Section 3, p34 1-7 odd, 17, 21, 25

Computations

1. What three things must be check to determine whether a function $\phi : S \mapsto S'$ is an isomorphism of a binary structure $\langle S, * \rangle$ with $\langle S', *' \rangle$?

ϕ must be *one-to-one*, *onto*, and *operation preserving*.

In Exercises 2 through 10, determine whether the given map ϕ is an isomorphism of the first binary structure with the second. If it is not an isomorphism, why not?

3. $\langle \mathbb{Z}, + \rangle$ with $\langle \mathbb{Z}, + \rangle$ where $\phi(n) = 2n$ for $n \in \mathbb{Z}$

An isomorphism must be one-to-one, onto, and operation preserving.

Proof. Onto: Let $y \in \mathbb{Z}$. Let us find $n \in \mathbb{Z}$ such that $y = \phi(x)$

$$\begin{aligned} y &= \phi(n) \\ y &= 2n \\ y/2 &= n \end{aligned}$$

$y/2$ is not always an integer, so we cannot say that ϕ is onto. Therefore, ϕ is not an isomorphism between $\langle \mathbb{Z}, + \rangle$ and $\langle \mathbb{Z}, + \rangle$. □

5. $\langle \mathbb{Q}, + \rangle$ with $\langle \mathbb{Q}, + \rangle$ where $\phi(x) = x/2$ for $x \in \mathbb{Q}$

An isomorphism must be one-to-one, onto, and operation preserving.

Proof. (a) One-to-one: Assume $\phi(x_1) = \phi(x_2)$ for some $x_1, x_2 \in \mathbb{Q}$.

$$\begin{aligned} \phi(x_1) &= \phi(x_2) \\ \frac{x_1}{2} &= \frac{x_2}{2} \\ x_1 &= x_2 \end{aligned}$$

Thus ϕ is one-to-one.

(b) Onto: Let $y \in \mathbb{Q}$. Let us find $x \in \mathbb{Q}$ such that $y = \phi(x)$

$$\begin{aligned} y &= \phi(x) \\ y &= \frac{x}{2} \\ 2y &= x \end{aligned}$$

Choose $x = 2y$. Thus ϕ is onto.

(c) Operation Preserving: Need to show that $\phi(x + y) = \phi(x) + \phi(y)$.

$$\begin{aligned} \phi(x+y) &= \frac{x+y}{2} \\ &= \frac{x}{2} + \frac{y}{2} \\ &= \phi(x) + \phi(y) \end{aligned}$$

Thus ϕ is operation preserving.

Since ϕ is one-to-one, onto, and operation preserving, it is an isomorphism between $\langle \mathbb{Q}, + \rangle$ and $\langle \mathbb{Q}, + \rangle$. □

7. $\langle \mathbb{R}, \cdot \rangle$ with $\langle \mathbb{R}, \cdot \rangle$ where $\phi(x) = x^3$ for $x \in \mathbb{R}$

An isomorphism must be one-to-one, onto, and operation preserving.

Proof. (a) One-to-one: Assume $\phi(x_1) = \phi(x_2)$ for some $x_1, x_2 \in \mathbb{R}$.

$$\begin{aligned} \phi(x_1) &= \phi(x_2) \\ x_1^3 &= x_2^3 \\ x_1 &= x_2 \end{aligned}$$

Thus ϕ is one-to-one.

(b) Onto: Let $y \in \mathbb{R}$. Let us find $x \in \mathbb{R}$ such that $y = \phi(x)$.

$$\begin{aligned} y &= \phi(x) \\ y &= x^3 \\ \sqrt[3]{y} &= x \end{aligned}$$

Choose $x = \sqrt[3]{x}$. Thus ϕ is onto.

(c) Operation Preserving: Need to show that $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$.

$$\begin{aligned} \phi(x \cdot y) &= (xy)^3 \\ &= x^3 \cdot y^3 \\ &= \phi(x) \cdot \phi(y) \end{aligned}$$

Thus ϕ is operation preserving.

Since ϕ is one-to-one, onto, and operation preserving, it is an isomorphism between $\langle \mathbb{R}, \cdot \rangle$ and $\langle \mathbb{R}, \cdot \rangle$. $\square$

17. The map $\phi : \mathbb{Z} \mapsto \mathbb{Z}$ defined by $\phi(n) = n + 1$ for $n \in \mathbb{Z}$ is one to one and onto $\mathbb{Z}$.

Give the definition of a binary operation $*$ on $\mathbb{Z}$ such that ϕ is an isomorphism mapping

a. $\langle \mathbb{Z}, \cdot \rangle$ onto $\langle \mathbb{Z}, * \rangle$

Since ϕ is already one-to-one and onto, we just need to define $*$ so that ϕ is operation preserving. Consider $a * b = ab - a - b + 2$

$$\begin{aligned} \phi(x) * \phi(y) &= (x+1) * (y+1) \\ &= (x+1)(y+1) - (x+1) - (y+1) + 2 \\ &= xy + x + y + 1 - x - 1 - y - 1 + 2 \\ &= xy + 1 \\ &= \phi(x \cdot y) \end{aligned}$$

$\phi(x) * \phi(y) = \phi(x \cdot y)$, meaning ϕ is operation preserving. Along with being one-to-one and onto, ϕ is thus an isomorphism between $\langle \mathbb{Z}, \cdot \rangle$ and $\langle \mathbb{Z}, * \rangle$.

b. $\langle \mathbb{Z}, * \rangle$ onto $\langle \mathbb{Z}, \cdot \rangle$

Since ϕ is already one-to-one and onto, we just need to define $*$ so that ϕ is operation preserving. Consider $a * b = ab + a + b$

$$\begin{aligned}\phi(x * y) &= \phi(xy + x + y)\\ &= xy + x + y + 1\\ &= (x+1)(y+1)\\ &= \phi(x) \cdot \phi(y)\end{aligned}$$

$\phi(x) * \phi(y) = \phi(x \cdot y)$, meaning ϕ is operation preserving. Along with being one-to-one and onto, ϕ is thus an isomorphism between $\langle \mathbb{Z}, \cdot \rangle$ and $\langle \mathbb{Z}, * \rangle$.

In Exercises 21 and 22, correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

21. A function $\phi : S \mapsto S'$ is an *isomorphism* if and only if $\phi(a * b) = \phi(a) *' \phi(b)$.

A function $\phi : S \mapsto S'$ is an *operation preserving* if and only if $\phi(a * b) = \phi(a) *' \phi(b)$.

25. Continuing the ideas of Exercise 24 can a binary structure have the left identity element e_L and a right identity element e_R where $e_L \neq e_R$? If so, given an example, using an operation on a finite set S. If not, prove that it is impossible.

We shall conduct a proof by contradiction

Proof. Let e_L and e_R be left- and right-sided identities, respectively, such that $e_L \neq e_R$. Let us consider $e_L * e_R$.

$$\begin{aligned}e_L * e_R &= e_L && \text{since } e_R \text{ is an identity from the right}\\ e_L * e_R &= e_R && \text{since } e_L \text{ is an identity from the left}\end{aligned}$$

Since a binary structure is uniquely defined, we must conclude that $e_L = e_R$. However, this contradicts our assertion that $e_L \neq e_L$. Therefore, there *cannot* be a binary structure such that the left identity and right identity are distinct. $\square$