LUNDS TEKNISKA HÖGSKOLA
MATEMATIK

TENTAMENSSKRIVNING
Matristeori
4-14 December 2023

Hand in solutions of 7 from 8 problems below if you follow any of the 7.5hp courses (FMAN71, MATC70 and FMA122F) or 6 problems if you follow any of the 6hp courses (FMAN70, FMA121F). For a passing grade (3.5 for 7.5 points course and 3.0 for 6 points course), at least one problem of the last four have to be solved. Credits can be given for partially solved problems. Write your solutions (in English or Swedish) neatly and explain your calculations. Both the content and the format of your solutions, and also how difficult problems you choose, will affect your grade.

The exam should be submitted through the canvas page by December 14. **Write your name, section-year, and the code for the course you are following on the first page. If you have preferences for when to take the oral exam, also write this on the first page.** The solutions should be submitted as a pdf-file. We encourage you to hand in scanned hand-written solutions, but make sure that the solutions are easily readable after scanning. Programs that you have written (e.g. matlab or maple files) should also be submitted through the canvas page. Make sure to reference these and explain how you have used them in your main solution-pdf.

You may use books and computer programs (e.g. Matlab and Maple), but it is not permitted to get help from other persons (or chat bots). You do not need to explain the elementary matrix operation such as matrix multiplications made by the computer, but should explain steps in more complicated calculations such as jordanization. Ask if you are not sure!

You can use any theorem in the book without proof but not exercises without explaining the solutions.

All matrices below are complex matrices if nothing else is specified explicitly.

Good luck!

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1. Compute the singular value decomposition of the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & 2 & -4 \end{pmatrix},$$

and find a vector X that minimizes $\|AX - B\|^2$, where $B = \begin{pmatrix} 18 \\ 9 \\ 0 \end{pmatrix}$. Is the solution unique?

2. Let

$$X = \begin{pmatrix} 10 & -20 \\ 4 & -8 \end{pmatrix} \quad Y = \begin{pmatrix} 6 & -10 \\ 2 & -3 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} -10 & 25 \\ -4 & 10 \end{pmatrix}.$$

If

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

determine which of $f(X)$, $f(Y)$ and $f(Z)$ are defined, and compute these. Show, for a general square matrix A, that if $f(A)$ is defined then

$$Af(A)A = A \quad \text{and} \quad f(A)Af(A) = f(A).$$

If in addition A is Hermitian show that $f(A)A$ and $Af(A)$ are Hermitian.

3. Suppose A and B are square matrices, p_A and p_B are their characteristic polynomials and π_A and π_B their minimal polynomials. Prove or disprove (e.g. by counter example) the following statements:

1. If $p_A(x) = p_B(x)$ and $\pi_A(x) = \pi_B(x)$ then A and B are similar.
2. If $p_A(x) = p_B(x) = \pi_A(x) = \pi_B(x)$ then A and B are similar.
3. $p_A(x) = \pi_A(x)$ if and only if A is similar to a companion matrix C.

4. Suppose that A is an $n \times n$ Hermitian positive definite matrix. Show that A has the condition number

$$\kappa(A) = \left(\max_{0 \neq X \in \mathbb{C}^n} \frac{X^H A X}{X^H X} \right) \cdot \left(\max_{0 \neq X \in \mathbb{C}^n} \frac{X^H X}{X^H A X} \right).$$

5. Let A be an $n \times n$ matrix and p and q polynomials. If q has the roots $\gamma_1, \gamma_2, ..., \gamma_n$, show that

1. $q(A)$ is invertible if and only if $\det(A - \gamma_i I) \neq 0$ for all i.
2. μ is an eigenvalue of $p(A)$ if and only if $\mu = p(\lambda)$, where λ is an eigenvalue of A.

6. Suppose that X has size $m \times n$ and rank r. Show that there are matrices A of size $m \times r$ and B of size $n \times r$ such that $X = AB^H$. Furthermore, show that for any such matrices we have $\|X\|_F \leq \frac{\|A\|_F^2 + \|B\|_F^2}{2}$, and equality can be achieved if and only if X has rank 1.

7. A function $f : M_n(\mathbb{R}) \to \mathbb{R}$ has the properties

1. $f(\alpha A + \beta B) = \alpha f(A) + \beta f(B)$, for all $\alpha, \beta \in \mathbb{R}$.
2. $f(AB) = f(BA)$.

   Show that $f(A) = \frac{1}{n} f(I) \operatorname{tr} A$.

8. Show that a square matrix A is Hermitian if and only if $A^H A = A^2$.