Math 115A, Lecture 4 - HW7

Due on Gradescope by 11:59pm on Thursday, May 23, 2024.

What you can use in your proofs: *you may use basic facts which are not "course material", including facts from calculus or basic number theory. If you use such a result, you must clearly state the fact that you are using, although you do not justify it. You may also use any theorems in the textbook or discussed in class, unless the problem is specifically proving a theorem from the text or from class.*

1. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by reflection across the line

$$L = \{(x, y, z) \in \mathbb{R}^3 \mid x = y,\, z = 0\}.$$

Concretely, this equation is given as follows: for any vector $v \in \mathbb{R}^3$, there exist unique vectors $x \in L$ and y perpendicular to L such that $v = x + y$. Then $T(v) = x - y$. In words, T does not change the component of x parallel to L, and negates the component of v orthogonal to L.

Let $\beta = \{(1, 1, 0),\ (1, -1, 0),\ (0, 0, 1)\}$.

(a) **Check that $(1, 1, 0)$ spans L and that $(1, -1, 0)$ and $(0, 0, 1)$ are orthogonal to $(1, 1, 0)$ (and therefore are orthogonal to the entire line L).**
Let $\beta = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\} = \{\beta_1, \beta_2, \beta_3\}$
L is spanned by the line $(x, y, z) = (x, x, 0)$ for $x \in \mathbb{R}$. Therefore $\beta_1 \in L$.
Since we know that $\beta_1 \in L$, it suffices to check that $\beta_2 \cdot \beta_1 = 0$ and $\beta_3 \cdot \beta_1 = 0$.

$$\beta_1 \cdot \beta_2 = (1 \cdot 1) + (1 \cdot -1) + (0 \cdot 0) = 0$$

$$\beta_1 \cdot \beta_3 = (1 \cdot 0) + (1 \cdot 0) + (0 \cdot 1) = 0$$

(b) **Compute $[T]_\beta$.**
$[T(\beta_1)]_\beta = [\beta_1]_\beta = [1\ 0\ 0]$
$[T(\beta_2)]_\beta = [(1, -1, 0)]_\beta = [0\ -1\ 0]$
$[T(\beta_3)]_\beta = [(0, 0, -1)]_\beta = [0\ 0\ -1]$

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(c) **Let $S = \{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}$ be the standard basis for $\mathbb{R}^3$. Compute $Q^{-1} = [I_{\mathbb{R}^3}]_\beta^S$.**
$[I(1, 1, 0)]_S = [(1, 1, 0)]_S = [1\ 1\ 0]$
$[I(1, -1, 0)]_S = [(1, -1, 0)]_S = [1\ -1\ 0]$
$[I(0, 0, 1)]_S = [(0, 0, 1)]_S = [0\ 0\ 1]$

$$Q^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) **Compute $Q = ([I_{\mathbb{R}^3}]_S^\beta$.**
$[I(1,0,0)]_\beta = [(1,0,0)]_\beta = [\frac{1}{2}\ \frac{1}{2}\ 0]$
$[I(0,1,0)]_\beta = [(0,1,0)]_\beta = [\frac{1}{2}\ -\frac{1}{2}\ 0]$
$[I(0,0,1)]_\beta = [(0,0,1)]_\beta = [0\ 0\ 1]$

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(e) **Recall that the change of basis formula states that $[T]_S = Q^{-1}[T]_\beta Q$. Use the change of basis formula and the previous items to compute $[T]_S$.**

$$[T]_S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[T]_S = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[T]_S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

2. **Determinants of triangular matrices.** Recall that a $n \times n$ matrix A with entries in F is *upper-triangular* if $A_{ij} = 0$ whenever $i > j$. A is *lower triangular* if $A_{ij} = 0$ whenever $j > i$.

