(i) To derive the coordinates X, assume the unique answer to be centered

i.e. $\sum_{i=1}^{n} x_{ik} = 0$

To get the relation with distance matrix D, try to find $B = XX^T$ ($X_{n\times p}$ is coordinate matrix)

$\Rightarrow$ we have $d_{ij}^2 = \|x_i - x_j\|^2$

$$= x_i x_i' + x_j x_j' - 2x_i x_j'$$

$$= b_{ii} + b_{jj} - 2b_{ij}$$

$\because \sum_{i=1}^{n} x_{ik} = 0 \Rightarrow \sum_{i=1}^{n} b_{ij} = \sum_{i=1}^{n}\sum_{k=1}^{q} x_{ik}x_{jk} = \sum_{k=1}^{q} x_{jk} \cdot \left(\sum_{i=1}^{n} x_{ik}\right) = 0$

$\sum_{i=1}^{n} d_{ij}^2 = \sum_{i=1}^{n} (b_{ii} + b_{jj} - 2b_{ij}) = \sum_{i=1}^{n} b_{ii} + \sum_{i=1}^{n} b_{jj} - \cancel{2\sum_{i=1}^{n} b_{ij}} = T + n b_{jj}$ ($T = \text{trace}(B)$)

$\sum_{j=1}^{n} d_{ij}^2 = T + n b_{ii}$

$\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ij}^2 = 2n \cdot T$

$\Rightarrow b_{ij} = \frac{1}{2}(d_{ij}^2 - b_{ii} - b_{jj}) = -\frac{1}{2}(d_{ij}^2 - d_{\cdot j}^2 - d_{i\cdot}^2 + d_{\cdot\cdot}^2)$

$\Rightarrow B = -\frac{1}{2} J D^{(2)} J$, where $J = I - \frac{1}{n} \cdot 11^T$

(ii) $\because B = XX' \Rightarrow$ we can use eigenvalue decomposition to derive X

$\Rightarrow B = E\Lambda E'$, E: eigen vectors matrix

Λ: $\text{diag}(\lambda_1, \lambda_2 \ldots \lambda_n)$

$\Rightarrow XX' = E\Lambda E' = (E\Lambda^{\frac{1}{2}}) \cdot (E\Lambda^{\frac{1}{2}})'$

$\Rightarrow X = E\Lambda^{\frac{1}{2}} \approx E_m \Lambda_m^{\frac{1}{2}}$ (Select m largest eigenvalues to lower the dimension, same concept as PCA)