Math 115AH Homework 5

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May 10, 2024

1. **Invertible functions.** Let $f\colon X \to Y$ be a function between sets X and Y.

   (a) **Prove that, if f is invertible, the inverse is unique.** [1] **Explicitly, show that if $g_1\colon Y \to X$ and $g_2\colon Y \to X$ are both such that $g_i \circ f = 1_X$ and $f \circ g_i = 1_Y$ for $i = 1, 2$, then $g_1 = g_2$ as functions.**

   If f is invertible, then $\exists\ g\colon Y \to X$ such that $g \circ f = 1_X$ for all $y \in Y$ Suppose there exist $g_1\colon Y \to X$ and $g_2\colon Y \to X$ such that $g_i \circ f = 1_X$ and $f \circ g_i = 1_Y$ for $i = 1, 2..$
   Notice that g_1 and g_2 have the same domain and codomain. WTS that for all $y \in Y, g_1(y) = g_2(y)$. Observe that if $f \circ g = 1_Y$ is defined, then g must be onto. Then for all $x \in X, \exists\ y \in Y$ such that $g(y) = x$. By similar logic, f must also be onto $\implies \exists\ x \in X$ such that $f(x) = y \quad \forall y \in Y$. Therefore we can represent every element in Y as $f(x) \in Y$ for some $x \in X$. Then $g_1(f(x)) = x = g_2(f(x)) \implies g_1 = g_2$.

   (b) **Give an example of sets X and Y and $f\colon X \to Y$ so that f is not invertible, but there exists $g\colon Y \to X$ such that $g \circ f = 1_X$.**
   Suppose $X = \{1\}$ and $Y = \{a, b\}$. Define f such that $f(1) = a$ and $g(a) = g(b) = 1$. We know that f is injective but not surjective, because then f would be invertible. Clearly, $(g \circ f)(1) = 1$, which is all we need to check because we defined $X = \{1\}$

   (c) **Prove that f is invertible if and only if f is one-to-one and onto.**

   If f is invertible, there exists an inverse to $f : X \to Y$ defined by $f^-1 = g\colon Y \to X$ such

[1] This justifies writing f^{-1} for the unique inverse of an invertible function f.

that $g(f(x) = x \ \forall \ x \in X$.

(Invertibility $\implies$ Bijection)

To show injectivtity:

Suppose $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Then

$$g(f(x_1)) = g(f(x_2))$$

$$\implies x_1 = x_2$$

To show surjectivity:

For all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

Notice that for all $y \in Y, \exists \ f^{-}1(y) \in X$. Then we know that $f(f^{-}1(y)) = y$.

(Bijection $\implies$ Invertibility)

If f is bijective, then for all $y \in Y, \exists \ x \in X$ such that $f(x) = y$. Additionally, suppose $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$, then $x_1 = x_2$. WTS that this implies the existence of a function $g\colon Y \to X$ such that $g(f(x)) = x$ for all $x \in X$. Since f is bijective, each element in Y has a single distinct preimage in X. This means that $g(y)$ can be represented as $g(f(x))$ for all $y \in Y$, since $f(x) = y$. Additionally, this also implies that if $g(f(x_1)) = g(f(x_2))$, then $g(f(x_1)) = x_1 = g(f(x_2)) = x_2 \implies x_1 = x_2$. This means that the function $f^{-}1 = g\colon Y \to X$ is well defined on the domain and codomain $\implies f^{-}1$ exists and f is invertible.

2. **In this problem, you will prove directly that matrix multiplication is associative and distributive. Let $A, B \in \mathrm{M}_{m \times n}(F)$ and $C, C' \in \mathrm{M}_{n \times k}(F)$, and $D \in \mathrm{M}_{k \times l}(F)$. Let $\lambda \in F$.**

   (a) **Prove that $(A + B)(\lambda C) = \lambda(AC + BC)$.**

   $(A + B)_{ij} = A_{ij} + B_{ij}$ where A_{ij} and B_{ij} are entries in A and B respectively.

