Cheat Sheet for Numerical Analysis SP 22. $\boxed{\text{midterm}}$

Root Finding.

Thm. Continuously contraction $\to$ Unique FP. Convergence. (Fixed Point).

Iteration formula Given x_0. L. x_0.

$$k_0(\varepsilon) = \left[\frac{\ln(x_1 - x_0) - \ln(\varepsilon(1-L))}{\ln(\frac{1}{L})}\right] + 1$$

Iteration does not depend on ξ.

Thm. Local Contraction $\Leftarrow\Rightarrow |f'(\xi)| < 1$

Speed of Convergence: $\lim_{n\to\infty} \frac{|x_{n+1} - \xi|}{|x_n - \xi|} = \mu$. $\begin{cases} \geq 1 & \text{Sublinear} \\ < 1 & \text{linear} \\ = 0 & \text{superlinear.} \end{cases}$ asymptotic rate. $\rho = -\log_{10}\mu$.

For ~~fixed~~ FP. $\lim_{n\to\infty} \frac{|x_{n+1} - \xi|}{|x_n - \xi|} = |g'(\xi)|$

[Newton's Method]

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Speed of Convergence:

$$\frac{|x_{k+1} - \xi|}{|x_k - \xi|^q} \to \mu < +\infty$$

$q = 2$ quadratic.
every iteration doubling its digits.

Main Thm of Newton's Method.

f is C_2 on $I_\delta = [\xi - \delta, \xi + \delta]$ $\delta > 0$. $f(\xi) = 0$ $f'(\xi) \neq 0$ suppose $\exists A > 0$ s.t. $\left|\frac{f''(y)}{f'(x)}\right| \leq A$ for all $x, y \in I_\delta$. Then. If $|\xi - x_0| \leq h$. $h = \min(\delta, \frac{1}{A})$. (x_k) converges to ξ quadratically.

$$\frac{|x_{k+1} - \xi|}{|x_k - \xi|^2} = \frac{|f''(x_k)|}{2|f'(x_k)|} \xrightarrow{k\to\infty} \frac{|f''(\xi)|}{2|f'(\xi)|}$$

[Secant Method]

Similarly: x_0, x_1 are close enough to ξ. the sequence x_k from Secant converges linear to ξ.

Pros: No need for f'

Cons: SLOW. $q \sim 1.4$

Bisection: Pros: only continuity is needed.
Cons: Meet then 1 root. Dummo.

Linear System:

Existence of / Idea of LU. $L_N \cdots L_1 A = U$ Unit lower. This is the idea of LU decomposition. Gauss.

$N = \frac{n(n-1)}{2}$

$$l_{ij} = \frac{1}{u_{jj}}\left\{a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}\right\}$$

$i = 2, \dots, n$

$j = 1, \dots, i-1$

Then $A = LU$

$Ax = b \Leftarrow\Rightarrow LUx = b$

$\Rightarrow \begin{cases} Ly = b \\ Ux = y \end{cases}$

$$L = \begin{bmatrix} 1 & & \\ l_{21} & \ddots & \\ l_{n1} & \cdots & 1 \end{bmatrix} \quad \begin{bmatrix} u_{11} & & \\ & \ddots & \\ & & u_{nn} \end{bmatrix}$$

$u_{1j} = a_{1j} \cdots$

$l_{i1} = \frac{a_{i1}}{u_{11}}$

$u_{2j} = a_{2j} - l_{21}u_{1j}$ $j = 2 \dots n$.

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj}$$

$i = 1, \dots, n$

$j = i, \dots, n$

FLOPs:

LU-Decompose:

$$\sum_{i=2}^{n}\sum_{j=1}^{i-1}(2j-1)+\sum_{i=1}^{n}\sum_{j=i}^{n}(2i-2)=\frac{1}{6}n(n-1)(4n+1)\sim\frac{2}{3}n^3$$

Back / Forward Substitution:

$$\sum_{i=1}^{n}2(i-1)=\frac{2n(n-1)}{2}=n^2-n\sim n^2$$

Complexity of Solving $Ax=b$.

$\frac{2}{3}n^3+2n^2$.

