Homework 8

2.1

3. Let

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

Find the following:

a. M_{13} and C_{13}.

Work.

$$M_{13} = \begin{vmatrix} 0 & 0 & 3 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = 3 \cdot (-1)^{1+3} \begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix} = 3 \cdot 1 \cdot 0 = 0$$
$$C_{13} = (-1)^{1+3} \cdot M_{13} = 1 \cdot 0 = 0$$

□

b. M_{23} and C_{23}.

Work.

$$\begin{aligned} M_{23} &= \begin{vmatrix} 4 & -1 & 6 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} \\ &= 4 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - 1 \cdot (-1)^{1+2} \begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 6 \cdot (-1)^{1+3} \begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix} \\ &= 4 \cdot -12 - (-1) \cdot (-48) + 6 \cdot 0 \\ M_{23} &= -48 - 48 + 0 = -96 \\ C_{23} &= (-1)^{2+3} M_{23} = -1 \cdot -96 = 96 \end{aligned}$$

□

11. Use the arrow technique to evaluate the determinant of the given matrix.

$$\begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{bmatrix}$$

Work.

$$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = ((-2 \cdot 5 \cdot 2) + (1 \cdot -7 \cdot 1) + (4 \cdot 3 \cdot 6)) - ((-2 \cdot -7 \cdot 6) + (1 \cdot 3 \cdot 2) + (4 \cdot 5 \cdot 1))$$
$$= ((-20) + (-7) + (72)) - ((84) + (6) + (20))$$
$$= (45) - (110)$$
$$= -65$$

□

18. Find all values of λ for which $\det(A) = 0$.

$$A = \begin{bmatrix} \lambda - 4 & 4 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda - 5 \end{bmatrix}$$

Work.

$$\begin{vmatrix} \lambda - 4 & 4 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda - 5 \end{vmatrix} = (\lambda - 5) \cdot (-1)^{3+3} \begin{vmatrix} \lambda - 4 & 4 \\ -1 & \lambda \end{vmatrix} = (\lambda - 5) \cdot (\lambda(\lambda - 4) - (4 \cdot -1))$$
$$= (\lambda - 5) \cdot (\lambda^2 - 4\lambda + 4) = (\lambda - 5) \cdot (\lambda - 2)^2$$

$$\lambda = 5,\ 2$$

□

21. Evaluate $\det(A)$ by a cofactor expansion along a row or column of your choice.

$$A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$$

Work.

$$\det(A) = 0 + 5 \cdot (-1)^{2+2} \begin{vmatrix} -3 & 7 \\ -1 & 5 \end{vmatrix} + 0 = 5 \cdot 1 \cdot -8 = -40$$

□

31. Evaluate the determinant of the given matrix by inspection.

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Work.

$$\begin{vmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 1 \cdot 1 \cdot 2 \cdot 3 = 6 \qquad \text{via Theorem 1 of Lecture Notes 31}$$

□

38. What is the maxiumum number of zeros that a 3×3 matrix can have without having a zero determinant? Explain your reasoning.

Proof. Consider $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We know that $\det I = 1 \neq 0$. This means that 6 zeros can be achieved without having a zero determinant.
If $A_{3\times 3}$ has 7 zero entries, then A has at most 2 non-zero entries. Since there are 3 rows, there is a row of all zeros. Expanding along this row to compute the determinant will produce an expansion with coefficients of all zeros, meaning that $\det A = 0$. Therefore, 6 zeros is the maximum number of zeros that a 3×3 matrix can have without having a zero determinant. □

2.2

14. Evaluate the determinant of the given matrix by reducing the matrix to row echelon form.

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{bmatrix}$$

Work.

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} \overset{R_2-5R_1}{\equiv} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} \overset{R_3+R_1}{\equiv} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 2 & 8 & 6 & 1 \end{vmatrix} \overset{R_4-2R_1}{\equiv}$$

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{vmatrix} \overset{R_4-12R_2}{\equiv} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{vmatrix} \overset{R_4+36R_2}{\equiv} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -13 \end{vmatrix} \overset{-\frac{1}{3}R_3}{\equiv} -3$$

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & -13 \end{vmatrix} \overset{-\frac{1}{13}R_4}{\equiv} -13 \cdot -3 \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{vmatrix} = -13 \cdot -3 \cdot 1 = 39$$

□

27. Evaluate the determinant given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6.$$

$$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$

Work.

$$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix} \overset{R_3+4R_2}{=\!=\!=} \begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g & h & i \end{vmatrix} \overset{-\frac{1}{3}R_1}{=\!=\!=} -3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -3 \cdot -6 = 18$$

□

29. Use row reduction to show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$

Work.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \overset{R_2-aR_1}{=\!=\!=} \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ a^2 & b^2 & c^2 \end{vmatrix} \overset{R_3-a^2R_1}{=\!=\!=} \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} \overset{R_3-(b+a)R_2}{=\!=\!=}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(b+a)(c-a) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c+a)(c-a)-(b+a)(c-a) \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{vmatrix} = 1 \cdot (b-a) \cdot (c-a)(c-b) = (b-a)(c-a)(c-b)$$

□

30. Confirm without evaluating the determinant directly:

$$\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1-t^2) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Work.

