

# MAT 369 Introduction to Graph Theory

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# 1 Introduction

## 1.1 Graphs and Graph Models

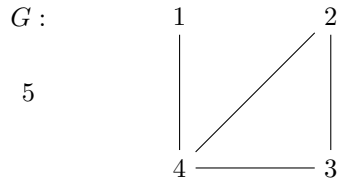
### Graph Definition

A (simple) **graph** is an ordered pair  $(V, E)$  where

- $V$  is a nonempty set of objects called "vertices"
- $E$  is a set containing some two-subsets of  $V$  called "edges".  $E$  may be empty.

Graphs are often represented pictorially. For example consider

$$G = (V, E) \text{ where } V = \{1, 2, 3, 4, 5\} \text{ and } E = \{\{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$$



- Vertices 1 and 4 are **adjacent** because they are joined by an edge.
- Vertex 2 and edge  $2 - 3$  are **incident**.
- Edges  $2 - 3$  and  $3 - 4$  are **adjacent**.

### Order Definition

The **order** of a graph  $G$  is  $|V(G)|$ , or the number of vertices.

### Size Definition

The **size** of a graph  $G$  is  $|E(G)|$ , or the number of edges.

The graph  $G$  from above has order 5 and size 4.

## 1.2 Connected Graphs

### Subgraph Definition

Let  $G$  and  $H$  be graphs.  $H$  is a **subgraph** of  $G$ , notated as  $H \subseteq G$ , if

$$V(H) \subseteq V(G) \text{ and } E(H) \subseteq E(G).$$

### Proper Subgraph Definition

$H$  is a **proper subgraph** of  $G$  if  $H \subseteq G$  and either

$$V(H) \subsetneq V(G) \text{ or } E(H) \subsetneq E(G).$$

### Spanning Subgraph Definition

Graph  $H$  is a **spanning subgraph** if  $H \subseteq G$  and  $V(H) = V(G)$ .

**Induced Subgraph Definition**

Graph  $H$  is a **induced subgraph** if  $H \subseteq G$  and if

$$u, v \in V(H) \text{ and } u, v \in E(G) \implies u, v \in E(H).$$

Essentially,  $H$  contains all valid edges it can take from  $G$ . Notation for **induced subgraph** is

$$G[S], \text{ where } S \text{ is a set of vertices from } G.$$

**Edge-induced Subgraph Definition**

$G[X]$  is an **edge-induced subgraph** of  $G$  if  $G[X]$  has edge set  $X \subseteq E(G)$  and a vertex set of all vertices incident with at least one edge of  $X$ . Interesting fact:  $G[E(G)]$  removes any isolated vertices.

**More on Spanning and Induced Subgraphs**

Let  $G$  be a graph with vertex  $v$  and edge  $e$ . Then,

- $G - e$  is the *spanning subgraph* of  $G$  whose edge set is  $E(G) - \{e\}$ .

This definition can be expanded to  $G - X$  for  $X \subseteq E(G)$ .

- $G - v$  is the *induced subgraph* of  $G$  whose vertex set is  $V(G) - \{v\}$  and edge set includes all edges of  $G$  except those incident with  $v$ .

This definition can be expanded to  $G - U$  for  $U \subseteq V(G)$ .

Let  $G$  be a graph,  $u, v \in V(G)$  and  $e = uv \notin E(G)$ . Then  $G + e$  is the graph with vertex set  $V(G)$  and edge set  $E(G) \cup \{e\}$ .  $G$  is a *spanning subgraph* of  $G + e$

**Walk, Trail, Path, Circuit, and Cycle Definitions**

Let  $u, v \in V(G)$ . A  $u - v$  **walk** in  $G$  is a sequence of vertices

$$(u = v_0, v_1, \dots, v_k = v)$$

beginning with  $u$ , ending with  $v$ , and consecutive vertices are adjacent.

A **trail** is a walk in which *no edges* are repeated. A **path** is a walk in which *no vertices* are repeated. Every *path* is a *trail* is a *walk*.

A **circuit** is a closed trail of length  $\geq 3$ . A **cycle** is a circuit with no repeated vertices, except for the first and the last, which are the same. A  $k$ -**cycle** is a cycle of length  $k$ . Every *cycle* is a *circuit* is a *walk*.

