Complex Analysis Done Simple

1. Complex Numbers

Before the actual complex analysis

(1) Definition of a complex number and its real/im parts, modular, etc.).
(2) Algebra on complex number
(3) Euler's Formula
(4) Every complex number can be associated with a matrix

$$z = x + iy \iff [z] = \begin{bmatrix} x & -y \\ y & -x \end{bmatrix}$$

and the following properties are well-projected

(a) Multiplication/Addition $\iff$ Matrix multiplication/addition
(b) Conjugacy $\iff$ Transpose
(c) $z \in (i)\mathbb{R} \iff$ Matrix is (anti-)symmetric
(d) Polar coordinates $\iff$ Polar decomposition $A = |A|u$ where u is unitary.

Then it's time to do analysis.

2. Differentiation

We recalled that the complex derivative of a function $f : \mathbb{C} \to \mathbb{C}$ is given by

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

Equivalently, there exists a function $\varphi_z : B_\varepsilon(z) \to \mathbb{C}$ such that

$$f(z+h) = f(z) + f'(z)h + \varphi_z(h) \text{ and } \lim_{h \to 0} \frac{\varphi_z(h)}{|h|} = 0.$$

We can canonically identify functions $f : \mathbb{C} \to \mathbb{C}$ with functions $F : \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$F(x, y) = (\mathrm{Re}(f(x + iy)), \mathrm{Im}(f(x + iy))).$$

A function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is called differentiable at $X = (x, y)$ if there exists φ_X such that

$$F(X + h) = F(X) + DF(X)h + \varphi_X(h) \text{ and } \lim_{h \to 0} \frac{\varphi_X(h)}{\|h\|} = 0.$$

Here $DF(x) = \begin{pmatrix} \partial_x F_1(x, y) & \partial_y F_1(x, y) \\ \partial_x F_2(x, y) & \partial_y F_2(x, y) \end{pmatrix}$ is the Jacobi matrix. A sufficient criterion for F to be differentiable at a point X is that all partial derivatives in the Jacobi matrix exist and are continuous function in $B_\varepsilon(X)$.

Theorem 2.1 (complex differentiable)**.** *$f:\mathbb{C}\to\mathbb{C}$ is differentiable at a point $z\in\mathbb{C}$ if and only if F is differentiable and the Cauchy-Riemann equations hold (which enforce the Jacobi matrix to correspond to a complex number)*

$$\partial_x F_1(x,y)=\partial_y F_2(x,y) \text{ and } \partial_y F_1(x,y)=-\partial_x F_2(x,y).$$

One can then introduce *Wirtinger derivatives* https://en.wikipedia.org/wiki/Wirtinger_derivatives

$$\partial_z=\tfrac{1}{2}(\partial_x-i\partial_y) \text{ and } \partial_{\bar z}=\tfrac{1}{2}(\partial_x+i\partial_y).$$

The Cauchy-Riemann equations then just simplify to $\partial_{\bar z}f(z)=0$ and for the complex derivative, we have $f'(z)=\partial_z f(z)$.

In addition, we have $4\partial_z\partial_{\bar z}=\Delta$ where $\Delta=\partial_x^2+\partial_y^2$ is the Laplace operator. Thus, every complex differentiable function is in fact harmonic, i.e. satisfies $\Delta f=0$.

Definition 2.1 (Domain)**.** *We call an open connected subset D of $\mathbb{C}$ a* domain.

We then showed that

Theorem 2.2. *If $f:D\to\mathbb{C}$ is complex differentiable and real-valued where D is a domain, then f is constant.*

Then we started discussing about what a power series is.

Definition 2.2 (Power series)**.** *The power series of f (if applicable) is in the form*

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n.$$

It converges for $|z-z_0|<R:=\limsup_{n\to\infty}|a_n|^{-1/n}$ and diverges for $|z-z_0|>R$. Here, R is called the radius of convergence.

In particular, we have

Theorem 2.3. *Let $f:B_R(z_0)\to\mathbb{C}$ be given by $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$, where R is the radius of convergence, then f is differentiable.*

Proof. Without loss of generality, we take $z_0=0$ and consider $z,w\in B_R(0)$. A candidate for the derivative is $g(z)=\sum_{n=1}^{\infty}a_n z^{n-1}$.

