1. **Testing binomial distributions I** [4 points]

Imagine we want to test whether a given coin is fair.

(a) Suppose that there is *no* additional information. What would be a reasonable test for at level $\alpha = 0.05$ to determine whether the coin is fair, when we flip it $n = 200$ times? Formulate H_0 and H_1 and determine the critical region.

(a)

$H_0 = \{\text{the coin is fair}\} = \{p = 0.5\}$

$H_1 = \{\text{the coin is not fair}\} = \{p \neq 0.5\}$.

Let X_i be the Bernoulli Random Variable with parameter p.

$X_i = \begin{cases} 0 & \text{Head} \\ 1 & \text{Tail.} \end{cases}$

$\Rightarrow S(\underline{X}) = \sum_{i=1}^{n=200} X_i \sim \text{Bin}(200, p)$

$S(\underline{X})$ takes values in $\{0, \cdots, 200\}$.

Let $[100-k+1, 100+k-1] \cap \mathbb{N}$ be C_α^c

Let $\phi(S(x)) = \begin{cases} 0 & \text{if } S(x) \in C_\alpha^c \\ 1 & \text{if } S(x) \in C_\alpha \end{cases}$

Type I error $\mathbb{P}_{p=0.5}[\phi(S(x)) = 1] = \mathbb{P}[S(x) \geq 100+k] + \mathbb{P}[S(x) \leq 100-k]$

$= \sum_{i=100+k}^{200} \binom{200}{i} (\frac{1}{2})^{200}$

Note that we know these two parts are symmetric

$+ \sum_{j=0}^{100-k} \binom{200}{j} (\frac{1}{2})^{200} \leq 0.05$

$\Rightarrow \sum_{j=0}^{100-k} \binom{200}{100-j} (\frac{1}{2})^{200} \leq 0.025$

Thanks to Online Calculator, we have $k=15$

Critical region: $[0, 85] \cup [115, 200]$

(b) Suppose now that we suspect that the coin has a higher chance to land on heads. How would the test in (a) change?

we should makes the test like this $\phi(x) = \begin{cases} 1 & \sum x_i \geq k_\alpha \\ 0 & \sum x_i < k_\alpha \end{cases}$

$\Rightarrow \mathbb{P}_{0.5}(\sum x_i \geq k_\alpha) \leq 0.05$

The biggest k_α satisfies the Condition is

$\Rightarrow k_\alpha = 113$

(c) Consider both situations in (a) and (b), but use an approximation of the binomial distribution coming from the central limit theorem. How do the critical values change?

$Var\, X_j = 200 \cdot \frac{1}{2} \cdot \frac{1}{2} = 50$

For (a), If we use CLT.

we are to consider

$$\mathbb{P}_{0.5}\left[\sum_{j=1}^{n} X_j \le 100-k\right] \le 0.025$$

Approximate it by Central limit Thm.

$$\frac{\sum_{j=1}^{n} X_j - 100}{\sqrt{200 \times \frac{1}{4}}} \le \frac{100-k-100}{\sqrt{200 \times \frac{1}{4}}} \approx \Phi\left(\frac{-k}{\sqrt{50}}\right) \le 0.025$$

$$\Rightarrow \frac{-k}{\sqrt{50}} = -1.96 \Rightarrow k = 13.86 \approx k=14$$

we could use the trick to substitute $100-k$ by $100-k-\frac{1}{2}$

Then $k = 13.86 + 0.5 = 14.36$ which is more accurate.

for (b)

$$\mathbb{P}_{0.5}\left[\sum_{j=1}^{n} X_j \ge k_\alpha\right] \le 0.05$$

$$\Rightarrow \frac{\sum_{j=1}^{n} X_j - 100}{\sqrt{50}} \ge \frac{k_\alpha - 100}{\sqrt{50}} \approx 1 - \Phi\left(\frac{k_\alpha - 100}{\sqrt{50}}\right) \le 0.05$$

$$\Phi\left(\frac{k_\alpha - 100}{\sqrt{50}}\right) \ge 0.95$$

$$\Rightarrow \frac{k_\alpha - 100}{\sqrt{50}} = 1.644$$

$$\Rightarrow k_\alpha = 111.62 \Rightarrow k_\alpha = 112$$

2. **Testing binomial distributions II** [4 Points]

