Problem 1

Let V be a vector space, and let $\vec{u}, \vec{v}, \vec{w} \in V$. Prove that if $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$ then $\vec{u} = \vec{v}$.

Proof. Consider $\vec{v}, \vec{u}, \vec{w} \in V$, and assume $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$.

$$\begin{aligned}
\vec{u} \oplus \vec{w} &= \vec{v} \oplus \vec{w} && \text{Assertion}\\
-\vec{u} \oplus \vec{u} \oplus \vec{w} &= -\vec{u} \oplus \vec{v} \oplus \vec{w} && \text{Axiom 3}\\
\mathbf{id} \oplus \vec{w} &= -\vec{u} \oplus \vec{v} \oplus \vec{w} && \text{Axiom 5}\\
\vec{w} &= (-\vec{u} \oplus \vec{v}) \oplus \vec{w} && \text{Axiom 4}\\
\mathbf{id} &= -\vec{u} \oplus \vec{v} && \text{Axiom 4}\\
\vec{v} &= -(-\vec{u}) && \text{Axiom 5}\\
\vec{v} &= \vec{u} && \text{Axiom 5}
\end{aligned}$$

$\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$

by Axiom 5, $-w \in V$

$(\vec{u} \oplus \vec{w}) \oplus -\vec{w} = (\vec{v} \oplus \vec{w}) \oplus -\vec{w}$

Axiom 3

$\vec{u} \oplus (\vec{w} \oplus -\vec{w}) = \vec{v} \oplus (\vec{w} \oplus -\vec{w})$

definition of additive inverse.

$\vec{u} \oplus \underline{id} = \vec{v} \oplus id$

definition of additive identity.

$\vec{u} = \vec{v}$

□

Problem 2

Prove Theorem B.

Proof. Let $\vec{u} \in V$ and $k \in \mathbb{R}$. Consider $\mathbf{id} = \vec{u}$:

Thm A implies $0 \odot \underline{id} = id$

$$\begin{aligned}
\mathbf{id} \oplus \vec{u} &= \vec{u} && \text{Axiom 4}\\
k \odot (\mathbf{id} \oplus \vec{u}) &= k \odot \vec{u} &&\\
k \odot \mathbf{id} \oplus k \odot \vec{u} &= k \odot \vec{u} && \text{Axiom 7}\\
\therefore k \odot \mathbf{id} &= \mathbf{id} && \text{Axiom 4}
\end{aligned}$$

$k \odot \underline{id} =$

$= k \odot (0 \odot \underline{id})$ Thm A

$= (k \cdot 0) \odot \underline{id}$ Axiom 9

$= 0 \odot \underline{id}$ prop of $\mathbb{R}$.

$= \underline{id}$ Thm A

□

Problem 3

Prove Theorem D. If $k \odot \vec{u} = \mathbf{id}$, then $k = 0$ and/or $u = \mathbf{id}$

Proof. ~~Consider~~ Let $k \odot \vec{u} = \mathbf{id}$ and $k \neq 0$. Since $\frac{1}{k} \neq \frac{1}{0}$, $\frac{1}{k}$ is well-defined.

$$\begin{aligned}
k \odot \vec{u} &= \mathbf{id} && \text{Assertion}\\
\frac{1}{k} \odot (k \odot \vec{u}) &= \frac{1}{k} \odot (\mathbf{id}) &&\\
(\frac{1}{k} \cdot k) \odot \vec{u} &= \frac{1}{k} \odot \mathbf{id} && \text{Axiom 9}\\
1 \odot \vec{u} &= \mathbf{id} && \text{Thm B}\\
\vec{u} &= \mathbf{id} && \text{Axiom 10}
\end{aligned}$$

~~Now consider $k \odot \vec{u} = \mathbf{id}$ and $\vec{u} \neq \mathbf{id}$. Since k is defined, $-k$ is also defined~~

$$\cancel{\begin{aligned}
k \odot \vec{u} &= \mathbf{id} && \text{Assertion}\\
k \odot \vec{u} &= \mathbf{id} &&\\
k \odot \vec{u} \oplus (-k) \odot \vec{u} &= \mathbf{id} \oplus (-k) \odot \vec{u} &&\\
(k + (-k)) \odot \vec{u} &= (-k) \odot \vec{u} && \text{Axiom 8}\\
0 \odot \vec{u} &= (-k) \odot \vec{u} &&
\end{aligned}}$$

~~Since $u \neq \mathbf{id}$,~~

$$0 = -k, \quad \therefore k = 0$$

□

Problem 4

Prove that there does not exist a real vector space of size 2. Show that there cannot be a vector space of size 2.

Proof. Let $V = \{\vec{u}, \vec{v}\}$ be a vectorspace where $\vec{u} \neq \vec{v}$. That is, it satisfies all 10 Axioms.
Axiom 4 states that **id** exists, and is unique. Therefore either $\vec{u} = \mathbf{id}$ or $\vec{v} = \mathbf{id}$. Both cannot be **id**, ~~so therefore $\vec{u} \neq \vec{v}$~~ since $\vec{u} \neq \vec{v}$.
Without the loss of generality, let $\vec{u} = \mathbf{id}$.

$$\vec{u} \oplus \vec{v} = \vec{v} \qquad \text{Axiom 4}$$
$$\mathbf{id} \oplus \vec{v} = \vec{v}$$

Now consider Axiom 5: ~~additive inverse exists for all $\vec{u} \in V$~~ $-v \in V$ (Note that $v \oplus u = v$, so we know $-\vec{v} \neq u$
~~Consider $-\vec{v} \oplus \vec{v} = \mathbf{id}$. Since $\vec{v} \neq \mathbf{id}$, $-v \neq v$. Since there is only one other element in V, $-\vec{v} = \vec{u}$ must be true. Remember that $\vec{u} = \mathbf{id}$.~~ therefore $\vec{v} \oplus \vec{v} = \vec{u}$

~~$\vec{u} \oplus \vec{v} = \vec{u}$~~ Now, $\vec{v} \oplus \vec{v} = \vec{u}$

~~Therefore we have from Axiom 4 and 5:~~ $1 \odot v \oplus 1 \odot v = u$ (Axiom 10)

~~$\vec{u} \oplus \vec{v} = \vec{u}$~~ $(1+1) \odot v = u$ (Axiom 8)

~~$\vec{u} \oplus \vec{v} = \vec{v}$~~ $2 \odot v = u = id$

~~$\therefore \vec{u} = \vec{v}$~~ by Thm A, either $2 = 0$ or $v = id$. We know $2 \neq 0$ and $v \neq id$ since $u = id$, a contradiction. $\therefore$

However, this contradicts with our assertion that $\vec{u} \neq \vec{v}$.
$\therefore$ A vectorspace of size 2 cannot exist. □