1. **Convergence in probability and in distribution** [4 Points]

(a) Consider a sequence $(X_n)_{n\geq 1}$ of random variables with $\mathbf{P}[X_n = n^\alpha] = \frac{1}{n}$ and $\mathbf{P}[X_n = 0] = 1 - \frac{1}{n}$ for every $n \in \mathbb{N}$. For which $\alpha \in \mathbb{R}$ does one have $X_n \xrightarrow[n\to\infty]{\mathbf{P}} 0$, and for which does one have $X_n \xrightarrow[n\to\infty]{d} 0$? For which ones does one have $\mathbf{E}[X_n] \xrightarrow[n\to\infty]{} 0$?

proof.

(a). $X_n \xrightarrow{\mathbb{P}} 0$

$\forall \alpha \in \mathbb{R}$. $\forall \varepsilon > 0$ $\lim_{n\to\infty} \mathbb{P}[|X_n - 0| > \varepsilon] = \lim_{n\to\infty} \frac{1}{n} = 0$

which shows (X_n) converges to 0 in Probability $\forall \alpha$.

(b) As Convergence in Probability implies Convergence in Distribution.

$X_n \xrightarrow{d} 0$

(c)

$\mathbb{E}[X_n] = n^\alpha \cdot \frac{1}{n} = n^{\alpha-1}$ which Converges to 0 if $\alpha - 1 < 0$

$\Rightarrow \alpha < 1$

(b) Suppose that $(X_n)_{n\geq 1}$ is an i.i.d. sequence of $Ber(p)$-distributed random variables. Determine the quantity a in

$$\frac{1}{n}\sum_{i=1}^{n}\exp(X_i) \xrightarrow[n\to\infty]{\mathbf{P}} a.$$

Proof. As $\exp(X_i)$ remains to be i.i.d.

By weak Law of Large Numbers.

$$\frac{1}{n}\sum_{i=1}^{n} \exp(X_i) \xrightarrow{\mathbb{P}} \mathbb{E}[\exp(X_i)] = p \cdot e$$

(c) Suppose that $(X_n)_{n\geq 1}$ is an i.i.d. sequence of random variables with $\mathbf{E}[X_1] = 0$ and $\mathrm{Var}[X_1] = \sigma^2 \in (0,\infty)$. Argue in detail, why

$$\frac{1}{\log(n)\sqrt{n}}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{\mathbf{P}} 0.$$

Hint: Use the properties of convergence in probability / in distribution from the notes.

Proof.

By CLT.

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i = \sqrt{n}\,\bar{X}_n \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

Also

$$\frac{1}{\log(n)} \xrightarrow{\mathbb{P}} 0$$ as $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\left|\frac{1}{\log(n)} - 0\right| < \varepsilon$ if $n > N$

Therefore. By the properties of converges in $\mathbb{P}$ and d.

$$\left(\frac{1}{\sqrt{n}} \cdot \sum_{i=1}^{n} X_i\right) \cdot \left(\frac{1}{\log(n)}\right) \xrightarrow{\mathbb{P}} 0$$ As desired.

2. **Some applications of the central limit theorem** [4 Points]

(a) Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. $\mathcal{U}(\{1,2,3,4\})$ distributed random variables. Consider the expression

$$Z_n = \log\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right).$$

1 4 9 16

Find the approximate distribution of Z_n for large n.
Hint: Use the δ-method.

Proof.

$$\mu = \mathbb{E}(X_i) = \frac{5}{2}$$

$$\sigma^2 = \operatorname{var}[X_i] = \mathbb{E}(X^2) - \mathbb{E}^2[X] = \frac{30}{4} - \frac{25}{4} = \frac{5}{4}$$

Therefore we have: $\sqrt{n} \cdot \frac{\bar{X}_n - \mu}{\sigma} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,1)$ by CLT

$$\Rightarrow \sqrt{n}(\bar{X}_n - \mu) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \sigma^2)$$

Let $g(x) = \log(x)$. Though δ method require g to be differentiable on $\mathbb{R}$. as X_i only takes positive value we can ignore the incommensurateness of Domain.

