MATH 272B: Numerical PDE
HW2

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Problem 1: Analysis of SOR Method for the 1D Poisson Equation (50 points)

Consider the one-dimensional Poisson equation on $[0, 1]$:

$$-u''(x) = f(x), \quad \text{for } x \in (0, 1),$$

with boundary conditions:

$$u(0) = u(1) = 0.$$

Using a second-order finite difference discretization with N interior points and mesh size $h = 1/(N + 1)$, we obtain a linear system $Ax = b$.
Given results:

- The eigenvalues of the Jacobi iteration matrix are $\lambda_j = \cos(j\pi/(N + 1))$ for $j = 1, \ldots, N$.
- The spectral radius of the Jacobi iteration matrix is $\rho(M_J) = \cos(\pi/(N + 1))$.
- The Gauss-Seidel method has eigenvalues that are squares of the Jacobi eigenvalues.

(a) (25 points) For the SOR method with relaxation parameter ω, prove that if μ is an eigenvalue of the SOR iteration matrix and λ is an eigenvalue of the Jacobi iteration matrix, then:

$$(\mu + \omega - 1)^2 = \omega^2 \lambda^2 \mu.$$

Solution. The SOR iteration matrix is given by:

$$M_{\text{SOR}} = (D + \omega L)^{-1}[(1 - \omega)D - \omega U],$$

where D, L, and U are the diagonal, lower triangular, and upper triangular parts of the matrix $A = D + L + U$, respectively. Note that here we use without proof that for some $\alpha, k \neq 0$

$$\det(kD - \alpha L - \alpha^{-1}U) = \det(kD - L - U) \tag{1}$$

Let μ be an eigenvalue of M_{SOR}. Determinant argument yields

$$\det((D+\omega L)^{-1}[(1-\omega)D-\omega U]-\mu I)=0 \tag{2}$$
$$\det((1-\omega)D-\omega U-\mu(D+\omega L))=0 \tag{3}$$
$$\det((1-\omega)-\omega D^{-1}U-\mu-\mu\omega D^{-1}L)=0 \tag{4}$$
$$\det(D^{-1})\det((1-\omega-\mu)D-\omega U-\mu\omega L)=0 \tag{5}$$
$$\det(\omega\sqrt{\mu}D^{-1})\det(\frac{(1-\omega-\mu)}{\omega}D-\frac{1}{\sqrt{\mu}}U-\sqrt{\mu}L)=0 \tag{6}$$
$$\det(G_J-\frac{(1-\omega-\mu)}{\omega\sqrt{\mu}}I)=0 \tag{7}$$

It follows that $\frac{(1-\omega-\mu)}{\omega\sqrt{\mu}}=\lambda$, and conclusion follows.

(b) (8 points) Using the result from (a), show that the optimal ω satisfies:

$$\omega_{\text{opt}}=\frac{2}{1+\sqrt{1-\rho(M_J)^2}},$$

where $\rho(M_J)$ is the spectral radius of the Jacobi iteration matrix.

Solution. The solution below is lagrely based on the reference.

By rearraging the quadratic relation, we have

$$\mu^2+2(\omega-1)\mu+(\omega-1)^2=\omega^2\lambda^2\mu \tag{8}$$
$$\mu^2+(2(\omega-1)-\omega^2\lambda^2)\mu+(\omega-1)^2=0 \tag{9}$$

By solving the quadratic equation, we have

$$\mu=\frac{1}{4}\left(\omega\lambda\pm\sqrt{(\omega\lambda)^2-4(\omega-1)}\right)^2. \tag{10}$$

We are going to divide the cases based on whether $\Delta=(\omega\lambda)^2-4(\omega-1)$ is positive or not (the split point is $\tilde{\omega}:=\frac{2(1-\sqrt{1-\lambda^2})}{\lambda^2}$)

- $\Delta<0$ i.e. $\tilde{\omega}<\omega<2$. μ is then complex. $\mu=\omega-1$ and is not dependent on λ.
- $\Delta>0$ i.e. $\tilde{\omega}\geq\omega>0$. μ is real and

$$|\mu|=\frac{1}{4}\left(\omega|\lambda|+\sqrt{(\omega|\lambda|)^2-4(\omega-1)}\right)^2. \tag{11}$$

Note that here μ increases with λ so that we only need to consider when $\lambda=\rho(M_J)$. We do $\frac{d|\mu|}{dw}$ and

$$\frac{d\rho(w)}{w}=\frac{1}{2}\left(\omega\rho(M_J)+\sqrt{(\omega\rho(M_J))^2-4(\omega-1)}\right)\left(\rho(M_J)+\frac{\rho(M_J)^2w-2}{\sqrt{(\omega\rho(M_J))^2-4(\omega-1)}}\right) \tag{12}$$

Apparently, the first term is positive and its sign is determined by the second term soley. After some tedious calculation, we found $\frac{d\rho(w)}{w}<0$ at the interval of ω $(0,\tilde{w}]$. It implies the minimal spectral radius is attained at $w=\tilde{w}(\rho(M_J))$.

Overall, the optimal ω is $\frac{2(1-\sqrt{1-\rho^2})}{\rho^2}$. Note that this is differ by our goal by multiplication at both sides by $1+\sqrt{1-\rho^2}$.

(c) (9 points) Show that with this optimal ω, the spectral radius of the SOR iteration matrix is:

$$\rho(M_{\omega_{\text{opt}}}) = \frac{1-\sin(\pi/(N+1))}{1+\sin(\pi/(N+1))}.$$

Solution. Directly plug $\omega = \frac{2}{1+\sqrt{1-\cos(\pi/(N+1)^2)}} = \frac{2}{1+\sin(\pi/N+1)}$ to (11) and we have the results.

$$\begin{aligned}
\rho(M_\omega) &= \frac{1}{4}(\frac{2\cos}{1+\sin} + \sqrt{\frac{4\cos^2}{(1+\sin)^2} - 4\frac{2-1-\sin}{1+\sin}})^2 && (13)\\
&= \frac{\cos^2}{(1+\sin)^2} && (14)\\
&= \frac{1-\sin^2}{(1+\sin)^2} && (15)\\
&= \frac{1-\sin}{1+\sin} && (16)
\end{aligned}$$

Here sin and cos denote $\sin(\pi/(N+1))$ and $\cos(\pi/(N+1))$ respectively.