

## Section 0, p8 12, 16, 17, 23, 25, 29, 31, 33

**12** Let  $A = \{1, 2, 3\}$  and  $B = \{2, 4, 6\}$ . For each relation between  $A$  and  $B$  given as a subset of  $A \times B$ , decide whether it is a function mapping  $A$  into  $B$ . If it is a function, decide whether it is one to one and whether it is onto  $B$ .

- a.  $\{(1, 4), (2, 4), (3, 6)\}$

Since no two ordered pairs have the same first term, this **is a function**. Since  $(1, 4)$  and  $(2, 4)$  share a second term, this function is **not one to one**. Since 2 does not appear in the relation in the second term, this function is **not onto  $B$** .

- b.  $\{(1, 4), (2, 6), (3, 4)\}$

Since no two ordered pairs have the same first term, this **is a function**. Since  $(1, 4)$  and  $(3, 4)$  share a second term, this function is **not one to one**. Since 2 does not appear in the relation in the second term, this function is **not onto  $B$** .

- c.  $\{(1, 6), (1, 2), (1, 4)\}$

Since  $(1, 6)$  and  $(1, 2)$  share a first term, this is **not a function**.

- d.  $\{(2, 2), (1, 6), (3, 4)\}$

Since no two ordered pairs have the same first term, this **is a function**. Since no two ordered pairs share the same second term, this function is **one to one**. Since every element of  $B$  appears in the range of the function, it is **onto  $B$** .

- e.  $\{(1, 6), (2, 6), (3, 6)\}$

Since no two ordered pairs have the same first term, this **is a function**. Since  $(1, 6)$  and  $(2, 6)$  share a second term, this function is **not one to one**. Since 2 does not appear in the relation in the second term, this function is **not onto  $B$** .

- f.  $\{(1, 2), (2, 6), (2, 4)\}$

Since  $(2, 6)$  and  $(2, 4)$  share a first term, this is **not a function**.

**16** List the elements of the power set  $\mathcal{P}$  of the given set and give the cardinality of the power set.

- a.  $\emptyset$

$$|\{\emptyset\}| = 1$$

- b.  $\{a\}$

$$|\{\emptyset, \{a\}\}| = 2$$

c.  $\{a, b\}$

$$|\{\emptyset, \{a\}, \{b\}, \{a, b\}\}| = 4$$

d.  $\{a, b, c\}$

$$|\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}| = 8$$

**17** Let  $A$  be a finite set, and let  $|A| = s$ . Based on the preceding exercise, make a conjecture about the value of  $|\mathcal{P}(A)|$ . Then try to prove your conjecture.

Based on the previous exercise, I conjecture that  $|\mathcal{P}(A)| = 2^s$ .

*Proof.* We shall conduct a *proof through induction*.

**Base case:** Let  $A = \emptyset$  and  $s = 0$ .  $\mathcal{P}(A) = \{\emptyset\}$ , and  $|\mathcal{P}(A)| = 1 = 2^0$ . Thus the base case holds.

**Inductive hypothesis:** Let  $|A| = k$  for some  $k \in \mathbb{Z}^{\geq 0}$ , and assume that  $|\mathcal{P}(A)| = 2^k$ .

**Inductive case:** Consider  $|A| = k$ . Let us add an element  $e$  to  $A$  to create  $A'$ . To consider  $\mathcal{P}(A')$ , start with all of the sets in  $\mathcal{P}(A)$ , which has a cardinality of  $2^k$  via the inductive hypothesis.

For each of these sets, we have a choice when we add it to  $\mathcal{P}(A')$  to include element  $e$  or to leave it alone, which are two choices. We will end with the original sets of the  $\mathcal{P}(A)$  without  $e$ , along with these sets with  $e$ , in  $\mathcal{P}(A')$ . Each of these groups has a cardinality of  $2^k$ , so the cardinality of  $\mathcal{P}(A')$  must be

$$2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

Thus, the inductive case holds, and the conjecture is correct.  $\square$

In Exercises 23 through 27, find the number of different partitions of a set having the given number of elements.

**23.** 1 element

There is a single partition,  $\{\{e\}\}$

**25.** 3 elements

There are 5 partitions,

$$\{\{e_1\}, \{e_2\}, \{e_3\}\}, \quad \{\{e_1\}, \{e_2, e_3\}\}, \quad \{\{e_1, e_2\}, \{e_3\}\}, \quad \{\{e_1, e_3\}, \{e_2\}\}, \quad \{\{e_1, e_2, e_3\}\}$$

In Exercises 29 through 34, determine whether the given relation is an equivalence relation on the set. Describe the partition arising from each equivalence relation.

**29.**  $n\mathcal{R}m$  in  $\mathbb{Z}$  if  $nm > 0$

Consider  $0 \in \mathbb{Z}$ . We know  $0 \cdot 0 \not> 0$ , thus 0 is not related to itself. So the relation is not reflexive, and cannot be an equivalence relation.

**31.**  $x\mathcal{R}y$  in  $\mathbb{R}$  if  $|x| = |y|$

A equivalence relation must be reflexive, symmetric, and transitive.

(a) Reflexive: Consider  $x \in \mathbb{R}$ . We know  $|x| = |x|$ , thus  $\mathcal{R}$  is reflexive.

- (b) Symmetric: Assume  $x\mathcal{R}y$ . We know  $|x| = |y|$  also implies  $|y| = |x|$ , thus  $y\mathcal{R}x$ . So  $\mathcal{R}$  is symmetric.
- (c) Transitive: Assume  $x\mathcal{R}y$  and  $y\mathcal{R}z$ . We  $|x| = |y|$  and  $|y| = |z|$  which implies  $|x| = |z|$ , thus  $z\mathcal{R}x$ . So  $\mathcal{R}$  is transitive.

The partition resulting from this equivalence relation will be sets  $\bar{x} = \{x, -x\}$  for each  $x \in \mathbb{R}$ .

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**33.**  $n\mathcal{R}m$  in  $\mathbb{Z}^+$  if  $n$  and  $m$  have the same number of digits in the usual base ten notation

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A equivalence relation must be reflexive, symmetric, and transitive.

- (a) Reflexive: Consider  $x \in \mathbb{Z}^+$ . We know a number has the same digits as itself, thus  $\mathcal{R}$  is reflexive.
- (b) Symmetric: Assume  $x\mathcal{R}y$ . We know  $x$  and  $y$  have the same number of digits, implying  $y$  and  $x$  have the same number of digits, thus  $y\mathcal{R}x$ . So  $\mathcal{R}$  is symmetric.
- (c) Transitive: Assume  $x\mathcal{R}y$  and  $y\mathcal{R}z$ . We know  $x$  and  $y$  have the same number of digits, and that  $y$  and  $z$  have the same number of digits, which implies  $x$  and  $z$  have the same number of digits, thus  $z\mathcal{R}x$ . So  $\mathcal{R}$  is transitive.

The partition resulting from this equivalence relation will be classes where each member has the same number of digits, i.e. all single digit numbers will be in a class, all 2 digit numbers will be in a class, all three digit numbers will be in a class, etc.

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