Problem Set 3

Submission:
Thursday, 02/24/2022, until 1 PM, to be uploaded on the NYU Brightspace course homepage.

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1. **On the multivariate normal distribution** [4 Points]

   Let $X = (X_1, X_2)$ be a random vector with $X \sim \mathcal{N}_2(\mu, \Sigma)$, where

   $$\mu = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

   (a) Write down the probability density function of the law of X.

   (b) Find the distribution of $Y = X_1 - 3X_2$ and of $Z = (X_1 + X_2, X_1 - X_2)$.

   (c) Consider $U \sim \mathcal{N}(0,1)$ and $R \sim Ber(\frac{1}{2})$, independent from U. Set

   $$V = \begin{cases} U, & R = 1, \\ -U, & R = 0. \end{cases}$$

   It can be proved that $V \sim \mathcal{N}(0,1)$.[1] Show that $\mathrm{Cov}[U,V] = 0$, and that U and V are *not* independent. Why is this not a contradiction to Remark 2.3 (i)?

2. **On the χ^2-distribution** [4 Points]

   (a) Calculate the density of a χ^2-distribution with one degree of freedom. For this, recall if $X \sim \mathcal{N}(0,1)$, then $Z = X^2 \sim \chi^2_1$.
   Hint: Look at $F_Z(z) = \mathbf{P}[X^2 \leq z]$.

   (b) Verify that the χ^2-distribution with n degrees of freedom has density

   $$f_{\chi^2_n}(x) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} e^{-\frac{x}{2}} x^{\frac{n}{2}-1} \mathbb{1}_{(0,\infty)}(x).$$

   Hint: You may use without proof the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Then identify the density of χ^2_1 with a Γ-distribution and use the result from Problem 2 (b) on Problem Set 1.

   (c) Suppose the coordinates (X_1, X_2, X_3) of a particle undergoing diffusive motion can be described at some time t by i.i.d. $\mathcal{N}(0,1)$-distributed random variables. What is the probability that the particle at time t is located within a ball of radius $\frac{1}{2}$?
   Hint: Use freely available online calculators for the cumulative distribution function of the χ^2 distribution.

3. **Derivation of the density of a t-distribution** [4 Points]

   In this problem, we show step-by-step that the density of the t-distribution with n degrees of freedom is given by

   $$f_{t_n}(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}.$$

[1] For instance, $\mathbf{P}[V \leq v] = \frac{1}{2}\mathbf{P}[U \leq v] + \frac{1}{2}\mathbf{P}[U \geq -v] = \mathbf{P}[U \leq v]$ by symmetry.

(a) Recall that $T = \frac{X}{\sqrt{Y/n}}$ has the t_n-distributiuon if $X \sim \mathcal{N}(0,1)$ and $Y \sim \chi^2_n$ are independent. Write down the probability density function of $\sqrt{Y/n}$.

Hint: Look at $F_{\sqrt{Y/n}}(y) = \mathbf{P}[\sqrt{Y/n} \leq y]$. The density of Y is known from Problem 2 (b).

(b) For the quotient of two independent, continuous, real random variables X and Y where Y is positive, one can also show that $Z = \frac{X}{Y}$ has density

$$f_Z(z) = \int_0^\infty y f_X(zy) f_Y(y) \mathrm{d}y, z \in \mathbb{R}.$$

Use this in combination with (a) to demonstrate that

$$(\star) \qquad f_{t_n}(z) = \frac{2^{1-\frac{n}{2}}}{\sqrt{2\pi}\Gamma(\frac{n}{2})} n^{\frac{n}{2}} \int_0^\infty y^n e^{-\frac{1}{2}(n+z^2)y^2} \mathrm{d}y.$$

(c) Recall now that the Γ-function is defined for $\alpha > 0$ by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \mathrm{d}x.$$

Validate the formula for f_{t_n} by using a substitution in the remaining integral in (b).

4. **Properties of estimators** [4 Points]

Let $X_1, \ldots, X_n$ be i.i.d. real random variables with $X_1 \sim Pois(\theta)$, with $\theta > 0$. We want to estimate $\gamma = \mathbf{P}_\theta[X_1 = 0] = e^{-\theta}$ based on the data $X_1, \ldots, X_n$. We consider the two estimators for γ:

$$\widehat{\gamma}_1 = \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{\{X_i = 0\}}, \qquad \widehat{\gamma}_2 = \exp\left(-\frac{1}{n}\sum_{i=1}^n X_i\right).$$

(a) Show that $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ are both consistent estimators for γ.

Hint: Use the weak law of large numbers and the continuous mapping theorem.

(b) Determine whether $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ are unbiased.

(c) Calculate the mean square error $\mathrm{MSE}_\theta(\widehat{\gamma}_1) = \mathbf{E}_\theta\left[(\widehat{\gamma}_1 - \gamma)^2\right]$.