Discrete Math for Computer Science

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Contents

1 Integer Properties

1.1 The Division Algorithm

In **integer division**, the input and output values must always be integers. For example, when 9 is divided by 4, the answer is 2 with a remainder of 1, instead of 2.25.

Divides

Let x and y be two integers. Then x *divides* y, $x \mid y$, if and only if $x \neq 0$ and there is an integer k such that $y = kx$. If there is no such integer or if $x = 0$, then x does not divide y, $x \nmid y$. If $x \mid y$, then y is said to be a *multiple* of x, and x is a *factor* or *divisor* of y.

Theorem: Divisibility and linear combinations

Let $x, y,$ and z be integers. If $x \mid y$ and $x \mid z$, then $x \mid (sy + tz)$ for any integers s and t.

Proof. Since $x \mid y$, then $y = kx$ for some integer k. Similarly, since $x \mid z$, then $z = jx$ for some integer j. A linear combination of y and z can be expressed as:

$$sy + tz = s(kx) + t(jx) = (sk + tj)x.$$

For some integers s and t. Since $sy + tz$ is an integer multiple of x, then $x \mid (sy + tz)$. □

Quotients and remainders

If $x \nmid y$, then there is s non-zero remainder when x is divided into y. The **Division Algorithm**, states that the result of the division and the remainder are unique.

Theorem: The Division Algorithm

Let n be an integer and let d be a positive integer. Then, there are unique integers q and r, with $0 \leq r \leq d-1$, such that $n = qd + r$.

Integer division definitions

In the Division Algorithm, q is called the **quotient** and r is called the **remainder**. The operations **div** and **mod** produce the quotient and the remainder as a function of n and d.

$$q = n \operatorname{div} d$$
$$r = n \bmod d$$

Here is some examples of computing div and mod:

$$15 \bmod 6 = 3 \qquad -11 \bmod 4 = 1$$
$$15 - 2 \cdot 6 = 3 \qquad -11 - (-3) \cdot 4 = 1$$

$$15 \operatorname{div} 6 = 2 \qquad -11 \operatorname{div} 4 = -3$$
$$\frac{15 - 3}{2} = 2 \qquad \frac{-11 - 1}{4} = -3$$

1.2 Modular arithmetic

Given a finite set of integers, we can define addition and multiplication on the elements in the set such that after every operation, we apply a modular function equal to the cardinality of the set.

- **addition mod** m

  the operation defined by adding two numbers and applying $\bmod m$ to the result

- **multiplication mod** m

  the operation defined by multiplying two numbers and applying $\bmod m$ to the result

The set $\{0, 1, 2, \ldots, m-1\}$ along with addition and multiplication mod m defines a closed mathematical system with m elements called a **ring**. The ring $\{0, 1, 2, \ldots, m-1\}$ with addition and multiplication mod m is denoted by $\mathbb{Z}_m$.

Applications

A common way to organize data is to maintain an array called a **hash table** which is slightly larger than the number of data items to be stored. A bldhash function is used to map each data item to a location in the array. Modulus is used to keep the results from a hash function in the range of the hash table.

Computers use functions called bldpseudo-random number generators that produce numbers having many of the statistical properties of random numbers but are in fact deterministically generated. Modulus is used to keep these pseudo-random number generators in a certain range when used.

Congruence mod m

Let $m \in \mathbb{Z} > 1$. Let x and y be any two integers. Then bld$x$ is congruent to $y \bmod m$ if $x \bmod m = y \bmod m$. The fact that x is congruent to $y \bmod m$ is denoted

$$x \equiv y \pmod{m}.$$

Theorem: Alternate characterization of congruence mod m

Let $m \in \mathbb{Z} > 1$. Let x and y be any two integers. Then $x \equiv y \pmod{m}$ if and only if $m \mid (x - y)$.

Proof. First suppose that $x \equiv y \pmod{m}$. By definition $xtmodm = y \bmod m$. We define the variable r to be the value of $x \bmod m = y \bmod m$. Therefore, $x = r + km$ for some integer k and $y = r + jm$ for some integer j. Then

$$x - y = (r + km) - (r + jm) = (k - j)m.$$

Since $(k - j)$ is an integer, $m \mid (x - y)$.
Now suppose that $m \mid (x - y)$. Then $(x - y) = tm$ for some integer t. Let r be the value of $x \bmod m$. Then $x = r + km$ for some integer k. The integer y can be expressed as

$$y = x - (x - y) = (r + km) - tm = r + (k - t)m.$$

Since r is an integer in the range from 0 to $m - 1$, r is the unique remainder when y is divided by m. Therefore $r = y \bmod m = x \bmod m$, and by definition $x \equiv y \pmod{m}$. □

Precedence of the mod operation

$$6 + 2 \bmod y = 6 + (2 \bmod 7) = 8$$
$$6 \cdot 2 \bmod 7 = (6 \cdot 2) \bmod 7 = 5$$

However, in general it is best to just use parentheses in order to clarify which operations should be performed first.

Theorem: Computing arithmetic operations mod m

Let m be an integer larger than 1. Let x and y be any integers. Then

$$[(x \bmod m) + (y \bmod m)] \bmod m = [x + y] \bmod m$$
$$[(x \bmod m)(y \bmod m)] \bmod m = [x + y] \bmod m$$

1.3 Prime factorizations

A number p is **prime** if it is an integer greater than 1 and its only factors are 1 and p. A positive integer is **composite** if it has a factor other than 1 or itself. Every integer greater than 1 is either prime of composite. Every positive integer greater than one can be expressed as a product of primes called its **prime factorization**. Moreover, the prime factorization is unique up to ordering of the factors.

Theorem: The Fundamental Theorem of Arithmetic

Every positive integer other than 1 can be expressed uniquely as a product of prime numbers where the primes factors are written in non-decreasing order.

Examples of prime factorizations in non-decreasing order

$$112 = 2^4 \cdot 7$$
$$612 = 2^2 \cdot 3^3 \cdot 17$$
$$243 = 3^5$$
$$17 = 17$$

Greater common divisors and least common multiples

- The **greatest common divisor (gcd)** of integers x and y that are not both zero is the largest integer that is a factor of both x and y.
- The **least common multiples (lcm)** of non-zero integers x and y is the smallest positive integer that is an integer multiple of both x and y.

