Homework 7

1.4 *Inverses; Algebraic Properties of Matrices*

28. *Show that if a square matrix A satisfies $A^2 - 3A + 1 = 0$, then $A^{-1} = 3I - A$.*

work. Consider $3I - A$.

$$A(3I - A) = 3AI - A^2 = 3A - A^2$$

If $A^2 - 3A + I = 0$, then we can simplify further to determine exactly what $3A - A^2$ equals.

$$\begin{aligned}
A^2 - 3A + I &= 0 \\
(3A - A^2) + A^2 - 3A + I &= (3A - A^2) + 0 \\
(3A + (-A^2 + A^2) - 3A) + I &= (3A - A^2) \\
(3A - 3A) + I &= 3A - A^2 \\
I &= 3A - A^2 \\
\therefore\ I &= A(3I - A) \\
\therefore\ I &= (3I - A)A
\end{aligned}$$

Since $I = A(3I - A)$ and $I = (3I - A)A$, therefore $A^{-1} = 3I - A$ if $A^2 - 3A + 1 = 0$. □

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31. *Assuming that all matrices are $n \times n$ and invertible, solve for D:*

$$C^T B^{-1} A^2 BAC^{-1} DA^{-2} B^T C^{-2} = C^T.$$

work.

$$\begin{aligned}
C^T B^{-1} A^2 BAC^{-1} DA^{-2} B^T C^{-2} &= C^T \\
(C^T B^{-1} A^2 BAC^{-1})^{-1} C^T B^{-1} A^2 BAC^{-1} DA^{-2} B^T C^{-2} &= (C^T B^{-1} A^2 BAC^{-1})^{-1} C^T \\
DA^{-2} B^T C^{-2} &= CA^{-1} B^{-1} A^{-2} BC^{T^{-1}} C^T
\end{aligned}$$

□

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39. *Using Matrix Inversion, find the unique solution of the given linear system.*

$$\begin{aligned}
3x_1 - 2x_2 &= -1 \\
4x_1 + 5x_2 &= 3
\end{aligned}$$

53a. *Show that if A, B and $A + B$ are invertible matrices with the same size, then*

$$A(A^{-1} + B^{-1})B(A + B)^{-1} = I.$$

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55. *Show that if A is a square matrix such that $A^k = 0$ for some positive integer k, then the matrix $(I - A)$ is invertible and*

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}.$$

1.5 *Elementary Matrices and a Method for Finding A-1*

15. *Use the inverse algorithm to find the inverse of the given matrix, if the inverse exists.*

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

25. *Find the inverse of the following 4×4 matrices, where $k_1, k_2, k_3, k_4,$ and k are all non-zero.*

a. $$\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}.$$

b. $$\begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

27. *Find all values of c, if any, for which the given matrix is invertible.*

29. *Write the given matrix as a product of elementary matrices.*

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$$

41. *Prove that if A and B are $m \times n$ matrices, then A and B are row equivalent if and only if A and B have the same reduced row eschelon form.*

1.6 *More on Linear Systems and Invertible Matrices*

15. *Determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.*

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$$

21. *Let $A\vec{x} = \vec{0}$ be a homogenous system of n linear equations in n unknown that has only the trivial solution. Show that if k is any positive integer, then the system $A^k\vec{x} = \vec{0}$ also has only the trivial solution.*