MAT 283 Calculus III

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11.1 Three-Dimensional Cartesian Space

Cartesian Coordinates in Three Dimensions

The **projection** of a point (x, y, z) in $\mathbb{R}^3$ onto a plane is the point in that plane closest to (x, y, z). The projection of the points constituting a given object onto a coordinate plane is often useful in helping to visualize or better understand the object.

Distance in Three Dimensions

The distance between two points, (x_1, y_1, z_1) and (x_2, y_2, z_2) is found by applying the *Pythagorean Theorem* successively, and is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

11.2 Vectors and Vector Algebra

Vector Terminology and Notation

In two- and three-dimensional space, vectors are often depicted as **directed line segments**. Such a directed line segment begins at an **initial point** P and ends at a **terminal point** Q, and the notation $\overrightarrow{PQ}$ is used to refer to the vector.

A subtle but very important point is that a vector is characterized *entirely* by its direction and magnitude, not by its initial and terminal points.

If a vector $\vec{u}$ is depicted with the origin as its initial point, the vector is said to be in **standard position**. The **component form** of $\vec{u}$ takes the form

$$\vec{u} = \langle u_1, \dots, u_n \rangle$$

Additionally, the length or **norm** of a vector $\vec{u}$ is

$$\|\vec{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}.$$

Vector Algebra

Vectors are added and scaled component-wise. Assume $\vec{u}, \vec{v},$ and $\vec{w}$ represent vectors, while $a, b \in \mathbb{R}$.

Scalar Multiplication Properties	Vector Addition Properties
$a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$	$\vec{u} + \vec{v} = \vec{v} + \vec{u}$
$(a + b)\vec{u} = a\vec{v} + b\vec{u}$	$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
$1\vec{u} = \vec{u};\ 0\vec{u} = \vec{0};\ a\vec{0} = \vec{0}$	$\vec{u} + (-\vec{u}) = \vec{0}$
$\|a\vec{u}\| = |a| \cdot \|\vec{u}\|$	

11.3 The Dot Product

The Dot Product and Its Properties

Given two vectors $\vec{u} = \langle u_1, \dots, u_n \rangle$ and $\vec{v} = \langle v_1, \dots, v_n$, the **dot product**, denoted as $\vec{u} \cdot \vec{v}$, is the scalar defined by

$$\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n$$

Properties of the Dot Product

Assume $\vec{u}, \vec{v},$ and $\vec{w}$ represent vectors, while $a \in \mathbb{R}$.

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \qquad \vec{0} \cdot \vec{u} = \vec{0}$$

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \qquad a(\vec{u} \cdot \vec{v}) = (a\vec{u}) \cdot \vec{v} = \vec{u} \cdot (a\vec{v})$$

$$\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$$

Dot Product and the Angle between Two Vector

If two nonzero vectors $\vec{u}$ and $\vec{v}$ are depicted so that their initial points coincide, and if θ represents the smaller of the two angles formed by $\vec{u}$ and $\vec{v}$, so that $0 \leq \theta \leq \pi$, then

$$\vec{u} \cdot \vec{v} = \|\vec{v}\|\|\vec{u}\| \cos\theta$$

Orthogonal Vectors

Two vectors $\vec{u}$ and $\vec{v}$ are **orthogonal**, **perpendicular**, or **normal**, if $\vec{u} \cdot \vec{v} = 0$

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