PDE Models Reductions

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Outline

1. Introduction

2. Linear cases: the standard POD-based method

μ-parameterized time dependent PDE

Given $\mu \in \mathcal{P}$, find $u(\mu; x, t) \in V$ such that:

$$\frac{\partial u}{\partial t}(\mu; x, t) = F(\mu; u(\mu; x, t)) \qquad in \qquad \Omega \times [0, T], \tag{1}$$

- $u(\mu; x, t)$ is the state variable, the map $u \mapsto F(\cdot; u)$ is a PDE operator, linear or not,
- the map $\mu \mapsto F(\mu; \cdot)$ is non-affine.

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- Classical numerical solvers (FV, FE) $\rightarrow$ High Fidelity (HF) solvers.
- CPU-time and memory consuming.

Goal: to develop a Reduced Order Model (ROM) of the HF model.

Introduction

Real world application: 2018 flood of the Aude river

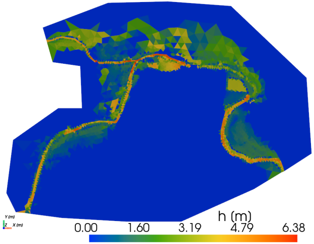


Fig. 1: 2D Shallow Water (SW) system with parameterization: $\mu \in \mathbb{R}^5$ inflow discharge parameters. FV scheme, 7267 cells.

	HF	ROM
CPU time	118 s	5 s
Memory cost	1287 Mo	123 Mo

Fig. 2: Simulations performed in 11th Gen Intel(R) Core(TM) i9-11950H, 2.60GHz with 32Gb of RAM capacity.

Linear steady case

The steady-state parametrized PDE model

Let V a Hilbert space equipped with a scalar product $(u, v)_V$, $\forall\, u, v \in V$, and the induced norm $||u||_V = \sqrt{(u, u)_V}$, $\forall u \in V$ and let Ω be a bounded open set of $\mathbb{R}^{N_h}$. Let us introduce $\mathcal{P} \subset L^\infty(\Omega)$ the parameter space such that $\mu \in \mathcal{P}$ with $\dim(\mathcal{P}) = N_\mu$. Let $P_s = \{\mu_s\}_{s=1}^{N_s}$, be a parameters set of dimension N_s obtained by sampling in some way the parameter space $\mathcal{P}$.

Steady μ-parametrized PDE

Given $\mu \in \mathcal{P}$, find $u(\mu) \in V$ such that:

$$a(\mu; u, v) = l(\mu; v), \qquad \forall v \in V. \tag{2}$$

Linear steady case, Galerkin FE

The corresponding High-Fidelity (HF) Finite Element (FE) model

V_h: the approximation space, using conforming FE method, i.e. $V_h \subset V$.

Steady μ-parametrized discrete PDE

Given $\mu \in \mathcal{P}$, find $u_h(\mu) \in V_h$ satisfying:

$$a(\mu; u_h(\mu), v_h) = l(\mu; v_h) \quad \forall v_h \in V_h. \tag{3}$$

Linear steady case, Galerkin FE

The corresponding High-Fidelity (HF) Finite Element model

The corresponding FE linear system

$\Phi(x) = \{\varphi_i(x)\}_{i=1}^{N_h}$ denotes the FE functions basis vector, $V_h = \text{span}\Phi(x)$. One has:

$$u_h(\mu; x) = \Phi^T(x)\mathbf{u}_h^\mu = \sum_{i=1}^{N_h} (u_h^\mu)_i \varphi_i(x), \tag{4}$$

the vector of dof $\mathbf{u}_h^\mu = ((u_h^\mu)_1, \cdots, (u_h^\mu)_{N_h})^T$ satisfies the linear system:

$$\mathbf{A}_h^\mu \mathbf{u}_h^\mu = \mathbf{F}_h^\mu, \tag{5}$$

with $\mathbf{u}_h^\mu \in \mathbb{R}^{N_h}$ and $\mathbf{A}_h^\mu$ is the stiffness matrix such that: $(\mathbf{A}_h^\mu)_{ij} = (a_{ij})_{i,j=1,\cdots,N_h}$ with $a_{ij} = a(\mu; \varphi_j, \varphi_i)$. $\mathbf{F}_h^\mu$ is the RHS: $(\mathbf{F}_h^\mu)_i = l(\mu; \varphi_i)_{i=1,\cdots,N_h}$.

