Homework 1

1. Solve and then prove that your closed form solution is correct by performing an inductive proof.

$$S_n = \{1, 4, 8, 13, 19, \dots\}$$

Start with subscript 0, i.e. Base Case: $T_0 = 1$

It seems that S_n is just the triangular numbers minus two, offset by one index. The recurrence relation which follows S_n is

$$T_n = T_{n-1} + (n+2).$$

In order to find the closed form, we can expand the terms. Expansions: $n - 1 + 0 + 1 = n$

$$\begin{aligned}
T_n &= T_{n-1} + (n+2) \\
&= T_{n-2} + (n+1) + (n+2) \\
&= T_{n-3} + n + (n+1) + (n+2) \\
&\vdots \\
&= T_0 + 3 + 4 + \cdots + n + (n+1) + (n+2) \\
&= (1 + 2 + 3 + \cdots + n + (n+1) + (n+2)) - 2 \\
&= \sum_{i=1}^{n+2} i \ - 2 \\
T_n &= \frac{(n+2)(n+3)}{2} - 2
\end{aligned}$$

Now we have a candidate for a closed form solution. We will prove this candidate through mathematical induction.

Proof through Induction. Proof that $T_n = \frac{(n+2)(n+3)}{2} - 2$ for all $n \in \mathbb{Z}^{\geq 0}$.
Base Case: $n = 0$

$$T_0 = \frac{(0+2)(0+3)}{2} - 2 = \frac{6}{2} - 2 = 3 - 2 = 1 \checkmark$$

Inductive Hypothesis: Assume that

$$T_k = \frac{(k+2)(k+3)}{2} - 2 \text{ for some } k \in \mathbb{Z}^{\geq 0}.$$

Inductive step: $n = k + 1$

$$\begin{aligned}
T_{k+1} &= T_k + ((k+1) + 2) \\
&= T_k + (k+3) \\
&= \frac{(k+2)(k+3)}{2} - 2 + (k+3) && \text{via Inductive Hypothesis} \\
&= \frac{(k+2)(k+3)}{2} + \frac{2(k+3)}{2} - 2 \\
&= \frac{(k+2)(k+3) + 2(k+3)}{2} - 2 \\
&= \frac{(k+3)(k+2+2)}{2} - 2 \\
T_{k+1} &= \frac{((k+1)+2)((k+1)+3)}{2} - 2
\end{aligned}$$

The inductive step holds. Therefore, through mathematical induction, $T_n = \frac{(n+2)(n+3)}{2} - 2$ for all $n \in \mathbb{Z}^{\geq 0}$. $\square$

2. Find a closed form solution. Extra credit (4 points): Perform and inductive proof.

$$S_n = \{1, 8, 36, 148, 596, \ldots\}.$$

Start with subscript 1, i.e. Base Case: $T_1 = 1$

The derived recurrence relation is

$$T_n = 4T_{n-1} + 4.$$

In order to find the closed form, we can expand the terms. Expansions: $n - 1 - 1 + 1 = n - 1$

$$\begin{aligned} T_n &= 4T_{n-1} + 4 & &= 4T_{n-1} + 4 \\ &= 4(4T_{n-2} + 4) + 4 & &= 4^2 T_{n-2} + 4^2 + 4 \\ &= 4(4(4T_{n-3} + 4) + 4) + 4 & &= 4^3 T_{n-3} + 4^3 + 4^2 + 4 \\ &\vdots \\ &= 4^{n-1} + \sum_{i=1}^{n-1} 4^i \\ &= 4^{n-1} + \frac{4^n - 4}{3} \end{aligned}$$

Now we have a candidate for a closed form solution. We will prove this candidate through mathematical induction.

Proof through Induction. Proof that $T_n = 4^{n-1} + \frac{4^n - 4}{3}$ for all $n \in \mathbb{Z}^+$.
Base Case: $n = 1$

$$T_1 = 4^{1-1} + \frac{4^1 - 4}{3} = 4^0 + \frac{0}{3} = 1 \checkmark$$

Inductive Hypothesis: Assume that

$$T_k = 4^{k-1} + \frac{4^k - 4}{3} \text{ for some } k \in \mathbb{Z}^+.$$

Inductive step: $n = k + 1$

$$\begin{aligned} T_{k+1} &= 4T_k + 4 \\ &= 4(4^{k-1} + \frac{4^k - 4}{3}) + 4 & & \text{via Induction Hypothesis} \\ &= 4^k + \frac{4^{k+1} - 16}{3} + \frac{12}{3} \\ &= 4^k + \frac{4^{k+1} - 16 + 12}{3} \\ T_{k+1} &= 4^k + \frac{4^{k+1} - 4}{3} \end{aligned}$$

The inductive step holds. Therefore, through mathematical induction, $T_n = 4^{n-1} + \frac{4^n - 4}{3}$ for all $n \in \mathbb{Z}^+$. $\square$