Homework 7

1.4 *Inverses; Algebraic Properties of Matrices*

28. *Show that if a square matrix A satisfies $A^2 - 3A + 1 = 0$, then $A^{-1} = 3I - A$.*

Proof. Consider $3I - A$.

$$A(3I - A) = 3AI - A^2 = 3A - A^2$$

If $A^2 - 3A + I = 0$, then we can simplify further to determine exactly what $3A - A^2$ equals.

$$\begin{aligned}
A^2 - 3A + I &= 0 \\
(3A - A^2) + A^2 - 3A + I &= (3A - A^2) + 0 \\
(3A + (-A^2 + A^2) - 3A) + I &= (3A - A^2) \\
(3A - 3A) + I &= 3A - A^2 \\
I &= 3A - A^2 \\
\therefore\ I &= A(3I - A) \\
\therefore\ I &= (3I - A)A
\end{aligned}$$

Since $I = A(3I - A)$ and $I = (3I - A)A$, therefore $A^{-1} = 3I - A$ if $A^2 - 3A + 1 = 0$. □

31. *Assuming that all matrices are $n \times n$ and invertible, solve for D:*

$$C^T B^{-1} A^2 BAC^{-1} DA^{-2} B^T C^{-2} = C^T.$$

Work.

$$\begin{aligned}
C^T B^{-1} A^2 BAC^{-1} DA^{-2} B^T C^{-2} &= C^T \\
(C^T B^{-1} A^2 BAC^{-1})^{-1} C^T B^{-1} A^2 BAC^{-1} DA^{-2} B^T C^{-2} &= (C^T B^{-1} A^2 BAC^{-1})^{-1} C^T \\
DA^{-2} B^T C^{-2} &= CA^{-1} B^{-1} A^{-2} BC^{T^{-1}} C^T \\
DA^{-2} B^T C^{-2} (A^{-2} B^T C^{-2})^{-1} &= CA^{-1} B^{-1} A^{-2} BC^{T^{-1}} C^T (A^{-2} B^T C^{-2})^{-1} \\
D &= CA^{-1} B^{-1} A^{-2} BC^{T^{-1}} C^T C^2 B^{T^{-1}} A^2 \\
D &= CA^{-1} B^{-1} A^{-2} BC^2 B^{T^{-1}} A^2
\end{aligned}$$

□

39. *Using Matrix Inversion, find the unique solution of the given linear system.*

$$\begin{aligned}
3x_1 - 2x_2 &= -1 \\
4x_1 + 5x_2 &= 3
\end{aligned}$$

Work.

$$\begin{bmatrix}3 & -2\\4 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$$
$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}3 & -2\\4 & 5\end{bmatrix}^{-1}\begin{bmatrix}-1\\3\end{bmatrix}$$
$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \frac{1}{3\cdot 5 - -2\cdot 4}\begin{bmatrix}5 & 2\\-4 & 3\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix}$$
$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \frac{1}{23}\begin{bmatrix}5 & 2\\-4 & 3\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix}$$
$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\frac{5}{23} & \frac{2}{23}\\-\frac{4}{23} & \frac{3}{23}\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix}$$
$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\frac{1}{23}\\\frac{13}{23}\end{bmatrix}$$

□

53a. *Show that if A, B and $A+B$ are invertible matrices with the same size, then*

$$A(A^{-1}+B^{-1})B(A+B)^{-1} = I.$$

Work.

$$\begin{aligned}A(A^{-1}+B^{-1})B(A+B)^{-1} &= (I+AB^{-1})B(A+B)^{-1}\\ &= (B+A)(A+B)^{-1} = (A+B)(A+B)^{-1} = I\end{aligned}$$

$$\therefore\ A(A^{-1}+B^{-1})B(A+B)^{-1} = I$$

□

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55. *Show that if A is a square matrix such that $A^k = 0$ for some positive integer k, then the matrix $(I-A)$ is invertible and*

$$(I-A)^{-1} = I + A + A^2 + \cdots + A^{k-1}.$$

Proof. Consider square matrix A such that $A^k = 0$ for some positive integer k. Now consider the matrices $(I-A)$ and $(I+A+A^2+\cdots+A^{k-1})$.

$$\begin{aligned}(I-A)(I+A+A^2+\cdots+A^{k-1}) &= (I+A+A^2+\cdots+A^{k-1}) - (A+A^2+\cdots+A^k)\\ &= I + A - A + A^2 - A^2 + \cdots + A^{k-1} - A^{k-1} + A^k\\ &= I + A^k\\ &= I + \vec{0} = I\end{aligned}$$

