

Section 6, p66 #1-4,45,46

In Exercises 1 through 4, find the quotient and remainder, according to the division algorithm, where n is divided by m .

1. $n = 42, m = 9$

$$42 = 4(9) + 6$$

2. $n = -42, m = 9$

$$-42 = -5(9) + 3$$

3. $n = -50, m = 8$

$$-50 = -7(8) + 6$$

4. $n = 50, m = 8$

$$50 = 6(8) + 2$$

Theory

45. Let r and s be positive integers. Show that $X = \{nr + ms : n, m \in \mathbb{Z}\}$ is a subgroup of \mathbb{Z}

A subgroup is closed under the operation of the group and closed under inverses.

Proof. Consider a , and $b \in X$, where $a = n_1r + m_1s$ and $b = n_2r + m_2s$.

$$\begin{aligned} a + b &= (n_1r + m_1s) + (n_2r + m_2s) \\ &= (n_1r + n_2r) + (m_1s + m_2s) \\ &= (n_1 + n_2)r + (m_1 + m_2)s \end{aligned}$$

Since $n_1 + n_2 \in \mathbb{Z}$ and $m_1 + m_2 \in \mathbb{Z}$, $a + b$ takes the form of elements in X . Now consider $a = nr + ms \in X$. We know a^{-1} exists since a is also in \mathbb{Z} .

$$\begin{aligned} a^{-1} &= (nr + ms)^{-1} \\ &= -(nr + ms) && \text{since we are using } \mathbb{Z} \text{ with addition} \\ &= (-n)r + (-m)s \end{aligned}$$

We know $-n$ and $-m \in \mathbb{Z}$, thus $a^{-1} \in X$. Thus X is a subgroup of \mathbb{Z} , $X < \mathbb{Z}$ □

46. Let a and b be elements of a group G . Show that if ab has finite order n , then ba also has order n

When we say a has order n , it means that in the cyclic group that $a^n = e$.

Proof. Assume $\langle ab \rangle$ has order n . This means that

$$\begin{aligned} (ab)^n &= e \\ a^n b^n &= e \end{aligned}$$

If we assume that $a^n = c$, for some $c \in G$, then it must follow that $b^n = c^{-1}$, since $a^n b^n = c c^{-1} = e$. Now let us consider the order of $\langle ba \rangle$.

$$\begin{aligned}(ba)^n &= b^n a^n \\ &= c^{-1} c && \text{since } a^n = c, b^n = c^{-1} \\ &= e\end{aligned}$$

Since $(ba)^n = e$, we can say that ba has order n . □
