MAT 260 LINEAR ALGEBRA
LECTURE 34

WING HONG TONY WONG

4.3 — Linear independence

Let V be a vector space, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$.

Definition 1. $\mathbf{v}_n$ is **linearly dependent** on $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$ if there exists $k_1, k_2, \ldots, k_{n-1} \in \mathbb{R}$ such that

$$\mathbf{v}_n = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_{n-1}\mathbf{v}_{n-1}.$$

Otherwise, $\mathbf{v}_n$ is **linearly independent** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$.

Definition 2. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are **linearly dependent** if there exists $k_1, k_2, \ldots, k_n \in \mathbb{R}$, not all k_i's are zero, such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{id}.$$

Otherwise, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are **linearly independent**.

Definition 3. $S \subseteq V$ is **linearly dependent** if there exists $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in S$ and $k_1, k_2, \ldots, k_n \in \mathbb{R}$, not all k_i's are zero, such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{id}.$$

Otherwise, S is **linearly independent**. If $S = \emptyset$, then S is also defined to be **linearly independent**.

Recall that a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n.$$

A **trivial linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n,$$

which is obviously equal to $\mathbf{id}$. Hence, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent if there is a **nontrivial linear combination** of them to form $\mathbf{id}$.

Theorem 4. *Let S be a subset of vectors in a vector space V. Then*
(a) if $\mathbf{id} \in S$, S is linearly dependent;
(b) if S contains only one vector, then S is linearly independent if and only if $\mathbf{id} \notin S$;
(c) if S contains exactly two vectors, then S is linearly independent if and only if neither vector is a scalar multiple of the other.

Date: Monday, April 17, 2023.

Theorem 5. *Let S be a subset of two or more vectors in a vector space V. Then S is linearly independent if and only if no vector in S is a linear combination of the others in S.*

If we want to prove that a set S is linearly independent, we often use proof by contrapositive or proof by contradiction. This is because linear independence is a "for all" statement:

All nontrivial linear combinations of vectors in S are not equal to **id**.

Hence, with proof by contrapositive or proof by contradiction, we can start with "there exists" assumption:

There exists a nontrivial combination of vectors in S that equals **id**.

Theorem 6. *Let $V = \mathbb{R}^n$, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$. If $m > n$, then S is linearly dependent.*

Example 7. In $\mathbb{R}^\infty$, let $\mathbf{e}_1 = (1, 0, 0, \ldots)$, $\mathbf{e}_2 = (0, 1, 0, \ldots)$, $\ldots$. Then $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$ is linearly independent.

Example 8. In $V = \mathbb{R}^4$, let $S = \{(1, 2, 3, 4), (5, 6, 7, 8), (9, 10, 11, 12)\}$, $S' = \{(1, 1, 1, 1), (1, 2, 4, 8), (1, 3, 9, 27), (1, 4, 16, 64)\}$, and $S'' = \{(1, 1, 1, 1), (1, 2, 4, 8), (1, 3, 9, 27), (1, 4, 16, 64), (1, 5, 25, 125)\}$. Then S and S'' are linearly dependent, while S' is linearly independent.

Example 9. In $V = P_\infty$, $S = \{1, x, x^2, \ldots\}$ is linearly independent, while $S' = \{1 - x, 5 + 3x - 2x^2, 1 + 3x - x^2\}$ is linearly dependent.

Theorem 10. *If $f_1, f_2, \ldots, f_n \in F(-\infty, \infty)$ have $n-1$ derivatives, then $\{f_1, f_2, \ldots, f_n\}$ are linearly independent if (but NOT only if) the Wronskian of these functions is not identically zero, where the Wronskian is*

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f^{(n-1)}(x) & f^{(n-1)}(x) & \cdots & f^{(n-1)}(x) \end{vmatrix}.$$

Example 11. In $V = F(-\infty, \infty)$, $S = \{2, \sin x\}$ is linearly independent, while $S' = \{2, \sin^2 x, \cos^2 x\}$ is linearly dependent.