

MAT 369 Introduction to Graph Theory

Peter Schaefer

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1 Introduction

1.1 Graphs and Graph Models

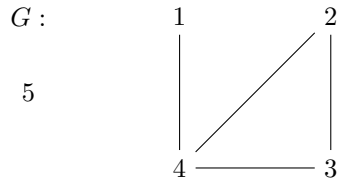
Graph Definition

A (simple) **graph** is an ordered pair (V, E) where

- V is a nonempty set of objects called "vertices"
- E is a set containing some two-subsets of V called "edges". E may be empty.

Graphs are often represented pictorially. For example consider

$$G = (V, E) \text{ where } V = \{1, 2, 3, 4, 5\} \text{ and } E = \{\{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$$



- Vertices 1 and 4 are **adjacent** because they are joined by an edge.
- Vertex 2 and edge $2 - 3$ are **incident**.
- Edges $2 - 3$ and $3 - 4$ are **adjacent**.

Order Definition

The **order** of a graph G is $|V(G)|$, or the number of vertices.

Size Definition

The **size** of a graph G is $|E(G)|$, or the number of edges.

The graph G from above has order 5 and size 4.

1.2 Connected Graphs

Subgraph Definition

Let G and H be graphs. H is a **subgraph** of G , notated as $H \subseteq G$, if

$$V(H) \subseteq V(G) \text{ and } E(H) \subseteq E(G).$$

Proper Subgraph Definition

H is a **proper subgraph** of G if $H \subseteq G$ and either

$$V(H) \subsetneq V(G) \text{ or } E(H) \subsetneq E(G).$$

Spanning Subgraph Definition

Graph H is a **spanning subgraph** if $H \subseteq G$ and $V(H) = V(G)$.

Induced Subgraph Definition

Graph H is a **induced subgraph** if $H \subseteq G$ and if

$$u, v \in V(H) \text{ and } u, v \in E(G) \implies u, v \in E(H).$$

Essentially, H contains all valid edges it can take from G . Notation for **induced subgraph** is

$$G[S], \text{ where } S \text{ is a set of vertices from } G.$$

Edge-induced Subgraph Definition

$G[X]$ is an **edge-induced subgraph** of G if $G[X]$ has edge set $X \subseteq E(G)$ and a vertex set of all vertices incident with at least one edge of X . Interesting fact: $G[E(G)]$ removes any isolated vertices.

More on Spanning and Induced Subgraphs

Let G be a graph with vertex v and edge e . Then,

- $G - e$ is the *spanning subgraph* of G whose edge set is $E(G) - \{e\}$.

This definition can be expanded to $G - X$ for $X \subseteq E(G)$.

- $G - v$ is the *induced subgraph* of G whose vertex set is $V(G) - \{v\}$ and edge set includes all edges of G except those incident with v .

This definition can be expanded to $G - U$ for $U \subseteq V(G)$.

Let G be a graph, $u, v \in V(G)$ and $e = uv \notin E(G)$. Then $G + e$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup \{e\}$. G is a *spanning subgraph* of $G + e$

Walk, Trail, Path, Circuit, and Cycle Definitions

Let $u, v \in V(G)$. A $u - v$ **walk** in G is a sequence of vertices

$$(u = v_0, v_1, \dots, v_k = v)$$

beginning with u , ending with v , and consecutive vertices are adjacent.

A **trail** is a walk in which *no edges* are repeated. A **path** is a walk in which *no vertices* are repeated. Every *path* is a *trail* is a *walk*.

A **circuit** is a closed trail of length ≥ 3 . A **cycle** is a circuit with no repeated vertices, except for the first and the last, which are the same. A **k-cycle** is a cycle of length k . Every *cycle* is a *circuit* is a *walk*.

Closed and Open Walks

A $u - v$ walk with $u = v$ is called a **closed** walk. A $u - v$ walk with $u \neq v$ is called a **open** walk.

Walk and Path Theorem

If G contains a $u - v$ walk of length ℓ , then G contains a $u - v$ path of length $\leq \ell$.

Connectivity Definition

A graph G is said to be **connected** if $\forall u, v \in V(G)$, G contains a $u - v$ path. If this is not true, i.e. $\exists u, v \in V(G)$ where there is no $u - v$ path, then G is said to be **disconnected**.

Component Definition

A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is a **component** of G . The number of components of a graph G is denoted by $k(G)$. A graph G is connected if and only if $k(G) = 1$. Additionally, a graph is the union of its components.

Components and Equivalence Relations Theorem

Define a relation R on the $V(G)$ so that uRv if G contains a $u - v$ walk. Then R is an equivalence relation.

Subtractive Connectivity Theorem (weak)

Let G be a graph of order ≥ 3 . If $\exists u, v \in V(G)$ such that $G - u$ and $G - v$ are connected, then G is connected.

Distance, Geodesic, Diameter, and Girth Definitions

The **distance** between vertices u and v , denoted as $d(u, v)$ or $d_G(u, v)$ is the smallest length of any $u - v$ path in G . If u and v are in different components, then $d(u, v)$ is undefined.

