The Kalman Filter, part 2

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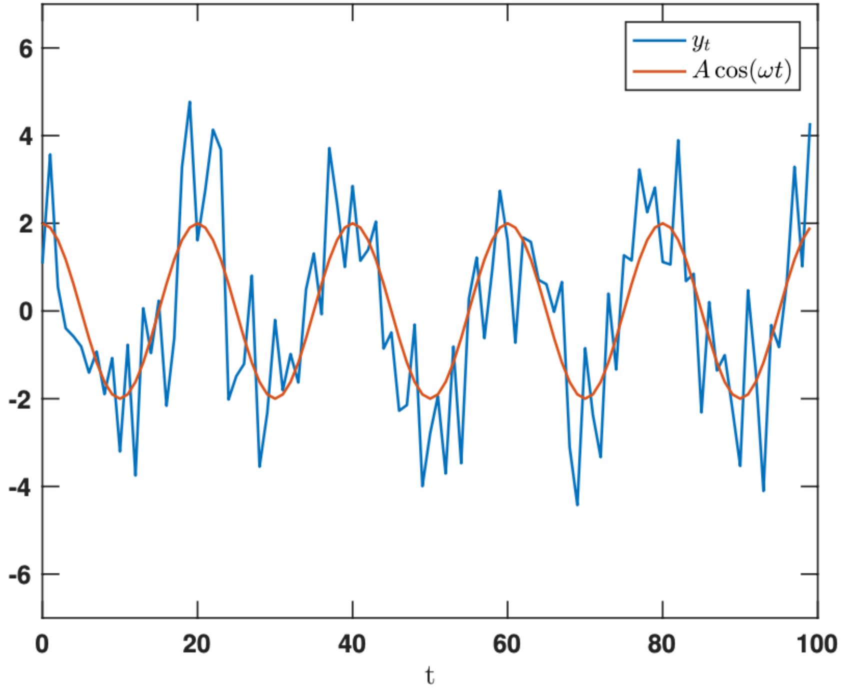
The Kalman filter



Example:
Consider the signal

$$y_t = A\cos(\omega t) + w_t$$

where w_t is a Gaussian white noise with variance σ_w^2. Assuming that the frequency, ω, is known, how can one estimate the amplitude, A, using a Kalman filter?

State dom

The Kalman filter

Let the hidden state be $x_t = A$. Then,

State form

$$\begin{aligned} x_t &= x_{t-1} \\ y_t &= \cos(\omega t) x_t + w_t \end{aligned}$$

To simplify notation, let $c_t = \cos(\omega t)$. Thus,

$$\begin{aligned} \hat{x}_{t+1|t} &= \hat{x}_{t|t} \\ \hat{y}_{t+1|t} &= c_{t+1} \hat{x}_{t+1|t} = c_{t+1} \hat{x}_{t|t} \end{aligned}$$

where

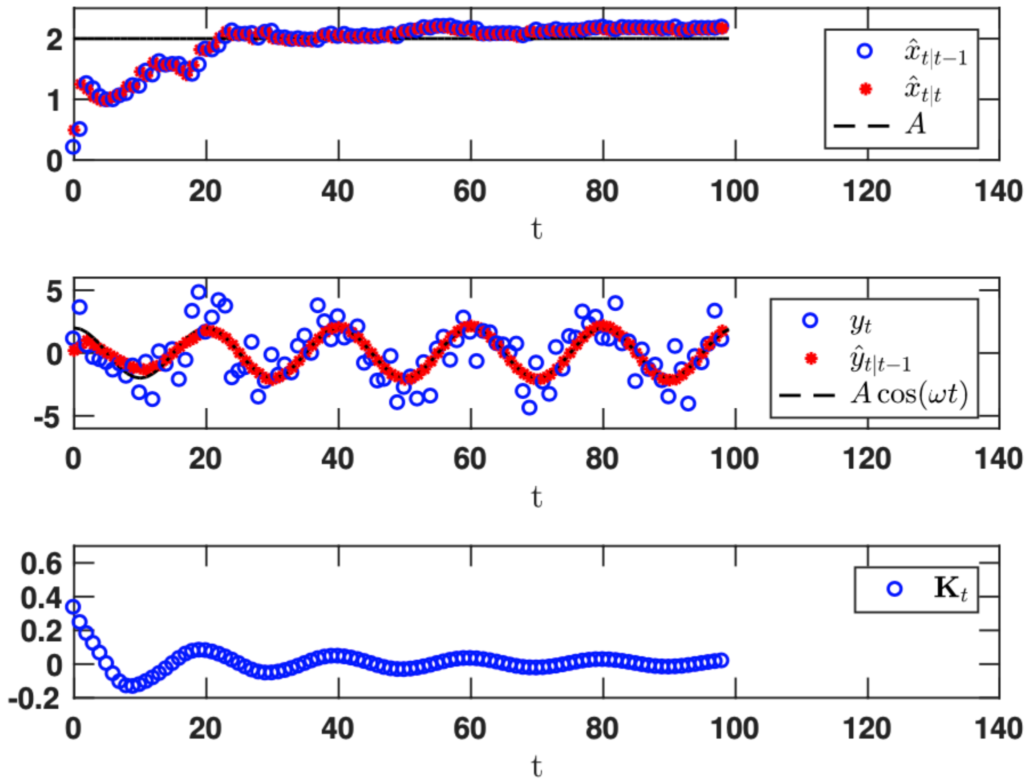
$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t (y_t - \hat{y}_{t|t-1})$$