**Closed and Open Walks**

A  $u - v$  walk with  $u = v$  is called a **closed** walk. A  $u - v$  walk with  $u \neq v$  is called a **open** walk.

**Walk and Path Theorem**

If  $G$  contains a  $u - v$  walk of length  $\ell$ , then  $G$  contains a  $u - v$  path of length  $\leq \ell$ .

*Proof.* Let  $P = (u = u_0, u_1, \dots, u_k = v)$  be a  $u - v$  walk of smallest length  $k \leq \ell$ .

*Claim.*  $P$  is a  $u - v$  path.

If **not**, then  $u_i = u_j$ , for some  $i \neq j$ . Then  $(u = u_0, u_1, \dots, u_i = u_j, \dots, u_k = v)$  will be a smaller  $u - v$  than  $P$ . This contradicts our assertion that  $P$  was the *smallest* walk.  $\square$

Hence,  $P$  is a  $u - v$  path of length  $k \leq \ell$ .  $\square$

**Connectivity Definition**

A graph  $G$  is said to be **connected** if  $\forall u, v \in V(G)$ ,  $G$  contains a  $u - v$  path. If this is not true, i.e.  $\exists u, v \in V(G)$  where there is no  $u - v$  path, then  $G$  is said to be **disconnected**.

**Component Definition**

A connected subgraph of  $G$  that is not a proper subgraph of any other connected subgraph of  $G$  is a **component** of  $G$ . The number of components of a graph  $G$  is denoted by  $k(G)$ . A graph  $G$  is connected if and only if  $k(G) = 1$ . Additionally, a graph is the union of its components.

**Components and Equivalence Relations Theorem**

Define a relation  $R$  on the  $V(G)$  so that  $uRv$  if  $G$  contains a  $u - v$  walk. Then  $R$  is an equivalence relation.

*Proof.* An equivalence relation must be reflexive, symmetric, and transitive.

1. Reflexive:  $\forall u \in V(G)$ ,  $(u)$  is a  $u - u$  walk, so  $uRu$ .
2. Symmetric: Suppose  $uRv$ . There is a  $u - v$  walk  $(u = u_0, u_1, \dots, u_n = v)$ . Then reversing the walk gives the  $v - u$  walk  $(v = u_n, \dots, u_1, u_0 = u)$ . So  $vRu$ .
3. Transitive: Suppose  $uRv$  and  $vRw$ . There is a  $u - v$  walk  $(u = u_0, u_1, \dots, u_n = v)$ , and  $v - w$  walk  $(v = v_0, v_1, \dots, v_m = w)$ . Then there is a  $u - w$  walk  $(u = u_0, u_1, \dots, u_n = v = v_0, \dots, v_m = w)$ . So  $uRw$ .

□

What are the equivalence classes?

$$\begin{aligned} [u] &= \{v \in V(G) \mid uRv\} \\ &= \{v \in V(G) \mid \text{there is a } u - v \text{ walk in } G\} \\ &= \text{the connected component containing } u \end{aligned}$$

**Subtractive Connectivity Theorem (weak)**

Let  $G$  be a graph of order  $\geq 3$ . If  $\exists u, v \in V(G)$  such that  $G - u$  and  $G - v$  are connected, then  $G$  is connected.

*Proof.* Suppose  $G$  has order at least 3 and  $\exists u, v \in V(G)$  such that  $G - u$  and  $G - v$  are connected. Let  $x, y \in V(G)$ .

Case 1  $\{x, y\} \neq \{u, v\}$ , meaning at least one is different. Say (WLOG)  $u \notin \{x, y\}$ .

Then  $x, y \in V(G - u)$ , which is connected, contains an  $x - y$  walk.

Case 2  $\{x, y\} = \{u, v\}$ . Say (WLOG)  $x = u$  and  $y = v$ .

Consider  $z \in V(G - u)$  and  $z \in V(G - v)$ .

Then  $x, z \in V(G - v)$  contains a  $x - z$  walk  $(x = x_0, \dots, x_n = z)$ , since it is connected.

Then  $z, y \in V(G - u)$  contains a  $z - y$  walk  $(z = z_0, \dots, z_m = y)$ , since it is connected.

