# Homework 4

## Problem 9

Let $V = \mathbb{R}^3$ be a vector space with standard addition and scalar multiplication. Use Theorem 3 of Lecture Note 8 to determine whether the following sets are subspaces of $V$.

**b.** *The set of vectors of the form* $(a, 1, 1)$*,* $W$

*Proof.* Axiom 1: Consider $\vec{a} = (a, 1, 1)$ and $\vec{b} = (b, 1, 1)$ for $a, b \in \mathbb{R}$. $\vec{a} \in W$ and $\vec{b} \in W$.

$$\begin{aligned} \vec{a} \oplus_W \vec{b} &= (a, 1, 1) \oplus (b, 1, 1) = (a + b, 1 + 1, 1 + 1) \\ &= (a + b, 2, 2) \notin W \end{aligned}$$

Therefore $W$ is **not** closed under addition.
Since Axiom 1 does not hold for $W$, $W$ is not a subspace of $V$. □

**c.** *The set of vectors of the form* $(a, b, c)$*, where* $b = a + c$*,* $W$

*Proof.* Axiom 1: Consider $\vec{v} = (a_1, b_1, c_1)$ and $\vec{u} = (a_2, b_2, c_2)$ for $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$. $\vec{v} \in W$ and $\vec{u} \in W$.

$$\begin{aligned} \vec{v} \oplus \vec{u} &= (a_1, b_1, c_1) \oplus (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2) \\ &\in W \end{aligned}$$

$$b_1 + b_2 = (a_1 + c_1) + (a_2 + c_2) = (a_1 + a_2) + (c_1 + c_2) \checkmark$$

Therefore $W$ is closed under addition.
Axiom 6: Consider $\vec{v} = (a, b, c)$ such that $a, b, c \in \mathbb{R}$ and $k \in \mathbb{R}$. Let $b = a + c$.

$$\begin{aligned} k \odot \vec{v} &= k \odot (a, b, c) = k \odot (a, a + c, c) = (ka, k(a + c), kc) = (ka, ka + kc, kc) \\ &\in W \end{aligned}$$

$$kb = k(a + c) = ka + kc \checkmark$$

Therefore $W$ is closed under scalar multiplication.
Since Axiom 1 and Axiom 6 hold for $W$, $\oplus$ and $\odot$ are inherited from $V$, and $W \subseteq V$, through the use of Theorem 3, $W$ is a subspace of $V$. □

**d.** *The set of vectors of the form* $(a, b, 0)$*,* $W$

*Proof.* Axiom 1: Consider $\vec{v} = (a_1, b_1, 0)$ and $\vec{u} = (a_2, b_2, 0)$ for $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

$$\begin{aligned} \vec{v} \oplus \vec{u} &= (a_1, b_1, 0) \oplus (a_2, b_2, 0) = (a_1 + a_2, b_1 + b_2, 0 + 0) = (a_1 + a_2, b_1 + b_2, 0) \\ &\in W \text{ since it takes the form } (a, b, 0) \end{aligned}$$

Therefore $W$ is closed under addition.
Axiom 6: Consider $\vec{v} = (a, b, 0)$ such that $a, b \in \mathbb{R}$ and $k \in \mathbb{R}$.

$$\begin{aligned} k \odot \vec{v} &= k \odot (a, b, 0) = (ka, kb, k0) = (ka, kb, 0) \\ &\in W \text{ since it takes the form } (a, b, 0) \end{aligned}$$

Therefore $W$ is closed under scalar multiplication.
Since Axiom 1 and Axiom 6 hold for $W$, $\oplus$ and $\odot$ are inherited from $V$, and $W \subseteq V$, through the use of Theorem 3, $W$ is a subspace of $V$. □

## Problem 10

Let $V = P_3$ be the vector space of all polynomials with degree *up to* 3, with standard addition and scalar multiplication. Use Theorem 3 of Lecture Note 8 to determine whether the following sets are subspaces of $V$.

