Math 115AH Homework 2

April 17, 2024

1. **Problem 1**

   (a) **Using the Cancellation Law for Vector Addition (textbook Theorem 1.1), prove that additive inverses are unique.**
   Let $x \in V$. Suppose $\exists\ x_1, x_2 \in V$ such that $x + x_1 = 0$ and $x + x_2 = 0$. Then by (VS2) $(x + x_1) + x_2 = 0 + x_2 \Longrightarrow x_1 + (x + x_2) = x_1 + 0 \Longrightarrow 0 + x_2 = 0 + x_1$. Therefore by cancellation law, $x_2 = x_1 \Longrightarrow$ the additive inverse is unique.

2. **Problem 2**

   (a) Textbook 1.2, 18, **Let V =** $\{(a_1, a_2) : a_1, a_2 \in R\}$. **For** $(a_1, a_2), (b_1, b_2) \in V$ **and** $c \in R$ **define**

   $$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \quad \textbf{and} \quad c(a_1, a_2) = (ca_1, ca_2) \tag{1}$$

   Is V a vector space over R with these operations? Justify your answer.
   No, V fails (VS1) as $(b_1, b_2) + (a_1, a_2) = (b_1 + 2a_1, b_2 + 3a_2)$. Since commutativity is defined on R, $b_1 + 2a_1 = 2a_1 + b_1 \neq a_1 + 2b_1$ and $b_2 + 3a_2 = 3a_2 + b_2 \neq a_2 + 3b_2$. Therefore $(a_1, a_2) + (b_1, b_2) \neq (b_1, b_2) + (a_1, a_2) \Longrightarrow$ V is not a vector space

3. **Problem 3**

   (a) **Prove that $\mathbb{C}$ is a vector space of $\mathbb{R}$**
   Use $(a + bi) + (c + di) = (a + c) + (b + d)i$
   Use $(a + bi)(c + di) = (ac - bd) + (bc + ad)i$
   Additive Identity $0 + 0i$
   Multiplicative Identity $1 + 0i$
   (VS1) $\forall (a+bi), (c+di) \in \mathbb{C}; a, b, c, d \in \mathbb{R}, (a+bi)+(c+di) = (a+c)+(b+d)i = (c+a)+(d+b)i = (c+di)+(a+bi)$
   (VS2) $\forall (a+bi), (c+di), (e+fi) \in \mathbb{C}; a, b, c, d, e, d \in \mathbb{R}, (a+bi)+((c+di)+(e+fi)) = (a+bi)+((c+e)+(d+f)i) = (a+(c+e))+(b+(d+f))i = ((a+c)+e)+((b+d)+f)i = ((a+c)+(b+d)i)+(e+fi) = ((a+bi)+(c+di))+(e+fi)$
   (VS3) $(a + bi) + (0 + 0i) = (a + 0) + (b + 0)i = a + bi\ \forall (a + bi) \in \mathbb{C}$
   (VS4) $\forall (a + bi) \in \mathbb{C}, \exists\ (-a) + (-b)i$ such that $((a + bi)) + ((-a) + (-b)i) = (a - a) + (b - b)i = 0 + 0i = 0$
   (VS5) $\forall (a + bi) \in \mathbb{C}, (1 + 0i)(a + bi) = (1a - 0b) + (0a + 1b)i = a + bi$
   (VS6) $\forall (a + bi) \in \mathbb{C}, \text{and } \forall c, c' \in \mathbb{R}, c \cdot (c' \cdot (a + bi)) = c(c'a + c'bi) = cc'a + cc'bi = c'(ca + cbi) = c'c(a + bi)$
   (VS7) $\forall (a + bi), (c + di) \in \mathbb{C}\text{and } \forall k \in \mathbb{R}, k((a + bi) + (c + di)) = k((a + c) + (b + d)i) = k(a + c) + k(b + d)i = ka + kc + kbi + kdi = k(a + bi) + k(c + di)$
   (VS8) $\forall (a+bi) \in \mathbb{C}, \text{and } \forall c, c' \in \mathbb{R}, (c+c')(a+bi) = (c+c')a+(c+c')bi = ca+c'a+cbi+c'bi = c(a+bi)+c'(a+bi)$

4. **Problem 4**
   Let S denote a set and V a vector space over a field F. We consider the set $\text{Fun}(S, V)$ of all functions $f : S \to V$

   (a) **Define addition on elements of Fun(S, V).**
   Let $f, g : S \to V \in \text{Fun}(S, V)$. Then $(f + g)(s) = f(s) + g(s)\ \forall s \in S$

   (b) **Define scalar multiplication of elements of Fun(S, V) by elements of F**
   Let $f : S \to V \in \text{Fun}(S, V)$ and $c \in F$. Then $(cf)(s) = c \cdot f(s)\ \forall s \in S$

(c) **Show that the two operations defined above make Fun(S, V) a vector space**

Since $(f+g)(s) = f(s) + g(s)$ is addition in the vector space V, $f(s)+g(s) \in V \implies \text{Fun}(S,V)$ is closed under addition.

