

Problem 1.

Let $V = \mathbb{R}^+ \times \mathbb{R}$ be a set. In other words, every element of V is in the form (u_1, u_2) , where u_1 is a positive real number and $u_2 \in \mathbb{R}$. For all (u_1, u_2) and $(v_1, v_2) \in V$, and for all $k \in \mathbb{R}$,

$$(u_1, u_2) \oplus (v_1, v_2) = (2u_1v_1, u_2 + v_2 - 3) \text{ and } k \odot (u_1, v_1) = (u_1^k, ku_2).$$

Verify the axioms 4, 5, and 7.

Ax4. *Proof.* Consider $\vec{u}, \vec{v} \in V$ such that $\vec{u} = (u_1, u_2)$ and $\vec{v} = (\frac{1}{2}, 3)$. (u_1 is positive real number).

$$\begin{aligned} \vec{u} \oplus \vec{v} &= (u_1, u_2) \oplus \left(\frac{1}{2}, 3\right) = \left(2u_1 \frac{1}{2}, u_2 + 3 - 3\right) \\ &= (u_1, u_2) = \vec{u} \\ \vec{v} \oplus \vec{u} &= \left(\frac{1}{2}, 3\right) \oplus (u_1, u_2) = \left(2 \frac{1}{2} u_1, 3 + u_2 - 3\right) \\ &= (u_1, u_2) = \vec{u} \end{aligned}$$

Since $\vec{u} \oplus \vec{v} = \vec{u}$ and $\vec{v} \oplus \vec{u} = \vec{u}$ for all $\vec{u} \in V$, therefore $\vec{v} = (\frac{1}{2}, 3)$ is the additive identity, **id**, for V .
 \therefore additive identity exists for V . \square

Ax5. *Proof.* Consider $\vec{u}, \vec{v} \in V$ such that $\vec{u} = (u_1, u_2)$ and $\vec{v} = (\frac{1}{4u_1}, 6 - u_2)$. Since by definition u_1 is a positive real number, $\frac{1}{4u_1}$ will always be defined and positive.

$$\begin{aligned} \vec{u} \oplus \vec{v} &= (u_1, u_2) \oplus \left(\frac{1}{4u_1}, 6 - u_2\right) = \left(2u_1 \frac{1}{4u_1}, u_2 + (6 - u_2) - 3\right) \\ &= \left(\frac{2}{4} \cdot \frac{u_1}{u_1}, u_2 - u_2 + 6 - 3\right) = \left(\frac{1}{2}, 3\right) = \mathbf{id} \\ \vec{v} \oplus \vec{u} &= \left(\frac{1}{4u_1}, 6 - u_2\right) \oplus (u_1, u_2) = \left(2 \frac{1}{4u_1} u_1, (6 - u_2) + u_2 - 3\right) \\ &= \left(\frac{2}{4} \cdot \frac{u_1}{u_1}, u_2 - u_2 + 6 - 3\right) = \left(\frac{1}{2}, 3\right) = \mathbf{id} \end{aligned}$$

\therefore additive inverse exists for all $\vec{u} \in V$. \square

Ax7. *Proof.* Consider $k \in \mathbb{R}$ and $(u_1, u_2), (v_1, v_2) \in V$.

$$\begin{aligned} k \odot ((u_1, u_2) \oplus (v_1, v_2)) &= k \odot (2u_1v_1, u_2 + v_2 - 3) \\ &= ((2u_1v_1)^k, k(u_2 + v_2 - 3)) \\ &= (4u_1^k v_1^k, ku_2 + kv_2 - 3k) \end{aligned}$$

$$\begin{aligned} k \odot (u_1, u_2) \oplus k \odot (v_1, v_2) &= (u_1^k, ku_2) \oplus (v_1^k, kv_2) \\ &= (2u_1^k v_1^k, ku_2 + kv_2 - 3) \end{aligned}$$

$$(4u_1^k v_1^k, ku_2 + kv_2 - 3k) \neq (2u_1^k v_1^k, ku_2 + kv_2 - 3) \text{ when } k \neq 1$$

Since $k \odot ((u_1, u_2) \oplus (v_1, v_2))$ does not always equal $k \odot (u_1, u_2) \oplus k \odot (v_1, v_2)$, Axiom 7 does not hold for V . \square

Problem 2.

