MAT 260 LINEAR ALGEBRA
LECTURE 40

WING HONG TONY WONG

4.5 — Dimension

Lemma 1. *Let V be a vector space with $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a spanning set. Let T be a subset of vectors in V with more than n vectors. Then T is linearly dependent.*

Theorem 2. *Let V be a vector space with $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis. Let T be a subset of vectors in V.*
(a) If T has more than n vectors, then T is linearly dependent.
(b) If T has less than n vectors, then $\mathrm{span}(T) \neq V$.

Corollary 3. *Let V be a vector space with $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis. Then every basis of V has exactly n vectors.*

Let V be a vector space with a basis of size n. Then V is **finite-dimensional** with **dimension** n, denoted by $\dim(V)$.

Example 4. The zero vector space has the empty set as the basis, so it has dimension 0.

Example 5. $\dim(\mathbb{R}^n) = n$, $\dim(P_n) = n + 1$, $\dim(M_{mn}(\mathbb{R})) = mn$.

Example 6. If V is a vector space with S a finite linearly independent subset, then $\dim(\mathrm{span}(S)) = |S|$.

Example 7. In the matrix equation $A\mathbf{x} = \mathbf{0}$, the dimension of the solution space is the number of free variables. For example, if the solution space is $\{(-s+2t, s, t, s-t) \in \mathbb{R}^4 : s, t \in \mathbb{R}\}$, then a basis of this solution space is $\{(-1, 1, 0, 1), (2, 0, 1, -1)\}$.

Theorem 8 (Plus/Minus Theorem)**.** *Let V be a vector space and S be a subset of V.*
(a) If S is linearly independent, and if $\mathbf{v} \in V \backslash \mathrm{span}(S)$, then $S \cup \{\mathbf{v}\}$ is linearly independent.
(b) If $\mathbf{v} \in S$ can be expressed as a linear combination of other vectors in S, then $\mathrm{span}(S) = \mathrm{span}(S \backslash \{v\})$.

Date: Monday, May 1, 2023.

Corollary 9. *Let V be a finite-dimensional vector space and S be a finite subset of V.*
(a) If S is linearly independent but not a spanning set, then S can be enlarged into a basis.
(b) If S is a spanning set but linearly dependent, then S can be reduced into a basis.

Theorem 10. *Let V be an n-dimensional vector space with S a subset of size n. Then S is a basis if and only if S is linearly independent or S is a spanning set.*

Theorem 11. *Let V be a finite-dimensional vector space with W a subspace. Then*
(a) W is finite-dimensional.
(b) $\dim(W) \leq \dim(V)$.
(c) $W = V$ if and only if $\dim(W) = \dim(V)$.