1. **Expectation and cumulative distribution function** [4 Points]

A real random variable X is *Rayleigh-distributed* with parameter $\sigma > 0$ if its law is characterized by the probability density function

$$f_X(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathbb{1}_{0,\infty)}(x).$$

We write $X \sim Ra(\sigma)$.

(a) Calculate the cumulative distribution function of a $Ra(\sigma)$-distributed random variable X.

(a)

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & x < 0 \\ \int_0^x \frac{t}{\sigma^2} \exp\left(-\frac{t^2}{2\sigma^2}\right) dt & x \geq 0 \end{cases}$$

Then we do the integration:

as $\left[\exp\left(-\frac{t^2}{2\sigma^2}\right)\right]' = -\frac{t}{\sigma^2} \exp\left(-\frac{t^2}{2\sigma^2}\right)$

$$\int_0^x \frac{t}{\sigma^2} \exp\left(-\frac{t^2}{2\sigma^2}\right) dt = -\int_0^x \left[\exp\left(-\frac{t^2}{2\sigma^2}\right)\right]' dt$$

$$= -\exp\left(-\frac{t^2}{2\sigma^2}\right)\Big|_0^x = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

Therefore, the CDF of $Ra(\sigma)$ is $F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) & x \geq 0 \end{cases}$

(b)

(b) Let $U \sim \mathcal{U}([0,1])$. Using the result from (a), show that for $\sigma > 0$ one has $Y = \sigma\sqrt{-2\log(U)} \sim Ra(\sigma)$.

Hint: Calculate $F_Y(x) = \mathbf{P}[Y \leq x]$ for $x \in \mathbb{R}$.

proof.

$$F_Y(x) = \mathbb{P}[Y \leq x]$$

$$= \mathbb{P}[\sigma\sqrt{-2\log(U)} \leq x]$$

Here if $x < 0$, then $\mathbb{P}[Y \leq x] = 0$

else if $x \geq 0$

$$F_Y(x) = \mathbb{P}[-2\log(U) \leq \frac{x^2}{\sigma^2}]$$

$$= \mathbb{P}[U \geq \exp(-2\frac{x^2}{\sigma^2})] =$$

CDF of U is $F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$

$$F_Y(x) = 1 - \exp(-2\frac{x^2}{\sigma^2}) \quad (x \geq 0)$$

Therefore. $F_Y(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp(-2\frac{x^2}{\sigma^2}) & x \geq 0 \end{cases}$

As $F_Y(x) = F_X(x)$, $Y = \sigma\sqrt{-2\log(U)} \sim Ra(\sigma)$

(c) Calculate $\mathbf{E}[X]$ and $\mathrm{Var}[X]$ for $X \sim Ra(\sigma)$.

$$\mathbb{E}[X] = \int_{\mathbb{R}} f_X(x) \cdot x \, dx$$

$$= \int_0^{+\infty} x \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$$

$$= -\int_0^{+\infty} x \left(-\frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) dx$$

$$= -\int_0^{+\infty} x \underbrace{\frac{d \exp\left(-\frac{x^2}{2\sigma^2}\right)}{dx} dx}$$

Do Integration By Parts here:

$$0 = x \exp\left(-\frac{x^2}{2\sigma^2}\right) \Big|_0^{+\infty} = \int_0^{+\infty} x \frac{d \exp\left(-\frac{x^2}{2\sigma^2}\right)}{dx} dx$$

$$+ \underbrace{\int_0^{+\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx}$$

$$\Rightarrow \mathbb{E}[X] = -\int_0^{+\infty} x \frac{d \exp\left(-\frac{x^2}{2\sigma^2}\right)}{dx} dx = \int_0^{+\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$$

$$= \frac{1}{2} \sigma\sqrt{2\pi} \int_{-\infty}^{+\infty} \underline{\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)} dx = \frac{1}{2} \sigma\sqrt{2\pi} = \sigma\sqrt{\frac{\pi}{2}}$$

The integrand is the density function of $N(0, \sigma)$

thus the integral $= 1$

Var [x]

$$\mathbb{E}[x^2] = -\int_0^{+\infty} x^2 \frac{d \exp(-\frac{x^2}{2\sigma^2})}{dx} dx$$