Linear steady case: Reduced Order Model (ROM)

- The reduced space is defined by:

$$V_{\text{rb}} = \text{span}\,\Xi(x), \quad \Xi(x) = (\xi_1(x), \dots, \xi_{N_{\text{rb}}}(x)) \tag{3.1.4}$$

- It is a subspace of the full-order space:

$$V_{\text{rb}} \subset V_h, \quad \text{with } V_h = \text{span}\,\Phi(x) \tag{3.1.5}$$

- Each reduced basis function belongs to the FE space:

$$\xi_n(x) \in V_h, \quad \forall\, n = 1, \dots, N_{\text{rb}} \tag{3.1.6}$$

- The reduced space has much lower dimension:

$$\dim(V_{\text{rb}}) = N_{\text{rb}} \ll N = \dim(V_h) \tag{3.1.7}$$

Galerkin reduced basis μ-parametrized PDE

Given $\mu \in \mathcal{P}$, find $u_{rb}(\mu) \in V_{rb}$ satisfying:

$$a(\mu;\, u_{rb}(\mu),\, v_{rb}) = l(\mu;\, v_{rb}), \qquad \forall v_{rb} \in V_{rb}. \tag{6}$$

Linear steady case: Reduced Order Model (ROM)

The corresponding reduced basis FE system

$\Xi(x) = \{\xi_n(x)\}_{n=1}^{N_{rb}}$ denotes the reduced basis functions, $V_{rb} = \text{span}\Xi(x)$. Let $\mathbf{B}_{rb} = [\xi_1|\cdots|\xi_{N_{rb}}] \in \mathbb{R}^{N_h \times N_{rb}}$ be the change of basis between V_h and V_{rb}. The vector ξ_n denotes the coordinates vector of the function $\xi_n(x)$ in the FE basis $\Phi(x)$. Therfore: $\Xi(x) = \mathbf{B}_{rb}^T\Phi(x)$, one has:

$$u_{rb}(\mu; x) = \Xi^T(x)\mathbf{u}_{rb}^{\mu} = \Phi^T(x)\mathbf{B}_{rb}\mathbf{u}_{rb}^{\mu} = \sum_{n=1}^{N_{rb}} (u_{rb}^{\mu})_n \xi_n(x) \qquad 1 \leq i \leq N_{rb}. \tag{7}$$

The vector of dof $\mathbf{u}_{rb}^{\mu} = ((u_{rb}^{\mu})_1, \cdots, (u_{rb}^{\mu})_{N_{rb}}) \in \mathbb{R}^{N_{rb}}$ satisfies the **linear reduced system**:

$$\mathbf{B}_{rb}^T\mathbf{A}_h^{\mu}\mathbf{B}_{rb}\mathbf{u}_{rb}^{\mu} = \mathbf{B}_{rb}^T\mathbf{F}_h^{\mu} \tag{8}$$

Linear steady case: Reduced Order Model (ROM)

The corresponding reduced basis FE system

The reduced matrix $\mathbf{A}_{rb}^{\mu}$ and the reduced RHS $\mathbf{F}_{rb}^{\mu}$ are obtained from the linear discrete parametrized problem given by Eq. (6). Indeed, for $\xi_m, \xi_n \in V_h$, $1 \leq m, n \leq N_{rb}$ we have:

$$a(\mu; \xi_m, \xi_n) = \sum_{i=1}^{N_h} \sum_{j=1}^{N_h} (\xi_m(x))_j a(\mu; \varphi_j, \varphi_i)(\xi_n(x))_i \text{ and } l(\mu; \xi_n) = \sum_{i=1}^{N_h} l(\mu; \varphi_i)(\xi_n(x))_i.$$

Equivalently, in matrix form:

$$\mathbf{B}_{rb}^{T} \mathbf{A}_{h}^{\mu} \underbrace{\mathbf{B}_{rb} \mathbf{u}_{rb}^{\mu}}_{\tilde{\mathbf{u}}_{h}^{\mu}} = \mathbf{B}_{rb}^{T} \mathbf{F}_{h}^{\mu} \tag{9}$$

Goal: find $\tilde{u}_h(\mu) \approx u_h(\mu)$ such that: $\tilde{\mathbf{u}}_h^{\mu} = \mathbf{B}_{rb} \mathbf{u}_{rb}^{\mu}$. **Question**: how to construct $\mathbf{B}_{rb}$?

