# The Optimal Sampling Pattern for Linear Control Systems

Enrico Bini, *Senior Member, IEEE*, and Giuseppe M. Buttazzo

***Abstract*—In digital control systems, the state is sampled at given sampling instants and the input is kept constant between two consecutive instants. With the *optimal sampling problem*, we mean the selection of sampling instants and control inputs, such that a given function of the state and input is minimized. In this paper, we formulate the optimal sampling problem and we derive a necessary condition for the optimality of a set of sampling instants in the linear quadratic regulator problem. Since the numerical solution of the optimal sampling problem is very time consuming, we also propose a new *quantization-based* sampling strategy that is computationally tractable and capable of achieving near-optimal cost. Finally, and probably most interesting of all, we prove that the quantization-based sampling is optimal in first-order systems for a large number of samples. Experiments demonstrate that quantization-based sampling has near-optimal performance even when the system has a higher order. However, it is still an open question whether quantization-based sampling is asymptotically optimal in any case.**

***Index Terms*—Control design, least-squares approximations, linear feedback control systems, linear systems, optimal control, processor scheduling, real-time systems.**

## I. Introduction

**R**EDUCING the number of sampling instants in digital controllers may have a beneficial impact on many system features: the computing power required by the controller, the amount of needed communication bandwidth, the energy consumed by the controller, etc. In this paper, we investigate the effect of sampling on the optimal linear quadratic regulator (LQR) problem. We formulate the problem as follows:

$$\begin{aligned} &\mathsf{minimize}_{\bar{u}} \int_0^T (x'Qx + \bar{u}'R\bar{u})\,dt + x(T)'Sx(T) \\ &\mathsf{s.t.} \begin{cases} \dot{x} = Ax + B\bar{u} \\ x(0) = x_0 \end{cases} \end{aligned} \tag{1}$$

where $x$ and $\bar{u}$ are the state and input signals (moving over $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively), $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{n\times n}$ are matrices, with $Q$ and $S$ positive semidefinite, and $R$ is positive definite. (To denote the transpose of any matrix $M$, we use the compact Matlab-like notation $M'$.) The control input signal $\bar{u}$ is constrained to be piecewise constant

$$\bar{u}(t) = u_k \qquad \forall t \in [t_{k-1}, t_k)$$

with $0 = t_0 < t_1 < \cdots < t_N = T$. The sequence $\{t_0, t_1, \ldots, t_{N-1}, t_N\}$ is called *sampling pattern*, while $t_k$ are called *sampling instants*. Often, we represent a sampling pattern by the values that separates two consecutive instants that are called *interarrivals* $\tau_k$. The sampling instants and the interarrivals are related to one another through to the relations

$$\begin{cases} t_0 = 0 \\ t_k = \sum_{i=0}^{k-1} \tau_i \quad k \geq 1, \end{cases} \qquad \tau_k = t_{k+1} - t_k.$$

In periodic sampling, we have $\tau_k = \tau$ for all $k$, with $\tau = T/N$ the period of the sampling.

In our formulation, we intentionally ignore disturbances to the system. While accounting for disturbances would certainly make the problem more adherent to the reality, it would also prevent us from deriving the analytical results that we propose in this paper. The extension to the case with disturbances is left as future work.

In continuous-time systems, the optimal control $u$ that minimizes the cost in (1) can be found by solving the Riccati differential equation

$$\begin{cases} \dot{K} = \mathrm{KBR}^{-1}B'K - A'K - KA - Q \\ K(T) = S \end{cases} \tag{2}$$

and then setting the input $u$ as

$$u(t) = -R^{-1}B'K(t)x(t). \tag{3}$$

In this case, the achieved cost is

$$J_\infty = x_0'K(0)x_0.$$

For *given* sampling instants, the *optimal* values $u_k$ of the input that minimize the cost (1) can be analytically determined through the classical discretization process described below. If we set

$$\Phi(\tau) = e^{A\tau}, \qquad \bar{A}_k = \Phi(\tau_k) \tag{4}$$

$$\Gamma(\tau) = \int_0^\tau e^{A(\tau-t)}\,dt\,B, \qquad \bar{B}_k = \Gamma(\tau_k) \tag{5}$$

$$\bar{Q}(\tau) = \int_0^\tau \Phi'(t)Q\Phi(t)\,dt, \qquad \bar{Q}_k = \bar{Q}(\tau_k) \tag{6}$$

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E. Bini is with Lund University, Lund 22100, Sweden (e-mail: bini@control.lth.se).

G. M. Buttazzo is with the University of Pisa, Pisa 56127, Italy (e-mail: buttazzo@dm.unipi.it).

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$$\bar{R}(\tau) = \tau R + \int_0^\tau \Gamma'(t) Q \Gamma(t)\, dt, \qquad \bar{R}_k = \bar{R}(\tau_k) \quad (7)$$

$$\bar{P}(\tau) = \int_0^\tau \Phi(t)' Q \Gamma(t)\, dt, \qquad \bar{P}_k = \bar{P}(\tau_k) \quad (8)$$

then the problem of minimizing the cost (1) can be written as a discrete time-variant problem

$$\begin{cases} x_{k+1} = \bar{A}_k x_k + \bar{B}_k u_k \\ \text{given } x_0 \end{cases}$$

with the cost

$$J = x_N' S x_N + \sum_{k=0}^{N-1} (x_k' \bar{Q}_k x_k + u_k' \bar{R}_k u_k + 2x_k' \bar{P}_k u_k).$$

This problem is then solved using dynamic programming [1], [2]. The solution requires the backward recursive definition of the sequence of matrices $\bar{K}_k$

$$\begin{cases} \bar{K}_k = \hat{Q}_k - \hat{B}_k \hat{R}_k^{-1} \hat{B}_k' \\ \bar{K}_N = S \end{cases} \quad (9)$$

with $\hat{Q}_k$, $\hat{R}_k$, and $\hat{B}_k$, functions of $\bar{K}_{k+1}$ as well, defined by

$$\hat{Q}_k = \bar{Q}_k + \bar{A}_k' \bar{K}_{k+1} \bar{A}_k,$$
$$\hat{R}_k = \bar{R}_k + \bar{B}_k' \bar{K}_{k+1} \bar{B}_k,$$
$$\hat{B}_k = \bar{P}_k + \bar{A}_k' \bar{K}_{k+1} \bar{B}_k.$$

Then, the optimal input sequence $u_k$ is determined by

$$u_k = -\hat{R}_k^{-1} \hat{B}_k' x_k \quad (10)$$

with minimal cost equal to

$$J = x_0' \bar{K}_0 x_0. \quad (11)$$

Equation (10) enables computing the optimal input signal $u_k$ for *given sampling instants* $t_0, t_1, \ldots, t_N$. In fact, the optimal input sequence depends on $\bar{A}_k$, $\bar{B}_k$, $\bar{Q}_k$, $\bar{R}_k$, $\bar{P}_k$, $\hat{Q}_k$, $\hat{R}_k$, and $\hat{B}_k$ which are all functions of the intersample separations $\tau_k = t_{k+1} - t_k$. However, to our best knowledge, the problem of determining the optimal sampling pattern is still open.

This paper is organized as follows. In Sections II-B and II-C, we recall some natural sampling techniques. In Section III, we formulate the problem of optimal sampling and we report some results. Since solving the optimal sampling problem is very time consuming, in Section IV we propose a new sampling method that we call *quantization-based sampling* being related to quantization theory. In Section V, we prove that quantization-based sampling is optimal for first-order systems when the number $N$ of samples tends to $\infty$, while in Section VI, we investigate second-order systems.

### A. Related Works

Triggering the activation of controllers by *events*, rather than by time, is an attempt to reduce the number of sampling instants per time unit. A first example of event-based controller was proposed by Årzén [3]. In the self-triggered controller [4], the control task determines the next instant when it will be activated. Wang and Lemmon addressed self-triggered linear $\mathcal{H}_\infty$ controllers [5]. Self-triggered controllers have also been analyzed and proved stable also for state-dependent homogeneous systems and polynomial systems [6]. Very recently, Rabi *et al.* [7] described the optimal envelope around the state that should trigger a sampling instant. Similar to this paper, they consider the constraint of $N$ given samples over a finite time horizon. In our paper, however, we aim at establishing a connection between quantization-based sampling (properly defined later in Section IV) and optimal sampling in absence of disturbances.

The connections between "quantization" and the control have been studied deeply in the past. Often, the quantization was intended as the selection of the control input over a discrete set (rather than dense). Elia and Mitter [8] computed the optimal quantizer of the input, which was proved to be logarithmic. Xu and Cao [9] proposed a method to optimally design a control law that selects among a finite set of control inputs. The input is applied when the (scalar) state reaches a threshold. The number of thresholds is finite. In a different, although very related, research area, Baines [10] proposed algorithms to find the best fitting of any function $u$ with a piecewise linear function $\bar{u}$, which minimizes the $L^2$ norm of $u - \bar{u}$. However, in all of these works, the instants $t_1, \ldots, t_{N-1}$ at which the approximating function changes are not optimization variables, while, in this paper, we explicitly investigate the optimal selection of the sampling instants. Moreover, in our method, the control inputs $u_0, \ldots, u_{N-1}$ are not determined by a quantization procedure (as in [10]), but rather by the solution of an optimal discrete time-varying LQR problem.

Finally, a problem related to the one considered here was addressed by Kowalska and von Mohrenschildt [11], who proposed the variable time control (VTC). Similar to our approach, they also perform cost minimization over the sampling instants as well. However, the authors perform a linearization of the discrete-time system in a neighborhood of every sampling instant, then losing optimality.

The contributions of this paper are:

- the determination of a necessary condition for the optimality of a sampling pattern;
- the introduction of the *quantization-based sampling*, which is capable of providing cost very close to the optimal one with small computational effort;
- the proof that quantization-based sampling is optimal for first-order systems with a large number of samples;
- a numerical evaluation that shows that quantization-based sampling is near optimal for second- and higher-order systems as well.

## II. Sampling Methods

For any given sampling method, the temporal distribution of the sampling instants is evaluated by the sampling density, while the capacity to reduce the cost is measured by the normalized cost. Both metrics are formally defined below.