$$\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \overset{R_1-R_2}{===} \begin{vmatrix} a_1+b_1t-a_1t-b_1 & a_2+b_2t-a_2t-b_2 & a_3+b_3t-a_3t+b_3 \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$\begin{vmatrix} a_1-b_1+b_1t-a_1t & a_2-b_2+b_2t-a_2t & a_3+b_3+b_3t-a_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$\begin{vmatrix} -1(b_1-a_1)+t(b_1-a_1) & -1(b_2-a_2)+t(b_2-a_2) & -1(b_3-a_3)+t(b_3-a_3) \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$\begin{vmatrix} (t-1)(b_1-a_1) & (t-1)(b_2-a_2) & (t-1)(b_3-a_3) \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (t-1)\begin{vmatrix} b_1-a_1 & b_2-a_2 & b_3-a_3 \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \overset{R_2-R_1}{===}$$

$$(t-1)\begin{vmatrix} b_1-a_1 & b_2-a_2 & b_3-a_3 \\ a_1t+b_1-b_1+a_1 & a_2t+b_2-b_2+a_2 & a_3t+b_3-b_3+a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$(t-1)\begin{vmatrix} b_1-a_1 & b_2-a_2 & b_3-a_3 \\ a_1t+a_1 & a_2t+a_2 & a_3t+a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (t-1)\begin{vmatrix} b_1-a_1 & b_2-a_2 & b_3-a_3 \\ a_1(t+1) & a_2(t+1) & a_3(t+1) \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$(t-1)(t+1)\begin{vmatrix} b_1-a_1 & b_2-a_2 & b_3-a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (t^2-1)\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

□

34. Find the determinant of the following matrix:

$$\begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$$

Work.

$$\begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix} = a^4 - 3b^4 - 6a^2b^2 + 8ab^3$$

For full derivation, see attached sheet. □

2.3

19. Decide whether the matrix is invertible, and if so, use the adjoint method to find its inverse.

$$A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$$

Work.

$$\begin{aligned}\det(A) &= 2\cdot(1)\cdot(-1\cdot 3-0\cdot 4)+(-1)\cdot(-1)\cdot(5\cdot 3-5\cdot 4)+2\cdot(1)\cdot(5\cdot 0-5\cdot(-1))\\ &= 2(-3)+1(-5)+2(10) = -6-5+10=-1\end{aligned}$$

Since $\det(A)\neq 0$, by the Big Theorem, A is invertible. To find the inverse via the adjoint method, first find the cofactors of A.

$$\begin{aligned}
&C_{11}=(-3-0)=-3 && C_{12}=-(-3-0)=3 && C_{13}=(-4+2)=-2\\
&C_{21}=-(0-5)=5 && C_{22}=(6-10)=-4 && C_{23}=-(8-10)=2\\
&C_{31}=(0+5)=5 && C_{32}=-(0+5)=-5 && C_{33}=(-2+3)=3
\end{aligned}$$

According to Theorem 9 of Lecture Notes 32,

$$\begin{aligned}
A^{-1} &= \frac{1}{\det(A)}\operatorname{adj}(A)\\
A^{-1} &= -1\cdot\begin{bmatrix}-3 & 3 & -2\\ 5 & -4 & 2\\ 5 & -5 & 3\end{bmatrix}^T\\
A^{-1} &= -1\cdot\begin{bmatrix}-3 & 5 & 5\\ 3 & -4 & -5\\ -2 & 2 & 3\end{bmatrix}\\
A^{-1} &= \begin{bmatrix}3 & -5 & -5\\ -3 & 4 & 5\\ 2 & -2 & -3\end{bmatrix}
\end{aligned}$$

□

27. Solve by Cramer's rule, where it applies:

$$\begin{aligned}1x_1-3x_2+1x_3&=4\\ 2x_1-1x_2+0x_3&=-2\\ 4x_1+0x_2-3x_3&=0\end{aligned}$$

Work.

$$\begin{bmatrix}1 & -3 & 1\\ 2 & -1 & 0\\ 4 & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}4\\ -2\\ 0\end{bmatrix}$$

According to Cramer's rule, $x_1=\frac{\det A_1}{\det A}, x_2=\frac{\det A_2}{\det A}$, and $x_3=\frac{\det A_3}{A}$, where A_i is the matrix obtained by replacing the i-th column of A by $\vec{b}$.

$$\begin{aligned}
x_1 &= \frac{\det A_1}{\det A}=\frac{30}{-11}=-\frac{30}{11}\\
x_2 &= \frac{\det A_2}{\det A}=\frac{38}{-11}=-\frac{38}{11}\\
x_3 &= \frac{\det A_3}{\det A}=\frac{-40}{-11}=\frac{40}{11}
\end{aligned}$$

□

33. Prove that if $\det(A) = 1$ and all the entries in A are integers, then all the entries in A^{-1} are integers.

Proof. Consider matrix A where $\det(A) = 1$ and all the entries in A are integers. Lets look at the adjoint formula for the inverse.

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$$

We know that $\det(A) = 1$, so $\frac{1}{\det(A)}$ is also 1. Since 1 is an integer, if all of entries of the adjacency matrix for A are integers, then all of the entries for A^{-1} will be integers. Lets look at the adjoint matrix formula.

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$$

If all of these cofactor entries of $\text{adj}(A)$ are integers, then all of the entries of A^{-1} will be integers. Consider how these cofactor entries are calculated, and which operations they use. A quick inspection reveals that they use only the following operations; $+, -,$ and $\cdot$. All of these operations are operations that keep integers as integers. This means that if all of the entries of A are integers, then all of the cofactors of the adjacency matrix for A will also be integers. Finally, this implies that all of the entries of A^{-1} will also be integers, so long as $\det(A) = 1$. $\square$