Graded Assignment #1

1 [2 points each] Which of the following are binary operations on the given sets? If it is not an operation, explain why.

(a). $S = \mathbb{R}^+$ with $a * b = a \ln b$

A binary operation must be uniquely defined and closed.

Proof. Consider $a = 1$ and $b = \frac{1}{e}$. Both a and $b \in \mathbb{R}^+$, but

$$a * b = a \ln b = 1 \ln \frac{1}{e} = 1 \cdot -1 = -1 \notin \mathbb{R}^+.$$

Thus S is not closed under $*$, and $*$ cannot be a binary operation. □

(b). $S = \mathbb{R}$ where $a * b$ is the root of the equation $x^2 - a^2b^2 = 0$

A binary operation must be uniquely defined and closed.

Proof. Consider $a = 2$ and $b = 1$.

$$\begin{aligned} a * b = x^2 - a^2b^2 &= 0 \\ x^2 - 2^2 1^2 &= \\ x^2 - 4 &= 0 \\ (x-2)(x+2) &= 0 \\ x &= \pm 2 \end{aligned}$$

Since there are two solutions, S is not uniquely defined under $*$, and $*$ cannot be a binary operation. □

2 [2 points each] Consider the binary operation $*$ defined on $\mathbb{R}^+$ by $a * b = \frac{ab}{a+b+1}$

(a). Is $*$ commutative? Explain.

$*$ is commuatative.

Proof. Consider $a * b$ and $b * a$ for $a, b \in \mathbb{R}^+$:

$$a * b = \frac{ab}{a+b+1} = \frac{ba}{b+a+1} = b * a$$

Since $a * b = b * a$ for all $a, b \in \mathbb{R}^+$, $*$ is commutative. □

(b). Is $*$ associative? Explain.

$*$ is associative.

Proof. Consider $a, b, c \in \mathbb{R}^+$:

$$\begin{aligned}(a * b) * c &= \frac{ab}{a+b+1} * c = \frac{\frac{ab}{a+b+1}c}{\frac{ab}{a+b+1} + c + 1} = \frac{abc}{(a+b+1)(\frac{ab}{a+b+1} + c + 1)} \\ &= \frac{abc}{ab + ac + bc + a + b + c + 1}\end{aligned}$$

$$\begin{aligned}a * (b * c) &= a * \frac{bc}{b+c+1} = \frac{a\frac{bc}{b+c+1}}{a + \frac{bc}{b+c+1} + 1} = \frac{abc}{(b+c+1)(a + \frac{bc}{b+c+1} + 1)} \\ &= \frac{abc}{ab + ac + bc + a + b + c + 1}\end{aligned}$$

Thus $(a * b) * c = a * (b * c)$, and $*$ is associative. □

---

3 [3 points] Let E denote the set of all even integers. Prove that $\langle \mathbb{Z}, + \rangle \simeq \langle E, + \rangle$.

Proof. An isomorphism must be one-to-one, onto, and operation preserving. Consider $\phi : \mathbb{Z} \to E$ such that $\phi(n) = 2n$.

1. One-to-one: Assume $\phi(n_1) = \phi(n_2)$ for $n_1, n_2 \in \mathbb{Z}$.

$$\begin{aligned}\phi(n_1) &= \phi(n_2) \\ 2n_1 &= 2n_2 \\ n_1 &= n_2\end{aligned}$$

   Thus ϕ is one-to-one.

2. Onto: Let $m \in E$ Let us find $n \in \mathbb{Z}$ such that $m = \phi(n)$. Since m is an even integer, it can be represented as $m = 2k$, where $k \in \mathbb{Z}$.

$$\begin{aligned}m &= \phi(n) \\ 2k &= 2n \\ k &= n\end{aligned}$$

   Choose $n = k$. Thus ϕ is onto.

3. Operation Preserving: Need to show that $\phi(n + m) = \phi(n) + \phi(m)$

$$\begin{aligned}\phi(n + m) &= 2(n + m) \\ &= 2n + 2m \\ &= \phi(n) + \phi(m)\end{aligned}$$

   Thus ϕ is operation preserving.