By δ-method.

$$\sqrt{n}\left(\log(\bar{X}_n) - \log(\mu)\right) \xrightarrow{d} \frac{1}{\mu}\mathcal{N}(0, \sigma^2)$$

$$\Rightarrow \log(\bar{X}_n) \xrightarrow{d} \frac{1}{\mu\sqrt{n}}\mathcal{N}(0, \sigma^2) + \log(\mu)$$

$$= \mathcal{N}\left(0, \frac{\sigma^2}{\mu^2 n}\right) + \log(\mu)$$

$$= \mathcal{N}\left(\log\left(\frac{5}{2}\right), \frac{5}{4n}\right)$$

Therefore as n is large.

$$\log(\bar{X}_n) \sim \mathcal{N}\left(\log\left(\frac{5}{2}\right), \frac{5}{4n}\right)$$

(b) Suppose that the number of goals in a soccer match is Poisson-distributed with mean 3. Assume also that during a season, there are $n = 300$ matches. Use the central limit theorem to find the approximate probability for the event that during a season, there are at least 860 goals, but less than 930 goals in total. You may assume that the numbers of goals in different matches are independent.

proof. Let $(x_i)_{i=1}^{n=300}$ denote the random variable representing the # of goals.

$x_i \sim \text{poiss}(3)$ i.e. $\mathbb{P}(x_i = k) = \frac{3^k e^{-3}}{k!}$

Here $\mu = \mathbb{E}[x_i] = 3$ $\sigma^2 = Var[x_i] = 3$

According to Central limit Thm.

$$\underbrace{\sqrt{n}\,\frac{\frac{1}{n}\sum_{i=1}^{n} x_n - \mu}{\sigma}}_{\text{denoted } Y_n} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,1)$$

Therefore.

$$\mathbb{P}\left[860 \le \sum_{i=1}^{n} x_n \le 930\right]$$

$$= \mathbb{P}\left[\sqrt{300}\,\frac{\frac{1}{300}\times 860 - 3}{\sqrt{3}} \le Y_{300} \le \sqrt{300}\cdot\frac{\frac{1}{300}\times 930 - 3}{\sqrt{3}}\right]$$

$$= \mathbb{P}\left[-1.3 \le Y_{300} \le 1\right]$$

$$\approx \Phi[0.577350] - \Phi[0.-769800] = 0.7445$$

3. **Asymptotic normality of the t-statistics** [4 Points]

In this problem, we study the t-statistics, which is defined as follows: Let $X_1, \dots, X_n$ be i.i.d. real random variables with $\mathbf{E}[X_1] = \mu$, and $\mathrm{Var}[X_1] = \sigma^2 \in (0, \infty)$. The t-statistics is

$$T_{n-1} = \frac{\overline{X}_n - \mu}{S_n/\sqrt{n}}, \qquad \text{where } \overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i,\ S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2.$$

The goal of this problem is to show step-by-step that $T_{n-1} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,1)$.

(a) First argue that one can write $\sum_{i=1}^n (X_i - \overline{X}_n)^2$ as $\sum_{i=1}^n X_i^2 - (\overline{X}_n)^2$.

(a) Proof. $\sum_{i=1}^{n} (x_i - \bar{x}_n)^2 = \sum_{i=1}^{n} x_i^2 - 2x_i \bar{x}_n + (\bar{x}_n)^2$

$= \sum_{i=1}^{n} x_i^2 + (\bar{x}_n)^2 - \frac{2}{n} x_i \sum_{j=1}^{n} (x_j)$

$= \sum_{i=1}^{n}\left(x_i^2 + (\bar{x}_n)^2\right) - \frac{2}{n}\left(\sum_{j=1}^{n} x_j\right)^2 = \sum_{i=1}^{n}\left(x_i^2 + (\bar{x}_n)^2\right) - 2n(\bar{x}_n)^2$

$= \sum_{i=1}^{n} x_i^2 + (\bar{x}_n)^2 - 2(\bar{x}_n)^2 = \sum_{i=1}^{n} x_i^2 - (\bar{x}_n)^2$

(b) Show that $\frac{1}{n}\sum_{i=1}^n (X_i - \overline{X}_n)^2$ converges in probability to σ^2.
Hint: Apply the weak law of large numbers to both $\frac{1}{n}\sum_{i=1}^n X_i^2$ and to $\overline{X}_n$. Then use the continuous mapping theorem and the fact that if $Y_n \xrightarrow[n\to\infty]{\mathbf{P}} Y$ and $Z_n \xrightarrow[n\to\infty]{\mathbf{P}} Z$, also $Y_n + Z_n \xrightarrow[n\to\infty]{\mathbf{P}} Y + Z$.

