Problem 1.

Let $V = \mathbb{R}^+ \times \mathbb{R}$ be a set. In other words, every element of V is in the form (u_1, u_2), where u_1 is a positive real number and $u_2 \in \mathbb{R}$. For all (u_1, u_2) and $(v_1, v_2) \in V$, and for all $k \in \mathbb{R}$,

$$(u_1, u_2) \oplus (v_1, v_2) = (2u_1v_1, u_2 + v_2 - 3) \text{ and } k \odot (u_1, v_1) = (u_1^k, ku_2).$$

Verify the axioms 4, 5, and 7.

Ax4. *Proof.* Consider $\vec{u}, \vec{v} \in V$ such that $\vec{u} = (u_1, u_2)$ and $\vec{v} = (\frac{1}{2}, 3)$. ($u_1$ is positive real number).

$$\begin{aligned}
\vec{u} \oplus \vec{v} &= (u_1, u_2) \oplus \left(\frac{1}{2}, 3\right) = \left(2u_1\frac{1}{2}, u_2 + 3 - 3\right) \\
&= (u_1, u_2) = \vec{u} \\
\vec{v} \oplus \vec{u} &= \left(\frac{1}{2}, 3\right) \oplus (u_1, u_2) = \left(2\frac{1}{2}u_1, 3 + u_2 - 3\right) \\
&= (u_1, u_2) = \vec{u}
\end{aligned}$$

Since $\vec{u} \oplus \vec{v} = \vec{u}$ and $\vec{v} \oplus \vec{u} = \vec{u}$ for all $\vec{u} \in V$, therefore $\vec{v} = (\frac{1}{2}, 3)$ is the additive identity, **id**, for V.

$\therefore$ additive identity exists for V. □

Ax5. *Proof.* Consider $\vec{u}, \vec{v} \in V$ such that $\vec{u} = (u_1, u_2)$ and $\vec{v} = (\frac{1}{4u_1}, 6 - u_2)$. Since by definition u_1 is a positive real number, $\frac{1}{4u_1}$ will always be defined and positive.

$$\begin{aligned}
\vec{u} \oplus \vec{v} &= (u_1, u_2) \oplus \left(\frac{1}{4u_1}, 6 - u_2\right) = \left(2u_1\frac{1}{4u_1}, u_2 + (6 - u_2) - 3\right) \\
&= \left(\frac{2}{4} \cdot \frac{u_1}{u_1}, u_2 - u_2 + 6 - 3\right) = \left(\frac{1}{2}, 3\right) = \mathbf{id} \\
\vec{v} \oplus \vec{u} &= \left(\frac{1}{4u_1}, 6 - u_2\right) \oplus (u_1, u_2) = \left(2\frac{1}{4u_1}u_1, (6 - u_2) + u_2 - 3\right) \\
&= \left(\frac{2}{4} \cdot \frac{u_1}{u_1}, u_2 - u_2 + 6 - 3\right) = \left(\frac{1}{2}, 3\right) = \mathbf{id}
\end{aligned}$$

$\therefore$ additive inverse exists for all $\vec{u} \in V$. □

Ax7. *Proof.* Consider $k \in \mathbb{R}$ and $(u_1, u_2), (v_1, v_2) \in V$.

$$\begin{aligned}
k \odot ((u_1, u_2) \oplus (v_1, v_2)) &= k \odot (2u_1v_1, u_2 + v_2 - 3) \\
&= ((2u_1v_1)^k, k(u_2 + v_2 - 3)) \\
&= (4u_1^k v_1^k, ku_2 + kv_2 - 3k)
\end{aligned}$$

$$\begin{aligned}
k \odot (u_1, u_2) \oplus k \odot (v_1, v_2) &= (u_1^k, ku_2) \oplus (v_1^k, kv_2) \\
&= (2u_1^k v_1^k, ku_2 + kv_2 - 3)
\end{aligned}$$

$$(4u_1^k v_1^k, ku_2 + kv_2 - 3k) \neq (2u_1^k v_1^k, ku_2 + kv_2 - 3) \text{ when } k \neq 1$$

Since $k \odot ((u_1, u_2) \oplus (v_1, v_2))$ does not always equal $k \odot (u_1, u_2) \oplus k \odot (v_1, v_2)$, Axiom 7 does not hold for V. □

Problem 2.

