Determine whether each set equipped with the given operations is a vector space. If it is a vector space, show that all 10 axioms hold; if not, find ALL axioms that fail.

Problem 13

The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by:

$$k \odot (x, y, z) = (k^2 x, k^2 y, k^2 z)$$

Axiom 1. *Proof.* $V = \mathbb{R}^3$. Let $\vec{v} = (v_1, v_2, v_3), \vec{u} = (u_1, u_2, u_3)$. $\forall\ \vec{v}, \vec{u} \in V$:

$$\vec{v} \oplus \vec{u} = (v_1, v_2, v_3) \oplus (u_1, u_2, u_3) = (v_1 + u_1, v_2 + u_2, v_3 + u_3)$$
$$v_1 + u_1, v_2 + u_2, \text{ and } v_3 + u_3 \in \mathbb{R}$$

$\therefore \forall\ \vec{v}, \vec{u} \in V :\ \vec{v} \oplus \vec{u} \in V$ □

Axiom 2. *Proof.* $\forall\ \vec{v}, \vec{u} \in V$:

$$\vec{v} \oplus \vec{u} = (v_1, v_2, v_3) \oplus (u_1, u_2, u_3) = (v_1 + u_1, v_2 + u_2, v_3 + u_3)$$
$$\vec{u} \oplus \vec{v} = (u_1, u_2, u_3) \oplus (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$
$$v_1 + u_1 = u_1 + v_1,\ v_2 + u_2 = u_2 + v_2,\ v_3 + u_3 = u_3 + v_3$$

$\therefore \forall\ \vec{v}, \vec{u} \in V :\ \vec{v} \oplus \vec{u} = \vec{u} \oplus \vec{v}$ □

Axiom 3. *Proof.* $\forall\ \vec{v}, \vec{u}, \vec{w} \in V$:

$$\begin{aligned} \vec{v} \oplus (\vec{u} \oplus \vec{w}) &= (v_1, v_2, v_3) \oplus ((u_1, u_2, u_3) \oplus (w_1, w_2, w_3)) \\ &= (v_1, v_2, v_3) \oplus (u_1 + w_1, u_2 + w_2, u_3 + w_3) \\ &= (v_1 + (u_1 + w_1), v_2 + (u_2 + w_2), v_3 + (u_3 + w_3)) \end{aligned}$$

$$\begin{aligned} (\vec{v} \oplus \vec{u}) \oplus \vec{w} &= ((v_1, v_2, v_3) \oplus (u_1, u_2, u_3)) \oplus (w_1, w_2, w_3) \\ &= (v_1 + u_1, v_2 + u_2, v_3 + u_3) \oplus (w_1, w_2, w_3) \\ &= ((v_1 + u_1) + w_1, (v_2 + u_2) + w_2, (v_3 + u_3) + w_3) \end{aligned}$$

Through the use of the properties of $\mathbb{R}$,

$$\begin{aligned} v_1 + (u_1 + w_1) &= (v_1 + u_1) + w_1 \\ v_2 + (u_2 + w_2) &= (v_2 + u_2) + w_2 \\ v_3 + (u_3 + w_3) &= (v_3 + u_3) + w_3 \end{aligned}$$

$\therefore \forall\ \vec{v}, \vec{u}, \vec{w} \in V :\ \vec{v} \oplus (\vec{u} \oplus \vec{w}) = (\vec{v} \oplus \vec{u}) \oplus \vec{w}$ □

Axiom 4. *Proof.* Let $\vec{v} = (0, 0, 0)$. $\forall\ \vec{u} \in V$:

$$\vec{v} \oplus \vec{u} = (0, 0, 0) \oplus (u_1, u_2, u_3) = (0 + u_1, 0 + u_2, 0 + u_3) = (u_1, u_2, u_3) = \vec{u}$$
$$\vec{u} \oplus \vec{v} = (u_1, u_2, u_3) \oplus (0, 0, 0) = (u_1 + 0, u_2 + 0, u_3 + 0) = (u_1, u_2, u_3) = \vec{u}$$

