Homework 8

4.3

4 Which of the following sets of vector in P_2 are linearly dependent?

a. $2 - x + 4x^2, \quad 3 + 6x + 2x^2, \quad 2 + 10x - 4x^2$

Work. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1(2-x+4x^2) + k_2(3+6x+2x^2) + k_3(2+10x-4x^2) = \mathbf{id}$. From this, we can get a linear system of equations, and an augmented matrix.

$$\begin{aligned} 2k_1 + 3k_2 + 2k_3 &= 0 \\ -xk_1 + 6xk_2 + 10xk_3 &= 0 \\ 4x^2k_1 + 2x^2k_2 - 4x^2k_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ -1 & 6 & 10 & 0 \\ 4 & 2 & -4 & 0 \end{array}\right] \xrightarrow[R_3+4R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} 0 & 15 & 22 & 0 \\ -1 & 6 & 10 & 0 \\ 0 & 26 & 36 & 0 \end{array}\right] \xrightarrow{R_3-2R_1} \left[\begin{array}{ccc|c} 0 & 15 & 22 & 0 \\ -1 & 6 & 10 & 0 \\ 0 & -4 & -8 & 0 \end{array}\right] \xrightarrow[-\frac{1}{4}R_3]{R_1+4R_3}$$

$$\left[\begin{array}{ccc|c} 0 & -1 & -10 & 0 \\ -1 & 6 & 10 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[R_2-6R_3]{R_1+R_3} \left[\begin{array}{ccc|c} 0 & 0 & -8 & 0 \\ -1 & 0 & -12 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[-R_2]{-\frac{1}{8}R_1} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 12 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-12R_1}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R_1\leftrightarrow R_2]{R_1\leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since the only solution is the trivial solution, therefore $\{2 - x + 4x^2, \quad 3 + 6x + 2x^2, \quad 2 + 10x - 4x^2\}$ are linearly independent. $\square$

c. $3 + x + x^2, \quad 2 - x + 5x^2, \quad 4 - 3x^2$

Work. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1(3+x+x^2) + k_2(2-x+5x^2) + k_3(4-3x^2) = \mathbf{id}$. From this, we can get a linear system of equations, and an augmented matrix.

$$\begin{aligned} 3k_1 + 2k_2 + 4k_3 &= 0 \\ xk_1 - xk_2 + 0xk_3 &= 0 \\ x^2k_1 - 5x^2k_2 - 3x^3k_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -5 & -3 & 0 \end{array}\right] \xrightarrow[R_3-R_2]{R_1-3R_2} \left[\begin{array}{ccc|c} 0 & 5 & 4 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -4 & -3 & 0 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -4 & -3 & 0 \end{array}\right] \xrightarrow{R_3+4R_1}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_2+R_1]{R_1-R_3} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1\leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since the only solution is the trivial solution, therefore $\{3 + x + x^2, \quad 2 - x + 5x^2, \quad 4 - 3x^2\}$ are linearly independent. $\square$

9 For which real values of λ do the following vectors form a linearly dependent set in $\mathbb{R}^3$?

$$v_1 = \left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right), \quad v_2 = \left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right), \quad v_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right)$$

13 Show that if $S = \{v_1, v_2, \dots, v_r\}$ is a linearly dependent set of vectors in a vector space V, and if $v_{r+1}, \dots, v_n$ are any vectors in V that are not in S, then $\{v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_n\}$ is also linearly dependent.

21 The functions $f_1(x) = x$ and $f_2(x) = \cos x$ are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wroński's test.

4.4

4 Which of the following form bases for P_2?

a. $1 - 3x + 2x^2, \quad 1 + x + 4x^2, \quad 1 - 7x$

answer

7 Find the coordinate vector of $\vec{w}$ relative to the basis $S = \{\vec{u}_1, \vec{u}_2\}$ for $\mathbb{R}^2$.

b. $\vec{u}_1 = (2, -4), \quad \vec{u}_2 = (3, 8); \quad \vec{w} = (1, 1)$

answer

c. $\vec{u}_1 = (1, 1), \quad \vec{u}_2 = (0, 2); \quad \vec{w} = (a, b)$

answer

12 Show that $\{A_1, A_2, A_3, A_4\}$ is a basis for $\mathcal{M}_{22}$, and express A as a linear combination of the basis vectors.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$$

4.5

3 Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.

$$1x_1 - 4x_2 + 3x_3 - 1x_4 = 0$$
$$2x_1 - 8x_2 + 6x_3 - 2x_4 = 0$$

7 Find bases for the following subspaces of $\mathbb{R}^3$.

a. The plane $3x - 2y + 5z = 0$.

answer

b. The plane $x - y = 0$.

answer

c. The lines $x = 2t, \quad y = -t, \quad z = 4t$.

answer

c. All the vectors of the form (a, b, c), where $b = a + c$.

answer

11

a. Show that the set W of all polynomials in P_2 such that $p(1) = 0$ is a subspace of P_2.

answer

b. Make a conjecture about the dimesion of W.

answer

c. Confirm you conjecture by finding a basis for W.

answer

18 Let S be a basis for an n-dimesional vector space V. show that if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ form a linearly independent set of vectors in V, then the coordinate vectors $(\vec{v}_1)_S, (\vec{v}_2)_S, \ldots, (\vec{v}_r)_S$ form a linearly independent set in $\mathbb{R}^n$, and conversely.