MATH-UA.0234 Mathematical Statistics Spring 2022
Dr. M. Nitzschner

Problem Set 1

Submission:
Thursday, 02/10/2022, until 1 PM, to be uploaded on the NYU Brightspace course homepage.

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1. **Expectation and cumulative distribution function** [4 Points]

   A real random variable X is *Rayleigh-distributed* with parameter $\sigma > 0$ if its law is characterized by the probability density function

   $$f_X(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathbb{1}_{0,\infty)}(x).$$

   We write $X \sim Ra(\sigma)$.

   (a) Calculate the cumulative distribution function of a $Ra(\sigma)$-distributed random variable X.

   (b) Let $U \sim \mathcal{U}([0,1])$. Using the result from (a), show that for $\sigma > 0$ one has $Y = \sigma\sqrt{-2\log(U)} \sim Ra(\sigma)$.
   Hint: Calculate $F_Y(x) = \mathbf{P}[Y \leq x]$ for $x \in \mathbb{R}$.

   (c) Calculate $\mathbf{E}[X]$ and $\mathrm{Var}[X]$ for $X \sim Ra(\sigma)$.

2. **Sums and quotients of random variables** [4 Points]

   In this problem we recall the important notion of ***convolution*** *for two probability distributions.*

   - *If X and Y are independent discrete real random variables with values in $\Omega_X, \Omega_Y \subseteq \mathbb{R}$ respectively, then $Z = X + Y$ has probability mass function*

     $$p_Z(k) = \sum_{\ell \in \Omega_Y} p_X(k - \ell) p_Y(\ell),$$

     for $k \in \Omega_X + \Omega_Y$, where p_X and p_Y are the probability mass functions of X and Y, and we set $p_X(r) = 0$ if $r \notin \Omega_X$.

   - *If X and Y are independent continuous real random variables then $Z = X + Y$ has probability density function*

     $$f_Z(z) = \int_{-\infty}^{\infty} f_X(x - y) f_Y(y) \mathrm{d}y.$$

     where f_X and f_Y are the probability density functions of X and Y respectively.

   (a) Let $\lambda, \mu > 0$. Show that if $X \sim Pois(\lambda)$ and $Y \sim Pois(\mu)$ are independent, then $X + Y \sim Pois(\lambda + \mu)$.

   (b) Let $X_1, ..., X_n$ be i.i.d. real random variables with $X_1 \sim \Gamma(\alpha, \beta)$ and $\alpha, \beta > 0$. Show that $\sum_{i=1}^n X_i \sim \Gamma(n\alpha, \beta)$.
   Hint: You can use that

   $$\int_0^1 u^{a-1}(1-u)^{b-1} \mathrm{d}u = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \qquad a, b > 0.$$

(c) For the quotient of two independent, continuous, positive real random variables X and Y one can also show that $Z = \frac{X}{Y}$ has density

$$f_Z(z) = \int_0^\infty y f_X(zy) f_Y(y) \mathrm{d}y, \qquad z > 0.$$

Using this, determine the law of the quotient of two independent, $\mathcal{U}([0,1])$-distributed random variables.

3. **Variances and covariances** [4 Points]

(a) Let X and Y be jointly continuous with joint density

$$f_{X,Y}(x,y) = (x+y)\mathbb{1}_{\{(x,y)\in[0,1]^2\}}.$$

Calculate the covariance matrix Σ and the correlation $\rho(X,Y)$. Then use Σ to calculate $\mathrm{Var}[X-2Y]$.

(b) Let $X_1, \ldots, X_n$ be i.i.d. real random variables with $\mu = \mathbf{E}[X_1]$ and $\sigma^2 = \mathrm{Var}[X_1]$. We define

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i \text{ (sample mean)}, \qquad S_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2 \text{ (sample variance)}.$$

Show that $\mathbf{E}[\overline{X}_n] = \mu$, $\mathrm{Var}[\overline{X}_n] = \frac{\sigma^2}{n}$ and $\mathbf{E}[S_n^2] = \sigma^2$.

4. **Moment generating functions** [4 Points]

In this problem we recall the ***moment generating functions*** *for random variables. For a real random variable X, the moment generating function is defined by*

$$\psi_X(t) = \mathbf{E}\left[e^{tX}\right], \qquad t \in \mathbb{R},$$

whenever this expression exists. One has:

- *Whenever $\psi_X(t) = \psi_Y(t)$ for $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, for two random variables X and Y, then $X \stackrel{d}{=} Y$.*
- *$\psi_X'(0) = \mathbf{E}[X]$.*

(a) Calculate the moment generating function of $X \sim \mathcal{N}(0,1)$.

(b) Suppose that X and Y are independent. Explain why $\psi_{X+Y}(t) = \psi_X(t)\psi_Y(t)$ (assuming all these expressions exist for a given t). Use this to show that if $X \sim \mathcal{N}(0,\sigma_1^2)$ and $Y \sim \mathcal{N}(0,\sigma_2^2)$, then $X+Y \sim \mathcal{N}(0, \sigma_1^2+\sigma_2^2)$, where $\sigma_1, \sigma_2 > 0$.

(c) The moment generating function of $X \sim Geo(p)$ with $p \in (0,1)$ is given by

$$\psi_X(t) = \frac{pe^t}{1-(1-p)e^t}, \qquad t < -\log(1-p).$$

Use this to verify that $\mathbf{E}[X] = \frac{1}{p}$.