MAT 260 LINEAR ALGEBRA
LECTURE 2

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4.1 — Real vector spaces

A "real vector space" V is simply a **nonempty set** that satisfies 10 properties, called the "10 axioms of a real vector space". This nonempty set can contain elements as creative as you can imagine. For instance, an element $\mathbf{v} \in V$ can be as crazy as $\mathbf{v} = apple$, and another element $\mathbf{u}$ from the same set V can be $\mathbf{u} = table$, so you can NEVER assume that an element $\mathbf{v} \in V$ is an ordered pair (e.g. $(-4, 0)$) or an ordered triple (e.g., $(3, -1, 5)$ or $\langle 1, -2, -4 \rangle$).

In this lecture note, as well as in the textbook, any **BOLD** letter $\mathbf{u}$, $\mathbf{v}$, or $\mathbf{w}$, etc. denotes **an element in the set V**. In handwriting, we often write $\overrightarrow{u}$, $\overrightarrow{v}$, or $\overrightarrow{w}$, etc. However, they should NEVER be assumed to be an ordered pair or an ordered triple. (For example, please distinguish between u and $\mathbf{u}$. u is NOT in V, while $\mathbf{u}$ is.)

In this nonempty set V, we define two operators: "addition" (denoted by $\oplus$) and "scalar multiplication" (denoted by $\odot$). They should NEVER be mixed up with the common addition and multiplication in $\mathbb{R}$.

In this lecture note, I will stick to using $\oplus$ and $\odot$ when denoting operators in V. However, if you read the book, they do NOT make any distinction between operators in V and operators in $\mathbb{R}$. Therefore, you MUST pay extra attention when you read the book whether the "+" sign means addition in V or addition in $\mathbb{R}$. They are drastically different and will completely mess you up if you are not careful.

Addition

$$\oplus : V \times V \to V$$

This is a "binary operator", **simply a map**. For example, $V = \{apple, orange, banana\}$. We can define the map $\oplus$ in any way we want. For example,

$\oplus$	*apple*	*orange*	*banana*
apple	*banana*	*banana*	*apple*
orange	*orange*	*apple*	*banana*
banana	*banana*	*orange*	*orange*

How do we read this table? Easy! For example, from

Date: Wednesday, January 25, 2023.

$\oplus$	*apple*	*orange*	*banana*
apple	*banana*	"$\underline{banana}$"	*apple*
orange	*orange*	*apple*	*banana*
banana	*banana*	*orange*	*orange*

we see that $\oplus(apple, orange) = banana$, and from

$\oplus$	*apple*	*orange*	*banana*
apple	*banana*	*banana*	*apple*
orange	*orange*	*apple*	*banana*
banana	*banana*	*orange*	"$\underline{orange}$"

we see that $\oplus(banana, banana) = orange$.

For simplification of symbols, we often write

$$apple \oplus orange$$

in place of

$$\oplus(apple, orange).$$

Note that "$apple \oplus orange$" and "$orange \oplus apple$" in THIS particular example are NOT the same, since one of them is *banana*, while the other is *orange*. However, YOU can define the map differently from mine, so that "$apple \oplus orange$" and "$orange \oplus apple$" are the same.

You see, this map is defined completely arbitrarily. There is no reason or pattern to anything I defined for this map, and you can totally define it in a completely different way as I did. Bear in mind, "addition" is **merely a map**, and it is up to the person who define it to specify the action.

One thing to pay attention though. In our example, you can NEVER define

$$\oplus(apple, orange) = watermelon,$$

since *watermelon* is NOT in V.

That's it for addition. You may be a little lost, since you still don't really know what's going on. That's the point! Addition is merely trying to map a pair of (i.e., two) elements from V to an element in V, and that's it!

Scalar Multiplication

$$\odot : \mathbb{R} \times V \to V$$

The fact that we use $\mathbb{R}$ as part of the domain is the whole reason why we call it "real" vector spaces at the first place. In our course, the word "scalar" always refers to a real number in $\mathbb{R}$.

Once again, scalar multiplication is **simply a map**. However, please pay EXTRA attention: the first element that is plucked into $\odot$ MUST be a scalar, and the second element MUST be from the nonempty set V. In other words,

$$\odot(x, \mathbf{u})$$

is LEGAL, but

$$\odot(x, u),\ \odot(\mathbf{x}, \mathbf{u}), \text{ and } \odot(\mathbf{u}, x)$$

are all ILLEGAL (bear in mind that *italicized* means scalars, and **bolded** means elements in V).

