2.6.1

a. Prove that $\frac{\sqrt{2}}{2}$ is irrational.

Proof. Assume that $\frac{\sqrt{2}}{2}$ is rational. That is, it takes the form $\frac{p}{q}$, for some $p, q \in \mathbb{Z}$, where $q \nmid p$. This also means that $\sqrt{2}$ must be rational, that is it takes the form $\frac{m}{n}$, for some $m, n \in \mathbb{Z}$, where $n \nmid m$.

$$\begin{aligned}\sqrt{2} &= \frac{m}{n}\\ \sqrt{2}^2 &= (\frac{m}{n})^2\\ 2 &= \frac{m^2}{n^2}\\ 2n^2 &= m^2\end{aligned}$$

Since $2n^2 = m^2$, m^2 is an even number. Since the square of an even integer is also even, this means that m is also even; it takes the form $2k$, for some $k \in \mathbb{Z}$. Therefore,

$$m^2 = (2k)^2 = 4k^2$$

$$2n^2 = 4k^2$$

This implies that there is a common factor between n and m, and $m \mid n$. This contradicts the assertion that $m \nmid n$. Therefore $\sqrt{2}$ cannot be rational, and must be irrational. □

b. Prove that $2 - \sqrt{2}$ is irrational.

Proof. Assume that $2 - \sqrt{2}$ is rational. That is, it takes the form $\frac{p}{q}$, for some $p, q \in \mathbb{Z}$. This assertion also implies that 2 is rational and $\sqrt{2}$ is rational. This was contradicted in problem 2.6.1a. Therefore $2 - \sqrt{2}$ cannot be rational, and must be irrational. □

2.6.2

a. Prove that if n is an integer such that n^3 is even, then n is even.

Proof. Assume that n^3 is odd. That is, $n^3 = 2k + 1$, for some $k \in \mathbb{Z}$. Consider $k = 4j^3 + 6j^2 + 2j$, for some $j \in \mathbb{Z}$

$$\begin{aligned}n^3 &= 2k + 1 = 2(4j^3 + 6j^2 + 2j) + 1\\ &= 8j^3 + 12j^2 + 4j + 1\\ &= 8j^3 + 8j^2 + 2j + 4j^2 + 4j + 1\\ &= (4j^2 + 4j + 1)(2j + 1)\\ &= (2j + 1)^3\end{aligned}$$

Therefore n takes the form $2k + 1$, an odd integer. This contradicts the conclusion that n is even. Therefore, if n is an integer such that n^3 is even, then n is even. □

2.6.6

a. If a group of 9 kids have won a total of 100 trophies, then at least one of the 9 kids has won as least 12 trophies.

Proof. Assume that if a group of 9 kids have won a total of 100 trophies, then all of the 9 kids have won at fewer than 12 trophies. This means that the total number of trophies must be:

$$\begin{aligned} \text{trophies} &\leq 9 \cdot 11 \\ &\leq 99 \end{aligned}$$

This contradicts the assertion that 9 kids have won a total of 100 trophies. Therefore, if a group of 9 kids have won a total of 100 trophies, then at least one of the 9 kids has won as least 12 trophies □

b. If a person buys at least 400 cups of coffee in a year, then there is at least one day in which the person has bought at least two cups of coffee.

Proof. Assume that if a persons buys at least 400 cups of coffee in a year, then there are no days in which the person has bought at least two cups of coffee. Therefore, the total number of coffees must be:

$$\begin{aligned} \text{cups of coffee} &\leq 1 \cdot 365 \\ &\leq 365 \end{aligned}$$

This contradicts with the assertion that a person buys at least 400 cups of coffee in a year. Therefore, if a person buys at least 400 cups of coffee in a year, then there is at least one day in which the person has bought at least two cups of coffee □