

Homework 9

4.3

4 Which of the following sets of vector in P_2 are linearly dependent?

a. $2 - x + 4x^2, \quad 3 + 6x + 2x^2, \quad 2 + 10x - 4x^2$

Work. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1(2 - x + 4x^2) + k_2(3 + 6x + 2x^2) + k_3(2 + 10x - 4x^2) = \mathbf{id}$. From this, we can get a linear system of equations, and an augmented matrix.

$$\begin{aligned} 2k_1 + 3k_2 + 2k_3 &= 0 \\ -xk_1 + 6xk_2 + 10xk_3 &= 0 \\ 4x^2k_1 + 2x^2k_2 - 4x^2k_3 &= 0 \end{aligned}$$

$$\begin{aligned} \left[\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ -1 & 6 & 10 & 0 \\ 4 & 2 & -4 & 0 \end{array} \right] &\xrightarrow[R_3+4R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} 0 & 15 & 22 & 0 \\ -1 & 6 & 10 & 0 \\ 0 & 26 & 36 & 0 \end{array} \right] &\xrightarrow{R_3-2R_1} \left[\begin{array}{ccc|c} 0 & 15 & 22 & 0 \\ -1 & 6 & 10 & 0 \\ 0 & -4 & -8 & 0 \end{array} \right] &\xrightarrow[-\frac{1}{4}R_3]{R_1+4R_3} \\ \left[\begin{array}{ccc|c} 0 & -1 & -10 & 0 \\ -1 & 6 & 10 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] &\xrightarrow[R_2-6R_3]{R_1+R_3} \left[\begin{array}{ccc|c} 0 & 0 & -8 & 0 \\ -1 & 0 & -12 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] &\xrightarrow[-R_2]{-\frac{1}{8}R_1} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 12 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] &\xrightarrow[R_3-2R_1]{R_2-12R_1} \\ \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] &\xrightarrow[R_1 \leftrightarrow R_2]{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \end{aligned}$$

Since the only solution is the trivial solution, therefore $\{2 - x + 4x^2, \quad 3 + 6x + 2x^2, \quad 2 + 10x - 4x^2\}$ are linearly independent. \square

c. $3 + x + x^2, \quad 2 - x + 5x^2, \quad 4 - 3x^2$

Work. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1(3 + x + x^2) + k_2(2 - x + 5x^2) + k_3(4 - 3x^2) = \mathbf{id}$. From this, we can get a linear system of equations, and an augmented matrix.

$$\begin{aligned} 3k_1 + 2k_2 + 4k_3 &= 0 \\ xk_1 - xk_2 + 0xk_3 &= 0 \\ x^2k_1 - 5x^2k_2 - 3x^3k_3 &= 0 \end{aligned}$$

$$\begin{aligned} \left[\begin{array}{ccc|c} 3 & 2 & 4 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -5 & -3 & 0 \end{array} \right] &\xrightarrow[R_3-R_2]{R_1-3R_2} \left[\begin{array}{ccc|c} 0 & 5 & 4 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -4 & -3 & 0 \end{array} \right] &\xrightarrow{R_3+R_1} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -4 & -3 & 0 \end{array} \right] &\xrightarrow{R_3+4R_1} \\ \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] &\xrightarrow[R_2+R_1]{R_1-R_3} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] &\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \end{aligned}$$

Since the only solution is the trivial solution, therefore $\{3 + x + x^2, \quad 2 - x + 5x^2, \quad 4 - 3x^2\}$ are linearly independent. \square

9 For which real values of λ do the following vectors form a linearly dependent set in \mathbb{R}^3 ?

$$v_1 = \left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right), \quad v_2 = \left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right), \quad v_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right)$$

Work. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1(\lambda, -\frac{1}{2}, -\frac{1}{2}) + k_2(-\frac{1}{2}, \lambda, -\frac{1}{2}) + k_3(-\frac{1}{2}, -\frac{1}{2}, \lambda) = (0, 0, 0)$. From this, we can get a linear system of equations, and matrix equation.

$$\begin{aligned} \lambda k_1 - \frac{1}{2}k_2 - \frac{1}{2}k_3 &= 0 \\ -\frac{1}{2}k_1 + \lambda k_2 - \frac{1}{2}k_3 &= 0 \\ -\frac{1}{2}k_1 - \frac{1}{2}k_2 + \lambda k_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider the determinant of the square matrix.

