Math 115AH Homework 3

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1. **Prove that the linear transformation $T : M_{n\times m}(F) \to M_{m\times n}(F)$ given by $T(A) = A^t$ is linear.**

   Recall that A^t is the matrix given by the formula $(A^t)_{ij} = A_{ji}$.

   Let $w, v \in M_{n\times m}(F)$ and $\lambda \in F$.

   To show that T is linear, we must show that $T(w + \lambda v) = T(w) + \lambda T(v)$.

   Suppose

$$w = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1m} \\ a_{21} & a_{22} & ... & a_{2m} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nm} \end{bmatrix} \text{ and } \quad \lambda v = \begin{bmatrix} \lambda b_{11} & \lambda b_{12} & ... & \lambda b_{1m} \\ \lambda b_{21} & \lambda b_{22} & ... & \lambda b_{2m} \\ ... & ... & ... & ... \\ \lambda b_{n1} & \lambda b_{n2} & ... & \lambda b_{nm} \end{bmatrix}$$

   Then

$$w + \lambda v = \begin{bmatrix} a_{11} + \lambda b_{11} & a_{12} + \lambda b_{12} & ... & a_{1m} + \lambda b_{1m} \\ a_{21} + \lambda b_{21} & a_{22} + \lambda b_{22} & ... & a_{2m} + \lambda b_{2m} \\ ... & ... & ... & ... \\ a_{n1} + \lambda b_{n1} & a_{n2} + \lambda b_{n2} & ... & a_{nm} + \lambda b_{nm} \end{bmatrix}$$

$$\implies T(w + \lambda v) = \begin{bmatrix} a_{11} + \lambda b_{11} & a_{21} + \lambda b_{21} & ... & a_{n1} + \lambda b_{n1} \\ a_{12} + \lambda b_{12} & a_{22} + \lambda b_{22} & ... & a_{n2} + \lambda b_{n2} \\ ... & ... & ... & ... \\ a_{1m} + \lambda b_{1m} & a_{2m} + \lambda b_{2m} & ... & a_{nm} + \lambda b_{nm} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & ... & a_{n1} \\ a_{12} & a_{22} & ... & a_{n2} \\ ... & ... & ... & ... \\ a_{1m} & a_{n2} & ... & a_{nm} \end{bmatrix} + \begin{bmatrix} \lambda b_{11} & \lambda b_{21} & ... & \lambda b_{n1} \\ \lambda b_{12} & \lambda b_{22} & ... & \lambda b_{n2} \\ ... & ... & ... & ... \\ \lambda b_{1m} & \lambda b_{2m} & ... & \lambda b_{nm} \end{bmatrix} = T(w) + \lambda T(v)$$

2. **Counting elements in vector spaces over finite fields.**

   (a) Let p be a prime number. How many elements does $\mathbb{Z}/p\mathbb{Z}$ have? How about $(\mathbb{Z}/p\mathbb{Z})^2$ and $(\mathbb{Z}/p\mathbb{Z})^3$?

   $\mathbb{Z}/p\mathbb{Z}$ contains p elements, $(\mathbb{Z}/p\mathbb{Z})^2$ contains p^2 elements, and $(\mathbb{Z}/p\mathbb{Z})^3$ contains p^3 elements.

   (b) Explain why, for every $i \geq 1, (\mathbb{Z}/p\mathbb{Z})^{i+1} = (\mathbb{Z}/p\mathbb{Z})^i \times (\mathbb{Z}/p\mathbb{Z})$

   First, observe that i must be at least one, because otherwise

$$(\mathbb{Z}/p\mathbb{Z})^0 = \emptyset \implies (\mathbb{Z}/p\mathbb{Z})^0 \times (\mathbb{Z}/p\mathbb{Z})^1 = \{(a, b)\} \text{ for } a \in \emptyset \text{ and } b \in \mathbb{Z}/p\mathbb{Z}$$

