Section 8.4

8.4.1

Define P(n) to be the assertion that: $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

a. *Verify that P(3) is true.*

$$P(3): 1^2 + 2^2 + 3^2 = \frac{3(3+1)(2 \cdot 3 + 1)}{6}$$
$$1 + 4 + 9 = \frac{84}{6}$$
$$14 = 14 \checkmark$$

b. *Express P(k).* $P(k) : \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

c. *Express P(k+1).* $P(k) : \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$

d. *In an inductive proof that for every positive integer n,* $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$*, what must be proven in the base case?* P(1) is true.

e. *In an inductive proof that for every positive integer n,*$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$*, what must be proven in the inductive step?* If P(k) is true, then P(k+1) is true.

f. *What would be the inductive hypothesis in the inductive step from your previous answer?* Assume that $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$ is true, for some $k \in \mathbb{Z}^+$.

g. *Prove by induction that for any positive integer n,* $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. Base Case: $n = 1$

$$P(1) : 1^2 = \frac{1(2)(3)}{6} \Rightarrow 1 = 1 \checkmark$$

Inductive Hypothesis: Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$
Inductive Case: $n = k + 1$

$$\begin{aligned}
\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (k+1)^2 && \text{separating out last term} \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{by inductive hypothesis} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\
&= \frac{(k+1)[(k+2)(2k+3)]}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} && \text{by algebra}
\end{aligned}$$

Therefore $P(k+1)$ is true. Since $P(1)$ is true, and $P(k+1)$ is true, therefore $P(n)$ is true for all $n \in \mathbb{Z}^+$. $\square$

8.4.2

Prove each of the following statements using mathematical induction.

a. *Prove that for any positive integer n, $\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2$*

Proof. Base Case: $n = 1$

$$P(1) : 1^3 = \left(\frac{1(1+1)}{2}\right)^2 \Rightarrow 1 = 1 \checkmark$$

Inductive Hypothesis: Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$
Inductive Case: $n = k + 1$

$$\begin{aligned}
\sum_{j=1}^{k+1} j^3 &= \sum_{j=1}^{k} j^3 + (k+1)^3 && \text{separating out last term} \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{by inductive hypothesis} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2[k^2 + 4k + 4]}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2 && \text{by algebra}
\end{aligned}$$

Therefore $P(k+1)$ is true. Since $P(1)$ is true, and $P(k+1)$ is true, therefore $P(n)$ is true for all $n \in \mathbb{Z}^+$. □

8.4.3

Prove each of the following statements using mathematical induction.

a. *Prove that for $n \geq 2$, $3^n > 2^n + n^2$.*

Proof. Base Case: $n = 2$

$$P(2) : 3^2 > 2^2 + 2^2 \Rightarrow 9 > 8 \checkmark$$

Inductive Hypothesis: Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$
Inductive Case: $n = k + 1$

$$\begin{aligned}
3^{k+1} = 3^k 3 &> 3 \cdot 2^k + 3 \cdot k^2 && \text{by inductive hypothesis} \\
&> 2 \cdot 2^k + k^2 + 2k + 1 && \text{since } k > 1 \\
&> 2^{k+1} + (k+1)^2
\end{aligned}$$

Therefore $P(k+1)$ is true. Since $P(1)$ is true, and $P(k+1)$ is true, therefore $P(n)$ is true for all $n \in \mathbb{Z}^+$. □