Since ϕ is one-to-one, onto, and operation preserving, thus ϕ is an isomorphism of $\langle \mathbb{Z}, + \rangle$ and $\langle E, + \rangle$, and $\langle \mathbb{Z}, + \rangle \simeq \langle E, + \rangle$. □

---

4 [3 points each] Prove that isomorphism is an equivalence relation among binary structures. To do this, you need to prove the following three properties:

(a). Reflexive: Every binary structure is isomorphic to itself. Hint: let $\langle S, * \rangle$ be a binary structure and define $\phi : S \to S$ by $\phi(x) = x$. Prove that ϕ is an isomorphism.

An isomorphism must be one-to-one, onto, and operation preserving.

Proof. Consider $\phi : S \to S$ such that $\phi(x) = x$.

1. One-to-one: Assume $\phi(x_1) = \phi(x_2)$ for some $x_1, x_2 \in S$.

$$\begin{aligned} \phi(x_1) &= \phi(x_2) \\ x_1 &= x_2 \end{aligned}$$

   Thus ϕ is one-to-one.

2. Onto: Let $y \in S$. Let us find $x \in S$ such that $y = \phi(x)$.

$$\begin{aligned} y &= \phi(x) \\ y &= x \end{aligned}$$

   Choose $x = y$. This ϕ is onto.

3. Operation Preserving: Need to show that $\phi(x * y) = \phi(x) * \phi(y)$.

$$\begin{aligned} \phi(x * y) &= x * y \\ &= \phi(x) * \phi(y) \end{aligned}$$

   Thus ϕ is operation preserving.

Since ϕ is one-to-one, onto, and operation preserving, ϕ is an isomorphism of $\langle S, * \rangle$ and $\langle S, * \rangle$, and $\langle S, * \rangle \simeq \langle S, * \rangle$. □

(b). Symmetric: For binary structures $\langle S_1, * \rangle$ and $\langle S_2, *' \rangle$, if $S_1 \simeq S_2$ then $S_2 \simeq S_1$. Hint: assume $\phi : S_1 \to S_2$ is an isomorphism and prove that $\phi^{-1} : S_2 \to S_1$ is also an isomorphism.

An isomorphism must be one-to-one, onto, and operation preserving.

Proof. Consider $\phi^{-1} : S_2 \to S_1$. Such an operation must exist, since ϕ is bijective by definition.

1. One-to-one: Assume $\phi^{-1}(x_1) = \phi^{-1}(x_2)$ for $x_1, x_2 \in S_2$.

$$\begin{aligned} \phi^{-1}(x_1) &= \phi^{-1}(x_2) \\ \phi(\phi^{-1}(x_1)) &= \phi(\phi^{-1}(x_2)) \\ x_1 &= x_2 \end{aligned}$$

   Thus ϕ^{-1} is one-to-one.

2. Onto: Let $y \in S_1$. Let us find $x \in S_2$ such that $y = \phi^{-1}(x)$.

$$\begin{aligned} y &= \phi^{-1}(x) \\ \phi(y) &= \phi(\phi^{-1}(x)) \\ \phi(y) &= x \end{aligned}$$

   Choose $x = \phi(y)$. Thus ϕ^{-1} is onto.

3. Operation Preserving: Need to show that $\phi^{-1}(x *' y) = \phi^{-1}(x) * \phi^{-1}(y)$. But first, consider $a, b \in S_1$.

$$\begin{aligned} \phi(a * b) &= \phi(a) *' \phi(b) && \text{since } \phi \text{ is an isomorphism and thus operation preserving} \\ \phi^{-1}\phi(a * b) &= \phi^{-1}(\phi(a) *' \phi(b)) \\ a * b &= \phi^{-1}(\phi(a) *' \phi(b)) \end{aligned}$$

We can use the equation $a*b = \phi^{-1}(\phi(a)*'\phi(b))$ to help us show that $\phi^{-1}(x*'y) = \phi^{-1}(x)*\phi^{-1}(y)$.

$$\begin{aligned} \phi^{-1}(x) * \phi^{-1}(y) &= \phi^{-1}(\phi(\phi^{-1}(x)) *' \phi(\phi^{-1}(y))) \\ &= \phi^{-1}(x *' y) \end{aligned}$$

Thus ϕ^{-1} is operation preserving.

