Homework 7

1.4 Inverses; Algebraic Properties of Matrices

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28. Show that if a square matrix A satisfies $A^2 - 3A + 1 = 0$, then $A^{-1} = 3I - A$.

Proof. Consider $3I - A$.

$$A(3I - A) = 3AI - A^2 = 3A - A^2$$

If $A^2 - 3A + I = 0$, then we can simplify further to determine exactly what $3A - A^2$ equals.

$$A^2 - 3A + I = 0$$
$$(3A - A^2) + A^2 - 3A + I = (3A - A^2) + 0$$
$$(3A + (-A^2 + A^2) - 3A) + I = (3A - A^2)$$
$$(3A - 3A) + I = 3A - A^2$$
$$I = 3A - A^2$$
$$\therefore \ I = A(3I - A)$$
$$\therefore \ I = (3I - A)A$$

Since $I = A(3I - A)$ and $I = (3I - A)A$, therefore $A^{-1} = 3I - A$ if $A^2 - 3A + 1 = 0$. $\square$

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31. Assuming that all matrices are $n \times n$ and invertible, solve for D:

$$C^T B^{-1} A^2 BAC^{-1} DA^{-2} B^T C^{-2} = C^T.$$

Work.

$$C^T B^{-1} A^2 BAC^{-1} DA^{-2} B^T C^{-2} = C^T$$
$$(C^T B^{-1} A^2 BAC^{-1})^{-1} C^T B^{-1} A^2 BAC^{-1} DA^{-2} B^T C^{-2} = (C^T B^{-1} A^2 BAC^{-1})^{-1} C^T$$
$$DA^{-2} B^T C^{-2} = CA^{-1} B^{-1} A^{-2} BC^{T^{-1}} C^T$$
$$DA^{-2} B^T C^{-2}(A^{-2} B^T C^{-2})^{-1} = CA^{-1} B^{-1} A^{-2} BC^{T^{-1}} C^T (A^{-2} B^T C^{-2})^{-1}$$
$$D = CA^{-1} B^{-1} A^{-2} BC^{T^{-1}} C^T C^2 B^{T^{-1}} A^2$$
$$D = CA^{-1} B^{-1} A^{-2} BC^2 B^{T^{-1}} A^2$$

   $\square$

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39. Using Matrix Inversion, find the unique solution of the given linear system.

$$3x_1 - 2x_2 = -1$$
$$4x_1 + 5x_2 = 3$$

Work.

$$\begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{3 \cdot 5 - -2 \cdot 4} \begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{23} & \frac{2}{23} \\ -\frac{4}{23} & \frac{3}{23} \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{23} \\ \frac{13}{23} \end{bmatrix}$$

$\square$

53a. Show that if A, B and $A + B$ are invertible matrices with the same size, then

$$A(A^{-1} + B^{-1})B(A + B)^{-1} = I.$$

Work.

$$A(A^{-1} + B^{-1})B(A + B)^{-1} = (I + AB^{-1})B(A + B)^{-1}$$
$$= (B + A)(A + B)^{-1} = (A + B)(A + B)^{-1} = I$$

$$\therefore \ A(A^{-1} + B^{-1})B(A + B)^{-1} = I$$

$\square$

55. Show that if A is a square matrix such that $A^k = 0$ for some positive integer k, then the matrix $(I - A)$ is invertible and
$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}.$$

Proof. Consider square matrix A such that $A^k = 0$ for some positive integer k. Now consider the matrices $(I - A)$ and $(I + A + A^2 + \cdots + A^{k-1})$.

$$
\begin{aligned}
(I - A)(I + A + A^2 + \cdots + A^{k-1}) &= (I + A + A^2 + \cdots + A^{k-1}) - (A + A^2 + \cdots + A^k) \\
&= I + A - A + A^2 - A^2 + \cdots + A^{k-1} - A^{k-1} + A^k \\
&= I + A^k \\
&= I + \vec{0} = I
\end{aligned}
$$

Therefore $(I - A)(I + A + A^2 + \cdots + A^{k-1}) = I$.

$$
\begin{aligned}
(I + A + A^2 + \cdots + A^{k-1})(I - A) &= (I + A + A^2 + \cdots + A^{k-1}) - (A + A^2 + \cdots + A^k) \\
&= I + A - A + A^2 - A^2 + \cdots + A^{k-1} - A^{k-1} + A^k \\
&= I + A^k \\
&= I + \vec{0} = I
\end{aligned}
$$

