Math 115AH - HW8

Due Tuesday, June 4, by 11:59pm

Edi Zhang

1. Textbook 6.1, #2. For this problem you'll have to read some things in the textbook that we didn't go over in class (specifically, the Cauchy–Schwarz inequality and the triangle inequality).

   Let $x = (2, 1+i, i)$ and $y = (2-i, 2, 1+2i)$ be vectors in $\mathbb{C}^3$ Compute $\langle x, y\rangle, \|x\|, \|y\|$, and $\|x+y\|$. Then verify both the Cauchy-Schwarz inequality and the triangle inequality.

   $\langle x, y\rangle$:

$$\begin{aligned}\langle x, y\rangle &= \langle (2, 1+i, i), (2-i, 2, 1+2i)\rangle \\ &= 2(2+i) + (1+i)(2) + i(1-2i) \\ &= 4 + 2i + 2 + 2i + 2 + i \\ &= 8 + 5i\end{aligned}$$

   $\|x\|$:

$$\begin{aligned}\|x\| &= \langle x, x\rangle^{\frac{1}{2}} \\ &= \langle (2, 1+i, i), (2, 1+i, i)\rangle^{\frac{1}{2}} \\ &= (2(2) + (1+i)(1-i) + i(-i))^{\frac{1}{2}} \\ &= (4 + 1 - i^2 - i^2)^{\frac{1}{2}} \\ &= (7)^{\frac{1}{2}} \\ &= \sqrt{7}\end{aligned}$$

   $\|y\|$:

$$\begin{aligned}\|y\| &= \langle y, y\rangle^{\frac{1}{2}} \\ &= \langle (2-i, 2, 1+2i), (2-i, 2, 1+2i)\rangle^{\frac{1}{2}} \\ &= ((2-i)(2+i) + 2(2) + (1+2i)(1-2i))^{\frac{1}{2}} \\ &= (4 - i^2 + 4 + 1 - 4i^2)^{\frac{1}{2}} \\ &= (4 + 1 + 4 + 1 + 4)^{\frac{1}{2}} \\ &= \sqrt{14}\end{aligned}$$

   $\|x+y\|$:

$$\begin{aligned}\|x+y\| &= \langle x+y, x+y\rangle^{\frac{1}{2}} \\ &= \langle (2, 1+i, i) + (2-i, 2, 1+2i), (2, 1+i, i) + (2-i, 2, 1+2i)\rangle^{\frac{1}{2}} \\ &= \langle (4-i, 3+i, 1+3i), (4-i, 3+i, 1+3i)\rangle^{\frac{1}{2}}\end{aligned}$$

$$= ((4-i)(4+i) + (3+i)(3-i) + (1+3i)(1-3i))^{\frac{1}{2}}$$
$$= (16+1+9+1+1+9)^{\frac{1}{2}}$$
$$= \sqrt{37}$$

Cauchy-Schwarz inequality:

$$|\langle x, y\rangle| \ \leq \|x\| \cdot \|y\|$$
$$|8+5i| \leq \|\sqrt{7}\| \cdot \|\sqrt{14}\|$$
$$|8+5i| \leq \sqrt{7} \cdot \sqrt{14}$$
$$\sqrt{64-25} \leq \sqrt{98}$$
$$\sqrt{39} \leq \sqrt{98}$$

Triangle inequality:

$$\|x+y\| \leq \|x\| + \|y\|$$
$$\sqrt{37} \leq \sqrt{7} + \sqrt{14}$$
$$37 \leq 7 + 14 + 2\sqrt{98}$$
$$37 \leq 21 + 2\sqrt{98}$$
$$16 \leq 2\sqrt{98}$$
$$8 \leq \sqrt{98}$$
$$64 \leq 98$$

2. Textbook 6.1, #4.

   (a) **Complete the proof in Example 5 that $\langle \cdot, \cdot \rangle$ (the Frobenius inner product) is an inner product on $M_{n\times n}(F)$**

   Need to verify (b and c)

   $\langle cx, y\rangle = c\langle x, y\rangle$

$$\langle \lambda A, B\rangle$$
$$= \operatorname{tr}(B^* \lambda A)$$
$$= \lambda(\operatorname{tr}(B^* A))$$
$$= \lambda\langle A, B\rangle$$