Let V be a set with a binary operator $\oplus$ defined, so that Axioms (1), (3), and (4) hold for V (note that other axioms may not hold). Let $\vec{v} \in V$. Prove that **if** $\vec{v}$ has an additive inverse, then this additive inverse is unique. (*Hint*: Let $\vec{w}$ and $\vec{x}$ be two different additive inverses of $\vec{v}$. Show that this will lead to a contradiction.)

Proof. Let $\vec{w}, \vec{x}, \vec{v} \in V$ such that $\vec{w}$ and $\vec{x}$ are two different additive inverses of $\vec{v}$, This implies that $\vec{w} \neq \vec{x}$.

$$\begin{aligned}
\vec{v} \oplus \vec{w} &= \mathbf{id} && \text{def. of additive inverse} \\
\vec{x} \oplus (\vec{v} \oplus \vec{w}) &= \vec{x} \oplus \mathbf{id} && \\
(\vec{x} \oplus \vec{v}) \oplus \vec{w} &= \vec{x} \oplus \mathbf{id} && \text{axiom 3} \\
\mathbf{id} \oplus \vec{w} &= \vec{x} \oplus \mathbf{id} && \text{def. of additive inverse} \\
\vec{w} &= \vec{x} && \text{def. of additive identity}
\end{aligned}$$

However, $\vec{w} = \vec{x}$ contradicts our assertion that $\vec{w} \neq \vec{x}$. Therefore, through contradiction, if $\vec{v}$ has an additive inverse, then this additive inverse is unique. □

Problem 3.

Let $V = P_3$, i.e., the set of all polynomials of degree up to 3, with standard addition and scalar multiplication. Let

$$W = \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in V : a_0 \cdot a_1 = 0 \right\}.$$

Verify whether W is a subspace of V.

Proof. According to Theorem 3 from Lecture 10, assuming that addition and scalar multiplication in W are inherited from V, W is a subspace of V if and only if Axioms 1 and 6 hold for W.

Ax1. *Proof.* Let $\vec{a}, \vec{b} \in W$ such that $\vec{a} = 0 + 1x + 0x^2 + 0x^3$ and $\vec{b} = 1 + 0x + 0x^2 + 0x^3$. Now consider $\vec{a} \oplus \vec{b}$:

$$\begin{aligned}
\vec{a} \oplus \vec{b} &= (0 + 1x + 0x^2 + 0x^3) \oplus (1 + 0x + 0x^2 + 0x^3) \\
&= 0 + 1x + 0x^2 + 0x^3 + 1 + 0x + 0x^2 + 0x^3 \\
&= (0+1) + (1+0)x + (0+0)x^2 + (0+0)x^3 \\
&= 1 + 1x + 0x^2 + 0x^3
\end{aligned}$$

$$1 \cdot 1 = 1 \neq 0$$

Therefore $\vec{a} \oplus \vec{b} \notin W$, even though $\vec{a} \in W$ and $\vec{b} \in W$.

This means that W is not closed under addition. □

Since Axiom 1 does not hold for W, W cannot be a subspace of V. □

Problem 5.

Let

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 3 & 2 \\ 5 & -4 \end{pmatrix}.$$

Express $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ as a linear combination of $A,\ B,$ and C. Use Gauss-Jordan elimination.

Proof. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1 A \oplus k_2 B \oplus k_3 C = M$.