However, there exists no $a \in \emptyset$ so $(\mathbb{Z}/p\mathbb{Z})^0 \times (\mathbb{Z}/p\mathbb{Z})^1 = \emptyset \neq (\mathbb{Z}/p\mathbb{Z})^1$

$$(\mathbb{Z}/p\mathbb{Z})^{i+1} = \{(a_1, a_2, ..., a_{i+1})\} \text{ for each } a_j \in \mathbb{Z}/p\mathbb{Z}$$

Similarly,

$$(\mathbb{Z}/p\mathbb{Z})^{i} = \{(b_1, b_2, ..., b_{i})\} \text{ for each } b_j \in \mathbb{Z}/p\mathbb{Z}$$

$$(\mathbb{Z}/p\mathbb{Z}) = \{c\} \text{ for } c \in \mathbb{Z}/p\mathbb{Z}$$

The cartesian product

$$(\mathbb{Z}/p\mathbb{Z})^{i} \times (\mathbb{Z}/p\mathbb{Z})$$

then becomes

$$\{(b_1, b_2, ..., b_i, c)\} \text{ for } b_j, c \in \mathbb{Z}/p\mathbb{Z}$$

Notice this contains the same number of elements as $(\mathbb{Z}/p\mathbb{Z})^{i+1}$, which is precisely $i+1$ elements. Each element is also from the same field, which means any $i+1$-tuple in $(\mathbb{Z}/p\mathbb{Z})^{i+1}$ can be represented by a tuple in $(\mathbb{Z}/p\mathbb{Z})^{i} \times (\mathbb{Z}/p\mathbb{Z})$. This shows that $(\mathbb{Z}/p\mathbb{Z})^{i+1} \subset (\mathbb{Z}/p\mathbb{Z})^{i} \times (\mathbb{Z}/p\mathbb{Z})$. The same can be said vice versa. If $v \in (\mathbb{Z}/p\mathbb{Z})^{i} \times (\mathbb{Z}/p\mathbb{Z}), v \in (\mathbb{Z}/p\mathbb{Z})^{i+1} \implies (\mathbb{Z}/p\mathbb{Z})^{i} \times (\mathbb{Z}/p\mathbb{Z}) \subset (\mathbb{Z}/p\mathbb{Z})^{i+1}$. Therefore $(\mathbb{Z}/p\mathbb{Z})^{i} \times (\mathbb{Z}/p\mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})^{i+1}$.

(c) How many elements does the vector space $(\mathbb{Z}/p\mathbb{Z})^n$ have? Prove using induction

Consider the base case $n = 0$:

$(\mathbb{Z}/p\mathbb{Z})^0$ is the empty set, which contains $n = 0$ elements.

This is likely a trivial case, so consider $n = 1:$

$(\mathbb{Z}/p\mathbb{Z})^1 = \mathbb{Z}/p\mathbb{Z} = \{a\}, a \in \mathbb{Z}/p\mathbb{Z}$, which contains p elements.

For the sake of showing an example of a tuple, consider $n = 2:$

$(\mathbb{Z}/p\mathbb{Z})^2 = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} = \{(a_1, a_2)\}, a_i \in \mathbb{Z}/p\mathbb{Z}$, which from part (a), we know contains p^2 elements. Notice that this is equal to p^n.

Now for the inductive step:

Assume that $(\mathbb{Z}/p\mathbb{Z})^n$ has p^n elements. WTS that this implies $(\mathbb{Z}/p\mathbb{Z})^{n+1}$ has p^{n+1} elements.

From part (b), we know that $(\mathbb{Z}/p\mathbb{Z})^{n+1} = (\mathbb{Z}/p\mathbb{Z})^n \times (\mathbb{Z}/p\mathbb{Z})$ and that $\mathbb{Z}/p\mathbb{Z}$ contains p elements. By our assumption that $(\mathbb{Z}/p\mathbb{Z})^n$ contains p^n elements, $(\mathbb{Z}/p\mathbb{Z})^{n+1} = (\mathbb{Z}/p\mathbb{Z})^n \times (\mathbb{Z}/p\mathbb{Z})$ contains $p^n * p = p^{n+1}$ elements.

Therefore $(\mathbb{Z}/p\mathbb{Z})^n$ contains p^n elements by induction.

