Homework 6

1.2

Determine the values for which the system has no solutions, exactly one solution, or infinitely many solutions

25
$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array}$$

work.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right) \xrightarrow[(-4,-8,12,-16)]{R_3-4R_1} \begin{pmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 0 & -7 & a^2-2 & a-14 \end{pmatrix} \xrightarrow[(-3,-6,9,-12)]{R_2-3R_1}$$

$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{pmatrix} \xrightarrow[(0,7,-14,10)]{R_3-R_2,-\frac{1}{7}R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{array}\right)$$ (same thing as 27)

R_3 represents the equation $(a^2-16)z = a-4$. If $a=4$, there are infinitely many solutions, since R_3 leads to a full row of 0's. If $a=-4$, then there is no solution, since R_3 leads to $0=-8$. If $a \neq \pm 4$, then there is exactly one solution to the system of equations. □

a contradiction

(same as 27)

27
$$\begin{array}{rcrcl} x & + & 2y & = & 1 \\ 2x & + & (a^2-5)y & = & a-1 \end{array}$$

work.

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & a^2-5 & a-1 \end{array}\right) \xrightarrow[(-2,-4,-2)]{R_2-2R_1} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & a^2-9 & a-3 \end{array}\right)$$

the augmented matrix is $\begin{pmatrix}1&2&1\\0&0&0\end{pmatrix}$. Since the last column and the second last column do not have a leading 1, the system has...

R_2 represents the equation $(a^2-9)y = a-3$. If $a=3$, ~~there are~~ infinitely many solutions, since R_2 leads to a full row of 0's. If $a=-3$, then there is no solution, since R_2 leads to $0=-6$. If $a \neq \pm 3$, then there is exactly one solution to the system of equations. □

a contradiction

$\begin{pmatrix}1&2&1\\0&a^2-9&a-3\end{pmatrix} \xrightarrow{\frac{1}{a^2-9}R_2} \begin{bmatrix}1&2&1\\0&1&\frac{1}{a+3}\end{bmatrix}$. This system has a unique solution, since every column except the last has a leading 1.

32 *Reduce* $\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$ *to rref without introducing fractions at any intermediate stage.*

work.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[(-2,-1,-3)]{R_3-R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[(-2,-6,-4)]{R_1-2R_3} \begin{bmatrix} 0 & -5 & -1 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[(0,2,29)]{R_1-R_2}$$

$$\begin{bmatrix} 0 & -3 & 28 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{bmatrix} 0 & -3 & 28 \\ 0 & 1 & -57 \\ 1 & 0 & 30 \end{bmatrix} \xrightarrow[(0,3,-171)]{R_1+3R_2} \begin{bmatrix} 0 & 0 & -143 \\ 0 & 1 & -57 \\ 1 & 0 & 30 \end{bmatrix} \xrightarrow[R_2+57R_1,R_3-30R_1]{-\frac{1}{143}R_1,R_1\leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

1.3

5h *Calculate* $(C^TB)A^T$, *where* $A = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}$.

work.

$$\begin{pmatrix}1 & 4 & 2\\ 3 & 1 & 5\end{pmatrix} \xrightarrow{C^T} \begin{pmatrix}1 & 3\\ 4 & 1\\ 2 & 5\end{pmatrix} \xrightarrow{C^TB} \begin{pmatrix}1\cdot 4+3\cdot 0 & 1\cdot -1+3\cdot 2\\ 4\cdot 4+1\cdot 0 & 4\cdot -1+1\cdot 2\\ 2\cdot 4+5\cdot 0 & 2\cdot -1+5\cdot 2\end{pmatrix} = \begin{pmatrix}4 & 5\\ 16 & -2\\ 8 & 8\end{pmatrix} \xrightarrow{C^TBA^T}$$

$$\begin{pmatrix}4 & 5\\ 16 & -2\\ 8 & 8\end{pmatrix}\begin{pmatrix}3 & -1 & 1\\ 0 & 2 & 1\end{pmatrix} \xrightarrow{C^TBA^T} \begin{pmatrix}4\cdot 3+5\cdot 0 & 4\cdot -1+5\cdot 2 & 4\cdot 1+5\cdot 1\\ 16\cdot 3-2\cdot 0 & 16\cdot -1-2\cdot 2 & 16\cdot 1-2\cdot 1\\ 8\cdot 3+8\cdot 0 & 8\cdot -1+8\cdot 2 & 8\cdot 1+8\cdot 1\end{pmatrix} = \begin{pmatrix}12 & 6 & 9\\ 48 & -20 & 14\\ 24 & 8 & 16\end{pmatrix}$$

