Determine whether each set equipped with the given operations is a vector space. If it is a vector space, show that all 10 axioms hold; if not, find ALL axioms that fail.

Problem 13

The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by:

$$k \odot (x, y, z) = (k^2x, k^2y, k^2z)$$

Axiom 1. *Proof.* Let $\vec{v} = (v_1, v_2, v_3), \vec{u} = (u_1, u_2, u_3)$. $\forall\ \vec{v}, \vec{u} \in V$:

$$\begin{gathered}\vec{v} \oplus \vec{u} = (v_1, v_2, v_3) \oplus (u_1, u_2, u_3) = (v_1 + u_1, v_2 + u_2, v_3 + u_3) \\ v_1 + u_1, v_2 + u_2, v_3 + u_3 \in \mathbb{R}\end{gathered}$$

$\therefore \forall\ \vec{v}, \vec{u} \in V :\ \vec{v} \oplus \vec{u} \in V$ □

Axiom 2. *Proof.* Let $\vec{v} = (v_1, v_2, v_3), \vec{u} = (u_1, u_2, u_3)$. $\forall\ \vec{v}, \vec{u} \in V$:

$$\begin{gathered}\vec{v} \oplus \vec{u} = (v_1, v_2, v_3) \oplus (u_1, u_2, u_3) = (v_1 + u_1, v_2 + u_2, v_3 + u_3) \\ \vec{u} \oplus \vec{v} = (u_1, u_2, u_3) \oplus (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \\ v_1 + u_1 = u_1 + v_1 \text{ by prop. of } \mathbb{R} \\ v_2 + u_2 = u_2 + v_2 \text{ by prop. of } \mathbb{R} \\ v_3 + u_3 = u_3 + v_3 \text{ by prop. of } \mathbb{R}\end{gathered}$$

$\therefore \forall\ \vec{v}, \vec{u} \in V :\ \vec{v} \oplus \vec{u} = \vec{u} \oplus \vec{v}$ □

Axiom 3. *Proof.* Let $\vec{v} = (v_1, v_2, v_3), \vec{u} = (u_1, u_2, u_3), \vec{w} = (w_1, w_2, w_3)$. $\forall\ \vec{v}, \vec{u}, \vec{w} \in V$:

$$\begin{aligned}\vec{v} \oplus (\vec{u} \oplus \vec{w}) &= (v_1, v_2, v_3) \oplus ((u_1, u_2, u_3) \oplus (w_1, w_2, w_3)) \\ &= (v_1, v_2, v_3) \oplus (u_1 + w_1, u_2 + w_2, u_3 + w_3) \\ &= (v_1 + (u_1 + w_1), v_2 + (u_2 + w_2), v_3 + (u_3 + w_3))\end{aligned}$$

$$\begin{aligned}(\vec{v} \oplus \vec{u}) \oplus \vec{w} &= ((v_1, v_2, v_3) \oplus (u_1, u_2, u_3)) \oplus (w_1, w_2, w_3) \\ &= (v_1 + u_1, v_2 + u_2, v_3 + u_3) \oplus (w_1, w_2, w_3) \\ &= ((v_1 + u_1) + w_1, (v_2 + u_2) + w_2, (v_3 + u_3) + w_3)\end{aligned}$$

Through the use of the properties of $\mathbb{R}$,

$$\begin{gathered}v_1 + (u_1 + w_1) = (v_1 + u_1) + w_1 \\ v_2 + (u_2 + w_2) = (v_2 + u_2) + w_2 \\ v_3 + (u_3 + w_3) = (v_3 + u_3) + w_3\end{gathered}$$

$\therefore \forall\ \vec{v}, \vec{u}, \vec{w} \in V :\ \vec{v} \oplus (\vec{u} \oplus \vec{w}) = (\vec{v} \oplus \vec{u}) \oplus \vec{w}$ □

Axiom 4. *Proof.* Let $\vec{v} = (0, 0, 0)$. $\forall\ \vec{u} \in V$:

$$\begin{gathered}\vec{v} \oplus \vec{u} = (0, 0, 0) \oplus (u_1, u_2, u_3) = (0 + u_1, 0 + u_2, 0 + u_3) = (u_1, u_2, u_3) = \vec{u} \\ \vec{u} \oplus \vec{v} = (u_1, u_2, u_3) \oplus (0, 0, 0) = (u_1 + 0, u_2 + 0, u_3 + 0) = (u_1, u_2, u_3) = \vec{u}\end{gathered}$$

Using properties of $\mathbb{R}$. $\therefore \vec{v} = (0, 0, 0)$ is the additive identity for V, **id**. □

