Math 115AH Homework 3

April 26, 2024

1. **Problem 1**

   (a) **Textbook 1.6, 3(b)**
   Determine which of the following sets are bases for $P_2(R)$

   $S = \{1 + 2x + x^2, 3 + x^2, x + x^2\}$
   $a(1 + 2x + x^2) + b(3 + x^2) + c(x + x^2) = \vec{0} = 0 + 0x + 0x^2$
   $(a + 3b) + (2a + c)x + (a + b + c)x^2 = 0 + 0x + 0x^2$
   $a + 3b = 0, 2a + c = 0, a + b + c = 0$
   Let $a = -3b = \frac{-c}{2}$, then $-3b + b + c = -2b + c = 0 \implies c = 2b$. Substituting, we get $2a + 2b = 0 \implies a = -b$. Since $a = -3b, a, b = 0 \implies c = 0 \implies S$ is linearly independent. Additionally, let $v \in P_2(R)$, then

   $$v = (a + 3b) + (2a + c)x + (a + b + c)x^2 \text{ (linear combination of elements in S rearranged as above)}$$

   Since S spans $P_2(R)$, it is also a basis.

   (b) **Textbook 1.6, 6**
   Give three different bases for F^2 and for $M_{2\times 2}(F)$

   F^2

   i. $\{(1, 0), (0, 1)\}$
   ii. $\{(1, 0), (1, 1)\}$
   iii. $\{(1, 2), (1, 1)\}$

   $M_{2\times 2}(F)$

   i. $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$
   ii. $\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$
   iii. $\left\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$

   (c) **Textbook 1.6, 16**
   The set of all upper triangular $n \times n$ matrices is a subspace W of $M_{2\times 2}(F)$. Find a basis for W. What is the dimension of W

   $\mathfrak{B}_W = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$
   $\dim(W) = 3$

2. **Problem 2**

   Let V be a vector space over an arbitrary field F and $\lambda \in F$ be an arbitrary scalar. Let $S = \{u_1, ..., u_n\}$ be a given finite subset of V. Define a new subset $S' \subset V$ by

$$S' = (S \backslash \{u_2\}) \cup \{u_2 - \lambda u_1\}$$

   Obtain S' from S by removing the element u_2 and replacing it with $u_2 - \lambda u_1$.

   (a) **Prove that if S is linearly independent, then S' is linearly independent**
   If S is linearly independent, then $a_1u_1 + a_2u_2 + ... + a_nu_n = \vec{0}$ for $a_i \in F$ if and only if $a_1 = a_2 = ... = a_n = 0$
   Observe that by removing the element u_2, $S \backslash \{u_2\}$ remains linearly independent.
   Let $S' = (S \backslash \{u_2\}) \cup \{u_2 - \lambda u_1\} = a_1u_1 + a_2(u_2 - \lambda u_1) + a_3u_3 ... + a_nu_n = \vec{0}$ for $a_i \in F$.
   This becomes

$$a_1u_1 + a_2u_2 + ... + a_nu_n - a_2\lambda u_1 = \vec{0}$$

$$= (a_1 - a_2\lambda)u_1 + a_2u_2 + ... + a_nu_n = \vec{0}$$

   Here, $a_1 - a_2\lambda$ becomes an arbitrary scalar. Notice that this is a linear combinations of elements in S which we defined as linearly independent. This implies that $a_1 - a_2\lambda = a_2 = a_3 = ... = a_n = 0$. Therefore S' must be linearly independent if S is linearly independent.