Do integration By Parts

$$0 = x^2 \exp(-\frac{x^2}{2\sigma^2}) \Big|_0^{+\infty} = \int_0^{\infty} x^2 \frac{d \exp(-\frac{x^2}{2\sigma^2})}{dx} dx + \int_0^{+\infty} 2x \exp(-\frac{x^2}{2\sigma^2}) dx$$

$$\Rightarrow \mathbb{E}[x^2] = -\int_0^{+\infty} x^2 \frac{d \exp(-\frac{x^2}{2\sigma^2})}{dx} dx$$

$$= \int_0^{+\infty} 2x \exp(-\frac{x^2}{2\sigma^2}) dx$$

$$= -2\sigma^2 \int_0^{+\infty} -\frac{x}{\sigma^2} \exp(-\frac{x^2}{2\sigma^2}) dx$$

$$= -2\sigma^2 \exp(-\frac{x^2}{2\sigma^2}) \Big|_0^{+\infty} = 2\sigma^2$$

$$\Rightarrow \mathrm{Var}[x] = \mathbb{E}[x^2] - \mathbb{E}^2[x]$$

$$= 2\sigma^2 - \frac{\pi}{2}\sigma^2$$

$$= (2 - \frac{\pi}{2})\sigma^2$$

2.

2. **Sums and quotients of random variables** [4 Points]

In this problem we recall the important notion of ***convolution*** *for two probability distributions.*

- *If X and Y are independent discrete real random variables with values in $\Omega_X, \Omega_Y \subseteq \mathbb{R}$ respectively, then $Z = X + Y$ has probability mass function*

$$p_Z(k) = \sum_{\ell \in \Omega_Y} p_X(k-\ell) p_Y(\ell),$$

for $k \in \Omega_X + \Omega_Y$, where p_X and p_Y are the probability mass functions of X and Y, and we set $p_X(r) = 0$ if $r \notin \Omega_X$.

- *If X and Y are independent continuous real random variables then $Z = X + Y$ has probability density function*

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x-y) f_Y(y) \mathrm{d}y.$$

where f_X and f_Y are the probability density functions of X and Y respectively.

(a) Let $\lambda, \mu > 0$. Show that if $X \sim Pois(\lambda)$ and $Y \sim Pois(\mu)$ are independent, then $X + Y \sim Pois(\lambda + \mu)$.

(a)

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \qquad \mathbb{P}(Y=k) = \frac{\mu^k e^{-\mu}}{k!}$$

$$\mathbb{P}(X+Y=k) = \sum_{i=0}^{k} \mathbb{P}(X=i, Y=k-i)$$

$$= \sum_{i=0}^{k} \mathbb{P}(X=i)\, \mathbb{P}(Y=k-i)$$

$$= \sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!} \cdot \frac{\mu^{k-i} e^{-\mu}}{(k-i)!}$$

$$= e^{-(\lambda+\mu)} \sum_{i=0}^{k} \frac{\lambda^i \mu^{k-i}}{i!(k-i)!} = e^{-(\lambda+\mu)} \sum_{i=0}^{k} \frac{\binom{k}{i} \lambda^i \mu^{k-i}}{k!}$$

$$= \frac{e^{-(\lambda+\mu)}}{k!} \sum_{i=0}^{k} \binom{k}{i} \lambda^i \mu^{k-i} = \frac{e^{-(\lambda+\mu)} (\lambda+\mu)^k}{k!}$$

which shows $X+Y \sim Pois(\mu+\lambda)$

(b)

(b) Let $X_1, ..., X_n$ be i.i.d. real random variables with $X_1 \sim \Gamma(\alpha, \beta)$ and $\alpha, \beta > 0$. Show that $\sum_{i=1}^{n} X_i \sim \Gamma(n\alpha, \beta)$.
Hint: You can use that

$$\int_0^1 u^{a-1}(1-u)^{b-1}\mathrm{d}u = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \qquad a, b > 0.$$

proof.

Proof by induction:

(Base Case)

$n=1$

$X_1 \sim \Gamma(1 \cdot \alpha, \beta)$ is trivial

(Inductive Step)

Assume $n-1$

prove for n.

