

Homework 3

Problem 1

Let V be a vector space, and let $\vec{u}, \vec{v}, \vec{w} \in V$. Prove that if $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$ then $\vec{u} = \vec{v}$.

Proof. Consider $\vec{v}, \vec{u}, \vec{w} \in V$, and assume $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$.

$\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$	Assertion
$(\vec{u} \oplus \vec{w}) \oplus -\vec{w} = (\vec{v} \oplus \vec{w}) \oplus -\vec{w}$	Axiom 5 states $-\vec{w} \in V$
$\vec{u} \oplus (\vec{w} \oplus -\vec{w}) = \vec{v} \oplus (\vec{w} \oplus -\vec{w})$	Axiom 3
$\vec{u} \oplus \mathbf{id} = \vec{v} \oplus \mathbf{id}$	Def. of additive inverse
$\vec{u} = \vec{v}$	Def. of additive identity

\therefore if $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$ then $\vec{u} = \vec{v}$ □

Problem 2

Prove Theorem B.

Proof. Let $k \in \mathbb{R}$. Recall that Theorem A implies that $0 \odot \mathbf{id} = \mathbf{id}$:

$k \odot \mathbf{id} = k \odot (0 \odot \mathbf{id})$	Thm A
$= (k \cdot 0) \odot \mathbf{id}$	Axiom 9
$= 0 \odot \mathbf{id}$	
$= \mathbf{id}$	Thm A

$\therefore \forall k \in \mathbb{R}, k \odot \mathbf{id} = \mathbf{id}$ □

Problem 3

Prove Theorem D. If $k \odot \vec{u} = \mathbf{id}$, then $k = 0$ or $\vec{u} = \mathbf{id}$

Proof. Let $k \odot \vec{u} = \mathbf{id}$ and $k \neq 0$. Since $\frac{1}{k} \neq \frac{1}{0}$, $\frac{1}{k}$ is well-defined.

$k \odot \vec{u} = \mathbf{id}$	Assertion
$\frac{1}{k} \odot (k \odot \vec{u}) = \frac{1}{k} \odot (\mathbf{id})$	
$(\frac{1}{k} \cdot k) \odot \vec{u} = \frac{1}{k} \odot \mathbf{id}$	Axiom 9
$1 \odot \vec{u} = \mathbf{id}$	Thm B
$\vec{u} = \mathbf{id}$	Axiom 10

\therefore if $k \odot \vec{u} = \mathbf{id}$, then $k = 0$ or $\vec{u} = \mathbf{id}$ □

Problem 4

Prove that there does not exist a real vector space of size 2. Show that there cannot be a vector space of size 2.

Proof. Let $V = \{\vec{u}, \vec{v}\}$ be a vectorspace where $\vec{u} \neq \vec{v}$. Axiom 4 states that \mathbf{id} exists, and is unique. Therefore either $\vec{u} = \mathbf{id}$ or $\vec{v} = \mathbf{id}$.

Without the loss of generality, let $\vec{u} = \mathbf{id}$.

$\vec{u} \oplus \vec{v} = \vec{v}$	Axiom 4
$\mathbf{id} \oplus \vec{v} = \vec{v}$	

Now consider Axiom 5: $-\vec{v} \in V$. Note that $\vec{v} \oplus \vec{u} = V$, so we know $-\vec{v} \neq \vec{u}$, so therefore $\vec{v} = -\vec{v}$.

$\vec{v} \oplus \vec{v} = \vec{u} = \mathbf{id}$	Def. of Additive Inverse
$1 \odot \vec{v} \oplus 1 \odot \vec{v} = \vec{u}$	Axiom 10
$(1 + 1) \odot \vec{v} = \vec{u}$	Axiom 8
$2 \odot \vec{v} = \vec{u} = \mathbf{id}$	

By Theorem A, either $2 = 0$ or $\vec{v} = \mathbf{id}$. We know that $2 \neq 0$ and $\vec{v} \neq \mathbf{id}$ since $\vec{u} = \mathbf{id}$, a contradiction.
 \therefore A vectorspace of size 2 cannot exist.

□