Discrete Math for Computer Science

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Contents

1 Logic

1.1 Propositions and Logical Operations

Proposition: a statement that is either <u>true</u> or <u>false</u>.
Some examples include "It is raining today" and "$3 \cdot 8 = 20$".
However, not all statements are propositions, such as "open the door"

Name	**Symbol**	**alternate name**
NOT	$\neg$	negation
AND	$\wedge$	conjunction
OR	$\vee$	disjunction
XOR	$\oplus$	exclusive or

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T	F	T	T	F
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	F	F	F

XOR is very useful for encryption and binary arithmetic.

1.2 Evaluating Compound Propositions

p : The weather is bad.
q : The trip is cancelled.
r : The trip is delayed.

$p \wedge q$: The weather is bad *and* the trip is cancelled
$p \vee q$: The weather is bad *or* the trip is cancelled
$p \wedge (q \oplus r)$: The weather is bad *and* either the trip is cancelled *or* delayed

Order of Evaluation $\neg$, then $\wedge$, then $\vee$, but parenthesis always help for clarity.

Example Truth Table:

p	q	$p \wedge q$	$\neg q$	$(p \wedge q) \oplus \neg q$
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	T	T

1.3 Conditional Statements

$p \to q$ where p is the <u>hypothesis</u> and q is the <u>conclusion</u>

Format	Terminology
$p \to q$	given
$\neg q \to \neg p$	contrapositive
$q \to p$	converse
$\neg p \to \neg q$	inverse

given	$p \to q$	$\equiv$	$\neg q \to \neg p$	contrapositive
inverse	$\neg p \to \neg q$	$\equiv$	$q \to p$	converse

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p is a <u>sufficient</u> condition for q
q is a <u>necessary</u> condition for p

Phrase	Logic
q if p	$p \to q$
q only if p	$q \to p$
q if and only if p	$p \leftrightarrow q$

Order of Operations: $p \wedge q \to r \equiv (p \wedge q) \to r$

1.4 Logical Equivalence

Tautology: a proposition that is always <u>true</u> **Contradiction**: a proposition that is always <u>false</u>

Logically equivalent: same truth value regardless of the truth values of their individual propositions

DeMorgan's Laws:
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Verbally,

It is not true that the patient has migraines *or* high blood pressure ≡
≡ The patient does not have migraines *and* does not have high blood pressure

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It is not true that the patient has migraines *and* high blood pressure ≡
≡ The patient does not have migraines *or* does not have high blood pressure

1.5 Laws of Propositional Logic

You can use **substitution** on logically equivalent propositions.

Law Name	$\vee$ or	$\wedge$ and
Idempotent	$p \vee p \equiv p$	$p \wedge p \equiv p$
Associative	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Distributive	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity	$p\vee$ F $\equiv p$	$p\wedge$ T $\equiv p$
Domination	$p\vee$ T $\equiv$ T	$p\wedge$ F $\equiv$ F
Double Negation	$\neg\neg p \equiv p$	
Complement	$p \vee \neg p \equiv$ T	$p \wedge \neg p \equiv$ F
DeMorgan	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Conditional	$p \rightarrow q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

1.6 Predicates and Quantifiers

Predicate: a logical statement where truth value is a function of a variable.

P(x): x is an even number. P(5): false P(2): true

Domain: the set of all possible values for a variable in a predicate.

Ex. $\mathbb{Z}^+$ is the set of all positive integers.
*If domain is not clear from context, it should be given as part of the definition of the predicate.

Quantifier: converts a predicate to a proposition.

Quantifier	Symbol	Meaning
Universal	$\forall$	"for all"
Existential	$\exists$	"there exists"

$\exists x(x + 1 < x)$ is false.

Counter Example: universally quantified statement where an element in the domain for which the predicate is false. Useful to prove a $\forall$ statement false.

1.7 Quantified Statements

Consider the two following two predicates:

$$P(x) : x \text{ is prime, } x \in \mathbb{Z}^+$$
$$O(x) : x \text{ is odd}$$

Proposition made of predicates: $\exists x(\mathrm{P}(x) \wedge \neg \mathrm{O}(x))$

Verbally: there exists a positive integer that is prime but is not odd.

Free Variable: a variable that is free to be any value in the domain.
Bound Variable: a variable that is bound to a quantifier.

P(x): x came to the party
S(x): x was sick

$\mathrm{P}(x) \overset{?}{\equiv} \neg\mathrm{S}(x)$
$\mathrm{P}(x) \not\equiv \neg\mathrm{S}(x)$

	P(x)	S(x)	$\neg$S(x)
Joe	T	F	T
Theo	F	T	F
Gert	T	F	T
Sam	F	F	T

1.8 DeMorgan's law for Quantified Statements

Consider the predicate: $\mathrm{F}(x)$: "x can fly", where x is a bird. According to the DeMorgan Identity for Quantified Statements,

$$\neg\forall x \mathrm{F}(x) \equiv \exists x \neg \mathrm{F}(x)$$
$$\text{"not every bird can fly} \equiv \text{"there exists a bird that cannot fly}$$

Example using DeMorgan Identities:

$$\begin{aligned}\neg\exists x(\mathrm{P}(x) \to \neg\mathrm{Q}(x)) &\equiv \forall x \neg(\mathrm{P}(x) \to \neg\mathrm{Q}(x)) \\ &\equiv \forall x(\neg\neg\mathrm{P}(x) \wedge \neg\neg\mathrm{Q}(x)) \\ &\equiv \forall x(\mathrm{P}(x) \wedge \mathrm{Q}(x))\end{aligned}$$

1.9 Nested Quantifiers

A logical expression with more than one quantifier that binds different variables in the same predicate is said to have **Nested Quantifiers**.

Logic	Variable Boundedness
$\forall x \exists y\ \mathrm{P}(x, y)$	x, y bound
$\forall x\ \mathrm{P}(x, y)$	x bound, y free
$\exists x \exists y\ \mathrm{T}(x, y, z)$	x, y bound, z free

Logic	Meaning
$\forall x \forall y\ \mathrm{M}(x, y)$	"everyone sent an email to everyone"
$\forall x \exists y\ \mathrm{M}(x, y)$	"everyone sent an email to someone"
$\exists x \forall y\ \mathrm{M}(x, y)$	"someone sent an email to everyone"
$\exists x \exists y\ \mathrm{M}(x, y)$	"someone sent an email to someone"

There is a two-player game analogy for how quantifiers work:

Player	Action	Goal
Existential Player $\exists$	selects value for existentially-bound variables	tries to make expression <u>true</u>
Universal Player $\forall$	selects value for universally-bound variables	tries to make expression <u>false</u>

Consider the predicate $\mathrm{L}(x, y)$: "x likes y".

$\exists x \forall y \mathrm{L}(x, y)$ means "there is a student who likes everyone in the school".

$\neg\exists x \forall y \mathrm{L}(x, y)$ means "there is no student who likes everyone in the school".

After applying DeMorgan's Laws,

$\forall x \exists y \neg\mathrm{L}(x, y)$ means "there is no student who likes everyone in the school".

1.10 More Nested Quantifiers

1.11 Logical Reasoning

1.12 Rules of Inference with Propositions

1.13 Rules of Inference with Quantifiers