Homework 8

4.3

4 Which of the following sets of vector in P_2 are linearly dependent?

a. $2 - x + 4x^2, \quad 3 + 6x + 2x^2, \quad 2 + 10x - 4x^2$

Work. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1(2 - x + 4x^2) + k_2(3 + 6x + 2x^2) + k_3(2 + 10x - 4x^2) = \textbf{id}$. From this, we can get a linear system of equations, and an augmented matrix.

$$\begin{aligned} 2k_1 + 3k_2 + 2k_3 &= 0 \\ -xk_1 + 6xk_2 + 10xk_3 &= 0 \\ 4x^2k_1 + 2x^2k_2 - 4x^2k_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ -1 & 6 & 10 & 0 \\ 4 & 2 & -4 & 0 \end{array}\right] \xrightarrow[R_3+4R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} 0 & 15 & 22 & 0 \\ -1 & 6 & 10 & 0 \\ 0 & 26 & 36 & 0 \end{array}\right] \xrightarrow{R_3-2R_1} \left[\begin{array}{ccc|c} 0 & 15 & 22 & 0 \\ -1 & 6 & 10 & 0 \\ 0 & -4 & -8 & 0 \end{array}\right] \xrightarrow[-\frac{1}{4}R_3]{R_1+4R_3}$$
$$\left[\begin{array}{ccc|c} 0 & -1 & -10 & 0 \\ -1 & 6 & 10 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[R_2-6R_3]{R_1+R_3} \left[\begin{array}{ccc|c} 0 & 0 & -8 & 0 \\ -1 & 0 & -12 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[-R_2]{-\frac{1}{8}R_1} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 12 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-12R_1}$$
$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R_1\leftrightarrow R_2]{R_1\leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since the only solution is the trivial solution, therefore $\{2 - x + 4x^2, \quad 3 + 6x + 2x^2, \quad 2 + 10x - 4x^2\}$ are linearly independent. □

c. $3 + x + x^2, \quad 2 - x + 5x^2, \quad 4 - 3x^2$

Work. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1(3 + x + x^2) + k_2(2 - x + 5x^2) + k_3(4 - 3x^2) = \textbf{id}$. From this, we can get a linear system of equations, and an augmented matrix.

$$\begin{aligned} 3k_1 + 2k_2 + 4k_3 &= 0 \\ xk_1 - xk_2 + 0xk_3 &= 0 \\ x^2k_1 - 5x^2k_2 - 3x^3k_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -5 & -3 & 0 \end{array}\right] \xrightarrow[R_3-R_2]{R_1-3R_2} \left[\begin{array}{ccc|c} 0 & 5 & 4 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -4 & -3 & 0 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -4 & -3 & 0 \end{array}\right] \xrightarrow{R_3+4R_1}$$
$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_2+R_1]{R_1-R_3} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1\leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since the only solution is the trivial solution, therefore $\{3 + x + x^2, \quad 2 - x + 5x^2, \quad 4 - 3x^2\}$ are linearly independent. □

9 For which real values of λ do the following vectors form a linearly dependent set in $\mathbb{R}^3$?

$$v_1 = \left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right), \quad v_2 = \left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right), \quad v_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right)$$

Work. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1\left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right) + k_2\left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right) + k_3\left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right) = (0, 0, 0)$. From this, we can get a linear system of equations, and matrix equation.

$$\begin{aligned} \lambda k_1 - \frac{1}{2}k_2 - \frac{1}{2}k_3 &= 0 \\ -\frac{1}{2}k_1 + \lambda k_2 - \frac{1}{2}k_3 &= 0 \\ -\frac{1}{2}k_1 - \frac{1}{2}k_2 + \lambda k_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider the determinant of the square matrix.

$$\begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} \underset{R_1+R_3}{\overset{R_1+R_2}{===}} \begin{vmatrix} \lambda-1 & \lambda-1 & \lambda-1 \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} === (\lambda-1)\begin{vmatrix} 1 & 1 & 1 \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} \underset{R_3+\frac{1}{2}R_1}{\overset{R_2+\frac{1}{2}R_1}{===}}$$

$$(\lambda-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & \lambda+\frac{1}{2} & 0 \\ 0 & 0 & \lambda+\frac{1}{2} \end{vmatrix} = (\lambda-1)(1)(\lambda+\frac{1}{2}) = (\lambda-1)(\lambda+\frac{1}{2})^2$$