For simplification of symbols, we often write

$$a \odot \mathbf{x}$$

in place of

$$\odot(a, \mathbf{x}).$$

Coming back to our example of $V = \{apple, orange, banana\}$. Once again, we can define the map $\odot$ completely arbitrarily. For example,

$$k \odot apple = orange \quad \text{for all } k \in \mathbb{R},$$

$$k \odot orange = \begin{cases} orange & \text{if } k \leq 2, \\ banana & \text{if } k > 2, \end{cases}$$

$$k \odot banana = \begin{cases} banana & \text{if } k < -5\sqrt{2}, \\ apple & \text{if } -5\sqrt{2} \leq k < 1.2, \\ banana & \text{if } k = 1.2, \\ orange & \text{if } k > 1.2. \end{cases}$$

In this example, we have $1.5 \odot orange = orange$, and $2 \odot orange = orange$. In other words,

$$2 \odot (1.5 \odot orange) = orange.$$

On the other hand, $3 \odot orange = banana$. Therefore, "$2 \odot (1.5 \odot orange)$" and "$3 \odot orange$" are NOT the same in THIS particular example. Once again, however, YOU can define the map $\odot$ differently from mine, so that "$2 \odot (1.5 \odot orange)$" and "$3 \odot orange$" are the same.

Let me reiterate here about the notations. If I write u or x, then it represents a scalar; if I write $\mathbf{u}$ or $\mathbf{x}$, then it represents an element in the set. It is very important not to mix them up, since you see that $x \odot \mathbf{u}$ is LEGAL, but $\mathbf{x} \odot u$ is ILLEGAL.

Exercise 1. Let $V = \{\mathbf{0}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$, with some addition $\oplus$ and scalar multiplication $\odot$ defined in a certain way.

(a) If we know that $\mathbf{a} \oplus \mathbf{b} = \mathbf{d}$, what is $\mathbf{b} \oplus \mathbf{a}$?
(b) If we know that $\mathbf{0} \oplus \mathbf{a} = \mathbf{a}$, $\mathbf{0} \oplus \mathbf{b} = \mathbf{b}$, $\mathbf{0} \oplus \mathbf{c} = \mathbf{c}$, what is $\mathbf{0} \oplus \mathbf{0}$ and $\mathbf{0} \oplus \mathbf{d}$?
(c) What is $\mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c}$?
(d) Is it true that $3 \odot (\mathbf{b} \oplus \mathbf{d}) = 3 \odot \mathbf{b} \oplus 3 \odot \mathbf{d}$? (*Note*: We assume that $\odot$ always operates before $\oplus$.)
(e) What is $1 \odot \mathbf{c}$?

10 Good Properties of Addition and Scalar Multiplication

By now, we should realize that "addition" and "scalar multiplication" on a nonempty set V CAN BE defined arbitrarily, and there can be no rules or patterns to govern them.

The only trouble is, such $\oplus$ and $\odot$ are VERY HARD to study, and in mathematics, we are not interested in them. We are ONLY ABLE to study those nonempty sets V with **nicely defined** "addition" and "scalar multiplication".

10 properties that it would be **nice** for the "addition" and "scalar multiplication" on V to have:

(1) **For all** $\mathbf{u}$ and $\mathbf{v}$ in V,

$$\mathbf{u} \oplus \mathbf{v} \in V.$$

(We say "V is closed under addition" IF the above happens.)

Question: Do we need to worry about $\mathbf{v} \oplus \mathbf{u}$? How about $\mathbf{v} \oplus \mathbf{v}$?

(2) **For all** $\mathbf{u}$ and $\mathbf{v}$ in V,

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

(We say "commutativity law holds for addition in V" IF the above happens.)

(3) **For all** $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V,

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

(We say "associativity law holds for addition in V" IF the above happens.)

(4) **There exists** an element $\mathbf{u}$ in V that BEHAVES as an "additive identity" of V. In other words,

$$\underline{\textbf{for all}}\ \mathbf{v} \text{ in } V,\ \mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u} = \mathbf{v}.$$

(We say "additive identity exists in V" IF the above happens.)

Important: Whether an element is an additive identity ONLY depends on its behavior, i.e., we need to add it to other elements and see whether we get back those other elements every time.