□

10 $A = \begin{bmatrix}3 & -2 & 7\\ 6 & 5 & 4\\ 0 & 4 & 9\end{bmatrix}$, $B = \begin{bmatrix}6 & -2 & 4\\ 0 & 1 & 3\\ 7 & 7 & 5\end{bmatrix}$, $AB = \begin{bmatrix}67 & 41 & 41\\ 64 & 21 & 59\\ 63 & 67 & 57\end{bmatrix}$, *and* $BA = \begin{bmatrix}6 & -6 & 70\\ 6 & 17 & 31\\ 63 & 41 & 122\end{bmatrix}$.

a *express each column vector of AB as a linear combination of the column vectors of A.*

work. Since each column vector of AB is computed using a row of A and a column of B, the linear combination will simply be the corresponding column of B.

1. $6\begin{pmatrix}3\\6\\0\end{pmatrix} + 0\begin{pmatrix}-2\\5\\4\end{pmatrix} + 7\begin{pmatrix}7\\4\\9\end{pmatrix} = \begin{pmatrix}67\\64\\63\end{pmatrix}$.
2. $-2\begin{pmatrix}3\\6\\0\end{pmatrix} + 1\begin{pmatrix}-2\\5\\4\end{pmatrix} + 7\begin{pmatrix}7\\4\\9\end{pmatrix} = \begin{pmatrix}41\\21\\67\end{pmatrix}$.
3. $4\begin{pmatrix}3\\6\\0\end{pmatrix} + 3\begin{pmatrix}-2\\5\\4\end{pmatrix} + 5\begin{pmatrix}7\\4\\9\end{pmatrix} = \begin{pmatrix}41\\59\\57\end{pmatrix}$.

□

b *express each column vector of BA as a linear combination of the column vectors of B.*

work. Since each column vector of BA is computed using a row of B and a column of A, the linear combination will simply be the corresponding column of A.

1. $3\begin{pmatrix}6\\0\\7\end{pmatrix} + 6\begin{pmatrix}-2\\1\\7\end{pmatrix} + 0\begin{pmatrix}4\\3\\5\end{pmatrix} = \begin{pmatrix}6\\6\\63\end{pmatrix}$.
2. $-2\begin{pmatrix}6\\0\\7\end{pmatrix} + 5\begin{pmatrix}-2\\1\\7\end{pmatrix} + 4\begin{pmatrix}4\\3\\5\end{pmatrix} = \begin{pmatrix}-6\\17\\41\end{pmatrix}$.
3. $7\begin{pmatrix}6\\0\\7\end{pmatrix} + 4\begin{pmatrix}-2\\1\\7\end{pmatrix} + 9\begin{pmatrix}4\\3\\5\end{pmatrix} = \begin{pmatrix}70\\31\\122\end{pmatrix}$.

□

15 *Find all values of k, if any, that satisfy* $\begin{bmatrix}k & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 2\\ 0 & 2 & -3\end{bmatrix}\begin{bmatrix}k\\1\\1\end{bmatrix}$. $= [0]$

work.

$$\begin{bmatrix}k & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 2\\ 0 & 2 & -3\end{bmatrix}\begin{bmatrix}k\\1\\1\end{bmatrix} = \begin{bmatrix}k+1 & k+2 & -1\end{bmatrix}\begin{bmatrix}k\\1\\1\end{bmatrix} = [k(k+1)+k+2-1] = \left[(k+1)^2\right] = [0]$$

$$k+1 = 0$$
$$k = -1$$

□

22 *description*

24 *description*

27 *description*

1.4

17 *description*

19e *description*

27 ~~*description*~~

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{ii} \end{pmatrix}$$

let

$$B = \begin{pmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{ii}} \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I_i$$

$$BA = \begin{pmatrix} 1 & & \cdots & 0 \\ & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I_i$$

$$\therefore \ B = A^{-1}$$