with

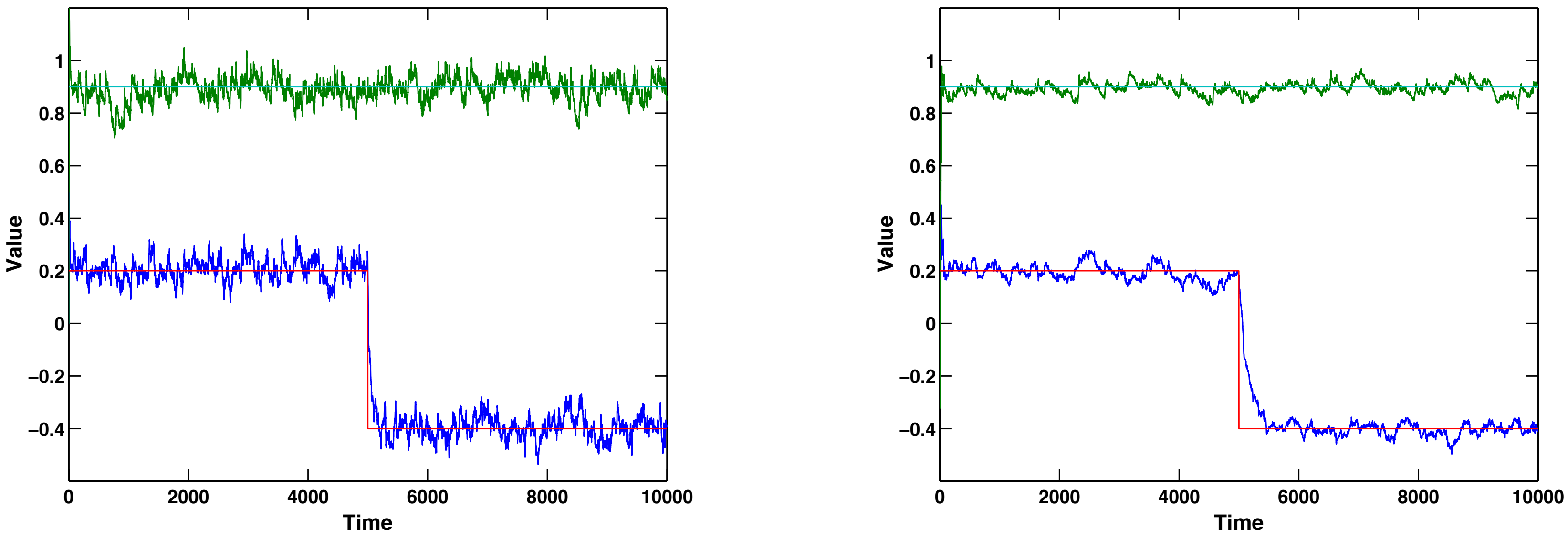
$$\begin{aligned} \mathbf{R}^{x,x}_{t+1|t} &= \mathbf{R}^{x,x}_{t|t} \\ \mathbf{R}^{y,y}_{t+1|t} &= c^2_{t+1} \mathbf{R}^{x,x}_{t+1|t} + \sigma^2_w \\ \mathbf{R}^{x,y}_{t+1|t} &= c_{t+1} \mathbf{R}^{x,x}_{t+1|t} \\ \mathbf{R}^{x,x}_{t|t} &= \mathbf{R}^{x,x}_{t|t-1} - K^2_t \mathbf{R}^{y,y}_{t+1|t} \\ K_t &= \frac{\mathbf{R}^{x,y}_{t|t-1}}{\mathbf{R}^{y,y}_{t|t-1}} = \frac{c_t \mathbf{R}^{x,x}_{t-1|t-1}}{c^2_t \mathbf{R}^{x,x}_{t-1|t-1} + \sigma^2_w} \end{aligned}$$

