Problem Set 5

Submission:
Thursday, 03/10/2022, until 1 PM, to be uploaded on the NYU Brightspace course homepage.

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1. **Maximum-Likelihood estimator I** [4 Points]

   Let $X_1, ..., X_n$ be i.i.d. random variables and θ unknown. Calculate the Maximum-Likelihood estimator (MLE) $\widehat{\theta}_n$ for θ, if under $\mathbf{P}_\theta$,

   (a) $X_1 \sim \mathcal{N}(\mu, \sigma^2)$, where $\theta = (\mu, \sigma^2)$,

   (b) $X_1 \sim Pois(\theta)$.

   (c) $X_1 \sim Ra(\theta)$, which means that the law of X_1 has the probability density function

   $$f(x) = \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) \mathbb{1}_{[0,\infty)}(x),$$

   with $\theta > 0$.

2. **Maximum-Likelihood estimator II** [4 Points]

   Let $X_1, ..., X_n$ be i.i.d. random variables such that $X_1 \sim \mathcal{U}([0, \theta])$, with unknown $\theta > 0$.

   (a) Calculate the Maximum-likelihood estimator (MLE) $\widehat{\theta}_n$ for θ.
   Hint: Do *not* try to differentiate the (log-)likelihood function with respect to θ. Instead argue directly, for which value of θ the likelihood function is maximized.

   (b) Check whether the estimator $\widehat{\theta}_n$ is consistent and unbiased.

3. **Maximum-Likelihood estimator III** [4 Points]

   In a pharmacological experiment, the effectiveness of an antibiotic is investigated: In n trials, dose t_1 is administered to a probe of bacterium A, then in n further trials, dose t_2 is administered. In each round it is counted, in how many of the probes the antibiotic was effective in reducing the growth of the bacteria. The probability that the growth of the bacteria is slowed by the antibiotic in a given probe is unknown, but can be modelled by $p_t = 1 - e^{-\beta t}$, where $\beta \in (0, \infty)$ is unknown.

   (a) Suppose that for $j \in \{1, ..., 2n\}$,

   $$X_j = \begin{cases} 1, & \text{the antibiotic reduced the growth of the bacteria in the } j\text{th trial,} \\ 0, & \text{the antibiotic did not reduce the growth of the bacteria in the } j\text{th trial.} \end{cases}$$

   Write down the log-likelihood function $L_{2n}(\beta)$ for the $2n$ probes. Find an equation that must be solved by the Maximum-likelihood estimator (MLE) $\widehat{\beta}_{2n}$ for β. Which assumption(s) do you have to make in this calculation?

   (b) Suppose that there are $n = 10$ trials with dose $t_1 = 1$, and $n = 10$ further trials with dose $t_2 = 2$. A reduction of bacteria growth is observed in 3 trials of the first group, and 5 trials of the second group. Find $\widehat{\beta}_{20}$ based on the given information.

   (c) Based on the model $p_t = 1 - e^{-\beta t}$, what would be an estimate of the probability be that the antibiotic reduces the growth of the bacteria when dose $t_3 = 4$ is administered?

4. **Caveats for unbiased estimators** [4 Points]

In this problem, we illustrate what can "go wrong" by focussing too much on unbiasedness of estimators, by giving two perhaps surprising examples.

(a) Let $X_1, ..., X_n$ be i.i.d. random variables with finite variance $\sigma^2 = \text{Var}[X_1] \in (0, \infty)$. Recall that

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \overline{X}_n)^2, \qquad \text{where } \overline{X}_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

is an unbiased estimator for $\text{Var}[X_1]$. Consider another estimator for σ^2 defined for $c \in (0, \infty)$ by

$$T_{n,c}^2 = c \sum_{i=1}^n (X_i - \overline{X}_n)^2$$

Show that $\text{MSE}[T_{n,c}^2]$ is minimal for $c = \frac{1}{n+1}$. Interpret this result.

(b) Now let $X_1, ..., X_n \overset{\text{i.i.d.}}{\sim} Ber(p)$ with unknown $p \in (0, 1)$. We will explain that

For $\gamma(p) = \frac{1}{p}$, no unbiased estimator exists.

We argue by contradiction: Suppose that there exists an unbiased estimator based on $X_1, ..., X_n$. One can show[1] that there also exists an unbiased estimator $\widehat{\gamma}_n$ that must factorize over the sufficient statistic $T(X_1, ..., X_n) = \sum_{i=1}^n X_i$, in other words, one can write

$$(\star) \qquad \widehat{\gamma}_n = f\left(\sum_{i=1}^n X_i\right)$$

for some function $f : \{0, ..., n\} \to \mathbb{R}$.

(i) Suppose $\mathbf{E}_p[\widehat{\gamma}_n] = \gamma(p) = \frac{1}{p}$. Use $(\star)$ to argue that we must have

$$(\star\star) \qquad \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} f(k) = \frac{1}{p}.$$

(ii) Explain that $(\star\star)$ implies that $f(k) = 0$ for every $k \in \{0, ..., n\}$. What does this mean for the estimator $\widehat{\gamma}_n$?
Hint: Multiply $(\star\star)$ by p. Use the fact that a polynomial of degree $N \in \mathbb{N}$ can only have N distinct zeroes, unless it is constant 0.

[1]This follows from the Rao-Blackwell theorem, which we will see later in this course.