## Section 6, p66 #9,11,17,19,23-25,27,35,37,50

In Exercises 8 through 11, find the number of generators of a cyclic group having the given order

**9.** 8

8 has prime factorization of  $2^3$ , so generators are  $\{1, 3, 5, 7\}$ .

**11.** 60

60 has prime factorization of  $2^2 \cdot 3 \cdot 5$ . Generators are

$$\{1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59\}$$

In Exercises 17 through 21, find the number of elements in the indicated cyclic group.

17. The cyclic subgroup of  $\mathbb{Z}_{30}$ , generated by 25

$$\langle 25 \rangle = \{25, 20, 15, 10, 5, 0\}$$

19. The cyclic subgroup  $\langle i \rangle$  of the group  $\mathbb{C}^*$  of nonzero complex numbers under multiplication

$$\{i,-1,-i,1\}$$

In Exercises 22 through 24, find all the subgroups of the given group, and draw the subgroup diagram for the subgroups.

**23.**  $\mathbb{Z}_{36}$ 

Subgroups and their generators:

- $\langle 1 \rangle = \langle 5 \rangle = \langle 7 \rangle = \langle 11 \rangle = \langle 13 \rangle = \langle 17 \rangle = \langle 19 \rangle = \langle 23 \rangle = \langle 25 \rangle = \langle 29 \rangle = \langle 31 \rangle = \langle 35 \rangle = \mathbb{Z}_{36}$
- $\langle 2 \rangle = \langle 10 \rangle = \langle 14 \rangle = \langle 22 \rangle = \langle 26 \rangle = \langle 34 \rangle = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34\}$
- $\langle 3 \rangle = \langle 15 \rangle = \langle 21 \rangle = \langle 33 \rangle = \{0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33\}$
- $\langle 4 \rangle = \langle 8 \rangle = \langle 16 \rangle = \langle 20 \rangle = \langle 28 \rangle = \langle 32 \rangle = \{0, 4, 8, 12, 16, 20, 24, 28, 32\}$
- $\langle 6 \rangle = \langle 30 \rangle = \{0, 6, 12, 18, 24, 30\}$
- $\langle 9 \rangle = \langle 27 \rangle = \{0, 9, 18, 27\}$
- $\langle 12 \rangle = \langle 24 \rangle = \{0, 12, 24\}$
- $\langle 18 \rangle = \{0, 18\}$

**24.**  $\mathbb{Z}_8$ 

Subgroups and their generators:

- $\langle 1 \rangle = \langle 3 \rangle = \langle 5 \rangle = \langle 7 \rangle = \mathbb{Z}_8$
- $\langle 2 \rangle = \langle 6 \rangle = \{0, 2, 4, 6\}$
- $\langle 4 \rangle = \{0, 4\}$

In Exercises 25 through 29, find all the orders of the subgroups of the given group.

**25.**  $\mathbb{Z}_6$ 

$$|\langle 1 \rangle| = |\langle 5 \rangle| = 6.$$
  $|\langle 2 \rangle| = \langle 4 \rangle = 3.$   $|\langle 3 \rangle| = 2.$ 

**27.**  $\mathbb{Z}_{12}$ 

$$|\langle 1 \rangle| = |\langle 5 \rangle| = |\langle 7 \rangle| = |\langle 11 \rangle| = 12. \ |\langle 2 \rangle| = |\langle 10 \rangle| = 6. \ |\langle 3 \rangle| = |\langle 9 \rangle| = 4. \ |\langle 4 \rangle| = |\langle 8 \rangle| = 3. \ |\langle 6 \rangle| = 2.$$

In Exercises 33 through 37, either give an example of a group with the property described, or explain why no example exists.

**35.** A cyclic group having only one generator

Consider  $\mathbb{Z}_2 = \{0, 1\}$ . The only generator for this group is  $\langle 1 \rangle$ .

**37.** A finite cyclic group having four generators

Consider  $\mathbb{Z}_8$ . This group has four generators:  $\langle 1 \rangle, \langle 3 \rangle, \langle 5 \rangle, \langle 7 \rangle$ .

Theory

**50.** Let G be a group and suppose  $a \in G$  generates a cyclic subgroup of order 2 and is the *unique* such element. Show that ax = xa for all  $x \in G$ . Hint: Consider  $(xax^{-1})^2$ 

Assume the above.

*Proof.* Consider  $(xax^{-1})^2$ .

$$(xax^{-1})^2 = xax'xax'$$
$$= xa^2x'$$
$$= xx' = e$$

because by def.  $a^2 = e$ 

Since  $(xax^{-1})^2 = e$ , but  $a^2 = e$  and a is the *unique* such element to do so, it follows that

$$xax^{-1} = a.$$

We can now manipulate this equation.

$$xax^{-1} = a$$
$$xax^{-1}x = ax$$
$$xa = ax$$

Thus we conclude that xa = ax, for all  $x \in G$ .

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