Similarly, since $(cf)(s) = c \cdot f(s)$ is scalar multiplication in the vector space V, $c \cdot f(s) \in V \implies \text{Fun}(S,V)$ is closed under scalar multiplication

For all $f, g \in \text{Fun}(S,V)$, $(f+g)(s) = f(s)+g(s) = g(s)+f(s) = (g+f)(s)$

For all $f, g, h \in \text{Fun}(S,V)$ and $s \in S$, $(f+g)+h = f(s)+g(s)+h(s) = f+(g+h)$.

Let $0(s) = \mathbf{0}$ for all $s \in S$ (where $\mathbf{0}$ is the zero vector in V). For any $f \in \text{Fun}(S,V)$, $f+0 = f$.

For all $f \in \text{Fun}(S,V)$, Let $c = -1 \in F$, then $(cf)(s) = -f(s) \in V$. Then there exists a function $-f(s)$ such that $f(s) + (-f(s)) = 0$

If you then let $c = 1$, then $1(f(s)) = f(s)$

For all $f \in \text{Fun}(S,V)$ and $a, b \in F, ab(f(s)) = a(b(f(s))$, and $(a+b)f(s) = af(s) + bf(s)$

For all $f, g \in \text{Fun}(S,V)$ and $a \in F, a((f+g)(s)) = a(f(s)+g(s)) = af(s) + ag(s)$

5. **Problem 5**

(a) **Prove that if (x, y) and $(z, w) \in R_U$, then $[x +_V z] = [y +_V w]$**

Observe that R_U is an equivalence relation. If $(x,y) \in R_U, (x,x) \in R_U \implies x \in [x] \implies [x] = [y]$. Similar logic yields $[z] = [w]$. Then $[x +_V z] = [x] + [z] = [y] + [w] = [y +_V w]$.

(b) **Prove that if $\lambda \in F$ and $(x, y) \in R_U$, then $[\lambda x] = [\lambda y]$**

From part (a), use the fact that $[x] = [y]$. Then $[\lambda x] = [\lambda \cdot_V x] = \lambda[x] = \lambda[y] = [\lambda \cdot_V y] = [\lambda y]$

(c) **Combining the previous two items, prove that (1) and (2) give well-defined addition and scalar multiplication operations on V/U**

Let $[v_1], [v_2] \in V/U$, then $[v_1] + [v_2] = [v_1 + v_2] \in V/U, \implies$ closure under addition

Let $[v_1] \in V/U$ and $c \in F$, then $c[v_1] = [cv_1] \in V/U \implies$ closure under scalar multiplication

Recall from (a) and (b) that if $(x,y), (z,w) \in R_U$, such that x and y are not necessarily equal and z and w are not necessarily equal, then $(x-y), (z-w) \in U$ and $[x] = [y], [z] = [w]$. Then because U is a subspace, $(x-y)+(z-w) = (x+z)-(y+w) \in U \implies (x+z, y+w) \in R_U \implies [x+z] = [y+w]$. Additionally, if $(x,y) \in R_U$ and $c \in F$, $[x] = [y]$ but x does not necessarily equal y and $x - y \in U$. However, because U is a subspace, $c(x-y) = cx - cy \in U \implies (cx, cy) \in R_U \implies [cx] = [cy]$.

Therefore, addition and scalar multiplication must be well-defined because even when $[x] = [y]$ and $[z] = [w]$ even when $x \neq y, z \neq w$, $[x+z] = [y+w]$, and $[cx] = [cy]$ for some $c \in F$.

6. **Problem 6**

Let V be a vector space over a field F. For any subsets $S, T \subset V$, we define the sum of S and T is the subset

$$S + T := \{v \in V \mid \exists s \in S, t \in T : v = s + t\}$$

I.e. the set of all elements in V which can be written as a sum of one element in S and one element in T

(a) **Let $V = \mathbb{R}^2, S = \{(1,0)\}$and $T = \{(-1,0\}$. Compute $S + T$ and $S \cup T$**

$S + T = (1,0) + (-1,0) = (0,0)$

$S \cup T = \{(1,0), (-1,0)\}$

(b) **Give an example of sets S and T such that S and T are subspaces, but $S \cup T$ is not a subspace.**

Let $S = \{(a,a) : a \in \mathbb{R}\}$ and $T = \{(b,-b) : b \in \mathbb{R}\}$

(c) **Prove that, if U_1 and U_2 are subspaces of V, then $U_1 + U_2$ is a subspace of V**

Let $u \in U_1 + U_2$ such that $u = u_1 + u_2$ for $u_1 \in U_1$ and $u_2 \in U_2$. Since U_1 and U_2 are subspaces, they both contain the zero vector. Therefore, by vector addition, $u = 0 + 0 = \vec{0}$, so $\vec{0} \in U_1 + U_2$.