$\therefore \vec{v} = (0, 0, 0)$ is the additive identity for V, **id**. □

Axiom 5. *Proof.* Let $\vec{v} = (v_1, v_2, v_3)$ and $\vec{u} = (-v_1, -v_2, -v_3)$. $\forall\ \vec{v}, \vec{u} \in V$:

$$\begin{aligned} \vec{v} \oplus \vec{u} &= (v_1, v_2, v_3) \oplus (-v_1, -v_2, -v_3) \\ &= (v_1 + -v_1, v_2 + -v_2, v_3 + -v_3) \\ &= (0, 0, 0) = \ \mathbf{id} \end{aligned} \qquad \begin{aligned} \vec{u} \oplus \vec{v} &= (-v_1, -v_2, -v_3) \oplus (v_1, v_2, v_3) \\ &= (-v_1 + v_1, -v_2 + v_2, -v_3 + v_3) \\ &= (0, 0, 0) = \ \mathbf{id} \end{aligned}$$

$\therefore \vec{u}$ is the additive inverse of $\vec{v}$, $\forall\ \vec{u} \in V$ □

Axiom 6. *Proof.* Let $\vec{v} = (x, y, z)$. $\forall\ \vec{v} \in V,\ k \in \mathbb{R}$:

$$k \odot \vec{v} = k \odot (x, y, z) = (k^2x, k^2y, k^2z)$$
$$k^2x, k^2y, k^2z \in \mathbb{R}$$

$\therefore \forall\ \vec{v} \in V,\ k \in \mathbb{R}:\ k \odot \vec{v} \in V$ □

Axiom 7. *Proof.* $\forall\ \vec{v}, \vec{u} \in V,\ k \in \mathbb{R}$:

$$\begin{aligned} LHS = k \odot (\vec{v} \oplus \vec{u}) &= k \odot ((v_1, v_2, v_3) \oplus (u_1, u_2, u_3)) \\ &= k \odot (v_1 + u_1, v_2 + u_2, v_3 + u_3) \\ &= (k^2(v_1 + u_1), k^2(v_2 + u_2), k^2(v_3 + u_3)) \\ &= (k^2v_1 + k^2u_1, k^2v_2 + k^2u_2, k^2v_3 + k^2u_3) \end{aligned}$$

$$\begin{aligned} RHS = k \odot \vec{v} \oplus k \odot \vec{u} &= k \odot (v_1, v_2, v_3) \oplus k \odot (u_1, u_2, u_3) \\ &= (k^2v_1, k^2v_2, k^2v_3) \oplus (k^2u_1, k^2u_2, k^2u_3) \\ &= (k^2v_1 + k^2u_1, k^2v_2 + k^2u_2, k^2v_3 + k^2u_3) = LHS \end{aligned}$$

$\therefore \forall\ \vec{v}, \vec{u} \in V,\ k \in \mathbb{R}:\ k \odot (\vec{v} \oplus \vec{u}) = k \odot \vec{v} \oplus k \odot \vec{u}$ □

Axiom 8. *Proof.* $\forall\ \vec{v} \in V,\ k, \ell \in \mathbb{R}$:

$$\begin{aligned} LHS = (k + \ell) \odot \vec{v} &= (k + \ell) \odot (v_1, v_2, v_3) \\ &= ((k + \ell)^2v_1, (k + \ell)^2v_2, (k + \ell)^2v_3) \\ &= (k^2v_1 + 2k\ell v_1 + \ell^2v_1, k^2v_2 + 2k\ell v_2 + \ell^2v_2, k^2v_3 + 2k\ell v_3 + \ell^2v_3) \end{aligned}$$

$$\begin{aligned} RHS = k \odot \vec{v} \oplus \ell \odot \vec{v} &= k \odot (v_1, v_2, v_3) \oplus \ell \odot (v_1, v_2, v_3) \\ &= (k^2v_1, k^2v_2, k^2v_3) \oplus (\ell^2v_1, \ell^2v_2, \ell^2v_3) \\ &= (k^2v_1 + \ell^2v_1, k^2v_2 + \ell^2v_2, k^2v_3 + \ell^2v_3) \neq LHS \\ &\text{if } k \neq 0 \text{ and } \ell \neq 0 \text{ and } \vec{v} \neq (0, 0, 0) \end{aligned}$$