   Let $M = (A + B)(\lambda C)$

$$M_{ij} = \sum_{d=1}^{n} (A + B)_{id} \lambda C_{dj}$$

$$M_{ij} = \lambda \sum_{d=1}^{n} (A + B)_{id} C_{dj}$$

   Now let $M' = \lambda(AC + BC)$

$$M'_{ij} = \lambda(\sum_{e=1}^{n} A_{ie} C_{ej} + \sum_{f=1}^{n} B_{if} C_{fj})$$

$$M'_{ij} = \lambda(\sum_{e=1}^{n} A_{ie}C_{ej} + B_{ie}C_{ej})$$

$$= \lambda \sum_{e=1}^{n} (A_{ie} + B_{ie})C_{ej} \quad \text{(Distributivity on } F\text{)}$$

$$= \lambda \sum_{e=1}^{n} (A + B)_{ie}C_{ej}$$

$$\implies M = M'$$

(b) **Prove that** $B(CD) = (BC)D$.

Let the entries of B, C, D be defined by B_{ij}, C_{ij}, D_{ij}

$$(CD)_{ij} = \sum_{\alpha=1}^{k} C_{i\alpha}D_{\alpha j}$$

Let $M = B(CD)$

Then

$$M_{ij} = \sum_{\beta=1}^{n} B_{i\beta}(CD)_{\beta j}$$

$$M_{ij} = \sum_{\beta=1}^{n} B_{i\beta} \sum_{\alpha=1}^{k} C_{\beta\alpha}D_{\alpha j}$$

$$M_{ij} = \sum_{\beta=1}^{n} \sum_{\alpha=1}^{k} B_{i\beta}C_{\beta\alpha}D_{\alpha j}$$

Let $M' = (BC)D$

$$(BC)_{ij} = \sum_{\alpha=1}^{n} B_{i\alpha}C_{\alpha j}$$

Then

$$M'_{ij} = \sum_{\beta=1}^{n} (BC)_{i\beta}D_{\beta j}$$

$$M'_{ij} = \sum_{\beta=1}^{k} \sum_{\alpha=1}^{n} B_{i\alpha}C_{\alpha\beta}D_{\beta j}$$

$$M'_{ij} = \sum_{\alpha=1}^{k} \sum_{\beta=1}^{n} B_{i\beta}C_{\beta\alpha}D_{\alpha j}$$

$M = M'$ because of commutativity on F, so the order of summation does not matter.

(c) **Recall that, for an $n \times m$ matrix A with entries in F, the transpose of A is the $m \times n$ matrix with entries in F given by $(A^t)_{ij} = Aji$. Prove that $(BC)^t = C^tB^t$. Use this and (a) to deduce that $B(C + C') = BC + BC'$.**

Let $M = C^tB^t$

$$M_{ij} = \sum_{\alpha=1}^{n} C^t_{i\alpha}B^t_{\alpha j}$$

$$M_{ij} = \sum_{\alpha=1}^{n} C_{\alpha i}B_{j\alpha}$$

Now let $M' = (BC)^t$

First consider that

$$(BC)_{ij} = \sum_{\alpha=1}^{n} B_{i\alpha}C_{\alpha j}$$

Then

$$M'_{ij} = (BC)^t_{ij} = (BC)_{ji} = \sum_{\alpha=1}^{n} B_{j\alpha}C_{\alpha i}$$

$M = M'$ from commutativity over F.

To prove that $B(C + C') = BC + BC'$, also show that

$$(A + B)^t = A^t + B^t$$

If $P = (A + B)^t$, then $P_{ij} = (A + B)^t_{ij} = (A + B)_{ji} = A_{ji} + B_{ji} = A^t_{ij} + B^t_{ij}$