When $\lambda = 1$ or $\lambda = -\frac{1}{2}$, the determinant of this matrix will be zero. By Theorem 4 of Lecture Notes 32, when the determinant is zero, the coefficient matrix is singular. By the Big Theorem, if the coefficient matrix is singular, then $A\vec{x} = \vec{0}$ does not only have the trivial solution, meaning that there exists $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1\left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right) + k_2\left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right) + k_3\left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right) = (0, 0, 0)$, where not all k_i are zero. This means that these vectors are linearly dependent when $\lambda = 1$ or $\lambda = -\frac{1}{2}$. □

13 Show that if $S = \{v_1, v_2, \ldots, v_r\}$ is a linearly dependent set of vectors in a vector space V, and if $v_{r+1}, \ldots, v_n$ are any vectors in V that are not in S, then $\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ is also linearly dependent.

Work. By definition, if S is a linearly dependent set of vectors in V, then $\exists\ k_1, k_2, \ldots, k_r \in \mathbb{R}$ such that $k_1v_1 + k_2v_2 + \cdots + k_rv_r = \mathbf{id}$, where not all $k_i = 0$. Now consider $S' = S \cup \{v_{r+1}, \ldots, v_n\}$, where $v_{r+1}, \ldots, v_n$ are not in S. Consider $k_1, k_2, \ldots, k_r, k_{r+1}, \ldots, k_n \in \mathbb{R}$ such that $k_1v_1 + k_2v_2 + \cdots + k_rv_r + k_{r+1}v_{r+1} + \cdots + k_nv_n = \mathbf{id}$.

$$\begin{aligned} k_1v_1 + k_2v_2 + \cdots + k_rv_r + k_{r+1}v_{r+1} + \cdots + k_nv_n &= \mathbf{id} && \neg\forall k_i = 0 \\ k_1v_1 + k_2v_2 + \cdots + k_rv_r + 0v_{r+1} + \cdots + 0v_n &= \mathbf{id} && \neg\forall k_i = 0 \\ k_1v_1 + k_2v_2 + \cdots + k_rv_r + \mathbf{id} + \cdots + \mathbf{id} &= \mathbf{id} && \neg\forall k_i = 0 \\ k_1v_1 + k_2v_2 + \cdots + k_rv_r &= \mathbf{id} && \neg\forall k_i = 0 \end{aligned}$$

This final equation is known to exist, since we asserted at the beginning that S was linearly dependent. Therefore, S' must also be linearly dependent. Therefore, if $S = \{v_1, v_2, \dots, v_r\}$ is a linearly dependent set of vectors in a vector space V, and if $v_{r+1}, \dots, v_n$ are any vectors in V that are not in S, then $\{v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_n\}$ is also linearly dependent. $\square$

21 The functions $f_1(x) = x$ and $f_2(x) = \cos x$ are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wroński's test.

Work. According to the Wroński's test, f_1 and f_1 are linearly independent if $W(x)$ is not identically zero.

$$W(x) = \begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix} = -x \sin x - \cos x$$

When $x = 0$, $W(x) = -0 \sin 0 - \cos 0 = 0 - 1 = -1 \neq 0$. Therefore W is not identically zero. Therefore, $f_1(x) = x$ and $f_2(x) = \cos x$ are linearly independent. $\square$

4.4

4 Which of the following form bases for P_2?

a. $1 - 3x + 2x^2, \quad 1 + x + 4x^2, \quad 1 - 7x$

Work. In order for $\{1-3x+2x^2, \ 1+x+4x^2, \ 1-7x\}$ to be a basis of P_2, it must be linearly independent and it must span P_2. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1(1 - 3x + 2x^2) + k_2(1 + x + 4x^2) + k_3(1 - 7x) = (b_1 + b_2x + b_3x^2)$.

