Faster Wavelet Tree Queries

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Abstract

Given a text, rank and select queries return the number of occurrences of a character up to a position (rank) or the position of a character with a given rank (select). These queries have applications in, e.g., compression, computational geometry, and most notably pattern matching in the form of the backward search—the backbone of many compressed full-text indices. Currently, in practice, for text over non-binary alphabets, the wavelet tree is probably the most used data structure for rank and select queries.

In this paper, we present techniques to speed up queries by a factor of two (access and select) up to three (rank), compared to the wavelet tree implementation contained in the widely used Succinct Data Structure Library (SDSL). To this end, we change the underlying tree structure from a binary tree to a 4-ary tree and reduce cache misses by approximating rank queries using a predictive model to prefetch all data required for the actual rank query.

1 Introduction

Wavelet trees [24] are a compressible self-indexing rank and select data structure, i.e., they can answer rank (number of occurrences of symbol up to position i) and select (position of i-th occurrence of symbol) queries, while still allowing to access the text. This makes them an important building block for compressed full-text indices, e.g., the FM-index [17] or the r-index [20], where they are used to answer rank queries during the pattern matching algorithm—the backwards search—cf. Algorithm 1.1.

Wavelet trees have many applications, which are discussed in multiple surveys [16, 25, 33, 35]. Due to the plethora of applications, a lot of research has been focused on the efficient construction of wavelet trees in both practice and theory. We give an overview of the state-of-the-art in Section 3. However, there exists barely any research focusing on the query performance of wavelet trees. While there exist alternative representations of the wavelet tree (namely the wavelet matrix, see Section 2) that provide better practical query performance, the better query performance is more of a byproduct of a space efficient representation for large alphabets.

The main building block of wavelet trees (and wavelet matrices) are bit vectors with binary rank and select support. There exist many different approaches tuning the rank and select support for query time and/or space overhead. Faster binary rank and select queries directly translate to faster queries on wavelet trees. We refer to Section 3 for an overview of binary rank and select support for bit vectors. However, improving only the binary rank and select data structure still not fully utilizes the full range of optimizations when it comes to answering queries using wavelet trees.

Wavelet trees usually utilize binary trees as underlying tree structure. We show that using a 4-ary tree as underlying tree structure (see Section 4) results in a query speedup of up to 2 for all queries compared to its competitor implemented in the widely used Succinct Data Structure Library (SDSL) [21]. Furthermore, we introduce the rank with additive approximation problem (see Section 6) and show how utilize a small prediction model to locate data necessary during rank queries. We use this information to improve rank queries (which are required for pattern matching) even more, achieving a total speedup of up to 3, by prefetching all data necessary to answer the query.

In our experimental evaluation (see Section 7), we not only show these impressive speedups for such a well-researched data structure but also that our data structure requires less space and is faster to construct—making it strictly superior to its competitors.

2 Preliminaries

A bit vector is a text over the alphabet $\{0,1\}$. Given a text T of length n over an alphabet $\Sigma = [0,\sigma)$. For $i \in [0,n)$ and $\alpha \in \Sigma$, we want to answer:

- $rank_{\alpha}(i) = |\{j < i : T[j] = \alpha\}|$ and
- $select_{\alpha}(i) = \min\{j : rank_{\alpha}(j) = i\}.$

Rank and select queries on bit vectors of length n can be answered in O(1) time with o(n) additional bits [8, 26]. The most significant bit (MSB) of a character is the bit with the highest value. For simplicity, we assume that the MSB is the leftmost bit. The i-th MSB is the bit with the i-th highest value. A length- ℓ bit-prefix of a character are the character's ℓ MSBs.

A wavelet tree [24] is a binary tree, where each node represents a subsequence of the text. Each node contains character with a specific length-k bit-prefix. The root of a wavelet tree represents all characters with the length-0 bit empty prefix, i.e., all characters. Then, whenever we visit a left child of a node that represents characters with bit-prefix α , the child represents character with-bit prefix α 0. The right child represents characters with bit-prefix

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\begin{array}{c|cccc} \mathbf{1} & \mathbf{Function} & BackwardsSearch(P[1..m], C, wt) \\ \mathbf{2} & & s=1, e=n \\ \mathbf{3} & & \mathbf{for} & i=m, \dots, 1 & \mathbf{do} \\ \mathbf{4} & & & s=C[P[i]]+wt.rank_{P[i]}(s-1)+1 \\ \mathbf{5} & & & e=C[P[i]]+wt.rank_{P[i]}(e) \\ \mathbf{6} & & & \mathbf{if} & s>e & \mathbf{then} \\ \mathbf{7} & & & & \mathbf{return} & \emptyset \\ \mathbf{8} & & & \mathbf{return} & [s,e] \end{array}
```

Algorithm 1.1. Backward search for a pattern P of length m using the wavelet tree wt over the Burrows-Wheeler transform of the text and the exclusive prefix sum over the histogram of characters C.

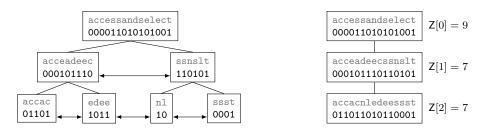


Figure 1: Wavelet tree (left) and a wavelet matrix (right) for the text accessandselect over the alphabet {a (000)₂, c (001)₂, d (010)₂, e (011)₂, l (100)₂, n (101)₂, s (110)₂, t (111)₂} (bit representation of characters given in gray). Note that we depict the text for better readability only; the text is not part of the wavelet tree or wavelet matrix. By concatenating all bit vectors in nodes of the wavelet tree that are connected by arrows, we obtain a level-wise wavelet tree. On each level in the wavelet matrix, there are the same intervals as in the wavelet tree.

 α 1. Alternatively, you can say that the left child represents characters that are in the lower half of the alphabet represented in its parent and the right child represents characters in the upper half. Here, the root represents characters in the whole alphabet.

On the ℓ -th level of the tree (the root has level 1), characters are represented by their ℓ -th MSB. Within a node, all represented characters are stored in a bit vector. If we concatenate the bit vectors of all nodes on the same level, we obtain a *level-wise* wavelet tree. We say that all characters that have been represented in a node of a non-level-wise wavelet tree are in the same interval. See Figure 1 for an example. All intervals in a wavelet tree can be identified by the bit prefix of the characters represented by that interval. In the following, we consider a level-wise wavelet trees.

The wavelet matrix [9] is an alternative representation of the wavelet tree. The first level of the wavelet matrix are the MSBs of the characters, the same as the first level of the wavelet tree. Then, to compute the next level ℓ , starting with the second, the text is stably sorted using the $(\ell-1)$ -th MSB as key. Just as with the wavelet tree, the characters are represented using their ℓ -th MSB on each level ℓ . The order of the characters on each level is given by the stably sorted text. Sorting the text looses the tree structure of the wavelet tree. However, the same intervals as in the wavelet tree occur on each level, just in a bit-reversal permutation¹ order. A comparison of the structure of a wavelet tree and a wavelet matrix can be found in Figure 1. The number of zero in each level is stored in the array Z, which are needed to answer queries using one less binary rank and/or select query per level compared to wavelet trees. We give the rank query algorithms for wavelet trees and matrices in Algorithms 2.1 and 2.2, resp. In the following, we use wavelet tree to refer to both wavelet tree and wavelet matrix. A wavelet trees for a text over an alphabet of size σ can answer access, rank, and select queries in $O(\log \sigma)$ time.

¹See https://oeis.org/A030109, last accessed 2023-11-08.