Two numbers are **relatively prime** if their greatest common divisor is 1.

Theorem: GCD and LCM from prime factorizations

Let x and y be two positive integers with prime factorizations expressed using a common set of primes as:

$$x = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$$
$$y = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_r^{\beta_r}$$

The p_i's are all distinct prime numbers. The exponents α_i's and β_i's are non-negative integers. Then:

- $x \mid y$ if and only if $\alpha_i \leq \beta_i$ for all $1 \leq i \leq r$
- $\gcd(x, y) = p_1^{\min(\alpha_1,\beta_1)} \cdot p_2^{\min(\alpha_2,\beta_2)} \cdots p_r^{\min(\alpha_r,\beta_r)}$
- $\operatorname{lcm}(x, y) = p_1^{\max(\alpha_1,\beta_1)} \cdot p_2^{\max(\alpha_2,\beta_2)} \cdots p_r^{\max(\alpha_r,\beta_r)}$

1.4 Factoring and primality testing

A **brute force algorithm** solves a problem by exhaustively searching all positive solutions without using an understanding of the mathematical structure in the problem to eliminate steps.

Theorem: Small Factors

If N is a composite number, then N has a factor greater than 1 and at most $\sqrt{N}$

Theorem: Infinite number of primes

There are an infinite number of primes.

Proof. Suppose that there are a finite number of primes. Since there are only a finite number, they can be listed:

$$p_1, p_2, \ldots, p_k$$

Take the product of all the primes and add 1. Call the resulting number N:

$$N = (p_1 \cdot p_2 \cdots p_k) + 1$$

The number N is larger than all of the primes numbers that were listed, so it must not be prime. Since N is a composite number, it is the product of at least two primes by the Fundamental Theorem of Arithmetic. There N is divisible by some prime p_j. Let

$$\frac{N}{p_j} = \frac{(p_1 \cdot p_2 \cdots p_k)}{p_j} + \frac{1}{p_j}$$

Note that p_j is one of the prime factors in $(p_1 \cdot p_2 \cdots p_k)$, so $(p_1 \cdot p_2 \cdots p_k)/p_j$ is an integer. However, $1/p_j$ is not an integer. Since N/p_j is the sum of two terms, one of which is an integer and the other of which is not an integer, then N/p_j is not an integer. This contradicts the fact that p_j evenly divides N. □

The Prime Number Theorem

Let $\pi(x)$ be the number of prime numbers in the range from 2 through x. Then

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\ln x} = 1.$$

Another way to interpret the Prime Number Theorem is that if a random number is selected from the range 2 to x, then the likelihood that the selected number is prime is roughly $1/\ln x$.

1.5 Greatest common factor divisor and Euclid's algorithm

There is an efficient way to compute the gcd of two numbers without finding their prime factorizations. The algorithm presented in this subsection is in wide use today and is attributed to the Greek mathematician Euclid who lived around 300 B.C. The basis of the algorithm is the following theorem:

GCD Theorem

Let x and y be two positive integers. Then $\gcd(x, y) = \gcd(y \bmod x, x)$.

Euclid's Algorithm for finding the greatest common divisor

```
Input: Two positive integers, x and y.
Output: gcd(x, y)

If(y < x)
  Swap x and y.
r := y mod x

While(r != 0)
  y := x
```

```
  x := r
  r := y mod x
End−While

Return(x)
```

Sample execution of Euclid's algorithm for $\gcd(675, 210)$:

$$\begin{array}{cccccc}
675 & 210 & 675 \bmod 210 & & & \\
 & 210 & 45 & 210 \bmod 45 & & \\
 & & 45 & 30 & 45 \bmod 30 & \\
 & & & 30 & 15 & 30 \bmod 15 \\
 & & & & \mathbf{15} & 0
\end{array}$$

The last non-zero number was 15, so $\gcd(675, 210) = 15$.

Expressing $\gcd(x, y)$ as a linear combination of x and y

Let x and y be integers, then there are integers s and t such that

$$\gcd(x, y) = sx + ty.$$

The values for s and t in the theorem above can be found by a series of substitutions using the equation from each iteration. The algorithm used to find the coefficient, s and t, such that $\gcd(x, y) = sx + ty$, is called the **Extended Euclidean Algorithm**.

The Extended Euclidean Algorithm

$$\begin{array}{cccccl}
 & & y & x & r & \\
675 & 210 & 45 & 30 & 15 & \\
 & & & & r = y \bmod x & \\
 & & & & r = y - (y \operatorname{div} x) \cdot x & \\
 & & & & 15 = 45 - (45 \operatorname{div} 30) \cdot 30 & \\
 & & & & \mathbf{15 = 45 - 1 \cdot 30} & \\
 & & & 30 & = 210 - (210 \operatorname{div} 45) \cdot 45 & \\
 & & & \mathbf{30} & \mathbf{= 210 - 4 \cdot 45} & \\
 & & 45 & = & 675 - (675 \operatorname{div} 210) \cdot 210 & \\
 & & \mathbf{45} & = & \mathbf{675 - 3 \cdot 210} &
\end{array}$$

We can use the bolded equations to solve for $15 = c \cdot 210 + d \cdot 675$.

$$\begin{aligned}
15 &= 45 - 30 \\
&= 45 - (210 - 4 \cdot 45) \\
&= 5 \cdot 45 - 210 \\
&= 5 \cdot (675 - 3 \cdot 210) - 210 \\
&= 5 \cdot 675 - 16 \cdot 210
\end{aligned}$$

Now we have the full answer and expansion for $\gcd(675, 210)$.

$$\gcd(675, 210) = 15 = 5 \cdot 675 - 16 \cdot 210.$$

The Multiplicative Inverse mod n

A **multiplicative inverse mod** n, or just **inverse mod** n, of an integer x, is an integer $s \in \{1, 2, \ldots, n-1\}$ such that $sx \bmod n = 1$.

For example, 3 is an inverse of 7 mod 10 because $3 \cdot 7 \bmod 10 = 1$. The number 7 is an inverse of 5 mod 17 because $7 \cdot 5 \bmod 17 = 1$. It is possible for a number to be its own multiplicative inverse mod n. For example, 7 is the inverse of $7 \bmod 8$ because $7 \cdot 7 \bmod 8 = 1$.