Now consider  $(x = x_0, \dots, x_n = z = z_0, \dots, z_m = y)$ . This is an  $x - y$  walk in  $G$ .

□

### Distance, Geodesic, Diameter, and Girth Definitions

The **distance** between vertices  $u$  and  $v$ , denoted as  $d(u, v)$  or  $d_G(u, v)$  is the smallest length of any  $u - v$  path in  $G$ . If  $u$  and  $v$  are in different components, then  $d(u, v)$  is undefined.

A  $u - v$  path of shortest length  $d(u, v)$  is called a **geodesic**. The **diameter** of a connected graph  $G$ , denoted as  $\text{diam}(G)$ , is the largest *geodesic* between any two vertices of  $G$ . The **girth** of a connected graph  $G$  is the length of the shortest cycle in  $G$ .

### Subtractive Connectivity Theorem (strong)

Let  $G$  be a graph of order  $\geq 3$ . Then  $G$  is connected if and only if  $\exists u, v \in V(G)$  such that  $G - u$  and  $G - v$  are connected.

*Proof.*  $\implies$  is already proven by the weak version of this theorem.

$\impliedby$  Suppose  $G$  is connected. Then  $\exists u, v \in V(G)$  such that  $d(u, v) = \text{diam}(G)$ .

Suppose to the contrary, WLOG, that  $G - v$  is disconnected. Then *exists*  $x, y \in V(G - v)$  such that there is no  $x - y$  walk in  $G - v$ . But  $G$  is connected, so there exist  $x - u$  and  $u - y$  paths in  $G$ .

Let  $P'$  be an  $x - u$  geodesic in  $G$  and  $P''$  be a  $u - y$  geodesic in  $G$ .

$v$  cannot be on  $P'$  or  $P''$  because if it was, then  $d(x, u) > d(u, v)$  or  $d(y, u) > d(u, v)$ , violating our assertion that  $d(u, v) = \text{diam}(G)$ .

Then  $P'P''$   $x - u - y$  is an  $x - y$  walk in  $G - v$ .

This contradicts our selection that  $x$  and  $y$  do not have a walk in  $G - v$ . Hence  $G - v$  and  $G - u$  are connected.  $\square$

## 1.3 Common Classes of Graphs

Name	Symbol	Order	Size
Path	$P_n$	$n$	$n - 1$
Cycle	$C_n$	$n \geq 3$	$n$
Complete	$K_n$	$n$	$\binom{n}{2}$
Complete Bipartite	$K_{s,t}$	$s + t$	$s \cdot t$

### Bipartite Graph Definition

$G$  is bipartite if  $V(G)$  can be partitioned into partite sets  $U$  and  $W$  so that every edge joins a vertex of  $U$  and a vertex of  $W$ .

### Odd Cycle and Bipartite-ness

$G$  is bipartite if and only if  $G$  contains no odd cycles.

### K-partite Definition

$G$  is a **k-partite** graph if  $V(G)$  can be partitioned into partite sets  $U_1, \dots, U_k$  so that every edge joins a vertex from  $U_i$  and a vertex of  $U_j$  where  $i \neq j$ .

## Constructing New Graphs from Old Graphs

### Disjoint Union

For two graphs  $G$  and  $H$ ,  $G \cup H$  is defined as...

$$\begin{aligned} V(G \cup H) &= V(G) \cup V(H) \\ E(G \cup H) &= E(G) \cup E(H) \end{aligned}$$

**Complement**

For one graph  $G$ ,  $\overline{G}$  is defined as...

$$V(\overline{G}) = V(G)$$

$$E(\overline{G}) = \{uv \mid u, v \in V(G), u \neq v, uv \notin E(G)\}$$

**Join**

For two graph,  $G$  and  $H$ ,  $G + H$  is defined as...

Start with  $G \cup H$  and draw all edges join a vertex of  $G$  and a vertex of  $H$

**Cartesian Product**

For two graphs,  $G$  and  $H$ ,  $G \times H$  is defined as...

$$V(G \times H) = \{(u, v) \mid u \in V(G) \text{ and } v \in V(H)\}$$

$$(u, v) - (x, y) \text{ if } u = x \text{ and } vy \in E(H) \vee v = y \text{ and } ux \in E(G)$$

A cartesian product between two graphs has the practical effect of duplicating one graph, and connecting the duplicates in the way of the other graph.