```
1 Function Wavelet Tree Rank<sub>\alpha</sub>(i)
       start = 0, size = n, bit\_mask = 1 \ll (levels - 1)
 \mathbf{2}
       for level = 0, \ldots, \lceil \log \sigma \rceil and i > 0 do
 3
           before = bv.rank_1(start)
 4
           position = bv.rank_1(start + i) - before
 5
           in = bv.rank_1(start + size) - before
 6
           if \alpha \& bit\_mask then
               start = start + (size - in)
 8
               size = in
 9
               i = position
10
           else
11
               size = size - in
12
               i = i - position
13
           start + = n
14
           bit\_mask = bit\_mask \gg 1
15
       return i
16
```

Algorithm 2.1. Rank query for a wavelet tree over a text of length n over an alphabet of size σ where all levels are stored in one consecutive bit vector bv.

```
1 Function WaveletMatrixRank_{\alpha}(i)
 2
       start = 0, size = n, bit\_mask = 1 \ll (levels - 1)
       for level = 0, ..., \lceil \log \sigma \rceil and i > 0 do
3
           before = bv.rank_1(start)
 4
           position = bv.rank_1(start + i) - before
 5
           in = before - Z_1[level]
 6
           if \alpha \& bit\_mask then
 7
               i = before
 8
               start = ((level + 1)n) + Z_0[level] + in
 9
           else
10
               i = i - before
11
               start = ((level + 1)n) + (start - (level \cdot n)) - in
           start+=n
13
           bit\_mask = bit\_mask \gg 1
14
       return i
15
```

Algorithm 2.2. Rank query for a wavelet matrix over a text of length n over an alphabet of size σ where all levels are stored in one consecutive bit vector bv. Additionally, $Z_0[\ell]$ denotes the zeros on level ℓ and $Z_1[\ell]$ denotes the number of ones before level ℓ in bv.

3 Related Work

The compact and compressed representation of texts with support for access, rank, and select queries (among others) is an active field of research. For example, bit vectors with rank and select support, i.e., binary rank and select data structures, are often a building block for

succinct data structures.

Binary Rank and Select Data Structures

For bit vectors of length n, rank and select data structures with constant query time can be constructed in linear time requiring o(n) space [8, 26]. Practical and well-performing implementations of these data structures can be found in the SDSL [21]. The currently most space efficient rank and select support for a size-u bit vector that contains n ones requires only $\log\binom{u}{n} + \frac{u}{\log u} + \tilde{O}(u^{\frac{3}{4}})$ bits (including the bit vector) [38]. In practice, the currently fastest select data structures are by Vigna [46]. Allowing for multiple configurations using a tuning parameter, they outperform all other select data structures while being space-efficient. However, they still require much more space than the currently most space-efficient data structures by Zhou et al. [47] that have recently been improved w.r.t. query throughput by Kurpicz [31]. There exist many more practical rank and select data structures that are outperformed by the ones mentioned above [23, 29, 36]. Another line of research considers compressed [2, 6, 7, 28, 44] and mutable [40, 41] bit vectors with rank and select support.

Wavelet Trees and Wavelet Matrices

Wavelet tree construction is a well studied field. Let T be a text of length n over an alphabet of size σ . The asymptotically best sequential wavelet tree construction algorithms require $O(n \log \sigma / \sqrt{\log n})$ time [3, 34]. These approaches make use of vectorized instructions, i.e., SIMD (single instruction, multiple data), to achieve their running time. There also exist implementations that make use of these instructions which are available in modern CPUs [12, 27] and are reported to be the fastest in practice. In shared memory, wavelet trees can be computed in $O(\sigma + \log n)$ time requiring only $O(n \log \sigma / \sqrt{\log n})$ work [45]. In practice, the fastest construction algorithms are based on domain decomposition [32, 19], where partial wavelet trees are computed in parallel and are then merged also in parallel, using a bottom-up construction for the partial wavelet tree construction [10]. Wavelet trees can also be computed in other models of computation, e.g., distributed [11] and external memory [14]. To compress a wavelet tree, it is constructed for the Huffman-compressed text.² The bit vectors in the Huffman-shaped wavelet tree requires $n[H_0(T)]$ bits of space, where H_0 is the zeroth order entropy of the text. A fully functional wavelet tree requires binary rank and select support on the bit vectors and needs $n\lceil \log \sigma \rceil (1+o(1))$ bits of space $(n[H_0(T)](1+o(1))$ bits of space for the Huffman-shaped wavelet tree). There are also wavelet trees for degenerate strings [1]. In theoretical work, multi-ary wavelet trees have been considered before with the main goal to reduce query time in the RAM model to $\Theta(\log_{\log n} \sigma)$ [18].

Alternative Representations of Sequences

There exist other compressed self-indices that can answer rank and select queries. Recently, a practical block tree implementation has been introduced [4]. A block tree is especially useful for highly compressible text, as they require only $O(z \log(n/z))$ words space, where z is the

²Bit-wise negated canonical Huffman codes are required [10].



2: Figure (left) wavelet 4-arv wavelet tree and a 4-arv matrix (right) for the accessandselect the alphabet text over $\{a\ (000)_2, c\ (001)_2, d\ (010)_2, e\ (011)_2, l\ (100)_2, n\ (101)_2, s\ (110)_2, t\ (111)_2\}\ (bit\ rep$ resentation of the characters is given in gray), i.e., the same input text as in Figure 1.

number of Lempel-Ziv factors of the text. Unfortunately, even highly tuned implementations are slow to compute [30]. Further dictionary-compressed representations allow for rank and select support in optimal time in compressed space [43] with respect to the size of a string attractor [42] of the text. For a grammar of size g and an alphabet of size g, rank and select support requires $O(\sigma g)$ space [5, 39]. Here, queries can be answered in $O(\log n)$ time.

4 4-Ary Wavelet Trees

When answering queries using a wavelet tree in practice, the query is translated to $O(\log \sigma)$ binary rank and select queries. Most of the time to answer a query on the wavelet tree is spent answering these binary rank and select queries. Additionally, on each level of the wavelet tree, the binary rank and select queries will result in at least one cache miss, which again is where most of the time for answering a binary rank or select query is used for. To reduce the number of cache misses, we have to reduce the number of levels. To this end, we make use of 4-ary wavelet trees. By doubling the number of children, we (roughly) halve the number of levels. If $\lceil \log \sigma \rceil$ is odd, the 4-ary wavelet tree has $\lceil \lceil \log \sigma \rceil/2 \rceil$ levels.