(a) **Using induction, prove a general formula for the characteristic polynomial of an upper-triangular matrix.**
Since a 1×1 matrix is a trivial, case consider the 2×2 upper triangular matrix as the base case:
$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$ (entries in F)

Then $\lambda I_2 - A = \begin{pmatrix} \lambda - a_{11} & a_{12} \\ 0 & \lambda - a_{22} \end{pmatrix}$ the characteristic polynomial is
$\det(\lambda I_2 - A) = (\lambda - a_{11})(\lambda - a_{22}) - (0)(a_{12})$
Assume the characteristic polynomial of an $n \times n$ upper triangular matrix A is

$$\det(\lambda I_n - A) = \prod_{i=1}^{n} (\lambda - A_{ii})$$

If we then consider the $(n+1)\times(n+1)$ upper triangular matrix A'. The characteristic polynomial is $\det(\lambda I_{n+1} - A)$. If we expand across the bottom row, all summands will be zero since the bottom row is all zero except for $A_{n+1,n+1}$. Therefore

$$\det(\lambda I_{n+1} - A) = (\lambda - A_{n+1,n+1})(\overline{\det(A_{n+1,n+1}})) $$
$$= (\lambda - A_{n+1,n+1}) \det(\lambda I_n - A)$$
$$= (\lambda - A_{n+1,n+1}) \prod_{i=1}^{n} (\lambda - A_{ii})$$
$$\prod_{i=1}^{n+1} (\lambda - A_{ii})$$

(b) **Using properties of matrices given in class, give a formula for the characteristic polynomial of a lower-triangular matrix.**

Let $\lambda I_n - A$ be an upper triangular matrix, then $(\lambda I_n - A)^t$ is a lower triangular matrix. Recall that $\det A = \det A^t$

$$\implies \det \lambda I_n - A = \det(\lambda I_n - A)^t$$

Then if $\det(\lambda I_n - A) = \det(\lambda I_n - A)^t$, they have the same characteristic polynomial, so the characteristic polynomial is

$$\prod_{i=1}^{n+1} (\lambda - A_{ii})$$

(c) Textbook 5.2 #9.
Let T be a linear operator on a finite-dimensional vector space V, and suppose there exists an ordered basis β for V such that $[T]_\beta$ is an upper triangular matrix.

i. **Prove that the characteristic polynomial for T splits**
If $[T]_\beta$ is an upper triangular matrix, then $[\lambda 1_V - T]_\beta$ is also an upper triangular matrix, and $\det[\lambda 1_V - T]_\beta$ (characteristic polynomial) is the product of the diagonal entries as from part (b), which by definition implies splitting.
Recall that if β is an ordered basis for V, then $\det T = \det[T]_\beta$. This means that $\det[\lambda 1_V - T]_\beta = \det[\lambda 1_V - T]$, which by definition is the characteristic polynomial of T. Therefore since $\det[\lambda 1_V - T]_\beta$ splits and $\det[\lambda 1_V - T]_\beta = [\lambda 1_V - T]$, then the characteristic polynomial of T must split as well.

ii. **State and prove an analogous result for matrices**

Let A be an $n \times n$ upper-triangular matrix. Then,

$$f_A = \det(\lambda I_n - A)$$

Since A is upper-triangular, this is just the product of the diagonal entries.

$$f_A = (\lambda - A_{11})(\lambda - A_{22}) \dots (\lambda - A_{nn})$$

Therefore we see that the characteristic polynomial of A splits.

3. Suppose that A and B are both square matrices with entries in a field F.

(a) **Prove that if A is similar to B, then $\det A = \det B$. (Hint: use the definition of similarity and basic facts about determinants stated in class/ in section 4.4 of the textbook.)**
If A is similar to B, then there exists the invertible matrix P such that

$$PAP^{-1} = B$$

$$\det(PAP^{-1}) = \det(B) \quad \text{(Take determinant of both sides)}$$
$$\det(P)\det(A)\det(P^{-1}) = \det(B) \quad \text{(Properties of determinant)}$$
$$\det(P)\det(P^{-1})\det(A) = \det(B) \quad \text{(Commutativity over } F\text{)}$$

Recall that for any invertible matrix P, $\det(P^{-1}) = \frac{1}{\det(P)}$. Therefore we get

$$\det(A) = \det(B)$$

(b) **Are the following two matrices A and B with entries in $\mathbb{R}$ similar?**

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

No, because $\det(A) = 0$ and $\det(B) = (2)(4)(1) = 8$ and if A and B are similar, then $\det(A) = \det(B)$