We shall try to show then that

$$\lim_{z\to w}\left(\frac{f(z)-f(w)}{z-w}-g(w)\right)=0.$$

First, observe that for $n\geq 1$

$$\frac{z^n-w^n}{z-w}=\sum_{k=0}^{n-1}z^{n-1-k}w^k.$$

This implies that

$$\frac{f(z)-f(w)}{z-w} - g(w) = \sum_{n=1}^{\infty} a_n \Big(\sum_{k=0}^{n-1} z^{n-1-k} - nw^{n-1} \Big)$$

Simplifying the expression in brackets, we find

$$\begin{aligned}
\sum_{k=0}^{n-1} z^{n-1-k} w^k - nw^{n-1} &= \sum_{k=0}^{n-2} z^{n-1-k} w^k - (n-1) w^{n-1} \\
&= \sum_{k=0}^{n-2} (k+1) z^{n-1-k} w^k - \sum_{k=0}^{n-2} k z^{n-1-k} - (n-1) w^{n-1} \\
&= \sum_{k=0}^{n-2} (k+1) z^{n-1-k} w^k - \sum_{k=0}^{n-1} k z^{n-1-k} \qquad (2.1) \\
&= \sum_{k=1}^{n-1} k z^{n-1-k} w^{k-1} - \sum_{k=0}^{n-1} k z^{n-1-k} \\
&= (z-w) \sum_{k=1}^{n-1} k z^{n-1-k} w^{k-1}
\end{aligned}$$

Let now $|w|, |z| < r < R$, then by using $|(z-w)\sum_{k=1}^{n-1} kz^{n-1-k}w^{k-1}| \leq |z-w| \sum_{k=1}^{n-1} kr^n \leq |z-w| r^n n^2$, we find

$$\Big| \sum_{n=1}^{\infty} a_n \Big(\sum_{k=0}^{n-1} z^{n-1-k} - nw^{n-1} \Big) \Big| \leq |z-w| \sum_{n=1}^{\infty} |a_n| n^2 r^{n-1}.$$

Observe that now that by the root test and since $r < R$ the last series converges. Thus, as $z \to w$ we find

$$\Big| \sum_{n=1}^{\infty} a_n \Big(\sum_{k=0}^{n-1} z^{n-1-k} - nw^{n-1} \Big) \Big| \to 0.$$

$\square$

3. Integration

We define the integral for complex-valued functions and $a, b \in \mathbb{R}$ by

$$\int_a^b f(s)\ ds := \int_a^b \mathrm{Re}(f(s))\ ds + i \int_a^b \mathrm{Im}(f(s))\ ds.$$

This way, the complex integral is still linear and satisfies the triangle inequality.

We call a curve a piecewise differentiable map $\gamma : [a, b] \to \mathbb{C}$. The range of a curve is called its trace.

We then define the line or contour integral

$$\int_\gamma f(s)\ ds := \int_a^b f(\gamma(t))\gamma'(t)\ dt.$$

Let $\psi : [\alpha, \beta] \to \mathbb{C}$ be a bijective continuously differentiable function with $\psi(\alpha) = a$ and $\psi(\beta) = b$, then $\bar{\gamma} := \gamma \circ \psi$ satisfies

$$\int_{\bar{\gamma}} f(z)\ dz = \int_\gamma f(z)\ dz.$$

If ψ satisfies $\psi(\alpha) = b$ and $\psi(\beta) = a$, then

$$\int_{\bar{\gamma}} f(z)\ dz = -\int_\gamma f(z)\ dz.$$

For example: $\gamma_1 = t(1 + i)$ and $\gamma_2(t) = t$ for $t \in [0, 1]$ and $\gamma_2(t) = 1 + (t - 1)i$ for $t \in [1, 2]$, then

$$\int_{\gamma_1} z\ dz = i \text{ and } \int_{\gamma_2} z\ dz = i.$$

In curves taking values in $\mathbb{R}^2$, one defines

$$\int_a^b f(\gamma(t))|\gamma'(t)|\ dt.$$

When using the definition involving the absolute value, one writes in complex analysis instead

$$\int_\gamma f(z)|dz| = \int_a^b f(\gamma(t))|\gamma'(t)|\ dt.$$

The triangle inequality reads then

$$\left| \int_\gamma f(z)\ dz \right| \leq \int_a^b |f(z)||dz|.$$

Theorem 3.1. *Let D be a domain, $F : D \to \mathbb{C}$ analytic and γ a closed curve. Let $f = F'$ also be continuous, then*

$$\int_\gamma f(z)\ dz = 0.$$

Definition 3.1 (Compactness in $\mathbb{C}$)**.** *A set $A \subset \mathbb{C}$ is called* compact *if any of the following definitions hold:*

- *A is closed an bounded (Heine-Borel)*
- *Every sequence $z_n \in A$ has a convergent subsequence (Bolzano-Weierstrass)*
- *Let $I_n \subset A$ be a sequence of closed sets, then if $\bigcap I_n = \emptyset$ then already finitely many of them have empty intersection.*
- *Every open cover has a finite subcover.*

Recall that Continuous functions on compact sets are uniformly continuous and attain a maximum and minimum value.

