Homework 8

2.1

3. Let

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

Find the following:

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a. M_{13} and C_{13}.

Work.

$$M_{13} = \begin{vmatrix} 0 & 0 & 3 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = 3 \cdot (-1)^{1+3} \begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix} = 3 \cdot 1 \cdot 0 = 0$$

$$C_{13} = (-1)^{1+3} \cdot M_{13} = 1 \cdot 0 = 0$$

□

- -

b. M_{23} and C_{23}.

Work.

$$M_{23} = \begin{vmatrix} 4 & -1 & 6 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix}$$

$$= 4 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - 1 \cdot (-1)^{1+2} \begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 6 \cdot (-1)^{1+3} \begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix}$$

$$= 4 \cdot -12 - (-1) \cdot (-48) + 6 \cdot 0$$

$$M_{23} = -48 - 48 + 0 = -96$$

$$C_{23} = (-1)^{2+3} M_{23} = -1 \cdot -96 = 96$$

□

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11. Use the arrow technique to evaluate the determinant of the given matrix.

$$\begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{bmatrix}$$

Work.

$$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = ((-2 \cdot 5 \cdot 2) + (1 \cdot -7 \cdot 1) + (4 \cdot 3 \cdot 6)) - ((-2 \cdot -7 \cdot 6) + (1 \cdot 3 \cdot 2) + (4 \cdot 5 \cdot 1))$$

$$= ((-20) + (-7) + (72)) - ((84) + (6) + (20))$$
$$= (45) - (110)$$
$$= -65$$

□

18. Find all values of λ for which $\det(A) = 0$.

$$A = \begin{bmatrix} \lambda - 4 & 4 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda - 5 \end{bmatrix}$$

Work.

$$\begin{vmatrix} \lambda - 4 & 4 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda - 5 \end{vmatrix} = (\lambda - 5) \cdot (-1)^{3+3} \begin{vmatrix} \lambda - 4 & 4 \\ -1 & \lambda \end{vmatrix} = (\lambda - 5) \cdot (\lambda(\lambda - 4) - (4 \cdot -1))$$

$$= (\lambda - 5) \cdot (\lambda^2 - 4\lambda + 4) = (\lambda - 5) \cdot (\lambda - 2)^2$$

$$\lambda = 5,\ 2$$

□

21. Evaluate $\det(A)$ by a cofactor expansion along a row or column of your choice.

$$A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$$

Work.

$$\det(A) = 0 + 5 \cdot (-1)^{2+2} \begin{vmatrix} -3 & 7 \\ -1 & 5 \end{vmatrix} + 0 = 5 \cdot 1 \cdot -8 = -40$$

□

31. Evaluate the determinant of the given matrix by inspection.

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Work.

$$\begin{vmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 1 \cdot 1 \cdot 2 \cdot 3 = 6 \qquad\qquad \text{via Theorem 1 of Lecture Notes 31}$$

□

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38. What is the maxiumum number of zeros that a $3{\times}3$ matrix can have without having a zero determinant? Explain your reasoning.

Proof. Consider $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We know that $\det I = 1 \neq 0$. This means that 6 zeros can be achieved without having a zero determinant.

If $A_{3\times 3}$ has 7 zero entries, then A has at most 2 non-zero entries. Since there are 3 rows, there is a row of all zeros. Expanding along this row to compute the determinant will produce an expansion with coefficients of all zeros, meaning that $\det A = 0$. Therefore, 6 zeros is the maximum number of zeros that a 3×3 matrix can have without having a zero determinant. □

2.2

14. Evaluate the determinant of the given matrix by reducing the matrix to row echelon form.

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{bmatrix}$$

Work.