A $u - v$ path of shortest length $d(u, v)$ is called a **geodesic**. The **diameter** of a connected graph G , denoted as $\text{diam}(G)$, is the largest *geodesic* between any two vertices of G . The **girth** of a connected graph G is the length of the shortest cycle in G .

Subtractive Connectivity Theorem (strong)

Let G be a graph of order ≥ 3 . Then G is connected if and only if $\exists u, v \in V(G)$ such that $G - u$ and $G - v$ are connected.

1.3 Common Classes of Graphs

Name	Symbol	Order	Size
Path	P_n	n	$n - 1$
Cycle	C_n	$n \geq 3$	n
Complete	K_n	n	$\binom{n}{2}$
Complete Bipartite	$K_{s,t}$	$s + t$	$s \cdot t$

Bipartite Graph Definition

G is bipartite if $V(G)$ can be partitioned into partite sets U and W so that every edge joins a vertex of U and a vertex of W .

Odd Cycle and Bipartite-ness

G is bipartite if and only if G contains no odd cycles.

K-partite Definition

G is a **k-partite** graph if $V(G)$ can be partitioned into partite sets U_1, \dots, U_k so that every edge joins a vertex from U_i and a vertex of U_j where $i \neq j$.

Constructing New Graphs from Old Graphs**Disjoint Union**

For two graphs G and H , $G \cup H$ is defined as...

$$\begin{aligned} V(G \cup H) &= V(G) \cup V(H) \\ E(G \cup H) &= E(G) \cup E(H) \end{aligned}$$

Complement

For one graph G , \overline{G} is defined as...

$$\begin{aligned} V(\overline{G}) &= V(G) \\ E(\overline{G}) &= \{uv | u, v \in V(G), u \neq v, uv \notin E(G)\} \end{aligned}$$

Join

For two graphs, G and H , $G + H$ is defined as...

Start with $G \cup H$ and draw all edges join a vertex of G and a vertex of H

Cartesian Product

For two graphs, G and H , $G \times H$ is defined as...

$$V(G \times H) = \{(u, v) | u \in V(G) \text{ and } v \in V(H)\}$$

$$(u, v) - (x, y) \text{ if } u = x \text{ and } vy \in E(H) \vee v = y \text{ and } ux \in E(G)$$

A cartesian product between two graphs has the practical effect of duplicating one graph, and connecting the duplicates in the way of the other graph.

Complement Connectivity Theorem

If G is disconnected, then \overline{G} is connected.

1.4 Multigraphs and Digraphs**Multigraph Definition**

A **multigraph** is a graph where a pair of vertices may be joined by any finite number of edges.

- Multiple edges: OK
- Loops: NOT OK

Pseudograph Definition

A **pseudograph** is a *multigraph* where loops are allowed

- Multiple edges: OK
- Loops: OK

Digraph Definition

A **directed graph** is a graph where $E(G)$ is a set of ordered pairs (rather than sets) of distinct vertices called directed edges, or arcs.

Oriented Graph Definition

An **oriented graph** is a *digraph* in $\forall u, v \in V(G)$, (u, v) and (v, u) are not both edges.

2 Degrees

2.1 Degree of a Vertex

Vertex Degree Definition

The **degree** of a vertex v , denoted as $\deg v$ or $\deg_G v$, is the number of edges incident with v . If the $\deg v = 0$, then v is an **isolated vertex**. If $\deg v = 1$, then v is a **leaf**.

- $\delta(G) = \min\{\deg v \mid v \in V(G)\}$, the minimum degree of G
- $\Delta(G) = \max\{\deg v \mid v \in V(G)\}$, the maximum degree of G

For any graph G and $v \in V(G)$,

$$0 \leq \delta(G) \leq \deg v \leq \Delta(G) \leq n - 1.$$

Neighborhood of a Vertex Definition

The **neighborhood** of a vertex v , denoted as $N(v)$, is the set of all vertices adjacent to v . So $|N(v)| = \deg v$.

Handshaking Theorem

For a graph G of size m , the total degree of G $\sum_{v \in V(G)} \deg v = 2m$.

Handshaking Corollary

Every graph has an even number of odd degree vertices.

Sum Degree and Connectivity Theorem

Consider graph G of order n . If $\deg u + \deg v \geq n - 1$ for all non-adjacent $(u, v) \in V(G)$, then G is connected.

Sum Degree and Connectivity Corollary

If G has order n and $\delta(G) \geq \frac{n-1}{2}$, then G is connected.

2.2 Regular Graphs

Regular Graph Definition

Graph G is **regular** if every vertex has the same degree. Graph G is **r -regular** if every vertex has degree r .

Regular Graph Existence Theorem

Let $r, n \in \mathbb{Z}$ such that $0 \leq r \leq n - 1$. Then there exists an r -regular graph of order n if and only if at least one of r and n is even.

Harary Graph

An Harary Graph, denoted as $H_{r,n}$, is an r -regular graph of order n .