**Complement Connectivity Theorem**

If  $G$  is disconnected, then  $\overline{G}$  is connected.

**1.4 Multigraphs and Digraphs****Multigraph Definition**

A **multigraph** is a graph where a pair of vertices may be joined by any finite number of edges.

- Multiple edges: OK
- Loops: NOT OK

**Pseudograph Definition**

A **pseudograph** is a *multigraph* where loops are allowed

- Multiple edges: OK
- Loops: OK

**Digraph Definition**

A **directed graph** is a graph where  $E(G)$  is a set of ordered pairs (rather than sets) of distinct vertices called directed edges, or arcs.

**Oriented Graph Definition**

An **oriented graph** is a *digraph* in  $\forall u, v \in V(G)$ ,  $(u, v)$  and  $(v, u)$  are not both edges.

## 2 Degrees

### 2.1 Degree of a Vertex

#### Vertex Degree Definition

The **degree** of a vertex  $v$ , denoted as  $\deg v$  or  $\deg_G v$ , is the number of edges incident with  $v$ . If the  $\deg v = 0$ , then  $v$  is an **isolated vertex**. If  $\deg v = 1$ , then  $v$  is a **leaf**.

- $\delta(G) = \min\{\deg v \mid v \in V(G)\}$ , the minimum degree of  $G$
- $\Delta(G) = \max\{\deg v \mid v \in V(G)\}$ , the maximum degree of  $G$

For any graph  $G$  and  $v \in V(G)$ ,

$$0 \leq \delta(G) \leq \deg v \leq \Delta(G) \leq n - 1.$$

#### Neighborhood of a Vertex Definition

The **neighborhood** of a vertex  $v$ , denoted as  $N(v)$ , is the set of all vertices adjacent to  $v$ . So  $|N(v)| = \deg v$ .

#### Handshaking Theorem

For a graph  $G$  of size  $m$ , the total degree of  $G$   $\sum_{v \in V(G)} \deg v = 2m$ .

#### Handshaking Corollary

Every graph has an even number of odd degree vertices.

#### Sum Degree and Connectivity Theorem

Consider graph  $G$  of order  $n$ . If  $\deg u + \deg v \geq n - 1$  for all non-adjacent  $(u, v) \in V(G)$ , then  $G$  is connected.

#### Sum Degree and Connectivity Corollary

If  $G$  has order  $n$  and  $\delta(G) \geq \frac{n-1}{2}$ , then  $G$  is connected.

### 2.2 Regular Graphs

#### Regular Graph Definition

Graph  $G$  is **regular** if every vertex has the same degree. Graph  $G$  is  **$r$ -regular** if every vertex has degree  $r$ .

#### Regular Graph Existence Theorem

Let  $r, n \in \mathbb{Z}$  such that  $0 \leq r \leq n - 1$ . Then there exists an  $r$ -regular graph of order  $n$  if and only if at least one of  $r$  and  $n$  is even.

#### Harary Graph

An Harary Graph, denoted as  $H_{r,n}$ , is an  $r$ -regular graph of order  $n$ .

#### Induced Regular Subgraph Theorem

For every graph  $G$ , and every integer  $r \geq \Delta(G)$ , there exists an  $r$ -regular graph  $H$ , containing  $G$  as an induced subgraph.



## 2.3 Degree Sequences

### Degree Sequence Definition

A **degree sequence** is a sequence of the degree of the vertices of a graph, typically, written in largest to smallest order.

### Graphical Degree Sequence Definition

A finite sequence of non-negative integers is **graphical** if it is the degree sequence of some graph.

### Graphical Degree Sequence Theorem

A non-increasing sequence  $S : d_1, d_2, \dots, d_n$ , where  $n \geq 1$ , of non-negative integers is graphical if and only if

$$S_1 : d_2 - 1, d_3 - 1, \dots, d_{d_1+1}, d_{d_1+2}, d_n$$

is graphical.