Assume $\sum_{i=1}^{n-1} X_i \sim \Gamma[(n-1)\alpha, \beta]$

As $\sum_{i=1}^{n} X_i = \sum_{i=1}^{n-1} X_i + X_n$,

If $x>0$ $\quad f_{\sum_{i=1}^{n} X_i}(x) = \int_0^x \frac{(x-y)^{(n-1)\alpha-1} e^{-\beta(x-y)}}{\Gamma((n-1)\alpha)} \beta^{(n-1)\alpha} \frac{y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)} \beta^{\alpha} \, dy$

$$= \frac{e^{-\beta x} \beta^{n\alpha}}{\Gamma((n-1)\alpha)\Gamma(\alpha)} \int_0^x (x-y)^{(n-1)\alpha-1} \cdot y^{\alpha-1} \, dy$$

$$= \frac{e^{-\beta x} \beta^{n\alpha}}{\Gamma((n-1)\alpha)\Gamma(\alpha)} \underbrace{\int_0^x (x-y)^{(n-1)\alpha-1} y^{\alpha-1} \, dy}_{A}$$

~~Let~~ let $y = xu$ $\quad u = \frac{y}{x}$

$$A = \int_0^x (x - xu)^{(n-1)\alpha - 1} (xu)^{\alpha-1} \, d\, xu$$

$$= \int_0^1 (x - xu)^{(n-1)\alpha-1} (xu)^{\alpha-1} \cdot x \, du$$

$x = xu$
$\Rightarrow u = 1$ $\quad = x^{n\alpha-1} \int_0^1 (1-u)^{(n-1)\alpha-1} u^{\alpha-1} \, du$

Therefore.

$$f_{\sum_{i=1}^n X_i}(x) = \frac{e^{-\beta x} \beta^{n\alpha} x^{n\alpha-1}}{\Gamma((n-1)\alpha)\,\Gamma(\alpha)} \cdot \int_0^1 (1-u)^{(n-1)\alpha-1} u^{\alpha-1} \, du$$

$$= \frac{e^{-\beta x} \beta^{n\alpha} x^{n\alpha-1}}{\Gamma((n-1)\alpha + \alpha)} = \frac{e^{-\beta x} \beta^{n\alpha} x^{n\alpha-1}}{\Gamma(n\alpha)}$$

$$= f(x, n\alpha, \beta)$$

$$\Rightarrow \sum_{i=1}^n X_i \sim \Gamma(n\alpha, \beta)$$

By Inductive step and Base step

We have shown that

$$\sum_{i=1}^n X_i \sim \Gamma(n\alpha, \beta) \quad \forall n \in \mathbb{N}.$$

(c)

(c) For the quotient of two independent, continuous, positive real random variables X and Y one can also show that $Z = \frac{X}{Y}$ has density

$$f_Z(z) = \int_0^\infty y f_X(zy) f_Y(y) \mathrm{d}y, \qquad z > 0.$$

Using this, determine the law of the quotient of two independent, $\mathcal{U}([0,1])$-distributed random variables.

$X \sim U([0,1])$ X and Y are independent.

$Y \sim U([0,1])$

when X Y positive.

$$f_{\frac{X}{Y}}(z) = \int_0^{+\infty} y f_X(zy) f_Y(y) dy$$

$$= \int_0^{+\infty} y f_X(zy) f_Y(y) dy$$

Case 1: If $0 \leq z \leq 1$

Then $f_{\frac{X}{Y}}(z) = \int_0^1 y \cdot 1 \cdot 1 \, dy$

$= \frac{1}{2} y^2 \Big|_0^1 = \frac{1}{2}$

Case 2: If $z > 1$

Then $f_{\frac{X}{Y}}(z) = \int_0^{\frac{1}{z}} y \cdot 1 \cdot 1 \, dy$

$= \frac{1}{2z^2}$

$$\Rightarrow f_{\frac{X}{Y}}(z) = \begin{cases} 0 & z < 0 \\ \frac{1}{2} & 0 \leq z \leq 1 \\ \frac{1}{2z^2} & z \geq 1 \end{cases}$$

3. **Variances and covariances** [4 Points]

(a) Let X and Y be jointly continuous with joint density

$$f_{X,Y}(x,y) = (x+y)\mathbb{1}_{\{(x,y)\in[0,1]^2\}}.$$

Calculate the covariance matrix Σ and the correlation $\rho(X,Y)$. Then use Σ to calculate $\mathrm{Var}[X-2Y]$.

