

## Homework 3

### Problem 1

Let  $V$  be a vector space, and let  $\vec{u}, \vec{v}, \vec{w} \in V$ . Prove that if  $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$  then  $\vec{u} = \vec{v}$ .

*Proof.* Consider  $\vec{v}, \vec{u}, \vec{w} \in V$ , and assume  $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$ .

$\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$	Assertion
$(\vec{u} \oplus \vec{w}) \oplus -\vec{w} = (\vec{v} \oplus \vec{w}) \oplus -\vec{w}$	Axiom 5 states $-\vec{w} \in V$
$\vec{u} \oplus (\vec{w} \oplus -\vec{w}) = \vec{v} \oplus (\vec{w} \oplus -\vec{w})$	Axiom 3
$\vec{u} \oplus \mathbf{id} = \vec{v} \oplus \mathbf{id}$	Def. of additive inverse
$\vec{u} = \vec{v}$	Def. of additive identity

$\therefore$  if  $\vec{u} \oplus \vec{w} = \vec{v} \oplus \vec{w}$  then  $\vec{u} = \vec{v}$  □

### Problem 2

Prove Theorem B.

*Proof.* Let  $k \in \mathbb{R}$ . Recall that Theorem A implies that  $0 \odot \mathbf{id} = \mathbf{id}$ :

$k \odot \mathbf{id} = k \odot (0 \odot \mathbf{id})$	Thm A
$= (k \cdot 0) \odot \mathbf{id}$	Axiom 9
$= 0 \odot \mathbf{id}$	
$= \mathbf{id}$	Thm A

$\therefore \forall k \in \mathbb{R}, k \odot \mathbf{id} = \mathbf{id}$  □

### Problem 3

Prove Theorem D. If  $k \odot \vec{u} = \mathbf{id}$ , then  $k = 0$  or  $\vec{u} = \mathbf{id}$

*Proof.* Let  $k \odot \vec{u} = \mathbf{id}$  and  $k \neq 0$ . Since  $\frac{1}{k} \neq \frac{1}{0}$ ,  $\frac{1}{k}$  is well-defined.

$k \odot \vec{u} = \mathbf{id}$	Assertion
$\frac{1}{k} \odot (k \odot \vec{u}) = \frac{1}{k} \odot (\mathbf{id})$	
$(\frac{1}{k} \cdot k) \odot \vec{u} = \frac{1}{k} \odot \mathbf{id}$	Axiom 9
$1 \odot \vec{u} = \mathbf{id}$	Thm B
$\vec{u} = \mathbf{id}$	Axiom 10

$\therefore$  if  $k \odot \vec{u} = \mathbf{id}$ , then  $k = 0$  or  $\vec{u} = \mathbf{id}$  □

### Problem 4

Prove that there does not exist a real vector space of size 2. Show that there cannot be a vector space of size 2.

*Proof.* Let  $V = \{\vec{u}, \vec{v}\}$  be a vectorspace where  $\vec{u} \neq \vec{v}$ . Axiom 4 states that  $\mathbf{id}$  exists, and is unique. Therefore either  $\vec{u} = \mathbf{id}$  or  $\vec{v} = \mathbf{id}$ .

Without the loss of generality, let  $\vec{u} = \mathbf{id}$ .

$\vec{u} \oplus \vec{v} = \vec{v}$	Axiom 4
$\mathbf{id} \oplus \vec{v} = \vec{v}$	

Now consider Axiom 5:  $-\vec{v} \in V$ . Note that  $\vec{v} \oplus \vec{u} = V$ , so we know  $-\vec{v} \neq \vec{u}$ , so therefore  $\vec{v} = -\vec{v}$ .

$$\begin{array}{ll} \vec{v} \oplus \vec{v} = \vec{u} = \mathbf{id} & \text{Def. of Additive Inverse} \\ 1 \odot \vec{v} \oplus 1 \odot \vec{v} = \vec{u} & \text{Axiom 10} \\ (1 + 1) \odot \vec{v} = \vec{u} & \text{Axiom 8} \\ 2 \odot \vec{v} = \vec{u} = \mathbf{id} & \end{array}$$

By Theorem A, either  $2 = 0$  or  $\vec{v} = \mathbf{id}$ . We know that  $2 \neq 0$  and  $\vec{v} \neq \mathbf{id}$  since  $\vec{u} = \mathbf{id}$ , a contradiction.  
 $\therefore$  A vectorspace of size 2 cannot exist.

□