Not every number has an inverse mod n. For example, 4 does not have an inverse mod 6. The condition is that x has an inverse mod n if and only if x and n are relatively prime.

The Extended Euclidean Algorithm can be used to find the multiplicative inverse of $x \bmod n$ when is exists.

- If $\gcd(x, n) \neq 1$, then x does not have a multiplicative inverse mod n.
- If x and n are relatively prime, then the Extended Euclidean Algorithm finds integers s and t such that $1 = sx + tn$.
- $sx - 1 = -tn$. Therefore, $(sx \bmod n) = (1 \bmod n)$. If $A - B$ is a multiple of n then $(A \bmod n) = (B \bmod n)$.
- $(s \bmod n)$ is the unique multiplicative inverse of x in $\{0, 1, \ldots, n-1\}$.

For example, suppose that Euclid's Algorithm returns

$$\gcd(31, 43) = 1 = 13 \cdot 34 - 18 \cdot 31$$

The coefficient of 31 is -18. Therefore, the multiplicative inverse of $31 \bmod 43$ is $(-18 \bmod 43) = 25$.

1.6 Number representation

A digit in binary is called a **bit**. In binary notation, each place value is a power of 2. Numbers represented in **base** b require b distinct symbols and each place value is a power of b.

Theorem: Number representation

For an integer $b > 1$. Every positive integer n can be expressed uniquely as

$$n = a_k \cdot b^k + a_{k-1} \cdot b^{k-1} + \cdots + a_1 \cdot b^1 + a_0 \cdot b^0,$$

where k is a non-negative integer, and each a_i is an integer in the range from 0 to $b-1$, and $a_k \neq 0$. The representation of n base b is called the **base b expansion of n** and is denoted by $(a_k a_{k-1} \ldots a_1 a_0)_b$.

Hexadecimal Numbers

In **hexadecimal** notation (or **hex** for short), numbers are represented in base 16. Typically, the set of symbol, in order of value, is

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}.$$

Additionally, here are the first 15 hexadecimal digits and correspond encodings in decimal and binary.

Hex	0	1	2	3	4	5	6	7
Decimal	0	1	2	3	4	5	6	7
Binary	0	1	10	11	100	101	110	111

Hex	8	9	A	B	C	D	E	F
Decimal	8	9	10	11	12	13	14	15
Binary	1000	1001	1010	1011	1100	1101	1110	1111

Since both hexadecimal and binary are powers of 2, there is an easy way to translate between the binary expansion and the hexadecimal expansion of a number. Groups of 4 binary digits can be directly translated into hexadecimal digits. Here is an example

$$\begin{array}{ccccc} 1 & 1101 & 0101 & 1110 & 1000 \\ 1 & D & 5 & E & 8 \end{array} \qquad 1,1101,0101,1110,1000_2 = 1D5E8_{16}$$

Hexadecimal notation is particularly useful in computer science because each hexadecimal digit can be used to represent a 4-bit binary number. A byte, which consists of 8 bits, can be represented by a 2-digit hexadecimal number. Two hexadecimal digits is easier for a human to recognize and remember than 8 bits.

Converting decimal numbers to base b

$$[\text{Base } b \text{ expansion of } (n \operatorname{div} b)]\,[n \bmod b]$$

Here is an example of 1161_{10} converted to base 7:

$$\begin{array}{cccc}
 & & \ulcorner & 1161 \\
 & & 1161 \operatorname{div} 7 & 1161 \bmod 7 \\
 & \ulcorner & 165 & 6 \\
 & 165 \operatorname{div} 7 & 165 \bmod 7 & | \\
\ulcorner & 23 & 4 & | \\
23 \operatorname{div} 7 & 23 \bmod 7 & | & | \\
3 & 2 & | & | \\
3 & 2 & 4 & 6
\end{array}$$

The number of digits required to represent a number

$$n = (\underbrace{[b-1][b-1]\dots[b-1]}_{k\text{digits}})_b = (1\underbrace{00\dots0}_{k0's})_b - 1 = b^k - 1$$

The number of digits required to expressed the base b expansion of a positive integer n is $\lceil \log_b(n+1) \rceil$. For example, the number of digits required to express the base 2 expansion of 13 is:

$$\lceil \log_2(13+1) \rceil = \lceil \log_2(14) \rceil = \lceil 3.8 \rceil = 4.$$

1.7 Fast exponentiation

Computing a power of x by repeated multiplication by x

$$p := x \xrightarrow{p:=p*x} p = x^2 \xrightarrow{p:=p*x} p = x^3 \xrightarrow{p:=p*x} p = x^4 \cdots$$

Computing a power of x by repeated squaring

$$p := x \xrightarrow{p:=p*p} p = x^2 \xrightarrow{p:=p*p} p = x^4 \xrightarrow{p:=p*p} p = x^8 \cdots$$

In order to compute x^y, examine the binary expansion of y:

$$y = a_k \cdot 2^k + a_{k-1} \cdot 2^{k-1} + \cdots + a_1 \cdot 2^1 + a_0 \cdot 2^0$$

We can use this expanded form of y to represent x as a sum of perfect squares

$$x^y = x^{a_k \cdot 2^k + a_{k-1} \cdot 2^{k-1} + \cdots + a_1 \cdot 2^1 + a_0 \cdot 2^0} = x^{a_k \cdot 2^k} \cdot x^{a_{k-1} \cdot 2^{k-1}} \cdots x^{a_1 \cdot 2^1} \cdot x^{a_0 \cdot 2^0}$$

Each coefficient a_j is either 0 or 1. If $a_j = 0$ for some j, then

$$x^{a_j \cdot 2^j} = x^0 = 1$$

and the corresponding factor can be omitted from the product. Here is an example, computing 7^{11}.

$$\begin{aligned}
7^{11} &= 7^{[(1\cdot 2^3)+(0\cdot 2^2)+(1\cdot 2^1)+(1\cdot 2^0)]} \\
&= 7^{(1\cdot 8)} \cdot 7^{(0\cdot 4)} \cdot 7^{(1\cdot 2)} \cdot 7^{(1\cdot 7)} \\
&= 7^8 \cdot 1 \cdot 7^2 \cdot 7^1 \\
7^1 1 &= 5,764,801 \cdot 49 \cdot 7 \\
&= 1,977,326,743
\end{aligned}$$

