MAST20004 Assignment 2, S1 2024: Due 4 pm, Friday 12 April

Question 1

Recall Old McDonald's farm from Assignment 1, where we choose an animal uniformly at random from that farm, and $\Omega = \{\text{pig, cow, chicken, dog, sheep}\}$. Let X be the number of legs of the animal that is selected. Let Y be the number of letters in the name of the selected animal. Suppose that all of the animals have their typical number of legs.

(a) What are the values of $X(\text{cow})$ and $Y(\text{cow})$?

(b) Find $\mathbb{P}(X = Y)$ (i.e. find the probability of the set of outcomes ω such that $X(\omega) = Y(\omega)$).

(c) Find $\mathbb{P}(X > Y)$.

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(d) Determine the probability mass function and distribution function of X.

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(e) Define a random variable Z by $Z(\omega) = X(\omega) + Y(\omega)$. Find $\mathbb{E}[Z]$ by first finding the probability mass function of Z, and verify that $\mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y]$.

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Question 2

The grand old Duke of York had 10000 men, and rather unusual training methods.
Each day he gets each man who is still in his service to roll a fair 6-sided die. Those who roll a 6 march to the top of a hill (which takes about half a day) where there is a village. Each of those who marched up that day rolls the die again and if they get another 6 then they have to march back down again (while the others remain in the village and leave the Duke's service).

(a) Let Y be the number of men who march both up and down the hill on day 1. What is the exact distribution of Y? Give an expression for $\mathbb{P}(Y \leq 300)$, and evaluate the probability in R.

(b) Let Y' be a Poisson random variable with the same expected value as Y. In R, evaluate $\mathbb{P}(Y' \leq 300)$.

(c) Let Y'' be a normal random variable with the same expected value and variance as Y. In R, evaluate $\mathbb{P}(Y'' \leq 300)$.

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(d) Estimate $\mathbb{P}(Y \leq 300)$ by simulating 1000 realisations of Y. Show your code. Make sure that your first line of code is `set.seed(ID)`, where ID is your student ID number.

(e) You are interested in the distribution of the number N of days until the Duke has no men left in his service. One of your MAST20004 lecturers gives you the following R code, claiming that it simulates one realisation of N:

```
in.service=function(people){
   sum( sample( 0:1, people, replace = TRUE, prob = c(5/36, 31/36) ) )}
men=10000
N=0
while(men > 0){ men = in.service(men)
               N=N+1 }
N
```

The other MAST20004 lecturer gives you the following R code, claiming that it simulates one realisation of N:

```
in.service=function(people){ rbinom(1,people, 31/36) }
men=10000
N=0
while(men > 0){ men = in.service(men)
               N=N+1 }
N
```

Which code is correct? Simulate (at least) 1000 realisations of N and store them in a vector called `N.vec` (start your code with set.seed(ID) as above). Estimate the expected value of N and plot the estimated probability mass function of N (use e.g. `plot(table(N.vec)/length(N.vec))` for the latter).

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Question 3

Two evenly matched basketball teams (call them A and B) compete in a best of 7 championship (first team to win 4 games wins the championship). Once the champion has been determined, no more games are played. In each game there is a home team, and an away team. The home team wins the game with probability $p \geq 1/2$, independent of all previous games. Suppose that the first three games will be held at the home of team A and the last 4 (or fewer if they are not needed) are played at the home of team B.

(a) Let X be the number of games won by team A out of the first 3 games. Specify the distribution of X.

(b) Find the probability that only 4 games are played.

(c) Which of the two teams is more likely to win the trophy? Explain why.

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(d) Give an expression for the probability that team A wins the trophy, and evaluate it when $p = 0.55$.

(e) Let Y be the number of games won by team A. Find the probability mass function for Y.

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(f) Evaluate the expected number of games won by team A, and the expected number of games played when $p = 1/2$.

(g) Observe (via computations or simulation) that when $p = 0.55$ the expected number of games won by team A is larger than that of team B, even though team B is more likely to win the trophy.

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Question 4

A tortoise and a hare are going to have a 10 metre race. Each minute the tortoise can move 1 metre. The hare moves 3 metres per minute, so in order to make the race somewhat fair, the hare always has to roll a fair 6-sided die each minute before moving. Whenever the hare rolls a 1,2,3, or 4, the hare moves 3 metres forwards. Otherwise (i.e. when rolling a 5 or 6), the hare must move 3 metres backwards.

(a) What is the expected displacement (from the common starting position) of each of the tortoise and hare after 1 minute?

(b) What is the expected displacement (from the common starting position) of each of the tortoise and hare after 10 minutes?

(c) Find the probability that the hare reaches the finish line (at a displacement of +10m from the start) before the tortoise (you should assume that within any given minute, the two animals each move at constant speed).

(d) Just before the race begins, an owl offers to compete. The owl can fly 10 metres in 1 minute, but suggests the following: before making each move, the owl will roll a fair 20 sided die. Whenever the owl rolls 1 through 9 then the owl moves 10 metres forward, otherwise the owl moves 10 metres backwards. Find the expected displacement of the owl after 10 minutes, and show that the owl probably wins the race.