Solution manifolds

The continuous solution manifold

Let $\mathcal{M}$ denotes the space of all solutions u of the μ-parametrized problem given by Eq. (2):

$$\mathcal{M} = \{u(\mu), \quad u(\mu) \text{ solution of } (2) \text{ with } \mu \in \mathcal{P}\} \subset V. \tag{10}$$

The discrete solution manifold

Let us define the corresponding discrete space of all solutions as follows:

$$\mathcal{M}_h = \{u_h(\mu), \quad u_h(\mu) \text{ solution of } (3) \text{ with } \mu \in \mathcal{P}\} \subset V_h. \tag{11}$$

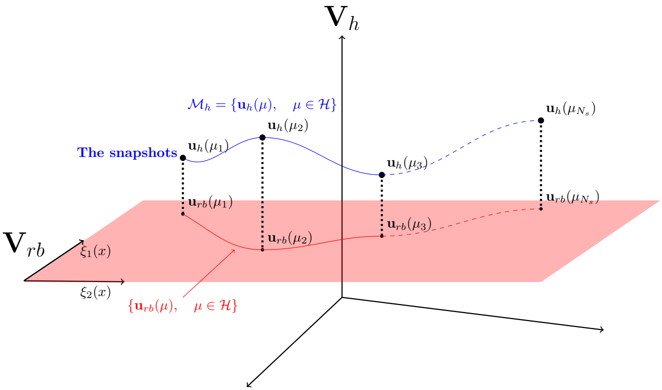
The discrete solution manifold

The spaces $\mathcal{M}$ and $\mathcal{M}_h$ are called the solution manifolds or . We set $\mathcal{M}_{h,s}$ such that:

$$\begin{aligned} \mathcal{M}_{h,s} &= \{u_h(\mu_s), \quad u_h(\mu_{N_s}) \in V_h \text{ solution of } (3) \text{ with } \mu_s \in P_s\}; \\ &= \{u_h(\mu_1), \cdots, u_h(\mu_{N_s})\}. \end{aligned} \tag{12}$$

Projection-based ROMs in Hilbert spaces. $V_{rb} \subset V_h \subset V$

ROM



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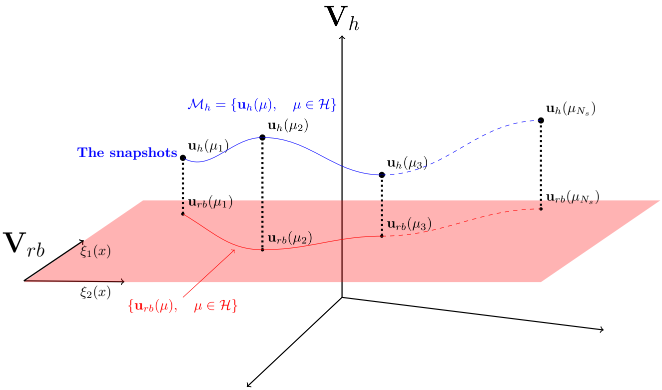

- $P_s = \{\mu_1, \cdots, \mu_{N_s}\} \in (\mathcal{P})^{N_s} \subset \mathbb{R}^{N_\mu}$

$\mathcal{P}$ is the parameter space.

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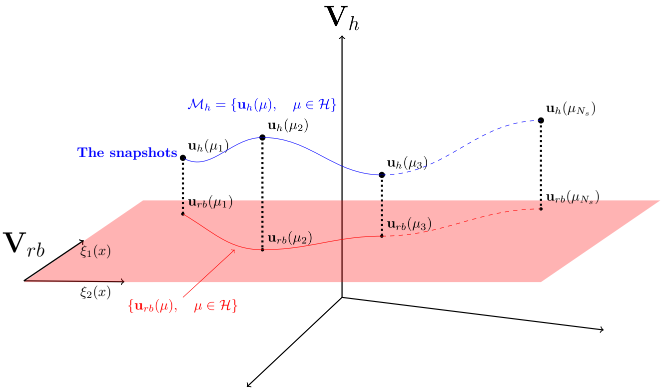

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$\mathcal{P}$ is the parameter space. $\mathcal{M}_h$ is the set of HF solutions (snapshots)

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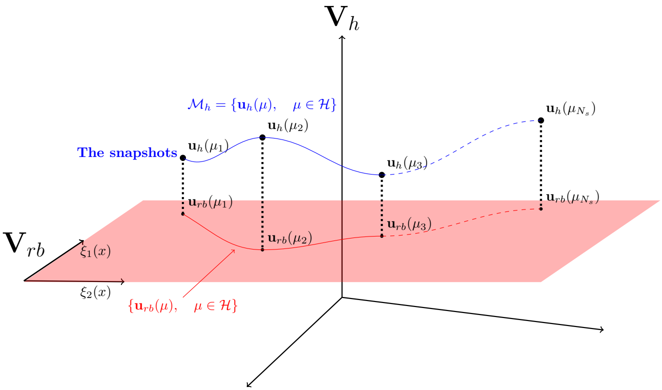

