Section 0, p8 12, 16, 17, 23, 25, 29, 31, 33

12 Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. For each relation between A and B given as a subset of $A \times B$, decide whether it is a function mapping A into B. If it is a function, decide whether it is one to one and whether it is onto B.

a. $\{(1, 4), (2, 4), (3, 6)\}$

Since no two ordered pairs have the same first term, this **is a function**. Since $(1, 4)$ and $(2, 4)$ share a second term, this function **is *not* one to one**. Since 2 does not appear in the relation in the second term, this function **is *not* onto** B.

b. $\{(1, 4), (2, 6), (3, 4)\}$

Since no two ordered pairs have the same first term, this **is a function**. Since $(1, 4)$ and $(3, 4)$ share a second term, this function **is *not* one to one**. Since 2 does not appear in the relation in the second term, this function **is *not* onto** B.

c. $\{(1, 6), (1, 2), (1, 4)\}$

Since $(1, 6)$ and $(1, 2)$ share a first term, this **is *not* a function**.

d. $\{(2, 2), (1, 6), (3, 4)\}$

Since no two ordered pairs have the same first term, this **is a function**. Since no two ordered pairs share the same second term, this function **is one to one**. Since every element of B appears in the range of the function, it **is onto** B.

e. $\{(1, 6), (2, 6), (3, 6)\}$

Since no two ordered pairs have the same first term, this **is a function**. Since $(1, 6)$ and $(2, 6)$ share a second term, this function **is *not* one to one**. Since 2 does not appear in the relation in the second term, this function **is *not* onto** B.

f. $\{(1, 2), (2, 6), (2, 4)\}$

Since $(2, 6)$ and $(2, 4)$ share a first term, this **is *not* a function**.

16 List the elements of the power set $\mathcal{P}$ of the given set and give the cardinality of the power set.

a. $\emptyset$

$|\{\emptyset\}| = 1$

b. $\{a\}$

$|\{\emptyset, \{a\}\}| = 2$

c. $\{a, b\}$

$|\{\emptyset, \{a\}, \{b\}, \{a, b\}\}| = 4$

d. $\{a, b, c\}$

$|\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, c, b\}\}| = 8$

17 Let A be a finite set, and let $|A| = s$. Based on the preceding exercise, make a conjecture about the value of $|\mathcal{P}(A)|$. Then try to prove your conjecture.

Based on the previous exercise, I conjecture that $|\mathcal{P}(A)| = 2^s$.

Proof. We shall conduct a *proof through induction.*

Base case: Let $A = \emptyset$ and $s = 0$. $\mathcal{P}(A) = \{\emptyset\}$, and $|\mathcal{P}(A)| = 1 = 2^0$. Thus the base case holds.

Inductive hypothesis: Let $|A| = k$ for some $k \in \mathbb{Z}^{\geq 0}$, and assume that $|\mathcal{P}(A)| = 2^k$.

Inductive case: Consider $|A| = k$. Let us add an element e to A to create A'. To consider $\mathcal{P}(A')$, start with all of the sets in $\mathcal{P}(A)$, which has a cardinality of 2^k via the inductive hypothesis.

For each of these sets, we have a choice when we add it to $\mathcal{P}(A')$ to include element e or to leave it alone, which are two choices. We will end with the original sets of the $\mathcal{P}(A)$ without e, along with these sets with e, in $\mathcal{P}(A')$. Each of these groups has a cardinality of 2^k, so the cardinality of $\mathcal{P}(A')$ must be

$$2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

Thus, the inductive case holds, and the conjecture is correct. □

In Exercises 23 through 27, find the number of different partitions of a set having the given number of elements.

23. 1 element

There is a single partition, $\{\{e\}\}$

25. 3 elements

There are 5 partitions,

$$\{\{e_1\}, \{e_2\}, \{e_3\}\}, \quad \{\{e_1\}, \{e_2, e_3\}\}, \quad \{\{e_1, e_2\}, \{e_3\}\}, \quad \{\{e_1, e_3\}, \{e_2\}\}, \quad \{\{e_1, e_2, e_3\}\}$$

In Exercises 29 through 34, determine whether the given relation is an equivalence relation on the set. Describe the partition arising from each equivalence relation.

29. $n\mathcal{R}m$ in $\mathbb{Z}$ if $nm > 0$

Consider $0 \in \mathbb{Z}$. We know $0 \cdot 0 \not> 0$, thus 0 is not related to itself. So the relation is not reflexive, and cannot be an equivalence relation.

31. $x\mathcal{R}y$ in $\mathbb{R}$ if $|x| = |y|$

A equivalence relation must be reflexive, symmetric, and transitive.

(a) Reflexive: Consider $x \in \mathbb{R}$. We know $|x| = |x|$, thus $\mathcal{R}$ is reflexive.

(b) Symmetric: Assume $x\mathcal{R}y$. We know $|x| = |y|$ also implies $|y| = |x|$, thus $y\mathcal{R}x$. So $\mathcal{R}$ is symmetric.

(c) Transitive: Assume $x\mathcal{R}y$ and $y\mathcal{R}z$. We $|x| = |y|$ and $|y| = |z|$ which implies $|x| = |z|$, thus $z\mathcal{R}x$. So $\mathcal{R}$ is transitive.

The partition resulting from this equivalence relation will be sets $\overline{x} = \{x, -x\}$ for each $x \in \mathbb{R}$.

33. $n\mathcal{R}m$ in $\mathbb{Z}^+$ if n and m have the same number of digits in the usual base ten notation

A equivalence relation must be reflexive, symmetric, and transitive.

(a) Reflexive: Consider $x \in \mathbb{Z}^+$. We know a number has the same digits as itself, thus $\mathcal{R}$ is reflexive.

(b) Symmetric: Assume $x\mathcal{R}y$. We know x and y have the same number of digits, implying y and x have the same number of digits, thus $y\mathcal{R}x$. So $\mathcal{R}$ is symmetric.

(c) Transitive: Assume $x\mathcal{R}y$ and $y\mathcal{R}z$. We know x and y have the same number of digits, and that y and z have the same number of digits, which implies x and z have the same number of digits, thus $z\mathcal{R}x$. So $\mathcal{R}$ is transitive.

The partition resulting from this equivalence relation will be classes where each member has the same number of digits, i.e. all single digit numbers will be in a class, all two digit numbers will be in a class, all three digit numbers will be in a class, etc.