Proof. Apply WLLN

$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \xrightarrow[\mathbb{P}]{n\to\infty} \mathbb{E}[x_i^2] = \sigma^2 + \mu^2$

$\bar{x}_n \xrightarrow[\mathbb{P}]{n\to\infty} \mathbb{E}[x_n] = \mu.$

By Continuous Mapping Thm.

$(\bar{x}_n)^2 \xrightarrow[\mathbb{P}]{n\to\infty} \mu^2$

By the Property of Convergence in Probability

$$\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x}_n)^2$$

$$=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\frac{1}{n}\sum_{i=1}^{n}(\bar{x}_n)^2=\frac{1}{n}\sum_{i=1}^{n}x_i^2-(\bar{x}_n)^2\xrightarrow{\mathbb{P}}\sigma^2+\mu^2-\mu^2=\sigma^2$$

(c) Conclude from (b) that $S_n^2 \xrightarrow[n\to\infty]{\mathbf{P}} \sigma^2$. Use the continuous mapping theorem *again* to argue that $\frac{1}{S_n} \xrightarrow[n\to\infty]{\mathbf{P}} \frac{1}{\sigma}$.

$$S_n^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x}_n)^2$$

$$\Rightarrow \text{As } n\to\infty \quad S_n^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x}_n)^2\xrightarrow{\mathbb{P}}\sigma^2$$

$$\text{as } \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x}_n)^2\xrightarrow{\mathbb{P}}\sigma^2$$

By Continuous mapping Thm. as $g(x)=x^{-\frac{1}{2}}$ is continuous on $(0,+\infty)$

Therefore, $g(S_n^2)\xrightarrow{\mathbb{P}}g(\sigma^2)$

$$\Rightarrow \frac{1}{S_n}\xrightarrow{\mathbb{P}}\frac{1}{\sigma}$$

(d) Finally use Slutsky's theorem and the central limit theorem to conclude that $T_{n-1} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,1)$.

Hint: Note that $T_{n-1} = \sqrt{n}\frac{\overline{X}_n-\mu}{\sigma}\cdot\frac{\sigma}{S_n}$.

$$T_{n-1}=\frac{\bar{x}_n-\mu}{S_n/\sqrt{n}}=\sqrt{n}\,\frac{\bar{x}_n-\mu}{\sigma}\cdot\frac{\sigma}{S_n}.$$

On the one hand:

$$\sqrt{n}\,\frac{\bar{x}_n-\mu}{\sigma}\xrightarrow[n\to\infty]{d}\mathcal{N}(0,1)$$

On the other hand:

$$\frac{\sigma}{S_n} \xrightarrow[n\to\infty]{P} \frac{1}{\sigma}\cdot\sigma = 1$$

By Slutsky's Thm.

$$T_{n-1} = \sqrt{n}\,\frac{\bar{X}_n - \mu}{\sigma}\cdot\frac{\sigma}{S_n} \xrightarrow{d} \mathcal{N}(0,1)$$

HW2-Q4

February 17, 2022

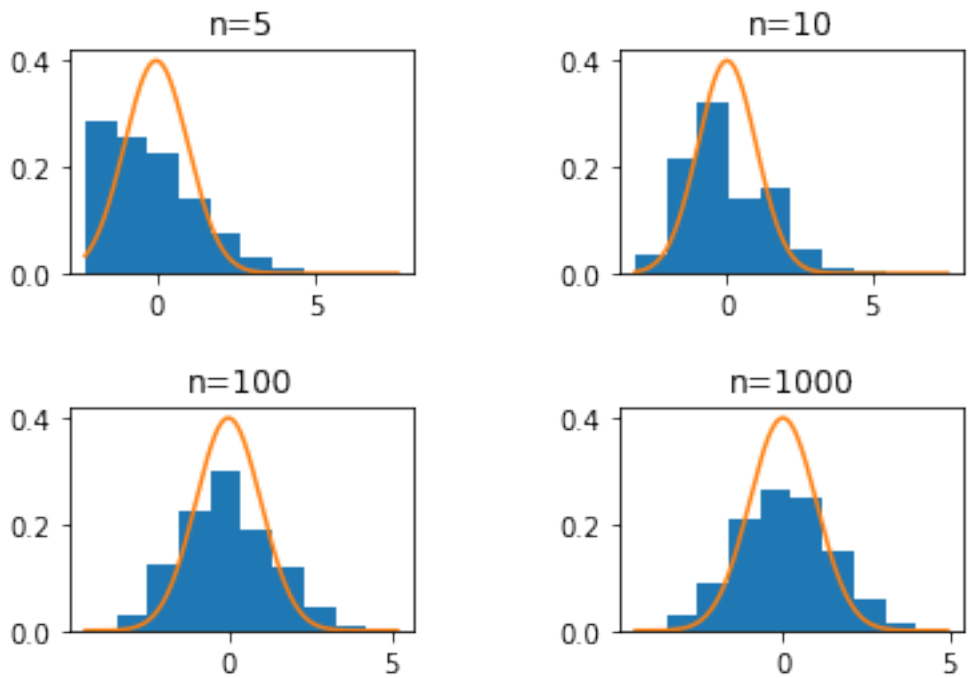
[1]:
```
import numpy as np
import scipy.stats as stats
import matplotlib.pyplot as plt
```