$\therefore$ Axiom 8 does not hold. □

Axiom 9. *Proof.* $\forall\ \vec{v} \in V,\ k, \ell \in \mathbb{R}$:

$$\begin{aligned} LHS = (k \cdot \ell) \odot \vec{v} &= (k \cdot \ell) \odot (v_1, v_2, v_3) \\ &= ((k \cdot \ell)^2v_1, (k \cdot \ell)^2v_2, (k \cdot \ell)^2v_3) \\ &= (k^2\ell^2v_1, k^2\ell^2v_2, k^2\ell^2v_3) \end{aligned}$$

$$\begin{aligned} RHS = k \odot (\ell \odot \vec{v}) &= k \odot (\ell \odot (v_1, v_2, v_3)) \\ &= k \odot (\ell^2v_1, \ell^2v_2, \ell^2v_3) \\ &= (k^2(\ell^2v_1), k^2(\ell^2v_2), k^2(\ell^2v_3)) = LHS \end{aligned}$$

$\therefore \forall\ \vec{v} \in V,\ k, \ell \in \mathbb{R} : (k \cdot \ell) \odot \vec{v} = k \odot (\ell \odot \vec{v})$ □

Axiom 10. *Proof.* $\forall\ \vec{v} \in V$:

$$1 \odot \vec{v} = 1 \odot (v_1, v_2, v_3) = (1^2v_1, 1^2v_2, 1^2v_3) = (v_1, v_2, v_3) = \vec{v}$$

$\therefore \forall\ \vec{v} \in V : 1 \odot \vec{v} = \vec{v}$ □

All Axioms *except* Axiom 8 work, therefore this is not a real vector space.

Problem 14

The set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 0$, and the addition and scalar multiplication operations are the same as those introduced in Example 6:

$$(\vec{f} \oplus \vec{g})(x) = \vec{f}(x) + \vec{g}(x)$$
$$(k \odot \vec{f})(x) = k\vec{f}(x)$$

Let $F = \{f : \mathbb{R} \to \mathbb{R} \text{ such that } f(1) = 0\}$.

Axiom 1. *Proof.* $\forall\ \vec{f}, \vec{g} \in F,\ x \in \mathbb{R}$:

$$\begin{aligned} (\vec{f} \oplus \vec{g})(x) &= \vec{f}(x) + \vec{g}(x) \in \mathbb{R}, \\ &\text{since } \vec{f}(x) \in \mathbb{R} \text{ and } \vec{g}(x) \in \mathbb{R}. \\ &\text{when } x = 1 : (\vec{f} \oplus \vec{g})(1) = \vec{f}(1) + \vec{g}(1) = 0 + 0 = 0 \checkmark \end{aligned}$$

$\therefore \forall\ \vec{f}, \vec{g} \in F : \vec{f} \oplus \vec{g} \in F$ □

Axiom 2. *Proof.* $\forall\ \vec{f}, \vec{g} \in F,\ x \in \mathbb{R}$:

$$\begin{aligned} LHS &= (\vec{f} \oplus \vec{g})(x) = \vec{f}(x) + \vec{g}(x) \\ RHS &= (\vec{g} \oplus \vec{f})(x) = \vec{g}(x) + \vec{f}(x) = \vec{f}(x) + \vec{g}(x) = LHS \end{aligned}$$

$\therefore \forall\ \vec{f}, \vec{g} \in F : \vec{f} \oplus \vec{g} = \vec{g} \oplus \vec{f}$ □

Axiom 3. *Proof.* $\forall\ \vec{f}, \vec{g}, \vec{h} \in F,\ x \in \mathbb{R}$:

$$\begin{aligned} LHS = (\vec{f} \oplus (\vec{g} \oplus \vec{h}))(x) &= \vec{f}(x) + (\vec{g} \oplus \vec{h})(x) \\ &= \vec{f}(x) + (\vec{g}(x) + \vec{h}(x)) \end{aligned}$$

$$\begin{aligned} RHS = ((\vec{f} \oplus \vec{g}) \oplus \vec{h})(x) &= (\vec{f} \oplus \vec{g})(x) + \vec{h}(x) \\ &= (\vec{f}(x) + \vec{g}(x)) + \vec{h}(x) \\ &= \vec{f}(x) + (\vec{g}(x) + \vec{h}(x)) = LHS \end{aligned}$$