An iterative algorithm for fast exponentiation

```
Input: Positive integers x and y.
Output: x^y

p := 1  // p holds the partial result
s := x  // s holds the current x^{2^j}
r := y  // r is used to compute the binary expansion of y

while(r > 0)
  if(r mod 2 = 1)
    p := p * s
  s := s * s
  r := r div 2
End-While

Return(p)
```

An iterative algorithm for fast modular exponentiation

```
Input: Positive integers x, y and n.
Output: x^y mod n

p := 1  // p holds the partial result
s := x  // s holds the current x^{2^j}
r := y  // r is used to compute the binary expansion of y

while(r > 0)
  if(r mod 2 = 1)
    p := p * s mod n
  s := s * s mod n
  r := r div 2
End-While

Return(p)
```

1.8 Introduction to cryptography

Cryptography is the science of protecting and authenticating data and communication. Cryptography is ubiquitous in the electronic age in which sensitive information such as credit card numbers and passwords are sent over the internet on a daily basis.

A cryptosystem is a system by which a **sender** sends a message to a **receiver**. The sender **encrypts** the message so that if an eavesdropper learns the transmitted message, they will be unable to recover the original message. The unencrypted message is called **plaintext** and the encrypted message is called the **ciphertext**. The receiver must have a **secret key** that allows him to **decrypt** the ciphertext to obtain the original plaintext.

Sending an Encrypted Text Message via a Secret Key

1. Alice wants to send the message "MEET AT DAWN" to Bob. Alice converts the text message to the number 130505202701292704012314.

2. Alice encrypts the numerical message with her copy of the secret key.

3. Alice sends encrypted message to Bob. Eve cannot read the encrypted message without the secret key.
4. Bob decrypts the encrypted message using the secret key to get 1305052027012927040123 14 which he then converts to "MEET AT DAWN".

Encryption and Decryption Functions

Consider a simple cryptosystem in which the set of all possible plaintexts come from $\mathbb{Z}_N$ for some integer N. Alice and Bob share a secret number $k \in \mathbb{Z}_N$. The security of their encryption scheme rests on the assumption that no one besides them knows the number k. To encrypt a plaintext $m \in \mathbb{Z}_n$, Alice computes:

$$c = (m + k) \bmod N \qquad \text{(encryption)}$$

Alice sends the ciphertext c to Bob. When Bob receives the ciphertext c, he decrypts c as follows:

$$m = (c - k) \bmod N \qquad \text{(decryption)}$$

The simple encryption scheme presented here is an example of symmetric key cryptography. In a **symmetric key cryptosystem**, Alice and Bob must meet in advance to decide on the value of a shared secret key.

Simple encryption scheme requirements

- If $m \neq m'$ and $m,\ m' \in \mathbb{Z}_N$ then $(m + k) \bmod N \neq (m' + k) \bmod N$ (no two distinct plaintexts map to same ciphertext).
- If $m \in \mathbb{Z}_N$ then $(((m+k) \bmod N)-k) \bmod N = m$ (decryption scheme is inverse of encryption scheme).

However, this encryption method is not terribly secure, and is not used in real world conditions because better methods exist.

1.9 The RSA cryptosystem

In **public key cryptography**, Bob has an **encryption key** that he provides *publicly* so that anyone can use it to send him an encrypted message. Bob holds a matching **decryption key** that he keeps *privately* to decrypt messages. While anyone can use the public key to encrypt a message, the security of the scheme depends on the fact that it is difficult to decrypt the message without having the matching private decryption key.

1. Alice encrypts her message to Bob using the public key.
2. Alice sends the encrypted message to Bob. Eve cannot read the encrypted message.
3. Bob decrypts the message using his private key.

Preparation of public and private keys in RSA

1. Bob selects two large prime numbers, p and q.
2. Bob computes $N = pq$ and $\phi = (p-1)(q-1)$.
3. Bob finds an integer e such that $\gcd(e, \phi) = 1$.
4. Bob computes the multiplication inverse of $e \bmod \phi$: an integer d such that $(ed \bmod \phi) = 1$.
5. Public (encryption) key: N and e.
6. Private (decryption) key: d.

The RSA scheme requires that m, the message, is an integer in $\mathbb{Z}_N$ and is not a multiple of p or q. Since p and q are primes with hundreds of digits, it is extremely unlikely that m is a multiple of primes p or q. Alice encrypts her plaintext using e and N to produce ciphertext c as follows:

$$c = m^e \bmod N \qquad \text{(encryption)}$$

Alice transmits c to Bob. Bob decrypts the ciphertext using d to recover m from c:

$$m = c^d \bmod N \qquad \text{(decryption)}$$

Number theoretic fact to establish correctness of RSA

Let p and q be prime numbers and $pq = N$. Suppose that $m \in \mathbb{Z}_n$ and $\gcd(m, N) = 1$. Then $m^{(p-1)(q-1)} \bmod N = 1$.

2 Introduction to Counting

2.1 Sum and Product Rules

The two most basic rules of counting are the sum and product rule. The **product rule** provides a way to count sequences.

Theorem: The Product Rule

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$$

Counting Strings

If Σ is a set of characters (called an **alphabet**) then Σ^n is the set of all string of length n whose characters come from the set Σ. The product rule can be applied directly to determine the number of strings of a given length over a finite alphabet:

$$|\Sigma^n| = |\underbrace{\Sigma \times \Sigma \times \cdots \Sigma}_{n \text{ times}}| = \underbrace{|\Sigma| \cdot |\Sigma| \cdots |\Sigma|}_{n \text{ times}} = |\Sigma|^n$$

Theorem: The Sum Rule

Consider n sets, $A_1, A_2, \ldots, A_n$. If the sets are pairwise disjoint, meaning that $A_i \cap A_j = \emptyset$ for $i \neq i$, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

2.2 The Bijection Rules

One way to approach difficult counting problems is to show that the cardinality of the set to be counted is equal to the cardinality of a set that is easy to count. The **bijection rule** says that if there is a bijection from one set to another then the two sets have the same cardinality.