## 2.4 Graph and Matrices

### Adjacency Matrix Definition

The **adjacency matrix** of  $G$  is the  $n \times n$  matrix  $A = [a_{ij}]$ , where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise;} \end{cases}$$

The entry  $a_{ij}$  in  $A^n$  is the number of walks of length  $n$  from  $v_i$  to  $v_j$ .

## 3 Isomorphic Graphs

### 3.1 The Definition of Isomorphism

#### Graph Equality Definition

Two graphs are **equal**, denoted as  $G = H$ , if  $V(G) = V(H)$  and  $E(G) = E(H)$ .

#### Graph Isomorphic Definition

Two (labels) graphs  $G$  and  $H$  are **isomorphic**, denoted as  $G \cong H$ , if they have the same structure, meaning there is a *bijection*  $\phi : V(G) \rightarrow V(H)$  such that for  $u, v \in V(G)$ ,  $\phi(u)\phi(v) \in E(H)$  if and only if  $uv \in E(G)$ .

#### Isomorphic Degree Theorem

If  $G \cong H$ , with isomorphic  $\phi : V(G) \rightarrow V(H)$ , then  $\deg_G u = \deg_H \phi(u)$ .

#### Isomorphic Degree Corollary

If  $G \cong H$ , their degree sequences are equal.

#### Graph Invariants

- To prove  $G \cong H$ , find an isomorphism.
- To prove  $G \not\cong H$ , find a graph invariant, where  $G$  and  $H$  differ.

#### Graph Invariants

- Order and Size
- Degree Sequence
- Cycles
- Diameter
- $k$  (number of components)
- $k$ -partite-ness
- (Other things)

#### Adjacency and Non-adjacency under Isomorphism Theorem

$G \cong H$  if and only if  $\overline{G} \cong \overline{H}$ .

### 3.2 Isomorphism as a Relation

#### Equivalence Relations and Isomorphism Theorem

Isomorphism is an equivalence relation.

## 4 Trees

### 4.1 Cut Edges

#### Cut-edge and Bridge Definition

An edge  $e$  of graph  $G$  is a **cut-edge**, or **bridge**, if  $G - e$  has more components than  $G$ .

#### Cut-edges and Cycles Theorem

An edge  $e$  of a graph  $G$  is a cut-edge if and only if  $e$  lies on *no* cycle in  $G$ .

### 4.2 Trees

#### Tree and Forest Definitions

A **tree** is an acyclic connected graph. A **forest** is an acyclic graph, where each component is a *tree*. A **Rooted tree** is a tree with a specific vertex designated as a root and drawn down.

Every edge of a tree is a cut-edge.

#### Unique Path in Trees Theorem

Graph  $G$  is a tree if and only if every 2 vertices are connected by a unique path.

#### Leaf Theorem

Every nontrivial tree has at least 2 leaves.

#### Autumn Theorem

If tree  $T$  has order  $t \geq 1$ , then  $T - v$ , where  $v$  is a leaf, is a tree of order  $t - 1$ .

#### Tree Size Theorem

Every tree of order  $n$  has size  $n - 1$ .

#### Forest Size Theorem

Every forest of order  $n$  with  $k$  components has size  $n - k$ .

#### Minimum Size of a Connected Graph Theorem

The size of every connected graph of order  $n$  is at least  $n - 1$ . Trees has minimal size among connected graphs of given order.

#### Tree Requirements Graph

Graph  $G$  of order  $n$  and size  $m$ . Then  $G$  is a tree if it satisfies any 2 of these properties:

1.  $G$  is connected
2.  $G$  acyclic
3.  $m = n - 1$

#### Tree Isomorphic Subgraph Theorem

Let  $T$  be a tree of order  $k$ . Then for any graph  $G$  with  $\delta(G) \geq k - 1$ ,  $T$  is isomorphic to a subgraph of  $G$ .

### 4.3 Minimum Spanning Tree

#### Spanning Tree Definition

Let  $G$  be a connected graph. A spanning subgraph of  $G$  that is a tree is called a **spanning tree**.

#### Spanning Tree Existence Theorem

Every connected graph contains a spanning tree.

#### Minimum Spanning Tree Definition

A **minimum spanning tree** is a spanning tree of minimum weight.

### Algorithms For Constructing Minimum Spanning Trees

#### Kruskal's Algorithm

1. Pick an edge of minimum weight.
2. Repeat, never allowing the chosen edges to produce a cycle.
3. Stop once you have a spanning tree.