*Definition 1:* Given a problem, specified by $x_0$, $A$, $B$, $Q$, $R$, and $S$, an interval length $T$, and a number of samples $N$, we define the *sampling density* $\sigma_{N,\mathsf{m}} : [0,T] \to \mathbb{R}^+$ of any sampling method $\mathsf{m}$ as

$$\sigma_{N,\mathsf{m}}(t) = \frac{1}{N\tau_k} \qquad \forall t \in [t_k, t_{k+1}).$$

Notice that the sampling density is normalized since

$$\int_0^T \sigma_{N,\mathsf{m}}(t)\,dt = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \sigma_{N,\mathsf{m}}(t)\,dt = \sum_{k=0}^{N-1} \frac{1}{N\tau_k}\tau_k = 1.$$

To remove the dependency on $N$, we also define the following density.

*Definition 2:* Given a problem, specified by $x_0$, $A$, $B$, $Q$, $R$, and $S$, and an interval length $T$, we define the *asymptotic sampling density* $\sigma_{\mathsf{m}} : [0,T] \to \mathbb{R}^+$ of any sampling method $\mathsf{m}$ as

$$\sigma_{\mathsf{m}}(t) = \lim_{N\to\infty} \sigma_{N,\mathsf{m}}(t).$$

The density $\sigma_{\mathsf{m}}$ provides significant information only when $N$ is large compared to the size of the interval $[0,T]$ while, in reality, it is often more desirable to have a small $N$ to reduce the number of executions of the controller. Nonetheless, the asymptotic density can still provide informative results that can guide the design of more efficient sampling techniques, even when $N$ is not large.

While the density $\sigma_{\mathsf{m}}$ provides an indication of how samples are distributed over time, the quantity defined below returns a measure of the cost associated with any sampling method.

*Definition 3:* Given a problem, specified by $x_0$, $A$, $B$, $Q$, $R$, $S$, an interval length $T$, and a number of samples $N$, we define the *normalized cost* of any sampling method $\mathsf{m}$ as

$$c_{N,\mathsf{m}} = \frac{N^2}{T^2}\frac{J_{N,\mathsf{m}} - J_\infty}{J_\infty}$$

where $J_{N,\mathsf{m}}$ is the minimal cost of the sampling method $\mathsf{m}$ with $N$ samples, and $J_\infty$ is the minimal cost of the continuous-time systems.

The scaling factor $N^2/T^2$ is motivated by the observation (proved by Melzer and Kuo [12]) that the cost $J(\tau)$ of periodic sampling with period $\tau$ can be approximated by $J_\infty + k\tau^2 + o(\tau^2)$ for small values of the period $\tau$ (the Taylor expansion in a neighborhood of $\tau \to 0$ does not have a first-order term).

To remove the dependency on $N$, we also compute the limit of the normalized cost.

*Definition 4:* Given a problem, specified by $x_0$, $A$, $B$, $Q$, $R$, and $S$, an interval length $T$, we define the *asymptotic normalized cost* of any sampling method $\mathsf{m}$ as

$$c_{\mathsf{m}} = \lim_{N\to\infty} c_{N,\mathsf{m}}. \tag{12}$$

The asymptotic normalized cost (12) is also very convenient from an "engineering" point of view. In fact, it can be readily used to estimate the number of samples to achieve a bounded cost increase with respect to the continuous-time case. If for a given sampling method $\mathsf{m}$, we can tolerate, at most, a (small) factor $\epsilon$ of cost increase with regard to the continuous-time optimal controller, then

$$(1+\epsilon)J_\infty \geq J_{N,\mathsf{m}} = T^2\frac{c_{N,\mathsf{m}}J_\infty}{N^2} + J_\infty$$

from which we deduce

$$N \geq T\sqrt{\frac{c_{N,\mathsf{m}}}{\epsilon}} \approx T\sqrt{\frac{c_{\mathsf{m}}}{\epsilon}}. \tag{13}$$

Relation (13) constitutes a good hint for assigning the number of samples in a given interval.

Notice that the cost $J_{N,\mathsf{m}}$ of any method $\mathsf{m}$ with $N$ samples, can be written by Taylor expansion as

$$J_{N,\mathsf{m}} = J_\infty\left(1+T^2N^{-2}c_{N,\mathsf{m}}\right) = J_\infty\left(1+T^2N^{-2}c_{\mathsf{m}}\right) + o(N^{-2})$$

with the remainder $o(N^{-2})$ such that $\lim_{N\to\infty} N^2 o(N^{-2}) = 0$. Hence, for a small value of $N$, the approximation of (13) is tight as long as the remainder $o(N^{-2})$ is small. While we do not provide any analytical result in this sense, later in the experiments of Section VII, we show that this approximation is quite tight for all of the considered examples.

Below, we recall the characteristics of the existing sampling methods.

### A. Periodic Sampling

The simplest (and almost universally used) sampling method is the one obtained by dividing the interval $[0,T]$ in $N$ intervals of equal size; it corresponds to the choice $t_k = kT/N$ and it is called *periodic sampling* (abbreviated $\mathsf{per}$). We then have that all of the intersampling periods are equal: $\tau_k = \tau = T/N$ for all $k$, and the sampling density is, obviously, constant with

$$\sigma_{\mathsf{per},N}(t) = 1/T, \qquad \forall N.$$

For the periodic case, it is possible to determine analytically the asymptotic normalized cost $c_{\mathsf{per}}$. In 1971, Melzer and Kuo [12] approximated the solution $\bar{K}(\tau)$ of the Discrete Algebraic Riccati Equation to the second order of the sampling period $\tau$, in a neighborhood of $\tau = 0$. They showed that

$$\bar{K}(\tau) = K_\infty + X\frac{\tau^2}{2} + o(\tau^2)$$

with $K_\infty$ being the solution of the ARE of the continuous-time problem (2) and $X$ as the second-order derivative of $\bar{K}(\tau)$ in 0, that is, the solution of the following Lyapunov equation:

$$\mathcal{A}'X + X\mathcal{A} + \frac{1}{6}\mathcal{A}'K_\infty BR^{-1}B'K_\infty\mathcal{A} = 0 \tag{14}$$

with

$$\mathcal{A} = A - BR^{-1}B'K_\infty.$$

Melzer and Kuo [12] also proved that such a solution is positive semidefinite. Hence, the normalized asymptotic cost in the periodic case is

$$\begin{aligned} c_{\mathsf{per}} &= \lim_{N\to\infty}\frac{N^2}{T^2}\frac{x_0'(K_\infty + X\frac{T^2}{2N^2} + o(N^{-2}))x_0 - x_0'K_\infty x_0}{x_0'K_\infty x_0} \\ &= \frac{x_0'Xx_0}{2x_0'K_\infty x_0}. \end{aligned}$$

In the case of a first-order system ($n = 1$), assuming without loss of generality $B = R = 1$, the Lyapunov equation (14) has the solution $X = \frac{1}{12}(K_\infty - A)K_\infty^2$ and the ARE has the solution $K_\infty = A + \sqrt{A^2 + Q}$. Hence, the asymptotic cost becomes

$$c_{\mathsf{per}} = \frac{1}{24}(A\sqrt{A^2+Q} + A^2 + Q). \tag{15}$$

### B. Deterministic Lebesgue Sampling

As the number of samples $N \to \infty$, the optimal sampled-time control input $\bar{u}$ tends to the optimal continuous-time input $u$. It is then natural to set the sampling instants so that $\bar{u}$ approximates $u$ as close as possible.

A tentative sampling method, that we describe here for the only purpose of a comparison with our proposed sampling method, which will be described later in Section IV, is to set a threshold $\Delta$ on the optimal input $u$, so that after any sampling instant $t_k$, the next one $t_{k+1}$ is determined such that

$$\|u(t_{k+1}) - u(t_k)\| = \Delta$$

with $u$ being the optimal continuous-time input. Through this sampling rule, however, we cannot establish a clear relationship between $\Delta$ and the number $N$ of sampling instants in $[0, T]$. If we assume that the dimension of the input space is $m = 1$ (which allows us to replace the notation of the norm $\|\cdot\|$, with the notation of the absolute value $|\cdot|$), we can enforce a constant $|u(t_{k+1}) - u(t_k)|$ (except when $\dot{u}$ changes its sign in $(t_k, t_{k+1})$) and a given number $N$ of sampling instants in $[0, T]$ by the following rule:

$$\forall k = 0, \ldots, N-1, \quad \int_{t_k}^{t_{k+1}} |\dot{u}(t)|\, dt = \frac{1}{N}\int_0^T |\dot{u}(t)|\, dt, \tag{16}$$

where $u$ is the optimal continuous-time control input. We call this sampling method *deterministic Lebesgue sampling* (abbreviated as dls), because of its similarity to the (stochastic) Lebesgue sampling proposed by Åström and Bernhardsson [13], which applied an impulsive input at any instant when the state, affected by disturbances, was hitting a given threshold.

Following this sampling rule, by construction, the asymptotic density is:

$$\sigma_{\mathsf{dls}}(t) = \frac{|\dot{u}(t)|}{\int_0^T |\dot{u}(s)|\, ds}.$$

After the sampling instants $t_1, t_2, \ldots, t_{N-1}$ are determined according to (16), the values of the control input $u_k$ are optimally assigned according to (10).

For the dls method, we are unable to determine the normalized cost $c_{\mathsf{dls}}$, in general. However, in Section V, we analytically compute $c_{\mathsf{dls}}$ for first-order systems ($n = 1$).

