

Homework 6

1.2

Determine the values for which the system has no solutions, exactly one solution, or infinitely many solutions

$$\begin{array}{rclcl} x & + & 2y & - & 3z & = & 4 \\ 25. \quad 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2 - 14)z & = & a + 2 \end{array}$$

work.

$$\begin{aligned} & \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2 - 14 & a + 2 \end{array} \right) \xrightarrow[(-4, -8, 12, -16)]{R_3 - 4R_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 0 & -7 & a^2 - 2 & a - 14 \end{array} \right) \xrightarrow[(-3, -6, 9, -12)]{R_2 - 3R_1} \\ & \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2 - 2 & a - 14 \end{array} \right) \xrightarrow[(0, 7, -14, 10)]{R_3 - R_2, -\frac{1}{7}R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2 - 16 & a - 4 \end{array} \right) \end{aligned}$$

R_3 represents the equation $(a^2 - 16)z = a - 4$. If $a = 4$, there are infinitely many solutions, since R_3 leads to a full row of 0's. If $a = -4$, then there is no solution, since R_3 leads to $0 = -8$. If $a \neq \pm 4$, then there is exactly one solution to the system of equations. \square

$$\begin{array}{rclcl} x & + & 2y & = & 1 \\ 27. \quad 2x & + & (a^2 - 5)y & = & a - 1 \end{array}$$

work.

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & a^2 - 5 & a - 1 \end{array} \right) \xrightarrow[(-2, -4, -2)]{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & a^2 - 9 & a - 3 \end{array} \right)$$

R_2 represents the equation $(a^2 - 9)y = a - 3$. If $a = 3$, there are infinitely many solutions, since R_2 leads to a full row of 0's. If $a = -3$, then there is no solution, since R_2 leads to $0 = -6$. If $a \neq \pm 3$, then there is exactly one solution to the system of equations. \square

$$32. \text{ Reduce } \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix} \text{ to rref without introducing fractions at any intermediate stage.}$$

work.

$$\begin{aligned} & \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[(-2, -1, -3)]{R_3 - R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[(-2, -6, -4)]{R_1 - 2R_3} \begin{bmatrix} 0 & -5 & -1 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[(0, 2, 29)]{R_1 - R_2} \\ & \begin{bmatrix} 0 & -3 & 28 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[R_3 + R_1]{R_2 - R_1} \begin{bmatrix} 0 & -3 & 28 \\ 0 & 1 & -57 \\ 1 & 0 & 30 \end{bmatrix} \xrightarrow[(0, 3, -171)]{R_1 + 3R_2} \begin{bmatrix} 0 & 0 & -143 \\ 0 & 1 & -57 \\ 1 & 0 & 30 \end{bmatrix} \xrightarrow[R_2 + 57R_1, R_3 - 30R_1]{-\frac{1}{143}R_1, R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

\square

1.3

$$5h. \text{ Calculate } (C^T B)A^T, \text{ where } A = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix}, C = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}.$$

work.

$$\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix} \xrightarrow{C^T} \begin{pmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{pmatrix} \xrightarrow{C^T B} \begin{pmatrix} 1 \cdot 4 + 3 \cdot 0 & 1 \cdot -1 + 3 \cdot 2 \\ 4 \cdot 4 + 1 \cdot 0 & 4 \cdot -1 + 1 \cdot 2 \\ 2 \cdot 4 + 5 \cdot 0 & 2 \cdot -1 + 5 \cdot 2 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 16 & -2 \\ 8 & 8 \end{pmatrix} \xrightarrow{C^T B A^T} \\ \begin{pmatrix} 4 & 5 \\ 16 & -2 \\ 8 & 8 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{C^T B A^T} \begin{pmatrix} 4 \cdot 3 + 5 \cdot 0 & 4 \cdot -1 + 5 \cdot 2 & 4 \cdot 1 + 5 \cdot 1 \\ 16 \cdot 3 - 2 \cdot 0 & 16 \cdot -1 - 2 \cdot 2 & 16 \cdot 1 - 2 \cdot 1 \\ 8 \cdot 3 + 8 \cdot 0 & 8 \cdot -1 + 8 \cdot 2 & 8 \cdot 1 + 8 \cdot 1 \end{pmatrix} = \begin{pmatrix} 12 & 6 & 9 \\ 48 & -20 & 14 \\ 24 & 8 & 16 \end{pmatrix}$$

□

10. $A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$, $AB = \begin{bmatrix} 67 & 41 & 41 \\ 64 & 21 & 59 \\ 63 & 67 & 57 \end{bmatrix}$, and $BA = \begin{bmatrix} 6 & -6 & 70 \\ 6 & 17 & 31 \\ 63 & 41 & 122 \end{bmatrix}$.

a. express each column vector of AB as a linear combination of the column vectors of A .

work. Since each column vector of AB is computed using a row of A and a column of B , the linear combination will simply be the corresponding column of B .

$$\begin{aligned} 1. & 6 \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} + 7 \begin{pmatrix} 7 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 67 \\ 64 \\ 63 \end{pmatrix}. \\ 2. & -2 \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} + 7 \begin{pmatrix} 7 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 41 \\ 21 \\ 67 \end{pmatrix}. \\ 3. & 4 \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} + 5 \begin{pmatrix} 7 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 41 \\ 59 \\ 57 \end{pmatrix}. \end{aligned}$$

□

b. express each column vector of BA as a linear combination of the column vectors of B .

work. Since each column vector of BA is computed using a row of B and a column of A , the linear combination will simply be the corresponding column of A .

$$\begin{aligned} 1. & 3 \begin{pmatrix} 6 \\ 0 \\ 7 \end{pmatrix} + 6 \begin{pmatrix} -2 \\ 1 \\ 7 \end{pmatrix} + 0 \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 63 \end{pmatrix}. \\ 2. & -2 \begin{pmatrix} 6 \\ 0 \\ 7 \end{pmatrix} + 5 \begin{pmatrix} -2 \\ 1 \\ 7 \end{pmatrix} + 4 \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 17 \\ 41 \end{pmatrix}. \\ 3. & 7 \begin{pmatrix} 6 \\ 0 \\ 7 \end{pmatrix} + 4 \begin{pmatrix} -2 \\ 1 \\ 7 \end{pmatrix} + 9 \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 70 \\ 31 \\ 122 \end{pmatrix}. \end{aligned}$$

□

15. Find all values of k , if any, that satisfy $\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix}$.

work.

$$\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k+1 & k+2 & -1 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = [k(k+1) + k+2-1] = [(k+1)^2] = [0]$$

$$\begin{aligned} k+1 &= 0 \\ k &= -1 \end{aligned}$$

□

22.

- a.** Show that if A has a row of zeros and B is any matrix for which AB is defined, then AB also has a row of zeros.
- b.** Find a similar result involving a column of zeros.

24. Find the 4×4 matrix $A = [a_{ij}]$ whose entries satisfy the stated condition.

- a.** $a_{ij} = i + j$
- b.** $a_{ij} = i^{j-1}$
- c.** $a_{ij} = \begin{cases} 1 & \text{if } |i - j| > 1 \\ -1 & \text{if } |i - j| \leq 1 \end{cases}$

27. How many 3×3 matrices A can you find such that

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ x - y \\ 0 \end{bmatrix}$$

for all choices of x, y , and z ?

1.4

17. Use the given information to find A : $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$.

19e. Given $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, compute $p(A)$, where $p(x) = 2x^2 - x + 1$.

27. Consider the matrix

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

where $a_{11} \cdot a_{22} \cdots a_{nn} \neq 0$. Show that A is invertible and find its inverse.