3. **Textbook 2.1, 18**

Give an example of linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $\text{Ker}(T) = \text{Im}(T)$.

By rank-nullity $\text{Ker}(T) = \text{Im}(T) = 1$.

Suppose we define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that

$T(e_1) = \vec{0}$ and $T(e_2) = e_1$

Then we have that $\text{span}(e_1) = \text{span}(1,0) = \text{Ker}(T)$. Notice that $\vec{0} = (0,0) \in \text{span}(e_1)$.

Since $T(e_2) = e_1 \neq \vec{0}$, $\text{span}(e_2) \notin \text{Ker}(T)$. Furthermore, $T(e_2)$ spans $e_1 \implies \text{span}(e_1) = \text{Im}(T)$. For the case of the zero vector, since T is linear, $T(\vec{0}) = \vec{0}$, so $\vec{0} \in \text{Im}(T)$.

Let $v \in \text{Im}(T)$. Then $v = (r_1e_1 + r_2e_2), r_1, r_2 \in \mathbb{R}$. Then

$$T(v) = T(r_1e_1 + r_2e_2)$$

$$= T(r_1e_1) + T(r_2e_2)$$

$$= r_1T(e_1) + r_2T(e_2)$$

$$= r_1\vec{0} + r_2e_1$$

$$= \vec{0} + r_2e_1$$

$$= r_2e_1 \implies v \in \text{Ker}(T)$$

$$\implies \text{Im}(T) \subset \text{Ker}(T)$$

Now let $v \in \text{Ker}(T)$ and let $r_i \in \mathbb{R}$. Then

$$v = (r_1e_1)$$

$$= r_1T(e_2)$$

$$= T(r_1e_2 \implies v \in \text{Im}(T)$$

$$\implies \text{Ker}(T) \subset \text{Im}(T)$$

Therefore, $\text{Im}(T) = \text{Ker}(T)$.

4. **Consider the set of all linear transformations from a vector space V over a field F to a vector space W over a field F. Prove that**

   (a) If that $S : V \to W$ and $T : V \to W$ are linear transformations from V to W, then the function $S + T : V \to W$ defined by $(S + T)(x) = S(x) + T(x)$ for all $x \in V$ is also linear.

   Consider $(S + T)(x + \lambda y)$

   Then:

$$(S + T)(x + \lambda y) = S(x + \lambda y) + T(x + \lambda y) \quad \text{(By definition of } S + T)$$

$$= S(x) + \lambda S(y) + T(x) + \lambda T(y) \quad \text{(By linearity of } S \text{ and } T)$$

$$= S(x) + T(x) + \lambda(S(y) + T(y)) \quad \text{(By associativity and distributivity on } W)$$

$$= (S + T)(x) + \lambda(S + T)(y) \implies S + T \text{ is linear.}$$

   (b) If T is a linear transformation from V to W and $\lambda \in F$, then the function $\lambda T : V \to W$ defined by $(\lambda T)(x) = \lambda T(x)$ for all $x \in V$ is also linear.

   Consider $(\lambda T)(x + \alpha y)$

   Then:

$$(\lambda T)(x + \alpha y) = \lambda(T)(x + \alpha y) \quad \text{(By definition of } \lambda T)$$

$$= \lambda(T(x) + \alpha T(y)) \quad \text{(By linearity of } T\text{)}$$

$$= \lambda T(x) + \alpha\lambda T(y) \quad \text{(By distributivity on } W\text{)}$$

$$= (\lambda T)(x) + \alpha(\lambda T)(y) \implies \lambda T \text{ is linear.}$$

(c) The function $\vec{0} : V \to W$ defined by $\vec{0}(x) = \vec{0}_W$ for all $x \in V$ is a linear transformation.
First notice that $\vec{0}$ sends $\vec{0}_V$ to $\vec{0}_W$.
Now suppose we have $\vec{0}(a + \lambda b)$ for $a, b \in V$ and $\lambda \in F$.

$$\vec{0}(a + \lambda b) = \vec{0}_W = \vec{0}_W + \vec{0}_W = \vec{0}_W + \lambda\vec{0}_W = \vec{0}(a) + \lambda\vec{0}(b)$$

(d) Consider the set $L(V, W)$ of linear transformations with domain V and codomain W. Prove that $L(V, W)$ is a a vector space over F.