Since ϕ^{-1} is one-to-one, onto, and operation preserving, ϕ^{-1} is an isomorphism of $\langle S_2, *' \rangle$ and $\langle S_1, * \rangle$, and $\langle S_2, *' \rangle \simeq \langle S_1, * \rangle$. □

(c). Transitive: For binary structures $\langle S_1, * \rangle$, $\langle S_2, *' \rangle$, and $\langle S_3, *'' \rangle$, if $S_1 \simeq S_2$ and $S_2 \simeq S_3$ then $S_1 \simeq S_3$. Hind: assume $\phi_1 : S_1 \to S_2$ and $\phi_2 : S_2 \to S_3$ are isomorphisms and prove that $\phi_2 \circ \phi_1 : S_1 \to S_3$ is also an isomorphism.

An isomorphism must be one-to-one, onto, and operation preserving.

Proof. Consider $\phi_2 \circ \phi_1 : S_1 \to S_3$.

1. One-to-one: Assume $\phi_2 \circ \phi_1(x_1) = \phi_2 \circ \phi_1(x_2)$ for some $x_1, x_2 \in S_3$.

$$\begin{aligned} \phi_2 \circ \phi_1(x_1) &= \phi_2 \circ \phi_1(x_2) \\ \phi_2(\phi_1(x_1)) &= \phi_2(\phi_1(x_2)) \\ \phi_1(x_1) &= \phi_1(x_2) && \text{since } \phi_2 \text{ is an isomorphism and thus one-to-one} \\ x_1 &= x_2 && \text{since } \phi_1 \text{ is an isomorphism and thus one-to-one} \end{aligned}$$

Thus $\phi_2 \circ \phi_1$ is one-to-one.

2. Onto: Let $y \in S_3$. Let us find $x \in S_1$ such that $y = \phi_2 \circ \phi_1(x)$.

$$\begin{aligned} y &= \phi_2 \circ \phi_1(x) \\ \phi_2^{-1}(y) &= \phi_2^{-1} \circ \phi_2 \circ \phi_1(x) && \text{since } \phi_2 \text{ is bijective, and has a well-defined inverse } \phi_2^{-1} \\ \phi_1^{-1} \circ \phi_2^{-1}(y) &= \phi_1^{-1} \circ \phi_2^{-1} \circ \phi_2 \circ \phi_1(x) && \text{since } \phi_1 \text{ is bijective, and has a well-defined inverse } \phi_1^{-1} \\ \phi_1^{-1} \circ \phi_2^{-1}(y) &= \phi_1^{-1} \circ \phi_1(x) \\ \phi_1^{-1} \circ \phi_2^{-1}(y) &= x \end{aligned}$$

Choose $x = \phi_1^{-1} \circ \phi_2^{-1}(y)$. Thus $\phi_2 \circ \phi_1$ is onto.

3. Operation Preserving: Need to show that $\phi_2 \circ \phi_1(x * y) = \phi_2 \circ \phi_1(x) *'' \phi_2 \circ \phi_1(y)$. Let us use the following equations to help with this. Let $a, b \in S_1$ and $\alpha, \beta \in S_2$

$$\begin{aligned} \phi_1(a * b) &= \phi_1(a) *' \phi_1(b) && \text{since } \phi_1 \text{ is an isomorphism and operation preserving} \\ \phi_2(\alpha *' \beta) &= \phi_2(\alpha) *'' \phi_2(\beta) && \text{since } \phi_2 \text{ is an isomorphism and operation preserving} \end{aligned}$$

We can compose these two equations to help us show $\phi_2 \circ \phi_1(x * y) = \phi_2 \circ \phi_1(x) *'' \phi_2 \circ \phi_1(y)$:

$$\begin{aligned}
\phi_2 \circ \phi_1(x * y) &= \phi_2(\phi_1(x) *' \phi_1(x)) \\
&= \phi_2(\phi_1(x)) *'' \phi_2(\phi_1(y)) \\
&= \phi_2 \circ \phi_1(x) *'' \phi_2 \circ \phi_1(y)
\end{aligned}$$

Thus $\phi_2 \circ \phi_1$ is operation preserving.

Since $\phi_2 \circ \phi_1$ is one-to-one, onto, and operation preserving, $\phi_2 \circ \phi_1$ is an isomorphism of $\langle S_1, * \rangle$ and $\langle S_3, *'' \rangle$, thus $\langle S_1, * \rangle \simeq \langle S_3, *'' \rangle$. $\square$