Section 2.4

2.4.1 Proving Statements about odd and even integers with direct proofs

a. The sum of an odd and an even integer is odd

Proof. Let x be an even integer and y be an odd integer. $x = 2k$ for some integer k and $y = 2j + 1$ for some integer j.

$$\begin{aligned} x + y &= 2k + 2j + 1 \\ &= 2(k + j) + 1 \end{aligned}$$

Since k and j are integers, $k + j$ is an integer. Therefore, $2(k + j) + 1$ is an odd integer.
$\therefore$ the sum of an odd and an even integer is odd □

e. If x is an even integer and y is an odd integer, then $x^2 + y^2$ is odd

Proof. Since x is an even integer, $x = 2k$ for some $k \in \mathbb{Z}$. Since y is an odd integer, $x = 2j + 1$ for some $j \in \mathbb{Z}$.

$$\begin{aligned} x^2 + y^2 &= (2k)^2 + (2j + 1)^2 \\ &= 4k^2 + 4j^2 + 4j + 1 \\ &= 2(2k^2 + 2j^2 + 2j) + 1 \end{aligned}$$

Since $k, j \in \mathbb{Z}$, $2k^2 + 2j^2 + 2j \in \mathbb{Z}$. Therefore $2(2k^2 + 2j^2 + 2j) + 1$ is an odd integer.
$\therefore$ if x is an even integer and y is an odd integer, then $x^2 + y^2$ is odd □

2.4.2 Proving statements about rational numbers with direct proofs

c. If x and y are rational numbers, then $3x + 2y$ is also a rational number

Proof. Since $x \in \mathbb{Q}$, $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Since $y \in \mathbb{Q}$, $y = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ with $d \neq 0$.

$$\begin{aligned} 3x + 2y &= 3\frac{a}{b} + 2\frac{c}{d} \\ &= \frac{3ad}{bd} + \frac{2bc}{bd} \\ &= \frac{3ad + 2bc}{bd}, b \neq 0, d \neq 0 \end{aligned}$$

Since both $b \neq 0$ and $d \neq 0$, $bd \neq 0$. $3ad + 2bc \in \mathbb{Z}$ by properties of $\mathbb{Z}$. $bd \in \mathbb{Z}$ by properties of $\mathbb{Z}$. Therefore $\frac{3ad+2bc}{bd}$ takes the form of a rational number.
$\therefore$ if x and y are rational numbers, then $3x + 2y$ is also a rational number □

f. The average of two rational number is also rational

Proof. Let $x, y \in \mathbb{Q}$.
Since $x \in \mathbb{Q}$, $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Since $y \in \mathbb{Q}$, $y = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ with $d \neq 0$.

The average of two numbers is found by $\frac{x+y}{2}$.

$$\begin{aligned}
\frac{x+y}{2} &= \frac{\frac{a}{b}+\frac{c}{d}}{2} \\
&= \frac{\frac{a}{b}}{2} + \frac{\frac{c}{d}}{2} \\
&= \frac{a}{2b} + \frac{c}{2d} \\
&= \frac{ad}{2bd} + \frac{bc}{2bd} \\
&= \frac{ad+bc}{2bd}, b \neq 0, d \neq 0
\end{aligned}$$

Since both $b \neq 0$ and $d \neq 0$, $bd \neq 0$. $ad + bc \in \mathbb{Z}$ by properties of $\mathbb{Z}$. $2bd \in \mathbb{Z}$ by properties of $\mathbb{Z}$. Therefore $\frac{ad+bc}{2bd}$ takes the form of a rational number.
$\therefore$ The average of two rational number is also rational □

2.4.3 Proving algebraic statements with direct proofs

a. For any positive real numbers x and y, $(x + y)^2 \geq xy$

Proof. Let $x \in \mathbb{R}$ such that $x > 0$. Let $y \in \mathbb{R}$ such that $y > 0$. Since $x > 0$ and $y > 0$, $x^2 > 0$, $xy > 0$, and $y^2 > 0$. Therefore their sum is also greater than 0.

$$\begin{aligned}
x^2 + xy + y^2 &\geq 0 \\
x^2 + xy + y^2 &\geq 0 \\
x^2 + 2xy + y^2 &\geq xy \\
(x+y)^2 &\geq xy
\end{aligned}$$

$\therefore$ for any positive real numbers x and y, $(x + y)^2 \geq xy$ □

b. If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$

Proof. Let $x \in \mathbb{R}$ such that $x \leq 3$.

$$\begin{aligned}
&x - 3 \leq 0 \\
&(x-3)(x-4) \geq 0, \text{ sign changes since } x \leq 3 \text{ and therefore } x - 4 < 0 \\
&12 - 7x + x^2 \geq 0
\end{aligned}$$

$\therefore$ if x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$ □

c. If n is a real number and $n > 1$, then $n^2 > n$

Proof. Let $n \in \mathbb{R}$ such that $n > 1$.

$$\begin{aligned}
&n > 1 \\
&n \cdot n > 1 \cdot n, \text{ sign stays since } n \text{ is positive} \\
&n^2 > n
\end{aligned}$$

$\therefore$ if n is a real number and $n > 1$, then $n^2 > n$ □

d. If x is a real number such that $0 < x < 1$, then $\frac{1}{x(1-x)} \geq 4$

Proof. Let $x \in \mathbb{R}$ such that $0 < x < 1$.

$$\begin{aligned}
(x - \frac{1}{2})^2 &\geq 0, \text{ since } n^2 \geq 0 \text{ for } n \in \mathbb{R} \\
x^2 - x + \frac{1}{4} &\geq 0 \\
4x^2 - 4x + 1 &\geq 0 \\
1 &\geq 4x - 4x^2 \\
\frac{1}{x - x^2} &\geq 4, \text{ sign stays when } 0 < x < 1 \\
\frac{1}{x(1-x)} &\geq 4
\end{aligned}$$

$\therefore$ if x is a real number such that $0 < x < 1$, then $\frac{1}{x(1-x)} \geq 4$ $\square$