   (b) **Prove that span(S) = span(S').** From part(a), we know that all linear combinations of elements in S' are in the form $c_1u_1 + c_2u_2 + ... + c_nu_n$ for $c_i \in F$, which is precisely also what all linear combinations of elements in S are in the form of. This means that span(S) = span(S')

3. **Problem 3**

   Show that the set

$$B = \{e_1, ..., e_n\}$$

   is linearly independent and spans F^n, i.e. is a basis for F^n

   (a) If B is linearly independent, then $a_1e_1 + a_2e_2 + ... + a_ne_n = \vec{0}$ for $a_i \in F$ if and only if $a_1 = a_2 = ... = a_n = 0$.
   WLOG consider $e_1 \in B$, which is the only element in B that contains a 1 and doesn't contain a 0 in the first coordinate. This means that the first coordinate of any linear combination of elements in B is a sum of the form $a_1(1) + a_2(0) + ... + a_n(0)$. Since this is addition and scalar multiplication in the field F, this sum can only equal 0 if $a_1 = 0$. Notice that the choice of e_1 is completely arbitrary, and the same result holds respectively for all elements in B. This means $a_1 = a_2 = ... = a_n = 0 \Longrightarrow B$ must be linearly independent.
   (b) To show that B spans F^n, recall that by definition, F^n is the set of all n-element ordered sets with entries in field F. Recall from (a) that the jth element in some $S \in F^n$ is equal to the sum $a_1 \cdot 0 + a_2 \cdot 0 + ... + a_j \cdot 1 + ... + a_n \cdot 0 = a_j$ for $a_i \in F$.
   Therefore, observe that span$(B) = \{(a_1 \cdot 1, a_2 \cdot 1, ..., a_n \cdot 1) : a_i \in F\} = \{(a_1, a_2, ..., a_n) : a_i \in F\} = F^n$

4. **Problem 4**

   (a) For any field F, explain why F^3 is the direct sum of the subspaces

$$U_1 = \{(a_1, a_2, a_3) \in F^3 | \ a_2 = a_3 = 0\} \quad \text{and} \quad U_2 = \{(a_1, a_2, a_3) \in F^3 | \ a_1 = 0\}$$

   Notice that e_1 spans U_1 and e_2, e_3 span U_2, so U_1 and U_2 are linearly independent subspaces $\Longrightarrow S \cap T = \{\vec{0}_V\}$
   The direct sum would be

$$U_1 \oplus U_2 = \{(a_1e_1, a_2e_2, a_3e_3) | \ a_i \in F\}$$

$$= \{(a_1, a_2, a_3) | \ a_i \in F\}$$

$$= F^3$$

(b) Let V be a vector space over a field F and let U_1, U_2 be a subspace of V. Prove that $V = U_1 \oplus U_2$ if and only if every vector $v \in V$ can be written uniquely as $v = u_1 + u_2$ for $u_1 \in U_1$ and $u_2 \in U_2$.

($\Rightarrow$) **If $V = U_1 \oplus U_2$, then every vector $v \in V$ can be written uniquely as $v = u_1 + u_2$ for $u_1 \in U_1$ and $u_2 \in U_2$.**
If $V = U_1 \oplus U_2$, then $U_1 \cap U_2 = \{\vec{0}\}$ and $U_1 + U_2 = V$. Then because $U_1 \cap U_2 = \{\vec{0}\}$, $u_1 \in U_1$ cannot be written as λu_2 for any $\lambda \in F$ and $u_2 \in U_2$. Then by definition, for all $v \in V, v = u_1 + u_2$ for some $u_1 \in U_1, u_2 \in U_2$.
($\Leftarrow$) **If every vector $v \in V$ can be written uniquely as $v = u_1 + u_2$ for $u_1 \in U_1$ and $u_2 \in U_2$, then $V = U_1 \oplus U_2$.**
By our assumption that every vector $v \in V$ can be written uniquely as $v = u_1 + u_2$ for $u_1 \in U_1$ and $u_2 \in U_2$, we know that $U_1 + U_2 = V$.
Now assume for the sake of contradiction that any u_1 and u_2 in U_1 and U_2 respectively are linearly dependent. Then $u_1 = \lambda u_2$ for some $\lambda \in F$. Then $v = \lambda u_2 + u_2 = (\lambda + 1)u_2$, which contradicts the fact that v must be written uniquely as $v = u_1 + u_2$. Therefore u_1 and u_2 must be linearly independent for all $u_1 \in U_1$ and $u_2 \in U_2 \implies U_1 \cap U_2 = \{\vec{0}\}$. If $U_1 + U_2 = V$ and $U_1 \cap U_2 = \{\vec{0}\}, V = U_1 \oplus U_2$.