Notation: If we have ALREADY known, either given to us or through our verification, that an element $\mathbf{u}$ is an additive identity, then we denote $\mathbf{u} = \mathbf{id}$. (In fact, we need to finish Problem 9 first.)

Warning: The book uses $\mathbf{0}$ instead of $\mathbf{id}$. They are the same, just the symbols are different. DO NOT mix up $\mathbf{0}$ with the real number 0.

(5) **For all** $\mathbf{v}$ in V, $\mathbf{v}$ has a CORRESPONDING "additive inverse" in V. In other words,

$$\underline{\textbf{there exists}}\ \mathbf{w} \text{ in } V,\ \mathbf{v} \oplus \mathbf{w} = \mathbf{w} \oplus \mathbf{v} = \mathbf{id}.$$

(We say "additive inverse ALWAYS exists in V" IF the above happens.)

Important: This Property (5) can hold ONLY AFTER Property (4) is ALREADY known to hold.

Important: CORRESPONDING means different $\mathbf{v}$ may have a different additive inverse.

Important: If some $\mathbf{v}$ in V has an additive inverse but some don't, then Property (5) does NOT hold.

Notation: If $\mathbf{w}$ is an additive inverse of $\mathbf{v}$, then we write $\mathbf{w} = -\mathbf{v}$. (This $-\mathbf{v}$ is pronounced as *bar-v*. The "$-$" is the "additive inverse sign", and you MUST NOT mix it up with the negative sign for $\mathbb{R}$.)

Note: We also have $\mathbf{v} = -\mathbf{w}$ at the same time! (Why?)

(6) **<u>For all</u>** k in $\mathbb{R}$ and **<u>for all</u>** $\mathbf{v}$ in V,

$$k \odot \mathbf{v} \in V.$$

(We say "V is closed under scalar multiplication" IF the above happens.)

(7) **<u>For all</u>** k in $\mathbb{R}$, and **<u>for all</u>** $\mathbf{u}$ and $\mathbf{v}$ in V,

$$k \odot (\mathbf{u} \oplus \mathbf{v}) = k \odot \mathbf{u} \oplus k \odot \mathbf{v}.$$

(We say "distributivity law over $\oplus$ holds for V" IF the above happens.)

(8) **<u>For all</u>** k and ℓ in $\mathbb{R}$, and **<u>for all</u>** $\mathbf{v}$ in V,

$$(k + \ell) \odot \mathbf{v} = k \odot \mathbf{v} \oplus \ell \odot \mathbf{v}.$$

(We say "distributivity law over $+$ holds for V" IF the above happens.)

Question: Why do we use "$+$" on the left hand side?

(9) **<u>For all</u>** k and ℓ in $\mathbb{R}$, and **<u>for all</u>** $\mathbf{v}$ in V,

$$(k \cdot \ell) \odot \mathbf{v} = k \odot (\ell \odot \mathbf{v}).$$

(We say "associativity law holds for scalar multiplication in V" IF the above happens.)

Question: Why do we use "$\cdot$" on the left hand side?

(10) **<u>For all</u>** $\mathbf{v}$ in V,

$$1 \odot \mathbf{v} = \mathbf{v}.$$

(We say "1 fixes every element in V by scalar multiplication" IF the above happens.)

Definition 2. Let V be a nonempty set, with $\oplus$ and $\odot$ defined on V as **<u>maps</u>**. If $\oplus$ and $\odot$ satisfy ALL 10 properties listed above, then V is a **<u>"real vector space"</u>**.

Conversely, if V is GIVEN to be a real vector space, then even if we don't know $\oplus$ and $\odot$ exactly are, we know they must satisfy all 10 properties, or all **<u>"10 axioms"</u>**.

Any element in a real vector space is called a **<u>"vector"</u>**.

Question 3. How to REMEMBER all 10 axioms?

Answer.