$\implies (A + B)^t = A^t + B^t$

Now consider

$$B(C + C') = BC + BC'$$

$$(B(C + C'))^t = (BC + BC')^t \quad \text{apply transpose}$$

$$(C + C')^tB^t = (BC + BC')^t \quad \text{from} \quad (BC)^t = C^tB^t$$

$$(C^t + C'^t)B^t = (BC)^t + (BC')^t \quad \text{from} \quad (A + B)^t = A^t + B^t$$

$$(C^t + C'^t)B^t = C^tB^t + C'^tB^t \quad \text{from} \quad (BC)^t = C^tB^t$$

$$C^tB^t + C'^tB^t = C^tB^t + C'^tB^t \quad \text{from part (a)}$$

$$\implies B(C + C') = BC + BC'$$

3. **Textbook 2.2 # 2 f,g only.**

   Let β and γ be the standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For each linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$, compute $[T]_\beta^\gamma$

   (f) $T\colon \mathbb{R}^n \to \mathbb{R}^n$ **defined by** $T(a_1, a_2, ..., a_n) = (a_n, a_{n-1}, ..., a_1)$

   Let $\beta = \{e_1, ..., e_n\} = \gamma$, and let $A = [T]_\beta^\gamma$

   The i-th column of $A = [T(e_i)]_\gamma = [e_{n-i+1}]_\gamma = e_{n-i+1}$. This is the $n \times n$ matrix with the bottom-left to top-right diagonal entries being all 1 and all other entries 0.

   (g) $T\colon \mathbb{R}^n \to \mathbb{R}$ **defined by** $T(a_1, a_2, ..., a_n) = a_1 + a_n$

   Let $\beta = \{e_1, ..., e_n\} = \gamma$, and let $A = [T]_\beta^\gamma$

   Then $T(v_i)$ only contains multiples of a_1 and a_n for $v_i \in \beta$. Therefore $A = [1\ 0\ ...\ 0\ 1]$

4. **Let $A \in \mathrm{M}_{m\times n}(F)$, and let e_i be the length n column vector with i-th entry 1, and all other entries zero. Prove that the i-th column of A is $A \cdot e_i$.**

   $(Ae_i)_k = \sum_{j=1}^n A_{kj}(e_i)_j$

   $$(e_i)_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

   So $(Ae_i)_k = A_{ki} \implies$ the k-th entry of $Ae_i =$ the k-th entry of the i-th column of $A \implies A = Ae_1$.

5. **Textbook 2.2 # 4**

   Define

   $$T\colon M_{2\times 2}(R) \to P_2(R) \quad \textbf{by} \quad T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2$$

   Let

   $$\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\} \quad \textbf{and} \quad \gamma = \{1, x, x^2\}$$

   Compute $[T]_\beta^\gamma$

   Let $A = [T]_\beta^\gamma$ and $\beta = \{\beta_1, \beta_2, \beta_3, \beta_4\}$

   The i-th column of A is $[T(\beta_i)]_\gamma$. Then

   $$[T]_\beta^\gamma = [\ [T(\beta_1)]_\gamma\ [T(\beta_2)]_\gamma\ [T(\beta_3)]_\gamma\ [T(\beta_4)]_\gamma\ ]$$

   $$= [\ [1]_\gamma\ [1 + x^2]_\gamma\ [0]_\gamma\ [2x]_\gamma\ ]$$

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

6. For an $m \times n$ matrix A, we define a linear transformation $L_A \colon \mathbb{R}^n \to \mathbb{R}^m$ by

$$L_A(\vec{x}) = A \cdot \vec{x},$$

where $\vec{x}$ is a column vector in $\mathbb{R}^n$ and $\cdot$ denotes matrix multiplication.

This linear transformation is called *left multiplication by A*. Define

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 8 & 9 \end{pmatrix}.$$

(a) **Let β_3 and β_2 denote the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Compute $[L_A]_{\beta_3}^{\beta_2}$.**

$\beta_3 = \{e_1, e_2, e_3\}$ and $\beta_2 = \{e_1, e_2\}$. Recall from Problem 4 that $L_A(e_i) = Ae_i =$ the i-th column of A.

$[L_A]_{\beta_3}^{\beta_2} = [[Ae_1]_{\beta_2}[Ae_2]_{\beta_2}[Ae_3]_{\beta_2}] = A$

(b) **Let**

$$\gamma_3 := \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\gamma_2 := \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\}.$$

Compute $[L_A]_{\gamma_3}^{\gamma_2}$.

