Test 3

Problem 1 Find the determinant of the 6×6 matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

Justify your answer.

Work. We can use row operations and their effects on the determinant to help us solve the 6×6 matrix.

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{vmatrix} \xlongequal{R_1 - R_2} \begin{vmatrix} 0 & -1 & -1 & -1 & -1 & -1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{vmatrix} \xlongequal{R_2 - R_3} \begin{vmatrix} 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{vmatrix} \xlongequal{R_3 - R_4}$$

$$\begin{vmatrix} 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{vmatrix} \xlongequal{R_4 - R_5} \begin{vmatrix} 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{vmatrix} \xlongequal{R_5 - R_6}$$

$$\begin{vmatrix} 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{vmatrix} \xlongequal[R_6 + R_5 + R_4 + R_3]{R_6 + R_2 + R_1} \begin{vmatrix} 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \xlongequal{R_1 - R_2}$$

$$\begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \xlongequal{R_2 - R_3} \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \xlongequal{R_3 - R_4}$$

$$\begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \xlongequal[R_6 + R_5]{R_4 - R_5} \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \xlongequal{R_6 \leftrightarrow R_5}$$

$$-\begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix} \xlongequal{R_5 \leftrightarrow R_4} \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix} \xlongequal{R_4 \leftrightarrow R_3}$$

$$-\begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix} \overset{R_3 \leftrightarrow R_2}{=\!=\!=} \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix} \overset{R_2 \leftrightarrow R_1}{=\!=\!=}$$

$$-\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix} = -(1)(-1)(-1)(-1)(-1)(-1) = (-1)^6 = 1$$

Therefore, the determinant of the 6×6 matrix is 1:

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{vmatrix} = 1.$$

□

Problem 2 Prove that

$$\det \begin{pmatrix} a & b & c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{pmatrix} = -(a-b)(b-c)(c-a)(a+b+c).$$

Proof. We can use row operations and their effects on the determinant to help us solve the 6×6 matrix.

$$\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix} \overset{R_2 + R_1}{=\!=\!=} \begin{vmatrix} a & b & c \\ a+b+c & a+b+c & a+b+c \\ a^2 & b^2 & c^2 \end{vmatrix} =$$

$$(a+b+c)\begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ a^2 & b^2 & c^2 \end{vmatrix} \overset{R_3 - R_1}{=\!=\!=} (a+b+c)\begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ a^2-1 & b^2-1 & c^2-1 \end{vmatrix} =$$

$$(a+b+c)\begin{vmatrix} a & 1 & a^2-1 \\ b & 1 & b^2-1 \\ c & 1 & c^2-1 \end{vmatrix} \underset{R_3 - R_1}{\overset{R_2 - R_1}{=\!=\!=}} (a+b+c)\begin{vmatrix} a & 1 & a^2-1 \\ b-a & 0 & b^2-1-a^2-1 \\ c-a & 0 & c^2-1-a^2-1 \end{vmatrix} =$$

$$(a+b+c)\begin{vmatrix} a & 1 & a^2-1 \\ b-a & 0 & b^2-a^2 \\ c-a & 0 & c^2-a^2 \end{vmatrix} = (a+b+c)\begin{vmatrix} a & 1 & a^2-1 \\ b-a & 0 & (b+a)(b-a) \\ c-a & 0 & (c+a)(c-a) \end{vmatrix} =$$

$$(b-a)(c-a)(a+b+c)\begin{vmatrix} a & 1 & a^2-1 \\ 1 & 0 & (b+a) \\ 1 & 0 & (c+a) \end{vmatrix} = (b-a)(c-a)(a+b+c)(1)(-1)[(c+a)-(b+a)] =$$

$$(b-a)(c-a)(a+b+c)(1)(-1)(c-b) = (b-a)(c-a)(a+b+c)(b-c) = -(a-b)(b-c)(c-a)(a+b+c)$$

Therefore, the determinant of the 3×3 matrix is $-(a-b)(b-c)(c-a)(a+b+c)$:

$$\det \begin{pmatrix} a & b & c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{pmatrix} = -(a-b)(b-c)(c-a)(a+b+c).$$

$\square$

Problem 3 Find the adjoint of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{pmatrix}.$$

Then use the adjoint to find A^{-1}.