Axiom 5. *Proof.* Let $\vec{v} = (v_1, v_2, v_3)$ and $\vec{u} = (-v_1, -v_2, -v_3)$. $\forall\ \vec{v}, \vec{u} \in V$:

$$\begin{aligned}\vec{v} \oplus \vec{u} &= (v_1, v_2, v_3) \oplus (-v_1, -v_2, -v_3) \\ &= (v_1 + -v_1, v_2 + -v_2, v_3 + -v_3) \\ &= (0, 0, 0) = \ \mathbf{id}\end{aligned} \qquad \begin{aligned}\vec{u} \oplus \vec{v} &= (-v_1, -v_2, -v_3) \oplus (v_1, v_2, v_3) \\ &= (-v_1 + v_1, -v_2 + v_2, -v_3 + v_3) \\ &= (0, 0, 0) = \ \mathbf{id}\end{aligned}$$

$\therefore \vec{u}$ is the additive inverse of $\vec{v}$, $\forall\ \vec{u} \in V$ $\square$

Axiom 6. *Proof.* Let $\vec{v} = (x, y, z)$. $\forall\ \vec{v} \in V,\ k \in \mathbb{R}$:

$$k \odot \vec{v} = k \odot (x, y, z) = (k^2x, k^2y, k^2z)$$
$$k^2x, k^2y, k^2z \in \mathbb{R} \text{ by prop. of } \mathbb{R}$$

$\therefore \forall\ \vec{v} \in V,\ k \in \mathbb{R}:\ k \odot \vec{v} \in V$ $\square$

Axiom 7. *Proof.* $\forall\ \vec{v}, \vec{u} \in V,\ k \in \mathbb{R}$:

$$\begin{aligned}LHS = k \odot (\vec{v} \oplus \vec{u}) &= k \odot ((v_1, v_2, v_3) \oplus (u_1, u_2, u_3)) \\ &= k \odot (v_1 + u_1, v_2 + u_2, v_3 + u_3) \\ &= (k^2(v_1 + u_1), k^2(v_2 + u_2), k^2(v_3 + u_3)) \\ &= (k^2v_1 + k^2u_1, k^2v_2 + k^2u_2, k^2v_3 + k^2u_3)\end{aligned}$$

$$\begin{aligned}RHS = k \odot \vec{v} \oplus k \odot \vec{u} &= k \odot (v_1, v_2, v_3) \oplus k \odot (u_1, u_2, u_3) \\ &= (k^2v_1, k^2v_2, k^2v_3) \oplus (k^2u_1, k^2u_2, k^2u_3) \\ &= (k^2v_1 + k^2u_1, k^2v_2 + k^2u_2, k^2v_3 + k^2u_3) = LHS\end{aligned}$$

$\therefore \forall\ \vec{v}, \vec{u} \in V,\ k \in \mathbb{R}:\ k \odot (\vec{v} \oplus \vec{u}) = k \odot \vec{v} \oplus k \odot \vec{u}$ $\square$

Axiom 8. *Proof.* $\forall\ \vec{v} \in V,\ k, \ell \in \mathbb{R}$:

$$\begin{aligned}LHS = (k + \ell) \odot \vec{v} &= (k + \ell) \odot (v_1, v_2, v_3) \\ &= ((k + \ell)^2v_1, (k + \ell)^2v_2, (k + \ell)^2v_3) \\ &= (k^2v_1 + 2k\ell v_1 + \ell^2v_1, k^2v_2 + 2k\ell v_2 + \ell^2v_2, k^2v_3 + 2k\ell v_3 + \ell^2v_3)\end{aligned}$$

$$\begin{aligned}RHS = k \odot \vec{v} \oplus \ell \odot \vec{v} &= k \odot (v_1, v_2, v_3) \oplus \ell \odot (v_1, v_2, v_3) \\ &= (k^2v_1, k^2v_2, k^2v_3) \oplus (\ell^2v_1, \ell^2v_2, \ell^2v_3) \\ &= (k^2v_1 + \ell^2v_1, k^2v_2 + \ell^2v_2, k^2v_3 + \ell^2v_3) \neq LHS\end{aligned}$$

