

## Homework 6

**1.2** Determine the values for which the system has no solutions, exactly one solution, or infinitely many solutions

$$\begin{array}{rclcl} x & + & 2y & - & 3z & = & 4 \\ \text{25.} & 3x & - & y & + & 5z & = & 2 \\ & 4x & + & y & + & (a^2 - 14)z & = & a + 2 \end{array}$$

work.

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2 - 14 & a + 2 \end{array} \right) \xrightarrow[(-4, -8, 12, -16)]{R_3 - 4R_1} \left( \begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 0 & -7 & a^2 - 2 & a - 14 \end{array} \right) \xrightarrow[(-3, -6, 9, -12)]{R_2 - 3R_1} \\ & \left( \begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2 - 2 & a - 14 \end{array} \right) \xrightarrow[(0, 7, -14, 10)]{R_3 - R_2, -\frac{1}{7}R_2} \left( \begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2 - 16 & a - 4 \end{array} \right) \end{aligned}$$

$R_3$  represents the equation  $(a^2 - 16)z = a - 4$ . If  $a = 4$ , there are infinitely many solutions, since  $R_3$  leads to a full row of 0's. If  $a = -4$ , then there is no solution, since  $R_3$  leads to  $0 = -8$ . If  $a \neq \pm 4$ , then there is exactly one solution to the system of equations.  $\square$

$$\begin{array}{rclcl} x & + & 2y & = & 1 \\ \text{27.} & 2x & + & (a^2 - 5)y & = & a - 1 \end{array}$$

work.

$$\left( \begin{array}{cc|c} 1 & 2 & 1 \\ 2 & a^2 - 5 & a - 1 \end{array} \right) \xrightarrow[(-2, -4, -2)]{R_2 - 2R_1} \left( \begin{array}{cc|c} 1 & 2 & 1 \\ 0 & a^2 - 9 & a - 3 \end{array} \right)$$

$R_2$  represents the equation  $(a^2 - 9)y = a - 3$ . If  $a = 3$ , there are infinitely many solutions, since  $R_2$  leads to a full row of 0's. If  $a = -3$ , then there is no solution, since  $R_2$  leads to  $0 = -6$ . If  $a \neq \pm 3$ , then there is exactly one solution to the system of equations.  $\square$

$$\text{32. Reduce } \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix} \text{ to rref without introducing fractions at any intermediate stage.}$$

work.

$$\begin{aligned} & \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[(-2, -1, -3)]{R_3 - R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[(-2, -6, -4)]{R_1 - 2R_3} \begin{bmatrix} 0 & -5 & -1 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[(0, 2, 29)]{R_1 - R_2} \\ & \begin{bmatrix} 0 & -3 & 28 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[R_3 + R_1]{R_2 - R_1} \begin{bmatrix} 0 & -3 & 28 \\ 0 & 1 & -57 \\ 1 & 0 & 30 \end{bmatrix} \xrightarrow[(0, 3, -171)]{R_1 + 3R_2} \begin{bmatrix} 0 & 0 & -143 \\ 0 & 1 & -57 \\ 1 & 0 & 30 \end{bmatrix} \xrightarrow[R_2 + 57R_1, R_3 - 30R_1]{-\frac{1}{143}R_1, R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$\square$

## 1.3

$$\text{5h. Calculate } (C^T B)A^T, \text{ where } A = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix}, C = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}.$$

work.