## III. Optimal Sampling

We now investigate the optimal solution of the problem (1). Let us introduce a notation that is useful in the context of this section. For any vector $x \in \mathbb{R}^n$, let us denote by $\boldsymbol{x}$ the following vector in $\mathbb{R}^{\boldsymbol{n}}$, with $\boldsymbol{n} = \frac{n(n+1)}{2}$,

$$\boldsymbol{x} = [x_1^2,\ 2x_1x_2, \ldots,\ 2x_1x_n,\\ x_2^2,\ 2x_2x_3, \ldots,\ 2x_2x_n, \ldots,\ x_{n-1}^2,\ 2x_{n-1}x_n,\ x_n^2]'$$

and for any matrix $M \in \mathbb{R}^{n\times n}$, let us denote by $\boldsymbol{M} \in \mathbb{R}^{\boldsymbol{n}}$ the vector

$$\boldsymbol{M} = [M_{1,1},\ M_{1,2}, \ldots,\ M_{1,n},\ M_{2,2},\ M_{2,3}, \ldots,\ M_{2,n}, \ldots,\\ \ldots,\ M_{n-1,n-1},\ M_{n-1,n},\ M_{n,n}]'.$$

This notation allows writing the cost (11) as $J = \boldsymbol{x}_0'\bar{\boldsymbol{K}}_0$ and the Riccati recursive equation (9) as

$$\begin{cases} \bar{\boldsymbol{K}}_k = r(\tau_k, \bar{\boldsymbol{K}}_{k+1}) \\ \bar{\boldsymbol{K}}_N = \boldsymbol{S} \end{cases} \tag{17}$$

with $r : \mathbb{R} \times \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}^{\boldsymbol{n}}$ properly defined from (9).

Since we search for a stationary point of $J$, let us investigate the partial derivatives $\frac{\partial \bar{\boldsymbol{K}}_k}{\partial \tau_h}$. First

$$h < k \qquad \Rightarrow \qquad \frac{\partial \bar{\boldsymbol{K}}_k}{\partial \tau_h} = 0$$

because $\bar{K}_k$ depends only on the current and the future sampling intervals $\{\tau_k, \tau_{k+1}, \ldots, \tau_{N-1}\}$. Then, we have

$$\begin{cases} \dfrac{\partial \bar{\boldsymbol{K}}_k}{\partial \tau_k} = \dfrac{\partial r}{\partial \tau}(\tau_k, \bar{\boldsymbol{K}}_{k+1}) \\[2ex] \dfrac{\partial \bar{\boldsymbol{K}}_k}{\partial \tau_h} = \dfrac{\partial r}{\partial \bar{\boldsymbol{K}}}(\tau_k, \bar{\boldsymbol{K}}_{k+1})\dfrac{\partial \bar{\boldsymbol{K}}_{k+1}}{\partial \tau_h} \qquad h > k \end{cases}$$

from which it follows:

$$\frac{\partial \bar{\boldsymbol{K}}_k}{\partial \tau_h} = \left[\prod_{i=k}^{h-1} \frac{\partial r}{\partial \bar{\boldsymbol{K}}}(\tau_i, \bar{\boldsymbol{K}}_{i+1})\right] \frac{\partial r}{\partial \tau}(\tau_h, \bar{\boldsymbol{K}}_{h+1}) \qquad h \geq k. \tag{18}$$

Notice that $\frac{\partial r}{\partial \bar{\boldsymbol{K}}}(\tau_i, \bar{\boldsymbol{K}}_{i+1}) \in \mathbb{R}^{\boldsymbol{p}\times\boldsymbol{p}}$.

Since the problem (1) is constrained by $\sum_{k=0}^{N-1}\tau_k = T$, from the KKT conditions, it follows that at the optimal point, the gradient $\nabla J$ must be proportional to $[1, 1, \ldots, 1]$, meaning that all components of $\nabla J$ have to be equal to each other. Hence, a necessary condition for the optimum is that for every $h = 0, \ldots, N-2$

$$\frac{\partial J}{\partial \tau_h} = \frac{\partial J}{\partial \tau_{h+1}} \qquad \Longleftrightarrow \qquad \boldsymbol{x}_0'\frac{\partial \bar{\boldsymbol{K}}_0}{\partial \tau_h} = \boldsymbol{x}_0'\frac{\partial \bar{\boldsymbol{K}}_0}{\partial \tau_{h+1}}$$

which can be rewritten as

$$\boldsymbol{x}_0'\left[\prod_{i=0}^{h-1}\frac{\partial r}{\partial \bar{\boldsymbol{K}}}(\tau_i, \bar{\boldsymbol{K}}_{i+1})\right]\left[\frac{\partial r}{\partial \tau}(\tau_h, \bar{\boldsymbol{K}}_{h+1}) - \frac{\partial r}{\partial \bar{\boldsymbol{K}}}(\tau_h, \bar{\boldsymbol{K}}_{h+1})\frac{\partial r}{\partial \tau}(\tau_{h+1}, \bar{\boldsymbol{K}}_{h+2})\right] = 0. \tag{19}$$

Finding the analytical solution of (19) is an overwhelming task. Hence, we propose some special cases that provide some insights on how the general solution should be. In Section III-C, we describe a numerical algorithm to find the solution.

### A. Two Sampling Instants: Optimality of Periodic Sampling

If $N = 2$, then (19) must be verified only for $h = 0$. In this special case, condition (19) becomes

$$\boldsymbol{x}_0' \left[ \frac{\partial r}{\partial \tau}(T - \tau_1, r(\tau_1, \boldsymbol{S})) - \frac{\partial r}{\partial \bar{\boldsymbol{K}}}(T - \tau_1, r(\tau_1, \boldsymbol{S})) \frac{\partial r}{\partial \tau}(\tau_1, \boldsymbol{S}) \right] = 0 \quad (20)$$

where the only unknown is $\tau_1$.

Equation (20) also allows checking whether periodic sampling can be optimal or not. Let $\bar{K}_\tau$ be the solution of the DARE associated with the discretized system with period $\tau$. Then, a necessary condition for the optimality of the periodic sampling with period $\tau$ is

$$\boldsymbol{x}_0' \left[ I_p - \frac{\partial r}{\partial \bar{\boldsymbol{K}}}(\tau, \bar{\boldsymbol{K}}_\tau) \right] \frac{\partial r}{\partial \tau}(\tau, \bar{\boldsymbol{K}}_\tau) = 0 \quad (21)$$

where $I_{\boldsymbol{n}}$ denotes the identity matrix in $\mathbb{R}^{\boldsymbol{n} \times \boldsymbol{n}}$. If (21) is false, then we are certain that when the system state is $x_0$, periodic sampling with period $\tau$ is not optimal.

### B. First-Order Systems

In the case of a first-order system ($n = 1$ and then $\boldsymbol{n} = 1$) we can avoid using the **bold face** notation introduced at the beginning of Section III, since $M$ and $\boldsymbol{M}$ are the same scalar value.

The necessary condition for optimality (19) requires computing the Riccati recurrence function $r(\tau, \bar{K})$ and its partial derivatives. For first-order order, the Riccati recurrence function $r$ is

$$r(\tau, \bar{K}) = \frac{\bar{Q}_k \bar{R}_k - \bar{P}_k^2 + (\bar{A}_k^2 \bar{R}_k - 2\bar{A}_k \bar{B}_k \bar{P}_k + \bar{B}_k^2 \bar{Q}_k)\bar{K}}{\bar{R}_k + \bar{B}_k^2 \bar{K}} \quad (22)$$

with the following partial derivatives:

$$\frac{\partial r}{\partial \bar{K}}(\tau, \bar{K}) = \frac{(\bar{A}_k \bar{R}_k - \bar{B}_k \bar{P}_k)^2}{(\bar{R}_k + \bar{B}_k^2 \bar{K})^2}$$

$$\frac{\partial r}{\partial \tau}(\tau, \bar{K}) = \frac{1}{(\bar{R}_k + \bar{B}_k^2 \bar{K})^2} \Big( R(\bar{P}_k + \bar{A}_k \bar{B}_k \bar{K})^2 + (\bar{A}_k \bar{R}_k - \bar{B}_k \bar{P}_k)(Q(\bar{A}_k \bar{R}_k - \bar{B}_k \bar{P}_k) + 2\bar{A}_k (A\bar{R}_k - B\bar{P}_k)K + 2\bar{A}_k \bar{B}_k (A\bar{B}_k - B\bar{A}_k)\bar{K}^2) \Big).$$

Let us now investigate the condition on $\tau_0, \ldots, \tau_{N-1}$ to satisfy (19). Since we assume $x_0 \neq 0$, we have that at least one of the two factors in (19) is equal to zero.

*Remark 5:* First, we observe that if $\tau_k$ is such that $\bar{A}_k \bar{R}_k = \bar{B}_k \bar{P}_k$, then $\frac{\partial r}{\partial \bar{K}}(\tau_k, \bar{K}) = 0$ for any possible $\bar{K}$. Let us set $k^*$ as the minimum indices among the $k$ such that $\bar{A}_k \bar{R}_k = \bar{B}_k \bar{P}_k$. From (18), it follows that $\frac{\partial \bar{K}_0}{\partial \tau_h} = 0$, for all $h \geq k^* + 1$. In fact, for such special $\tau_{k^*}$, the value of $\bar{K}_{k^*}$ is

$$\bar{K}_{k^*} = \frac{\bar{Q}_{k^*} \bar{R}_{k^*} - \bar{P}_{k^*}^2}{\bar{R}_{k^*}}$$

that is independent of $\bar{K}_{k^*+1}$ and then independent of any $\tau_{k^*+1}, \ldots, \tau_{N-1}$. These are all potential critical points that need to be explicitly tested.

If instead all intersample separations are such that $\bar{A}_k \bar{R}_k$ is never equal to $\bar{B}_k \bar{P}_k$ (this happens if the minimum $\tau_k$ such that $\bar{A}_k \bar{R}_k = \bar{B}_k \bar{P}_k$ is larger than $T$, or when $\tau_k$ is small enough since $\bar{A}_k \bar{R}_k = \tau_k R + o(\tau_k)$ and $\bar{B}_k \bar{P}_k = \frac{B^2 Q}{2} \tau_k^3 + o(\tau_k^3)$), then from (19), it follows that an optimal sampling pattern must satisfy the condition:

$$\frac{\partial r}{\partial \tau}(\tau_h, r(\tau_{h+1}, \bar{K}_{h+2})) - \frac{\partial r}{\partial \bar{K}}(\tau_h, r(\tau_{h+1}, \bar{K}_{h+2})) \frac{\partial r}{\partial \tau}(\tau_{h+1}, \bar{K}_{h+2}) = 0. \quad (23)$$

This relationship allows finding all intersample separations $\tau_0, \ldots, \tau_{N-2}$ starting from any $\tau_{N-1}$ using the backward recursive equations (17) and (23). In order to fulfil the equality $\sum_{k=0}^{N-1} \tau_k = T$, we have then to choose $\tau_{N-1}$ appropriately; this is made through an iterative procedure that we implemented to scale the value $\tau_{N-1}$ until the constraint $\sum_{k=0}^{N-1} \tau_k = T$ is verified. In Section V, this condition will be exploited to find the asymptotic behavior of optimal sampling.