(c) **Prove that if A is similar to B, then $\det(tI_n - A) = \det(tI_n - B)$.**
If A is similar to B, then there exists the invertible matrix P such that

$$P^{-1}AP = B$$

Then, substitute to get

$$\det(tI_n - P^{-1}AP)$$
$$= \det(P^{-1})\det(PtI_n - AP) \quad \text{Factor out } P^{-1}$$
$$= \det(P^{-1})\det(P)\det(tI_n - A) \quad \text{Factor out } P$$
$$= \det(tI_n - A)$$

(d) **Let V be an n-dimensional vector space and let $T \colon V \to V$ be a linear transformation. Prove that the characteristic polynomial of T does not depend on the choice of matrix representation: if β and γ are two bases for V, then $f_{[T]_\beta}(t) = f_{[T]_\gamma}(t)$.**

Let β and γ be arbitrary distinct bases for V. Then

$$f_{[T]_\beta}(t) = \det(t1_V - [T]_\beta)$$

From part (c), we know that if A and B are similar, then $\det(tI_n - A) = \det(tI_n - B)$. Therefore, it suffices to show that $[T]_\beta$ and $[T]_\gamma$ are similar. By definition, since γ and β are both bases for V, there exists the invertible matrix $Q = [1_V]_\beta^\gamma$ such that

$$Q^{-1} = [1_V]_\gamma^\beta$$

and

$$Q[T]_\beta Q^{-1} = [1_V]_\beta^\gamma [T]_\beta [1_V]_\gamma^\beta = [T]_\gamma$$

This implies that $[T]_\beta$ and $[T]_\gamma$ are similar. Therefore, by part (c),

$$\det(t1_V - [T]_\beta) = \det(t1_V - [T]_\gamma)$$
$$\implies f_{[T]_\beta}(t) = f_{[T]_\gamma}(t)$$

4. **Textbook 5.1 #2 (a), (c) - only compute the characteristic polynomials of the given transformations.**

(a)

$$T = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}$$

$$f_T(t) = \det(tI_2 - T) = (t-2)(t-3) + 5 = t^2 - 5t + 11$$

(c)

$$T = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$f_T(t) = \det(tI_4 - T) = \det \begin{pmatrix} t-1 & 0 & 1 & 0 \\ 1 & t-1 & 0 & -1 \\ -1 & -1 & t & 1 \\ 0 & 0 & 1 & t \end{pmatrix}$$

$$= (t-1)\det \begin{pmatrix} t-1 & 0 & -1 \\ -1 & t & 1 \\ 0 & 1 & t \end{pmatrix} + 1 \det \begin{pmatrix} 1 & t-1 & -1 \\ -1 & -1 & 1 \\ 0 & 0 & t \end{pmatrix}$$

$$= (t-1)[(t-1)(t^2-1) - (-1)(0+1)] + [t((-1) - (-1)(t-1))]$$
$$= (t-1)[(t-1)(t^2-1) + 1] + [t((-1) + t - 1))]$$
$$= (t-1)^2(t^2-1) + t - 1 + [t(t-2)]$$
$$= (t-1)^2(t^2-1) + t - 1 + t^2 - 2t$$
$$= (t-1)^2(t^2-1) - 1 + t^2 - t$$
$$= (t-1)^2(t^2-1) + t^2 - 1 - t$$
$$= (t^2 - 2t + 1)(t^2-1) + t^2 - 1 - t$$
$$= t^4 - t^2 - 2t^3 + 2t + t^2 - 1 + t^2 - 1 - t$$
$$= t^4 - 2t^3 + t^2 - 2$$

5. **Textbook 5.1 #3, (a),(f).**
(a)

$$[T]_\beta = [[T(\beta_1)]_\beta [T(\beta_2)]_\beta]$$
$$T(1,2) = (10-12, 17-20) = (-2,-3)$$
$$T(2,3) = (20-18, 34-30) = (2,4)$$
$$[T]_\beta = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} 10 & -6 \\ 17 & -10 \end{pmatrix}$$
$$f_T(t) = (t-10)(t+10) + 102 = t^2 + 2$$

V is a field over real numbers, but T has complex eigenvalues, β is not a basis of eigenvectors