- The first 5 axioms (i.e., Axioms (1) to (5)) deal with addition only, and the next 5 (i.e., Axioms (6) to (10)) involves scalar multiplication.
- Both the first axiom in the first 5 and the first axiom in the next 5 (i.e., Axioms (1) and (6)) deal with closure.
- Axiom (2) and (3) deal with commutativity and associativity.
- Axiom (4) deals with the **existence** of additive identity, and Axiom (5) deals with the **existence** of additive inverse.
- Axioms (7) and (8) are still close to Axioms (1) to (5), so they also involve addition. The only possibility is they deal with distributivity.
- Axiom (7) is yet closer to Axioms (1) to (5), so it involves $\mathbf{u} \oplus \mathbf{v}$ on the left hand side.
- Axiom (8) is slightly further from Axioms (1) to (5), so it involves $k + \ell$ on the left hand side.
- Axioms (8) and (9) are the only axioms that involve 2 real numbers.
- Axiom (9) deals with associativity for scalar multiplication.
- Axiom (10) is $1\mathbf{v} = \mathbf{v}$.

$\square$

Verify whether a Nonempty Set V Satisfies each of the 10 Properties

We can classify the 10 axioms into THREE types:

(a) Axioms (1) and (6);
(b) Axioms (4) and (5);
(c) Axioms (2), (3), (7), (8), (9), (10).

The way to verify each type of axioms is drastically different.

Example 4. Let $V = \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$. We define $\oplus$ and $\odot$ in the following way.

$$\text{For all } \mathbf{u} = (u_1, u_2, \ldots, u_n),\ \mathbf{v} = (v_1, v_2, \ldots, v_n) \in V \text{ and for all } k \in \mathbb{R},$$
$$\mathbf{u} \oplus \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \quad \text{and} \quad k \odot \mathbf{u} = (ku_1, ku_2, \ldots, ku_n).$$

Verify all 10 axioms.

Note: This $\oplus$ and $\odot$ are called the "standard" addition and scalar multiplication **for $\mathbb{R}^n$**.

Solution. Axiom (1):

For all $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in V$, since u_i and v_i are in $\mathbb{R}$ for each $i = 1, 2, \ldots, n$, by the closure of addition for $\mathbb{R}$, we have

$$u_i + v_i \in \mathbb{R}.$$

Hence, $\mathbf{u} \oplus \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \in \mathbb{R}^n = V$. Therefore, Axiom (1) holds.

Axiom (2):

For all $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in V$,

$$\begin{aligned}
LHS &= \mathbf{u} \oplus \mathbf{v} \\
&= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \qquad \text{(by definition of } \oplus\text{)} \\
&= (v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n) \qquad \text{(by commutativity of addition for } \mathbb{R}\text{)} \\
&= \mathbf{v} \oplus \mathbf{u} \qquad \text{(by definition of } \oplus\text{)} \\
&= RHS.
\end{aligned}$$

Therefore, Axiom (2) holds.

Axiom (3):

For all $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in V$,

$$\begin{aligned}
LHS &= \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) \\
&= (u_1, u_2, \ldots, u_n) \oplus (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n) \qquad \text{(by definition of } \oplus\text{)} \\
&= \big(u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n)\big). \qquad \text{(by definition of } \oplus\text{)}
\end{aligned}$$

$$\begin{aligned}
RHS &= (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} \\
&= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \oplus (w_1, w_2, \ldots w_n) \qquad \text{(by definition of } \oplus\text{)} \\
&= \big((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \ldots, (u_n + v_n) + w_n\big) \qquad \text{(by definition of } \oplus\text{)} \\
&= \big(u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n)\big) \qquad \text{(by associativity of addition for } \mathbb{R}\text{)} \\
&= LHS.
\end{aligned}$$

Therefore, Axiom (3) holds.

Note: The proofs of Axioms (2) and (3) are very similar. This is because they are of the same type.

Note: I do NOT start with $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$, NOR do I start with $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$. If you do, you are ASSUMING THE CONCLUSION, and the proof falls apart.

Axiom (4):

Consider $\mathbf{u} = (0, 0, \ldots, 0) \in V$. For all $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in V$,

$$\begin{aligned}
\mathbf{u} \oplus \mathbf{v} &= (0 + v_1, 0 + v_2, \ldots, 0 + v_n) \qquad \text{(by definition of } \oplus\text{)} \\
&= (v_1, v_2, \ldots, v_n) \\
&= \mathbf{v}.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathbf{v} \oplus \mathbf{u} &= (v_1 + 0, v_2 + 0, \ldots, v_n + 0) \qquad \text{(by definition of } \oplus\text{)} \\
&= (v_1, v_2, \ldots, v_n) \\
&= \mathbf{v}.
\end{aligned}$$

Therefore, Axiom (4) holds, with $\mathbf{id} = (0, 0, \ldots, 0)$.

