Part III Essay

Classifying Toposes

University of Cambridge

Contents

1 Introduction

The work in mathematics foundations had two champions in the 20th century. In one hand, we had the axiomatization of set theory, overcoming the Russell's Paradox, that continued with the results relating to the Axiom of Choice and the Continuum Hypothesis. The Part III introductory subject for this area is Model Theory. In that subject, it is discussed how axioms translate into actual mathematical models and vice versa. In the second half of the century, Category Theory emerged as a powerful tool for abstracting mathematical notions, potentially showing how concepts and results in different areas can be seen as different expressions of wider (and usually more abstract) results. This had important implications for all abstract algebra, mainly via the work on sheaves by Grothendieck (of from we will have a taste in this essay) that lead to the proof of the Weil conjectures.
However, the reality is that these two areas can clash. Usually, model theory uses set-theoretic basis with classical logic as its way of expressing truth. In the other hand, category theory lets us notice that **Set** can be seen as a well-behaved category, but that much of its strengths can be abstracted to more general settings. It has also been proposed as an alternative basis for math formalization. In this work we will explore how to make a *Categorical Model Theory*, we aim to construct alternatives to the category of sets as basics objects of mathematics, and we will see how we can construct models and reason about them in this setting.

We will begin in Section 2 exploring elementary toposes and Grothendieck toposes. We will start defining elementary toposes: toposes are mathematical structures where most of maths foundations work can be done, similar to what we usually do in sets. We will define them thinking about what good properties do sets have and abstracting them into categorical terms. After that, we will see how to construct a certain class of these toposes, called Grothendieck toposes. We will do so by abstracting ideas coming from topology into generalized constructions in categories. We will finish these ideas with the definition of geometric morphisms, which we will use later as they will give maps that conserve models in toposes. The first section is very definition heavy, but we try to give insight on why those are the definitions that we use.
In Section 3 our objective is to see that Grothendieck toposes are elementary toposes. We will do so by finding constructions for each property that define elementary toposes. We will furthermore show how Grothendieck toposes have some properties that will be important for interpreting theories within them. This section will contain the longest proofs, that will help us to see how the definitions of section 2 fit our objectives.
In Section 4 we introduce the syntax of a class of theories, called geometric theories. These are a superset of algebraic theories that are naturally suited for Grothendieck toposes. In particular, they are preserved by inverse images of geometric morphisms. Again, this section is definition heavy.
Finally, in Section 5, we will see that toposes are powerful enough to provide classifying spaces for a geometric theory, i.e. that we can construct a topos that *completly* models a given theory. We will

define the notion of *classifying topos* and construct a *syntactic category* for a theory. We will sketch the proof of the theorem that states that the syntactic category of a theory provides its classifying topos.

We mainly follow *Theories, Sites, Toposes* from O. Caramello [1] and *Sheaves in Geometry and Logic* from S. Mac Lane and I. Moerdijk [4]. A short document from E. Riehl [5] was a good introduction to the topic and several nLab pages have of essential help, specially for finding examples. For the sake of breveity, we will assume all results from *Category Theory* and *Model Theory and Non-Classical Logic* from the Part III of the University of Cambridge (2023-2024). Nonetheless, we will redefine some notions of those courses for completeness and clarity.

2 Toposes

In this section we define an elementary topos as a well-behaved category, comparing them with **Set**. After that, we will construct a certain class of toposes, named Grothendieck toposes, that arise from a topological abstraction.
Finally, we will define a class of well-behaved maps from a Grothendieck topos to another Grothendieck topos: a geometric morphism. It will later become useful because they will raise into functors that preserve the structures we define on toposes.

2.1 Elementary toposes

As mentioned, we are trying to characterize *well-behaved categories*, where our model for a well behaved category is **Set**.

Definition 2.1 (**Set** category)**.** **Set** is the category where the objects are sets (constructed as in ZFC) and morphisms are functions between them.

We have to clarify what we mean by well-behaved. As we said we want to be able to model theories in toposes. Well behaved means that we are able to interpret formulae, in a way similar to what we would do in **Set**, in that category. The properties we ask for in this section will help to do so, and it will become clear in Section 4 how we use them.

The ZFC axioms tell us which new sets we can construct given some sets. One of the most used construction is the Cartesian product. It is basic for the interpretation of functions of *several variables* as it allows us to construct the domain of it.
We can check that the Cartesian product always exists, using the Union Axiom, the Power Set Axiom and the Axiom of Extensionality. In fact, we can extend the definition to the product of n sets. We can translate this construction to a characterization in the category **Set**. In order translate from set-theoretical constructions to categorical definitions we usually take a *universal property* that captures a characterization for our desired object. This universal property needs to be expressed in categorical fashion, only making use of objects and morphisms, that are the things that we can *see*

through categorical lenses. In particular, we should avoid using the notion of elements in the sets A, B that we used in our set-theoretical definition.

Definition 2.2 (Finite product)**.** Given objects $A_1, \dots, A_n$, we say that $A_1 \times \cdots \times A_n$ is their product if there exist projections $\pi_i : A_1 \times \cdots \times A_n \to A_i$ such that given another object B and morphisms $f_i : B \to A_i$ there exists an unique morphism $f_1 \times \cdots \times f_n : B \to A_1 \times \cdots \times A_n$ with $\pi_i \circ (f_1 \times \cdots \times f_n) = f_i$

$$\begin{tikzcd}
& B \arrow[d, dashed, "f_1\times\cdots\times f_n"] \arrow[ldd, "f_1"'] \arrow[rdd, "f_n"] & \\
& A_1 \times \cdots \times A_n \arrow[ld, "\pi_1"] \arrow[rd, "\pi_n"'] & \\
A_1 & \dots & A_n
\end{tikzcd}$$

We say that a category has finite products if given n objects, its product always exists. The product can be of 0 objects, in that case we get a terminal object. As the Cartesian product of sets satisfies the universal property, with the morphism π_i being the i-th coordinate projection, the category **Set** has finite limits.

Another important property that holds in **Set** is that the collection of all morphisms between two given sets is itself a set. Note that this property also holds for other types of mathematical structures, notably in vector spaces and modules. Given two sets X, Y, we call Y^X the set of all functions $X \to Y$. We can again find a universal property which captures the concept. In the previous case the universal property talked about projections. In this case, we can draw inspiration from computer science, as the universal property is *currying*. Currying states that given a function in two variables $f : A \times B \to C$ we can think of it as a function on the first variable to functions on the second variable. In symbols: $f : A \to (B \to C)$ or: $f(x, y) = g_x(y)$ where $g_x(y) = f(x, y)$. Categorically, defining $\mathrm{Hom}(A, B)$ as the collection of morphisms from A to B we have $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, Z^Y)$ with the bijection being natural in X, Y and Z. We define the exponentials of two objects with the universal property:

Definition 2.3 (Exponential)**.** Given a category with finite products, and two objects X, Z, we say Z^X is their exponential if there exists a morphism $e : Z^X \times X \to Z$ (that accounts for evaluations) such that for any other object Y and morphism $Y \times X \to Z$, there exists an unique f' such that $e \circ (f' \times 1) = f$.

$$\begin{tikzcd}
& Y \times X \arrow[ld, dashed, "f'\times 1"'] \arrow[d, "f"] \\
Z^X \times X \arrow[r, "e"'] & Z
\end{tikzcd}$$

We call the categories with exponentials and finite products Cartesian closed categories.

Definition 2.4 (Cartesian closed)**.** A category with finite products such that for any two objects there exists their exponential is called Cartesian closed.

A property of **Set** that might not seem so obvious, but has important consequences for interpreting theories, is the existence of *equalizers*. Given two functions f, g between sets A, B we can construct the subset of A where they agree. If we call E this subset and $e : E \to A$ the injection from E to A, we have that $f \circ e = g \circ e$. In addition, any other e' such that $f \circ e' = g \circ e'$ will factor (uniquely) through e. Note that in the construction, we used that sets have elements to easily construct the equalizer (E, e). But we can abstract it into an universal property that describes it categorically.

Definition 2.5 (Equalizer)**.** Given objects A, B and morphism f, g an equalizer (E, e) of f and g is an object E and a morphism $e : E \to A$ such that $f \circ e = g \circ e$ and for any other e' with $f \circ e' = g \circ e'$ we have that e' factorizes uniquely through e, that is, there exists a unique h such that $e \circ h = e'$.

$$\begin{tikzcd}
C \arrow[d, dashed, "h"'] \arrow[dr, "e'"] & & \\
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

From the discussion above, it is clear that notion of equalizer in essential to interpret theories, as it allows us to capture the meaning of equality. We will soon see that the interpretation for the equality in a general topos is directly given by equalizers.

The notions of finite products and equalizers can be described as concrete cases of a wider class of constructions, finite limits. Intuitively, a limit takes a finite diagram in a category and constructs a universal object for it. This allows us to capture several useful constructions, namely initial objects, products, equalizers, and pullbacks. Pullbacks will have a very relevant role in the interpretation of theories, as they generalize the intersection of sets.
As we hinted, products and equalizers are especially important, as they allow us to construct any other finite limits from them.

Theorem 2.6 (Finite products and equalizers are enough)**.** *Any category with finite products and equalizers has all finite limits.*

The proof is an exercise done in the category theory course. Hence, this shows that sets have all finite limits.
The last construction that we will consider is the subobject classifier. The idea here is to generalise the fact that sets have well behaved subsets. We briefly saw subobjects in the definition of equalizer, but here we should discuss more about them.
Subobjects are in fact the construction of set theory that closely relates to propositions. Given a preposition and a set S and a proposition ϕ by the Axiom of Separation we can construct the subset

$S' = \{x \in S | \phi(x)\}$. In a general topos, the interpretation of a proposition will also come from the construction of a subobject. We will discuss more about subobjects constructions in toposes in Section 3.2, but we begin discussing the subobjects classifier.

The idea behind the subobject classifier comes from thinking about a subset S of a set A in two ways. First, is to define the characteristic function $\chi_S : A \to \{0, 1\}$ such that $\chi_S(x) = 1$ if $x \in S$ and 0 otherwise. Here, we interpret the set $\Omega = \{0, 1\}$ as the set $\{\text{false}, \text{true}\}$. Second, definition involves characterizing any object S with a injective function $i : S \to A$ as a subset of A, identifying $x \in S$ with $i(x) \in A$. We can combine these two definitions in the following diagram:

$$\begin{tikzcd} S \arrow[r, "!"'] \arrow[d, tail, "i"] & 1 \arrow[d, "\top"'] \\ A \arrow[r, "\chi_S"] & \Omega \end{tikzcd}$$

being a pullback square. Here ! is the unique morphism to the terminal set 1 and $\top$ is the morphism from 1 to Ω with the image of the unique element of 1 being 1 (or true). Note that this is a pullback since given another S' with $i' : S' \to A$ such that $\chi_S \circ i' = \top$, we know that $i(S') \subseteq i(S)$, so we have a unique $m : S' \to S$ with $i \circ m = i'$, namely $m = i^{-1} \circ i'$ where we can take i^{-1} as it is injective and $i(S') \subseteq i(S)$. In order to generalize this property to any category $\mathcal{C}$ with finite limits, we need to be able to construct an object Ω, a morphism $\top$ and a morphism χ_S for each monomorphism, making any square as above a pullback.

Definition 2.7 (Subobject classifier)**.** In a category with finite limits, an object Ω is a subobject classifier if there exists an morphism $\top : 1 \to \Omega$ and for each monomorphism $i : S \rightarrowtail A$ there exists a morphism $\chi_S : A \to \Omega$ such that the square:

$$\begin{tikzcd} S \arrow[r, "!"'] \arrow[d, tail, "i"] & 1 \arrow[d, "\top"'] \\ A \arrow[r, "\chi_S"] & \Omega \end{tikzcd}$$

is a pullback square.

Definition 2.8 (Elementary topos)**.** An elementary topos is a Cartesian closed category with finite limits and a subobject classifier.

Remark 2.9. There are several equivalent definitions for an elementary topos. We could for example ask only for equalisers and finite products to exists, since would imply all finite limits exists. We can also weaken the condition on exponentials as they can be constructed from exponentials of the subobject classifier.

Example 2.10. The category **FinSet** of finite sets is an elementary topos. This follows from the fact that Cartesian products and exponentials of finite sets are finite.

We conclude here our characterisation of well-behaved categories. As we already said, this will allow us to interpret theories with structures others than sets.
In next section we see a concrete way of construction a subclass of elementary toposes, Grothendieck toposes, that come with relevant topological intuition.

2.2 Grothendieck toposes

In this section we aim to define Grothendieck toposes. We will start discussing the notion of sheaves on a topology. Sheaves are the generalization of functions that we can define *locally*. After this we will generalize the notion of topology to the notion of Grothendieck Topology, which will admit a generalization of the definition of sheaves.

The main intuition for building sheaves is to look at objects such that they are essentially defined locally, and that this local property extends globally.
A natural example comes from the space of differentiable functions. Take an space X and define $C^k(U, \mathbb{R})$ the space of k times differentiable functions from an open subset U of X to $\mathbb{R}$. These sets are related for comparable subsets. This is, if $V \subset U$ we can restrict the function in U to functions of V. We can define the restriction map $r_V^U : C^k(U, \mathbb{R}) \to C^k(V, \mathbb{R})$ that takes $f : U \to \mathbb{R}$ to $r_V^U(f) = f_{|V}$. This restriction functions are functorial in the sense that $r_U^U = id$ and if $W \subseteq U \subseteq V$ then $r_W^V = r_W^U \circ r_U^V$. Note that we can construct the same restriction function with other classes of *nice* functions, for example, continuous or analytic functions. We generalize this example to the notion of presheaf, beginning with the definition of the poset of open subsets.

Definition 2.11 (Poset of open subsets)**.** Given a topological space X, we define the category $\mathcal{O}(X)$ with objects the open subsets of X and a unique morphism $V \to U$ if $V \subseteq U$.

Definition 2.12 (Presheaf on topological space)**.** Given a topological space X a presheaf F is a functor $F : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$.

Looking at the previous example, we can observe that our sets $C^k(U, \mathbb{R})$ have even more structure. They satisfy what we call *gluing property*: consider open subsets U, V and two functions $f : U \to \mathbb{R}, g : V \to \mathbb{R}$ such that $f_{|U \cap V} = g_{|U \cap V}$, there exists a unique $h \in C^k(U \cup V, \mathbb{R})$ such that $h_{|U} = f$ and $h_{|V} = g$. This does in fact hold not only for two functions but for *arbitrarily* many. Translating the property into categorical terms we get.