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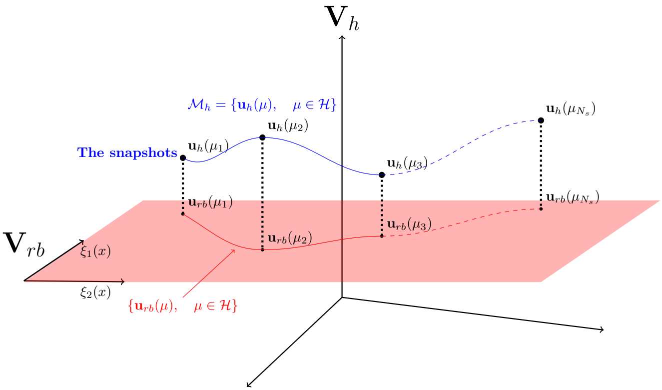

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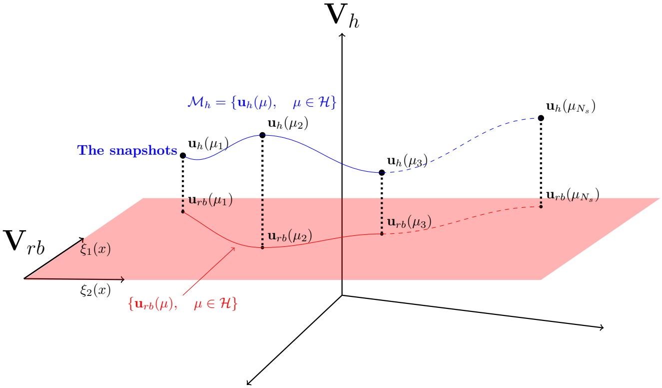

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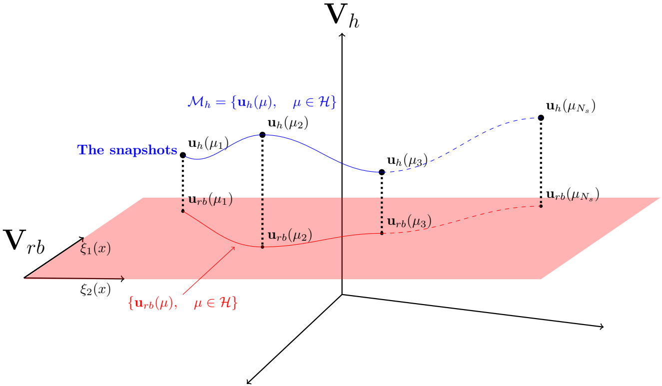

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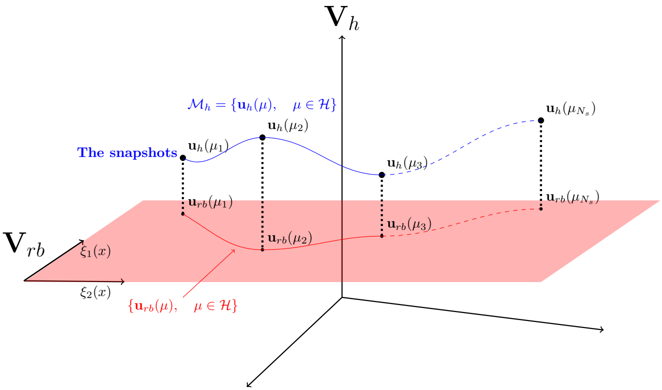

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We build $\mathbf{S} = [\mathbf{u}_h(\mu_1)| \cdots |\mathbf{u}_h(\mu_{N_s})] \in \mathbb{R}^{N_h \times N_s}$ the snapshots matrix.

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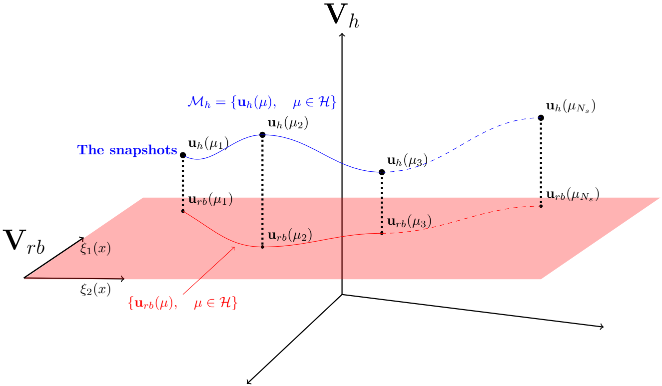

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We build $\mathbf{S} = [\mathbf{u}_h(\mu_1)| \cdots |\mathbf{u}_h(\mu_{N_s})] \in \mathbb{R}^{N_h \times N_s}$ the snapshots matrix. In the sequel, V_{rb} is constructed by the POD method.

