MAT 260 LINEAR ALGEBRA
LECTURE 23

WING HONG TONY WONG

1.5 — Elementary matrices and a method for finding A^{-1}

Recall that the three elementary row operations of a matrix A are
1. Switch row i and row j;
2. Multiply row i by a nonzero constant k;
3. Replace row i by row $i + k \times$ row j.

In fact, they correspond to multiplying **elementary matrices** in front of A. To obtain these elementary matrices, we do the same operations on the identity matrix.

1. $$\begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 0 & & & & 1 & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & 1 & & & & 0 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix};$$

2. $$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & k & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix};$$

3. $$\begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 1 & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & k & & & & 1 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix}.$$

Matrices A and B are called **row equivalent** if we can obtain one from the other by elementary row operations.

Theorem 1. *Performing a row operation on matrix A is equivalent to multiplying the corresponding elementary matrix E in front of A.*

Date: Wednesday, March 22, 2023.

Hence, if A and B are row equivalent, then there are elementary matrices $E_1, E_2, \ldots, E_t$ such that $E_t \ldots E_2 E_1 A = B$.

Theorem 2. *Any elementary matrix E is invertible, and the inverse E^{-1} is also an elementary matrix.*

Theorem 3. *Let A be an $n \times n$ matrix. The following are equivalent.*
(i) A is invertible.
(ii) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(iii) $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$.
(iv) $A\mathbf{x} = \mathbf{b}$ is consistent (i.e. either has a unique solution or infinitely many solutions) for all $\mathbf{b}$.
(v) The reduced row echelon form of A is I_n.
(vi) A can be expressed as a product of elementary matrices, i.e. $A = E_1 E_2 \ldots E_t$.

Proof. $(i) \Rightarrow (ii)$: Assume that A is invertible. When we solve the equation $A\mathbf{x} = \mathbf{0}$, we can multiply A^{-1} to the front of both sides of the equation. This yields

$$\begin{aligned} A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{0} \\ (A^{-1}A)\mathbf{x} &= \mathbf{0} \\ I\mathbf{x} &= \mathbf{0} \\ \mathbf{x} &= \mathbf{0}. \end{aligned}$$

Hence, the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

$(ii) \Rightarrow (iii)$: Assume that the equation $A\mathbf{x} = \mathbf{0}$ has <u>only</u> the trivial solution. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two distinct solutions for the the equation $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2 \neq \mathbf{0}$, and

$$\begin{aligned} A\mathbf{x}_1 = A\mathbf{x}_2 &= \mathbf{b} \\ A\mathbf{x}_1 - A\mathbf{x}_2 &= \mathbf{0} \\ A(\mathbf{x}_1 - \mathbf{x}_2) &= \mathbf{0}. \end{aligned}$$

Hence, $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ is also a solution to the equation $A\mathbf{x} = \mathbf{0}$, contradicting that $\mathbf{x} = \mathbf{0}$ is the only solution to the equation $A\mathbf{x} = \mathbf{0}$.

$(iii) \Rightarrow (v)$: Assume that the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$. We can solve the equation $A\mathbf{x} = \mathbf{b}$ by performing Gauss-Jordan elimination on the augmented matrix $(A|\mathbf{b})$ to obtain $\mathrm{rref}(A|\mathbf{b})$, the reduced row echelon form of $(A|\mathbf{b})$. This is equivalent to multiplying elementary matrices $E_1, E_2, \ldots, E_t$ such that

$$E_t \cdots E_2 E_1 (A|\mathbf{b}) = \mathrm{rref}(A|\mathbf{b}).$$

Since the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, in $\mathrm{rref}(A|\mathbf{b})$, there is a leading 1 in every column except the last. Since A is a square matrix of size $n \times n$, we have $\mathrm{rref}(A|\mathbf{b}) = (I|\mathbf{c})$ for some $\mathbf{c}$. If we restrict the above matrix multiplication to the first n columns, then we have $E_t \cdots E_2 E_1 A = I = \mathrm{rref}(A)$.