Note: I do NOT use **id** in the entire proof until the last line. This is because we can only declare an element as **id** AFTER we have verified that it behaves as an additive identity through checking the DEFINITION.

Note: Axiom (4) is about the EXISTENCE of additive identity, so all we need to do is to show that one element in V behaves as an additive identity.

Axiom (5):

For all $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in V$, consider $\mathbf{w} = (-v_1, -v_2, \ldots, -v_n) \in V$.

$$\begin{aligned} \mathbf{v} \oplus \mathbf{w} &= \big(v_1 + (-v_1), v_2 + (-v_2), \ldots, v_n + (-v_n)\big) \qquad \text{(by definition of } \oplus) \\ &= (0, 0, \ldots, 0) \\ &= \mathbf{id}. \qquad \text{(proved in Axiom (4))} \end{aligned}$$

Also,

$$\begin{aligned} \mathbf{w} \oplus \mathbf{v} &= (-v_1 + v_1, -v_2 + v_2, \ldots, -v_n + v_n) \qquad \text{(by definition of } \oplus) \\ &= (0, 0, \ldots, 0) \\ &= \mathbf{id}. \qquad \text{(proved in Axiom (4))} \end{aligned}$$

Therefore, Axiom (5) holds, with $-\mathbf{v} = (-v_1, -v_2, \ldots, -v_n)$.

Note: I do NOT use $-\mathbf{v}$ in the entire proof until the last line.

Note: Axiom (5) is about the EXISTENCE of additive inverse for each ARBITRARY $\mathbf{v}$, so all we need to do is to show that one element in V behaves as an additive inverse of $\mathbf{v}$.

Note: When we prove $-\mathbf{v} = (-v_1, -v_2, \ldots, -v_n)$, notice that we do NOT try to do crazy things such as multiplying the "$-$" into the parentheses. (Why can't we do that?)

Axiom (6):

For all $k \in \mathbb{R}$ and for all $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in V$, since v_i is in $\mathbb{R}$ for each $i = 1, 2, \ldots, n$, by the closure of multiplication (NOT scalar multiplication, why?) for $\mathbb{R}$, we have

$$kv_i \in \mathbb{R}.$$

Hence, $k \odot \mathbf{v} = (kv_1, kv_2, \ldots, kv_n) \in \mathbb{R}^n = V$. Therefore, Axiom (6) holds.

Axiom (7):

For all $k \in \mathbb{R}$ and for all $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in V$,

$$\begin{aligned} LHS &= k \odot (\mathbf{u} \oplus \mathbf{v}) \\ &= k \odot (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \qquad \text{(by definition of } \oplus) \\ &= \big(k(u_1 + v_1), k(u_2 + v_2), \ldots, k(u_n + v_n)\big). \qquad \text{(by definition of } \odot) \end{aligned}$$

$$\begin{aligned} RHS &= k \odot \mathbf{u} \oplus k \odot \mathbf{v} \\ &= (ku_1, ku_2, \ldots, ku_n) \oplus (kv_1, kv_2, \ldots, kv_n) \qquad \text{(by definition of } \odot) \\ &= (ku_1 + kv_1, ku_2 + kv_2, \ldots, ku_n + kv_n) \qquad \text{(by definition of } \oplus) \\ &= \big(k(u_1 + v_1), k(u_2 + v_2), \ldots, k(u_n + v_n)\big) \qquad \text{(by distributivity for } \mathbb{R}) \\ &= LHS. \end{aligned}$$

Therefore, Axiom (7) holds.

Axiom (8):

For all $k, \ell \in \mathbb{R}$ and for all $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in V$,

$$\begin{aligned} LHS &= (k+\ell) \odot \mathbf{v} \\ &= \big((k+\ell)v_1, (k+\ell)v_2, \ldots, (k+\ell)v_n\big). \qquad \text{(by definition of } \odot) \end{aligned}$$

$$\begin{aligned} RHS &= k \odot \mathbf{v} \oplus \ell \odot \mathbf{v} \\ &= (kv_1, kv_2, \ldots, kv_n) \oplus (\ell v_1, \ell v_2, \ldots, \ell v_n) \qquad \text{(by definition of } \odot) \\ &= (kv_1 + \ell v_1, kv_2 + \ell v_2, \ldots, kv_n + \ell v_n) \qquad \text{(by definition of } \oplus) \\ &= \big((k+\ell)v_1, (k+\ell)v_2, \ldots, (k+\ell)v_n\big) \qquad \text{(by distributivity for } \mathbb{R}) \\ &= LHS. \end{aligned}$$

Therefore, Axiom (8) holds.

