Problem 1

Let V be a vector space, and let $\vec{u}, \vec{v}, \vec{w} \in V$. Prove that if $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$ then $\vec{u} = \vec{v}$.

Proof. Consider $\vec{v}, \vec{u}, \vec{w} \in V$, and assume $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$.

$$\begin{aligned} \vec{u} \oplus \vec{w} &= \vec{v} \oplus \vec{w} && \text{Assertion} \\ -\vec{u} \oplus \vec{u} \oplus \vec{w} &= -\vec{u} \oplus \vec{v} \oplus \vec{w} && \text{Axiom 3} \\ \mathbf{id} \oplus \vec{w} &= -\vec{u} \oplus \vec{v} \oplus \vec{w} && \text{Axiom 5} \\ \vec{w} &= (-\vec{u} \oplus \vec{v}) \oplus \vec{w} && \text{Axiom 4} \\ \mathbf{id} &= -\vec{u} \oplus \vec{v} && \text{Axiom 4} \\ \vec{v} &= -(-\vec{u}) && \text{Axiom 5} \\ \therefore \vec{v} &= \vec{u} && \text{Axiom 5} \end{aligned}$$

□

Problem 2

Prove Theorem B.

Proof. Let $\vec{u} \in V$ and $k \in \mathbb{R}$. Consider $\mathbf{id} = \vec{u}$:

$$\begin{aligned} \mathbf{id} \oplus \vec{u} &= \vec{u} && \text{Axiom 4} \\ k \odot (\mathbf{id} \oplus \vec{u}) &= k \odot \vec{u} && \\ k \odot \mathbf{id} \oplus k \odot \vec{u} &= k \odot \vec{u} && \text{Axiom 7} \\ \therefore k \odot \mathbf{id} &= \mathbf{id} && \text{Axiom 4} \end{aligned}$$

□

Problem 3

Prove Theorem D. If $k \odot \vec{u} = \mathbf{id}$, then $k = 0$ and/or $u = \mathbf{id}$

Proof. Consider $k \odot \vec{u} = \mathbf{id}$ and $k \neq 0$. Since $\frac{1}{k} \neq \frac{1}{0}$, $\frac{1}{k}$ is well-defined.

$$\begin{aligned} k \odot \vec{u} &= \mathbf{id} && \text{Assertion} \\ \frac{1}{k} \odot k \odot \vec{u} &= \frac{1}{k} \odot \mathbf{id} && \\ (\frac{1}{k} \cdot k) \odot \vec{u} &= \frac{1}{k} \odot \mathbf{id} && \text{Axiom 9} \\ 1 \odot \vec{u} &= \mathbf{id} && \text{Thm B} \\ \vec{u} &= \mathbf{id} && \text{Axiom 10} \end{aligned}$$

Now consider $k \odot \vec{u} = \mathbf{id}$ and $\vec{u} \neq \mathbf{id}$. Since k is defined, $-k$ is also defined.

$$\begin{aligned} k \odot \vec{u} &= \mathbf{id} && \text{Assertion} \\ k \odot \vec{u} &= \mathbf{id} && \\ k \odot \vec{u} \oplus (-k) \odot \vec{u} &= \mathbf{id} \oplus (-k) \odot \vec{u} && \\ (k + (-k)) \odot \vec{u} &= (-k) \odot \vec{u} && \text{Axiom 8} \\ 0 \odot \vec{u} = (-k) \odot \vec{u} && \end{aligned}$$

Since $u \neq \mathbf{id}$,

$$0 = -k, \ \therefore k = 0$$

□

Problem 4

Prove that there does not exist a real vector space of size 2. Show that there cannot be a vector space of size 2.

Proof. Let $V = \{\vec{u}, \vec{v}\}$ be a vectorspace. That is, it satisfies all 10 Axioms.
Axiom 4 states that **id** exist, and is unique, therefore either $\vec{u} = \mathbf{id}$ or $\vec{v} = \mathbf{id}$. Both cannot be **id**, so therefore $\vec{u} \neq \vec{v}$
Without the loss of generality, let $\vec{u} = \mathbf{id}$.

$$\begin{aligned} \vec{u} \oplus \vec{v} &= \vec{v} && \text{Axiom 4} \\ \mathbf{id} \oplus \vec{v} &= \vec{v} \end{aligned}$$

Now consider Axiom 5: additive inverse exists for all $\vec{u} \in V$.
Consider $-\vec{v} \oplus \vec{v} = \mathbf{id}$. Since $\vec{v} \neq \mathbf{id}$, $-\vec{v} \neq \vec{v}$. Since there is only one other element in V, $-\vec{v} = \vec{u}$ must be true. Remember that $\vec{u} = \mathbf{id}$.

$$\vec{u} \oplus \vec{v} = \vec{u}$$

Therefore we have from Axiom 4 and 5:

$$\begin{aligned} \vec{u} \oplus \vec{v} &= \vec{u} \\ \vec{u} \oplus \vec{v} &= \vec{v} \\ \therefore \vec{u} &= \vec{v} \end{aligned}$$

However, this contradicts with our assertion that $\vec{u} \neq \vec{v}$.
$\therefore$ A vectorspace of size 2 cannot exist.

$\square$