That is, $k_1 \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix} \oplus k_2 \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix} \oplus k_3 \begin{pmatrix} 3 & 2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. From this equation, we get a linear system of equations:

$$\begin{aligned} 2k_1 + 1k_2 + 3k_3 &= 1 \\ 1k_1 - 1k_2 + 2k_3 &= 2 \\ 4k_1 + 3k_2 + 5k_3 &= 3 \\ 0k_1 + 4k_2 - 4k_3 &= 4 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & -1 & 2 & 2 \\ 4 & 3 & 5 & 3 \\ 0 & 4 & -4 & 4 \end{array}\right) \xrightarrow[(-4,-2,-6,-2)]{R_3-2R_1} \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & -1 & 2 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 4 & -4 & 4 \end{pmatrix} \xrightarrow[(-2,2,-4,-4)]{R_1-2R_2} \begin{pmatrix} 0 & 3 & -1 & -3 \\ 1 & -1 & 2 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 4 & -4 & 4 \end{pmatrix} \xrightarrow[(0,1,-1,1)]{R_2+R_3}$$

$$\begin{pmatrix} 0 & 3 & -1 & -3 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 4 & -4 & 4 \end{pmatrix} \xrightarrow[(0,-3,3,-3)]{R_1-3R_3} \begin{pmatrix} 0 & 0 & 2 & -6 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 4 & -4 & 4 \end{pmatrix} \xrightarrow[(0,-4,4,-4)]{R_4-4R_1} \begin{pmatrix} 0 & 0 & 2 & -6 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1}$$

$$\begin{pmatrix} 0 & 0 & 1 & -3 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[(0,0,-1,3)]{R_2-R_1} \begin{pmatrix} 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 6 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_3+R_1} \begin{pmatrix} 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This augmented matrix represents the following equations:

$$\begin{aligned} k_1 &= 6 \\ k_2 &= -2 \\ k_3 &= -3 \end{aligned}$$

This means that $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a linear combination of $A,\ B,$ and C, when $k_1 = 6,\ k_2 = -3,$ and $k_3 = -3$ $\square$

Problem 6.

Decide whether

$$\vec{u} = 2 + x + 4x^2,\ \vec{v} = 1 - x - 7x^2,\ \text{and}\ \vec{w} = 3 + 2x + 9x^2.$$

spans P_2. Justify your answer using Gauss-Jordan elimination.

Proof. Let $\vec{y} = y_0 + y_1x + y_2x^2$, and let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1\vec{u} + k_2\vec{v} + k_3\vec{w} = \vec{y}$. In other words,

$$k_1(2 + x + 4x^2) + k_2(1 - x - 7x^2) + k_3(3 + 2x + 9x^2) = y_0 + y_1x + y_2x^2.$$

From this equation, we get the following system of linear equations.

$$\begin{aligned} 2k_1 + 1k_2 + 3k_3 &= y_0 \\ 1k_1 - 1k_2 + 2k_3 &= y_1 \\ 4k_1 - 7k_2 + 9k_3 &= y_2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & y_0 \\ 1 & -1 & 2 & y_1 \\ 4 & -7 & 9 & y_2 \end{array}\right) \xrightarrow[(-4,-2,-6,-2y_0)]{R_3-2R_1} \begin{pmatrix} 2 & 1 & 3 & y_0 \\ 1 & -1 & 2 & y_1 \\ 0 & -9 & 3 & y_2 - 2y_0 \end{pmatrix} \xrightarrow[(-2,2,-4,-2y_1)]{R_1-2R_2} \begin{pmatrix} 0 & 3 & -1 & y_0 - 2y_1 \\ 1 & -1 & 2 & y_1 \\ 0 & -9 & 3 & y_2 - 2y_0 \end{pmatrix}$$

$$\xrightarrow[(0,9,-3,2y_0-4y_1)]{R_3+2R_3} \begin{pmatrix} 0 & 3 & -1 & y_0 - 2y_1 \\ 1 & -1 & 2 & y_1 \\ 0 & 0 & 0 & y_2 - 2y_0 + 2y_0 - 4y_1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & -1 & y_0 - 2y_1 \\ 1 & -1 & 2 & y_1 \\ 0 & 0 & 0 & y_2 - 4y_1 \end{pmatrix}$$

The last row represents the equation $0 = y_2 - 4y_1$. If $0 \neq y_2 - 4y_1$, then there is no solution to the system of linear equations. Therefore, there exists $\vec{y} \in P_2$ that cannot be spanned by $\{\vec{u}, \vec{v}, \vec{w}\}$. $\square$

Problem 9.

Let V be a real vector space. Prove that V cannot have exactly 3 elements.