### C. Numerical Solution

In general, finding the $\tau_0, \ldots, \tau_{N-1}$ that solves (19) is very hard. Hence, we did implement a gradient descent algorithm, which iteratively performs the following steps:

Step 1) computes the gradient $\nabla J = \left( \frac{\partial J}{\partial \tau_0}, \ldots, \frac{\partial J}{\partial \tau_{N-1}} \right)$ at the current solution;

Step 2) project $\nabla J$ onto the equality constraint $\sum_{k=0}^{N-1} \tau_k = T$ by removing the component that is orthogonal to the constraint;

Step 3) performs a step along the negative projected gradient and then updates the solution if the cost has been reduced or reduces the length of the step if the cost is not reduced.

As will be shown later in Section VII, this numerical optimization procedure is capable of finding solutions that are much better than periodic and dls sampling. However, this considerable cost reduction has a price. The major drawback of the numerical algorithm is certainly its complexity. Moreover, with the problem being nonconvex, the gradient-descent algorithm does not guarantee reaching the global minimum.

Only the computation of the gradient of the matrix $\bar{K}_0$ with respect to all sampling instants has the complexity of $O(N^2 n^3)$. This step needs to be computed over and over until numerical stopping criteria of the gradient descent algorithm are reached. While still giving interesting insights on the optimal sampling pattern problem, this considerable computational cost prevents computing the asymptotic behavior for large $N$ and practical applications of this result. For this reason, we propose below another solution which demonstrated surprising properties (proved later in Section V).

## IV. Quantization-Based Sampling

In this section, we describe a sampling method that is capable of providing near-minimal cost (considerably lower than dls sampling) without requiring execution of heavy optimization routines. The basic idea is to approximate the optimal continuous-time control input $u$ with a piecewise constant function.

This approach is well studied under the name of *quantization*, a discretization procedure which aims to approximate a function, in the $L^p$ sense, by means of piecewise constant functions. Given a function $u \in L^p(\Omega)$, the goal is to find a piecewise constant function $\bar{u}$ taking only $N$ values, which realizes the best approximation of $u$, in the sense that the $L^p(\Omega)$ norm

$$\int_\Omega \|u(x) - \bar{u}(x)\|_p \, dx$$

is minimal. In our case, if the dimension of the input space is $m = 1$, the quantization problem can be formulated as minimizing the quantization error $E_{\mathsf{qnt}}$

$$E_{\mathsf{qnt}} = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} |u(t) - u_k|^2 \, dt \tag{24}$$

with $t_0 = 0$ and $t_N = T$.

In this problem, the unknowns are the constants $\{u_0, \dots, u_{N-1}\}$ to approximate the function $u$ as well as the intermediate instants $\{t_1, \dots, t_{N-1}\}$. If we differentiate the quantization error $E_{\mathsf{qnt}}$ with respect to $u_k$, we find that

$$u_k = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} u \, dt \tag{25}$$

which, not surprisingly, states that the constant $u_k$ that better approximates $u$ in the interval $[t_k, t_{k+1}]$ is its average value over the interval. Thanks to (25), the quantization cost can be rewritten as

$$E_{\mathsf{qnt}} = \int_0^T |u|^2 \, \mathrm{d}t - \sum_{k=0}^{N-1} (t_{k+1} - t_k)|u_k|^2.$$

Now we differentiate the error with respect to $t_k$, with $k = 1, \dots, N-1$. We find

$$\begin{aligned}
\frac{\partial E_{\mathsf{qnt}}}{\partial t_k} &= -\frac{\partial}{\partial t_k}\left(\frac{1}{t_{k+1} - t_k}\left|\int_{t_k}^{t_{k+1}} u \, dt\right|^2\right) - \\
&\qquad \frac{\partial}{\partial t_k}\left(\frac{1}{t_k - t_{k-1}}\left|\int_{t_{k-1}}^{t_k} u \, dt\right|^2\right) \\
&= -\frac{1}{(t_{k+1} - t_k)^2}\left|\int_{t_k}^{t_{k+1}} u \, dt\right|^2 + 2u'(t_k)u_k + \\
&\qquad \frac{1}{(t_k - t_{k-1})^2}\left|\int_{t_{k-1}}^{t_k} u \, dt\right|^2 - 2u'(t_k)u_{k-1} \\
&= -|u_k|^2 + 2u'(t_k)u_k + |u_{k-1}|^2 - 2u'(t_k)u_{k-1} \\
&= |u_{k-1} - u(t_k)|^2 - |u_k - u(t_k)|^2
\end{aligned}$$

from which it follows that the sampling sequence that minimizes the quantization error must be such that

$$|u_{k-1} - u(t_k)|^2 = |u_k - u(t_k)|^2. \tag{26}$$

We can then define the quantization-based sampling method (abbreviated with $\mathsf{qnt}$) as follows:

1) the optimal continuous-time input $u$ is computed;
2) the piecewise-constant function $\bar{u}$ that minimizes $E_{\mathsf{qnt}}$ of (24) is found by applying the gradient condition of (26);
3) for the sampling instants $t_0(= 0), t_1, \dots, t_{N-1}, t_N(= T)$ of this solution $\bar{u}$, we compute the optimal input sequence from (10), since the inputs of (25) are not optimal for the minimization of $J$.

An efficient implementation and a proof of convergence of this algorithm is beyond the scope of this paper. The interested reader can find our implementation of this function at github.com/ebni/samplo. Finally, we remark that the method $\mathsf{qnt}$ is applicable to any linear system with dimension of the input space $m = 1$ and any dimension $n$ of the state space.

### A. Asymptotic Behavior

As shown in [14] and [15], the quantization problem of a function $u \in L^p(\Omega)$ is equivalent to minimize the Wasserstein's distance $W_p(\mu, \nu)$ where $\mu$ is the image measure $u^{\#}(dx/|\Omega|)$ and $\nu$ is a sum of Dirac masses

$$\nu = \frac{1}{N} \sum_{k=1}^{N} \delta_{y_k}.$$

As $N \to \infty$, the asymptotic density of points $y_k$ can be computed and is equal to

$$\frac{f(y)^{m/(m+p)}}{\int f(y)^{m/(m+p)} \, dy}$$

where $m$ is the dimension of the space of values of $u$ and $f$ is the density of the absolutely continuous part of the measure $\mu$.

If the input space has dimension $m = 1$, then we find $f(y) = 1/|\dot{u}|\big(u^{-1}(y)\big)$, with $u$ being the solution of the Riccati (3). From the asymptotic density of values $y_k$, which is

$$\frac{|\dot{u}|^{-1/(1+p)}\big(u^{-1}(y)\big)}{\int |\dot{u}|^{-1/(1+p)}\big(u^{-1}(y)\big) \, dy}$$

we can pass to the asymptotic sampling density, which is then

$$\sigma_{\mathsf{qnt}}(t) = \frac{|\dot{u}(t)|^{p/(1+p)}}{\int_0^T |\dot{u}(t)|^{p/(1+p)} \, dt}.$$

Taking $p = 2$, that is, minimizing the $L^2$ norm $\int_0^T |u - \bar{u}|^2 \, dt$, we end up with the asymptotic sampling density

$$\sigma_{\mathsf{qnt}}(t) = \frac{|\dot{u}(t)|^{2/3}}{\int_0^T |\dot{u}(s)|^{2/3} \, ds}. \tag{27}$$

Equation (27) provides a very interesting intuition, which can be used as follows to determine the sampling instants. The steps, also illustrated in Fig. 1, are described below:

1) the optimal continuous-time input $u$ is computed;
2) the sampling instants $t_1, \dots, t_{N-1}$ are determined such that their asymptotic density is (27) by construction, that is, we choose $t_1, \dots, t_{N-1}$ such that
$$\forall k = 0, \dots, N-1, \quad \int_{t_k}^{t_{k+1}} |\dot{u}(t)|^{2/3} \, dt = \frac{1}{N} \int_0^T |\dot{u}(t)|^{2/3} \, dt \tag{28}$$
with the usual hypothesis of $t_0 = 0$, $t_N = T$;
3) for such a sampling sequence $t_0(= 0), t_1, \dots, t_{N-1}, t_N(= T)$, we compute the optimal input sequence from (10), which guarantees to minimize the control cost $J$ for given sampling instants.

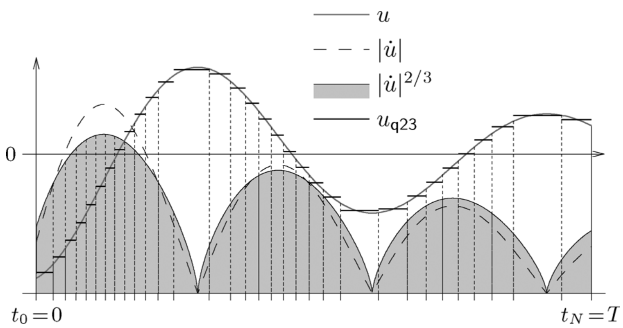


Fig. 1. Sampling according to the asymptotic density.

After the optimal continuous-time input $u$ is computed (which has the complexity of solving a Riccati differential equation), the complexity of computing the sampling instants from $u$ is $O(N)$. Notice that the difference between this method and the dls method [whose instants are selected according to (16)] is only in the exponent of $|\dot{u}|$.

This method is abbreviated with q23 to remind the $2/3$ exponent in (28). Although method q23 follows from the asymptotic sampling density (i.e., with $N \to \infty$) of method qnt, as will be shown in Sections V and VI, this method produces sampling sequences with a near-optimal cost also for reasonably small values of $N$.

Finally, we remark that similar to the method qnt, the method q23 is applicable to any linear system with a dimension of the input space $m = 1$ and any dimension $n$ of the state space.

