MAT 369 Introduction to Graph Theory

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Fall 2023

Contents

1 Introduction

1.1 Graphs and Graph Models

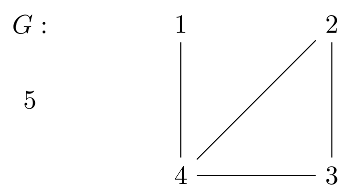
Graph Definition

A (simple) **graph** is an ordered pair (V, E) where

- V is a nonempty set of objects called "vertices"
- E is a set containing some two-subsets of V called "edges". E may be empty.

Graphs are often represented pictorially. For example consider

$$G = (V, E) \text{ where } V = \{1, 2, 3, 4, 5\} \text{ and } E = \{\{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$$



- Vertices 1 and 4 are **adjacent** because they are joined by an edge.
- Vertex 2 and edge $2 - 3$ are **indicent**.
- Edges $2 - 3$ and $3 - 4$ are **adjacent**.

Order Definition

The **order** of a graph G is $|V(G)|$, or the number of vertices.

Size Definition

The **size** of a graph G is $|E(G)|$, or the number of edges.
The graph G from above has order 5 and size 4.

1.2 Connected Graphs

Subgraph Definition

Let G and H be graphs. H is a **subgraph** of G, notated as $H \subseteq G$, if

$$V(H) \subseteq V(G) \text{ and } E(H) \subseteq E(G).$$

Proper Subgraph Definition

H is a **proper subgraph** of G if $H \subseteq G$ and either

$$V(H) \subsetneq V(G) \text{ or } E(H) \subsetneq E(G).$$

Spanning Subgraph Definition

Graph H is a **spanning subgraph** if $H \subseteq G$ and $V(H) = V(G)$.

Induced Subgraph Definition

Graph H is a **induced subgraph** if $H \subseteq G$ and if

$$u, v \in V(H) \text{ and } u, v \in E(G) \implies u, v \in E(H).$$

Essentially, H contains all valid edges it can take from G. Notation for **induced subgraph** is

$$G[S], \text{ where } S \text{ is a set of vertices from } G.$$

Edge-induced Subgraph Definition

$G[X]$ is an **edge-induced subgraph** of G if $G[X]$ has edge set $X \subseteq E(G)$ and a vertex set of all vertices incident with at least one edge of X. Interesting fact: $G[E(G)]$ removes any isolated vertices.

More on Spanning and Induced Subgraphs

Let G be a graph with vertex v and edge e. Then,

- $G - e$ is the *spanning subgraph* of G whose edge set is $E(G) - \{e\}$.

  This definition can be expanded to $G - X$ for $X \subseteq E(G)$.

- $G - v$ is the *induced subgraph* of G whose vertex set is $V(G) - \{v\}$ and edge set includes all edges of G except those incident with v.

  This definition can be expanded to $G - U$ for $U \subseteq V(G)$.

Let G be a graph, $u, v \in V(G)$ and $e = uv \notin E(G)$. Then $G + e$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup \{e\}$. G is a *spanning subgraph* of $G + e$

Walk, Trail, Path, Circuit, and Cycle Definitions

Let $u, v \in V(G)$. A $u - v$ **walk** in G is a sequence of vertices

$$(u = v_0, v_1, \ldots, v_k = v)$$

beginning with u, ending with v, and consecutive vertices are adjacent.

A **trail** is a walk in which *no edges* are repeated. A **path** is a walk in which *no vertices* are repeated. Every *path* is a *trail* is a *walk.*

A **circuit** is a closed trail of length ≥ 3. A **cycle** is a circuit with no repeated vertices, except for the first and the last, which are the same. A k**-cycle** is a cycle of length k. Every *cycle* is a *circuit* is a *walk.*

Closed and Open Walks

A $u - v$ walk with $u = v$ is called a **closed** walk. A $u - v$ walk with $u \neq v$ is called a **open** walk.

Walk and Path Theorem

If G contains a $u - v$ walk of length ℓ, then G contains a $u - v$ path of length $\leq \ell$.

Connectivity Definition

A graph G is said to be **connected** if $\forall u, v \in V(G)$, G contains a $u - v$ path. If this is not true, i.e. $\exists u, v \in V(G)$ where there is no $u - v$ path, then G is said to be **disconnected**.

Component Definition

A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is a **component** of G. The number of components of a graph G is denoted by $k(G)$. A graph G is connected if and only if $k(G) = 1$. Additionally, a graph is the union of its components.

Components and Equivalence Relations Theorem

Define a relation R on the $V(G)$ so that uRv if G contains a $u-v$ walk. Then R is an equivalence relation.

