MAT 260 LINEAR ALGEBRA
LECTURE 7

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4.1 — Real vector spaces (continued)

Previously, we learnt what a real vector space is, and we learnt how to verify whether a nonempty set V with addition "$\oplus$" and scalar multiplication "$\odot$" explicitly defined is a real vector space or not.

This time, you are GIVEN that V is a real vector space. However, you have NO IDEA what the set looks like, what elements are in V, nor what $\oplus$ and $\odot$ are. All you know is that $\oplus$ and $\odot$ satisfies all 10 axioms.

However, regardless what V is, the following theorems still hold. How can we prove them? The only tool we are allowed is the DEFINITION of a real vector space, i.e., the 10 axioms.

Recall

- $\mathbf{id}$ is the additive identity of V, which EXISTS in V due to Axiom (4);
- FOR ALL $\mathbf{u} \in V$, $-\mathbf{u}$ ("bar" $\mathbf{u}$) is the additive inverse of $\mathbf{u}$, which EXISTS in V due to Axiom (5).

Theorem A (Textbook Theorem 4.1.1(a))**.** *Let V be a vector space.* ***For all*** $\mathbf{u} \in V$,

$$0 \odot \mathbf{u} = \mathbf{id}.$$

Theorem B (Textbook Theorem 4.1.1(b))**.** *Let V be a vector space.* ***For all*** $k \in \mathbb{R}$,

$$k \odot \mathbf{id} = \mathbf{id}.$$

Theorem C (Textbook Theorem 4.1.1(c))**.** *Let V be a vector space.* ***For all*** $\mathbf{u} \in V$,

$$(-1) \odot \mathbf{u} = -\mathbf{u}.$$

Theorem D (Textbook Theorem 4.1.1(d))**.** *Let V be a vector space.* ***If*** $k \odot \mathbf{u} = \mathbf{id}$, ***then***

$$k = 0 \quad OR \quad \mathbf{u} = \mathbf{id}.$$

Date: Monday, February 6, 2023.

Proof of Theorem A.

$$
\begin{aligned}
0 \odot \mathbf{u} \oplus \mathbf{u} &= 0 \odot \mathbf{u} \oplus 1 \odot \mathbf{u} \qquad \text{(by Axiom (10))} \\
&= (0+1) \odot \mathbf{u} \qquad \text{(by Axiom (8))} \\
&= 1 \odot \mathbf{u} \\
&= \mathbf{u}. \qquad \text{(by Axiom (10))}
\end{aligned}
$$

By Axiom (5), since $\mathbf{u} \in V$, $-\mathbf{u} \in V$. Therefore,

$$
\begin{aligned}
(0 \odot \mathbf{u} \oplus \mathbf{u}) \oplus (-\mathbf{u}) &= \mathbf{u} \oplus (-\mathbf{u}) \\
0 \odot \mathbf{u} \oplus \big(\mathbf{u} \oplus (-\mathbf{u})\big) &= \mathbf{u} \oplus (-\mathbf{u}) \qquad \text{(by Axiom (3))} \\
0 \odot \mathbf{u} \oplus \mathbf{id} &= \mathbf{id} \qquad \text{(by the definition of additive inverse)} \\
0 \odot \mathbf{u} &= \mathbf{id}. \qquad \text{(by the definition of additive identity)}
\end{aligned}
$$

□

Another Proof of Theorem A.

$$
\begin{aligned}
0 \odot \mathbf{u} \oplus 0 \odot \mathbf{u} &= (0+0) \odot \mathbf{u} \qquad \text{(by Axiom (8))} \\
&= 0 \odot \mathbf{u}.
\end{aligned}
$$

By Axiom (6), since $0 \in \mathbb{R}$ and $\mathbf{u} \in V$, we have $0 \odot \mathbf{u} \in V$.
By Axiom (5), since $0 \odot \mathbf{u} \in V$, we have $-(0 \odot \mathbf{u}) \in V$. Therefore,

$$
\begin{aligned}
(0 \odot \mathbf{u} \oplus 0 \odot \mathbf{u}) \oplus \big(-(0 \odot \mathbf{u})\big) &= 0 \odot \mathbf{u} \oplus \big(-(0 \odot \mathbf{u})\big) \\
0 \odot \mathbf{u} \oplus \Big(0 \odot \mathbf{u} \oplus \big(-(0 \odot \mathbf{u})\big)\Big) &= 0 \odot \mathbf{u} \oplus \big(-(0 \odot \mathbf{u})\big) \qquad \text{(by Axiom (3))} \\
0 \odot \mathbf{u} \oplus \mathbf{id} &= \mathbf{id} \qquad \text{(by the definition of additive inverse)} \\
0 \odot \mathbf{u} &= \mathbf{id}. \qquad \text{(by the definition of additive identity)}
\end{aligned}
$$

□

How do we COME UP with these two proofs of Theorem A? The key is, we need to UNDERSTAND the question.

In the statement of Theorem A, we are asked to prove $0 \odot \mathbf{u} = \mathbf{id}$. However, this is NOT asking us to "calculate" the LHS and show that it is equal to the RHS. Really?! Then what are we asked to show?

