Section 2 Binary Operations, p25 1,3,5,27,28,36

Exercises 1 through 4 concern the binary operation $*$ defined on $S = \{a, b, c, d, e\}$ by means of Table 2.26 (not shown).

1. Compute $b * d$, $c * c$, and $[(a * c) * e] * a$

Here are the computations:

$$\begin{aligned} b * d &= e \\ c * c &= b \\ [(a * c) * e] * a = [c * e] * a = a * a &= a \end{aligned}$$

3. Compute $(b * d) * c$ and $b * (d * c)$. Can you say on the basis of these computations whether $*$ is associative?

Examples can only tell us if $*$ is not associative.

$$\begin{aligned} (b * d) * c = e * c = a \\ b * (d * c) = b * b = c \end{aligned}$$

Since $a \neq c$, we know that $*$ is not associative.

5. Complete Table 2.27 so as to define a commutative binary operation $*$ on $S = \{a, b, c, d\}$.

2.28 Table

$*$	a	b	c	d
a	a	b	c	**d**
b	b	d	**a**	c
c	c	a	d	b
d	d	**c**	**b**	a

In Exercise 27 and 28, either prove the statement or give a counterexample.

27. Every binary operation on a set consisting of a single element (is) commutative and associative.

There is only one unique set consisting of a single element.

Proof. Consider $S = \{s\}$ where $s * s = s$.

(a) Commutative: $s * s = s = s * s$. Thus S is commutative under $*$.

(b) Associative: $s * (s * s) = s * s = s = s * s = (s * s) * s$. Thus S is associative under $*$.

Thus any binary operation on a set consisting of a single element is commutative and associative. □

28. Every commutative binary operation on a set having just two elements is associative.

We shall conduct a proof through counterexample.

Proof. Consider $S = \{a, b\}$ with $*$ such that $\begin{array}{c|cc} * & a & b \\ \hline a & b & a \\ b & a & a \end{array}$.

$$a * (a * b) = a * a = b$$
$$(a * a) * b = b * b = a$$

Since we assert that $b \neq a$, thus $a * (a * b) \neq (a * a) * b$, so S is a binary operation, which is commutative, but not associative. □

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36 Suppose that $*$ is an *associative binary operation* on a set S. Let $H = \{a \in S : a * x = x * a \text{ for all } x \in S\}$. Show that H is closed under $*$. (We think of H as consisting of all elements of S that *commute* with every element in S.)

Proof. Consider $a, b, c, d \in H$. We need to show that $(a * b) * (c * d) = (c * d) * (a * b)$ for H to be closed under $*$, since that is the defining property of H.

$$\begin{aligned}
LHS &= (a * b) * (c * d) \\
&= a * (b * c) * d && \text{(1) } * \text{ is associative} \\
&= a * (c * b) * d && \text{(2) elements in } H \text{ are commutative} \\
&= (a * c) * (b * d) && (1) \\
&= (c * a) * (d * b) && (2) \\
&= c * (a * d) * b && (1) \\
&= c * (d * a) * b && (2) \\
&= (c * d) * (a * b) && (1) \\
&= RHS
\end{aligned}$$

Thus we can conclude that $(a * b) * (c * d) = (c * d) * (a * b)$, and H is closed under $*$. □