$$\begin{aligned} 1k_1 + 1k_2 + 1k_3 &= b_1 \\ -3xk_1 + 1xk_2 - 7xk_3 &= b_2 \\ 2x^2k_1 + 4x^2k_2 + 0x^2k_3 &= b_3 \end{aligned}$$

Consider the augmented matrix of this linear system of equations.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ -3 & 1 & -7 & b_2 \\ 2 & 4 & 0 & b_3 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 + 3R_1} \begin{bmatrix} 1 & 1 & 1 & b_1 \\ 0 & 4 & -4 & b_2 + 3b_1 \\ 0 & 2 & -2 & b_3 - 2b_1 \end{bmatrix} \xrightarrow{R_2 - 2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 0 & b_2 + 3b_1 - b_3 + 4b_1 \\ 0 & 2 & -2 & b_3 - 2b_1 \end{array}\right]$$

The second row of the augmented matrix represents $0 = b_2 + 3b_1 - b_3 + 4b_1$. If $0 \neq b_2 + 3b_1 - b_3 + 4b_1$, then there is no solution to the linear system of equations, and thus no coefficients for k_1, k_2, k_3 that satisfy the above condition. Therefore $\{1 - 3x + 2x^2, \ 1 + x + 4x^2, \ 1 - 7x\}$ does not span P_2, and thus cannot be a basis for P_2. $\square$

7 Find the coordinate vector of $\vec{w}$ relative to the basis $S = \{\vec{u}_1, \vec{u}_2\}$ for $\mathbb{R}^2$.

b. $\vec{u}_1 = (2, -4), \quad \vec{u}_2 = (3, 8); \quad \vec{w} = (1, 1)$

Work. Let $k_1, k_2 \in \mathbb{R}$ such that $k_1(2, -4) + k_2(3, 8) = (1, 1)$.

$$2k_1 + 3k_2 = 1$$
$$-4k_1 + 8k_3 = 1$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ -4 & 8 & 1 \end{array}\right] \xrightarrow{R_2+2R_2} \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 14 & 3 \end{array}\right] \xrightarrow{\frac{1}{14}R_2} \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1 & \frac{3}{14} \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{cc|c} 2 & 0 & \frac{5}{14} \\ 0 & 1 & \frac{3}{14} \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{array}\right]$$

Therefore, $\frac{5}{28}(2, -4) + \frac{3}{14}(3, 8) = (1, 1)$, or in other terms $(1, 1)_S = (\frac{5}{28}, \frac{3}{14})$. □

c. $\vec{u}_1 = (1, 1), \quad \vec{u}_2 = (0, 2); \quad \vec{w} = (a, b)$

Work. Let $k_1, k_2 \in \mathbb{R}$ such that $k_1(1, 1) + k_2(0, 2) = (a, b)$.

$$1k_1 + 0k_2 = a$$
$$1k_1 + 2k_2 = b$$

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 1 & 2 & b \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 2 & b-a \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{array}\right]$$

Therefore, $a(1, 1) + \frac{b-a}{2}(0, 2) = (a, b)$, or in other terms $(a, b)_S = (a, \frac{b-a}{2})$. □

12 Show that $\{A_1, A_2, A_3, A_4\}$ is a basis for $\mathcal{M}_{22}$, and express A as a linear combination of the basis vectors.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$$

Proof of basis. Let $k_1, k_2, k_3, k_4 \in \mathbb{R}$ such that $k_1 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$.

$$1k_1 + 1k_2 + 1k_3 + 0k_4 = 6$$
$$0k_1 + 1k_2 + 0k_3 + 0k_4 = 2$$
$$1k_1 + 0k_2 + 0k_3 + 1k_4 = 5$$
$$0k_1 + 0k_2 + 1k_3 + 0k_4 = 3$$

Now consider only the coefficient matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_3+R_4, R_3+R_2]{R_3-R_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_1-R_3]{R_1-R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since every column has a leading 1, $\{A_1, A_2, A_3, A_4\}$ are linearly independent. Since every row has a leading 1, $\{A_1, A_2, A_3, A_4\}$ spans $\mathcal{M}_{22}$. Therefore, since $\{A_1, A_2, A_3, A_4\}$ is linearly independent and spans $\mathcal{M}_{22}$, $\{A_1, A_2, A_3, A_4\}$ is a basis for $\mathcal{M}_{22}$. □

