Homework 6

1.2

Determine the values for which the system has no solutions, exactly one solution, or infinitely many solutions

25.
$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array}$$

work.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right) \xrightarrow[(-4,-8,12,-16)]{R_3-4R_1} \begin{pmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 0 & -7 & a^2-2 & a-14 \end{pmatrix} \xrightarrow[(-3,-6,9,-12)]{R_2-3R_1}$$
$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{pmatrix} \xrightarrow[(0,7,-14,10)]{R_3-R_2,-\frac{1}{7}R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{array}\right)$$

R_3 represents the equation $(a^2-16)z = a-4$. If $a = 4$, there are infinitely many solutions, since R_3 leads to a full row of 0's. If $a = -4$, then there is no solution, since R_3 leads to $0 = -8$. If $a \neq \pm 4$, then there is exactly one solution to the system of equations. □

27.
$$\begin{array}{rcrcl} x & + & 2y & = & 1 \\ 2x & + & (a^2-5)y & = & a-1 \end{array}$$

work.

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & a^2-5 & a-1 \end{array}\right) \xrightarrow[(-2,-4,-2)]{R_2-2R_1} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & a^2-9 & a-3 \end{array}\right)$$

R_2 represents the equation $(a^2-9)y = a-3$. If $a = 3$, there are infinitely many solutions, since R_2 leads to a full row of 0's. If $a = -3$, then there is no solution, since R_2 leads to $0 = -6$. If $a \neq \pm 3$, then there is exactly one solution to the system of equations. □

32. *Reduce* $\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$ *to rref without introducing fractions at any intermediate stage.*

work.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[(-2,-1,-3)]{R_3-R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[(-2,-6,-4)]{R_1-2R_3} \begin{bmatrix} 0 & -5 & -1 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[(0,2,29)]{R_1-R_2}$$
$$\begin{bmatrix} 0 & -3 & 28 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{bmatrix} 0 & -3 & 28 \\ 0 & 1 & -57 \\ 1 & 0 & 30 \end{bmatrix} \xrightarrow[(0,3,-171)]{R_1+3R_2} \begin{bmatrix} 0 & 0 & -143 \\ 0 & 1 & -57 \\ 1 & 0 & 30 \end{bmatrix} \xrightarrow[R_2+57R_1,R_3-30R_1]{-\frac{1}{143}R_1,R_1\leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

1.3

5h. *Calculate* $(C^TB)A^T$*, where* $A = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}$.

work.

$$\begin{pmatrix}1 & 4 & 2\\ 3 & 1 & 5\end{pmatrix} \xrightarrow{C^T} \begin{pmatrix}1 & 3\\ 4 & 1\\ 2 & 5\end{pmatrix} \xrightarrow{C^TB} \begin{pmatrix}1\cdot 4+3\cdot 0 & 1\cdot -1+3\cdot 2\\ 4\cdot 4+1\cdot 0 & 4\cdot -1+1\cdot 2\\ 2\cdot 4+5\cdot 0 & 2\cdot -1+5\cdot 2\end{pmatrix} = \begin{pmatrix}4 & 5\\ 16 & -2\\ 8 & 8\end{pmatrix} \xrightarrow{C^TBA^T}$$

$$\begin{pmatrix}4 & 5\\ 16 & -2\\ 8 & 8\end{pmatrix}\begin{pmatrix}3 & -1 & 1\\ 0 & 2 & 1\end{pmatrix} \xrightarrow{C^TBA^T} \begin{pmatrix}4\cdot 3+5\cdot 0 & 4\cdot -1+5\cdot 2 & 4\cdot 1+5\cdot 1\\ 16\cdot 3-2\cdot 0 & 16\cdot -1-2\cdot 2 & 16\cdot 1-2\cdot 1\\ 8\cdot 3+8\cdot 0 & 8\cdot -1+8\cdot 2 & 8\cdot 1+8\cdot 1\end{pmatrix} = \begin{pmatrix}12 & 6 & 9\\ 48 & -20 & 14\\ 24 & 8 & 16\end{pmatrix}$$