## V. First-Order Systems: Quantization is Asymptotically Optimal

These quantization-based sampling techniques (methods qnt and q23) follow the intuitive idea that the optimal discrete-time input should mimic the optimal continuous-time input. As will be shown later in Section VII, their excellent capability to reduce the cost appears in all of the performed experiments. Unfortunately, we were unable to prove a general result that relates the costs $J_{N,\mathsf{qnt}}$ or $J_{N,\mathsf{q23}}$ to the minimal continuous-time cost $J_\infty$ or to the cost of optimal sampling $J_{N,\mathsf{opt}}$. Nonetheless, for first-order systems ($n = 1$), we did actually prove that the asymptotic sampling density $\sigma_{\mathsf{qnt}}$ of the quantization ((27)) is actually equal to the asymptotic density of the optimal sampling $\sigma_{\mathsf{opt}}$. The only additional hypothesis we are using is that the weight $S$ of the final state at the instant $T$ is set equal to the continuous ARE solution $K_\infty$, which implies $K(t) = K_\infty$, $\forall t \in [0, T]$. As will be shown later in the proof, this assumption is needed only to simplify the expression of the optimal continuous-time control input. Proving the asymptotic optimality of quantization-based sampling in a more general hypothesis required a too involved mathematical development. Moreover, we observe that assuming the weight of the final state equal to the solution of the ARE is not very stringent, since the state at time $t = T$ is going to be small anyway, especially for large $T$.

Also, throughout this section, we assume that if $Q = 0$, then $A > 0$; otherwise, the optimal input is obviously $\bar{u}(t) = 0$, which is a constant function independent of the sampling instants.

For such a first-order system, we are actually able to compute analytically the asymptotic optimal sampling density $\sigma_{\mathsf{opt}}$ and asymptotic normalized cost $c_{\mathsf{opt}}$.

*Lemma 6:* Consider a first-order system ($n = 1$) with weight of the final state $S$ equal to the solution of the continuous ARE. Then, the optimal sampling pattern has asymptotic density

$$\sigma_{\mathsf{opt}} \propto |\dot{u}|^{2/3} \tag{29}$$

with $u$ being the optimal continuous-time input.

*Proof:* Up to suitable normalizations of the cost function and system dynamics, we can assume, without loss of generality, that $B = 1$ and $R = 1$. From (2), it follows that the solution of the ARE, and then the weight of the final state, is:

$$S = A + \sqrt{A^2 + Q}.$$

This assumption enables us to have a simple expression for the optimal continuous-time input $u$, that is

$$u(t) = -x_0(A + \sqrt{A^2 + Q})e^{-\sqrt{A^2+Q}\,t}. \tag{30}$$

From (4)–(8), the discretised system with an intersample separation of $\tau_k$ gives the following discrete-time model:

$$\begin{aligned}
\bar{A}_k &= 1 + A\tau_k + \frac{A^2}{2}\tau_k^2 + \frac{A^3}{6}\tau_k^3 + \frac{A^4}{24}\tau_k^4 + o(\tau_k^4)\\
\bar{B}_k &= \tau_k + \frac{A}{2}\tau_k^2 + \frac{A^2}{6}\tau_k^3 + \frac{A^3}{24}\tau_k^4 + o(\tau_k^4)\\
\bar{Q}_k &= Q(\tau_k + A\tau_k^2 + \frac{2A^2}{3}\tau_k^3 + \frac{A^3}{3}\tau_k^4) + o(\tau_k^4)\\
\bar{R}_k &= \tau_k + Q(\frac{1}{3}\tau_k^3 + \frac{A}{4}\tau_k^4 + \frac{7A^2}{60}\tau_k^5) + o(\tau_k^5)\\
\bar{P}_k &= Q(\frac{1}{2}\tau_k^2 + \frac{A}{2}\tau_k^3 + \frac{7A^2}{24}\tau_k^4) + o(\tau_k^4).
\end{aligned}$$

Since we investigate the asymptotic optimal sampling density ($N \to \infty$ and then $\tau_k \to 0$), we realize that $\bar{A}_k\bar{R}_k = \tau_k + o(\tau_k)$ is never equal to $\bar{B}_k\bar{P}_k = \frac{Q}{2}\tau_k^3 + o(\tau_k^3)$, for small $\tau_k$. With Remark 5, the optimal solution must satisfy (23) which establishes a relationship between $\tau_h$ and $\tau_{h+1}$. Since they both tend to zero, we write $\tau_h$ as a function of $\tau_{h+1}$

$$\tau_h = \alpha\tau_{h+1} + \beta\tau_{h+1}^2 + o(\tau_{h+1}^2) \tag{31}$$

with $\alpha$ and $\beta$ being suitable constants to be found from (23).

Approximating the Riccati recurrence function $r$ of (22) to the fourth order[1] with regard to $\tau$, we find

$$\begin{aligned}
r(\tau, \bar{K}) &= \bar{K} - (\bar{K}^2 - 2A\bar{K} - Q)\tau\\
&\quad + \big(\bar{K}^3 - 3A\bar{K}^2 + (2A^2 - Q)\bar{K} + AQ\big)\tau^2\\
&\quad - \Big(\bar{K}^4 - 4A\bar{K}^3 + (\frac{55}{12}A^2 - \frac{4}{3}Q)\bar{K}^2\\
&\quad + (-\frac{4}{3}A^3 + \frac{5}{2}QA)\bar{K} - \frac{2}{3}QA^2 + \frac{1}{4}Q^2\Big)\tau^3\\
&\quad + \Big(\bar{K}^5 - 5A\bar{K}^4 + (\frac{49A^2}{6} - \frac{5}{3}Q)\bar{K}^3 + \frac{19}{4}(-A^3 + QA)\bar{K}^2\\
&\quad + (\frac{2A^4}{3} - \frac{13QA^2}{4} + \frac{7Q^2}{12})\bar{K} + \frac{QA^3}{3} - \frac{Q^2A}{2}\Big)\tau^4\\
&\quad + o(\tau^4)
\end{aligned} \tag{32}$$

[1]For computing this and the following expressions, we made use of the symbolic manipulation tool "Maxima" (http://maxima.sourceforge.net/).

with the partial derivatives

$$
\begin{aligned}
\frac{\partial r}{\partial \tau} &= -(\bar{K}^2 - 2A\bar{K} - Q) \\
&\quad + 2\big(\bar{K}^3 - 3A\bar{K}^2 + (2A^2 - Q)\bar{K} + AQ\big)\tau \\
&\quad - 3\Big(\bar{K}^4 - 4A\bar{K}^3 + (\frac{55}{12}A^2 - \frac{4}{3}Q)\bar{K}^2 \\
&\qquad + (-\frac{4}{3}A^3 + \frac{5}{2}QA)\bar{K} - \frac{2}{3}QA^2 + \frac{1}{4}Q^2\Big)\tau^2 \\
&\quad + 4\Big(\bar{K}^5 - 5A\bar{K}^4 + (\frac{49A^2}{6} - \frac{5}{3}Q)\bar{K}^3 + \frac{19}{4}(-A^3 + QA)\bar{K}^2 \\
&\qquad + (\frac{2A^4}{3} - \frac{13QA^2}{4} + \frac{7Q^2}{12})\bar{K} + \frac{QA^3}{3} - \frac{Q^2A}{2}\Big)\tau^3 \\
&\quad + o(\tau^3) \\
\frac{\partial r}{\partial \bar{K}} &= 1 - 2(\bar{K} - A)\tau + (3\bar{K}^2 - 6A\bar{K} + 2A^2 - Q)\tau^2 \\
&\quad - \Big(4\bar{K}^3 - 12A\bar{K}^2 + 2(\frac{55}{12}A^2 - \frac{4}{3}Q)\bar{K} \\
&\qquad - \frac{4}{3}A^3 + \frac{5}{2}QA\Big)\tau^3 + o(\tau^3).
\end{aligned}
$$

We now replace in the necessary condition for optimality (23) the expressions above. If we write $\tau_h$ as a function of $\tau_{h+1}$ ((31)), we find

$$
\begin{aligned}
&- \tau_{h+1}^2 \frac{Q + A\bar{K}_{h+2}}{12}\Big[\Big(6(\alpha-1)(\alpha+1)^2 A\bar{K}_{h+2}^2 \\
&\quad + 2\big[(\alpha+1)\big((2\alpha^2 - 2\alpha - 1)Q + (-4\alpha^2 - 2\alpha + 5)A^2\big) \\
&\qquad - 3\alpha\beta A\big]\bar{K}_{h+2} \\
&\quad - 6\big((\alpha+1)^2(\alpha-1)A + \alpha\beta\big)Q\Big)\tau_{h+1} \\
&- 3(\alpha^2 - 1)(Q + A\bar{K}_{h+2})\Big] + o(\tau_{h+1}^3) = 0
\end{aligned}
$$

from which we have

$$
\begin{aligned}
&\Big(6(\alpha-1)(\alpha+1)^2 A\bar{K}_{h+2}^2 \\
&\qquad + 2\big[(\alpha+1)\big((2\alpha^2 - 2\alpha - 1)Q \\
&\qquad\qquad + (-4\alpha^2 - 2\alpha + 5)A^2\big) - 3\alpha\beta A\big]\bar{K}_{h+2} \\
&\qquad - 6\big((\alpha+1)^2(\alpha-1)A + \alpha\beta\big)Q\Big)\tau_{h+1} \\
&- 3(\alpha^2 - 1)(Q + A\bar{K}_{h+2}) + o(\tau_{h+1}) = 0.
\end{aligned}
\tag{33}
$$

From (33), we have that both coefficients of the zero-order term and of the first-order term in $\tau_{h+1}$ are zero. By setting the constant (that is, $-3(\alpha^2 - 1)(Q + A\bar{K}_{h+2})$) equal to zero, we find $\alpha^2 = 1$. However, from (31), we observe that $\alpha = -1$ is not feasible, since it will lead to negative intersample separations. Hence, we have $\alpha = 1$. By replacing $\alpha = 1$ in the coefficient of $\tau_{h+1}$ in (33) and setting it equal to zero, we find

$$(2Q + 2A^2 + 3\beta A)\bar{K}_{h+2} + 3\beta Q = 0$$

from which we find

$$\beta = -\frac{2}{3}\frac{(Q + A^2)\bar{K}_{h+2}}{Q + A\bar{K}_{h+2}}.$$

Recalling the expression (31), we can now assert that a necessary condition for the optimality of a sampling pattern is that

$$\tau_h = \tau_{h+1} - \frac{2}{3}\frac{(Q + A^2)\bar{K}_{h+2}}{Q + A\bar{K}_{h+2}}\tau_{h+1}^2 + o(\tau_{h+1}^2). \tag{34}$$

We are now going to exploit (34) to find the asymptotic sampling density of the optimal pattern.