A function f from a set S to a set T is called a bijection if and only if f has a well-defined *inverse*, f^{-1}.

The Bijection Rule

Let S and T be two finite sets. If there is a bijection from S to T, then $|S| = |T|$

The k-to-1 Rule

Let X and Y be finite sets. The function $f : X \to Y$ is a **k-to-1 correspondence** if for every $y \in Y$, there are exactly k difference $x \in x$ such that $f(x) = y$.

Suppose there is a k-to-1 correspondence from a finite set A to a finite set B. Then

$$|B| = \frac{|A|}{k}.$$

2.3 The generalized product rule

The **generalized product rule** says that in selecting an item from a set, if the number of choices at each step does not depend on previous choices made, then the number of items in the set is a product of the number of choices in each step.

Generalized Product Rule

Consider a set S of sequences of k items. Suppose there are:

- n_1 choices for the first item.
- For every possible choice for the first item, there are n_2 choice for the second item.
- For every possible choice for the first and second items, there are n_3 choices for the third item.
- $\vdots$
- For every possible choice for the first $k-1$ items, there are n_k choices for the k-th item.

Then $|S| = n_1 \cdot n_2 \cdots n_k$.

2.4 Counting permutations

An **r-permutation** is a sequence of r items with **no repetitions**, all taken from the same set. In a sequence, order matters, so (a, b, c) is different from (b, a, c).

The number of r-permutations from a set with n elements

Let r and n be positive integers with $r \leq n$. The number of r-permutations from a set with n elements is denoted by $P(n, r)$:

$$P(n,r) = \frac{n!}{(n-r)!} = n(n-1)\cdots(n-r+1)$$

2.5 Counting subsets

A subset of size r is called an **r-subset**. An r-subset is sometimes referred to as an **r-combination**. In a subset, order does not matter, so $\{a, b, c\}$ is the same as $\{b, a, c\}$. The counting rules for sequences and subsets are commonly referred to as "*permutations* and *combinations*". The term "combination" is the context of counting is another word for "subset".

Counting Subsets: 'n choose r' notation

The number of ways to select an r-subset from a set of size n is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

$\binom{n}{r}$ is read as "n choose r". The notation $C(n, r)$ is sometimes used for $\binom{n}{r}$.

We can calculated an expression for $\binom{n}{n-r}$ by replacing r with $n-r$ in the expression for $\binom{n}{r}$.

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r!)r!} = \binom{n}{r}$$

This is an **identity** for r-subsets.

2.6 Subset and permutation examples

Two different cat selection problems: Subset vs. Permutations

Consider two closely related counting problems:

1. A family goes to the animal shelter to adopt 3 cats. The shelter has 20 different cats from which to select. How many ways are there for the family to their selection?

2. Three different families go to the animal shelter to adopt a cat. Each family will select one cat. How many ways are there for the families to make their selection? (Note that which family gets which cat matters).

In the first problem, the number is ways to make the selection is $\binom{20}{3}$ because the order in which the cats are selected is not important. The outcome is a 3-subset.

In the second problem, the specific cat selected by each family is important. Additionally, no cat can belong to two families. Thus, the answer is $P(20,3) = 20 \cdot 19 \cdot 18$. The outcome is a 3-permutation.

2.7 Counting by complement

Counting by complement is a technique for counting the number of elements in a set S that have a property by counting the total number of elements in S and subtracting the number of elements in S that do not have the property.

$$|P| = |S| - |\overline{P}|$$

Suppose we want to count the number of people in a room with red hair. We know that there are 20 people in the room and exactly 12 of them do not have red hair. Then we can deduce that the number of people in the room with red hair is 20 - 12 = 8.

2.8 Permutations with repetitions

A **permutation with repetition** is an ordering of a set of items in which some of the items may be identical to each other. To illustrate with a smaller example, there are $3! = 6$ permutations of the letters CAT, because the letters in CAT are all different. However, there are only 3 different ways to scramble the letters in DAD: ADD, DAD, DDA.

Formula for Counting Permutations with Repetition

The number of distinct sequences with $n_1 1's, n_2's, \dots, n_k k's$, where $n = n_1 + n_2 + \cdots + n_k$ is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

The formula for permutations with repetition is derived from repeated use of the formula for counting r-subsets:

$$\begin{aligned}
&\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} \\
=&\frac{n!}{n_1!(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdot\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\cdots\frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_k!0!} \\
=&\frac{n!}{n_1!n_2!\cdots n_k!}
\end{aligned}$$

2.9 Counting multisets

A set is a collection of distinct items. A **multiset** is a collection that can have multiple instances of the same kind of item. When $\{1,2,2,3\}$ is viewed as a set, the repetitions don't matter and $\{1,2,2,3\} = \{1,2,3\}$. However, when $\{1,2,2,3\}$ is viewed as a multiset, then the fact there are two occurrences of 2 is important, and $\{1,2,2,3\} \neq \{1,2,3\}$. Two multisets are equal if they have the same number of each type of element. For multisets, the order of elements still does not matter.

Rules for encoding a selection of n objects from m varieties

Selections	Code words
n = number of items to select	n = number of 0's in code word
m = number of varieties	$m-1$ = number of 1's in code word
Number selected from the first variety	Number of 0's before the first 1
Number selected from the i-th variety, for $1 < i < m$	Number of 0's between the i-1st and i-th 1
Number selected from the last variety	Number of 0's after the last 1

If the mapping of selections to code words is a bijection, then by the bijection rule, the number of distinct code words is equal to the number of distinct selections. If the number of objects to select is n, and the number of varieties of object is m, each code word has n 0's and $m-1$ 1's, for a total of $n+m-1$ bits. The binary string of length $n+m-1$ with exactly $m-1$ 1's is

$$\binom{n+m-1}{m-1}$$

Theorem: Counting Multisets

The number of ways to select n objects from a set of m varieties is

$$\binom{n+m-1}{m-1},$$

if there is no limitation on the number of each variety available and objects of the same variety are indistinguishable.

A set of identical items are called **indistinguishable** because it is impossible to distinguish one of the item from another. A set of different or distinct items are called **distinguishable** because it is possible to distinguish one of the items from the others.