0.1 Possion Distribution

0.1.1 CLT - Possion

[2]:
```
M = 5000
n_choices= [5, 10, 100, 1000]

def rv_pos(n):
    Sn = sum(np.array([np.random.poisson(lam=1/2) for i in range(n)]))
    rv = (Sn - n*(1/2))/((n**(1/2))*0.5)

    return rv


hist_1 = np.zeros((4, M))

for i in range(len(n_choices)):
    hist_1[i] = np.array([rv_pos(n_choices[i]) for j in range(M)])
```

[3]:
```
fig, axs = plt.subplots(2, 2)
plt.subplots_adjust(wspace=0.6, hspace=0.6)

for i in range(len(n_choices)):
    x_space = np.linspace(min(hist_1[i]), max(hist_1[i]), 1000)
    axs[i//2, i%2].hist(hist_1[i], density=True)
    axs[i//2, i%2].plot(x_space, stats.norm(0, 1).pdf(x_space))
    axs[i//2, i%2].set_title("n=" + str(n_choices[i]))
```

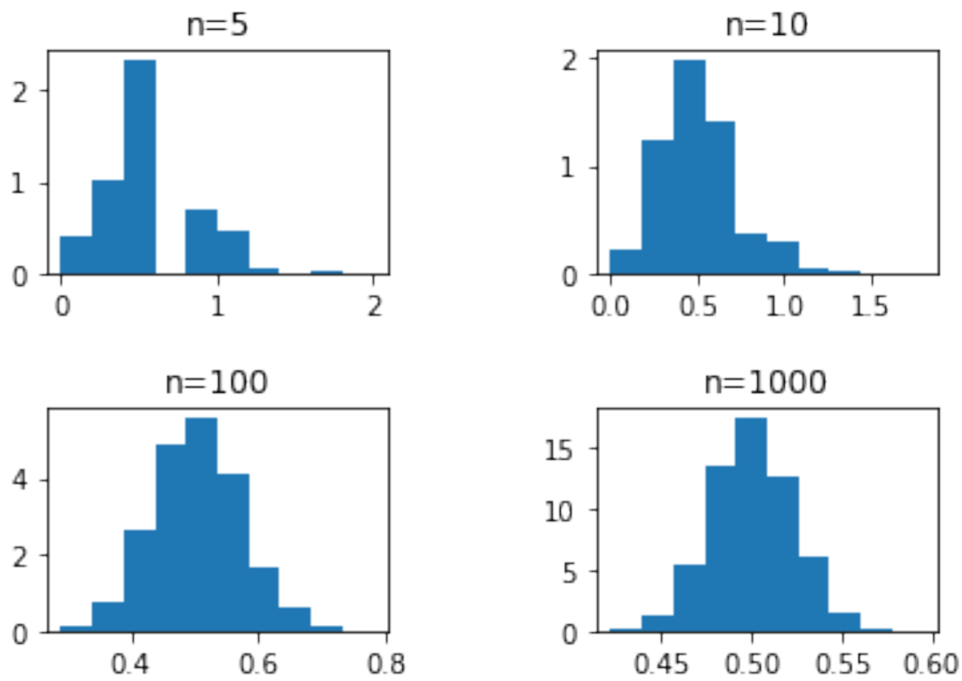


As we can see, when n becomes larger, the distribution of the generated random variable $\frac{S_n - n\cdot\mu}{\sqrt{n}\cdot\sigma}$ is more close to normal distribution, which is the essense of central limit theorem.