5. **Problem 5**

(a) Let $T : V \to W$ be a linear transformation. Let $S \subset V$ be such that $\text{span}(S) = V$. Prove that

$$\text{Im}(T) = \text{span}\{T(s) \mid s \in S\}$$

STP that both are subsets of each other
Let $y \in \text{Im}(T)$, then by definition, $y = T(v)$ for some $v \in S$. Since S spans V,

$$v = \sum_{i=0}^{n} a_i v_i$$

for $a_1, ..., a_n \in F$ and $v_1, ..., v_n \in S$
Then

$$y = T(v) = T(\sum_{i=0}^{n} a_i v_i)$$

$$= \sum_{i=0}^{n} a_i T(v_i) \quad \text{(By linearity)}$$

$$\implies y \in \text{span}\{T(s) \mid s \in S\}$$

$$\implies \text{Im}(T) \subset \text{span}\{T(s) \mid s \in S\}$$

Now let $y \in \text{span}\{T(s) \mid s \in S\}$
Then

$$= \sum_{i=0}^{n} a_i T(v_i)$$

for $a_1, ..., a_n \in F$ and $v_1, ..., v_n \in S$

$$= T(\sum_{i=0}^{n} a_i v_i) \quad \text{(By linearity)}$$

$$y \in \text{Im}(T)$$

$$\implies \text{span}\{T(s) \mid s \in S\} \subset \text{Im}(T)$$

Therefore, $\text{span}\{T(s) \mid s \in S\} = \text{Im}(T)$

(b) Note that $T(s) \in W$ for all $s \in S$, so $\mathrm{Im}(T) \subset W$. Notice from part (a) that $\mathrm{Im}(T) = \mathrm{span}\{T(s) \mid s \in S\}$ is closed under addition and scalar multiplication and contains the zero vector (because T is a linear transformation). Therefore $\mathrm{Im}(T)$ is a subspace of W.

6. **Problem 6**

(a) Proof, by induction on n, that

$$\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

First check the base case: $n = 1$

$$\sum_{j=1}^{1} j^3 = 1^3 = 1 = \frac{1^2(1+1)^2}{4}$$

Inductive Step: Show that $P_n \implies P_{n+1}$

Assume that

$$\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

Then

$$\begin{aligned}\sum_{j=1}^{n+1} j^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\ &= \frac{(n+1)^2(n^2 + 4(n+1))}{4} \\ &= \frac{(n+1)^2(n^2 + 4n + 4)}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4} = P_{n+1}\end{aligned}$$

(b) Conclude that $\sum_{j=1}^{n} j^3 = (\sum_{j=1}^{n} j)^2$

Recall that

$$1 + 2 + ... + n = \frac{n(n+1)}{2} = \sum_{j=1}^{n} j$$

Then

$$(\sum_{j=1}^{n} j)^2 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$$

From part (a), we know that

$$\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

Therefore,

$$\sum_{j=1}^{n} j^3 = (\sum_{j=1}^{n} j)^2$$

7. **Problem 7**

Let V be an n-dimensional vector space over a field F.

(a) Prove that every spanning set $G \subset V$ contains at least n elements.

Let $L \subset V$ be linearly independent. By the replacement theorem, $\dim G \geq \dim L$. Assume for the sake of contradiction that G is a spanning set that contains less than n elements. Then $\dim G \leq n - 1$. Now let L be a basis of $V \implies L$ contains n linearly independent vectors that span $V \implies \dim L = n$. However, this means that $\dim G \leq \dim L$, which is a contradiction. Therefore G cannot contain less than n elements.

(b) If S spans V and has exactly n elements, then S is actually a basis for V.