2.10 Assignment problems: Balls in bins

	No restrictions (any positive m and n)	**Max 1 ball per bin** (m must be at least n)	**Same # of balls per bin** (m must evenly divide n)
Indistinguishable	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	1
Distinguishable	m^n	$P(m,n)$	$\frac{n!}{((n/m)!)^m}$

2.11 Inclusion-exclusion principle

The **principle of inclusion-exclusion** is a technique for determining the cardinality of the union sets that uses the cardinality of each individual set as well as the cardinality of their intersections.

The inclusion-exclusion principle with two sets

Let A and B be two finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$

The inclusion-exclusion principle with three sets

Let $A, B,$ and C be three finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

The inclusion-exclusion principle with an arbitrary number of sets

Let $A_1, A_2, \ldots, A_n$ be a set of n finite sets.

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| = &\sum_{j=1}^{n} |A_j| \\ &- \sum_{1 \leq j \leq k \leq n} |A_j \cap A_k| \\ &+ \sum_{1 \leq j \leq j \leq l \leq n} |A_j \cap A_k \cap A_l| \\ &\vdots \\ &+ (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|\end{aligned}$$

The inclusion-exclusion principle and the sum rule

A collection of sets is **mutually disjoint** if the intersection of every pair of sets in the collection is empty. If we apply the principle of inclusion-exclusion to determine the union of a collection of mutually disjoint sets, then all the terms with the intersections are zero. Thus, for a collection of mutually disjoint sets, the cardinality of the union of the sets is just equal to the sum of the cardinality of each of the individual sets:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

The equation above is a restatement of the sum rule which only applies when the sets are mutually disjoint.

Determining the Cardinality of a Union by Complement

Counting by complement can be used to express the size of the union as:

$$|U| - |\overline{P_1 \cup P_2 \cup \cdots \cup P_n}| = |P_1 \cup P_3 \cup \cdots \cup P_n|$$

3 Advanced Counting

3.1 Generating permutations

There are situations in which it is necessary to generate, not just count, all permutations of a set or subsets of a given size.

Lexicographic Order

A well-defined order imposed on the n-tuples is useful to systematically generate all the elements in a set of n-tuples. Generating the n-tuples in the set from smallest to largest ensures that each n-tuple is generated exactly once.

Lexicographic order is a way or ordering n-tuples in which two n-tuples are compared according to the first entry where they differ. An example of such ordering is the word in a dictionary.

Generating Permutations

A **permutation** of the set $\{1, 2, \ldots, n\}$ is an ordered n-tuple in which each number in $\{1, 2, \ldots, n\}$ appears exactly once. For example, $(2, 5, 1, 4, 3)$ is a permutation of the set $\{1, 2, 3, 4, 5\}$.

Generating r-subsets of a set

Unlike sequences or n-tuples, the order in which the elements of a set or subset are written does not matter. Sets can be ordered lexicographically by first sorting the elements in increasing order and then comparing the two sets as if they were ordered sequences. For example, $\{2, 3, 11\} < \{2, 5, 6\}$, because the first element is the same in both sets but in the second element $3 < 5$.

3.2 Binomial coefficients and combinatorial identities

An **identity** is a theorem stating that two mathematical expressions are equal.

Theorem: A Simple Combinatorial Identity

For any non-negative integers n and k such that $k \leq n$:

$$\binom{n}{k} = \binom{n}{n-k}$$

A proof that makes use of counting principles is called a **combinatorial proof**. Combinatorial proofs usually involve defining a set S and counting the number of elements in S to get a mathematical expression for the number of items in the set. Every combinatorial proof of an identity uses a bijection implicitly as part of the argument.

Theorem: The Binomial Theorem

For any non-negative integer n and any real numbers a and b:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

The coefficients $\binom{n}{k}$ are called binomial coefficients.

For the case $n = 5$, the Binomial Theorem says that

$$\begin{aligned}(a+b)^5 &= \binom{5}{0}a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + \binom{5}{5}b^5 \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

The Binomial Theorem can also be used to obtain combinatorial identities. For example, by plugging in $a = b = 1$, the Binomial Theorem yields the identity below.

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Similarly, by letting $a = -1$ and $b = 1$, and requiring that n be positive, the left hand side of the Binomial Theorem becomes 0. The right hand side of the Binomial Theorem becomes:

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

In the expanded form,

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$$

Pascal's Triangle

The 17^{th} century French mathematician, Blaise Pascal, developed a triangular chart that contains all the number of the form $\binom{n}{k}$. The chart, now known as Pascal's Triangle, can be used to derive the value of a particular $\binom{n}{k}$. The n^{th} row of Pascal's Triangle contains the $n+1$ binomial coefficients of the form $\binom{n}{k}$ as shown in the figure below.

$$\begin{array}{lccccccccc}
n=0 & & & & & \binom{0}{0} & & & & \\
n=1 & & & & \binom{1}{0} & & \binom{1}{1} & & & \\
n=2 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
n=3 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
n=4 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

Theorem: Pascal's Identity

For any positive n and k such that $k < n$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

3.3 The pigeonhole principle

The pigeonhole principle is a mathematical tool used to establish that repetitions are guaranteed to occur in certain sets and sequences. The **pigeonhole principle** says that if $n+1$ pigeons are placed in n boxes, then there must be at least one box with more than one pigeon. The diagram below shows 10 pigeons, represented as P, in 9 boxes.

$$\begin{array}{ccc}
P & P & P \\
P & P & P, P \\
P & P & P
\end{array}$$

Theorem: The Pigeonhole Principle

If a function f has a domain of size at least $n+1$ and a target of size at most n, where n is a positive integer, then there are two elements in the domain that map to the same element in the target (i.e., the function is not one-to-one)

Theorem: The Generalized Pigeonhole Principle

Consider a function whose domain has at least n elements and whose target has k elements, for n and k positive integers. Then there is an element y in the target such that f maps at least $\lceil n/k \rceil$ elements in the domain to y.

Theorem: Converse of the Generalized Pigeonhole Principle

Suppose that a function f maps a set of n elements to a target with k elements, where n and k are positive integers. In order to guarantee that there is an element y in the target to which f maps to at least b elements from the domain, then n must be at least $k(b-1)+1$.