Theorem 3.2. *Let D be a domain, $f : D \to \mathbb{C}$ analytic and $\Delta \subset D$ a compact triangle with boundary curve γ, then*

$$\int_\gamma f(z)\ dz = 0.$$

Proof. The proof rests on finding a nested sequence of subtriangles:

$$\Delta \supset \Delta_1 \supset \Delta_2 ...$$

with boundary lengths $L(\gamma_n) = 2^{-n} L(\gamma), \operatorname{diam}(\Delta_n) = 2^{-n} \operatorname{diam}(\Delta)$ and

$$\Big| \int_\gamma f(z)\ dz \Big| \leq 4^n \Big| \int_{\gamma_n} f(z)\ dz \Big|.$$

Compactness ensures the existence of one point in all triangles. Then one splits the integral for a sufficiently small triangle

$$\int_{\gamma_n} f(z)\ dz = \int_{\gamma_n} \underbrace{(f(z) - f(z_0) - f'(z_0)(z - z_0))}_{\leq \varepsilon |z - z_0|}\ dz + \int_{\gamma_n} f(z_0) + f'(z_0)(z - z_0)\ dz.$$

□

Recall that a set $G \subset \mathbb{C}$ is convex, if for all

$$p, q \in G \Rightarrow \{tp + (1-t)q; t \in [0,1]\} \subset G.$$

Theorem 3.3. *Let $G \subset \mathbb{C}$ be a convex domain and $f : G \to \mathbb{C}$ analytic. Then for every closed curve $\gamma \subset G$ we find*

$$\int_\gamma f(z)\ dz = 0.$$

Proof. Proof follows from constructing the anti-derivative of f. □

Let $G \subset \mathbb{C}$ and γ_0, γ_1 be two closed curves defined on $[0,1]$, then they are called homotopic, if there exists a continuous map $H : [0,1]^2 \to G$ such that

$$\begin{aligned} H(s,0) &= H(s,1) \quad \forall s \in [0,1] \\ H(0,t) &= \gamma_0(t) \quad \forall t \in [0,1] \\ H(1,t) &= \gamma_1(t) \quad \forall t \in [0,1]. \end{aligned} \tag{3.1}$$

A closed curve that is homotopic to a point is called *null-homotopic*. A set in which every closed curve is null-homotopic is called *simply connected*. Examples of simply connected domains that are not convex are e.g. star domains, i.e. sets G such that there exists a $p_0 \in G$ such that $tp_0 + (1-t)p \in G$ for all p and $t \in [0,1]$. In this case, let γ be a curve, then

$$H(s,t) = p_0 + (1-s)(\gamma(t) - p_0)$$

is a homotopy. This allows us to formula the general Cauchy integral theorem

Theorem 3.4 (Cauchy's Integral Theorem)**.** *Let $G \subset \mathbb{C}$ be a domain and $f : G \to \mathbb{C}$ analytic. For two closed curves γ_0, γ_1 that are homotopic, we have*

$$\int_{\gamma_0} f(z)\ dz = \int_{\gamma_1} f(z)\ dz.$$

In particular, if γ is nullhomotopic in G, then we have

$$\int_{\gamma} f(z)\ dz = 0.$$

Proof. Let H be the homotopy. Step 1: $\varphi(z) = d(z, \mathbb{C} \setminus G)$ is continuous and therefore attains its infimum on $K = H([0,1])^2$. (Bolzano-Weierstrass). This implies that there is $\varepsilon > 0$ such that

$$z \in K, |w - z| < \varepsilon, \Rightarrow w \in G.$$

Step 2: Since H is continuous on a compact set it is uniformly continuous, i.e. for fixed $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that

$$|s - s'|, |t - t'| \leq 1/m, \Rightarrow |H(s,t) - H(s',t')| < \varepsilon.$$

Step 3: We define for $k = 0, .., m$ a polygonal chain π_k defined by the boundary points

$$H(k/m, 0), H(k/m, 1/m), ..., H(k/m, 1).$$

Then, it is clear by Step 1 and Step 2 that $\pi_k \subset G$, as π_k is $\varepsilon > 0$ close to the actual curve.

Step 4: The boundary integrals coincide by looking at neighbourhoods $B_\varepsilon(H(0, l/m))$ for l between 0 and m. Thus, let σ_l be the curve composed of $\gamma_0|_{[l/m,(l+1)/m]}$ and the straight line between $H_0(0, l/m)$ and $H_0(0, (l+1)/m)$, then $\int_{\sigma_l} f(z)\ dz = 0$ by the previous versions of Cauchy's theorem. Iterating this yields:

$$\int_{\gamma_0} f(z)\ dz = \int_{\pi_0} f(z)\ dz, \text{ and similarly } \int_{\gamma_1} f(z)\ dz = \int_{\pi_m} f(z)\ dz.$$

Step 5: σ_{kl} is the closed path along

$$H(k/m, l/m), H(k/m, (l+1)/m), H((k+1)/m, l/m), H((k+1)/m, (l+1).$$

Then by a telescopic sum argument the proof follows, as $\int_{\sigma_{kl}} f(z) = 0$. $\square$

4. Important theorems, starting from Cauchy's formula

We start with a version of Cauchy's integral formula.

