Spring 2022: Numerical Analysis
Assignment 1 (due Feb. 13, 2022 at 11:59pm ET)

Homework submission. Homework assignments must be submitted through Gradescope. Please hand in cleanly handwritten or typed (preferably with LaTeX—I provide the source files of these assignments if you want to use them to learn LaTeX) homeworks. If you are required to hand in code or code listings, this will explicitly be stated on that homework assignment (if nothing is stated, you are not required to hand in code listings).

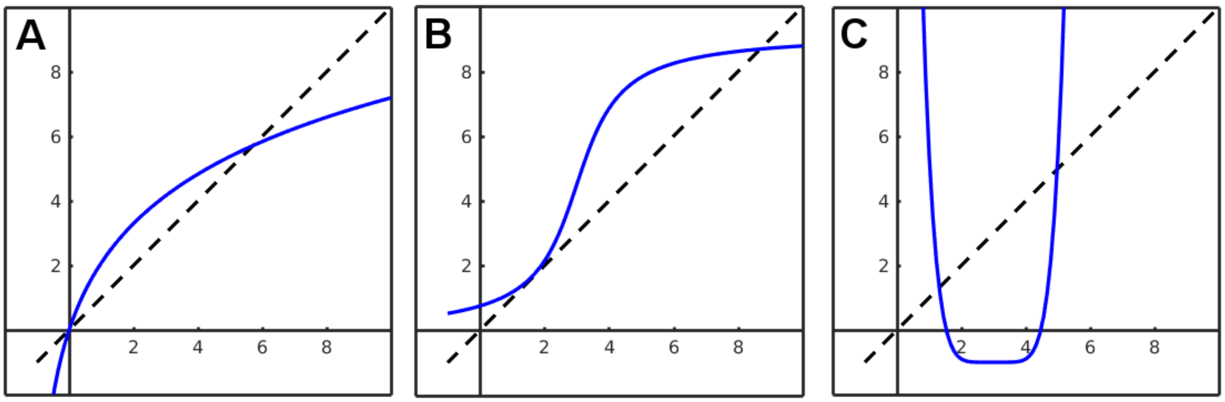
Collaboration. NYU's integrity policies will be enforced. You are encouraged to discuss the problems with other students in person or on Campuswire. However, you must write (i.e., type) every line of code yourself and also write up your solutions independently. Copying of any portion of someone else's solution/code or allowing others to copy your solution/code is considered cheating.

Plotting and formatting. Plot figures carefully and think about what you want to illustrate with a plot. Choose proper ranges and scales (`semilogx, semilogy, loglog`), always label axes, and give meaningful titles. Sometimes, using a table can be useful, but never submit pages filled with numbers. Discuss what we can observe in and learn from a plot. Use `format compact` and other format commands to control outputs. When you create figures in MATLAB (or Python, Julia), please export them in a vector graphics format (.eps, .pdf, .dxf) rather than raster graphics or bitmaps (.jpg, .png, .gif, .tif). Vector graphics-based plots avoid pixelation and thus look much cleaner.

Programming. This is an essential part of this class. We will use MATLAB for demonstration purposes in class, but you are free to use other languages (Python, Julia). Basic programming skills are crucial for many jobs, so this is also a good chance to get more comfortable with it, if you aren't already. In your programs, please use meaningful variable names and try to write clean, concise and easy-to-read code. Comments for explanation are a crucial part of every program.

1. **[1+1+1+1pt]** Let $f(x) = e^x - x^2 - 2x - 1$ and $g(x) = 2\ln(x+1)$, where $x \in (-1, \infty)$.
   (a) Verify that the roots of $f(x)$ are the same as the fixed points of $g(x)$.
   (b) Sketch $y = g(x)$, $y = x$ and indicate all fixed points (sketch by hand or plot with a computer). You don't need to calculate them. (Hint for the sketch: Note that $g'(0) > 1$).
   (c) Use Brouwer's fixed point theorem to argue the existence of a fixed point ξ in the interval $[a, b] = [e - 1, e^2 - 1]$.
   (d) Use the contraction mapping theorem to show that ξ is the only fixed point in the interval $[e - 1, e^2 - 1]$.

2. **[2+2pt]** Stability of fixed points.
   (a) For each of the three functions (solid lines) depicted below,
      (i) Write down the approximate values of the fixed points (as estimated by eye).
      (ii) State for each fixed point, whether it is stable, unstable or neither of the two.
   (b) You are given the first ten iterates of two sequences, x_k and y_k, both of which converge to zero:



k	x_k	y_k
0	1.0000000000000	1.0000000000000
1	0.3000000000000	0.6648383611734
2	0.0900000000000	0.4404850619261
3	0.0270000000000	0.2895527955097
4	0.0081000000000	0.1869046766665
5	0.0024300000000	0.1155100169867
6	0.0007290000000	0.0638472856062
7	0.0002187000000	0.0254178900244
8	0.0000656100000	0.0032236709627
9	0.0000196830000	0.0000080907744
10	0.0000059049000	0.0000000000001

(i) What is (most likely) the order of convergence of x_k? Explain your answer.

(ii) What is (most likely) the order of convergence of y_k? Explain your answer[1].

3. **[2+1+1pt]** Let g be defined on $[5\pi/8, 11\pi/8]$.

$$g(x) = x + 0.8\sin x.$$

(a) Determine the (smallest possible) Lipschitz constant L.

(b) How many iterations are required to increase the accuracy by a factor of 100, i.e., given some $x_0 \in [1,2]$, when what is k such that you can guarantee that $|x_k - \xi| \leq 10^{-2}|x_0 - \xi|$?

(c) Starting with initial guess $x_0 = 5\pi/8$, compute the first fixed point iterate x_1 and use it, together with the Lipschitz constant you found, to compute after how many fixed point iterations k you can be certain that $|\xi - x_k| < 10^{-10}$.