In fact, we need to DEMONSTRATE that $0 \odot \mathbf{u}$ shows some BEHAVIOR like the additive identity. In other words, we must add "something" to $0 \odot \mathbf{u}$, and hope to get that "something" back. That's why we decided add $\mathbf{u}$ to $0 \odot \mathbf{u}$ in the first proof, and add $0 \odot \mathbf{u}$ to $0 \odot \mathbf{u}$ in the second.

Once we have demonstrated that after adding "something" to $0 \odot \mathbf{u}$, we actually get that "something" back, we need to "cancel" out that "something" from both sides, and that's why we use Axiom (5) to call up the additive inverse to help.

Important: What I am doing here after the two proofs is the **evaluation** step for writing mathematical proofs, the FIFTH step that I introduced on the very first day of class:

(1) UNDERSTAND the question,
(2) Collect all the tools, such as DEFINITIONS, theorems, conditions given in the problem, previous HW exercises, etc.
(3) Build the logical BRIDGE between the "conditions" and the "desired conclusion".
(4) PRESENT your ideas.
(5) EVALUATE: think about what tripped you up, what new things you learnt from this proof, what are the keys steps, and what prompts you to use this particular method, etc.

Next, we will look at the proof of Theorem C. What would you do at the first step? We are asked to prove $-1 \odot \mathbf{u} = -\mathbf{u}$. Are we going to "distribute" -1 into (u_1, u_2) and get $(-u_1, -u_2)$ and hence done?

I have seen many students doing that. Unfortunately, this method is terribly WRONG. It is so wrong that I can find "4 layers of wrong"!

- First, we DON'T know what V is, so we CAN'T assume that $\mathbf{u} \in V$ is in the form (u_1, u_2).
- Second, even if V is $\mathbb{R}^2$, we CAN'T assume that scalar multiplication of -1 into (u_1, u_2) gives $(-u_1, -u_2)$.
- Third, even if $-1 \odot (u_1, u_2) = (-u_1, -u_2)$, this is NOT a "distributive" property, since "distributive" property must be distributing OVER another operation, but NOT into each entry of the coordinate pair.
- Fourth, even if we get $(-u_1, -u_2)$, we have NO CLUE whether $(-u_1, -u_2)$ is $-\mathbf{u}$, since $(-u_1, -u_2)$ may not be the additive inverse of (u_1, u_2).

Then how are we going to prove Theorem C. We try to LEARN from the experience in proving Theorem A. Instead of "calculating" the LHS and show that it is equal to the RHS, we are going to check the DEFINITION of $-\mathbf{u}$.

Proof of Theorem C.

$$\begin{aligned}
-1 \odot \mathbf{u} \oplus \mathbf{u} &= -1 \odot \mathbf{u} \oplus 1 \odot \mathbf{u} \qquad \text{(by Axiom (10))} \\
&= (-1+1) \odot \mathbf{u} \qquad \text{(by Axiom (8))} \\
&= 0 \odot \mathbf{u} \\
&= \mathbf{id}. \qquad \text{(by Theorem } A\text{)}
\end{aligned}$$

Also,

$$\begin{aligned}
\mathbf{u} \oplus (-1) \odot \mathbf{u} &= 1 \odot \mathbf{u} \oplus (-1) \odot \mathbf{u} \qquad \text{(by Axiom (10))} \\
&= (1 + (-1)) \odot \mathbf{u} \qquad \text{(by Axiom (8))} \\
&= 0 \odot \mathbf{u} \\
&= \mathbf{id}. \qquad \text{(by Theorem } A\text{)}
\end{aligned}$$

Therefore, by **the definition of additive inverse**,

$$-1 \odot \mathbf{u} = -\mathbf{u}.$$

$\square$

$\boxed{\textbf{Homework}}$

Problem 1 (Textbook 4.1.23)**.** Let V be a vector space, and let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. Prove that if

$$\mathbf{u} \oplus \mathbf{w} = \mathbf{v} \oplus \mathbf{w},$$

then

$$\mathbf{u} = \mathbf{v}.$$

Note: This is the "cancellation law" for a real vector space V.

Problem 2 (Textbook 4.1.25)**.** Prove Theorem B.

Problem 3 (Textbook 4.1.27)**.** Prove Theorem D.

Hint: Compare the statement of Theorem D

$$\text{If } k \odot \mathbf{u} = \mathbf{id}, \text{ then } k = 0 \text{ or } \mathbf{u} = \mathbf{id}.$$

with

$$\text{If } k \odot \mathbf{u} = \mathbf{id} \text{ and } k \neq 0, \text{ then } \mathbf{u} = \mathbf{id}.$$

Note that they are **equivalent**, then prove the second statement instead. By the way, generally speaking in mathematics, when do we need to so cautious about a number k being nonzero?

Problem 4. Prove that there does not exist a real vector space of size 2.

Hint: Let $V = \{\mathbf{u}, \mathbf{v}\}$, and assume that there are $\oplus$ and $\odot$ defined on V such that V becomes a vector space. There will be some consequences following from this assumption, and if you find some contradictory results, then you have successfully proved that such a vectors space cannot exist.