

## Graded Assignment #1

**1** [2 points each] Which of the following are binary operations on the given sets? If it is not an operation, explain why.

(a).  $S = \mathbb{R}^+$  with  $a * b = a \ln b$

A binary operation must be uniquely defined and closed.

*Proof.* Consider  $a = 1$  and  $b = \frac{1}{e}$ . Both  $a$  and  $b \in \mathbb{R}^+$ , but

$$a * b = a \ln b = 1 \ln \frac{1}{e} = 1 \cdot -1 = -1 \notin \mathbb{R}^+.$$

Thus  $S$  is not closed under  $*$ , and  $*$  cannot be a binary operation. □

(b).  $S = \mathbb{R}$  where  $a * b$  is the root of the equation  $x^2 - a^2b^2 = 0$

A binary operation must be uniquely defined and closed.

*Proof.* Consider  $a = 2$  and  $b = 1$ .

$$\begin{aligned} a * b &= x^2 - a^2b^2 = 0 \\ x^2 - 2^21^2 &= \\ x^2 - 4 &= 0 \\ (x - 2)(x + 2) &= 0 \\ x &= \pm 2 \end{aligned}$$

Since there are two solutions,  $S$  is not uniquely defined under  $*$ , and  $*$  cannot be a binary operation. □

**2** [2 points each] Consider the binary operation  $*$  defined on  $\mathbb{R}^+$  by  $a * b = \frac{ab}{a+b+1}$

(a). Is  $*$  commutative? Explain.

$*$  is commutative.

*Proof.* Consider  $a * b$  and  $b * a$  for  $a, b \in \mathbb{R}^+$ :

$$a * b = \frac{ab}{a+b+1} = \frac{ba}{b+a+1} = b * a$$

Since  $a * b = b * a$  for all  $a, b \in \mathbb{R}^+$ ,  $*$  is commutative. □

(b). Is  $*$  associative? Explain.

$*$  is associative.

*Proof.* Consider  $a, b, c \in \mathbb{R}^+$ :

$$\begin{aligned}(a * b) * c &= \frac{ab}{a+b+1} * c = \frac{\frac{ab}{a+b+1}c}{\frac{ab}{a+b+1} + c + 1} = \frac{abc}{(a+b+1)(\frac{ab}{a+b+1} + c + 1)} \\ &= \frac{abc}{ab + ac + bc + a + b + c + 1} \\ a * (b * c) &= a * \frac{bc}{b+c+1} = \frac{a\frac{bc}{b+c+1}}{a + \frac{bc}{b+c+1} + 1} = \frac{abc}{(b+c+1)(a + \frac{bc}{b+c+1} + 1)} \\ &= \frac{abc}{ab + ac + bc + a + b + c + 1}\end{aligned}$$

Thus  $(a * b) * c = a * (b * c)$ , and  $*$  is associative. □

**3** [3 points] Let  $E$  denote the set of all even integers. Prove that  $\langle \mathbb{Z}, + \rangle \simeq \langle E, + \rangle$ .

*Proof.* An isomorphism must be one-to-one, onto, and operation preserving. Consider  $\phi : \mathbb{Z} \rightarrow E$  such that  $\phi(n) = 2n$ .

1. One-to-one: Assume  $\phi(n_1) = \phi(n_2)$  for  $n_1, n_2 \in \mathbb{Z}$ .

$$\begin{aligned}\phi(n_1) &= \phi(n_2) \\ 2n_1 &= 2n_2 \\ n_1 &= n_2\end{aligned}$$

Thus  $\phi$  is one-to-one.

2. Onto: Let  $m \in E$ . Let us find  $n \in \mathbb{Z}$  such that  $m = \phi(n)$ . Since  $m$  is an even integer, it can be represented as  $m = 2k$ , where  $k \in \mathbb{Z}$ .

$$\begin{aligned}m &= \phi(n) \\ 2k &= 2n \\ k &= n\end{aligned}$$

Choose  $n = k$ . Thus  $\phi$  is onto.

3. Operation Preserving: Need to show that  $\phi(n + m) = \phi(n) + \phi(m)$

$$\begin{aligned}\phi(n + m) &= 2(n + m) \\ &= 2n + 2m \\ &= \phi(n) + \phi(m)\end{aligned}$$

Thus  $\phi$  is operation preserving.