   $\overline{\langle x, y\rangle} = \langle y, x\rangle$

$$\overline{\langle A, B\rangle} = \overline{\operatorname{tr}(B^* A)}$$
$$= \overline{\sum_{i=1}^{n} (B^* A)_{ii}}$$
$$= \overline{\sum_{i=1}^{n} \sum_{k=1}^{n} (B^*)_{ik} A_{ki}}$$

$$= \overline{\sum_{i=1}^{n}\sum_{k=1}^{n}\overline{B}_{ki}A_{ki}}$$

$$= \overline{\sum_{i=1}^{n}\sum_{k=1}^{n}A_{ki}\overline{B}_{ki}} \quad \text{Commutativity over } F$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\overline{A}_{ki}\overline{\overline{B}}_{ki}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\overline{A}_{ki}B_{ki}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n}(A)^*_{ik}B_{ki}$$

$$= \sum_{i=1}^{n}(A^*B)_{ii}$$

$$= \operatorname{tr}(A^*B)$$

$$= \langle B, A\rangle$$

(b) **Use the Frobenius inner product to compute $\|A\|, \|B\|$, and $\langle A, B\rangle$ for**

$$A = \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix}$$

$$A^* = \begin{pmatrix} 1 & 3 \\ 2-i & -i \end{pmatrix} \quad \text{and} \quad B^* = \begin{pmatrix} 1-i & -i \\ 0 & i \end{pmatrix}$$

$\|A\|$:

$$\langle A, A\rangle^{\frac{1}{2}} = \sqrt{\operatorname{tr}(A^*A)}$$

$$A^*A = \begin{pmatrix} 10 & 2+4i \\ 2-4i & 6 \end{pmatrix}$$

$$\implies \sqrt{\operatorname{tr}(A^*A)} = \sqrt{16}$$

$$= 4$$

$\|B\|$:

$$\langle B, B\rangle^{\frac{1}{2}} = \sqrt{\operatorname{tr}(B^*B)}$$

$$B^*B = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\implies \sqrt{\text{tr}(B^*B)} = \sqrt{4}$$
$$= 2$$

$\langle A, B \rangle$:

$$\langle A, B \rangle = \text{tr}(B^*A)$$
$$B^*A = \begin{pmatrix} 1-4i & 4-i \\ 3i & -1 \end{pmatrix}$$
$$\implies \text{tr}(B^*A) = -4i$$

3. Textbook 6.1, #8.

   (a) Recall that if the above expression is an inner product, then
   $$\langle x, x \rangle \geq 0 \ \forall x \in \mathbb{R}^2$$
   If we consider
   $$\langle (a, b), (a, b) \rangle$$
   Then if the expression is an inner product,
   $$\langle (a, b), (a, b) \rangle = a^2 - b^2$$
   This isn't always non-negative because b can be greater than a. Therefore the expression given is not an inner product.

   (b) For this expression, we can apply the same axiom. Suppose A is a diagonal matrix with diagonals all equal to -1, and B is the zero matrix. Then $\text{tr}(A+B) = -n$. Since $n \geq 1, \text{tr}(A+B)$ is negative, which violates the inner product axioms that states
   $$\langle x, x \rangle \geq 0 \ \forall x \in F$$

   (c) Show
   $$\langle f(x), g(x) \rangle = \int_0^1 f'(x)g(x)dx$$
   is not an inner product. Recall that if the above expression is an inner product, then
   $$\langle x, y \rangle = \langle y, x \rangle \ \forall x, y \in \mathbb{R}$$
   Therefore let $f(x) = x^2$ and $g(x) = 2$ Then we get:
   $$\langle f(x), g(x) \rangle$$
   $$= \int_0^1 f'(x)g(x)dx$$
   $$= \int_0^1 4x$$
   $$\int_0^1 f'(x)g(x)dx$$
   $$= \int_0^1 4x$$

$$= 2$$

To show failure of symmetry, consider

$$\langle g(x), f(x) \rangle$$
$$= \int_0^1 g'(x) f(x) dx$$
$$= \int_0^1 dx$$
$$= 1 \neq 2$$
$$\implies \langle g(x), f(x) \rangle \neq \langle f(x), g(x) \rangle$$
$$\implies \text{Not an inner product}$$

4. Textbook 6.1, #9.
Let β be a basis for a finite-dimensional inner product space.

(a) **Prove that if $\langle x, z \rangle = 0$ for all $z \in \beta$, then $x = 0$**
Since z is arbitrary, we can set $z = x$
Then we get

$$\langle x, x \rangle = 0$$
$$\implies x = 0 \quad \text{By IP(4)}$$

Alternatively, we can also prove this way
Let $x = (x_1, ..., x_n)$ and $z = (z_1, ..., z_n)$.
By definition, $\langle x, z \rangle =$

$$\sum_{i=1}^{n} x_i \overline{z}_i$$

We require that

$$\sum_{i=1}^{n} x_i \overline{z}_i = 0$$

Let $\beta = \{e_1, ...e_n\}$ be the standard basis over F, and let $z_i = e_i$.
Then $\langle x, z_j \rangle =$

$$\sum_{i=1}^{n} x_i \overline{z}_j = 0$$

Since $x_i \overline{z}_j = x_i$ for $i = j$ and $x_i \overline{z}_j = 0$ for $i \neq j$, we are left with x_j.
Therefore, $x_j = 0$ for all $1 \leq j \leq n \implies x = (0, 0, ..., 0) \implies x = 0$