**b.** *The set of polynomials* $a_0 + a_1x + a_2x^2 + a_3x^3$ *for which* $a_0 + a_1 + a_2 + a_3 = 0$*,* $W$*.*

*Proof.* Axiom 1: Consider $\vec{a} = a_0+a_1x+a_2x^2+a_3x^3$ and $\vec{b} = b_0+b_1x+b_2x^2+b_3x^3$ where $a_{0-3}, b_{0-3} \in \mathbb{R}$.
Let $a_0 + a_1 + a_2 + a_3 = 0$ and $b_0 + b_1 + b_2 + b_3 = 0$.

$$\begin{aligned}\vec{a} \oplus \vec{b} &= (a_0 + a_1x + a_2x^2 + a_3x^3) \oplus (b_0 + b_1x + b_2x^2 + b_3x^3)\\ &= a_0 + a_1x + a_2x^2 + a_3x^3 + b_0 + b_1x + b_2x^2 + b_3x^3\\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3\\ &\in W\end{aligned}$$

$$\begin{aligned}(a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) &= a_0 + a_1 + a_2 + a_3 + b_0 + b_1 + b_2 + b_3\\ &= 0 + 0 = 0 \checkmark\end{aligned}$$

Therefore $W$ is closed under addition.
Axiom 6: Consider $\vec{a} = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $a_{0-3} \in \mathbb{R}$ and $k \in \mathbb{R}$.
Let $a_0 + a_1 + a_2 + a_3 = 0$.

$$\begin{aligned}k \odot \vec{a} &= k \odot (a_0 + a_1x + a_2x^2 + a_3x^3) = k(a_0 + a_1x + a_2x^2 + a_3x^3)\\ &= ka_0 + ka_1x + ka_2x^2 + ka_3x^3\\ &\in W\end{aligned}$$

$$ka_0 + ka_1x + ka_2x^2 + ka_3x^3 = k(a_0 + a_1x + a_2x^2 + a_3x^3) = k(0) = 0 \checkmark$$

Therefore $W$ is closed under scalar multiplication.
Since Axiom 1 and Axiom 6 hold for $W$, $\oplus$ and $\odot$ are inherited from $V$, and $W \subseteq V$, through the use of Theorem 3, $W$ is a subspace of $V$. □

**c.** *The set of polynomials* $a_0 + a_1x + a_2x^2 + a_3x^3$ *in which* $a_0, a_1, a_2,$ *and* $a_3$ *are integers.*

*Proof.* Axiom 6: Consider $\vec{a} = 1 + 1x + 1x^2 + 1x^3$ and $k = 0.66$.

$$\begin{aligned}k \odot \vec{a} &= 0.66 \odot (1 + 1x + 1x^2 + 1x^3) = 0.66(1 + 1x + 1x^2 + 1x^3)\\ &= 0.66 + 0.66x + 0.66x^2 + 0.66x^3\\ &\notin W, \text{ since } 0.66 \notin \mathbb{Z}\end{aligned}$$

Therefore $W$ is **not** closed under scalar multiplication.
Since Axiom 6 does not hold for $W$, $W$ is not a subspace of $V$. □

## Problem 11

Let $V = F(-\infty, \infty)$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$, with standard addition and scalar multiplication. Use Theorem 3 of Lecture Note 8 to determine whether the following sets are subspaces of $V$.

