Problem 1

Let V be a vector space, and let $\vec{u}, \vec{v}, \vec{w} \in V$. Prove that if $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$ then $\vec{u} = \vec{v}$.

Proof. Consider $\vec{v}, \vec{u}, \vec{w} \in V$, and assume $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$.

$$\begin{aligned}
\vec{u} \oplus \vec{w} &= \vec{v} \oplus \vec{w} && \text{Assertion} \\
(\vec{u} \oplus \vec{w}) \oplus -\vec{w} &= (\vec{v} \oplus \vec{w}) \oplus -\vec{w} && \text{Axiom 5 states } -\vec{w} \in V \\
\vec{u} \oplus (\vec{w} \oplus -\vec{w}) &= \vec{v} \oplus (\vec{w} \oplus -\vec{w}) && \text{Axiom 3} \\
\vec{u} \oplus \mathbf{id} &= \vec{v} \oplus \mathbf{id} && \text{Def. of additive inverse} \\
\vec{u} &= \vec{v} && \text{Def. of additive identity}
\end{aligned}$$

$\therefore$ if $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$ then $\vec{u} = \vec{v}$ $\square$

Problem 2

Prove Theorem B.

Proof. Let $k \in \mathbb{R}$. Recall that Theorem A implies that $0 \odot \mathbf{id} = \mathbf{id}$:

$$\begin{aligned}
k \odot \mathbf{id} &= k \odot (0 \odot \mathbf{id}) && \text{Thm } A \\
&= (k \cdot 0) \odot \mathbf{id} && \text{Axiom 9} \\
&= 0 \odot \mathbf{id} \\
&= \mathbf{id} && \text{Thm } A
\end{aligned}$$

$\therefore \forall\, k \in \mathbb{R},\ k \odot \mathbf{id} = \mathbf{id}$ $\square$

Problem 3

Prove Theorem D. If $k \odot \vec{u} = \mathbf{id}$, then $k = 0$ and/or $u = \mathbf{id}$

Proof. Consider $k \odot \vec{u} = \mathbf{id}$ and $k \neq 0$. Since $\frac{1}{k} \neq \frac{1}{0}$, $\frac{1}{k}$ is well-defined.

$$\begin{aligned}
k \odot \vec{u} &= \mathbf{id} && \text{Assertion} \\
\frac{1}{k} \odot k \odot \vec{u} &= \frac{1}{k} \odot \mathbf{id} \\
(\frac{1}{k} \cdot k) \odot \vec{u} &= \frac{1}{k} \odot \mathbf{id} && \text{Axiom 9} \\
1 \odot \vec{u} &= \mathbf{id} && \text{Thm B} \\
\vec{u} &= \mathbf{id} && \text{Axiom 10}
\end{aligned}$$

Now consider $k \odot \vec{u} = \mathbf{id}$ and $\vec{u} \neq \mathbf{id}$. Since k is defined, $-k$ is also defined.

$$\begin{aligned}
k \odot \vec{u} &= \mathbf{id} && \text{Assertion} \\
k \odot \vec{u} &= \mathbf{id} \\
k \odot \vec{u} \oplus (-k) \odot \vec{u} &= \mathbf{id} \oplus (-k) \odot \vec{u} \\
(k + (-k)) \odot \vec{u} &= (-k) \odot \vec{u} && \text{Axiom 8} \\
0 \odot \vec{u} &= (-k) \odot \vec{u}
\end{aligned}$$

Since $u \neq \mathbf{id}$,

$$0 = -k, \quad \therefore k = 0$$

$\square$

Problem 4

Prove that there does not exist a real vector space of size 2. Show that there cannot be a vector space of size 2.

Proof. Let $V = \{\vec{u}, \vec{v}\}$ be a vectorspace. That is, it satisfies all 10 Axioms.
Axiom 4 states that **id** exist, and is unique, therefore either $\vec{u} = \mathbf{id}$ or $\vec{v} = \mathbf{id}$. Both cannot be **id**, so therefore $\vec{u} \neq \vec{v}$
Without the loss of generality, let $\vec{u} = \mathbf{id}$.

$$\begin{aligned} \vec{u} \oplus \vec{v} &= \vec{v} && \text{Axiom 4} \\ \mathbf{id} \oplus \vec{v} &= \vec{v} \end{aligned}$$

Now consider Axiom 5: additive inverse exists for all $\vec{u} \in V$.
Consider $-\vec{v} \oplus \vec{v} = \mathbf{id}$. Since $\vec{v} \neq \mathbf{id}$, $-\vec{v} \neq \vec{v}$. Since there is only one other element in V, $-\vec{v} = \vec{u}$ must be true. Remember that $\vec{u} = \mathbf{id}$.

$$\vec{u} \oplus \vec{v} = \vec{u}$$

Therefore we have from Axiom 4 and 5:

$$\begin{aligned} \vec{u} \oplus \vec{v} &= \vec{u} \\ \vec{u} \oplus \vec{v} &= \vec{v} \\ \therefore \vec{u} &= \vec{v} \end{aligned}$$

However, this contradicts with our assertion that $\vec{u} \neq \vec{v}$.
$\therefore$ A vectorspace of size 2 cannot exist.

□