Therefore $(I + A + A^2 + \cdots + A^{k-1})(I - A) = I$. Since $(I + A + A^2 + \cdots + A^{k-1})(I - A) = I$ and $(I - A)(I + A + A^2 + \cdots + A^{k-1}) = I$, therefore $(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}$ $\qquad\square$

1.5 Elementary Matrices and a Method for Finding A-1

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15. Use the inverse algorithm to find the inverse of the given matrix, if the inverse exists.

$$
\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}.
$$

Proof.

$$
\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix} \xrightarrow[-R_1]{R_2+2R_1} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 10 & -7 \\ -4 & 2 & -9 \end{bmatrix} \xrightarrow{R_4+4R_1} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 10 & -7 \\ 0 & -10 & 7 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 10 & -7 \\ 0 & 0 & 0 \end{bmatrix}
$$

The last matrix has a row of all zeros, which makes it impossible for $\mathrm{rref}(A) = I$, thus $\mathrm{rref}(A) \neq I$. Therefore, by the Big Theorem from Lecture Note 23, A is not invertible. $\qquad\square$

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25. Find the inverse of the following 4×4 matrices, where $k_1, k_2, k_3, k_4,$ and k are all non-zero.

a. $\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}.$

Work.

$$
\left[\begin{array}{cccc|cccc}
k_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & k_2 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & k_3 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & k_4 & 0 & 0 & 0 & 1
\end{array}\right]
\xrightarrow[\frac{1}{k_3}R_3,\ \frac{1}{k_4}R_4]{\frac{1}{k_1}R_1,\ \frac{1}{k_2}R_2}
\left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & \frac{1}{k_1} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_4}
\end{array}\right]
$$

$$
\therefore\
\left[\begin{array}{cccc}
k_1 & 0 & 0 & 0 \\
0 & k_2 & 0 & 0 \\
0 & 0 & k_3 & 0 \\
0 & 0 & 0 & k_4
\end{array}\right]^{-1}
=
\left[\begin{array}{cccc}
\frac{1}{k_1} & 0 & 0 & 0 \\
0 & \frac{1}{k_2} & 0 & 0 \\
0 & 0 & \frac{1}{k_3} & 0 \\
0 & 0 & 0 & \frac{1}{k_4}
\end{array}\right]
$$

□

b. $\left[\begin{array}{cccc}
k & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & k & 1 \\
0 & 0 & 0 & 1
\end{array}\right].$

Work.

$$
\left[\begin{array}{cccc|cccc}
k_1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & k_3 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right]
\xrightarrow[R_3-R_4]{R_1-R_2}
\left[\begin{array}{cccc|cccc}
k & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & k & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\xrightarrow[\frac{1}{k}R_3]{\frac{1}{k}R_1}
\left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & \frac{1}{k} & -\frac{1}{k} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k} & -\frac{1}{k} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\therefore\
\left[\begin{array}{cccc}
k & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & k & 1 \\
0 & 0 & 0 & 1
\end{array}\right]^{-1}
=
\left[\begin{array}{cccc}
\frac{1}{k} & -\frac{1}{k} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{k} & -\frac{1}{k} \\
0 & 0 & 0 & 1
\end{array}\right]
$$

□

27. Find all values of c, if any, for which the given matrix is invertible.

$$
\left[\begin{array}{ccc}
c & c & c \\
1 & c & c \\
1 & 1 & c
\end{array}\right]
$$

Work.

$$\begin{bmatrix} c & c & c & | & 1 & 0 & 0 \\ 1 & c & c & | & 0 & 1 & 0 \\ 1 & 1 & c & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2-R_3]{R_1-R_2} \begin{bmatrix} c-1 & 0 & 0 & | & 1 & -1 & 0 \\ 0 & c-1 & 0 & | & 0 & 1 & -1 \\ 1 & 1 & c & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{c-1}R_2]{\frac{1}{c-1}R_1} \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{c-1} & -\frac{1}{c-1} & 0 \\ 0 & 1 & 0 & | & 0 & \frac{1}{c-1} & -\frac{1}{c-1} \\ 1 & 1 & c & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3-R_1]{R_3-R_2} \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{c-1} & -\frac{1}{c-1} & 0 \\ 0 & 1 & 0 & | & 0 & \frac{1}{c-1} & -\frac{1}{c-1} \\ 0 & 0 & c & | & -\frac{1}{c-1} & 0 & \frac{c}{c-1} \end{bmatrix}$$

$$\xrightarrow{\frac{1}{c}} \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{c-1} & -\frac{1}{c-1} & 0 \\ 0 & 1 & 0 & | & 0 & \frac{1}{c-1} & -\frac{1}{c-1} \\ 0 & 0 & 1 & | & -\frac{1}{c(c-1)} & 0 & \frac{1}{c-1} \end{bmatrix}$$