(Commutativity) Let $S, T \in L$ such that $S : V \to W$ and $T : V \to W$. Then for all $v \in V, (S + T)(v) = S(v) +_W T(v) = T(v) +_W S(v) = (T + S)(v)$.
(Associativity) Let $R, S, T \in L$. Then for all $v \in V$, $(R+S)(v)+T(v) = R(v)+_W S(v)+_W T(v) = R(v)+_W (S+T)(v)$.
(Additive Identity) Let $Z, S \in L$, define $Z(v) = \vec{0}$ for all $v \in V$. Precisely, let Z be the function $\vec{0}$ from part (c) which we proved to be linear. Then $(Z + S)(v) = Z(v) +_W S(v) = \vec{0}_W +_W S(v) = S(v)$.
(Additive Inverse) Let $S \in L$. Define $T : V \to W$ such that for all $v \in V$, $T(v) = (-1) \cdot_W S(v)$. To show that T is linear, first observe that $T(\vec{0}_V) = (-1) \cdot_W S(\vec{0}_V) = (-1) \cdot_W \vec{0}_W = \vec{0}_W$. Additionally, consider $T(v + \lambda u)$ for $v, u \in V$.

$$T(v + \lambda u) = -S(v + \lambda u)$$

$$= -S(v) + -S(\lambda u) \quad \text{(By linearity of } S\text{)}$$

$$= -S(v) + (\lambda)(-S(u)) \quad \text{(By linearity of } S\text{)}$$

$$= T(v) + \lambda T(u)$$

Therefore T is linear so $\exists\ T \in L$. Then $(S + T)(v) = S(v) + T(v) = S(v) + -S(v) = 0$.

(e) (Multiplicative Identity) Let $T \in L$ and $v \in V$. Then

$$1 \cdot T = (1 \cdot T)(v) = 1$$

$$= 1 \cdot_W T(v) \quad \text{(By Linearity of } T\text{)}$$

$$= T(v)$$

(f) (Associativity of scalars) Let $a, b \in F$, let $v \in V$, and let $T \in L$. Then

$$a \cdot (b \cdot T)(v) = a \cdot (b \cdot T(v)) \quad \text{(By Linearity of } T\text{)}$$

$$= (a \cdot b) \cdot T(v) \quad \text{(By associativity on } W\text{)}$$

$$= ((a \cdot b) \cdot T)(v) \quad \text{(By linearity of } T\text{)}$$

(g) (Distributivity over vectors) Let $a \in F$, let $v \in V$, and let $S, T \in L$. Then

$$a \cdot (S+T)(v) = a \cdot (S(v) + T(v)) \quad \text{(By addition of linear functions)}$$

$$= a \cdot S(v) + a \cdot T(v) \quad \text{(By distributivity on } W\text{)}$$

$$= (a \cdot S)(v) + (a \cdot T)(v) \quad \text{(By linearity of } S \text{ and } T\text{)}$$

(h) (Distributivity over scalars) Let $a, b \in F$, let $v \in V$, and let $T \in L$. Then

$$((a+b) \cdot T)(v) = (a+b) \cdot T(v) \quad \text{(By linearity of } T\text{)}$$

$$= a \cdot T(v) + b \cdot T(v) \quad \text{(By distributivity on } W\text{)}$$

$$= (a \cdot T)(v) + (b \cdot T)(v) \quad \text{(By linearity of } T\text{)}$$

5. **Textbook 2.1, 2,6**

Find bases for $\text{Ker}(T)$ and $\text{Im}(T)$, compute the nullity and rank of T, and verify dimension theorem. Finally, use the appropriate theorems to determine whether T is injective or surjective.

