Problem 1.

Let $V = \mathbb{R}^+ \times \mathbb{R}$ be a set. In other words, every element of V is in the form (u_1, u_2), where u_1 is a positive real number and $u_2 \in \mathbb{R}$. For all (u_1, u_2) and $(v_1, v_2) \in V$, and for all $k \in \mathbb{R}$,

$$(u_1, u_2) \oplus (v_1, v_2) = (2u_1v_1, u_2 + v_2 - 3) \text{ and } k \odot (u_1, v_1) = (u_1^k, ku_2).$$

Verify the axioms 4, 5, and 7.

Ax4. *Proof.* Consider $\vec{u}, \vec{v} \in V$ such that $\vec{u} = (u_1, u_2)$ and $\vec{v} = (\frac{1}{2}, 3)$. ($u_1$ is positive real number).

$$\begin{aligned}
\vec{u} \oplus \vec{v} &= (u_1, u_2) \oplus \left(\frac{1}{2}, 3\right) = \left(2u_1\frac{1}{2}, u_2 + 3 - 3\right) \\
&= (u_1, u_2) = \vec{u} \\
\vec{v} \oplus \vec{u} &= \left(\frac{1}{2}, 3\right) \oplus (u_1, u_2) = \left(2\frac{1}{2}u_1, 3 + u_2 - 3\right) \\
&= (u_1, u_2) = \vec{u}
\end{aligned}$$

Since $\vec{u} \oplus \vec{v} = \vec{u}$ and $\vec{v} \oplus \vec{u} = \vec{u}$ for all $\vec{u} \in V$, therefore $\vec{v} = (\frac{1}{2}, 3)$ is the additive identity, **id**, for V.
$\therefore$ additive identity exists for V. $\square$

Ax5. *Proof.* Consider $\vec{u}, \vec{v} \in V$ such that $\vec{u} = (u_1, u_2)$ and $\vec{v} = (\frac{1}{4u_1}, 6 - u_2)$. Since by definition u_1 is a positive real number, $\frac{1}{4u_1}$ will always be defined and positive.

$$\begin{aligned}
\vec{u} \oplus \vec{v} &= (u_1, u_2) \oplus \left(\frac{1}{4u_1}, 6 - u_2\right) = \left(2u_1\frac{1}{4u_1}, u_2 + (6 - u_2) - 3\right) \\
&= \left(\frac{2}{4} \cdot \frac{u_1}{u_1}, u_2 - u_2 + 6 - 3\right) = \left(\frac{1}{2}, 3\right) = \mathbf{id} \\
\vec{v} \oplus \vec{u} &= \left(\frac{1}{4u_1}, 6 - u_2\right) \oplus (u_1, u_2) = \left(2\frac{1}{4u_1}u_1, (6 - u_2) + u_2 - 3\right) \\
&= \left(\frac{2}{4} \cdot \frac{u_1}{u_1}, u_2 - u_2 + 6 - 3\right) = \left(\frac{1}{2}, 3\right) = \mathbf{id}
\end{aligned}$$

$\therefore$ additive inverse exists for all $\vec{u} \in V$. $\square$

Ax7. *Proof.* Consider $k \in \mathbb{R}$ and $(u_1, u_2), (v_1, v_2) \in V$.

$$\begin{aligned}
k \odot ((u_1, u_2) \oplus (v_1, v_2)) &= k \odot (2u_1v_1, u_2 + v_2 - 3) \\
&= ((2u_1v_1)^k, k(u_2 + v_2 - 3)) \\
&= (4u_1^kv_1^k, ku_2 + kv_2 - 3k)
\end{aligned}$$

$$\begin{aligned}
k \odot (u_1, u_2) \oplus k \odot (v_1, v_2) &= (u_1^k, ku_2) \oplus (v_1^k, kv_2) \\
&= (2u_1^kv_1^k, ku_2 + kv_2 - 3)
\end{aligned}$$

$$(4u_1^kv_1^k, ku_2 + kv_2 - 3k) \neq (2u_1^kv_1^k, ku_2 + kv_2 - 3) \text{ when } k \neq 1$$

Since $k \odot ((u_1, u_2) \oplus (v_1, v_2))$ does not always equal $k \odot (u_1, u_2) \oplus k \odot (v_1, v_2)$, Axiom 7 does not hold for V. $\square$

Problem 2.

Let V be a set with a binary operator $\oplus$ defined, so that Axioms (1), (3), and (4) hold for V (note that other axioms may not hold). Let $\vec{v} \in V$. Prove that **if** $\vec{v}$ has an additive inverse, then this additive inverse is unique. (*Hint*: Let $\vec{w}$ and $\vec{x}$ be two different additive inverses of $\vec{v}$. Show that this will lead to a contradiction.)

Problem 3.

Let $V = P_3$, i.e., the set of all polynomials of degree up to 3, with standard addition and scalar multiplication. Let

$$W = \left\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in V : a_0 \cdot a_1 = 0\right\}.$$

Verify whether W is a subspace of V.

Problem 5.

Let

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 3 & 2 \\ 5 & -4 \end{pmatrix}.$$

Express $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ as a linear combination of A, B, and C. Use Gauss-Jordan elimination.

Problem 6.

Decide whether

$$\vec{u} = 2 + x + 4x^2,\ \vec{v} = 1 - x - 7x^2,\ \text{and}\ \vec{w} = 3 + 2x + 9x^2.$$

spans P_2. Justify your answer using Gauss-Jordan elimination.

Problem 9.

Let V be a real vector space. Prove that V cannot have exactly 3 elements.