# MAT 260 LINEAR ALGEBRA
# LECTURE 38

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## 4.4 — Basis and coordinates

Let $V$ be a vector space.

**Definition 1.** $S \subseteq V$ is a **basis** of $V$ if

- $S$ is linearly independent, and
- $\text{span}(S) = V$.

Note that a basis of a vector space is not unique in general.

**Definition 2.** $V$ is **finite-dimensional** if $V$ has a basis of finite size. Otherwise, $V$ is **infinite-dimensional**.

**Example 3.** In $\mathbb{R}^n$, the set of standard unit vectors, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ forms a basis. This is called the standard basis of $\mathbb{R}^n$. In $\mathbb{R}^3$, the set $\{(1,1,1),(1,2,4),(1,3,9)\}$ also forms a basis.

**Example 4.** In $P_n$, the set of standard unit vectors, $\{1, x, x^2, \ldots, x^n\}$ forms a basis. This is called the standard basis of $P_n$. In $P_2$, the set $\{1, 1+x, 1+x+x^2\}$ also forms a basis.

**Example 5.** In $\mathbb{R}^\infty$, the set of standard unit vectors, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$ does not form a basis. However, in $P_\infty$, the set of standard unit vectors $\{1, x, x^2, \ldots\}$ forms a basis. In these two vector spaces, it is easy to see they are both infinite-dimensional.

**Example 6.** In $M_{mn}(\mathbb{R})$, let $A_{ij}$ the matrix with 1 at the $ij$-th entry and 0 everywhere else. Then the set of all $A_{ij}$ forms a basis of $M_{mn}(\mathbb{R})$, and it is called the standard basis of $M_{mn}(\mathbb{R})$.

**Theorem 7.** *If $S$ is a basis of $V$, then every nonzero vector $\mathbf{v} \in V$ can be expressed uniquely in the form $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ for some $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in S$.*

Let $V$ be a finite-dimensional vector space, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of $V$. By Theorem 7, there is a map $V \to \mathbb{R}^n$ defined by

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \mapsto (\mathbf{v})_S = (a_1, a_2, \ldots, a_n).$$

$(\mathbf{v})_S$ is called the **coordinate vector of v relative to** $S$.

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**Example 8.** In $P_2$, if $S = \{1, x, x^2\}$, then $(1 - x^2)_S = (1, 0, -1)$. If we take $S' = \{1, 1 + x, 1 + x + x^2\}$ instead, then $(1 - x^2)_{S'} = (1, 1, -1)$.

**Example 9.** In $\mathbb{R}^3$, if $S = \{(1, 1, 1), (1, 0, 1), (1, 0, -1)\}$, then $(3, 2, 1)_S = (2, 0, 1)$.