Manifolds $\mathcal{M}$

Plot: a 3D manifold $\mathcal{M}_h = \{u_h(\mu);\ \mu \in \mathcal{P} = [0, 10] \subset \mathbb{R}\}$, for a steady linear advection-diffusion equation.
3 components of $\mathbf{u}_h(\mu)$ for various values of μ.

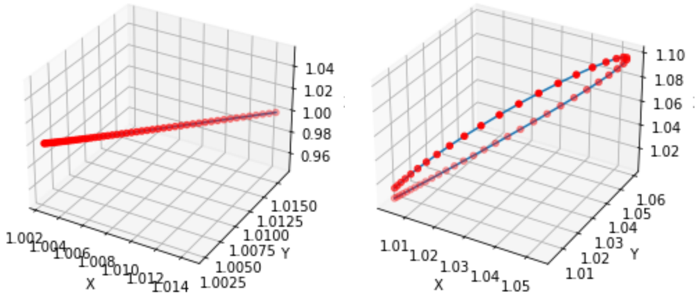

Fig. 3: (Left) Affine case: $(\lambda(\mu) = (\mu + \mu_0))$. (Right) Non-affine case: $(\lambda(\mu) = \exp(\mu_0(\mu + 1)))$.

Definition

For $\mathbf{A} \in \mathcal{M}_{N_h \times N_s}$ a (real) matrix, there exist two orthogonal matrices $\mathbf{U} = (\boldsymbol{\xi}_1 | \cdots | \boldsymbol{\xi}_{N_h}) \in \mathcal{M}_{N_h \times N_h}$ and $\mathbf{Z} = (\boldsymbol{\psi}_1 | \cdots | \boldsymbol{\psi}_M) \in \mathcal{M}_{N_s \times N_s}$ such that:

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{Z}^T \text{ with } \boldsymbol{\Sigma} = diag(\sigma_1, \cdots, \sigma_p) \in \mathcal{M}_{N_h \times N_s} \tag{13}$$

and $\sigma_1 \geq \cdots \geq \sigma_p \geq 0$, $p = \min(N_h, N_s)$.

The SVD and correlation matrix

- **Singular vector relations:**

$$\mathbf{S}\boldsymbol{\psi}_m = \sigma_m \boldsymbol{\xi}_m, \qquad \mathbf{S}^T \boldsymbol{\xi}_m = \sigma_m \boldsymbol{\psi}_m \quad \text{for } m = 1, \ldots, N_s$$

- **Equivalent eigenvalue problems:**

$$\mathbf{S}^T\mathbf{S}\boldsymbol{\psi}_m = \sigma_m^2 \boldsymbol{\psi}_m, \qquad \mathbf{S}\mathbf{S}^T \boldsymbol{\xi}_m = \sigma_m^2 \boldsymbol{\xi}_m$$

- **Correlation matrix:** Define $\mathbf{C} \in \mathbb{R}^{N_s \times N_s}$ by:

$$\mathbf{C}_{mn} = (u_{\mu,m}, u_{\mu,n})_\Box \quad \text{for } 1 \leq m, n \leq N_s$$

- **The case where $\Box = L^2$ scalar product:**

$$\mathbf{C} = \mathbf{S}^T\mathbf{S}$$

- **The case where $\Box = V$-scalar product:**

$$\mathbf{C} = \mathbf{S}^T\mathbf{V}_h\mathbf{S} \quad \text{with } \mathbf{V}_h \text{ symmetric positive definite}$$

- **Spectral properties:** $\mathbf{C}$ is symmetric positive definite $\Rightarrow$ eigenvalues $\lambda_m = \sigma_m^2 > 0$

$$\mathbf{C}\boldsymbol{\psi}_m = \lambda_m \boldsymbol{\psi}_m$$

The Proper Orthogonal Decomposition (POD) reduced space

Definition

The POD reduced space V_{POD} is defined as:

$$V_{\text{POD}} \equiv V_{\text{rb}} = \text{span}\left\{\boldsymbol{\xi}_1(x), \ldots, \boldsymbol{\xi}_{N_{\text{rb}}}(x)\right\} \tag{3.2.10}$$

where each $\boldsymbol{\xi}_n(x) \in V_h$ is the n-th left singular vector of $\mathbf{S}$, i.e., the reduced basis consists of the first N_{rb} left singular vectors $\{\boldsymbol{\xi}_m\}_{1 \le m \le N_{\text{rb}}}$ of $\mathbf{S}$.