The first digit of various numerical data sets (such as electricity bills, street addresses, stock prices,...) typically follows the *Benford law*. This is a distribution on $(\{1, ..., 9\}, \mathcal{P}(\{1, ..., 9\})$ with probability mass function

$$p(k) = \log_{10}\left(1 + \frac{1}{k}\right), \qquad 1 \le k \le 9.$$

Here $\log_{10} = \frac{\log}{\log(10)}$ is the logarithm with base 10.

(a) Show that $(p(k))_{k\in\{1,...,9\}}$ defines a probability mass function on $\{1, ..., 9\}$.

By Benford's law, 1 should be the first digit in roughly 30% of the numbers in a valid statistical data set. Suppose we suspect a given sample of 100 independent data points to be fraudulent, since the first digit 1 shows up only 17 times in the sample.

(a) To show $(p(k))$ defines a probability mass function

we show it is

- Nonnegative: $p(k) = \log_{10}(1+\frac{1}{k}) \geq \log(1) = 0$

- Sum up to 1:

$$\sum_{k=1}^{9} p(k) = \log_{10} \prod_{i=1}^{n=9} (1+\frac{1}{k}) = \log_{10} \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{9}{8}\cdot\frac{10}{9} = \log_{10} 10 = 1$$

(b)

(b) Formulate a testing problem and construct the Neyman-Pearson test ϕ^* at level $\alpha = 0.05$ for the hypotheses

H_0 : the probability for 1 as first digit is 30%, against

H_1 : the probability for 1 as first digit is smaller than 30%.

Find the critical region and determine whether H_0 can be rejected with our observation.

Here we don't know the distribution of 1 as first digit.

Therefore, we model each observation as a Bernoulli Random Variable $X_i \sim Ber(\theta)$

In that case, $H_0: \theta = 0.3$
$H_1: \theta < 0.3$

We assume the # of observation is $n = 100$

Suppose $\theta_0 > \theta_1$, which is the case here.

$\theta^{x_i}(1-\theta)^{x_i}$

$$L_{\theta_0,\theta_1}(x_1,\cdots,x_n) = \frac{\theta_1^{\sum_{i=1}^n x_i}(1-\theta_1)^{(n-\sum_{i=1}^n x_i)}}{\theta_0^{\sum_{i=1}^n x_i}(1-\theta_0)^{(n-\sum_{i=1}^n x_i)}}$$

Note that $\theta_0 = 0.3$
$\theta_1 < \theta_0$

$$= \left(\frac{\theta_1}{\theta_0}\right)^{\sum_{i=1}^n x_i}\left(\frac{1-\theta_1}{1-\theta_0}\right)^{n-\sum_{i=1}^n x_i}$$

where we can find a monotune decreasing function for $T(x) = \sum_{i=1}^n x_i$

By Theorem 7.11, we have $\phi^*(x) = \mathbb{1}_{\{T(x) \le k_\alpha^*\}}$ is a UMP test at level $\alpha \in (0,1)$ where $\alpha = \mathbb{P}_{\theta_0}[T(x) \le k_\alpha^*]$ for

$$H_0 = \Theta_0 = [\theta_0, +\infty) \cap \Theta = \theta_0$$

$$H_1 = \Theta_0 = (-\infty, \theta_0) \cap \Theta$$

It remains to decide k_α^* for $n=100$

$$\mathbb{P}_{\theta_0=0.3}\left[\sum_{i=1}^n x_i \le k_\alpha^*\right] \le 0.05$$

$$k_\alpha^* = 22$$

Therefore, the critical region of ϕ^* is $\{0, \cdots, 22\}$

and the test can be formulated as

$$\phi^*(x) = \begin{cases} 0 & \sum_{i=1}^n x_i > 22 \\ 1 & \sum_{i=1}^n x_i \le 22 \end{cases}$$