Proof.

$$f_X(x) = \int_{-\infty}^{+\infty} (x+y)\,\mathbb{1}_{\{(x,y)\in[0,1]^2\}}\,dy$$

$$= \int_0^1 (x+y)\,dy = x + \frac{1}{2} \qquad x \in [0,1]$$

other wise $f_X(x) = 0$

Similarly

$$f_Y(y) = y + \frac{1}{2}$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\cdot\mathbb{E}[Y]$$

$$= \underbrace{\int_{[0,1]^2} xy\,(x+y)\,dx\,dy}_{A} - \underbrace{\left[\int_{[0,1]} x\,(x+\tfrac{1}{2})\,dx\right]^2}_{B^2}$$

$$A = \int_0^1\int_0^1 x^2y + xy^2\,dx\,dy \qquad B = \int_0^1 x^2 + \frac{1}{2}x\,dx$$

$$= \int_0^1 \left(\frac{1}{3}x^3y + \frac{1}{2}x^2y^2\right)\Big|_{x=0}^{x=1} dy \qquad = \frac{1}{3}x^3 + \frac{1}{4}x^2\Big|_0^1 = \frac{7}{12}$$

$$= \int_0^1 \frac{1}{3}y + \frac{1}{2}y^2\,dy$$

$$= \frac{1}{6}y^2 + \frac{1}{6}y^3\Big|_0^1 = \frac{1}{3}$$

$$\Rightarrow \mathrm{Cov}(X,Y) = A - B^2 = -\frac{1}{144}$$

$$\mathrm{var}[X] = \mathbb{E}[X^2] - \mathbb{E}^2[X]$$

$$= \int_0^1 x^2 (x + \tfrac{1}{2})\, dx - \frac{49}{144}$$

$$= \frac{x^4}{4} + \frac{1}{6}x^3 \Big|_0^1 - \frac{49}{144}$$

$$= \frac{1}{4} + \frac{1}{6} - \frac{49}{144} = \frac{36+24-49}{144} = \frac{11}{144}$$

So does $\mathrm{var}[Y]$

Therefore. $\Sigma = \begin{bmatrix} \frac{11}{144} & -\frac{1}{144} \\ -\frac{1}{144} & \frac{11}{144} \end{bmatrix}$

$$X - 2Y = \underbrace{[1 \ \ -2]}_{:= A} \cdot \begin{bmatrix} X \\ Y \end{bmatrix}$$

Therefore. $\mathrm{var}[X-2Y] = A \cdot \Sigma \ A^T$

$$= [1 \ \ -2] \begin{bmatrix} \frac{11}{144} & -\frac{1}{144} \\ -\frac{1}{144} & \frac{11}{144} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= [1 \ \ -2] \begin{bmatrix} \frac{13}{144} \\ \frac{-23}{144} \end{bmatrix} = \frac{13+46}{144} = \frac{59}{144}$$

(b) Let $X_1, ..., X_n$ be i.i.d. real random variables with $\mu = \mathbf{E}[X_1]$ and $\sigma^2 = \mathrm{Var}[X_1]$. We define

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ (sample mean)}, \qquad S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 \text{ (sample variance)}.$$

Show that $\mathbf{E}[\overline{X}_n] = \mu$, $\mathrm{Var}[\overline{X}_n] = \frac{\sigma^2}{n}$ and $\mathbf{E}[S_n^2] = \sigma^2$.

$$\mathbb{E}[\bar{X}_n] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E} X_i = n \cdot \frac{1}{n} \cdot \mu = \mu$$

$$\mathrm{Var}[\bar{X}_n] = \mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}\, X_i = \frac{1}{n^2}\, n \cdot \sigma^2 = \frac{\sigma^2}{n}$$

$$\mathbb{E}[S_n^2] = \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right]$$

$$= \frac{1}{n-1}\sum_{i=1}^{n} \mathbb{E}\left(X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2\right)]$$

$$= \frac{1}{n-1}\sum_{i=1}^{n} \mathbb{E}\left(X_i^2 - 2X_i\frac{\sum_{i=1}^{n} X_i}{n} + \bar{X}_n^2\right)$$

$$= \frac{1}{n-1}\sum_{i=1}^{n} \mathbb{E}\left(X_i^2 - \frac{2}{n}X_i^2\right) - \frac{2(n-1)}{n}\mu^2 + \mathbb{E}[\bar{X}_n]^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n} \frac{n-2}{n}\mathbb{E}[X_i^2] - \frac{2(n-1)}{n}\mu^2 + \mathbb{E}[\bar{X}_n]^2$$