Definition 2.13 (Sheaf on topological space)**.** A sheaf on a topological space is a presheaf F such that for each open subset U and open covering $\{U_i\}$ of U, we have that the following diagram is an equalizer diagram:

$$FU \xrightarrow{\quad e \quad} \prod_i FU_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j)$$

where $e(t) = \{t|_{U_i}^U\}_i$, $p(\{t_i\}_i) = \{t_i|_{U_i \cap U_j}^{U_i}\}_{i,j}$ and $q(\{t_i\}_i) = \{t_j|_{U_i \cap U_j}^{U_j}\}_{i,j}$. In elementary terms, it says that FU is the *biggest* set such that we can define a function e that equalizes p, q. It reads that, for all i, j we have the expected equality $t|^{U}_{U_i}|^{U_i}_{U_i \cap U_j} = t|^{U}_{U_j}|^{U_j}_{U_i \cap U_j}$.

As we discussed above, sheaves carry local information that defines them globally, by gluing. If we restricted only to presheaves, we would not have this properties, as the following example shows:

Example 2.14. For an open subset U of $\mathbb{R}$, define $B(U)$ the set of bounded functions from $U \to \mathbb{R}$. This assignment defines a presheaf, but does not define a sheaf as can be seen taking $U_n = (-n, n)$ and the identity function in each U_n. The restrictions agree everywhere, but the gluing is not bounded. This exemplifies that we cannot capture non-local behaviours, as boundeness, by sheaves.

Definition 2.15 (Category of sheaves on a topological space)**.** The category of sheaves on a topological space has shaves as objects and natural transformations between sheaves, seen as functors $F : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$, as arrows.

We need to generalize the concept of topology in order to define sheaves in a general category. The idea is to focus again in the power of sheaves. Sheaves define objects that work locally, so the local information faithfully shows all the information on the object. If we remember the manifolds theory, or more concretely, Riemann surfaces, this should sound familiar. Recall that a Riemann surface is an space where we can define a cover on it such that the transitions between covering parts are well behaved (holomorphic in this case). Hence, understanding the open covers of an space in similar to understanding the space as a whole. Our objective is to generalize the notion of open cover of a space.

We start exploring the translation of open covers of an space X into the category $\mathcal{O}(X)$. We want to define to each open subset S a collection of open covers. A cover of S should translate in $\mathcal{O}(X)$ to a collection of arrows $S_i \to S$ such that $\cup S_i = S$. We will not ask for this last condition to hold in *all* covers, but we will see that we will define Grothendieck topologies making some of these *complete* covers appear. We will ask for every open subset of a member of a cover to be a member of the cover as well. This translates into:

Definition 2.16 (Sieve)**.** A sieve S on an object c of a category $\mathcal{C}$ is a family of arrows with codomain c such that for any $f \in S$, we have $f \circ g \in S$ for any composable g.

In the category $\mathcal{O}(X)$, the definition of sieve is saying that if U is part of a cover then any open subset of U is also in the cover. We want to select some sieves for each object c in our category such that they provide a family similar to the one of covers of an open subset. The first condition we will ask for is that the collection of all open subsets of an open subset is a cover. For the next property, we have first to notice an important closeness property of sieves:

Definition 2.17 (Pullback sieve)**.** Given an arrow $f : b \to c$ and a sieve S on c, the collection $f^*(S) = \{g | f \circ g \in S\}$ is a sieve on b, that we call the pullback of S through f.

The definition above gives us a sieve, as if g factors through f, gh also factors through f for any h. We will require that if S is considered a cover, for any f the sieve $f^*(S)$ is also a cover. In $\mathcal{O}(X)$, the property states that for a cover C on S and a open subset S' we have that $C \cap S'$ is a cover of S', as expected.

The last condition we will ask is the "moral" converse of the last one: if for a sieve S we have that all the pushbacks through a cover C are a cover, then we have that S is also a cover. The translation to open covers is again very natural. The property states that if you have a cover C and a sieve S such that for each open subset O in the cover C, $O \cap S$ is a cover of O, necessarily S will also be cover. With the stated conditions, we can define the collection of covers on a category, that we will call Grothendieck topology.

Definition 2.18 (Grothendieck topology)**.** A Grothendieck topology on a category $\mathcal{C}$ is a function J which assigns to each object c of $\mathcal{C}$ a collection $J(c)$ of sieves on c, such that

- the maximal sieve $t_c = \{f \mid \text{cod } f = c\}$ is in $J(c)$
- (stability axiom) if $S \in J(c)$, then $h^*(S) \in J(d)$ for any arrow $h : d \to c$
- (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $h^*(R) \in J(\text{dom } h)$ for all $h \in S$, then $R \in J(c)$

We call a sieve $S \in J(c)$ a J-covering sieve, or say that S covers c.

Example 2.19. Given a topological space X and the poset $\mathcal{O}(X)$ we can define a Grothendieck topology on $\mathcal{O}(X)$. We take for each open $O \in \mathcal{O}(X)$, $J(O)$ to be the set of all open covers of O. As we discussed before, this definition checks the three conditions. The interesting aspect of this construction is that we move from an space with points to the category $\mathcal{O}(X)$ where we lost the points. But indeed many important topological properties can be described only with the knowledge of open covers, most importantly the notion of compactness.
We could also take less interesting topologies. For instance consider if $J(O)$ only contains the maximal sieve on O, J it is a Grothendieck topology, but not very interesting.

Before re-stating the sheaves definitions, we define the pair of a category and a Grothendieck topology on it:

Definition 2.20 (Site)**.** A site in a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a category and J a Grothendieck topology on $\mathcal{C}$. If $\mathcal{C}$ is small, we say it is an small site.

From this, the definitions of presheaf and sheaf are the expected ones:

Definition 2.21 (Presheaf)**.** A presheaf on a category $\mathcal{C}$ is a functor $F : \mathcal{C}^{\text{op}} \to \mathbf{Set}$. We denote the category of presheaves and natural transformations between them as $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$.

Definition 2.22 (Sheaf)**.** A J-sheaf on a site $(\mathcal{C}, J)$ is a presheaf F on $\mathcal{C}$ such that for each sieve $S \in J(U)$, the following diagram is an equalizer diagram:

$$FU \xrightarrow{\quad e \quad} \prod_{f \in S} F(\text{dom } f) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{f,g \quad f \in S, \text{dom } f = \text{cod } g} F(\text{dom } g)$$

where $e(x) = \{P(f)(x)\}_f$, $p(\{x_f\}_f) = \{x_{fg}\}_{f,g}$ and $q(\{x_f\}_f) = \{P(g)(x_f)\}_{f,g}$. So the equalizer gives the identity $P(fg)(x) = P(g)(P(f)(x))$ for all $f \in S$, composable g and $x \in FU$.

Remark 2.23. As in the case of topological sheaves, we can rewrite the above in elementary terms of matching families. For a sieve S on C, with $S \in J(C)$, a matching family is $\{x_f\}_{f \in S}$ such that $x_f \in P(D)$ if $f : D \to C$. An amalgamation is a $x \in P(C)$ such that for all $f \in S$ we have $P(f)(x) = x_f$. A presheaf is a sheaf if there exists exactly one amalgamation for each matching family.

We can see how the example of topological sheaves generalizes here:

Example 2.24. Given an space X, a topological sheaf on $\mathcal{O}(X)$ is immediately translated into a sheaf on $\mathcal{O}(X)$ taking the sieves of $J(C)$ to be all the open coverings of C as we saw in the last example.

Definition 2.25 (Category of sheaves)**.** The category of sheaves $\mathbf{Sh}(\mathcal{C}, J)$ on a site $(\mathcal{C}, J)$ has sheaves as objects and natural transformations as morphisms, when we regard a sheaf as functors $\mathcal{C}^{\text{op}} \to \mathbf{Set}$.

Finally we can define our new class of toposes.

Definition 2.26 (Grothendieck topos)**.** A Grothendieck topos is a category equivalent to a category $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves over a small site $(\mathcal{C}, J)$. We then say that $(\mathcal{C}, J)$ is a site of definition of the topos.

Example 2.27. The category **Set** is a Grothendieck topos. Its site of definition is $(\{*\}, J)$ where $\{*\}$ is the one point space and J is the trivial topology over it.
On the other hand, **FinSet** is an Elementary topos that is not Grothendieck. We will see in next section that a Grothendieck topos has all small limits, and **FinSet** only has finite limits.

2.3 Geometric morphisms

Ending our generalization from topology, we are interested in the *good* maps between Grothendieck toposes. In topology, the morphisms are the continuous maps, and we will generalize them into geometric morphisms between toposes. This class of morphisms will provide us with functors, the inverse image of the morphism, that preserve the properties of the models inside Grothendieck toposes. We will see this in next sections.
The intuition for the definition comes from the fact that we want that them preserve interpretations. We already established that toposes have a nice structure, similar to sets. If we defined a morphism between toposes E and F only as a functor from $E \to F$, it would not necessarily preserve limits. Our way of forcing the preservation is asking for the existence of adjoints.

Definition 2.28 (Geometric morphism)**.** A geometric morphism $f : F \to E$ between toposes is a pair of adjoint functors $f^* : E \to F \dashv f_* : F \to E$, respectively called the inverse image and the direct image of f, such that the left adjoint f^* preserves finite limits.

Remark 2.29. Notice that f^* always preserves colimits as it has a right adjoint, while f_* always preserves limits as it has a left adjoint.

The definition of geometric transformations is the expected.

Definition 2.30 (Geometric transformation)**.** A geometric transformation $\alpha : f \to g$ between two geometric morphisms $f, g : F \to E$ is a natural transformation from f^* to g^*.

Remark 2.31. As f^* and f_* are adjoint the above definition could also be given as a natural transformation from g_* to f_*.

We can now define the category of geometric morphisms and transformations between them.

Definition 2.32 (Geometric morphisms category)**.** Given two toposes E, F the geometric morphisms category $\mathbf{Geom}(E, F)$ has as objects geometric morphisms $E \to F$ and as arrows geometric transformations between them.

Example 2.33. Geometric functors generalize continuous functions in the sense that given a continuous map $f : X \to Y$ of topological spaces, it induces a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ with direct image $\mathbf{Sh}(f)_*$ given by $\mathbf{Sh}(f)_*(P)(V) = P(f^{-1}(V))$ for each $V \in \mathcal{O}(Y)$. Note that if f is open then we can define $\mathbf{Sh}(f)^*(P)(W) = P(f(W))$ for each $W \in \mathcal{O}(X)$. Then $\mathbf{Sh}(f)^*$ and $\mathbf{Sh}(f)_*$ are inverses and in particular adjoints. This is not the case in general, so we need to modify the definition of $\mathbf{Sh}(f)^*$ and we only get an adjunction. For constructing $\mathbf{Sh}(f)^*$ we define $\mathbf{Etale}(X) = \{f : E \to X | f \text{ is a local homomorphism}\}$. Between this etale spaces f gives a morphism $f^+ : \mathbf{Etale}(Y) \to \mathbf{Etale}(X)$ defined by pullback. With this we can construct

$$f^* : \mathbf{Sh}(Y) \xrightarrow{\Lambda} \mathbf{Etale}(Y) \xrightarrow{f^+} \mathbf{Etale}(X) \xrightarrow{\Gamma} \mathbf{Sh}(X)$$

that turns up to be the inverse image. This construction extends further, as the functors Λ and Γ constructed via the consideration of etales can be defined from presheaves:

$$\Gamma\Lambda : \mathbf{Set}^{\mathcal{O}(X)^{\mathrm{op}}} \xrightarrow{\Lambda} \mathbf{Etale}(X) \xrightarrow{\Gamma} \mathbf{Sh}(X)$$

giving the left adjoint to the inclusion $\mathbf{Sh}(X) \hookrightarrow \mathbf{Set}^{\mathcal{O}(X)^{\mathrm{op}}}$ and hence carrying each presheaf to its "closest" sheaf. This construction is known as sheafification and can be done not only for presheaves over a topological space but for any presheaf. We will use this adjunction later for seeing the existence of colimits in a Grothendieck topos.
The correspondence of continuous functions to geometric morphisms is a bijection if the spaces satisfy separability constraints, for example Hausdorff is enough, so it justifies the saying that geometric morphisms *generalize* continuous functions. The fully worked out version of this example can be found on II of [4].

3 Constructions on Grothendieck toposes

This section has two parts. Firstly, we will see that Grothendieck toposes are Elementary toposes by constructing the needed objects explicitly. At the end of the subsection we will show that Grothendieck toposes also have all (small) colimits. On the second part, we will talk about the lattices of subobjects in Grothendieck toposes, that will be important for interpreting logic within them.

3.1 Grothendieck toposes are elementary toposes

For seeing that Grothendieck toposes are elementary toposes we shall see that they are Cartesian closed categories, with finite limits and with a subobject classifier. We will give a short proof of them being Cartesian, sketch a proof of the existence of exponentials and give a complete proof on the construction of the subobject classifier, where we will be able to see how the definition of Grothendieck toposes gives us a rich structure. To end the subsection, we will show that Grothendieck toposes have all colimits.

We start with the finite limits:

Proposition 3.1 (Finite limits for Grothendieck topos)**.** *A Grothendieck topos has all small limits, so in particular has all finite limits.*

Proof. Note that the category sheaves can be seen as subcategories of $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$, the category of presheaves. Recall that all small limits exists in $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ and that they are computed pointwise. Then, we just need to prove that the a limit of sheaves is a sheaf itself.

Lemma 3.2. *Let $I \to \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ be a diagram of sheaves P_i. Then $\lim P_i$ is a sheaf.*

Proof. Take $P = \lim P_i$. Since limits are computed pointwise for each $A \in \mathcal{C}$ we have $P(A) = \lim P_i(A)$. If S is a cover of C we have an equalizer for each $i \in I$:

$$P_i(C) \longrightarrow \prod_{f\in S} P_i(\text{dom } f) \rightrightarrows \prod_{f,g} P_i(\text{dom } g)$$

Note that equalizers itself are limits, so we want to be able to "interchange" the order of limits. This is true in general. If a category that has all limits of shape D and all limits of shape D', then this limits commute. This can be seen identifying limits as right adjoints and using the result that asserts that right adjoints preserve limits (IX.2 in [3]). Then we get an equalizer

$$P(C) \longrightarrow \prod_{f\in S} P(\text{dom } f) \rightrightarrows \prod_{f,g} P(\text{dom } g)$$

so P is a sheaf. □

This ends the proof. □

Remark 3.3. The initial object 1 is a sheaf that just assigns $1(C) = \{*\}$ for all objects C. This is what we expected, as limits of presheaves are computed pointwise.