□

10. $A = \begin{bmatrix}3 & -2 & 7\\ 6 & 5 & 4\\ 0 & 4 & 9\end{bmatrix}$, $B = \begin{bmatrix}6 & -2 & 4\\ 0 & 1 & 3\\ 7 & 7 & 5\end{bmatrix}$, $AB = \begin{bmatrix}67 & 41 & 41\\ 64 & 21 & 59\\ 63 & 67 & 57\end{bmatrix}$, *and* $BA = \begin{bmatrix}6 & -6 & 70\\ 6 & 17 & 31\\ 63 & 41 & 122\end{bmatrix}$.

a. *express each column vector of AB as a linear combination of the column vectors of A.*

work. Since each column vector of AB is computed using a row of A and a column of B, the linear combination will simply be the corresponding column of B.

1. $6\begin{pmatrix}3\\6\\0\end{pmatrix} + 0\begin{pmatrix}-2\\5\\4\end{pmatrix} + 7\begin{pmatrix}7\\4\\9\end{pmatrix} = \begin{pmatrix}67\\64\\63\end{pmatrix}$.
2. $-2\begin{pmatrix}3\\6\\0\end{pmatrix} + 1\begin{pmatrix}-2\\5\\4\end{pmatrix} + 7\begin{pmatrix}7\\4\\9\end{pmatrix} = \begin{pmatrix}41\\21\\67\end{pmatrix}$.
3. $4\begin{pmatrix}3\\6\\0\end{pmatrix} + 3\begin{pmatrix}-2\\5\\4\end{pmatrix} + 5\begin{pmatrix}7\\4\\9\end{pmatrix} = \begin{pmatrix}41\\59\\57\end{pmatrix}$.

□

b. *express each column vector of BA as a linear combination of the column vectors of B.*

work. Since each column vector of BA is computed using a row of B and a column of A, the linear combination will simply be the corresponding column of A.

1. $3\begin{pmatrix}6\\0\\7\end{pmatrix} + 6\begin{pmatrix}-2\\1\\7\end{pmatrix} + 0\begin{pmatrix}4\\3\\5\end{pmatrix} = \begin{pmatrix}6\\6\\63\end{pmatrix}$.
2. $-2\begin{pmatrix}6\\0\\7\end{pmatrix} + 5\begin{pmatrix}-2\\1\\7\end{pmatrix} + 4\begin{pmatrix}4\\3\\5\end{pmatrix} = \begin{pmatrix}-6\\17\\41\end{pmatrix}$.
3. $7\begin{pmatrix}6\\0\\7\end{pmatrix} + 4\begin{pmatrix}-2\\1\\7\end{pmatrix} + 9\begin{pmatrix}4\\3\\5\end{pmatrix} = \begin{pmatrix}70\\31\\122\end{pmatrix}$.

□

15. *Find all values of k, if any, that satisfy* $\begin{bmatrix}k & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 2\\ 0 & 2 & -3\end{bmatrix}\begin{bmatrix}k\\1\\1\end{bmatrix}$.

work.

$$\begin{bmatrix}k & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 2\\ 0 & 2 & -3\end{bmatrix}\begin{bmatrix}k\\1\\1\end{bmatrix} = \begin{bmatrix}k+1 & k+2 & -1\end{bmatrix}\begin{bmatrix}k\\1\\1\end{bmatrix} = [k(k+1)+k+2-1] = \left[(k+1)^2\right] = [0]$$

$$k+1 = 0$$
$$k = -1$$

$\square$

22.

a. *Show that if A has a row of zeros and B is any matrix for which AB is defined, then AB also has a row of zeros.*

Proof. Consider

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} B_{m\times k} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mk} \end{bmatrix}.$$