Work. To find the adjoint of matrix A, we have to find the cofactors for each element of the matrix, and then transpose the matrix.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{pmatrix} \xrightarrow{\text{(cofactor)}} \begin{pmatrix} (0\cdot 3 - 1\cdot 2) & -(-1\cdot 3 - 3\cdot 2) & (-1\cdot 1 - 3\cdot 0) \\ -(1\cdot 3 - 1\cdot 1) & (2\cdot 3 - 3\cdot 1) & -(2\cdot 1 - 3\cdot 1) \\ (1\cdot 2 - 0\cdot 1) & -(2\cdot 2 - (-1)\cdot 1) & (2\cdot 0 - (-1)\cdot 1) \end{pmatrix} =$$
$$\begin{pmatrix} (-2) & -(-9) & (-1) \\ -(2) & (3) & -(-1) \\ (2) & -(5) & (1) \end{pmatrix} = \begin{pmatrix} -2 & 9 & -1 \\ -2 & 3 & 1 \\ 2 & -5 & 1 \end{pmatrix} \xrightarrow{\text{(transpose)}} \begin{pmatrix} -2 & -2 & 2 \\ 9 & 3 & -5 \\ -1 & 1 & 1 \end{pmatrix}$$

We can then use the equation $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$ to find A^{-1} using $\operatorname{adj}(A)$ and $\det(A)$.

$$\det(A) = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{vmatrix} \overset{R_1+R_2}{=\!=\!=} \begin{vmatrix} 1 & 1 & 3 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{vmatrix} \underset{R_2+R_1}{\overset{R_3-3R_1}{=\!=\!=}} \begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 5 \\ 0 & -2 & -6 \end{vmatrix} =$$
$$(1)(1)(1\cdot -6 - (-2)\cdot 5) = (-6+10) = 4$$

Therefore, $A^{-1} = \frac{1}{4} \operatorname{adj}(A)$:

$$A^{-1} = \frac{1}{4} \operatorname{adj}(A) = \frac{1}{4} \begin{pmatrix} -2 & -2 & 2 \\ 9 & 3 & -5 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{9}{4} & \frac{3}{4} & -\frac{5}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$\square$

Problem 6 Determine ALL values a and b such that

$$\left\{ \begin{pmatrix} a & b \\ b & b \end{pmatrix}, \begin{pmatrix} b & a \\ b & b \end{pmatrix}, \begin{pmatrix} b & b \\ a & b \end{pmatrix}, \begin{pmatrix} b & b \\ b & a \end{pmatrix} \right\}$$

is linearly independent.

Work. Let $k_1, k_2, k_3, k_4 \in \mathbb{R}$ such that

$$k_1 \begin{pmatrix} a & b \\ b & b \end{pmatrix} + k_2 \begin{pmatrix} b & a \\ b & b \end{pmatrix} + k_3 \begin{pmatrix} b & b \\ a & b \end{pmatrix} + k_4 \begin{pmatrix} b & b \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

From this, we can get a linear system of equations, and a matrix equation.

$$\begin{aligned}
ak_1 + bk_2 + bk_3 + bk_4 &= 0\\
bk_1 + ak_2 + bk_3 + bk_4 &= 0\\
bk_1 + bk_2 + ak_3 + bk_4 &= 0\\
bk_1 + bk_2 + bk_3 + ak_4 &= 0
\end{aligned}$$

$$\begin{pmatrix} a & b & b & b\\ b & a & b & b\\ b & b & a & b\\ b & b & b & a \end{pmatrix}\begin{pmatrix} k_1\\ k_2\\ k_3\\ k_4 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ 0 \end{pmatrix}$$

Consider the determinant of the square matrix.

$$\begin{vmatrix} a & b & b & b\\ b & a & b & b\\ b & b & a & b\\ b & b & b & a \end{vmatrix} \overset{R_1+R_2+R_3}{\underset{R_1+R_4}{=\!=\!=\!=\!=}} \begin{vmatrix} a+3b & a+3b & a+3b & a+3b\\ b & a & b & b\\ b & b & a & b\\ b & b & b & a \end{vmatrix} = (a+3b)\begin{vmatrix} 1 & 1 & 1 & 1\\ b & a & b & b\\ b & b & a & b\\ b & b & b & a \end{vmatrix} \overset{R_2-aR_1,R_4-R_1}{\underset{R_3-aR_1}{=\!=\!=\!=\!=}}$$

$$(a+3b)\begin{vmatrix} 1 & 1 & 1 & 1\\ b-a & 0 & b-a & b-a\\ b-a & b-a & 0 & b-a\\ b-a & b-a & b-a & 0 \end{vmatrix} = (a+3b)(b-a)^3\begin{vmatrix} 1 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 0 \end{vmatrix} \overset{R_2-R_1,R_4-R_1}{\underset{R_3-R_1}{=\!=\!=\!=\!=}}$$

$$(a+3b)(b-a)^3\begin{vmatrix} 1 & 1 & 1 & 1\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{vmatrix} = (a+3b)(b-a)^3(1)(-1)(-1)(-1) = -(a+3b)(b-a)^3$$