(f)
$$[T]_\beta = [[T(\beta_1)]_\beta [T(\beta_2)]_\beta] [T(\beta_3)]_\beta [T(\beta_4)]_\beta$$

$$T((1,1),(0,0)) = ((-7+4, -8+5),(0,0) = ((-3,-3),(0,0)$$
$$T((-1,0),(2,0)) = ((7-8, 8-8),(2,0)) = ((-1,0),(2,0))$$
$$T((1,2),(0,0)) = ((-7+8, -8+10),(0,0)) = ((1,2),(0,0))$$
$$T((-1,0),(0,2)) = ((7-8, 8-8),(0,2)) = ((-1,0),(0,2))$$

In β-coordinates, we get
$$[T]_\beta = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then,
$$f_{[T]_\beta}(t) = \det(tI_4 - [T]_\beta) = (t+3)(t-1)^3$$

Yes, β is a basis of eigenvectors

6. **Textbook 5.2 #3, (a), (b), (f).**
(a)
$$T(f(x)) = f'(x) + f''(x)$$
$$T(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2 + 2c + 6dx = b + 2c + 2cx + 6dx + 3dx^2$$

Let β be the standard basis for $P_3(\mathbb{R})$. We can use this as a consequence of what we proved in 3(d)

$$T(\beta_1) = T(1) = 0$$
$$T(\beta_2) = T(x) = 1$$
$$T(\beta_3) = T(x^2) = 2x + 2$$
$$T(\beta_4) = T(x^3) = 3x^2 + 6x$$

In β coordinates, this is the matrix

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} f_{[T]_\beta}(t) &= \det(tI_4 - [T]_\beta) \\ &= \det \begin{pmatrix} t & -1 & -2 & 0 \\ 0 & t & -2 & -6 \\ 0 & 0 & t & -3 \\ 0 & 0 & 0 & t \end{pmatrix} \\ &= t^4 \end{aligned}$$

Since $t = 0$, there are no eigenvectors and T is not diagonalizable.

(b)

$$T(ax^2 + bx + c) = cx^2 + bx + a$$

Let β be the standard basis for $P_2(\mathbb{R})$

$$T(\beta_1) = T(1) = x^2$$
$$T(\beta_2) = T(x) = x$$
$$T(\beta_3) = T(x^2) = 1$$

In β coordinates, this is the matrix

$$[T]_\beta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$f_{[T]_\beta}(t) = \det(tI_3 - [T]_\beta)$$

$$= \det \begin{pmatrix} t & 0 & -1 \\ 0 & t-1 & 0 \\ -1 & 0 & t \end{pmatrix}$$

$$= t((t-1)t) + (-1)(0 + (t-1))$$
$$= t^2(t-1) + (-1)(t-1)$$
$$= (t^2-1)(t-1)$$
$$= (t+1)(t-1)^2$$
$$t = 1, -1$$

1 and -1 have algebraic multiplicity of 2 and 1 respectively Clearly, the characteristic polynomial splits, so we need to check that for $t = 1, -1$, the multiplicity of $T = \text{nullity}(tI - [T]_\beta)$
For $t = 1$,

$$= \ker \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Clearly, this is spanned by $(1, 0, 1)$ and $0, 1, 0$, so the nullity is 2.
For $t = -1$,

$$= \ker \begin{pmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

This is spanned by $(1, 0, -1)$, so the nullity is 1 The respective multiplicities agree with the respective nullities, therefore, $[T]_\beta$ is diagonalizable $\implies T$ is diagonalizable . The basis β' for V such that $[T]'_\beta$ is diagonal is the eigenbasis. Therefore we need to find the eigenvectors for each eigenvalue. Given that $T(v) = \lambda v$, first consider $\lambda = 1$:

$$T(v) = 1v = v$$
$$v = \text{span}\{(1, 0, 1), (0, 1, 0)\}$$

Then for $\lambda = -1$:

$$T(v) = -1v = -v$$
$$v = \text{span}\{(1, 0, -1)\}$$

Therefore, the basis $\beta' = \{(1, 0, 1), (0, 1, 0), (1, 0, -1)\}$.