Let $\gamma_3 = \{\alpha_1, \alpha_2, \alpha_3\}$

$[L_A]_{\gamma_3}^{\gamma_2} = [[A\alpha_1]_{\gamma_2}[A\alpha_2]_{\gamma_2}[A\alpha_3]_{\gamma_2}] = [[\begin{smallmatrix} 4 \\ 17 \end{smallmatrix}]_{\gamma_2}[\begin{smallmatrix} 3 \\ 18 \end{smallmatrix}]_{\gamma_2}[\begin{smallmatrix} 0 \\ 9 \end{smallmatrix}]_{\gamma_2}] = [\begin{smallmatrix} 8.5 & 9 & 4.5 \\ -1.5 & -2 & -1.5 \end{smallmatrix}]$

7. **Section 2.3 # 16**

Let V be a finite-dimensional vector space, and let $T\colon V \to V$ be linear.

(a) **If** $\operatorname{rank}(T) = \operatorname{rank}(T^2)$, **prove that** $\operatorname{Im}(T) \cap \ker(T) = \{\vec{0}\}$

If $\operatorname{rank}(T) = \operatorname{rank}(T^2)$, then

$$\dim(\operatorname{Im}(T)) = \dim(\operatorname{Im}(T^2))$$

$$\implies \dim(\ker(T)) = \dim(\ker(T^2)) \text{ by rank-nullity}$$

Notice that if $k \in \ker(T)$

$$\implies T(k) = \vec{0}$$

$$\implies T(T(k)) = \vec{0} \text{ (Since } T \text{ is linear)}$$

$$\implies k \in \ker(T^2)$$

Therefore $\ker(T) \subset \ker(T^2)$. But since we know that $\dim(\ker(T)) = \dim(\ker(T^2))$, it must follow that $\ker(T) = \ker(T^2)$. (Recall from Homework 3 that if $U \subset V$ is a subspace and $\dim U = \dim V$, then $U = V$)

Let $v \in \operatorname{Im}(T)$ and $v \in \ker(T)$. Then

$$T(v) = \vec{0} \text{ and } v = T(u) \quad \text{for some} \quad u \in V$$

$$T(T(u)) = \vec{0}$$

$$u \in \ker(T^2)$$

Then $u \in \ker(T) \implies T(u) = v = \vec{0}$. Therefore, if $v \in \operatorname{Im}(T)$ and $v \in \ker(T)$, then $v = \vec{0} \implies \operatorname{Im}(T) \cap \ker(T) = \{\vec{0}\}$.

(b) **(Part (a) continued) Deduce that** $V = \operatorname{Im}(T) \oplus \ker(T)$

Since $\operatorname{Im}(T) \subset V$ and $\ker(T) \subset V$ are both subspaces of V, so $\operatorname{Im}(T) + \ker(T) \subset V$.

From rank-nullity, $\dim(V) = \dim(\operatorname{Im}(T)) + \dim(\ker(T))$

Therefore, $\operatorname{Im}(T) + \ker(T) = V$

(c) **Prove that** $V = \operatorname{Im}(T^k) \oplus \ker(T^k)$ **for some positive integer** k

From part (a), we know that $V = \operatorname{Im}(T) \oplus \ker(T)$. Therefore we can express any vector $v \in V$ uniquely as the sum of a vector $w \in \operatorname{Im}(T)$ and $u \in \ker(T)$

Let $v = w + u$

Then

$$T^k(v) = T^k(w+u) = T^k(w) + T^k(u) = T^k(w) + 0$$

Notice that $T^k(v) \in V, T^k(w) \in \mathrm{Im}(T^k)$, and $0 \in \ker(T^k)$. Therefore we've represented a vector in V as the sum of an element from $\mathrm{Im}(T^k)$ and an element from $\ker(T^k)$. We previously defined that $u + w = v$ was unique since V is a direct sum of $\mathrm{Im}(T)$ and $\ker(T)$. Additionally, since $T^k(v) = T^k(w), T^k(v)$ is solely dependent on $T^k(w) \implies T^k(v)$ is a unique composition of elements in $\mathrm{Im}(T^k)$ and $\ker(T^k)$.