Proposition 3.4 (Exponentials for Grothendieck Topos)**.** *A Grothendieck topos has all exponentials.*

Proof. This proof is similar to the last one, but the process of checking that the constructed exponential is a sheaf is more involved. One can find a full proof at III.6 of [4]. We begin constructing exponentials in the presheaves category $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$.

Lemma 3.5. *The category $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ has exponentials. Given two presheaves Q, P their exponential is* *defined on objects as:*

$$Q^P(C) = \mathrm{Hom}(\mathbf{y}(C) \times P, Q)$$

where $\mathbf{y} : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ *is the Yoneda embedding* $\mathbf{y}(C) = \mathrm{Hom}(-, C)$.

Proof. The intuition for the definition comes from remembering that the exponential forces $\mathrm{Hom}(R \times P, Q) \simeq \mathrm{Hom}(R, Q^P)$. If R is a representable functor $R = \mathrm{Hom}(-, C) = \mathbf{y}C$, using that the Yoneda Lemma states $X(C) \simeq \mathrm{Hom}(\mathbf{y}(C), X)$, the equivalence reads $Q^P(C) \simeq \mathrm{Hom}(\mathbf{y}(C), Q^P) \simeq \mathrm{Hom}(\mathbf{y}(C) \times P, Q)$.
For proving that this is an exponential, we need to construct a natural transformation $e : Q^P \times P \to Q$. Let us take the definition pointwise as $e_C(\theta, y) = \theta_C(id_C, y)$. The naturality reads that for all $f : C \to D$ we have $f \circ \theta(1, y) = \theta(f, f \circ y)$ that is true by naturality of θ. Now, we have to check that for each natural transformation $\phi : R \times P \to Q$ there exists a unique natural transformation $\phi' : R \to \mathrm{Hom}(\mathbf{y}(-) \times P, Q)$ making the following diagram commute:

$$\begin{tikzcd}
R \times P \arrow[d, dashed, "\phi' \times id"'] \arrow[dr, "\phi"] & \\
\mathrm{Hom}(\mathbf{y}(-) \times P, Q) \times P \arrow[r, "e"'] & Q
\end{tikzcd}$$

We construct it pointwise: for each $C \in \mathcal{C}$ and $u \in RC$ we need a natural transformation $\phi'(u) : \mathrm{Hom}(-, C) \times P \to Q$. Again we have to take the definition pointwise: for each $D \in \mathcal{C}$, $f : D \to C$ and $x \in PD$ we define:

$$\begin{aligned} \phi'_C(u)_D : \mathrm{Hom}(D, C) \times PD &\to QD \\ (f, x) &\to \phi_D(R(f)(u), x) \end{aligned}$$

The naturality of ϕ' in D follows from the naturality of ϕ. The diagram reads for elements $(x, y) \in RD \times PD$:

$$e_D(\phi'(x), y) = \phi'(x)(id_D, y) = \phi_D(R(id_D)(x), y) = \phi_D(x, y)$$

So the diagram commutes. This together with naturality forces ϕ' to be the only choice, so Q^P is the exponential. $\square$

Given that we know how to compute exponentials for presheaves, we are left to show that this definition makes the exponentials of sheaves to be a sheaf again, i.e. we have to check the amalgamation property for Q^P if Q and P are sheaves. We will prove the uniqueness and refer the reader to III.6 of [4] for the proof of existence, that involves sheafification. Note that since elements τ of $Q^P(C)$ are natural transformations $\mathrm{Hom}(-, C) \times P \to Q$, we have

$$\tau(gh, P(h)(x)) = Q(h)(\tau(g, x))$$

for $g : D \to C, x \in P(D)$ and $h : E \to D$.
Suppose that τ and σ are elements of $Q^P(C)$ and $S \in J(C)$ is a sieve, such that $Q^P(f)(\tau) = Q^P(f)(\sigma)$ for all $f \in S$. Then for all $g : D \to C$ and $x \in P(D)$ we have that $\tau(fg, x) = \sigma(fg, x)$ so taking $g = id$ sets that $\tau(f, x) = \sigma(f, x)$.
Now, if $k : C' \to C$ for every $g \in k^*(S)$ and $x \in P(C')$ we have:

$$Q(g)(\tau(k, x)) = \tau(kg, P(g)(x)) = \sigma(kg, P(g)(x)) = Q(g)(\sigma(k, x))$$

since $k^*(S) \in J(C')$ by the stability axiom of Grothendieck topologies and Q is a sheaf we have that $\tau(k, x) = \sigma(k, x)$. But this applies to all k and x so in particular $\tau = \sigma$ so an amalgamation is unique if it exists. □

The last construction we will tackle is the subobject classifier. To do so, let us introduce the concept of closed sieve. We begin by thinking what the subobject classifier for sheaves on topological spaces should be. As the most "general" object, it is natural to consider the object Ω such that $\Omega(U) = \{V|V \text{ open subset of } U\}$, mapping each open set to all its open subsets. If we want a sieve, we can change V by the principal sieve $\downarrow(V) = \{W|W \subseteq V\}$. Then, we are mapping each open subset to all his principal sieves. Saying that a sieve S is principal is equivalent to saying that if there is a covering of U by elements of S then U is also in S. We can generalize this property for sieves that do not come from topological spaces with the concept of closed sieve:

Definition 3.6 (Closed sieve)**.** Given a site $(\mathcal{C}, J)$ a sieve S on A is closed if for all arrows $f : A' \to A$ we have:

$$f^*(S) \in J(A') \implies f \in S$$

Our subobject classifier will be the the sheaf Ω such that $\Omega(c) = \{S|S \text{ is closed sieve on c}\}$. We have to see that it is a sheaf, so we start seeing that it is a presheaf:

Lemma 3.7. *The assignation Ω such that $\Omega(c) = \{S|S \text{ is closed sieve on } c\}$ yields a functor $\Omega : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ defined on arrows $f : A \to B$ as $\Omega f : \Omega B \to \Omega A$ as $\Omega f(S) = f^*(S)$.*

Proof. We need to check that f^* sends closed sieves to closed sieves. Take $f : A \to B$ and S a closed sieve, so we have that for all $g : C \to B$, $g^*(S) \in J(C) \implies g \in S$. So for a $h : D \to A$ if we have that $h^*(f^*(S)) \in J(A)$ it is the same as saying $(fh)^*(S) \in J(A)$ so $fh \in S$ since S is closed. Then, by definition $h \in f^*(S)$, so $f^*(S)$ is closed, as we wanted.
The associativity follows from the definition of pullback sieve. □

Lemma 3.8. *The presheaf Ω is a sheaf.*

Proof. We have to check that there is one amalgamation for each matching family. We will work with a sieve $S \in J(C)$ and a matching family $X = \{x_f\}_{f \in S}$ with $x_f \in \Omega(D)$ for $f : D \to C$ and $g^*(x_f) = x_{fg}$.
First, let us begging seeing that there is at most one amalgamation. Suppose that there exists two closed sieves $M, N \in \Omega(C)$ such that $\Omega(f)(M) = f^*(M) = x_f = f^*(N) = \Omega(f)(N)$ for all $f \in S$. Note that if $f \in M$ we have that $f^*(M)$ is the maximal sieve m, containing all arrows with

codomain M, and that the converse is also true since if $id \in f^*(M)$ it means that $f \circ id = f \in M$. From that observation we have that $M \cap S = N \cap S$, since if $f \in S$ then $f \in M \iff f^*(M) = m = f^*(N) \iff f \in N$. Now take $f : A \to C \in M$, we have that $f^*(M \cap S) = f^*(M) \cap f^*(S) = m \cap f^*(S) \in J(A)$ where the last inclusion follows from $S \in J(C)$ and the stability axiom of a Grothendieck topology. Then $f^*(M \cap S) = f^*(N \cap S) \in J(A)$. We can apply the transitivity axiom of Grothendieck topologies for seeing that since for all $h \in f^*(N \cap S)$ we have $h^*(f^*(N)) = (fh)^*(N) = m \in J(\text{cod } h)$ since $fh \in N \cap S \subset N$ so we have that $f^*(N) \in J$. Since N is closed, this means that $f \in N$. Reversing M and N we have that $M = N$ as we wanted.
In order to see that there exists an amalgamation, we will construct it. Let us introduce a new definition:

Definition 3.9 (Closure of a sieve)**.** Given a sieve S on c we define its clousure $\overline{S} = \{h | \text{cod } h = c \text{ and } h^*(S) \in J(\text{dom } h)\}$. $\overline{S}$ is a closed sieve and we have for any g with cod $g = c$:

$$\overline{g^*(S)} = g^*(\overline{S})$$

Proof. For this definition to work, we need to see that $\overline{S}$ is a sieve. If h is such that $h^*(S) \in J$ then, for any g, $(hg)^*(S) = g^*(h^*(S)) \in J$ from the pullback stability axiom. We also see that $\overline{S}$ is the smallest closest sieve that contains S by definition.
Then, we have that $g^*(S) \subseteq g^*(\overline{S})$ implies $\overline{g^*(S)} \subseteq g^*(\overline{S})$. In the other direction we have that $f \in g^*(\overline{S})$ implies $gf \in \overline{S}$, so $(gf)^*(S) \in J(\text{dom } f)$ and $f^*(g^*(S)) \in J(\text{dom } f)$, that means that $f \in \overline{g^*(S)}$. □

Remember that we are given a matching family: $X = \{x_f\}_{f \in S}$ with $x_f \in \Omega(D)$ for $f : D \to C$ and $g^*(x_f) = x_{fg}$. We have to combine somehow the closed sieves x_f and the sieve S, to construct a closed sieve M in C such that $f^*(M) = x_f$. We do it considering the sieve

$$M = \{f \circ g | f \in S, g \in x_f\}$$

on C, it is a sieve as each x_f is. But since it does not need to be closed, we take $\overline{M}$ instead. We are left to show that $f^*(\overline{M}) = x_f$. We will start showing

$$f^*(M) = x_f.$$

We have that $x_f \subseteq f^*(M)$ for $g \in x_f$, so by definition of M we have $f \circ g \in M$. In the other direction, take $u \in f^*(M)$. This means that it exists $f' \in S, g \in x_{f'}$ such that $fu = f'g$. Then $x_{fu} = x_{f'g} \implies u^*(x_f) = g^*(x_{f'})$ from the matching condition. Since $g \in x_{f'}$ we have $f^*(x_{f'})$ is the maximal sieve. So $u \in x_f$. Finally, we have that

$$f^*(\overline{M}) = \overline{f^*(M)} = \overline{x_f} = x_f$$

from the property discussed in the definition of closure of a sieve and the fact that x_f is closed.
This proves that Ω is a sheaf. □

Once we have proven that Ω is a sheaf, we would like to see that it verifies the pullback diagram in the definition of a subobject classifier. We will also need to define the maps $\top$ and χ_S, so we need to briefly discuss the notion of subsheaf:

Definition 3.10. A subsheaf of a sheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf $G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ such that for all objects c we have $G(c) \subseteq F(c)$ and for all arrows $f : a \to b$ we have $G(f) = F(f)_{|G(a)}$. Then the inclusion $G \rightarrowtail F$ is a monic natural transformation.

Proposition 3.11 (Subobject classifier for Grothendieck topos)**.** *The sheaf Ω of all closed sieves is a subobject classifier for the category $\mathbf{Sh}(C, J)$.*

Proof. For the definition of $\top : 1 \to \Omega$, we have to choose a closed sheaf for each object C. It is not surprising that m_C, the maximal sieve on C, is closed, by vacuity of the condition. So $\top(C) = m_C$ for all C. Since $f^*(m_C) = m_B$ for all $f : B \to C$ it is a valid natural transformation.
Let F be a sheaf and G be a subsheaf. By definition of $\chi_G : F \to \Omega$, for each object C we need to construct a closed sieve using an element of FC and the subsheaf G. Let us take

$$(\chi_G)_C(x) = \{f : D \to C | F(f)(x) \in G(D)\}.$$

We have to check that it is a sieve and that it is closed.
To check that it is sieve, note that if $F(f)(x) \in G(D)$ then for a $g : E \to D$, $F(fg)(x) = F(g)(F(f)(x)) = G(g)(F(f)(x)) \in G(E)$, since $F(f)(x) \in G(D)$ allows us to change F for G and we are calculating everything inside G. It follows that, if $f \in S$ then $fg \in S$ so $(\chi_G)_C(x)$ is a sieve.
We can see that it is closed using the following lemma.

Lemma 3.12. *If G is a subpresheaf of F it is a subsheaf of F if and only if for each sieve $S \in J(C)$ and $e \in F(C)$ we have*

$$F(f)(e) \in G(D) \quad \forall f : D \to C, f \in S \implies e \in G(C)$$

Proof. A subpresheaf is a subsheaf if and only if for each matching family there exists an amalgamation. We know the amalgamation exists in F so the condition reduces to see if it lies in G, which is exactly the stated condition. $\square$

We use the reverse direction of the lemma. The condition of closeness says that for all $f : D \to C$,

$$f^*((\chi_G)_C(x)) \in J(D) \implies f \in (\chi_G)_C(x).$$

Note

$$f^*((\chi_G)_C(x)) = \{g : A \to D | fg \in (\chi_G)_C(x)) = \{g : A \to D | F(fg)(x) \in G(A) \in (\chi_G)_C(x)) =$$

$$= \{g : A \to D | F(g)(F(f)(x)) \in G(A)\}$$

so if $f^*((\chi_G)_C(x)) \in J(D)$ applying the previous lemma we get $F(f)(x) \in G(D)$ or in other words $f \in (\chi_G)_C(x)$ as we wanted.
We saw that χ_G is well defined on objects. Remember that an arrow $f : A \to B$ acted on closed sieves as $\Omega f : \Omega B \to \Omega A$ with $\Omega(f)(S) = f^*(S)$. Then, for checking we have a natural transformation, we need to verify that $(\chi_G)_B(f) = f^*(\chi_G)$. Let us do it pointwise: on the left hand side, $g : C \to A \in (\chi_G)_B(F(f)(x)) \iff F(fg)(x) \in G(C)$. On the right hand side, $g \in f^*(\chi_G(x)) \iff fg \in \chi_G(x) \iff F(fg)(x) \in G(C)$ so both sides coincide.
Finally, we need to verify that the square

$$\begin{tikzcd} G \arrow[r] \arrow[d, tail, "i"] & 1 \arrow[d, "m_*"'] \\ F \arrow[r, "\chi_G"] & \Omega \end{tikzcd}$$

is a pullback. We can do it, again, pointwise, so the condition will be the pullback condition for sets. For an object C the top-right arrows compose to m_C; the left-bottom composes to $(\chi_G)_C$ so we need to check that $G(C)$ is the largest set such that $x \in G(C) \iff (\chi_G)_C(x) = m_C \iff F(g)(x) \in G(A) \quad \forall g : A \to C$. The *only if* direction is clear. The *if* direction follows from taking $g = id : C \to C$, and this also shows that $G(C)$ is the largest set that fulfills the condition.
This ends the proof for the construction of the subobject classifier for Grothendieck toposes. □

Remark 3.13. Note that in the example **Set**, the subobject classifier only had two elements, namely *true* and *false*. In general, this is not the case, as there can be many closed sieves covering an object. This translates to the fact that the logic in an arbitrary Grothendieck topos is not classical logic, but intuitionistic logic, so we have many different levels of truth for each object.
A good example of this phenomena appears when we consider the topos **Set** $\times$ **Set**. The subobject classifier has 4 elements, given by the 2×2 table of possible truth values: for $S \subseteq X, S' \subseteq X'$ we have the characteristic function $(\phi_S : X \to \{0,1\}, \phi_{S'} : X' \to \{0,1\})$ that for each pair $(x, x') \in X \times X$ assigns the corresponding truth value $(x \in S, x' \in S')$.
In the case of sheaves over a topological space, for each open subset O the classifying object maps O to all its open subsets. So for $G \leq F$ the truthiness of an element x of $F(O)$ under χ_G is the largest open subset O' such that $x \in G(O')$. For instance, if $O = \mathbb{R}^n$, F is the sheaf of continuous functions and G the sheaf of derivable functions, for each continuous function $f : \mathbb{R}^n \to \mathbb{R}$ we have that $\chi_G(f)$ is the largest open set of $\mathbb{R}^n$ where it is derivable.