(f)

$V = M_{2\times 2}(\mathbb{R})$ and T is defined by $T(A) = A^t$
Let $S = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the standard basis for $M_{2\times 2}$
$T(E_{11}) = E_{11}$
$T(E_{12}) = E_{21}$
$T(E_{21}) = E_{12}$

$T(E_{22}) = E_{22}$
In S coordinates, this is the matrix

$$[T]_S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$f_{[T]_S}(t) = \det \begin{pmatrix} t-1 & 0 & 0 & 0 \\ 0 & t & -1 & 0 \\ 0 & -1 & t & 0 \\ 0 & 0 & 0 & t-1 \end{pmatrix}$$

$$(t-1)[t(t)(t-1) - (-1)(-1)(t-1)]$$
$$= (t-1)[t^2(t-1) - (t-1)]$$
$$= (t-1)(t^2-1)(t-1)$$
$$= (t+1)(t-1)^3$$
$$t = -1, 1(\text{multiplicity 1,3 respectively})$$

Since the characteristic polynomial splits, STP that the multiplicity equals the nullity as in part (b)
For $t = 1$,

$$= \ker \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is spanned by $\{(1,0,0,0), (0,1,1,0), (0,0,0,1)\} \implies$ nullity $= 3$

For $t = -1$,

$$= \ker \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

This is spanned by $\{(0,1,-1,0)\} \implies$ nullity $= 1$

Therefore, $[T]_\beta$ is diagonalizable $\implies T$ is diagonalizable
To find the eigenbasis β, use $T(v) = \lambda v$ for each λ
For $\lambda = 1$

$$T(v) = v$$

$$\text{span}\{(1,0,0,0),(0,1,1,0),(0,0,0,1)\}$$

For $\lambda = -1$

$$T(v) = -v$$

$$\text{span}\{(0,-1,1,0)\}$$

$$\beta = \{(1,0,0,0),(0,1,1,0),(0,0,0,1),(0,-1,1,0)\}$$

7. Textbook 5.2 #8.
Suppose that $A \in M_{n\times n}(F)$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda 1}) = n-1$. Prove that A is diagonalizable.

If $\dim(E_{\lambda 1}) = n-1$, then $\dim(E_{\lambda 2}) = 1$, since $\dim(E_{\lambda 1}) + \dim(E_{\lambda 2}) = n$. We need to show that the basis vectors for $E_{\lambda 1}$ and $E_{\lambda 2}$ for a basis.
Let $\{v_1, ..., v_{n-1}\}$ be a basis for $E_{\lambda 1}$, and let w be a basis for $E_{\lambda 2}$
Clearly, $M_{n\times n}$ is spanned by n vectors, and $\dim(E_{\lambda 1}) + \dim(E_{\lambda 2}) = n$.. Therefore, we only need to show that $\{v_1, ..., v_{n-1}, w\}$ is linearly independent.
Suppose for the sake of contradiction that w is linearly dependent.
Let $v \in E_{\lambda 1}$. Then $T(v) = \lambda_1 v$
Since w is linearly independent from $E_{\lambda 1}, v = \alpha w$ for some $\alpha \in F$

$$\implies T(v) = T(\alpha w)$$

$$\implies \lambda_1 v = \alpha T(w)$$

$$\implies \lambda_1 v = \alpha \lambda_1 w$$

However, since $w \in E_{\lambda 2}, T(w) = \lambda_2 w$. Since λ_1, λ_2 are distinct eigenvalues, this is not possible. Therefore, the set $\{v_1, ..., v_{n-1}, w\}$ is linearly independent and forms a basis, which implies that A has n eigenvectors and is diagonalizable.

8. **Let V be a vector space over F and $T: V \to V$ linear with eigenvalues $\lambda_1, \dots, \lambda_n$, and such that $\lambda_i \neq \lambda_j$ for $i \neq j$. (You should assume that T has no other eigenvalues.)**

   (a) **Let $E = E_{\lambda_1}$ denote the λ_1 eigenspace. Let V/E denote the quotient vector space. Prove that the function $S: V/E \to V/E$ defined for $[v] \in V/E$ by $S([v]) = [T(v)]$ is well-defined and linear.**

   Notice that $[v] \in V/E = \{\text{vectors in the same eigenspace}\}$.
   Let $u, w \in V/E$ such that u and w are distinct and in E. WTS that

$$[u] = [w] \implies S([u]) = S([w])$$

$$S([u]) = [T(u)]$$

$$= [\lambda_1 u]$$

$$= \lambda_1 [u]$$

$$S([w]) = [T(w)]$$
$$= [\lambda_1 w]$$
$$= \lambda_1 [w]$$

Therefore, $[u] = [w] \implies S([u]) = S([w])$ Therefore S is well defined.