Let us compute the derivative of the asymptotic density $\sigma_{\mathrm{opt}}$ of the optimal sampling at a generic instant $t_{h+1}$. By Definitions 1 and 2, we have

$$
\begin{aligned}
\dot{\sigma}_{\mathrm{opt}}(t_{h+1}) &= \lim_{N\to\infty} \frac{\sigma_{\mathrm{opt}}(t_{h+2}) - \sigma_{\mathrm{opt}}(t_{h+1})}{\tau_{h+1}} \\
&= \lim_{N\to\infty} \frac{\frac{1}{N\tau_{h+1}} - \frac{1}{N\tau_h}}{\tau_{h+1}} \\
&= \lim_{N\to\infty} \frac{1 - \frac{1}{1 - \frac{2}{3}\frac{(Q+A^2)\bar{K}_{h+2}}{Q+A\bar{K}_{h+2}}\tau_{h+1}}}{N\tau_{h+1}^2} \\
&= \lim_{N\to\infty} \frac{-\frac{2}{3}\frac{(Q+A^2)\bar{K}_{h+2}}{Q+A\bar{K}_{h+2}}\tau_{h+1}}{N\tau_{h+1}^2} \\
&= -\frac{2}{3}\frac{(Q + A^2)\bar{K}_{h+2}}{Q + A\bar{K}_{h+2}} \lim_{N\to\infty} \frac{1}{N\tau_{h+1}} \\
&= -\frac{2}{3}\frac{(Q + A^2)\bar{K}_{h+2}}{Q + A\bar{K}_{h+2}}\sigma_{\mathrm{opt}}(t_{h+2})
\end{aligned}
$$

from which we obtain the differential equation

$$\dot{\sigma}_{\mathrm{opt}}(t) = -\frac{2}{3}\frac{(Q + A^2)K(t)}{Q + AK(t)}\sigma_{\mathrm{opt}}(t) \tag{35}$$

with $K(t)$ being the solution of the Riccati differential equation (2). If $S = A + \sqrt{Q + A^2}$, then $K(t)$ is constantly equal to $S$. Then in such a special case, (35) becomes

$$
\begin{aligned}
\dot{\sigma}_{\mathrm{opt}}(t) &= -\frac{2}{3}\frac{(Q + A^2)(A + \sqrt{Q + A^2})}{Q + A(A + \sqrt{Q + A^2})}\sigma_{\mathrm{opt}}(t) \\
&= -\frac{2}{3}\sqrt{Q + A^2}\,\sigma_{\mathrm{opt}}(t)
\end{aligned}
$$

which is solved by

$$\sigma_{\mathrm{opt}}(t) = c\, e^{-\frac{2}{3}\sqrt{Q + A^2}\, t}$$

with $c$ being a suitable constant such that $\int_0^T \sigma_{\mathrm{opt}}(t)\, dt = 1$.

From (30) of the optimal continuous-time input, we obtain $\sigma_{\mathrm{opt}} \propto |\dot{u}|^{2/3}$. The Lemma is then proved. ■

Basically, Lemma 6 states that by tolerating the weak assumption that the weight $S$ of the final state $x(T)$ is equal to the solution of the continuous ARE, the asymptotic density $\sigma_{\mathrm{opt}}$ of the optimal sampling is the same as the asymptotic density $\sigma_{\mathrm{q23}}$ of the quantization-based sampling. In addition to this result, the next Lemma also provides an exact computation of the asymptotic normalized cost $c_{\mathrm{opt}}$ (see Definition 4) of the optimal sampling. This result allows quantifying the benefit of optimal sampling.

The following Lemma provides a more general result from which the asymptotic normalized cost $c_{\mathsf{opt}}$ of the optimal sampling is derived later in Corollary 8.

*Lemma 7:* Consider a first-order system ($n = 1$). Let us assume up to suitable normalizations that $B = 1$ and $R = 1$ and that the weight of the final state is equal to the solution of the continuous ARE, $S = A + \sqrt{A^2 + Q}$. Then, the asymptotic normalized cost of the sampling method $\mathsf{m}\alpha$ with asymptotic sampling density

$$\sigma_{\mathsf{m}\alpha}(t) = \frac{\alpha(S-A)}{1 - e^{-\alpha(S-A)T}} e^{-\alpha(S-A)t} \propto |\dot{u}(t)|^{\alpha} \tag{36}$$

is

$$c_{\mathsf{m}\alpha} = \frac{S}{12(S-A)T^2} \frac{1 - e^{-2(1-\alpha)(S-A)T}}{2(1-\alpha)} \left(\frac{1 - e^{-\alpha(S-A)T}}{\alpha}\right)^2. \tag{37}$$

*Proof:* Under our hypotheses, the solution of the Riccati differential equation is $K(t) = S = A + \sqrt{Q + A^2}$, for all $t \in [0, T]$. Hence, the optimal continuous-time cost is

$$J_\infty = x_0^2 S.$$

Since we are investigating the normalized cost $c_{\mathsf{opt}}$ (see Definition 3), we consider the sequence:

$$\xi_k = \frac{N^2}{T^2}\left(\frac{\bar{K}_k}{S} - 1\right)$$

so that $c_{\mathsf{opt}} = \lim_{N\to\infty} \xi_0$. From the definition of $\xi_k$, it follows that:

$$\bar{K}_k = S(\frac{T^2}{N^2}\xi_k + 1).$$

From (32), by approximating $\bar{K}_k$ to the third order of $\tau_k$, we have

$$\begin{aligned} \bar{K}_k = {} & \bar{K}_{k+1} - (\bar{K}_{k+1} - S)(\bar{K}_{k+1} - 2A + S)\tau_k \\ & + (\bar{K}_{k+1} - A)(\bar{K}_{k+1} - S)(\bar{K}_{k+1} - 2A + S)\tau_k^2 \\ & - \Big(\bar{K}_{k+1}^4 - 4A\bar{K}_{k+1}^3 + (\frac{55}{12}A^2 - \frac{4}{3}Q)\bar{K}_{k+1}^2 \\ & + (-\frac{4}{3}A^3 + \frac{5}{2}QA)\bar{K}_{k+1} - \frac{2}{3}QA^2 + \frac{1}{4}Q^2\Big)\tau_k^3 + o(\tau_k^3) \end{aligned}$$

which allows finding a recurrent relationship that defines $\xi_k$

$$\begin{cases} \xi_k = \xi_{k+1} - \xi_{k+1}(S\frac{T^2}{N^2}\xi_{k+1} + 2(S-A))\tau_k \\ \quad + \xi_{k+1}(S\frac{T^2}{N^2}\xi_{k+1} + 2(S-A))(S\frac{T^2}{N^2}\xi_{k+1} + S - A)\tau_k^2 \\ \quad + \frac{S(S-A)^2}{12}\frac{N^2}{T^2}\tau_k^3 + o(\tau_k^3) \\ \xi_N = 0. \end{cases} \tag{38}$$

From (38), it follows that the discrete derivative of $\xi_k$ is:

$$\begin{aligned} \frac{\xi_{k+1} - \xi_k}{\tau_k} = {} & \xi_{k+1}(S\frac{T^2}{N^2}\xi_{k+1} + 2(S-A)) \\ & - \xi_{k+1}(S\frac{T^2}{N^2}\xi_{k+1} + 2(S-A))(S\frac{T^2}{N^2}\xi_{k+1} + S - A)\tau_k \\ & - \frac{S(S-A)^2}{12}\frac{N^2}{T^2}\tau_k^2 + o(\tau_k^2). \end{aligned}$$

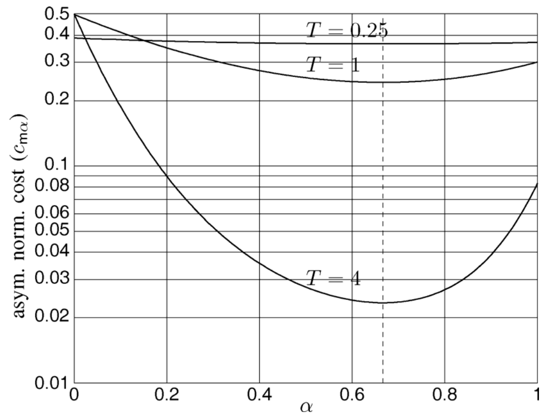


Fig. 2. Asymptotic normalized cost as a function of $\alpha$.

By definition of asymptotic sampling density (see Definitions 1 and 2), as $N \to \infty$, the intersample separation $\tau_k$ tends to zero with

$$\tau_k = \frac{1}{N\sigma_{\mathsf{m}\alpha}(t_k)} + o\Big(\frac{1}{N}\Big).$$

From this observation, as $N \to \infty$, the discrete derivative of $\xi_k$ becomes the differential equation

$$\dot{\delta}(t) = 2(S-A)\delta(t) - \frac{S(S-A)^2}{12T^2}\sigma_{\mathsf{m}\alpha}^{-2}(t)$$

where $\delta(t)$ is the limit of $\xi_k$. With the sampling density $\sigma_{\mathsf{m}\alpha}(t)$ of (36), the differential equation above becomes

$$\begin{cases} \dot{\delta}(t) = 2(S-A)\delta(t) - \frac{S(1-e^{-\alpha(S-A)T})^2}{12\alpha^2T^2} e^{2\alpha(S-A)t} \\ \delta(T) = 0 \end{cases}$$

which is a first-order linear nonhomogeneous differential equation, whose explicit solution is

$$\begin{aligned} \delta(t) = \frac{S(1 - e^{-\alpha(S-A)T})^2}{24(S-A)(1-\alpha)\alpha^2T^2}\Big(& e^{2\alpha(S-A)t} \\ & - e^{-2(1-\alpha)(S-A)T}e^{2(S-A)t}\Big). \end{aligned}$$

Since the asymptotic normalized cost $c_{\mathsf{m}\alpha}$ coincides with $\delta(0)$, we obtain (37) and the Lemma is proved. ■

The reason for assuming a sampling density as in (36) is quite simple: periodic, $\mathsf{dls}$, and optimal sampling (q23) are all special cases of the asymptotic density (36). In fact:

- the periodic sampling has constant sampling density; hence, it corresponds to the case $\alpha = 0$;
- the $\mathsf{dls}$ sampling corresponds, by construction, to the case $\alpha = 1$;
- from Lemma 6, the optimal sampling corresponds to the case $\alpha = 2/3$.