In order to ensure that $-(a+3b)(b-a)^3 \neq 0$, $a+3b \neq 0$ and $b-a \neq 0$. These conditions will ensure that the determinant of this matrix will not be zero. By Theorem 4 of Lecture Notes 32, as long as the determinant of a square matrix is not zero, the inverse of the matrix exists. Then, by the Big Theorem, if the coefficient matrix is invertible, then $A\vec{x} = \vec{0}$ has only the trivial solution. This means that there does not exist $k_1, k_2, k_3, k_3 \in \mathbb{R}$ such that

$$k_1\begin{pmatrix} a & b\\ b & b \end{pmatrix} + k_2\begin{pmatrix} b & a\\ b & b \end{pmatrix} + k_3\begin{pmatrix} b & b\\ a & b \end{pmatrix} + k_4\begin{pmatrix} b & b\\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 0\\ 0 & 0 \end{pmatrix}.$$

This means that

$$\left\{\begin{pmatrix} a & b\\ b & b \end{pmatrix}, \quad \begin{pmatrix} b & a\\ b & b \end{pmatrix}, \quad \begin{pmatrix} b & b\\ a & b \end{pmatrix}, \quad \begin{pmatrix} b & b\\ b & a \end{pmatrix}\right\}$$

are linearly independent when $a+3b \neq 0$ and $b-a \neq 0$. □

Problem 7 Prove or disprove the following statement:

$$\det(A^T A) \geq 0 \text{ for all } n \times n \text{ matrices } A.$$

Proof. We can use Theorem 1 and Theorem 6 from Lecture notes 32 to help proof the statement.

$$\begin{aligned}
&\text{Thm 1.} \quad \det A = \det A^T\\
&\text{Thm 6.} \quad \det AB = \det A \det B
\end{aligned}$$

Case 1: $\det A \geq 0$. Through Theorem 1, this implies that $\det A^T \geq 0$, since $\det A = \det A^T$.

$$\begin{aligned} \det A &\geq 0 \\ \det A^T \cdot \det A &\geq 0 \cdot 0 && \text{Thm 1.} \\ \det(A^T A) &\geq 0 && \text{Thm 6.} \end{aligned}$$

Case 2: $\det A < 0$. Through Theorem 1, this implies that $\det A^T < 0$, since $\det A = \det A^T$.

$$\begin{aligned} \det A &< 0 \\ \det A^T \cdot \det A &> 0 \cdot 0 && \text{Thm 1.} && \text{(inequality switches because of mult. by a negative)} \\ \det(A^T A) &> 0 && \text{Thm 6.} \\ \det(A^T A) &\geq 0 && && (> \text{ also implies } \geq) \end{aligned}$$

Since $\det A$ must either be greater than or equal to zero, or less than zero, and the statement is true for both possibilities, this means that

$$\det(A^T A) \geq 0 \text{ for all } n \times n \text{ matrices } A.$$

□

Problem 8 Let $V = \mathcal{M}_{nn}$, the vector space containing all $n \times n$ matrices. Prove or disprove:

$$W = \{A \in V : \det(A) = 0\} \text{ forms a subspace of } V.$$

Proof. To prove whether or not $W = \{A \in V : \det(A) = 0\}$ is a subspace of $V = \mathcal{M}_{nn}$, both Axioms 1 and Axioms 6 must hold.

Proof. Axiom 1: Closed under vector addition

Consider $A_{n\times n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$ and $B_{n\times n} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$. Since A and B are diagonal matrices, through Theorem 1 of Lecture Notes 31,

$$\det A = 1 \cdot 0 \cdot 0 \cdots 0 = 0, \text{ and}$$
$$\det B = 0 \cdot 1 \cdot 1 \cdots 1 = 0.$$

This means that both A and $B \in W$. Now consider $C = A + B$:

$$C_{n\times n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

C is also a diagonal matrix. Therefore, through Theorem 1 of Lecture Notes 31,

$$\det(A + B) = \det(C) = 1 \cdot 1 \cdot 1 \cdots 1 = 1 \neq 0$$

W is not closed under addition, meaning that Axiom 1 does not hold. □

Since Axiom 1 does not hold for W, it cannot possibly be a subspace of $V = \mathcal{M}_{nn}$. □