In a 4-ary wavelet tree, we represent the characters on each level using two bits that we store in a quad vector, i.e., a vector over the alphabet $\{0,1,2,3\}$ with access, rank, and select support, see Section 5. If $\lceil \log \sigma \rceil$ is odd, characters on the last level are represented using a single bit in a bit vector. In the first level, each character is now represented by its two MSBs and all characters share a length-0 bit-prefix. When visiting the first child of a node that represents characters with bit prefix α , its four children represents characters with bit-prefix α 00, α 01, α 10, and α 11 (first to fourth child). Alternatively, you can say that the *i*-th child represents characters that are in the *i*-th quarter of the alphabet represented by its parent. See Figure 2 for an example.

There also exists a "4-ary" wavelet matrix representation of the 4-ary wavelet tree. Here, we also use two bits to represent the characters at each level. Again, the first level of the wavelet matrix is the same as the first level of the 4-ary wavelet tree. Then, to compute the next level ℓ starting with the second, the text is stably sorted using the $(2\ell-1)$ -th and 2ℓ -th MSBs as key. As with the "normal" wavelet tree and wavelet matrix, this results in the same intervals, where characters with the same bit-prefix are represented, just in a different order. Also, queries for the wavelet matrix can be adopted to work with the (4-ary) wavelet matrix in the same way we adopted the queries of the wavelet tree. To do so, we now have to store the exclusive prefix sum of the histogram of all entries of all levels (as a replacement of Z in

the "normal" wavelet matrix).

Queries in 4-Ary Wavelet Trees

Querying a 4-ary wavelet tree works similarly to querying a "normal" wavelet tree. Since there are now four children, more book keeping is necessary to keep track of the interval that is visited during the query. This does not result in more rank and select queries on the quad vectors (and possibly bit vector on the last level). Overall, the additional book keeping is less expensive than the cache misses on each level as we can clearly see in our experiments in Section 7.1. Then again, querying the 4-ary wavelet matrix works similarly to querying the "normal" wavelet matrix. See Algorithm 6.1 for an example.

5 Quad Vectors

At the heart of our 4-ary wavelet trees is a space-efficient and fast rank and select data structure for quad vectors. Our data structure uses a block-based design and follows the popular memory layout for block-based rank and select data structures for bit vectors [31, 47] adapted to quad vectors. In a block-based design, the number of occurrences of different symbols is stored for blocks of different size. The number is stored either for the whole input up to the block or for the input contained in a bigger block. For our quad vector, we store the following information for each symbol $\alpha \in \{00, 01, 10, 11\}$:

Super Blocks cover 4096 symbols and store the number of occurrences before the start of the super block.

Blocks cover 512 symbols and store the number of occurrences before the start of the block within the super block.

For each super block, we only have to store seven blocks, as there are no occurrences of any symbol before the first block within the super block, i.e., all counters are zero. The counter within each block can be stored in just 12 bits, as the maximum number of occurrences of a single symbol within one super block before the last block is 3584 ([log 3584] = 12). Therefore, the counters of the seven blocks fit into 84 bits and can use 44 bits for the counter of the super block, when using 128 bits for both super block and the pertinent blocks.³ Additionally, storing super blocks and pertinent blocks interleaved, reduces the number of cache misses and allows for the usage of vectorized instructions [31].

Lemma 5.1. A quad vector with rank support has a space-overhead of 6.25 %. Adding select support introduces an additional overhead of $4\lceil \log n \rceil/8192$.

Proof. We require 128 bits for each super blocks including its blocks for each symbol. Resulting in a space-overhead of $4 \cdot 128/8192 = 6.25 \%$. Since we store the block of every 8192-nd occurrences of a symbol to answer select queries more efficiently, this introduces another $4 \cdot 32/8192 = 1.5625 \%$ space-overhead (allowing select support for quad vectors of length up to 2^{41} quads).

³In practice, computer words have size 8, 16, 32, 64, 128, and on modern machines even 256 and 512 bits. Aligning the size of (super-)blocks with computer words improves the performance.

We can reduce the number of cache misses by doubling the required space. A cache line on nearly all hardware has size 64 bytes. To make a super block and its pertinent blocks fit into one cache line, the number of symbols per super block and block can be halved. This doubles the number of counters we have to store, but we also guaranteed at most two cache misses per rank query and three per select query. However, by doing so, the space-overhead increases to 12.5%.

In theory, we can save space by storing only information for three symbols and using sophisticated data structures to represent the information. However, we did not implement the following variant, as preliminary experiments showed that the space-saving features heavily impacted the query performance.

Lemma 5.2. A quad vector with rank support requires only 2.41 % space-overhead.

Proof. We only save the information for three of the four symbols, as we can compute all information for the fourth symbol using the information of the other three symbols. Removing the information for one symbol saves 25% of space, hence the space-overhead is now only 4.6875%.

Using the Elias-Fano encoding [13, 15], a monotonic increasing sequence of k integers in a universe of size u can be stored using only $k(2 + \log(u/k))$ bits while allowing constant time access to all integers. Since the number of occurrences of symbols withing super blocks are monotonic increasing sequences, we can use Elias-Fano encoding to store them. To this end, we introduce mega~blocks that cover 2^{18} symbols. We store the number of occurrences of each symbol from the beginning of the text to the beginning of each mega block and encode only the information for the remaining three symbols. We now can store the following information for each super block: Three 18-bit counters for three symbols storing the number of occurrences from the beginning of the mega block and the Elias-Fano encoded sequences that require at most 141 bits. Overall, we require 195 bits for all three symbols. Therefore, this variant has a space overhead of 2.41%.

5.1 Answering Queries

Answering queries using this approach is similar to the bit vector case. Assume we want to get the rank of the symbol α at position i. We simply have to identify the super block (i/4096) and the block $(i \mod 4096)/512)$ where the position occurs in. Adding up these counters, we only have to scan $(i \mod 512)$ positions within the block and add the number of occurrences of α in the block up to that position to the result. All this can be done in constant time. To find the position where the j-th α occurs, we first identify the closest smaller sampled position. Starting from there, we do scan super blocks until we have identified the super block containing the position. Then, we continue with scanning block until we have identified the block containing the position. Finally, we scan the quad vector (within the block) until we have found the correct position and return the index. While this may not be a constant time query, it is very fast in practice, see Section 7.

