Graded Assignment #2

1 [2 points each]

a. List the elements of $\langle f \rangle$ in S_6 where $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 2 & 5 & 4 \end{pmatrix}$

$\langle f \rangle$ consists of $\{f, f^2, f^3, \mathbf{id}\}$ where

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 2 & 5 & 4 \end{pmatrix} \qquad f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 1 & 5 & 2 \end{pmatrix}$$

$$f^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 6 & 5 & 1 \end{pmatrix} \qquad \mathbf{id} = f^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

b. If $f(x) = x + 1$, describe the cyclic subgroup $\langle f \rangle$ of $\langle S_{\mathbb{R}}, \circ \rangle$.

$\langle f \rangle$ consists of $\{\mathbf{id} = f^0, f^1, f^2, \ldots\}$, where for $x \in \mathbb{R}$

$$f^n(x) = x + n$$

c. If $f(x) = x + 1$, describe the cyclic subgroup $\langle f \rangle$ of $\langle F(\mathbb{R}), + \rangle$.

$\langle f \rangle$ consists of $\{\mathbf{id} = f^0, f^1, f^2, \ldots\}$, where for $x \in \mathbb{R}$

$$f^n(x) = nx + n$$

2 [4 points] The subgroup of S_5 generated by $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}$ has six elements. List them, using f and/or g as appropriate, and write the operation table of this subgroup.

Since each of these permutations are disjoint, the resulting subgroup generated by them will be Abelian. Let us first look at the permutations of f and g individually.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}$$

$$\mathbf{id} = f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \qquad g^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 3 & 4 \end{pmatrix}$$

$$\mathbf{id} = g^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

Knowing these identities, let us determine the final two elements of this subgroup.

$$fg = gf = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix} \qquad fg^2 = g^2 f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}$$

The subgroup generated is the group $\{\mathbf{id}, f, g, g^2, fg, fg^2\}$, with operation table

$\circ$	$\mathbf{id}$	f	g	g^2	fg	fg^2
$\mathbf{id}$	$\mathbf{id}$	f	g	g^2	fg	fg^2
f	f	$\mathbf{id}$	fg	fg^2	g	g^2
g	g	fg	g^2	$\mathbf{id}$	fg^2	f
g^2	g^2	fg^2	$\mathbf{id}$	g	f	fg
fg	fg	g	fg^2	f	g^2	$\mathbf{id}$
fg^2	fg^2	g^2	f	fg	$\mathbf{id}$	g

3 [2 points each] You must justify your answers for parts (b) and (c).

a. Compute the following product in S_9 and write your answer as a permutation tabular form: $(1,4,7)(1,6,7,8)(7,4,1,3,2)$.

$(1,4,7)(1,6,7,8)(7,4,1,3,2) =$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 2 & 6 & 5 & 1 & 7 & 4 & 9 \end{pmatrix}$$

b. Determine whether the following is even or odd: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 4 & 1 & 5 & 6 & 2 & 3 & 8 \end{pmatrix}$.

First, let's write the permutation as a product of disjoint cycles.

$$(1,7,3)(2,4,5,6)$$

Then let's write each cycle as a product of transpositions.

$$(1,7,3) = (1,3)(1,7)$$
$$(2,4,5,6) = (2,6)(2,5)(2,4)$$

Then we can combine the products of transpositions, and see the resulting parity.

$$(1,7,3)(2,4,5,6) = (1,3)(1,7)(2,6)(2,5)(2,4)$$

From this, we see that this permutation is a product of an **odd** number of transpositions.

c. Determine whether the following is even or odd: $(1,2,7,6)(3,2,4,1)(7,8,1,2)$.

Firstly, let's write the permutation as a product of *disjoint* cycles.

$$(1,2,7,6)(3,2,4,1)(7,8,1,2) = (1,4,2,6)(3,7,8)$$

Then let's write each cycle as a product of transpositions.

$$(1,4,2,6) = (1,6)(1,2)(1,4)$$
$$(3,7,8) = (3,8)(3,7)$$

Then we can combine the products of transpositions, and see the resulting parity.

$$(1,2,7,6)(3,2,4,1)(7,8,1,2) = (1,4,2,6)(3,7,8) = (1,6)(1,2)(1,4)(3,8)(3,7)$$

From this, we see that this permutation is a product of an **odd** number of transpositions.

4 [4 points] Let H and K be subgroups of a group G. Define $\sim$ on G by $a \sim b$ if and only if $a = hbk$ for some $h \in H$ and some $k \in K$. Prove that $\sim$ is an equivalence relation on G. The equivalence classes of this equivalence relation are called *double cosets*.

Proof. An equivalence relation is reflexive, symmetric, and transitive.

1. Reflexive: Consider $a \in G$. Since both H and K are subgroups of G, we know $e \in H$ and $e \in K$. Thus, a can be expressed as

   $$a = eae$$

   And so we can see that $a \sim a$, for all $a \in G$. Thus, $\sim$ is reflexive.

2. Symmetric: Assume that $a \sim b$, for some $a, b \in G$. This means that $a = hbk$ for some $h \in H$ and some $k \in K$. Since H and K are subgroups, we know they are closed under inverses, so $h^{-1} \in H$ and $k^{-1} \in K$. So we can manipulate $a = hbk$ as such:

   $$\begin{aligned} a &= hbk \\ h^{-1}ak^{-1} &= h^{-1}hbkk^{-1} \\ h^{-1}ak^{-1} &= b \\ b &= h^{-1}ak^{-1} \end{aligned}$$

   Therefore, we can see that if $a \sim b$, then $b \sim a$. Thus $\sim$ is symmetric.

3. Transitive: Assume that $a \sim b$ and $b \sim c$ for some $a, b, c \in G$. This means that $a = h_1bk_1$ and $b = h_2ck_2$, for $h_1, h_2 \in H$ and $k_1, k_2 \in K$.

   $$a = h_1bk_1 = h_1(h_2ck_2)k_1 = (h_1h_2)c(k_2k_1)$$

   We know that $h_1h_2 \in H$ and $k_2k_1 \in K$, meaning that $a \sim c$. Thus if $a \sim b$ and $b \sim c$, then $a \sim c$. This means that $\sim$ is transitive.

Because $\sim$ is reflexive, symmetric, and transitive, it is a equivalence relation on G. □