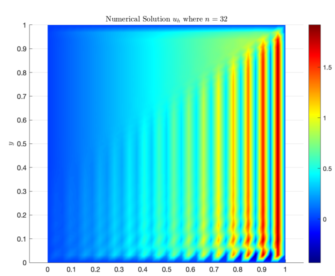
(a) $n = 4$



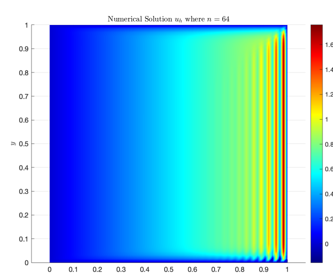
(b) $n = 8$



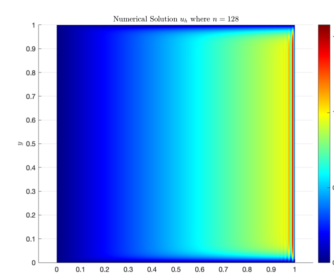
(c) $n = 16$



(d) $n = 32$



(e) $n = 64$



(f) $n = 128$

Figure 4: Numerical results for different values of n.