**b.** *The set of functions* $f$ *in* $F(-\infty, \infty)$ *for which* $f(0) = 1$*.*

*Proof.* Axiom 1: Consider $\vec{f}(x) = e^x$ and $\vec{g}(x) = e^x$. $\vec{f}, \vec{g} \in W$ since $e^0 = 0$.

$$\begin{aligned}(\vec{f} \oplus \vec{g})(x) = \vec{f}(x) + \vec{g}(x) = e^x + e^x = 2e^x \notin W \\ \text{when } x = 0 : 2e^0 = 2 \cdot 1 = 2 \neq 1\end{aligned}$$

Therefore $W$ is **not** closed under addition.
Since Axiom 1 does not hold for $W$, $W$ is not a subspace of $V$. ☐

**c.** *The set of functions $\vec{f}$ in $F(-\infty, \infty)$ for which $f(-x) = x$, $W$*

*Proof.* Axiom 1: Consider $\vec{f}, \vec{g} \in W$.

$$\begin{aligned}(\vec{f} \oplus \vec{g})(x) &= \vec{f}(x) + \vec{g}(x) \\ \text{when plugging in } -x &: \\ \vec{f}(-x) + \vec{g}(-x) &= x + x = 2x \\ &\neq x \text{ if } x \neq 0\end{aligned}$$

Therefore $W$ is **not** closed under addition.
Since Axiom 1 does not hold for $W$, $W$ is not a subspace of $V$. ☐

## Problem 13

Let $V$ be a vector space. Let $I$ be a nonempty set (often called the "index set"), and let $W_i$ be a subspace of $V$ for all $i \in I$. Prove that $\bigcap\limits_{i \in I} W_i$, is a subspace of V.

*Proof.* Since $W_i$ is a subspace $\forall\ i \in I$, this implies the following:

1. $\forall\ i \in I : W_i \subseteq V \Leftrightarrow \bigcap\limits_{i \in I} W_i \subseteq V$
2. $\oplus_{\bigcap_{i \in I} W_i}$ and $\odot_{\bigcap_{i \in I} W_i}$ are inherited from $V$

Therefore, by Theorem 3, only Axiom 1 and Axiom 6 must be proven for $\bigcap\limits_{i \in I} W_i$ to be a subspace of $V$.

Axiom 1: Consider $\vec{u}, \vec{v} \in \bigcap\limits_{i \in I} W_i$. This implies the following:

$$\begin{aligned}\vec{u} \in W_i \ \ \forall\ i \in I \\ \vec{v} \in W_i \ \ \forall\ i \in I\end{aligned}$$

Consider if $\vec{u} \oplus \vec{v} \notin W_j$, for some $j \in I$. Since $\vec{u} \in W_j$ and $\vec{v} \in W_j$ but $\vec{u} \oplus \vec{v} \notin W_j$, by definition $W_j$ is not closed under addition, and thus not a subspace of $V$. This contradicts our assertion that $\forall\ i \in I$, $W_i$ is a subspace of $V$. Therefore, through contradiction, $\vec{u} \oplus \vec{v} \in W_j \ \ \forall j \in I$. This statement is equivalent to

$$\vec{u} \oplus \vec{v} \in \bigcap_{i \in I} W_i$$

Therefore $\bigcap\limits_{i \in I} W_i$ is closed under addition.

Axiom 6: Consider $\vec{v} \in \bigcap\limits_{i \in I} W_i$ and $k \in \mathbb{R}$. This implies:

$$\vec{v} \in W_i \ \ \forall\ i \in I$$

Consider if $k \odot \vec{v} \notin W_j$, for some $j \in I$. Since $k \in \mathbb{R}$ and $\vec{v} \in W_j$ but $k \odot \vec{v} \notin W_j$, by definition $W_j$ is not closed under scalar multiplication, and thus not a subspace of $V$. This contradicts our assertion that $\forall\ i \in I,\ W_i$ is a subspace of $V$. Therefore, through contradiction, $k \odot \vec{v} \in W_j\ \ \forall j \in I$. This statement is equivalent to

$$\vec{u} \oplus \vec{v} \in \bigcap_{i \in I} W_i$$

Therefore $\bigcap\limits_{i \in I} W_i$ is closed under scalar multiplication.

Since Axiom 1 and Axiom 6 hold for $\bigcap\limits_{i \in I} W_i$, through Theorem 3, $\bigcap\limits_{i \in I} W_i$ is a subspace of $V$. $\square$