(b) **Prove that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in \beta$, then $x = y$**
If $\langle x, z \rangle = \langle y, z \rangle$, then

$$\langle x, z \rangle - \langle y, z \rangle = 0$$
$$\langle x - y, z \rangle = 0 \quad \text{Linear with first argument}$$

Since z is arbitrary, if we plug in $z = x - y$, we get

$$\langle x - y, x - y \rangle = 0$$

$\implies x - y = 0$ by IP(4) $\implies x = y$

5. Textbook 6.2, #2 (a), (c), (d).
Apply Gram-Schmidt process, normalize vectors to obtain an orthonormal basis β, and compute Fourier coefficients relative to β. Finally, use Theorem 6.5 to verify result.
(a)
$V = \mathbb{R}^3, S = \{(1,0,1),(0,1,1),(1,3,3)\}, \text{and } x = (1,1,2)$
$w_1 = (1,0,1), w_2, = (0,1,1), w_3 = (1,3,3)$
$v_1 = w_1 = (1,0,1)$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2}v_1$$
$$= (0,1,1) - \frac{\langle (0,1,1),(1,0,1)\rangle}{2}(1,0,1)$$
$$= (0,1,1) - \frac{1}{2}(1,0,1)$$
$$v_2 = (-\frac{1}{2}, 1, \frac{1}{2})$$
$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2}v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2}v_2$$
$$= (1,3,3) - \frac{\langle (1,3,3),(1,0,1)\rangle}{\|(1,0,1)\|^2}(1,0,1) - \frac{\langle (1,3,3),(-\frac{1}{2},1,\frac{1}{2})\rangle}{\|(-\frac{1}{2},1,\frac{1}{2})\|^2}(-\frac{1}{2},1,\frac{1}{2})$$
$$= (1,3,3) - \frac{4}{2}(1,0,1) - \frac{4}{\frac{3}{2}}(-\frac{1}{2},1,\frac{1}{2})$$
$$= (1,3,3) - (2,0,2) - \frac{8}{3}(-\frac{1}{2},1,\frac{1}{2})$$
$$= (1,3,3) - (2,0,2) - (-\frac{4}{3},\frac{8}{3},\frac{4}{3})$$
$$v_3 = (\frac{1}{3},\frac{1}{3},-\frac{1}{3})$$

Normalize each vector:

$$u_1 = \frac{1}{\|v_1\|}v_1 = \frac{1}{\sqrt{2}}(1,0,1)$$
$$u_2 = \frac{1}{\|v_2\|}v_2 = \frac{\sqrt{6}}{6}(-1,2,1)$$
$$u_3 = \frac{1}{\|v_3\|}v_3 = \frac{\sqrt{3}}{3}(1,1,-1)$$
$$\beta = \{u_1,u_2,u_3\}$$
$$x = (1,1,2) = \sum_{i=1}^{3}\langle x, u_i\rangle u_i$$
$$= \langle x,u_1\rangle u_1 + \langle x,u_2\rangle u_2 + \langle x,u_3\rangle u_3$$
$$= \frac{3}{\sqrt{2}}u_1 + \frac{\sqrt{6}}{2}u_2 + 0u_3$$

Fourier Coefficients:

$$(\frac{3}{\sqrt{2}}, \frac{\sqrt{6}}{2}, 0)$$

Confirm Theorem 6.5:

$$x = \frac{3}{\sqrt{2}}u_1 + \frac{\sqrt{6}}{2}u_2 + 0u_3$$

$$(1,1,2) = \frac{3}{\sqrt{2}}\frac{1}{\sqrt{2}}(1,0,1) + \frac{\sqrt{6}}{2}\frac{\sqrt{6}}{6}(-1,2,1)$$

$$(1,1,2) = \frac{3}{2}(1,0,1) + \frac{1}{2}(-1,2,1)$$

$$(1,1,2) = (1,1,2)$$

(c)