Multiple $\{b_i\}_{i=1}^{k}$. $\frac{2}{3}n^3+2kn^2$

Inverse: $\frac{2}{3}n^3+n\cdot 2n^2=\frac{2}{3}n^3+2n^3$

Multiply Ax: $2n^2=2n^2+n^2$

AB: $=2n^3$

Thm: $A\in\mathbb{R}^{n\times n}$. Every leading principle submatrix $A^{(k)}$ $k\in\{1,\dots n-1\}$ nonsingular $\Rightarrow \exists$ LU decomposition. (necessity not articulated)

Pivoting & Procedure: Pivoting makes LU decomposition possible for every matrix).

Thm. $PA=LU$ Put the biggest to the diagonal. Trick. $\underbrace{P_k M_1 M_2 P_k}_{\text{lower.}} P_k A=U$

Condition Number:

Norm. Induced Norm $\begin{cases}1\text{ norm } \max(\text{Column Sum})\\ \infty\text{ norm } \max(\text{row Sum})\\ 2\text{ norm } \max\sqrt{\lambda_i\, A^TA}.\end{cases}$

(3)

$\circ\ \lambda\ \Delta$

Define: $k(A)=\|A\|\cdot\|A^{-1}\|$.

$\|b\|=\|Ax\|\le\|A\|\,\|x\|$

$\|\Delta x\|=\|A^{-1}\Delta b\|\le\|A^{-1}\|\|\Delta b\|$ $\Rightarrow \frac{\|\Delta x\|}{\|x\|}\le\|A\|\,\|A^{-1}\|\frac{\|\Delta b\|}{\|b\|}$

whether it is poorly / well- depends on the norm chosen.

Least Square $\underset{m\times n\ \ m>n.}{Ax=b}$ $\min_x\|Ax-b\|_2^2 \Leftrightarrow A^TAx=A^Tb$

Floating Points.

Mantissa. (ABS)

Exponent.

sign

Single precision: $N_s+N_p+N_f=1+8+23=32$ bits.

Double precision: $N_s+N_p+N_f=1+11+52=64$ bits.

$$\frac{|\hat{x}-x|}{|x|}\le u=2^{-(N_f+1)}=\begin{cases}2^{-24}\approx 6.0\times10^{-8} & 8\text{ or }7\text{ digits}\\ 2^{-53}\approx 1.1\times10^{-16} & 16\text{ digits.}\end{cases}$$

$$\varepsilon=\begin{cases}2^{-23} & 1.2\times10^{-7}\\ 2^{-52} & 2.2\times10^{-16}\end{cases}$$

what is dangerous:

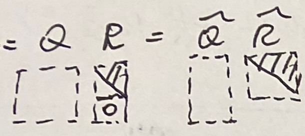
$\{$ Addition of ~~1 and 10~~ widely-differing magnitude.

Substraction of close numbers.

Cheat Sheet For Numerical Analysis {p 22} Final !!

QR-decomposition

$A = Q\ R = \hat{Q}\ \hat{R}$

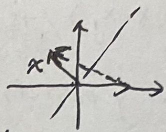


- why: to do least square estimation. $\min_x \|Ax-b\|_2^2 \cong \min_x \|Rx - Q^T b\|$ and we can solve that very quick. 2° condition num better.

How: use Householder matrix.

$H_v = I - \frac{2}{v^T v} v v^T$ (symmetric, orthonormal)

In practice, we find H that transform x to e_i (punchline is projection)



set $v = x + c e_1$ where $c = \begin{cases} \text{sgn}(\beta) \|x\| & \text{if } \beta \neq 0 \\ \|x\| & \beta = 0 \end{cases}$

Then we can deal with sub matrices (punchline is augment by 1 on top left corner).

$H_n \cdots H_1 A = R \Rightarrow A = \underbrace{(H_1 \cdots H_n)}_{Q} R$. and economy form follows.

Eigenvalues

For symmetric Real matrix, it is real O.N. diagonalizable.

G Thm: Eigenvalue lies in the disc. $D_i = \{z \in \mathbb{C} \mid |z - a_{ii}| \le R_i\}$ with $R_i = \sum_{j=1, j\neq i}^{n} |a_{ij}|$

In each cluster, we have # of eigenvalues # of discs.

3 ways to compute eigenvalues.

1° Power method. (for $|\lambda_1| > |\lambda_2| \ge |\lambda_3| \cdots$)

with: x_0 which is not orthogonal to the eigenvector of λ_1.