Lemma 4.1. *Let $f : B_R(z_0) \to \mathbb{C}$ be analytic and $r \in (0, R)$, let $\gamma_r(t) = z_0 + re^{it}$ with $t \in [0, 2\pi]$, then for any $z \in B_r(z_0)$ we have*

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{w-z} \, dw.$$

Proof. f is continuous, therefore we may use a homotopy argument to reduce everything to radii $r < \delta$. Hence, by the integral theorem

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} \, dw = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{w-z} \, dw.$$

Then, it is easy to see that

$$\Big| \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{w-z} \, dw - f(z) \Big| < \varepsilon.$$

□

A sequence of functions $f_n : K \to \mathbb{C}$ is said to converge uniformly to f if

$$\lim_{n\to\infty} \sup_{z\in K} |f_n(z) - f(z)| = 0.$$

Example: $\sqrt{x^2 + 1/n}$ on $[-1, 1]$, counterexample x^n on $[0, 1]$ is not uniformly convergent.

Lemma 4.2. *Let $\gamma : [0, 1] \to K$ be a curve and g_n continuous functions that converge uniformly to g on K, then*

$$\lim_{n\to\infty} \int_{\gamma} g_n(z) \, dz = \int_{\gamma} g(z) \, dz.$$

Theorem 4.1. *Let $f : D \to \mathbb{C}$ be analytic, then f has a power series expansion with positive radius of convergence*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n,$$

which converges uniformly in every open disc around points a fully contained in D.

Proof. Use Cauchy's integral formula and the geometric series. □

Corollary 4.3. *Let $f : D \to \mathbb{C}$ be analytic, then f is infinitely many times differentiable and for any $a \in G$ we have that if $\gamma(t) = a + re^{2\pi it} \in D$ that for all $z \in B_r(a)$*

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^{n+1}} \, dw.$$

Theorem 4.2. *Let $f, g : D \to \mathbb{C}$, the following are equivalent*

(1) $f = g$,
(2) *There is* $a \in D$ *such that* $f^{(n)}(a) = g^{(n)}(a)$ *for all* $n \in \mathbb{N}_0$,
(3) $\{z \in D : f(z) = g(z)\}$ *has an accumulation point.*

Proof. (1) implies (2) and (3) is clear. That (2) implies (1) follows from the power series representation. That (3) implies (2) follows also from the power series representation and a simple contradiction argument. □

Theorem 4.3. *Any analytic function on a simply connected domain has an antiderivative.*

Proof. Use contour integral and Cauchy's integral theorem. □

Definition 4.4 (Entire)**.** *A function that is analytic on all of* $\mathbb{C}$ *is called entire.*

Entire functions are good enough, but bounded Entire functions are even better. Since we have

Theorem 4.4 (Liouville)**.** *Every bounded entire function is constant.*

Proof. We derive Cauchy estimates

$$\begin{aligned}
|f^{(n)}(z)| &= |\frac{n!}{2\pi i}\int_{|w|=r}\frac{f(w)}{(w-z)^{n+1}}dw| \\
&\leq \frac{n!}{2\pi}|\int_{|w|=r}\frac{f(w)}{(w-z)^{n+1}}dw| \\
&\leq \frac{n!}{2\pi}\sup\frac{|f(w)|}{|w-z|^{n+1}}2\pi r \\
&= n!\cdot r\cdot\sup\frac{|f(w)|}{|w-z|^{n+1}}
\end{aligned}$$

We set $z = 0$ and it follows that, given $|f| < \infty$, for $n \geq 1$

$$|f^n(0)| \leq \frac{n!}{r^n}\sup|f(w)| \xrightarrow{r\to\infty} 0$$

In particular, we have $f' = 0$, which implies f constant.(Or alternatively, do the series expansion). □

As a corollary one readily obtains the fundamental theorem of algebra.

Corollary 4.5 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial on* $\mathbb{C}$ *has a root.*

Proof. If not, then $1/p$ would be bounded and entire, hence constant, which contradicts our assumptions. □

We next discuss the maximum principle

Theorem 4.5. *Let D be a domain and $f : D \to \mathbb{C}$ analytic. If there is $a \in D$ and $\varepsilon > 0$ such that*

$$|z - a| < \varepsilon \Rightarrow |f(z)| \leq |f(a)|$$

then f is constant.

Proof. Use $|f(a)| \geq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{it})|^2 \, dt$ and the power series expansion of f.

$$\begin{aligned} |f(z)|^2 &= \frac{1}{2\pi} \int_0^{2\pi} |f(z)|^2 dt \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z + re^{it})|^2 dt \end{aligned}$$

We do the series expansion of $f(z + re^{it})$ and use

(1) $|x|^2 = x\bar{x}$
(2) Uniform convergence legitimizes exchange of limit i.e.

$$(\sum_{i=0}^{\infty} \rho)(\sum_{j=0}^{\infty} \phi) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \rho\phi$$

Thus, we have

$$\begin{aligned} |f(z)|^2 &\geq \sum_{m}^{\infty} \sum_{n}^{\infty} c_n \bar{c_m} r^{n+m} \cdot \frac{\int_0^{2\pi} e^{it(n-m)}}{2\pi} \\ &= \sum_{m}^{\infty} \sum_{n}^{\infty} c_n \bar{c_m} r^{n+m} \delta_{n,m} \\ &= \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \\ &= |f(z)|^2 + \frac{|f^{(1)}(z)|^2}{1!} r^2 + \dots \end{aligned}$$