Subtractive Connectivity Theorem (weak)

Let G be a graph of order ≥ 3. If $\exists u, v \in V(G)$ such that $G-u$ and $G-v$ are connected, then G is connected.

Distance, Geodesic, Diameter, and Girth Definitions

The **distance** between vertices u and v, denoted as $d(u,v)$ or $d_G(u,v)$ is the smallest length of any $u-v$ path in G. If u and v are in different components, then $d(u,v)$ is undefined.

A $u-v$ path of shortest length $d(u,v)$ is called a **geodesic**. The **diameter** of a connected graph G, denoted as $\operatorname{diam}(G)$, is the largest *geodesic* between any two vertices of G. The **girth** of a connected graph G is the length of the shortest cycle in G.

Subtractive Connectivity Theorem (strong)

Let G be a graph of order ≥ 3. Then G is connected if and only if $\exists u, v \in V(G)$ such that $G-u$ and $G-v$ are connected.

1.3 Common Classes of Graphs

Name	**Symbol**	**Order**	**Size**
Path	P_n	n	$n-1$
Cycle	C_n	$n \geq 3$	n
Complete	K_n	n	$\binom{n}{2}$
Complete Bipartite	$K_{s,t}$	$s+t$	$s \cdot t$

Bipartite Graph Definition

G is bipartite if $V(G)$ can be partitioned into partite sets U and W so that every edge joins a vertex of U and a vertex of W.

Odd Cycle and Bipartite-ness

G is bipartite if and only if G contains no odd cycles.

K-partite Definition

G is a k**-partite** graph if $V(G)$ can be partitioned into partite sets $U_1, \ldots, U_k$ so that every edge joins a vertex from U_i and a vertex of U_j where $i \neq j$.

Constructing New Graphs from Old Graphs

Disjoint Union

For two graphs G and H, $G \cup H$ is defined as...

$$V(G \cup H) = V(G) \cup V(H)$$
$$E(G \cup H) = E(G) \cup E(H)$$

Complement

For one graph G, $\overline{G}$ is defined as...

$$V(\overline{G}) = V(G)$$
$$E(\overline{G}) = \{uv | u, v \in V(G), u \neq v, uv \notin E(G)\}$$

Join

For two graph, G and H, $G + H$ is defined as...

Start with $G \cup H$ and draw all edges join a vertex of G and a vertex of H

Cartesian Product

For two graphs, G and H, $G \times H$ is defined as...

$$V(G \times H) = \{(u, v) | u \in V(G) \text{ and } v \in V(H)\}$$
$$(u, v) - (x, y) \text{ if } u = x \text{ and } vy \in E(H) \vee v = y \text{ and } ux \in E(G)$$

A cartesian product between two graphs has the practical effect of duplicating one graph, and connecting the duplicates in the way of the other graph.

Complement Connectivity Theorem

If G is disconnected, then $\overline{G}$ is connected.

1.4 Multigraphs and Digraphs

Multigraph Definition

A **multigraph** is a graph where a pair of vertices may be joined by any finite number of edges.

- Multiple edges: OK
- Loops: NOT OK

Pseudograph Definition

A **pseudograph** is a *multigraph* where loops are allowed

- Multiple edges: OK
- Loops: OK

Digraph Definition

A **directed graph** is a graph where $E(G)$ is a set of ordered pairs (rather than sets) of distinct vertices called directed edges, or arcs.

Oriented Graph Definition

An **oriented graph** is a *digraph* in $\forall u, v \in V(G)$, (u, v) and (v, u) are not <u>both</u> edges.

2 Degrees

2.1 Degree of a Vertex

Vertex Degree Definition

The **degree** of a vertex v, denoted as $\deg v$ or $\deg_G v$, is the number of edges incident with v. If the $\deg v = 0$, then v is an **isolated vertex**. If $\deg v = 1$, then v is a **leaf**.

- $\delta(G) = \min\{\deg v \mid v \in V(G)\}$, the minimum degree of G
- $\Delta(G) = \max\{\deg v \mid v \in V(G)\}$, the maximum degree of G

For any graph G and $v \in V(G)$,

$$0 \leq \delta(G) \leq \deg v \leq \Delta(G) \leq n - 1.$$

Neighborhood of a Vertex Definition

The **neighborhood** of a vertex v, denoted as $N(v)$, is the set of all vertices adjacent to v. So $|N(v)| = \deg v$.

Handshaking Theorem

For a graph G of size m, the total degree of G $\sum_{v \in V(G)} = 2m$.