(c) Determine the p-value of the test ϕ^* from the previous subexercise. Explain its interpretation in the context of the problem.

$$\text{p-value} = \inf\{\alpha \in \mathcal{A} : T(x) \in C_\alpha\}$$

In this case, the p-value: $\mathbb{P}_{\theta_0}\{\sum_{i=1}^{n} X_i \le 22\} = 0.0478$

In this case, it means if we set $k_\alpha^* = 22$, the probability that a true H_0 get rejected is 0.0478. Alternatively. we could set the α to be 0.0478, and the test is still valid at that significance.

3. **Neyman-Pearson test for Poisson distributions** We need the data of several years. [4 Points]

The number of claims reported to an insurance company during a year is Poisson-distributed with some parameter $\lambda > 0$. From previous years, the insurance company uses $\lambda = \lambda_0$ as a model parameter. The company notices that the number of claims has increased, and therefore wants to test

$$H_0: \quad \lambda = \lambda_0, \text{ against}$$
$$H_1: \quad \lambda = \lambda_1,$$

where $\lambda_1 > \lambda_0$.

(a) Find a Neyman-Pearson test ϕ^* at level $\alpha \in (0,1)$ for this testing problem.

(a) $H_0: \lambda = \lambda_0$ against

$H_1: \lambda = \lambda_1$

$$f(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$L_{\lambda_1, \lambda_0}(x) = \frac{f_{\lambda_1}(x_1, \ldots, x_n)}{f_{\lambda_0}(x_1, \ldots, x_n)} = \frac{\dfrac{\lambda_1^{\sum x_i} e^{-n\lambda_1}}{x_1! \cdots x_n!}}{\dfrac{\lambda_0^{\sum x_i} e^{-n\lambda_0}}{x_1! \cdots x_n!}}$$

$$= \left(\frac{\lambda_1}{\lambda_0}\right)^{\sum_{i=1}^{n} x_i} \cdot e^{-n(\lambda_1 - \lambda_0)}$$

$$\phi^* \circ X = \begin{cases} 1 & L_{\lambda_0, \lambda_1}(x) \geq c \\ 0 & L_{\lambda_0, \lambda_1}(x) < c \end{cases}$$

and $\mathbb{P}_{\lambda_0}[\phi^* \circ (x) = 1] \leq \alpha$.

(b) Is the test from the previous subexercise a uniformly most powerful test for H_0 against

$$\widetilde{H}_1 : \quad \lambda \in \{\lambda' ; \lambda' > \lambda_0\}?$$

The Neyman-Pearson test we got in last subexersize is only guaranteed to be the most powerful test for $H_0 : \{\lambda_0\}$ against $H_1 : \{\lambda_1\}$.

(c) Suppose now that $\lambda_0 = 9000$ and the company observed 9876 claims. Decide whether H_0 is rejected at a level $\alpha = 0.05$.

Hint: Use that $\sum_{k=0}^{9155} \frac{(9000)^k}{k!} e^{-9000} \approx 0.9491$ and $\sum_{k=0}^{9156} \frac{(9000)^k}{k!} e^{-9000} \approx 0.9502$.

(c)

Here we could see the determinant factor is $\sum_{i=1}^{n} X_i$

where, in this case, we only have the observation for 1 year.