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} \underset{R_2-5R_1}{=\!=\!=} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} \underset{R_3+R_1}{=\!=\!=} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 2 & 8 & 6 & 1 \end{vmatrix} \underset{R_4-2R_1}{=\!=\!=}$$

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{vmatrix} \underset{R_4-12R_2}{=\!=\!=} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{vmatrix} \underset{R_4+36R_2}{=\!=\!=} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -13 \end{vmatrix} \underset{=\!=\!=}{\overset{-\frac{1}{3}R_3}{}} -3$$

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & -13 \end{vmatrix} \underset{=\!=\!=}{\overset{-\frac{1}{13}R_4}{}} -13 \cdot -3 \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{vmatrix} = -13 \cdot -3 \cdot 1 = 39$$

□

27. Evaluate the determinant given that

$$
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{vmatrix} = -6.
$$

$$
\begin{vmatrix}
-3a & -3b & -3c \\
d & e & f \\
g - 4d & h - 4e & i - 4f
\end{vmatrix}
$$

Work.

$$
\begin{vmatrix}
-3a & -3b & -3c \\
d & e & f \\
g - 4d & h - 4e & i - 4f
\end{vmatrix}
\overset{R_3 + 4R_2}{=\!=\!=}
\begin{vmatrix}
-3a & -3b & -3c \\
d & e & f \\
g & h & i
\end{vmatrix}
\overset{-\frac{1}{3}R_1}{=\!=\!=} -3
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{vmatrix} = -3 \cdot -6 = 18
$$

□

29. Use row reduction to show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$

Work.

$$
\begin{vmatrix}
1 & 1 & 1 \\
a & b & c \\
a^2 & b^2 & c^2
\end{vmatrix}
\overset{R_2 - aR_1}{=\!=\!=}
\begin{vmatrix}
1 & 1 & 1 \\
0 & b - a & c - a \\
a^2 & b^2 & c^2
\end{vmatrix}
\overset{R_3 - a^2 R_1}{=\!=\!=}
\begin{vmatrix}
1 & 1 & 1 \\
0 & b - a & c - a \\
0 & b^2 - a^2 & c^2 - a^2
\end{vmatrix}
\overset{R_3 - (b+a)R_2}{=\!=\!=}
$$

$$
\begin{vmatrix}
1 & 1 & 1 \\
0 & b - a & c - a \\
0 & 0 & c^2 - a^2 - (b+a)(c-a)
\end{vmatrix}
=
\begin{vmatrix}
1 & 1 & 1 \\
0 & b - a & c - a \\
0 & 0 & (c+a)(c-a) - (b+a)(c-a)
\end{vmatrix}
=
$$

$$
\begin{vmatrix}
1 & 1 & 1 \\
0 & b - a & c - a \\
0 & 0 & (c-a)(c-b)
\end{vmatrix}
= 1 \cdot (b-a) \cdot (c-a)(c-b) = (b-a)(c-a)(c-b)
$$

□

30. Confirm without evaluating the determinant directly:

$$
\begin{vmatrix}
a_1 + b_1 t & a_2 + b_2 t & a_3 + b_3 t \\
a_1 t + b_1 & a_2 t + b_2 & a_3 t + b_3 \\
c_1 & c_2 & c_3
\end{vmatrix} = (1 - t^2)
\begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{vmatrix}
$$

Work.

$$\begin{vmatrix} a_1 + b_1 t & a_2 + b_2 t & a_3 + b_3 t \\ a_1 t + b_1 & a_2 t + b_2 & a_3 t + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \overset{R_1 - R_2}{=\!=\!=} \begin{vmatrix} a_1 + b_1 t - a_1 t - b_1 & a_2 + b_2 t - a_2 t - b_2 & a_3 + b_3 t - a_3 t + b_3 \\ a_1 t + b_1 & a_2 t + b_2 & a_3 t + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$\begin{vmatrix} a_1 - b_1 + b_1 t - a_1 t & a_2 - b_2 + b_2 t - a_2 t & a_3 + b_3 + b_3 t - a_3 t \\ a_1 t + b_1 & a_2 t + b_2 & a_3 t + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$\begin{vmatrix} -1(b_1 - a_1) + t(b_1 - a_1) & -1(b_2 - a_2) + t(b_2 - a_2) & -1(b_3 - a_3) + t(b_3 - a_3) \\ a_1 t + b_1 & a_2 t + b_2 & a_3 t + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$\begin{vmatrix} (t-1)(b_1 - a_1) & (t-1)(b_2 - a_2) & (t-1)(b_3 - a_3) \\ a_1 t + b_1 & a_2 t + b_2 & a_3 t + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (t-1) \begin{vmatrix} b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ a_1 t + b_1 & a_2 t + b_2 & a_3 t + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \overset{R_2 - R_1}{=\!=\!=}$$

$$(t-1) \begin{vmatrix} b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ a_1 t + b_1 - b_1 + a_1 & a_2 t + b_2 - b_2 + a_2 & a_3 t + b_3 - b_3 + a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$(t-1) \begin{vmatrix} b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ a_1 t + a_1 & a_2 t + a_2 & a_3 t + a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (t-1) \begin{vmatrix} b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ a_1(t+1) & a_2(t+1) & a_3(t+1) \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$(t-1)(t+1) \begin{vmatrix} b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (t^2 - 1) \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1 - t^2) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