(2)

Suppose we have $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

To solve for $\text{Ker}(T)$, first solve for $(a_1 - a_2, 2a_3) = (0, 0)$:

$a_1 - a_2 = 0 \implies a_1 = a_2$

$2a_3 = 0 \implies a_3 = 0$

Therefore a basis B_K for $\text{Ker}(T) = \{1, 1, 0\} \implies$ nullity $= 1$.

Observe that $\text{Im}(T)$ is just $\mathbb{R}^2$, so a basis B_I for $\text{Im}(T) = \{(1, 0), (0, 1)\} \implies$ rank $= 2$.

If V is the domain of T, then by the dimension theorem, $\dim V = \dim \text{Ker}(T) + \dim \text{Im}(T)$. Clearly, this is satisfied as $\dim V = 3$ and $\dim(\text{Ker}(T)) = 1, \dim(\text{Im}(T)) = 2$

Since $\text{Im}(T) = \mathbb{R}^2$, T must be surjective.

We know that T is injective if and only if $\text{Ker}(T) = \{\vec{0}\}$. Since we showed above that $K = \text{span}\{1, 1, 0\}, T$ is not injective.

(6)

Suppose we have $T : M_{n \times n}(F) \to F$ defined by $T(A) = \text{tr}(A)$.

Since all entries of M are in the field F, $\text{tr}(A)$ is the sum of n arbitrary elements in F. Therefore, every element in F can be represented as a sum of n arbitrary elements in $F \implies \text{Im}(T) = F. \implies \dim(\text{Im}(T)) = 1$. A basis B_I for $\text{Im}(T)$ can be written as $\text{span}(a)$ for $a \in F$.

For the kernel, first recognize that by rank nullity, $\dim(\text{Ker}(T)) = n^2 - 1$. Notice that each non-diagonal entry contribute a basis vector, being the matrix with that specific non-diagonal entry $a_{ij} = 1$ and all other entries being 0 for all $1 \leq i, j \leq n$ with $i \neq j$. These, in total, contribute $n^2 - n$ basis vectors.

Then we look at diagonal entries. Since the diagonal contains n entries, imagine we collapse it into one n column vector.

Then the basis vectors would be

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ ... \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ ... \\ 0 \end{bmatrix}, \quad ... \quad, \begin{bmatrix} 1 \\ 0 \\ 0 \\ ... \\ -1 \end{bmatrix}$$

Notice that there are exactly $n-1$ basis vectors here. Therefore the total number of basis vectors equals $n^2-n+(n-1) = n^2-1$, which matches the result from rank nullity. Let E_{ij} represent the $n \times n$ matrix such that the entry $e_{ij} = 1$ and all other entries equal 0, and let A_n represent the $n \times n$ matrix such that $a_{11} = 1, a_{nn} = -1$, and all other entries equal 0. Then the basis for $\text{Ker}(T)$ would look like:

$$\{E_{ij} : i \neq j, 1 \leq i, j \leq n\} \cup \bigcup_{i=2}^{n} A_i$$

T can only be injective if and only if $\text{Ker}(T) = \{\vec{0}\}$ which is clearly not the case, so T is not injective. For every element $k \in F$, there exists n other elements in F that sum to k. An example would be k and $n-1$ 0_F's. Therefore, there exists an $n \times n$ matrix with the diagonal summing to k for all $k \in F \implies T$ is surjective.

6. **Textbook 2.1, 17**

   Let V, W be finite-dimensional vector spaces and $T : V \to W$ be linear.

   (a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.

   Suppose for the sake of contradiction that T is onto. Then $\text{Im}(T) = W \implies \dim(\text{Im}(T)) = \dim(W)$. By the dimension theorem, $\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Im}(T)) = \dim(\text{Ker}(T)) + \dim(W)$. However, this is not possible as $\dim(V) < \dim(W)$ and there cannot be a negative dimension. Therefore T cannot be onto.