Handshaking Corollary

Every graph has an even number of odd degree vertices.

Sum Degree and Connectivity Theorem

Consider graph G of order n. If $\deg u + \deg v \geq n - 1$ for all non-adjacent $(u, v) \in V(G)$, then G is connected.

Sum Degree and Connectivity Corollary

If G has order n and $\delta(G) \geq \frac{n-1}{2}$, then G is connected.

2.2 Regular Graphs

Regular Graph Definition

Graph G is **regular** if every vertex has the same degree. Graph G is r-**regular** if every vertex has degree r.

Regular Graph Existence Theorem

Let $r, n \in \mathbb{Z}$ such that $0 \leq r \leq n - 1$. Then there exists an r-regular graph of order n if and only if at least one of r and n is even.

Harary Graph

An Harary Graph, denoted as $H_{r,n}$, is an r-regular graph of order n.

Induced Regular Subgraph Theorem

For every graph G, and every integer $r \geq \Delta(G)$, there exists on r-regular graph H, containing G as an induced subgraph.

2.3 Degree Sequences

Degree Sequence Definition

A **degree sequence** is a sequence of the degree of the vertices of a graph, typically, written in largest to smallest order.

Graphical Degree Sequence Definition

A finite sequence of non-negative integers is **graphical** if it is the degree sequence of some graph.

Graphical Degree Sequence Theorem

A non-increasing sequence $S : d_1, d_2, \ldots, d_n$, where $n \geq n$, of non-negative integers is graphical if and only if

$$S_1 : d_2 - 1, d_3 - 1, \ldots, d_{d_1+1}, d_{d_1+2}, d_n$$

is graphical.

2.4 Graph and Matrices

Adjacency Matrix Definition

The **adjacency matrix** of G is the $n \times n$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise;} \end{cases}$$

The entry a_{ij} in A^n is the number of walks of length n from v_i to v_j.

3 Isomorphic Graphs

3.1 The Definition of Isomorphism

Graph Equality Definition

Two graphs are **equal**, denoted as $G = H$, if $V(G) = V(H)$ and $E(G) = E(H)$.

Graph Isomorphic Definition

Two (labels) graphs G and H are **isomorphic**, denoted as $G \cong H$, if they have the same structure, meaning there is a *bijection* $\phi : V(G) \to V(H)$ such that for $u, v \in V(G), \phi(u)\phi(v) \in E(H)$ if and only if $uv \in E(G)$.

Isomorphic Degree Theorem

If $G \cong H$, with isomorphic $\phi : V(G) \to V(H)$, then $\deg_G u = \deg_H \phi(u)$.

Isomorphic Degree Corollary

If $G \cong H$, their degree sequences are equal.

Graph Invariants

- To prove $G \cong H$, find an isomorphism.
- To prove $G \ncong H$, find a graph invariant, where G and H differ.

Graph Invariants

- Order and Size
- Degree Sequence
- Cycles
- Diameter
- k (number of components)
- k-partite-ness
- (Other things)

Adjacency and Non-adjacency under Isomorphism Theorem

$G \cong H$ if and only if $\overline{G} \cong \overline{H}$.

3.2 Isomorphism as a Relation

Equivalence Relations and Isomorphism Theorem

Isomorphism is an equivalence relation.

4 Trees

4.1 Cut Edges

Cut-edge and Bridge Definition

An edge e of graph G is a **cut-edge**, or **bridge**, if $G - e$ has more components than G.

Cut-edges and Cycles Theorem

An edge e of a graph G is a cut-edge if and only if e lies on *no* cycle in G.

4.2 Trees

Tree and Forest Definitions

A **tree** is an acyclic connected graph. A **forest** is an acyclic graph, where each component is a *tree*. A **Rooted tree** is a tree with a specific vertex designated as a root and drawn down.

Every edge of a tree is a cut-edge.

Unique Path in Trees Theorem

Graph G is a tree if and only if every 2 vertices are connected by a unique path.

Leaf Theorem

Every nontrivial tree has at least 2 leaves.

Autumn Theorem

If tree T has order $t \geq 1$, then $T - v$, where v is a leaf, is a tree of order $t - 1$.

Tree Size Theorem

Every tree of order n has size $n - 1$.

Forest Size Theorem

Every forest of order n with k components has size $n - k$.

Minimum Size of a Connected Graph Theorem

The size of every connected graph of order n is at least $n - 1$. Trees has minimal size among connected graphs of given order.