$\therefore$ Axiom 8 does not hold. $\square$

Axiom 9. *Proof.* $\forall\ \vec{v} \in V,\ k, \ell \in \mathbb{R}$:

$$\begin{aligned}LHS = (k \cdot \ell) \odot \vec{v} &= (k \cdot \ell) \odot (v_1, v_2, v_3) \\ &= ((k \cdot \ell)^2v_1, (k \cdot \ell)^2v_2, (k \cdot \ell)^2v_3) \\ &= (k^2\ell^2v_1, k^2\ell^2v_2, k^2\ell^2v_3)\end{aligned}$$

$$\begin{aligned}RHS = k \odot (\ell \odot \vec{v}) &= k \odot (\ell \odot (v_1, v_2, v_3)) \\ &= k \odot (\ell^2v_1, \ell^2v_2, \ell^2v_3) \\ &= (k^2(\ell^2v_1), k^2(\ell^2v_2), k^2(\ell^2v_3)) = LHS\end{aligned}$$

$\therefore \forall\ \vec{v} \in V,\ k, \ell \in \mathbb{R} : (k \cdot \ell) \odot \vec{v} = k \odot (\ell \odot \vec{v})$ $\square$

Axiom 10. *Proof.* $\forall\ \vec{v} \in V$:

$$\begin{aligned} 1 \odot \vec{v} &= 1 \odot (v_1, v_2, v_3) \\ &= (1^2 v_1, 1^2 v_2, 1^2 v_3) \\ &= (v_1, v_2, v_3) = \vec{v} \end{aligned}$$

$\therefore \forall\ \vec{v} \in V : 1 \odot \vec{v} = \vec{v}$ □

All Axioms *except* Axiom 8 work, therefore this is not a real vector space.

Problem 14

The set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 0$, and the addition and scalar multiplication operations are the same as those introduced in Example 6:

$$\begin{aligned} (\vec{f} \oplus \vec{g})(x) &= \vec{f}(x) + \vec{g}(x) \\ (k \odot \vec{f})(x) &= k\vec{f}(x) \end{aligned}$$

Let F be the set of all functions $\vec{f} : \mathbb{R} \to \mathbb{R}$ such that $\vec{f}(x) = 0$.

Axiom 1. *Proof.* $\forall\ \vec{f}, \vec{g} \in F,\ x \in \mathbb{R}$:

$$\begin{aligned} \text{by definition } (\vec{f} \oplus \vec{g})(x) &= \vec{f}(x) + \vec{g}(x) \\ \text{when } x = 1 :\ \vec{f}(1) + \vec{g}(1) &= 0 + 0 = 0\ \checkmark \end{aligned}$$

$\therefore \forall\ \vec{f}, \vec{g} \in F : \vec{f}(x) + \vec{g}(x) \in F$ □

Axiom 2. *Proof.* $\forall\ \vec{f}, \vec{g} \in F,\ x \in \mathbb{R}$:

$$\begin{aligned} LHS = (\vec{f} \oplus \vec{g})(x) &= \vec{f}(x) + \vec{g}(x) \\ \text{when } x = 1,\ \vec{f}(1) + \vec{g}(1) &= 0 + 0 = 0\ \checkmark \\ RHS = (\vec{g} \oplus \vec{f})(x) &= \vec{g}(x) + \vec{f}(x) = LHS \\ \text{when } x = 1,\ \vec{g}(1) + \vec{f}(1) &= 0 + 0 = 0\ \checkmark \end{aligned}$$

$\therefore \forall\ \vec{f}, \vec{g} \in F : \vec{f}(x) + \vec{g}(x) = \vec{g}(x) + \vec{f}(x)$ □

Axiom 3. *Proof.* $\forall\ \vec{f}, \vec{g}, \vec{h} \in F,\ x \in \mathbb{R}$:

$$\begin{aligned} LHS = (\vec{f} \oplus (\vec{g} \oplus \vec{h}))(x) &= \vec{f}(x) + (\vec{g} \oplus \vec{h})(x) \\ &= \vec{f}(x) + (\vec{g}(x) + \vec{h}(x)) \\ \text{when } x = 1,\ \vec{f}(1) + (\vec{g}(1) + \vec{h}) &= 0 + (0 + 0) = 0\ \checkmark \\ \\ RHS = ((\vec{f} \oplus \vec{g}) \oplus \vec{h})(x) &= (\vec{f} \oplus \vec{g})(x) + \vec{h}(x) \\ &= (\vec{f}(x) + \vec{g}(x)) + \vec{h}(x) = LHS \\ \text{when } x = 1,\ (\vec{f}(1) + \vec{g}(1)) + \vec{h} &= (0 + 0) + 0 = 0\ \checkmark \end{aligned}$$

$\therefore \forall\ \vec{f}, \vec{g}, \vec{h} \in F : (\vec{f} \oplus (\vec{g} \oplus \vec{h}))(x) = ((\vec{f} \oplus \vec{g}) \oplus \vec{h})(x)$ □