$V = P_2(\mathbb{R})$, with the inner product $\langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)dt, \quad S = \{1, x, x^2\}$, and $h(x) = 1 + x$
$w_1 = 1, w_2 = x, w_3 = x^2$
$v_1 = w_1 = 1$

$$\begin{aligned} v_2 &= w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2}v_1 \\ &= x - \frac{\langle x, 1\rangle}{1}(1) \\ &= x - \int_0^1 xdx \\ v_2 &= x - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} v_3 &= w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2}v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2}v_2 \\ &= x^2 - \frac{\langle x^2, 1\rangle}{1}(1) - \frac{\langle x^2, x - \frac{1}{2}\rangle}{\|x - \frac{1}{2}\|^2}(x - \frac{1}{2}) \\ &= x^2 - \frac{1}{3} - \frac{\int_0^1 x^3 - \frac{x^2}{2}dx}{\|x - \frac{1}{2}\|^2}(x - \frac{1}{2}) \\ &= x^2 - \frac{1}{3} - \frac{\int_0^1 x^3 - \frac{x^2}{2}dx}{(\int_0^1 x^2 - x + \frac{1}{4}dx)}(x - \frac{1}{2}) \\ &= x^2 - \frac{1}{3} - \frac{\frac{1}{4} - \frac{1}{6}}{(\frac{1}{3} - \frac{1}{2} + \frac{1}{4})}(x - \frac{1}{2}) \\ &= x^2 - \frac{1}{3} - \frac{\frac{1}{12}}{(\frac{1}{12})}(x - \frac{1}{2}) \\ &= x^2 - \frac{1}{3} - (x - \frac{1}{2}) \\ &= x^2 - x - \frac{1}{3} + \frac{1}{2} \\ v_3 &= x^2 - x + \frac{1}{6} \end{aligned}$$

Normalize each vector:

$$u_1 = \frac{1}{\|v_1\|}v_1 = \frac{1}{\sqrt{1}}(1) = 1$$

$$u_2 = \frac{1}{\|v_2\|}v_2 = \frac{1}{\sqrt{\frac{1}{12}}}(x - \frac{1}{2}) = \sqrt{12}(x - \frac{1}{2})$$

$$u_3 = \frac{1}{\|v_3\|}v_3 = \sqrt{180}(x^2 - x + \frac{1}{6})$$

$$h(x) = \sum_{i=1}^{3} \langle h(x), u_i \rangle u_i$$
$$= \langle 1+x, u_1 \rangle u_1 + \langle 1+x, u_2 \rangle u_2 + \langle 1+x, u_3 \rangle u_3$$
$$= \frac{3}{2}u_1 + \frac{1}{2\sqrt{3}}u_2 + 0u_3$$

Fourier Coefficients:

$$(\frac{3}{2}, \frac{1}{2\sqrt{3}}, 0)$$

Confirm Theorem 6.5:

$$1 + x = \frac{3}{2}u_1 + \frac{1}{2\sqrt{3}}u_2 + 0u_3$$
$$1 + x = \frac{3}{2} + \frac{1}{2\sqrt{3}}(\sqrt{12})(x - \frac{1}{2})$$
$$1 + x = \frac{3}{2} + (x - \frac{1}{2})$$
$$1 + x = \frac{3}{2} - \frac{1}{2} + x)$$
$$1 + x = 1 + x$$

(d)
$V = \text{span}(S), S = \{(1, i, 0), (1 - i, 2, 4i)\}, \text{and } x = (3 + i, 4i, -4)$
$w_1 = (1, i, 0), w_2, = (1 - i, 2, 4i)$
$v_1 = w_1 = (1, i, 0)$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2}v_1$$
$$= (1 - i, 2, 4i) - \frac{\langle (1 - i, 2, 4i), (1, i, 0) \rangle}{\|(1, i, 0)\|^2}(1, i, 0)$$
$$= (1 - i, 2, 4i) - \frac{1 - i - 2i}{1 - i^2}(1, i, 0)$$
$$= (1 - i, 2, 4i) - \frac{1 - 3i}{2}(1, i, 0)$$
$$= (1 - i, 2, 4i) - \frac{1}{2}(1 - 3i, i + 3, 0)$$
$$v_2 = (\frac{1}{2} + \frac{i}{2}, \frac{1}{2} - \frac{i}{2}, 4i)$$

Normalize each vector:

$$u_1 = \frac{1}{\|v_1\|}v_1 = \frac{1}{\sqrt{2}}(1, i, 0)$$
$$u_2 = \frac{1}{\|v_2\|}v_2 = \frac{1}{\sqrt{\frac{1}{2}}}(\frac{1}{2} + \frac{i}{2}, \frac{1}{2} - \frac{i}{2}, 4i) = \sqrt{2}(\frac{1}{2} + \frac{i}{2}, \frac{1}{2} - \frac{i}{2}, 4i)$$

$$x = \sum_{i=1}^{2} \langle x, u_i \rangle u_i$$

$$(3+i,4i,-4) = \langle (3+i,4i,-4), u_1\rangle u_1 + \langle (3+i,4i,-4), u_2\rangle u_2$$
$$= \frac{3\sqrt{2}+4}{\sqrt{2}} u_1 + 17i\sqrt{2}u_2$$