The POD reduced basis

Definition

The POD reduced basis basis can be also defined from $\{\mathbf{w}_1|\dots|\mathbf{w}_{N_s}\}$ the eigenvectors of the correlation matrix $\mathbf{C}$ as follows:

$$\boldsymbol{\xi}_i = \frac{1}{\sigma_s}\mathbf{S}\mathbf{w}_i.$$

Therefore, in the matrix form, the reduced basis basis $\mathbf{B}_{rb}$ satisfies:

$$\mathbf{B}_{rb} = \mathbf{S}\mathbf{W}\boldsymbol{\Sigma}^{-1}, \tag{14}$$

where $\boldsymbol{\Sigma}^{-1} = diag(\sigma_1^{-1}, \cdots, \sigma_{N_s}^{-1})$ and $\mathbf{W} \in \mathcal{M}_{N_s \times N_s}$ is a matrix containing the eigenvectors of $\mathbf{C}$.

The orthogonal projector and error estimation

Definition

Given μ in $\mathcal{P}$, $\forall u_h(\mu) \in V_h$,

$$P_{POD}(u_h(\mu)) = \sum_{n=1}^{N_{rb}} (u_h(\mu), \xi_n(x))_{\square}\, \xi_n(x). \tag{15}$$

The orthogonal projector

The orthogonal projector matrix form

For each snapshot vector $\mathbf{u}_h(\mu) \in \mathbb{R}^{N_h}$, the POD projection operator denoted by $\mathbf{P}_{rb}$ of $\mathbf{u}_h(\mu)$ onto the span of $\mathbf{B}_{rb} = [\boldsymbol{\xi}_1|\cdots|\boldsymbol{\xi}_{N_{rb}}] \in \mathbb{R}^{N_h \times N_{rb}}$ or equivalently onto the reduced space V_{rb} in matrix form is given by:

- For L^2 **scalar product**:

$$\mathbf{P}_{rb}\mathbf{u}_h(\mu) = \mathbf{B}_{rb}\mathbf{B}_{rb}^T\mathbf{u}_h(\mu) = \mathbf{B}_{rb}\mathbf{u}_h^{N_{rb}}(\mu), \tag{16}$$

  where

$$\mathbf{u}_h^{N_{rb}}(\mu) = \mathbf{B}_{rb}^T\mathbf{u}_h(\mu) \in \mathbb{R}^{N_{rb}} \tag{17}$$

- For V **scalar product**:

$$\mathbf{P}_{rb}\mathbf{u}_h(\mu) = \mathbf{B}_{rb}\mathbf{V}_h\mathbf{B}_{rb}^T\mathbf{u}_h(\mu). \tag{18}$$

  where $\mathbf{V}_h \in \mathcal{M}_{N_h \times N_h}$ is symmetric and positive definite matrix.

The POD error estimation

Proposition 1

Among all semi-orthonormal bases of dimension N_{rb}, the POD basis is optimal in the least-squares sense. That is, it minimizes the total projection error:

$$\sum_{s=1}^{N_s} \|\mathbf{u}_{\mu,s} - \mathbf{P}_{POD}\mathbf{u}_{\mu,s}\|_{\square}^2 = \min_{\mathbf{B}\in\mathbf{B}_{N_{rb}}^{\perp}} \sum_{s=1}^{N_s} \|\mathbf{u}_{\mu,s} - \mathbf{P}_{B}\mathbf{u}_{\mu,s}\|_{\square}^2 = \sum_{s=N_{rb}+1}^{N_s} \lambda_s, \tag{19}$$

where $\mathbf{P}_B$ denotes the orthogonal projection onto the subspace $\mathbf{B}$, and λ_s are the eigenvalues associated with the POD decomposition.

POD reduced basis algorithm – offline Phase

- Compute N_s HF snapshots and their corresponding vectors $\mathbf{u}_{\mu,n}$:

$$u_{\mu,n} \equiv u_h(\mu_n), \quad 1 \leq n \leq N_s, \quad \mu_n \in \mathbb{R}^{N_\mu}$$

- Build the snapshot matrix:

$$\mathbf{S} = [\mathbf{u}_{\mu,1} \mid \cdots \mid \mathbf{u}_{\mu,N_s}] \in \mathbb{R}^{N_h \times N_s}$$

- Form the correlation matrix:

$$\mathbf{C} = \mathbf{S}^T \mathbf{V}_h \mathbf{S} \in \mathbb{R}^{N_s \times N_s}$$

  where $\mathbb{V}_h$ is the mass matrix (for L^2 or V-product). $\mathbf{C}$ is symmetric positive definite.
- Compute the N_{rb} largest eigenpairs:

$$\mathbf{C}\mathbf{w}_n = \lambda_n \mathbf{w}_n, \quad \|\mathbf{w}_n\|_V = 1, \quad 1 \leq n \leq N_{\text{rb}}$$

