

Homework 4

Problem 9

Let $V = \mathbb{R}^3$ be a vector space with standard addition and scalar multiplication. Use Theorem 3 of Lecture Note 8 to determine whether the following sets are subspaces of V .

- b. The set of vectors of the form $(a, 1, 1)$, W

Proof. Axiom 1: Consider $\vec{a} = (a, 1, 1)$ and $\vec{b} = (b, 1, 1)$ for $a, b \in \mathbb{R}$. $\vec{a} \in W$ and $\vec{b} \in W$.

$$\begin{aligned}\vec{a} \oplus_W \vec{b} &= (a, 1, 1) \oplus (b, 1, 1) = (a + b, 1 + 1, 1 + 1) \\ &= (a + b, 2, 2) \notin W\end{aligned}$$

Therefore W is **not** closed under addition.

Since Axiom 1 does not hold for W , W is not a subspace of V . □

- c. The set of vectors of the form (a, b, c) , where $b = a + c$, W

Proof. Axiom 1: Consider $\vec{v} = (a_1, b_1, c_1)$ and $\vec{u} = (a_2, b_2, c_2)$ for $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$. $\vec{v} \in W$ and $\vec{u} \in W$.

$$\begin{aligned}\vec{v} \oplus \vec{u} &= (a_1, b_1, c_1) \oplus (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2) \\ &\in W\end{aligned}$$

$$b_1 + b_2 = (a_1 + c_1) + (a_2 + c_2) = (a_1 + a_2) + (c_1 + c_2) \checkmark$$

Therefore W is closed under addition.

Axiom 6: Consider $\vec{v} = (a, b, c)$ such that $a, b, c \in \mathbb{R}$ and $k \in \mathbb{R}$. Let $b = a + c$.

$$\begin{aligned}k \odot \vec{v} &= k \odot (a, b, c) = k \odot (a, a + c, c) = (ka, k(a + c), kc) = (ka, ka + kc, kc) \\ &\in W\end{aligned}$$

$$kb = k(a + c) = ka + kc \checkmark$$

Therefore W is closed under scalar multiplication.

Since Axiom 1 and Axiom 6 hold for W , \oplus and \odot are inherited from V , and $W \subseteq V$, through the use of Theorem 3, W is a subspace of V . □

- d. The set of vectors of the form $(a, b, 0)$, W

Proof. Axiom 1: Consider $\vec{v} = (a_1, b_1, 0)$ and $\vec{u} = (a_2, b_2, 0)$ for $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

$$\begin{aligned}\vec{v} \oplus \vec{u} &= (a_1, b_1, 0) \oplus (a_2, b_2, 0) = (a_1 + a_2, b_1 + b_2, 0 + 0) = (a_1 + a_2, b_1 + b_2, 0) \\ &\in W \text{ since it takes the form } (a, b, 0)\end{aligned}$$

Therefore W is closed under addition.

Axiom 6: Consider $\vec{v} = (a, b, 0)$ such that $a, b \in \mathbb{R}$ and $k \in \mathbb{R}$.

$$\begin{aligned}k \odot \vec{v} &= k \odot (a, b, 0) = (ka, kb, k0) = (ka, kb, 0) \\ &\in W \text{ since it takes the form } (a, b, 0)\end{aligned}$$

Therefore W is closed under scalar multiplication.

Since Axiom 1 and Axiom 6 hold for W , \oplus and \odot are inherited from V , and $W \subseteq V$, through the use of Theorem 3, W is a subspace of V . □

Problem 10

Let $V = P_3$ be the vector space of all polynomials with degree up to 3, with standard addition and scalar multiplication. Use Theorem 3 of Lecture Note 8 to determine whether the following sets are subspaces of V .

- b. The set of polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$, W .

Proof. Axiom 1: Consider $\vec{a} = a_0 + a_1x + a_2x^2 + a_3x^3$ and $\vec{b} = b_0 + b_1x + b_2x^2 + b_3x^3$ where $a_{0-3}, b_{0-3} \in \mathbb{R}$.

Let $a_0 + a_1 + a_2 + a_3 = 0$ and $b_0 + b_1 + b_2 + b_3 = 0$.

$$\begin{aligned}\vec{a} \oplus \vec{b} &= (a_0 + a_1x + a_2x^2 + a_3x^3) \oplus (b_0 + b_1x + b_2x^2 + b_3x^3) \\ &= a_0 + a_1x + a_2x^2 + a_3x^3 + b_0 + b_1x + b_2x^2 + b_3x^3 \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \\ &\in W\end{aligned}$$

$$\begin{aligned}(a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) &= a_0 + a_1 + a_2 + a_3 + b_0 + b_1 + b_2 + b_3 \\ &= 0 + 0 = 0 \quad \checkmark\end{aligned}$$

Therefore W is closed under addition.

Axiom 6: Consider $\vec{a} = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $a_{0-3} \in \mathbb{R}$ and $k \in \mathbb{R}$.

Let $a_0 + a_1 + a_2 + a_3 = 0$.

$$\begin{aligned}k \odot \vec{a} &= k \odot (a_0 + a_1x + a_2x^2 + a_3x^3) = k(a_0 + a_1x + a_2x^2 + a_3x^3) \\ &= ka_0 + ka_1x + ka_2x^2 + ka_3x^3 \\ &\in W\end{aligned}$$

$$ka_0 + ka_1x + ka_2x^2 + ka_3x^3 = k(a_0 + a_1x + a_2x^2 + a_3x^3) = k(0) = 0 \quad \checkmark$$

Therefore W is closed under scalar multiplication.

Since Axiom 1 and Axiom 6 hold for W , \oplus and \odot are inherited from V , and $W \subseteq V$, through the use of Theorem 3, W is a subspace of V . \square

- c. The set of polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2 , and a_3 are integers.

Proof. Axiom 6: Consider $\vec{a} = 1 + 1x + 1x^2 + 1x^3$ and $k = 0.66$.

$$\begin{aligned}k \odot \vec{a} &= 0.66 \odot (1 + 1x + 1x^2 + 1x^3) = 0.66(1 + 1x + 1x^2 + 1x^3) \\ &= 0.66 + 0.66x + 0.66x^2 + 0.66x^3 \\ &\notin W, \text{ since } 0.66 \notin \mathbb{Z}\end{aligned}$$

Therefore W is **not** closed under scalar multiplication.

Since Axiom 6 does not hold for W , W is not a subspace of V . \square

Problem 11

Let $V = F(-\infty, \infty)$ be the vector space of all functions from \mathbb{R} to \mathbb{R} , with standard addition and scalar multiplication. Use Theorem 3 of Lecture Note 8 to determine whether the following sets are subspaces of V .

- b. The set of functions f in $F(-\infty, \infty)$ for which $f(0) = 1$.

Proof. Axiom 1: Consider $\vec{f}(x) = e^x$ and $\vec{g}(x) = e^x$. $\vec{f}, \vec{g} \in W$ since $e^0 = 0$.

$$(\vec{f} \oplus \vec{g})(x) = \vec{f}(x) + \vec{g}(x) = e^x + e^x = 2e^x \notin W$$

when $x = 0 : 2e^0 = 2 \cdot 1 = 2 \neq 0$

Therefore W is **not** closed under addition.

Since Axiom 1 does not hold for W , W is not a subspace of V . □

c. The set of functions f in $F(-\infty, \infty)$ for which $f(-x) = x$

Proof. □

Problem 13

Let V be a vector space. Let I be a nonempty set (often called the "index set"), and let W_i be a subspace of V for all $i \in I$. Prove that $\bigcap_{i \in I} W_i$ is a subspace of V .

Proof. □