It follows that the remainder terms are 0, which implies f is constant. □

Remark 4.1. *If given f nonzero analytic, minimum principle also holds by applying the above theorem to $\frac{1}{f}$.*

Corollary 4.6. *Let f be a bounded domain and $f : D \to \mathbb{C}$ analytic, then*

$$\sup_{z \in D} |f(z)| = \sup_{z \in \partial D} |f(z)|.$$

Proof. If it happens in the interior, then the function is constant and the statement holds. □

We also recall Morera's theorem

Theorem 4.6 (Morera's Theorem). *Let D be a domain and $f : D \to \mathbb{C}$ continuous such that*

$$\int_\Delta f(z)\, dz = 0 \text{ for all triangles } \Delta \subset D$$

then f is analytic.

Theorem 4.7 (Weierstrass Convergence Theorem). *Analycity is preserved under Uniform Convergence*

Proof. Uniform convergence allows exchange of limit and integral i.e. if $f_n \xrightarrow[u]{n\to\infty} f$, then $\lim_{n\to\infty} \int_a^b f = \int_a^b \lim_{n\to\infty} f_n = \int_a^b f$. Given this fact, we have

$$\int_\Delta f = \int_\Delta \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_\Delta f_n = 0$$

□

5. Winding Numbers, Cauchy's formula, and Logarithm

Definition 5.1 (Winding Number). *Let γ be a closed curve and $z \notin \gamma([0,1])$, then*

$$n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z}$$

is called the winding number of γ around z. It is an constant integer on connected components of $\mathbb{C} \setminus \gamma([0,1])$.

Proof. Check $\varphi(t) = e^{\int_0^t \frac{\gamma'(s)}{\gamma(s)-z}\, ds}$ satisfies

$$\varphi'(t) = \varphi(t) \frac{\gamma'(t)}{\gamma(t) - z}.$$

This way,

$$\frac{d}{dt}\left(\frac{\varphi(t)}{\gamma(t) - z}\right)(t) = 0.$$

Hence $\varphi/(\gamma - z) = \text{const}$ which implies that $\varphi(0) = \varphi(1)$ and thus $n(\gamma, z) \in \mathbb{Z}$. □

Theorem 5.1 (Cauchy's formula)**.** *Let $G \subset \mathbb{C}$ be open and $\gamma : [0,1] \to G$ nullhomotopic, $f : G \to \mathbb{C}$ analytic, then*

$$n(\gamma, z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw.$$

Proof. Consider the auxiliary function $g(\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z}, & \zeta \neq z \\ f'(z), & \zeta = z \end{cases}$. This function is analytic as g (power series), Cauchy's integral theorem therefore implies the result. □

Remark 5.1. *This formula can be viewed as a general version of the Cauchy integral formula, where the winding number is introduced into the story.*

Definition 5.2 (Logarithm)**.** *Let G be a domain. We call an analytic function $g : G \to \mathbb{C}$ a branch of the logarithm, if $e^{g(z)} = z$ for all $z \in G$.*

Since $e^z \neq 0$ for all $z \in \mathbb{C}$, we cannot have $0 \in G$.

However, this is still not sufficient. By the chain rule $e^{g(z)}g'(z) = 1$, hence $g'(z) = 1/z$ which cannot have an analytic anti-derivative on let's say $\mathbb{C} \setminus \{0\}$.

On the other hand, we have

Theorem 5.2. *Let G be simply connected with $0 \notin G$, then there exists a branch g of the logarithm. Any other branch is different by $2\pi i\mathbb{Z}$ different from g.*

Proof. Since $1/z$ has an antiderivative g, we can check that $\frac{d}{dz}ze^{-g(z)} = 0$, so we have that $ze^{-g(z)}$ is constant.

We may therefore define $g(z) = w_0 + \int_\gamma \frac{dw}{w}$, where γ is a curve starting at $e^{w_0} = z_0$, then $ze^{-g(z)} = 1$. This is a branch of the logarithm.

Any other branch satisfies $1 = \frac{z}{z} = \frac{e^{g(z)}}{e^{h(z)}} = e^{g(z)-h(z)}$. □

Definition 5.3 (General powers/roots z^α)**.** *For $z \in \mathbb{C}$, $z^\alpha := e^{g(z)\alpha}$ and are analytic functions of z.*

6. Singularities

There are three types of singularities, which are

- Removable Singularities
- Poles
- Essential Singularities

We then give definitions for each of them.

