Linear Algebra

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Contents

1 Brief Review

Commonly Used Sets

- $\mathbb{N}$: set of **natural numbers**

  could be *positive* integers

  could be *nonnegative* integers

- $\mathbb{Z}$: set of **integers**
- $\mathbb{Q}$: set of **rational numbers**
- $\mathbb{R}$: set of **real numbers**

Set Building

To denote sets too large to just list, we use **set builder** notation:

$$\{\text{candidate} : \text{condition}\}$$

Examples:

$$\{x \text{ is a fruit} : x \text{ is of yellow color}\}$$
$$\{x \text{ is a human being} : x \text{ is a president of the U.S.}\}$$
$$\{x \text{ is a city} : x \text{ is a capitol of a country}\}$$

Other Notations

- $\forall$: for all
- $\exists$: there exists
- s.t.: such that
- $\rightarrow\leftarrow$: contradiction
- WTS: want to show

2 Real Vector Spaces

A **real vector space** is simply a *nonempty set* that satisfies 10 properties called **10 axioms of a real vector space**.

- $\vec{v} \in$ vector space V can be *anything*
- **Never** assume that an element $\vec{v} \in V$ is an ordered pair

Addition

- denoted by $\oplus$
- simply a map

$$\oplus : V \times V \to V$$

Example of a definition of $\oplus$ for $V = \{\text{apple}, \text{orange}, \text{banana}\}$:

$\oplus$	apple	orange	banana
apple	banana	**banana**	apple
orange	orange	apple	banana
banana	banana	orange	orange

$$\oplus(\text{apple}, \text{orange}) = \text{banana} = \text{apple} \oplus \text{orange}$$

Scalar Multiplication

- denoted by $\odot$
- simply a map
- *must* be $r \times \vec{v}$ for $r \in \mathbb{R}, \vec{v} \in V$

$$\odot : \mathbb{R} \times V \to V$$

Example of a definition of $\odot$ for $V = \{\text{apple}, \text{orange}, \text{banana}\}$:

$$k \odot \text{apple} = \text{orange}, \forall\ k \in \mathbb{R}$$

$$k \odot \text{orange} = \begin{cases} \text{orange}, & \text{if } k \leq 2, \\ \text{banana}, & \text{if } k > 2, \end{cases}$$

$$k \odot \text{banana} = \begin{cases} \text{banana}, & \text{if } k < -5\sqrt{2}, \\ \text{apple}, & \text{if } -5\sqrt{2} \leq k < 1.2, \\ \text{banana}, & \text{if } k = 1.2, \\ \text{orange}, & \text{if } k > 2, \end{cases}$$

$$\odot(3, \text{orange}) = \text{banana} = 3 \odot \text{orange}$$

10 Good Properties of Addition and Scalar Multiplication

1. **Closed Under Addition** $\forall\ \vec{v}, \vec{u} \in V$,
$$\vec{u} \oplus \vec{v} \in V$$

2. **Commutativity Under Addition** $\forall\ \vec{v}, \vec{u} \in V$,
$$\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$$

3. **Associativity Under Addition** $\forall\ \vec{v}, \vec{u}, \vec{w} \in V$,
$$\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$$
4. **Additive Identity Exists** $\exists\ \vec{u}\ \forall\ \vec{v} \in V$,
$$\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u} = \vec{v}$$
$$\vec{u} \text{ is called the additive identity}, \mathbf{id}$$
5. **Additive Inverse Always Exists** $\forall\ \vec{v}\ \exists\ \vec{w} \in V$,
$$\vec{v} \oplus \vec{w} = \vec{w} \oplus \vec{v} = \mathbf{id}$$
$$\vec{w} = -\vec{v} \text{ and is pronounced as } \textit{bar-}\vec{v}$$
6. **Closed Under Scalar Multiplication** $\forall\ k \in \mathbb{R}, \vec{v} \in V$,
$$k \odot \vec{v} \in V$$
7. **Distributivity Over** $\oplus$ $\forall\ k \in \mathbb{R}, \vec{u}, \vec{v} \in V$,
$$k \odot (\vec{u} \oplus \vec{v}) = k \odot \vec{u} \oplus k \odot \vec{v}$$
8. **Distributivity Over** $+$ $\forall\ k, \ell \in \mathbb{R}, \vec{v} \in V$,
$$(k + \ell) \odot \vec{v} = k \odot \vec{v} \oplus \ell \odot \vec{v}$$
9. **Associativity Over Scalar Multiplication** $\forall\ k, \ell \in \mathbb{R}, \vec{v} \in V$,
$$(k \cdot \ell) \odot \vec{v} = k \odot (\ell \odot \vec{v})$$
10. **1 Fixes Every Element In V By** $\odot$ $\forall\ \vec{v} \in V$,
$$1 \odot \vec{v} = \vec{v}$$