A as a linear combination.

$$
\begin{aligned}
1k_1 + 1k_2 + 1k_3 + 0k_4 &= 6 \\
0k_1 + 1k_2 + 0k_3 + 0k_4 &= 2 \\
1k_1 + 0k_2 + 0k_3 + 1k_4 &= 5 \\
0k_1 + 0k_2 + 1k_3 + 0k_4 &= 3
\end{aligned}
$$

$$
\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right]
\xrightarrow[R_3+R_4, R_3+R_2]{R_3-R_1}
\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right]
\xrightarrow[R_1-R_3]{R_1-R_2}
$$

$$
\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right]
\xrightarrow{R_3 \leftrightarrow R_4}
\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]
$$

Therefore, $1A_1 + 2A_2 + 3A_3 + 4A_4 = A$. □

4.5

3 Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.

$$
\begin{aligned}
1x_1 - 4x_2 + 3x_3 - 1x_4 &= 0 \\
2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0
\end{aligned}
$$

Work.

$$
\left[\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 2 & -8 & 6 & -2 & 0 \end{array}\right]
\xrightarrow{R_2-2R_1}
\left[\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

Let $x_2 = t_2,\ x_3 = t_3,\ x_4 = t_4$, where t_2, t_3, t_4 are free parameters. $x_1 = 4t_2 - 3t_3 + 1t_4$. Therefore a solution to the homogeneous linear system takes the form

$$
\begin{aligned}
(x_1, x_2, x_3, x_4) &= (4t_2 - 3t_3 + 1t_4, t_2, t_3, t_4) \\
&= (4t_2, t_2, 0, 0) + (-3t_3, 0, t_3, 0) + (t_4, 0, 0, t_4) \\
&= t_2(4, 1, 0, 0) + t_3(-3, 0, 1, 0) + t_4(1, 0, 0, 1)
\end{aligned}
$$

Therefore a basis of the solution space is $\{(4, 1, 0, 0), (-3, 0, 1, 0), (1, 0, 0, 1)\}$. Since there are three basis vectors, the dimension is the solution space is 3. □

7 Find bases for the following subspaces of $\mathbb{R}^3$.

a. The plane $3x - 2y + 5z = 0$.

Work.

$$
\begin{aligned}
3x - 2y + 5z &= 0 \\
x - \frac{2}{3}y + \frac{5}{3}z &= 0
\end{aligned}
$$

Let $y = t_y$ and $z = t_z$, where t_y and t_z are free parameters. $x = \frac{2}{3}t_y - \frac{5}{3}t_z$. Therefore a point on the plane $3x - 2y + 5z = 0$ takes the form

$$
\begin{aligned}
(x, y, z) &= (\frac{2}{3}t_y - \frac{5}{3}t_z, t_y, t_z) \\
&= (\frac{2}{3}t_y, t_y, 0) + (-\frac{5}{3}t_z, 0, t_z) \\
&= t_y(\frac{2}{3}, 1, 0) + t_z(-\frac{5}{3}, 0, 1)
\end{aligned}
$$

Therefore a basis of the solution space is $\{(\frac{2}{3}, 1, 0), (-\frac{5}{3}, 0, 1)\}$. $\square$

b. The plane $x - y = 0$.

Work.

$$
\begin{aligned}
x - y &= 0 \\
x - y + z &= z
\end{aligned}
$$

Let $y = t_y$ and $z = t_z$, where t_y and t_z are free parameters $x = t_y$. Therefore a point on the plane $x - y = 0$ takes the form

$$
\begin{aligned}
(x, y, z) &= (t_y, t_y, t_z) \\
&= (t_y, t_y, 0) + (0, 0, t_z) \\
&= t_y(1, 1, 0) + t_z(0, 0, 1)
\end{aligned}
$$

Therefore a basis of the solution space is $\{(1, 1, 0), (0, 0, 1)\}$. $\square$

c. The lines $x = 2t, \quad y = -t, \quad z = 4t$.