$\therefore \forall\ \vec{f}, \vec{g}, \vec{h} \in F : \vec{f} \oplus (\vec{g} \oplus \vec{h}) = (\vec{f} \oplus \vec{g}) \oplus \vec{h}$ □

Axiom 4. *Proof.* Let $\vec{f} : \mathbb{R} \to \mathbb{R}$ such that $\forall\ x \in \mathbb{R} : \vec{f}(x) = 0$. $\forall\ \vec{g} \in F,\ \forall\ x \in \mathbb{R}$:

$$\text{since } \vec{f} : \mathbb{R} \to \mathbb{R} \text{ and } \vec{f}(1) = 0, \quad \therefore \vec{f} \in F$$

$$\begin{aligned} (\vec{f} \oplus \vec{g})(x) = \vec{f}(x) + \vec{g}(x) = 0 + \vec{g}(x) = \vec{g}(x), \quad \therefore \vec{f} \oplus \vec{g} &= \vec{g} \\ (\vec{g} \oplus \vec{f})(x) = \vec{g}(x) + \vec{f}(x) = \vec{g}(x) + 0 = \vec{g}(x), \quad \therefore \vec{g} \oplus \vec{f} &= \vec{g} \end{aligned}$$

$\therefore \vec{f}$ is the additive identity for F, **id**. □

Axiom 5. *Proof.* $\forall\ \vec{g} \in F$. Let $\vec{f} : \mathbb{R} \to \mathbb{R}$ such that $\forall\ x \in \mathbb{R} : \vec{f}(x) = -g(x)$. $\forall\ x \in \mathbb{R}$:

$$\vec{f}(1) = -\vec{g}(1) = -0 = 0, \quad \therefore \vec{f} \in F$$

$$\begin{aligned} (\vec{f} \oplus \vec{g})(x) &= \vec{f}(x) \oplus \vec{g}(x) = -\vec{g}(x) + \vec{g}(x) = 0. \\ &\therefore \vec{f} \oplus \vec{g} = \text{ constant 0 function } = \textbf{id} \\ (\vec{g} \oplus \vec{f})(x) &= \vec{g}(x) \oplus \vec{f}(x) = \vec{g}(x) - \vec{g}(x) = 0. \\ &\therefore \vec{g} \oplus \vec{f} = \text{ constant 0 function } = \textbf{id} \end{aligned}$$

$\therefore \vec{f}$ is the additive inverse of $\vec{g}$, $\forall\ \vec{g} \in F$ □

Axiom 6. *Proof.* $\forall\ k, x \in \mathbb{R}$ and $\forall\ \vec{f} \in F$:

$$(k \odot \vec{f})(x) = k\vec{f}(x) \qquad \text{(by definition)}$$
$$\text{when } x = 1: \ k\vec{f}(1) = k \cdot 0 = 0 \checkmark$$

$\therefore \forall\ k \in \mathbb{R}$ and $\forall\ \vec{f} \in F$, $k \odot \vec{f} \in F$ $\square$

Axiom 7. *Proof.* $\forall\ \vec{f}, \vec{g} \in F$ and $\forall\ k, x \in \mathbb{R}$:

$$LHS = (k \odot (\vec{f} \oplus \vec{g}))(x) = k(\vec{f} \oplus \vec{g})(x) = k(\vec{f}(x) + \vec{g}(x)) = k\vec{f}(x) + k\vec{g}(x)$$
$$RHS = (k \odot \vec{f} \oplus k \odot \vec{g})(x) = (k \odot \vec{f})(x) + (k \odot \vec{g})(x) = k\vec{f}(x) + k\vec{g}(x) = LHS$$

$\therefore \forall\ \vec{f}, \vec{g} \in F$ and $\forall\ k \in \mathbb{R}$, $k \odot (\vec{f} \oplus \vec{g}) = k \odot \vec{f} \oplus k \odot \vec{g}$ $\square$