In Fig. 2, we plot the asymptotic normalized cost as $\alpha$ varies. The system in the plot has $A = 1$ and $Q = 8$ (and $B = R = 1$, $S = A + \sqrt{A^2 + Q} = 4$). We plot the cost $c_{\mathsf{m}\alpha}$ for three different values of $T$: 0.25, 1, and 4. This experiment confirms the validity of Lemma 6: the minimal cost occurs when $\alpha = \frac{2}{3}$ (denoted in the figure by a dashed vertical line).

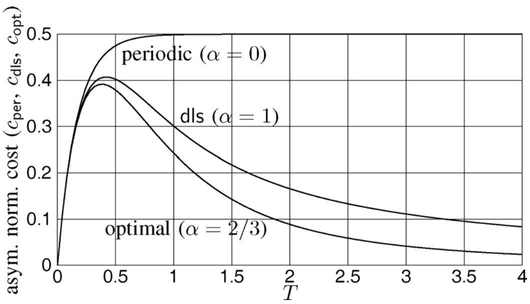


Fig. 3. Asymptotic normalized cost as a function of $T$.

By evaluating the cost of (37) with $\alpha$ being equal to 0, 1, and $\frac{2}{3}$, we can then find the explicit cost expression of the asymptotic normalized cost for the periodic, the dls, and the optimal sampling, respectively, as stated in the following Corollary.

*Corollary 8:* Consider a first-order system ($n = 1$). Let us assume, up to suitable normalizations, that $B = 1$ and $R = 1$ and that the weight of the final state is equal to the solution of the continuous ARE $S = A + \sqrt{A^2 + Q}$. Then, the asymptotic normalized costs of the periodic dls and optimal sampling are, respectively

$$c_{\mathrm{per}} = c_{\mathrm{m}0} = \frac{A\sqrt{A^2+Q} + A^2 + Q}{24}(1 - e^{-2\sqrt{A^2+Q}\,T}) \tag{39}$$

$$c_{\mathrm{dls}} = c_{\mathrm{m}1} = \frac{A + \sqrt{A^2+Q}}{12T}(1 - e^{-\sqrt{A^2+Q}\,T})^2, \tag{40}$$

$$c_{\mathrm{opt}} = c_{\mathrm{m}\frac{2}{3}} = \frac{9}{32T^2}(\frac{A}{\sqrt{A^2+Q}} + 1)(1 - e^{-\frac{2}{3}\sqrt{A^2+Q}\,T})^3. \tag{41}$$

Notice that as $T \to \infty$, the cost $c_{\mathrm{per}}$ coincides with the one derived earlier in (15), which was a consequence of the second-order approximation of the cost of periodic sampling (14) derived by Melzer and Kuo [12].

In Fig. 3, we draw the asymptotic normalized costs for the three sampling methods: periodic, dls, and optimal (method q23). As expected, the cost of optimal sampling is always lower than the other two methods. As $T \to \infty$, the cost $c_{\mathrm{per}}$ tends to the constant of (15), that is, $\frac{1}{2}$ in this case. The cost $c_{\mathrm{dls}}$ tends to zero as $\frac{1}{4T}$, while $c_{\mathrm{opt}}$ tends faster to zero with $\frac{3}{8T^2}$.

## VI. Second-Order Systems

In Section V, we show that the quantization-based sampling is optimal for first-order systems. First-order systems, however, are quite special cases. For example, they never exhibit oscillations in the optimal control input. Hence, the second natural investigation that we perform is on systems that can oscillate. For this purpose, we assume

$$A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad Q = qI, \quad R = 1. \tag{42}$$

Notice that we must assume $q > 0$; otherwise, the optimal input is always $u(t) = 0$. The system of (42) has the following solution of the ARE:

$$K_\infty = \begin{bmatrix} \omega\sqrt{\rho^2+2\rho-3} & \omega(1-\rho) \\ \omega(1-\rho) & \rho\omega\sqrt{\rho^2+2\rho-3} \end{bmatrix}$$

with $\rho$ defined as

$$\rho = \sqrt{1 + \frac{q}{\omega^2}} > 1.$$

For such a system, the characteristic polynomial $\chi(s)$ of the closed-loop system with optimal state feedback is

$$\chi(s) = \det(sI - (A - BR^{-1}B'K_\infty)) = s^2 + 2\omega_n\zeta s + \omega_n^2 \tag{43}$$

with the following damping ratio $\zeta$ and natural frequency $\omega_n$:

$$\zeta = \frac{1}{2}\sqrt{(\rho-1)\left(1 + 3\frac{1}{\rho}\right)}$$
$$\omega_n = \omega\sqrt{\rho}.$$

Hence, by properly choosing the problem parameters $q$ and $\omega$, we can construct overdamped, critically damped, and underdamped systems with any natural frequency.

If we assume $x_0 = [1\ 0]'$, then the cost of the optimal continuous-time input is

$$J_\infty = x_0'K_\infty x_0 = \omega\sqrt{\rho^2+2\rho-3}.$$

With this initial condition, if the closed-loop system is overdamped (that is, when $\rho > 3$), then the optimal input is

$$u(t) = 2\sqrt{2}\omega\frac{\sqrt{\rho-1}}{\sqrt{\rho-3}}\sinh\left(\omega_n t\sqrt{\zeta^2-1} - \log\frac{\sqrt{\rho^2-1}+\sqrt{\rho^2-9}}{\sqrt{8}}\right)e^{-\omega_n t\zeta}$$

if the system is underdamped ($1 < \rho < 3$), the optimal input is

$$u(t) = 2\sqrt{2}\omega\sqrt{\frac{\rho-1}{3-\rho}}\sin\left(\omega_n t\sqrt{1-\zeta^2} - \arctan\sqrt{\frac{9-\rho^2}{\rho^2-1}}\right)e^{-\omega_n t\zeta}$$

and, finally, if the system is critically damped ($\rho = 3$), then the optimal input simply is

$$u(t) = (4\omega^2 t - 2\omega\sqrt{3})e^{-\omega\sqrt{3}t}.$$

For such a second-order system, we are not capable of demonstrating that the asymptotic (with the number of samples $N \to \infty$) sampling density of the quantization problem (that, as a reminder, is proportional to $|\dot{u}|^{\frac{2}{3}}$) is the same as the asymptotic sampling density of the optimal LQR problem (1). Instead, we propose a numerical evaluation suggesting that the two asymptotic densities may coincide, even in the second-order case.

TABLE I
FIRST-ORDER SYSTEM ($A = B = R = 1,\ Q = 8$): NORMALIZED COSTS $c_{N,\mathsf{m}}$, WITH VARYING $N$

| $N$ | $c_{N,\mathsf{per}}$ | $c_{N,\mathsf{dls}}$ | $c_{N,\mathsf{q23}}$ | $c_{N,\mathsf{qnt}}$ | $c_{N,\mathsf{num}}$ |
|---|---|---|---|---|---|
| 4 | 0.4958 | 0.3200 | 0.2541 | 0.2539 | 0.2536 |
| 8 | 0.4980 | 0.3082 | 0.2454 | 0.2454 | 0.2454 |
| 16 | 0.4986 | 0.3033 | 0.2432 | 0.2432 | 0.2432 |
| 32 | 0.4987 | 0.3017 | 0.2426 | 0.2426 | 0.2426 |
| 64 | 0.4987 | 0.3012 | 0.2425 | 0.2425 | 0.2425 |
| 128 | 0.4988 | 0.3011 | 0.2424 | 0.2425 | 0.2424 |
| 256 | 0.4988 | 0.3010 | 0.2424 | 0.2427 | 0.2424 |
| 512 | 0.4988 | 0.3010 | 0.2424 | 0.2435 | 0.2424 |
| $\infty$ | $\frac{1}{2}(1-e^{-6})$ | $\frac{1}{3}(1-e^{-3})^2$ | $\frac{3}{8}(1-e^{-2})^3$ | — | $\frac{3}{8}(1-e^{-2})^3$ |
| $\epsilon = 2\%$ | $N \geq 4.99$ | $N \geq 3.88$ | $N \geq 3.48$ | $N \geq 3.49$ | $N \geq 3.48$ |

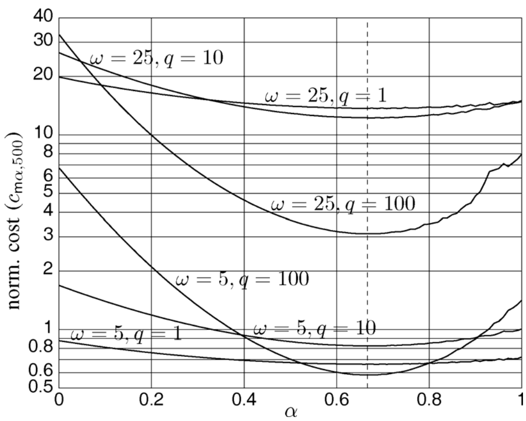


Fig. 4. Normalized cost $c_{\mathsf{m}\alpha,500}$ for second-order systems.

More precisely, let us define a sampling method $\mathsf{m}\alpha$ with the sampling instants $t_0(=0), t_1, \ldots, t_{N-1}, t_N(=T)$ such that

$$\forall k = 0, \ldots, N-1, \quad \int_{t_k}^{t_{k+1}} |\dot{u}(t)|^\alpha \, dt = \frac{1}{N} \int_0^T |\dot{u}(t)|^\alpha \, dt.$$

As observed in Section V, such a method is of interest, because periodic $\mathsf{dls}$ and $\mathsf{q23}$ sampling methods are all special cases for $\alpha$ equal to 0, 1, and 2/3, respectively.