Now let $u, u' \in U_1 + U_2$ such that $u = u_1 + u_2$ and $u' = u_3 + u_4$ for $u_1, u_3 \in U_1$ and $u_2, u_4 \in U_2$. Then $u + u' = (u_1 + u_2) + (u_3 + u_4) = (u_1 + u_3) + (u_2 + u_4)$ by vector associativity. (Note, since U_1, U_2 are subspaces,

$u_1 + u_3 \in U_1$ and $u_2 + u_4 \in U_2$). Therefore $(u_1 + u_3) + (u_2 + u_4) \in U_1 + U_2$.
Let $u \in U_1 + U_2$ such that $u = u_1 + u_2$ for $u_1 \in U_1, u_2 \in U_2$ and $c \in \mathbb{R}$. Then since U_1, U_2 are subspaces, $cu_1 \in U_1$ and $cu_2 \in U_2$. Then $cu = c(u_1 + u_2) = cu_1 + cu_2 \in U_1 + U_2$
By Theorem 1.3, $U_1 + U_2$ must be a subspace of V.

(d) **Prove that, if U_1, U_2, and W are all subspaces of V, $U_1 \subset W$, and $U_2 \subset W$, then $U_1 + U_2 \subset W$.**
Let $u_1 + u_2 = u \in U_1 + U_2$ for $u_1 \in U_1, u_2 \in U_2 \implies u_1, u_2 \in W \implies u_1 + u_2 \in W \implies u \in W$.

7. **Problem 7**
Textbook 1.3, 1

(a) **If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.**
False

(b) **The empty set is a subspace of every vector space.**
False, the empty set doesn't contain the zero vector

(c) **If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$**
True

(d) **The intersection of any two subsets of V is a subspace of V**
False

(e) **An $n \times n$ diagonal matrix can never have more than n nonzero entries**
True

(f) **The trace of a square matrix is the product of its diagonal entries**
False

(g) **Let W be the xy-plane in $\mathbb{R}^3$; that is, $W = \{(a_1, a_2, 0) : a_1, a_2 \in R\}$ Then $W = R^2$**
False

8. **Problem 8**
Let $V = \text{Mat}_{n\times n}(F)$. Let $U = \{A \in V | A = A^t\}$. Prove that U is a subspace of V.

U contains the zero vector, the matrix with all entries 0, since this matrix is symmetric.
Let $A, B \in U$. Then $A + B$ is the $n \times n$ matrix with entries

$$A_{i,j} + B_{i,j} \text{ for } 1 \leq i, j \leq n$$

Since U is the set of all symmetric $n \times n$ matrices, $A_{i,j} = A_{j,i}$ and $B_{i,j} = B_{j,i} \implies A_{i,j} + B_{i,j} = A_{j,i} + B_{j,i} \implies A + B$ is also a symmetric matrix $\implies A + B \in U$ Suppose now you have $c \in F$, then cA equals the matrix with entries

$$c(A_{i,j}) \text{ for } 1 \leq i, j \leq n$$

$A_{i,j} = A_{j,i} \implies c(A_{i,j}) = c(A_{j,i}) \implies cA \in U$ By Theorem 1.3, U must be a subspace.

9. **Problem 9**
Let V be a vector space over F and $U \subset V$ a subspace. Prove that $T : V \to V/U$ defined by $T(v) = [v]$ is linear. What is the kernel of T?

Let $v_1, v_2 \in V$, then $T(v_1) = [v_1], T(v_2) = [v_2]$, and $T(v_1 + v_2) = [v_1 + v_2] = [v_1] + [v_2] = T(v_1) + T(v_2)$
Furthermore, let $c \in \mathbb{R}$, then $T(cv_1) = [cv_1] = c[v_1] = c(T(v_1))$. Therefore T is linear because it preserves the operations of vector addition and scalar multiplication defined in V in V/U

The kernel of T is

$$\{v \in V : T(v) = [0]\}$$

This is the set of all elements in [0].

10. **Problem 10**

    Given vector spaces V, W over a field F and $T : V \to W$ a linear transformation, prove that $\mathrm{Im}(T)$ is a subspace.

    Since V is a vector space, $\vec{0}_V \in V,\ T(\vec{0}_V) = \vec{0}_W \in \mathrm{Im}(T)$
    Let $w_1, w_2 \in W$, then $\exists v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Since $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2) \in$ Im(T).
    Now let $c \in F$, then $c(w_1) = c(T(v_1)) = T(cv_1) \in$ Im(T).
    By Theorem 1.3, $\mathrm{Im}(T)$ is a subspace.