AB can be expressed as a series of row vectors. These row vectors can be expressed as linear combinations from numbers of the row vectors from A, and the corresponding row of B. Here is an example:

$$\begin{aligned} a_{11}\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \end{bmatrix} + a_{12}\begin{bmatrix} b_{21} & b_{22} & \cdots & b_{2k} \end{bmatrix} + \cdots + a_{1m}\begin{bmatrix} b_{m1} & b_{m2} & \cdots & b_{mk} \end{bmatrix} &= \quad (1) \\ = \begin{bmatrix} ab_{11} & ab_{11} & \cdots & ab_{1k} \end{bmatrix} & \quad (2) \end{aligned}$$

Since there is a row of zeros, one of these equations will be:

$$0\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \end{bmatrix} + 0\begin{bmatrix} b_{21} & b_{22} & \cdots & b_{2k} \end{bmatrix} + \cdots + 0\begin{bmatrix} b_{m1} & b_{m2} & \cdots & b_{mk} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}.$$

This represents a row of all zeros, which is contained within AB, at the same row at which it occurs in A. Therefore, if A has a row of zeros and B is any matrix for which AB is defined, then AB also has a row of zeros. $\square$

b. *Find a similar result involving a column of zeros.*

Proof. Consider

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{12} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} B_{m\times k} = \begin{bmatrix} b_{11} & b_{12} & \cdots & 0 & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & 0 & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & 0 & \cdots & b_{mk} \end{bmatrix}.$$

AB can be expressed as a series of column vectors. These column vectors can be expressed as linear combinations from numbers of the column vectors from B, and the corresponding column of A. Here is an example:

$$b_{11}\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + b_{21}\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{2m} \end{bmatrix} + \cdots + b_{m1}\begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} ab_{11} \\ ab_{21} \\ \vdots \\ ab_{n1} \end{bmatrix}.$$

Since there is a row of zeros, one of these equations will be:

$$0\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + 0\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{2m} \end{bmatrix} + \cdots + 0\begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This represents a column of all zeros, which is contained within AB, at the same column at which it occurs in A. Therefore, if B has a column of zeros and A is any matrix for which AB is defined, then AB also has a column of zeros. $\square$

24. *Find the 4×4 matrix $A = [a_{ij}]$ whose entries satisfy the stated condition.*

a. $a_{ij} = i + j$

work.

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

□

b. $a_{ij} = i^{j-1}$

work.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

□

c. $a_{ij} = \begin{cases} 1 & \textit{if } |i-j| > 1 \\ -1 & \textit{if } |i-j| \le 1 \end{cases}$

work.

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

□

27. *How many 3×3 matrices A can you find such that*

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ 0 \end{bmatrix}$$

for all choices of $x, y,$ and z?

1.4

17. *Use the given information to find A: $(I+2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$.*

work.

$$\begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix} \xrightarrow[I+2A]{(I+2A)^{-1^{-1}}} \frac{1}{-13} \begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{13} & \frac{2}{13} \\ \frac{4}{13} & \frac{1}{13} \end{bmatrix} \xrightarrow[2A]{I+2A-I} \begin{bmatrix} -\frac{18}{13} & \frac{2}{13} \\ \frac{4}{13} & -\frac{12}{13} \end{bmatrix} \xrightarrow[A]{\frac{1}{2}\cdot 2A} \begin{bmatrix} -\frac{9}{13} & \frac{1}{13} \\ \frac{2}{13} & -\frac{6}{13} \end{bmatrix}$$

□

19e. *Given $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, compute $p(A)$, where $p(x) = 2x^2 - x + 1$.*

work.

$$p(A) = 2A^2 - A + I = 2\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^2 + \begin{bmatrix} -3 & -1 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2\begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ -2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 22 & 8 \\ 16 & 6 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 7 \\ 14 & 6 \end{bmatrix}$$

□

27. *Consider the matrix*

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

where $a_{11} \cdot a_{22} \cdots a_{nn} \neq 0$. Show that A is invertible and find its inverse.

Proof. Consider $B = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$.

$$AB = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$
$$BA = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

Since $AB = I = BA$, therefore $B = A^{-1}$, the inverse of A. This also proves that A is invertible. □