Theorem 3.14 (Grothendieck toposes are elementary toposes)**.** *A Grothendieck topos is an elementary topos.*

Proof. Follows from last three propositions. □

Remark 3.15. The reverse inclusion is not true, a counterexample is **FinSet**, as we already discussed.

In fact, Grothendieck toposes also have all small colimits. This will be relevant as some of the constructions that we will need for interpreting theories in a topos will involve colimits.

Theorem 3.16. *A Grothendieck topos has all small colimits.*

Proof. The proof is not as elementary as the limit construction in Proposition 3.1, since in general limits do not commute with colimits. This follows from the fact that the inclusion $i : \mathbf{Sh}(\mathcal{C}, J) \hookrightarrow \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ has a left adjoint $\mathbf{a} : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Sh}(\mathcal{C}, J)$ given by sheafification, as hinted in example 2.33. The construction for the adjoint can be found at III.5 of [4]. The functor $\mathbf{a}$ is constructed making quotients of matching families in order to force each family to have one and only one amalgamation. It takes a presheaf to the closest sheaf that *forgot* the non-local conditions. For example, the presheaf of bounded continuous functions is sent to the sheaf of continuous functions. From this description it should not be surprising that this makes $\mathbf{Sh}(\mathcal{C}, J)$ into a reflective subcategory of $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Assuming the existence of this adjunction to proof is similar to the construction of limits in 3.1. We know that in $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ all colimits exist as they are computed pointwise. Now a colimit colimit F_j in $\mathbf{Sh}(\mathcal{C}, J)$ is equal to $\mathbf{a}(\mathrm{colim}\ i(F_j))$. This holds since colim $i(F_j)$ exists since it is computed and $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ and $\mathbf{a}$ preserves colimits since it is a left adjoint and left adjoints preserve colimits. □

3.2 Logic in a topos

In order to interpret theories in a topos, we need to discuss the lattices of subobjects and quantifiers. We will see that the subobject lattices are Heyting algebras and that we can see quantifiers as adjunctions.

We start with subobjects lattices. Given an object C on a Grothendieck topos, we will see that the lattice of subsheaves $S \rightarrowtail C$ can be made into a Heyting algebra. This will allow us to interpret formulas in the next chapter, making use of intuitive logic rather than classical logic, as we expected from the remark 3.13.

Definition 3.17 (Order on subsheaves)**.** Given G and G' subsheaves of F we say that $G \leq_F G' \iff G(a) \subseteq G'(a)$ for all objects a of C.

Remark 3.18. The definition above is equivalent to the usual factorization definition of subobjects for a general category.

We want to show that the lattice of subobjects with this order is a Heyting algebra. We will be able to explicitly construct operations. We remind some basic definitions of Heyting algebras:

Definition 3.19 (Heyting algebra)**.** A Heyting algebra is a bounded lattice H with an extra binary operation $\Rightarrow$ such that $\forall a, b \in H$ we have that $a \Rightarrow b$ is the greatest element x such that $a \wedge x \leq b$. So we need to define binary operations $\wedge, \vee, \Rightarrow$ and elements $\top, \bot$ such that they respect the order and usual associativity, commutativity and absorption laws:

- $a \vee \bot = a$
- $a \wedge \top = a$
- $a \vee (a \wedge b) = a, a \wedge (a \vee b) = a$

In addition, a Heyting algebra is complete if for all elements $x \in H$ and $S \subseteq H$ we have $x \wedge \bigvee_{s\in S} s = \bigvee_{s\in S}(x \wedge s)$. It follows that in a complete Heyting algebra $(a \Rightarrow b) = \bigvee\{c|a \wedge c \leq b\}$.

Theorem 3.20. *For any Grothendieck topos $\mathcal{C}$ and any object E of $\mathcal{C}$, the poset $Sub(E)$ of all subobjects of E in $\mathcal{C}$ is a complete Heyting algebra.*

Proof. We will define $\wedge, \vee, \Rightarrow, \top$ and $\bot$ and check that they satisfy all needed identities.
Indeed, $\top$ is just $id : E \to E$.

The other easy definition is the infimum, or $\wedge$. We can take it pointwise and define $\bigwedge_i(A_i)(c) = \bigcap_i A_i(c)$ for an arbitrary family $\{A_i\}$. Remembering Lemma 3.12, it follows that the pointwise definition gives a sheaf. Commutativity and associativity laws follow from a direct check.

We could define the supremum from the infima as $\bigvee_i A_i = \bigwedge\{B|A_i \leq B, \forall i\}$, but there is a more explicit construction, again, pointwise. We take

$$e \in (\bigvee_i A_i)(c) \iff \{f : d \to c|E(f)(e) \in A_i(d) \text{ for some } i\} \in J(c).$$

We need to check that the definition above gives a subsheaf of E and that it is the smallest containing all A_i.
For seeing that $(\bigvee_i A_i)$ is a subpresheaf, note that if $e \in (\bigvee_i A_i)(c)$ we have that $e \in E(c)$. We need to see that if $e \in (\bigvee_i A_i)(c)$, for any $g : c' \to c$ the element $E(g)(e)$ is in $(\bigvee_i A_i)(c')$. Note that if $S = \{f : d \to c|E(f)(e) \in A_i(d) \text{ for some } i\}$ is in $J(c)$ then $g^*(S) \in J(c')$ by the transitivity condition on Grothendieck topologies. But $g^*(S) = \{h : d \to c'|E(gh)(e) \in A_i(d) \text{ for some } i\} = \{f : d \to c|E(h)(E(g)(e)) \in A_i(d) \text{ for some } i\}$ so $E(g)(e)$ satisfies the condition.
For checking that it is a sheaf we can apply Lemma 3.12. We need that for all sieve $S \in J(c)$, we have that if $E(f)(e) \in (\bigvee A_i)(d) \quad \forall (f : d \to c) \in S$ then $e \in (\bigvee A_i)(c)$. Translating it to our terms:

$$\{h : a \to d|E(fh)(e) \in A_i(d)\} \in J(d) \quad \forall f \in S \implies \{g : a \to c|E(g)(e) \in A_i(d)\} \in J(c)$$

. However, note that setting $R = \{g : a \to c|E(g)(e) \in A_i(d)\}$ we have that $\{h : A \to d|E(fh)(e) \in A_i(d)\} = f^*(R)$, so by the transitivity condition of Grothendieck topologies, the condition on the left hand implies that $R \in J(c)$ as we wanted. Finally, with the purpose of seeing that it is the smallest containing all A_i, suppose we only had one A_i. Then, the condition for $e \in (\bigvee_i A_i)(c)$ is again the same condition of being a presheaf of Lemma 3.12 and the proposed condition is the right generalization. Commutativity and associativity laws follow from a direct check. Absorption follows from the first characterization.

Having defined the supremum, we can see what $\bot$ should be, as it is $\bigvee \emptyset$. The condition states that $e \in \bot(c) \iff \emptyset \in J(c)$, so it is the sheaf that assigns m_c if the empty sieve is in $J(c)$ and the empty set otherwise. Note that in the first case $E(c)$ contains only one element, from the fact that there is a unique amalgamation.

We can now check the identity for completeness, and it will give us the existence for the $\Rightarrow$ operator, that we will also define explicitly. We need that $B \wedge \bigvee_i A_i = \bigvee_i (B \wedge A_i)$. The inclusion $\supseteq$ always holds. For the inclusion $B \wedge \bigvee_i A_i \subseteq \bigvee_i (B \wedge A_i)$ take $e \in E(c)$ such that $e \in B(c)$ and $e \in \bigvee_i (A_i)$, so $S = \{f : d \to c | E(f)(e) \in A_i(d) \text{ for some } A_i(d)\}$ is in $J(c)$. Then, for all $f \in S$ we have $E(f)(e) \in A_i(d) \cap B(d)$ for some i, as it is in $B(d)$ since $e \in B(c)$. Then the same pick of i (using the Axiom of Choice) for each f proves that $e \in (\bigvee_i B \wedge A_i)(c)$.

We know that $A \Rightarrow B = \bigvee \{C | A \wedge C \leq B\}$ but we provide a more explicit construction. Define $A \Rightarrow B$ pointwise:

$$e \in (A \Rightarrow B)(c) \iff S_e = \{f : d \to c | E(f)(e) \in A(d) \implies E(f)(e) \in B(d)\} \in J(c)$$

It is a subpresheaf $S_{fe} = f^*(S_e)$ and the stability axiom of Grothendieck topologies. For seeing that it is a sheaf we again use Lemma 3.12 and the transitivity axiom.
In order to check that this defines the operator $\Rightarrow$ we have to check that

$$U \leq (A \Rightarrow B) \iff U \wedge A \leq B$$

For the *only if* direction note that $e \in A(c)$ then $S_e = \{f : d \to c : E(f)(e) \in B(d)\} \in S$ as the first condition is always satisfied. Now if $e \in U \leq (A \Rightarrow B)$, by definition $S_e \in J(c)$ and by the amalgamation property $e \in B(c)$, so $U \wedge A \leq B$.
For the if direction, take $e \in U(c)$. Then for any $f : D \to C$ we have that $E(f)(e) \in U(d)$. So if $E(f)(e) \in A(d)$ by hypothesis $E(f)(e) \in B(d)$. Then $S_e = m_c \in J(c)$ and $U \leq (A \Rightarrow B)$. This finishes the proof for the construction of $A \Rightarrow B$. Since we have constructed all operators of the Heyting algebra we are finished. □

Remark 3.21. Infima and suprema can be seen as pushbacks and pushouts respectively. Since we only care about finite infimum, this is immediate from the definition for the infimum of two subobjects. For an arbitrary supremum the definition is equivalent to the smallest sheaf that makes the diagram for a wide pushout, just the generalization of pushout for arbitrary number of objects, commute.

The last logical construction that we need to discuss is how to tackle quantifiers. We define quantifiers as adjoints. We get the intuition for this characterization from the case in sets.
We want to see quantifiers not just as an expression that evaluates to true or false, but as map. We can construct such a map taking the case where we have a proposition $P(x, y)$ in two variables. If x lives in X and y in Y, we will identify P with a subset $S \subseteq X \times Y$ with $(x, y) \in S$ if and only if $P(x, y)$ holds. This allows us to define a maps between sets $\forall_p(S) = \{x | \forall y \in Y, (x, y) \in S\}$ and similarly $\exists_p(S) = \{x | \exists y \in Y, (x, y) \in S\}$. Note that if $S' \subseteq S$, the inclusions $\forall_p(S') \subseteq \forall_p(S)$ and $\exists_p(S') \subseteq \exists_p(S)$ hold. Hence, the assignations above give two functors $\forall_p, \exists_p : P(X \times Y) \to P(X)$

between the posets categories of subsets of $X \times Y$ and subsets of X. We have a natural functor $P(X \times Y) \to P(X)$ given by the projection, but its inverse $p^{-1} : P(X) \to P(X \times Y)$ defined as $p^{-1}(X') = \{(x, y) | x \in X'\}$ is also a functor between posets. These functors are adjoints in the following order

$$\exists_p \dashv p^{-1} \dashv \forall_p$$

Since the categories involved are posets for the adjunction being true we only need to check:

$$\exists_p S \subseteq T \iff S \subseteq p^{-1}T \quad \text{and} \quad p^{-1}T \subseteq S \iff T \subseteq \forall_p S.$$

Both equivalences are immediate. For the first one

$$\exists_p S \subseteq T \iff ((x, y) \in S \implies x \in T) \iff S \subseteq p^{-1}T$$

and for the second one

$$p^{-1}T \subseteq S \iff (x \in T \implies (\forall y, (x, y) \in S)) \iff T \subseteq \forall_p S$$

We can get the same adjunctions for any function not only the projection, as in the following theorem in a general Grothendieck topos. We will only make use of the functor $\exists_f$ in the next chapters.