Axiom (9):

For all $k, \ell \in \mathbb{R}$ and for all $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in V$,

$$\begin{aligned} LHS &= (k \cdot \ell) \odot \mathbf{v} \\ &= \big((k \cdot \ell)v_1, (k \cdot \ell)v_2, \ldots, (k \cdot \ell)v_n\big). \qquad \text{(by definition of } \odot) \end{aligned}$$

$$\begin{aligned} RHS &= k \odot (\ell \odot \mathbf{v}) \\ &= k \odot (\ell v_1, \ell v_2, \ldots, \ell v_n) \qquad \text{(by definition of } \odot) \\ &= \big(k(\ell v_1), k(\ell v_2), \ldots, k(\ell v_n)\big) \qquad \text{(by definition of } \odot) \\ &= \big((k \cdot \ell)v_1, (k \cdot \ell)v_2, \ldots, (k \cdot \ell)v_n\big) \qquad \text{(by associativity of multiplication for } \mathbb{R}) \\ &= LHS. \end{aligned}$$

Therefore, Axiom (9) holds.

Axiom (10):

For all $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in V$,

$$\begin{aligned} LHS &= 1 \odot \mathbf{v} \\ &= (1v_1, 1v_2, \ldots, 1v_n) \qquad \text{(by definition of } \odot) \\ &= (v_1, v_2, \ldots, v_n) \\ &= RHS. \end{aligned}$$

Therefore, Axiom (10) holds.

$\square$

Important: In each of the above proofs, we must follow the operations $\oplus$, $\odot$, and parenetheses STRICTLY.

Example 5. Let $V = \{\mathbf{u}\}$ be a singleton set. We define $\oplus$ and $\odot$ in the following way.

$$\text{For all } k \in \mathbb{R}, \quad \mathbf{u} \oplus \mathbf{u} = \mathbf{u} \quad \text{and} \quad k \odot \mathbf{u} = \mathbf{u}.$$

Verify all 10 axioms.

Note: This vector space is called the "trivial vector space".

Solution. Axiom (1):

For all $\mathbf{v}, \mathbf{w} \in V$, $\mathbf{v} = \mathbf{w} = \mathbf{u}$, so

$$\mathbf{v} \oplus \mathbf{w} = \mathbf{u} \oplus \mathbf{u} = \mathbf{u} \in V. \qquad \text{(by definition of } \oplus\text{)}$$

Therefore, Axiom (1) holds.

Axiom (2):

For all $\mathbf{v}, \mathbf{w} \in V$, $\mathbf{v} = \mathbf{w} = \mathbf{u}$, so

$$LHS = \mathbf{v} \oplus \mathbf{w} = \mathbf{u} \oplus \mathbf{u} = \mathbf{w} \oplus \mathbf{v} = RHS.$$

Therefore, Axiom (2) holds.

Axiom (3):

For all $\mathbf{v}, \mathbf{w}, \mathbf{x} \in V$, $\mathbf{v} = \mathbf{w} = \mathbf{x} = \mathbf{u}$, so

$$LHS = \mathbf{v} \oplus (\mathbf{w} \oplus \mathbf{x}) = \mathbf{u} \oplus (\mathbf{u} \oplus \mathbf{u}) = \mathbf{u} \oplus \mathbf{u} = (\mathbf{u} \oplus \mathbf{u}) \oplus \mathbf{u} = (\mathbf{v} \oplus \mathbf{w}) \oplus \mathbf{x} = RHS.$$

Therefore, Axiom (3) holds.

Axiom (4):

Consider $\mathbf{u}$. For all $\mathbf{v} \in V$, $\mathbf{v} = \mathbf{u}$, so

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{u} \oplus \mathbf{u} = \mathbf{u} = \mathbf{v}$$

and

$$\mathbf{v} \oplus \mathbf{u} = \mathbf{u} \oplus \mathbf{u} = \mathbf{u} = \mathbf{v}.$$

Therefore, Axiom (4) holds, with $\mathbf{id} = \mathbf{u}$.

