1 Homework 5

Section 4.2

7

Which of the following are linear combinations of $\vec{u} = (0, -2, 2)$ and $\vec{v} = (1, 3, -1)$?

a. $(2, 2, 2)$

Proof. Let $k_1, k_2 \in \mathbb{R}$ such that $k_1\vec{u} + k_2\vec{v} = (2, 2, 2)$. That is, $k_1(0, -2, 2) + k_2(1, 3, -1) = (2, 2, 2)$. From this equation, we get a linear system of equations.

$$\begin{aligned} 0k_1 + 1k_2 &= 2 \\ -2k_1 + 3k_2 &= 2 \\ 2k_1 - 1k_2 &= 2 \end{aligned}$$

$$\left(\begin{array}{cc|c} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 2 & -1 & 2 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2} \begin{pmatrix} 0 & 1 & 2 \\ 1 & -\frac{3}{2} & -1 \\ 2 & -1 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -\frac{3}{2} & -1 \\ 0 & 1 & 2 \\ 2 & -1 & 2 \end{pmatrix} \xrightarrow[(-2,3,2)]{R_3 - 2R_1} \begin{pmatrix} 1 & -\frac{3}{2} & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow[(0,-2,-4)]{R_3 - 2R_2}$$

$$\begin{pmatrix} 1 & -\frac{3}{2} & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[(0,\frac{3}{2},3)]{R_1 + \frac{3}{2}R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

This augmented matrix represents the following equations:

$$\begin{aligned} k_1 + 0k_2 &= 2 \\ 0k_1 + k_2 &= 2 \\ 0 + 0 &= 0 \end{aligned} \qquad\qquad \begin{aligned} k_1 &= 2 \\ k_2 &= 2 \end{aligned}$$

This means that $(2, 2, 2)$ is a linear combination of $\{\vec{u}, \vec{v}\}$, when $k_1 = 2$ and $k_2 = 2$. $\square$

c. $(0, 4, 5)$

Proof. Let $k_1, k_2 \in \mathbb{R}$ such that $k_1\vec{u} + k_2\vec{v} = (0, 4, 5)$. That is, $k_1(0, -2, 2) + k_2(1, 3, -1) = (0, 4, 5)$. From this equation, we get a linear system of equations.

$$\begin{aligned} 0k_1 + 1k_2 &= 0 \\ -2k_1 + 3k_2 &= 4 \\ 2k_1 - 1k_2 &= 5 \end{aligned}$$

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 4 \\ 2 & -1 & 5 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{3}{2} & -2 \\ 2 & -1 & 5 \end{pmatrix} \xrightarrow[(-2,3,4)]{R_3 - 2R_2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{3}{2} & -2 \\ 0 & 2 & 9 \end{pmatrix} \xrightarrow[(0,-2,0)]{R_3 - 2R_2} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & -\frac{3}{2} & -2 \\ 0 & 0 & 9 \end{array}\right)$$

The last row from this matrix provides the equation $0k_1 + 0k_2 = 9$, meaning $0 + 0 = 9$, which is impossible. Therefore, $(0, 4, 5)$ is not spanned by $\{\vec{u}, \vec{v}\}$. $\square$

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Express the following combinations of $\vec{u} = (2, 1, 4), \vec{v} = (1, -1, 3)$, and $\vec{w} = (3, 2, 5)$

a. $(-9, -7, -15)$

Proof. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1\vec{u}+k_2\vec{v}+k_3\vec{w} = (-9,-7,-15)$. That is $k_1(2,1,4)+k_2(1,-1,3)+k_3(3,2,5) = (-9,-7,-15)$. From this equation, we get a linear system of equations.

$$\begin{aligned} 2k_1 + 1k_2 + 3k_3 &= -9 \\ 1k_1 - 1k_2 + 2k_3 &= -7 \\ 4k_1 + 3k_2 + 5k_3 &= -15 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right) \xrightarrow[(-4,-2,-6,18)]{R_3-2R_1} \begin{pmatrix} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 0 & 1 & -1 & 3 \end{pmatrix} \xrightarrow[(-2,2,-4,14)]{R_1-2R_2} \begin{pmatrix} 0 & 3 & -1 & 5 \\ 1 & -1 & 2 & -7 \\ 0 & 1 & -1 & 3 \end{pmatrix} \xrightarrow{R_2+R_1}$$

$$\begin{pmatrix} 0 & 3 & -1 & 5 \\ 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 3 \end{pmatrix} \xrightarrow[(0,-3,3,-9)]{R_1-3R_3} \begin{pmatrix} 0 & 0 & 2 & -4 \\ 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 3 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 0 & 0 & 1 & -2 \\ 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 3 \end{pmatrix} \xrightarrow[(0,0,-1,2)]{R_2-R_1}$$

$$\begin{pmatrix} 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & -1 & 3 \end{pmatrix} \xrightarrow[(0,0,1,-2)]{R_3+R_1} \begin{pmatrix} 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[R_2\leftrightarrow R_3]{R_1\leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