Tips To Remember The 10 Axioms

- first 5 axioms deal with addition ONLY, the next 5 axioms involve scalar multiplication
- first of the 5 axioms for addition and scalar multiplication deal with closure
- axioms 4 and 5 are about the existence of something
- axioms 8 and 9 are the only axioms that involve 2 real numbers

Verifying the 10 Axioms

a.. Axioms (1) and (6): proof of closure

b.. Axioms (4) and (5): show existence

c.. Axioms (2), (3), (7), (8), (9), (10): proof for all elements

Standard Addition and Scalar Multiplication for $\mathbb{R}^n$

$\forall\ \vec{u} = (u_1, u_2, \ldots, u_n), \vec{v} = (v_1, v_2, \ldots, v_n) \in V$ and $\forall\ k \in \mathbb{R}$,
$$\begin{aligned} \vec{u} \oplus \vec{v} &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \\ k \odot \vec{u} &= (ku_1, ku_2, \ldots, ku_n) \end{aligned}$$

3 Axiom-Based Theorems

Given that V is a real vector space, there are a number of theorems that are always true, because they are built upon the axioms.

Theorem A

Let V be a vector space. $\forall\ \vec{v} \in V$:

$$0 \odot \vec{v} = \mathbf{id}$$

Theorem B

Let V be a vector space. $\forall\ k \in \mathbb{R}$:

$$k \odot \mathbf{id} = \mathbf{id}$$

Theorem C

Let V be a vector space. $\forall\ \vec{v} \in V$:

$$(-1) \odot \vec{v} = -\vec{v}$$

Theorem D

Let V be a vector space. **If** $k \odot \vec{v} = \mathbf{id}$, then:

$$k = 0 \qquad \textit{and/or} \qquad \vec{v} = \mathbf{id}$$

4 Subspaces

Let V be a vector space, with $\oplus$ and $\odot$ denoting its addition and scalar multiplication operations respectively. A *nonempty set* W is a **subspace** of V if these three properties are satisfied.

1. $W \subseteq V$
2. Addition and scalar multiplication operations in W are *inherited* from V: $\oplus_W = \oplus_V$ and $\odot_W = \oplus_V$
3. W is a vector space

Theorem 3: Needed Axioms for a Subspace

Let V be a vector space, and let W be a nonempty subset V such that the addition and scalar multiplication are inherited from V. Then W is a subspace of V if and only if Axioms 1 and 6 hold for W.

Linear Combination and Span

Let V be a vector space, and let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subseteq V$. A **linear combination** of $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is

$$k_2\vec{v}_2 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n, \text{ where } k_i \text{ are scalars}$$

The **span** of $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is the set of ALL linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, which is

$$\text{span}(S) = \{\vec{v} \in V : \vec{v} = k_2\vec{v}_2 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n, \text{ where } k_1, k_2, \dots, k_n \in \mathbb{R}\}$$

If $S = \emptyset$, then we define $\text{span}(S) = \{\mathbf{id}\}$

Theorem 16: Smallest Subspace of a Vector Space

Let V be a vector space, and let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subseteq V$. Then span($S$) is the ***smallest*** *subspace of V containing S.*

Theorem 19: Span Equality

Let V be a vector space, and let S and T be two finite subsets of V. Then

$$\text{span}(S) = \text{span}(T) \iff S \subseteq \text{span}(T) \text{ and } T \subseteq \text{span}(S)$$

Gauss-Jordan Elimination