3.4 Generating functions

Generating functions are a powerful tool that can be used to solve a variety of problems related to counting and recurrence relations. A **generating function** is a way of representing a sequence of number as a n algebraic function in which each term in the sequence is a coefficient of an x^j term in the function. The advantage of representing sequences algebraically is that there are many techniques from algebra that can be used to manipulate functions which then leads to insight about the sequences they represent.

The sequence $f_0, f_1, f_2, f_3, \ldots$ is represented by the generating function

$$F(x) = f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots$$

The numbers in the sequence are the coefficients of the terms in $F(x)$. For the sequence $1, 1, 1, 1, \ldots$ is represented by the generating function

$$H(x) = 1 + x + x^2 + x^3 + \cdots$$

When $|x| < 1$ for $H(x)$ the sum is finite and has a closed form of

$$H(x) = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$$

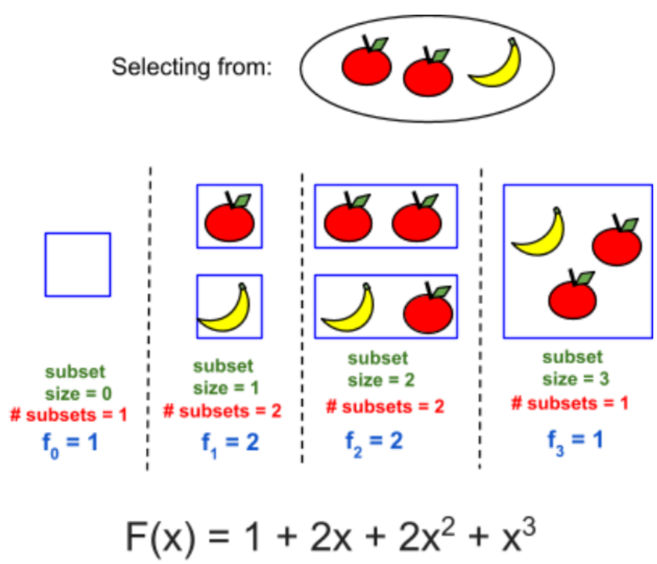
Additionally, for the partial sum up to x^k,

$$\sum_{j=0}^{k} x^j = \frac{1-x^k}{1-x}$$

Using Generating Functions for Counting

One of the primary uses for generating functions is to help solve counting problems. In general, f_j is the number of way to select a subset of j objects.

Consider the situation in which there is a set of two apples and one banana. The apples are indistinguishable, so selecting one apple is the same as selecting the other.



If the set from which the subsets are selected is an infinite set, then the resulting sequence is also infinite.

Summary of Generating Functions

Description	Long Form	Short Form
Infinite supply of one kind of item	$1 + x + x^2 + x^3 + \cdots$	$\dfrac{1}{1-x}$
Selecting from a set of n identical items	$1 + x + x^2 + x^3 + \cdots + x^n$	$\dfrac{1-x^{n-1}}{1-x}.$
Infinite supple of identical items grouped in batches of k	$1 + x^k + x^{2k} + x^{3k} + \cdots$	$\dfrac{1}{1-x^k}$
Infinite supple of 2 different kinds of items	$1 + 2x + 3x^2 + 4x^3 + \cdots$	$\dfrac{1}{(1-x)^2}$

Products of Generating Functions

Generating functions become very useful when different sets of objects are combined together into larger sets.

For example, if we are selecting subsets of apples from a set of 3 apples, the generating function is $A(x) = 1 + x + x^2 + x^3$, and if we are selecting bananas from a set of 2 bananas, the generating function is $B(x) = 1 + x + x^2$. Now suppose we pool the three apples and two bananas into a set of five pieces of fruit and ask 'how many ways can one select a set of fruit from the collection of five pieces of fruit?'

The power of generating functions is illustrated in the fact that if we take the product of $A(x)$, the generating function for apples, and $B(x)$, the generating function for bananas, to get $A(x)B(x)$, then the resulting product is the generating function for the pooled set of five pieces.

$$A(x)B(x) = (1 + x + x^2 + x^3)(1 + x + x^2) = 1 + 2x + 3x^2 + 3x^3 + 2x^4 + x^5$$

Note that the rule of multiplying generating function only works when the two sets of objects being combined are *distinct*. The rule would not work for combining two sets of indistinguishable apples.

Now suppose that we add oranges to the set of selections. There are three oranges, but they come wrapped in a single pack, so one can select zero or three oranges, but not one or two. The generating function for oranges is $O(x) = 1 + x^3$. Again, we can take the product of the generating functions to solve, 'how many ways are there to select subsets of fruit of a particular size?'

$$A(x)B(x)O(x) = 1 + 2x + 3x^2 + 4x^3 + 4x^4 + 4x^5 + 3x^6 + 2x^7 + x^8$$

The coefficient of x^5 in the generating function for the whole set of fruit is 4, so there are 4 ways to select 5 pieces of fruit.

4 Discrete Probability

4.1 Probability of an event

One of the primary applications of counting is to calculate probabilities of random events.

An **experiment** is a procedure that results in one out of a number of possible **outcomes**. The set of all possible outcomes is called the **sample space** of the experiment. A subset of the sample space is called an **event**.

Discrete vs. Continuous Probability

Discrete probability is concerned with experiments in which the sample space is finite or a countably infinite set. A set is **countably infinite** if there is a one-to-one correspondence between the elements of the set and the integers. An infinite set that is not countably infinite is said to be **uncountably infinite**.

Probability Distributions

A **probability distribution** over the outcomes of an experiment with a countable sample space S if a function $p : S \to [0, 1]$ with the property that

$$\sum_{s \in S} p(s) = 1.$$

The probability of outcome s is $p(s)$. If $E \subseteq S$ is an event, then the **probability of event** E is

$$p(E) = \sum_{s \in E} p(s).$$

The Uniform Distribution

The probability distribution in which every outcome has the same probability is called the **uniform distribution**. The uniform distribution reduces questions about probabilities to questions about counting because for every event E,

$$p(E) = \frac{|E|}{|S|}.$$

4.2 Unions and complements of events

Calculating Probabilities for Unions of Events

Two events are **mutually exclusive** if the two events are disjoint, meaning that the intersection of the two events is empty. If follows from the definition of the probability of an event that if E_1 and E_2 are mutually exclusive, then:

$$p(E_1 \cup E_2) = p(E_1) + p(E_2).$$

However, if two events are not mutually exclusive, the probability of the union of the events can be determined by a version of the Inclusion-Exclusion principle:

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

The statement holds for non-uniform as well as uniform distributions.

