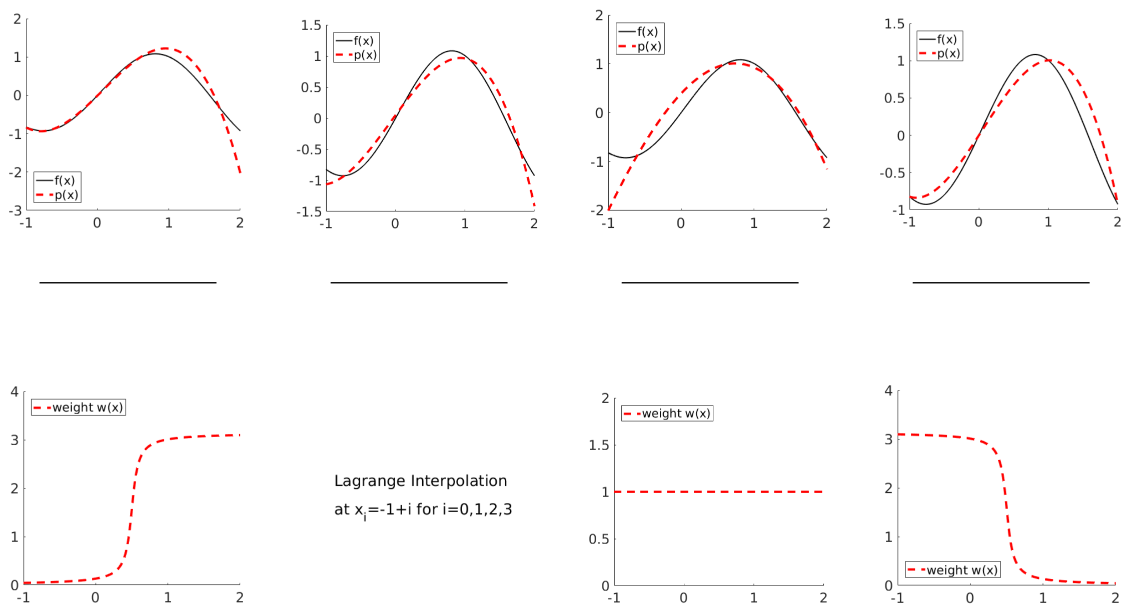
Spring 2022: Numerical Analysis
Assignment 7 (due May 12, 2022 at 11:59pm ET)

1. **[Best 2-norm approximation, 2+3pt]**

   (a) The upper row in the below figure shows a function f together with a polynomial approximation. For three plots, the optimal best 2-norm fit for three different weights $w(x)$ is used, and one is the result of an Lagrange interpolation. Match the approximations in the upper row with the information (weight functions or interpolation points) in the lower row.



(A) (B) (C) (D)

   (b) Let $\{\varphi_0, \varphi_1, \varphi_2\}$ be a system of ortho*normal* polynomials on $[-1, 1]$ with respect to the weight function $w(x) = \sqrt{1 - x^2}$ given by

$$\varphi_0(x) = \sqrt{\frac{2}{\pi}}, \quad \varphi_1(x) = 2x\sqrt{\frac{2}{\pi}}, \quad \varphi_2(x) = (4x^2 - 1)\sqrt{\frac{2}{\pi}}.$$

   Given $f(x) = \frac{2}{\sqrt{1-x^2}}$, find the polynomial best fit of degree 2 in the weighted 2-norm.

2. **[Interpolation and optimal 2-norm approximation, 2+2+2+2pt]** For an interval (a, b), $n \in \mathbb{N}$ and disjoint points $x_0, \dots, x_n$ in $[a, b]$, we define[1] for polynomials p, q

$$\langle p, q\rangle := \sum_{i=0}^{n} p(x_i)q(x_i).$$

   (a) Show that $\langle \cdot, \cdot \rangle$ is an inner product for each $\mathcal{P}_k$ with $k \le n$, where $\mathcal{P}_k$ denotes the space of polynomials of degree k or less.

