Math 272A homework 6

Note: Please read the collaboration policy on the syllabus.

1 (3pts).

1a) Recall the weak formulation of Poisson's equation $-\Delta u = f$ on $\Omega$ with Neumann boundary condition $\frac{\partial u}{\partial \boldsymbol{n}} = g$ on $\partial\Omega$. Show that the weak formulation is equivalent to the variational formulation

$$u = \mathrm{argmin}_{w\in H^1(\Omega)} \left( \frac{1}{2}\int_\Omega |\nabla w|^2 d\boldsymbol{x} - \int_\Omega f w d\boldsymbol{x} - \int_{\partial\Omega} g w dS \right).$$

[*Hint*: recall the variational formulation of the Dirichlet boundary value problems.]

1b) Derive the weak formulation for Poisson's equation with Robin boundary condition $a\frac{\partial u}{\partial \boldsymbol{n}} + bu = g$ on $\partial\Omega$ where $a, b$ are nonzero real numbers. [*Hint*: follow similar arguments for the derivation of the weak formulation for Dirichlet and Neumann boundary value problem.]

1c) Show that the weak formulation derived in 1b) is equivalent to the variational formulation

$$u = \mathrm{argmin}_{w\in H^1(\Omega)} \left( \frac{1}{2}\int_\Omega |\nabla w|^2 d\boldsymbol{x} - \int_\Omega f w d\boldsymbol{x} - \int_{\partial\Omega} \frac{1}{a} g w dS + \int_{\partial\Omega} \frac{b}{2a} w^2 dS \right).$$

2. (3pts) Recall (WF1) and (WF2) for the Stokes equation discussed on Nov 9.

2a) Show that (WF1) is equivalent to (WF2).

2b) Show that the bilinear form $B(\cdot,\cdot)$ on $H_0^1(\Omega;\mathbb{R}^d) \times L_0^2(\Omega)$ is not coercive, i.e., there does not exist $r > 0$ such that

$$B((\boldsymbol{v}, q), (\boldsymbol{v}, q)) \geq r\|(\boldsymbol{v}, q)\|_{H_0^1(\Omega;\mathbb{R}^d)\times L_0^2(\Omega)}$$

[*Hint*: Consider the case $\boldsymbol{v} = 0$.]

3. (4pts) (*programming problem*) Modify the FEM code for Poisson's equation and implement the linear FEM for the convection-diffusion equation

$$\begin{cases} -D\Delta u + \boldsymbol{b}\cdot\nabla u = f & \text{on } \Omega \\ \qquad\qquad\quad u = g & \text{on } \partial\Omega \end{cases}$$

where $D > 0$ and $\boldsymbol{b} = [1, 0]^T$.

3a) Use the exact solution $u(x_1, x_2) = (x_1+x_2)^2\cos(x_1+2x_2)$ (with $f$ and $g$ computed accordingly) to test your code. Test your code for $D = 1$ and $h = 1/4, 1/8, 1/16, 1/32$ (corresponding to $n = 4, 8, 16, 32$) and study the order of convergence using the two norms $|u - u_h|_{H^1}$ and $\|u - u_h\|_{L^2}$ as before.

3b) We study the behavior of the method for small $D > 0$.

- We first use the exact solution $u(x_1, x_2) = (x_1 + x_2)^2\cos(x_1 + 2x_2)$ and do the same test in 3a) for $D = 1\times 10^{-7}$. Compared with the convergence curve in 3a), what differences do you observe? Now visualize the numerical solutions. What do you observe?
- We next do a test where the exact solution is not explicit. Choose $f \equiv 1$, $g \equiv 0$ and $D = 1\times 10^{-3}$. By the elliptic regularity theory, the exact solution should be smooth. Test your code for $h = 1/4, 1/8, 1/16, 1/32$ (corresponding to $n = 4, 8, 16, 32$) and visualize the numerical solutions. Do you think the numerical results can be trusted? Try with an even larger $n$ to visualize the numerical solution. What do you think is the reason that in this case it is even harder to approximate the exact solution well?