The Complement of an Event

The **complement** of an event E is $S - E$ and is denoted by $\overline{E}$. Since $\overline{E}$ and E are disjoint events, $p(\overline{E}) + p(E) = 1$. It follows then that

$$p(\overline{E}) = 1 - p(E).$$

4.3 Conditional probability and independence

If the event F happens, the new probability of E is the **conditional probability** of E given F, denoted by $p(E|F)$. The conditional probability of E given F is

$$p(E|F) = \frac{p(E \cup F)}{p(F)}.$$

If the distribution is uniform, then $p(E) = |E|/|S|$ and the conditional probability becomes:

$$p(E|F) = \frac{p(E \cup F)}{p(F)} = \frac{|E \cup F|/|S|}{|F|/|S|} = \frac{|E \cup F|}{|F|}.$$

The Complement of an Event and Conditional Probability

If E and F are both events in the same sample space S, then the probability of E and the probability of $\overline{E}$ still sum to 1, even when conditioned on the event F.

$$p(E|F) + p(\overline{E}|F) = 1$$

Independent Events

Let E and F be two events in the same sample space. The following three conditions are equivalent.

1. $p(E|F) = \frac{p(E \cap F)}{p(F)} = p(E)$
2. $p(E \cap F) = p(E) \cdot p(F)$
3. $p(F|E) = \frac{p(E \cap F)}{p(E)} = p(F)$

If one of the three conditions hold, then events E and F are independent.

Calculating the Probabilities of Two Independent Events

If X and Y are events in the same sample space, and X and Y are independent, then

$$p(X \cap Y) = p(X) \cdot p(Y).$$

Mutual Independence

Events $A_1, \ldots, A_n$ in sample space S are **mutually independent** if the probability of the intersection of any subset of the events is equal to the product of the probabilities of the events in the subset. In particular, if $A_1, \ldots, A_n$ are mutually independent, then

$$p(A_1 \cap A_2 \cap \cdots \cap A_n) = p(A_1) \cdot p(A_2) \cdots p(A_n).$$

4.4 Bayes' Theorem

Suppose that F and X are events from a common sample space and $p(F) \neq 0$ and $p(X) \neq 0$. Then

$$p(F|X) = \frac{p(X|F)p(F)}{p(X|F)p(F) + p(X|\overline{F})p(\overline{F})}.$$

This is known as Bayes' Theorem. In other words, Bayes' theorem tells us how to update our initial beliefs about a hypothesis (represented by $p(F)$) based on new evidence (represented by $p(X|F)$), taking into account the prior probability of the hypothesis (represented by $p(F)$) and the overall probability of observing the evidence (represented by $p(X)$).

4.5 Random variables

A **radnom variable** X is a function from the sample space S of an experiment to the real numbers. $X(S)$ denoted the range of the function X.

Random Variables and Probabilities

If X is a random variable defined on the sample space S of an experiment and $r \in \mathbb{R}$, then $X = r$ is an event. The event $X = r$ consists of all outcomes s in the sample space such that $X(s) = r$. $p(X = r)$ is the sum of the $p(s)$ for all s such that $X(s) = r$.

Distribution over a Random Variable

The **distribution** of a random variable is the set of all pairs $(r, p(X = r))$ such that $r \in X(S)$.

4.6 Expectation of random variables

The **expected value** of a random variable X is denoted $E[X]$ and is defined as

$$E[X] = \sum_{s \in S} X(s)p(s),$$

where $p(s)$ is the probability of outcome s.

Alternatively, if X is a random variable defined over an experiment with a sample space S,

$$E[X] = \sum_{r \in X(S)} r \cdot p(X = r),$$

where $X(S)$ is the range of the function X.

4.7 Linearity of expectations

If X and Y are two random variables defined on the same sample space S, and $c \in \mathbb{R}$,

$$\begin{aligned} E[X + Y] &= E[X] + E[Y], \quad \text{and} \\ E[cX] &= xE[X]. \end{aligned}$$

Linearity of expectations can be shown by induction to apply to more than two variables. If $X_1, \ldots, X_n$ are n variables defined on the same sample space, then

$$E\left[\sum_{j=i}^{n} X_j\right] = \sum_{j=1}^{n} E[X_j]$$

4.8 Bernoulli trials and the binomial distribution

A **Bernoulli trial** is an experiment with two outcomes: **success** and **failure**. In a sequence of independent Bernoulli trials, called a **Bernoulli process**, the outcomes of the repeated experiments are assumed to be mutually independent and have the same probability of success and failure. Usually the probability of success is denoted by the variable p, and the probability of failure, $(1 - p)$, denoted by the variable q.

Bernoulli Trial and Probabilities

The probability of exactly k successes in a sequence of n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$ is

$$\binom{n}{k} p^k q^{n-k}.$$

The distribution over the random variable defined by the number of the successes in a sequence of independent Bernoulli trials is called the **binomial distribution**. The probability that the number of successes is k in a sequence of length n with probability of success p is denoted by $b(k; n, p)$. By the theorem above,

$$b(k; n, p) = \binom{n}{k} p^k.$$

The range of the random variable denoting the number of successes in a sequence of n Bernoulli trials is 0 through n. Since the values of $b(k;)$ are a probability distribution over the possible values for k, the probabilities should sum to 1 as k ranges from 0 through n:

$$\sum_{k=0}^{n} b(k; n, p) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^n = 1$$

5 Graphs

5.1 Introduction to Graphs

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5.3 Graph isomorphism

5.4 Walks, trails, circuits, paths, and cycles

5.5 Graph connectivity

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5.7 Hamiltonian cycles and paths

5.8 Planar coloring

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6 Trees

6.1 Introduction to trees

6.2 Tree application examples

6.3 Properties of trees

6.4 Tree traversals

6.5 Spanning trees and graph traversals

6.6 Minimum spanning trees