Here $\mathbb{E}[X^2] = \mathbb{E}^2[X] + \mathrm{Var}[X]$

$$= \frac{1}{n-1}\sum_{i=1}^{n} \frac{n-2}{n}(\mu^2 + \sigma^2) - \frac{2(n-1)}{n}\mu^2 + \mu^2 + \frac{\sigma^2}{n}$$

$$= \frac{1}{n-1}\cdot n \cdot \left(\frac{n-2}{n}(\mu^2 + \sigma^2) - \frac{2(n-1)}{n}\mu^2 + \mu^2 + \frac{\sigma^2}{n}\right)$$

$$= \frac{1}{n-1}\cdot n \cdot \left(\cancel{\frac{2n-2}{n}\mu^2} - \cancel{\frac{2(n-1)}{n}\mu^2} + \left(\frac{n-2}{n} + \frac{1}{n}\right)\sigma^2\right)$$

$$= \frac{n}{n-1}\cdot\frac{n-1}{n}\sigma^2 = \sigma^2$$

where we show $\mathbb{E}[S_n^2] = \sigma^2$

4. **Moment generating functions** [4 Points]

In this problem we recall the ***moment generating functions*** *for random variables. For a real random variable X, the moment generating function is defined by*

$$\psi_X(t) = \mathbf{E}\left[e^{tX}\right], \qquad t \in \mathbb{R},$$

whenever this expression exists. One has:

- *Whenever $\psi_X(t) = \psi_Y(t)$ for $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, for two random variables X and Y, then $X \stackrel{d}{=} Y$.*
- $\psi'_X(0) = \mathbf{E}[X]$.

(a) Calculate the moment generating function of $X \sim \mathcal{N}(0,1)$.

(a)

$$\psi_X(t) = \mathbb{E}\left[e^{tX}\right]$$

$$= \int_{-\infty}^{+\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2-2tx+t^2)} \cdot e^{\frac{1}{2}t^2} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{1}{2}t^2} \cdot \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-t)^2} \, dx$$

$$= e^{\frac{1}{2}t^2} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} \, dx$$

$$= e^{\frac{1}{2}t^2} \int_{-\infty}^{+\infty} \underline{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}} \, dx$$

The integrand here is the density function for $\mathcal{N}(0,1)$ which implies the integral is 1

$$\Rightarrow \underline{\psi_X(t) = e^{\frac{1}{2}t^2}}$$

(b) Suppose that X and Y are independent. Explain why $\psi_{X+Y}(t) = \psi_X(t)\psi_Y(t)$ (assuming all these expressions exist for a given t). Use this to show that if $X \sim \mathcal{N}(0, \sigma_1^2)$ and $Y \sim \mathcal{N}(0, \sigma_2^2)$, then $X + Y \sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$, where $\sigma_1, \sigma_2 > 0$.

Proof.

$$\psi_{X+Y}(t) = \mathbb{E}[e^{(X+Y)t}]$$
$$= \mathbb{E}[e^{Xt} \cdot e^{Yt}]$$

As X and Y are independent. So do e^{Xt} and e^{Yt}

Therefore, $\psi_{X+Y}(t) = \mathbb{E}[e^{Xt} \cdot e^{Yt}] = \mathbb{E}[e^{Xt}] \cdot \mathbb{E}[e^{Yt}]$
$= \psi_X(t) \cdot \psi_Y(t)$

For $X \sim N(0, \sigma_1^2)$ $Y \sim N(0, \sigma_2^2)$.

If $Z \sim N(0, \sigma^2)$, a similar Calculation will show $\psi_Z = e^{\frac{1}{2}\sigma^2 t^2}$

Therefore, $\psi_{X+Y} = \psi_X \cdot \psi_Y = e^{\frac{1}{2}\sigma_1^2 t} e^{\frac{1}{2}\sigma_2^2 t}$
$= e^{\frac{1}{2}(\sigma_1^2 + \sigma_2^2) t}$

Here we can see the moment generating function of $X+Y$ coincides with $N(0, \sigma_1^2 + \sigma_2^2)$

which implies they have the same distribution.

$\Rightarrow$ $X+Y \sim N(0, \sigma_1^2 + \sigma_2^2)$

(c) The moment generating function of $X \sim Geo(p)$ with $p \in (0,1)$ is given by

$$\psi_X(t) = \frac{pe^t}{1-(1-p)e^t}, \qquad t < -\log(1-p).$$

Use this to verify that $\mathbf{E}[X] = \frac{1}{p}$.

$$\mathbb{E}[X] = \psi_X'(0)$$

$$= \left. \frac{pe^t(1-(1-p)e^t) - pe^t(-(1-p)e^t)}{(1-(1-p)e^t)^2} \right|_{t=0}$$

$$= \frac{p^2 + p(1-p)}{p^2} = \frac{1}{p}$$