$(v) \Rightarrow (vi)$: Assume $\text{rref}(A) = I$. Then there exists elementary matrices $F_1, F_2, \dots, F_t$ such that

$$\begin{aligned} F_t \cdots F_2 F_1 A &= I \\ (F_t \cdots F_2 F_1)^{-1}(F_t \cdots F_2 F_1 A) &= (F_t \cdots F_2 F_1)^{-1} I \\ \big((F_t \cdots F_2 F_1)^{-1}(F_t \cdots F_2 F_1)\big) A &= (F_t \cdots F_2 F_1)^{-1} \\ IA &= (F_t \cdots F_2 F_1)^{-1} \\ A &= F_1^{-1} F_2^{-1} \cdots F_t^{-1}. \end{aligned}$$

By Theorem **??**, $E_1 = F_1^{-1}$, $E_2 = F_2^{-1}$, ..., $E_t = F_t^{-1}$ are elementary matrices, and

$$A = E_1 E_2 \cdots E_t.$$

$(vi) \Rightarrow (i)$: Assume that $A = E_1 E_2 \cdots E_t$, where $E_1, E_2, \dots, E_t$ are elementary matrices. By Theorem **??**, $E_1, E_2, \dots, E_t$ are invertible, and

$$A^{-1} = E_t^{-1} \cdots E_2^{-1} E_1^{-1}.$$

$(i) \Rightarrow (iv)$: Assume that A is invertible. For all $\mathbf{b}$, when we solve the equation $A\mathbf{x} = \mathbf{b}$, we can multiply A^{-1} to the front of both sides of the equation. This yields

$$\begin{aligned} A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{b} \\ (A^{-1}A)\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b}. \end{aligned}$$

Hence, the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$.

$(iv) \Rightarrow (v)$: Assume the negation of the statement (v), i.e., the reduced row echelon form of A is not I. By Theorem 1 in Lecture Note 22, $\text{rref}(A)$ has a row of zeros. Let i be the smallest integer such that the i-th row of $\text{rref}(A)$ is a row of zeros.

Let $E_1, E_2, \dots, E_t$ be elementary matrices such that $E_t \cdots E_2 E_1 A = \text{rref}(A)$. Let

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

such that $c_i = 1$ and $c_1 = c_2 = \cdots = c_{i-1} = c_{i+1} = \cdots = c_n = 0$. Let $\mathbf{b} = (E_t \cdots E_2 E_1)^{-1}\mathbf{c}$.

Now, consider the equation $A\mathbf{x} = \mathbf{b}$. To solve this equation, we perform Gauss-Jordan elimination on the augmented matrix $(A|\mathbf{b})$. Notice that

$$E_t \cdots E_2 E_1 (A|\mathbf{b}) = (E_t \cdots E_2 E_1 A | E_t \cdots E_2 E_1 \mathbf{b}) = (\text{rref}(A)|\mathbf{c}),$$

which is a reduced row echelon form. Hence, $\text{rref}(A|\mathbf{b}) = (\text{rref}(A)|\mathbf{c})$, which has a leading 1 in the last column. In other words, the equation $A\mathbf{x} = \mathbf{b}$ has no solution, which is the negation of the statement (iv).

□

If A is invertible, we can do Gauss-Jordan elimination to obtain I_n as the reduced row echelon form, i.e. $E_t \dots E_2 E_1 A = I$, or $A^{-1} = E_t \dots E_2 E_1$. If we start with the matrix $B = (A|I)$ and do Gauss-Jordan elimination, we are again multiplying $E_t \dots E_2 E_1$ to the left of B, so $E_t \dots E_2 E_1 (A|I) = (I|A^{-1})$. This is how we can find the inverse of A in general.

Example 1. Find the inverse of $\begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \\ -2 & 2 & 3 \end{pmatrix}$.

Example 2. (1.5.38) Show that $A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{pmatrix}$ is row equivalent to $B = \begin{pmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$, and find a sequence of elementary row operations that produces B from A.

Example 3. (1.5.40) Show that $A = \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{pmatrix}$ is not invertible for any values of the entries.