$$\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix} \xrightarrow{C^T} \begin{pmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{pmatrix} \xrightarrow{C^T B} \begin{pmatrix} 1 \cdot 4 + 3 \cdot 0 & 1 \cdot -1 + 3 \cdot 2 \\ 4 \cdot 4 + 1 \cdot 0 & 4 \cdot -1 + 1 \cdot 2 \\ 2 \cdot 4 + 5 \cdot 0 & 2 \cdot -1 + 5 \cdot 2 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 16 & -2 \\ 8 & 8 \end{pmatrix} \xrightarrow{C^T B A^T} \\ \begin{pmatrix} 4 & 5 \\ 16 & -2 \\ 8 & 8 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{C^T B A^T} \begin{pmatrix} 4 \cdot 3 + 5 \cdot 0 & 4 \cdot -1 + 5 \cdot 2 & 4 \cdot 1 + 5 \cdot 1 \\ 16 \cdot 3 - 2 \cdot 0 & 16 \cdot -1 - 2 \cdot 2 & 16 \cdot 1 - 2 \cdot 1 \\ 8 \cdot 3 + 8 \cdot 0 & 8 \cdot -1 + 8 \cdot 2 & 8 \cdot 1 + 8 \cdot 1 \end{pmatrix} = \begin{pmatrix} 12 & 6 & 9 \\ 48 & -20 & 14 \\ 24 & 8 & 16 \end{pmatrix}$$

□

10.  $A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}$ ,  $B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$ ,  $AB = \begin{bmatrix} 67 & 41 & 41 \\ 64 & 21 & 59 \\ 63 & 67 & 57 \end{bmatrix}$ , and  $BA = \begin{bmatrix} 6 & -6 & 70 \\ 6 & 17 & 31 \\ 63 & 41 & 122 \end{bmatrix}$ .

a. express each column vector of  $AB$  as a linear combination of the column vectors of  $A$ .

work. Since each column vector of  $AB$  is computed using a row of  $A$  and a column of  $B$ , the linear combination will simply be the corresponding column of  $B$ .

$$\begin{aligned} 1. & 6 \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} + 7 \begin{pmatrix} 7 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 67 \\ 64 \\ 63 \end{pmatrix}. \\ 2. & -2 \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} + 7 \begin{pmatrix} 7 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 41 \\ 21 \\ 67 \end{pmatrix}. \\ 3. & 4 \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} + 5 \begin{pmatrix} 7 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 41 \\ 59 \\ 57 \end{pmatrix}. \end{aligned}$$

□

b. express each column vector of  $BA$  as a linear combination of the column vectors of  $B$ .

work. Since each column vector of  $BA$  is computed using a row of  $B$  and a column of  $A$ , the linear combination will simply be the corresponding column of  $A$ .

$$\begin{aligned} 1. & 3 \begin{pmatrix} 6 \\ 0 \\ 7 \end{pmatrix} + 6 \begin{pmatrix} -2 \\ 1 \\ 7 \end{pmatrix} + 0 \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 63 \end{pmatrix}. \\ 2. & -2 \begin{pmatrix} 6 \\ 0 \\ 7 \end{pmatrix} + 5 \begin{pmatrix} -2 \\ 1 \\ 7 \end{pmatrix} + 4 \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 17 \\ 41 \end{pmatrix}. \\ 3. & 7 \begin{pmatrix} 6 \\ 0 \\ 7 \end{pmatrix} + 4 \begin{pmatrix} -2 \\ 1 \\ 7 \end{pmatrix} + 9 \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 70 \\ 31 \\ 122 \end{pmatrix}. \end{aligned}$$

□

15. Find all values of  $k$ , if any, that satisfy  $\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix}$ .

work.

$$\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k+1 & k+2 & -1 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = [k(k+1) + k+2 - 1] = [(k+1)^2] = [0]$$

$$\begin{aligned} k+1 &= 0 \\ k &= -1 \end{aligned}$$

□

22.

- a. Show that if  $A$  has a row of zeros and  $B$  is any matrix for which  $AB$  is defined, then  $AB$  also has a row of zeros.

*Proof.* Consider

$$A_{n \times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} B_{m \times k} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mk} \end{bmatrix}.$$

$AB$  can be expressed as a series of row vectors. These row vectors can be expressed as linear combinations from numbers of the row vectors from  $A$ , and the corresponding row of  $B$ . Here is an example:

$$\begin{aligned} a_{11} [b_{11} \ b_{12} \ \cdots \ b_{1k}] + a_{12} [b_{21} \ b_{22} \ \cdots \ b_{2k}] + \cdots + a_{1m} [b_{m1} \ b_{m2} \ \cdots \ b_{mk}] &= (1) \\ &= [ab_{11} \ ab_{12} \ \cdots \ ab_{1k}] \quad (2) \end{aligned}$$

