Section 2 Binary Operations, p25 7,9,11,17,19,21,23

In Exercises 7 through 11, determine whether the binary operation $*$ defined is commutative and whether $*$ is associative.

7. $*$ defined on $\mathbb{Z}$ by letting $a * b = a - b$

$*$ is neither commutative nor associative.

(a) Commutative: Consider 1 and 2. $1 - 2 = -1 \neq 2 - 1 = 1$. Thus $*$ is not commutative.

(b) Associative: Consider $1 - (4 - 3) = 0 \neq (1 - 3) - 3 = -5$. Thus $*$ is not commutative.

9. $*$ defined on $\mathbb{Q}$ by letting $a * b = ab/2$

$*$ is both commutative and associative.

(a) Commutative: $a * b = \frac{ab}{2} = \frac{ba}{2} = b * a$. Thus $*$ is commutative.

(b) Associative: Consider a, b, c.

$$a * (b * c) = a * \frac{bc}{2} = \frac{a\frac{bc}{2}}{2} = \frac{1}{4} \cdot abc$$
$$(a * b) * c = \frac{ab}{2} * c = \frac{\frac{ab}{2}c}{2} = \frac{1}{4} \cdot abc$$

Thus, $*$ is associative.

11. $*$ defined on $\mathbb{Z}^+$ by letting $a * b = a^b$

$*$ is neither commutative nor associative.

(a) Commutative: $2 * 3 = 2^3 = 8 \neq 3 * 2 = 3^2 = 9$. Thus $*$ is not commutative.

(b) Associative: Consider $2, 3, 3$.

$$2 * (3 * 3) = 2 * 9 = 2^9 = 512$$
$$(2 * 3) * 3 = 6 * 3 = 6^3 = 216$$

Thus $*$ is not associative.

In Exercises 17 through 22, determine whether the definition of $*$ does give a binary operation on the set. In the event that $*$ is *not* a binary operation, state whether condition 1 (uniquely defined), condition 2 (closed), or both of these conditions are violated.

17. On $\mathbb{Z}^+$, define $*$ by letting $a * b = a - b$.

$*$ is not a binary operation on $\mathbb{Z}^+$. Consider $a = 1$ and $b = 2$: $1 * 2 = 1 - 2 = -1 \notin \mathbb{Z}^+$. Thus $\mathbb{Z}^+$ is not closed under $*$. It is, however, uniquely defined for all $a, b \in \mathbb{Z}^+$.

19. On $\mathbb{R}$, define $*$ by letting $a * b = a - b$.

$*$ is a binary operation on $\mathbb{R}$. A real number minus a real number is always a real number, and always produces a single, definitive answer.

21. On $\mathbb{Z}^+$, define $*$ by letting $a * b = c$, where c is at least 5 more than $a + b$.

$*$ is not a binary operation on $\mathbb{Z}^+$. Consider $a = 1$ and $b = 1$: $1 * 1 = 7$ and 8. Thus $*$ is not uniquely defined. It is, however, closed.

23 Let H be the subset of $M_2(\mathbb{R})$ consisting of all matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{R}$. Is H closed under

a. matrix addition?

Consider $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$.

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix}$$

Since this take the form of H, H is closed under matrix addition.

b. matrix multiplication?

Consider $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$.

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -ad - bc \\ bc + ad & -bd + ac \end{bmatrix} = \begin{bmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{bmatrix}$$

Since this takes the form of H, H is closed under matrix multiplication.