$x_{k+1} = \frac{A x_k}{\|A x_k\|}$ $\Rightarrow$ it finally converges to eigenvector for λ_1 ($:= \eta_1$)

$x_k = A x_{k-1} = \cdots A^k x_0 = \sum_{i=1} \alpha_i A^k \eta_i = \sum_{i=1}^{n} \alpha_i \lambda_i^k \eta_i = \alpha_1 \lambda_1^k (\eta_1 + \sum_{i=2}^{n} \frac{\alpha_i}{\alpha_1} \left(\frac{\lambda_i}{\lambda_1}\right)^k \eta_i)$

Rate: linear.

Given y the eigenvector, $\lambda = \frac{y^T A y}{y^T y}$. $\frac{|\lambda_1|}{|\lambda_2|} \uparrow$ convergence speed $\uparrow$. only allows for the max (λ)

2° Inverse method. (punchline. use $(A - \theta I)^{-1}$ to do power method.) Now the biggest become $\min(\lambda_i - \theta)$ is the eigenvalue closes to θ.

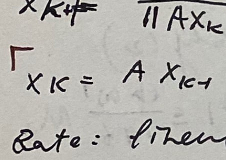
x_0 not orthogonal to the candidate limit.

$x_{k+1} = (A - \theta I)^{-1} x_k$ and then normalize. In practice, solve for x_{k+1} s.t. $(A - \theta I) x_{k+1} = x_k$

3° QR-Algorithm. (Step 1. Convert it to Tridiagonal one ; Step 2. $A_k = QR$ $A_{k+1} = RQ$)

Step 1: Tridiagonalization

we use a series of Householder to do A.



do $H_1 A H_1$

and then do it for the sub matrix.

Step 2. Do

$A_k = QR \rightarrow A_{k+1} = RQ$.

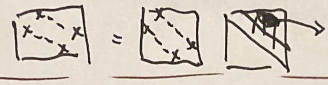
$R = Q^T A$ ~~For eigenvectors $A \to P^T A P = A'$~~

$RQ = Q^T A Q$.

How to find eigenvectors? Start with B symmetric $\to P_n^T B P_n = A$ Tridiagonal.

Step II $\bar{Q}^{(k)T} A \bar{Q}^{(k)} \to D$ diagonal $\Rightarrow \bar{Q}^{(k)T} P_n^T B P_n \bar{Q}^{(k)} \cong D$

$\Rightarrow B P_n \bar{Q}^{(k)} = P_n \bar{Q}^{(k)} D \Rightarrow B\theta_i = \lambda_i \theta_i$ where θ_i is the ith column of $P_n \bar{Q}^{(k)}$)

cost: cheap!  this part vanishes as ~~Q^T~~ A is symmetric.

Interpolation / Numeric Integral

Interpolation: (Lagrange polynomials form a basis)

1° Lagrange: $\{x_0 \cdots x_n\}$ points, interpolate these points.

How: use $L_i = \prod_{j=0, j\neq i}^{n}(x-x_j) / \prod_{j=0, j\neq i}^{n}(x_i - x_j)$ $\quad L_i(x_j) = \delta_{ij}$ and $P_n(x) = \sum_{i=0}^{n} L_i f(x_i)$

Error: $f(x) - P_{n}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{n+1}(x)$

$|f-P|_{(x)} \le \frac{M_{n+1}}{(n+1)!}|\Pi_{n+1}(x)|$

Exact: Poly (n)

Convergence: if $(n+1)!$ grows $> M_{n+1}|\Pi_{n+1}(x)|$

2° Hermite: $\{x_0, \cdots, x_n\}$ points, interpolate Value and ∇.

How: $H_k(x) = L_k^2(x)(1 - 2L_k'(x_k)(x-x_k)) \in P_{2n+1}$

$K_k(x) = L_k^2(x)(x - x_k) \in P_{2n+1}$ $\quad$ only $H_k(x_k) = 1$, $K_k'(x_k) = 1$

$P_{2n+1} = \sum_{k=0}^{n}(H_k(x) y_k + K_k(x) z_k)$

Error: $f(x) - P_{2n+1}(x) = \frac{f^{(2n+2)}(x)}{(2n+2)!}(\Pi_{n+1}(x))^2$

$|f(x) - P_{2n+1}(x)| \le \frac{M_{2n+2}}{(2n+2)!}(\Pi_{n+1}(x))^2$

Exact: Poly $(2n+1)$ Similarly, it induces a basis.