□

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34. Find the determinant of the following matrix:

$$\begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$$

Work.

$$\begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix} = a^4 - 3b^4 - 6a^2 b^2 + 8ab^3$$

For full derivation, see attached sheet.

□

2.3

19. Decide whether the matrix is invertible, and if so, use the adjoint method to find its inverse.

$$A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$$

Work.

$$\det(A) = 2 \cdot (1) \cdot (-1 \cdot 3 - 0 \cdot 4) + (-1) \cdot (-1) \cdot (5 \cdot 3 - 5 \cdot 4) + 2 \cdot (1) \cdot (5 \cdot 0 - 5 \cdot (-1))$$
$$= 2(-3) + 1(-5) + 2(10) = -6 - 5 + 10 = -1$$

Since $\det(A) \neq 0$, by the Big Theorem, A is invertible. To find the inverse via the adjoint method, first find the cofactors of A.

$$C_{11} = (-3 - 0) = -3 \qquad\qquad C_{12} = -(-3 - 0) = 3 \qquad\qquad C_{13} = (-4 + 2) = -2$$
$$C_{21} = -(0 - 5) = 5 \qquad\qquad C_{22} = (6 - 10) = -4 \qquad\qquad C_{23} = -(8 - 10) = 2$$
$$C_{31} = (0 + 5) = 5 \qquad\qquad C_{32} = -(0 + 5) = -5 \qquad\qquad C_{33} = (-2 + 3) = 3$$

According to Theorem 9 of Lecture Notes 32,

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

$$A^{-1} = -1 \cdot \begin{bmatrix} -3 & 3 & -2 \\ 5 & -4 & 2 \\ 5 & -5 & 3 \end{bmatrix}^T$$

$$A^{-1} = -1 \cdot \begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{bmatrix}$$

$\square$

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27. Solve by Cramer's rule, where it applies:

$$1x_1 - 3x_2 + 1x_3 = 4$$
$$2x_1 - 1x_2 + 0x_3 = -2$$
$$4x_1 + 0x_2 - 3x_3 = 0$$

Work.

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & -1 & 0 \\ 4 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$$

According to Cramer's rule, $x_1 = \frac{\det A_1}{\det A}, x_2 = \frac{\det A_2}{\det A}$, and $x_3 = \frac{\det A_3}{A}$, where A_i is the matrix obtained by replacing the i-th column of A by $\vec{b}$.

$$x_1 = \frac{\det A_1}{\det A} = \frac{30}{-11} = -\frac{30}{11}$$
$$x_2 = \frac{\det A_2}{\det A} = \frac{38}{-11} = -\frac{38}{11}$$
$$x_3 = \frac{\det A_3}{\det A} = \frac{-40}{-11} = \frac{40}{11}$$

$\square$

33. Prove that if $\det(A) = 1$ and all the entries in A are integers, then all the entries in A^{-1} are integers.

Proof. Consider matrix A where $\det(A) = 1$ and all the entries in A are integers. Lets look at the adjoint formula for the inverse.

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

We know that $\det(A) = 1$, so $\frac{1}{\det(A)}$ is also 1. Since 1 is an integer, if all of entries of the adjacency matrix for A are integers, then all of the entries for A^{-1} will be integers. Lets look at the adjoint matrix formula.

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$$

If all of these cofactor entries of $\text{adj}(A)$ are integers, then all of the entries of A^{-1} will be integers. Consider how these cofactor entries are calculated, and which operations they use. A quick inspection reveals that they use only the following operations; $+, -,$ and $\cdot$. All of these operations are operations that keep integers as integers. This means that if all of the entries of A are integers, then all of the cofactors of the adjacency matrix for A will also be integers. Finally, this implies that all of the entries of A^{-1} will also be integers, so long as $\det(A) = 1$. $\qquad \square$