Math 272A Take-home Final

Note:
1. There is no time limit to complete the problems. You must submit your answers via Gradescope by Wednesday, Dec 11, 11:59 pm.
2. You are allowed to collaborate with others, but you must follow the *collaboration policy* detailed in the syllabus.
3. Everyone needs to do the first four problems listed below. For the fifth problem, you are free to choose one from the three options.

1. (10pts) Use a finite difference method to discretize the problem

$$\begin{cases} -a(x)u''(x) + b(x)u'(x) = f(x) \quad x \in (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

where $a(x)$ and $b(x)$ are continuous functions and $a(x) \geq a_0 > 0$. Discuss the *truncation error* and *numerical stability* of your method (you are allowed to make assumptions on $b(x)$).

2. (10pts) Give a weak formulation and use the Lax-Milgram theorem to prove the existence and uniqueness of a weak solution of the Dirichlet problem

$$\text{(1)} \qquad \begin{cases} -\displaystyle\sum_{j,k=1}^{d} \frac{\partial}{\partial x_k}\left(a_{jk}(\boldsymbol{x})\frac{\partial u}{\partial x_j}(\boldsymbol{x})\right) + c(\boldsymbol{x})u(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega \\ \hfill u(\boldsymbol{x}) = g(\boldsymbol{x}) \quad \boldsymbol{x} \in \partial\Omega, \end{cases}$$

where $c(\boldsymbol{x}) \in L^\infty(\Omega)$ and $c(\boldsymbol{x}) \geq 0$ and $A(\boldsymbol{x}) := (a_{jk}(\boldsymbol{x}))_{j,k=1}^d \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ is symmetric and uniformly positive definite on Ω, with

$$\boldsymbol{\xi}^T A(\boldsymbol{x})\boldsymbol{\xi} \geq \lambda|\boldsymbol{\xi}|^2$$

where $\lambda > 0$ is a constant.

3. (10pts) State the Galerkin approximation to the weak formulation of equation (1). Do we have the quasi-optimal approximation property of the Galerkin approximation? Prove your claim. Suppose we use a p-th order finite element space (i.e., the space consists of continuous piecewise p-th order polynomials). What is the order of convergence of the finite element method (you may quote any result from any FEM book (e.g., Brenner-Scott))?

4. (10pts) Consider the saddle point problem: find $(u,p) \in V \times Q$ such that

$$\text{(2)} \qquad \begin{cases} a(u,v) + b(v,p) = \langle f, v\rangle \\ \hfill b(u,q) = \langle g, q\rangle \end{cases}$$

for all $(v,q) \in V \times Q$.
(4a). Suppose we take finite dimensional spaces $V_h := \text{span}\{\phi_j\}_{j=1}^n \subset V$ and $Q_h := \text{span}\{\varphi_j\}_{j=1}^m \subset Q$ and define Galerkin approximation with respect to $V_h \times Q_h$. Write down the resulting linear

system

$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} U = F. \tag{3}$$

(4b). For the Stokes system, verify that we have

$$a(v_h, v_h) \geq \gamma \|v_h\|_V^2 \quad \forall v_h \in V_h,$$

for some $\gamma > 0$. Show that for the Stokes system, the above inequality is equivalent to the positive definiteness of A_h (i.e., $\boldsymbol{x}^T A_h \boldsymbol{x} \geq c|\boldsymbol{x}|^2$ for some $c > 0$).
(4c). Under the condition that A_h is positive definite, what condition do we need on B_h for equation (3) to be solvable?

5. (10pts) (Choose one problem to complete).
(option 1) Prove the Lax-Milgram theorem using the Riesz representation theorem and the Banach fixed point theorem (for a bounded and coercive bilinear form $a : U \times U \to \mathbb{R}$.)
(option 2) State and prove Babuška's theory for the well-posedness of the problem (find $u \in U$ s.t. $b(u, v) = \langle f, v \rangle$ for all $v \in V$).
(option 3) Implement the linear FEM to solve Poisson's equation $-\Delta u = f$ on Ω with Dirichlet boundary condition $u = g$ on $\partial\Omega$ on an L-shaped domain $\Omega := (-1, 1)^2 \backslash [0, 1]^2$ (You may modify the FEM code for the 2d Poisson's equation on the square domain (HW 5)). Study the convergence rate of the method for case 1: manufactured smooth solutions (i.e., take a smooth function u and compute the corresponding right hand side data f and the boundary data g), and case 2: $f \equiv 1$ and $g \equiv 0$. In the second case, the exact solution is unknown. You may take the FEM solution u_h with a very small h and use it as (an approximation of) the exact solution for computing convergence rates in this case.