Fourier Coefficients:

$$(\frac{3\sqrt{2}+4}{\sqrt{2}}, 17i\sqrt{2})$$

Confirm Theorem 6.5:

$$(3+i,4i,-4) = \frac{3\sqrt{2}+4}{\sqrt{2}}(\frac{1}{\sqrt{2}}(1,i,0)) + 17i\sqrt{2}(\sqrt{2}(\frac{1}{2}+\frac{i}{2},\frac{1}{2}-\frac{i}{2},4i))$$
$$= \frac{3\sqrt{2}+4}{2}(1,i,0) + 34i(\frac{1}{2}+\frac{i}{2},\frac{1}{2}-\frac{i}{2},4i))$$

6. Textbook 6.2, #4.
Let $S = \{(1,0,i),(1,2,1)\}$ in $\mathbb{C}^3$. Compute $S^\perp$
Notice that we simply need to find an orthogonal complement to the vectors in S. Since $\dim(S) = 2, \dim(S^\perp) = 3-2 = 1$ by the dimension theorem. Therefore we can just take the cross product:

$$(1,0,i)\times(1,2,1)$$
$$= (0-2i, -(1-i), 2)$$
$$= (-2i, i-1, 2)$$
$$S^\perp = \{(-2i, i-1, 2)\}$$

The following problems on orthogonal projections and complements.

7. **Suppose that V is an inner product space over $F=\mathbb{R}$ or $F=\mathbb{C}$. Let $U\subset V$ be a subspace. Recall that $U^\perp = \{v\in V \mid \langle u,v\rangle = 0, \forall u\in U\}$.**

 (a) **Let $U = \operatorname{span}(S)$. Prove that $w\in U^\perp$ if and only if $\langle w,s\rangle = 0$ for all $s\in S$.**

 $(\Longrightarrow)$

 If $w\in U^\perp$, then by definition $\langle w,u\rangle = 0\ \forall u\in U$. Since $\operatorname{span}(S) = U, s\in S \implies s\in U$. Therefore $\langle w,s\rangle = 0$ for all $s\in S$.
 $(\Longleftarrow)$

 If $\langle w,s\rangle = 0$ for all $s\in S$, w is orthogonal to every element in S. Since $U = \operatorname{span}(S)$, this implies w is orthogonal to $U \implies w\in U^\perp$

 (b) **Prove that $U+U^\perp$ is always a direct sum.**

 Suppose $u,u'\in U$ and $u_1,u_1'\in U^\perp$
 WTS that $u+u_1 = u'+u_1' \implies u=u'$ and $u_1' = u_1'$

 First show that $U^\perp\cap U = \vec{0}$
 Let $v\in U^\perp\cap U$
 $\implies v\in U$ and $v\in U^\perp$
 By definition, $U^\perp = \{v\in V \mid \langle u,v\rangle = 0, \forall u\in U\}$.

$$\implies \langle v,v\rangle = 0$$

$$\implies v = 0$$
$$\implies U^{\perp} \cap U = \vec{0}$$

Assume that

$$u + u_1 = u' + u_1'$$
$$u - u' = u_1' - u_1$$

Since $u, u' \in U, u - u' \in U$. By the same logic, $u_1' - u_1 \in U^{\perp}$ Because they are equal, this implies that $u - u'$ and $u_1' - u_1$ are in both U and $U^{\perp}$

$$\implies u - u', u_1' - u_1 \in U^{\perp} \cap U$$

But since $U^{\perp} \cap U = \vec{0}$,

$$\implies u - u' = u_1' - u_1 = 0$$
$$\implies u = u' \quad \text{and} \quad u_1' = u_1$$

(c) **Let U be finite-dimensional. Explain why U also has a finite, orthonormal basis.**

If U is finite dimensional, we let $\dim(U) = n$. Then U has a basis $\beta = \{v_1, \ldots, v_n\}$. Then by definition, we can apply the Gram-Schmidt Process to a linearly independent set of vectors to produce an orthonormal basis of dimension n, which is finite.

(d) **Suppose that $\{u_1, \ldots, u_n\}$ is an orthogonal basis for U. Prove that $V = U \oplus U^{\perp}$.**

Recall from part (b) that $U^{\perp} \cap U = \vec{0}$.
WTS that $U^{\perp} + U = V$.