$\Rightarrow \mathbb{P}_{\lambda_0}\{\phi_0(x) \geq c\} \leq \alpha$

$\Leftrightarrow \mathbb{P}_{\lambda_0}[X \geq k_\alpha^*] \leq \alpha = 0.05$

$\Rightarrow \mathbb{P}_{\lambda_0}[X < k_\alpha^*] > 0.95$

$\Rightarrow$ we should set $k_\alpha^* = 9157$

Clearly. $x = 9876$ clearly reject the hypothesis H_0

4. **Testing for the variance in normal distributions** [4 Points]

Suppose that $X_1, ..., X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with *known* $\mu \in \mathbb{R}$ and unknown $\sigma^2 > 0$. Assume furthermore that $0 < \sigma_0^2 < \sigma_1^2$. Show that the test

$$\phi^*(\mathbf{X}) = \begin{cases} 1, & \text{if } \sum_{i=1}^n (X_i - \mu)^2 > \boxed{\sigma_0^2} \chi^2_{n,1-\alpha}, \\ 0, & \text{if } \sum_{i=1}^n (X_i - \mu)^2 \le \sigma_0^2 \chi^2_{n,1-\alpha}, \end{cases}$$

is the Neyman-Pearson test at level $\alpha \in (0,1)$ for $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 = \sigma_1^2$.

proof.

$$L_{\sigma_0,\sigma_1} = \frac{f_{\sigma_1^2}(x_1, ---, x_n)}{f_{\sigma_0^2}(x_1, --- x_n)} = \frac{\left(\frac{1}{\sigma_1\sqrt{2\pi}}\right)^n e^{\sum -\frac{1}{2}\frac{(x_i-\mu)^2}{\sigma_1^2}}}{\left(\frac{1}{\sigma_0\sqrt{2\pi}}\right)^n e^{\sum -\frac{1}{2}\frac{(x_i-\mu)^2}{\sigma_0^2}}}$$

$$= K(\sigma_1, \sigma_0) \cdot \exp\left(\underbrace{\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)}_{>0} \sum_{i=1}^n (x_i-\mu)^2\right)$$

Here we can see, L_{σ_0,σ_1} increases as $\sum_{i=1}^n (x_i-\mu)^2$ increases

Therefore. $L_{\sigma_0,\sigma_1} \ge C_\alpha^* \Leftrightarrow \sum_{i=1}^n (x_i-\mu)^2 \ge K_\alpha^*$

It remains to decide K_α^*

We want. $\mathbb{P}_{\sigma_0}\left(\sum_{i=1}^n (x_i-\mu)^2 > K_\alpha^*\right) = \alpha$

$\Rightarrow$ K_α^* should be the $1-\alpha$ quantile of $\sum_{i=1}^n (x_i-\mu)^2$

$$\sum_{i=1}^n (x_i-\mu)^2 \sim \sum_{i=1}^n N^2(0, \sigma_0^2)$$

$\times \frac{1}{\sigma_0^2}$ $$\frac{\sum_{i=1}^n (x_i-\mu)^2}{\sigma_0^2} \sim \sum_{i=1}^n N^2(0,1) = \chi_n^2$$

Therefore,

$$\Rightarrow \mathbb{P}\left(\sum_{i=1}^{n}(x_i-\mu)^2 > k_\alpha^*\right) = \alpha$$

$$\Leftrightarrow \mathbb{P}\left(\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma_0^2} \leq \frac{k_\alpha^*}{\sigma_0^2}\right) = 1-\alpha$$

$$\mathbb{P}\left(\chi_n^2 \leq \frac{k_\alpha^*}{\sigma_0^2}\right) = 1-\alpha$$

$$\Rightarrow \frac{k_\alpha^*}{\sigma_0^2} = \chi_{n,1-\alpha}^2 \quad \Rightarrow k_\alpha^* = \sigma_0^2\, \chi_{n,1-\alpha}^2$$

Hence, the UMP test is

$$\phi^*_0 X = \begin{cases} 1 & \sum_{i=1}^{n}(x_i-\mu)^2 > \sigma_0^2\, \chi_{n,1-\alpha}^2 \\ 0 & \sum_{i=1}^{n}(x_i-\mu)^2 \leq \sigma_0^2\, \chi_{n,1-\alpha}^2 . \end{cases}$$