Problem 1

Let V be a vector space, and let $\vec{u}, \vec{v}, \vec{w} \in V$. Prove that if $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$ then $\vec{u} = \vec{v}$.

Proof. Consider $\vec{v}, \vec{u}, \vec{w} \in V$, and assume $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$.

$$\begin{aligned}
\vec{u} \oplus \vec{w} &= \vec{v} \oplus \vec{w} && \text{Assertion} \\
(\vec{u} \oplus \vec{w}) \oplus -\vec{w} &= (\vec{v} \oplus \vec{w}) \oplus -\vec{w} && \text{Axiom 5 states } -\vec{w} \in V \\
\vec{u} \oplus (\vec{w} \oplus -\vec{w}) &= \vec{v} \oplus (\vec{w} \oplus -\vec{w}) && \text{Axiom 3} \\
\vec{u} \oplus \mathbf{id} &= \vec{v} \oplus \mathbf{id} && \text{Def. of additive inverse} \\
\vec{u} &= \vec{v} && \text{Def. of additive identity}
\end{aligned}$$

$\therefore$ if $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$ then $\vec{u} = \vec{v}$ □

Problem 2

Prove Theorem B.

Proof. Let $k \in \mathbb{R}$. Recall that Theorem A implies that $0 \odot \mathbf{id} = \mathbf{id}$:

$$\begin{aligned}
k \odot \mathbf{id} &= k \odot (0 \odot \mathbf{id}) && \text{Thm } A \\
&= (k \cdot 0) \odot \mathbf{id} && \text{Axiom 9} \\
&= 0 \odot \mathbf{id} \\
&= \mathbf{id} && \text{Thm } A
\end{aligned}$$

$\therefore \forall\, k \in \mathbb{R},\ k \odot \mathbf{id} = \mathbf{id}$ □

Problem 3

Prove Theorem D. If $k \odot \vec{u} = \mathbf{id}$, then $k = 0$ or $\vec{u} = \mathbf{id}$

Proof. Let $k \odot \vec{u} = \mathbf{id}$ and $k \neq 0$. Since $\frac{1}{k} \neq \frac{1}{0}$, $\frac{1}{k}$ is well-defined.

$$\begin{aligned}
k \odot \vec{u} &= \mathbf{id} && \text{Assertion} \\
\frac{1}{k} \odot (k \odot \vec{u}) &= \frac{1}{k} \odot (\mathbf{id}) \\
(\frac{1}{k} \cdot k) \odot \vec{u} &= \frac{1}{k} \odot \mathbf{id} && \text{Axiom 9} \\
1 \odot \vec{u} &= \mathbf{id} && \text{Thm B} \\
\vec{u} &= \mathbf{id} && \text{Axiom 10}
\end{aligned}$$

$\therefore$ if $k \odot \vec{u} = \mathbf{id}$, then $k = 0$ or $\vec{u} = \mathbf{id}$ □

Problem 4

Prove that there does not exist a real vector space of size 2. Show that there cannot be a vector space of size 2.

Proof. Let $V = \{\vec{u}, \vec{v}\}$ be a vectorspace where $\vec{u} \neq \vec{v}$.Axiom 4 states that **id** exists, and is unique. Therefore either $\vec{u} = \mathbf{id}$ or $\vec{v} = \mathbf{id}$.
Without the loss of generality, let $\vec{u} = \mathbf{id}$.

$$\begin{aligned}
\vec{u} \oplus \vec{v} &= \vec{v} && \text{Axiom 4} \\
\mathbf{id} \oplus \vec{v} &= \vec{v}
\end{aligned}$$

Now consider Axiom 5: $-\vec{v} \in V$. Note that $\vec{v} \oplus \vec{u} = V$, so we know $-\vec{v} \neq \vec{u}$, so therefore $\vec{v} = -\vec{v}$.

$$\begin{aligned} \vec{v} \oplus \vec{v} = \vec{u} &= \mathbf{id} && \text{Def. of Additive Inverse} \\ 1 \odot \vec{v} \oplus 1 \odot \vec{v} &= \vec{u} && \text{Axiom 10} \\ (1+1) \odot \vec{v} &= \vec{u} && \text{Axiom 8} \\ 2 \odot \vec{v} = \vec{u} &= \mathbf{id} \end{aligned}$$

By Theorem A, either $2 = 0$ or $\vec{v} = \mathbf{id}$. We know that $2 \neq 0$ and $\vec{v} \neq \mathbf{id}$ since $\vec{u} = \mathbf{id}$, a contradiction.
$\therefore$ A vectorspace of size 2 cannot exist.

$\square$