MAT 369 Introduction to Graph Theory

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1 Introduction

1.1 Graphs and Graph Models

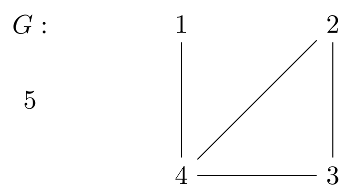
Graph Definition

A (simple) **graph** is an ordered pair (V, E) where

- V is a nonempty set of objects called "vertices"
- E is a set containing some two-subsets of V called "edges". E may be empty.

Graphs are often represented pictorially. For example consider

$$G = (V, E) \text{ where } V = \{1, 2, 3, 4, 5\} \text{ and } E = \{\{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$$



- Vertices 1 and 4 are **adjacent** because they are joined by an edge.
- Vertex 2 and edge $2 - 3$ are **indicent**.
- Edges $2 - 3$ and $3 - 4$ are **adjacent**.

Order Definition

The **order** of a graph G is $|V(G)|$, or the number of vertices.

Size Definition

The **size** of a graph G is $|E(G)|$, or the number of edges.
The graph G from above has order 5 and size 4.

1.2 Connected Graphs

Subgraph Definition

Let G and H be graphs. H is a **subgraph** of G, notated as $H \subseteq G$, if

$$V(H) \subseteq V(G) \text{ and } E(H) \subseteq E(G).$$

Proper Subgraph Definition

H is a **proper subgraph** of G if $H \subseteq G$ and either

$$V(H) \subsetneq V(G) \text{ or } E(H) \subsetneq E(G).$$

Spanning Subgraph Definition

Graph H is a **spanning subgraph** if $H \subseteq G$ and $V(H) = V(G)$.

Induced Subgraph Definition

Graph H is a **induced subgraph** if $H \subseteq G$ and if

$$u, v \in V(H) \text{ and } u, v \in E(G) \implies u, v \in E(H).$$

Essentially, H contains all valid edges it can take from G. Notation for **induced subgraph** is

$$G[S], \text{ where } S \text{ is a set of vertices from } G.$$

Edge-induced Subgraph Definition

$G[X]$ is an **edge-induced subgraph** of G if $G[X]$ has edge set $X \subseteq E(G)$ and a vertex set of all vertices incident with at least one edge of X. Interesting fact: $G[E(G)]$ removes any isolated vertices.

More on Spanning and Induced Subgraphs

Let G be a graph with vertex v and edge e. Then,

- $G - e$ is the *spanning subgraph* of G whose edge set is $E(G) - \{e\}$.

  This definition can be expanded to $G - X$ for $X \subseteq E(G)$.

- $G - v$ is the *induced subgraph* of G whose vertex set is $V(G) - \{v\}$ and edge set includes all edges of G except those incident with v.

  This definition can be expanded to $G - U$ for $U \subseteq V(G)$.

Let G be a graph, $u, v \in V(G)$ and $e = uv \notin E(G)$. Then $G + e$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup \{e\}$. G is a *spanning subgraph* of $G + e$

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