Work. A point on the lines $x = 2t, \quad y = -t, \quad z = 4t$ take the form

$$
\begin{aligned}
(x, y, z) &= (2t, -t, 4t) \\
&= t(2, -1, 4)
\end{aligned}
$$

Therefore a basis of the solution space is $\{(2, -1, 4)\}$. $\square$

c. All the vectors of the form (a, b, c), where $b = a + c$.

Work. A vector which satisfies the conditions takes the form

$$\begin{aligned}(a, b, c) &= (a, a + c, c)\\ &= (a, a, 0) + (0, c, c)\\ &= a(1, 1, 0) + c(0, 1, 1)\end{aligned}$$

Therefore a basis of the solution space is $\{(1, 1, 0), (0, 1, 1)\}$. □

11

a. Show that the set W of all polynomials in P_2 such that $p(1) = 0$ is a subspace of P_2.

Proof. Axiom 1: Consider p_a and $p_b \in W$. Now consider $p_a + p_b$.

$$(p_a + p_b)(x) = p_a(x) + p_b(x)$$

When $x = 1$, $(p_a + p_b)(1) = p_a(1) + p_b(1) = 0 + 0 = 0$. Therefore Axiom 1 holds.
Axiom 6: Consider $p \in W$ and $k \in \mathbb{R}$. Now consider $k \cdot p$.

$$(kp)(x) = kp(x)$$

When $x = 1$, $(kp)(x) = kp(x) = k \cdot 0 = 0$. Therefore Axiom 6 holds. Since Axiom 1 and Axiom 6 hold for W, W is a subspace of P_2. □

b. Make a conjecture about the dimesion of W.

Conjecture. I conjecture that the dimension of W is two. □

c. Confirm you conjecture by finding a basis for W.

Work. A polynomial which satisfies the condition $p(1) = 0$ must also satisfy the following identity.

$$p(1) = p_0 + p_1(1) + p_2(1^2) = 0 \qquad = p_0 + p_1 + p_2 = 0$$

Let $p_1 = t_1$ and $p_2 = t_2$, where t_1 and t_2 are free parameters. $p_0 = -t_1 - t_2$. Therefore a vector of coefficients takes the form

$$\begin{aligned}(p_0, p_1, p_2) &= (-t_1 - t_2, t_1, t_2) \\ &= (-t_1, t_1, 0) + (-t_2, 0, t_2) \\ &= t_1(-1, 1, 0) + t_2(-1, 0, 1)\end{aligned}$$

Therefore a basis of the solution space is $\{-1 + x, \ -1 + x^2\}$. There are two basis vectors, so the dimension of the subspace is two. □

18 Let S be a basis for an n-dimesional vector space V. show that if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ form a linearly independent set of vectors in V, then the coordinate vectors $(\vec{v}_1)_S, (\vec{v}_2)_S, \ldots, (\vec{v}_r)_S$ form a linearly independent set in $\mathbb{R}^n$, and conversely.

Proof. Let S be a basis for an n-dimensional vector space V, and consider an linearly independent set of vectors in V, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$. Then, by definition for $k_1, \ldots, k_r \in \mathbb{R}$,

$$k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r = \mathbf{id}$$

only has the trivial solution. Now consider a change of basis into base S. The equation now becomes

$$k_1(\vec{v}_1)_S + k_2(\vec{v}_2)_S + \cdots + k_r(\vec{v}_r)_S = (\mathbf{id})_S.$$

However, $(\mathbf{id})_S = \mathbf{id}$ for all basis, since it can only be obtained by a trivial linear combination, which produces a sum of $\mathbf{id}$. Therefore we have

$$k_1(\vec{v}_1)_S + k_2(\vec{v}_2)_S + \cdots + k_r(\vec{v}_r)_S = \mathbf{id}$$

which only has the trivial solution. Therefore the coordinate vectors $(\vec{v}_1)_S, (\vec{v}_2)_S, \ldots, (\vec{v}_r)_S$ form a linearly independent set in $\mathbb{R}^n$. Conversely, if S is changed to the original basis for V, then the converse can be achieved. □