Definition 3.22 (Pullback functor)**.** Given an arrow $f : A \to B$ we have a functor $f^* : Sub(B) \to Sub(A)$ defined for a subobject $B' \leq B$ following the pullback diagram:

$$\begin{tikzcd} f^*(B') \arrow[r] \arrow[d, tail] & B' \arrow[d, tail] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Theorem 3.23. *For any morphism of sheaves $f : A \to B$ in a Grothendieck topos $\mathcal{C}$, the pullback functor $f^* : Sub(B) \to Sub(A)$ has both a left adjoint $\exists_f : Sub(A) \to Sub(B)$ and a right adjoint $\forall_f : Sub(A) \to Sub(B)$.*

Proof. A full proof can be found on III.8 in [4]. We will only give the construction of $\exists_f$ as it is the one that we will use. As above, note first that the adjunction $\exists_f \dashv f^*$ is between posets, so it will be an adjuction iff

$$\exists_f(A') \leq B' \iff A' \leq f^*(B')$$

for all $A' \leq A$ and $B' \leq B$.
We define $\exists_f$ pointwise by, given $A' \leq A$, $e \in \mathcal{C}$ and $y \in B(e)$:

$$y \in \exists_f(A')(e) \iff \{g : e' \to e | \exists a \in A'(e'), f_{e'}(a) = B(g)(y)\} \in J(e)$$

We have to check that $\exists_f(A')$ is a subsheaf of B.
We can assert it is a subpresheaf since the sieve for $B(g)(e)$ is the pullback by g of the sieve of e, and it is on the topology by the stability axiom of Grothendieck topologies.
Moreover, it is a sheaf because the condition of Lemma 3.12 holds similarly to the cases in last proof by the transitivity axiom.
To prove the adjunction condition, we start by only if condition. Take $B' \leq B$ and suppose $\exists_f(A') \leq B'$. Then, by definition, for any $e \in E$ and $a \in A'(e)$ we have $f_e(a) \in \exists_f(A')(e)$ as the sieve of the condition is the maximum sieve taking the witness $B(g)(a)$ for each $g : e' \to e$ and using the naturality of f. Then $f_e(a) \in B'(e)$, or $a \in f_e^*(B') = f^*(B')(e)$, so $A' \leq f^*(B')$.
In the opposite direction, suppose $A' \leq f^*(B')$ and take $e \in E$ and $y \in \exists_f(A')(e)$. By definition, there exists a cover $S \in J(e)$ such that for each $(g : e' \to e) \in S$ we have $B(g)(y)$ is in the image of $f_{e'} : A'(e) \to B(e')$. Note that the hypothesis states that $A'(e') \subseteq f_{e'}^*(B')(e')$, so the image of $f_{e'}$ is in $B'(e')$. Then $B(g)(y) \in B'(e')$. By the amalgamation property we have that $y \in B'(e)$, and thus $\exists_f(A') \leq B'$. This finishes the proof. $\square$

We can characterize the existence of $\exists_f$ using limits and colimits. We do it providing a characterization of it using images and seeing that images can be constructed using limits and colimits.

Definition 3.24 (Images)**.** A category C with finite limits has images if for any morphism $f : A \to B$ there exists a subobject $Im(f) \leq B$ which is the least object through which f factors.

Proposition 3.25. *In a category with images, the left adjoint $\exists_f$ of the pullback functor $f^* : Sub(B) \to Sub(A)$, assigns to a subobject $m : A' \rightarrowtail A$ the image of the composite arrow $f \circ m$.*

Proof. As in the previous theorem we need to check that with this definition we have

$$\exists_f(A') \leq B' \iff A' \leq f^*(B')$$

holds for all $A' \leq A$ and $B' \leq B$. In our case this reads,

$$Im(f \circ m) \leq B' \iff A' \leq f^*(B')$$

holds for all $m : A' \rightarrowtail A$ and $B' \leq B$.
For the other direction we construct the commutative diagram:

$$\begin{tikzcd}
A' \arrow[r, two heads] \arrow[d, tail] \arrow[dd, tail, bend right, "m"'] & Im(f \circ m) \arrow[dd, tail, bend left] \\
f^*(B') \arrow[r] \arrow[d, tail] & B' \arrow[d, tail] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Note that by definition, $Im(f \circ m)$ is the smallest subobject of B for which $f \circ m$ factors, but it also factors via B', so we have an injective arrow $Im(f \circ m) \rightarrowtail B'$, as we wanted.
For the *only if* implication we have the similar commutative diagram:

$$\begin{tikzcd}
A' \arrow[r, two heads] \arrow[dd, bend right, tail, "m"'] & Im(f \circ m) \arrow[d, tail] \arrow[dd, bend left, tail] \\
f^*(B') \arrow[r] \arrow[d, tail] & B' \arrow[d, tail] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

For completing the diagram, we have to use the pullback universal property. Note that $f \circ m$ is the same as the composition via $Im(f \circ m)$ and B' from the commutativity of the subobjects morphisms. By the pullback property there exists $n : A' \to f^*(B')$, making the diagram still commute. Indeed n is injective since $n \circ g = n \circ h$ implies $m \circ g = m \circ h$ so $g = h$. This finishes the proof. □

Proposition 3.26. *If images are always regular monomorphisms, images can be constructed with pushouts and equalizers. In particular, a category with limits and colimits has images.*

Proof. Given $f : A \to B$ construct the pushout:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "i_1"] \\
B \arrow[r, "i_2"'] & B \sqcup_f B
\end{tikzcd}$$

and the equalizer:

$$\begin{tikzcd}
\mathrm{Im}\ f \arrow[r, tail, "m"'] & B \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & B \sqcup_f B
\end{tikzcd}$$

Note that since m is an equalizer and $i_i \circ f = i_2 \circ f$ we get a unique e such that $f = m \circ e$. Given another factorization, $f = m' \circ e'$ with m' being regular mono. Then, let $d, d' : B \to D$ such that m' is their equalizer. We have that $df = d'f$, so by the pushout diagram we have a unique $g : B \sqcup_f B \to D$ such that $d = gi_1$ and $d' = gi_2$. Since $i_1 m = i_2 m$ we have $dm = d'm$. By the equalizer property of m', there exists a unique $h : \mathrm{Im}\ f \to \mathrm{dom}\ m$ such that $m = m'h$ so m factorizes through m'. Then $m \leq m'$, as we wanted. □

4 Geometric theories

In this section we will define the syntax of geometric theories. It will be similar to the usual first-order logic studied in Model Theory, but will have infinitaries disjunctions. Then we will discuss how to interpret the syntax in a topos, taking advantage of the constructions that we asked a topos to have in order to mimic the way we interpreted first-order logic with sets.

We begin by describing the syntax that will be used for constructing our theories. A difference which diverges from the set-theoretic approach to a type-theory approach is the use of multi-typed signature.

Definition 4.1 (Signature)**.** A signature Σ consists of:

- A set Σ-types of types or sorts.
- A set of Σ-function symbols. Each symbol f consists of a finite lists of source types (maybe zero) and one output type. We write:

$$f : A_1 \times \cdots \times A_n \to B$$

  to write that the input types are $A_1, \ldots, A_n$ and the output is of type B.
- A set of Σ-relation symbols. Each symbol R consists of a finite lists of related types. We write:

$$R \to A_1 \times \cdots \times A_n$$

  to write that the types are $A_1, \ldots, A_n$.

The usual definitions follow. We use the classical notion of free variables.

Definition 4.2 (Term)**.** A term always has a type. Fixed a signature Σ, we write $t : A$ for saying that the term t has type A. A term is either defined as a variable of a certain type or as an application of a function symbols to previously defined terms. Given a function $f : A_1 \times \cdots \times A_n \to B$ and terms $a_1 : A_1, \ldots, a_n : A_n$, $f(a_1, \ldots, a_n)$ is a term of type B.

We will write $\{\mathbf{x}.t\}$ for a term t with a context of variables $\mathbf{x}$ which will always contain all free variables of t.

We will talk only about geometric formulae.

Definition 4.3 (Formula)**.** Given a signature Σ, the collection F of (geometric) formulae can be constructed inductively with rules:

- Relation: $R(t_1, ..., t_n)$ is in F for any terms $t_1 : A_1, \ldots, t_n : A_n$ and any relation symbol $R \to A_1 \times \cdots \times A_n$
- Equality: $(s = t)$ is in F for any terms s and t of the same type.
- Truth: $\top$ is in F
- False: $\bot$ is in F
- Existential quantification: $(\exists x)\phi$ is in F whenever ϕ is in F and x is a free variable in ϕ
- Binary conjunction: $(\phi \wedge \psi)$ is in F whenever ϕ and ψ are in F
- Binary disjunction: $(\phi \vee \psi)$ is in F whenever ϕ and ψ are in F

- Infinitary disjunction: $\bigvee_{i\in I}\phi_i$ is in F whenever I is a set, ϕ_i is in F for all $i \in I$ there only appear a finite number of free variables.

We will write $\{\mathbf{x}.\phi\}$ for a formula ϕ with a context of variables $\mathbf{x}$ which will always contain all free variables of ϕ.

All interpretations are the expected. We note that the rules for infinitiaries disjunctions are just the infinite equivalent to the distributive axiom:

$$\phi \wedge \bigvee_{i\in I}\psi_i \vdash \bigvee_{i\in I}(\phi \wedge \psi_i)$$

and the rule which says that from $\psi_i \vdash \phi$ for all $i \in I$ we can deduce $\bigvee_{i\in I}\psi_i \vdash \phi$.

Definition 4.4 (Sequent)**.** Given a signature Σ, a sequent is an expression $\phi \vdash_{\mathbf{x}} \psi$ where ϕ and ψ are formulae and $\mathbf{x}$ contains their free variables. We interpret it as: if ϕ holds then ψ also holds.

Definition 4.5 (Geometric theory)**.** Given a signature σ, a geometric theory is a set of sequents $\mathbb{T}$. The elements of $\mathbb{T}$ are called, non-logical, axioms.

Remark 4.6. The name geometric theory is not used because most of these theories arise from a *geometric* setting, but because they are preserved by geometric morphisms. The name geometric morphisms arose because they are a generalization of continuous functions, that are defined in topological settings and sometimes geometry and topology are used as synonyms.

Example 4.7. Usual algebraic structures as groups, commutative groups, rings or commutative rings are geometric, as algebraic theories are a subset of geometric theories. With the addition of existentials and infinitaries disjunctions we can construct some other interesting theories:

1. Local rings: these are rings where $0 \neq 1$ and for all x, y such that $x + y = 1$ we have that x or y are invertible. The geometric theory defining them is the usual ring theory with the addition of the sequents $(0 = 1) \vdash \bot$ and

$$\exists z.(x + y)z = 1 \vdash (\exists z.xz = 1) \vee (\exists z.yz = 1),$$

where we needed to use the existential quantification.

2. Fields: similar to previous example we can add the following sequent to the theory of rings.

$$\top \vdash (x = 0) \vee (\exists y.xy = 1)$$

3. Torsion abelian groups: the infinitaries disjunctions allow us to assert for each element x there exists a natural number n such that $n \cdot x = 0$ and thus,

$$\top \vdash \bigvee_{n\geq 1} n \cdot x = 0.$$

This can not be done without infinitaries disjunctions. Combining this example with the previous we get that the theory of fields with finite characteristic is also geometric.

4. Successor algebra: similar to previous but closer to foundations we can express the usual Peano axioms. For a type $\mathbb{N}$, a function $s : \mathbb{N} \to \mathbb{N}$ and a constant $0 : \mathbb{N}$ we add the following axioms

$$0 = s(n) \vdash \bot$$

$$s(n) = s(n') \vdash n = n'$$

$$\top \vdash \bigvee_{i \geq 0} n = \underbrace{s \circ \ldots \circ s}_{i}(0).$$

4.1 Interpreting theories in toposes

The next steps are to describe how to translate our syntax defined above into models in toposes. This is, how to interpret functions, relations and formulas into Σ-structures. In this section we will walk through the definitions of the previous sections and explain how to translate each one in toposes.

We will fix a signature Σ and a topos $\mathcal{C}$. For building a model of Σ in $\mathcal{C}$, the main idea is to assign an object in $\mathcal{C}$ to each type of Σ. Then, indeed functions will be arrows and relations will be subobjects, as it happens in **Set**.

Definition 4.8 (Σ-structure)**.** A Σ-structure M in a topos $\mathcal{C}$ will have:

- An object MA for each type A in Σ.
- An arrow $f : MA_1 \times \cdots \times MA_n \to MB$ for each function symbol $f : A_1 \times \cdots \times A_n \to B$ in Σ.
- A subobject $MR \rightarrowtail MA_1 \times \cdots \times MA_n$ for each relation symbol $R \to A_1 \times \cdots \times A_n$ in Σ.

Remark 4.9. Note that the above definition is valid since in a topos, as we already discussed, finite products always exists. The fact that the subobjects lattices are well-behaved will also be important later.

As usual, we can talk about Σ-structures morphisms:

Definition 4.10 (Σ-structure morphism)**.** Given structures M, N a Σ-structure morphism $h : M \to N$ is a collection of arrows $h_A : MA \to NA$ for each type in Σ such that make the definitions of function and relation symbols commute. For each function symbol $f : A_1 \times \cdots \times A_n \to B$

$$\begin{tikzcd} MA_1 \times \cdots \times MA_n \arrow[r, "Mf"] \arrow[d, "h_{A_1} \times \cdots \times h_{A_n}"] & MB \arrow[d, "h_B"] \\ NA_1 \times \cdots \times NA_n \arrow[r, "Nf"] & NB \end{tikzcd}$$

commutes. And for all each relation symbol $R \to A_1 \times \cdots \times A_n$

$$\begin{tikzcd}
MR \arrow[r, tail] \arrow[d] & MA_1 \times \cdots \times MA_n \arrow[d, "h_{A_1} \times \cdots \times h_{A_n}"] \\
NR \arrow[r, tail] & NA_1 \times \cdots \times NA_n
\end{tikzcd}$$

commutes.

Example 4.11. A functor F between toposes $\mathcal{C}$ and $\mathcal{D}$ that conserves finite products and monomorphisms induces a Σ-structure morphism for each model M in $\mathcal{C}$. This just defines a new model N in $\mathcal{D}$ defined pointwise as $N = FM$.

Now, given a structure, we want to be able to evaluate formulas on it, to see if they hold or not. An important observation is that all interpretation will happen in the same topos $\mathcal{C}$, and thus terms and formulas will be interpreted in the same context. In this sense, all our universe will be closed, as terms and logic will be *formed* of a unique *essence*, this is, objects and arrows in a certain topos. We start with the evaluation of terms, which their interpretation is straightforward from the sets case.

Definition 4.12 (Term interpretation)**.** For a term with context $\{\mathbf{x}.t\}$ and a model M, we will define an interpretation as an arrow: $\{\mathbf{x}.t\}_M : MA_1 \times \cdots \times MA_n \to B$ if $\mathbf{x} = (a_1 : A_1, \ldots, a_n : A_n)$ and $t : B$. We have two cases:

- Variable: if the term is a variable it needs to be $a_i : A_i$ for a unique a_i in $\mathbf{x}$. Then $\{\mathbf{x}.t\}_M = \pi_i$ is the usual projection on the i-th variable.

- Function evaluation: if the term is the evaluation of a function $t = f(t_1, \ldots, t_n)$ for some terms $t_1 : T_1, \ldots, t_m : T_m$ we define $\{\mathbf{x}.t\}_M$ as the following composition

$$MA_1 \times \cdots \times MA_n \xrightarrow{\langle \{\mathbf{x}.t_1\}_M, \ldots, \{\mathbf{x}.t_m\}_M \rangle} MT_1 \times \cdots \times MT_m \xrightarrow{Mf} MB.$$

Remark 4.13. Note that the above definition respects the inductive definition of terms, using the projection for variables, and that the construction only uses the existence of finite products.