Suppose for some arbitrary $v \in V$, let $P_U(v)$ is the orthogonal projection of v onto some subspace $U \subset V$.
Then, by definition,

$$v = P_U(v) + z \quad \text{for some} \quad z \in U^{\perp}$$

Since $P_U(v) \in U$, we have that

$$v \in U + U^{\perp}$$
$$\implies V \subset U + U^{\perp}$$

We know trivially that $U \subset V$ and $U^{\perp} \subset V$, therefore

$$U + U^{\perp} \subset V$$
$$\implies V = U + U^{\perp}$$

Combining our results, we have that

$$V = U \oplus U^{\perp}$$

(e) **With set-up as in part (b), prove that $p_U \colon V \to V$ can equivalently be defined by the the following formula: if $v = x + z$ for $z \in U^{\perp}$ and $x \in U$, then $p_U(v) = x$.**

Recall from part (b) that $U^{\perp} \cap U = \vec{0}$, and any representation of $U + U^{\perp}$ is unique. This means that $v = x + z$ is a unique representation.
This implies that x and z are linearly independent.
For some $v \in V$, we know that $v = P_U(v) + z$ from part (d). But because v has a unique representation.

$$x + z = P_U(v) + z$$
$$x = P_U(v)$$

8. Textbook 6.2 #6.
Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^{\perp}$, but $\langle x, y\rangle \neq 0$

By Theorem 6.6, for $y \in V$, there exists unique vectors $u \in W$ and $z \in W^{\perp}$ such that $y = u + z$.

Let $x, y \in V$ such that $x \notin W$ and $y \in W^{\perp}$
This means
$$x = u + z \text{ such that } z \neq 0$$
and
$$y = u' + z' \text{ such that } u' = 0$$
Then, taking the inner product, we get
$$\langle x, y\rangle$$
$$= \langle u + z, z'\rangle$$
$$= \langle u, z'\rangle + \langle z, z'\rangle$$
$$= 0 + \langle z, z'\rangle$$
Since $z, z' \in W^{\perp}$, $\langle z, z'\rangle \neq 0$
$$\implies \langle x, y\rangle \neq 0$$
Therefore if $x \notin W$, there exists $y \in V$ such that $y \in W^{\perp}$, but $\langle x, y\rangle \neq 0$

9. **Textbook 6.2 # 13(a)-(c). Note that even if S is not a subspace, we can define $S^{\perp} = \{v \in V \mid \langle v, s\rangle = 0\, \forall s \in S\}$.**
Let V be an inner product space, S and S_0 be subsets of V, and W be a finite-dimensional subspace of V. Prove the following results.
   (a) $S_0 \subset S$ **implies that** $S^{\perp} \subset S_0^{\perp}$

   Let $v \in S^{\perp}$. Then
   $$\langle v, s\rangle = 0\ \forall s \in S$$
   Since $S_0 \subset S$, this means that
   $$\langle v, s'\rangle = 0\ \forall s' \in S_0$$
   $$\implies v \in S_0^{\perp}$$
   $$\implies S^{\perp} \subset S_0^{\perp}$$
   (b) $S \subset (S^{\perp})^{\perp}$**; so** $\text{span}(S) \subset (S^{\perp})^{\perp}$

   By definition, $(S^{\perp})^{\perp}$ is the set of all vectors in V that are orthogonal to every vector in $S^{\perp}$:
   $$(S^{\perp})^{\perp} = \{v \in V \mid \langle v, w\rangle = 0\ \forall w \in S^{\perp}\}.$$
   Let $s \in S$. To show that $s \in (S^{\perp})^{\perp}$, need to show that
   $$\langle s, w\rangle = 0\ \forall w \in S^{\perp}.$$
   Since $w \in S^{\perp}$, by definition, $\langle w, s\rangle = 0\ \forall s \in S$. Because of conjugate symmetry ($\langle w, s\rangle = \overline{\langle s, w\rangle}$), we also have
   $$\langle s, w\rangle = 0\ \forall w \in S^{\perp}.$$
   Therefore, $s \in (S^{\perp})^{\perp}$, which implies
   $$S \subset (S^{\perp})^{\perp}.$$

Let $v \in \text{span}(S)$. Then $v \in S$
Since $S \subset (S^\perp)^\perp, v \in (S^\perp)^\perp$

$$\implies \text{span}(S) \subset (S^\perp)^\perp$$

(c) $W = (W^\perp)^\perp$

To show that $W = (W^\perp)^\perp$, need to show $W \subset (W^\perp)^\perp$ and $(W^\perp)^\perp \subset W$.

$W \subset (W^\perp)^\perp$:
Let $w \in W$. By definition, $(W^\perp)^\perp$ equals

$$(W^\perp)^\perp = \{v \in V \mid \langle v, z \rangle = 0 \ \forall z \in W^\perp\}.$$