Axiom (5):

For all $\mathbf{v} \in V$, $\mathbf{v} = \mathbf{u}$. Consider $\mathbf{w} = \mathbf{u} \in V$.

$$\mathbf{v} \oplus \mathbf{w} = \mathbf{u} \oplus \mathbf{u} = \mathbf{u} = \mathbf{id}$$

and

$$\mathbf{w} \oplus \mathbf{v} = \mathbf{u} \oplus \mathbf{u} = \mathbf{u} = \mathbf{id}.$$

Therefore, Axiom (5) holds, with $-\mathbf{v} = -\mathbf{u} = \mathbf{u}$.

Axiom (6):

For all $k \in \mathbb{R}$ and for all $\mathbf{v} \in V$, $\mathbf{v} = \mathbf{u}$, so

$$k \odot \mathbf{v} = k \odot \mathbf{u} = \mathbf{u} \in V. \qquad \text{(by definition of } \odot\text{)}$$

Therefore, Axiom (6) holds.

Axiom (7):

For all $k \in \mathbb{R}$ and for all $\mathbf{v}, \mathbf{w} \in V$, $\mathbf{v} = \mathbf{w} = \mathbf{u}$, so

$$LHS = k \odot (\mathbf{v} \oplus \mathbf{w}) = k \odot (\mathbf{u} \oplus \mathbf{u}) = k \odot \mathbf{u} = \mathbf{u}$$

and

$$RHS = k \odot \mathbf{v} \oplus k \odot \mathbf{w} = k \odot \mathbf{u} \oplus k \odot \mathbf{u} = \mathbf{u} \oplus \mathbf{u} = \mathbf{u} = LHS.$$

Therefore, Axiom (7) holds.

Axiom (8):
For all $k, \ell \in \mathbb{R}$ and for all $\mathbf{v} \in V$, $\mathbf{v} = \mathbf{u}$, so

$$LHS = (k + \ell) \odot \mathbf{v} = (k + \ell) \odot \mathbf{u} = \mathbf{u} \qquad \text{(Why?)}$$

and

$$RHS = k \odot \mathbf{v} \oplus \ell \odot \mathbf{v} = k \odot \mathbf{u} \oplus \ell \odot \mathbf{u} = \mathbf{u} \oplus \mathbf{u} = \mathbf{u} = LHS.$$

Therefore, Axiom (8) holds.

Axiom (9):
For all $k, \ell \in \mathbb{R}$ and for all $\mathbf{v} \in V$, $\mathbf{v} = \mathbf{u}$, so

$$LHS = (k \cdot \ell) \odot \mathbf{v} = (k \cdot \ell) \odot \mathbf{u} = \mathbf{u} \qquad \text{(Why?)}$$

and

$$RHS = k \odot (\ell \odot \mathbf{v}) = k \odot (\ell \odot \mathbf{u}) = k \odot \mathbf{u} = \mathbf{u} = LHS.$$

Therefore, Axiom (9) holds.

Axiom (10):
For all $\mathbf{v} \in V$, $\mathbf{v} = \mathbf{u}$, so

$$LHS = 1 \odot \mathbf{v} = 1 \odot \mathbf{u} = \mathbf{u} = RHS. \qquad \text{(Why?)}$$

Therefore, Axiom (10) holds.

□

Example 6. Let V be the set of functions with both domain and codomain being $\mathbb{R}$. In other words,

$$V = \{\mathbf{f} : \mathbb{R} \to \mathbb{R}\}.$$

For any two functions $\mathbf{f}$ and $\mathbf{g}$ in V, and for all $k \in \mathbb{R}$, define the operations of addition and scalar multiplication such that for all $x \in \mathbb{R}$,

$$(\mathbf{f} \oplus \mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x),$$
$$(k \odot \mathbf{f})(x) = k\mathbf{f}(x).$$

Verify all 10 axioms.

Note: This vector space is often denoted by $F(-\infty, \infty)$, and the addition and scalar multiplication defined in this example are "standard" **for** $F(-\infty, \infty)$.

Example 7. Let V be the set of all polynomials of x with real coefficients. In other words,

$$V = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : n \in \mathbb{Z}_{\geq 0}, a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}\}.$$

For any two polynomials $\mathbf{u} = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ and $\mathbf{v} = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m$ in V, without loss of generality, let $n \geq m$. Let $b_i = 0$ for all $m < i \leq n$. For all $k \in \mathbb{R}$, define the operations of addition and scalar multiplication such that

$$\mathbf{u} \oplus \mathbf{v} = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n,$$
$$k \odot \mathbf{u} = (ka_0) + (ka_1)x + (ka_2)x^2 + \cdots + (ka_n)x^n.$$

Verify all 10 axioms.