Now we define the way of interpret formulae. We will use the properties discussed in section 3.2.

We have to think about an interpretation with context $(a_1 : A_1, \ldots, a_n : A_n)$ as a subobject of $A_1 \times \cdots \times A_n$ that says in which "subset" the formula holds. Then, the interpretation of $\top, \bot$, $\phi \cup \psi$ and $\phi \cap \psi$ follows from the discussion on the Heyting algebra on subobjects. The case where the formula $\{\mathbf{x}.R(b_1, \ldots, b_m)\}$ is a relation can be seen to be a pullback. Thinking about the case **Set** we have a diagram:

$$\begin{tikzcd}
\{\mathbf{x}.R(b_1,\ldots,b_m)\}_M \arrow[d, tail, "\subseteq"'] & & R \arrow[d, tail, "\subseteq"] \\
A_1 \times \cdots \times A_n \arrow[rr, "f"'] & & B_1 \times \cdots \times B_m
\end{tikzcd}$$

Where f maps $\mathbf{x}$ to $\mathbf{b}$. It follows that the top arrow has to be the composition taking the inverse $R \subseteq B_1 \times \cdots \times B_m$ as the last step. Thus, we want $\{\mathbf{x}.R(b_1,\ldots,b_m)\}_M$ to be the biggest subobject such that the inverse is defined. This is exactly the universal property of a pullback.

The equality $t = s$ can be constructed similarly, thinking about it as the relation $id \times id : B \rightarrowtail B \times B$ with the bottom arrow being $\{\mathbf{x}.t\}_M \times \{\mathbf{x}.s\}_M$. This is another characterization for the equalizer of $\{\mathbf{x}.t\}_M$ and $\{\mathbf{x}.s\}_M$, this was expected as we saw that equalizers captured the notion of equality, as the name suggests.

The last case is the existential formulas, defined using the adjoint functor $\exists_f$ discussed in Section 3.2.

Definition 4.14 (Formula interpretation)**.** For a formula with context $\{\mathbf{x}.\phi\}$ and a model M we will define an interpretation as subobject:

$$\{\mathbf{x}.\phi\}_M : \{\mathbf{x}.\phi\}_M \rightarrowtail MA_1 \times \cdots \times MA_n$$

if $\mathbf{x} = (a_1 : A_1, \ldots, a_n : A_n)$. Note that we overload the notion of the subobject arrow and codomain. We have several cases:

- Relation: if $\phi = R(t_1, \ldots, t_n)$ and $\{\mathbf{x}.f\}_M : MA_1 \times \cdots \times MA_n \to B_1 \times \cdots \times B_m$ such that $f_M(\mathbf{x}) = \mathbf{t}$ then $\{\mathbf{x}.\phi\}_M$ is the pullback:

$$\begin{tikzcd}
\{\mathbf{x}.\phi\}_M \arrow[rr] \arrow[d, tail] & & MR \arrow[d, tail] \\
A_1 \times \cdots \times A_n \arrow[rr, "f_M"'] & & B_1 \times \cdots \times B_m
\end{tikzcd}$$

- Equality: if $\phi = (s = t)$ and $\{\mathbf{x}.f_M\} : MA_1 \times \cdots \times MA_n \to B$ such that $f_M(\mathbf{x}) = t$ and $\{\mathbf{x}.g_M\} : MA_1 \times \cdots \times MA_n \to B$ such that $g_M(\mathbf{x}) = s$ then $\{\mathbf{x}.\phi\}_M$ is the equalizer:

$$\begin{tikzcd}
\{\mathbf{x}.\phi\}_M \arrow[r, tail] & A_1 \times \cdots \times A_n \arrow[r, shift left, "f_M"] \arrow[r, shift right, "g_M"'] & B
\end{tikzcd}$$

- Truth: $\{\mathbf{x}.\top\}_M$ is the identity morphism on the object $A_1 \times \cdots \times A_n$, thought as $\{\mathbf{x}.\top\}_M : \{\mathbf{x}.\top\}_M \rightarrowtail A_1 \times \cdots \times A_n$.

- False: $\bot$ is the bottom subobject of the Heyting algebra on subobjects of $A_1 \times \cdots \times A_n$.

- Conjunction and disjunctions: are defined as in the Heyting algebra of subobjects of $A_1 \times \cdots \times A_n$.
- Existential quantification: If $\phi = (\exists y)\psi(\mathbf{x}, y)$ then $\{\mathbf{x}.\phi\}_M = \exists_p(\{\mathbf{x}, y.\psi\}_M)$, where p is the projection $MA_1 \times \cdots \times MA_n \times B \to MA_1 \times \cdots \times MA_n$ with $y : B$.

The last thing we constructed were sequents. From the previous discussion it should be clear that we should define a sequent $\sigma \equiv \phi \vdash_{\mathbf{x}} \psi$ to hold if and only if $\{\mathbf{x}.\phi\} \leq \{\mathbf{x}.\psi\}$ in the corresponding subobject lattice, this is the x for which ϕ holds are a "subset" of the x where ψ holds.

Definition 4.15 (Sequent interpretation)**.** Given a model M a sequent $\sigma \equiv \phi \vdash_{\mathbf{x}} \psi$ is satisfied in M if $\{\mathbf{x}.\phi\} \leq \{\mathbf{x}.\psi\}$ as subobjects of $MA_1 \times \cdots \times MA_n$ where $\mathbf{x} = (a_1 : A_1, \ldots, a_n : A_n)$. Then we write $M \vDash \sigma$.

Remark 4.16. Notice that a sequent has to be interpreted as $\forall x_1, \ldots, x_n(\phi \implies \psi)$ where ϕ and ψ are geometric formulas and $x_i : A_i$.

Definition 4.17 (Theory satisfability)**.** We say that a geometric theory $\mathbb{T}$ is satisfied in a model M if all axioms of $\mathbb{T}$ are satisfied in M. We call M a model of $\mathbb{T}$.

To end this section we have to talk about which morphisms preserve theories. This will be the central concept to define a classifying topos in the next section.

Definition 4.18 (Category of models)**.** Given a theory $\mathbb{T}$ and a topos C we define the category $\mathbf{Mod}(\mathbb{T}, C)$ with objects models of $\mathbb{T}$ and arrows Σ-structure morphisms.
A functor between $\mathbf{Mod}(\mathbb{T}, C)$ and $\mathbf{Mod}(\mathbb{T}, D)$ for two toposes C, D has to conserve the validity of the axioms of $\mathbb{T}$.

Given a functor $C \to D$ between toposes, we can ask if it lifts to a functor between $\mathbf{Mod}(T, C)$ and $\mathbf{Mod}(T, D)$. This is not the case in general. However, we already defined a class of functors that do so for any geometric theory $\mathbb{T}$. Those come from geometric morphism defined in subsection 2.3, which preserve the models thanks to the adjunction and the finite limits conditions in the definition.

Theorem 4.19. *Given a geometric morphism $f : D \to C$ and a geometric theory $\mathbb{T}$, the inverse image f^* provides a functor $f^* : \mathbf{Mod}(\mathbb{T}, C) \to \mathbf{Mod}(\mathbb{T}, D)$.*

Proof. Take f^* to be the pointwise application of the inverse image of f, namely f^*. Since it preserves finite limits and monomorphisms (since it preserves pullbacks) it preserves interpretations. Since it has a right adjoint it preserves all colimits.
For a model M in C, we get an interpretation of each formula $\{\mathbf{x}.\phi\}$, $\{\mathbf{x}.\phi\}^M \leq X_1^M \times \cdots \times X_n^M$ which gets mapped to $f^*(\{\mathbf{x}.\phi\}^M) \leq X_1^{f^*M} \times \cdots \times X_n^{f^*M}$. We could also get an interpretation by doing the calculations in D, this is as $\{\mathbf{x}.\phi\}^{f^*(M)} \leq X_1^{f^*M} \times \cdots \times X_n^{f^*M}$. We can check that both interpretations are equal by induction on the definition of the formula. All steps in our constructions of formulas are finite limits, that are preserved by hypothesis, except disjunctions and existential quantification. Remember that disjunctions (finite or arbitrary) can be described as pullbacks so they are preserved since (arbitrary) colimits are preserved. For existentials quantification, remember that the adjunction for $\exists_f$ can be constructed via images, and images can be constructed via pushouts and equalizers, so they are also preserved. $\square$

5 Classifying toposes

In this section we will start discussing the notion and importance of classifying the topos of a theory. Then, we will construct the syntactic category of a geometric theory and will sketch a proof stating that it is a classifying topos for that theory.

We saw that given a theory $\mathbb{T}$ and a topos E we have a category $\mathbf{Mod}(\mathbb{T}, E)$ of models of $\mathbb{T}$ in E. We can make this assignation into a functor $T_{\mathbb{T}}$ from the category $\mathbf{Topos}^{\mathrm{op}}$, the opposite of the category of all toposes with geometric morphisms as arrows, and the category $\mathbf{Mod}(\mathbb{T})$ with objects $\mathbf{Mod}(\mathbb{T}, E)$ for each topos E and functors between them as arrows. The definition of this functor on arrows follows from previous theorem 4.19. A representation for the functor $T_{\mathbb{T}}$ would be a topos $\mathcal{E}_{\mathbb{T}}$ such that $T_{\mathbb{T}} \simeq \mathrm{Hom}_{\mathbf{Topos}}(-, \mathcal{E}_{\mathbb{T}}) = \mathbf{Geom}(-, \mathcal{E}_{\mathbb{T}})$. In this sense, the topos $\mathcal{E}_{\mathbb{T}}$ represents all the models of the theory $\mathbb{T}$, or classifies it. This equivalence means that we can translate the task of understanding the models of a theory in a given category into understanding the geometric morphisms to a given category. This may or may not be more complicated, but is a question of a different nature.

Definition 5.1 (Classifying toposes)**.** Let $\mathbb{T}$ be a geometric theory over a given signature. A classifying topos of $\mathbb{T}$ is a Grothendieck topos $\mathcal{E}_{\mathbb{T}}$ such that for any Grothendieck topos E we have an equivalence of categories

$$\mathbf{Geom}(E, \mathcal{E}_{\mathbb{T}}) \simeq \mathbf{Mod}(\mathbb{T}, E)$$

that is natural in E.

The power of the classifying topos is better seen through the universal model. A universal model is a model in the classifying topos such that all models of $\mathbb{T}$ are created uniquely as the image of it through the inverse image of geometric morphism.

Definition 5.2 (Universal model)**.** A universal model of a geometric theory $\mathbb{T}$ is a model $U_{\mathbb{T}}$ of $\mathbb{T}$ in $\mathcal{E}_{\mathbb{T}}$ such that for any $\mathbb{T}$-model M in a Grothendieck topos F there exists a unique (up to isomorphism) geometric morphism $f_M : F \to \mathcal{E}_{\mathbb{T}}$ such that $f_M^*(U_{\mathbb{T}}) \simeq M$.

Remark 5.3. We can see $U_{\mathbb{T}}$ as the universal element of $T_{\mathbb{T}}$.

This two definitions are two ways of stating the same universal property.

Lemma 5.4. *A topos $\mathcal{E}_{\mathbb{T}}$ is a classifying topos for $\mathbb{T}$ if and only if there is a universal model $U_{\mathbb{T}}$ in $\mathcal{E}_{\mathbb{T}}$.*

Proof. We begin by expanding the naturality of E in the definition of a classifying topos. For all geometric morphisms $f : E \to F$ we get a commutative diagram:

$$\begin{tikzcd}
\mathbf{Geom}(F, \mathcal{E}_{\mathbb{T}}) \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "- \circ f"'] & \mathbf{Mod}(\mathbb{T}, F) \arrow[d, "f^*"] \\
\mathbf{Geom}(E, \mathcal{E}_{\mathbb{T}}) \arrow[r, leftrightarrow, "\simeq"'] & \mathbf{Mod}(\mathbb{T}, E)
\end{tikzcd}$$

For the direct implication suppose that $\mathcal{E}_{\mathbb{T}}$ is the classifying topos of $\mathbb{T}$ and take $F = \mathcal{E}_{\mathbb{T}}$. Write $U_{\mathbb{T}}$ for the model mapped to the identity $id : \mathcal{E}_{\mathbb{T}} \to \mathcal{E}_{\mathbb{T}}$ by the equivalence. For a model in $M \in \mathbf{Mod}(\mathbb{T}, E)$ take f_M the morphism mapped to M by the equivalence. Following id through the diagram we get in the top-right part $f^*(U_{\mathbb{T}})$ and in the left-bottom M so $U_{\mathbb{T}}$ is a universal model as we stated.
For the reverse implication we map each model M to the geometric morphism f_M such that $f_M^*(U_{\mathbb{T}}) = M$. Then in the naturality square for a geometric morphism g in the top-right part we get $f^*(g^*(U_{\mathbb{T}}))$ and in the left-bottom part $(g \circ f)^*(U_{\mathbb{T}})$ so they commute as we wanted to see. $\square$

Remark 5.5. The universal model is the minimal model of $\mathbb{T}$ in the sense that the formula $\forall x(\phi(x) \to \psi(x))$ holds in $U_{\mathbb{T}}$ if and only if it holds in every model of $\mathbb{T}$. The proof can be found in X.7 of [4].

5.1 Syntactic category

In order to construct a classifying topos for a theory $\mathbb{T}$, we need to construct a site $(C_{\mathbb{T}}, J_T)$ only from the syntactic information that we have. It is natural to consider objects in $C_{\mathbb{T}}$ as formulae with a context $\{\mathbf{x}.\phi\}$ (up to renaming variables). Then, we want to define arrows between formulas $\{\mathbf{x}.\phi\}$ and $\{\mathbf{y}.\psi\}$. We could try take an arrow if and only if $\phi \vdash \psi$, but since contexts do not need to match, we can not do it directly. For addressing it, we can think about the graph of a function. Given a function $f : X \to Y$, we can express it as a monomorphism $(1, f) : X \to X \times Y$. For any $X' \simeq X$ with isomorphism $\alpha : X' \rightarrowtail X$ we get a pair of diagrams:

$$\begin{tikzcd} X' \arrow[r, "\alpha"] \arrow[dr, "f'"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \qquad \mathrm{Graph}(f) : S \xrightarrow{(\alpha, f')} X \times Y$$

Now, S, as a subobject of $X \times Y$, has three important properties:

- It is exhaustive in X, so for all $x \in X$ there exists $y \in Y$ such that $(x, y) \in S$.
- It is the graph of f, so $(x, f(x)) \in S$.
- It is injective in the sense that if $(x, y) \in S$ and $(x, y') \in S$ then $y = y'$.