By definition, a basis of V contains n linearly independent vectors that span V. We need to show that S is linearly independent and is therefore a basis of V, since we know that S has exactly n elements and spans V. Suppose for the sake of contradiction that S spans V and has n elements but is not a basis for V. Then S must be linearly dependent. This means that there exists a nonempty set of vectors H such that $S \backslash H$ is linearly independent and still spans $V \implies S \backslash H$ is a basis for V. Since H is nonempty, $\dim H > 0$. Then $\dim S \backslash H < \dim S = \dim V = n$. This means that $S \backslash H$ has less than n elements but is a basis for V, which contradicts the fact that a basis of V by definition must contain exactly n linearly independent elements that span V. Therefore S must be a basis for V.

(c) Show that every spanning subset $G \subset V$ contains a basis.

From part (a), we know that G must contain at least n elements. We also know from part(b) that if G contains exactly n elements, G is a basis for V.
Now consider when G contains at least $n+1$ elements. Since a basis B of an n-dimensional vector space contains exactly n linearly independent vectors, there exists the nonempty set $H \in V$ such that $H \cup B = G$, as G spans V. Then by definition, if $x \in B$, then $x \in H \cup B \implies x \in G \implies B \subset G$. Therefore G contains a basis of V.

8. **Problem 8**

Prove that, if a vector space V over a field F contains an infinite linearly independent subset $S \subset V$, then every spanning set for V is infinite.

Let G be any spanning set for V. Then, by the replacement theorem, $\dim G \geq \dim S$. Therefore, if S contains infinite elements, then G must also contain infinite elements.

9. **Problem 9**

Suppose that V is a finite-dimensional vector space and $U \subset V$ is a subspace.

(a) Prove that $\dim(U)$ is finite and $\dim(U) \leq \dim(V)$
If V is finite and U is a subset of V, then U must also be a finite basis. If $U \subset V$, then for all $x \in U, x \in V$. Suppose for the sake of contradiction that $\dim(U) > \dim(V)$. Then there exists some $v \in U$ such that $v \notin V$, which contradicts the fact that $U \subset V$. Therefore $\dim(U) \leq \dim(V)$.

(b) Prove that if $U \subset V$ is a subspace and $\dim U = \dim V$, then $U = V$
It suffices to prove that $V \subset U$. Since $\dim U = \dim V$, U and V have the same number of elements in their bases. And since $U \subset V$, for all $u \in U, u \in V$. Therefore, every vector in U must also be in V. But since U and V have the same dimension, any vector in U can be represented by a set of basis vectors in V such that the number of basis vectors in V is the same as the number of basis vectors in U. Therefore for all $v \in V, v \in U$ which implies that $V \subset U$. Therefore $U = V$.

10. **Problem 10**

1.6, 31

Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$.

(a) Prove that $\dim(W_1 \cap W_2) \leq n$.

Suppose $x \in W_1 \cap W_2$, then $x \in W_1$ and $x \in W_2$. Notice that $\dim(W_1 \cap W_2)$ is maximized when $W_2 \subset W_1$. Therefore, consider when $W_2 \subset W_1$. Then, $W_1 \cap W_2 = W_2 \implies \dim(W_1 \cap W_2) = \dim W_2 = n$.
$\implies \dim(W_1 \cap W_2) \leq n$

(b) Prove that $\dim(W_1 + W_2) \leq m + n$.

Recall that $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$. Notice that this time, $\dim(W_1 + W_2)$ is minimized when $W_2 \subset W_1$ and maximized when $W_2 \cap W_1 = \{\vec{0}\}$. Therefore, consider when $W_2 \cap W_1 = \{\vec{0}\}$. This means that every element in W_2 is linearly independent from elements in W_1. Then $\dim(W_2+W_1) = \dim W_2 + \dim W_1 = m+n$. $\implies \dim(W_1+W_2) \leq m+n$.

11. **Problem 11**

1.6, 33(a)

Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, show that $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is a basis for V.