3° Best Approximation in L^2 sense.

If normal, $\varphi_n = x^n - \langle \varphi_{n-1}, x^n\rangle \varphi_{n-1} \cdots$

If not, $\varphi_n = x^n - \frac{\langle \varphi_{n-1}, x^n\rangle}{\langle \varphi_{n-1}, \varphi_{n-1}\rangle}\varphi_{n-1} \cdots$

Legendre polynomials: $\varphi_0 = 1$, $\varphi_1 = x$, $\varphi_2 = \frac{1}{2}(3x^2-1)$, $\varphi_3 = \frac{1}{2}(5x^3 - 3x)$, $\varphi_4 = \cdots$ for $[a,b]$: $a=-1$, $b=1$.

$\varphi_0 = 1$, $\varphi_1 = x - \frac{1}{2}$, $\varphi_2 = x^2 - x + \frac{1}{6}$, $\varphi_3 = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$ $\quad a=0$, $b=1$

Integration / Newton Cotes

- Lagrange: $\int_a^b f\,dx = \int_a^b \sum_{i=0}^{n} L_i(x) f(x_i) = \sum_{i=0}^{n} f(x_i)\underbrace{\int_a^b L_i(x)}_{=w_i}$

$n=1$ Trapezoidal: $w_0 = w_1 = \frac{a-b}{2} = \int_a^b L_i(x)dx \Rightarrow \int_a^b f\,dx \approx \frac{b-a}{2}(f(a) + f(b))$ $\quad$ Exact: Poly (1)

Error: $|E_1 f| \le \frac{M_2(b-a)^3}{12}$

$n=2$ Simpson: $w_0 = \frac{b-a}{6}$ $w_1 = \frac{4(b-a)}{6}$ $w_2 = \frac{(b-a)}{6} \Rightarrow \int_a^b f\,dx \approx \frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b))$ $\quad$ Exact: Poly (2)

Error: $|E_2 f| \le \frac{(b-a)^4}{196}M_3$

Improved $|E_2 f| \le \frac{(b-a)^5}{2880}M_4$

Composite Tra: $|E_1 f| \le \frac{(b-a)^3}{12m^2}M_2$

Composite Simpson: $|E_2 f| \le \frac{(b-a)^5}{2880 m^4}M_4$

- Hermite (Gauss Quadrature)

Derivation: $P_{2n+1}(x) = \sum_{k=0}^{n} H_k(x) f(x_k) + \sum_{k=0}^{n} K_k(x) f'(x_k)$

$\int_a^b w(x) f(x)\,dx \approx \int_a^b w(x) P_{2n+1}(x)\,dx = \sum_{k=0}^{n} f(x_k)\int_a^b w(x) H_k(x)\,dx + \sum_{k=0}^{n} f'(x_k)\underbrace{\int_a^b w(x) K_k(x)\,dx}$

$V_k = \int_a^b w(x) L_k^2(x)(x-x_k)\,dx = \int_a^b w(x)\,\Pi_{n+1} \cdots\cdots dx$ want Π_{n+1} orthogonal $V_k = 0$

to every polynomial degree $\le n$. find $\varphi_{(n+1)}$ and use its root as nodes.

Then $V_k = 0$. $\Rightarrow I = \sum_{k=0}^{n} f(x_k)\underbrace{\int_a^b w(x) H_k(x)\,dx}_{W_k}$ $\quad W_k = \int_a^b w(x) H_k(x)\,dx$

$= \int_a^b w(x) L_k^2(x)\,dx - 2L_k'(x_k)\underbrace{\int_a^b w(x) L_k^2(x-x_k)}_{=0}$

$= \int_a^b w(x) L_k^2(x)$

Space for $LU \sim \frac{2}{3}n^3$

B/F-Sub $\sim n^2$

Induced Norm.

1-Norm - Column Sum max

∞-norm row sum

2 norm $\max\sqrt{\lambda_i A^T A}$

$\kappa(A) = \|A\| \cdot \|A^{-1}\|$

$\|b\| = \|Ax\| \le \|A\|\|x\|$

$\|\Delta x\| = \|A^{-1}\Delta b\| \le \|A^{-1}\|\|\Delta b\| \to \frac{\|\Delta x\|}{\|x\|} \le \|A\|\,\|A^{-1}\|\frac{\|\Delta b\|}{\|b\|}$