Section 7.2

7.2.1

Characterize the rate of growth of each function f below by giving a function g such that $f = \Theta(g)$. The function g should be one of the functions in the table of common functions.

a. $f(n) = n^8 + 3n - 4 \quad \to \quad \Theta(n^8)$

b. $f(n) = 2 \cdot 3^n \quad \to \quad \Theta(3^n)$

c. $f(n) = 2^n + 3^n \quad \to \quad \Theta(3^n)$

d. $f(n) = 7(\log\log n) + 3(\log n) + 12n \quad \to \quad \Theta(n)$

e. $f(n) = 9(n\log n) + 5(\log\log n) + 5 \quad \to \quad \Theta(n\log n)$

f. $f(n) = n \cdot \log_{37} n \quad \to \quad \Theta(n\log n)$

g. $f(n) = n^{21} + (1.1)^n \quad \to \quad \Theta(1.1^n)$

h. $f(n) = 23n + n^3 - 2 \quad \to \quad \Theta(n^3)$

7.2.2

Give complete proofs for the growth rates of the polynomials below. You should provide specific values for c and n_0, and prove algebraically that the functions satisfy the definitions for $\mathcal{O}$ and Ω.

b. $f(n) = n^3 + 3n^2 + 4$. *Prove that* $f = \Theta(n^3)$

Proof. of $\mathcal{O}(n^3)$: Consider $n_0 = 1$ and $c = 8$: For $n \geq 1$, $1 \leq n^2$, so:

$$n^3 + 3n^2 + 4 \leq n^3 + 3n^2 + 4n^2.$$

For $n \geq 1$, $n^2 \leq n^3$, so:

$$n^3 + 3n^2 + 4n^2 \leq n^3 + 3n^3 + 4n^3.$$

$$f(n) = n^3 + 3n^2 + 4 \leq n^3 + 3n^2 + 4n^2 \leq n^3 + 3n^3 + 4n^3 = 8n^3 = 8(n^3).$$

Therefore, with witness $n_0 = 1$ and $c = 8$, $f = \mathcal{O}(n^3)$.

Proof. of $\Omega(n^3)$: Consider $n_0 = 1$ and $c = 1$: For $n \geq 1$, $3n^2 \geq 0$ and $4 \geq 0$. Adding these inequalities yields

$$3n^2 + 4 \geq 0$$

Adding n^3 to both sides to get

$$n^3 + 3n^2 + 4 \geq n^3$$

Therefore, with witness $n_0 = 1$ and $c = 1$, $f = \Omega(n^3)$.

Since $f = \mathcal{O}(n^3)$ and $f = \Omega(n^3)$, $\therefore f = \Theta(n^3)$. $\square$