Let $v \in \mathrm{Im}(T^k) \cap \ker(T^k)$

Then $v = T^k(u)$ for some $u \in V$, and $T^k(v) = 0$.

Then $T^k(T^k(u)) = 0 \implies T^k(u) = v = 0$

Therefore $\mathrm{Im}(T^k) \cap \ker(T^k) = \{\vec{0}\}$

$\implies V = \mathrm{Im}(T^k) \oplus \ker(T^k)$

8. **Let V and W be vector spaces over F with $\dim(V) = m$ and $\dim(W) = n$. Let $\beta = \{v_1, \dots, v_m\}$ be a basis for V and $\gamma = \{w_1, \dots, w_n\}$ be a basis for W.**

   (a) **Referencing facts about matrix representations from class and the textbook, prove that there is a linear transformation $\mathbb{T}\colon \mathcal{L}(V, W) \to \mathrm{M}_{n\times m}(F)$ given by $\mathbb{T}(T) = [T]_\beta^\gamma$.**

   Let $S, T \in \mathcal{L}$, and let WTS that $\mathbb{T}(T + \lambda S) = \mathbb{T}(T) + \lambda\mathbb{T}(S)$.

   Let $v \in V$, and let $Av = (T + \lambda S)(v)$. Recall that $(T + \lambda S)(v) = T(v) + \lambda S(v)$

   Then the j-th column of $\mathbb{T}(A) = [Av_j]_\gamma = [(T + \lambda S)(v_j)]_\gamma = [T(v_j) + \lambda S(v_j)]_\gamma$ for $1 \le j \le m$.

   Recall from lecture that this is equal to

$$[T(v_j)]_\gamma + [\lambda S(v_j)]_\gamma$$

$$\mathbb{T}(A)_j = [T(v_j)]_\gamma + \lambda[S(v_j)]_\gamma$$

$$\implies \mathbb{T}(A) = \mathbb{T}(T) + \lambda\mathbb{T}(S)$$

   (b) **Prove that $\mathbb{T}$ is an isomorphism.**

   To show that $\mathbb{T}$ is an isomorphism, it suffices to prove that $\mathbb{T}$ is a bijection from $\mathcal{L}(V, W) \to \mathrm{M}_{n\times m}(F)$.

Let $S, T \in \mathcal{L}$.

If $\mathbb{T}(S) = \mathbb{T}(T)$, then $[S]_\beta^\gamma = [T]_\beta^\gamma \implies$ they contain the same entries.

Therefore $[S(v_j)]_\gamma = [T(v_j)]_\gamma$ for $1 \leq j \leq m$. Recall that the function $f_\gamma \colon W \to F^n$ is a linear isomorphism $\implies S(v_j) = T(v_j) \quad \forall v_j \in \beta \implies S = T$ since S and T also have the same domain and codomain. Therefore $\mathbb{T}$ is injective.

To show surjectivity, we must show that any $n \times m$ matrix A over F can be written as $\mathbb{T}(T)$ for some function $T \in \mathcal{L}(V, W)$

Let A_j be the j-th column of A. Recall that the function $f_\gamma \colon W \to F^n$ is an isomorphism. Therefore $f_\gamma^{-1}(A_j) = w \in W$. Now define T such that $T(v_j) = f_\gamma^{-1}(A_j)$

This means that $[T(v_j)]_\gamma = f_\gamma(f_\gamma^{-1}(A_j)) = A_j \implies [T]_\beta^\gamma = A$

Precisely, we have defined a $T \in \mathcal{L}$ such that $\mathbb{T}(T) = A$. Therefore, $\mathbb{T}$ is surjective.

Since $\mathbb{T}$ is both injective and surjective, it is bijective and therefore an isomorphism.