To show that S is linear, need to show that $S([u] + \alpha[w]) = S([u]) + \alpha S([w])$
Recall that the definition of addition on quotient spaces that $[u] + [\alpha v] = [u + \alpha v]$. Therefore,

$$S([u] + \alpha[w]) = S([u + \alpha w])$$
$$= [T(u + \alpha w)] \quad \text{By the definition of } S$$
$$= [T(u) + \alpha T(w)] \quad \text{By linearity of } T$$
$$= [T(u)] + \alpha[T(w)] \quad \text{By addition and multiplication on } V/E$$
$$= S([u]) + \alpha S([w])$$

(b) **Prove that $\lambda_2, \ldots, \lambda_n$ are eigenvalues of S.**

Let $v_i \in E_{\lambda_i}$ for $2 \leq i \leq n$
Then $T(v_i) = \lambda_i v_i$.
Consider $S([v_i])$. By the definition of S, this equals

$$[T(v_i)]$$
$$= [\lambda_i v_i] \quad \text{Since } \lambda_i \text{ is the eigenvalue for } E_i$$
$$= \lambda_i [v_i] \quad \text{By scalar multiplication on } V/E$$

$\implies \lambda_i$ is an eigenvalue of S.

9. **Let A be an $n \times n$ square matrix with n distinct eigenvalues λ_i, $1 \leq i \leq n$. Suppose we have nonzero eigenvectors v_i, $1 \leq i \leq n$, such that $Av_i = \lambda_i v_i$. Show that the set $\beta = \{v_1, v_2, \ldots, v_n\}$ is linearly independent.**

Consider when

$$\sum_{i=1}^{n} \alpha_i v_i = 0$$
$$\implies A(\sum_{i=1}^{n} \alpha_i v_i) = T(0) = 0$$
$$\implies \sum_{i=1}^{n} \alpha_i A(v_i) = 0 \quad \text{By distributivity of matrices}$$
$$\implies \sum_{i=1}^{n} \alpha_i \lambda_i v_i = 0$$

Therefore, since we know that λ_i and v_i are nonzero for all $1 \leq i \leq n$, we know that α_i must be zero for all $1 \leq i \leq n$, $\implies \beta$ is linearly independent.
There is the edge case that one $\lambda_i = 0$. However, the corresponding v_i is still linearly independent from the remaining vectors. WLOG, suppose for the sake of contradiction that $\lambda_n = 0$ and v_n is linearly dependent. Let $x \in \{v_1, \ldots, v_{n-1}\}$ Then, $x = \gamma v_n$ for some $\gamma \in F$

$$\implies A(x) = A(\gamma v_n)$$
$$\implies A(x) = \gamma\lambda_n v_n$$
$$\implies A(x) = 0$$

This is not possible, because this implies $\text{Im}(A) = 0$, which is not the case because all other eigenvalues and eigenvectors are nonzero. Therefore, even if there exists a $\lambda_i = 0$, β is still linearly independent.

10. Recall that, for $A \in M_{n\times n}(F)$,
$$f_A(t) = \det(tI_n - A)$$

(a) **Let $A \in M_{n\times n}(F)$. Prove, by induction on n, that $\deg(f_A) = n$**

Consider the base case being a 2×2 matrix, since a 1×1 matrix is trivial.
Let
$$A = \begin{pmatrix} a_0 & a_2 \\ a_1 & a_3 \end{pmatrix} \quad a_i \in F$$

Then $f_A(t) = (t - a_o)(t - a_3) - (a_1)(a_2) \implies \deg(f_A(t)) = 2$

For the inductive step, show that $P_n \implies P_{n+1}$
Let A be an $n + 1 \times n + 1$ matrix. Then,

$$f_A(t) = (t - A_{11})(\det(\overline{A_{11}}))$$
$$= (t - A_{11})f_{A_{11}}(t)))$$
$$(t - A_{11})(\det(\overline{A_{11}}))$$