In Fig. 4, we illustrate the normalized cost for $\omega \in \{5, 25\}$ and $q \in \{1, 10, 100\}$, with $N = 500$ sampling instants, as $\alpha$ varies. Surprisingly, we have that in all cases, the normalized cost reaches the minimum at $\alpha = \frac{2}{3}$. This observation suggests that a result analogous to Lemma 6 may also hold for second-order systems. In addition, we observe that the normalized cost is higher in all cases when the optimal input has larger variations ($w = 25$). When the cost of the state (represented by $q$) is large compared to the cost of the input, then the choice of the sampling method has a stronger impact on the overall cost.

## VII. NUMERICAL EVALUATION

In this section, we investigate how the normalized cost varies with the number of samples $N$. We compare the following sampling methods:

- periodic sampling ($\mathsf{per}$);
- deterministic Lebesgue sampling ($\mathsf{dls}$), with sampling instants determined according to (16);
- quantization based on the theoretical asymptotic density of ($\mathsf{q23}$), with sampling instants determined according to (28);
- quantization based on the exact condition of gradient equal to zero of (26) (abbreviated with $\mathsf{qnt}$); for large $N$, this method tends to $\mathsf{q23}$;
- optimal numerical solution ($\mathsf{num}$), computed by the gradient-descent algorithm described in Section III-C.

TABLE II
SECOND-ORDER SYSTEM ($\omega = 5,\ q = 100$): NORMALIZED COSTS $c_{N,\mathsf{m}}$, WITH VARYING $N$

| $N$ | $c_{N,\mathsf{per}}$ | $c_{N,\mathsf{dls}}$ | $c_{N,\mathsf{q23}}$ | $c_{N,\mathsf{qnt}}$ | $c_{N,\mathsf{num}}$ |
|---|---|---|---|---|---|
| 4 | 8.9956 | 1.7499 | 1.9642 | 1.8199 | 1.6878 |
| 8 | 7.2440 | 5.7915 | 0.7064 | 0.5578 | 0.5497 |
| 16 | 6.8638 | 4.1775 | 0.6356 | 0.5638 | 0.5596 |
| 32 | 6.7721 | 1.9258 | 0.5936 | 0.5704 | 0.5656 |
| 64 | 6.7494 | 1.5956 | 0.5818 | 0.5747 | 0.5724 |
| 128 | 6.7437 | 1.4830 | 0.5828 | 0.5799 | 0.5774 |
| 256 | 6.7423 | 1.4355 | 0.5826 | 0.5845 | 0.5805 |
| 512 | 6.7419 | 1.4332 | 0.5839 | 0.5951 | 0.5823 |
| $\epsilon = 2\%$ | $N \geq 18.36$ | $N \geq 8.47$ | $N \geq 5.40$ | $N \geq 5.45$ | $N \geq 5.39$ |

In all experiments of this section, the length of the interval is $T = 1$. Also notice that in all cases, the optimal input signals $u_0, \ldots, u_{N-1}$ are selected according to (10), while the sampling sequence depends on the chosen method.

In the first experiment, we tested a first-order system with $A = 1$ and $Q = 8$. In Table I, we report the normalized costs as $N$ grows. In the row corresponding to $N = \infty$, we report the theoretical values, as computed from (39)–(41). We observe that in this case, the convergence to the asymptotic limit is quite fast. This supports the approximation made in (13) and more, in general, the adoption of the asymptotic normalized cost as a metric to judge sampling methods, even with low $N$. Also, in the last row, we report the bound on the number of samples for each sampling method, if a cost increase of, at most, $\epsilon$ =2% is tolerated with regard to the continuous-time case.

In Table II, we report similar data for a second-order system of the kind described in (42), with $\omega = 5$ and $q = 100$. Such a choice makes the closed-loop system underdamped. We observe again that the convergence to the limit is fast.

In the final experiment, we tested the following third-order system:

$$A = \begin{bmatrix} 1 & 12 & 0 \\ -12 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{44}$$

with initial condition $x_0 = [1\ 0\ 0]'$.

In Table III, we report the computed normalized costs corresponding to $Q = 0$ (upper part of the table), $Q = 10\,I$ (middle portion), and $Q = 100\,I$ (bottom of the table). The weights to

TABLE III
THIRD-ORDER SYSTEM (OF (44) WITH, FROM TOP TO BOTTOM, $Q = 0$, $Q = 10\,I$, AND $Q = 100\,I$): NORMALIZED COSTS $c_{N,\mathsf{m}}$, WITH $N \in \{10, 20, 40\}$

| $N$ | $c_{N,\mathsf{per}}$ | $c_{N,\mathsf{dls}}$ | $c_{N,\mathsf{q23}}$ | $c_{N,\mathsf{qnt}}$ | $c_{N,\mathsf{num}}$ |
|---|---|---|---|---|---|
| 10 | 14.488 | 9.4078 | 8.2823 | 5.9003 | 5.8911 |
| 20 | 13.382 | 8.0191 | 8.1115 | 6.6710 | 6.5765 |
| 40 | 13.136 | 8.4132 | 7.8431 | 7.2797 | 7.1273 |
| $\epsilon = 2\%$ | $N \geq 25.63$ | $N \geq 20.51$ | $N \geq 19.80$ | $N \geq 19.08$ | $N \geq 18.88$ |
| 10 | 16.615 | 7.2274 | 4.9684 | 2.9200 | 2.9139 |
| 20 | 14.640 | 6.3719 | 3.7374 | 3.2204 | 3.1786 |
| 40 | 14.221 | 4.8349 | 3.6936 | 3.6597 | 3.4165 |
| $\epsilon = 2\%$ | $N \geq 26.66$ | $N \geq 15.55$ | $N \geq 13.59$ | $N \geq 13.53$ | $N \geq 13.07$ |
| 10 | 30.959 | 10.161 | 1.7244 | 0.96569 | 0.94476 |
| 20 | 25.509 | 24.900 | 1.1534 | 1.1478 | 0.99627 |
| 40 | 24.352 | 4.5791 | 1.2179 | 1.2095 | 1.1631 |
| $\epsilon = 2\%$ | $N \geq 34.89$ | $N \geq 15.13$ | $N \geq 7.80$ | $N \geq 7.78$ | $N \geq 7.63$ |

input $u$ and to the final state $x(T)$ were always assumed constant ($R = I$ and $S = K_\infty$).

In the last row of each table case, we report again the estimate of the necessary number of samples in $[0, 1]$ if a cost increase of $r$ =2% is tolerated with regard to the continuous control input [as computed from (13)]. The interested reader can find the code for performing these experiments at github.com/ebni/samplo.

We now provide some comments on the data reported in this section.

- The runtime of the experiments of Table III took one day on a 2.40-GHz laptop. The weight of this simulation prevented us from performing it on higher dimension systems or with a larger number of samples.
- The experiments confirm the validity of the asymptotic density of the quantization-based sampling of (28), since the cost achieved by q23 tends to the cost of the numerical quantization qnt as $N$ grows.
- The capacity of quantization-based sampling and dls sampling to reduce the cost with regard to periodic sampling is much higher in all of those circumstances with high variation of the optimal continuous-time input $u$ (such as when $Q$ is larger compared to $R$). This behavior is actually proved for first-order systems. In fact, from (39)–(41), if $Q \to \infty$, we have $c_{\mathsf{per}} \approx Q$, $c_{\mathsf{dls}} \approx \sqrt{Q}$, and $c_{\mathsf{opt}} \approx 1/\sqrt{Q}$.
- The cost achieved by quantization-based sampling (qnt, and q23 for larger $N$) appears to be very close to the optimal one, even for higher order systems. However, it is still an open question whether Lemmas 6 and 7 can be proved in general or not.

## VIII. CONCLUSIONS AND FUTURE WORKS

In this paper, we investigated the effect of the sampling sequence over the LQR cost. We formulate the problem for determining the optimal sampling sequence and we derive a necessary optimality condition based on the study of the gradient of the cost with regard to the sampling instants. Hence, following a different path of investigation, we proposed *quantization-based* sampling, which selects the sampling instants (but not control sequence) in the way that better approximates the optimal control input. Surprisingly, this sampling method is demonstrated to be optimal for first-order systems and a large number of samples per time unit. For second-order systems, such an asymptotic optimality is apparent from our numerical experiments, although it is not formally proved.

Since this research is quite new, there are more open issues than questions with answers. Among the open problems, we mention:

- proving the asymptotic optimality of the quantization-based sampling even in general (higher order) cases;
- the application of the proposed methods to closed-loop feedback where the state is also affected by disturbances;
- possibly more efficient implementation of the gradient optimization procedure;
- the investigation of global minimization procedures which could lead to higher cost reduction (gradient descent algorithms could indeed fall into local minima);
- the investigation of different approaches to approximate the optimal control input.

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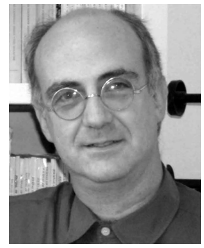
**Enrico Bini** (SM'13) received the Ph.D. degree in real-time systems from Scuola Superiore Sant'Anna, Pisa, Italy, in 2004 and the M.Sc. degree in mathematics from the University of Pisa, Pisa, Italy, in 2010.

While earning the Ph.D. degree, he visited Sanjoy Baruah at the University of North Carolina, Chapel Hill, NC, USA. After becoming Assistant Professor at the Scuola Superiore Sant'Anna, in 2012, he spent two years at Lund University, Lund, Sweden, with a Marie-Curie Intra-European Fellowship, working with Karl-Erik Årzén. He has published more than 70 papers (two best paper awards) on scheduling algorithms, real-time operating systems, and design and optimization techniques for real-time and control systems.



**Giuseppe M. Buttazzo** received the Ph.D. degree in mathematics from Scuola Normale di Pisa, Pisa, Italy, in 1980.

He also obtained his first permanent position as a Researcher with Scuola Normale di Pisa in 1980. He became Full Professor of Mathematical Analysis at the University of Ferrara, Ferrara, Italy, in 1986, and moved on in 1990 to the University of Pisa, where he is currently with the Department of Mathematics. His main research interests include calculus of variations, partial differential equations, optimization problems, as well as optimal and control theory.

Dr. Buttazzo wrote more than 170 scientific papers and several books on these subjects. He managed several international research projects and trained many Ph.D. students.