For an arrow $\theta : \{\mathbf{x}.\phi\} \to \{\mathbf{y}.\psi\}$ we will ask for the equivalent of the conditions above. So $\theta(\mathbf{x}, \mathbf{y})$ will be a equivalence class of formulaes with context $(\mathbf{x}, \mathbf{y})$ such that

- $\phi \vdash_{\mathbf{x}} (\exists \mathbf{y})\theta$
- $\theta \vdash_{\mathbf{x},\mathbf{y}} \phi \wedge \psi,$ and

- $\theta \wedge \theta[\mathbf{z}/\mathbf{y}] \vdash_{\mathbf{x},\mathbf{y},\mathbf{z}} \mathbf{y} = \mathbf{z}$

The composition can be defined as the equivalent to the graph of the composition of two functions, and identity as the equivalent to the graph of the identity.

Definition 5.6 (α-equivalence)**.** We say that $\phi(\mathbf{x})$ and $\psi(\mathbf{y})$ are α-equivalent if $\psi(\mathbf{y})$ is obtained from $\phi(\mathbf{x})$ by renaming free and bounded variables, in particular replacing every free occurrence of x_i in ϕ by y_i. We write $\{\mathbf{x}.\phi\}$ for the α-equivalence class of the formula-in-context $\phi(\mathbf{x})$.

Remark 5.7. Note that the types of $\mathbf{x}$ and $\mathbf{y}$ have to coincide in the definition above.

Definition 5.8 (Syntactic category for a geometric theory)**.** Let $\mathbb{T}$ be a geometric theory over a signature Σ. The syntactic category $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ has as objects the α-equivalence classes of geometric formulae-in-context $\{\mathbf{x}.\phi\}$ over Σ and as arrows $\{\mathbf{x}.\phi\} \to \{\mathbf{y}.\psi\}$ (taking disjoint contexts $\mathbf{x}$ and $\mathbf{y}$) the $\mathbb{T}$-provable-equivalence classes $[\theta]$ of geometric formulae $\theta(\mathbf{x}, \mathbf{y})$ such that the sequents:

- $\phi \vdash_{\mathbf{x}} (\exists \mathbf{y})\theta$
- $\theta \vdash_{\mathbf{x},\mathbf{y}} \phi \wedge \psi$, and
- $\theta \wedge \theta[\mathbf{y}'/\mathbf{y}] \vdash_{\mathbf{x},\mathbf{y},\mathbf{y}'} \mathbf{y} = \mathbf{y}'$

are provable in $\mathbb{T}$. The composite of two arrows

$$\{\mathbf{x}.\phi\} \xrightarrow{[\theta]} \{\mathbf{y}.\psi\} \xrightarrow{[\gamma]} \{\mathbf{z}.\chi\}$$

is defined as the $\mathbb{T}$-provable-equivalence class of the formula $(\exists \mathbf{y})(\theta \wedge \gamma)$. The identity arrow on an object $\{\mathbf{x}.\phi\}$ is the arrow $\{\mathbf{x}.\phi\} \xrightarrow{[\phi \wedge \mathbf{x}' = \mathbf{x}]} \{\mathbf{x}'.\phi[\mathbf{x}'/\mathbf{x}]\}$

This category is not yet a topos, but it has enough structure to interpret a geometric theory on it and it is related to $\mathbb{T}$ to the point of being a "quasi" classifying topos. The first equivalence that we will state using the syntactic category will use geometric functors, which are functors between categories that preserve geometric theories. Then, the models of $\mathbb{T}$ inside a topos will be in bijection with the functors from $\mathcal{C}_{\mathbb{T}}$.

Definition 5.9 (Geometric category)**.** A category C is geometric if we can interpret a geometric theory in it, as described in previous section. It needs to have finite limits, images that are stable under pullback and arbitrary unions of subobjects stable under pullback.

Remark 5.10. In Section 3 we saw that a Grothendieck topos was a geometric category.

Definition 5.11 (Geometric functor)**.** A geometric functor $F : C \to E$ between geometric categories C and E is a functor that preserves geometric constructions (as defined above). We call $\mathbf{GeomFunc}(C, E)$ the category with objects geometric functors between C and E and natural transformations between them as arrows.

Theorem 5.12. *For any geometric theory $\mathbb{T}$, $\mathcal{C}_{\mathbb{T}}$ is a geometric category.*

Proof. We will only describe the constructions without proof. The full proof can be found on D1.4.10 of [2].
For showing that it has all finite limits, it is enough to show it has finite products and equalizers. The terminal object is $\{\emptyset.\top\}$ as for any other object $\{\mathbf{x}.\phi\}$ an arrow θ has to satisfy $\theta \vdash \phi$ and $\phi \vdash \theta$, hence the $\mathbb{T}$-provable class of ϕ is the unique arrow.
The product of $\{\mathbf{x}.\phi\}$ and $\{\mathbf{y}.\psi\}$ is $\{\mathbf{x},\mathbf{y}.\phi\wedge\psi\}$ with projections $[\phi\wedge\psi\wedge\mathbf{x}=\mathbf{x}']$ and $[\phi\wedge\psi\wedge\mathbf{y}=\mathbf{y}'\}$. Given morphisms $[\theta] : \{\mathbf{z}.\chi\} \to \{\mathbf{x}.\phi\}$ and $[\gamma] : \{\mathbf{z}'.\chi\} \to \{\mathbf{y}.\psi\}$, the product morphism is the class $[\chi \wedge \psi]$.
The equalizer of a pair $[\theta], [\gamma] : \{\mathbf{x}.\phi\} \to \{\mathbf{y}.\psi\}$ is the object $\{\mathbf{x}'.\exists y\theta \wedge \gamma\}$ with morphism the class of $\exists y\theta \wedge \gamma \wedge (\mathbf{x} = \mathbf{x}')$.
The subobjects constructions correspond to the equivalent operations on formulas.
Finally, images are provided by the existence of epi-mono factorizations. For any arrow $[\theta(\mathbf{x}.\mathbf{y})]$: $\{\mathbf{x}.\phi\} \to \{\mathbf{y}.\psi\}$ we get a factorization $\{\mathbf{x}.\phi\} \twoheadrightarrow \{\mathbf{y}.\exists\mathbf{x}(\theta(\mathbf{x},\mathbf{y}))\} \rightarrowtail \{\mathbf{y}.\psi\}$, so $\{\mathbf{y}.\exists\mathbf{x}(\theta(\mathbf{x},\mathbf{y}))\}$ is the image of $[\theta]$. □

For seeing the wanted equivalence we focus on the universal model of $\mathbb{T}$ in $\mathcal{C}_{\mathbb{T}}$, that later will raise to be the universal model on the classifying topos of $\mathbb{T}$. Its construction is what we should expect, mapping the interpretation of each term to the corresponding construction that we defined in $\mathcal{C}_{\mathbb{T}}$.

Definition 5.13 (Universal model in $\mathcal{C}_{\mathbb{T}}$)**.** Given a geometric theory $\mathbb{T}$, the universal model of $\mathbb{T}$ in $\mathcal{C}_{\mathbb{T}}$ in the structure $M_{\mathbb{T}}$ that assigns

1. To a type A the object $\{x : A.\top\}$.
2. To a function symbol $f : A_1 \times \cdots \times A_n \to B$ the morphism
$$\{\mathbf{x}.\top\} \xrightarrow{[f(\mathbf{x})=y]} \{b.\top\}$$
3. To a relation symbol $R \to A_1 \times \cdots \times A_n$ the morphism
$$\{\mathbf{x}.R(\mathbf{x}\} \xrightarrow{[R(\mathbf{x})]} \{\mathbf{x}.\top\}$$

Note that the morphism are seen to exist by definition.

The fact that $M_{\mathbb{T}}$ is a structure that reflects the properties of $\mathbb{T}$ is well exposed by next lemma:

Lemma 5.14. *Let $\mathbb{T}$ be a geometric theory. Then*

1. *For any geometric formula-in-context $\{\mathbf{x}.\phi\}$, its interpretation $\{\mathbf{x}.\phi\}_{M_{\mathbb{T}}}$ in $M_{\mathbb{T}}$ is the subobject $[\phi] : \{\mathbf{x}.\phi\} \to \{\mathbf{x}.\top\}$.*
2. *A geometric sequent $\phi \vdash \psi$ is satisfied in $M_{\mathbb{T}}$ if and only if it is provable in $\mathbb{T}$.*

Proof. The first part is proven by induction on the structure of the formula.
The second immediately follows from the next proposition:

Proposition 5.15. *We have the following characterization for subobjects in $\mathcal{C}_{\mathbb{T}}$*

1. *A morphism $[\theta] : \{\mathbf{x}.\phi\} \to \{\mathbf{y}.\psi\}$ is an isomorphism in $\mathcal{C}_{\mathbb{T}}$ iff θ is an arrow from $\{\mathbf{y}.\psi\} \to \{\mathbf{x}.\phi\}$.*

2. *$[\theta]$ is a monomorphism in $\mathcal{C}_{\mathbb{T}}$ iff the sequent*

$$(\theta(\mathbf{x},\mathbf{y}) \wedge \theta(\mathbf{x}',\mathbf{y})) \vdash \mathbf{x} = \mathbf{x}'$$

   is provable in $\mathbb{T}$.

3. *Any subobject of $\{\mathbf{x}.\phi\}$ in $\mathcal{C}_{\mathbb{T}}$ is isomorphic to one of the form*

$$\{\mathbf{x}'.\psi\} \xrightarrow{\psi \wedge x' = x} \{\mathbf{x}.\phi\},$$

   where $\psi \vdash \phi$ is provable in $\mathbb{T}$. Moreover, for two such subobjects we have $\{\mathbf{x}.\chi\} \leq \{\mathbf{y}.\psi\}$ in $Sub(\{\mathbf{x}.\phi\})$ iff the sequent $\chi \vdash \psi$ is provable in $\mathbb{T}$.

Proof. 1. For the only if direction take $\tau : \{\mathbf{y}.\psi\} \to \{\mathbf{x}.\phi\}$ such that $[\tau] \circ [\theta] \equiv id \equiv \phi \wedge \mathbf{x} = \mathbf{x}'$. Then we have that $\exists y \theta(\mathbf{x},\mathbf{y}) \wedge \tau(\mathbf{y},\mathbf{x}') \dashv\vdash \phi(\mathbf{x}) \wedge \mathbf{x} = \mathbf{x}'$. We have the following sequents:

$$\theta(\mathbf{x},\mathbf{y}) \vdash \theta(\mathbf{x},\mathbf{y}) \wedge \theta(\mathbf{x}',\mathbf{y}) \wedge \mathbf{x} = \mathbf{x}' \vdash \phi(x) \wedge \theta(\mathbf{x}',\mathbf{y}) \wedge \mathbf{x} = \mathbf{x}' \vdash \exists \mathbf{y} \theta(\mathbf{x},\mathbf{y}') \wedge \tau(\mathbf{y}',\mathbf{x}) \wedge \theta(\mathbf{x},\mathbf{y})$$

where the second is by the condition of θ being a morphism and the third by hypothesis. Now by $\theta(\mathbf{x},\mathbf{y}') \wedge \theta(\mathbf{x},\mathbf{y})$ implies that $\mathbf{y}' = \mathbf{y}$ so we get $\tau(\mathbf{y},\mathbf{x})$. Similarly we can get $\tau(y,x) \vdash \theta(x,y)$ so we are finished.
For the if direction note that the condition translated to the graph of a function states that θ is an involution, so $\theta \circ \theta$ is the identity as we wanted.

---

2. Given a morphism $[\theta(\mathbf{x},\mathbf{y})] : \{\mathbf{x}.\phi\} \to \{\mathbf{y}.\psi\}$ the first part of his epi-mono factorization is $[\theta(\mathbf{x},\mathbf{y})] : \{\mathbf{x}.\phi\} \to \{\mathbf{y}.\exists \mathbf{x}' \theta(\mathbf{x}',\mathbf{y})\}$. By 1, this is a isomorphism if and only if $[\theta(\mathbf{y},\mathbf{x})] : \{\mathbf{y}.\exists \mathbf{x}' \theta(\mathbf{x}',\mathbf{y})\} \to \{\mathbf{x}.\phi\}$ is a morphism. The first two sequents are satisfied trivially and third is the stated condition.

---

3. By 2, any morphism of this form is monic. Given a monomorphism $[\theta(\mathbf{x},\mathbf{y})] : \{\mathbf{x}.\phi\} \to \{\mathbf{y}.\psi\}$ we construct the following diagram

$$\begin{tikzcd}
\{\mathbf{x}.\phi\} \arrow[rr, "\theta(\mathbf{x},\mathbf{y})"] \arrow[d, leftrightarrow, "\theta(\mathbf{x},\mathbf{y})"'] & & \{\mathbf{y}.\psi\} \\
\{\mathbf{y}'.\exists \mathbf{x}' \theta(\mathbf{x}',\mathbf{y})\} \arrow[urr, "y = y' \wedge \exists \mathbf{x}' \theta(\mathbf{x}',\mathbf{y})"'] & &
\end{tikzcd}$$

Then $y = y' \wedge \exists \mathbf{x}' \theta(\mathbf{x}', \mathbf{y})$ is easily seen to be a morphism and by 2 it is a monomorphism.
Also, following the same argument as in 2, we have that $\theta : \{\mathbf{x}.\phi\} \to \{\mathbf{y}'.\exists \mathbf{x}' \theta(\mathbf{x}', \mathbf{y})\}$ is an isomorphism.
For the second part note that the only possible morphism $\{\mathbf{x}.\chi\} \to \{\mathbf{y}.\psi\}$ is the class of $\chi \wedge \mathbf{x} = \mathbf{x}'$.
By the conditions on morphism this is a morphism if and only if $\chi \vdash \psi$ as we wanted. □

By the point 3 of the previous proposition and the definition of sequent satisfiability we are finished. □

This lemma states that $M_{\mathbb{T}}$ is a model of $\mathbb{T}$ as we should expect. Then, we can prove the first equivalence:

Theorem 5.16. *For any geometric theory $\mathbb{T}$ and geometric category E, we have an equivalence of categories $\boldsymbol{GeomFunc}(\mathcal{C}_{\mathbb{T}}, E) \simeq \boldsymbol{Mod}(\mathbb{T}, E)$ natural in E.*

Proof. Given a geometric functor $G : \mathcal{C}_{\mathbb{T}} \to E$ we map it to G. It is a model of $\mathbb{T}$ as $M_{\mathbb{T}}$ is a model and G preserves models as it is a geometric functor.
Given a model M of $\mathbb{T}$ in E we define the functor $F_M : C_{\mathbb{T}} \to E$ defined on objects $F_M(\{\mathbf{x}.\phi\}) = \{\mathbf{x}.\phi\}_M$. For an arrow $\theta : \{\mathbf{x}.\phi\} \to \{\mathbf{y}.\psi\}$, recall that we discussed that it defined a graph $\{\mathbf{x}, \mathbf{y}.(x, y)\}$ of an arrow f. We take $F_M(\theta) = f$. This defines a functor. The full proof of this being a geometric functor can be found at X.5 of [4]. The proof is similar to the proof of 5.12 in the sense that we can do the constructions following the syntactic properties of the theory.
Both compositions are identities basically by definition. The naturality follows by arrow chasing as we did in Lemma 5.4. □

5.2 Classifying topos existence for geometric theories

The last thing that we have to do is to lift $C_{\mathbb{T}}$ to a Grothendieck topos. We will then define a topology $J_{\mathbb{T}}$ on it giving us the topos $\mathbf{Sh}(C_{\mathbb{T}}, J_{\mathbb{T}})$. With this it suffices to show that $\mathbf{Geom}(E, \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})) \simeq \mathbf{GeomFunc}(\mathcal{C}_{\mathbb{T}}, E)$ and hence, that every geometric theory has a classifying topos. This equivalence comes from a more general result called Diaconescu's equivalence. We will not provide a full proof, that can be checked at [4].