Definition 6.1 (Removable singularities). *If an analytic function $f : D\backslash\{z_0\} \to \mathbb{C}$ has an analytic extension $\bar{f} : D \to \mathbb{C}$ such that $\bar{f}|_{D\backslash\{z_0\}} = f$, then z_0 is a removable singularity.*

Definition 6.2 (Poles). *If $\lim_{z\to z_0} |f(z)| = \infty$ then z_0 is a pole.*

Definition 6.3 (Essential Singularities). *If neither of the first two applies, then z_0 is an essential singularity of f.*

We will then give a series of examples of different types of singularities

Example 6.1. *We will give one example for each.*

(1) *Removable Singularity.*

$$f(z) = \frac{\sin(z)}{z}$$
$$\hat{f}(z) = \begin{cases} f(z) & z \neq 0 \\ 1 & z = 0 \end{cases}$$

(2) *Examples for poles are trivial*
(3) *Examples for essesntial singularities are highly nontrivial, and the construction somehow relies on the Laurent Series. One typical example is*

$$f(z) = e^{1/z}$$

In particular, we have Riemann's theorem (for removable singularities):

Theorem 6.1 (Riemann). *Let $z_0 \in D \subset \mathbb{C}$ be the point in question, if f is bounded in a neighbourhood of z_0 then z_0 is a removable singularity*

Proof. The rough idea is to let $h(z) = (z - z_0)^2 f(z)$ and show this function is analytic.

We define a function h with $h(z) = (z - z_0)^2 f(z)$ and $h(z_0) = \lim_{z\to z_0} h(z) = 0$. h is analytic, since at $z = z_0$, we have

$$\begin{aligned} \lim_{z\to z_0} \frac{h(z) - h(z_0)}{z - z_0} &= \lim_{z\to z_0} (z - z_0) f(z) \\ &= 0 \end{aligned}$$

We then do the series expansion of h around z_0(which seems to be the fate of the analytics). In particular, we have the first two coefficient(denoted by c)

$$c_0 = h(z_0) = 0$$
$$c_1 = h'(z_0) = 0$$

Therefore, we have

$$
\begin{aligned}
h(z) &= \sum_{n=0}^{\infty}(z-z_0)^n \\
&= (z-z_0)^2\sum_{n=2}^{\infty}c_n(z-z_0)^{n-2} + c_0 + c_1(z-z_0) \\
&= (z-z_0)^2\sum_{n=0} c_{n+2}(z-z_0)^n \\
&:= (z-z_0)^2\hat{f}(z)
\end{aligned}
$$

where $\hat{f}$ is the analytic function that allows z_0 to be removable.

□

The following result appears in the textbook.

Theorem 6.2. *If f has an isolated singularity at z_0 then the point is a removable singularity if and only if*

$$\lim_{z\to z_0}(z-z_0)f(z) = 0$$

Remark 6.1. *This result somehow makes removable singularity and poles comparable. The condition to meet a removable is stronger since it requires 0 on the right hand side but a general analytic function would suffice for a pole.*

The next result characterizes essential singularities:

Theorem 6.3 (Casorati-Weierstrass)**.** *Let $z_0 \in D \subset \mathbb{C}$ be an essential singularity for $f : D \setminus \{z_0\} \to \mathbb{C}$ analytic, then $f(B_\delta(z_0) \setminus \{z_0\})$ is dense in $\mathbb{C}$.*

Proof. Assume it is not dense, by definition, we have some $w \in \mathbb{C}$ that f cannot touch i.e.

$$|z-z_0| < \delta \implies |f(z)-w| \geq \varepsilon$$

Let $g(z) = \frac{1}{f(z)-w}$ on $B_{\delta(z_0)}\backslash z_0$ and $g(z_0) = \lim_{z\to z_0} g(z)$. Since $g < \infty$ by assumption, we have z_0 is a removable singularity and g is analytic on the ball with z_0 extracted. We have 2 cases then,

(1) $g(z_0) = 0$. It follows that $\lim_{z\to z_0}|f(z)| = \infty$ i.e. z_0 is a pole
(2) $g(z_0) = c \neq 0$. It follows that $\lim_{z\to z_0} f(z) = \frac{1}{c} + w$ i.e. z_0 is removable.

Density follows from the contracdiction.

□

Finally, for poles we have

Theorem 6.4 (Poles). *Let $z_0 \in D \subset \mathbb{C}$ be a pole for $f : D \setminus \{z_0\} \to \mathbb{C}$ analytic, then there is an analytic function $g : D \to \mathbb{C}$ and a natural number m, such that $g(z_0) \neq 0$ and*

$$f(z) = \frac{g(z)}{(z - z_0)^m}.$$

Proof. Given $\lim_{z \to z_0} |f(z)| = \infty$, $h = 1/f$ is bounded locally, thus having removable singularities. Thus, h is locally analytic and let $h(z_0) = 0$. As usual, we do series expansion. With c_n representing its coefficient, we have $c_0 = h(z_0) = 0$. Let $m \geq 1$ be the first natural number with $c_m \neq 0$, we have

$$\begin{aligned} h(z) &= \sum_{n=0}^{\infty} c_n (z - z_0)^n \\ &= \sum_{n=0}^{\infty} c_n (z - z_0)^{n-m} (z - z_0)^m \\ &= (z - z_0)^m \sum_{n=0}^{\infty} c_{n+m} (z - z_0)^n \\ &:= (z - z_0)^m \cdot l(z) \end{aligned}$$

It follows that

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

where $g(z) = \frac{1}{l(z)}$ □

People may be confused why we denote the series by $l(z)$. I don't know either, but I suspect it indicates the birth of Laurent Series. This characterization of poles actually gives us a opportunity to derive a series expansion of function around poles and removables, just like the Power Series to analytic functions.