$$\begin{aligned} & \begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} \begin{vmatrix} \frac{R_1+R_2}{R_1+R_3} \\ \frac{R_1+R_2}{R_1+R_3} \\ \frac{R_1+R_2}{R_1+R_3} \end{vmatrix} \begin{vmatrix} \lambda-1 & \lambda-1 & \lambda-1 \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} = (\lambda-1) \begin{vmatrix} 1 & 1 & 1 \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} \begin{vmatrix} \frac{R_2+\frac{1}{2}R_1}{R_3+\frac{1}{2}R_1} \\ \frac{R_2+\frac{1}{2}R_1}{R_3+\frac{1}{2}R_1} \\ \frac{R_2+\frac{1}{2}R_1}{R_3+\frac{1}{2}R_1} \end{vmatrix} \\ & (\lambda-1) \begin{vmatrix} 1 & 1 & 1 \\ 0 & \lambda+\frac{1}{2} & 0 \\ 0 & 0 & \lambda+\frac{1}{2} \end{vmatrix} = (\lambda-1)(1)(\lambda+\frac{1}{2}) = (\lambda-1)(\lambda+\frac{1}{2})^2 \end{aligned}$$

When $\lambda = 1$ or $\lambda = -\frac{1}{2}$, the determinant of this matrix will be zero. By Theorem 4 of Lecture Notes 32, when the determinant is zero, the coefficient matrix is singular. By the Big Theorem, if the coefficient matrix is singular, then $A\vec{x} = \vec{0}$ does not only have the trivial solution, meaning that there exists $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1(\lambda, -\frac{1}{2}, -\frac{1}{2}) + k_2(-\frac{1}{2}, \lambda, -\frac{1}{2}) + k_3(-\frac{1}{2}, -\frac{1}{2}, \lambda) = (0, 0, 0)$, where not all k_i are zero. This means that these vectors are linearly dependent when $\lambda = 1$ or $\lambda = -\frac{1}{2}$. \square

13 Show that if $S = \{v_1, v_2, \dots, v_r\}$ is a linearly dependent set of vectors in a vector space V , and if v_{r+1}, \dots, v_n are any vectors in V that are not in S , then $\{v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_n\}$ is also linearly dependent.

Work. By definition, if S is a linearly dependent set of vectors in V , then $\exists k_1, k_2, \dots, k_r \in \mathbb{R}$ such that $k_1v_1 + k_2v_2 + \dots + k_rv_r = \mathbf{id}$, where not all $k_i = 0$. Now consider $S' = S \cup \{v_{r+1}, \dots, v_n\}$, where v_{r+1}, \dots, v_n are not in S . Consider $k_1, k_2, \dots, k_r, k_{r+1}, \dots, k_n \in \mathbb{R}$ such that $k_1v_1 + k_2v_2 + \dots + k_rv_r + k_{r+1}v_{r+1} + \dots + k_nv_n = \mathbf{id}$.

$$\begin{aligned} k_1v_1 + k_2v_2 + \dots + k_rv_r + k_{r+1}v_{r+1} + \dots + k_nv_n &= \mathbf{id} & \neg \forall k_i &= 0 \\ k_1v_1 + k_2v_2 + \dots + k_rv_r + 0v_{r+1} + \dots + 0v_n &= \mathbf{id} & \neg \forall k_i &= 0 \\ k_1v_1 + k_2v_2 + \dots + k_rv_r + \mathbf{id} + \dots + \mathbf{id} &= \mathbf{id} & \neg \forall k_i &= 0 \\ k_1v_1 + k_2v_2 + \dots + k_rv_r &= \mathbf{id} & \neg \forall k_i &= 0 \end{aligned}$$

This final equation is known to exist, since we asserted at the beginning that S was linearly dependent. Therefore, S' must also be linearly dependent. Therefore, if $S = \{v_1, v_2, \dots, v_r\}$ is a linearly dependent set of vectors in a vector space V , and if v_{r+1}, \dots, v_n are any vectors in V that are not in S , then $\{v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_n\}$ is also linearly dependent. \square

21 The functions $f_1(x) = x$ and $f_2(x) = \cos x$ are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wronski's test.