Let us begging describing the topology $J_{\mathbb{T}}$. We define it from a more general construction on for Grothendieck topologies, but we remark an interesting fact relating to the theory $\mathbb{T}$.

Definition 5.17 (Covering family)**.** In a geometric category, a covering family is a family of arrows such that the union of their images is the maximal subobject.

Definition 5.18 (Syntactic topology)**.** The syntactic topology $J_{\mathbb{T}}$ is the Grothendieck topology on $\mathcal{C}_{\mathbb{T}}$ whose covering sieves are all those which contain small covering families.

Remark 5.19. A sieve $S = \{[\sigma^i(\mathbf{x^i}, \mathbf{y}) : \{\mathbf{x^i}.\phi\} \to \{\mathbf{y}.\psi\}] :\}_i$ on c in $J_{\mathbb{T}}(c)$ if and only if

$$\psi(\mathbf{y}) \vdash \bigvee_i \exists \mathbf{x^i} \sigma^i(\mathbf{x^i}, \mathbf{y})$$

This shows that it is a natural choice of cover, as we are that for each $\mathbf{y}$ that satisfies ψ we get a formula in the cover that is also satisfied.

Definition 5.20 (Syntactic site)**.** Given a geometric theory its syntactic site is the pair $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$.

With this definition we have everything needed to state the main theorem:

Theorem 5.21 (Classifying toposes exist for geometric theory)**.** *For any geometric theory $\mathbb{T}$, the topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ classifies $\mathbb{T}$.*

Proof. As we commented, the result follows from Diaunescu's equivalence. This uses the concept of flat functor, which we will not dig on.

Definition 5.22 (Flat functor)**.** A functor $A : C \to E$ from a small category with finite limits C to a Grothendieck topos E is said to be flat if $- \otimes A$ preserves finite limits.

Definition 5.23. Let E be a Grothendieck topos and (C, J) a site. A functor $F : C \to E$ is said to be J-continuous if it sends J-covering sieves to epimorphic families. We write $\mathbf{Flat}_J(C, E)$ for the category with objects flat J-continuous functors and arrows natural transformations between them.

Proposition 5.24 (Diaconescu's equivalence)**.** *For any essentially small site $(\mathcal{C}, J)$ and Grothendieck topos E, we have an equivalence of categories*

$$\boldsymbol{Geom}(E, \mathbf{Sh}(\mathcal{C}, J)) \simeq \boldsymbol{Flat}_J(\mathcal{C}, E)$$

natural in E.

Proof. The idea for the equivalence is to first consider the equivalence

$$\mathbf{Geom}(E, \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}) \simeq \mathbf{Flat}(\mathcal{C}, E)$$

where we drop the restrictions given by J. The intuition of why this hold comes from realizing that the finite limits condition is satisfied by hypothesis and that $- \otimes A$ having a left adjoint should not be very surprising given the categories are enough well-behaved, as in our case. In fact, the adjoint functor is $\mathrm{Hom}(A, -)$. The construction is as follows:

We can construct a geometric morphism from a flat functor. Given a flat functor A we construct a geometric morphism τ such that $\tau^* = - \otimes A$. This functor preserves finite limits since A does and it has a left adjoint $\tau_* = \mathrm{Hom}(A, -)$. In the other direction, given a geometric morphism $f : E \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ we have a natural functor $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \to E$ to consider for composing with the inverse image, this is the Yoneda embedding. So given f we construct a flat functor $A = f^* \circ \mathbf{y}$.

Now when we restrict from presheaves to sheaves we get the extra condition of continuity in the flat functors. A full proof can be checked at VII.7 of [4]. $\square$

Then, we are left to see that $\mathbf{Flat}_{J_\mathbb{T}}(\mathcal{C}_\mathbb{T}, E) \simeq \mathbf{GeomFunc}(\mathcal{C}_\mathbb{T}, E)$. In fact this is an equality:

Proposition 5.25. *Let E be a geometric category. A flat functor $F : \mathcal{C}_\mathbb{T} \to E$ is geometric if and only if it sends $J_\mathbb{T}$-covering sieves to epimorphic families.*

This finishes the sketch of the proof. □

This theorem also gives us a description of the universal model of $\mathbb{T}$, via the universal model $M_\mathbb{T}$ in $\mathcal{C}$.

Theorem 5.26. *The universal model $M_\mathbb{T}$ is raised via the equivalence into the universal model $U_\mathbb{T}$ of the theory $\mathbb{T}$ in $\mathbf{Sh}(\mathcal{C}_\mathbb{T}, J_\mathbb{T})$.*

Proof. We consider the following diagram:

$$\begin{tikzcd}
& \mathbf{GeomFunc}(\mathcal{C}_\mathbb{T}, \mathcal{C}_\mathbb{T}) \arrow[r, leftrightarrow] \arrow[d, "G_{id}\circ -"] & \mathbf{Mod}(\mathbb{T}, \mathcal{C}_\mathbb{T}) \arrow[d, "G_{id}(-)"] \\
\mathbf{Geom}(\mathbf{Sh}(\mathcal{C}, J), \mathbf{Sh}(\mathcal{C}, J)) \arrow[r, leftrightarrow] \arrow[d, "-\circ F_M"] & \mathbf{GeomFunc}(\mathcal{C}_\mathbb{T}, \mathbf{Sh}(\mathcal{C}, J)) \arrow[r, leftrightarrow] \arrow[d] & \mathbf{Mod}(\mathbb{T}, \mathbf{Sh}(\mathcal{C}, J)) \arrow[d, "F_M^*(-)"] \\
\mathbf{Geom}(E, \mathbf{Sh}(\mathcal{C}, J)) \arrow[r, leftrightarrow] & \mathbf{GeomFunc}(\mathcal{C}_\mathbb{T}, E) \arrow[r, leftrightarrow] & \mathbf{Mod}(\mathbb{T}, E)
\end{tikzcd}$$

where $G_{id} : \mathcal{C}_\mathbb{T} \to \mathbf{Sh}(\mathcal{C}, J)$ is the geometric functor corresponding to the identity $\mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{C}, J)$ under the equivalence and $F_M : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{C}, J)$ is the geometric morphism corresponding to a model M in E. This commutes by naturality. We can repeat the diagram with the elements that we get in each space, setting $M_\mathbb{T}$ in $\mathbf{Mod}(\mathbb{T}, \mathcal{C}_\mathbb{T})$:

$$\begin{tikzcd}
& id \arrow[r, leftrightarrow] \arrow[d] & M_\mathbb{T} \arrow[d] \\
id \arrow[r, leftrightarrow] \arrow[d] & G_{id} \arrow[r, leftrightarrow] \arrow[d] & G_{id}(M_\mathbb{T}) \arrow[d] \\
F_M \arrow[r, leftrightarrow] & F_M^* \circ G_{id} \arrow[r, leftrightarrow] & M = F_M^*(G_{id}(M_\mathbb{T}))
\end{tikzcd}$$

Thus, for each M there is a unique F_M such that $F_M(G_{id}(M_\mathbb{T})) = M$ so we have $G_{id}(M_\mathbb{T})$ is the universal model $U_\mathbb{T}$ we were searching. □

Finally, it is worth noticing that the reciprocal of the main theorem is true, with a theory constructed ad-hoc for each topos.

Theorem 5.27 (Reciprocal of main theorem 5.21)**.** *Every Grothendieck topos is the classifying topos of some geometric theory.*

Proof. The full proof can be found on Theorem 2.1.11 of [1]. The idea is to use Diaconescu's equivalence in "the reverse" direction used in the main theorem. Given a Grothendieck topos $\mathbf{Sh}(\mathcal{C}, J)$ we have an equivalence $\mathbf{Geom}(E, \mathbf{Sh}(\mathcal{C}, J)) \simeq \mathbf{Flat}_J(\mathcal{C}, E)$. Then, it is enough to construct a theory $\mathbb{T}^J_{\mathcal{C}}$ such that the models of $\mathbb{T}^J_{\mathcal{C}}$ in a Grothendieck topos can be identified with the J-continuous flat functors from $\mathcal{C}$ to E and the morphism of models with natural transformations. The signature of $\mathbb{T}^J_{\mathcal{C}}$ has a type $a_{\mathbb{T}}$ for each $a \in \text{ob}\,\mathcal{C}$ and a function symbol $f_{\mathbb{T}} : a_{\mathbb{T}} \to b_{\mathbb{T}}$ for each arrow $f : a \to b$. Thus, we can force that the interpretation gives exactly such a functor by asking functoriality, flatness and continuity. We can do it with geometric sequents. For example, for all arrow $f = g \circ h$ we add an axiom:

$$\top \vdash f_{\mathbb{T}}(x) = g_{\mathbb{T}}(h_{\mathbb{T}}(x))$$

or for continuity for each J-covering family $\{f_i : b_i \to a\}_I$ we ask:

$$\top \vdash_x \bigvee_I \exists y_i (f_i)_{\mathbb{T}}(y_i) = x$$

□

In general, concrete descriptions of the classifying toposes are difficult to get. We briefly describe some:

Example 5.28. We have the following pairs of theories and classifying toposes:

- For the theory of rings: the classifying topos for the theory of rings is the presheaf topos $\mathbf{Set}^{(\mathbf{fp-rings})}$ where $(\mathbf{fp} - \mathbf{rings})$ is the category of finitely presented rings. The universal object is $\text{Hom}(\mathbb{Z}[X], -)$ where $\mathbb{Z}[X]$ is the polynomial ring of one variable over the integers. The proof can be seen at VIII.5 of [4].

- For the theory of local rings: the classifying topos is the Zariski topos. The Zariski topos is the topos $\mathbf{Sh}(\mathbf{Rng}^{\text{op}}_{\text{f.g.}}, J)$ of sheaves on the opposite of the category $\mathbf{Rng}_{\text{f.g.}}$ of finitely generated rings with respect to the topology J on $Rng^{\text{op}}_{\text{f.g.}}$ defined as: for any cosieve S in $\mathbf{Rng}$ on an object A, $S \in J(A)$ if and only if S contains a finite family $\{\chi_i : A \to A[s_i^{-1}] | 1 \leq i \leq n\}$ of canonical inclusions $\chi_i : A \to A[s_i^{-1}]$ in $\mathbf{Rng}_{\text{f.g.}}$, where $\{s_1, \ldots, s_n\}$ is a set of elements of A which is not contained in any proper ideal of A. The proof can be found at VIII.6 of [4].

- For successor algebra: the classifying topos of the successor theory, which we previously saw, is $\mathbf{Set}$. Since $\mathbf{Set}$ is the terminal topos, it does not have subobjects. From this one we can deduce that this theory is complete in this context.

- For theories with empty signature: there are a family of theories also classified by $\mathbf{Set}$. Given a theory with empty signature, it is either complete or inconsistent. If it is complete, it is classified by **Set**, otherwise it is classified by the inconsistent topos. There a few theories with empty signature from category theory. Initial and terminal objects would be two examples.

6 Conclusion

In this essay we have constructed Grothendieck toposes and we saw that they are rich enough to interpret and classify interesting theories, namely geometric theories. From here we could say many more things.

The first thing would be talking about soundness and completeness for the pair geometric theories-Grothendieck toposes. As almost seen in the previous sections, both theorems hold.

A deeper follow-up could talk about the theory of bridges, as explained in [1]. The idea behind this theory is exploring what happens if two theories have the same classifying topos, although probably not the same universal object within it. This should hint that there is some relation between them. In a very categorical approach of seeing mathematics, we would say that they are the same thing seen through different lenses. If this happens for two theories $\mathbb{T}$ and $\mathbb{T}'$ we have an equivalence of categories $\mathbf{Mod}(\mathbb{T}, E) \simeq \mathbf{Mod}(\mathbb{T}', E)$ for all toposes E. This is the first interesting consequence and is easy to see. Deeper implications can argue about what categorical invariants exists in the sites of definition of the classifying topos. As we already hinted at the end of last section, just giving natural characterizations of the classifying topos, without resorting to the syntactic category, is a difficult task. Hence the general study of equivalences for usual theories is complicated. A long term project is to understand more about the sites of definitions of classifying toposes. Hopefully this will lead to make apparently distant theories related by their classifying toposes.

As a personal note, I think that this kind of mathematics are probably not very popular (likely because the lack of *practical* use), but deeply interesting for several reasons. It is a good way of seeing that the usual definitions and foundations of regular maths *could be different*, with the philosophical discussion that can follow. It is also very gratifying taking a concept and abstracting it to the absurd. In particular, the definition of Grothendieck topology and how with very little more you can define an structure, Grothendieck topos, which has a lot of properties that the *very natural* sets also satisfy, has been very enjoyable.

References

[1] O. Caramello. *Theories, Sites, Toposes.* Oxford University Press, 2017.

[2] P. Johnstone. *Sketches of an Elephant – A Topos Theory Compendium.* Oxford University Press, 2002.

[3] S. Mac Lane. *Categories for the Working Mathematician.* Springer, 1997.

[4] S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic.* Springer, 1994.

[5] E. Riehl. Sketches of an elephant: An introduction to topos theory. *https://math.jhu.edu/ eriehl/ct/sketches-of-an-elephant.pdf*, 2019.