Homework 6

1.2

Determine the values for which the system has no solutions, exactly one solution, or infinitely many solutions

25. $$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array}$$

work.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right) \xrightarrow[(-4,-8,12,-16)]{R_3-4R_1} \begin{pmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 0 & -7 & a^2-2 & a-14 \end{pmatrix} \xrightarrow[(-3,-6,9,-12)]{R_2-3R_1}$$
$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{pmatrix} \xrightarrow[(0,7,-14,10)]{R_3-R_2,-\frac{1}{7}R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{array}\right)$$

R_3 represents the equation $(a^2-16)z = a-4$. If $a=4$, there are infinitely many solutions, since R_3 leads to a full row of 0's. If $a=-4$, then there is no solution, since R_3 leads to $0=-8$. If $a \neq \pm 4$, then there is exactly one solution to the system of equations. ☐

27. $$\begin{array}{rcrcl} x & + & 2y & = & 1 \\ 2x & + & (a^2-5)y & = & a-1 \end{array}$$

work.

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & a^2-5 & a-1 \end{array}\right) \xrightarrow[(-2,-4,-2)]{R_2-2R_1} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & a^2-9 & a-3 \end{array}\right)$$

R_2 represents the equation $(a^2-9)y = a-3$. If $a=3$, there are infinitely many solutions, since R_2 leads to a full row of 0's. If $a=-3$, then there is no solution, since R_2 leads to $0=-6$. If $a \neq \pm 3$, then there is exactly one solution to the system of equations. ☐

32. *Reduce* $\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$ *to rref without introducing fractions at any intermediate stage.*

work.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[(-2,-1,-3)]{R_3-R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[(-2,-6,-4)]{R_1-2R_3} \begin{bmatrix} 0 & -5 & -1 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[(0,2,29)]{R_1-R_2}$$
$$\begin{bmatrix} 0 & -3 & 28 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{bmatrix} 0 & -3 & 28 \\ 0 & 1 & -57 \\ 1 & 0 & 30 \end{bmatrix} \xrightarrow[(0,3,-171)]{R_1+3R_2} \begin{bmatrix} 0 & 0 & -143 \\ 0 & 1 & -57 \\ 1 & 0 & 30 \end{bmatrix} \xrightarrow[R_2+57R_1,R_3-30R_1]{-\frac{1}{143}R_1,R_1\leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

☐

1.3

5h. *Calculate* $(C^T B)A^T$, *where* $A = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}$.

work.

$$\begin{pmatrix}1&4&2\\3&1&5\end{pmatrix} \xrightarrow{C^T} \begin{pmatrix}1&3\\4&1\\2&5\end{pmatrix} \xrightarrow{C^TB} \begin{pmatrix}1\cdot4+3\cdot0 & 1\cdot-1+3\cdot2\\4\cdot4+1\cdot0 & 4\cdot-1+1\cdot2\\2\cdot4+5\cdot0 & 2\cdot-1+5\cdot2\end{pmatrix} = \begin{pmatrix}4&5\\16&-2\\8&8\end{pmatrix} \xrightarrow{C^TBA^T}$$

$$\begin{pmatrix}4&5\\16&-2\\8&8\end{pmatrix}\begin{pmatrix}3&-1&1\\0&2&1\end{pmatrix} \xrightarrow{C^TBA^T} \begin{pmatrix}4\cdot3+5\cdot0 & 4\cdot-1+5\cdot2 & 4\cdot1+5\cdot1\\16\cdot3-2\cdot0 & 16\cdot-1-2\cdot2 & 16\cdot1-2\cdot1\\8\cdot3+8\cdot0 & 8\cdot-1+8\cdot2 & 8\cdot1+8\cdot1\end{pmatrix} = \begin{pmatrix}12&6&9\\48&-20&14\\24&8&16\end{pmatrix}$$

□

10. $A = \begin{bmatrix}3&-2&7\\6&5&4\\0&4&9\end{bmatrix}$, $B = \begin{bmatrix}6&-2&4\\0&1&3\\7&7&5\end{bmatrix}$, $AB = \begin{bmatrix}67&41&41\\64&21&59\\63&67&57\end{bmatrix}$, *and* $BA = \begin{bmatrix}6&-6&70\\6&17&31\\63&41&122\end{bmatrix}$.

a. *express each column vector of AB as a linear combination of the column vectors of A.*

work. Since each column vector of AB is computed using a row of A and a column of B, the linear combination will simply be the corresponding column of B.