By definition, every $z \in W^\perp$ satisfies $\langle z, w \rangle = 0$ for all $w \in W$. By conjugate symmetry of the inner product, we have that

$$\langle w, z \rangle = 0 \ \forall z \in W^\perp.$$

Therefore, $w \in (W^\perp)^\perp$

$$\implies W \subset (W^\perp)^\perp.$$

$(W^\perp)^\perp \subset W$:
Assume for the sake of contradiction that there exists $x \in (W^\perp)^\perp$ such that $x \notin W$. By problem (8), there exists $y \in V$ such that $y \in W^\perp$ and $\langle x, y \rangle \neq 0$.
However, since $x \in (W^\perp)^\perp$, by definition x must be orthogonal to every vector in $W^\perp$.
Therefore $\langle x, y \rangle = 0$ for all $y \in W^\perp$, contradicting our assumption that $\langle x, y \rangle \neq 0$.
Therefore, there is no $x \in (W^\perp)^\perp$ such that $x \notin W$, which implies that if $x \notin W, x \notin (W^\perp)^\perp$

$$(W^\perp)^\perp \subset W.$$

by contrapositive.
Then we conclude that

$$W = (W^\perp)^\perp.$$

10. **Let W_1, W_2 be subspaces of a finite-dimensional inner product space. Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$.**

Let $v \in (W_1 + W_2)^\perp$. Then

$$\langle v, w_1 + w_2 \rangle = 0 \quad w_1 \in W_1, w_2 \in W_2$$

$$\langle v, w_1 \rangle + \langle v, w_2 \rangle = 0$$

Since the above statement holds for all w_1, w_2, consider the cases when either $w_1 = 0$ or $w_2 = 0$
If $w_1 = 0$:

$$\langle v, 0 \rangle + \langle v, w_2 \rangle = 0$$

$$0 + \langle v, w_2 \rangle = 0$$

$$\langle v, w_2 \rangle = 0$$

$$\implies v \in W_2^\perp$$

If $w_2 = 0$:

$$\langle v, w_1 \rangle + \langle v, 0 \rangle = 0$$

$$\langle v, w_1 \rangle + 0 = 0$$

$$\langle v, w_1 \rangle = 0$$
$$\implies v \in W_1^\perp$$

Then $v \in W_1^\perp \cap W_2^\perp$

$$\implies (W_1 + W_2)^\perp \subset W_1^\perp \cap W_2^\perp$$

Now let $v \in W_1^\perp \cap W_2^\perp$. Then for any $w_1 \in W_1$ and $w_2 \in W_2$

$$\langle v, w_1 \rangle = 0 \text{ and } \langle v, w_2 \rangle = 0$$

Therefore

$$\langle v, w_1 \rangle + \langle v, w_2 \rangle$$
$$= 0 + 0 = 0$$
$$\implies \langle v, w_1 \rangle + \langle v, w_2 \rangle = 0$$
$$\implies \langle v, w_1 + w_2 \rangle = 0$$
$$\implies v \in (W_1 + W_2)^\perp$$
$$\implies W_1^\perp \cap W_2^\perp \subset (W_1 + W_2)^\perp$$

Combining results, we get

$$W_1^\perp \cap W_2^\perp = (W_1 + W_2)^\perp$$

11. **Textbook 6.2 #23.**

Let V be the vector space of all sequences σ in F such that $\sigma(n) \neq 0$ for only finitely many positive integers n. For $\sigma, \mu \in V$, we define

$$\langle \sigma, \mu \rangle = \sum_{n=1}^{\infty} \sigma(n)\overline{\mu(n)}$$

Since all but a finite number of terms of the series are zero, the series converges

(a) **Prove that $\langle \cdot, \cdot \rangle$ is an inner product on V, and hence V is an inner product space.**
Suffices to show that $\langle \cdot, \cdot \rangle$ satisfies the Inner Product Axioms:
Let $\sigma, \mu, \gamma \in V$ and $\lambda \in F$

i. $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

$$\langle \sigma + \mu, \gamma \rangle = \sum_{n=1}^{\infty} (\sigma + \mu)(n))\overline{\gamma(n)}$$

$$\langle \sigma + \mu, \gamma \rangle = \sum_{n=1}^{\infty} (\sigma(n) + \mu(n))\overline{\gamma(n)}$$

$$= \sum_{n=1}^{\infty} \sigma(n)\overline{\gamma(n)} + \sum_{n=1}^{\infty} \mu(n)\overline{\gamma(n)}$$
$$= \langle \sigma, \gamma \rangle + \langle \mu, \gamma \rangle$$

ii. $\langle \lambda\sigma, \mu\rangle = \lambda\langle \sigma, \mu\rangle$

$$\begin{aligned}\langle \lambda\sigma, \mu\rangle &= \sum_{n=1}^{\infty} \lambda\sigma(n)\overline{\mu(n)}\\ &= \lambda\sum_{n=1}^{\infty} \sigma(n)\overline{\mu(n)}\\ &= \lambda\langle \sigma, \mu\rangle\end{aligned}$$

