Test 2

Problem 1 Let A be an $n \times n$ matrix such that $A^4 + A^2 = O$. Show that

$$(A^2 - A + I)^{-1} = A^2 + A + I$$

Proof. Consider $A^2 + A + I$.

$$\begin{aligned}(A^2+A+I)(A^2-A+I) &= A^4 - A^3 + A^2 + A^3 - A^2 + A + A^2 - A + I^2 = A^4 + A^2 + I\\ (A^2-A+I)(A^2+A+I) &= A^4 + A^3 + A^2 - A^3 - A^2 - A + A^2 + A + I^2 = A^4 + A^2 + I\end{aligned}$$

If $A^4 + A^2 = O$, we can simplify further:

$$\begin{aligned} A^4 + A^2 + I &= O + I && \text{since } A^4 + A^2 = O\\ &= I\\ \therefore (A^2+A+I)(A^2-A+I) &= I\\ \therefore (A^2-A+I)(A^2+A+I) &= I\end{aligned}$$

Since $(A^2+A+I)(A^2-A+I) = I$ and $(A^2-A+I)(A^2+A+I) = I$, therefore if $A^4 + A^2 = O$, then $(A^2 - A + I)^{-1} = A^2 + A + I$. $\square$

Problem 2 In $\mathbb{R}^3$, let $\vec{v}_1 = (-3, 1, 4), \vec{v}_2 = (-4, 2, 5)$, and $\vec{v}_3 = (-1, 0, 2)$. Express $\vec{u} = (5, -4, 2)$ as a linear combination of $\vec{v}_1, \vec{v}_3$, and $\vec{v}_3$ by finding the MATRIX INVERSE.

Work. Let k_1, k_2, and $k_3 \in \mathbb{R}$ such that $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{u}$. That is, $k_1(-3, 1, 4) + k_2(-4, 2, 5) + k_3(-1, 0, 2) = (5, -4, 2)$. This equation can be converted into a matrix equation.

$$\begin{pmatrix}-3 & -4 & -1\\ 1 & 2 & 0\\ 4 & 5 & 2\end{pmatrix}\begin{pmatrix}k_1\\ k_2\\ k_3\end{pmatrix} = \begin{pmatrix}5\\ -4\\ 2\end{pmatrix}$$

In order to solve this equation, we must use the inverse algorithm.

$$\left(\begin{array}{ccc|ccc}-3 & -4 & -1 & 1 & 0 & 0\\ 1 & 2 & 0 & 0 & 1 & 0\\ 4 & 5 & 2 & 0 & 0 & 1\end{array}\right) \xrightarrow[R_3-4R_2]{R_1+3R_2} \left(\begin{array}{ccc|ccc}0 & 2 & -1 & 1 & 3 & 0\\ 1 & 2 & 0 & 0 & 1 & 0\\ 0 & -3 & 2 & 0 & -4 & 1\end{array}\right) \xrightarrow[R_2-R_1]{R_3+2R_1}$$

$$\left(\begin{array}{ccc|ccc}0 & 2 & -1 & 1 & 3 & 0\\ 1 & 0 & -1 & -1 & -2 & 0\\ 0 & 1 & 0 & 2 & 2 & 1\end{array}\right) \xrightarrow{R_1-2R_3} \left(\begin{array}{ccc|ccc}0 & 0 & -1 & -3 & -1 & -2\\ 1 & 0 & -1 & -1 & -2 & 0\\ 0 & 1 & 0 & 2 & 2 & 1\end{array}\right) \xrightarrow{R_2+R_1}$$

$$\left(\begin{array}{ccc|ccc}0 & 0 & -1 & -3 & -1 & -2\\ 1 & 0 & 0 & -4 & -3 & -2\\ 0 & 1 & 0 & 2 & 2 & 1\end{array}\right) \xrightarrow[R_1\leftrightarrow R_2\leftrightarrow R_3]{-R_1} \left(\begin{array}{ccc|ccc}1 & 0 & 0 & -4 & -3 & -2\\ 0 & 1 & 0 & 2 & 2 & 1\\ 0 & 0 & 1 & 3 & 1 & 2\end{array}\right)$$

Though the use of the inverse algorithm, we know that

$$\begin{pmatrix}-3 & -4 & -1\\ 1 & 2 & 0\\ 4 & 5 & 2\end{pmatrix}^{-1} = \begin{pmatrix}-4 & -3 & -2\\ 2 & 2 & 1\\ 3 & 1 & 2\end{pmatrix}.$$

We can use this identity to solve the original equation:

$$\begin{pmatrix} -3 & -4 & -1 \\ 1 & 2 & 0 \\ 4 & 5 & 2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -4 & -1 \\ 1 & 2 & 0 \\ 4 & 5 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -3 & -4 & -1 \\ 1 & 2 & 0 \\ 4 & 5 & 2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -3 & -4 & -1 \\ 1 & 2 & 0 \\ 4 & 5 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -4 & -3 & -2 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -20+12-4 \\ 10-8+2 \\ 15-4+4 \end{pmatrix}$$

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -12 \\ 4 \\ 15 \end{pmatrix}$$

This means that when $k_1 = -12$, $k_2 = 4$, and $k_3 = 15$, the equation $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{u}$ is true. That is, $-12(-3,1,4) + 4(-4,2,5) + 15(-1,0,2) = (5,-4,2)$. □

Problem 3 Solve for the matrix A if

$$(I-2A)^{-1} = \begin{pmatrix} 1 & -3 & 3 \\ -2 & 2 & -5 \\ 3 & -8 & 9 \end{pmatrix}.$$