Proof. Let V be a real vector space containing exactly 3 elements. Let the first element of V be $\mathbf{id}$, which is required to be in V through Axiom 4. Next, let the second element of V be $\vec{v}$ (Note that $\vec{v}$ cannot be $\mathbf{id}$, since the additive identity is unique). Finally, let the second element of V be $-\vec{v}$, the additive inverse of $\vec{v}$, which is required to be in V through Axiom 5. Therefore, we have $V = \{\mathbf{id}, \vec{v}, -\vec{v}\}$, where $\mathbf{id} \neq \vec{v}$ and $\mathbf{id} \neq -\vec{v}$ and $\vec{v} \neq -\vec{v}$.

Now consider $\vec{v} \oplus \vec{v}$:

Case 1. $\vec{v} \oplus \vec{v} = \mathbf{id}$

Proof. Consider $\vec{v} \oplus \vec{v} \oplus -\vec{v}$.

$$\begin{aligned}
\vec{v} \oplus \vec{v} \oplus -\vec{v} &= \vec{v} \oplus (\vec{v} \oplus -\vec{v}) && \text{axiom 3} \\
&= \vec{v} \oplus \mathbf{id} && \text{def. of additive inverse} \\
&= \vec{v} && \text{def. of additive identity}
\end{aligned}$$

$$\begin{aligned}
\vec{v} \oplus \vec{v} \oplus -\vec{v} &= (\vec{v} \oplus \vec{v}) \oplus -\vec{v} && \text{axiom 3} \\
&= \mathbf{id} \oplus -\vec{v} && \text{assertion} \\
&= -\vec{v} && \text{def. of additive identity}
\end{aligned}$$

This implies that $\vec{v} = -\vec{v}$, which contradicts our assertion that $\vec{v} \neq -\vec{v}$. Therefore, $\vec{v} \oplus \vec{v} \neq \mathbf{id}$. □

Case 2. $\vec{v} \oplus \vec{v} = \vec{v}$

Proof. Consider $\vec{v} \oplus \vec{v} \oplus -\vec{v}$.

$$\begin{aligned}
\vec{v} \oplus \vec{v} \oplus -\vec{v} &= \vec{v} \oplus (\vec{v} \oplus -\vec{v}) && \text{axiom 3} \\
&= \vec{v} \oplus \mathbf{id} && \text{def. of additive inverse} \\
&= \vec{v} && \text{def. of additive identity}
\end{aligned}$$

$$\begin{aligned}
\vec{v} \oplus \vec{v} \oplus -\vec{v} &= (\vec{v} \oplus \vec{v}) \oplus -\vec{v} && \text{axiom 3} \\
&= \vec{v} \oplus -\vec{v} && \text{assertion} \\
&= \mathbf{id} && \text{def. of additive identity}
\end{aligned}$$

This implies that $\mathbf{id} = \vec{v}$, which contradicts our assertion that $\mathbf{id} \neq \vec{v}$. Therefore, $\vec{v} \oplus \vec{v} \neq \vec{v}$ □

Case 3. $\vec{v} \oplus \vec{v} = -\vec{v}$

Proof. Consider the assertion that $\vec{v} \oplus \vec{v} = -\vec{v}$

$$\begin{aligned}
\vec{v} \oplus \vec{v} &= -\vec{v} && \text{assertion} \\
1 \odot \vec{v} \oplus 1 \odot \vec{v} &= -\vec{v} && \text{axiom 10} \\
1 \odot \vec{v} \oplus 1 \odot \vec{v} &= (-1) \odot \vec{v} && \text{theorem C} \\
(1+1) \odot \vec{v} &= (-1) \odot \vec{v} && \text{axiom 8}
\end{aligned}$$

This last equation implies that $1 + 1 = -1$, which is a contradiction. Therefore, $\vec{v} \oplus \vec{v} \neq \vec{v}$ □

Every possible result in V for $\vec{v} \oplus \vec{v}$ leads to a contradiction. Therefore, $\vec{v} \oplus \vec{v} \notin V$, meaning that V is not closed under addition. This means that V, containing exactly three elements, cannot be a real vector space. □