2.2.2 Prove each statement by exhaustion

a. For every integer n such that $0 \leq n < 2$, $(n+1)^2 > n^3$

Proof. Let $n \in \mathbb{Z}$ such that $0 \leq n < 2$,

$$\begin{aligned} n = 0: &\quad (0+1)^2 = 1 > 0 = 0^3 \checkmark \\ n = 1: &\quad (1+1)^2 = 4 > 1 = 1^3 \checkmark \\ n = 2: &\quad (2+1)^2 = 9 > 8 = 2^3 \checkmark \end{aligned}$$

$$\therefore \forall n \in \mathbb{Z} \text{ such that } 0 \leq n < 2, (n+1)^2 > n^3$$

□

2.2.3 Find a counter example

b. If n is an integer and n^2 is divisible by 4, then n is divisible by 4.

Counter example: Consider $n = 2$. $n^2, 4$ is divisible by 4, but 2 is not.

e. The multiplicative inverse of $x \in \mathbb{R}$ is a real number y such that $xy = 1$. Every real number has a multiplicative inverse.

Counter example: Consider $x = 0$. $\forall y \in \mathbb{R}, xy \neq 1$. 0 has no multiplicative inverse.

2.2.5 Proving existential statements

a. There are positive integers x and y such that $\frac{1}{x} + \frac{1}{y}$ is an integer.

Proof. Consider $x = y = 1$. $\frac{1}{x} = 1$ and $\frac{1}{y} = 1$ and $1 + 1 \in \mathbb{Z}$. □

c. There are integers m and n such that $\sqrt{m+n} = \sqrt{m} + \sqrt{n}$.

Proof. Consider $m = n = 0$. $\sqrt{0+0} = \sqrt{0} + \sqrt{0}$. □

h. $\forall x, y \in \mathbb{R}, \exists z \in \mathbb{R}$ such that $x - z = z - y$.

Proof. Consider $z = \frac{x+y}{2}$,

$$\begin{aligned} x - \frac{x+y}{2} &= \frac{x+y}{2} - y &\qquad x + y &= 2\left(\frac{x+y}{2}\right) \\ x + y &= x + y &\qquad 0 &= 0 \end{aligned}$$

$$\therefore \forall x, y \in \mathbb{R}, \exists z \in \mathbb{R} \text{ such that } x - z = z - y$$

□