Work.

$$\left(\begin{array}{ccc|ccc} 1 & -3 & 3 & 1 & 0 & 0 \\ -2 & 2 & -5 & 0 & 1 & 0 \\ 3 & -8 & 9 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3-3R_1]{R_2+2R_1} \left(\begin{array}{ccc|ccc} 1 & -3 & 3 & 1 & 0 & 0 \\ 0 & -4 & 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right) \xrightarrow[R_2+4R_3]{R_1+3R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & -8 & 0 & 3 \\ 0 & 0 & 1 & -10 & 1 & 4 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1-3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 22 & -3 & -9 \\ 0 & 0 & 1 & -10 & 1 & 4 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 22 & -3 & -9 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -10 & 1 & 4 \end{array}\right)$$

Therefore, through the inverse algorithm,

$$\begin{pmatrix} 1 & -3 & 3 \\ -2 & 2 & -5 \\ 3 & -8 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} 22 & -3 & -9 \\ -3 & 0 & 1 \\ -10 & 1 & 4 \end{pmatrix}$$

Using this fact, we can now solve the original equation.

$$(I-2A)^{-1} = \begin{pmatrix} 1 & -3 & 3 \\ -2 & 2 & -5 \\ 3 & -8 & 9 \end{pmatrix}$$

$$I-2A = \begin{pmatrix} 1 & -3 & 3 \\ -2 & 2 & -5 \\ 3 & -8 & 9 \end{pmatrix}^{-1}$$

$$I-2A = \begin{pmatrix} 22 & -3 & -9 \\ -3 & 0 & 1 \\ -10 & 1 & 4 \end{pmatrix}$$

$$-2A = \begin{pmatrix} 22 & -3 & -9 \\ -3 & 0 & 1 \\ -10 & 1 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$-2A = \begin{pmatrix} 21 & -3 & -9 \\ -3 & -1 & 1 \\ -10 & 1 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} -10\frac{1}{2} & 1\frac{1}{2} & 4\frac{1}{2} \\ 1\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 5 & -\frac{1}{2} & -1\frac{1}{2} \end{pmatrix}$$

□

Problem 4 Let

$$A = \begin{pmatrix} 1 & 0 & 3 \\ -2 & 0 & -5 \\ 0 & 2 & 0 \end{pmatrix}.$$

Write A as a product of elementary matrices.

Work.

$$\begin{pmatrix} 1 & 0 & 3 \\ -2 & 0 & -5 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{R_2+2R_1} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{R_1-3R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Each of these row operations can be expressed as a left multiplication of an elementary matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ -2 & 0 & -5 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ -2 & 0 & -5 \\ 0 & 2 & 0 \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ -2 & 0 & -5 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ -2 & 0 & -5 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

□

Problem 5 Find the conditions on $b_1, b_2,$ and b_3 such that the system

$$\begin{aligned} 1x_1 - 1x_2 + 1x_3 &= b_1 \\ -4x_1 + 7x_2 + 2x_3 &= b_2 \\ -2x_1 + 3x_2 + 0x_3 &= b_3 \end{aligned}$$

is consistent.

Work. The system of linear equations can be represented as an augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ -4 & 7 & 2 & b_2 \\ -2 & 3 & 0 & b_3 \end{array}\right)$$

Reducing this augmented matrix will provide the constraints on $b_1, b_2,$ and b_3 to make the system consistent.

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ -4 & 7 & 2 & b_2 \\ -2 & 3 & 0 & b_3 \end{array}\right) \xrightarrow[R_3+2R_1]{R_2+4R_1} \begin{pmatrix} 1 & -1 & 1 & b_1 \\ 0 & 3 & 6 & b_2+4b_1 \\ 0 & 1 & 2 & b_3+2b_1 \end{pmatrix}$$

$$\xrightarrow{R_2-3R_3} \begin{pmatrix} 1 & -1 & 1 & b_1 \\ 0 & 0 & 0 & b_2+4b_1-3b_3-6b_1 \\ 0 & 1 & 2 & b_3+2b_1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & -1 & 1 & b_1 \\ 0 & 1 & 2 & b_3+2b_1 \\ 0 & 0 & 0 & b_2-2b_1-3b_3 \end{pmatrix}$$

The last row represents the equation $0 = b_2 - 2b_1 - 3b_3$. If $0 = b_2 - 2b_1 - 3b_3$, then the augmented matrix can be put into reduced row echelon form without any leading 1's in the final column, and the matrix will be consistent. However, if $0 \neq b_2 - 2b_1 - 3b_3$, then the last row will have a leading entry in the last column, and thus there will no solution, meaning the system is not consistent. $\square$

Problem 7 Prove that for all $n \times n$ matrices A, the matrix $A^T A + 2AA^T$ is symmetric.

Proof. Consider square matrix A, and the matrix $A^T A + 2AA^T$. Now consider the transpose of this matrix, $(A^T A + 2AA^T)^T$.

$$\begin{aligned} (A^T A + 2AA^T)^T &= (A^T A)^T + (2AA^T)^T && \text{by properties of transpose} \\ &= A^T (A^T)^T + 2(A^T)^T A^T && \text{by properties of transpose} \\ (A^T A + 2AA^T)^T &= A^T A + 2AA^T \end{aligned}$$

Since $(A^T A + 2AA^T)^T = A^T A + 2AA^T$, by definition, $A^T A + 2AA^T$ is a symmetric matrix, for all $A_{n \times n}$. $\square$