MAT 260 LINEAR ALGEBRA
LECTURE 31

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2.1 — Determinants by cofactor expansion

In Chapter 1, we learnt that whether a square matrix A is invertible is an important information. From the Big Theorem, we can determine whether A is invertible by finding the reduced row echelon form of A and see if it turns into the identity matrix I. In this section, we will learn another indicator, called **determinant**, that determines whether A is invertible.

The determinant of a 1×1 matrix is the entry itself. The determinant of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$. We also write $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

If A is a square matrix of order $n \times n$, then we define its determinant inductively. The **minor** of the entry a_{ij}, denoted by M_{ij}, is the determinant of the submatrix of A obtained by deleting the i-th and the j-th column from A. The number $(-1)^{i+j} M_{ij}$, denoted by C_{ij}, is the **cofactor** of the entry a_{ij}. Note that the matrix $((-1)^{i+j})$ is

$$\begin{pmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The determinant of A is defined to be

$$\sum_{k=1}^{n} a_{ik} C_{ik}$$

or

$$\sum_{k=1}^{n} a_{kj} C_{kj}$$

for any i-th row or any j-th column of A. The surprising thing is, they are always the same.

Example 1. Find the determinant of $\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$.

Date: Monday, April 10, 2023.

Example 2. Find the determinant of $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

Theorem 1. *If A is a diagonal, upper triangular or lower triangular matrix, then* $\det A = a_{11}a_{22}\dots a_{nn}$.

There is a short cut for computing the determinant of 3×3 matrices, which can be shown by using arrow patterns. However, it is very important to remember that this trick fails for $n \times n$ matrices for all $n \geq 4$.