```
 \begin{array}{lll} \textbf{1 Function } Rank_{\alpha}(i) \\ \textbf{2} & | r_0 = i, \ b_0 = 0 \\ \textbf{3} & | \textbf{for } k = 1, \dots, \ell + 1 \ \textbf{do} \\ \textbf{4} & | \alpha_k = (\alpha >> 2*(\ell - 1 - k)) \& 3, \, \textbf{offset} = C_k[\alpha_k] \\ \textbf{5} & | b_k = Q[k].rank_{\alpha_k}(b_{k-1}) + \textbf{offset} \\ \textbf{6} & | r_k = Q[k].rank_{\alpha_k}(r_{k-1}) + \textbf{offset} \\ \textbf{7} & | \textbf{return } r_{\ell} - b_{\ell} \\ \end{array}
```

Algorithm 6.1. Rank query for a 4-ry wavelet matrix with ℓ levels. For level k, Q[k] is the quad vector and $C_k[\alpha_k]$ is the number of character $< \alpha_k$ on level k.

6 Faster Rank Queries with Prefetching

Modern CPUs can issue multiple memory requests concurrently, paving the way for proactive prefetching of cache lines predicted to be accessed in the near future. By issuing the memory requests for the accessed cache line and the anticipated ones simultaneously, prefecthing helps hiding memory latency and reducing the impact of memory access delays on the CPU's execution pipeline.

Prefetching manifests in two forms: hardware and software prefetching. Hardware prefetching is implemented within the CPU's microarchitecture and is driven by the hardware itself. Modern CPUs come equipped with dedicated prefetcher units that monitor memory access patterns and automatically issue prefetch requests based on these patterns. These units analyze the memory addresses being accessed and attempt to predict future memory accesses. They then fetch the predicted data into the cache in advance. For example, the sequential prefetching predicts that the next memory location to be accessed will be contiguous to the current one. It fetches additional cache lines in advance to take advantage of spatial locality. This is particularly effective for array traversal where data tends to be stored sequentially. The strided prefetching instead looks at the delta between the addresses of the memory accesses and looks for patterns within it. If a consistent pattern in the stride is detected, the CPU fetches cache lines based on this pattern, assuming that the algorithm will continue accessing memory addresses with the same stride.

Software prefetching, instead, is controlled by the programmer or the compiler through explicit instructions. Programmers can insert prefetch instructions (e.g., _mm_prefetch intrinsic for x86 CPUs) into their code to indicate which data should be pref etched and when. However, software prefetching requires a deep understanding of the algorithm's memory access patterns and the underlying memory hierarchy, because incorrect or excessive prefetching can lead to performance degradation.

The goal of this section is to show how to introduce software prefetching in the algorithm of the rank query. For the following discussion, we give the pseudo code for $rank_{\alpha}(i)$ query on a 4-ry wavelet matrix in Algorithm 6.1. A $rank_{\alpha}(i)$ query on a wavelet tree has to traverse each of the $\ell = \lceil \lceil \log \sigma \rceil / 2 \rceil$ levels. At each level k, we perform two rank queries on the quad vectors of that level for the character $\alpha_k \in [0,3]$ to compute b_k and r_k . These two rank queries use the results b_{k-1} and r_{k-1} of the two rank queries computed at the previous level. Below we discuss only how to perform the prefetching for r_k as the prefetching for b_k can be

done in a similar way.

Every rank query in a quad vector for a given position i needs to access only two cache lines: the one containing counters for the superblock and block of that position, and the one containing the data block with the i-th character. These two cache lines can be requested in parallel as they only depend on position i. Hence, we observe just the latency of at most one cache miss per level. The challenge in eliminating this cache miss is that both the cache lines we need to access at a certain level k depend on the position r_{k-1} , which is known only when the rank query at level k-1 has been computed. Solving this challenge could be possible with a predictive model capable of anticipating the cache lines required for ranking at position r_{k-1} across all levels k, way before position r_{k-1} is computed. Such an advanced prediction would enable us to initiate simultaneous requests for all these cache lines.

6.1 Predicting Cache Lines in a Quad Vector

This challenge led us to the definition of the *Rank with Additive Approximation* problem and our predictive model will take the form of a lightweight data structure.

Definition 6.1. Given a quaternary vector Q[1, n] and fixed an additive error ϵ , the goal is to build a data structure to answer additive approximated rank queries. Given a position i and a symbol $\alpha \in [0, 3]$, $rank_{\alpha}^{\approx}(i)$ approximates the correct rank query by returning any arbitrary value \tilde{r} within $[r, r + \epsilon]$, where $r = rank_{\alpha}(i)$.

A prediction model that correctly predicts the needed cache lines of a certain level, is actually solving the Rank with Additive Approximation problem on the quad vector of the previous level with ϵ equal to the cache line size, e.g., 512 bits (256 quads). Vice versa, if we have a solution for the problem with the same ϵ , we have a way to predict the required cache lines.

Lemma 6.1. Any data structure that solves the Rank with Additive Approximation problem on Q[1,n] with additive error ϵ needs at least $\Omega(n/\epsilon)$ bits of space.

Proof. Assume by contradiction that there exists a solution for the problem that uses $o(n/\epsilon)$ bits of space for any quad vector of length n. Then, we could use this data structure to represent any quad vector with less than 2n bits, which is impossible because of an information-theoretical lower bound.

Given any Q[1, n], we obtain its expanded version of $\hat{Q}[1, 3\epsilon n]$ by replacing each character with a run of 3ϵ of its copies. We use the above data structure to index \hat{Q} using o(n) bits of space. Now, we reconstruct Q by querying the data structure for any character at the beginning and the end of each run. The correct character in Q can be identified because the results of the two queries differ by at least 2ϵ , while the results for the other characters differ by at most ϵ .