The resulting inverse matrix is undefined when $c = 0$ or when $c = 1$, therefore $c \neq 0,\ 1$. $\qquad\square$

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29. Write the given matrix as a product of elementary matrices.

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$$

Work.

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 5 \\ 0 & -8 \end{bmatrix} \xrightarrow{-\frac{1}{8}R_2} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1-5R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Each of these row operations can be expressed as a left multiplication of a elementary matrix.

$$\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} = \left(\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{8} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

$\qquad\square$

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41. Prove that if A and B are $m \times n$ matrices, then A and B are row equivalent if and only if A and B have the same reduced row echelon form.

Proof. First, if A and B are row equivalent, there exists a sequence of elementary row operations that can transform B into A. Now consider the $rref(A)$; it is by definition obtainable by row operations starting at A. Therefore there exists a sequence of row operations which transform B into A, and then

5

A into $rref(A)$. The same can be said for A. Since reduced row echelon form is unique, thus $rref(A)$ is also the $rref(B)$. Therefore, A and B have the same reduced row echelon form.

Second, if A and B have the same reduced row echelon form, then there exists a sequence of elementary row operations that can transform A and B into the same reduced row echelon form. By definition there is a sequence of elementary row matrices to transform A into $rref(A)$, which is also $rref(B)$. And also by definition there are elementary row matrices to transform $rref(B)$ into B. Therefore, there is a sequence of elementary row operations to transform A into B. Therefore, by definition, A and B are row equivalent.

Combining both statements, we can conclude that A and B are row equivalent if and only if they have the same reduced row echelon form.   □

1.6 More on Linear Systems and Invertible Matrices

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15. Determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.

$$x_1 - 2x_2 + 5x_3 = b_1$$
$$4x_1 - 5x_2 + 8x_3 = b_2$$
$$-3x_1 + 3x_2 - 3x_3 = b_3$$

Work.

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -4 & 5 & -8 & b_3 - b_1 \end{array}\right] \xrightarrow[R_2 - 4R_1]{R_3 + R_2} \begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - b_1 + b_2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 1 & -4 & \frac{1}{3}(b_2 - 4b_1) \\ 0 & 0 & 0 & b_3 - b_1 + b_2 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 0 & -3 & b_1 + \frac{2}{3}(b_2 - 4b_1) \\ 0 & 1 & -4 & \frac{1}{3}(b_2 - 4b_1) \\ 0 & 0 & 0 & b_3 - b_1 + b_2 \end{bmatrix}$$

The last row represents $0 = b_3 - b_1 + b_2$, which is $b_1 = b_3 + b_2$. If $b_1 = b_3 + b_2$, then reduced row echelon form is complete, and there are infinitely many solutions to the linear system. If $b_1 \neq b_3 + b_2$, then the last row represents $0 = k$ where $k \neq 0$, and there is no solution to the linear system.   □

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21. Let $A\vec{x} = \vec{0}$ be a homogenous system of n linear equations in n unknown that has only the trivial solution. Show that if k is any positive integer, then the system $A^k\vec{x} = \vec{0}$ also has only the trivial solution.

Proof. Since $A\vec{x} = \vec{0}$ has only the trivial solution, through the Big Theorem of Lecture 23, this means that A is invertible. This also means that $(A^k)^{-1}$ exists, as it is known that $(A^k)^{-1} = A^{-k} = (A^{-1})^k$. Now Consider the system $A^k\vec{x} = \vec{0}$.

$$A^k\vec{x} = \vec{0}$$
$$(A^k)^{-1}A^k\vec{x} = (A^k)^{-1}\vec{0}$$
$$I\vec{x} = \vec{0}$$
$$\vec{x} = \vec{0}$$

Therefore $\vec{x} = \vec{0}$, and the only solution for the system is the trivial solution. $\square$