1. $6\begin{pmatrix}3\\6\\0\end{pmatrix} + 0\begin{pmatrix}-2\\5\\4\end{pmatrix} + 7\begin{pmatrix}7\\4\\9\end{pmatrix} = \begin{pmatrix}67\\64\\63\end{pmatrix}$.
2. $-2\begin{pmatrix}3\\6\\0\end{pmatrix} + 1\begin{pmatrix}-2\\5\\4\end{pmatrix} + 7\begin{pmatrix}7\\4\\9\end{pmatrix} = \begin{pmatrix}41\\21\\67\end{pmatrix}$.
3. $4\begin{pmatrix}3\\6\\0\end{pmatrix} + 3\begin{pmatrix}-2\\5\\4\end{pmatrix} + 5\begin{pmatrix}7\\4\\9\end{pmatrix} = \begin{pmatrix}41\\59\\57\end{pmatrix}$.

□

b. *express each column vector of BA as a linear combination of the column vectors of B.*

work. Since each column vector of BA is computed using a row of B and a column of A, the linear combination will simply be the corresponding column of A.

1. $3\begin{pmatrix}6\\0\\7\end{pmatrix} + 6\begin{pmatrix}-2\\1\\7\end{pmatrix} + 0\begin{pmatrix}4\\3\\5\end{pmatrix} = \begin{pmatrix}6\\6\\63\end{pmatrix}$.
2. $-2\begin{pmatrix}6\\0\\7\end{pmatrix} + 5\begin{pmatrix}-2\\1\\7\end{pmatrix} + 4\begin{pmatrix}4\\3\\5\end{pmatrix} = \begin{pmatrix}-6\\17\\41\end{pmatrix}$.
3. $7\begin{pmatrix}6\\0\\7\end{pmatrix} + 4\begin{pmatrix}-2\\1\\7\end{pmatrix} + 9\begin{pmatrix}4\\3\\5\end{pmatrix} = \begin{pmatrix}70\\31\\122\end{pmatrix}$.

□

15. *Find all values of k, if any, that satisfy* $\begin{bmatrix}k&1&1\end{bmatrix}\begin{bmatrix}1&1&0\\1&0&2\\0&2&-3\end{bmatrix}\begin{bmatrix}k\\1\\1\end{bmatrix}$.

work.

$$\begin{bmatrix}k&1&1\end{bmatrix}\begin{bmatrix}1&1&0\\1&0&2\\0&2&-3\end{bmatrix}\begin{bmatrix}k\\1\\1\end{bmatrix} = \begin{bmatrix}k+1 & k+2 & -1\end{bmatrix}\begin{bmatrix}k\\1\\1\end{bmatrix} = [k(k+1)+k+2-1] = \begin{bmatrix}(k+1)^2\end{bmatrix} = [0]$$

$$k+1 = 0$$
$$k = -1$$

$\square$

22.

- ***a.*** *Show that if A has a row of zeros and B is any matrix for which AB is defined, then AB also has a row of zeros.*
- ***b.*** *Find a similar result involving a column of zeros.*

24. *Find the 4×4 matrix $A = [a_{ij}]$ whose entries satisfy the stated condition.*

a. $a_{ij} = i + j$

work.

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

$\square$

b. $a_{ij} = i^{j-1}$

work.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

$\square$

c. $a_{ij} = \begin{cases} 1 & \text{if } |i-j| > 1 \\ -1 & \text{if } |i-j| \leq 1 \end{cases}$

work.

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$\square$

27. *How many 3×3 matrices A can you find such that*

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ 0 \end{bmatrix}$$

for all choices of $x, y,$ and z?

1.4

17. *Use the given information to find A: $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$.*

work.

$$\begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix} \xrightarrow[I+2A]{(I+2A)^{-1^{-1}}} \frac{1}{-13} \begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{13} & \frac{2}{13} \\ \frac{4}{13} & \frac{1}{13} \end{bmatrix} \xrightarrow[2A]{I+2A-I} \begin{bmatrix} -\frac{18}{13} & \frac{2}{13} \\ \frac{4}{13} & -\frac{12}{13} \end{bmatrix} \xrightarrow[A]{\frac{1}{2}\cdot 2A} \begin{bmatrix} -\frac{9}{13} & \frac{1}{13} \\ \frac{2}{13} & -\frac{6}{13} \end{bmatrix}$$

$\square$

19e. *Given* $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, *compute* $p(A)$, *where* $p(x) = 2x^2 - x + 1$.

work.

$$p(A) = 2A^2 - A + I = 2\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^2 + \begin{bmatrix} -3 & -1 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2\begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ -2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 22 & 8 \\ 16 & 6 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 7 \\ 14 & 6 \end{bmatrix}$$

□

27. *Consider the matrix*

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

where $a_{11} \cdot a_{22} \cdots a_{nn} \neq 0$. *Show that* A *is invertible and find its inverse.*

Proof. Consider $B = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$.

$$AB = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$
$$BA = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

Since $AB = I = BA$, therefore $B = A^{-1}$, the inverse of A. This also proves that A is invertible. □