Lemma 6.2. There is a data structure with constant query time requiring $\Theta(n/\epsilon)$ bits, i.e., matching the space lower bound, for the Rank with Additive Approximation problem on Q[1, n] with additive error ϵ .

Proof. The idea is to use a bit vector $B_{\alpha}[1, \lceil 2n/\epsilon \rceil]$, for each of the character $\alpha \in [0, 3]$. We split Q[1, n] into blocks of size $\epsilon/2$. The *i*th bit in B_{α} is set to 1 if and only if the *i*th block of Q contains the *j*th of α , for some j which is a multiple of $\epsilon/2$.

We add the required extra data structure to support rank queries on the bit vector B_{α} . A query $rank_{\alpha}^{\approx}(i)$ is solved as follows. Let $j = \lfloor 2i/\epsilon \rfloor$ be the block in Q that contains our target position i. We compute $k = rank_1(j-1)$ on the bit vector B_{α} . This way, we know that the number of occurrences in Q up to position i is at least $r \cdot \epsilon/2$. Moreover, the exact number of occurrences of α up to the block j is at most $k \cdot \epsilon/2 + \epsilon/2 - 1$. As the jth block as size $\epsilon/2$, we conclude that returning $\tilde{r} = k \cdot \epsilon/2$ gives the required estimate.

6.2 Predicting Cache Lines in a Wavelet Tree

Let us consider the rank query $rank_{\alpha}(i)$. Consider the rank query $rank_{\alpha}(i)$. For addressing this query through a 4-ary wavelet tree, we divide the character α into its quaternary components $\alpha_1, \alpha_2, \ldots, \alpha_\ell$. Then, at level k, we compute $r_k = rank\alpha_k(r_k - 1)$. See Algorithm 6.1. As we mentioned above we focus on prefetching for r_k s (line 5), as we can deal with b_k s in a similar way.

The prefetching is possible if can approximate each r_k with \tilde{r}_k , such that $\tilde{r}_k \in [r_k, r_k + \epsilon]$ with $\epsilon = 256$. Indeed, each cache line has size 512 bits and, thus, spans 256 positions of the quad vector at level k. The value \tilde{r}_k introduces uncertainty only within the span of two consecutive cache lines. Note that prefetching is effective only if we compute the approximated ranks \tilde{r}_k for all the levels. This way we issue the requests for all the required cache lines in parallel before starting to use these cache lines to compute the exact ranks r_k .

Unfortunately, solving the Rank with Additive Approximation problem with error ϵ for the quad vector at each level of the wavelet tree is not enough to guarantee that \tilde{r}_k is at most at distance ϵ from r_k (i.e., $\tilde{r}_k \in [r_k, r_k + \epsilon]$), for all the levels k. This is because the value \tilde{r}_k is computed with an approximated rank at position \tilde{r}_{k-1} because the exact position r_{k-1} is unknown, i.e., we can compute $rank_{\alpha_k}^{\approx}(\tilde{r}_{k-1})$ and not $rank_{\alpha_k}^{\approx}(r_{k-1})$. As the position \tilde{r}_{k-1} is already affected by some error, the errors of our approximations sum up level by level. Thus, at level k the error could be up to $(k-1)\epsilon$.

We can solve this issue by correcting the approximations at each level. This approach is inspired by a solution for the substring occurrence estimation on texts with compressed indexes [37]. The main idea is to refine the estimates at each level k with a correction term Δ . To compute Δ we need to store a set of discriminant positions $D_{k,\alpha}$ for each character $\alpha \in [0,3]$ at level k.

In the solution of Theorem 6.2 we store a bit vector B_{α} for each character $\alpha \in [0,3]$. A bit was set to one for each position corresponding to an occurrence of α which is a multiple of ϵ . The set $D_{k,\alpha}$ consists of the position in the quad vector corresponding to those occurrences. We note the positions in these sets can be stored within $\Theta(\log \epsilon)$ bits per position in several ways. The most suitable one for our purposes is to associate each position with its corresponding bit set to one in B_{α} and store its offset within the corresponding block

At query time, given r_{k-1} and the character α_k , we want to compute the discriminant position d_{k-1} which is the successor of r_{k-1} in the set D_{k,α_k} . This discriminant position can be computed in constant time with a rank and a select query on the bit vector of α . Once

we computed d_{k-1} , the correction term Δ is $\min(d_{k-1} - \tilde{r}_{k-1}, \epsilon - 1)$ and the approximated rank is computed as $\tilde{r}_k = rank_{\alpha_k}(d_{k-1}) - \Delta$. This correction is enough to guarantee that our approximations always remain at a distance at most ϵ from the correct ones over all the levels k of the wavelet tree.

Lemma 6.3. At any level k, we have $\tilde{r}_k \in [r_k, r_k + \epsilon)$.

Proof. The proof is by induction on k. For the first level k=1, as at the beginning $\tilde{r_0}=r_0$, we have $r_1 \in [r_1, r_1 + \epsilon)$ by Theorem 6.2. For general k, we assume that $\tilde{r}_{k-1} \in [r_{k-1}, r_{k-1} + \epsilon)$, and we prove $\tilde{r}_k \in [r_k, r_k + \epsilon)$. We want to prove that $\tilde{r}_k \leq r_k$ and $r_k - \tilde{r}_k \leq \epsilon$. There are two cases based on the relationship between d_{k-1} and r_{k-1} . By definition we know that $\tilde{r}_{k-1} \leq d_{k-1}$ and by inductive hyphotesis $\tilde{r}_{k-1} \leq r_{k-1}$.

The first case is $r_{k-1} \leq d_{k-1}$. Thus, we have $\tilde{r}_{k-1} \leq r_{k-1} \leq d_{k-1}$. Let z be the number of occurrences of the ranked character α_k in the interval $[r_{k-1}, d_{k-1}]$.

$$r_{k} - \tilde{r}_{k} = rank_{\alpha_{k}}(r_{k-1}) - rank_{\alpha_{k}}^{\approx}(\tilde{r}_{k-1})$$

$$= rank_{\alpha_{k}}(d_{k-1}) - z - (rank_{\alpha_{k}}(d_{k-1}) - \Delta)$$

$$= \Delta - z < \epsilon$$

The last inequality follows by $[r_{k-1}, d_{k-1}]$ being contained in $[\tilde{r}_{k-1}, d_{k-1}]$, bounding z by the minimum of the length of $[\tilde{r}_{k-1}, d_{k-1}]$ and $\epsilon - 1$. If the interval is larger than $\epsilon - 1$, there cannot be more than $\epsilon - 1$ of α_k since we sampled a discriminant position every ϵ occurrences of α_k . It also follows that $z \leq \Delta$ and, thus, $\tilde{r}_k \leq r_k$.