#### Prim's Algorithm

1. Choose any vertex  $u \in V(G)$ .
2. Let  $e$  be an edge of minimum weight incident with  $u$ .
3. Continue picking edges of minimum weight from the set of edges having exactly one of its vertices incident with an already selected edge.
4. Stop once you have a spanning tree.

### 4.4 Counting Labeled Trees

#### Cayley's Theorem

There are  $n^{n-2}$  distinct labeled trees on  $n$  vertices.

#### Prüfer Sequence

Encoding a Tree to a Sequence

1. Start with a labeled tree  $T$ , and  $i = 1$ .
2. Let  $b_i$  = smallest label on a leaf.
3. Let  $a_i$  = label of the adjacent vertex of  $b_i$ .
4. Remove  $b_i$  and record  $a_i$  in the sequence.
5. Repeat with  $b_{i+1}$  and  $a_{i+1}$ .
6. Stop once only vertices remain.

Decoding a Sequence to a Tree

1. Start with  $(a_1, \dots, a_{n-2})$  and  $i = 1$ .
2. Let  $b_i$  = smallest element of  $\{1, \dots, n\}$  **not** in the sequence.

3. Draw edge  $a_i b_i$ .
4. Remove  $a_i$  from the sequence and  $b_i$  from the set.
5. Repeat with  $b_{i+1}$  and  $a_{i+1}$ .
6. Stop once the sequence is empty, and draw an edge between the last two elements in the set.

## 5 Connectivity

### 5.1 Cut Vertices

#### Cut-edge and Cut-vertex Definition

- **Cut-edge:** Removing cut-edge  $e$  creates a new component.
- **Cut-vertex:** Removing cut-vertex  $v$  creates new components(s).

#### Leaves and Cut-vertices Theorem

Let  $G$  be a connected graph with cut-edge  $e = uv$ .  $v$  is a cut-vertex if and only if  $\deg v \geq 2$ , meaning that  $v$  is not a leaf.

#### Leaves and Cut-vertices Corollary 1

Every vertex of a non-trivial tree is either a leaf of a cut-vertex.

#### Leaves and Cut-vertices Corollary 2

Let  $G$  be a connected graph and of order at least 3. If  $G$  contains a cut-edge, then  $G$  contains a cut-vertex.

#### Paths and Cut-vertices Theorem

Let  $G$  be a connected graph with cut-vertex  $v$ . Let  $u, w$  be vertices in different components of  $G - v$ . Then  $v$  lies on every  $u - w$  path in  $G$ .

#### Paths and Cut-vertices Corollary

Let  $G$  be connected.  $v \in V(G)$  is a cut-vertex if and only if  $\exists u, w \in V(G) - \{v\}$  such that  $v$  lies on every  $u - w$  path in  $G$ .

#### Non-cut-vertex Theorem

Every nontrivial connected Graph contains at least 2 vertices that are not cut-vertices.

### 5.2 Blocks

#### Non-separable Definition

A graph is called **non-separable** if...

1. it is nontrivial,
2. it is connected,
3. it has no cut-vertices, meaning every edge is on a cycle.

Otherwise, it is called **separable**.

#### Common Cycle and Non-separability Theorem

A graph of order at least 3 is non-separable if and only if every 2 vertices (pairwise) lie on a common cycle.

#### Block Definition

A **block** of  $G$  is a maximal, *non-separable* subgraph of  $G$ .

**Blocks are Equivalence Relations Theorem**

Define a Relation  $R$  on  $E(G)$  where  $eRf$  if  $e = f$  or  $e$  and  $f$  lie on a common cycle of  $G$ .  $R$  is an equivalence relation, where equivalence classes of  $R$  are edge-induced blocks of  $G$ .

**Blocks are Equivalence Relations Corollary**

Let  $B_1$  and  $B_2$  be distinct blocks in a nontrivial connected graph  $G$ . Then,

1.  $E(B_1) \cap E(B_2) = \emptyset$ , meaning  $B_1$  and  $B_2$  are edge disjoint.
2.  $B_1$  and  $B_2$  have at most 1 vertex in common.
3. The common vertex, if it exists, is a cut-vertex.

