MAT 260 LINEAR ALGEBRA
LECTURE 28

WING HONG TONY WONG

1.6 — More on linear systems and invertible matrices

Theorem 1 (1.6.23). *Let $A\mathbf{x} = \mathbf{b}$ be a consistent system of linear equations, and let $\mathbf{x}_1$ be any fixed solution. Show that $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_0$ for some solution $\mathbf{x}_0$ to the equation $A\mathbf{x}_0 = \mathbf{0}$.*

If we are solving multiple equations $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$, ..., $A\mathbf{x} = \mathbf{b}_k$ at the same time, then we can row reduce $(A|\mathbf{b}_1|\mathbf{b}_2|\cdots|\mathbf{b}_k)$.

Example 2. Solve the systems

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 + 8x_3 &= 9 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 + 8x_3 &= -6 \end{aligned} \quad .$$

Theorem 3. *If AB is an invertible matrix, and A and B are square matrices, then both A and B are invertible.*

Example 4. What are the conditions on b_1, b_2, b_3 such that

$$\begin{aligned} x_1 + x_2 + 2x_3 &= b_1 \\ x_1 + x_3 &= b_2 \\ 2x_1 + x_2 + 3x_3 &= b_3 \end{aligned}$$

is consistent? How about

$$\begin{aligned} x_1 + x_2 + 2x_3 &= b_1 \\ x_1 + x_3 &= b_2 \ ? \\ x_2 + 3x_3 &= b_3 \end{aligned}$$

1.7 — Diagonal, triangular, and symmetric matrices

Example 5. Once again, recall the symbols for some subspaces of M_{nn}, the set of all $n \times n$ **square** matrices.

- Sets of $n \times n$ **upper-triangular** matrices: $\mathcal{U}_{nn} = \{A \in M_{nn} : a_{ij} = 0 \text{ for all } i > j\}$.
- Sets of $n \times n$ **lower-triangular** matrices: $\mathcal{L}_{nn} = \{A \in M_{nn} : a_{ij} = 0 \text{ for all } i < j\}$.
- Sets of $n \times n$ **diagonal** matrices: $\mathcal{D}_{nn} = \{A \in M_{nn} : a_{ij} = 0 \text{ for all } i \neq j\}$.
- Sets of $n \times n$ **symmetric** matrices: $\{A \in M_{nn} : a_{ij} = a_{ji} \text{ for all } i, j\}$, or $\{A \in M_{nn} : A^\top = A\}$.

Date: Monday, April 3, 2023.

- Sets of $n \times n$ **skew-symmetric** matrices: $\{A \in M_{nn} : a_{ij} = -a_{ji} \text{ for all } i, j\}$, or $\{A \in M_{nn} : A^\top = -A\}$.

Here are some properties of these matrices:

- If $A \in \mathcal{U}_{nn}$ (respectively $A \in \mathcal{L}_{nn}$ or $A \in \mathcal{D}_{nn}$), then A is invertible if and only if all diagonal entries are nonzero. Besides, if A is invertible, then $A^{-1} \in \mathcal{U}_{nn}$ (respectively $A^{-1} \in \mathcal{L}_{nn}$ or $A^{-1} \in \mathcal{D}_{nn}$), and all the diagonal entries in A^{-1} are reciprocals of those in A.

- If $A, B \in \mathcal{U}_{nn}$ (respectively $A, B \in \mathcal{L}_{nn}$ or $A, B \in \mathcal{D}_{nn}$), then $AB \in \mathcal{U}_{nn}$ (respectively $AB \in \mathcal{L}_{nn}$ or $AB \in \mathcal{D}_{nn}$). Also, the ii-th entry of AB is $a_{ii}b_{ii}$. Consequently, the ii-th entry of A^n is a_{ii}^n.

- If $A \in \mathcal{U}_{nn}$ (respectively $A \in \mathcal{L}_{nn}$), then $A^\top \in \mathcal{L}_{nn}$ (respectively $A^\top \in \mathcal{U}_{nn}$).

- If $A \in \mathcal{D}_{nn}$, then the i-th row of AB is the scalar product between a_{ii} and the i-th row of B, and the j-th column of BA is the scalar product between a_{ii} and the j-th column of B.

- $\mathcal{U}_{nn} \cap \mathcal{L}_{nn} = \mathcal{D}_{nn}$.

- If A and B are symmetric matrices, then $A^\top$, $A+B$, $A-B$, and kA are all symmetric.

- For all $m \times n$ matrices A, $A^\top A$ and $AA^\top$ are both symmetric matrices.

- For all $n \times n$ matrices A, $A + A^\top$ is a symmetric matrix, while $A - A^\top$ is a skew-symmetric matrix.

Theorem 6. *If A and B are symmetric matrices, then AB is symmetric if and only if A and B commute.*

Theorem 7. *If A is an invertible symmetric matrix, then A^{-1} is also symmetric.*