MAT 260 LINEAR ALGEBRA
LECTURE 32

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2.2, 2.3 — Determinants by row reduction and column reduction and properties of determinants

Remember that we can obtain the determinant of matrix A by expansion along any row or any column. So it is not surprising that we have the following theorem.

Theorem 1. $\det A = \det A^\top$.

In view of this theorem, any row operations on A becomes column operations on $A^\top$ and vice versa. Hence, it makes sense to see how row operations and column operations on A will affect $\det A$.

Theorem 2. (a) *Let B be a matrix obtained by switching two rows or two columns of A. Then* $\det B = -\det A$.
(b) *Let B be a matrix obtained by multiplying a scalar k to one row or one column of A. Then* $\det B = k \det A$.
(c) *Let B be a matrix obtained by adding k times of one row to another row, or adding k times of one column to another column, then* $\det B = \det A$.

To prove Theorem 2, we first prove the **multi-linearity** of the determinant function.

Theorem 3. *Let A, B and C be identical $n \times n$ matrices except the i-th row. Let the i-th row of A and B be $\mathbf{a}_i$ and $\mathbf{b}_i$ respectively, and let the i-th row of C be $k\mathbf{a}_i + \mathbf{b}_i$. Then*

$$\det C = k \det A + \det B.$$

Same results hold if we replace "rows" by "columns".

Example 1. Find the determinant of $A = \begin{pmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{pmatrix}$.

Example 2. Find the determinant of $A = \begin{pmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{pmatrix}$.

By Theorem $2(b)$, if A has a row of all zeros, then $\det A = 0$. By Theorem $2(c)$, if A has two rows proportional to each other (or two columns proportional to each other), then we also have $\det A = 0$.

The most important consequence of Theorem 2 is the following theorem.

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Theorem 4. *A is invertible if and only if* $\det A \neq 0$.

By Theorem $2(a)$, together with our knowledge about determinants for diagonal, upper-triangular and lower triangular matrices, we find that
(i) if E is an elementary matrix obtained by switching two rows of I, then $\det E = -1$;
(ii) if E is an elementary matrix obtained by multiplying a nonzero scalar k to one row of I, then $\det E = k$;
(iii) if E is an elementary matrix obtained by adding k times of one row to another row, then $\det E = 1$.

Hence, combining this observation with Theorem 2, we have the following theorem.

Theorem 5. *Let E be an elementary matrix. Then* $\det EA = \det E \det A$.

Theorem 5 can be further generalized into the following theorem.

Theorem 6. $\det AB = \det A \det B$.

Since $A \cdot A^{-1} = I$, we have

Theorem 7. $\det A^{-1} = \frac{1}{\det A}$.

Another property of determinants is as follows. It is a consequence of Theorem $2(b)$.

Theorem 8. $\det kA = k^n \det A$.

2.3 — Matrix inverse and Cramer's Rule

The **adjoint** of A is

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^{\top},$$

denoted by $\mathrm{adj}(A)$, where C_{ij} are the cofactors of A.

Theorem 9. $A^{-1} = \dfrac{1}{\det A}\mathrm{adj}(A)$.

When we solve a system of linear equations, we can also use Cramer's rule.

Theorem 10. *If* $\det A \neq 0$*, then the system of linear equations* $A\mathbf{x} = \mathbf{b}$ *has a unique solution given by*

$$x_1 = \frac{\det A_1}{\det A},\ x_2 = \frac{\det A_2}{\det A},\ \ldots,\ x_n = \frac{\det A_n}{\det A},$$

where A_i is the matrix obtained by replacing the i-th column of A by $\mathbf{b}$.