0.1.2 LNN - Possion

[4]:
```
def rv_pos_lln(n):
    Xn_bar = np.array([np.random.poisson(lam=1/2) for i in range(n)]).mean()
    return Xn_bar

hist_2 = np.zeros((4, M))

for i in range(len(n_choices)):
    hist_2[i] = np.array([rv_pos_lln(n_choices[i]) for j in range(M)])
```

[5]:
```
fig, axs = plt.subplots(2, 2)
plt.subplots_adjust(wspace=0.6, hspace=0.6)

for i in range(len(n_choices)):
    x_space = np.linspace(min(hist_2[i]), max(hist_2[i]), 1000)
    axs[i//2, i%2].hist(hist_2[i], density=True)
    axs[i//2, i%2].set_title("n=" + str(n_choices[i]))
```



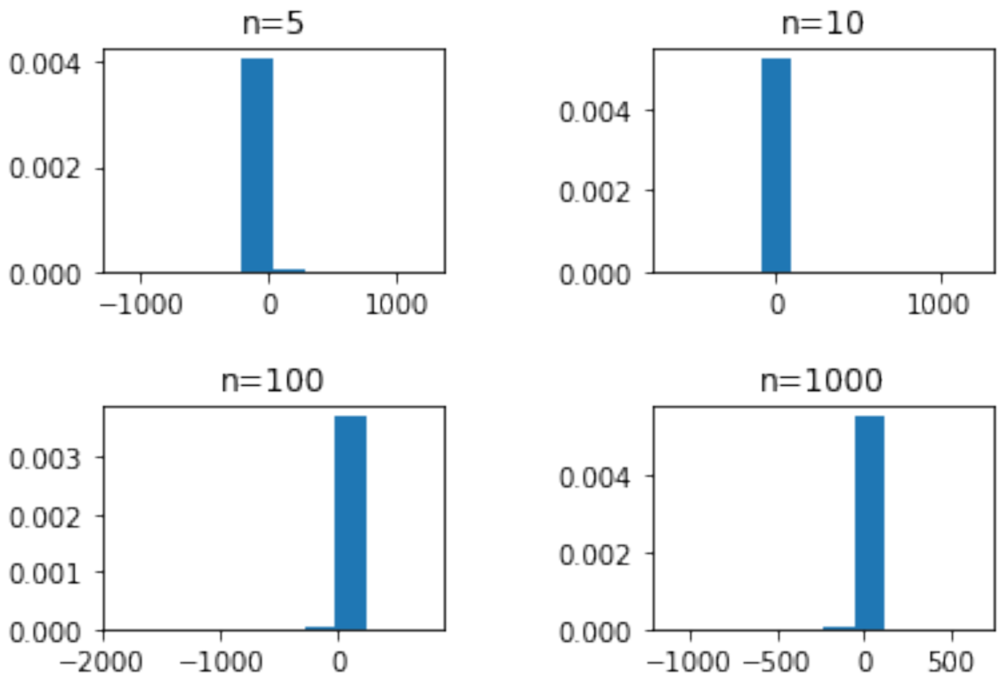
As we can see, as n grows, $\bar{X}_n$ is more concentrated to the mean of X_1, which is 1. ~ The reuslt is consistent with **Law of Large Numbers**

0.2 Cauchy

[6]:
```
def rv_cauchy(n):
    Sn = sum(np.array([np.random.standard_cauchy() for i in range(n)]))
    # rv = (Sn - n*(1/2))/((n**(1/2))*0.5)
    # Since mean and variance is essentially missing here
    # We do not normalize it
    return Sn/n

hist_3 = np.zeros((4, M))

for i in range(len(n_choices)):
    hist_3[i] = np.array([rv_cauchy(n_choices[i]) for j in range(M)])
```

[7]:
```
fig, axs = plt.subplots(2, 2)
plt.subplots_adjust(wspace=0.6, hspace=0.6)

for i in range(len(n_choices)):
    x_space = np.linspace(min(hist_3[i]), max(hist_3[i]), 1000)
    axs[i//2, i%2].hist(hist_3[i], density=True)
    # axs[i//2, i%2].plot(x_space, stats.norm(0, 1).pdf(x_space))
```

```
    axs[i//2, i%2].set_title("n=" + str(n_choices[i]))
```



As we can see, the mean of cauchy distributions do not behave like other distributions, which conerges to the mean as n grows large. This is perhaps because the mean and variance of cauchy distribution is essentially missing.