Since  $\phi$  is one-to-one, onto, and operation preserving, thus  $\phi$  is an isomorphism of  $\langle \mathbb{Z}, + \rangle$  and  $\langle E, + \rangle$ , and  $\langle \mathbb{Z}, + \rangle \simeq \langle E, + \rangle$ . □

**4** [3 points each] Prove that isomorphism is an equivalence relation among binary structures. To do this, you need to prove the following three properties:

- (a). Reflexive: Every binary structure is isomorphic to itself. Hint: let  $\langle S, * \rangle$  be a binary structure and define  $\phi : S \rightarrow S$  by  $\phi(x) = x$ . Prove that  $\phi$  is an isomorphism.

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An isomorphism must be one-to-one, onto, and operation preserving.

*Proof.* Consider  $\phi : S \rightarrow S$  such that  $\phi(x) = x$ .

1. One-to-one: Assume  $\phi(x_1) = \phi(x_2)$  for some  $x_1, x_2 \in S$ .

$$\begin{aligned}\phi(x_1) &= \phi(x_2) \\ x_1 &= x_2\end{aligned}$$

Thus  $\phi$  is one-to-one.

2. Onto: Let  $y \in S$ . Let us find  $x \in S$  such that  $y = \phi(x)$ .

$$\begin{aligned}y &= \phi(x) \\ y &= x\end{aligned}$$

Choose  $x = y$ . This  $\phi$  is onto.

3. Operation Preserving: Need to show that  $\phi(x * y) = \phi(x) * \phi(y)$ .

$$\begin{aligned}\phi(x * y) &= x * y \\ &= \phi(x) * \phi(y)\end{aligned}$$

Thus  $\phi$  is operation preserving.

Since  $\phi$  is one-to-one, onto, and operation preserving,  $\phi$  is an isomorphism of  $\langle S, * \rangle$  and  $\langle S, * \rangle$ , and  $\langle S, * \rangle \simeq \langle S, * \rangle$ .  $\square$

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- (b). Symmetric: For binary structures  $\langle S_1, * \rangle$  and  $\langle S_2, *' \rangle$ , if  $S_1 \simeq S_2$  then  $S_2 \simeq S_1$ . Hint: assume  $\phi : S_1 \rightarrow S_2$  is an isomorphism and prove that  $\phi^{-1} : S_2 \rightarrow S_1$  is also an isomorphism.

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An isomorphism must be one-to-one, onto, and operation preserving.

*Proof.* Consider  $\phi^{-1} : S_2 \rightarrow S_1$ . Such an operation must exist, since  $\phi$  is bijective by definition.

1. One-to-one: Assume  $\phi^{-1}(x_1) = \phi^{-1}(x_2)$  for  $x_1, x_2 \in S_2$ .

$$\begin{aligned}\phi^{-1}(x_1) &= \phi^{-1}(x_2) \\ \phi(\phi^{-1}(x_1)) &= \phi(\phi^{-1}(x_2)) \\ x_1 &= x_2\end{aligned}$$

Thus  $\phi^{-1}$  is one-to-one.

2. Onto: Let  $y \in S_1$ . Let us find  $x \in S_2$  such that  $y = \phi^{-1}(x)$ .

$$\begin{aligned}y &= \phi^{-1}(x) \\ \phi(y) &= \phi(\phi^{-1}(x)) \\ \phi(y) &= x\end{aligned}$$

Choose  $x = \phi(y)$ . Thus  $\phi^{-1}$  is onto.

3. Operation Preserving: Need to show that  $\phi^{-1}(x *' y) = \phi^{-1}(x) * \phi^{-1}(y)$ . But first, consider  $a, b \in S_1$ .

$$\begin{aligned}\phi(a * b) &= \phi(a) *' \phi(b) && \text{since } \phi \text{ is an isomorphism and thus operation preserving} \\ \phi^{-1}\phi(a * b) &= \phi^{-1}(\phi(a) *' \phi(b)) \\ a * b &= \phi^{-1}(\phi(a) *' \phi(b))\end{aligned}$$

We can use the equation  $a * b = \phi^{-1}(\phi(a) *' \phi(b))$  to help us show that  $\phi^{-1}(x *' y) = \phi^{-1}(x) * \phi^{-1}(y)$ .

$$\begin{aligned}\phi^{-1}(x) * \phi^{-1}(y) &= \phi^{-1}(\phi(\phi^{-1}(x)) *' \phi(\phi^{-1}(y))) \\ &= \phi^{-1}(x *' y)\end{aligned}$$

Thus  $\phi^{-1}$  is operation preserving.