This augmented matrix represents the following equations:

$$\begin{aligned} k_1 &= -2 \\ k_2 &= 1 \\ k_3 &= -2 \end{aligned}$$

This means that $(-9,-7,-15)$ is a linear combination of $\{\vec{u},\vec{v},\vec{w}\}$, when $k_1 = -2, k_2 = 1,$ and $k_3 = -2$. □

c. $(0,0,0)$

Proof. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1\vec{u}+k_2\vec{v}+k_3\vec{w} = (0,0,0)$. That is $k_1(2,1,4)+k_2(1,-1,3)+k_3(3,2,5) = (0,0,0)$. From this equation, we get a linear system of equations.

$$\begin{aligned} 2k_1 + 1k_2 + 3k_3 &= 0 \\ 1k_1 - 1k_2 + 2k_3 &= 0 \\ 4k_1 + 3k_2 + 5k_3 &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right) \xrightarrow[(-4,-2,-6,0)]{R_3-2R_1} \begin{pmatrix} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xrightarrow[(-2,2,-4,0)]{R_1-2R_2} \begin{pmatrix} 0 & 3 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xrightarrow{R_2+R_1}$$

$$\begin{pmatrix} 0 & 3 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xrightarrow[(0,-3,3,0)]{R_1-3R_3} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xrightarrow[(0,0,-1,0)]{R_2-R_1}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xrightarrow[(0,0,1,0)]{R_3+R_1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xrightarrow[R_2\leftrightarrow R_3]{R_1\leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

This augmented matrix represents the following equations:

$$\begin{aligned} k_1 &= 0 \\ k_2 &= 0 \\ k_3 &= 0 \end{aligned}$$

This means that $(0,0,0)$ is a linear combination of $\{\vec{u},\vec{v},\vec{w}\}$, when $k_1 = 0, k_2 = 0,$ and $k_3 = 0$. □

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Which of the following are linear combinations of

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}?$$

a. $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$

Proof. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1 A + k_2 B + k_3 C = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$.
That is, $k_1 \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + k_2 \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$. From this equation, we get a linear system of equations.

$$\begin{aligned} 4k_1 + 1k_2 + 0k_3 &= 6 \\ 0k_1 - 1k_2 + 2k_3 &= -8 \\ -2k_1 + 2k_2 + 1k_3 &= -1 \\ -2k_1 + 3k_2 + 4k_3 &= -8 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{array}\right) \xrightarrow[(-4,4,2,-2)]{R_1+2R_3} \begin{pmatrix} 0 & 5 & 2 & 4 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{pmatrix} \xrightarrow[(2,-2,-1,1)]{R_4-R_3} \begin{pmatrix} 0 & 5 & 2 & 4 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ 0 & 1 & 3 & -7 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3}$$

$$\begin{pmatrix} 0 & 5 & 2 & 4 \\ 0 & -1 & 2 & -8 \\ 1 & -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 3 & -7 \end{pmatrix} \xrightarrow[R_2+R_4]{R_3+R_4} \begin{pmatrix} 0 & 5 & 2 & 4 \\ 0 & 0 & 5 & -15 \\ 1 & 0 & 2\frac{1}{2} & -6\frac{1}{2} \\ 0 & 1 & 3 & -7 \end{pmatrix} \xrightarrow[(0,-5,-15,35)]{R_1-5R_4} \begin{pmatrix} 0 & 0 & -13 & 39 \\ 0 & 0 & 5 & -15 \\ 1 & 0 & 2\frac{1}{2} & -6\frac{1}{2} \\ 0 & 1 & 3 & -7 \end{pmatrix} \xrightarrow[\frac{1}{5}R_2]{R_1+2R_2}$$

