Section 6, p66 #9,11,17,19,23-25,27,35,37,50

In Exercises 8 through 11, find the number of generators of a cyclic group having the given order

9. 8

8 has prime factorization of 2^3, so generators are $\{1, 3, 5, 7\}$.

11. 60

60 has prime factorization of $2^2 \cdot 3 \cdot 5$. Generators are

$$\{1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59\}$$

In Exercises 17 through 21, find the number of elements in the indicated cyclic group.

17. The cyclic subgroup of $\mathbb{Z}_30$, generated by 25

$\langle 25\rangle = \{25, 20, 15, 10, 5, 0\}$

19. The cyclic subgroup $\langle i\rangle$ of the group $\mathbb{C}^*$ of nonzero complex numbers under multiplication

$\{i, -1, -i, 1\}$

In Exercises 22 through 24, find all the subgroups of the given group, and draw the subgroup diagram for the subgroups.

23. $\mathbb{Z}_{36}$

Subgroups and their generators:

- $\langle 1\rangle = \langle 5\rangle = \langle 7\rangle = \langle 11\rangle = \langle 13\rangle = \langle 17\rangle = \langle 19\rangle = \langle 23\rangle = \langle 25\rangle = \langle 29\rangle = \langle 31\rangle = \langle 35\rangle = \mathbb{Z}_{36}$
- $\langle 2\rangle = \langle 10\rangle = \langle 14\rangle = \langle 22\rangle = \langle 26\rangle = \langle 34\rangle = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34\}$
- $\langle 3\rangle = \langle 15\rangle = \langle 21\rangle = \langle 33\rangle = \{0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33\}$
- $\langle 4\rangle = \langle 8\rangle = \langle 16\rangle = \langle 20\rangle = \langle 28\rangle = \langle 32\rangle = \{0, 4, 8, 12, 16, 20, 24, 28, 32\}$
- $\langle 6\rangle = \langle 30\rangle = \{0, 6, 12, 18, 24, 30\}$
- $\langle 9\rangle = \langle 27\rangle = \{0, 9, 18, 27\}$
- $\langle 12\rangle = \langle 24\rangle = \{0, 12, 24\}$
- $\langle 18\rangle = \{0, 18\}$

24. $\mathbb{Z}_8$

Subgroups and their generators:

- $\langle 1\rangle = \langle 3\rangle = \langle 5\rangle = \langle 7\rangle = \mathbb{Z}_8$
- $\langle 2\rangle = \langle 6\rangle = \{0, 2, 4, 6\}$
- $\langle 4\rangle = \{0, 4\}$

In Exercises 25 through 29, find all the orders of the subgroups of the given group.

25. $\mathbb{Z}_6$

$|\langle 1\rangle| = |\langle 5\rangle| = 6$. $|\langle 2\rangle| = \langle 4\rangle = 3$. $|\langle 3\rangle| = 2$.

27. $\mathbb{Z}_{12}$

$|\langle 1\rangle| = |\langle 5\rangle| = |\langle 7\rangle| = |\langle 11\rangle| = 12$. $|\langle 2\rangle| = |\langle 10\rangle| = 6$. $|\langle 3\rangle| = |\langle 9\rangle| = 4$. $|\langle 4\rangle| = |\langle 8\rangle| = 3$. $|\langle 6\rangle| = 2$.

In Exercises 33 through 37, either give an example of a group with the property described, or explain why no example exists.

35. A cyclic group having only one generator

Consider $\mathbb{Z}_2 = \{0, 1\}$. The only generator for this group is $\langle 1\rangle$.

37. A finite cyclic group having four generators

Consider $\mathbb{Z}_8$. This group has four generators: $\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \langle 7\rangle$.

Theory

50. Let G be a group and suppose $a \in G$ generates a cyclic subgroup of order 2 and is the *unique* such element. Show that $ax = xa$ for all $x \in G$. Hint: Consider $(xax^{-1})^2$

Assume the above.

Proof. Consider $(xax^{-1})^2$.

$$\begin{aligned}(xax^{-1})^2 &= xax'xax' \\ &= xa^2x' \\ &= xx' = e && \text{because by def. } a^2 = e\end{aligned}$$

Since $(xax^{-1})^2 = e$, but $a^2 = e$ and a is the *unique* such element to do so, it follows that

$$xax^{-1} = a.$$

We can now manipulate this equation.

$$\begin{aligned}xax^{-1} &= a \\ xax^{-1}x &= ax \\ xa &= ax\end{aligned}$$

Thus we conclude that $xa = ax$, for all $x \in G$. □