- Recover the POD modes (left singular vectors of $\mathbf{S}$):

$$\boldsymbol{\xi}_n = \frac{1}{\sqrt{\lambda_n}} \mathbf{S}\mathbf{w}_n, \quad 1 \leq n \leq N_{\text{rb}}$$

- Construct the reduced basis matrix:

$$\mathbf{B}_{\text{rb}} = [\boldsymbol{\xi}_1 \mid \cdots \mid \boldsymbol{\xi}_{N_{\text{rb}}}] \in \mathbb{R}^{N_h \times N_{\text{rb}}}$$

POD reduced basis algorithm – online phase

The online phase (real-time computations)

Given a new parameter value $\mu_{\text{new}} \in \mathcal{P}$:

- Assemble the high-fidelity stiffness matrix:

$$\mathbf{A}_h^{\mu_{\text{new}}} \in \mathbb{R}^{N_h \times N_h}$$

- Compute the reduced-order matrices:

$$\mathbf{A}_{\text{rb}}^{\mu_{\text{new}}} = \mathbf{B}_{\text{rb}}^T \mathbf{A}_h^{\mu_{\text{new}}} \mathbf{B}_{\text{rb}}, \quad \mathbf{F}_{\text{rb}}^{\mu_{\text{new}}} = \mathbf{B}_{\text{rb}}^T \mathbf{F}_h^{\mu_{\text{new}}}$$

- Solve the reduced system:

$$\mathbf{A}_{\text{rb}}^{\mu_{\text{new}}} \mathbf{u}_{\text{rb}}(\mu_{\text{new}}) = \mathbf{F}_{\text{rb}}^{\mu_{\text{new}}}, \quad \mathbf{u}_{\text{rb}}(\mu_{\text{new}}) \in \mathbb{R}^{N_{\text{rb}}}$$

- Reconstruct the full solution in the FE basis:

$$\mathbf{u}_{\text{rb}}^{N_h} = \mathbf{B}_{\text{rb}} \mathbf{u}_{\text{rb}}(\mu_{\text{new}}) \quad \Rightarrow \quad \mathbf{u}_{\text{rb}}^{N_h} \in \mathbb{R}^{N_h}$$

- The reconstructed solution $u_{\text{rb}}(x; \mu_{\text{new}})$ can be expressed using the FE basis $\{\varphi_i(x)\}_{i=1}^{N_h}$, enabling visualization.
- According to Proposition 1, $\mathbf{u}_{\text{rb}}^{N_h}$ belongs to the optimal reduced basis of dimension N_{rb}.

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  - Solve the N_{rb}-dimensional system: $\mathbf{A}_{rb}^{\mu_{new}} \boldsymbol{u}_{rb}^{\mu_{new}} = \mathbf{F}_{rb}^{\mu_{new}}$.
  - Deduce the reduced solution in the HF basis: $\tilde{\mathbf{u}}_h^{\mu_{new}} = \mathbf{B}_{rb} \mathbf{u}_{rb}^{\mu_{new}}$.

Unsteady linear case: Galerkin FE

FE system using implicit Euler time discretization

$$\left(\frac{1}{\Delta t}\mathbf{M}_h + \mathbf{A}_h^{\mu}\right)\mathbf{u}_{h,k}^{\mu} = \frac{1}{\Delta t}\mathbf{M}_h\mathbf{u}_{h,k-1}^{\mu} + \mathbf{F}_h, \;\; 1 \leq k \leq N_t, \tag{20}$$

with $(\mathbf{M}_h)_{ij} = (\varphi_i, \varphi_j)_{L^2(\Omega)}$, $(\mathbf{A}_h^{\mu})_{ij} = a(\mu; \varphi_i, \varphi_j)$, $\;1 \leq i, j \leq N_h$ and $(\boldsymbol{F}_h)_i = l(\varphi_i)$, $\;1 \leq i \leq N_h$.