Tree Requirements Graph

Graph G of order n and size m. Then G is a tree if it satisfies any 2 of these properties:

1. G is connected
2. G acyclic
3. $m = n - 1$

Tree Isomorphic Subgraph Theorem

Let T be a tree of order k. Then for any graph G with $\delta(G) \geq k - 1$, T is isomorphic to a subgraph of G.

4.3 Minimum Spanning Tree

Spanning Tree Definition

Let G be a connected graph. A spanning subgraph of G that is a tree is called a **spanning tree**.

Spanning Tree Existence Theorem

Every connected graph contains a spanning tree.

Minimum Spanning Tree Definition

A **minimum spanning tree** is a spanning tree of minimum weight.

Algorithms For Constructing Minimum Spanning Trees

Kruskal's Algorithm

1. Pick an edge of minimum weight.
2. Repeat, never allowing the chosen edges to produce a cycle.
3. Stop once you have a spanning tree.

Prim's Algorithm

1. Choose any vertex $u \in V(G)$.
2. Let e be an edge of minimum weight incident with u.
3. Continue picking edges of minimum weight weight from the set of edges having exactly one of its vertices incident with an already selected edge.
4. Stop once you have a spanning tree.

4.4 Counting Labeled Trees

Cayley's Theorem

There are n^{n-2} distinct labeled trees on n vertices.

Prüfer Sequence

Encoding a Tree to a Sequence

1. Start with a labeled tree T, and $i = 1$.
2. Let b_i = smallest label on a leaf.
3. Let a_i = label of the adjacent vertex of b_1.
4. Remove b_i and record a_i in the sequence.
5. Repeat with b_{i+1} and a_{i+1}.
6. Stop once only vertices remain.

Decoding a Sequence to a Tree

1. Start with $(a_1, \dots, a_{n-2})$ and $i = 1$.
2. Let b_i = smallest element of $\{1, \dots, n\}$ **not** in the sequence.

3. Draw edge $a_i b_i$.
4. Remove a_i from the sequence and b_i from the set.
5. Repeat with b_{i+1} and a_{i+1}.
6. Stop once the sequence is empty, and draw an edge between the last two elements in the set.

5 Connectivity

5.1 Cut Vertices

Cut-edge and Cut-vertex Definition

- **Cut-edge**: Removing cut-edge e creates a new component.
- **Cut-vertex**: Removing cut-vertex v creates new components(s).

Leaves and Cut-vertices Theorem

Let G be a connected graph with cut-edge $e = uv$. v is a cut-vertex if and only if $\deg v \geq 2$, meaning that v is not a leaf.

Leaves and Cut-vertices Corollary 1

Every vertex of a non-trivial tree is either a leaf of a cut-vertex.

Leaves and Cut-vertices Corollary 2

Let G be a connected graph and of order at least 3. If G contains a cut-edge, then G contains a cut-vertex.

Paths and Cut-vertices Theorem

Let G be a connected graph with cut-vertex v. Let u, w be vertices in different components of $G - v$. Then v lies on every $u - w$ path in G.

Paths and Cut-vertices Corollary

Let G be connected. $v \in V(G)$ is a cut-vertex if and only if $\exists\, u, w \in V(G) - \{v\}$ such that v lies on every $u - w$ path in G.

Non-cut-vertex Theorem

Every nontrivial connected Graph contains at least 2 vertices that are not cut-vertices.

5.2 Blocks

Non-separable Definition

A graph is called **non-separable** if...

1. it is nontrivial,
2. it is connected,
3. it has no cut-vertices, meaning every edge is on a cycle.

Otherwise, it is called **separable**.

Common Cycle and Non-separability Theorem

A graph of order at least 3 is non-separable if and only if every 2 vertices (pairwise) lie on a common cycle.

Block Definition

A **block** of G is a maximal, *non-separable* subgraph of G.

Blocks are Equivalence Relations Theorem

Define a Relation R on $E(G)$ where eRf if $e = f$ or e and f lie on a common cycle of G. R is an equivalence relation, where equivalence classes of R are edge-induced blocks of G.

Blocks are Equivalence Relations Corollary

Let B_1 and B_2 be distinct blocks in a nontrivial connected graph G. Then,

1. $E(B_1) \cap E(B_2) = \emptyset$, meaning B_1 and B_2 are edge disjoint.
2. B_1 and B_2 have at most 1 vertex in common.
3. The common vertex, if is exists, is a cut-vertex.

5.3 Connectivity

Vertex-cut and Minimum Vertex-cut Definition

- A **vertex-cut** is a set $U \subseteq V(G)$ such that $G - U$ is disconnected.
- A **minimum vertex-cut** is a *vertex-cut* of minimum cardinality.