Therefore $(I-A)(I+A+A^2+\cdots+A^{k-1})=I$.

$$\begin{aligned}(I+A+A^2+\cdots+A^{k-1})(I-A) &= (I+A+A^2+\cdots+A^{k-1})-(A+A^2+\cdots+A^k)\\ &= I+A-A+A^2-A^2+\cdots+A^{k-1}-A^{k-1}+A^k\\ &= I+A^k\\ &= I+\vec{0}=I\end{aligned}$$

Therefore $(I+A+A^2+\cdots+A^{k-1})(I-A)=I$. Since $(I+A+A^2+\cdots+A^{k-1})(I-A)=I$ and $(I-A)(I+A+A^2+\cdots+A^{k-1})=I$, therefore $(I-A)^{-1}=I+A+A^2+\cdots+A^{k-1}$ □

1.5 *Elementary Matrices and a Method for Finding A-1*

15. *Use the inverse algorithm to find the inverse of the given matrix, if the inverse exists.*

$$\begin{bmatrix}-1&3&-4\\2&4&1\\-4&2&-9\end{bmatrix}.$$

Proof.

$$\begin{bmatrix}-1&3&-4\\2&4&1\\-4&2&-9\end{bmatrix}\xrightarrow[-R_1]{R_2+2R_1}\begin{bmatrix}1&-3&4\\0&10&-7\\-4&2&-9\end{bmatrix}\xrightarrow{R_4+4R_1}\begin{bmatrix}1&-3&4\\0&10&-7\\0&-10&7\end{bmatrix}\xrightarrow{R_3+R_2}\begin{bmatrix}1&-3&4\\0&10&-7\\0&0&0\end{bmatrix}$$

The last matrix has a row of all zeros, which makes it impossible for rref$(A)=I$, thus rref$(A)\neq I$. Therefore, by the Big Theorem from Lecture Note 23, A is not invertible. □

25. *Find the inverse of the following 4×4 matrices, where $k_1,k_2,k_3,k_4,$ and k are all non-zero.*

a. $\begin{bmatrix}k_1&0&0&0\\0&k_2&0&0\\0&0&k_3&0\\0&0&0&k_4\end{bmatrix}.$

Work.

$$\left[\begin{array}{cccc|cccc}k_1&0&0&0&1&0&0&0\\0&k_2&0&0&0&1&0&0\\0&0&k_3&0&0&0&1&0\\0&0&0&k_4&0&0&0&1\end{array}\right]\xrightarrow[\frac{1}{k_3}R_3,\ \frac{1}{k_4}R_4]{\frac{1}{k_1}R_1,\ \frac{1}{k_2}R_2}\left[\begin{array}{cccc|cccc}1&0&0&0&\frac{1}{k_1}&0&0&0\\0&1&0&0&0&\frac{1}{k_2}&0&0\\0&0&1&0&0&0&\frac{1}{k_3}&0\\0&0&0&1&0&0&0&\frac{1}{k_4}\end{array}\right]$$

$$\therefore\begin{bmatrix}k_1&0&0&0\\0&k_2&0&0\\0&0&k_3&0\\0&0&0&k_4\end{bmatrix}^{-1}=\begin{bmatrix}\frac{1}{k_1}&0&0&0\\0&\frac{1}{k_2}&0&0\\0&0&\frac{1}{k_3}&0\\0&0&0&\frac{1}{k_4}\end{bmatrix}$$

□

b. $\begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Work.

$$\left[\begin{array}{cccc|cccc} k_1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k_3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-R_4]{R_1-R_2} \left[\begin{array}{cccc|cccc} k & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{k}R_3]{\frac{1}{k}R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & -\frac{1}{k} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k} & -\frac{1}{k} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\therefore \begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{k} & -\frac{1}{k} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{k} & -\frac{1}{k} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

□

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27. *Find all values of c, if any, for which the given matrix is invertible.*

$$\begin{bmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{bmatrix}$$

Work.