$\overline{A_{11}}$ is an $n \times n$ matrix which we assumed to have degree n. Therefore,

$$f_A(t) = (t - A_{11})(\det(\overline{A_{11}}))$$

must have degree $n + 1$

(b) **Using the previous item and defintions, prove that if V is a vector space over F, $\dim(V) = n$ and $T\colon V \to V$ is linear, then $\deg(f_T) = n$**

Let β be an ordered basis for V. Since $\dim(V) = n$, it must follow that β has n elements.
Recall that $f_T(t) = f_{[T]_\beta}(t)$ if β is a basis for V
Since $[T]_\beta$ is an $n \times n$ matrix, $f_{[T]_\beta}(t)$ must have degree n from part (a) $\implies f_T(t)$ has degree n.

11. **We say that the sum $W_1 + \ldots + W_r$ is an (internal) direct sum if the representation of elements in $W_1 + \ldots + W_r$ is unique**

(a) **Recall that $W_1 + W_2$ is a direct sum if $W_1 \cap W_2 = \vec{0}$. Prove that $W_1 \cap W_2 = \vec{0} \iff$ the definition above with $r = 2$ is satisfied**
($\implies$)
If $W_1 \cap W_2 = \vec{0}$, for any $w_1 \in W_1$ and $w_2 \in W_2$, w_1 and w_2 are linearly independent. Recall that w_1 has a unique representation in W_1. Since all vectors in W_2 are linearly independent from vectors in W_1, the representation of any w_1 in $W_1 + W_2$ remains unique since no vector in W_1 can be represented as a linear combination of vectors in W_2. Therefore, W_1+W_2 is a direct sum

($\Longleftarrow$)
If $W_1 + W_2$ is a direct sum, then for $w, w' \in W_1 + W_2$,

$$w_1 + w_2 = w_1' + w_2' \implies w = w'$$

Suppose for the sake of contradiction that $W_1 \cap W_2 \neq \vec{0}$. Then, there exists some $w_i \in W_1$ such that $w_i = \alpha u_i$ for some $\alpha \in F$ and $u_i \in W_2$.
Then $w_i + u_i \in W_1 \cap W_2$
But $w_i + u_i = \alpha u_i + u_i$, which is a contradiction because we assumed that any element in $W_1 \cap W_2$ has a unique representation. Therefore, $W_1 \cap W_2 = \vec{0}$

(b) **Let $V = \mathbb{R}^2, W_1 = \text{span}\{(1,0)\}, W_2 = \text{span}\{(0,1)\},$ and $W_3 = \{(1,1)\}$. Explain why $W_1 + W_2, W_2 + W_3,$ and $W_1 + W_3$ are direct sums, but $W_1 + W_2 + W_3$ is not**

Let $w_1 \in W_1, w_2 \in W_2, w_3 \in W_3$. From observation, the pairs of vectors w_1 and w_2, w_1 and w_3, and w_2 and w_3 are linearly independent. From part (a), this implies that $W_1 + W_2, W_2 + W_3$, and $W_1 + W_3$ are direct sums. However, consider the vector $(1,1) \in W_1 + W_2 + W_3$ $(1,1)$ is both $1(1,0) + 0(0,1) + 0(1,1)$ and $0(1,0) + 0(0,1) + 1(1,1)$. Therefore, since the vector has multiple representations, $W_1 + W_2 + W_3$ is not a direct sum.

(c) **Let $W_1, \ldots, W_r$ be arbitrary subspaces of V (not necessarily with a sum that is direct). Consider the vector spaces $W_1 \times \ldots \times W_r = \{(w_1, \ldots, w_r) | w_i \in W_i\}$ with addition and scaling defined component-wise. Prove that the transformation**

$$T \colon W_1 \times \ldots \times W_r \to W_1 + \ldots + W_r$$

given by

$$T(v_1, \ldots, v_r) = v_1 + \cdots + v_r$$

is linear.

