Linear Algebra

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Contents

1   Brief Review

Commonly Used Sets

- $\mathbb{N}$: set of **natural numbers**

    could be *positive* integers

    could be *nonnegative* integers

- $\mathbb{Z}$: set of **integers**

- $\mathbb{Q}$: set of **rational numbers**

- $\mathbb{R}$: set of **real numbers**

Set Building

To denote sets too large to just list, we use **set builder** notation:

$$\{\text{candidate} : \text{condition}\}$$

Examples:

$$\{x \text{ is a fruit} : x \text{ is of yellow color}\}$$
$$\{x \text{ is a human being} : x \text{ is a president of the U.S.}\}$$
$$\{x \text{ is a city} : x \text{ is a capitol of a country}\}$$

Other Notations

- $\forall$: for all

- $\exists$: there exists

- s.t.: such that

- $\rightarrow\leftarrow$: contradiction

- WTS: want to show

2   Real Vector Spaces

A **real vector space** is simply a *nonempty set* that satisfies 10 properties called **10 axioms of a real vector space**.

- $\vec{v} \in$ vector space V can be *anything*

- **Never** assume that an element $\vec{v} \in V$ is an ordered pair

Addition

- denoted by $\oplus$

- simply a map

$$\oplus : V \times V \to V$$

Example of a definition of $\oplus$ for $V = \{\text{apple}, \text{orange}, \text{banana}\}$:

$\oplus$	apple	orange	banana
apple	banana	**banana**	apple
orange	orange	apple	banana
banana	banana	orange	orange

$$\oplus(\text{apple}, \text{orange}) = \text{banana} = \text{apple} \oplus \text{orange}$$

Scalar Multiplication

- denoted by $\odot$

- simply a map

- *must* be $r \times \vec{v}$ for $r \in \mathbb{R}, \vec{v} \in V$

$$\odot : \mathbb{R} \times V \to V$$

Example of a definition of $\odot$ for $V = \{\text{apple}, \text{orange}, \text{banana}\}$:

$$k \odot \text{apple} = \text{orange}, \forall k \in \mathbb{R}$$

$$k \odot \text{orange} = \begin{cases} \text{orange}, & \text{if } k \leq 2, \\ \text{banana}, & \text{if } k > 2, \end{cases}$$

$$k \odot \text{banana} = \begin{cases} \text{banana}, & \text{if } k < -5\sqrt{2}, \\ \text{apple}, & \text{if } -5\sqrt{2} \leq k < 1.2, \\ \text{banana}, & \text{if } k = 1.2, \\ \text{orange}, & \text{if } k > 2, \end{cases}$$

$$\odot(3, \text{orange}) = \text{banana} = 3 \odot \text{orange}$$

10 Good Properties of Addition and Scalar Multiplication

1. **Closed Under Addition** $\forall \ \vec{v}, \vec{u} \in V$,
$$\vec{u} \oplus \vec{v} \in V$$

2. **Commutativity Under Addition** $\forall \ \vec{v}, \vec{u} \in V$,
$$\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$$

3. **Associativity Under Addition** $\forall\ \vec{v}, \vec{u}, \vec{w} \in V$,

$$\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$$

4. **Additive Identity Exists** $\exists\ \vec{u}\ \forall\ \vec{v} \in V$,

$$\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u} = \vec{v}$$

$\vec{u}$ is called the additive identity, **id**

5. **Additive Inverse Always Exists** $\forall\ \vec{v}\ \exists\ \vec{w} \in V$,

$$\vec{v} \oplus \vec{w} = \vec{w} \oplus \vec{v} = \mathbf{id}$$

$\vec{w} = -\vec{v}$ and is pronounced as *bar-$\vec{v}$*

6. **Closed Under Scalar Multiplication** $\forall\ k \in \mathbb{R}, \vec{v} \in V$,

$$k \odot \vec{v} \in V$$

7. **Distributivity Over** $\oplus\ \forall\ k \in \mathbb{R}, \vec{u}, \vec{v} \in V$,

$$k \odot (\vec{u} \oplus \vec{v}) = k \odot \vec{u} \oplus k \odot \vec{v}$$

8. **Distributivity Over** $+\ \forall\ k, \ell \in \mathbb{R}, \vec{v} \in V$,

$$(k + \ell) \oplus \vec{v} = k \odot \vec{v} \oplus \ell \odot \vec{v}$$

9. **Associativity Over Scalar Multiplication** $\forall\ k, \ell \in \mathbb{R}, \vec{v} \in V$,

$$(k \cdot \ell) \odot \vec{v} = k \odot (\ell \odot \vec{v})$$

10. **1 Fixes Every Element In V By** $\odot\ \forall\ \vec{v} \in V$,

$$1 \odot \vec{v} = \vec{v}$$

Tips To Remember The 10 Axioms

- first 5 axioms deal with addition ONLY, the next 5 axioms involve scalar multiplication
- first of the 5 axioms for addition and scalar multiplication deal with closure
- axioms 4 and 5 are about the existence of something
- axioms 8 and 9 are the only axioms that involve 2 real numbers

Verifying the 10 Axioms

a. Axioms (1) and (6): proof of closure

b. Axioms (4) and (5): show existence

c. Axioms (2), (3), (7), (8), (9), (10): proof for all elements

Standard Addition and Scalar Multiplication for $\mathbb{R}^n$

$\forall\ \vec{u} = (u_1, u_2, \ldots, u_n), \vec{v} = (v_1, v_2, \ldots, v_n) \in V$ and $\forall\ k \in \mathbb{R}$,

$$\vec{u} \oplus \vec{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$
$$k \odot \vec{u} = (ku_1, ku_2, \ldots, ku_n)$$