Connectivity Definition

The **connectivity** of graph G is

$$\kappa(G) = \min\{|U| \mid U \subseteq V(G), \text{ such that } G - U \text{ is disconnected or trivial.}\}$$

Note that $0 \leq \kappa(G) \leq n - 1$.

k-connectivity Definition

G is called k-connected if $\kappa(G) \geq k$.

Edge-cut, Minimal, and Minimum Edge-cut Definition

- An **edge-cut** is a set $X \subseteq E(G)$ such that $G - X$ is disconnected.
- A **minimal edge-cut** is an *edge-cut* X where no proper subset of X is also an edge-cut.
- A **minimum edge-cut** is an *edge-cut* of minimum cardinality.

Edge-connectivity Definition

The **edge-connectivity** of a nontrivial graph G is

$$\lambda(G) = \min\{|X| \mid X \subseteq E(G), \text{ such that } G - X \text{ is disconnected or trivial.}\}$$

Note that $0 \leq \lambda(G) \leq n - 1$.

k-edge-connectivity Definition

G is called k-edge-connected if $\lambda(G) \geq k$.

Edge-connectivity of Complete Graphs Theorem

$$\forall\, n \in \mathbb{N},\ \lambda(K_n) = n - 1$$

Connectivity and Edge-connectivity Ordering Theorem

For a graph G,

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

IMPORTANT: These proofs involve taking minimum-edge or minimum-vertex cuts and comparing their cardinalities.

Cubic Connectivity Theorem

If graph G is 3-regular, also called *cubic*, then

$$\kappa(G) = \lambda(G)$$

Upper Bound for Connectivity Theorem

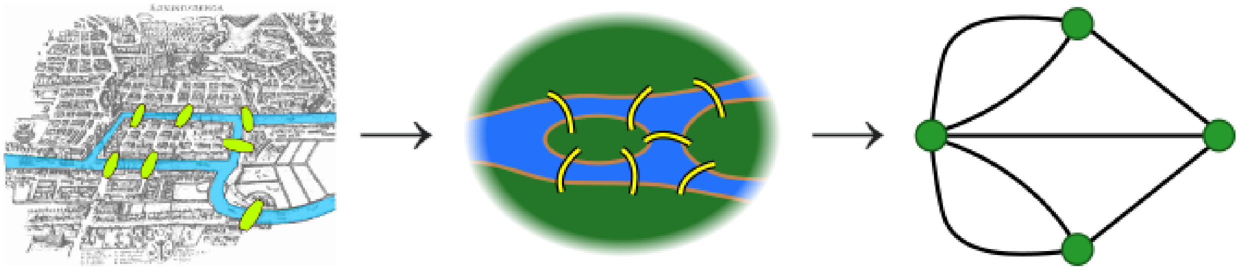
If G has order n and size $m \geq n - 1$, then

$$\kappa(G) \leq \lfloor \frac{2m}{n} \rfloor$$

6 Traversability

6.1 Eulerian Graphs

Seven Bridges of Königsberg Problem



Can you go for a walk, crossing each bridge exactly once? No

Eulerian Circuits and Trails Definition

- A **Eulerian Circuit** is a circuit containing every edge of graph G.
- A **Eulerian Trail** is an open trail containing every edge of graph G.
- A **Eulerian Graph** is a graph that contains an *Eulerian Circuit*

Even Degree and Eulerian Circuits Theorem

A nontrivial, connected graph is Eulerian if and only if *every* vertex has *even* degree.

Even Degree and Eulerian Trails Corollary

A connected graph G contains an Eulerian Trail, if and only if exactly 2 vertices of G have odd degree.

6.2 Hamiltonian Graphs

Hamiltonian Cycles and Paths Definition

- A **Hamiltonian Cycle** is a cycle containing every vertex of graph G.
- A **Hamiltonian Path** is a path containing every vertex of graph G.
- A **Hamiltonian Graph** is a graph that contains an *Hamiltonian Cycle*.

Degree Sum and Hamiltonian Graphs Theorem

Let G have order $n \geq 3$. If $\deg u + \deg v \geq n$ for all pairs of nonadjacent vertices $u, v \in V(G)$, then G is Hamiltonian. Note that this is only a *one way* statement.

Degree Sum and Hamiltonian Graphs Corollary

Let G have order $n \geq 3$. If $\deg v \geq \frac{n}{2}$ for all $v \in V(G)$, then G is Hamiltonian. Note that this is only a *one way* statement.

7 Digraphs

8 Matchings and Factorization

9 Planarity

10 Coloring Graphs

11 Ramsey Numbers

12 Distance

13 Domination