   (b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

   By the dimension theorem, $\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Im}(T))$. Suppose T is one-to-one.
   Recall that T is one-to-one if and only if $\text{Ker}(T) = \{\vec{0}\} \implies \dim(\text{Ker}(T)) = 0$. Then $\dim(V) = 0 + \dim(\text{Im}(T)) = \dim(\text{Im}(T))$. Because $\text{Im}(T) \subset W$, $\dim(\text{Im}(T)) \leq \dim(W) \implies \dim(V) \leq \dim(W)$. However, this is a contradiction because we defined that $\dim(V) > \dim(W)$. Therefore T cannot be one-to-one.

7. **Textbook 2.1, 21**

   (a) Prove that T and U are linear

   i. Consider $T(v + \lambda u)$ for $v, u \in V$ and $\lambda \in F$.

   More specifically suppose

   $$v = (a_1, a_2, ...) \quad \text{and} \quad u = (b_1, b_2, ...)$$

   Then by our operations defined,

   $$\lambda u = (\lambda b_1, \lambda b_2, ...)$$

   and

   $$v + \lambda u = (a_1 + \lambda b_1, a_2 + \lambda b_2, ...)$$

Then

$$T(v+\lambda u) = (a_2+\lambda b_2, a_3+\lambda b_3, ...)$$

$$= (a_2, a_3, ...) + (\lambda b_2, \lambda b_3, ...)$$

$$= (a_2, a_3, ...) + \lambda(b_2, b_3, ...)$$

$$= T(v) + \lambda T(u)$$

ii. Now consider $U(v+\lambda u)$ for $v, u \in U$ and $\lambda \in F$.

Let v and u be defined as in (i):

$$v = (a_1, a_2, ...) \quad \text{and} \quad u = (b_1, b_2, ...)$$

Then by our operations defined,

$$\lambda u = (\lambda b_1, \lambda b_2, ...)$$

and

$$v + \lambda u = (a_1 + \lambda b_1, a_2 + \lambda b_2, ...)$$

Then

$$U(v+\lambda u) = (0, a_1+\lambda b_1, a_2+\lambda b_2, ...)$$

$$= (0, a_1, a_2, ...) + (0, \lambda b_1, \lambda b_2, ...)$$

$$= (0, a_1, a_2, ...) + \lambda(0, b_1, b_2, ...)$$

$$= U(v) + \lambda U(u)$$

(b) Prove that T is onto, but not one-to-one

We need to show that $\text{Im}(T) = V$.

By definition, $\text{Im}(T) \subset V$, so we need to show $V \subset \text{Im}(T)$

Let $y \in V$, then $y = (b_1, b_2, ...)$ $\ b_i \in F$. Notice that $\exists\ x \in V$ where $x = (a_1, b_1, b_2, b_3, ...)$ $\ a_i, b_i \in F$ and $T(x) = y \implies y \in \text{Im}(T) \implies V \subset \text{Im}(T) \implies \text{Im}(T) = V \implies T$ is onto. Observe that in the definition of x, a_1 is a completely arbitrary element in F. Suppose $x = (a_1, b_1, b_2, b_3, ...)$ and $x_1 = (a_2, b_1, b_2, b_3, ...)$ such that $a_1 \neq a_2$. However, $T(x) = T(x_1) = (b_1, b_2, ...) \implies$ T is not one-to-one

(c) Prove that U is one-to-one, but not onto.

Let $x_1, x_2 \in V$ such that $x_1 = (a_1, a_2, ...)$ and $x_2, = (b_1, b_2, ...)$ $\ a_i, b_i \in F$. Then $U(x_1) = (0, a_1, a_2, ...)$ and $U(x_2) = (0, b_1, b_2, ...)$. Suppose $U(x_1) = U(x_2) \implies a_1 = b_1, a_2 = b_2, ... \implies x_1 = x_2$. Therefore, U is injective. For U to be onto, every element in V must be in $\text{Im}(U)$. However, let $y \in \text{Im}(U)$. Then $y = (0, c_1, c_2, ...), c_i \in F$. For some arbitrary element $v \in V$, $v = (d_1, d_2, ...)$ $\ d_i \in F$. But since d_i are arbitrary elements in F, $\exists v$ such that $d_1 \neq 0 \implies \exists v \notin \text{Im}(U) \implies V \not\subset \text{Im}(U) \implies U$ is not onto.

