MATH-UA.0234 Mathematical Statistics Spring 2022
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Problem Set 8

Submission:
Thursday, 04/21/2022, until 1 PM, to be uploaded on the NYU Brightspace course homepage.

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1. **Testing binomial distributions I** [4 points]

   Imagine we want to test whether a given coin is fair.

   (a) Suppose that there is *no* additional information. What would be a reasonable test for at level $\alpha = 0.05$ to determine whether the coin is fair, when we flip it $n = 200$ times? Formulate H_0 and H_1 and determine the critical region.

   (b) Suppose now that we suspect that the coin has a higher chance to land on heads. How would the test in (a) change?

   (c) Consider both situations in (a) and (b), but use an approximation of the binomial distribution coming from the central limit theorem. How do the critical values change?

2. **Testing binomial distributions II** [4 Points]

   The first digit of various numerical data sets (such as electricity bills, street addresses, stock prices,...) typically follows the *Benford law*. This is a distribution on $(\{1, ..., 9\}, \mathscr{P}(\{1, ..., 9\})$ with probability mass function

   $$p(k) = \log_{10}\left(1 + \frac{1}{k}\right), \qquad 1 \leq k \leq 9.$$

   Here $\log_{10} = \frac{\log}{\log(10)}$ is the logarithm with base 10.

   (a) Show that $(p(k))_{k\in\{1,...,9\}}$ defines a probability mass function on $\{1, ..., 9\}$.

   By Benford's law, 1 should be the first digit in roughly 30% of the numbers in a valid statistical data set. Suppose we suspect a given sample of 100 independent data points to be fraudulent, since the first digit 1 shows up only 17 times in the sample.

   (b) Formulate a testing problem and construct the Neyman-Pearson test ϕ^* at level $\alpha = 0.05$ for the hypotheses

   H_0 : the probability for 1 as first digit is 30%, against
   H_1 : the probability for 1 as first digit is smaller than 30%.

   Find the critical region and determine whether H_0 can be rejected with our observation.

   (c) Determine the p-value of the test ϕ^* from the previous subexercise. Explain its interpretation in the context of the problem.

   (d) Suppose that it is revealed later that the data set was in fact fraudulent, and generated by some method in which the probability of having first digit 1 was only 20%. Calculate the probability of a type II error for the test ϕ^*.

3. **Neyman-Pearson test for Poisson distributions** [4 Points]

The number of claims reported to an insurance company during a year is Poisson-distributed with some parameter $\lambda > 0$. From previous years, the insurance company uses $\lambda = \lambda_0$ as a model parameter. The company notices that the number of claims has increased, and therefore wants to test

$$\begin{aligned} H_0 : &\qquad \lambda = \lambda_0, \text{ against} \\ H_1 : &\qquad \lambda = \lambda_1, \end{aligned}$$

where $\lambda_1 > \lambda_0$.

(a) Find a Neyman-Pearson test ϕ^* at level $\alpha \in (0,1)$ for this testing problem.

(b) Is the test from the previous subexercise a uniformly most powerful test for H_0 against

$$\widetilde{H}_1 : \qquad \lambda \in \{\lambda' \,;\, \lambda' > \lambda_0\}?$$

(c) Suppose now that $\lambda_0 = 9000$ and the company observed 9876 claims. Decide whether H_0 is rejected at a level $\alpha = 0.05$.

Hint: Use that $\sum_{k=0}^{9155} \frac{(9000)^k}{k!} e^{-9000} \approx 0.9491$ and $\sum_{k=0}^{9156} \frac{(9000)^k}{k!} e^{-9000} \approx 0.9502$.

4. **Testing for the variance in normal distributions** [4 Points]

Suppose that $X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with *known* $\mu \in \mathbb{R}$ and unknown $\sigma^2 > 0$. Assume furthermore that $0 < \sigma_0^2 < \sigma_1^2$. Show that the test

$$\phi^*(\mathbf{X}) = \begin{cases} 1, & \text{if } \sum_{i=1}^n (X_i - \mu)^2 > \sigma_0^2 \chi^2_{n,1-\alpha}, \\ 0, & \text{if } \sum_{i=1}^n (X_i - \mu)^2 \leq \sigma_0^2 \chi^2_{n,1-\alpha}, \end{cases}$$

is the Neyman-Pearson test at level $\alpha \in (0,1)$ for $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 = \sigma_1^2$.