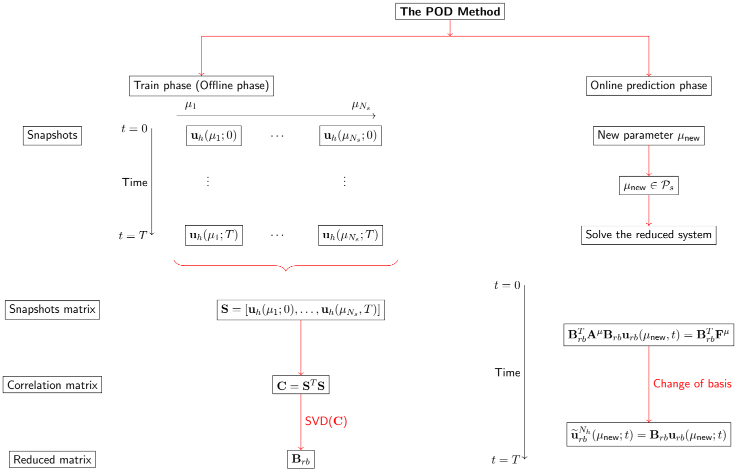
FE reduced system

$$\left(\frac{1}{\Delta t}\mathbf{M}_{rb} + \mathbf{A}_{rb}^{\mu}\right)\boldsymbol{u}_{rb,k}^{\mu} = \frac{1}{\Delta t}\mathbf{M}_{rb}\boldsymbol{u}_{rb,k-1}^{\mu} + \boldsymbol{F}_{rb}, \;\; 1 \leq k \leq N_t, \tag{21}$$

with $\mathbf{M}_{rb} = \mathbf{B}_{rb}^T\mathbf{M}_h\mathbf{B}_{rb}$, $\mathbf{A}_{rb}^{\mu} = \mathbf{B}_{rb}^T\mathbf{A}_h^{\mu}\mathbf{B}_{rb}$ and $\mathbf{F}_{rb} = \mathbf{B}_{rb}^T\mathbf{F}_h$

The POD Method: Offline and Online Phases
Unsteady Linear PDE Case

ROM



Numerical example: unsteady linear case, Galerkin FE

μ-parametrized unsteady linear advection-diffusion equation, $\mu = (\mu_1, \mu_2)$

$$\begin{cases} \partial_t u_h(\mu;t) - \mathrm{div}(\lambda(\mu_1)\nabla u_h(\mu;t)) + \boldsymbol{w}\cdot\nabla u_h(\mu;t) = f(\mu_2) & \text{in } \boldsymbol{Q}_T = (0,T)\times\Omega, \\ u_h(\mu;t) = 0 & \text{in } \Gamma_D, \\ -\lambda(\mu_1)\nabla u_h(\mu;t)\cdot n = 0 & \text{in } \Gamma_N, \\ u_h(\mu;0) = u_0(\mu) & \text{a.e in } \Omega. \end{cases}$$

with $\mu = (\mu_1, \mu_2)$, $\lambda(\mu_1) = \exp(\mu_1 - 11)$ and $f(\mu_2) = \cos(\mu_2 \mathrm{L} x)$

Numerics

- $\mu \in \mathcal{P} = [1, 10] \times [0, \frac{\pi}{L}]$.
- N_s snapshots, $N_s = (20 \times 20) \times N_t$ with $N_t = 20$.
- HF dimension $N_h = 1296$. $\epsilon^2_{POD} = 10^{-5} \implies$ RB dimension $N_{rb} = 32$.

Unsteady linear case

Standard POD method results

Case $\mu = (\mu_1, \mu_2)$, with non-affine parameterization, $\lambda(\mu_1) = \exp(\mu_1 - 11)$, $\mu_1 = 2.68$ and $f(\mu_2) = \cos(\mu_2 \mathrm{L}x)$ with $\mu_2 = 1.48$.

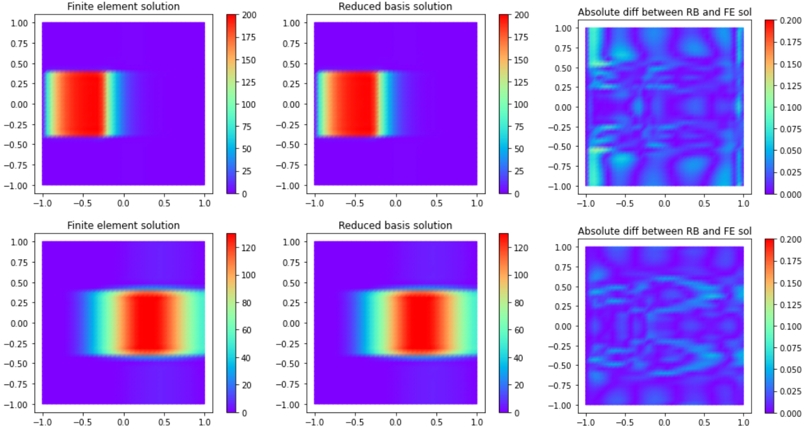


Fig. 4: (Left) The FE solution. (Middle) The POD RB solution. (Right) The absolute error between the FE and POD RB solutions. (Top) At time instant $t = 0$s. (Bottom) At time instant $t = 0.87$s.

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