Work. According to the Wronski's test, f_1 and f_2 are linearly independent if $W(x)$ is not identically zero.

$$W(x) = \begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix} = -x \sin x - \cos x$$

When $x = 0$, $W(x) = -0 \sin 0 - \cos 0 = 0 - 1 = -1 \neq 0$. Therefore W is not identically zero. Therefore, $f_1(x) = x$ and $f_2(x) = \cos x$ are linearly independent. \square

4.4

4 Which of the following form bases for P_2 ?

a. $1 - 3x + 2x^2, \quad 1 + x + 4x^2, \quad 1 - 7x$

Work. In order for $\{1 - 3x + 2x^2, 1 + x + 4x^2, 1 - 7x\}$ to be a basis of P_2 , it must be linearly independent and it must span P_2 . Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1(1 - 3x + 2x^2) + k_2(1 + x + 4x^2) + k_3(1 - 7x) = (b_1 + b_2x + b_3x^2)$.

$$\begin{aligned} 1k_1 + 1k_2 + 1k_3 &= b_1 \\ -3xk_1 + 1xk_2 - 7xk_3 &= b_2 \\ 2x^2k_1 + 4x^2k_2 + 0x^2k_3 &= b_3 \end{aligned}$$

Consider the augmented matrix of this linear system of equations.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ -3 & 1 & -7 & b_2 \\ 2 & 4 & 0 & b_3 \end{array} \right] \xrightarrow[R_3 - 2R_1]{R_2 + 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 4 & -4 & b_2 + 3b_1 \\ 0 & 2 & -2 & b_3 - 2b_1 \end{array} \right] \xrightarrow{R_2 - 2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 0 & b_2 + 3b_1 - b_3 + 4b_1 \\ 0 & 2 & -2 & b_3 - 2b_1 \end{array} \right]$$

The second row of the augmented matrix represents $0 = b_2 + 3b_1 - b_3 + 4b_1$. If $0 \neq b_2 + 3b_1 - b_3 + 4b_1$, then there is no solution to the linear system of equations, and thus no coefficients for k_1, k_2, k_3 that satisfy the above condition. Therefore $\{1 - 3x + 2x^2, 1 + x + 4x^2, 1 - 7x\}$ does not span P_2 , and thus cannot be a basis for P_2 . \square

7 Find the coordinate vector of \vec{w} relative to the basis $S = \{\vec{u}_1, \vec{u}_2\}$ for \mathbb{R}^2 .

b. $\vec{u}_1 = (2, -4)$, $\vec{u}_2 = (3, 8)$; $\vec{w} = (1, 1)$

Work. Let $k_1, k_2 \in \mathbb{R}$ such that $k_1(2, -4) + k_2(3, 8) = (1, 1)$.

$$\begin{aligned} 2k_1 + 3k_2 &= 1 \\ -4k_1 + 8k_2 &= 1 \end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ -4 & 8 & 1 \end{array} \right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 14 & 3 \end{array} \right] \xrightarrow{\frac{1}{14}R_2} \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1 & \frac{3}{14} \end{array} \right] \xrightarrow{R_1-3R_2} \left[\begin{array}{cc|c} 2 & 0 & \frac{5}{14} \\ 0 & 1 & \frac{3}{14} \end{array} \right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{array} \right]$$

Therefore, $\frac{5}{28}(2, -4) + \frac{3}{14}(3, 8) = (1, 1)$, or in other terms $(1, 1)_S = (\frac{5}{28}, \frac{3}{14})$. \square

c. $\vec{u}_1 = (1, 1)$, $\vec{u}_2 = (0, 2)$; $\vec{w} = (a, b)$

Work. Let $k_1, k_2 \in \mathbb{R}$ such that $k_1(1, 1) + k_2(0, 2) = (a, b)$.

$$\begin{aligned} 1k_1 + 0k_2 &= a \\ 1k_1 + 2k_2 &= b \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 1 & 2 & b \end{array} \right] \xrightarrow{R_2-R_1} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 2 & b-a \end{array} \right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{array} \right]$$

Therefore, $a(1, 1) + \frac{b-a}{2}(0, 2) = (a, b)$, or in other terms $(a, b)_S = (a, \frac{b-a}{2})$. \square

12 Show that $\{A_1, A_2, A_3, A_4\}$ is a basis for \mathcal{M}_{22} , and express A as a linear combination of the basis vectors.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$$

Proof of basis. Let $k_1, k_2, k_3, k_4 \in \mathbb{R}$ such that $k_1 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$.

$$\begin{aligned} 1k_1 + 1k_2 + 1k_3 + 0k_4 &= 6 \\ 0k_1 + 1k_2 + 0k_3 + 0k_4 &= 2 \\ 1k_1 + 0k_2 + 0k_3 + 1k_4 &= 5 \\ 0k_1 + 0k_2 + 1k_3 + 0k_4 &= 3 \end{aligned}$$

Now consider only the coefficient matrix.