7. Residue Theorem and his friends

We need a guy called Laurent Series to define residue. The Laurent series around poles turns out to be pretty nice, but we give a general definition first.

Definition 7.1. *The Laurent series for a complex function $f(z)$ about a point z_0 is given by*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

In particular, since g is holomorphic(analytic), $g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$, any function with only with removable singularities and poles can therefore be expanded into a somehow special *Laurent series* defined above.

Before we do the derivation, we first give the functions in scope an elegant name.

Definition 7.2 (meromorphic). *Functions with only removable singularties and poles are called meromorphic.*

For meromorphic functions, we have

$$f(z) = \frac{g(z)}{(z-z_0)^m} = \sum_{n=-m}^{\infty} c_n (z-z_0)^n.$$

One thing we may notice is that: for removable singularities, m is actually 0; for poles, $m > 0$ but is finite. We then have the following corollary

Corollary 7.3. *Let z_0 be a singular point of f and $f(z) = \sum_{-\infty}^{\infty} c_n(z-z_0)^n$ its Laurent Series. Then:*

(1) z_0 *is removable* $\iff$ $c_n = 0$ *for* $n < 0$
(2) z_0 *is a pole* $\iff$ $\exists m < \infty$ *such that* $a_{-n} = 0\ \forall n > m$.
(3) z_0 *is essential* $\iff$ $c_n \neq 0$ *for infinitely many negative indices.*

In particular, the coefficient c_{-1} is so crucial that it worth a name.

Definition 7.4 (residue). *The coefficient c_{-1} of the Laurent Series is called the residue of f at z_0, denoted by* $\mathrm{res}(f, z_0)$.

Corollary 7.5. *If f has a pole of order m and $g(z) = (z-z_0)^m f(z)$, then*

$$\mathrm{res}(f, z_0) = \frac{g^{(m-1)}(z_0)}{(m-1)!}.$$

Theorem 7.1 (Residue theorem). *Let D be a domain and γ a null-homotopic curve in D. Let f have poles $\{z_1, ..., z_N\}$ in D and $f : D \setminus \{z_1, ..., z_N\}$ holomorphic, then*

$$\frac{1}{2\pi i}\int_\gamma f(z)\ dz = \sum_{k=1}^{N} \mathrm{res}(f, z_k) n(\gamma, z_k).$$

Proof. With out loss of generality, we assume there's only one pole z_1. Laurent series about z_1 reads

$$f(z) = \sum_{n=-m}^{\infty} c_n(z - z_z)^n$$

We define $g(z)$ by

$$\begin{aligned} g(z) &= f(z) - \sum_{n=-m}^{-1} c_n (z-z_1)^n \\ &= \sum_{n=0}^{\infty} c_n (z-z_1)^n \end{aligned}$$

Note that $g(z)$ is analytic, by Cauchy's theorem, we have $\int_\gamma g(z) = 0$. It follows that

$$\int_\gamma f dz = \int_\gamma \sum_{n=-m}^{-1} c_n (z - z_1)^n dz = \sum_{n=-m}^{-1} c_n \int_\gamma \frac{1}{(z - z_1)^n}$$

where

$$\int_\gamma \frac{1}{(z - z_n)^n} dz = \begin{cases} 2\pi i \cdot n(\gamma, z_1) & n = 1 \\ 0 & n \geq 2 \end{cases}$$

We have the above assertion since, let $h(z) = 1$, by Cauchy's integral formula we have

$$n(\gamma, z) h^{(n)}(z) = \frac{1}{2\pi i} \int_\gamma \frac{h(w)}{(w - z)^{n+1}} dw = \frac{1}{2\pi i} \int_\gamma \frac{1}{(w - z)^{n+1}}$$

The theorem therefore follows. □

Example 7.1. *Here's an example of computing integral on the real by applying the residue theorem.*

We can then compute for $f(x) = e^x + e^{-x}$

$$\int_{-\infty}^{\infty} \frac{dx}{f(x)} = \pi/2.$$

This follows from integrating along $-R, R, R + \pi i, -R + \pi i$, using that $f(z + i\pi) = -f(z)$ and the residue theorem stating that with $g(z) = 1/f(z)$

$$\operatorname{res}(f, i\pi/2) = \lim_{z \to i\pi/2} \frac{(z - i\pi/2)}{f'(i\pi/2)(z - i\pi/2)} = \frac{1}{2i}.$$

The following theorems follows from the residue theorem.

