Section 8.5

8.5.1

Proving divisibilty results by induction

a. *Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$.*

Proof. Base Case: $n = 1$

$$3^{2(1)} - 1 = 8 = 4(2) \checkmark$$

Inductive Hypothesis: Assume that 4 evenly divides $3^{2k} - 1$, for some $k \in \mathbb{Z}^+$. This means that

$$\begin{aligned} 3^{2k} - 1 &= 4m, \text{ where } m \in \mathbb{Z}. \\ 3^{2k} &= 4m + 1 \end{aligned}$$

Inductive Case: $n = k + 1$

$$\begin{aligned} 3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\ &= (3^{2k} \cdot 9) - 1 \\ &= (4m + 1) \cdot 9 - 1 && \text{by the Inductive Hypothesis} \\ &= 36m + 8 \\ &= 4(9m + 2) \end{aligned}$$

Since m is an integer, $9m + 2$ is also an integer. Therefore, $3^{2(k+1)} - 1$ is equal to 4 times an integer. This means that $3^{2(k+1)} - 1$ is divisible by 4. Therefore, for any positive integer n, 4 evenly divides $3^{2n} - 1$. □

c. *Prove that for any positive integer n, 4 evenly divides $11^n - 7^n$.*

Proof. Base Case: $n = 1$

$$11^1 - 7^1 = 4 = 4(1) \checkmark$$

Inductive Hypothesis: Assume that 4 evenly divides $11^k - 7^k$, for some $k \in \mathbb{Z}^+$. This means that

$$\begin{aligned} 11^k - 7^k &= 4m, \text{ where } m \in \mathbb{Z}. \\ 11^k &= 4m + 7^k \end{aligned}$$

Inductive Case: $n = k + 1$

$$\begin{aligned} 11^{k+1} - 7^{k+1} &= 11 \cdot 11^k - 7 \cdot 7^k \\ &= 11 \cdot (4m + 7^k) - 7 \cdot 7^k && \text{by the Inductive Hypothesis} \\ &= 44m + 4 \cdot 7^k \\ &= 4(11m + 7^k) \end{aligned}$$

Since m and k are both integers, $11m + 7^k$ is also an integer. Therefore, $11^{k+1} - 7^{k+1}$ is equal to 4 times an integer, and thus 4 evenly divides $11^{k+1} - 7^{k+1}$. Therefore, for any positive integer n, 4 evenly divides $11^n - 7^n$. □

e. *Prove that for any positive integer n, 2 evenly divides $n^2 - 5n + 2$.*

Proof. Base Case: $n = 1$

$$1^2 - 5(1) + 2 = -2 = -2(1) \checkmark$$

Inductive Hypothesis: Assume that 2 evenly divides $k^2 - 5k + 2$, for some $k \in \mathbb{Z}^+$. This means that there exists some integer m such that $k^2 - 5k + 2 = 2m$.

Inductive Step: $n = k + 1$

$$\begin{aligned}
(k+1)^2 - 5(k+1) + 2 &= k^2 + 2k + 1 - 5k - 5 + 2 \\
&= k^2 - 5k + 2 + (2k - 4) \\
&= 2m + 2(k - 2) && \text{by the inductive hypothesis} \\
&= 2(m + k - 2)
\end{aligned}$$

Since m and k are integers, $m + k - 2$ is also an integer. Therefore, $(k+1)^2 - 5(k+1) + 2$ is equal to 2 times an integer, and thus is divisible by 2. This completes the inductive step.

By mathematical induction, for any positive integer n, 2 evenly divides $n^2 - 5n + 2$. □

8.5.3

Proving explicit formulas for recurrence relations by induction.

a. *Define the sequence* $\{b_n\}$ *as follows:*

- $b_0 = 1$
- $b_n = 2b_{n-1} + 1$ *for* $n \geq 1$

Prove that for $n \geq 0, b_n = 2^{n+1} - 1$.

Proof. Base Case: $n = 0$

$$b_0 = 1 = 2^{0+1} - 1 \checkmark$$

Inductive Hypothesis: Assume that for some $k \geq 0$, $b_k = 2^{k+1} - 1$.

Inductive Step: $n = k + 1$

$$\begin{aligned}
b_{k+1} &= 2b_k + 1 \\
&= 2(2^{k+1} - 1) + 1 && \text{by the Inductive Hypothesis} \\
&= 2^{k+2} - 2 + 1 \\
&= 2^{k+2} - 1
\end{aligned}$$

Therefore, $b_{k+1} = 2^{k+2} - 1$.

By mathematical induction, for all $n \geq 0, b_n = 2^{n+1} - 1$. □

8.4.3

Prove each of the following statements using mathematical induction.

a. *Prove that for* $n \geq 2,\ 3^n > 2^n + n^2$.

Proof. Base Case: $n = 2$

$$P(2) : 3^2 > 2^2 + 2^2 \Rightarrow 9 > 8 \checkmark$$

Inductive Hypothesis: Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$
Inductive Case: $n = k + 1$

$$\begin{aligned}
3^{k+1} = 3^k 3 &> 3 \cdot 2^k + 3 \cdot k^2 && \text{by inductive hypothesis} \\
&> 2 \cdot 2^k + k^2 + 2k + 1 && \text{since } k > 1 \\
&> 2^{k+1} + (k+1)^2
\end{aligned}$$

Therefore $P(k+1)$ is true. Since $P(1)$ is true, and $P(k+1)$ is true, therefore $P(n)$ is true for all $n \in \mathbb{Z}^+$. $\square$