The second case is $d_{k-1} < r_{k-1}$. Thus, $\tilde{r}_{k-1} \le d_{k-1} \le r_{k-1}$. Let z be the number of occurrences of α_k in the interval $[r_{k-1}, d_{k-1}]$.

$$r_k - \tilde{r}_k = rank_{\alpha_k}(r_{k-1}) - rank_{\alpha_k}^{\approx}(\tilde{r}_{k-1})$$

$$= rank_{\alpha_k}(d_{k-1}) + z - (rank_{\alpha_k}(d_{k-1}) - \Delta)$$

$$= z + \Delta \leq r_{k-1} - \tilde{r}_{k-1} \leq \epsilon$$

The first inequality follows by observing that Δ is at most the distance between \tilde{r}_{k-1} and d_{k-1} and z is at most the distance between d_{k-1} and r_{k-1} . So, their sum is at most $r_{k-1} - \tilde{r}_{k-1}$. The last inequality is by inductive hypothesis.

The space required by this predicting data structure is $\Theta((n/\epsilon)\log\epsilon)$ for each level of the wavelet tree. So, the overall space usage is $\Theta((n\log\sigma/\epsilon)\log\epsilon)$ bits. As we mentioned above, prefetching with the above data structure can be done by setting $\epsilon=256$. However, we are left with an issue. If the indexed sequence is too large, the predicting data structure itself does not fit in the cache and, thus, to avoid cache misses in the wavelet tree we would pay cache misses in the predicting data structure. This issue could be solved by introducing a hierarchy of predictors in which a predictor at a specific level takes on the responsibility of prefetching the necessary cache lines for the subsequent-level predictor. Each predictor allows an error that is roughly ϵ times less than the one at the next level, until the predictor at the head of the hierarchy fits in cache. Unfortunately, a larger hierarchy becomes impractical quite soon for two reasons. First, to fully exploit prefetching we would have to request all the predicted cache lines in parallel, and current CPUs can issue only 5–10 memory requests in parallel. Second, each level of the hierarchy introduces a cost of $\Theta(\log \sigma)$ to the query.

6.3 Practical Implementations

In our implementation, we relaxed the previous solution in several respects. First, we do not use the correcting term Δ and the discriminant positions. This is because in our tests we used sequences with an alphabet size σ up to 256, which requires a wavelet tree of at most 4 levels. Thus, the error growth here is very limited and it can be afforded by prefetching more cache lines. Second, we limit the hierarchy to just two levels of predictors. The first one implements the solution of Theorem 6.2 with error $\epsilon = 2048$. For the second level, we observe that super block and block counters can be used as a variant of the solution of Theorem 6.2 with error $\epsilon = 256$. Even if is more space inefficient, it is preferred because the space is already needed by the wavelet tree implementation. This way, we can use the first level to predict the super block containing r_k for each level and prefetch the cache lines containing the counters of those super blocks and their blocks. Then, we use these counters to refine the predictions to prefetch the cache lines with the correct blocks of data in the quad vectors.

The first level of the hierarchy has to store a bit vector of size n' = n/2048 bits for each character $\alpha \in [0,3]$ in each level of the wavelet tree. We enhance this bit vector with rank support with a space overhead of n'/4 bits [46]. So, the overhead of the predicting data structure is $\ell(4n/2048 + n/2048) = 5\ell n/2048$ bits, where ℓ is the number of levels in the 4-ary wavelet tree. The predictor at the second level of the hierarchy does not introduce any extra space overhead. Observe that the first predictor fits in a 32 MB L3 cache for sequences up to ≈ 100 Billions of characters over an alphabet of size 256.

We conclude by observing that cache lines needed by access and select queries cannot be predicted with proposed solutions. The reason is that each query on a certain level has a double dependency on the result of the previous one. Indeed, both position and symbol are known only when the previous query is solved.

7 Experimental Evaluation

We first discuss our experimental setup. Then, in Section 7.1, we show the benefit of using 4-ary wavelet trees and approximate rank queries. Finally, in Section 7.2, we compare quad and bit vectors.

Experimental Setup

For our experiments, we used a machine equipped with two AMD EPYC 7713 (64 cores, 2 threads per core, 2 GHz base clock, and cache sizes: 64 KB L1I and 64 KB L1D per core, 512 KB L2I+D per core, and 256 MB L3I+D, with 32 MB per 8 cores called *core complex*) and 2 TB DDR4 RAM running Ubuntu 20.04.3 LTS kernel version 5.4.0-155. All experiments were performed using a single thread, with hyperthreading and turbo boost disabled. C++ code of competitors was compiled with GCC 11.1.0 with flags 03 and march=native and Rust code was compiled using cargo build --release. Our Rust implementation is available at https://github.com/rossanoventurini/qwt. We ran each experiment ten times (10M queries for each run) and report the average running time. We generate all queries in advance as follows. Let S[0,n) be the indexed sequence. Each access query asks to access

Table 1: Latency of access, rank, and select queries (row 1–3) given in μs and the space (row 4) is given in GiB. The small number in parentheses is the speedup of QWM_{256}^{pfs} over the method represented by the column. All results for 8 GiB input files.

	input	sdsl_wm	sdsl_fbb	pasta_wm	sucds	$\rm QWM^{pfs}_{256}$
access	English CC DNA Wiki	$\begin{array}{ccc} 1270 & (1.7\times) \\ 1185 & (1.7\times) \\ 239 & (1.5\times) \\ 1216 & (1.7\times) \end{array}$	$\begin{array}{ccc} 1506 & (2.1\times) \\ 1897 & (2.7\times) \\ 665 & (4.2\times) \\ 1712 & (2.4\times) \end{array}$	$\begin{array}{ccc} 1618 & (2.2\times) \\ 1511 & (2.2\times) \\ 353 & (2.2\times) \\ 1681 & (2.4\times) \end{array}$	$\begin{array}{ccc} 1122 & (1.5\times) \\ 1210 & (1.7\times) \\ 316 & (2.0\times) \\ 1198 & (1.7\times) \end{array}$	$731 (1.0 \times)$ $700 (1.0 \times)$ $157 (1.0 \times)$ $712 (1.0 \times)$