[1] This problem highlights a relationship between optimal 2-norm interpolation and interpolation. You can think of the inner product as obtained from a weighted 2-norm inner product as limit of weight functions w that are very large at the node points and small or zero elsewhere.

(b) Why is $\langle \cdot\,, \cdot \rangle$ not an inner product for $k > n$?

(c) Show that the Lagrange polynomials L_i corresponding to the nodes $x_0, \dots, x_n$ are orthonormal with respect to the inner product $\langle \cdot\,, \cdot \rangle$.

(d) For a continuous function $f : [a, b] \to \mathbb{R}$, compute its optimal approximation in $\mathcal{P}_n$ with respect to the inner product $\langle \cdot\,, \cdot \rangle$ and compare with the interpolation of f.

3. **[Newton-Cotes vs. Gauss Quadrature, 2+2+2+1pt]** We discussed two methods to integrate functions numerically, namely the Newton-Cotes formulas and Gauss quadrature.

   (a) Recall that we calculated the first three orthogonal polynimals with respect to $w \equiv 1$ on $(0, 1)$ in class to be $\{\varphi_0, \varphi_1, \varphi_2\} = \{1, x - 1/2, x^2 - x + 1/6\}$. Calculate $\varphi_3(x)$ using the ansatz $\varphi_3(x) = x^3 - a_2\varphi_2(x) - a_1\varphi_1(x) - a_0\varphi_0(x)$, with appropriately computed $a_2, a_1, a_0 \in \mathbb{R}$.

   (b) Derive the Gaussian Quadrature formula for $n = 2$, i.e., calculate both the quadrature points x_0, x_1, x_2 (these are the roots of φ_3 and the corresponding weights W_0, W_1, W_2.[2]

   (c) Now we want to compare Gaussian quadrature derived in (b) with the Simpson's Rule. Use both methods to numerically find

   $$I_k = \int_0^1 x^k\, dx, \qquad \text{for} \quad k = 0, \dots, 7.$$

   Plot the errors arising in each method as a function of k. Note that to find the error, you will need to calculate the exact values for I_k (by hand).

   (d) Explain your findings using the results on the exact integration for polynomials up to certain degrees discussed in class.

4. **[Orthogonal polynomials on $[0, \infty)$, 2+2+2pt extra credit]**

   (a) Find orthogonal polynomials l_0, l_1, l_2, l_3 for the unbounded interval $[0, \infty)$ with the weight function $\omega(x) = \exp(-x)$.[3] Plot these polynomials (they are called *Laguerre polynomials*).

   (b) As these are orthogonal polynomials, they correspond to a quadrature rule for weighted integrals on $[0, \infty)$. The resulting quadrature points and weight are given in Table 1. Verify

   Table 1: Gauss quadrature points and weights for quadrature on $[0, \infty)$.

n	x_i	W_i
2	0.585786	0.853553
	3.41421	0.146447
3	0.415775	0.711093
	2.29428	0.278518
	6.28995	0.0103893
4	0.322548	0.603154
	1.74576	0.357419
	4.53662	0.0388879
	9.39507	0.000539295

   that for $n = 2$, $n = 3$, the quadrature nodes x_i are the roots of the polynomials $l_2(x), l_3(x)$ (up to round-off).

[2] See equation (10.7) in the book.

[3] Feel free to look up the values for the indefinite integrals $\int_0^\infty \exp(-t)t^k\, dx$ ($k = 0, 1, 2, 3$)—I use Wolfram Alpha for looking up things like that: http://www.wolframalpha.com/.

(c) Use the quadrature rules from Table 1 to approximate the integrals

$$\int_0^\infty \exp(-x)\exp(-x)\,dx \quad \text{and} \quad \int_0^\infty \exp(-x^2)\,dx.$$

Note that, to take into account the weight $\omega(x) = \exp(-x)$, for the first integral $f(x) = \exp(-x)$ and for the second $f(x) = \exp(-x^2 + x)$. Report the errors for $n = 2, 3, 4$ using that the exact values for the integrals are $1/2$ and $\sqrt{\pi}/2$.