Axiom 4. *Proof.* Let $\vec{f} : \mathbb{R} \to \mathbb{R}$ such that $\forall\ x \in \mathbb{R} : \vec{f}(x) = 0$. $\forall\ \vec{g} \in F$:

$$(\vec{f} \oplus \vec{g})(x) = \vec{f}(x) + \vec{g}(x) = 0 + \vec{g}(x) = \vec{g}(x) \text{ for } x \in \mathbb{R}$$
$$\text{when } x = 1,\ \vec{f}(1) + \vec{g}(1) = 0 + 0 = 0\ \checkmark$$

$$(\vec{g} \oplus \vec{f})(x) = \vec{g}(x) + \vec{f}(x) = \vec{g}(x) + 0 = \vec{g}(x) \text{ for } x \in \mathbb{R}$$
$$\text{when } x = 1,\ \vec{g}(1) + \vec{f}(1) = 0 + 0 = 0\ \checkmark$$

$\therefore \vec{f}(x) = 0$ is the additive identity for F, **id**. □

Axiom 5. *Proof.* Let $\vec{f} : \mathbb{R} \to \mathbb{R}$ such that $\forall\ x \in \mathbb{R} : \vec{f}(x) = -g(x)$. $\forall\ \vec{g} \in F$:

$$(\vec{f} \oplus \vec{g})(x) = \vec{f}(x) \oplus \vec{g}(x) = -\vec{g}(x) + \vec{g}(x) = 0 \text{ for } x \in \mathbb{R}$$
$$(\vec{g} \oplus \vec{f})(x) = \vec{g}(x) \oplus \vec{f}(x) = \vec{g}(x) - \vec{g}(x) = 0 \text{ for } x \in \mathbb{R}$$

$\therefore \vec{f}$ is the additive inverse of $\vec{g}$, $\forall\ \vec{g} \in F$ □

Axiom 6. *Proof.* $\forall\ k, x \in \mathbb{R}$ and $\forall\ \vec{f} \in F$:

$$(k \odot \vec{f}) = k\vec{f}(x) \qquad \text{by definition}$$
$$\text{when } x = 1:\ k\vec{f}(1) = k \cdot 0 = 0\ \checkmark$$

$\therefore \forall\ k \in \mathbb{R}$ and $\forall\ \vec{f} \in F$, $k \odot \vec{f} \in F$ □

Axiom 7. *Proof.* $\forall\ \vec{f}, \vec{g} \in F$ and $\forall\ k, x \in \mathbb{R}$:

$$\begin{aligned} LHS = (k \odot (\vec{f} \oplus \vec{g}))(x) &= k(\vec{f} \oplus \vec{g})(x) \\ &= k(\vec{f}(x) + \vec{g}(x)) \\ &= k\vec{f}(x) + k\vec{g}(x) \end{aligned}$$

$$\begin{aligned} RHS = (k \odot \vec{f} \oplus k \odot \vec{g})(x) &= (k \odot \vec{f})(x) + (k \odot \vec{g})(x) \\ &= k\vec{f}(x) + k\vec{g}(x) = LHS \end{aligned}$$

$$\text{when } x = 1:\ k\vec{f}(1) + k\vec{g}(1) = k0 + k0 = 0\ \checkmark$$

$\therefore \forall\ \vec{f}, \vec{g} \in F$ and $\forall\ k, x \in \mathbb{R}$, $(k \odot (\vec{f} \oplus \vec{g}))(x) = (k \odot \vec{f} \oplus k \odot \vec{g})(x)$ □

Axiom 8. *Proof.* $\forall\ \vec{f} \in F$ and $\forall\ k, \ell, x \in \mathbb{R}$:

$$\begin{aligned} LHS = ((k + \ell) \odot \vec{f})(x) &= (k + \ell)\vec{f}(x) \\ &= k\vec{f}(x) + \ell\vec{f}(x) \end{aligned}$$

$$\begin{aligned} RHS = (k \odot \vec{f} \oplus \ell \odot \vec{f})(x) &= (k \odot \vec{f})(x) + (\ell \odot \vec{f})(x) \\ &= k\vec{f}(x) + \ell\vec{f}(x) = LHS \end{aligned}$$

$$\text{when } x = 1:\ k\vec{f}(1) + \ell\vec{f}(1) = k0 + \ell0 = 0\ \checkmark$$

$\therefore \forall\ \vec{f} \in F$ and $\forall\ k, \ell, x \in \mathbb{R}$, $((k + \ell) \odot \vec{f})(x) = (k \odot \vec{f} \oplus \ell \odot \vec{f})(x)$ □