Section 5 Subgroups, p55 #21-25, 27-28, 43, 47, 55

21 Write at least 5 elements of each of the following cyclic groups.

a. $25\mathbb{Z}$ under addition

$\{25, 50, 75, 100, 125, \ldots\}$

b. $\{(\frac{1}{2})^n : n \in \mathbb{Z}\}$ under multiplication

$\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\}$

c. $\{\pi^n : n \in \mathbb{Z}\}$ under multiplication

$\{1, \pi, \pi^2, \pi^3, \pi^4, \ldots\}$

In Exercises 22 through 25, describe all the elements in the cyclic subgroup of $GL(2, \mathbb{R})$ generated by the given 2×2 matrix.

22. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\ldots, \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \ldots$

24. $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$\ldots, \begin{bmatrix} 3^{-2} & 0 \\ 0 & 2^{-2} \end{bmatrix}, \begin{bmatrix} 3^{-1} & 0 \\ 0 & 2^{-1} \end{bmatrix}, \begin{bmatrix} 3^0 & 0 \\ 0 & 2^0 \end{bmatrix}, \begin{bmatrix} 3^1 & 0 \\ 0 & 2^1 \end{bmatrix}, \begin{bmatrix} 3^2 & 0 \\ 0 & 2^2 \end{bmatrix}, \ldots$

25. $\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & -2^{2k+1} \\ -2^{2k+1} & 0 \end{bmatrix}, \begin{bmatrix} 2^{2k} & 0 \\ 0 & 2^{2k} \end{bmatrix}, k \in \mathbb{Z}$

In Exercises 27 through 35, find the cyclic subgroup of the given group generated by the indicated element.

27. The subgroup of $\mathbb{Z}_4$ generated by 3

$\langle 3 \rangle = \{3, 3^2 = 2, 3^3 = 1, 3^4 = 0\}$

28. The subgroup of V generated by c

V	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

The subgroup:

$$\langle c \rangle = \begin{array}{c|cc} & e & c \\ \hline e & e & c \\ c & c & e \end{array}$$

Theory

43. Show that if H and K are subgroups of an Abelian group G, then

$$\{hk : h \in H \text{ and } k \in K\}$$

is a subgroup of G.

To show this we must show closure and closure of inverses.

Proof. Let $L = \{hk : h \in H \text{ and } k \in K\}$

(a) Closure: Consider $h_1k_1, h_2k_2 \in L$.

$$h_1k_1h_2k_2 = h_1h_2k_1k_2 = (h_1h_2)(k_1k_2) \in L,$$

since we know that G is Abelian, and H and K are closed. Thus this takes the form of an element in L, and L is closed.

(b) Closure of Inverses: Consider $hk \in L$.

$$(hk)^{-1} = k^{-1}h^{-1} = h^{-1}k^{-1} \in L,$$

since we know G is Abelian, and H and K are closed for inverses. Thus this takes the form of an element in L, and L is closed for inverses.

Since L is closed and closed for inverses, it is a subgroup of G. □

47. Prove that if G is an Abelian group, written multiplicatively, with identity element e, then all elements x of G satisfying the equation $x^2 = e$ form a subgroup H of G.

To show this we must show closure and closure of inverses.

Proof. Assume the information above, with such a subset called X. Consider $x, y \in X$.

$$(xy)^2 = xyxy = x^2y^2 = ee = e,$$

since G is Abelian. Thus the defining property of this set is satisfied, and X is closed under the operation of G. Now consider $x^{-1} \in G$.

$$(x^{-1})^2 = x^{-2} = (x^2)^{-1} = e^{-1} = e.$$

Thus x^{-1} satisfies the defining property of X, and X is closed under inverses. Since X is closed under the operation of G and closed for inverses, X is a subgroup of G. □

55. Prove that every cylic group is Abelian.

A cyclic group takes the form $\langle a \rangle = \{a^n : n \in \mathbb{Z}\}$.

Proof. We must prove that $ab = ba$ for any $a, b \in \langle a \rangle$. Consider a^n and $a^m \in \langle a \rangle$ for $n, m \in \mathbb{Z}$.

$$a^n a^m = a^{n+m} = a^{m+n} = a^m a^n$$

Thus, $a^n a^m = a^m a^n$, and any cyclic group is also Abelian. $\square$