4. **[1+1+1+1pt]** The equation

$$f(x) := x^2 - 5 = 0,$$

has a single root $\xi = \sqrt{5} \approx 2.2361\ldots$ in the interval $[1,3]$. Consider the fixed point iteration $x_{k+1} = g(x_k)$, where g is defined as one of the following options:

- $g_1(x) = 5 + x - x^2$,

[1] Note that we typically thing of local convergence, i.e., close to the limit point.

- $g_2(x) = 1 + x - \frac{1}{5}x^2$,
- $g_3(x) = \frac{1}{2}x + \frac{5}{2x^2}$.

(a) Identify the fixed point functions for which the fixed point is also a root of f.

(b) For the cases where computing the fixed point is equivalent with solving $f(x) = 0$, discuss whether the fixed point iteration is guaranteed to converge in some neighborhood of ξ.

(c) If the iteration in b) is guaranteed to converge, compute the value of

$$\lim_{k\to\infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|}$$

Hint: Everything is easier with the mean value theorem!

(d) Give Newton's method for solving $f(x) = 0$ and, for given starting value $x_0 = 2$ compute the first Newton iterate x_1.

5. **[3pt]** Define the function g by $g(0) = 0$, $g(x) = -x\sin^2(1/x)$ for $0 < x \leq 1$. Show that g is continuous, and that 0 is the only fixed point of g in the interval $[0, 1]$. By considering the iteration $x_{n+1} = g(x_n)$, with $x_0 = 1/(k\pi)$ and $x_0 = 2/((2k+1)\pi)$, where k is an integer, show that using the definition of stability provided in class, $\xi = 0$ is neither stable nor unstable.

6. **[2+2pt]** Raytracing is an algorithm that involves finding the point at which a ray (a line with a direction and an origin) intersects a curve or surface. We will consider a ray intersecting with an ellipse. The general equation for an ellipse is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1 = 0$$

and the equation for a ray starting from the point $P_0 = [x_0, y_0]$ in the direction $\mathbf{V}_0 = [u_0, v_0]$, is

$$\mathbf{R}(t) = [x_0 + tu_0, y_0 + tv_0]$$

where $t \in [0, \infty)$ parameterizes the ray. In this problem we will take $a = 3$, $b = 2$, $P_0 = [0, b]$, $\mathbf{V}_0 = [1, -0.3]$. Using your favorite root finding algorithm write a code which computes the intersection of the given ray and the ellipse and plot your results. .

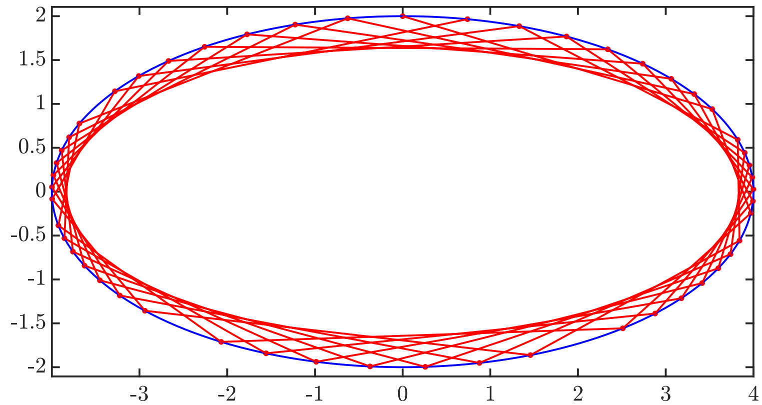
(a) Plug the equation for the ray, $\mathbf{R}(t)$, into the equation for the ellipse and analytically (with pen and paper) solve for the value of t which gives the point of intersection, call it t_i.

(b) Perform the same calculation numerically using your favorite root finder. Report your answer to within an error of 10^{-6} and justify how you found the minimum number of iterations required to achieve this tolerance. Also report the point of intersection $P_i = \mathbf{R}(t_i)$

7. **[4pt, extra credit]** If the walls of the ellipse are perfect mirrors, a ray of light will reflect infinitely around within the ellipse. We will write an algorithm to compute it's trajectory. Using the same parameters as the previous problem. Implement the following algorithm for 50 steps. At step k of the process, we are given a point on the ellipse $P_k = [x_k, y_k]$ and a ray direction $\mathbf{V}_k = [u_k, v_k]$

(a) Using P_k and $\mathbf{V}_k$ and your favorite root finder, calculate the value of t corresponding to the point of intersection call it t_i^k.

(b) Compute $P_{k+1} = \mathbf{R}(t_i^k)$.



(c) Compute the normal vector at $P_{k+1} = [x_{k+1}, y_{k+1}]$ as $\mathbf{N}_{k+1} = [\frac{b}{a}x_{k+1}, \frac{a}{b}y_{k+1}]$. Using this compute the UNIT normal $\mathbf{n}_{k+1} = \mathbf{N}_{k+1}/||\mathbf{N}_{k+1}||_2$.

(d) Compute the unit ray vector $\mathbf{w}_k = \mathbf{V}_k/||\mathbf{V}_k||_2$ and update the new ray vector using the reflection formula

$$\mathbf{V}_{k+1} = \mathbf{w}_k - 2\left(\mathbf{w}_k \cdot \mathbf{n}_{k+1}\right)\mathbf{n}_{k+1}$$

(e) repeat from step 1 using $\mathbf{V}_{k+1}$ and P_{k+1}.

Plot the trajectory (all of the points that you computed with line connecting them) as well as the ellipse. Also report the 50th point of intersection. Note that the shape that encloses all of the reflected rays is called a 'caustic'.