iii. $\overline{\langle \mu, \sigma\rangle} = \langle \sigma, \mu\rangle$

$$\begin{aligned}\overline{\langle \mu, \sigma\rangle} &= \overline{\sum_{n=1}^{\infty} \mu(n)\overline{\sigma(n)}}\\ &= \sum_{n=1}^{\infty} \overline{\mu(n)\overline{\sigma(n)}}\\ &= \sum_{n=1}^{\infty} \overline{\mu(n)}\sigma(n)\\ &= \sum_{n=1}^{\infty} \sigma(n)\overline{\mu(n)}\\ &= \langle \sigma, \mu\rangle\end{aligned}$$

iv. If $\sigma(n) \neq 0, \langle \sigma, \sigma\rangle \in \mathbb{F}_{>0}$

$$\langle \sigma, \sigma\rangle = \sum_{n=1}^{\infty} \sigma(n)\overline{\sigma(n)}$$

If $\sigma(n) = a + bi$, $\overline{\sigma(n)} = a - bi$, $(a \neq 0, b \neq 0$
$\sigma(n)\overline{\sigma(n)} = a^2 + b^2$
$\implies \sigma(n)\overline{\sigma(n)} > 0$

(b) **For each positive integer n, let e_n be the sequence defined by $e_n(k) = \delta_{nk}$, where δ_{nk} is the Kronecker delta. Prove that $\{e_1, e_2, ...\}$ is an orthonormal basis for V**

$$\langle e_n, e_m\rangle = \delta_{nm}.$$

$$\langle e_n, e_m\rangle = \sum_{k=1}^{\infty} e_n(k)\overline{e_m(k)} = \sum_{k=1}^{\infty} \delta_{nk}\overline{\delta_{mk}}.$$

Since δ_{nk} and δ_{mk} are both 1 only when $n = m$, respectively, we get:

$$\langle e_n, e_m\rangle = \delta_{nm}.$$

Therefore, $\{e_1, e_2, ...\}$ is orthonormal.

To show that $\{e_1, e_2, \ldots\}$ spans V, consider any $\sigma \in V$. Since $\sigma(n) \neq 0$ for only finitely many n, we can write:

$$\sigma = \sum_{n=1}^{\infty} \sigma(n)e_n.$$

Therefore, $\sigma \in V$ can be expressed as a linear combination of $\{e_n\}$.
Therefore, $\{e_1, e_2, \ldots\}$ is an orthonormal basis for V.

(c) **Let $\sigma_n = e_1 + e_n$ and $W = \mathrm{span}(\{\sigma_n : n \geq 2\})$**

i. **Prove that $e_1 \notin W$, so $W \neq V$**
Suppose for the sake of contradiction that $e_1 \in W$. Then e_1 can be written as a linear combination of $\{\sigma_n : n \geq 2\}$.

$$e_1 = \sum_{k=2}^{m} a_k \sigma_k = \sum_{k=2}^{m} a_k(e_1 + e_k)$$

$$= \left(\sum_{k=2}^{m} a_k\right) e_1 + \sum_{k=2}^{m} a_k e_k.$$

For e_1 to be in the span of $\{\sigma_n : n \geq 2\}$, the coefficient of e_1 must be 1, and the sum $\sum_{k=2}^{m} a_k e_k$ must be 0, which implies that all $a_k = 0$. But if all $a_k = 0$, then the sum $\sum_{k=2}^{m} a_k = 0$, which contradicts the requirement that the coefficient of e_1 is 1.
Therefore, $e_1 \notin W$, so $W \neq V$

ii. **Prove that $W^\perp = \{0\}$, and conclude that $W \neq (W^\perp)^\perp$**

To prove $W^\perp = \{0\}$, WTS that if $v \in W^\perp$, then $v = 0$.
Consider some $v \in W^\perp$. For v to be in $W^\perp$, it must be orthogonal to every σ_n:

$$\langle v, \sigma_n \rangle = 0 \quad \text{for all } n \geq 2.$$

Since $\sigma_n = e_1 + e_n$,

$$\langle v, \sigma_n \rangle = \langle v, e_1 \rangle + \langle v, e_n \rangle.$$

This implies that

$$\langle v, e_1 \rangle + \langle v, e_n \rangle = 0.$$

For each $n \geq 2$, we have:

$$\langle v, e_n \rangle = -\langle v, e_1 \rangle.$$

Since e_n for $n \geq 2$ are linearly independent are part of the orthonormal basis $\{e_1, e_2, \ldots\}$, the only solution is the trivial solution

$$\langle v, e_1 \rangle = 0 \quad \text{and} \quad \langle v, e_n \rangle = 0 \quad \text{for all } n \geq 2.$$

$\implies v = 0 \implies W^\perp = \{0\}$.
Since $W^\perp = \{0\}$, we have $(W^\perp)^\perp = V$. However, since $W \neq V$, we can conclude that $W \neq (W^\perp)^\perp$