The Kalman filter

Set $t = 0$, initial state $\hat{x}_{t|t}$, and initial $\mathbf{R}^{x,x}_{t|t}$
repeat
  Predict state and observation
  $\hat{x}_{t+1|t} = \hat{x}_{t|t}$, $\hat{y}_{t+1|t} = c_{t+1}\hat{x}_{t|t}$
  Update covariances for predictions
  $\mathbf{R}^{x,x}_{t+1|t} = \mathbf{R}^{x,x}_{t|t}$, $\mathbf{R}^{y,y}_{t+1|t} = c^2_{t+1}\mathbf{R}^{x,x}_{t|t} + \sigma^2_w$, $\mathbf{R}^{x,y}_{t+1|t} = c_{t+1}\mathbf{R}^{x,x}_{t|t}$
  Update Kalman gain
  $\mathbf{K}_{t+1} = \frac{c_{t+1}\mathbf{R}^{x,x}_{t|t}}{c^2_{t+1}\mathbf{R}^{x,x}_{t|t} + \sigma^2_w}$
  Update estimate of state
  $\hat{x}_{t+1|t+1} = \hat{x}_{t+1|t} + K_{t+1}\left(y_{t+1} - \hat{y}_{t+1|t}\right)$
  $t \leftarrow t + 1$
until end of signal



The Kalman filter



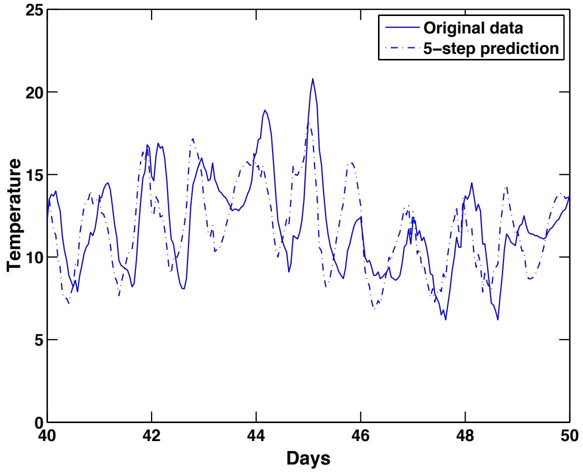
Example:
Consider an AR(2) with an unknown parameter a_1 and a_2

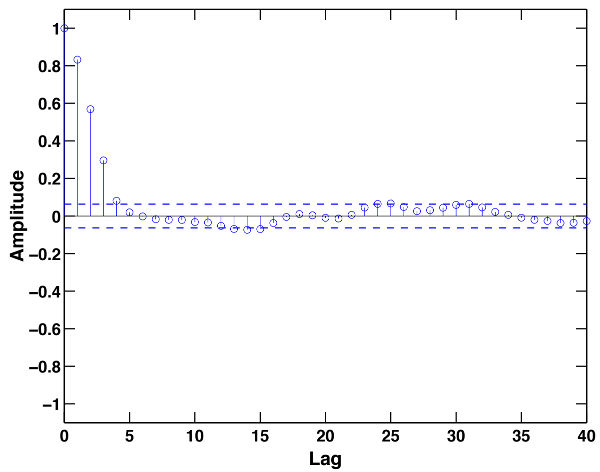
$$y_t + a_1 y_{t-1} + a_2 y_{t-2} = e_t$$

What should we set as states if we wish to estimate a_1 and a_2?

The figures show the estimates for $\mathbf{R}_e = 10^{-4}\mathbf{I}$ and $\mathbf{R}_e = 10^{-5}\mathbf{I}$.

The Kalman filter





Often, one is interested in predicting the process and the states k steps ahead. Such a prediction is formed as

$$\begin{aligned}\hat{\mathbf{x}}_{t+k+1|t} &= \mathbf{A}\hat{\mathbf{x}}_{t+k|t} + \mathbf{B}\hat{\mathbf{u}}_{t+k|t} \\ \mathbf{R}^{x,x}_{t+k+1|t} &= \mathbf{A}\mathbf{R}^{x,x}_{t+k|t}\mathbf{A}^T + \mathbf{B}\mathbf{R}^{u,u}_{t+k|t}\mathbf{B}^T + \mathbf{R}_e\end{aligned}$$

In the particular case when there is no input, the state prediction simplifies to

$$\hat{\mathbf{x}}_{t+k+1|t} = \mathbf{A}^k\hat{\mathbf{x}}_{t|t}$$

yielding the output prediction

$$\hat{\mathbf{y}}_{t+k|t} = \mathbf{C}\hat{\mathbf{x}}_{t+k|t} = \mathbf{C}\mathbf{A}^k\hat{\mathbf{x}}_{t|t}$$

The figures shows the 5-step prediction of the Svedala temperature example (see the book for details); note the MA$(k-1)$ structure of the residual.