Axiom 8. *Proof.* $\forall\ \vec{f} \in F$ and $\forall\ k, \ell, x \in \mathbb{R}$:

$$\begin{aligned} LHS = ((k + \ell) \odot \vec{f})(x) &= (k + \ell)\vec{f}(x) \\ &= k\vec{f}(x) + \ell\vec{f}(x) \end{aligned}$$

$$\begin{aligned} RHS = (k \odot \vec{f} \oplus \ell \odot \vec{f})(x) &= (k \odot \vec{f})(x) + (\ell \odot \vec{f})(x) \\ &= k\vec{f}(x) + \ell\vec{f}(x) = LHS \end{aligned}$$

$\therefore \forall\ \vec{f} \in F$ and $\forall\ k, \ell \in \mathbb{R}$, $(k + \ell) \odot \vec{f} = k \odot \vec{f} \oplus \ell \odot \vec{f}$ $\square$

Axiom 9. *Proof.* $\forall\ \vec{f} \in F$ and $\forall\ k, \ell, x \in \mathbb{R}$:

$$\begin{aligned} LHS = ((k \cdot \ell) \odot \vec{f})(x) &= (k \cdot \ell)\vec{f}(x) \\ &= k\ell\vec{f}(x) \end{aligned} \qquad \begin{aligned} RHS = (k \odot (\ell \odot \vec{f}))(x) &= k(\ell \odot \vec{f})(x) \\ &= k(\ell\vec{f}(x)) = k\ell\vec{f}(x) = LHS \end{aligned}$$

$\therefore \forall\ \vec{f} \in F$ and $\forall\ k, \ell \in \mathbb{R}$, $(k \cdot \ell) \odot \vec{f} = k \odot (\ell \odot \vec{f})$ $\square$

Axiom 10. *Proof.* $\forall\ \vec{f} \in F$:

$$(1 \odot \vec{f})(x) = 1 \cdot \vec{f}(x) = \vec{f}(x)$$

$\square$

$\therefore$ 1 fixes all elements in F.
Since all 10 Axioms hold for F, F is a real vector space.

Problem 16

Verify all 10 axioms for Example 8. Let $V = \mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. For all $\vec{u} = u, \vec{v} = v \in V$, $k \in \mathbb{R}$ define $\oplus$ and $\odot$ as:

$$\vec{u} \oplus \vec{v} = u \cdot v \qquad k \odot \vec{u} = u^k$$

Axiom 1. *Proof.* $\forall\ \vec{u} = u, \vec{v} = v \in V$:

$$\vec{u} \oplus \vec{v} = u \cdot v$$

Since $u > 0$ and $v > 0$, $u \cdot v \in \mathbb{R}^+$. $\therefore \forall\ \vec{u} = u, \vec{v} = v \in V : \vec{u} \oplus \vec{v} \in V$ $\square$

Axiom 2. *Proof.* $\forall\ \vec{u} = u, \vec{v} = v \in V$:

$$LHS = \vec{u} \oplus \vec{v} = u \cdot v$$
$$RHS = \vec{v} \oplus \vec{u} = v \cdot u = LHS$$

$\therefore \forall\ \vec{u} = u, \vec{v} = v \in V : \vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$ $\square$

Axiom 3. *Proof.* $\forall\ \vec{u} = u, \vec{v} = v, \vec{w} = w \in V$:

$$LHS = \vec{u} \oplus (\vec{v} \oplus \vec{w}) = \vec{u} \oplus (v \cdot w) = u \cdot (v \cdot w) = u \cdot v \cdot w$$

$$RHS = (\vec{u} \oplus \vec{v}) \oplus \vec{w} = (u \cdot v) \oplus \vec{w} = (u \cdot v) \cdot w = u \cdot v \cdot w = LHS$$

$\therefore \forall\ \vec{u} = u, \vec{v} = v, \vec{w} = w \in V : \vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$ □

Axiom 4. *Proof.* Let $\vec{u} = 1 \in \mathbb{R}^+$. $\forall\ \vec{v} = v \in V$:

$$\vec{u} \oplus \vec{v} = 1 \cdot v = v$$
$$\vec{v} \oplus \vec{u} = v \cdot 1 = v$$

$\therefore \forall\ \vec{v} = v \in V : \vec{u} = 1$ is the additive identity, **id**. □