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow[R_3+R_4, R_3+R_2]{R_3-R_1} \left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow[R_1-R_3]{R_1-R_2} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Since every column has a leading 1, $\{A_1, A_2, A_3, A_4\}$ are linearly independent. Since every row has a leading 1, $\{A_1, A_2, A_3, A_4\}$ spans \mathcal{M}_{22} . Therefore, since $\{A_1, A_2, A_3, A_4\}$ is linearly independent and spans \mathcal{M}_{22} , $\{A_1, A_2, A_3, A_4\}$ is a basis for \mathcal{M}_{22} . \square

A as a linear combination.

$$1k_1 + 1k_2 + 1k_3 + 0k_4 = 6$$

$$0k_1 + 1k_2 + 0k_3 + 0k_4 = 2$$

$$1k_1 + 0k_2 + 0k_3 + 1k_4 = 5$$

$$0k_1 + 0k_2 + 1k_3 + 0k_4 = 3$$

$$\begin{aligned} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 3 \end{array} \right] & \xrightarrow[R_3+R_4, R_3+R_2]{R_3-R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 3 \end{array} \right] & \xrightarrow[R_1-R_3]{R_1-R_2} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 3 \end{array} \right] & \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array} \right] \end{aligned}$$

Therefore, $1A_1 + 2A_2 + 3A_3 + 4A_4 = A$. □

4.5

3 Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.

$$1x_1 - 4x_2 + 3x_3 - 1x_4 = 0$$

$$2x_1 - 8x_2 + 6x_3 - 2x_4 = 0$$

Work.

$$\left[\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 2 & -8 & 6 & -2 & 0 \end{array} \right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Let $x_2 = t_2$, $x_3 = t_3$, $x_4 = t_4$, where t_2, t_3, t_4 are free parameters. $x_1 = 4t_2 - 3t_3 + 1t_4$. Therefore a solution to the homogeneous linear system takes the form

$$\begin{aligned} (x_1, x_2, x_3, x_4) &= (4t_2 - 3t_3 + 1t_4, t_2, t_3, t_4) \\ &= (4t_2, t_2, 0, 0) + (-3t_3, 0, t_3, 0) + (t_4, 0, 0, t_4) \\ &= t_2(4, 1, 0, 0) + t_3(-3, 0, 1, 0) + t_4(1, 0, 0, 1) \end{aligned}$$

Therefore a basis of the solution space is $\{(4, 1, 0, 0), (-3, 0, 1, 0), (1, 0, 0, 1)\}$. Since there are three basis vectors, the dimension of the solution space is 3. □

7 Find bases for the following subspaces of \mathbb{R}^3 .

a. The plane $3x - 2y + 5z = 0$.

Work.

$$3x - 2y + 5z = 0$$

$$x - \frac{2}{3}y + \frac{5}{3}z = 0$$

Let $y = t_y$ and $z = t_z$, where t_y and t_z are free parameters. $x = \frac{2}{3}t_y - \frac{5}{3}t_z$. Therefore a point on the plane $3x - 2y + 5z = 0$ takes the form

$$\begin{aligned}(x, y, z) &= \left(\frac{2}{3}t_y - \frac{5}{3}t_z, t_y, t_z\right) \\ &= \left(\frac{2}{3}t_y, t_y, 0\right) + \left(-\frac{5}{3}t_z, 0, t_z\right) \\ &= t_y\left(\frac{2}{3}, 1, 0\right) + t_z\left(-\frac{5}{3}, 0, 1\right)\end{aligned}$$

Therefore a basis of the solution space is $\{(\frac{2}{3}, 1, 0), (-\frac{5}{3}, 0, 1)\}$. □

b. The plane $x - y = 0$.

Work.

$$x - y = 0$$

$$x - y + z = z$$

Let $y = t_y$ and $z = t_z$, where t_y and t_z are free parameters $x = t_y$. Therefore a point on the plane $x - y = 0$ takes the form

$$\begin{aligned}(x, y, z) &= (t_y, t_y, t_z) \\ &= (t_y, t_y, 0) + (0, 0, t_z) \\ &= t_y(1, 1, 0) + t_z(0, 0, 1)\end{aligned}$$

Therefore a basis of the solution space is $\{(1, 1, 0), (0, 0, 1)\}$. □

c. The lines $x = 2t$, $y = -t$, $z = 4t$.

