Spring 2022: Numerical Analysis
Assignment 6 (due May 2, 2022 at 11:59pm ET)

1. **[Space of polynomials P_n, 2+2+2pts]** Let P_n be the space of functions defined on $[-1,1]$ that can be described by polynomials of degree less of equal to n with coefficients in $\mathbb{R}$. P_n is a linear space in the sense of linear algebra, in particular, for $p, q \in P_n$ and $a \in \mathbb{R}$, also $p+q$ and ap are in P_n. Since the monomials $\{1, x, x^2, \dots, x^n\}$ are a basis for P_n, the dimension of that space is $n+1$.

   (a) Show that for pairwise distinct points $x_0, x_1, \dots, x_n \in [-1,1]$, the Lagrange polynomials $L_k(x)$ are in P_n, and that they are linearly independent, that is, for a linear combination of the zero polynomial with Lagrange polynomials with coefficients α_k, i.e.,

   $$\sum_{k=0}^{n} \alpha_k L_k(x) = 0 \text{ (the zero polynomial)}$$

   necessarily follows that $\alpha_0 = \alpha_1 = \ldots = \alpha_n = 0$. Note that this implies that the $(n+1)$ Lagrange polynomials also form a basis of P_n.

   (b) Since both the monomials and the Lagrange polynomials are a basis of P_n, each $p \in P_n$ can be written as linear combination of monomials as well as Lagrange polynomials, i.e.,

   $$p(x) = \sum_{k=0}^{n} \alpha_k L_k(x) = \sum_{k=0}^{n} \beta_k x^k, \tag{1}$$

   with appropriate coefficients $\alpha_k, \beta_k \in \mathbb{R}$. As you know from basic matrix theory, there exists a basis transformation matrix that converts the coefficients $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_n)^T$ to the coefficients $\boldsymbol{\beta} = (\beta_0, \dots, \beta_n)^T$. Show that this basis transformation matrix is given by the so-called Vandermonde matrix $V \in \mathbb{R}^{n+1 \times n+1}$ given by

   $$V = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^n \end{pmatrix},$$

   i.e., the relation between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in (1) is given by $\boldsymbol{\alpha} = V\boldsymbol{\beta}$. An easy way to see this is to choose appropriate x in (1).

   (c) Note that since V transforms one basis into another basis, it must be an invertible matrix. Let us compute the condition number of V numerically.[1] Compute the 2-based condition number $\kappa_2(V)$ for $n = 5, 10, 20, 30$ with uniformly spaced nodes $x_i = -1 + (2i)/n$, $i = 0, \dots, n$. Based on the condition numbers, can this basis transformation be performed accurately?

2. **[Hermite interpolation, 1+2+1+2pts]** We are given distinct interpolation points x_i, $i = 0, \dots, n$. In class we introduced the Hermite interpolation polynomials $H_k(x)$ and $K_k(x)$ as follows:

   $$H_k(x) = [L_k(x)]^2(1 - 2L_k'(x_k)(x - x_k)), \qquad K_k(x) = [L_k(x)]^2(x - x_k),$$

   where L_k are the Lagrange polynomials.

[1] MATLAB provides the function `vander`, which can be used to assemble V (actually, the transpose of V). Alternatively, one can use a simple loop to construct V.

(a) Show that $H_k, K_k \in P_{2n+1}$, i.e., they are polynomials of degree $2n+1$ or lower.

(b) Show that H_k, K_k as defined above satisfy the following conditions:

$$H_k(x_i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k, \end{cases} \quad \text{and} \quad H_k'(x_i) = 0, \quad i = 0, \dots, n.$$

$$K_k'(x_i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k, \end{cases} \quad \text{and} \quad K_k(x_i) = 0, \quad i = 0, \dots, n.$$

(c) Argue that H_k, K_k are the unique polynomials in P_{2n+1} satisfying the conditions in (b).

(d) Find a (Hermite) polynomial $p_3 \in \boldsymbol{P}_3$ that interpolates $f(x) := 3\exp(x)$ and f' in $x_0 = 0, x_1 = 1/2$. Give the polynomial p_3 in the Hermite basis, plot f and p_3 in the same graph, and plot the four Hermite basis functions in another graph.

3. **[Newton-Cotes Rules, 3+2+2pts]**

(a) For $f(x) = x^2 + x + 1$ on $[0, 2]$, we are interested in numerical approximations of the integral $I := \int_0^2 f(x)\, dx$. By splitting the integral into two,

$$\int_0^2 f(x)\, dx = \int_0^1 f(x)\, dx + \int_1^2 f(x)\, dx,$$

and using the trapezoidal rule on the subintervals $[0, 1]$ and $[1, 2]$, find an approximation of I (i.e., use the composite Trapeziodal rule for two subintervals). Also, use the Simpson's rule on $[0, 2]$ to approximate I. Which of the two approximations is more accurate, and why?

(b) The error estimate for the Simpson's rule is given by

$$|E_2(f)| \leq \frac{(b-a)^5}{2880} M_4,$$

where $M_4 = \max_{x \in [a,b]} |f^{(iv)}(x)|$. Here, $f^{(iv)}$ denotes the 4-th derivative of f. Use the error estimate to explain which functions f are integrated exactly by the Simpson's rule.