Let V be a set with a binary operator \oplus defined, so that Axioms (1), (3), and (4) hold for V (note that other axioms may not hold). Let $\vec{v} \in V$. Prove that if \vec{v} has an additive inverse, then this additive inverse is unique. (*Hint*: Let \vec{w} and \vec{x} be two different additive inverses of \vec{v} . Show that this will lead to a contradiction.)

Proof. Let $\vec{w}, \vec{x}, \vec{v} \in V$ such that \vec{w} and \vec{x} are two different additive inverses of \vec{v} . This implies that $\vec{w} \neq \vec{x}$.

$$\begin{array}{ll}
 \vec{v} \oplus \vec{w} = \mathbf{id} & \text{def. of additive inverse} \\
 \vec{x} \oplus (\vec{v} \oplus \vec{w}) = \vec{x} \oplus \mathbf{id} & \\
 (\vec{x} \oplus \vec{v}) \oplus \vec{w} = \vec{x} \oplus \mathbf{id} & \text{axiom 3} \\
 \mathbf{id} \oplus \vec{w} = \vec{x} \oplus \mathbf{id} & \text{def. of additive inverse} \\
 \vec{w} = \vec{x} & \text{def. of additive identity}
 \end{array}$$

However, $\vec{w} = \vec{x}$ contradicts our assertion that $\vec{w} \neq \vec{x}$. Therefore, through contradiction, if \vec{v} has an additive inverse, then this additive inverse is unique. \square

Problem 3.

Let $V = P_3$, i.e., the set of all polynomials of degree up to 3, with standard addition and scalar multiplication. Let

$$W = \{a_0 + a_1x + a_2x^2 + a_3x^3 \in V : a_0 \cdot a_1 = 0\}.$$

Verify whether W is a subspace of V .

Proof. According to Theorem 3 from Lecture 10, assuming that addition and scalar multiplication in W are inherited from V , W is a subspace of V if and only if Axioms 1 and 6 hold for W .

Ax1. *Proof.* Let $\vec{a}, \vec{b} \in W$ such that $\vec{a} = 0 + 1x + 0x^2 + 0x^3$ and $\vec{b} = 1 + 0x + 0x^2 + 0x^3$. Now consider $\vec{a} \oplus \vec{b}$:

$$\begin{aligned}
 \vec{a} \oplus \vec{b} &= (0 + 1x + 0x^2 + 0x^3) \oplus (1 + 0x + 0x^2 + 0x^3) \\
 &= 0 + 1x + 0x^2 + 0x^3 + 1 + 0x + 0x^2 + 0x^3 \\
 &= (0 + 1) + (1 + 0)x + (0 + 0)x^2 + (0 + 0)x^3 \\
 &= 1 + 1x + 0x^2 + 0x^3
 \end{aligned}$$

$$1 \cdot 1 = 1 \neq 0$$

Therefore $\vec{a} \oplus \vec{b} \notin W$, even though $\vec{a} \in W$ and $\vec{b} \in W$.

This means that W is not closed under addition. \square

Since Axiom 1 does not hold for W , W cannot be a subspace of V . \square

Problem 5.

Let

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 3 & 2 \\ 5 & -4 \end{pmatrix}.$$

Express $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ as a linear combination of A , B , and C . Use Gauss-Jordan elimination.

Proof. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1 A \oplus k_2 B \oplus k_3 C = M$.

That is, $k_1 \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix} \oplus k_2 \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix} \oplus k_3 \begin{pmatrix} 3 & 2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. From this equation, we get a linear system of equations:

$$2k_1 + 1k_2 + 3k_3 = 1$$

$$1k_1 - 1k_2 + 2k_3 = 2$$

$$4k_1 + 3k_2 + 5k_3 = 3$$

$$0k_1 + 4k_2 - 4k_3 = 4$$

$$\begin{aligned} & \left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & -1 & 2 & 2 \\ 4 & 3 & 5 & 3 \\ 0 & 4 & -4 & 4 \end{array} \right) \xrightarrow[(-4, -2, -6, -2)]{R_3 - 2R_1} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & -1 & 2 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 4 & -4 & 4 \end{array} \right) \xrightarrow[(-2, 2, -4, -4)]{R_1 - 2R_2} \left(\begin{array}{ccc|c} 0 & 3 & -1 & -3 \\ 1 & -1 & 2 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 4 & -4 & 4 \end{array} \right) \xrightarrow[(0, 1, -1, 1)]{R_2 + R_3} \\ & \left(\begin{array}{ccc|c} 0 & 3 & -1 & -3 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 4 & -4 & 4 \end{array} \right) \xrightarrow[(0, -3, 3, -3)]{R_1 - 3R_3} \left(\begin{array}{ccc|c} 0 & 0 & 2 & -6 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 4 & -4 & 4 \end{array} \right) \xrightarrow[(0, -4, 4, -4)]{R_4 - 4R_1} \left(\begin{array}{ccc|c} 0 & 0 & 2 & -6 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{\frac{1}{2}R_1} \\ & \left(\begin{array}{ccc|c} 0 & 0 & 1 & -3 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow[(0, 0, -1, 3)]{R_2 - R_1} \left(\begin{array}{ccc|c} 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 6 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{R_3 + R_1} \left(\begin{array}{ccc|c} 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow[R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_2} \\ & \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{aligned}$$

This augmented matrix represents the following equations:

$$k_1 = 6$$

$$k_2 = -2$$

$$k_3 = -3$$

This means that $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a linear combination of A , B , and C , when $k_1 = 6$, $k_2 = -3$, and $k_3 = -3$ \square

Problem 6.

Decide whether

$$\vec{u} = 2 + x + 4x^2, \vec{v} = 1 - x - 7x^2, \text{ and } \vec{w} = 3 + 2x + 9x^2.$$

spans P_2 . Justify your answer using Gauss-Jordan elimination.

Proof. Let $\vec{y} = y_0 + y_1x + y_2x^2$, and let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1\vec{u} + k_2\vec{v} + k_3\vec{w} = \vec{y}$. In other words,

$$k_1(2 + x + 4x^2) + k_2(1 - x - 7x^2) + k_3(3 + 2x + 9x^2) = y_0 + y_1x + y_2x^2.$$

From this equation, we get the following system of linear equations.

$$2k_1 + k_2 + 3k_3 = y_0$$

$$k_1 - k_2 + 2k_3 = y_1$$

$$4k_1 - 7k_2 + 9k_3 = y_2$$

$$\begin{aligned} \left(\begin{array}{ccc|c} 2 & 1 & 3 & y_0 \\ 1 & -1 & 2 & y_1 \\ 4 & -7 & 9 & y_2 \end{array} \right) &\xrightarrow[(-4, -2, -6, -2y_0)]{R_3 - 2R_1} \left(\begin{array}{ccc|c} 2 & 1 & 3 & y_0 \\ 1 & -1 & 2 & y_1 \\ 0 & -9 & 3 & y_2 - 2y_0 \end{array} \right) \xrightarrow[(-2, 2, -4, -2y_1)]{R_1 - 2R_2} \left(\begin{array}{ccc|c} 0 & 3 & -1 & y_0 - 2y_1 \\ 1 & -1 & 2 & y_1 \\ 0 & -9 & 3 & y_2 - 2y_0 \end{array} \right) \\ &\xrightarrow[(0, 9, -3, 2y_0 - 4y_1)]{R_3 + 2R_3} \left(\begin{array}{ccc|c} 0 & 3 & -1 & y_0 - 2y_1 \\ 1 & -1 & 2 & y_1 \\ 0 & 0 & 0 & y_2 - 2y_0 + 2y_0 - 4y_1 \end{array} \right) = \left(\begin{array}{ccc|c} 0 & 3 & -1 & y_0 - 2y_1 \\ 1 & -1 & 2 & y_1 \\ 0 & 0 & 0 & y_2 - 4y_1 \end{array} \right) \end{aligned}$$

The last row represents the equation $0 = y_2 - 4y_1$. If $0 \neq y_2 - 4y_1$, then there is no solution to the system of linear equations. Therefore, there exists $\vec{y} \in P_2$ that cannot be spanned by $\{\vec{u}, \vec{v}, \vec{w}\}$. \square

Problem 9.