Induced Regular Subgraph Theorem

For every graph G , and every integer $r \geq \Delta(G)$, there exists an r -regular graph H , containing G as an induced subgraph.

2.3 Degree Sequences

Degree Sequence Definition

A **degree sequence** is a sequence of the degree of the vertices of a graph, typically, written in largest to smallest order.

Graphical Degree Sequence Definition

A finite sequence of non-negative integers is **graphical** if it is the degree sequence of some graph.

Graphical Degree Sequence Theorem

A non-increasing sequence $S : d_1, d_2, \dots, d_n$, where $n \geq 1$, of non-negative integers is graphical if and only if

$$S_1 : d_2 - 1, d_3 - 1, \dots, d_{d_1+1}, d_{d_1+2}, d_n$$

is graphical.

2.4 Graph and Matrices

Adjacency Matrix Definition

The **adjacency matrix** of G is the $n \times n$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise;} \end{cases}$$

The entry a_{ij} in A^n is the number of walks of length n from v_i to v_j .

3 Isomorphic Graphs

3.1 The Definition of Isomorphism

Graph Equality Definition

Two graphs are **equal**, denoted as $G = H$, if $V(G) = V(H)$ and $E(G) = E(H)$.

Graph Isomorphic Definition

Two (labels) graphs G and H are **isomorphic**, denoted as $G \cong H$, if they have the same structure, meaning there is a *bijection* $\phi : V(G) \rightarrow V(H)$ such that for $u, v \in V(G)$, $\phi(u)\phi(v) \in E(H)$ if and only if $uv \in E(G)$.

Isomorphic Degree Theorem

If $G \cong H$, with isomorphic $\phi : V(G) \rightarrow V(H)$, then $\deg_G u = \deg_H \phi(u)$.

Isomorphic Degree Corollary

If $G \cong H$, their degree sequences are equal.

Graph Invariants

- To prove $G \cong H$, find an isomorphism.
- To prove $G \not\cong H$, find a graph invariant, where G and H differ.

Graph Invariants

- Order and Size
- Degree Sequence
- Cycles
- Diameter
- k (number of components)
- k -partite-ness
- (Other things)

Adjacency and Non-adjacency under Isomorphism Theorem

$G \cong H$ if and only if $\overline{G} \cong \overline{H}$.

3.2 Isomorphism as a Relation

Equivalence Relations and Isomorphism Theorem

Isomorphism is an equivalence relation.

4 Trees

4.1 Cut Edges

Cut-edge and Bridge Definition

An edge e of graph G is a **cut-edge**, or **bridge**, if $G - e$ has more components than G .

Cut-edges and Cycles Theorem

An edge e of a graph G is a cut-edge if and only if e lies on *no* cycle in G .

4.2 Trees

Tree and Forest Definitions

A **tree** is an acyclic connected graph. A **forest** is an acyclic graph, where each component is a *tree*. A **Rooted tree** is a tree with a specific vertex designated as a root and drawn down.

Every edge of a tree is a cut-edge.

Unique Path in Trees Theorem

Graph G is a tree if and only if every 2 vertices are connected by a unique path.

Leaf Theorem

Every nontrivial tree has at least 2 leaves.

Autumn Theorem

If tree T has order $t \geq 1$, then $T - v$, where v is a leaf, is a tree of order $t - 1$.

Tree Size Theorem

Every tree of order n has size $n - 1$.

Forest Size Theorem

Every forest of order n with k components has size $n - k$.

Minimum Size of a Connected Graph Theorem

The size of every connected graph of order n is at least $n - 1$. Trees has minimal size among connected graphs of given order.

Tree Requirements Graph

Graph G of order n and size m . Then G is a tree if it satisfies any 2 of these properties:

1. G is connected
2. G acyclic
3. $m = n - 1$

Tree Isomorphic Subgraph Theorem

Let T be a tree of order k . Then for any graph G with $\delta(G) \geq k - 1$, T is isomorphic to a subgraph of G .

4.3 Minimum Spanning Tree

Spanning Tree Definition

Let G be a connected graph. A spanning subgraph of G that is a tree is called a **spanning tree**.

Spanning Tree Existence Theorem

Every connected graph contains a spanning tree.

Minimum Spanning Tree Definition

A **minimum spanning tree** is a spanning tree of minimum weight.

Algorithms For Constructing Minimum Spanning Trees

Kruskal's Algorithm

1. Pick an edge of minimum weight.
2. Repeat, never allowing the chosen edges to produce a cycle.
3. Stop once you have a spanning tree.

Prim's Algorithm

1. Choose any vertex $u \in V(G)$.
2. Let e be an edge of minimum weight incident with u .
3. Continue picking edges of minimum weight from the set of edges having exactly one of its vertices incident with an already selected edge.
4. Stop once you have a spanning tree.

4.4 Counting Labeled Trees

Cayley's Theorem

There are n^{n-2} distinct labeled trees on n vertices.

5 Connectivity

6 Traversability

7 Digraphs

8 Matchings and Factorization

9 Planarity

10 Coloring Graphs

11 Ramsey Numbers

12 Distance

13 Domination