rank	English CC DNA Wiki	$\begin{array}{ccc} 1498 & (3.2\times) \\ 1350 & (2.8\times) \\ 321 & (1.8\times) \\ 1402 & (2.9\times) \end{array}$	$\begin{array}{ccc} 1474 & (3.1\times) \\ 1913 & (3.9\times) \\ 665 & (3.8\times) \\ 1649 & (3.4\times) \end{array}$	$\begin{array}{ccc} 1797 & (3.8\times) \\ 1725 & (3.5\times) \\ 394 & (2.2\times) \\ 1855 & (3.8\times) \end{array}$	$\begin{array}{cc} 1408 & (3.0\times) \\ 1424 & (2.9\times) \\ 503 & (2.9\times) \\ 1442 & (3.0\times) \end{array}$	$472 (1.0 \times)$ $490 (1.0 \times)$ $176 (1.0 \times)$ $488 (1.0 \times)$
select	English CC DNA Wiki	4849 (2.2×) 4483 (2.0×) 1032 (2.0×) 4546 (2.1×)	 	4882 (2.2×) 4646 (2.1×) 910 (1.7×) 4956 (2.3×)	$\begin{array}{ccc} 4245 & (1.9\times) \\ 4396 & (1.9\times) \\ 1440 & (2.8\times) \\ 4349 & (2.0\times) \end{array}$	$\begin{array}{ccc} 2229 & (1.0\times) \\ 2260 & (1.0\times) \\ 521 & (1.0\times) \\ 2185 & (1.0\times) \end{array}$
space	English CC DNA Wiki	11.9 11.9 3.0 11.9	5.0 5.8 2.2 5.8	8.0 8.0 2.0 8.0	10.5 10.5 3.9 10.5	9.0 9.0 2.3 9.0

the symbol at a random position in the sequence. For rank queries, we generate a random position $i \in [0, n)$ and use $\langle i, S[i] \rangle$ as a rank query. For select queries, we select a symbol c at random following their distribution in the sequence, i.e., more frequent symbols have a higher probability of being selected. Then, we generate a random value $r \in [1, occ(c)]$, where occ(c) is the number of occurrences of c, and use $\langle r, c \rangle$ as a select query.

7.1 Evaluation of 4-Ary Wavelet Trees

We compare our 4-ary wavelet trees with other wavelet trees. Note that we compare wavelet matrices if available, as those are faster in practice than wavelet trees, however, all implementations mentioned below contain also support for both wavelet trees. To the best of our knowledge, there exists no other k-ary wavelet tree implementation for k > 2 and no other (uncompressed) wavelet tree implementation with access, rank, and select support. In the following, $sdsl_wm$ denotes wavelet matrices built on bit vectors of the SDSL library (wm_int) [21]. We also included the fastest compressed wavelet tree implementation in the SDSL— $sdsl_fbb$ [22]. A wavelet matrix implementation built on bit vectors of the PASTA-toolbox library, using the most space-efficient rank and select data structures [31], is denoted by $pasta_wm$. Additionally, sucds is the wavelet matrix implementation in the sucds library⁴. QWM_{256} and QWM_{512} are our implementations of wavelet matrices built on quad vectors

⁴https://github.com/kampersanda/sucds, last accessed 2023-11-08.

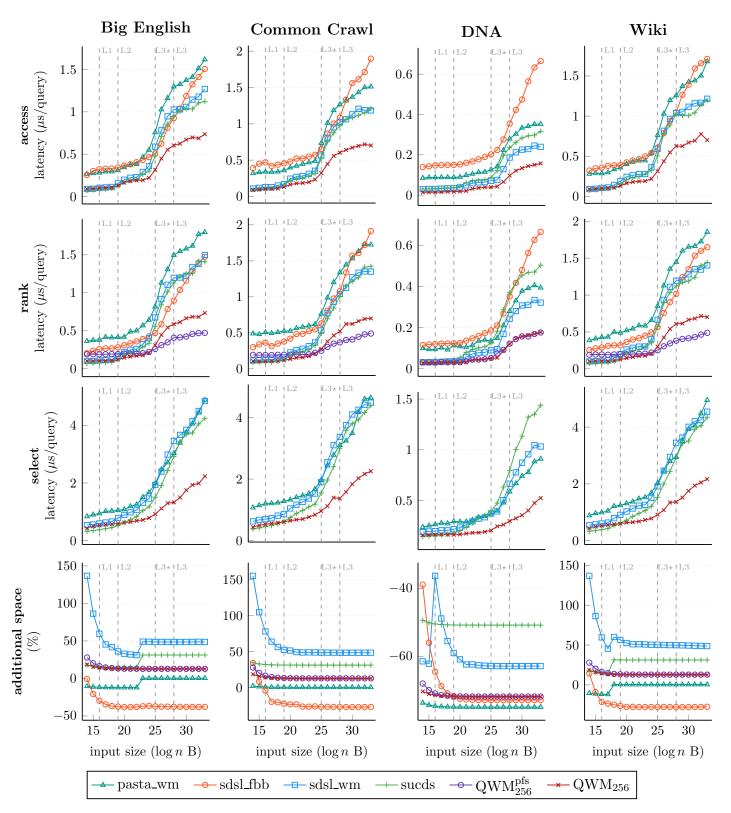


Figure 3: Comparison of our new wavelet trees with competitors. The first three rows give the access, rank, and select query latency of the implementations, and the last row shows the space overhead of the wavelet tree compared to the input (storing one character per byte). The vertical dashed gray lines indicate the L1, L2, L3*, and L3 cache sizes of the CPU used for these experiments (L3* indicates the L3 cache size per core complex).

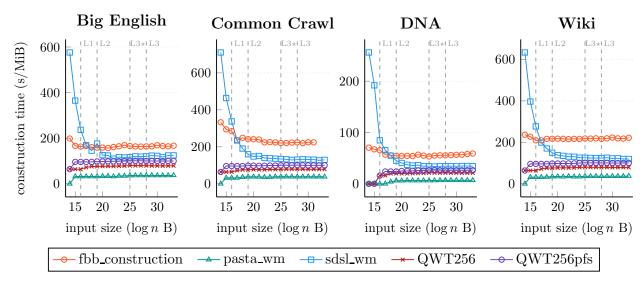


Figure 4: Wavelet tree construction times normalized by input size. Vertical dashed gray lines indicate the L1, L2, L3 \star , and L3 cache sizes of the CPU used for these experiments (L3 \star indicates the L3 cache size per core complex).

with blocks of size 256 and 512 symbols per block, cf. Section 4. QWM^{pfs} denotes the usage of our predictive model (see Section 6). We wanted to include a wavelet matrix based on learned compressed rank and select data structures [7], however, here query times are significantly greatermaking the plots harder to read. Furthermore, the experiments for inputs $> 1 \, \text{GiB}$ did not finish in reasonable time. Therefore, we excluded this implementation from the results.

As inputs, we use text prefixes between 16 KiB and 8 GiB in size, generated from the following datasets. *English* is the concatenation of all 35 750 English text files from the Gutenberg Project. We removed the headers related to the project, leaving just the real text. *DNA* are FASTQ files from the 1000 Genomes Project, where we considered only the raw sequence and kept only the character A, C, G, and T, to obtain a very small alphabet. *CC* is a concatenation of the WET files of Common Crawl corpus, i.e., a web crawl without HTML tags. Here, we removed all meta information added by the corpus. *Wiki* is a concatenation of XML data of the English Wikipedia from June 2023. Note that we did not use the famous Pizza&Chili corpus, as we needed inputs larger than 2 GiB.

Access, Rank, and Select Queries

The first three rows of Figure 3 show the query result of our experimental evaluation. In the main part, we only list results for our wavelet tree with block size 256 as they are never slower than those with block size 512, see Figure 7. Overall, sucds almost always at most as fast as sdsl_wm, therefore, to save space, we only discuss sdsl_wm in the following. For access and select queries, the behavior of all algorithms is very similar. Here, pasta_wm is always the slowest. For inputs for which the wavelet tree fits into the L2 cache, sdsl_wm and QWM₂₅₆ have a similar query time. This is to be expected, as there are still not many cache misses during querying. There is a steep increase in query time as soon as the wavelet tree

does not fit into the L3 cache of the core complex. However, QWM₂₅₆'s query time does not increase as fast, resulting in a speedup of up to 1.73 (access) and 2.17 (select) compared to sdsl_wm.

Finally, for rank queries the behavior of the three previously discussed algorithms is similar. But here, we also have our wavelet tree QWM₂₅₆ using our prediction mode. We can actually see it being effective as soon as the wavelet trees doe not fit into the L3 cache of the core complex, i.e., as soon as we expect more and more cache misses during querying. Here, QWM₂₅₆ becomes faster than QWM₂₅₆, resulting in a speedup of up to 1.55 compared to our wavelet tree without prediction model and up to 3.17 compared to sdsl_wm. A summary of the results for the largest inputs can be found in Table 1.