Since  $\phi^{-1}$  is one-to-one, onto, and operation preserving,  $\phi^{-1}$  is an isomorphism of  $\langle S_2, *' \rangle$  and  $\langle S_1, * \rangle$ , and  $\langle S_2, *' \rangle \simeq \langle S_1, * \rangle$ .  $\square$

- (c). Transitive: For binary structures  $\langle S_1, * \rangle$ ,  $\langle S_2, *' \rangle$ , and  $\langle S_3, *'' \rangle$ , if  $S_1 \simeq S_2$  and  $S_2 \simeq S_3$  then  $S_1 \simeq S_3$ . Hind: assume  $\phi_1 : S_1 \rightarrow S_2$  and  $\phi_2 : S_2 \rightarrow S_3$  are isomorphisms and prove that  $\phi_2 \circ \phi_1 : S_1 \rightarrow S_3$  is also an isomorphism.

An isomorphism must be one-to-one, onto, and operation preserving.

*Proof.* Consider  $\phi_2 \circ \phi_1 : S_1 \rightarrow S_3$ .

1. One-to-one: Assume  $\phi_2 \circ \phi_1(x_1) = \phi_2 \circ \phi_1(x_2)$  for some  $x_1, x_2 \in S_3$ .

$$\begin{aligned}\phi_2 \circ \phi_1(x_1) &= \phi_2 \circ \phi_1(x_2) \\ \phi_2(\phi_1(x_1)) &= \phi_2(\phi_1(x_2)) \\ \phi_1(x_1) &= \phi_1(x_2) && \text{since } \phi_2 \text{ is an isomorphism and thus one-to-one} \\ x_1 &= x_2 && \text{since } \phi_1 \text{ is an isomorphism and thus one-to-one}\end{aligned}$$

Thus  $\phi_2 \circ \phi_1$  is one-to-one.

2. Onto: Let  $y \in S_3$ . Let us find  $x \in S_1$  such that  $y = \phi_2 \circ \phi_1(x)$ .

$$\begin{aligned}y &= \phi_2 \circ \phi_1(x) \\ \phi_2^{-1}(y) &= \phi_2^{-1} \circ \phi_2 \circ \phi_1(x) && \text{since } \phi_2 \text{ is bijective, and has a well-defined inverse } \phi_2^{-1} \\ \phi_1^{-1} \circ \phi_2^{-1}(y) &= \phi_1^{-1} \circ \phi_2^{-1} \circ \phi_2 \circ \phi_1(x) && \text{since } \phi_1 \text{ is bijective, and has a well-defined inverse } \phi_1^{-1} \\ \phi_1^{-1} \circ \phi_2^{-1}(y) &= \phi_1^{-1} \circ \phi_1(x) \\ \phi_1^{-1} \circ \phi_2^{-1}(y) &= x\end{aligned}$$

Choose  $x = \phi_1^{-1} \circ \phi_2^{-1}(y)$ . Thus  $\phi_2 \circ \phi_1$  is onto.

3. Operation Preserving: Need to show that  $\phi_2 \circ \phi_1(x * y) = \phi_2 \circ \phi_1(x) *'' \phi_2 \circ \phi_1(y)$ . Let us use the following equations to help with this. Let  $a, b \in S_1$  and  $\alpha, \beta \in S_2$

$$\begin{aligned}\phi_1(a * b) &= \phi_1(a) *' \phi_1(b) && \text{since } \phi_1 \text{ is an isomorphism and operation preserving} \\ \phi_2(\alpha *' \beta) &= \phi_2(\alpha) *'' \phi_2(\beta) && \text{since } \phi_2 \text{ is an isomorphism and operation preserving}\end{aligned}$$

We can compose these two equations to help us show  $\phi_2 \circ \phi_1(x * y) = \phi_2 \circ \phi_1(x) *'' \phi_2 \circ \phi_1(y)$ :

$$\begin{aligned}\phi_2 \circ \phi_1(x * y) &= \phi_2(\phi_1(x) *' \phi_1(y)) \\ &= \phi_2(\phi_1(x)) *'' \phi_2(\phi_1(y)) \\ &= \phi_2 \circ \phi_1(x) *'' \phi_2 \circ \phi_1(y)\end{aligned}$$

Thus  $\phi_2 \circ \phi_1$  is operation preserving.

Since  $\phi_2 \circ \phi_1$  is one-to-one, onto, and operation preserving,  $\phi_2 \circ \phi_1$  is an isomorphism of  $\langle S_1, * \rangle$  and  $\langle S_3, *'' \rangle$ , thus  $\langle S_1, * \rangle \simeq \langle S_3, *'' \rangle$ .  $\square$

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