To show linearity, we must show that for any $u, w \in W_1 \times \ldots \times W_r$ and $\lambda \in F$,

$$T(u + \lambda w) = T(u) + \lambda T(w)$$

$u = (u_1, \ldots, u_r), w = (w_1, \ldots, w_r)$
By the definition of addition and scaling component-wise,

$$u + \lambda w = (u_1 + \lambda w_1, \ldots, u_r + \lambda w_r)$$

$$T(u + \lambda w) = u_1 + \lambda w_1 + \ldots + u_r + \lambda w_r$$

Since the addition of subspaces is commutative, this is equal to

$$u_1 + \ldots + u_r + \lambda w_1 + \ldots + \lambda w_r$$

$$u_1 + \ldots + u_r + \lambda(w_1 + \ldots + w_r)$$
$$= T(u) + \lambda T(w)$$

(d) **Prove that T is an isomorphism if and only if the sum $W_1 + \ldots + W_r$ is an internal direct sum.**

For simplicity, let $\beta = W_1 \times \ldots \times W_r$ and $\gamma = W_1 + \ldots + W_r$

($\Longrightarrow$)
Suppose T is an isomorphism, then there is a bijection from $\beta \to \gamma$. Furthermore, there exists the function $T^{-1}\colon \gamma \to \beta$
Let $a, a' \in \gamma$ such that $a = a'$ and

$$a = a_1 + \ldots + a_r \quad \text{for } a_i \in W_i \ \forall 1 \le i \le r$$

and

$$a' = a'_1 + \ldots + a'_r \quad \text{for } a'_i \in W_i \ \forall 1 \le i \le r$$

Because $a = a', T^{-1}(a) = T^{-1}(a')$. Then

$$T^{-1}(a_i) = T^{-1}(a'_i)$$

But since T^{-1} is also an isomorphism, $a_i = a'_i$. Therefore, $\gamma = W_1 + \ldots + W_r$ is a direct sum.

($\Longleftarrow$)
Suppose γ is a direct sum. Then, referencing above,

$$a_i = a'_i \ \forall 1 \le i \le r$$

Therefore, if we have $b, b' \in \beta$ and $T(b) = a, T(b') = a$

$$T(b) = T(b_1, \ldots, b_r) = b_1 + \ldots + b_r = a$$
$$T(b') = T(b'_1, \ldots, b'_r) = b'_1 + \ldots + b'_r = a$$

Since γ is a direct sum, $b_i = b'_i \ \forall 1 \le i \le r$. Therefore, $b = b' \implies T$ is one-to-one.
To show that T is onto, consider $a \in \gamma$.

$$a = a_1 + \ldots + a_r \quad \text{for } a_i \in W_i \ \forall 1 \le i \le r$$

Then by the definition of T,

$$T(a_1, \ldots, a_r) = a$$

Therefore any arbitrary $a \in \gamma$ has a preimage in β, so T is onto.
Since T is both one-to-one and onto, T is an isomorphism.

(e) **Suppose that V is finite-dimensional, so that $W_1, \ldots, W_r$ are also finite-dimensional vectors spaces. Prove that $\dim(W_1 \times \cdots \times W_r) = \dim(W_1) + \ldots + \dim(W_r)$. Using the previous item, conclude that if $W_1 + \ldots + W_r$ is a direct sum, then $\dim(W_1 + \ldots + W_r) = \dim(W_1) + \ldots + \dim(W_r)$**

Suppose we construct a basis β for some W_i such that $\beta = \{e_1, e_2, \ldots, e_{di}\}$. Then the dimension $W_1 \times \cdots \times W_r$ would be the sum of the dimensions of each W_i by definition of the Cartesian product.
Let $\dim(W_i) = d_i$. Then $\dim(W_1 \times \cdots \times W_r) = d_1 + \ldots + d_r = \dim(W_1) + \ldots + \dim(W_r)$

Suppose $W_1 + \ldots + W_r$ is a direct sum. Then $T\colon W_1 \times \cdots \times W_r \to W_1 + \ldots + W_r$ is an isomorphism by part (d).
Therefore $\dim(W_1 \times \cdots \times W_r) = \dim(W_1 + \ldots + W_r)$
Since we also know that $\dim(W_1 \times \cdots \times W_r) = \dim(W_1) + \ldots + \dim(W_r)$, we conclude that

$$\dim(W_1 + \ldots + W_r) = \dim(W_1) + \ldots + \dim(W_r)$$