Section 4 Groups, p45 #2,3,5,10,11-16 all

In Exercises 1 through 6, determine whether the binary operation $*$ gives a group structure on the given set. If no group results, give the first axiom in order $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3$ from Definition 4.1 that does not hold.

2. Let $*$ be defined on $\mathbb{Z}$ by letting $a * b = ab$.

$\mathfrak{G}_2$ (identity) does not hold. One might consider 1 to be the identity, but $1 \cdot 0 = 0$. In fact, $n \cdot 0 = 0$ for any such $n \in \mathbb{Z}$. So no identity can exist with this $*$ on $\mathbb{Z}$.

3. Let $*$ be defined on $2\mathbb{Z} = \{2n : n \in \mathbb{Z}\}$ by letting $a * b = ab$.

$\mathfrak{G}_2$ (identity) does not hold. There is no such element e where $e * n = n$, for any $n \in 2\mathbb{Z}$.

5. Let $*$ be defined on the set $\mathbb{R}^*$ of nonzero real numbers by letting $a * b = a/b$.

$\mathfrak{G}_2$ (identity) is only partially held. 1 is a right identity, as $x * 1 = x/1 = x$ for all $x \in \mathbb{R}^*$. However, this does not apply to the left as $1 * x = 1/x \neq x$ unless $x = 1$. Since both are required for this axiom to apply, it is not satisfied.

10. Let n be a positive integer and let $n\mathbb{Z} = \{nm | m \in \mathbb{Z}\}$.

Show the following:

a. $\langle n\mathbb{Z}, +\rangle$ is a group.

A group must be closed, associative, have an identity, and have an inverse for every element.

Proof. Consider $\langle n\mathbb{Z}, +\rangle$.

i. Closed: Consider nm and np for $m, p \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$.

$$nm + np = n(m + p)$$

Thus $\langle n\mathbb{Z}, +\rangle$ is closed under addition.

ii. Associativity: Consider $nm, np, nq \in n\mathbb{Z}$.

$$\begin{aligned}(nm + np) + nq &= n(m + p) + nq \\ &= n(m + p + q)\end{aligned}$$

$$\begin{aligned}nm + (np + nq) &= nm + n(p + q) \\ &= n(m + p + q)\end{aligned}$$

Thus $\langle n\mathbb{Z}, +\rangle$ is associative.

iii. Identity: Consider $n0$ and $nm \in n\mathbb{Z}$.

$$\begin{aligned}n0 + nm &= n(0 + m) = nm \\ nm + n0 &= n(m + 0) = nm\end{aligned}$$

Thus $\langle n\mathbb{Z}, +\rangle$ has an identity.

iv. Inverse: Consider $nm + n\overline{m} = n0$

$$\begin{aligned}nm + n\overline{m} &= n0 \\ n\overline{m} &= n0 + (-nm) \\ n\overline{m} &= -nm\end{aligned}$$

Thus every element nm has inverse $-nm$.

Because $\langle n\mathbb{Z}, +\rangle$ is closed, associative, has an identity, and an inverse for every element, it is a group. $\square$

b. $\langle n\mathbb{Z}, +\rangle \simeq \langle \mathbb{Z}, +\rangle$.

answer

In exercises 11 through 18, determine whether the given set of matrices under the specified operation, matrix addition or multiplication, is a group.

11. All $n \times n$ diagonal matrices under matrix addition.

answer

12. All $n \times n$ diagonal matrices under matrix multiplication.

answer

13. All $n \times n$ diagonal matricies with no zero diagonal entry under matrix multiplication.

answer

14. All $n \times n$ diagonal matricies with all diagonal entries 1 or -1 under matrix multiplication

answer

15. All $n \times n$ upper-triangular matricies under matrix multiplication.

answer

16. All $n \times n$ upper-triangular matricies under matrix addition.

answer