Since there is a row of zeros, one of these equations will be:

$$0 [b_{11} \ b_{12} \ \cdots \ b_{1k}] + 0 [b_{21} \ b_{22} \ \cdots \ b_{2k}] + \cdots + 0 [b_{m1} \ b_{m2} \ \cdots \ b_{mk}] = [0 \ 0 \ \cdots \ 0].$$

This represents a row of all zeros, which is contained within  $AB$ , at the same row at which it occurs in  $A$ . Therefore, if  $A$  has a row of zeros and  $B$  is any matrix for which  $AB$  is defined, then  $AB$  also has a row of zeros. □

- b. Find a similar result involving a column of zeros.

*Proof.* Consider

$$A_{n \times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{12} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} B_{m \times k} = \begin{bmatrix} b_{11} & b_{12} & \cdots & 0 & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & 0 & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & 0 & \cdots & b_{mk} \end{bmatrix}.$$

$AB$  can be expressed as a series of column vectors. These column vectors can be expressed as linear combinations from numbers of the column vectors from  $B$ , and the corresponding column of  $A$ . Here is an example:

$$b_{11} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + b_{21} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{2m} \end{bmatrix} + \cdots + b_{m1} \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} ab_{11} \\ ab_{21} \\ \vdots \\ ab_{n1} \end{bmatrix}.$$

Since there is a row of zeros, one of these equations will be:

$$0 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + 0 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{2m} \end{bmatrix} + \cdots + 0 \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This represents a column of all zeros, which is contained within  $AB$ , at the same column at which it occurs in  $A$ . Therefore, if  $B$  has a column of zeros and  $A$  is any matrix for which  $AB$  is defined, then  $AB$  also has a column of zeros. □

24. Find the  $4 \times 4$  matrix  $A = [a_{ij}]$  whose entries satisfy the stated condition.

a.  $a_{ij} = i + j$

work.

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

□

b.  $a_{ij} = i^{j-1}$

work.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

□

c.  $a_{ij} = \begin{cases} 1 & \text{if } |i - j| > 1 \\ -1 & \text{if } |i - j| \leq 1 \end{cases}$

work.

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

□

27. How many  $3 \times 3$  matrices  $A$  can you find such that

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ x - y \\ 0 \end{bmatrix}$$

for all choices of  $x, y$ , and  $z$ ?

work.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Any other combination for the entries of  $A$  will cause a different value to be in the resulting column matrix. □

## 1.4

17. Use the given information to find  $A$ :  $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$ .

work.

$$\begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix} \xrightarrow[I+2A]{(I+2A)^{-1} \cdot} \frac{1}{-13} \begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{13} & \frac{2}{13} \\ \frac{4}{13} & \frac{1}{13} \end{bmatrix} \xrightarrow[2A]{I+2A-I} \begin{bmatrix} -\frac{18}{13} & \frac{2}{13} \\ \frac{4}{13} & -\frac{12}{13} \end{bmatrix} \xrightarrow[A]{\frac{1}{2} \cdot 2A} \begin{bmatrix} -\frac{9}{13} & \frac{1}{13} \\ \frac{2}{13} & -\frac{6}{13} \end{bmatrix}$$

□

**19e.** Given  $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ , compute  $p(A)$ , where  $p(x) = 2x^2 - x + 1$ .

work.

$$\begin{aligned} p(A) &= 2A^2 - A + I = 2 \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^2 + \begin{bmatrix} -3 & -1 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2 \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 22 & 8 \\ 16 & 6 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 7 \\ 14 & 6 \end{bmatrix} \end{aligned}$$

□

**27.** Consider the matrix

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

where  $a_{11} \cdot a_{22} \cdots a_{nn} \neq 0$ . Show that  $A$  is invertible and find its inverse.

*Proof.* Consider  $B = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$ .

$$\begin{aligned} AB &= \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I \\ BA &= \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I \end{aligned}$$

Since  $AB = I = BA$ , therefore  $B = A^{-1}$ , the inverse of  $A$ . This also proves that  $A$  is invertible. □