Axiom 5. *Proof.* $\forall\ \vec{v} = v \in V$, Let $\vec{u} = \frac{1}{v}$ (Note $\frac{1}{v} \in \mathbb{R}^+$ since $v > 0$):

$$\vec{u} \oplus \vec{v} = \frac{1}{v} \cdot v = 1 = \mathbf{id}$$
$$\vec{v} \oplus \vec{u} = v \cdot \frac{1}{v} = 1 = \mathbf{id}$$

$\therefore \forall\ \vec{v} = v \in V : \vec{u} = \frac{1}{v} = -\vec{v}$ □

Axiom 6. *Proof.* $\forall\ \vec{v} = v \in V$ and $\forall\ k \in \mathbb{R}$:

$$k \odot \vec{v} = u^k < 0 \qquad \text{since } u > 0$$

$\therefore \forall\ \vec{v} = v \in V$ and $\forall\ k \in \mathbb{R} : k \odot \vec{v} \in V$ □

Axiom 7. *Proof.* $\forall\ \vec{v} = v, \vec{u} = u \in V$ and $\forall\ k \in \mathbb{R}$:

$$LHS = k \odot (\vec{v} \oplus \vec{u}) = k \odot (u \cdot v) = (u \cdot v)^k = u^k \cdot v^k$$

$$RHS = k \odot \vec{v} \oplus k \odot \vec{u} = (v^k) \oplus (u^k) = v^k \cdot u^k = LHS$$

$\therefore \forall\ \vec{v} = v, \vec{u} = u \in V$ and $\forall\ k \in \mathbb{R} : k \odot (\vec{v} \oplus \vec{u}) = k \odot \vec{v} \oplus k \odot \vec{u}$ □

Axiom 8. *Proof.* $\forall\ \vec{v} = v \in V$ and $\forall\ k, \ell \in \mathbb{R}$:

$$LHS = (k + \ell) \odot \vec{v} = v^{k+\ell} = v^k v^\ell$$

$$RHS = k \odot \vec{v} \oplus \ell \odot \vec{v} = v^k \oplus v^\ell = v^k v^\ell = LHS$$

$\therefore \forall\ \vec{v} = v \in V$ and $\forall\ k, \ell \in \mathbb{R} : (k + \ell) \odot \vec{v} = k \odot \vec{v} \oplus \ell \odot \vec{v}$ □

Axiom 9. *Proof.* $\forall\ \vec{v} = v \in V$ and $\forall\ k, \ell \in \mathbb{R}$:

$$LHS = (k \cdot \ell) \odot \vec{v} = v^{k \cdot \ell} = v^{k\ell}$$

$$RHS = k \odot (\ell \odot \vec{v}) = k \odot (v^k) = v^{k^\ell} = v^{k\ell} = LHS$$

$\therefore \forall\ \vec{v} = v \in V$ and $\forall\ k, \ell \in \mathbb{R} : (k \cdot \ell) \odot \vec{v} = k \odot (\ell \odot \vec{v})$ □

Axiom 10. *Proof.* $\forall\ \vec{v} = v \in V$:

$$1 \odot \vec{v} = v^1 = v = \vec{v}$$

$\therefore \forall\ \vec{v} = v \in V : 1 \odot \vec{v} = \vec{v}$ □

Since V holds under all 10 Axioms, V is a real vector space.

Problem 17

a. A vector is a directed line segment (an arrow)

False, A vector can be anything you can imagine, from a function to fruit.

b. A vector is an n-tuple of real numbers.

False, A vector can be anything you can imagine, including an n-tuple of real numbers, but it doesn't have to be.

c. A vector is any element of a vector space

True, A vector **is** any element of a vector space

e. The set of polynomials with degree *exactly* 1 is a vector space under the operation defined in Example 7.

False. Consider $\vec{v} = 2x + 1$, $\vec{u} = -2x + 3$. Both $\vec{v}$ and $\vec{u} \in V$ since they are both exactly degree 1. Now consider $\vec{v} \oplus \vec{u} = 2x + 1 - 2x + 3 = 4$. 4 is not exactly degree 1; it is degree 0, and therefore not in V. Thus V is not closed under addition.