In Figure 7, we show a comparison of our different wavelet tree configurations. Here, we can see that for access and select queries the query time is very similar for QWM_{256} and QWM_{512} . However, for rank queries, QWM_{256}^{pfs} is always faster than QWM_{512}^{pfs} . Therefore, we only include QWM_{256} and QWM_{256}^{pfs} in the main part of this paper. Note that the space requirements are as expected for all wavelet tree variants.

In Figure 5, we show a comparison of the throughput of all tested wavelet trees. Here, we can see that our approach has a similar benefit. We currently cannot explain the outliers in our measurements for small inputs, however, they do not occur on other hardware and are not easy to reproduce even on our experimental setup. We are sure that no other heavy process was running at the time of the experiment.

Space-Overhead and Construction

The last row in Figure 3 shows the additional space required by the wavelet tree compared to the input. Additional space < 0% is due to the compression of the alphabet, where the wavelet tree requires less than 8 levels and the input requires one byte per character. Similarly, stark increases are due to a new level being requires, which is not the case for our wavelet trees, as we do not use a bit vector for the last level, even if it would suffice, initially wasting some space in the progress. The spikes in the beginning are due to the general overhead of the data structures. Overall, the space-overhead is mostly based on the used rank and select data structures. Since our quad vectors support rank and select using less memory than the bit vectors in the SDSL, our wavelet trees are also more memory efficient—around 75% less space-overhead. Without surprise, pasta_wm is the most space-efficient wavelet tree, however, this comes with slower (wavelet tree) queries.

The construction time of the wavelet trees is shown in Figure 4. Here, pasta_wm is the fastest to construct, our QWM_{256} and QWM_{256}^{pfs} require similar time but the additional time required for the prediction model is visible, and sdsl_wm is the slowest to construct. Note that we are not focusing on the construction and highly tuned construction algorithms for wavelet trees exist [12].

7.2 Evaluation of Quad Vectors

We now compare our quad vectors with the bit vectors used in the wavelet trees in Section 7.1 to show that the speedup is actually due to the improvements presented in this paper and not only due to better rank and select support for quad vectors. sdsl_bv is the implementation

of bit vectors of the SDSL library [21]. For rank queries, the bit vector is enhanced with $rank_support_v$, and it uses $select_support_mcl$ to support select queries, i.e., Clark and Munro's approach to compute select queries [8]. pasta_bv is the implementation of bit vectors of the PASTA library [31]. The bit vector is enhanced using the FlatRankSelect data structure to support rank and select queries. QV_{256} and QV_{512} are our implementations of quad vectors with blocks of size 256 and 512 symbols (see Section 5).

In these experiments, we generate random bit/quad sequences containing from $16\,\mathrm{K}$ to $8\,\mathrm{G}$ symbols, i.e., the number of symbols contained in each level of the wavelet trees. Each bit has a $50\,\%$ chance of being set and each quad has a $25\,\%$ percent chance of appearing.

The running time of access, rank, and select queries of the bit vectors is depicted in Figure 6. There, we can see that access queries as well as rank and select queries for larger inputs require roughly the same time. However, for all input sizes, sdsl_bv requires roughly the same time (rank) or is faster (select) than QV_{256} , the quad vector used in the comparison of the wavelet trees.

The additional space usage is as expected and similar to the wavelet trees. Again, sdsl_bv has a spike for smaller inputs, which can be explained by the general overhead of the data structure starting at this input size.

Overall, the evaluation of the bit and quad vectors show that the improvements of three wavelet tree queries considered here are solely due to the algorithmic ideas presented in this paper.

8 Conclusion

We have presented two improvements for wavelet trees that achieve a speedup for access and select queries of up to 2 and for rank queries—which are the most important queries in many applications—up to 3. To this end, we first changed the underlying tree structure from a binary tree to a 4-ary tree, reducing the number of cache misses in the process. Furthermore, we introduced the Rank with Additive Approximation problem and showed a small predictive model that solved this problem in practice. This predictive models allows us to prefetch all data necessary for rank queries, resulting in a better speedup compared to access and select queries.

It remains an open problem to combine the 4-ary wavelet tree layout with the sublinear construction algorithm based on vectorized instructions. Another interesting line of future research are compressed 4-ary wavelet trees and compressed quad vectors. Additionally, we want to explore bit vectors for even larger alphabets, as our experiments indicate that the techniques proposed in this paper benefit from wavelet trees with more levels.

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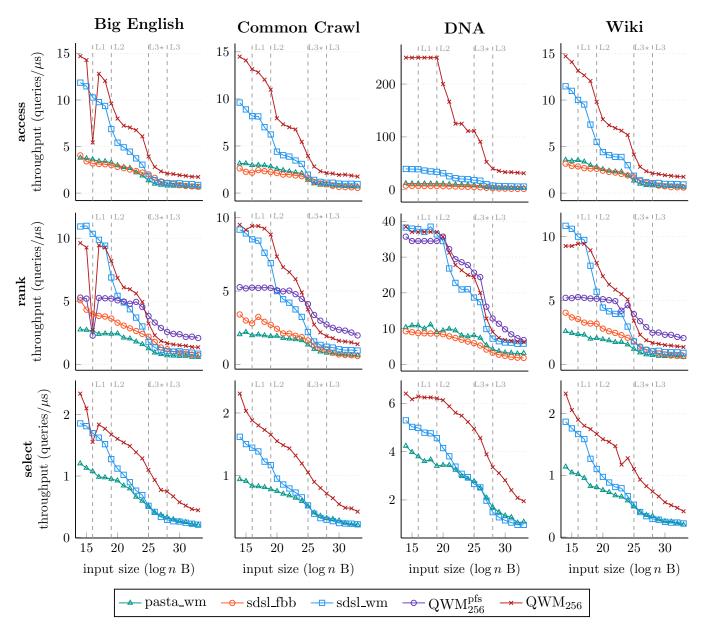


Figure 5: Comparison of the *throughput* our new wavelet trees with competitors. The vertical dashed gray lines indicate the L1, L2, L3 \star , and L3 cache sizes of the CPU used for these experiments (L3 \star indicates the L3 cache size per core complex). The additional space is the same as reported in our latency experiments in Figure 3.

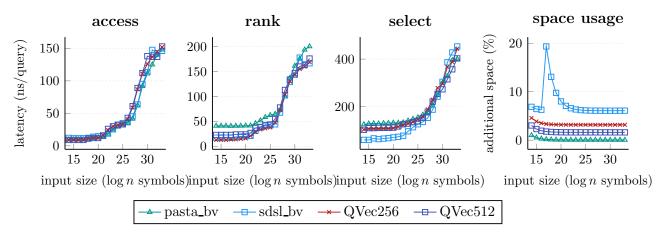


Figure 6: Comparison of access, rank, and select query latency for our quad vectors with bit vectors.

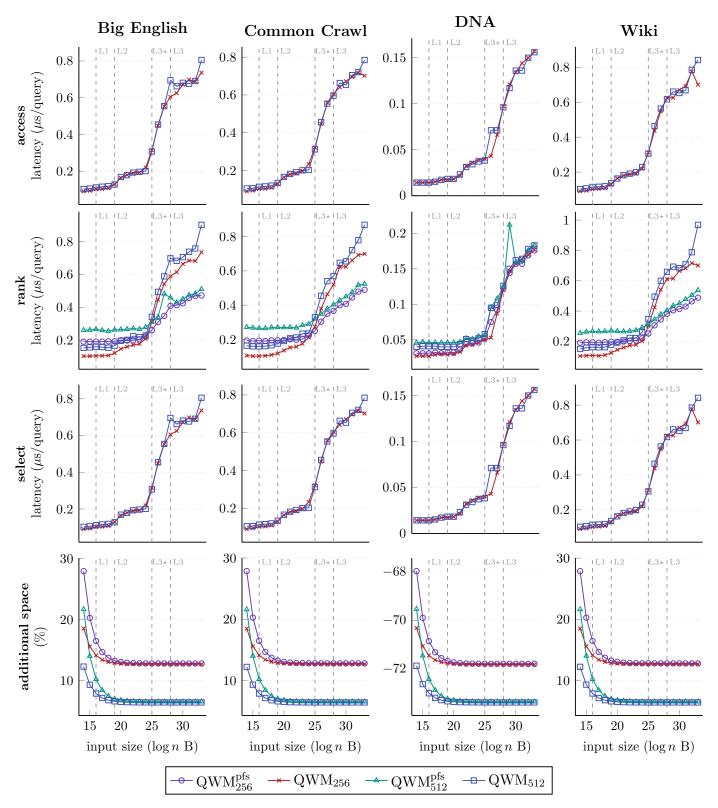


Figure 7: Comparison of all our new wavelet trees implementations. The first three rows give the access, rank, and select query latency of the implementations, and the last row shows the space overhead of the wavelet tree compared to the input (storing one character per byte). The vertical dashed gray lines indicate the L1, L2, L3 \star , and L3 cache sizes of the CPU used for these experiments (L3 \star indicates the L3 cache size per core complex).