Let V be a real vector space. Prove that V cannot have exactly 3 elements.

Proof. Let V be a real vector space containing exactly 3 elements. Let the first element of V be \mathbf{id} , which is required to be in V through Axiom 4. Next, let the second element of V be \vec{v} (Note that \vec{v} cannot be \mathbf{id} , since the additive identity is unique). Finally, let the second element of V be $-\vec{v}$, the additive inverse of \vec{v} , which is required to be in V through Axiom 5. Therefore, we have $V = \{\mathbf{id}, \vec{v}, -\vec{v}\}$, where $\mathbf{id} \neq \vec{v}$ and $\mathbf{id} \neq -\vec{v}$ and $\vec{v} \neq -\vec{v}$.

Now consider $\vec{v} \oplus \vec{v}$:

Case 1. $\vec{v} \oplus \vec{v} = \mathbf{id}$

Proof. Consider $\vec{v} \oplus \vec{v} \oplus -\vec{v}$.

$$\begin{aligned} \vec{v} \oplus \vec{v} \oplus -\vec{v} &= \vec{v} \oplus (\vec{v} \oplus -\vec{v}) && \text{axiom 3} \\ &= \vec{v} \oplus \mathbf{id} && \text{def. of additive inverse} \\ &= \vec{v} && \text{def. of additive identity} \end{aligned}$$

$$\begin{aligned} \vec{v} \oplus \vec{v} \oplus -\vec{v} &= (\vec{v} \oplus \vec{v}) \oplus -\vec{v} && \text{axiom 3} \\ &= \mathbf{id} \oplus -\vec{v} && \text{assertion} \\ &= -\vec{v} && \text{def. of additive identity} \end{aligned}$$

This implies that $\vec{v} = -\vec{v}$, which contradicts our assertion that $\vec{v} \neq -\vec{v}$. Therefore, $\vec{v} \oplus \vec{v} \neq \mathbf{id}$. \square

Case 2. $\vec{v} \oplus \vec{v} = \vec{v}$

Proof. Consider $\vec{v} \oplus \vec{v} \oplus -\vec{v}$.

$$\begin{aligned} \vec{v} \oplus \vec{v} \oplus -\vec{v} &= \vec{v} \oplus (\vec{v} \oplus -\vec{v}) && \text{axiom 3} \\ &= \vec{v} \oplus \mathbf{id} && \text{def. of additive inverse} \\ &= \vec{v} && \text{def. of additive identity} \end{aligned}$$

$$\begin{aligned} \vec{v} \oplus \vec{v} \oplus -\vec{v} &= (\vec{v} \oplus \vec{v}) \oplus -\vec{v} && \text{axiom 3} \\ &= \vec{v} \oplus -\vec{v} && \text{assertion} \\ &= \mathbf{id} && \text{def. of additive identity} \end{aligned}$$

This implies that $\mathbf{id} = \vec{v}$, which contradicts our assertion that $\mathbf{id} \neq \vec{v}$. Therefore, $\vec{v} \oplus \vec{v} \neq \vec{v}$. \square

Case 3. $\vec{v} \oplus \vec{v} = -\vec{v}$

Proof. Consider the assertion that $\vec{v} \oplus \vec{v} = -\vec{v}$

$$\begin{aligned} \vec{v} \oplus \vec{v} &= -\vec{v} && \text{assertion} \\ 1 \odot \vec{v} \oplus 1 \odot \vec{v} &= -\vec{v} && \text{axiom 10} \\ 1 \odot \vec{v} \oplus 1 \odot \vec{v} &= (-1) \odot \vec{v} && \text{theorem C} \\ (1 + 1) \odot \vec{v} &= (-1) \odot \vec{v} && \text{axiom 8} \end{aligned}$$

This last equation implies that $1 + 1 = -1$, which is a contradiction. Therefore, $\vec{v} \oplus \vec{v} \neq -\vec{v}$. \square

Every possible result in V for $\vec{v} \oplus \vec{v}$ leads to a contradiction. Therefore, $\vec{v} \oplus \vec{v} \notin V$, meaning that V is not closed under addition. This means that V , containing exactly three elements, cannot be a real vector space. \square