8. **Textbook 2.1, 24**

Let $T : V \to W$ be linear, $b \in W$, and $K = \{x \in V : T(x) = b\}$ be nonempty. Prove that if $s \in K$, then $K = \{s\} + \text{Ker}(T)$.

If $s \in K$, then $T(s) = b$. $\text{Ker}(T) = \{x : T(x) = 0\}$. By the definition of nonempty sets, $\{s\} + \text{Ker}(T). = \{j + k : j \in \{s\}, k \in \text{Ker}(T)\}$. Since T is linear, $T(j + k) = T(j) + T(k) = T(j) + 0 = T(j) = b$. Therefore for all $y \in \{s\} + \text{Ker}(T), T(y) = b \implies y \in \{j + k : j \in \{s\}, k \in \text{Ker}(T)\} \implies y \in K \implies \{s\} + \text{Ker}(T) \subset K$. Similarly, if $y \in K$, then $T(y) = b = b + 0 = T(z) + T(x)$ for all $x \in \text{Ker}(T)$ and $z \in \{s\}$. Then $y \in \{s\} + \text{Ker}(T) \implies K \subset \{s\} + \text{Ker}(T)$ Therefore $K = \{s\} + \text{Ker}(T)$

9. **Let V and W be vector spaces over F and let $B = \{v_1, ...v_n\}$ be a basis for V and $G = \{w_1, ...w_n\}$ be a basis for W.**

(a) Prove that the linear transformation $T : V \to W$ determined by $T(v_i) = w_i$ for $1 \le i \le n$ is one-to-one and onto

Notice that V and W are both n-dimensional. Let $a, b \in V$ such that $a = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + ...\alpha_n v_n$ and $b = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + ...\beta_n v_n$

Then

$$T(a) = T(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + ...\alpha_n v_n)$$

$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + ... + \alpha_2 T(v_n) \quad \text{(Linearity of T)}$$

$$= \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + ...\alpha_n w_n$$

Similarly

$$T(b) = T(\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + ...\beta_n v_n)$$

$$= \beta_1 T(v_1) + \beta_2 T(v_2) + ... + \beta_2 T(v_n)$$

$$= \beta_1 w_1 + \beta_2 w_2 + \beta_3 w_3 + ...\beta_n w_n$$

If $T(a) = T(b) \implies \alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_n = \beta_n \implies a = b$, so T is one-to-one.

To prove surjectivity, we must show that $\forall\ w \in W, \exists\ v \in V$ such that $T(v) = w$.

Let $w \in W$, then $w = k_1 w_1 + k_2 w_2 + ... + k_n w_n$. Since we know that $T(v_i) = w_i$, we have that

$$w = k_1 T(v_1) + k_2 T(v_2) + ... + k_n T(v_n)$$

$$= T(k_1 v_1 + k_2 v_2 + ... + k_n v_n) \quad \text{(By Linearity of T)}$$

Since $v = k_1 v_1 + k_2 v_2 + ... + k_n v_n$ is a linear combination of elements in B, we know that $\exists\ v \in V \implies F$ is surjective.

(b) Let F be a field. Using the previous item, define an explicit linear map $T : P_3(F) \to \text{Mat}_{2\times 2}(F)$ that is a bijection.

$P_3(F) = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_i \in F\}$

Let $\mathfrak{B}$ be a basis for $P_3(F)$. Then

$$\mathfrak{B} = \{1, x, x^2, x^3\} = \{v_1, v_2, v_3, v_4\} \quad \text{from part (a)}$$

Let $p \in P_3(F)$. Then $p = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ for some $a_i \in F$. Define T such that $T(p) = \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}$.

Then the basis $\mathfrak{R}$ of $T(p) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \{w_1, w_2, w_3, w_4\}$ from part (a)

We know from part (a) that this is a bijection, since $\mathfrak{R}$ and $\mathfrak{B}$ are both 4-dimensional, and $T(v_i) = w_i$ for $1 \leq i \leq n$.