(c) Let $f(x) = \frac{1}{4}x^4 + \sin(x)$. According to the error estimate, what is the maximal error you will make when integrating f over $[0, \pi]$? You do not need to calculate the approximate integral.

4. **[Errors in polynomial interpolation, extra credit, 3pt]** Interpolate the function

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

on the domain $[-1, 1]$ using Lagrange polynomials with Chebyshev points.[2] You can use the following MATLAB function `lagrange_interpolant` to compute the values of the Lagrange interpolants p_n.

[2]Recall that the Chebyshev points on the interval $[a, b]$ are

$$x_i = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\left(\frac{i+\frac{1}{2}}{n+1}\pi\right) \text{ for } i = 0, \dots, n.$$

```
    function y0 = lagrange_interpolant(x, y, x0)
    % x is the vector of abscissas.
    % y is the matching vector of ordinates.
    % x0 represents the target to be interpolated
    % y0 represents the solution from the Lagrange interpolation
y0 = 0;
n = length(x);
for j = 1 : n
    t = 1;
    for i = 1 : n
        if i~=j
            t = t * (x0-x(i))/(x(j)-x(i));
        end
    end
    y0 = y0 + t*y(j);
end
```

Describe qualitatively what you see for $n = 2, 4, 8, 16, 32, 64, 128, 256$ interpolation points. Provide a table of the maximum errors[3]

$$||p_n - f||_\infty = \max_{x\in[-1,1]} |p_n(x) - f(x)|,$$

and the L_2-errors[4]

$$||p_n - f||_2 = \sqrt{\int_{-1}^{1} (p_n(x) - f(x))^2 dx}$$

for each $n = 2, 4, 8, 16, 32, 64, 128, 256$. Do you expect convergence in the maximum norm? How about in the L_2 norm?

5. **[Composite trapezoidal and Simpson sum, extra credit, 2+2+2pt]** Write codes[5] to approximate integrals of the form

$$I(f) = \int_a^b f(t)\, dt$$

using the trapezoidal and Simpson's rule on the sub-intervals $[x_{i-1}, x_i]$, $i = 1, \ldots, m$, where $x_i = a + ih$, $i = 0, \ldots, m$ with $h = (b-a)/m$.[6]

[3] You can approximate the maximum error by evaluating the error $p_n - f$ at a large number of uniformly distributed points, e.g., at $\sim 10n$ points, and determining the difference using the maximum absolute value, i.e.

$$||p_n - f||_\infty = \max_{x\in[-1,1]} |p_n(x) - f(x)| \approx \max_{j=0,\ldots,10n} |p_n(\xi_j) - f(\xi_j)|,$$

where $\xi_j = -1 + \frac{2}{10n} j$ for $j = 0, \ldots, 10n$.

[4] You can approximate the L_2-error by evaluating the error $p_n - f$ at a large number of uniformly distributed points, e.g., at $\sim 10n$ points, and computing

$$||p_n - f||_2 = \sqrt{\int_{-1}^{1} (p_n(x) - f(x))^2 dx} \approx \sqrt{\frac{2}{10n} \sum_{j=0}^{10n} (p_n(\xi_j) - f(\xi_j))^2},$$

where $\xi_j = -1 + \frac{2}{10n} j$ for $j = 0, \ldots, 10n$.

[5] Ideally, you write functions `trapez(f,a,b,m)` and `simpson(f,a,b,m)`, where f is a function handle (see http://www.mathworks.com/help/matlab/matlab_prog/creating-a-function-handle.html if you are not familiar with that concept) or f is the vector $(f(x_0), \ldots, f(x_m))$.

[6] For these composite rules, see Definitions 7.1 and 7.2 in the book.

(a) Hand in listings of your codes, and use them to approximate the integral

$$\int_{0.1}^{1} \sqrt{x}\, dx.$$

Compare the numerical errors $\mathcal{E}$ for both quadrature rules (the exact value of the integral is $\frac{2}{3} - \frac{1}{15\sqrt{10}}$). Try different m (e.g., $m = 10, 20, 40, 80, \ldots$) and plot the quadrature errors versus m in a double-logarithmic plot.

(b) To numerically study how the errors $\mathcal{E}$ decrease with m, we assume that the errors behaves like Cm^{κ}, with to-be-determined $C, \kappa \in \mathbb{R}$. Applying the logarithm to $\mathcal{E} = Cm^{\kappa}$ results in

$$\log(\mathcal{E}) = D + \kappa \log(m), \tag{2}$$

where $D = \log(C)$. Use the values for m and $\log(\mathcal{E})$ you computed in (a) to find the best-fitting values for D and κ in (2) by solving a least squares problem. Compare your findings for κ with the theoretical estimates for the composite trapezoidal and Simpson's rules.[7]

(c) Repeat steps (a) and (b) using $a = 0$ instead of $a = 0.1$ as lower integration bound. Can the theoretical estimates for the composite rules still be applied and why/why not?

[7] Compare with (7.16) and (7.18) in the book. You can ignore the constants, just compare κ, the exponent of m, with the theoretical results.