$$\left[\begin{array}{ccc|ccc} c & c & c & 1 & 0 & 0 \\ 1 & c & c & 0 & 1 & 0 \\ 1 & 1 & c & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2-R_3]{R_1-R_2} \left[\begin{array}{ccc|ccc} c-1 & 0 & 0 & 1 & -1 & 0 \\ 0 & c-1 & 0 & 0 & 1 & -1 \\ 1 & 1 & c & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{c-1}R_2]{\frac{1}{c-1}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{c-1} & -\frac{1}{c-1} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{c-1} & -\frac{1}{c-1} \\ 1 & 1 & c & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-R_1]{R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{c-1} & -\frac{1}{c-1} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{c-1} & -\frac{1}{c-1} \\ 0 & 0 & c & -\frac{1}{c-1} & 0 & \frac{c}{c-1} \end{array}\right]$$

$$\xrightarrow{\frac{1}{c}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{c-1} & -\frac{1}{c-1} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{c-1} & -\frac{1}{c-1} \\ 0 & 0 & 1 & -\frac{1}{c(c-1)} & 0 & \frac{1}{c-1} \end{array}\right]$$

The resulting inverse matrix is undefined when $c = 0$ or when $c = 1$, therefore $c \neq 0,\ 1$. □

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29. *Write the given matrix as a product of elementary matrices.*

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$$

Work.

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 5 \\ 0 & -8 \end{bmatrix} \xrightarrow{-\frac{1}{8}R_2} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1-5R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Each of these row operations can be expressed as a left multiplication of a elementary matrix.

$$\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} = \left(\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{8} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

□

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41. *Prove that if A and B are $m \times n$ matrices, then A and B are row equivalent if and only if A and B have the same reduced row echelon form.*

Proof. First, if A and B are row equivalent, there exists a sequence of elementary row operations that can transform B into A. Now consider the $rref(A)$; it is by definition obtainable by row operations starting at A. Therefore there exists a sequence of row operations which transform B into A, and then A into $rref(A)$. The same can be said for A. Since reduced row echelon form is unique, thus $rref(A)$ is also the $rref(B)$. Therefore, A and B have the same reduced row echelon form.

Second, if A and B have the same reduced row echelon form, then there exists a sequence of elementary row operations that can transform A and B into the same reduced row echelon form. By definition there is a sequence of elementary row matrices to transform A into $rref(A)$, which is also $rref(B)$. And also by definition there are elementary row matrices to transform $rref(B)$ into B. Therefore, there is a sequence of elementary row operations to transform A into B. Therefore, by definition, A and B are row equivalent.

Combining both statements, we can conclude that A and B are row equivalent if and only if they have the same reduced row echelon form. □

1.6 *More on Linear Systems and Invertible Matrices*

15. *Determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.*

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$$

Work.

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{array}\right] \xrightarrow{R_3-R_1} \begin{bmatrix} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -4 & 5 & -8 & b_3-b_1 \end{bmatrix} \xrightarrow[R_2-4R_1]{R_3+R_2} \begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & b_2-4b_1 \\ 0 & 0 & 0 & b_3-b_1+b_2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 1 & -4 & \frac{1}{3}(b_2-4b_1) \\ 0 & 0 & 0 & b_3-b_1+b_2 \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 1 & 0 & -3 & b_1+\frac{2}{3}(b_2-4b_1) \\ 0 & 1 & -4 & \frac{1}{3}(b_2-4b_1) \\ 0 & 0 & 0 & b_3-b_1+b_2 \end{bmatrix}$$

The last row represents $0 = b_3 - b_1 + b_2$, which is $b_1 = b_3 + b_2$. If $b_1 = b_3 + b_2$, then reduced row echelon form is complete, and there are infinitely many solutions to the linear system. If $b_1 \neq b_3 + b_2$, then the last row represents $0 = k$ where $k \neq 0$, and there is no solution to the linear system. □

21. *Let $A\vec{x} = \vec{0}$ be a homogenous system of n linear equations in n unknown that has only the trivial solution. Show that if k is any positive integer, then the system $A^k\vec{x} = \vec{0}$ also has only the trivial solution.*

Proof. Since $A\vec{x} = \vec{0}$ has only the trivial solution, through the Big Theorem of Lecture 23, this means that A is invertible. This also means that $(A^k)^{-1}$ exists, as it is known that $(A^k)^{-1} = A^{-k} = (A^{-1})^k$. Now Consider the system $A^k\vec{x} = \vec{0}$.

$$\begin{aligned} A^k\vec{x} &= \vec{0} \\ (A^k)^{-1}A^k\vec{x} &= (A^k)^{-1}\vec{0} \\ I\vec{x} &= \vec{0} \\ \vec{x} &= \vec{0} \end{aligned}$$

Therefore $\vec{x} = \vec{0}$, and the only solution for the system is the trivial solution. □