Note: This vector space is often denoted by P_∞, and the addition and scalar multiplication defined in this example are "standard" for P_∞.

Example 8. Let $V = \mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. For all $\mathbf{u} = u$ and $\mathbf{v} = v$ in V, and for all $k \in \mathbb{R}$, define the operations of addition and scalar multiplication such that

$$\mathbf{u} \oplus \mathbf{v} = u \cdot v \quad \text{and} \quad k \odot \mathbf{u} = u^k.$$

Verify all 10 axioms.

Note: This is a very strange vector space, but it definitely opens up your mind about what vector spaces are like.

Homework

Problem 9. Let V be a nonempty set, and let $\oplus : V \times V \to V$ be an addition operator defined on V. Assume that an additive identity exists in V. Prove that the additive identity is unique. (In other words, show we cannot have two distinct elements in V such that both behave as an additive identity.)

Problem 10 (Textbook 4.1.1)**.** Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} \oplus \mathbf{v} = (u_1 + v_1, u_2 + v_2), \quad \text{and} \quad k \odot \mathbf{u} = (0, ku_2).$$

(a) Compute $\mathbf{u} \oplus \mathbf{v}$ and $k \odot \mathbf{u}$ for $\mathbf{u} = (-1, 2)$, $\mathbf{v} = (3, 4)$, and $k = 3$.
(b) Explain why V is closed under addition and scalar multiplication.
(c) Since addition on V is the standard addition operation on $\mathbb{R}^2$, certain vector space axioms hold for V because they are known to hold for $\mathbb{R}^2$ (by Example 4). Which axioms are they?
(d) Show Axioms (7), (8), and (9) hold.
(e) Show that Axiom (10) fails and hence that V is not a vector space under the given operations.

Problem 11 (Textbook 4.1.2)**.** Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} \oplus \mathbf{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1), \quad \text{and} \quad k \odot \mathbf{u} = (ku_1, ku_2).$$

(a) Compute $\mathbf{u} \oplus \mathbf{v}$ and $k \odot \mathbf{u}$ for $\mathbf{u} = (0, 4)$, $\mathbf{v} = (1, -3)$, and $k = 2$.
(b) Show that $(0, 0) \neq \mathbf{id}$.
(c) Show that $(-1, -1) = \mathbf{id}$.
(d) Show that Axiom (5) holds by producing an ordered pair $\mathbf{v}$ such that $\mathbf{u} \oplus \mathbf{v} = \mathbf{id}$ for $\mathbf{u} = (u_1, u_2)$. (This $\mathbf{v}$ will then become $-\mathbf{u}$.)
(e) Find two vector space axioms that fail to hold.

In Problems 12 to 15, determine whether each set equipped with the given operations is a vector space. If it is a vector space, show that all 10 axioms hold; if not, find ALL axioms that fail. In other words, you have to check each of the 10 axioms for every single problem.

Problem 12 (Textbook 4.1.5)**.** The set of all pairs of real numbers of the form (x, y), where $x \geq 0$, with the standard operations on $\mathbb{R}^2$.

Problem 13 (Textbook 4.1.7)**.** The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by

$$k \odot (x, y, z) = (k^2 x, k^2 y, k^2 z).$$

Problem 14 (Textbook 4.1.10)**.** The set of all functions $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{f}(1) = 0$, and the addition and scalar multiplication operations are the same as those introduced in Example 6.

Problem 15 (Textbook 4.1.11)**.** The set of all pairs of real numbers of the form $(1, x)$ with the operations

$$(1, y) \oplus (1, y') = (1, y + y') \quad \text{and} \quad k \odot (1, y) = (1, ky).$$

Problem 16 (Textbook 4.1.16)**.** Verify all 10 axioms for Example 8.

Problem 17 (Textbook 4.1.TF)**.** In the following parts, determine whether the statement is true of false, and justify your answer.

(a) A vector is a directed line segment (an arrow).
(b) A vector is an n-tuple of real numbers.
(c) A vector is any element of a vector space.
(d) There is a vector space consisting of exactly two distinct vectors.
(e) The set of polynomials with degree exactly 1 is a vector space under the operations defined in Example 7.