Work. A point on the lines $x = 2t$, $y = -t$, $z = 4t$ take the form

$$\begin{aligned}(x, y, z) &= (2t, -t, 4t) \\ &= t(2, -1, 4)\end{aligned}$$

Therefore a basis of the solution space is $\{(2, -1, 4)\}$. □

- c. All the vectors of the form (a, b, c) , where $b = a + c$.

Work. A vector which satisfies the conditions takes the form

$$\begin{aligned}(a, b, c) &= (a, a + c, c) \\ &= (a, a, 0) + (0, c, c) \\ &= a(1, 1, 0) + c(0, 1, 1)\end{aligned}$$

Therefore a basis of the solution space is $\{(1, 1, 0), (0, 1, 1)\}$. □

11

- a. Show that the set W of all polynomials in P_2 such that $p(1) = 0$ is a subspace of P_2 .

Proof. Axiom 1: Consider p_a and $p_b \in W$. Now consider $p_a + p_b$.

$$(p_a + p_b)(x) = p_a(x) + p_b(x)$$

When $x = 1$, $(p_a + p_b)(1) = p_a(1) + p_b(1) = 0 + 0 = 0$. Therefore Axiom 1 holds.

Axiom 6: Consider $p \in W$ and $k \in \mathbb{R}$. Now consider $k \cdot p$.

$$(kp)(x) = kp(x)$$

When $x = 1$, $(kp)(1) = kp(1) = k \cdot 0 = 0$. Therefore Axiom 6 holds. Since Axiom 1 and Axiom 6 hold for W , W is a subspace of P_2 . □

- b. Make a conjecture about the dimension of W .

Conjecture. I conjecture that the dimension of W is two. □

- c. Confirm your conjecture by finding a basis for W .

Work. A polynomial which satisfies the condition $p(1) = 0$ must also satisfy the following identity.

$$p(1) = p_0 + p_1(1) + p_2(1^2) = 0 \qquad \qquad \qquad = p_0 + p_1 + p_2 = 0$$

Let $p_1 = t_1$ and $p_2 = t_2$, where t_1 and t_2 are free parameters. $p_0 = -t_1 - t_2$. Therefore a vector of coefficients takes the form

$$\begin{aligned}(p_0, p_1, p_2) &= (-t_1 - t_2, t_1, t_2) \\ &= (-t_1, t_1, 0) + (-t_2, 0, t_2) \\ &= t_1(-1, 1, 0) + t_2(-1, 0, 1)\end{aligned}$$

Therefore a basis of the solution space is $\{-1 + x, -1 + x^2\}$. There are two basis vectors, so the dimension of the subspace is two. \square

18 Let S be a basis for an n -dimensional vector space V . show that if $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ form a linearly independent set of vectors in V , then the coordinate vectors $(\vec{v}_1)_S, (\vec{v}_2)_S, \dots, (\vec{v}_r)_S$ form a linearly independent set in \mathbb{R}^n , and conversely.

Proof. Let S be a basis for an n -dimensional vector space V , and consider an linearly independent set of vectors in V , $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$. Then, by definition for $k_1, \dots, k_r \in \mathbb{R}$,

$$k_1 \vec{v}_1 + k_2 \vec{v}_2 + \dots + k_r \vec{v}_r = \mathbf{id}$$

only has the trivial solution. Now consider a change of basis into base S . The equation now becomes

$$k_1(\vec{v}_1)_S + k_2(\vec{v}_2)_S + \dots + k_r(\vec{v}_r)_S = (\mathbf{id})_S.$$

However, $(\mathbf{id})_S = \mathbf{id}$ for all basis, since it can only be obtained by a trivial linear combination, which produces a sum of \mathbf{id} . Therefore we have

$$k_1(\vec{v}_1)_S + k_2(\vec{v}_2)_S + \dots + k_r(\vec{v}_r)_S = \mathbf{id}$$

which only has the trivial solution. Therefore the coordinate vectors $(\vec{v}_1)_S, (\vec{v}_2)_S, \dots, (\vec{v}_r)_S$ form a linearly independent set in \mathbb{R}^n . Conversely, if S is changed to the original basis for V , then the converse can be achieved. \square