The Kalman filter

One need to take a bit of care when predicting ARMA processes, as the required noise realisations are typically not available. For example, consider the ARMAX process

$$A(z)y_t = B(z)z^{-d}x_t + C(z)e_t$$

where

$$\begin{aligned} A(z) &= 1 + a_1 z^{-1} + \ldots + a_6 z^{-6} \\ B(z) &= b_0 + b_1 z^{-1} + b_2 z^{-2} + b_4 z^{-4} + b_5 z^{-5} \\ C(z) &= 1 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} \end{aligned}$$

The process may then be written on state space form as

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{A}\mathbf{x}_t + \mathbf{e}_t \\ y_t &= \mathbf{C}_{t|t-1}\mathbf{x}_{t|t-1} + w_t \end{aligned}$$

where $x_{t|t-1}$ contains the model parameters using measurements up to y_{t-1},

$$\begin{aligned} \mathbf{C}_{t|t-1} = \big[& -y_{t-1} \quad -y_{t-2} \quad \ldots \quad -y_{t-6} \quad e_{t-1} \quad e_{t-2} \quad e_{t-3} \\ & u_t \quad u_{t-1} \quad u_{t-2} \quad u_{t-4} \quad u_{t-5} \ \big] \end{aligned}$$

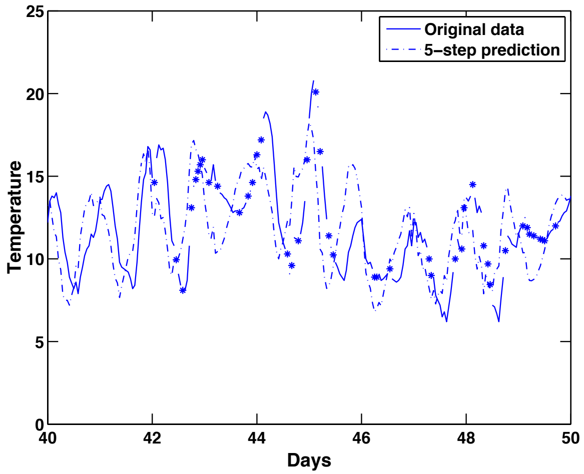
with $\mathbf{A} = \mathbf{I}$ and $w_t = e_t$. This allows for the forming of the k-step predictions

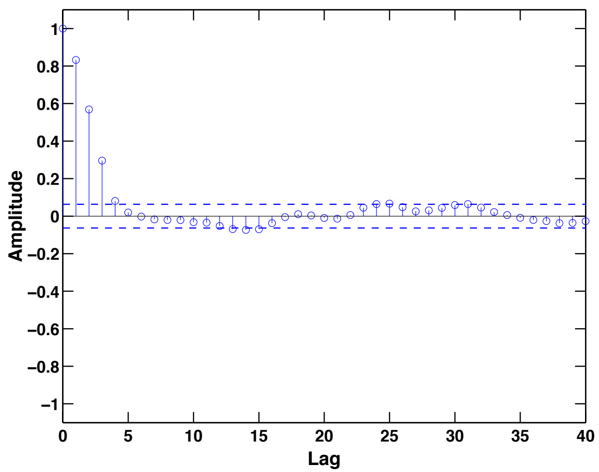
$$\begin{aligned} \hat{\mathbf{y}}_{t+1|t} &= \mathbf{C}_{t+1|t}\hat{\mathbf{x}}_{t+1|t} = \mathbf{C}_{t+1|t}\hat{\mathbf{x}}_{t|t} \\ \hat{\mathbf{y}}_{t+2|t} &= \mathbf{C}_{t+2|t}\hat{\mathbf{x}}_{t|t} \end{aligned}$$

where

$$\begin{aligned} \mathbf{C}_{t+2|t} = \big[& -\hat{y}_{t+1|t} \quad -y_t \quad \ldots \quad -y_{t-4} \quad 0 \quad e_t \quad e_{t-1} \\ & u_{t+2} \quad u_{t+1} \quad u_t \quad u_{t-2} \quad u_{t-3} \ \big] \end{aligned}$$

The Kalman filter





One can also use the Kalman filter to predict missing samples; this can be done by changing the update equations to

$$\begin{aligned}\hat{\mathbf{x}}_{t|t} &= \hat{\mathbf{x}}_{t|t-1}\\ \mathbf{R}^{x,x}_{t|t} &= \mathbf{R}^{x,x}_{t|t-1}\end{aligned}$$

for the case of a sample being missing.

The figures shows the 5-step prediction of the Svedala temperature example with missing samples (marked with a star).