Theorem 7.2 (Argument Principle)**.** *Let D be a domain and $f : D \to \mathbb{C}$ meromorphic, γ a (simply) closed curve, then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\ dz = N(f) - P(f),$$

where $N(f)$ is the number of zeros (counting multiplicities) and $P(f)$ is the number of poles (counting multiplicities) enclosed by γ.

Proof. The punchline is to use residue theorem to do the integral. There are two kinds of poles of function $\frac{f'}{f}$, one is the poles of f, and the other is the zeros of f(since it is placed as the denominator). Whatever it is, it can be written into

$$f(z) = (z - z_0)^m g(z)$$

where m is either the multiplicity of the zero or the negative multiplicity of the pole, and g is analytic.

With this expression, we have

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}$$

Though people may get frustrated that f has other zeros/poles other than z_0, the splendid fact is that, the remaining term, $\frac{g'}{g}$ can be expressed in the same way again about another zero/pole.

With that process, we have the following expression

$$\frac{f'(z)}{f(z)} = \sum_{z_i \text{ zeros}} \frac{n_i}{z - z_i} - \sum_{z_j \text{ poles}} \frac{p_j}{z - z_j} + \frac{g'(z)}{g(z)}$$

where $n_i(p_j)$ is the multiplicity of each zero(pole), g is analytic and nonzero, making $\frac{g'}{g}$ analytic.

By residue theorem(or Cauchy integral formula), we have

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{z_i \text{ zeros}} n_i - \sum_{z_j \text{ poles}} p_j = N(f) - P(f)$$

$\square$

Theorem 7.3 (Rouché)**.** *Let $f, g : D \to \mathbb{C}$ be analytic and γ a simply closed curve. Let neither f nor g have any zeros on $\gamma([0,1])$, then under the assumption $|f(z)+g(z)| < |f(z)| + |g(z)|$ for $z \in \gamma([0,1])$, f and g have the same number of zeros inside γ.*

Proof. Let

$$\left|\frac{f(z)}{g(z)} + 1\right| < \left|\frac{f(z)}{g(z)}\right| + 1$$

The above inequality implies $f(z)/g(z)$ cannot be a positive real number. We then study

$$h_t(z) = \frac{f(z)}{f(z)} - t$$

where $t \in \mathbb{R}_{\geq 0}$, and

$$\varphi(t) = \frac{1}{2\pi i} \int_\gamma \frac{h_0'(z)}{h_0(z) - t}\, dt = \frac{1}{2\pi i} \int_\gamma \frac{h_t'(z)}{h_t(z)} = N(h_t) - P(h_t)$$

Note that φ is continuous and the image is in $\mathbb{Z}$. Given $\lim_{t\to\infty} \varphi(t) = 0$, by continuity, $\varphi(0) = 0$. Thus, we have

$$0 = \varphi(0) = N(h_t) - P(h_t) = N(f) - N(g)$$

and theorem follows.

$\square$

The following example is another version of Fundamental Theorem of Algebra by applying Rouché.

Example 7.2. *let f and g be:*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$
$$g(z) = -a_n z^n$$

let $|z| = R > 1$ and provided R is large enough, we have

$$\begin{aligned} |f(z) + g(z)| &= |a_0 + \cdots + a_{n-1} z^{n-1}| \\ &\leq |a_0| + |a_1| R + \cdots + |a_{n-1}| R^{n-1} \\ &\leq \max |a_0|, |a_1|, \ldots, |a_{n-1}| n R^{n-1} \\ &< |a_n| R^n |g(z)| \\ &\leq |g(z)| + |f(z)| \end{aligned}$$

By Rouché, f has n zeros inside $|z| = R$ as g does.

The following theorem is again a corollary of Rouché

Theorem 7.4. *Let $f : D \to \mathbb{C}$ be an analytic function such that $f(z) - w_0$ has at z_0 a zero of order k. There exist $\varepsilon > 0$, $\delta > 0$ such that for every $|w - w_0| < \varepsilon$ but $w \neq w_0$, there are precisely k distinct points $|z_i - z_0| < \delta$ with $f(z_i) = w$*

Proof. We have the following 2 assertions. There exists $\delta > 0$ with

(1) $f'(z) \neq 0$ for $\delta > |z - z_0| > 0$.
(2) $f'(z) \neq w_0$ for $\delta > |z - z_0| > 0$

These two assertions are guaranteed by analycity and the function cannot be constant(otherwise $f = w_0$ would have a accumulation point/$f^n (= 0)$ which imply constantness.). Let $g(z) = f(z) - w_0$, we have $|g(z)| \geq \varepsilon$ for z on the circle $|z - z_0| = \delta$(Compactness + Continuity). Choose w such that $|w - w_0| < \varepsilon$ and take the curve $|z - z_0| = \delta$. We have

$$|(w - f(z)) + g(z)| = |w - w_0| < \varepsilon \leq |g(z)| \leq |g(z)| + |w - g(z)|$$

By Rouché, we have the number of zeros of g is the same as the number of zeros of $w - f(z)$, and the distinctness is guaranteed by non-vanishing gradient. $\square$