Recall that if $V = W_1 \oplus W_2$, then $W_1 + W_2 = V$, $W_2 \cap W_1 = \{\vec{0}\}$, and V is the set:

$$\{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$$

First show that $\beta_1 \cap \beta_2 = \emptyset$. For the sake of contradiction, suppose that $\beta_1 \cap \beta_2$ is a nonempty set. Then $\exists\ v \in \beta_1 \cap \beta_2$. $\implies \exists\ w \in V$ that can be written both as a linear combination of vectors in β_1 and a linear combination of vectors in β_2 (set every coefficient to 0 except for the coefficient of v). Because $\beta_1 \subset W_1$ and $\beta_2 \subset W_2$, this further implies that $w \in W_1$ and $w \in W_2 \implies w \in W_1 \cap W_2$. Note that by definition, v must be nonzero because it is part of a basis, which means w must therefore be a nonzero vector. However, we defined that $W_2 \cap W_1 = \{\vec{0}\}$. This is a contradiction! Therefore, $\beta_1 \cap \beta_2 = \emptyset$ by contradiction.

To show that $\beta_1 \cup \beta_2$ is a basis for V, we must show that $\beta_1 \cup \beta_2$ is linearly independent and that $\beta_1 \cup \beta_2$ spans V.

Let $\beta_1 = \{u_1, u_2, ...u_m\}$ and $\beta_2 = \{v_1, v_2, ...v_n\}$

Then suppose

$$0 = a_1u_1 + a_2u_2 + ... + a_mu_m \quad \text{for} \quad a_1, ...a_m \in F$$

Because β_1 is linearly independent, (*) $a_1 = a_2 = ... = a_m = 0$.

Similarly, suppose

$$0 = b_1v_1 + b_2v_2 + ... + b_nv_n \quad \text{for} \quad b_1, ...b_n \in F$$

Then because β_2 is linearly independent, (**) $b_1 = b_2 = ... = b_n = 0$.

Now consider the set $\beta_1 \cup \beta_2 = \{u_1, ...u_m, v_1, ..., v_n\}$

Then suppose

$$0 = a_1u_1 + a_2u_2 + ... + a_mu_m + b_1v_1 + ... + b_nv_n \quad \text{for} \quad a_i, b_i \in F$$

Then this implies that

$$c_1u_1 + c_2u_2 + ... + c_mu_m = b_1v_1 + ... + b_nv_n \quad \text{for} \quad c_i \in F$$

This means that there exists some vector $z \in V$ such that $z \in W_1$ and $z \in W_2 \implies z \in W_1 \cap W_2$. However, we defined $W_1 \cap W_2 = \{\vec{0}\}$, which means z must be the zero vector. Therefore,

$$0 = c_1u_1 + c_2u_2 + ... + c_mu_m = b_1v_1 + ... + b_nv_n$$

By (*) and (**),

$$c_1 = c_2 = ... = c_m = b_1 = ... = b_n = 0$$

$\implies \beta_1 \cup \beta_2$ is linearly independent.

To show that $\beta_1 \cup \beta_2$ spans, first show that for any $w_1 \in W_1$,

$$w_1 = a_1u_1 + a_2u_2 + ... + a_mu_m \quad \text{for} \quad a_1, ...a_m \in F \quad \text{and} \quad u_1, ..., u_m \in \beta_1$$

Similarly for any $w_2 \in W_2$,

$$w_2 = b_1v_1 + b_2v_2 + ... + b_nv_n \quad \text{for} \quad b_1, ...b_n \in F \quad \text{and} \quad v_1, ..., v_n \in \beta_2$$

Then any vector $z \in W_1 + W_2 = V$ can be written as

$$z = a_1u_1 + a_2u_2 + ... + a_mu_m + b_1v_1 + b_2v_2 + ... + b_nv_n$$

Therefore, the set $\{u_1, ...u_m, v_1, ..., v_n\} = \beta_1 \cup \beta_2$ spans $W_1 + W_2$. Combined with the fact that $\beta_1 \cup \beta_2$ is linearly independent, $\beta_1 \cup \beta_2$ is a basis for V.