

MAT 311 Abstract Algebra

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1 Sets and Relations

Definition: What *is* Abstract Algebra

- Algebra: procedures for performing operations, i.e. $+$, $-$, \times , \div , and methods for solving equations. It uses bldspecific operations on **specific** objects.
- Abstract Algebra: discuss **general** structures and the relationships between the elements of these structures.

1.1 Sets

Definition: Set

A set is a collection of objects. These objects are called "elements". A set is typically uppercase, and elements are typically lowercase.

Set Notation

1. List Notation:

$$B = \{\text{John, Paul, Ringo, George}\}$$

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

2. Set-builder Notation:

$$B = \{b : b \text{ is a Beatle}\}$$

Well-Defined Sets

Sets must be **well-defined**. That is, given set S and any element x , either $x \in S$ or $x \notin S$.

Definition: Subset

A set A is a subset of set B , written as $A \subseteq B$, if every element of A is also in B .

Note: every non-empty set has at least two subsets:

- The set itself
- \emptyset

Definition: Proper Subset

If $A \subseteq B$ but $A \neq B$, then A is a **proper subset** of B , written $A \subset B$ or $A \subsetneq B$.

Note: A set B is an *improper subset* of itself.

Definition: Cartesian Product

Let A and B be sets. The set $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$ is the cartesian product of A and B .

Note: $A \times B = B \times A \iff A = B$, or $A \times B = \emptyset$.

Example

Let $A = \{c : c \text{ is a primary color}\}$ and let $B = \{\epsilon, \delta\}$. Find:

1. $B \times B = \{(\epsilon, \epsilon), (\epsilon, \delta), (\delta, \epsilon), (\delta, \delta)\}$
2. $A \times \emptyset = \emptyset$

1.2 Relations

Definition: Relation

A **relation** between sets A and B is a subset \mathcal{R} of $A \times B$. It is a collection of ordered pairs. Note: $(a, b) \in \mathcal{R} \equiv a\mathcal{R}b$ means "a is related to b".

Definition: Function

A **function** is a relation in which no two of the ordered pairs have the same first term. Note: if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function, then it passes the vertical-line test.

Definition: One-to-One

A function is **one-to-one**, or **injective**, if no two ordered pairs have the same second term.

To prove f is one-to-one, first assume that $f(x_1) = f(x_2)$, then show that $x_1 = x_2$.

Definition: Onto

A function $f : X \rightarrow Y$ is **onto**, or **surjective**, if the codomain is equal to the range, meaning every element $y \in Y$ has some $x \in X$ such that $f(x) = y$.

Definition: One-to-One Correspondence

A function $f : X \rightarrow Y$ is a **one-to-one correspondence**, or a **bijection**, if it is both one-to-one and onto.

1.3 Partitions and Equivalence Relations

Definition: Partition

A **partition** of a set S is a collection of non-empty subsets of S such that:

1. The union of these subsets is S .
2. These subsets are pairwise disjoint.

Note: these subsets are called **cells** of the partition.

Definition: Equivalence Relation

An **equivalence relation** \mathcal{R} on a set S must be:

1. Reflexive, meaning $x\mathcal{R}x \quad \forall x \in S$.
2. Symmetric, meaning if $x\mathcal{R}y$, then $y\mathcal{R}x$.
3. Transitive, meaning if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$.

Definition: Equivalence Class

$\bar{x} = \{y \in S : x\mathcal{R}y\}$ is the equivalence class of x

Example

Let $S = \mathbb{R}$. Define $x\mathcal{R}y$ iff $x \geq y$. Is \mathcal{R} an equivalence relation on S ?

1. Is \mathcal{R} reflexive? $\forall x \in S, x\mathcal{R}x$, so YES.
2. Is \mathcal{R} symmetric? Consider 5 and 1: $5 \geq 1$ but $1 \not\geq 5$, so NO.
3. Is \mathcal{R} transitive? If $x \geq y$ and $y \geq z$ then $x \geq z$, so YES.

Since \mathcal{R} is not symmetric, it is not an equivalence relation on S .

Note on Partition Cells and Equivalence Classes

Partitions give rise to equivalence relations and vice versa. The *cells* of the partition are analogous to the *equivalence classes* of the equivalence relation.

2 Binary Operations

Definition: Binary Operation

A **binary operation** $*$ on a set S is a function from $S \times S$ into S , $*$: $S \times S \rightarrow S$. That is, $*$ is a rule which assigns to each ordered pair $(a, b) \in S \times S$ exactly one element $a * b \in S$.

Condition 1: Uniquely Defined

For all $a, b \in S \times S$, $a * b$ must be **uniquely defined**. This means that $*$ cannot be undefined for any $a * b$, and each $a * b$ must have exactly one result, not two or more.

Condition 2: Closed under $*$

S must be **closed** under $*$. That is,

$$\forall a, b \in S, \quad a * b \in S.$$

Definition: Commutative

A binary operation $*$ on a set S is commutative if

$$\forall a, b \in S, \quad a * b = b * a.$$

Definition: Associative

A binary operation $*$ on a set S is associative if

$$\forall a, b, c \in S, \quad a * (b * c) = (a * b) * c.$$