**5.3 Connectivity****Vertex-cut and Minimum Vertex-cut Definition**

- A **vertex-cut** is a set  $U \subseteq V(G)$  such that  $G - U$  is disconnected.
- A **minimum vertex-cut** is a *vertex-cut* of minimum cardinality.

**Connectivity Definition**

The **connectivity** of graph  $G$  is

$$\kappa(G) = \min\{|U| \mid U \subseteq V(G), \text{ such that } G - U \text{ is disconnected or trivial.}\}$$

Note that  $0 \leq \kappa(G) \leq n - 1$ .

 **$k$ -connectivity Definition**

$G$  is called  $k$ -connected if  $\kappa(G) \geq k$ .

**Edge-cut, Minimal, and Minimum Edge-cut Definition**

- An **edge-cut** is a set  $X \subseteq E(G)$  such that  $G - X$  is disconnected.
- A **minimal edge-cut** is an *edge-cut*  $X$  where no proper subset of  $X$  is also an edge-cut.
- A **minimum edge-cut** is an *edge-cut* of minimum cardinality.

**Edge-connectivity Definition**

The **edge-connectivity** of a nontrivial graph  $G$  is

$$\lambda(G) = \min\{|X| \mid X \subseteq E(G), \text{ such that } G - X \text{ is disconnected or trivial.}\}$$

Note that  $0 \leq \lambda(G) \leq n - 1$ .

 **$k$ -edge-connectivity Definition**

$G$  is called  $k$ -edge-connected if  $\lambda(G) \geq k$ .

**Edge-connectivity of Complete Graphs Theorem**

$$\forall n \in \mathbb{N}, \lambda(K_n) = n - 1$$

**Connectivity and Edge-connectivity Ordering Theorem**

For a graph  $G$ ,

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

**IMPORTANT:** These proofs involve taking minimum-edge or minimum-vertex cuts and comparing their cardinalities.

**Cubic Connectivity Theorem**

If graph  $G$  is 3-regular, also called *cubic*, then

$$\kappa(G) = \lambda(G)$$

**Upper Bound for Connectivity Theorem**

If  $G$  has order  $n$  and size  $m \geq n - 1$ , then

$$\kappa(G) \leq \lfloor \frac{2m}{n} \rfloor$$



## 6 Traversability

### 6.1 Eulerian Graphs

#### Seven Bridges of Königsberg Problem



Can you go for a walk, crossing each bridge exactly once? No

#### Eulerian Circuits and Trails Definition

- A **Eulerian Circuit** is a circuit containing every edge of graph  $G$ .
- A **Eulerian Trail** is an open trail containing every edge of graph  $G$ .
- A **Eulerian Graph** is a graph that contains an *Eulerian Circuit*

#### Even Degree and Eulerian Circuits Theorem

A nontrivial, connected graph is Eulerian if and only if *every* vertex has *even* degree.

#### Even Degree and Eulerian Trails Corollary

A connected graph  $G$  contains an Eulerian Trail, if and only if exactly 2 vertices of  $G$  have odd degree.

### 6.2 Hamiltonian Graphs

#### Hamiltonian Cycles and Paths Definition

- A **Hamiltonian Cycle** is a cycle containing every vertex of graph  $G$ .
- A **Hamiltonian Path** is a path containing every vertex of graph  $G$ .
- A **Hamiltonian Graph** is a graph that contains an *Hamiltonian Cycle*.

#### Degree Sum and Hamiltonian Graphs Theorem

Let  $G$  have order  $n \geq 3$ . If  $\deg u + \deg v \geq n$  for all pairs of nonadjacent vertices  $u, v \in V(G)$ , then  $G$  is Hamiltonian. Note that this is only a *one way* statement.

#### Degree Sum and Hamiltonian Graphs Corollary

Let  $G$  have order  $n \geq 3$ . If  $\deg v \geq \frac{n}{2}$  for all  $v \in V(G)$ , then  $G$  is Hamiltonian. Note that this is only a *one way* statement.

## 7 Digraphs

## 8 Matchings and Factorization

## 9 Planarity

## 10 Coloring Graphs

## 11 Ramsey Numbers

## 12 Distance

## 13 Domination