$$\begin{pmatrix} 0 & 0 & -3 & 9 \\ 0 & 0 & 1 & -3 \\ 1 & 0 & 2\frac{1}{2} & -6\frac{1}{2} \\ 0 & 1 & 3 & -7 \end{pmatrix} \xrightarrow[R_1\leftrightarrow R_4]{R_1+3R_2} \begin{pmatrix} 0 & 1 & 3 & -7 \\ 0 & 0 & 1 & -3 \\ 1 & 0 & 2\frac{1}{2} & -6\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[(0,0,-3,9)]{R_1-3R_2} \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 1 & 0 & 2\frac{1}{2} & -6\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[(0,0,-2\frac{1}{2},7\frac{1}{2})]{R_3-2\frac{1}{2}R_2}$$

$$\begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This augmented matrix represents the following equations:

$$\begin{aligned} k_1 &= 1 \\ k_2 &= 2 \\ k_3 &= -3 \end{aligned}$$

This means that $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is a linear combination of $\{A, B, C\}$, when $k_1 = 1, k_2 = 2$, and $k_3 = -3$. □

c. $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$

Proof. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1A + k_2B + k_3C = \begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$.
That is, $k_1 \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + k_2 \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$. From this equation, we get a linear system of equations.

$$\begin{aligned} 4k_1 + 1k_2 + 0k_3 &= 6 \\ 0k_1 - 1k_2 + 2k_3 &= 0 \\ -2k_1 + 2k_2 + 1k_3 &= 3 \\ -2k_1 + 3k_2 + 4k_3 &= 8 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 3 \\ -2 & 3 & 4 & 8 \end{array}\right) \xrightarrow[(-4,4,2,6)]{R_1+2R_3} \begin{pmatrix} 0 & 5 & 2 & 12 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 3 \\ -2 & 3 & 4 & 8 \end{pmatrix} \xrightarrow[(2,-2,-1,-3)]{R_4-R_3} \begin{pmatrix} 0 & 5 & 2 & 12 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 3 \\ 0 & 1 & 3 & 5 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3}$$

$$\begin{pmatrix} 0 & 5 & 2 & 12 \\ 0 & -1 & 2 & 0 \\ 1 & -1 & -\frac{1}{2} & -1\frac{1}{2} \\ 0 & 1 & 3 & 5 \end{pmatrix} \xrightarrow[R_2+R_4]{R_3+R_4} \begin{pmatrix} 0 & 5 & 2 & 12 \\ 0 & 0 & 5 & 5 \\ 1 & 0 & 2\frac{1}{2} & 3\frac{1}{2} \\ 0 & 1 & 3 & 5 \end{pmatrix} \xrightarrow[(0,-5,-15,-25)]{R_1-5R_4} \begin{pmatrix} 0 & 0 & -13 & -13 \\ 0 & 0 & 5 & 5 \\ 1 & 0 & 2\frac{1}{2} & 3\frac{1}{2} \\ 0 & 1 & 3 & 5 \end{pmatrix} \xrightarrow[(0,0,10,10)]{R_1+2R_2}$$

$$\begin{pmatrix} 0 & 0 & -3 & -3 \\ 0 & 0 & 5 & 5 \\ 1 & 0 & 2\frac{1}{2} & 3\frac{1}{2} \\ 0 & 1 & 3 & 5 \end{pmatrix} \xrightarrow[-\frac{1}{3}R_1]{R_4+R_1} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 5 & 5 \\ 1 & 0 & 2\frac{1}{2} & 3\frac{1}{2} \\ 0 & 1 & 0 & 2 \end{pmatrix} \xrightarrow[R_2-5R_1]{R_3-2\frac{1}{2}R_1} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix} \rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This augmented matrix represents the following equations:

$$\begin{aligned} k_1 &= 1 \\ k_2 &= 2 \\ k_3 &= 1 \end{aligned}$$

This means that $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$ is a linear combination of $\{A, B, C\}$, when $k_1 = 1, k_2 = 2,$ and $k_3 = 1$. □

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In each part express the vector as a linear combination of $\vec{p}_1 = 2 + x + 4x^2, \vec{p}_2 = 1 - x + 3x^2,$ *and* $\vec{p}_3 = 3 + 2x + 5x^2$.

a. $-9 - 7x - 15x^2$

d. $7 + 8x + 9x^2$

Section 1.2

4

In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given reduced row echelon form. Solve the system.

a. $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$